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# ELEMENTARY STRUCTURAL ANALYSIS

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# PREFACE

In this book the authors have covered that subject matter which they believe to be of importance in a well-organized sequence of undergraduate courses dealing with the theory of structures for civil engineers. Some material is also included which, because of restricted schedules, may have to be left out of formal assignments. Such portions of the book include the material in fine print throughout the text; Chap. 16, which deals with structures not directly related to civil engineering; and Chaps. 17–19, which treat the analysis of structures by means of models. Such material will, however, serve to develop the structural knowledge of the student whose interest has become genuinely aroused, and it will be valuable in connection with thesis work.

The material covered by this book has been confined almost entirely to methods of stress analysis. Design procedure, where mentioned at all, is covered only incidentally, since the length of the book as written indicates the desirability of a separate book dealing with design, and since our practice, which is also followed in a number of other schools, consists of teaching the theory of stress analysis and the principles of design as separate subjects.

The authors have attempted in this presentation to accomplish two results: (1) to tie in the various procedures of structural analysis with the principles of applied mechanics on which they are based, thus showing that the theory of structural analysis is but one phase of advanced applied mechanics, and (2) to show that the methods of analysis derived for civil engineering structures are applicable in principle to structures lying outside the field of practice of most civil engineers. With these thoughts in mind, they hope this book will prove to be of help both to students of structural engineering and to young practicing engineers.

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# CHAPTER 1

## INTRODUCTION

**1·1  Engineering Structures.**  The design of bridges, buildings, towers, and other fixed structures is very important to the civil engineer. Such structures are composed of interconnected members and are supported in a manner such that they are capable of holding applied external forces in static equilibrium.  A structure must also hold in equilibrium the gravity forces that are applied as a consequence of its own weight. A transmission tower, for example, is acted upon by its own weight, by wind and ice loads applied directly to the tower, and by the forces applied to the tower by the cables that it supports.  The members of the tower must be so arranged and designed that they will hold these forces in static equilibrium and thus transfer their effects to the foundations of the tower.

There are many kinds of structures in addition to those mentioned above.  Dams, piers, pavement slabs for airports and highways, penstocks, pipe lines, standpipes, viaducts, and tanks are all typical *civil engineering* structures.  Nor are structures of importance only to the *civil engineer.*  The structural frame of an aircraft is important to the *aeronautical engineer;* the structure of a ship receives particular attention from the *naval architect;* the *chemical engineer* is concerned with the structural design of high-pressure vessels and other industrial equipment; the *mechanical engineer* must design machine parts and supports with due consideration of structural strength; and the *electrical engineer* is similarly concerned with electrical equipment and its housing.

The analysis of all these structures is based, however, on the same fundamental principles.  In this book the illustrations used to demonstrate the application of these principles are drawn largely from civil engineering structures, but the methods of analysis described can be used for structures that are important in other branches of engineering.

**1·2  General Discussion of Structural Design.**  A structure is designed to perform a certain function.  To perform this function satisfactorily it must have sufficient strength and rigidity.  Economy and good appearance are further objectives of major importance in structural design.

The complete design of a structure is likely to involve the following five stages:

1. Establishing the general layout to fit the functional requirements of the structure

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2. Consideration of the several possible solutions that may satisfy the functional requirements
3. Preliminary structural design of the various possible solutions
4. Selection of the most satisfactory solution, considering an economic, functional, and aesthetic comparison of the various possible solutions
5. Detailed structural design of the most satisfactory solution

Both the preliminary designs of stage 3 and the final detailed design of stage 5 may be divided into three broad phases, although in practice these three phases are usually interrelated. First, the loads acting on the structure must be determined. Next, the maximum stresses in the members and connections of the structure must be analyzed. Finally, the members and connections of the structure must be dimensioned, *i.e.*, the make-up of each part of the structure must be determined.

That these three steps are interrelated may be seen from considerations such as the following: The weight of the structure itself is one of the loads that a structure must carry, and this weight is not definitely known until the structure is fully designed; in a statically indeterminate structure, the stresses depend on the elastic properties of the members, which are not known until the main members are designed. Thus, in a sense, the design of any structure proceeds by successive approximations. For example, it is necessary to assume the weights of members in order that they may be properly designed. After the structure is designed, the true weights may be computed; and unless the true weights correspond closely to those assumed, the process must be repeated.

In designing a structure, it is important to realize that each part must have sufficient strength to withstand the maximum stress to which it can be subjected. To compute such maximum stresses, it is necessary to know, not only *what* loads may act, but the exact *position* of these loads on the structure that will cause the stress under consideration to have its maximum value.

Thus, when a railroad locomotive crosses a bridge, a given portion of the bridge receives its maximum stress with the locomotive at a given position on the bridge. A second part of the structure may be subjected to its maximum stress with the locomotive in another position.

In this book, the emphasis is placed on the stress analysis of structures. But, in order to discuss stress analysis satisfactorily, it is desirable to give some attention to the loads acting on a structure and to the design of members and connections.

**1·3   Dead Loads.** The dead load acting on a structure consists of the weight of the structure itself and of any other immovable loads

that are constant in magnitude and permanently attached to the structure. Thus, for a highway bridge, the dead load consists of the main supporting trusses or girders, the floor beams and stringers of the floor system, the roadway slabs, the curbs, sidewalks, fences or railings, lampposts, and other miscellaneous equipment.

Since the dead load acting on a member must be assumed before the member is designed, one should design the members of a structure in such a sequence that, to as great an extent as is practicable, the weight of each member being designed is a portion of the dead load carried by the next member to be designed. Thus, for a highway bridge, one would first design the road slab, then the stringers that carry the slab loads to the floor beams, then the floor beams that carry the stringer loads to the main girders or trusses, and finally the main girders or trusses.

In designing a member such as a floor slab, stresses due to dead loads are likely to be only a small percentage of the total stress in a member, so that, even if dead loads are not very accurately estimated, the total stress can be predicted with fair accuracy and hence the first design be quite satisfactory. For main trusses and girders, however, the dead

<div align="center">Table 1·1</div>

| Material | Weight, lb per cu ft |
|---|---|
| Steel or cast steel | 490 |
| Cast iron | 450 |
| Aluminum alloys | 175 |
| Timber (treated or untreated) | 50 |
| Concrete (plain or reinforced) | 150 |
| Compacted sand, earth, gravel, or ballast | 120 |
| Loose sand, earth, and gravel | 100 |
| Macadam or gravel, rolled | 110 |
| Cinder filling | 60 |

loads constitute a greater portion of the total load to be carried, so that it is more important to make a reasonably accurate first estimate of dead weights. Often data concerning the dead weights of other similar structures will serve as a guide to the designer. Many investigations have been carried out with the purpose of presenting such data in a convenient form.[1] It should be emphasized, however, that the original dead-weight estimate is tentative, whatever the source of the data may be. After a

[1] The student is referred to p. 72 of "Structural Theory" (John Wiley & Sons, Inc., New York, 1942) by H. SUTHERLAND and H. L. BOWMAN for tables giving the weights of roof trusses and to p. 84 of the same book for an excellent summary of formulas giving the weights of bridges. Charts dealing with the weights of railroad bridges, highway bridges, and signal bridges are given in Chap. I of C. M. Spofford's "Theory of Structures," 4th ed., McGraw-Hill Book Company, Inc., New York, 1939.

structure is designed, its actual dead weight should be accurately computed and the stress analysis and design revised as necessary.   This is necessary for safety and desirable for economy.

If the dimensions of a structure are known, dead loads may be computed on the basis of unit weights of the materials involved.  Unit weights for some of the materials commonly used in engineering structures are given in Table 1·1.

Unit weights for other materials are readily available in many books and handbooks.[1]

**1·4  Live Loads—General.**  As contrasted to dead loads, which remain fixed in both magnitude and location, it is usually necessary to consider live loads, *i.e.*, loads that vary in position.  It is sometimes convenient to classify live loads into movable loads and moving loads. Movable loads are those which may be moved from one position to another on a structure, such as the contents of a storage building.  They are usually applied gradually and without impact.  Moving loads are those which move under their own power, such as a railroad train or a series of trucks.  They are usually applied rather rapidly and therefore exert an impact effect on the structure.

When live loads are involved, attention must be given to the placing of such loads on a structure so that the stress in the structural member or connection under consideration will have its maximum possible value. Thus, while we speak of dead stresses due to dead loads, we refer to maximum live stresses due to live loads.

**1·5  Live Loads for Highway Bridges.**  The live load for highway bridges consists of the weight of the applied moving load of vehicles and pedestrians.  The live load for each lane of the roadway consists of a train of heavy trucks following each other closely.  The weight and weight distribution of each truck vary with the specification under which one designs, but a typical example is afforded by the H-series trucks specified by the American Association of State Highway Officials (AASHO).

These H-series trucks are illustrated in Fig. 1·1.  They are designated H, followed by a number indicating the gross weight in tons for the standard truck.  The choice as to which of the H-series trucks shall be used for the design of a given structure depends on circumstances such as the importance of the bridge and the expected traffic.  Actually, the traffic over a highway bridge will consist of a multitude of different types of vehicles.  It is designed, however, for a train of standard trucks, so chosen that the bridge will prove safe and economical in its actual performance.

[1] The student is referred, for example, to the section on Weights and Specific Gravities in "Steel Construction," American Institute of Steel Construction, New York.

It is seen that the loading per lane of roadway consists of a series of concentrated wheel loads.   The stress analysis involved in computing maximum live stresses due to a series of concentrated live loads may become rather complicated.   Under some conditions it is permissible to substitute for purposes of stress analysis an equivalent loading consisting of a uniform load per foot of lane, plus a single concentrated load.   Thus, for the H-20 loading, the equivalent live load consists of a uniform load of 640 lb per lin ft of lane, plus a concentrated load of either 18,000 lb or 26,000 lb, depending on whether live moments or shears, respectively, are being computed.   This equivalent live load is not exactly equivalent
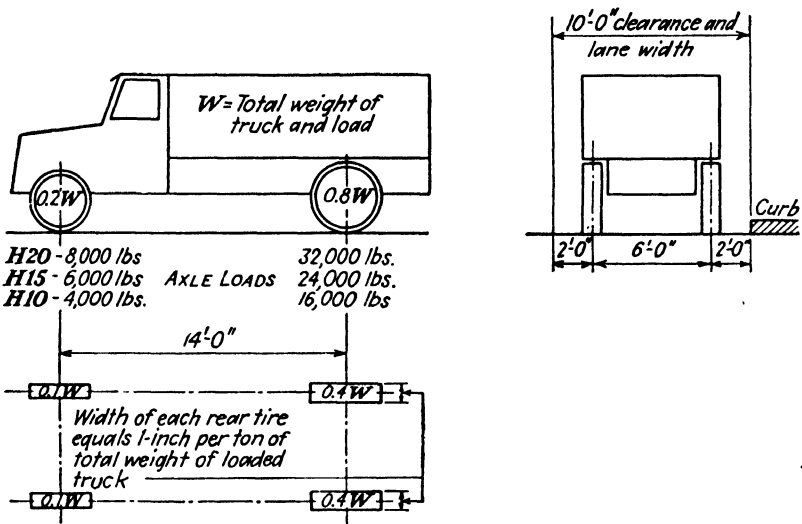


Fig. 1·1

to the series of concentrated wheel loads, but it permits a simpler computation of maximum stresses that correspond closely enough to those which would be computed from the actual loads to be used for design purposes.

It is often necessary to design a highway bridge to carry electric-railway cars.   Specifications define the wheel loads and spacings to be used for this purpose.

**1·6   Live Loads for Railroad Bridges.**   The live load for railroad bridges consists of the locomotives and cars that cross them.   The live load for each track is usually taken as that corresponding to two locomotives followed by a uniform load which represents the weight of the cars.   To standardize such loadings a series of E-loadings was devised by Theodore Cooper.   These loadings are designated by the letter E,

followed by a number indicating the load in kips[1] on the driving axle. The loads on other axles always bear the same ratio to the load on the driving axle. The uniform load following the two locomotives always has an intensity per foot of track equal to one-tenth the load on the driving axle. The wheel spacings do not vary with the Cooper's rating. Figure 1·2 illustrates a Cooper's E-50 loading. Modern railroad bridges are designed for at least an E-60 loading and often for an E-70 or even a heavier loading. It should be noted that the wheel loads for these
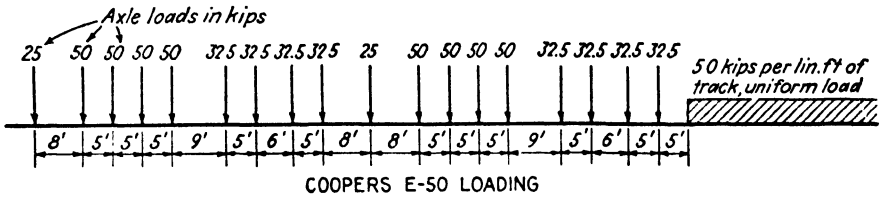


COOPERS E-50 LOADING
Fig. 1·2

heavier Cooper's loadings can be obtained from Fig. 1·2 by direct proportion.

Simplified equivalent loadings are sometimes used in place of actual wheel loadings to represent live loads on railroad bridges.

**1·7 Live Loads for Buildings.** Live loads for buildings are usually considered as movable loads of uniform intensities. The intensity of the floor loads to be used depends on the purpose for which the building is designed, as indicated in Table 1·2.[2]

**Table 1·2**

|  | Minimum Live Load, lb per sq ft |
|---|---|
| Human occupancy: |  |
| Private dwellings, apartment houses, etc | 40 |
| Rooms of offices, schools, etc. | 50 |
| Aisles, corridors, lobbies, etc., of public buildings | 100 |
| Industrial or commercial occupancy: |  |
| Storage purposes (general) | 250 |
| Manufacturing (light). | 75 |
| Printing plants | 100 |
| Wholesale stores (light merchandise). | 100 |
| Retail salesrooms (light merchandise) | 75 |
| Garages |  |
| All types of vehicles. | 100 |
| Passenger cars only | 80 |
| Sidewalks . . . . . 250 lb per sq ft or 8,000 lb concentrated, whichever gives the larger moment or shear |  |

[1] To facilitate computations, loads are usually given in units of kips, 1 kip being equal to 1,000 lb.

[2] "Steel Construction," *op. cit.*

When floors are to carry special live loads of known intensities greater than those suggested above, these special loads should of course be used in design.

**1·8  Impact.**  Unless live load is applied gradually, the distortion of the structure to which the live load is applied is greater than it would be if the live load were considered as a static load.  Since the distortion is greater, the stresses in the structure are higher.  The increase in stress due to live load over and above the value that this stress would have if the live load were applied gradually is known as impact stress.  Impact stresses are usually associated with moving live loads.  For purposes of structural design, impact stresses are usually obtained by multiplying the live-load stresses by a fraction called the impact fraction, which is specified rather empirically.  The determination of a wholly rational fraction for this purpose would be very complicated, since it depends on the time function with which the live load is applied, the portion of the structure over which the live load is applied, and the elastic and inertia properties of the structure itself.

For highway bridges, the impact fraction $I$ is given in the specifications of the AASHO by

$$I = \frac{50}{L + 125} \qquad \text{but not to exceed 0.300} \qquad (1\cdot1)$$

in which $L$ is the length in feet of the portion of the span loaded to produce the maximum stress in the member considered.  For example, suppose that the maximum live positive shear at the center of a 100-ft longitudinal girder of a highway bridge equals 1,000,000 lb and occurs with the live load extending over half the 100-ft span.  Then the loaded length $L$ is 50 ft; the impact fraction $I = 50/(50 + 125) = 0.286$; the impact shear is obtained by multiplying the live shear by the impact fraction and therefore equals $1,000,000 \times 0.286 = 286,000$ lb.  The total effect of the live load, *i.e.*, live shear plus impact shear, is equal to 1,000,-000 lb plus 286,000 lb, or 1,286,000 lb.

The Specifications for the Design and Construction of Steel Railway Bridges, published by the American Railway Engineering Association (AREA), treat impact as follows (note that impact percentage discussed equals 100 times the impact fraction as previously defined):

"To the maximum computed static live-load stresses, there shall be added the impact, consisting of

*a.* The lurching effect:

A percentage of the static live-load stress equal to . . . . . . . . . . . . $\dfrac{100}{S}$

$S$ = spacing, in feet, between centers of longitudinal girders, stringers, or trusses; or length, in feet, of floor beams or transverse girders.

*b.* The direct vertical effect:

With steam locomotives (hammer blow, track irregularities, and car impact) a percentage of the live-load stress equal to

For $L$ less than 100 ft. . . . . . . . . . . . . . . . . . . . $100 - 0.60L$

For $L$ 100 ft or more . . . . . . . . . . . . . . . . . . $\dfrac{1,800}{L - 40} + 10$

With electric locomotives (track irregularities and car impact), a percentage of static live-load stress equal to . . . . . . . . . . . . . $\dfrac{360}{L} + 12.5$

$L$ = length, ft, center to center of supports for stringers, longitudinal girders, and trusses (chords and main members)

or $L$ = length of floor beams or transverse girders, ft, for floor beams, floor-beam hangers, subdiagonals of trusses, transverse girders, and supports for transverse girders"

To illustrate the application of the foregoing impact specification, let us assume that the longitudinal girder described in the previous example is one of the two main girders of a steam-railroad bridge and that these two girders are spaced at 18 ft center to center. Then, for the lurching effect, since $S$ = 18, the percentage impact equals $^{100}\!/_{18}$ = 5.5%; for the direct vertical effect, since $L$ = 100, the percentage impact equals 1,800/(100 − 40) + 10 = 40.0% (note that in this case $L$ = 100 because this is the span of the girder, whereas in the previous example we used $L$ = 50 because the loaded length of the girder was 50 ft); thus the total percentage impact equals 5.5 + 40.0 = 45.5%; the impact shear equals 1,000,000 × 0.455 = 455,000 lb; the total effect of the load, *i.e.*, live shear plus impact shear = 1,000,000 + 455,000 = 1,455,000 lb.

Other specifications give still other rules for determining impact, but the two methods discussed are perhaps the most important of those in common use. They illustrate, moreover, the type of impact equations specified elsewhere.

It is usually unnecessary to consider impact stresses in designing for movable live loads such as the live loads for buildings. Moreover, when a structure is designed of timber, impact is often ignored. This is largely because timber, as a material, is much stronger in resisting loads of short duration than in resisting permanently applied loads, and it therefore can use this reserve of strength to carry impact loads.

**1·9 Snow and Ice Loads.** Snow loads are often of importance, particularly in the design of roofs. Snow should be considered as a movable load, for it will not necessarily cover the entire roof, and some of the members supporting the roof may receive maximum stresses with the snow covering only a portion of the roof. The density of snow, of

course, will vary greatly, as will the fall of snow to be expected in different regions. In a given locality, the depth of snow that will gather on a given roof will depend on the slope of the roof and on the roughness of the roof surface. On flat roofs in areas subjected to heavy snowfalls, snow load may be as large as 45 lb per sq ft. Whether or not snow and wind loads should be assumed to act simultaneously on a roof is problematical, since a high wind is likely to remove much of the snow.

Ice loads may also be of importance, as, for example, in designing a tower built up of relatively small members which have proportionately large areas on which ice may gather. Ice having a density equal approximately to that of water may build up to a thickness of 2 or more inches on such members. It may also build up to much greater thicknesses, but when it does it is apt to contain snow or rime and hence have a lower density. When ice builds up on a member, it alters the shape and the projected area of the member. This should be considered in computing wind loads acting on members covered with ice.

**1·10 Lateral Loads—General.** The loadings previously discussed usually act vertically, although it is not necessary that live loads and their associated impact loads shall act in that direction. In addition, there are certain loads that are almost always applied horizontally, and these must often be considered in structural design. Such loads are called lateral loads. We shall now consider some of the more important kinds of lateral loads.

Wind loads, soil pressures, hydrostatic pressures, forces due to earthquakes, centrifugal forces, and longitudinal forces usually come under this classification.

**1·11 Wind Loads.** Wind loads are of importance, particularly in the design of large structures, such as tall buildings, radio towers, and long-span bridges, and for structures, such as mill buildings and hangars, having large open interiors and walls in which large openings may occur. The wind velocity that should be considered in the design of a structure depends on the geographical location and on the exposure of the structure. For most locations in the United States, a design to withstand a wind velocity of 100 mph is satisfactory.

The AASHO specifies that the wind force on a highway bridge shall be assumed as a movable horizontal load equal to 30 lb per sq ft acting on $1\frac{1}{2}$ times the area of the structure as seen in elevation, including the floor system and railings, and on one-half of the area of all trusses or girders in excess of two in the span. This amounts to specifying 30 lb for each square foot of projected area on the windward truss or girder but only half that amount for other trusses or girders, since they are partly shielded from the wind by the windward portion of the structure.

Specifications for railroad bridges usually call for wind loads comparable to those specified in the foregoing paragraph for highway bridges.

For buildings, the specifications of the American Institute of Steel Construction (AISC) state that the frame of a building must be designed to carry a wind pressure of not less than 20 lb per sq ft on the vertical projection of exposed surfaces during erection and 15 lb per sq ft on the vertical projection of the finished structure.

For buildings with gabled or rounded roofs, more exact data for wind loads on roofs should be used. The student is referred to a report entitled Wind Bracing in Steel Buildings,[1] where, for convenience, roof pressures (and suctions) are expressed in terms of $q$, the velocity pressure, which is defined by

$$q = \tfrac{1}{2}mV^2 \tag{1 2}$$

in which $m$ is the mass of a unit volume of air and $V$ is the velocity of the wind in units corresponding to $m$. For average conditions, and where the wind velocity is expressed in mph, the velocity pressure in pounds per square foot may be taken as

$$q = 0.002558V^2 \tag{1·3}$$

For gabled roofs, it is recommended that the following loadings shall be adopted for the windward slope:

1. For slopes of 20°, or less, a suction (uplift) of $0.7q$
2. For slopes between 20 and 30°, a suction of

$$p = (0.07\alpha - 2.10)q \tag{1 4}$$

in which $\alpha$ is the roof slope in degrees.

3. For slopes between 30 and 60°, a pressure of

$$p = (0.03\alpha - 0.90)q \tag{1 5}$$

4. For slopes steeper than 60°, a pressure of $0.90q$

It is suggested that the wind load on the leeward slope of a gabled roof shall be taken as a suction of $0.6q$ for all slopes. All pressures and suctions are perpendicular to the surfaces on which they act.

The American Society of Civil Engineers (ASCE) report to which reference has been made also discusses the magnitude and distribution of wind pressures on rounded roofs such as are used on hangars.

[1] *Proc. ASCE*, March, 1936, p. 397.

The recommendations of this report are undoubtedly more consistent with the actual aerodynamic forces acting on a roof than the wind forces often used in the design of roofs. The practice often followed in determining wind pressures on surfaces which are not vertical is to follow a formula such as that of Duchemin, whereby $P_n = P \dfrac{2 \sin i}{1 + \sin^2 i}$, in which $P_n$ is the intensity of normal pressure on a given surface, $P$ is the intensity of pressure on a vertical surface, and $i$ is the angle made by the surface with the horizontal. Such a procedure leads to wind pressure on the windward slope of a gable roof and makes no attempt to account for suction on the leeward slope.

In the AISC specifications for wind loads on buildings, it is assumed that the entire pressure specified acts on the windward vertical surface of the building. This procedure is satisfactory for the design of typical tall buildings, although the actual lateral load due to wind consists of pressure on the windward side and suction on the down-wind side.   This matter also is discussed in the ASCE report.

**1·12   Soil Pressures.**   Loads on retaining walls, on walls of buildings, and on other structures due to the pressure of soil must frequently be considered by the structural engineer.   The lateral pressure caused by soil on a wall varies when the wall yields.   After a small movement of the wall, the soil pressure reaches a minimum value known as the active pressure.   If, on the other hand, the wall is forced into the backfill, the pressure between the wall and the backfill increases to a maximum value known as the passive pressure.   Under usual conditions, the active pressure at any depth is about $\frac{1}{4}$ the vertical pressure, and the passive pressure is about 4 times the vertical pressure.   Since the lateral pressure would equal the vertical pressure if the material were a fluid, the approximate values of $\frac{1}{4}$ and 4 are sometimes called "hydrostatic-pressure ratios" for the active and passive cases, respectively.



FIG. 1·3

According to the above discussion, any wall that may yield without detrimental results may be designed on the basis of active pressure, although the pressure that will actually act on the wall will in general be somewhat above this value.   Further, the distribution of pressure over such a wall may be assumed to be triangular, although this assumption is not strictly correct.

For cohesionless soils, the total resultant force corresponding to active soil pressure acting on a strip of wall 1 ft long may be computed on the basis of the theory developed by Coulomb, which assumes failure of soil by rupture along an inclined plane through the soil.   Referring to Fig. 1·3, this total resultant force is denoted by $P$, which acts two-thirds of the way down from the top of the wall, in a direction making the angle $\phi'$ with a line perpendicular to the face of the wall, $\phi'$ being the friction angle of the soil on the masonry.   $H$ is the vertical depth of soil above the base of the wall.   The angle that the surface of the soil makes with the horizontal is defined by $i$.   The batter of the wall is defined by the angle $\theta$.
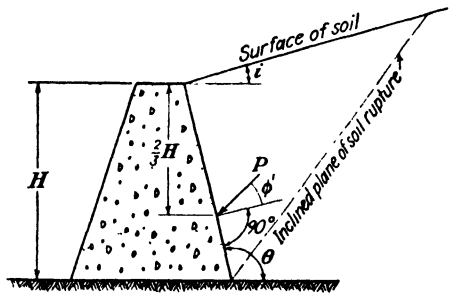
The total force $P$ in pounds, according to Coulomb's theory, is expressed by

$$P = \frac{1}{2}\gamma H^2 \left[ \frac{\csc\theta \sin(\theta - \phi)}{\sqrt{\sin(\theta + \phi')} + \sqrt{\dfrac{\sin(\phi + \phi')\sin(\phi - i)}{\sin(\theta - i)}}} \right]^2 \quad (1\cdot6)$$

where $\gamma$ is the weight of the earth in pounds per cubic foot, $\phi$ is the angle of internal friction of the soil (easily obtained by shear tests in a laboratory, and with typical values of from 30 to 40°), and $\phi'$, the friction angle of soil on masonry, has about the same value as $\phi$ for a rough wall but is somewhat less than $\phi$ for a smooth wall.

If $i = 0$, $\theta = 90°$, and $\phi = \phi'$, Eq. (1·6) reduces to

$$P = \frac{1}{2}\gamma H^2 \left[ \frac{\cos\phi}{(1 + \sqrt{2}\sin\phi)^2} \right] \quad (1\cdot7)$$

If, for example, we consider a sand for which $\phi = \phi' = 30°$, and $\gamma = 100$ lb per cu ft, acting on the back of a vertical wall 10 ft high, the resultant thrust on a strip of wall 1 ft long, by Eq. (1·7), is given by

$$P = \frac{1}{2}(100)(10)^2 \left\{ \frac{0.867}{[1 + 1.414(0.500)]^2} \right\} = 1,490 \text{ lb}$$

This resultant force acts 3.33 ft above the base of the wall, in a direction toward the wall and downward, making an angle of 30° with the horizontal.

For a more complete treatment of soil pressures, the student is referred to books on soil mechanics.

**1·13  Hydrostatic Pressures.**  Dams, tanks, etc., are subjected to hydrostatic loads that as a rule may be easily computed in accordance with the elementary principles of hydraulics.  Hydrostatic loads should in general be considered as movable loads, inasmuch as critical stresses in a structure do not necessarily occur when the liquid involved is at its highest possible level.  In some structures, the presence of certain hydrostatic pressures actually relieves stresses in a structure.  Thus an underground tank might be more likely to collapse when empty than when full, or a tank built above the ground might undergo a critical-stress condition when it is only partly filled.

It is sometimes necessary to consider hydrodynamic loads, such as occur when water traveling at a high velocity strikes a bridge pier or a caisson.

**1·14  Earthquake Forces.**  Important structures located in regions subject to severe earthquakes are often designed to resist earthquake effects.  During an earthquake, structural damage may result from the fact that the foundation of the structure undergoes accelerations.  Such

accelerations are largely horizontal, and vertical components of accelera-
tion are usually neglected. The rate of horizontal acceleration of the
foundations may be of the order of magnitude of one-tenth the accelera-
tion due to gravity, that is, 3.2 ft per sec², and is commonly referred to
as 0.1$g$. If the structure is assumed to act as a rigid body, it will accel-
erate horizontally at the same rate as its foundations. Hence each part
of the structure will be acted upon by a horizontal inertia force equal to
its mass multiplied by its horizontal acceleration, or

$$\text{Lateral force} = \text{mass} \times \text{acceleration} = \frac{\text{weight}}{g} \times 0.1g$$
$$= \text{one-tenth of its weight}$$

Structures are often designed to resist earthquakes on the foregoing
basis, although it is quite approximate, inasmuch as the assumption that
the entire structure accelerates as a rigid body is usually not particularly
valid. Actually, the structure will undergo elastic distortions that will
affect the acceleration of its various members.

The horizontal acceleration of a structure such as a dam will produce
not only horizontal inertia forces due to the mass of the dam but also
hydrodynamic forces as the dam moves rapidly into the water that it
retains.

**1·15 Centrifugal Forces.** In designing a bridge on which the
tracks or roadway are curved, vehicles crossing the structure exert
centrifugal force that may be of sufficient magnitude to require consid-
eration in design. Such centrifugal forces are lateral loads and should
be considered as moving loads.

If a weight $W$ travels at velocity $V$ around a curve of radius $R$, the
centrifugal $C$ (which acts through the center of gravity of the object of
weight $W$) is expressed by

$$C = \text{mass} \times \text{circular acceleration} = \frac{W}{g} \cdot \frac{V^2}{R} = \frac{WV^2}{32.2R} \qquad (1·8)$$

**1·16 Longitudinal Forces.** For a bridge, horizontal forces acting
in the direction of the longitudinal axis of the structure, *i.e.*, in the direc-
tion of the roadway, are called *longitudinal forces*. Such forces are
applied whenever the vehicles crossing the structure increase or decrease
their speed. Since they are inertia forces resulting from the acceleration
or deceleration of vehicles, they act through the centers of gravities of
the vehicles. The magnitude of such forces is limited by the frictional
forces that can be developed between the contact surfaces of the wheels
of the vehicles applying these forces to the roadway or track and the
surface of the roadway or track.

For railway bridges, the AREA specifications state that longitudinal forces shall be assumed to act 6 ft above the top of the rail and that such forces shall be considered as being applied by the live load on one track only. For that track, longitudinal forces shall be taken as the larger of the following:

1. Due to braking, 15% of the total live load, without impact
2. Due to traction, 25% of the weight on the driving wheels of the locomotive, without impact

For highway bridges, the AASHO specifications state that provision shall be made for the effect of a longitudinal force of 10 per cent of the live load on the structure, acting 4 ft above the roadway.

**1·17  Thermal Forces.** Changes in temperature cause strains in the members of a structure and hence produce distortions in the structure as a whole. If the changes in shape due to temperature encounter restraint, as is often the case in a statically indeterminate structure, stresses will be set up within the structure. The forces set up in a structure as a result of temperature changes are often called thermal forces. In addition to considering the forces set up by changes in temperature, it is important to take into consideration the expansion and contraction of a structure, particularly in connection with support details.

In a moderate climate, one should consider a variation in temperature of 0 to 120°F. In cold climates, this range should be extended to from −30 to 120°F.

**1·18  Make-up of Girders.** In a structure built of structural steel, rolled sections such as standard I beams or wide-flanged beams are commonly used to support loads across a span. When rolled sections can be used, they are economical because they require less fabrication than built up girders. For longer spans and for heavier loadings, however, the bending moments and shears will be found to be too large to be carried safely by rolled sections, and built-up girders must be used. The most important parts of a typical built-up plate girder, as shown in Fig. 1·4, are the web plate, the top flange, which is composed of flange angles and cover plates, and the bottom flange, which is similarly composed. Over the end bearing plates are vertical angles called end stiffeners, while at other points along the span it is usually necessary to have further vertical angles, which are called intermediate stiffeners. The component parts of the girder shown in Fig. 1·4 are riveted together. Welding is often used instead of riveting, in which case the details of the girder differ somewhat, but the essential component parts, viz., the web, top and bottom flange, and web stiffeners, must still be provided.
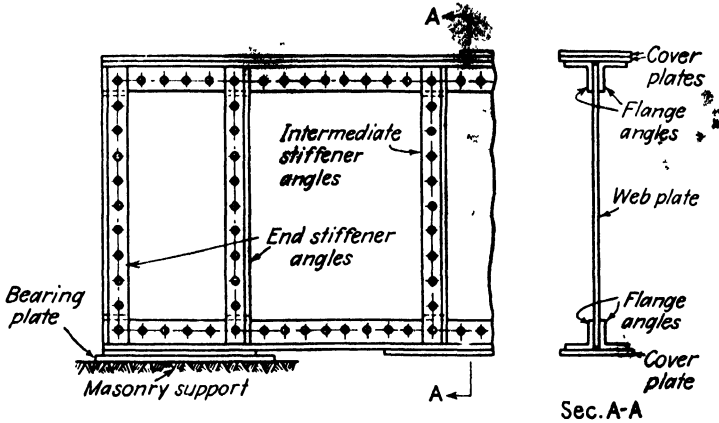
FIG. 1·4

**1·19   Make-up of Trusses.**   Fabrication, shipping, and erection considerations usually limit the depth of built-up girders to about 10 ft. When bending moments and shears are so large that they cannot be carried by a girder of that depth, it becomes necessary to employ a truss. The layout of a typical truss is shown in Fig. 1·5a.   Members $L_0L_1$, $L_1L_2$, . . . , $L_5L_6$ are called bottom chords; members $U_1U_2$, $U_2U_3$, . . . , $U_4U_5$ are called top chords; members $L_0U_1$ and $U_5L_6$ are called end posts
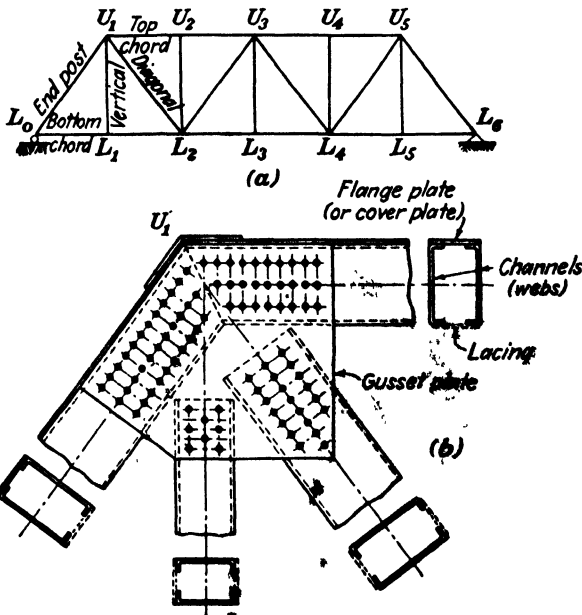


FIG. 1·5

and are often included as top-chord members; members $U_1L_2$, $L_2U_3$, ..., $L_4U_5$ are called diagonals; members $U_1L_1$, $U_2L_2$, ..., $U_5L_5$ are called verticals. Figure 1·5b shows a typical connection detail for the members of a riveted truss and the make-up of typical truss members. The plates to which the members intersecting at a joint are connected are called gusset plates. For long-span trusses, members are sometimes connected at a joint by a pin passing through the webs of the members themselves or through pin plates connected to these webs. Trusses may be welded rather than riveted.



FIG. 1·6
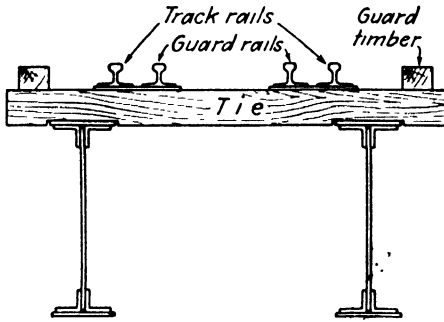
**1·20 Make-up of Floor Systems.** For a plate-girder railroad bridge, if the girders are not too far apart and if the track is located at the top of the girders, it is economical to have the ties rest directly on the cover plates of the top flanges of the girders, as shown in Fig. 1·6. Such bridges are called deck structures. As the distance between girders
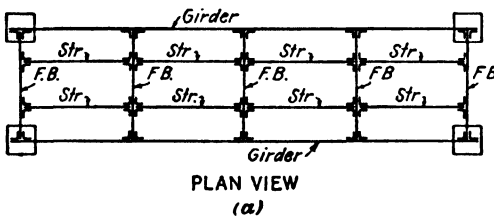


PLAN VIEW
(a)
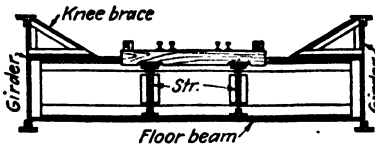


TRANSVERSE SECTION
(Railroad bridge)
(b)
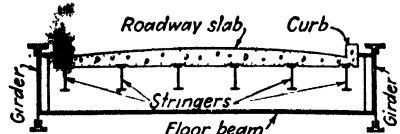
TRANSVERSE SECTION
(Highway bridge)
(c)

FIG. 1·7

becomes larger, such construction becomes uneconomical. It is moreover obvious that it cannot be followed if the tracks are to be located below the tops of the girders. Under either of the foregoing circumstances, it becomes necessary to build up a floor system composed of

stringers and floor beams, as shown in Fig. 1·7. If such a built-up floor
system is at the top of the girders or trusses, the bridge is still called a
deck structure; if it is at the bottom of the girders or trusses, the bridge
is called a through structure; if it is at an intermediate elevation it is
called a half-through structure. Figures 1·7a and b illustrate a half-
through single-track railroad bridge and show the make-up of floor sys-
tems for such bridges. The ties rest directly on members called stringers,
which are parallel to the main girders. These stringers frame into
members called floor beams, which are transverse to, and frame into,
the main girders. Thus a load applied to the rails is transferred by the
ties to the stringers, which carry the load to the
floor beams. The floor beams carry the load to
the girders, which in turn transfer the load to the
foundations of the structure.

Figure 1·7c shows a transverse section through
a typical highway bridge with a floor system.
Such a floor system is similar to that described
for a railroad bridge, with the floor slab of the
bridge resting on stringers that frame into floor
beams, which in turn frame into the main girders.

Similar framing may be used for the floor of a
building, as shown in Fig. 1·8. In this case the
floor slab would rest on the floor beams and girders
as well as on the stringers, but loads applied to the



PLAN VIEW
Floor framing of building
Fig. 1·8

floor slab directly over a stringer would be carried by the stringer to
the floor beams, thence to the girders, thence to the columns, and finally
to the foundations.

For bridges in which the main load-carrying elements are trusses
rather than girders, floor systems are always used. The floor beams
are located at the panel points (joints) of the loaded chord, so that the
members of the truss will not be subjected to transverse loads and thus
undergo primary bending.

**1·21  Bracing Systems.** Trusses, girders, floor beams, stringers,
columns, etc., which are designed primarily to carry vertical loads, are
often referred to as the main members of a structure. In addition to
the main members, most structures require bracing systems, which serve
a number of purposes, the most important of which is that of resisting
lateral loads. The most important bracing systems of a typical through
truss bridge are shown in Fig. 1·9. The top-chord lateral system lies
in the plane of the top chords and consists of a cross strut at each top-
chord panel point and diagonals connecting the ends of these cross struts.
The bottom-chord lateral system lies in the plane of the bottom chords

and consists of the floor beams, which occur at each bottom-chord panel point and which serve as cross struts, and diagonals connecting the ends of the floor beams. In the planes of the end posts is the so-called "portal" bracing, which stiffens the end posts laterally.
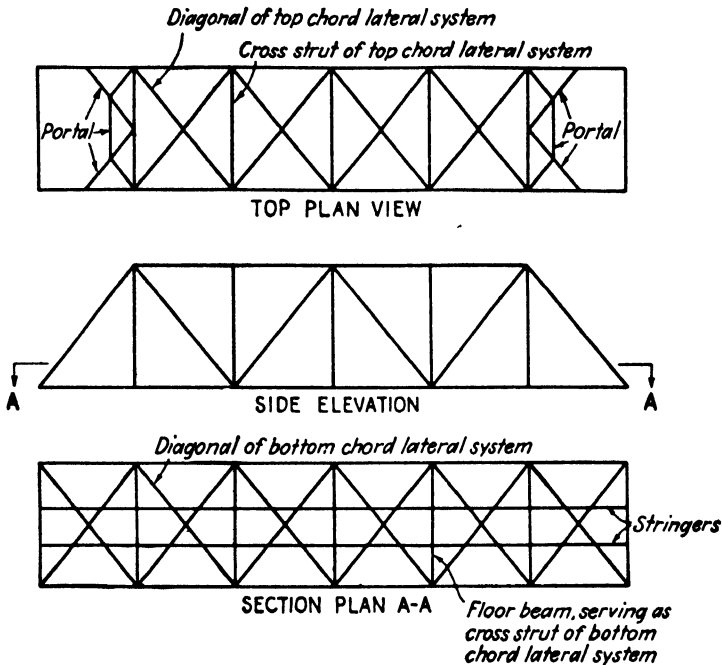


FIG. 1·9

**1·22 Fiber Stresses.** Sometimes stress denotes force per unit area. In structural analysis, however, it is customary to speak of the total tensile or compressive force acting in a truss member as the *stress* in the member. The term *stress intensity* or *fiber stress* is usually employed to denote force per unit area. In structural design, the total stresses and moments in members are first computed, so that the stress intensities can then be investigated. In order that the structure shall perform satisfactorily, it is necessary that these stress intensities, or fiber stresses, be kept within certain specified limits. The fiber stresses to be permitted are called the "working stresses," the "permissible stresses," or the "allowable stresses" and depend, of course, upon the physical properties of the material used.

A transverse section through a structural member is likely to pass through rivets. Since a well-driven rivet is tight and substantially fills the rivet hole, it is customary to assume that in transferring compressive

stresses the member is not weakened by rivet holes.  The area of a transverse section, with no deduction made for rivet holes, is called the "gross area."  Even though the rivet is well driven, however, it is not customary to assume that tensile stresses can be transferred across the rivet hole.  The area of a transverse section, reduced by the area of those rivet holes located where the section is in tension, is called the "net area."  In defining permissible fiber stresses, it is necessary to define whether they are applicable to the net or gross area (or section).

In riveted fabrication, the rivet hole is punched $\frac{1}{16}$ in. in diameter greater than the diameter of the rivet itself, and it is assumed that the material punched is damaged to a diameter $\frac{1}{16}$ in. greater than the hole itself.  Thus, in computing net areas, the effective diameter of the hole is taken as $\frac{1}{8}$ in. greater than the diameter of the rivet.

The following allowable stresses for structural steel are specified by the AISC for structural-steel buildings, with all values in pounds per square inch:

*Tension:* Structural steel, net section. . . . . . . . . . . . . . . 20,000
      Rivets, on area based on nominal diameter . . . . . . . . . . 20,000
      Bolts and other threaded parts, on nominal area at root of thread 20,000
*Compression:* Columns, gross section
      For axially loaded columns with values of $l/r$ not greater than 120
$$17,000 - 0.485 \frac{l^2}{r^2}$$
      For axially loaded columns with values of $l/r$ greater than 120
$$\frac{18,000}{1 + \frac{l^2}{18,000 r^2}}$$
      in which $l$ is the unbraced length of the column and $r$ is the corresponding radius of gyration of the section, both in inches
*Bending:* (based on gross moment of inertia) Tension on extreme
      fibers of rolled sections, plate girders, and built-up members  . 20,000
      Compression on extreme fibers of rolled sections, plate girders,
      and built-up members,
          with $ld/bt$ not in excess of 600 . . . . . . . . . . . . . . 20,000
          with $ld/bt$ in excess of 600. . . . . . . . . . . . . . $\dfrac{12,000,000}{ld/bt}$
      in which $l$ is the unsupported length, and $d$ the depth, of the
      member; $b$ is the width, and $t$ the thickness, of its compression
      flange; all in inches.
*Shearing:* Rivets . . . . . . . . . . . . . . . . . . . . . . 15,000
      Pins, and turned bolts in reamed or drilled holes . . . . . . . 15,000
      Unfinished bolts . . . . . . . . . . . . . . . . . . . . . 10,000
      Webs of beams and plate girders, gross section. . . . . . . 13,000

|                                              | Double Shear | Single Shear |
|----------------------------------------------|--------------|--------------|
| *Bearing:* Rivets                            | 40,000       | 32,000       |
| Turned bolts in reamed or drilled holes      | 40,000       | 32,000       |
| Unfinished bolts                             | 25,000       | 20,000       |
| Pins                                         | 32,000       | 32,000       |
| Contact areas                                |              |              |
| Milled stiffeners and other milled surfaces  |              | 30,000       |
| Fitted stiffeners                            |              | 27,000       |

Permissible structural-steel stresses for highway and railroad bridges vary somewhat from the foregoing values. For such values, the student is referred to the specifications of the AASHO and the AREA, respectively. Permissible stresses in concrete, timber, and other structural materials are available in various specifications and handbooks.

**1·23   Factor of Safety.** It should be obvious that for a number of reasons the permissible fiber stress should be less than the ultimate or breaking strength of the material used. Loads, particularly loads of certain types, cannot be predicted with too much accuracy. Certain simplifying assumptions are almost always made in stress analysis, so that even for the assumed loads the computed stresses will be somewhat in error. Even though care is used in the control of structural materials employed, some variation in the properties of these materials will exist, even when the materials are new. With the passage of time, partial disintegration must be expected.

It is, moreover, desirable to keep the actual stresses in a structure within the yield-point stresses, since otherwise plastic flow will permit deformations without increased loads. Usually an attempt is made to keep actual stresses within the elastic limit.

Hence allowable fiber stresses must be appreciably lower than ultimate fiber stresses. For rolled sections of structural steel, the ultimate tensile strength in pounds per square inch may be 60,000 to 72,000; the yield point is about half the ultimate tensile strength. The ultimate stress divided by the permissible stress is defined as the "factor of safety." Thus for a 60,000-psi ultimate-strength steel, where a permissible stress of 20,000 psi is specified, the factor of safety is 3.

It should be clearly understood, however, that this does not mean that the loads could be increased by a factor of 3 without damaging the structure. Suppose that the elastic limit of this steel is 30,000 psi, and suppose further that, owing to the neglecting of so-called "secondary stresses" and other factors, the actual fiber stresses are 25 per cent higher than the computed fiber stresses. Then the factor by which

the loads could be increased without exceeding the elastic limit of the steel is actually given by

$$\frac{30{,}000}{1.25(20{,}000)} = 1.2$$

This factor of 1.2 is probably of greater importance than the factor of safety of 3.

**1·24  Practical and Ideal Structures.**   Rarely if ever does an actual structure correspond to the idealized structure that is considered in its analysis.   The materials of which the structure is built do not have the exact properties assumed, nor do the dimensions correspond exactly with their theoretical values.   Structural details such as lacing bars and



<!-- Figure labels: b, b; Gusset plate; Symbol representing rigid joint; P; h; h; Anchorage detail; (a); (b) -->
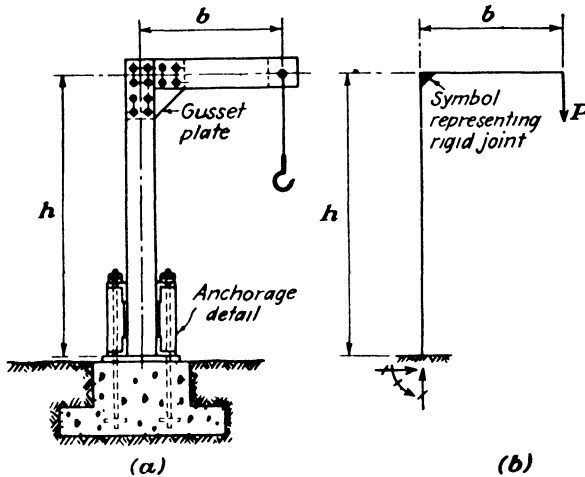
Fig. 1·10

gusset plates introduce effects that might make an analysis very complicated indeed; but because they have little actual effect, they are usually neglected in the analysis of stresses in main members.   Because of the width of members, considerable difference may exist between clear spans and center-to-center spans which are ordinarily used in analyses. Support details may vary considerably from the idealized type assumed for purposes of analysis.   A member may not actually be prismatic, and yet it may be assumed to be, to simplify computations.

The practical structure of Fig. 1·10a, for example, might be analyzed on the basis of the idealized structure of Fig. 1·10b, in which the footing has been assumed to be perfectly fixed, although it could not be so in nature; the extra column area of the anchorage detail and the material

of the gusset plate have been ignored; the gusset plate has been assumed as a truly rigid connection, whereas it will actually permit some rotational yielding between the column and the horizontal girder; the effective height of the column has been taken as the distance from the top of the bedplate to the center line of the girder; and the effective span of the girder has been measured from the center line of the column to the center of the applied load.

It is necessary to idealize a structure in order to carry out a practical analysis. Experience and judgment are necessary in determining the idealized structure that should be used in a given case. In important structures, where doubt exists as to the most logical assumptions to be made in idealizing a structure, it is sometimes desirable to compute stresses on the basis of more than one possible idealized form, and to design the structure to resist the stresses corresponding to all the analyses.

### 1·25   Problems for Solution.

**Problem 1·1**   An end-supported girder with a span of 50 ft carries a dead load of 500 lb per ft and a live load of 600 lb per ft. The dead load extends over the entire span of the girder, and maximum live moment at the center of the girder occurs with the live load extending over the entire span.

*a.* Assuming that the intensity of dead load varies directly with the girder span but that live-load intensity remains constant, compute the total maximum moment (dead plus live) at mid-span, for spans varying from 20 to 100 ft, based on 10-ft intervals.

*b.* Assuming that by the use of lightweight alloys the dead load is reduced by 40 per cent, what is the percentage reduction in total moment at mid-span for each span specified in part *a?*

**Problem 1·2**   The rear tire of an H-series truck has a width of 20 in.

*a.* What is the load on this rear tire?

*b.* What is the total weight of the truck, and how is the truck designated?

**Problem 1·3**   For a Cooper's E-70 loading,

*a.* What is the load on one driving wheel?

*b.* What is the spacing between driving wheels?

*c.* What is the equivalent uniform load of the cars, in kips per foot of track?

**Problem 1·4**   Compute the impact moments corresponding to Prob. 1·1*a* in accordance with the specifications of the AASHO.

**Problem 1·5**   Figure 1·11 shows the bent of a mill building. These bents are spaced 20 ft, center to center. Determine the wind loads for which an intermediate bent should be designed, basing your load
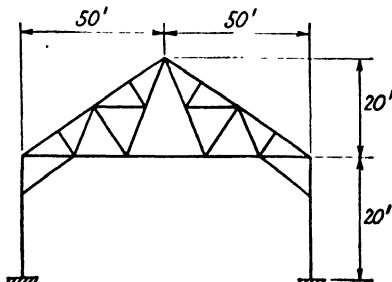


Fig. 1·11

determination for vertical surfaces on the specifications of the AISC and the loadings for inclined surfaces on the recommendations of the ASCE for a wind velocity of 100 mph.

**Problem 1·6**   The retaining wall of Fig. 1·12 is acted upon by sand weighing 120 lb per cu ft and having an angle of internal friction of 35°.   The friction angle between this sand and the vertical contact face of the wall is 30°.   Determine the magnitude, direction, and point of application of the resultant force exerted by the sand on a strip of wall 20 ft in length, in accordance with Coulomb's theory.
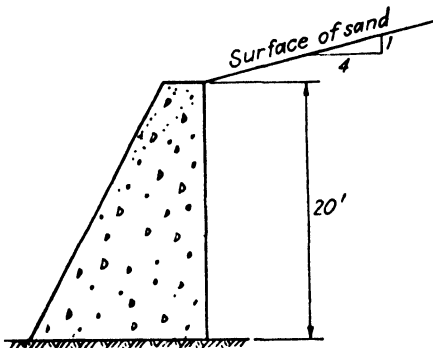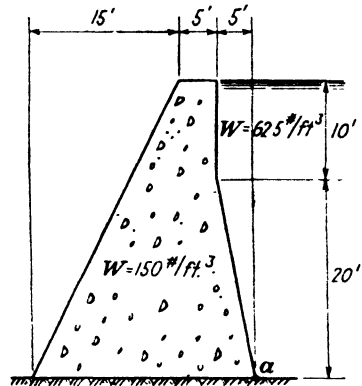


Fig. 1·12                              Fig. 1·13

**Problem 1·7**   Considering a strip 1 ft long of the dam of Fig. 1·13, determine the moment, about point $a$, of the total hydrostatic pressure acting on the upstream face of the dam.

**Problem 1·8**   The dam of Fig. 1·13 is subjected to an earthquake involving horizontal accelerations of the foundation of 0.1$g$.   Treating the dam as a rigid body and assuming the reservoir to be empty, determine the moment, about point $a$, of the lateral forces to which a 1-ft strip of the dam is subjected.

**Problem 1·9**   The compression flange of a girder consists of two angles, each 6 × 6 × ½″, and one cover plate, 14 × ½″.   The girder is 10 ft deep. It has a span of 96 ft, and the top flange is braced laterally at intervals of 16 ft.   In accordance with the AISC specifications, what fiber stress would be permissible in the design of the compression flange?

# CHAPTER 2

# REACTIONS

**2·1 Definitions.** The major portion of this book will be devoted to the analysis of so-called *planar structures, i.e.,* structures that may be considered to lie in a plane that also contains the lines of action of all the forces acting on the structure. Such a structure is by far the most common in structural analysis. The analysis of three-dimensional, or space, structures involves no new fundamental principles beyond those required for planar structures, but the numerical computations are greatly complicated by the additional geometry introduced by the third dimension. For these reasons, the emphasis will be placed on planar structures, and a limited discussion of space structures will be reserved for later portions of the book.

Since the discussion in this chapter is limited to planar structures, all the force systems will be so-called *coplanar force systems, i.e.,* systems consisting of several forces the lines of action of which all lie in one plane. Some of these systems have special characteristics, and it will be found
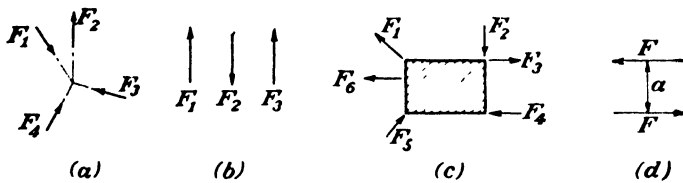


Fig. 2·1

convenient to classify them accordingly and to identify them by special names.

A *concurrent coplanar force system* is shown in Fig. 2·1a. This system consists of several forces the lines of action of which all intersect at a common point. A *parallel coplanar force system* consists of several forces the lines of action of which are all parallel as shown in Fig. 2·1b. A *general coplanar force system* consists of several forces the lines of action of which are in various directions and do not intersect in a common point, as shown in Fig. 2·1c. Another important system, a couple, is shown in Fig. 2·1d. A *couple* consists of two equal and opposite parallel forces that do not have a common line of action.

Since, in a planar structure, the lines of action of all the forces lie in the plane of the structure, each of the forces $F$ could be resolved into

24

two components $F_x$ and $F_y$, where the $x$ and $y$ reference axes may be taken in any directions as long as they do not coincide. It is almost always desirable to select $x$ and $y$ axes that are mutually perpendicular, in which case $F_x$ and $F_y$ are called rectangular components. Further, it is usually most convenient to take the $x$ axis horizontally and the $y$ axis vertically.

**2·2  General—Conventional Supports.** Most structures are either partly or completely restrained so that they cannot move freely in space. Such restrictions on the free motion of a body are called restraints and are supplied by supports that connect the structure to some stationary body. For example, consider a planar structure such as the bar $AB$ shown in Fig. 2·2. If this bar were a free body and were acted upon by a force $P$, it would move freely in space with some combined translatory and rotational motion. If, however, a restraint were introduced in the form of a hinge that connected the bar to some stationary body at point $A$, then the motion of the bar would be partly restricted and could consist only of a rotational movement about the hinge. During such a rotation, point $B$ would move along an arc with point $A$ as the center. Instantaneously, point $B$ could be considered to move normal to the line



FIG. 2·2

$AB$, or, in this case, vertically. If, therefore, another restraint were introduced that would not allow point $B$ to move instantaneously in a vertical direction, the rotation about the hinge at point $A$ would be prevented and thus the free motion of the bar would be completely restricted. It is evident that this type of restraint would be supplied by the supports shown at $B$ in either $c$ or $d$ of Fig. 2·2.

The supports at $A$ and $B$, in restricting the free motion of the bar, are called upon to resist the action that the force $P$ imposes upon them through the bar. The resistances that they thus develop to counteract
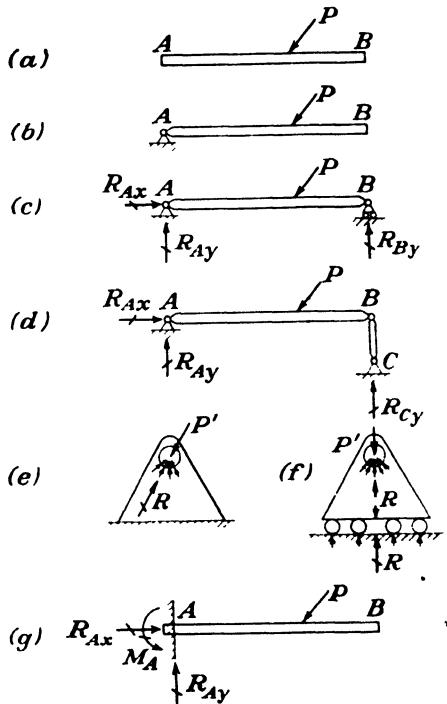
the action of the bar upon them are called *reactions*.   The effect of the supports may therefore be replaced by the reactions that they supply the structure.

In future discussions, it will be necessary to deal continually with the reactions that different types of supports supply, and it therefore will be convenient to use a conventional symbol for describing these different types.   A *hinge support*, as shown in Fig. 2·2e, is represented by the symbol ⊹. In such a support, if it is assumed that the pin of the hinge is frictionless in the pinhole, then the contact pressures between the pin and its hole must remain normal to the circular contact surface and must therefore be directed through the center of the pin.   The reaction $R$ that the support supplies to the structure must completely counteract the action of the force $P'$, and therefore $R$ and $P'$ must be collinear and numerically equal but must act opposite to one another.   It is therefore evident that a hinge support supplies a reactive force the line of action of which is known to pass through the center of the hinge pin but the magnitude and direction of which are unknown.[1]   These two unknown elements of such a reaction could also be represented by the unknown magnitudes of its horizontal and vertical components, $R_x$ and $R_y$, respectively, both acting through the center of the hinge pin.

A *roller support*, as shown in Fig. 2·2f, is represented by either the symbol ⊹ or ⊟. In the same manner as above, it may be reasoned that the reactive force of a roller support must be directed through the center of the pin.   In addition, however, if the rollers are frictionless, they can transmit only a pressure which is normal to the surface on which they roll.   Hence, a roller support supplies a resultant reactive force which acts normal to the surface on which the rollers roll and is directed through the center of the hinge pin.   It is therefore evident that a roller support supplies a reactive force which is applied at a known point and acts in a known direction, but the magnitude of which is unknown.   Roller supports are usually detailed so that they can supply a reaction acting either away from or toward the supporting surface.

Consideration of the *link support BC* shown in Fig. 2·2d shows that for small movements it effectively reproduces the action of the roller support shown at $B$ in Fig. 2·2c.   By the same approach as used above, it may be reasoned that, if the pins at the ends of this link are frictionless, the force transmitted by the link must act through the centers of the pins at each end.   Therefore, a link support also supplies a reaction of a

[1] According to this terminology, the "direction of a force" is intended to define the slope of its line of action, while the "magnitude" indicates not only its numerical size but also the sense in which the force acts along this line of action, i.e., whether toward or away from a body.

known direction and a known point of application but of an unknown magnitude.   A link support is denoted by the symbol ⚓ .

One other common type of support used for planar structures is that shown at $A$ in Fig. 2·2$g$ and called a *fixed support*.   Such a support encases the member so that both translation and rotation of the end of the member are prevented.   A fixed support therefore supplies a reaction, the magnitude, point of application, and direction of which are all unknown.   These three unknown elements may also be considered to be a force, which acts through a specific point but has an unknown magnitude and direction, and a couple of unknown magnitude.   For example, the three unknown elements could be selected as a couple and a horizontal and a vertical force, the two latter acting through the center of gravity of the end cross section.   A fixed support is designated by the symbol ⊨ .

The external forces acting on a body therefore consist of two distinct types—the applied loads $P$ (shown thus →) and the reactions $R$ (shown thus ↠).   A reactive force at support $a$ will be designated as $R_a$, while its $x$ and $y$ components will be called $R_{ax}$ and $R_{ay}$.   A reactive couple at support $a$ will be called $M_a$.

**2·3   Equations of Static Equilibrium—Planar Structures.**   If the supports of a planar structure are considered to be replaced by the reactions that they supply, the structure will be acted upon by a general coplanar force system consisting of the known applied loads and the unknown reactions.   In general, the resultant effect of a general coplanar force system could be either a resultant force acting at some point and in some direction in the plane or a resultant couple.

A body that is initially at rest and remains at rest when acted upon by a system of forces is said to be in a state of *static equilibrium*.   For such a state to exist, it is necessary that the combined resultant effect of the system of forces shall be neither a force nor a couple; otherwise, there will be a tendency for motion of the body.   In order that the combined resultant effect of a general system of forces acting on a planar structure shall not be equivalent to a resultant force, it is necessary that the algebraic sum of all the $F_x$ components shall be equal to zero and likewise that the algebraic sum of all the $F_y$ components shall be equal to zero.   In order that the combined resultant effect shall not be equivalent to a couple, it is necessary that the algebraic sum of the moments of all the forces about any axis normal to the plane of the structure shall also be equal to zero.   The three following conditions must therefore be fulfilled simultaneously by the loads and reactions of a planar structure for the structure to remain in a state of static equilibrium:

$$\left.\begin{array}{l} \Sigma F_x = 0 \\ \Sigma F_y = 0 \\ \Sigma M = 0 \end{array}\right\} \tag{2·1}$$

These equations are called the *equations of static equilibrium* of a planar structure subjected to a general system of forces.

In the special case where a planar structure is acted upon by a concurrent system of loads and reactions, it is impossible for the resultant effect of the system to be a couple, for the lines of action of all the forces of a concurrent system intersect in a common point. Therefore, for the structure to remain in static equilibrium in such a case it is necessary only that the two following conditions be satisfied:

$$\begin{array}{l} \Sigma F_x = 0 \\ \Sigma F_y = 0 \end{array} \tag{2 2}$$

These equations are the equations of static equilibrium for the special case of a planar structure subjected to a concurrent system of forces.

Quite often in the discussions that follow, structural members will be referred to as being *rigid bodies*. In an exact sense, a rigid body is one in which there is no relative movement between any two particles of the body. Of course, any structural element is never absolutely rigid since it is made of materials that deform slightly under the loads imposed on them. However, such deformation is so slight that the changes of dimension, the shifting of the lines of action of forces, etc., may usually be neglected during the investigation of the condition of equilibrium of the body. Thus, in most problems, in applying the equations of static equilibrium to structural elements, it will be assumed that they are rigid bodies for all practical purposes and hence that the geometry after application of the loads is essentially the same as before.

**2·4 Equations of Condition.** Many structures consist of simply one rigid body—a truss, frame, or beam—restrained in space by a certain number of supports. Sometimes, however, the structure may be built up out of several rigid bodies partly connected together in some manner, the whole assemblage then being mounted on a certain number of supports. In either type of structure, the force system consisting of the loads and reactions must satisfy the equations of static equilibrium if the structure is to remain at rest. In the latter type of structure, however, the details of the method of construction used to connect the separate bodies together may enforce further restrictions on the force system acting on the structure. The separate parts may be connected together by hinges, links, or rollers in some way, and in each case these details can transmit only a certain type of force from one part of the structure to the other.

Figure 2·3 illustrates this type of structure. It is composed of two rigid members *ab* and *bc* connected together by a frictionless hinge at point *b* and supported by hinge supports at points *a* and *c*. Since a frictionless hinge cannot transmit a couple, its insertion imposes the condition that the action of one portion of the structure on the other portion connected by the hinge can consist only of a force acting through the center of the hinge pin. Therefore, the algebraic sum of the moments of the loads and reactions applied to any *one* such portion of the structure taken about an axis through the center of the hinge pin must be equal to zero. Such conditions introduced by the method of construction (other than the manner in which the supports are detailed) result in so-called *equations of construction or condition*.
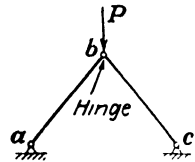


FIG. 2·3

**2·5  Static Stability and Instability—Statically Determinate and Indeterminate Structures.** Consider first a planar structure which is acted upon by a general system of loads and into which no equations of construction have been introduced. If the supports are replaced by the reactions that they supply to the structure, the structure will be acted upon by a general system of forces consisting of the known loads and the unknown reactions. If the structure is in static equilibrium under these forces, the three equations of static equilibrium may be written in terms of the known loads and the unknown elements defining the reactions. The simultaneous solution of these three equations will in certain cases determine the magnitude of the unknown reaction elements. Whether or not these three equations are sufficient for the complete determination of the reactions, they must be satisfied for the structure to be in static equilibrium and therefore they form a partial basis for the solution to obtain the reactions of any structure that is in static equilibrium.

If there are fewer than three unknown independent reaction elements, there are not enough unknowns to satisfy the three equations of static equilibrium simultaneously. Fewer than three unknown reaction elements are therefore insufficient to keep a planar structure in equilibrium when it is acted upon by a general system of loads. Under such conditions, a structure is said to be *statically unstable*.

Under certain special conditions, a planar structure having fewer than three unknown independent reaction elements may be in static equilibrium. Of course, if the system of applied loads acting on the structure is in equilibrium itself, no reactions are required; also, if the loads and reactions have certain mutual characteristics, fewer than three reaction elements may be sufficient for equilibrium. For example, considering the

bar shown in Fig. 2·2b, if the resultant effect of the applied loads is a force whose line of action goes through the center of the hinge pin at point A, the forces acting on the structure are concurrent and the horizontal and vertical components of the reaction at A will be capable of maintaining static equilibrium. Moreover, if the bar were supported at both points A and B by rollers which rolled on horizontal surfaces, the reactions supplied by such supports could maintain in a state of static equilibrium any system of applied loads of which the resultant effect is either a couple or a vertical force. Such structures, although stable under special types of loading but unstable under the general case of loading, are said to be in a state of *unstable equilibrium* and are still classed as unstable structures.

Since three unknowns can be obtained from the solution of three independent simultaneous equations, the reactions of a stable planar structure having exactly three unknown reaction elements may be obtained from the simultaneous solution of the three equations of static equilibrium. In such a case, the reactions of the structure are said to be *statically determinate*. However, if there are more than three unknown independent reaction elements supplied to a stable planar structure, the three equations of static equilibrium are not sufficient to determine the unknown reactions. This is evident since all but three of the unknowns could be assigned arbitrary values and then the remaining three determined from the simultaneous solution of the three equations of static equilibrium. In such cases, there are an infinite number of related sets of values for the unknown reactions that could satisfy the conditions of static equilibrium. The correct values for the reactions cannot therefore be determined simply from these three equations but must also satisfy certain distortion conditions of the structure, as will be discussed later in this book. If the unknown reaction elements cannot be determined simply by the equations of static equilibrium, the reactions of the structure are said to be *statically indeterminate*. The structure is then said to be indeterminate to a degree equal to the number by which the unknowns exceed the available equations of statics.

From the above discussion it may be concluded that at least three independent reaction elements are *necessary* to satisfy the conditions of static equilibrium for a planar structure acted upon by any general system of loads. It may easily be demonstrated, however, that three or more elements are not always *sufficient*, and therefore a planar structure having three or more independent reaction elements may still be unstable. This is the reason why a *stable* planar structure was specified in the discussion of the previous paragraph.

The question of sufficiency of the reactions for stability may be dis-

cussed by an extension of the approach used in the early part of Art. 2·2. For example, the bars shown in Figs. 2·2c and d are stable when supported by the horizontal and vertical components of the reaction at A and the vertical reaction at B. These structures are equivalent to that shown in Fig. 2·4, where the hinge support has been replaced by two links attached to the hinge pin at point A. These two links may be in any directions as long as they are not collinear. This structure would be stable as long as the line of action of the link at B did not also pass through the center of the hinge pin at A. If this line of action did pass through A, the structure would be only partly restrained and therefore unstable under the general system of loads because there would be nothing to prevent the instantaneous rotation of the structure about the hinge at point A. In such cases, where there are nominally sufficient reaction elements but the geometrical arrangement is such that the structure is unstable, the structure is said to be *geometrically unstable.*
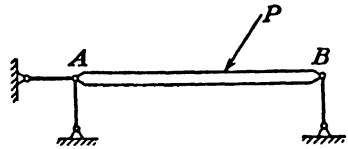


FIG. 2·4

One other case where three independent reaction elements are not sufficient for stability should be discussed. Consider a bar supported by three parallel links as shown in Fig. 2·5. It is apparent that there is no restraint to prevent a small translation of the structure normal to the direction of the links. Hence, if the resultant effect of the applied loads has a component in this direction, motion will be produced and such a structure must be classed as geometrically unstable. These and similar considerations lead to the following conclusion: *If the reactions are equivalent to those supplied by a system of three or more link supports that are either concurrent or parallel, they are not sufficient to maintain static equilibrium of a planar structure subjected to a general system of loads even if there are three or more unknown reaction elements.*
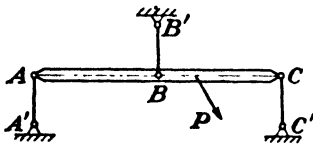


FIG. 2·5

In other words, the stability of a structure is determined not only by the number of reaction elements but also by their arrangement.

It should be noted that unstable structures having three or more independent reaction elements usually could also be classed as statically indeterminate. Consider the structure shown in Fig. 2·5. While it is unstable and starts to translate horizontally under the load P, it is not completely unrestrained. Instantaneously the bar translates horizontally, and the three links rotate about points A', B', and C', respectively. After a finite rotation of the links, points A, B, and C will have moved vertically as well as horizontally. A vertical movement of these points

can be accomplished only by making the bar $AC$ bend.    The final equilib-
rium position is determined not only by the geometry of the structure
but also by its elastic distortion due to the stresses developed in the links
and the bar.    In this final position, the links will have moved through
such finite rotations that the algebraic sum of the horizontal components
of the stresses in the links is equal to the horizontal component of the
load.    Hence, the analysis of this structure in its final equilibrium position
involves not only the equations of static equilibrium but also its distortion
properties, and the structure may therefore be classed as statically inde-
terminate.    The structure shown in Fig. 2·4 also falls into this same
category if the line of action of the link at $B$ passes through the hinge
at $A$.

    If the structure shown in Fig. 2·5 were acted upon by a system of
vertical loads, there would be no tendency for it to move horizontally,
*i.e.*, the structure would be in unstable equilibrium.    In such a case, note
that the reactions would also be statically indeterminate.

**2·6  Stability and Determinancy of Structures Involving
Equations of Condition.**    So far the discussion has been restricted to
structures in which no special construction conditions have been intro-
duced.    If, however, such conditions are introduced into a planar struc-
ture, a like number of equations of condition are added to the three
equations of statics.    It is then necessary for the loads and reactions acting
on the structure to satisfy simultaneously both the equations of static
equilibrium and the equations of condition.    If the number of unknown
reaction elements is fewer than the total number of equations, the struc-
ture is statically unstable or possibly in unstable equilibrium for certain
special conditions of loading.    If the number of reaction elements is more
than the total number of equations, the structure is classed as statically
indeterminate.    If, however, there are the same number of reaction
elements as there are equations, the structure is statically determinate
unless the reactions are so located and arranged that geometrical instabil-
ity is possible.

    Such a condition of instability may be simply illustrated by a con-
sideration of a structure of the type shown in Fig. 2·3.    The structure
shown there is, of course, stable; but if the hinge at $b$ lay on the line
joining points $a$ and $c$, there would be no restraint to the instantaneous
rotation of members $ab$ and $bc$ about points $a$ and $c$, respectively.    After
a certain finite rotation of both members, direct stresses will be developed
in members $ab$ and $bc$ that in view of the new slopes of the members will
have vertical components that will keep the load $P$ in equilibrium.    The
computation of the equilibrium position of joint $b$ and the resulting direct
stresses that are developed involves a consideration of the distortion

properties of the structure. Therefore, this structure is not only unstable in its original position but also statically indeterminate.

Geometrical instability similar to that just illustrated is most likely to occur whenever equations of construction are introduced into an originally stable structure. It is therefore evident that care must be used in introducing construction hinges, etc., so that geometrical instability is not produced. Such a condition will always be apparent, for solution of the combined equations of static equilibrium and equations of condition will yield inconsistent, infinite, or indeterminate values for the unknown reaction elements.[1]

**2·7 Free-body Sketches.** In the previous discussion, it is shown that the unknown reaction elements of a statically determinate and stable structure may be computed from the simultaneous solution of the equations of static equilibrium, augmented, in certain cases, by equations of condition. All the equations involve some or all of the forces acting on the structure, including both the applied loads and the reactions supplied by the supports. To assist in formulating these equations, it is desirable to draw so-called *free-body sketches* of either the entire structure or some portions of it. The importance of drawing an adequate number of these free-body sketches cannot be overemphasized to the student. Such sketches are the basis of the successful stress analysis of structures. The student should be admonished that *it is impossible to draw too many free-body sketches* and that time spent in so doing is never wasted.

A free-body sketch of the entire structure is drawn by isolating the structure from its supports and showing it acted upon by all the applied loads and all the possible reaction components that the supports may supply to the structure. Such a sketch is illustrated in Fig. 2·7b. In this same manner, any portion of the structure can be isolated by passing any desired section through the structure and a free-body sketch drawn showing this portion acted upon by the applied loads and reactions, together with any forces that may act on the faces of the members cut by the isolating section. Any force the magnitude of which is unknown may be assumed to act in either sense along its line of action. The assumed sense is used in writing any equation involving such a force. When the magnitude of such a force is determined from the solution, if the sign is positive, the force is then known to act in the assumed sense; if negative, in the opposite sense.

Sometimes it becomes desirable to isolate several portions of the structure and to draw free-body sketches of these portions. In such

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[1] For a discussion of this subject, see W. M. FIFE and J. B. WILBUR, "Theory of Statically Indeterminate Structures," Art. 8, p. 9, McGraw-Hill Book Company, Inc., New York, 1937.

cases, it is necessary to show the internal forces acting on the internal faces that have been exposed by the isolating section. If free-body sketches are drawn for two adjacent por-tions of the structure and the internal forces have been assumed to act in certain senses on an internal face of one portion, the cor-responding forces must be assumed to act with the same numerical values but in *opposite* senses on the matching face of the adjacent portion. This is evident since the action and reaction of one body on another must be numerically equal but opposite in sense. Such free-body sketches are shown
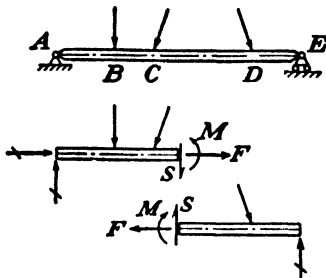


Fig. 2·6

in Fig. 2·6. If this practice is not followed, equations of static equi-librium, written using two such free-body sketches, will not be consistent with one another and it will be impossible to obtain a correct solution from them. It is of course obvious that any particular reaction which is assumed to act in a certain sense on one sketch must be shown in the same sense on all other sketches in which it appears.

**2·8 Computation of Reactions.** If no equations of condition have been introduced into a statically determinate structure, the com-putation of the reactions involves only a straightforward application of the equations of static equilibrium. Such computations may be illus-trated by considering the simple end-supported beam shown in Fig. 2 7a. The unknown reaction elements may be taken as the vertical and hori-zontal components of the reaction at A and the vertical component of the reaction at D. These may be assumed to act as shown in the free-body sketch, Fig. 2·7b.

To obtain these three unknowns there are available the three equa-tions of static equilibrium $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$, and therefore such a structure is statically determinate. It is possible to write the following three equations of static equilibrium and solve them simul-taneously for the three unknowns $R_{Ax}$, $R_{Ay}$, and $R_{Dy}$:

$$\Sigma F_x = 0, \xrightarrow{+}, R_{Ax} - 36 = 0$$
$$\Sigma F_y = 0, \uparrow+, R_{Ay} + R_{Dy} - 60 - 48 = 0$$
$$\Sigma M_C = 0, +\rangle, 12R_{Ay} - 6R_{Dy} - (60)(6) = 0$$

While such a solution is always possible, it is not very ingenious and it is inefficient, particularly in a complicated structure. Consider the advantages of proceeding as follows: By taking the summation of moments about an axis through point A, the only unknown entering the equation will be $R_{Dy}$, and a direct solution for it will be possible.

$\Sigma M_A = 0, +\!\!\curvearrowright, \ (60)(6) + (48)(12) - (R_{Dy})(18) = 0$

$$\therefore R_{Dy} = {}^{936}\!/_{18} = 52 \text{ kips } \uparrow$$

In the same manner, $R_{Ay}$ may be found directly from

$\Sigma M_D = 0, +\!\!\curvearrowright, \ (R_{Ay})(18) - (60)(12) - (18)(6) = 0$

$$\therefore R_{Ay} = \frac{1,008}{18} = 56 \text{ kips } \updownarrow$$

Of course, these values should satisfy the equation $\Sigma F_y = 0$, and the following check is obtained:

$$\Sigma F_y = 0, \ \uparrow +, \ 56 + 52 - 60 - 48 = 0 \qquad \therefore \text{ O.K.}$$

From $\Sigma F_x = 0$, $R_{Ax}$ is obtained directly,

$$\Sigma F_x = 0, \ \xrightarrow{+}, \ R_{Ax} - 36 = 0 \qquad \therefore R_{Ax} = 36 \text{ kips } \rightarrow$$

Thus, by ingenuity in applying the equations of static equilibrium, the solution for the reactions is facilitated.
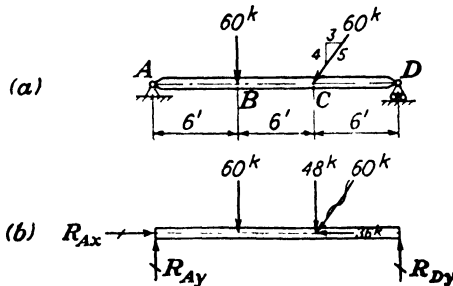


Fig. 2·7

To enlarge on this discussion, it should be noted that the three conventional equations of static equilibrium $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$ may always be replaced by the independent moment equations $\Sigma M_A = 0$, $\Sigma M_B = 0$, and $\Sigma M_C = 0$, when points $A$, $B$, and $C$ are three points which do not lie on a straight line. This may be verified in the following manner: If a system of forces satisfies any one of these moment equations, such as $\Sigma M_A = 0$, then the resultant effect of the system cannot be a couple but may be a resultant force acting through point $A$. If the system also satisfies the equation $\Sigma M_B = 0$, then the resultant effect of the system may still be a resultant force; but, if so, such a force can act only along the line joining points $A$ and $B$. If in addition, however, the system also satisfies the equation $\Sigma M_C = 0$ (where $C$ does not lie on a line going through $A$ and $B$), this eliminates the possibility of the resultant force existing and acting along the line $AB$. Therefore, if the system

satisfies all three moment equations, its resultant effect can be neither a couple nor a resultant force and the system must be in a condition of static equilibrium.[1]

This principle may often be used to advantage in writing equations of static equilibrium, for it is often possible to select a moment axis so that only one unknown reaction element is involved in the summation of moments about that axis, thus leading to a direct solution for that one unknown. The student should study the illustrative examples in Art. 2·11 in conjunction with this article, since methods of arranging the computations and applying the above general approach will be discussed with a view to facilitating the numerical computations.

**2·9   Computation of Reactions—Structures Involving Equations of Condition.**   Whenever special conditions of construction have been introduced into a structure, the evaluation of the reactions involves both the equations of static equilibrium and the equations of condition. Even when such structures are statically determinate, the computation of the reactions is more difficult and requires more ingenuity than cases where no equations of condition have been introduced.
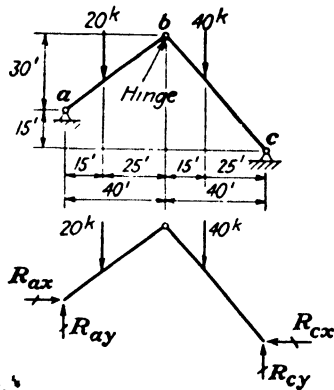


FIG. 2·8

To illustrate the method of attack in such cases, consider the structure shown in Fig. 2 8. This structure is of a type already discussed and may be shown to be statically determinate and stable. There are four unknown reaction elements, but in addition to the three equations of static equilibrium there is the equation of condition introduced by the frictionless hinge at point b. As discussed previously (Art. 2·4), such a hinge is neither capable of transmitting a couple from the portion ab to the portion bc, nor vice versa. It is therefore necessary that the algebraic sum of the moments about an axis through point b of all the forces acting on either the portion ab or the portion bc must add up to zero. Stating this in equation form,

$$\sum_{}^{ab} M_b = 0 \qquad or \qquad \sum_{}^{bc} M_b = 0$$

[1] *Question:* If two moment equations were obtained, first from $\Sigma M_A = 0$ and then from $\Sigma M_B = 0$, would a third independent equation be obtained by summing up the force components perpendicular to the line $AB$ and setting this sum equal to zero? Why? Would summing up the force components in any direction other than parallel to $AB$ result in a satisfactory independent equation?

At first glance, a student might infer from this that such a hinge really introduces two independent equations of condition. This is not so. There is only *one* independent equation introduced, as may easily be shown by the following reasoning: One of the equations of static equilibrium requires that for the entire structure the summation of the moments about any axis of all the forces shall be equal to zero, and hence, with $b$ as an axis, $\Sigma M_b = 0$. If one then writes the equation of condition that, for the portion $ab$, $\overset{ab}{\sum} M_b = 0$, it immediately implies that the algebraic sum of the moments about $b$ of all the remaining forces acting on the remainder of the structure $bc$ must add up to zero. Therefore $\overset{bc}{\sum} M_b = 0$ is not an independent relation—it is simply equal to the equation $\Sigma M_b = 0$ minus the equation $\overset{ab}{\sum} M_b = 0$. The student should constantly keep these ideas in mind when setting up the equations of condition of the structure. He should never be fooled into thinking that he has more independent equations than he actually has. In this case any two of these three equations—$\Sigma M_b = 0$, $\overset{ab}{\sum} M_b = 0$, $\overset{bc}{\sum} M_b = 0$—can be used independently, but the remaining one is not an independent relation.

To proceed now with the solution of this example, the reactions may be obtained easily if ingenuity is used in setting up the equations of statics.

$$\Sigma M_a = 0, \; \curvearrowright, \; (20)(15) + (10)(55) - 80R_{cy} + 15R_{cx} = 0$$

$$\therefore R_{cy} = 31.25 + {}^3\!/_{16}R_{cx} \quad (a)$$

$$\overset{bc}{\sum} M_b = 0, \; \curvearrowright, \; (10)(15) - 10R_{cy} + 15R_{cx} = 0$$

$$\therefore R_{cx} = {}^8\!/_9 R_{cy} - 13.33^* \quad (b)$$

Substituting for $R_{cy}$ from Eq. $(a)$ into Eq. $(b)$,

$$R_{cx} = {}^8\!/_9(31.25 + {}^3\!/_{16}R_{cx}) - 13.33 \qquad \therefore R_{cx} = 17.33 \text{ kips} \leftrightarrow$$

And then substituting back in $(a)$,

$$R_{cy} = 34.5 \text{ kips} \; \Updownarrow$$

In a similar manner,

$$\Sigma M_c = 0, \; \curvearrowright, \; 80R_{ay} + 15R_{ax} - (20)(65) - (10)(25) = 0$$

$$\therefore R_{ay} = 28.75 - {}^3\!/_{16}R_{ax} \quad (c)$$

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* Numbers such as 13.333 . . . will often be written 13.3, the dot over the last digit indicating that it may be repeated indefinitely.

$$\Sigma M_b = 0, \curvearrowright, \quad 40R_{ay} - 30R_{az} - (20)(25) = 0$$

$$\therefore R_{az} = \tfrac{4}{3}R_{ay} - 16.67 \quad (d)$$

Then substituting from Eq. (c) into Eq. (d),

$$R_{az} = \tfrac{4}{3}(28.75 - \tfrac{3}{16}R_{az}) - 16.67 \qquad \therefore R_{az} = 17.33 \text{ kips} \rightarrow$$

And substituting back in (c),

$$R_{ay} = 25.5 \text{ kips} \updownarrow$$

Using equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for checks on the solution,

$$\Sigma F_x = 0, \overrightarrow{+}, 17.33 - 17.33 = 0 \qquad \therefore \text{ O.K.}$$
$$\Sigma F_y = 0, \uparrow +, 25.5 - 20 - 40 + 34.5 = 0 \qquad \therefore \text{ O.K.}$$

If the supports of $a$ and $c$ of this structure had been on the same elevation, it is evident that the solution of the reactions would have been much easier, for in that case a direct solution for the vertical components of the reactions would be obtained from the equations $\Sigma M_a = 0$ and $\Sigma M_c = 0$. From a consideration of some of the illustrative examples in Art. 2·11, it will be apparent that the solution of the reactions of complicated structures involving special conditions of construction may be expedited in some cases by isolating internal portions as free bodies and applying the equations of static equilibrium to those portions. It should again be emphasized to the student, however, that such a technique does not add any new independent equations besides the three equations of static equilibrium for the entire structure plus the equations of condition resulting from the special conditions of construction. The equations used may be in a different form, but they are not new independent ones.

**2·10 Examples for Classification.** In this article, examples are discussed illustrating the methods of determining whether a structure is stable or unstable and whether it is statically determinate or indeterminate with respect to its reactions. Note that all beams are represented by straight lines coinciding with their centroidal axes and that their depth is not shown in the sketches of Fig. 2·9. This will be common practice in the remainder of the book whenever the depth does not essentially affect the solution of the problem.

Consider the beam shown in Fig. 2·9A(a). The unknown independent reaction elements are the magnitude and direction of the reaction at $A$ and the magnitude of the reaction at $B$ or a total of three. These elements may also be considered as the magnitude of the horizontal and vertical components of the reaction at $A$ and of either the horizontal or vertical component at $B$. Note that, if the point of application and the

direction of a reaction are known, the one unknown element of this reaction may be considered as the magnitude of the resultant reaction itself or the magnitude of either its vertical or horizontal component, since either component may be expressed in terms of the other, using the known direction of the reaction.  Since the reaction supplied by the
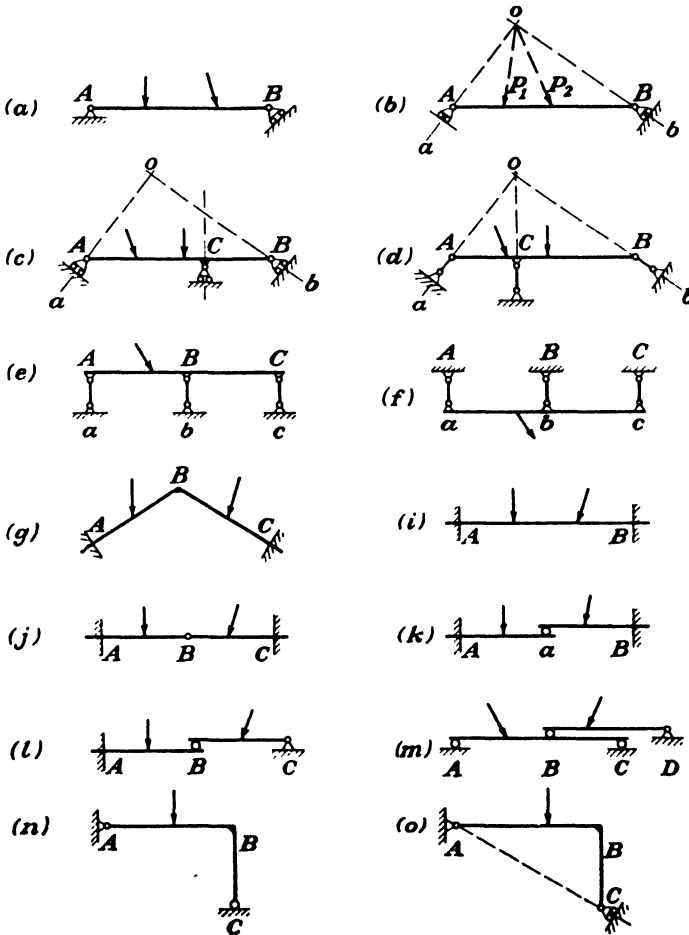


Fig. 2·9A

support at $B$ does not pass through point $A$, this structure is stable.  The three unknown reaction elements may therefore be found from the three available equations of static equilibrium, and the structure is statically determinate.

Consideration of the beam in Fig. 2·9A(b) shows that there are only two unknown reaction elements—the magnitude of the reactions at $A$ and $B$. These two unknown elements may also be considered as either the hori-

zo**⟨⟩**al or the vertical component of each reaction. Since the three equa-
tions of static equilibrium cannot be satisfied simultaneously by these two
independent unknown reaction elements, this structure is statically
unstable under a general system of applied loads. The lines of action
of the two reactions intersect at point $O$. If the resultant effect of the
applied loads is a resultant force whose line of action also passes through
point $O$, the two reactions will be capable of keeping such a special system
of loads in equilibrium. While the structure is still classed as unstable, it
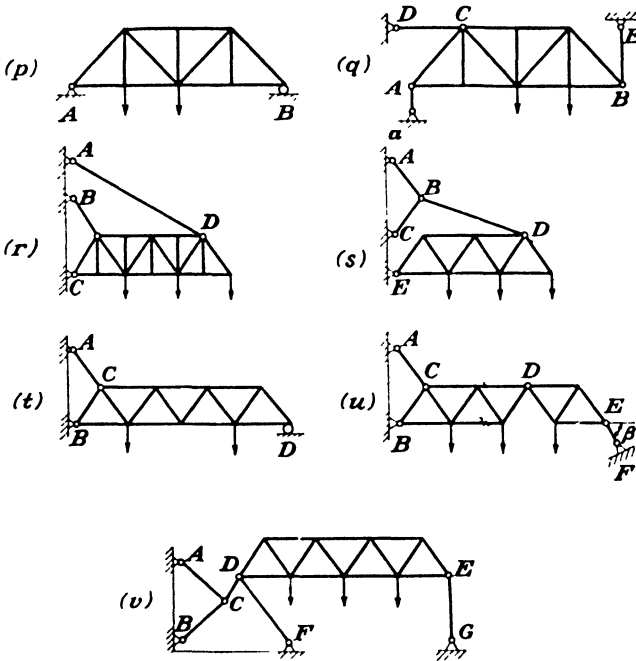


Fig. 2·9B

will be in a state of unstable equilibrium under this loading and its
reactions can be determined by the equations of static equilibrium.

If a roller support is added at point $C$, as shown in Fig. 2·9A(c), the
reactions of the structure will be equivalent to three links whose lines
of action are neither concurrent nor parallel. The structure will therefore
be stable and also statically determinate, for the three unknown reaction
elements—the three unknown magnitudes—can be found from the three
equations of static equilibrium. On the other hand, if the link support is
applied at point $C$ in Fig. 2·9A(d), the structure will be geometrically
unstable since the supports will consist of three links whose lines of action
all intersect at point $O$. The structure is not completely unrestrained,
of course, but will rotate instantaneously about point $O$ through some

finite angle until equilibrium is attained. In this new position, the reactions will be statically indeterminate.

The structure shown in Fig. 2·9$A$($g$) is obviously stable and has a total of six unknown reaction elements—a horizontal and vertical component and a couple at each support. Since there are only three equations of static equilibrium available, the structure is indeterminate to the third degree. The insertion of the hinge in the structure shown in Fig. 2·9$A$($j$) introduces one condition equation and therefore makes the structure indeterminate to the second degree. The insertion of the roller in Fig. 2·9$A$($k$) makes it possible to transmit only a vertical force from one part of the structure to the other. This in effect introduces two equations of condition, one that the sum of the moments about $a$ of the forces acting on *either* portion shall be equal to zero and the other that the sum of the horizontal components of the forces acting on *either* portion shall be equal to zero. As a result, this structure is indeterminate to the first degree only.

In the figures from $p$ on, all the truss portions should be considered as rigid bodies whose bar stresses are statically determinate once the reactions have been calculated. The arrangement of the bars of a truss necessary for stability, etc., is discussed in detail in Chap. 4. For the purposes of the present discussion, the trussed portions may be considered as solid bodies. The structures in Figs. 2·9$B$($p$) and ($q$) are easily seen to be stable and statically determinate under any general system of loads. In Fig. 2·9$B$($r$), the structure has four unknown reaction elements—the magnitude of the reactions at the link supports $A$ and $B$—and both the magnitude and direction of the reaction of the hinge support at $C$. To solve for these four unknowns there are only three equations of static equilibrium. The structure is therefore stable and statically indeterminate to the first degree.

For structures similar to that shown in Fig. 2·9$B$($s$), it is often better to use a different approach to determine their stability or statical determinancy. In such structures, the counting of the available equations may be awkward and obscure. Suppose instead that the structure is broken up into separate parts and the solution for the forces connecting each part to the rest of the structure is considered. If in this case an isolating section is passed through the link $BD$ and the hinge support at $E$ is replaced by the horizontal and vertical reaction components that it supplies, the truss portion will be isolated as a free body acted upon by the applied loads and three unknown forces—the link stress and the two components of the reaction at $E$. If this free body is to be in equilibrium, these three unknowns may be determined so as to satisfy the three equations of static equilibrium. Taking $\Sigma M_E = 0$ on the free body will yield an equation involving the link stress as the only unknown. Know-

ing the stress in the link *BD*, we can find the stresses in links *AB* and *BC* by $\Sigma F_x = 0$ and $\Sigma F_y = 0$ applied to hinge *B* isolated as a free body. The reactions of this structure may therefore be found by the equations of static equilibrium, and the structure is thus said to be stable and statically determinate.

**2·11 Illustrative Examples of Computation of Reactions.** The student should study the following examples carefully. All the structures are statically determinate, but the student should investigate them independently for practice.

**Example 2·1** *Determine the reactions of the beam ab:*

*Method A:*
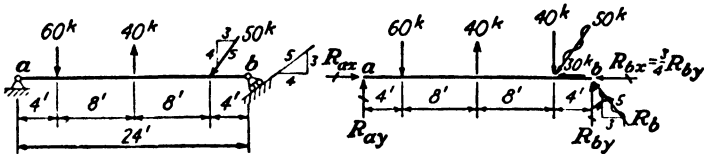


$\Sigma M_a = 0, \curvearrowright +$

$\quad (60)(4) - (40)(12) + (40)(20) - (R_{by})(24) = 0 \qquad \therefore R_{by} = {}^{560}\!/_{24} = 23.3^k \updownarrow$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R_{bx} = \tfrac{3}{4}R_{by} = 17.5^k \leftrightarrow$

$\Sigma M_b = 0, \curvearrowright +$

$\quad (R_{ay})(24) - (60)(20) + (40)(12) - (40)(4) = 0 \qquad \therefore R_{ay} = {}^{880}\!/_{24} = 36.6^k \updownarrow$

$\Sigma F_x = 0, \xrightarrow{+}, R_{ax} - 30 - 17.5 = 0 \qquad \therefore R_{ax} = 47.5^k \leftrightarrow$

*Check:*

$\qquad\qquad \Sigma F_y = 0, \uparrow +, 36.6 - 60 + 40 - 40 + 23.3 = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 = 0 \qquad \therefore O.K.$

*Method B:*



| $\Sigma M_b = 0, \curvearrowright +$ | | | $\Sigma M_a = 0, \curvearrowright +$ | | |
|---|---|---|---|---|---|
| | + | − | | + | − |
| $-(60)(20) =$ | | 1,200 | $+(60)(4) =$ | 240 | |
| $+(40)(12) =$ | 480 | | $-(40)(12) =$ | | 480 |
| $-(40)(4) =$ | | 160 | $+(40)(20) =$ | 800 | |
| | $+\overline{480}$ | $-\overline{1,360}$ | | $+\overline{1,040}$ | $-\overline{480}$ |
| | $+$ | 480 | | $-$ | 480 |
| $R_{ay} = 36.6^k \updownarrow = -\dfrac{880}{24}$ | | | $R_{by} = +\dfrac{560}{24} = 23.3^k \updownarrow$ | | |
| $R_{ax} = 47.5^k \leftrightarrow$ | | | $R_{bx} = -\tfrac{3}{4} \times 23.3 = 17.5^k \leftrightarrow$ | | |

*Method C:*

$$\Sigma M_b = 0: R_{ay} \qquad \uparrow \quad \downarrow$$
$$40 \times \tfrac{4}{24} = \quad 6.6$$
$$40 \times \tfrac{12}{24} = \qquad\quad 20$$
$$60 \times \tfrac{20}{24} = 50$$
$$\overline{\qquad\quad 56.6 \quad 20}$$
$$20$$
$$R_{ay} = \overline{36.6^k} \Uparrow$$
$$R_{ax} = \overline{47.5^k} \rightarrow\!\rightarrow$$

$$\Sigma M_a = 0: R_{by} \qquad \uparrow \quad \downarrow$$
$$60 \times \tfrac{4}{24} = 10$$
$$40 \times \tfrac{12}{24} = \qquad\quad 20$$
$$40 \times \tfrac{20}{24} = 33.\dot{3}$$
$$\overline{\qquad\quad 43.3 \quad 20}$$
$$20$$
$$R_{by} = \overline{23.3^k} \Uparrow$$
$$R_{bx} = \overline{17.5^k} \leftarrow\!\leftarrow$$

**Discussion:**

The three methods of solving for the reactions are all fundamentally the same and differ only in the details of the arrangement of the computations. Method A is probably the best way to organize the computations for an unusual or complicated structure. However, the systematic arrangement of either methods B or C will be found most useful for the simple or conventional types of beams and trusses.

Note that it is usually extremely convenient to replace inclined forces by their horizontal and vertical components and to use these components instead of the forces in writing the equations of static equilibrium.[1]

Note also that it is helpful in checking work to indicate the directions of couples or forces which were assumed as plus in writing the equations.

The computer should clearly indicate his results by underlining them and also by indicating the units and directions of the forces. Remember that if an answer comes out plus, the force was assumed acting in the correct direction on the free-body sketch; if the answer comes out minus, the force acts opposite to the direction assumed.

Note that in this problem two moment equations and one force equation were used in the solution. The check equation $\Sigma F_y = 0$ gives a check on the vertical components of the reactions but does not check the horizontal components. If $\Sigma M = 0$ were written about any axis that did not lie on a line through a and b, a check would be obtained on the values of the horizontal reactions.

[1] If both a force and its components are shown on a free-body sketch, a wavy line should be drawn along the shank of the force arrow, thereby indicating that the force has been replaced by its components

**Example 2·2** *Determine the reactions of this structure:*



**Isolate ab:**

$\Sigma M_b = 0, \curvearrowleft$
$(9)(R_{ay}) - (40)(5) = 0$
$\therefore R_{ay} = \underline{22.2^k} \Updownarrow$

$\Sigma M_a = 0, \curvearrowleft$
$(40)(4) - (S)(9) = 0$
$S = \underline{17.7^k} \uparrow$

$\Sigma F_x = 0$
$\therefore R_{ax} = \underline{30^k} \Longleftarrow +$

**Isolate bd:**

$\Sigma M_d = 0, \curvearrowleft$
$-(17.7)(18) = -320$
$-(40)(15) = -600$
$-(4 \times 8)(8) = -256$
$-(30)(2) = -60$
$R_{cy} = 103^k \Updownarrow = \dfrac{-1{,}236}{12}$

$\Sigma F_x = 0$
$\therefore F = \underline{30^k} \longleftarrow$

$\Sigma M_c = 0, \curvearrowleft \qquad + \qquad -$
$-(17.7)(6) = \qquad 106.6$
$-(40)(3) = \qquad 120$
$+(4 \times 8)(4) = 128$
$+(30)(10) = 300$
$\qquad\qquad +428 \quad -226.6$
$\qquad\qquad\qquad -226.6$
$R_{dy} = \dfrac{+201.3}{12} = 16.7^k \Updownarrow$

**Check:** $\Sigma F_y = 0, \uparrow +,$ *for entire structure*

$22.2 - 40 - 40 + 103 - 32 - 30 + 16.7 = 0$
$142 - 142 = 0 \qquad \therefore O.K.$

**Discussion:**

 When isolating the two portions ab and bd from one another, imagine that the hinge pin is removed. The hinge may transmit a force acting in any direction through the center of the hinge pin. If the horizontal and vertical components of this force are designated by F and S and assumed to act as shown on the portion ab, they must act in the opposite senses on the portion bd. Note that an independent check could be obtained on both the horizontal and the vertical reactions by taking $\Sigma M = 0$ about some axis which does not lie on the line through a and d.

**Example 2·3** *Determine the reactions of this beam:*
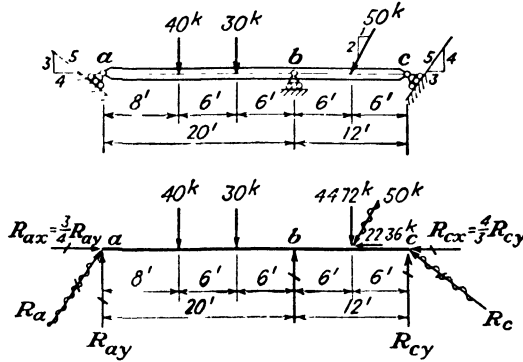


$\Sigma M_b = 0$,

$(R_{ay})(20) - (40)(12) - (30)(6) + (44.72)(6) - (R_{cy})(12) = 0$,

$$20R_{ay} = 12R_{cy} + 391.68$$

$\Sigma F_x = 0$, $\xrightarrow{}$, $\tfrac{3}{4}R_{ay} - 22.36 - \tfrac{4}{3}R_{cy} = 0$, $R_{cy} = 0.5625R_{ay} - 16.77$

$\therefore 20R_{ay} = 6.75R_{ay} - 201.24 + 391.68$, $R_{ay} = +14.37^k \updownarrow$, $R_{ax} = 10.78^k \rightarrow$

*and*

$R_{cy} = 8.08 - 16.77 = -8.69$ $R_{cy} = -8.69^k \updownarrow$, $R_{cx} = -11.59^k \rightarrow$

$\Sigma M_a = 0$,

$(40)(8) + (30)(14) - (R_{by})(20) + (44.72)(26) + (8.69)(32) = 0$,

$$R_{by} = 109.04^k \updownarrow$$

*Check:* $\Sigma F_y = 0$, $\uparrow +$

$14.37 - 40 - 30 + 109.04 - 44.72 - 8.69 = 0$

$123.41 - 123.41 = 0$ $\therefore$ O.K.

*Check:* $\Sigma F_x = 0$, $\xrightarrow{}$

$10.78 - 22.36 + 11.59 = 0$

$-22.36 + 22.37 = 0$ $\therefore$ O.K.

**Example 2·4** *Determine the reactions of this structure:*
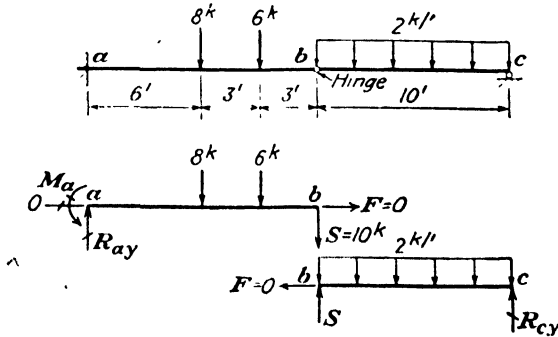
*Isolate bc:*

$$\Sigma M_b = 0, \; \widehat{+}, \; (2)(10)(5) - (R_{cy})(10) = 0 \qquad \therefore R_{cy} = 10^k \; \updownarrow$$

$$\Sigma M_c = 0, \; \widehat{+}, \; (S)(10) - (2)(10)(5) = 0 \qquad \therefore S = 10^k \uparrow$$

*Isolate ab:*

$$\Sigma M_a = 0, \; \widehat{+}, \; (10)(12) + (6)(9) + (8)(6) - M_a = 0 \qquad \therefore M_a = 222^{k1} \{$$

$$\Sigma F_y = 0, \; \uparrow +, \; R_{ay} - 8 - 6 - 10 = 0, \; R_{ay} = 24^k \; \updownarrow$$

*Check:* $\Sigma F_y = 0,$ *applied to entire beam,*

$$24 - 8 - 6 - 20 + 10 = 0$$
$$\therefore 0 = 0 \qquad \therefore O.K.$$

**Example 2·5** *Determine the reactions of this truss:*
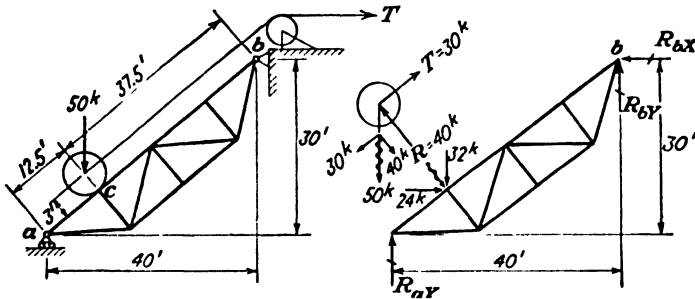


$$\Sigma F_x = 0, \qquad R_{bx} = 24^k \; \leftleftarrows$$

$$\Sigma M_b = 0, \; \widehat{+}, \; (R_{ay})(40) - (40)(37.5) = 0, \qquad R_{ay} = 37.5^k \; \updownarrow$$

$$\Sigma M_a = 0, \; \widehat{+}, \; (40)(12.5) - (24)(30) - (R_{by})(40) = 0, \qquad R_{by} = -5.5^k \; \updownarrow$$

*Check:* $\qquad \Sigma F_y = 0, \; \uparrow +, \; 37.5 - 32 - 5.5 = 0 \qquad \therefore O.K.$

**Discussion:**

In problems such as this, first compute the loads acting on the structure. This may be done by isolating the roller, that is in equilibrium under its own weight, the tension in the cable, and the reaction of the truss on the roller. The loads acting on the structure having first been determined, it is a straigh'forward problem to determine the reactions. Note that it has been assumed that there is no friction between the roller and truss and therefore that the interacting force between the two acts normal to the line ab and is directed through the center of the roller.

**Example 2·6**  *Determine the reactions of this structure:*
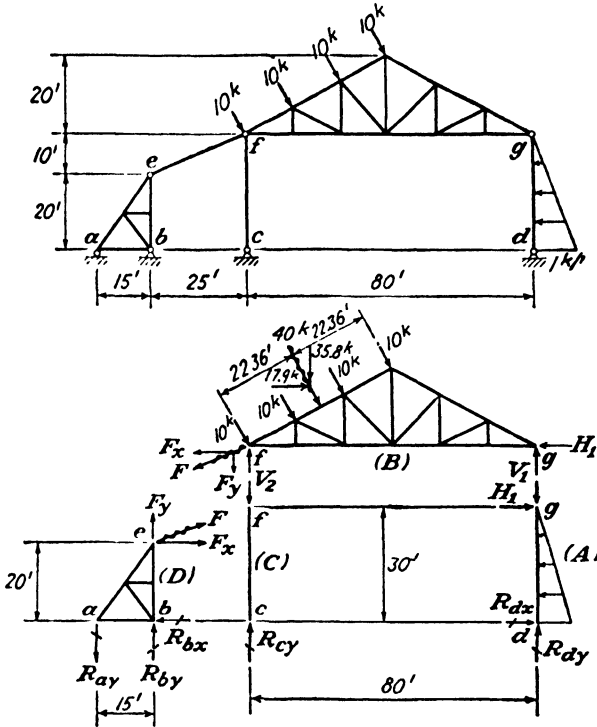


*Free-body sketch A:*

$$\Sigma M_d = 0, \;\overset{\curvearrowright}{+}, \; (H_1)(30) - \frac{(1)(30)}{2}(10) = 0, \qquad H_1 = 5^k \rightarrow\!\!\rightarrow$$

$$\Sigma M_g = 0, \;\overset{\curvearrowright}{+}, \; \frac{(1)(30)}{2}(20) - (R_{dx})(30) = 0, \qquad R_{dx} = 10^k \rightarrow\!\!\rightarrow$$

*Free-body sketch B:*

$$\Sigma M_f = 0, \;\overset{\curvearrowright}{+}, \; (40)(22.36) - (V_1)(80) = 0, \qquad V_1 = 11.18^k \uparrow \qquad \therefore R_{dy} = \underline{11.18^k} \;\Updownarrow$$

$$\Sigma F_x = 0, \;\overset{\rightarrow}{+}, \; 17.9 - 5 - F_x = 0, \qquad F_x = 12.9 \leftarrow \qquad \therefore F_y = \tfrac{2}{5}(12.9) = 5.16^k \;\Updownarrow$$

$$\Sigma M_g = 0, \;\overset{\curvearrowright}{+}, \; (17.9)(10) - (35.8)(60) - (5.16)(80) + (V_2)(80) = 0,$$

$$V_2 = 29.77 \uparrow$$
$$\therefore R_{cy} = \underline{29.77^k} \;\Updownarrow$$

**Check:**  $\Sigma F_y = 0, \;\uparrow+, \; -5.16 + 29.77 - 35.8 + 11.18 = 0$

$$\therefore -0.01 = 0 \qquad \therefore O.K.$$

*Free-body sketch D:*

$\Sigma M_b = 0$, $\curvearrowright +$, $(12.9)(20) - (R_{ay})(15) = 0$    $\therefore R_{ay} = 17.2^k \updownarrow$

$\Sigma M_a = 0$, $\curvearrowright +$, $(12.9)(20) - (5.16)(15) - (R_{by})(15) = 0$    $\therefore R_{by} = 12.04^k \updownarrow$

$\Sigma F_x = 0$,     $R_{bx} = 12.9^k \leftrightarrow$

*Check on structure as a whole:*

$\Sigma F_x = 0$, $\rightarrow +$, $-12.9 + 17.9 + 10 - \dfrac{(1)(30)}{2} = 0$    $\therefore 0 = 0$    O.K.

$\Sigma F_y = 0$, $\uparrow +$, $-17.2 + 12.04 + 29.77 - 35.8 + 11.18 = 0$,    $-0.01 = 0$    O.K.

$\Sigma M_a = 0$, $\curvearrowright +$, $-(12.04)(15) - (29.77)(40) - (11.18)(120) + (17.9)(40)$

$$+ (35.8)(60) - \frac{(1)(30)}{2}(10) = 0$$

$$-2,863 + 2,864 = 0    O.K.$$

**Discussion:**

*In problems of this type, the structure is broken up into its separate structural elements, and the free-body sketches are drawn for each element. The interacting forces between elements may be assumed to act in either sense, but they must act in opposite senses on two adjacent elements. For example, if force F is assumed to act down to the left on sketch B, it must act up to the right on sketch D. Since the structure as a whole is in equilibrium, each of its elements must be in equilibrium. The equations of static equilibrium for each element must be satisfied, and they therefore form a basis for the solution of the unknown reactions and unknown interacting forces.*

*· Note that when all the unknown reactions have been obtained the results may be checked by applying the equations of static equilibrium to the structure as a whole to see whether or not they are satisfied.*

**2·12 Superposition of Effects.** At this point, a brief discussion of the *principle of superposition* is appropriate. This principle is used continually in structural computations. As a matter of fact, its use has already been implied in the solution of Example 2·1 by method C. There the reactions due to each of the three loads were computed separately and added algebraically to obtain the reactions due to all three loads acting simultaneously. In other words, the separate effects were superimposed to obtain the total effect.

Such a procedure is usually permissible. However, there are two important cases in which the principle of superposition is *not* valid: (1) when the geometry of the structure changes an essential amount during the application of the loads; (2) when the strains in the structure are not directly proportional to the corresponding stresses, even though the effect of change in geometry can be neglected. The latter case occurs whenever the material of the structure is stressed beyond the elastic limit or when it does not follow Hooke's law through any portion of its stress-strain curve.

In Art. 2·3, it is pointed out that usually the deformations of a structure are so slight that it is permissible to consider the structure as a rigid body in applying the equations of static equilibrium and therefore to

neglect the effect of slight changes in geometry on the lever arms of forces, the inclinations of members of the structure, etc.   However, consider the structure shown in Fig. 2·10.   In this case, all three hinges lie along the same straight line in the unloaded structure.   It will be found necessary to consider the alteration of the geometry caused by the deformation of the structure, for the lever arms, slopes of the members, etc., are changed by an important amount.   As a result, it will be found that the stresses and deflections in the structure are not directly proportioned to the load $P$ even though the material of the structure may follow Hooke's law.   This is there- fore an illustration of the first case noted above where the principle of superposi- tion is not valid.   In this structure, the



FIG. 2·10

effects of a load $2P$ are not twice the effects of a load of $P$, nor are the effects of a load equal to $P_1 + P_2$ equal to the algebraic sum of the effects of $P_1$ and of $P_2$ acting separately.   From applied mechanics, a more important case may be recalled where superposition is not valid—the case of a slender strut acted upon by both axial and transverse loads.   There the stresses, moments, deflections, etc., due to an axial load of $P_1 + P_2$ are not equal to the algebraic sum of the values caused by $P_1$ and $P_2$ acting separately.   Fortunately, most cases of this type where superposition is not valid are easily recognized.

It is mentioned above that superposition is not valid in cases where, although the effect of change in geometry can be neglected, the material of the structure does not follow Hooke's law.   If such structures are also statically determinate, then quantities that can be found by statically determinate stress analysis (such as reactions, shears, and bending moments) may be superimposed but fiber stresses and deflections cannot be superimposed.   For example, in the case of an end-supported cast-iron beam, the reactions and the shear and bending moment on the transverse cross sections are statically determinate quantities and may be super- imposed.   However, the fiber stresses and deflections produced by the bending moment due to a load $2P$ are not equal to twice those due to load $P$, and hence such quantities cannot be superimposed.   If the reactions of a cast-iron beam are statically indeterminate, then none of these quan- tities may be superimposed, since the stress analysis is then a function of the distortion of the structure.

## 2·13   Problems for Solution.

Problem 2·1   Classify the structures shown in Figs. 2·9 $e, f, i. l$ to $o$, and $t$ to $v$ as stable or unstable, and statically determinate or indeterminate.   Discuss and state reasons for your answer.
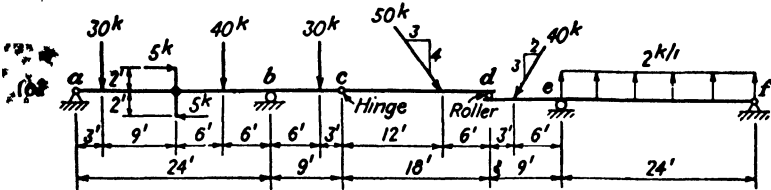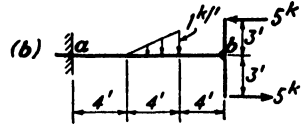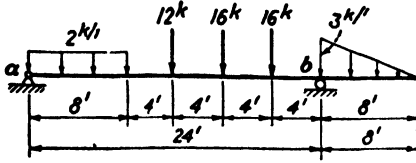
Hint: Only a vertical force can be transmitted through
      the roller at point d
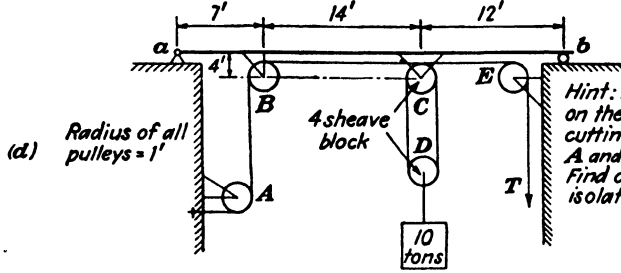


(d) Radius of all
    pulleys = 1'

4 sheave
block

Hint: Find the reactions
on the beam a b by
cutting cable between
A and B and C and E
Find cable tension by
isolating pulley D

10
tons
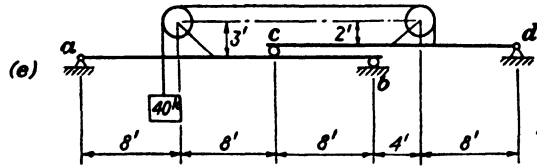


(e)

Radius of both
pulleys = 1'
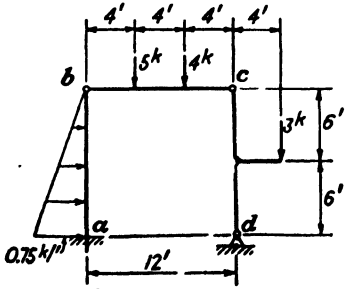


(f)
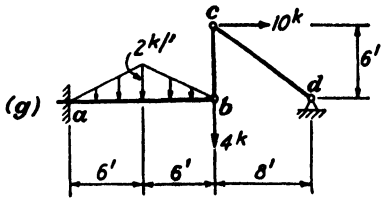
(g)

FIG. 2·11

**Problem 2·2**   Determine the reactions of the beams of Fig. 2·11.
**Problem 2·3**   Determine the reactions of the structures of Fig. 2·12.
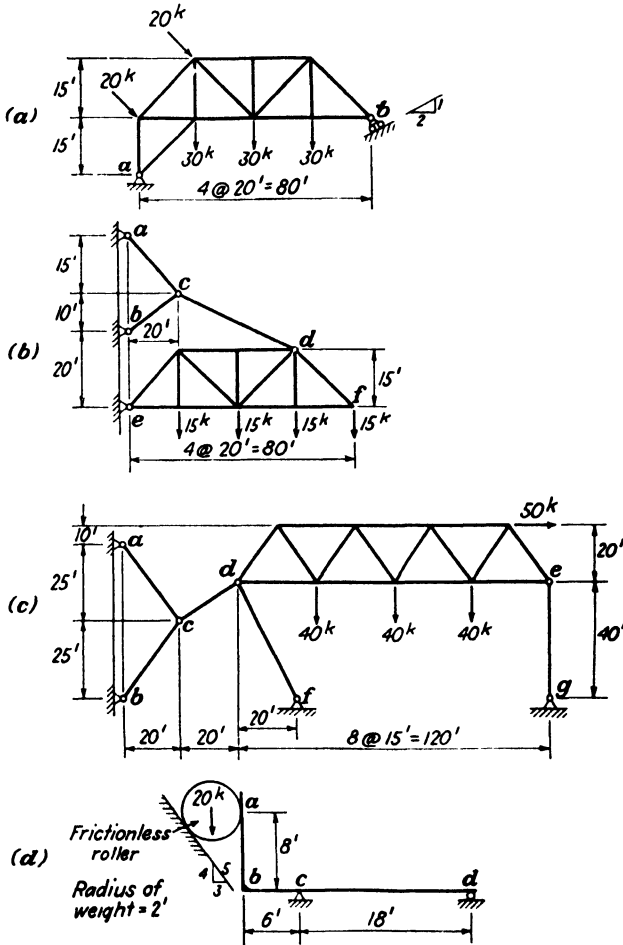


Fig. 2·12

**Problem 2·4**   It is sometimes difficult to convince a student that there are only a certain number of independent equations.   To demonstrate this, consider the structure shown in Fig. 2·8.   Draw three free-body sketches—one of the entire structure, one of the portion *ab*, and one of the portion *bc*.   Write three equations of static equilibrium for each sketch.   Compare and combine these nine equations containing six unknowns, and show that there are only four independent equations containing four unknowns among them.

# CHAPTER 3

## SHEAR AND BENDING MOMENT

**3·1 General.** The ultimate aim of all stress analysis is to determine the adequacy of a structure to carry the loads for which it is being designed. The criteria for determining this involves a comparison of the fiber stresses developed by the applied loads with the allowable stresses for the structural material being used. The stresses acting on any cross section may be studied by passing an imaginary section that cuts through the structure along this cross section and isolates *any* convenient portion of the structure as a free body. If all the other forces acting on this isolated portion have already been determined, the required resultant effects of the fiber stresses acting on the cross section being investigated may easily be computed by the equations of static equilibrium.

One of the commonest structural elements to be investigated in this manner is a *beam*, *i.e.*, a member that is subjected to bending or flexure by loads acting transversely to its centroidal axis or sometimes by loads acting both transversely and parallel to this axis. The following discussions are limited to straight beams, *i.e.*, beams in which the axis joining the centroids of the cross sections (centroidal axis) is a straight line. It is also assumed that all the loads and reactions lie in a single plane which also contains the centroidal axis of the member. If this were not so, the beam might be subjected to both twisting and bending.

**3·2 Determination of Stresses in Beams.** Suppose that, in determining the adequacy of the statically determinate beam in Fig. 3·1, it is necessary to compute the fiber stresses on a transverse cross section *mn*. The reactions necessary for static equilibrium may be computed easily and are shown in free-body sketch *A*. The portions of the beam to the left and right of cross section *mn* may be imagined to be isolated from one another by cutting the structure in two along this section. Free-body sketches *B* and *C* may then be drawn showing all the forces acting on these two portions of the beam.

When one considers the external forces acting on either of the portions *B* or *C*, it is immediately apparent that the portions are not in static equilibrium under the external forces acting alone. If the beam as a whole is in equilibrium, however, then each and every portion of it must be in equilibrium. It is therefore necessary that there shall be internal forces or stresses distributed over the internal faces which have been exposed by the imaginary cut. These stresses must be of such a magni-

tude that their resultant effect balances that of the external forces acting
on the isolated portion and therefore maintains the portion in a state of
static equilibrium.

The fiber stresses acting on the exposed internal faces may be broken
down into two components, one component acting normal to the face
and called the *normal fiber stresses* and the other acting parallel to the
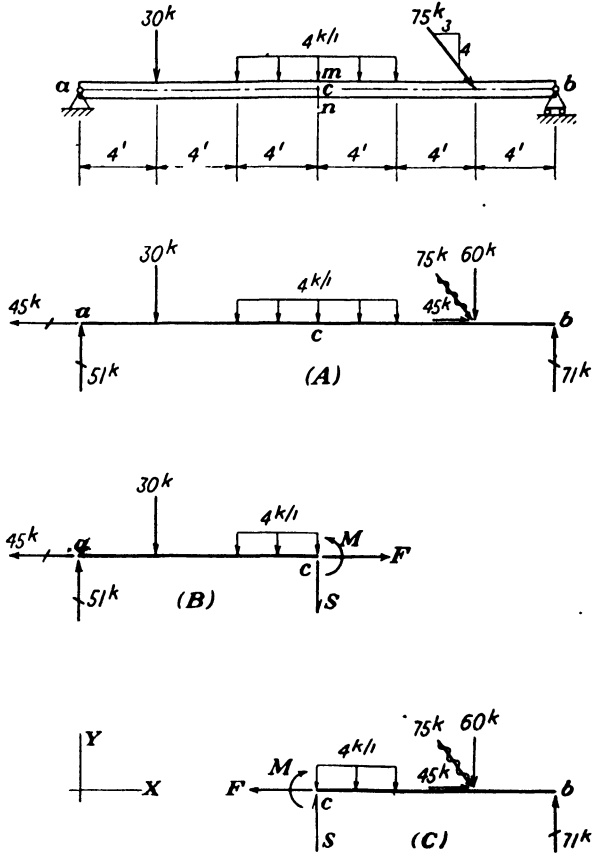


FIG. 3·1

face and called the *shear fiber stresses*.   In the free-body sketches *B* and
*C*, these fiber stresses have been replaced by their resultant effect as
represented by the stresses *S* and *F*, acting through the centroid of the
cross section, and the couple *M*.   Note that the resultant effects *S*, *F*, and
*M* of the fiber stresses acting on the portion in sketch *B* are shown to be
numerically equal but opposite in sense to the corresponding effects
shown in sketch *C*.   That this must be so is apparent from the following
considerations:

Consider the free-body sketch A showing the entire beam acted upon by all the external loads and reactions. Suppose that the resultant of all the external forces applied to the beam on the left of section *mn* is computed in magnitude and position. Suppose similar computations are made for the resultant of the remaining external forces applied to the right of *mn*. Now if the beam as a whole is in equilibrium under *all* the forces acting on it, it must be apparent that the resultant of those forces applied to the left of *mn* must be collinear, numerically equal, but opposite in sense to the resultant of the remaining forces applied to the right of *mn*. It therefore follows that the resultant of the external forces in sketch B must be numerically equal but opposite in sense to the resultant of the external forces in sketch C and hence the resultant effects of the fiber stresses in sketches B and C must likewise bear the same relation.

It is convenient to assign names to $F$, $S$, and $M$, the resultant effects of the fiber stresses acting on a cross section of a member. The axial force $F$ acts through the centroid of the cross section and will be called the *axial stress*. The transverse force $S$ will be called the *shear stress* and the couple $M$ the *resisting moment*.

In order to satisfy the three equations of static equilibrium for either the portion of the beam in sketch B or the portion in sketch C, the magnitudes of $F$, $S$, and $M$ must be such as to counteract the resultant of the external forces acting on the portion considered. Which portion is used makes no difference theoretically. The portion having the fewer external forces is ordinarily used so as to simplify the computations. Static equilibrium will be maintained by values computed in the following manner: The axial stress $F$ and the shear stress $S$ must be equal and opposite to the axial and transverse components, respectively, of the resultant of the external forces acting on the portion of the beam under consideration. Upon taking moments about an axis through the centroid of the cross section, *i.e.*, the point of intersection of the stresses $F$ and $S$, it is then apparent that the resisting moment $M$ must be equal in magnitude but opposite in direction to the moment of the resultant of the external forces acting on this portion.

Once the axial stress, the shear stress, and the resisting moment have been determined at any section, the intensities of the normal and shear stresses at any point on the cross section may be computed by using well-known equations given in standard textbooks on the strength of materials.

**3·3  Shear and Bending Moment Defined; Sign Convention.**
From the previous discussion, it is evident that, in order to determine the magnitudes of the axial stress, the shear stress, and the resisting moment acting on a cross section of a beam, it is advisable to compute first the

magnitude and position of the resultant of the external forces acting on the portion of the beam on either side of that cross section.  It is usually convenient to represent this resultant by its axial component, its transverse component, and its moment about an axis through the centroid of the cross section under consideration.  These three elements are statically equivalent to this resultant and are given the following three names, respectively: *axial force, shear force,* and *bending moment.*  The definition of these three terms may therefore be summarized as follows:

*Axial Force F:* The axial force at any transverse cross section of a straight beam is the algebraic sum of the components acting parallel to the axis of the beam of all the loads and reactions applied to the portion of the beam on *either* side of that cross section.

*Shear Force (Shear) S:* The shear force at any transverse cross section of a straight beam is the algebraic sum of the components acting transverse to the axis of the beam of all the loads and reactions applied to the portion of the beam on *either* side of the cross section.

(+) Axial force

*Bending Moment M:* The bending moment at any transverse cross section of a straight beam is the algebraic sum of the moments, taken about an axis passing through the centroid of the cross section, of all the loads and reactions applied to the portion of the beam on either side of the cross section.  The axis about which the moments are taken is, of course, normal to the plane of loading.

(+) Shear force

(+) Bending moment

Fig. 3·2

While it is not the purpose of the authors to encourage the memorizing of structural principles, formulas, etc., these definitions recur constantly and are so fundamental to structural engineering that students should study and understand them so thoroughly and completely that they are indelibly impressed on their minds.

With these definitions introduced, this discussion may now be summarized by saying that the axial stress acting on a cross section will be equal but opposite to the axial force at that section; the shear stress will be equal and opposite to the shear force (or shear); and the resisting moment will be equal and opposite to the bending moment.

In subsequent calculations, the following sign convention will be used to designate the directions of axial force, shear, and bending moment at any transverse cross section of a beam.  The convention is that ordinarily used in structural engineering.  It is clear and simple to employ and will be referred to as the *beam convention.*  As shown in Fig. 3·2, axial force is plus when it tends to pull two portions of a member apart, *i.e.,* when it tends to produce a tensile stress on the cross section.  Shear force is plus
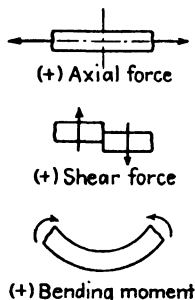
when it tends to push the left portion upward with respect to the right.
Bending moment is plus when it tends to produce tension in the lower
fibers of the beam and compression in the upper fibers, *i.e.*, to bend the
beam concave upward.   Many beams are horizontal, and this convention
may be applied without confusion.   When a member is not horizontal,
however, either side may be selected as the "lower side" and the beam
convention applied to correspond.

   **3·4   Method of Computation of Shear and Bending Moment.**
The procedure for computing the axial force, shear force, and bending
moment at any section of a beam is straightforward and may be explained
easily by the illustrative problem shown in Fig. 3·3.   In this problem, it is
desired to compute the axial force, shear, and bending moment at the
cross sections at points *b*, *c*, and *d*.   The computation of the axial force
is simple and needs no explanation in this case, as is true of this computa-
tion with respect to most beams.   If the couple applied by the bracket
at point *c* is assumed to be applied along the cross section at point *c*,
there will be an abrupt change in the bending moment at this point and
it is necessary to compute the bending moment, first on a cross section
an infinitesimal distance to the left of *c* and then on a cross section an
infinitesimal distance to the right of *c*.

   It will be recalled that the shear and bending moment at any cross
section may be computed by considering *all* the external loads and
reactions applied to the portion of the beam on *either* side of the cross
section under consideration.   *Either* portion may be used, but the com-
putations may usually be performed more efficiently by using that involv-
ing the smaller number of forces.   One portion having been selected,
*only the loads and reactions acting on that portion* are included in the
summation of the force components or moments.

   The left-hand portion of the beam was chosen for computing the shear
and bending moment at section *b*, and at the sections to the left and right
of point *c*.   Free-body sketches *B*, *C*, and *D*, respectively, show the por-
tion used in each case.   To illustrate the advantage of using one portion
instead of the other, the shear and bending moment at section *d* are
computed by using first the left- and then the right-hand portion, as
shown in free-body sketches *E* and *F*, respectively.   Note the simplicity
of the computations for sketch *F* as compared with sketch *E*.

   To explain typical computations, consider those for the shear and
bending moment on the cross section just to the left of point *c*, as shown in
sketch *C*.   The shear force at this section is the algebraic sum of the
57-kip reaction, the uniformly distributed load totaling 16 kips, and the
concentrated load of 30 kips, where the reaction causes positive shear
(tends to push left portion up) and the two loads cause negative shear

(tend to push the left portion down). Hence, the shear force is

$$S = +57 - 16 - 30 = +11 \text{ kips}$$

having a resultant positive effect tending to push the left portion up with respect to the right. For equilibrium, the total shear stress obviously must be 11 kips acting down on the cross section in sketch C. In the same manner, the bending moment is the algebraic sum of the moments
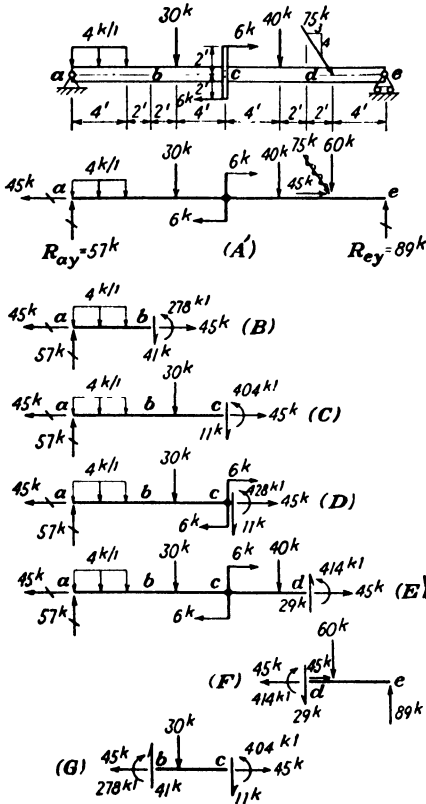


Sketch A:

$$\Sigma M_e = 0 +$$
| | |
|---|---|
| $(4 \times 4)(22)$ = | $-$ 352 |
| $(30)(16)$ = | $-$ 480 |
| $(40)(8)$ = | $-$ 320 |
| $(60)(4)$ = | $-$ 240 |
| | $\overline{-1392}$ |
| $(6)(4)$ = | $+$ 24 |
| | $\overline{-1368}$ |

$$\therefore R_{ay} = \frac{1368}{24} = 57^k \uparrow$$

$$\Sigma M_a = 0 +$$
| | |
|---|---|
| $(4 \times 4)(2)$ = | $+$ 32 |
| $(30)(8)$ = | 240 |
| $(6)(4)$ = | 24 |
| $(40)(16)$ = | 640 |
| $(60)(20)$ = | 1200 |
| | $\overline{+2136}$ |

$$\therefore R_{ey} = \frac{2136}{24} = 89^k \uparrow$$

Check: $\Sigma F_y = 0, \uparrow +$

$$57 + 89 - 16 - 30 - 40 - 60 = 0 \quad \text{o.k}$$

Sketch B: $S = 57 - (4)(4) = +41^k$
$M = (57)(6) - (16)(4) = +278^{k'}$

Sketch C: $S = 57 - 16 - 30 = +11^k$
(to left of $\quad M = (57)(12) - (16)(10) - (30)(4)$
point C) $\quad = +404^{k'}$

Sketch D: $S = 57 - 16 - 30 = +11^k$
to right of $M = (57)(12) - (16)(10) - (30)(4)$
(point C) $\quad + 24 = +428^{k'}$

Sketch E: $S = 57 - 16 - 30 - 40 = -29^k$
$M = (57)(18) - (16)(16) - (30)(10)$
$\quad - (40)(2) + 24 = +414^{k'}$

Sketch F: $S = -89 + 60 = -29^k$
$M = (89)(6) - (60)(2) = +414^{k'}$

Sketch G: $S = 41 - 30 = +11^k$
$M = 278 + (41)(6) - (30)(4)$
$\quad = +404^{k'}$

FIG. 3·3

about the centroid of the cross section of the same three forces. Since the reaction tends to produce tension in the lower fibers at the section and the two loads tension in the upper fibers, the bending moment is

$$M = (57)(12) - (16)(10) - (30)(4) = +404 \text{ kip-ft}$$

having a resultant positive tendency to produce tension in the lower fibers. Again for equilibrium, it is necessary for the resisting moment to be 404 kip-ft acting counterclockwise on the cross section in sketch C. The remaining computations are self-explanatory.

·It is important to note that, if the axial force, shear, and bending moment are known at one cross section, similar items may be computed at any other cross section, by using these known quantities rather than working with all the external forces on the entire portion of the beam on either side of the new cross section. For example, the axial force, shear, and bending moment at c could be computed with the quantities already computed at b. This is apparent since the shear, axial force, and bending moment at b are statically equivalent to the resultant of the external forces applied to the left of b. Hence, the contribution of these forces to the resultant effect of all the forces to the left of c may be evaluated by using their statical equivalent rather than the forces themselves. The advantage of this procedure increases with the number of external forces applied to the left of point b. Such computations are illustrated under sketch G of Fig. 3·3, and the forces acting on the isolated portion bc are shown in this sketch.

**3·5  Shear and Bending-moment Curves.** When a beam is being analyzed or designed for a stationary system of loads, it is helpful to have curves or diagrams available from which the value of the shear and bending moment at any cross section may readily be obtained. Such curves may be constructed by drawing a base line corresponding in length to the axis of the beam and then plotting ordinates at points along this base line, which indicate the value of the shear or bending moment at that cross section of the beam. Plus values of shear or bending moment are plotted as upward ordinates from the base line and minus values as downward ordinates. Curves drawn connecting the ends of all such ordinates along the base line are called shear and bending-moment curves. In Fig. 3·4, shear and bending-moment curves are shown for the beam in Fig. 3·3.

The construction of these curves is quite straightforward, but needs some explanation. The shear on a cross section an infinitesimal distance to the right of point a is +57 kips, and therefore the shear curve rises abruptly from zero to +57 at this point. In the portion af, the shear on any cross section a distance x from point a is

$$S = 57 - 4x$$

which indicates that the shear curve in this portion is a straight line decreasing from an ordinate of +57 at a to +41 at point f. Since no additional external loads are applied between points f and g, the shear remains +41 on any cross section throughout this interval and the shear curve is a horizontal line as shown. An infinitesimal distance to the left of point g the shear is +41, but at an infinitesimal distance to the right of this point the 30-kip load has caused the shear to be further reduced to

+11. Therefore, at point $g$, there is an abrupt change in the shear curve from +41 to +11. In this same manner, the remainder of the shear curve may easily be verified. It should be noted that, in effect, a concentrated load is assumed to be applied at a point and hence at such a point the ordinate to the shear curve changes abruptly by an amount equal to the load. Physically, it is impossible for a load to be applied at a point without developing an infinite contact pressure, and it is therefore necessary for such loads to be distributed over a small area. However,

45$^k$  4$^{k/t}$  30$^k$  6$^k$  40$^k$ 75$^k$ 60$^k$

$57 \times 4 =$      228
$-16 \times 2 = -$ 32
                196 $= M @ f$
$41 \times 4 =$     164
$-30$               360 $= M @ g$
$11 \times 4 =$      44
                    404 $= M @ c$
                     24
                    428 $= M @ c$
$11 \times 4 =$      44
$-40$               472 $= M @ h$
$-29 \times 4 = -116$
$-60$               356 $= M @ j$
$-89 \times 4 = -356$
                      0 $= M @ e$
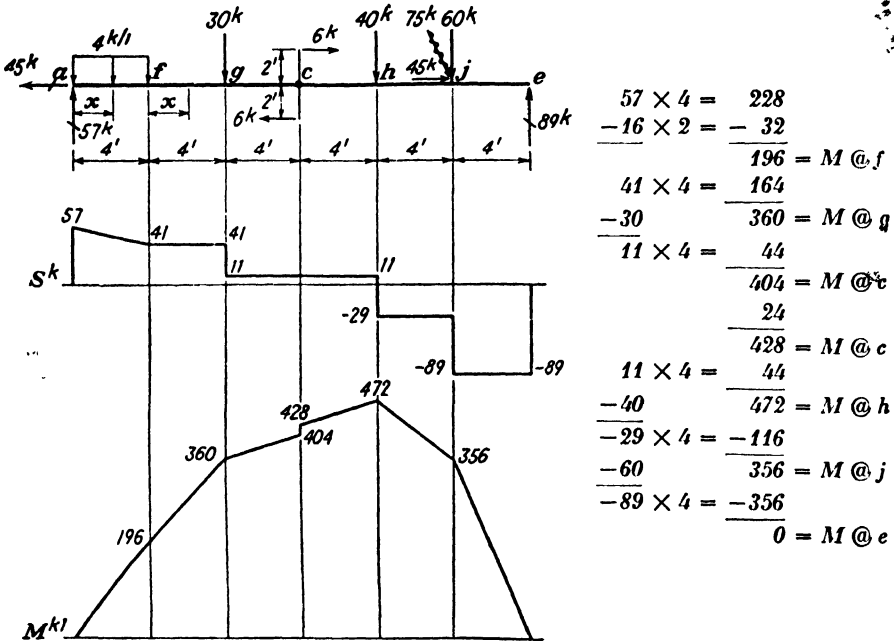
FIG. 3·4

for computations such as for shear and bending moment, such inconsistencies are ignored, and it is considered mathematically possible that concentrated loads may be applied at a point.

In the portion $af$, the bending moment at a cross section a distance $x$ from point $a$ is $M = 57x - 2x^2$. Therefore, the bending-moment curve starts at 0 at point $a$ and increases along a curved line to an ordinate of +196 kip-ft at point $f$. In the portion $fg$, the bending moment at any point a distance $x$ from point $f$ is $M = 196 + 41x$. Hence, the bending-moment curve in this portion is a straight line increasing from an ordinate of 196 at $f$ to 360 at $g$. Likewise, in the portion $gc$, the bending-moment curve is a straight line increasing to a value of 404 at a cross section an infinitesimal distance to the left of point $c$. However, at a cross section

an infinitesimal distance to the right of point $c$, the bending moment has increased by 24 to 428. Assuming the external couple of 24 kip-ft to be applied exactly on the cross section at point $c$, there will be an abrupt change in the bending-moment curve similar to the abrupt changes in the shear curve already discussed. In an analogous manner, the remainder of the bending-moment curve may easily be verified. The computations for the controlling ordinates of the curve are shown in Fig. 3·4.

**3·6. Relations between Load, Shear, and Bending Moment.** In those cases where a beam is subjected to transverse loads, the construction of the shear and bending-moment curves may be facilitated by recognizing certain relationships that exist between load, shear, and bending moment. For example, consider the beam shown in Fig. 3·5. Suppose that the shear $S$ and the bending moment $M$ have been computed at the cross section for any point $m$. Point $m$ is located by the distance $x$, which is measured from point $a$, being positive when measured to the right from that point. Suppose that the shear and bending moment are now computed at the cross section at point $n$, a differential distance $dx$ to the right of point $m$. Assuming that a uniformly distributed *upward* load of an intensity $p$ per unit length of beam has been applied to the beam between $m$ and $n$, the shear and bending moment will have increased by differential amounts to values of $S + dS$ and $M + dM$, respectively.



FIG. 3·5

The new values of shear and bending moment at point $n$ may be computed by using the values already computed at point $m$, as is discussed in Art. 3·4. Thus,

$$S + dS = S + p\,dx \qquad\qquad (a)$$

$$M + dM = M + S\,dx + p\,dx\,\frac{dx}{2} \qquad\qquad (b)$$

Therefore, from $(a)$ it is evident that

$$\frac{dS}{dx} = p \qquad\qquad (c$$

and, differential quantities of the second order being neglected, from Eq. (b) it may be found that

$$\frac{dM}{dx} = S \qquad\qquad (d)$$

It should be particularly noted that, in addition to the usual beam convention being used for shear and bending moment, upward loads have been considered as positive and $x$ has been assumed to increase from left to right.

The relationships stated mathematically in Eqs. (c) and (d) are tremendously helpful in constructing shear and bending-moment curves. Consider first Eq. (c). It states that the rate of change of shear at any point is equal to the intensity of load applied to the beam at that point, *i.e.*, that the slope of the shear curve at any point is equal to the intensity of the load applied to the beam at that point. The change in shear $dS$ between two cross sections a differential distance $dx$ apart is

$$dS = \frac{dS}{dx}\,dx = p\,dx$$

Therefore, the difference in shear at two cross sections $A$ and $B$ is

$$S_B - S_A = \int_{x_A}^{x_B} p\,dx \qquad \text{or} \qquad S_B = S_A + \int_{x_A}^{x_B} p\,dx$$

Thus, the difference in the ordinates of the shear curve at points $A$ and $B$ is equal to the total load applied to the beam between these two points.

According to Eq. (c) and the sign convention used in its derivation, if the load is upward, or positive, at a point on the beam, the shear is changing at a positive rate at this point. This means that if the shear is computed on a cross section just to the right of this point, *i.e.*, at a slightly greater distance $x$ from the left support, it will tend to be *more* positive, or algebraically larger, than it was at the first point. Of course, if the load is downward, or negative, at a point, just the reverse will be true. If we think of this interpretation in terms of slope of the shear curve and use the ordinary calculus convention for slope of a curve, $dS/dx$ is plus, the curve slopes upward to the right, since positive values of $S$ are plotted upward and $x$ increases from left to right. If $dS/dx$ is minus, the shear curve slopes downward to the right.

To apply these ideas, if a uniformly distributed load is applied to a portion of a beam, $p$ will be constant and therefore the shear will change at a constant rate and the shear curve will be a straight sloping line in such a portion. However, if the load is distributed but its intensity varies continuously, the shear curve will be a curved line whose slope changes continuously to correspond. If no load is applied to a beam

between two points, the rate of change of shear will be zero, *i.e.*, the shear will remain constant and the shear curve will be straight and parallel to the base line in this portion. At a point where a concentrated load is applied to a beam, the intensity of load will be infinite, and therefore the slope of the shear curve will be infinite, or vertical. At such a point, there will be a discontinuity in the shear curve, and the difference in ordinates from one side of the load to the other will be equal to the concentrated load. These ideas conform to the discussion of the previous article.

Equation (*d*) may be interpreted in the same manner. It states that the rate of change of bending moment at any point is equal to the shear at that point in the beam, *i.e.*, the slope of the bending-moment curve at any point is equal to the ordinate of the shear curve at that point. The change in bending moment $dM$ between two cross sections a differential distance $dx$ apart is

$$dM = \frac{dM}{dx} dx = S\,dx$$

Therefore, the difference in the bending moment at two cross sections $A$ and $B$ is

$$\int_{M_A}^{M_B} dM = M_B - M_A = \int_{x_A}^{x_B} S\,dx \qquad \therefore\; M_B = M_A + \int_{x_A}^{x_B} S\,dx$$

or the difference in the ordinates of the bending-moment curve at points $A$ and $B$ is equal to the area under the shear curve between the two points.

From Eq. (*d*) it is evident that, if the shear is positive at a point in a beam, the rate of change of bending moment is also positive at this point. This means that if the bending moment is computed on a cross section just to the right of this point, *i.e.*, at a slightly greater distance $x$ from the left support, it will tend to be *more* positive, or algebraically larger, than it was at the first point. If the shear is negative, just the reverse will be true. In terms of slope of the bending-moment curve, it may be said that if $dM/dx$ is positive (or negative), the slope of the bending-moment curve at this point is upward (or downward) to the right, since positive values of $M$ are plotted upward and $x$ increases from left to right.

If the shear is constant in a portion of the beam, the bending-moment curve will be a straight line in this portion. However, if the shear varies in any manner within a portion, the bending-moment curve will be a curved line. At a point where a concentrated load is applied, there is an abrupt change in the ordinate of the shear curve, and therefore an abrupt change in the slope of the bending-moment curve at such a point. At a point where the shear curve goes through zero and the ordinate

the left of the point are positive and those to the right negative; the slope of the bending-moment curve will change from positive at the left of the point to negative at the right of the point. Therefore, the ordinate of the bending-moment curve will be a maximum at such a point. If, on the other hand, the shear curve goes through zero in the reverse manner, the ordinate at that point on the bending-moment curve will be a minimum.

**3·7 Construction of Shear and Bending-moment Curves.** The ideas of Art. 3·6 may be utilized most efficiently in constructing shear and bending-moment curves for beams subjected to transverse loads if the following procedure is adopted: After computing the reactions of a beam, first plot a load curve. The load curve is a curve the ordinates of which show the intensity of the distributed load applied to the beam at any point. In addition all concentrated loads should be indicated. Upward, or positive, loads should be drawn above the base line and negative loads below. Then the shear and bending-moment curves may be constructed in turn, proceeding from left to right across the beam, and establishing the *shape* of the curves by using the following principles, which are summarized from the above discussion:

1. The slope of the shear curve at any point is equal to the intensity of the distributed load at that point.

2. Abrupt changes in the ordinates of the shear curve occur at points of application of concentrated loads.

3. The slope of the bending-moment curve at any point is equal to the ordinate of the shear curve at that point.

4. At points where concentrated loads are applied, there are abrupt changes in the ordinates of the shear curve, and hence abrupt changes in the slopes of the bending-moment curve.

It is usually necessary to compute the numeral values of the ordinates of the shear and bending-moment curves only at the points where the shapes of the curves change or at points where the maximum or mini- mum values occur. Such values may usually be most easily computed by direct computation as in Fig. 3·3. Such computations may be checked using the following principles if the value of one ordinate of a curve is known:

5. The difference in the ordinates of the shear curve between any two points is equal to the total load applied to the beam between these two points, i.e., the area under the load curve between these two points plus any concentrated loads applied within this portion.

6. The difference in the ordinates of the bending-moment curve between any two points is equal to the area under the shear curve between these two points.

This method of constructing shear and bending-moment curves will be illustrated in the examples that follow.

Although all these relations and this discussion apply specifically to the case of a beam loaded by transverse loads, it should not be inferred that they are useless in analyzing a beam subjected to a more general condition of loading. The method of handling such cases will be discussed in the examples that follow. For cases of loading involving anything more complicated than transverse loads, it will be seen that a load curve loses its utility and becomes impractical. While some of the above relations may be used to advantage, it will be found that, in most of the more complicated cases of loading, they must be revised. To illustrate, in Example 3·5 it will be seen that abrupt changes in the ordinates of the moment curve occur at points where external couples are applied to the beam. Therefore, the difference in the ordinates of the bending-moment curve between any two points will be equal to the area under the shear curve between these two points plus or minus the sum of any external couples applied to the beam within this portion. However, the student will find that the experience gained in drawing the shear and bending-moment curves for the simpler cases of transverse loadings will enable him to proceed to the more complicated problems with very little difficulty.

**3·8 Illustrative Examples—Statically Determinate Beams.** The following examples will illustrate the construction of shear and bending-moment curves for statically determinate beams, utilizing the ideas and principles discussed above:

**Example 3·1**
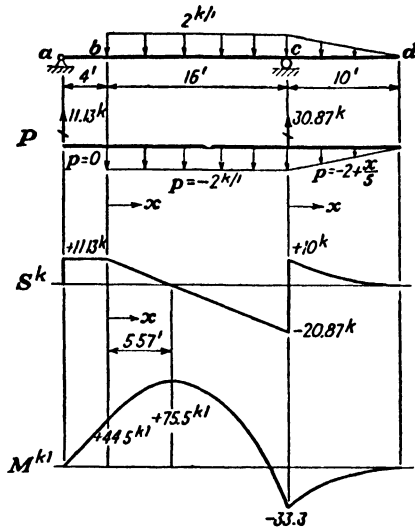


$$\Sigma M_c = 0, \curvearrowright +$$
$$-(2)(16)(8) = -256$$
$$+(2)(1\tfrac{9}{2})(1\tfrac{9}{3}) = +33.3$$
$$\overline{-222.6}$$
$$\therefore R_{ay} = 11.13^k \updownarrow$$
$$\Sigma M_a = 0, \curvearrowright +$$
$$(2)(16)(12) = +384$$
$$(2)(1\tfrac{9}{2})(23.3) = +233.3$$
$$\overline{+617.3}$$
$$\therefore R_{cy} = 30.87^k \updownarrow$$
$$\Sigma F_y = 0, \uparrow +,$$
$$11.13 + 30.87 - 32 - 10 = 0 \quad \therefore O.K.$$

**Shear:**
$$S_c(left) = 11.13 - (2)(16) = -20.87^k$$
*Location of point of $S = 0$ between $b$ and $c$,*
$$S_x = 11.13 - 2x = 0 \qquad \therefore x = 5.57'$$

**Bending moment:**
$$M_b = +(11.13)(4) = +44.52^{k'}$$
$$M_c = -\frac{(2)(10)}{2}\left(\frac{10}{3}\right) = -33.3^{k'}$$
*Between $b$ and $c$,*
$$M_{max} = (11.13)(9.57) - \frac{(2)(5.57)^2}{2}$$
$$= +75.48^{k'}$$

**Discussion:**

In establishing the shape of the shear and bending-moment curves, follow the ideas of Art. 3·7. Starting at the left end of the shear curve, the curve rises abruptly to a value of $+11.13$. From $a$ to $b$, since $p = 0$, the shear curve is horizontal. From $b$ to $c$, since $p = -2$, the shear curve slopes downward to the right at a constant slope to a value of $-20.87$ just to the left of $c$. At $c$, the reaction causes an abrupt increase in the ordinate of the shear curve to $+10$ just to the right of point $c$. From $c$ to $d$, since $p = -2 + (x/5)$, the shear curve slopes downward to the right with a slope that varies linearly from $-2$ at $c$ to $0$ at $d$.

In the same way, the bending-moment curve starts at $0$ at $a$ and progresses from $a$ to $b$ with a constant positive slope (upward to the right). To the right of $b$, the slope decreases linearly from a slope of $+11.13$ at $b$ to zero at the point of maximum moment, and further to a slope of $-20.87$ at $c$. There is an abrupt change in slope at $c$ to a value of $+10$ just to the right of $c$. Between $c$ and $d$, the slope decreases from $+10$ to $0$ at $d$.

The numerical value of the controlling ordinates of the shear and bending moment may be most easily computed by direct computation in the manner discussed in Art. 3·4.

**Example 3·2**
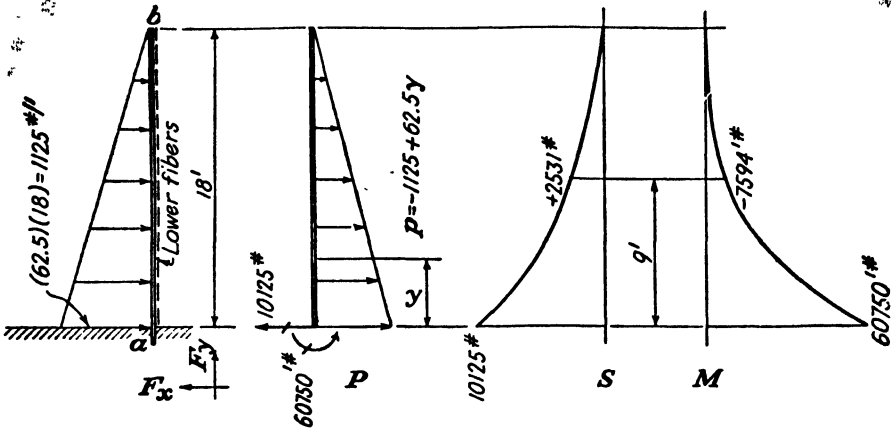


$$\Sigma F_x = 0, \overleftarrow{+}, \; R_{ax} - (1,125)(1\%) = 0, \qquad R_{ax} = 10,125 \; ^{\#} \text{++}$$
$$\Sigma M_a = 0, \overset{\curvearrowright}{+}, \; (1,125)(1\%)(6) - M_a = 0, \qquad M_a = \overline{60,750'^{\#}} \text{⟍}$$

*Shear and bending moment:*

$$At \; y = 9', \qquad p = -1,125 + (62.5)(9) = -1,125 + 562.5 = -562.2^{\#/''}$$
$$S = +(562.5)(\%) = +2,531.3^{\#}$$
$$M = -(562.5)(\%)(3) = -7,593.8'^{\#}$$

**Discussion:**

After the reactions have been determined, the load, shear, and bending-moment curves may be drawn, the fibers on the right-hand side of the cantilever being considered as the "lower fibers" in applying the beam convention. Then, the uniformly varying load would be considered downward or negative in plotting the load curve.

The shear curve rises abruptly at a to a value of +10,125. Progressing toward b, the shear curve starts downward with a negative slope of 1,125 but gradually flattens out to zero slope as well as a zero ordinate at b.

On the other hand, the bending-moment curve starts from an ordinate of −60,750 at a with a positive slope of +10,125. Proceeding toward b, the magnitude of slope remains positive but steadily decreases until at b both the slope and the ordinate of the bending-moment curve are zero.

The shape of these curves having been established, an ordinate of either the shear or the bending-moment curve at any intermediate point such as y = 9 may most easily be computed directly by considering the portion of the beam between that point and b.

## Example 3·3



$$\Sigma M_b = 0, \curvearrowright$$
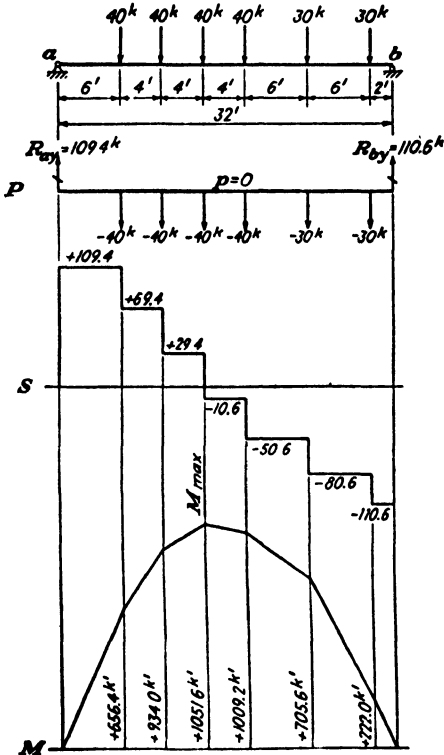
$$
\begin{array}{rcr}
30 \times 2 = & 60 \\
30 \times 8 = & 240 \\
40 \times 14 = & 560 \\
40 \times 18 = & 720 \\
40 \times 22 = & 880 \\
\underline{40 \times 26 =} & \underline{1{,}040} \\
\overline{220} & \overline{3{,}500}
\end{array}
$$

$$\therefore R_{ay} = 109.37^k \;\updownarrow$$

$$\Sigma M_a = 0, \curvearrowright$$

$$
\begin{array}{rcr}
30 \times 30 = & 900 \\
30 \times 24 = & 720 \\
40 \times 18 = & 720 \\
40 \times 14 = & 560 \\
40 \times 10 = & 400 \\
\underline{40 \times\ \ 6 =} & \underline{240} \\
& \overline{3{,}540}
\end{array}
$$

$$\therefore R_{by} = 110.63^k \;\updownarrow$$

$$\Sigma F_y = 0,\ \uparrow,\ 109.37 + 110.63 - 220 = 0$$
$$+$$
$$\therefore O.K.$$

$$
\begin{array}{lrl}
S = & 109.4 \times 6 = & 656.4 \\
 & -40 & \overline{656.4} = M_6 \\
S = & 69.4 \times 4 = & +277.6 \\
 & -40 & \overline{934.0} = M_{10} \\
S = & 29.4 \times 4 = & +117.6 \\
 & -40 & \overline{1051.6} = M_{14} \\
S = & -10.6 \times 4 = & -42.4 \\
 & -40 & \overline{1009.2} = M_{18} \\
S = & -50.6 \times 6 = & -303.6 \\
 & -30 & \overline{705.6} = M_{24} \\
S = & -80.6 \times 6 = & -483.6 \\
 & -30 & \overline{222.0} = M_{30} \\
S = & -110.6 \times 2 = & -221.2 \\
 & +110.6 & \overline{0.8} = M_{32} \\
 & \overline{0} & 0
\end{array}
$$

$$\therefore O.K. \qquad\qquad \therefore O.K.$$

NOTE: $M_{max}$ occurs where shear passes through zero (14 ft from a).

## Discussion:

In computing ordinates to the shear and bending-moment curves for concentrated load systems, it is convenient to arrange the computations in this manner, the ordinates being computed successively from left to right by means of principles 5 and 6 stated near the end of Art. 3·7. Note that a check of the computations is obtained if both curves come back to zero at point b.

**Example 3·4**

$5^{k/'}$

$a$ ... $d$

$8'$ $|6$   $18'$   $|c$ $6'$

$\downarrow 88.8^k$   $\downarrow 71.1^k$

$P$

$p = -5$

$x$

$+48.8'$

$+30$

$S$   $m$

$-40$   $-41.1$

$9.76'$

$+79.02^{k'}$

$M$   $m$

$-90^{k'}$
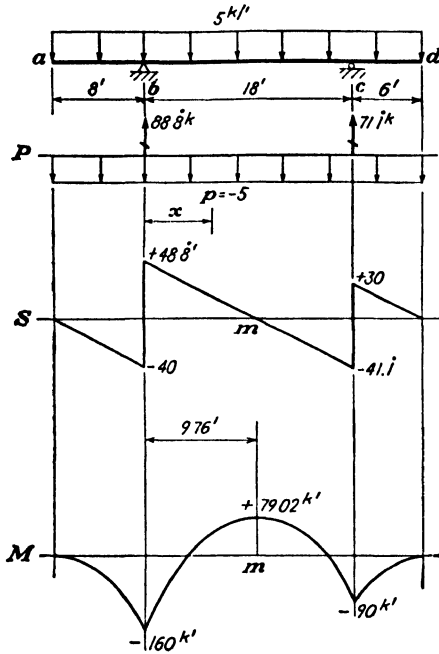
$-160^{k'}$

$\Sigma M_c = 0,\ \curvearrowright +$

$(R_{by})(18) - (32)(5)(10) = 0$

$\therefore R_{by} = 88.8^k$ ⇟

$\Sigma M_b = 0,\ \curvearrowright +$

$-(R_{cy})(18) + (32)(5)(8) = 0$

$\therefore R_{cy} = 71.1^k$ ⇟

$\Sigma F_y = 0, \uparrow +,\ 88.8 + 71.1 - 160 = 0$

$\therefore O.K.$

$M_b = -(5)(8)(4) = -160^{k'}$

$M_c = -(5)(6)(3) = -90^{k'}$

*In portion b to c,*

$$S = 48.8 - 5x$$

*When* $S = 0 = 48.8 - 5x, \therefore x = 9.7'$

$$\therefore M_{max} - M_b = \frac{(48.8)(9.7)}{2}$$

$$M_{max} = M_b + 239.02$$
$$= -160 + 239.02$$
$$= +79.02^{k'}$$

**Discussion:**

*Note that the magnitude of the maximum ordinate of the bending-moment curve between b and c may easily be computed by adding algebraically the area under the shear curve between b and m to the ordinate $M_b$. In this case, this is particularly easy because the area under the shear curve is one triangle.*

## Example 3·5

*All pulleys have 2' diameter*
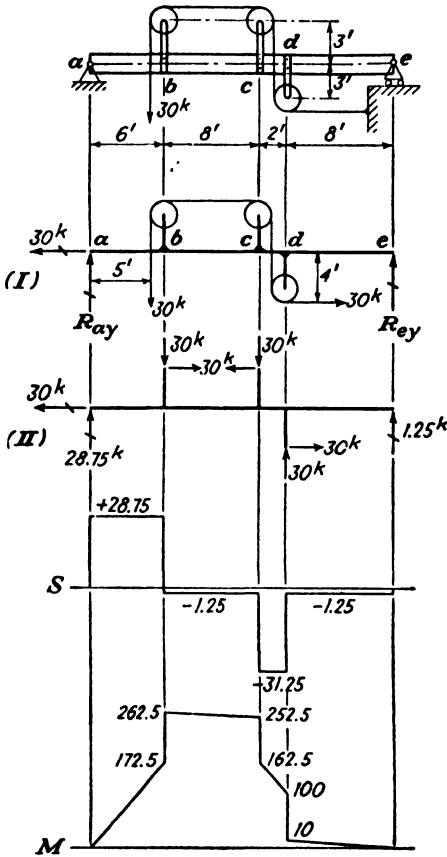


$\Sigma M_a = 0, \overset{\frown}{+}$
$\quad (5)(30) - (4)(30) - (R_{ey})(24) = 0$
$\qquad\qquad\qquad\qquad \therefore R_{ey} = 1.25^k$

$\Sigma M_e = 0, \overset{\frown}{+}$
$\quad (R_{ay})(24) - (30)(19) - (30)(4) = 0$
$\qquad\qquad\qquad\qquad \therefore R_{ay} = 28.75^k$

$\Sigma F_y = 0, \uparrow+, \ 28.75 + 1.25 - 30 = 0$
$\qquad\qquad\qquad\qquad\qquad \therefore O K.$

**Bending moments:**

At b (just to left),
$\quad M_{b_L} = +(28.75)(6) = +172.5^{k'}$

At b (just to right),
$M_{b_R} = (28.75)(6) + (30)(3)$
$\qquad\qquad\qquad\qquad = +262.5^{k'}$

At c (just to left),
$\quad M_{c_L} = 262.5 - (1.25)(8) = +252.5^{k'}$

At c (just to right),



isolate element of beam between two cross sections—one just to left, one just to right of point c. This element will have a differential length, say, of $0+$ ft.
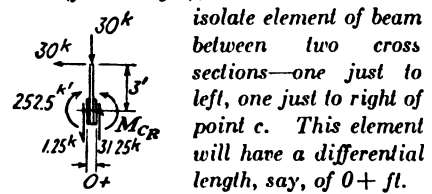
$\Sigma M_{c_R} = 0, \overset{\frown}{+}$
$\quad +252.5 - (30)(3) - (1.25)(0+)$
$\qquad\qquad\qquad\qquad\qquad - M_{c_R} = 0$
$\quad M_{c_R} = 252.5 - 90 - 0 = +162.5^{k'}$

At d (just to left),
$\quad M_{d_L} = (1.25)(8) + (30)(3) = +100^{k'}$

At d (just to right),
$\qquad M_{d_R} = (1.25)(8) = +10^{k'}$

**Discussion:**

As is evident in Chap. 2, the computation of the reactions of such structures may be carried out without first computing the forces that the individual pulleys apply to the beam. However, for the construction of shear and bending-moment curves and the computation of internal stresses in the beam, it is necessary to compute the detailed manner in which the pulleys apply the loads. The reactions from sketch I and the pulley loads having been computed, sketch II may be drawn, showing the precise manner in which the structure is loaded.

As explained in the latter part of Art. 3·7, it is unwise to attempt to draw a load curve in such problems. Instead, the load curve is replaced by a free-body sketch such as sketch II. The student may now proceed to draw the shear and bending-moment curves, using

*the fundamental definitions and methods of computation and, of course, utilizing the experience gained in simple problems such as Examples 3·1 to 3·4. All the above calculations may be followed without difficulty.*

---

**Example 3·6**
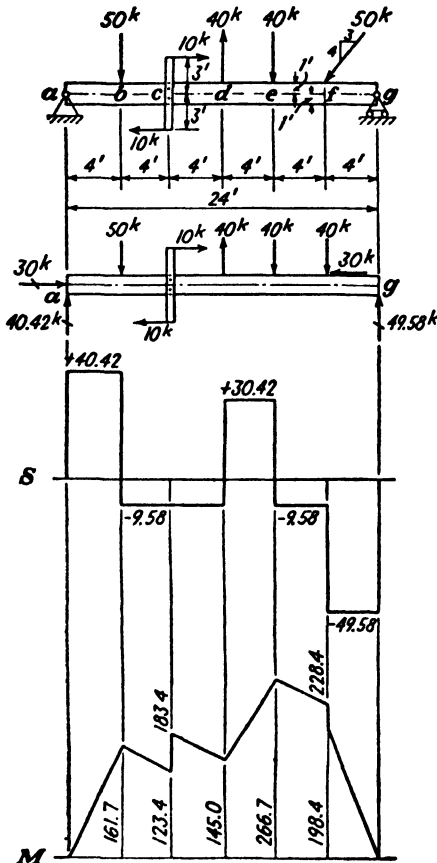


$\Sigma M_a = 0,\ \curvearrowright+$

$(50)(4) + (10)(6) - (40)(12)$
$\quad + (40)(16) + (40)(20) - (30)(1)$
$\qquad\qquad\qquad\qquad - (R_{gv})(24) = 0$

$200 + 60 - 480 + 640 + 800 - 30$
$\qquad\qquad\qquad\qquad\qquad = 24R_{gv}$

$\therefore R_{gv} = \underline{49.58^k}\ \Updownarrow$

$\Sigma M_g = 0,\ \curvearrowright+$

$(R_{av})(24) - (50)(20) + (10)(6)$
$\quad + (40)(12) - (40)(8) - (40)(4)$
$\qquad\qquad\qquad\qquad - (30)(1) = 0$

$24R_{av} = 1{,}000 - 60 - 480 + 320$
$\qquad\qquad\qquad\qquad + 160 + 30$

$\therefore R_{av} = 40.42^k\ \Updownarrow$

$\Sigma F_y = 0,\ \uparrow+,$

$40.42 + 49.58 - 50 + 40 - 40 - 40$
$\qquad\qquad\qquad = 0 \qquad \therefore O.K$

**Bending moment:**

$M_b = 40.42 \times 4 = \quad 161.68 = M_b$

$\qquad\qquad -9.58 \times 4 = \quad \underline{-38.32}$
$\qquad\qquad\qquad\qquad\qquad \overline{123.36} = M_c(left)$
$\qquad\qquad\qquad\qquad\qquad \underline{+60}$
$\qquad\qquad\qquad\qquad\qquad \overline{+183.36} = M_c(right)$

$\qquad\qquad -9.58 \times 4 = \quad \underline{-38.32}$
$\qquad\qquad\qquad\qquad\qquad \overline{145.04} = M_d$

$\qquad\qquad 30.42 \times 4 = \quad \underline{121.68}$
$\qquad\qquad\qquad\qquad\qquad \overline{+266.72} = M_e$

$\qquad\qquad -9.58 \times 4 = \quad \underline{-38.32}$
$\qquad\qquad\qquad\qquad\qquad \overline{+228.40}$
$\qquad\qquad\qquad\qquad\qquad = M_f\ (left)$
$\qquad\qquad\qquad\qquad\qquad \underline{-30}$
$\qquad\qquad\qquad\qquad\qquad \overline{+198.40}$
$\qquad\qquad\qquad\qquad\qquad = M_f\ (right)$

$-49.58 \times 4 = \quad \underline{-198.32}$
$\qquad\qquad\qquad\qquad\qquad \overline{0.08} = M_g$
$\qquad\qquad\qquad\qquad\qquad\quad 0$

**Discussion:**

    *This example is similar to Example 3·5, and the same comments are applicable. In addition, it should be noted that inclined loads are sometimes applied to the top or bottom edges of a beam, such as the 50-kip load at f. In such cases, it is evident that the horizontal component of such loads will produce a couple which will cause an abrupt change in the ordinates of the bending-moment curve at this point.*

**Example 3·7**
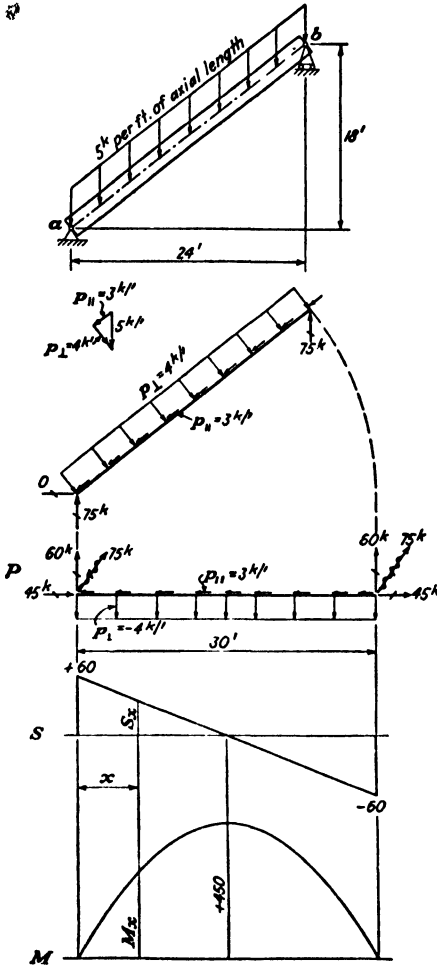


$$\Sigma M_a = 0, \; \curvearrowright +$$
$$(5)(30)(12) - (R_{by})(24) = 0$$
$$\therefore R_{by} = 75^k$$

$$\Sigma M_b = 0, \; \curvearrowright +$$
$$-(5)(30)(12) + (R_{ay})(24) = 0$$
$$\therefore R_{ay} = 75^k$$

**Shear and bending moment:**

At a distance $x$ from $a$,
$$S_x = 60 - 4x$$
$$M_x = 60x - 2x^2$$

When $x = 15'$,
$$S_{15} = 60 - 4(15) = 0$$
$$M_{15} = (60)(15) - (2)(15)^2 = +450^{k'}$$

**Discussion:**

When inclined beams are acted upon by a uniformly distributed vertical load per unit of axial length, such as their own dead weight, the load intensity may be resolved into components perpendicular and parallel to the axis of the beam. If the load is applied to the axis of the beam, only the component perpendicular to the axis contributes to shear and bending moment, the component parallel to the axis contributing only to the axial force. If the reactions are also resolved into components parallel and perpendicular to the axis, the shear and bending-moment curves may easily be constructed in the usual manner. These curves may be drawn about axes drawn parallel to the member or, for convenience, in the manner illustrated above. The load curve may also be used to advantage in such problems.

**3·9   Illustrative Example—Girders with Floor Beams.**   In all the previous examples, the loads have been applied directly to the beam itself.   Quite often, however, the loads are applied indirectly through a floor system that is supported by girders.   A typical construction of this kind is shown diagrammatically in Fig. 1·7 and also in Fig. 3·6.   In such a structure, the loads $P$ are applied to the longitudinal members $S$, which are called *stringers*.   These are supported by the transverse members $FB$, called *floor beams*. The floor beams are in turn supported by the girders $G$. Therefore, no matter whether the loads are applied to the stringers as a uniformly distributed load or as some system of concentrated loads, their effect on the girder is that of concentrated loads
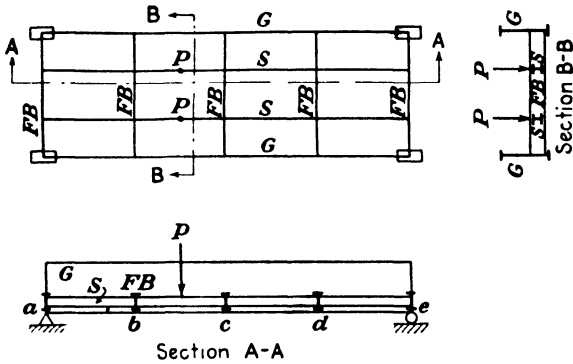


Fig. 3·6

applied by the floor beams at points $a$, $b$, $c$, etc.

To illustrate the construction of the shear and bending-moment curves for girders loaded in this manner, consider Example 3·8.   For simplicity, it will be assumed in this example that the loads are applied to stringers supported on the top flange of the girder as shown.   The stringers and girder will be assumed to lie in the same plane.   It will further be assumed that the stringers are supported as simple end supported beams, with a hinge support at one end and a roller at the other.   As a first step, it is necessary to obtain the stringer reactions and from them to determine the concentrated forces acting on the girder.   From this point on, the construction of the shear and bending-moment curves for the girder proceed as for any beam acted upon by a system of concentrated loads.

The student should study the following questions concerning this type of structure: How do the shear and bending-moment curves differ in the two cases, *i.e.*, with and without the stringers?   Is there any notable similarity?   If a uniformly distributed load is applied to the structure, how will the shear and bending-moment curves compare with and without stringers?   If the stringers are not supported as simple end-supported beams by the girder, will the answers to the previous questions be altered?   Problems at the end of the chapter will emphasize some of these points.

**Example 3·8**
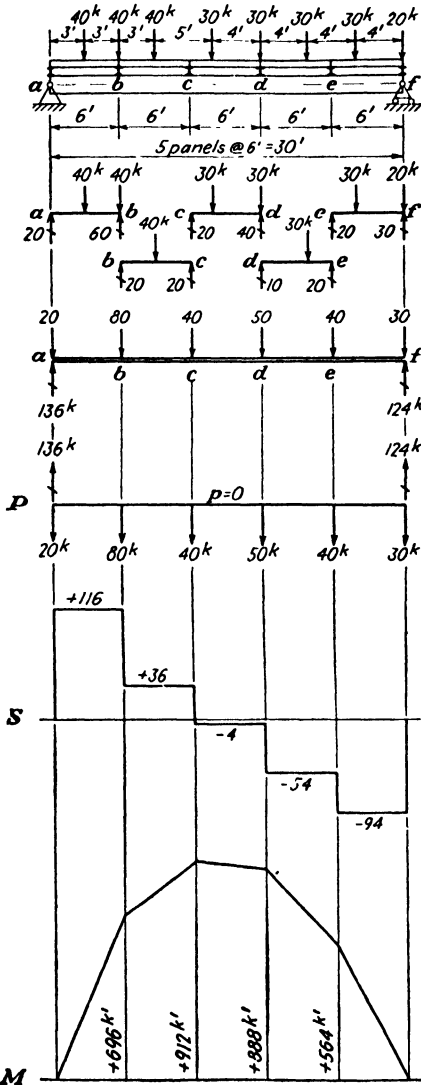


Considering isolated girder acted upon by stringer reactions,

$$\Sigma M_a = 0, \;\curvearrowright$$
$$(80)(1P) = 80P$$
$$(40)(2P) = 80P$$
$$(50)(3P) = 150P$$
$$(40)(4P) = 160P$$
$$(5P)(R_{fy}) = \overline{470P}$$
$$Net\ R_{fy} = \underset{\overline{\phantom{xx}}}{94^k}\ \Updownarrow$$
$$\phantom{Net\ R_{fy} =}\ 30^k$$
$$Gross\ R_{fy} = \overline{124^k}\ \Updownarrow$$

$$\Sigma M_f = 0, \;\curvearrowleft$$
$$(80)(4P) = 320P$$
$$(40)(3P) = 120P$$
$$(50)(2P) = 100P$$
$$(40)(1P) = \underline{40P}$$
$$(5P)(R_{ay}) = \overline{580P}$$
$$Net\ R_{ay} = \underset{\overline{\phantom{xx}}}{116^k}\ \Updownarrow$$
$$\phantom{Net\ R_{ay} =}\ 20^k$$
$$Gross\ R_{ay} = \overline{136^k}\ \Updownarrow$$

$$\Sigma F_y = 0, \;\uparrow +$$
$$124 + 136 - (3)(40) - (4)(30)$$
$$- 20 = 0$$
$$260 - 120 - 120 - 20 = 0 \;\therefore O.K.$$

*These reactions may also be checked by using the applied loads directly.*
**Shear and bending moment on girder:**

$$Gross\ R_{ay} = \quad 136^k$$
$$\quad\quad\quad\quad\quad \underline{-20} \quad\quad\quad\quad 0 = M_a$$
$$S_{a\to b} = \overline{116^k} \times 6 = +696^{k'}$$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad \overline{+696^{k'}} = M_b$$
$$\quad\quad\quad\quad -80$$
$$S_{b\to c} = \overline{\phantom{x}36} \times 6 = +216$$
$$\quad\quad\quad\quad\quad\quad\quad\quad \overline{+912} = M_c$$
$$\quad\quad\quad\quad -40$$
$$S_{c\to d} = \overline{-4} \times 6 = \underline{-24}$$
$$\quad\quad\quad\quad\quad\quad\quad\quad \overline{+888} = M_d$$
$$\quad\quad\quad\quad -50$$
$$S_{d\to e} = \overline{-54} \times 6 = -324$$
$$\quad\quad\quad\quad\quad\quad\quad\quad \overline{+564} = M_e$$
$$\quad\quad\quad\quad -40$$
$$S_{e\to f} = \overline{-94} \times 6 = -564$$
$$\quad\quad\quad\quad\quad\quad\quad\quad\quad \overline{0} = M_f$$
$$\quad\quad\quad\quad -30$$
$$\quad\quad\quad\quad \overline{-124}$$
$$Gross\ R_{fy} = +124$$
$$\quad\quad\quad\quad\quad \overline{0}$$

**Discussion:**

*Note the terms "gross" and "net" reactions. The gross reaction is the total force supplied by the support and includes any load applied to the beam immediately over the support point. The net reaction is the reaction at a support due to all loads except the one applied right at this support. Note that only the net reaction enters the computations for shear and bending moment.*

---

### 3·10   Illustrative Examples—Statically Indeterminate Beams.

From the discussion in Chap. 2, it will be recalled that the stress analysis of indeterminate structures involves the satisfaction of not only the equations of static equilibrium but also certain conditions of distortion. The analysis of such structures is discussed in detail later in this book. In these later chapters, it will be seen that, after some of the unknown stress components (such as reactions, shears, and moments) have been found so as to satisfy the conditions of distortion, the remaining unknowns may be found so as to satisfy the equations of static equilibrium.   That is, the remaining portion of the problem is statically determinate and may be handled by means of the techniques explained in Chaps. 2 and 3 for statically determinate structures.

Once the reactions of a statically indeterminate beam have been determined, the shear and bending moment may be computed at any desired cross section in the same manner as that used for statically determinate beams.   The same principles may also be followed in constructing shear and bending-moment curves.

**Example 3·9**  *The following bending moments have been computed by using methods discussed later for analysis of statically indeterminate structures:*
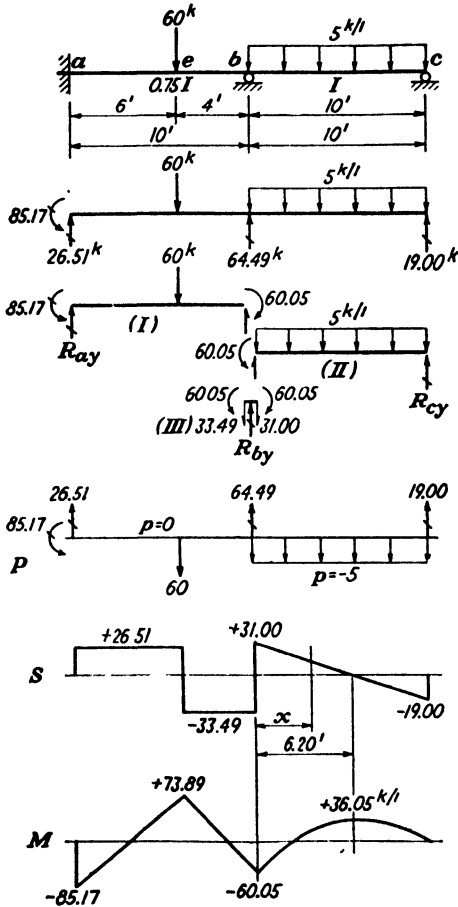
$$M_a = -85.17^{k'}$$
$$M_b = -60.05^{k'}$$
$$M_c = 0$$



**Sketch I:**

$$\Sigma M_b = 0, \overset{\curvearrowright}{+}, (R_{ay})(10) + 60.05 - 85.17$$
$$- (60)(4) = 0$$
$$\therefore R_{ay} = \underline{26.51^k} \; \updownarrow$$
$$\therefore S_{bL} = 26.51 - \overline{60} = \underline{-33.49^k}$$

**Sketch II:**

$$\Sigma M_b = 0, \overset{\curvearrowright}{+},$$
$$(5)(10)(5) - 60.05 - (R_{cy})(10) = 0$$
$$\therefore R_{cy} = \underline{19.00^k} \; \updownarrow$$
$$\therefore S_{bR} = -19.00 \; \overline{+ 50} = \underline{+31.00^k}$$

**Sketch III:**

$$\Sigma F_y = 0, \; \uparrow +, \; R_{by} = 33.49 + 31.00$$
$$= \underline{64.49^k} \; \updownarrow$$

**Bending moments:**

$$M_e = -85.17 + (26.51)(6)$$
$$= \underline{+73.89^{k'}}$$

*Between b and c, find* $M_{max}$,

$$S_x = 31.00 - 5x$$

*When* $S_x = 0 = 31 - 5x$,
$$\therefore x = 6.20'$$

$$M_{max} = -60.05 + (31.00)\left(\frac{6.20}{2}\right)$$
$$= \underline{+36.05^{k'}}$$
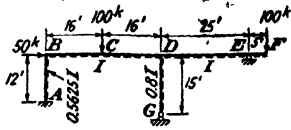
**Example 3·10**  *The following bending moments have been computed by methods of analysis of indeterminate structures:*

"Lower fibers" considered as being on the side of the member by the dashed lines. (+) Bending moments cause tension in lower fibers

At A end of AB, $M = -260.9$
At B end of AB, $M = -62.0$
At B end of BD, $M = -62.0$
At D end of BD, $M = -430.0$
At D end of DE, $M = +72.6$
At D end of DG, $M = +502.7$



$$\Sigma M_B^{(I)} = 0, \quad \begin{array}{r} 260.9 \\ 62.0 \\ \hline 198.9 \end{array}$$
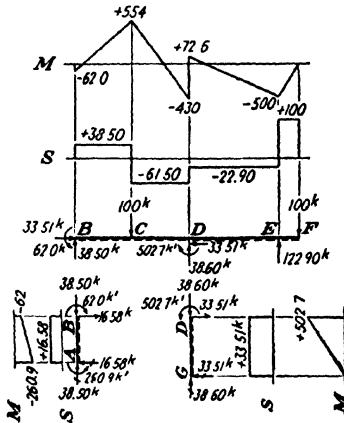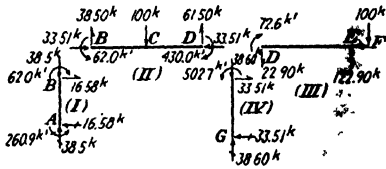
$$R_{Ax} = \frac{198.9}{12} = 16.58$$

$$\Sigma M_B^{(II)} = 0, \quad 100 \times 16 = 1{,}600$$
$$\begin{array}{r} 62 \\ \hline 1{,}662.0 \\ -430.0 \\ \hline 1{,}232.0 \end{array}$$

$$S_B = 38.5 = \frac{1{,}232.0}{32}$$

$$\Sigma M_D^{(II)} = 0, \quad 100 \times 16 = 1{,}600$$
$$\begin{array}{r} 430 \\ \hline 2{,}030 \\ -62 \\ \hline 1{,}968 \end{array}$$

$$S_D = 61.5 = \frac{1{,}968}{32}$$

$$\Sigma M_D^{(III)} = 0, \quad 100 \times 30 = 3{,}000$$
$$\begin{array}{r} 72.6 \\ \hline 3{,}072.6 \end{array}$$

$$R_{Fy} = 122.9 = \frac{3{,}072.6}{25}$$

### 3·11 Problems for Solution.

**Problem 3·1**  Draw the shear and bending-moment curves for the conditions of loading of a simple end-supported beam as shown in Fig. 3·7.

*Suggestion:* What is the maximum bending moment in each case?  If, in part *b*, *k* equals 0.5, what is the maximum bending moment?
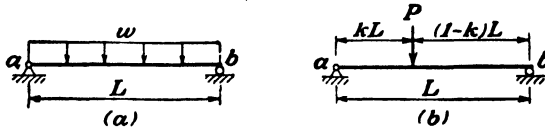


FIG. 3·7

**Problem 3·2**  Draw the load, shear, and bending-moment curves for the beam of Fig. 3·8.
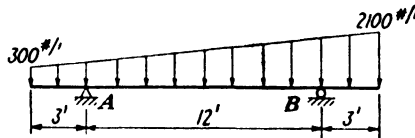


FIG. 3·8

**Problem 3·3**  Draw the shear and bending-moment curves for the beam of Fig. 3·9.
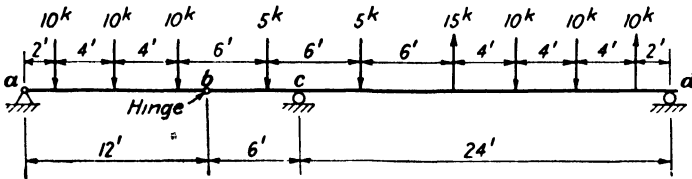


FIG. 3·9

**Problem 3·4**  Draw the shear and bending-moment curves for the beams shown in Probs. 2·2*a* and *b*.

**Problem 3·5**  Draw the shear and bending-moment curves for beam *AB* of Fig. 3·10.
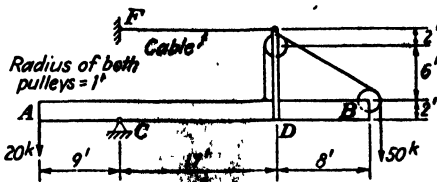


FIG. 3·10
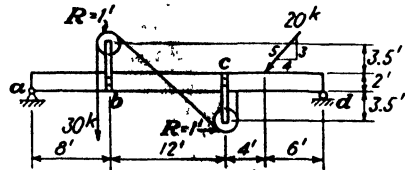
FIG. 3·11

**Problem 3·6**  Draw the shear and bending-moment curves for the beam of Fig. 3·11

**Problem 3·7**   Draw the shear and bending-moment curves for beam *ab* of Prob. 2 2*d* and for beams *ab* and *cd* of Prob. 2·2*e*.

**Problem 3·8**   Draw the shear and bending-moment curves for members *ab* and *bc* of the structure of **Fig. 3·12**.
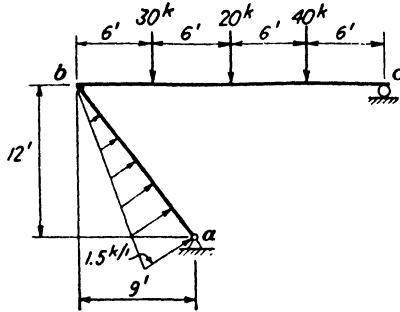


Fɪɢ. 3·12

**Problem 3·9**   Draw the shear and bending-moment curves for girder *ab* of Fig. 3·13.
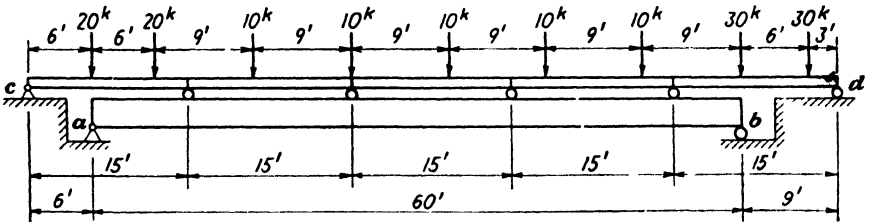


Fɪɢ. 3·13

**Problem 3·10**   Draw the shear and bending-moment curves for girder *ab* of Fig. 3·14.
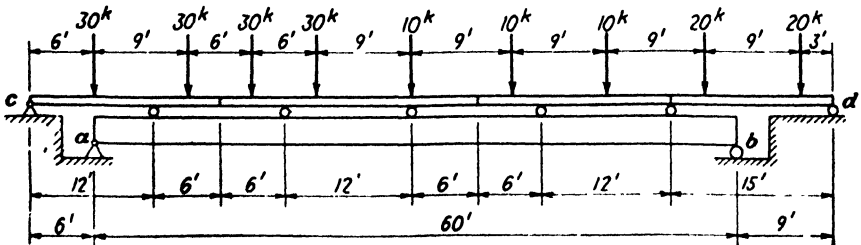


Fɪɢ. 3·14

# CHAPTER 4

## TRUSSES

**4·1  General—Definitions.**  In this chapter, the general theory of the stress analysis of trusses is discussed.   Consideration is also given to the manner in which the members (bars) of a truss must be arranged in order to obtain a stable structure.   In a subsequent chapter, the stress analysis of some of the more important types of bridge and roof trusses under design loading conditions is considered in detail.

A *truss* may be defined as a structure composed of a number of bars, all lying in one plane and hinged together at their ends in such a manner as to form a rigid framework.   For the purposes of the discussion in this chapter, it will be assumed that the following conditions exist: (1) The members are connected together at their ends by frictionless pin joints. (2) Loads and reactions are applied to the truss only at the joints.   (3) The centroidal axis of each member is straight, coincides with  the line connecting the joint centers at each end of the member, and lies in a plane that also contains the lines of action of all the loads and reactions. Of course, it is physically impossible for all these conditions to be satisfied exactly in an actual truss, and therefore a truss in which these idealized conditions are assumed to exist is called an *ideal truss.*

Any member of an ideal truss may be isolated as a free body by disconnecting it from the joints at each end.   Since all external loads and reactions are applied to the truss only. at the joints and no loads are applied between the ends of the members themselves, the isolated member would be acted upon by only two forces, one at each end, each force representing the action on the member by the joint at that end.   Since all the pin joints are assumed to be frictionless, each of these two forces must be directed through the center of its corresponding pin joint. For these two forces to sátisfy the three conditions of static equilibrium for the isolated member, $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$, it is apparent that the two forces must both act along the line joining the joint centers at each end of the member and must be numerically equal but opposite in sense.   Since the centroidal axes of the members of an ideal truss are straight and coincide with the line connecting the joints at each end of the member, every transverse cross section of a member will be subjected to the same axial force but to no bending moment or shear force.   The stress analysis of an ideal truss is completed, therefore, when the axial stresses have been determined for all the members of the truss.

Three-dimensional structures composed of a number of bars hinged together in such a manner as to form a rigid framework are called *space frameworks*. Such structures are discussed in detail in a later chapter.

**4·2 Ideal vs. Actual Trusses.** While it is true that the ideal truss is hypothetical and can never exist physically, the stress analysis of an actual truss based on the assumption that it acts as an ideal truss usually furnishes a satisfactory solution for the axial stresses in the members of the actual truss. The axial stresses in the members, or bars, of a truss will be referred to as the *bar stresses*. The stress intensities due to the bar stresses computed on the basis that the truss acts as an ideal truss are referred to as *primary stress intensities*.

The pins of an actual pin-jointed truss are never really frictionless; moreover, most modern trusses are made with riveted or welded joints so that there can be no essential change in the angles between the members meeting at a joint. As a result, even when the external loads are applied at the joint centers, the action of the joints on the ends of a member may consist of both an axial and transverse force and a couple The transverse cross sections of a member may be subjected, therefore, to an axial force, a shear force, and a bending moment. In addition, the dead weight of the members themselves must necessarily be distributed along the members and therefore contributes to further bending of the members. If good detailing practice is followed and care is taken to see that the centroidal axes of the members coincide with the lines connecting the joint centers, additional bending of the members due to possible eccentricities of this type may be eliminated or minimized.

All these departures from the conditions required for an ideal truss not only may develop a bending of the members of an actual truss but also may cause bar stresses that are somewhat different from those in an ideal truss. The difference between the stress intensities in the members of an actual truss and the primary stress intensities computed for the corresponding ideal truss are called *secondary stress intensities*. It may be demonstrated, however, that in the case of the usual truss, where it is detailed so that the centroidal axes of the members meet at the joint centers and where the members are relatively slender, the secondary stress intensities are small in comparison with the primary stress intensities.[1] The primary stress intensities computed on the basis that the truss acts as an ideal truss are therefore usually satisfactory for practical design purposes.

In subsequent discussions, the term *truss* will be used to denote a

[1] See Parcel, J. I., and G. A. Maney, "Statically Indeterminate Stresses," 1st ed., Chap. VII, John Wiley & Sons, Inc., New York, 1926.

framework that either is actually an ideal pin-jointed truss or may be assumed to act as if it were an ideal truss.

**4·3   Arrangement of Members of a Truss.**   In Art. 4·1, it is stated that the members of a truss must be hinged together in such a manner as to form a rigid framework.   The term *rigid* as used in this instance has the same significance as when used previously in Art. 2·3, *i.e.*, a framework is said to be rigid if there is no relative movement between any of its particles beyond that caused by the small elastic deformations of the members of the framework.   In this sense, a rigid framework may be obtained by arranging the truss members in many different ways.   When the bars have been satisfactorily arranged in one of these ways, the entire truss can be supported in some manner and used to carry loads just like a beam.

Suppose that it is necessary to form a truss with pin joints at points *a*, *b*, *c*, and *d*.   If this is attempted by pin-connecting four bars together as shown in Fig. 4·1a, the resulting framework will not be rigid and may collapse in the manner shown owing to the loads *P*, until, in this case, joints *a*, *d*, and *c*, are lying along a straight line. After a little thought, it is apparent that any attempt such as this to connect four or more joints together with a like number of bars hinged at their ends will result in a framework which will collapse under all but a few special conditions of loadings.   If, however, points *a* and *b* are first connected by a bar *ab*, then two other bars of lengths *ad* and *bd* can be hinged at *a* and *b*, respectively.   If the *d* ends of these bars are then hinged together at point *d*, a rigid triangle will be formed connecting joints *a*, *b*, and *d*.   Bars of lengths *dc* and *bc* can then be connected to the pin joints at *d* and *b*, respectively.   The *c* ends of these bars can then be made to coincide at point *c* and pinned together at this point, thus rigidly connecting joint *c* to the triangle *abd* and resulting in a rigid framework of five bars with joints at *a*, *b*, *c*, and *d*.   As alternate arrangements, point *c* can be connected to joints *a* and *b* by bars *ac* and *bc* or to joints *a* and *d* by bars *ac* and *dc*.   Several other alternate arrangements can be used by first forming a triangle with joints *a*, *b*, and *c*; or with *a*, *d*, and *c*; or with *b*, *c*, and *d*.   Any of these arrangements will result in a rigid framework capable of withstanding any system of joint loads without collapsing.
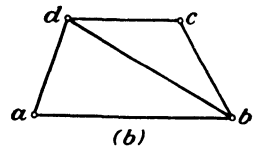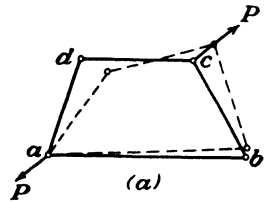
In this same manner, any number of pin joints can be connected together with bars to form a rigid framework.   The procedure is first to select three joints that do not lie along a straight line.   These three points

can then be connected by three bars pinned together to form a triangle. Each of the other joints can then be connected in turn, two bars being used to connect it to any two suitable joints on the framework already constructed. Of course, the new joint and the two joints to which it is connected should never lie along the same straight line. Each of the trusses shown in Fig. 4·2 has been formed in this manner by starting with a rigid triangle *abc* and using two additional bars to connect each of the other joints in alphabetical order.
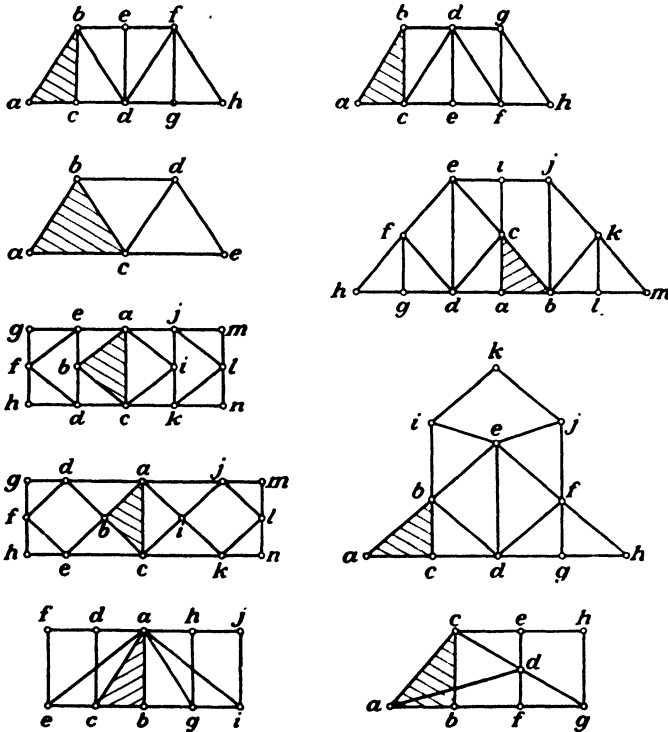


FIG. 4·2

Trusses the bars of which have been arranged in this manner are called *simple trusses*, for this is the simplest and commonest type of bar arrangement encountered in practice.

In all truss diagrams such as those shown in Fig. 4·2, the members will be represented by single lines and the pin joints connecting them by small circles. Sometimes bars may cross each other but may be arranged in such a manner that they are not connected together by a joint at their point of intersection.

When the members have been arranged to form a simple truss, the

entire framework may then be supported in the same manner as a beam. . In order to approach the conditions of an ideal truss, the supports should be detailed so that the reactions are applied at the joints of the truss. Upon recalling the discussion in Art. 2·5, it is apparent that, if the supports of the truss are arranged so that they are equivalent to three-link supports neither parallel nor concurrent, then the structure is stable and its reactions are statically determinate under a general condition of loading. Illustrations of a simple truss supported in a stable and statically determinate manner are shown in Fig. 4·3.

Sometimes it is desirable to connect two or more simple trusses together to form one rigid framework. In such cases, the composite framework built up in this manner is called a *compound truss*. One simple truss can be supported adequately by another simple truss if the two trusses are connected together at certain points by three links neither parallel nor concurrent or by the equivalent of this type of connection. Hence, two trusses connected in this manner will form a composite framework that is completely rigid. Additional simple trusses can be connected in a similar manner to the framework already assembled to form a more elaborate compound truss.

Several examples of compound trusses are shown in Fig. 4·4. In all these cases the simple trusses that have been connected together are shown crosshatched. In trusses *a*, *e*, and *f*, the simple trusses have been connected together by bars 1, 2, and 3.



FIG. 4·3

In cases *b* and *c*, the trusses have been hinged together at one common joint, thus requiring only one additional bar to form a rigid composite framework. In case *d*, the additional bar connecting trusses *A* and *B* together has been replaced by the simple truss *C*.

The members having been arranged to form a compound truss, the entire framework may be supported in the same manner as a simple truss.

**4·4 Notation and Sign Convention Used in Truss Stress Analysis.** Before the stress analysis of trusses is discussed, it is necessary to establish a definite notation and sign convention for designating the bar stresses in the members of a truss.
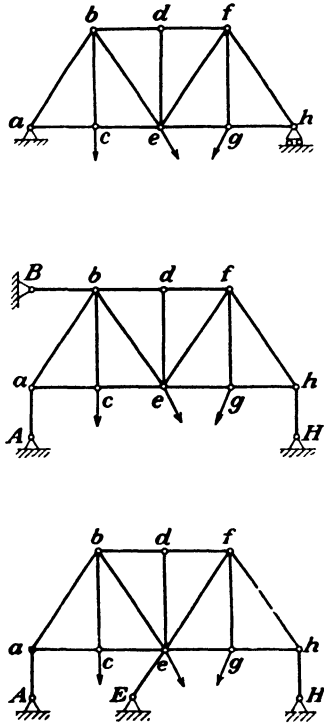
The various members of a truss will be designated by the names of the joints at each end of the member. The letter $F$ will be used to denote the bar stress in a member. Thus, subscripts being used to denote the bar, $F_{ah}$ denotes the bar stress in member $ah$. The values of the bar stresses in the members of a truss are often tabulated or written alongside the various members on a line diagram of the truss. For this purpose, it is convenient to have a definite convention for designating the character of stress in a member, *i.e.*, whether the internal stress is tension or compression. The most convenient convention is to use a *plus* (+) *sign to designate a tension and a minus* (−) *sign to designate a compression.* Thus,
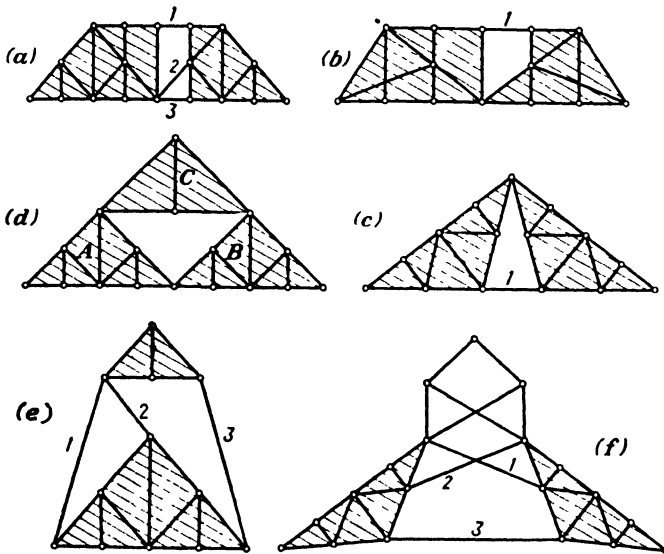


FIG. 4·4

+10 means a tension of 10, and −10 a compression of 10. A plus sign is used to designate tension because such a bar stress causes an elongation, or an *increase* in the length, of the bar. Thus, a plus stress causes a plus change in length. On the other hand, a compression, or minus, bar stress causes a decrease, or minus change, in length of a bar.

In the stress analysis of trusses, it is often convenient to work with two rectangular components of a bar stress rather than the bar stress itself. For this purpose, two orthogonal directions $x$ and $y$ are selected (usually horizontal and vertical, respectively), and the two corresponding components in bar $ah$ are designated to $X_{ah}$ and $Y_{ah}$. It is particularly important for the student to be completely familiar with the various relationships between a force and its two rectangular components. These relationships are so important in truss analysis that a student must

have complete facility with them. For this reason, some of them are reviewed at this time. When it is realized that a bar stress acts along the axis of a member, the following statements are self-evident:

1. *The horizontal (or vertical) component of a bar stress is equal to the bar stress multiplied by the ratio of the horizontal (or vertical) projection of a member to its axial length.*

2. *The bar stress of a member is equal to its horizontal (or vertical) component multiplied by the ratio of the axial length of the member to its horizontal (or vertical) projection.*

3. *The horizontal component of a bar stress is equal to the vertical component multiplied by the ratio of the horizontal to the vertical projection of the axial length, and vice versa.*

The following principle is also useful and important in dealing with bar stress computations:

4. *Any force may be replaced by its rectangular components as long as the components are both assumed to act at some one convenient point along the line of action of the force.*

**4·5   Theory of Stress Analysis of Trusses.** To determine the adequacy of a truss to withstand a given condition of loading, it is first necessary to compute the bar stresses developed in the members of the truss to resist the prescribed loading. The fundamental approach to studying the internal forces, or stresses, in any body is the same whether it be a beam, a truss, or some other type of structure. In the case of a truss, this approach consists in passing an imaginary section that cuts through certain of the bars and isolates some convenient portion of the truss as a free body. Acting on the internal cross sections exposed by the isolating section will be internal forces, or stresses. In the case of a member of an ideal truss, the resultant of these stresses is simply an axial force referred to as the bar stress in the member.

If the truss as a whole is in static equilibrium, then any isolated portion of it must likewise be in equilibrium. Any particular isolated portion of a truss will be acted upon by a system of forces which may consist of certain external forces and the bar stresses acting on the exposed faces of those members which have been cut by the isolating section. It is often possible to isolate portions of a truss so that each portion is acted upon by a limited number of unknown bar stresses, which may then be determined so as to satisfy the equations of static equilibrium for that portion.

This procedure may be explained quite easily by considering a specific example such as the simple truss shown in Fig. 4·5. This truss is supported in such a manner that the reactions are statically determinate and are easily computed as shown in the figure. Proceeding now with the

determination of the bar stresses in this truss, suppose an imaginary section is passed around joint $a$, cutting through bars $ah$ and $ab$ and thus completely isolating joint $a$ from the rest of the truss as shown in free-body sketch $A$ of this figure.

Such an isolated joint will be a body acted upon by a concurrent system of forces since the bar stresses in the cut bars of an ideal truss and

$8^k$
$12^k$
$g$
$h$          $f$          $10'$
$32'$
$a$
$b \downarrow 20^k$   $c \downarrow 40^k$   $d \downarrow 40^k$   $e$
$4 @ 24' = 96'$

$R_{ay} = \frac{3}{4} \times (20+12) = 24$
$\frac{1}{2} \times (40+8) = 24$
$\frac{1}{4} \times 40 = 10$
$\overline{58^k} \uparrow$

Check:
32
48
40
$\overline{120^k} \downarrow$

58
62
$\overline{120^k} \uparrow$

$R_{ey} = \frac{3}{4} \times 40 = 30$
$\frac{1}{2} \times (40+8) = 24$
$\frac{1}{4} \times (20+12) = 8$
$\overline{62^k} \uparrow$
∴OK

$y$
$x$
$12^k$   $8^k$
$-20$   $-48$
$h$   $-52$
$-43.5$   $+45$
$-58$   $+72.5$   $+20$   $+75$   $+45$
$0$   $a$
$+43.5$   $b$  $+43.5^k$   $c$   $d$   $e$
$58^k \uparrow$   $20^k$   $40^k$   $40^k$   $62^k \uparrow$

$Y_{ah} = \frac{4}{5}F_{ah}$
$F_{ah}$
$X_{ah} = \frac{3}{5}F_{ah}$
$a$   $F_{ab}$
$58^k$   $(A)$

$58^k$   $\frac{3}{5} \times 58 = 72.5^k$
$\frac{3}{4} \times 58^k = 43.5^k$
$a$   $43.5^k$
$58^k$   $(B)$

$20^k$
$43.5^k$   $b$   $43.5^k$
$20^k$   $(C)$

$12^k$   $\frac{5}{13}F_{hg}$   $Y_{hg}$
$F_{hg}$   $g$   $X_{hg}$
$h$   $\frac{12}{13}F_{hg}$
$43.5^k$   $\frac{3}{5}F_{hc}$
$72.5$   $58$   $20$   $\frac{4}{5}F_{hc}$   $F_{hc}$
$a$   $b$   $c$
$(D)$

$12^k$   $Y_{hg}$
$F_{hg}$   $g$   $X_{hg}$
$h$
$F_{hc}$
$a$   $b$   $F_{bc}$   $c$
$58^k$   $20^k$
$(E)$

FIG. 4·5

the external forces are all forces the lines of action of which are directed through the center of the isolated joint. The resultant of a concurrent system of forces cannot be a couple, and hence the force system will be in equilibrium if $\Sigma F_x = 0$ and $\Sigma F_y = 0$. Therefore, if there are only two unknown bar stresses acting on a given isolated joint and these do not have the same line of action,[1] the two conditions for static equilibrium will yield two independent equations that may be solved simultaneously for the two unknown stresses. If there are more than two unknown bar stresses, the values of all of the unknowns cannot be determined immediately from these two equations alone.

In this particular case, however, the isolated joint $a$ is acted upon by the known reaction and only two unknown bar stresses $F_{ah}$ and $F_{ab}$. The slopes of the members being known, the horizontal and vertical components of the two unknown bar stresses may be expressed in terms of the bar stress as shown in sketch $A$. The two equations of static equilibrium may then be written, assuming both $F_{ah}$ and $F_{ab}$ as being tensions,

$$\Sigma F_y = 0, \; +\uparrow, \; 58 + \tfrac{4}{5}F_{ah} = 0 \qquad (a)$$
$$\Sigma F_x = 0, \; \overset{\longrightarrow}{+}, \; \tfrac{3}{5}F_{ah} + F_{ab} = 0 \qquad (b)$$

Then, from Eq. $(a)$,

$$F_{ah} = -72.5 \text{ kips} \qquad \text{(compression)}$$

and hence, from Eq. $(b)$,

$$F_{ab} = -\tfrac{3}{5}F_{ah} = -(\tfrac{3}{5})(-72.5) = +43.5 \text{ kips} \qquad \text{(tension)}$$

Then the components of $F_{ah}$ are

$$X_{ah} = (\tfrac{3}{5})(-72.5) = -43.5 \text{ kips}$$
$$Y_{ah} = (\tfrac{4}{5})(-72.5) = -58 \text{ kips}$$

Thus, the minus sign indicates that $F_{ah}$ is opposite to the assumed sense (a compression), while the plus sign indicates that $F_{ab}$ is in the assumed sense (a tension). The sign of the results, therefore, automatically conforms to the sign convention that has been adopted to indicate the character of stress. These results may now be recorded on the line diagram of the truss as $-72.5$ and $+43.5$, the signs indicating the proper character of stress. The components of the stress in $ah$ may be recorded conveniently as shown on the line diagram.

Such conformity in the signs of the results will always be obtained if, in drawing the free-body sketches and setting up the equations of statics

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[1] If the two unknown bar stresses have the same line of action, will the two equations be consistent and independent?

the sense of the unknown bar stresses is assumed to be tension. If this is done, then a plus sign for an answer indicates that the assumed sense is correct and therefore tension, while a minus sign indicates that the assumed sense is incorrect and therefore compression. Thus the signs of the results will automatically conform with the established sign convention.

The procedure just used may be applied in principle to solve for the unknown bar stresses at any isolated joint that is acted upon by only two unknown bar stresses. In this particular truss, the remaining unknown bar stresses could be computed easily by isolating the remaining joints one after the other, always selecting the next joint to be isolated so that the stresses in all but two (or fewer) of the cut bars have previously been computed. Of course, it is also necessary for these two unknown bar stresses to have different lines of action. This technique of passing a section so as to isolate a single joint of a truss is called the *method of joints*.

Sometimes it is more expedient to pass a section that isolates a portion containing several joints of a truss. This latter technique of passing the cutting section is called the *method of sections*. An isolated portion consisting of several joints of a truss will be a body that is acted upon by a nonconcurrent system of forces that may consist of certain external forces and the bar stresses in those bars cut by the isolating section. For equilibrium of such a portion, the three equations $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$ must be satisfied by the forces acting on this part of the truss. Therefore, if there are only three unknown bar stresses acting on this part and these three bars are neither parallel nor concurrent, the values of the three unknown stresses may be obtained from these three equilibrium equations.

A typical application of the method of sections is shown in free-body sketch $E$ of Fig. 4·5. In this case, the cutting section is passed through bars $hg$, $hc$, and $bc$, thus isolating the portion of the truss to the left of this section. The unknown stresses in these three cut bars may now be determined by solving the three equations of equilibrium for the isolated portion. In previous discussions of the computations of reactions, it has been demonstrated that it is often possible to expedite the solution in such cases of nonconcurrent force systems by using ingenuity in writing the equations of statics. For example, to solve for $F_{hg}$, take moments about point $c$, the point of intersection of $F_{hc}$ and $F_{bc}$, and resolve $F_{hg}$ into its horizontal and vertical components at point $g$. Then only $X_{hg}$ enters the moment equation, and

$$\Sigma M_c = 0, \; \overset{\curvearrowright}{+}, \; (X_{hg})(42) + (58)(48) - (32)(24) = 0$$

whence $X_{hg} = -48$ and by proportion $Y_{hg} = -20$ and $F_{hg} = -52$. In

a similar manner,

$$\Sigma M_h = 0, \;\widehat{+}, \;(58)(24) - (F_{,c})(32) = 0$$

whence $F_{bc} = +43.5$. Then either $\Sigma F_z = 0$ or $\Sigma F_y = 0$ may be used to obtain the horizontal or vertical component, respectively, of $F_{hc}$.

$$\Sigma F_x = 0, \;\overrightarrow{+}, \;X_{hc} + 43.5 - 48 = 0$$

whence $X_{hc} = +4.5$ and by proportion $Y_{hg} = +6$ and $F_{hg} = +7.5$. Of course, any three independent equations of statics could be written and solved for these three unknown stresses. However, if ingenuity is not used, the three equations may all contain all three unknowns and have to be solved simultaneously, whereas, as just shown, it is possible to write three equations, each of which contains only one unknown.

**4·6  Application of Method of Joints and Method of Sections.** In the previous article, the equations involved in applying both the method of joints and the method of sections were set up in a rather formal manner. However, it is often unnecessary to do this. For example, consider joint $a$, which was used in illustrating the method of joints in the previous article. At the present time, consider this isolated joint as shown in free-body sketch $B$ of Fig. 4·5. By inspection, it is obvious that, for $\Sigma F_y = 0$ to be satisfied, the vertical component in bar $ah$ must push downward with a force of 58 kips in order to balance the reaction. Then, by proportion, the horizontal component and the bar stress itself in this bar are equal to 43.5 and a compression of 72.5, respectively, acting in the directions shown. The horizontal component in $ah$ being known, it is now apparent that, for $\Sigma F_x = 0$ to be satisfied, the stress in $ab$ must be a tension of 43.5 acting to the right to balance the horizontal component in $ah$ acting to the left.

Since the bar stress in $ab$ is known, it is a simple matter to find the stresses in $bc$ and $bh$ by passing a section that isolates joint $b$ as shown in free-body sketch $C$ of Fig. 4·5. Again, this simple case may be easily solved in an informal manner to obtain the two unknown bar stresses $F_{bc}$ and $F_{bh}$. To satisfy $\Sigma F_x = 0$, it is apparent that $F_{bc}$ must a tension of 43.5 and, to satisfy $\Sigma F_y = 0$, $F_{bh}$ must be a tension of 20.

If joint $h$ is isolated in a similar manner, as shown in free-body sketch $D$, the isolating section will cut through four bars, two in which the bar stresses are known and two in which they are unknown. Again these two unknowns may be found from the equilibrium conditions $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for the isolated joint. Assuming the unknown stresses to be tension, the two equations may be written as follows:

$$\Sigma F_x = 0, \;\overrightarrow{+}, \;{}^{12}\!/_{13}F_{hg} + {}^3\!/_5 F_{hc} + 43.5 = 0$$
$$\Sigma F_y = 0, \;+\!\uparrow, \;{}^5\!/_{13}F_{hg} - {}^4\!/_5 F_{hc} - 12 - 20 + 58 = 0$$

In this case, unfortunately, both equations contain both unknowns, and it is necessary to solve the equations simultaneously for these two values. Of course, the two unknowns can be obtained quite easily in this manner, but consider the advantages of proceeding as follows:

In the discussion of computation of reactions in Chap. 2, it will be recalled that it was often advantageous to replace either or both $\Sigma F_x = 0$ and $\Sigma F_y = 0$ by one or two moment equations. A similar technique is likewise desirable in the present case of the isolated joint $h$. Suppose that in free-body sketch $D$ the positions of joints $a$, $c$, and $g$ are located in space as shown. Then, $\Sigma M_c = 0$ could be used instead of the equation $\Sigma F_y = 0$ or $\Sigma F_x = 0$. Taking moments about point $c$ not only eliminates $F_{hc}$ from the equation but also makes it possible to simplify the computation of the moments of the stresses in bars $ah$ and $hg$. These two bar stresses may now be resolved into their vertical and horizontal components at joints $a$ and $g$, respectively, and then only the vertical component of $F_{ah}$ and the horizontal component of $F_{hg}$ will enter the moment equation and the lever arms of both these components are easily obtained. In this way,

$$\Sigma M_c = 0, \; \curvearrowright, \; (X_{hg})(42) + (58)(48) - (32)(24) = 0 \qquad \therefore \; X_{hg} = -48$$

and by proportion $Y_{hg} = -20$ and $F_{hg} = -52$.

$F_{hg}$ and its two components being known, it is an easy matter to use either $\Sigma F_x = 0$ or $\Sigma F_y = 0$ and obtain, respectively, either the horizontal or vertical components of the stress in $hc$ directly. For example, since $X_{hg} = -48$, the horizontal component in $hc$ must act to the right with 4.5 kips so as to balance the 43.5 kips in $ah$ and make $\Sigma F_x = 0$. This means that this bar is in tension and by proportion the bar stress and vertical component are $+7.5$ and $+6.0$, respectively. All these computations may now be checked by seeing whether the results satisfy $\Sigma F_y = 0$.

In the following illustrative examples, additional techniques and "tricks" will be used to expedite the application of the method of joints and the method of sections. In the first few examples, all the free-body sketches will be shown in detail. To train the student to visualize free-body sketches when possible, such sketches will be omitted in the later examples. When desirable, the numerical computations will be carried out in an informal manner. If the student finds it difficult to follow these short cuts, he should draw the necessary free-body sketches and set up the equilibrium equations in a fundamental manner. The student should recognize that it is desirable to develop a facility for visualizing free-body sketches and solving equilibrium equations in an informal manner; but he should also recognize that even the expert has to go back to fundamentals—draw sketches and write equations—whenever he is confused or faced with a difficult problem.

The student should also note carefully the technique of drawing free-body sketches of isolated portions of a truss. Any bar stress that is known in magnitude[1] from previous computations should be shown acting with this known magnitude on any sketch that is drawn subsequently. For example, in drawing free-body sketch $D$ of Fig. 4·5, the stresses in bars $ah$ and $bh$ have previously been computed and recorded on the line diagram of the truss as $-72.5$ and $+20$, respectively. Hence, the bar stresses acting on the stub ends of these two bars should be shown pushing the stub end of $ah$ into joint $h$ and pulling the stub end of $bh$ out of this joint. Thus the sense of these known bar stresses having been indicated by arrows, the forces should be labeled with their numerical value only, *viz.*, simply by 72.5 and 20, not by $-72.5$ and $+20$. As suggested in the previous article, the free-body sketch is completed by showing the unknown bar stresses as being tensions.

When the value of a bar stress is recorded on the line diagram of the truss, it will be found helpful also to draw arrows at each end of the member, indicating the direction in which the force in the member acts *on the joint*. This procedure will be followed in recording the bar stresses in the remaining illustrations of this chapter.

[1] Note that magnitude has previously been defined as including the sense in which a force acts.

**Example 4·1**  *Compute the bar stresses in members Cc, CD, cd, cD, and DE of this truss, due to the loads shown.*



$$\Sigma M_b = 0, \qquad \curvearrowleft +$$
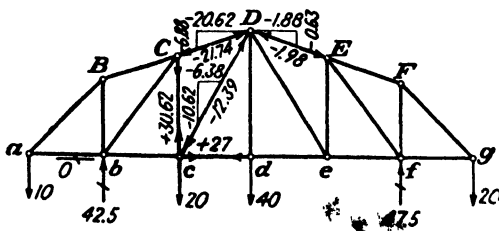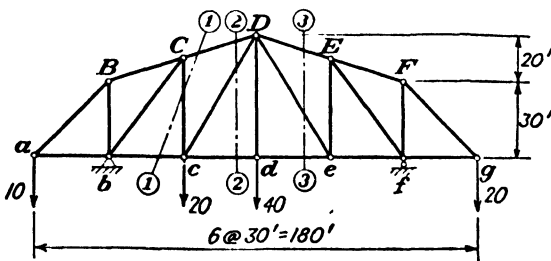
$$
\begin{array}{rcr}
20 \times 1 = & & 20 \\
40 \times 2 = & & 80 \\
20 \times 5 = & & 100 \\
\hline
& & 200 \\
-10 \times 1 = & & -10 \\
\hline
& & 190 \\
R_{fv} = & 47.5 \uparrow
\end{array}
$$

$$\Sigma M_f = 0, \qquad \curvearrowright +$$

$$
\begin{array}{rcr}
40 \times 2 = & & -80 \\
20 \times 3 = & & -60 \\
10 \times 5 = & & -50 \\
\hline
& & -190 \\
20 \times 1 = & & 20 \\
\hline
& & -170 \\
R_{bv} = & 42.5 \uparrow
\end{array}
$$

$$\Sigma F_y = 0, \uparrow +$$
$$47.5 + 42.5 - 10 - 20 - 40$$
$$- 20 = 0$$

*Bar Cc Section* ①-①
*Bar CD*
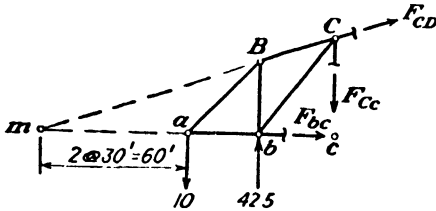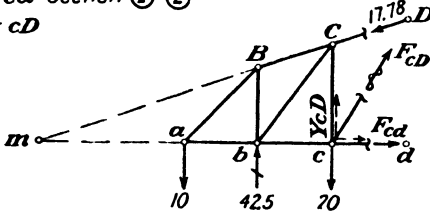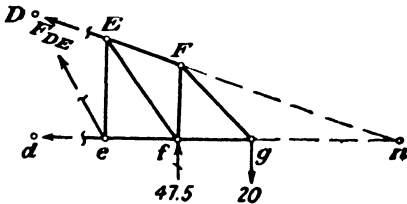


$\Sigma M_m = 0,$

$10 \times 2 \quad = \quad +20$

$42.5 \times 3 = -127.5$

$\overline{\qquad -107.5 \curvearrowleft}$

$\therefore F_{Cc} = + \dfrac{107.5}{4} = +26.88$

$\Sigma M_c = 0,$

$42.5 \times 1 = +42.5$

$10 \times 2 \quad = -20$

$\overline{\qquad +22.5 \curvearrowright}$

$\therefore X_{CD} = \dfrac{-(22.5)(30)}{40}$

$= -16.88$

$\therefore F_{CD} = -(16.88) \left( \dfrac{31.63}{30} \right)$

$= -17.80$

*Bar cd Section* ②-②
*Bar cD*



$\Sigma M_m = 0,$

$10 \times 2 = \quad +20$

$20 \times 4 = \quad \underline{80}$

$+100$

$42.5 \times 3 = -127.5$

$\overline{\qquad -27.5 \curvearrowleft}$

$\therefore Y_{cD} = \dfrac{-27.5}{4} = -6.88$

$\therefore F_{cD} = -6.88 \times \dfrac{58.3}{50}$

$= -8.02$

$\Sigma M_D = 0,$

$20 \times 1 = -20$

$10 \times 3 = -30$

$\overline{\qquad -50}$

$42.5 \times 2 = +85$

$\overline{\qquad +35}$

$\therefore F_{cd} = +35 \times \dfrac{30}{50} = +21$

$\Sigma M_e = 0,$

$47.5 \times 1 = -47.5$

$20 \times 2 \quad = +40.$

$\overline{\qquad - 7.5 \curvearrowleft}$

$\therefore X_{DE} = -7.5 \times \dfrac{30}{40}$

$= -5.62$

$\therefore F_{DE} = -5.62 \times \dfrac{31.6}{30}$

$= -5.92$

*Bar DE Section* ③-③

**Discussion:**

Note that, after the stress in Cc has been found, the vertical component of the stress in cD can easily be found by isolating joint c.   Note also that, the stress in cD being known, the vertical component in CD can be found from $\Sigma F_y = 0$ rather than $\Sigma M_r = 0$ for the free-body sketch for section 2-2.   The stress in CD being known, the stress in cd can be obtained from $\Sigma F_x = 0$ applied to this free-body sketch.

The stress in member DE can be obtained by isolating the portion of the truss to either the right or the left of section 3-3.   The portion to the right was chosen because it has fewer external forces acting on it.

In all these computations, the moments of the vertical forces are computed in terms of panel lengths.   When necessary, the panel length of 30 ft is substituted at the end of the computation.   This trick simplifies the numerical work in such computations.

**Example 4·2**   Determine the bar stresses in members dg, eg, gh, and hm.
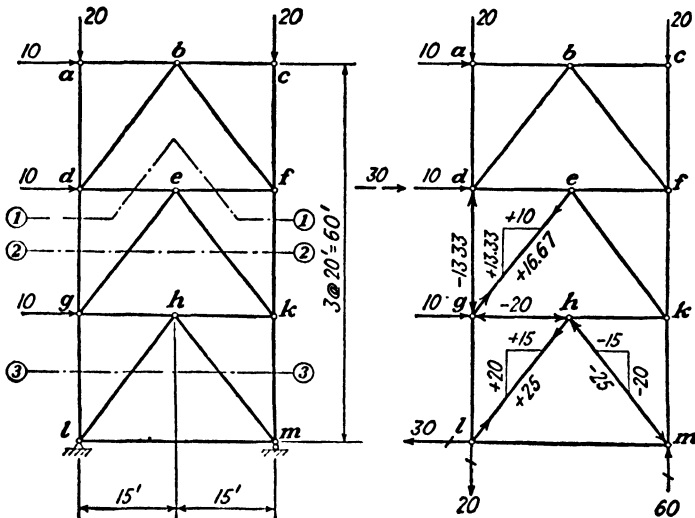


$$\Sigma M_l = 0, \;\curvearrowright_{+}$$
$$(20)(30) = \phantom{0}600$$
$$(30)(40) = \underline{1,200}$$
$$\phantom{(30)(40) = }1,800 \curvearrowright$$
$$\therefore R_{my} = 60 \updownarrow$$
$$\Sigma M_m = 0, \;\curvearrowright_{+}$$
$$(30)(40) = +1,200$$
$$(20)(30) = \underline{- \phantom{0}600}$$
$$\phantom{(20)(30) = }+ \phantom{0}600 \curvearrowright$$
$$\therefore R_{ly} = 20 \updownarrow$$

Check by $\Sigma F_y = 0, \;+\uparrow$.

$$-20 - 20 + 60 - 20 = 0$$

*Bar dg: Section ①-①*
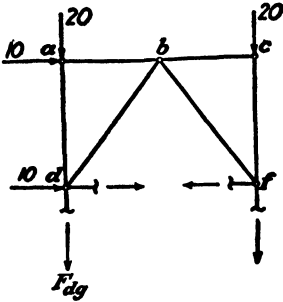


$$\Sigma M_f = 0, \curvearrowleft +$$
$$(10)(20) = +200$$
$$(20)(30) = -600$$
$$\overline{\phantom{(20)(30) =} -400} \curvearrowleft$$
$$\therefore F_{dg} = -\left(\frac{400}{30}\right) = \underline{-13.33}$$

*Bar eg; Section ②-②*
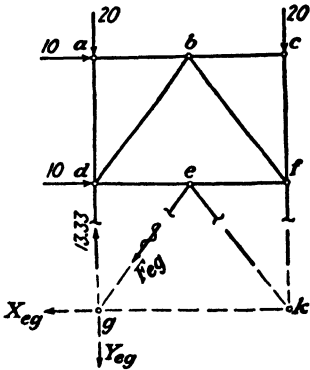


$$\Sigma M_k = 0, \curvearrowleft +$$
$$(13.33)(30) = +400$$
$$(10)(20) = +200$$
$$(10)(40) = +400$$
$$\overline{\phantom{(10)(40) =} +1,000}$$
$$(20)(30) = -600$$
$$\overline{\phantom{(20)(30) =} +400} \curvearrowright$$
$$\therefore Y_{eg} = \frac{400}{30} = \underline{+13.33} \qquad X_{eg} = \underline{+10}$$
$$F_{eg} = \underline{+16.67}$$

*Bar gh: Isolate joint g.*
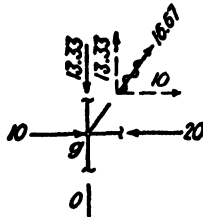
*Bar hm: Isolate joint h.   Then, consider portion above isolated by section 3-3.   From* $\Sigma F_v = 0$, $Y_{lh}$ *must be equal and opposite to* $Y_{hm}$. *Assume* $Y_{lh} = A$; *then* $Y_{hm} = -A$, *etc.*
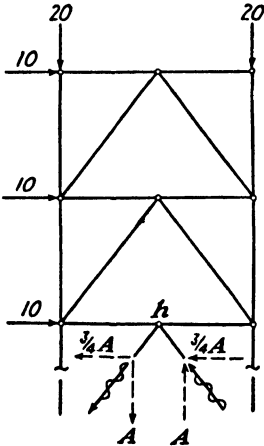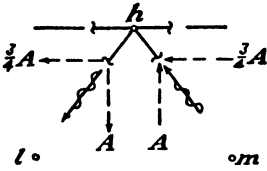
*Then* $\Sigma F_x = 0$, $\overrightarrow{+}$

$$10 + 10 + 10 - \tfrac{3}{4}A - \tfrac{3}{4}A = 0, \qquad A = 20$$
$$\therefore \ X_{lh} = +15 \qquad \text{and} \qquad X_{hm} = -15$$
$$Y_{lh} = +20 \qquad \qquad Y_{hm} = -20$$
$$F_{lh} = \underline{+25} \qquad \qquad F_{hm} = \underline{-25}$$

**Discussion:**

    *If the stresses in all the bars are required, the method of joints can be applied, the joints being isolated successively in the following order: a, c, b, d, f, e, g, k, h, l, m.   This is probably the most efficient method of finding all the bar stresses in this particular truss. If only a few particular bar stresses are required, the computation can be carried out as illustrated in this example.*

**Example 4·3**  *Compute the bar stresses in members bc, BC, aC, and bC.*



$\Sigma M_a = 0,$ $\quad\curvearrowright +$
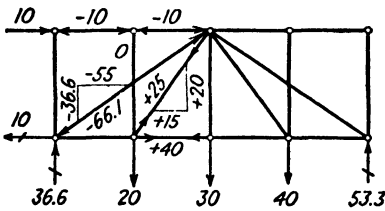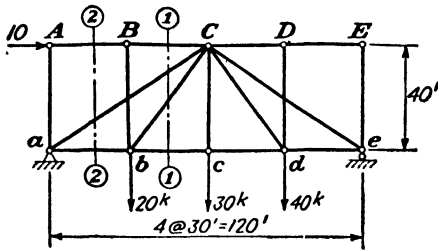
$10 \times \tfrac{4}{3}p = \quad +13.3p$

$20 \times 1p = \quad 20 \ p$

$30 \times 2p = \quad 60 \ p$

$40 \times 3p = \quad 120 \ p$

$\overline{\quad\quad +213.3p}$

$\therefore R_{ey} = 53.3\Uparrow$

$\Sigma M_e = 0,$ $\quad\curvearrowright +$

$20 \times 3p = \quad -60p$

$30 \times 2p = \quad -60p$

$40 \times 1p = \quad -40p$

$\overline{\quad\quad -160p}$

$10 \times \tfrac{4}{3}p = \quad +13.3p$

$\overline{\quad\quad -146.6p}$

$\therefore R_{ay} = 36.6\Uparrow$

*Check:*

$\Sigma F_y = 0,\ \uparrow +,\ 36.6 - 20 - 30$
$\quad\quad\quad\quad\quad - 40 + 53.3 = 0$

**Bar bc:** *Portion to left of 1-1,* $\Sigma M_C = 0, \curvearrowright +,\ 36.6 \times 2p = \quad 73.3p$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 20 \times 1\ p = \quad -20.0p$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overline{53.3p} \quad\quad \therefore F_{bc} = +40$

**Bar BC:** *Isolate joint A, then B.* $\quad F_{BC} = -10$

**Bar aC:** *Portion to left of 2-2,* $\quad \Sigma F_y = 0 \quad \therefore Y_{aC} = -36.6$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \therefore F_{aC} = \dfrac{\sqrt{3^2 + 2^2}}{2}(-36.6) = -66.1$

**Bar bC:** *Portion to left of 1-1, with stress in aC known, then, from* $\Sigma F_y = 0,$

$\therefore Y_{bC} = +20$

*or, from joint B,* $F_{Bb} = 0;$ *then, from joint b,* $\Sigma F_y = 0$

$\therefore Y_{bC} = +20$

**Example 4·4**   *Compute the bar stresses in members cd, BC, bm, nf, nF, and md.*
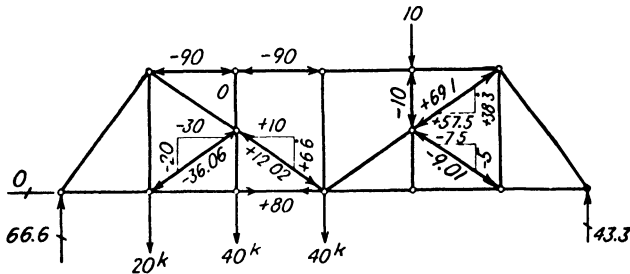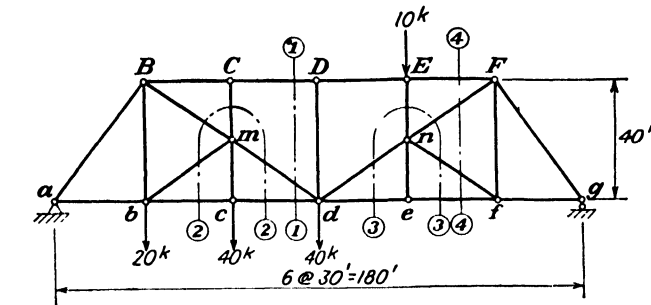


$$\Sigma M_a = 0, \overset{\curvearrowright}{+} \qquad \Sigma M_g = 0, \overset{\curvearrowleft}{+}$$

| | | |
|---|---|---|
| $20 \times 1 =$ | $20$ | $10 \times 2 =$ | $20$ |
| $40 \times 2 =$ | $80$ | $40 \times 3 =$ | $120$ |
| $40 \times 3 =$ | $120$ | $40 \times 4 =$ | $160$ |
| $10 \times 4 =$ | $40$ | $20 \times 5 =$ | $100$ |
| | $\overline{260}$ | | $\overline{400}$ |

$$R_{gy} = 43.3 \uparrow \qquad \therefore R_{ay} = 66.6 \uparrow$$

**Bar cd:** Portion to left of 1-1, $\Sigma M_B = 0, \overset{\curvearrowright}{+}$, $66.6 \times 1p + 40 \times 1p = 106.6p$
$$\therefore F_{cd} = \underline{-80}$$

**Bar BC:** Same portion, $\Sigma M_d = 0, \overset{\curvearrowright}{+}$,
$$66.6 \times 3p - 40 \times 1p - 20 \times 2p = 120p \qquad \therefore F_{CD} = \underline{-90}$$
$$\therefore \text{Isolate joint } C, F_{BC} = \underline{-90}$$

**Bar bm:** Isolating joint $C$ shows $F_{Cm} = 0$. Hence, considering portion isolated by section 2-2,
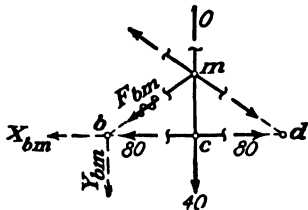


$$\Sigma M_d = 0, \overset{\curvearrowleft}{+}, +40 \times 1p = +40p$$
$$\therefore Y_{bm} = -20$$
$$\therefore X_{bm} = -30$$
$$\therefore F_{bm} = \underline{-36.06}$$

**Bar' nf:** *Isolating joint E shows* $F_{En} = -10$. *Hence, considering portion isolated by section 3-3,*

$$\Sigma M_d = 0, \overset{\curvearrowright}{+}, \quad 10 \times 1p = 10p \qquad \therefore Y_{nf} = \underline{-5}, \qquad X_{nf} = \underline{-7.5}, \qquad F_{nf} = \underline{-9.01}$$

**Bar nF:** *Portion to right of 4-4*, $\Sigma F_y = 0 \qquad \therefore Y_{nF} = \underline{+38.\dot{3}}$

**Bar md:** *Portion to right of 1-1*, $\Sigma F_y = 0 \qquad \therefore Y_{md} = \underline{+6.\dot{6}}$

---

**Example 4·5** *Determine the bar stresses in all members of this truss:*
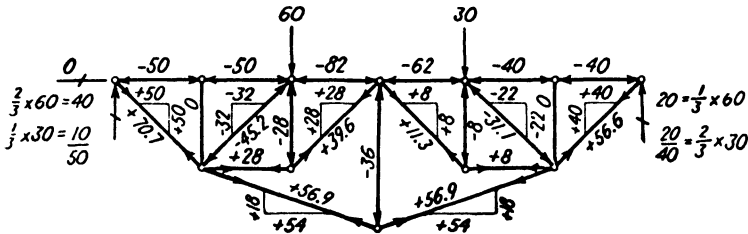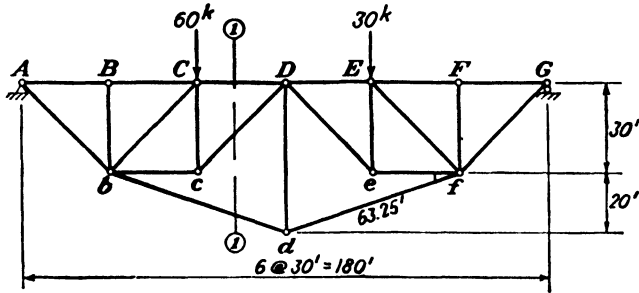


Portion to left 1-1

$$\Sigma M_D = 0, \overset{\curvearrowright}{+}$$
$$50 \times 3p = 150p$$
$$60 \times 1p = -60p$$
$$\overline{\frac{90p}{50}} = \frac{90 \times 30}{50} = 54 \qquad \therefore X_{bd} = \underline{+54}$$

**Discussion:**

In the solution of this problem, it is possible to start out and apply the method of joints successfully at joints A, B, G, and F. However, if one attempts to apply this same procedure at any one of the remaining joints, he finds it impossible since there are more than two unknown bar stresses at each of these joints. It is therefore desirable to resort to the method of sections. In this problem, the bar stress in member bd is found by considering the portion to the left of section 1-1. The method of joints can then be applied at joint b and then successively at each of the remaining joints.

It should be noted that this is a compound truss. In such trusses it is usually impossible to solve for all the bar stresses by using simply the method of joints. As illustrated in this problem, it is usually necessary to make at least one application of the method of sections.

**Example 4·6**  *Determine the bar stresses in all the members of this truss.*
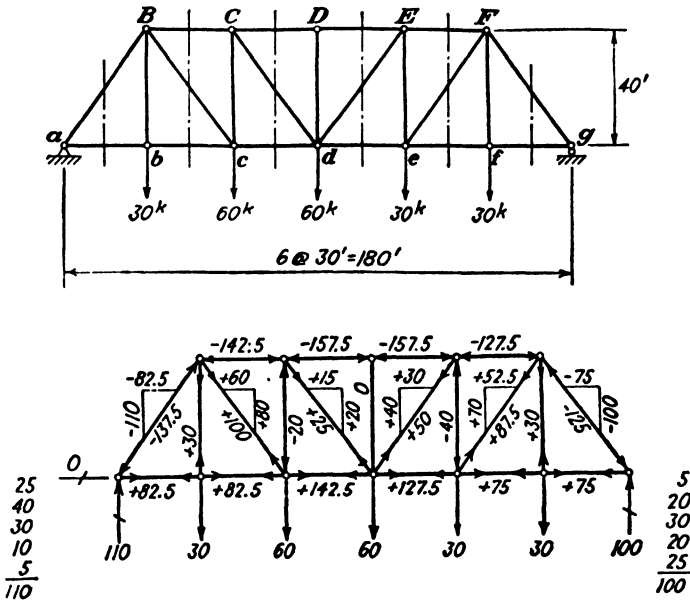


**Discussion:**

In the solution of this example, it is possible to start at one end of the truss and work through to the other end, using only the method of joints to compute the bar stresses. This is a simple truss, and it is always possible to find all the bar stresses in such a truss in this manner once the reactions have been computed.

It also should be noted that the vertical components of the bar stresses in the diagonals can also be easily obtained by using $\Sigma F_y = 0$ on the portions either to the right or to the left of the indicated vertical sections through the panels. The stress in the verticals can then be obtained from $\Sigma F_y = 0$ at the joints, and the stress in the chords from $\Sigma F_x = 0$ at the joints, working across from end to end of the truss.

## 4·7  Discussion of Method of Joints and Method of Sections.

The examples in the previous article illustrate that isolations of portions of the truss using both the method of joints and the method of sections must be employed in the stress analysis of trusses. Experience in such computations will teach the student how to combine these two methods most effectively. It is the purpose of this particular article to summarize and clarify the important points concerning them.

In the previous discussion of the method of joints, it is pointed out that this procedure enables one to determine immediately all the unknown bar stresses acting on an isolated joint, provided that there are not more

than two unknown stresses and that these two unknown stresses have
different lines of action. It sometimes happens that there is only one
unknown bar stress acting on an isolated joint. In such a case, one of
the two available equilibrium equations may be used to solve for the one
unknown stress, and the other may be used as a check that must be
satisfied by all the forces acting on this joint. If there are more than two
unknown stresses acting on an isolated joint, it is usually impossible to
obtain an immediate solution for any of the unknowns from the two equa-
tions of equilibrium that are available at that joint. In such cases, it is
necessary to isolate additional joints and write two additional equations
for each joint. In this manner, it is sometimes possible to obtain $n$
independent equations, involving $n$ unknown bar stresses. Then these
$n$ equations can be solved simultaneously for the $n$ unknowns.
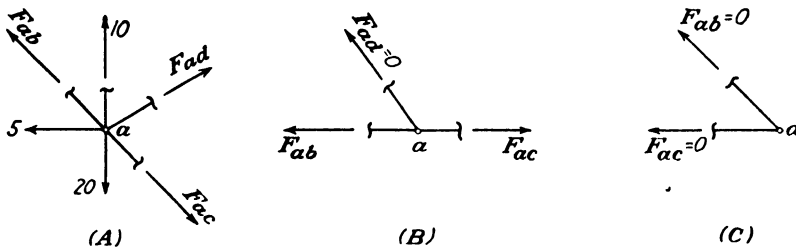


FIG. 4·6

There is one important case where there are more than two unknown
bar stresses, acting on an isolated joint but so arranged that it is possible
to obtain the value of one of them immediately. If all the unknown
stresses except one have the same line of action, then the stress in that
one particular bar may be determined immediately. Such a case is
shown in sketch $A$ of Fig. 4·6. If the $x$ axis is taken as being parallel to
the line $bac$ and the $y$ axis perpendicular to this line, then the $y$ component
of the unknown stress $F_{ad}$ may be determined immediately from the
equation $\Sigma F_y = 0$. No immediate solution may be obtained from
$\Sigma F_x = 0$ at such a joint since this equation involves both the unknowns
$F_{ab}$ and $F_{ac}$. A special case of this type is shown in sketch $B$ of Fig. 4·6,
where the joint is acted upon by only the three unknown bar stresses.
If $F_{ab}$ and $F_{ac}$ have the same line of action, then it is apparent that the
only remaining force at the joint, $F_{ad}$, must be zero. It is also of interest
to consider the case shown in sketch $C$ of Fig. 4·6. In this case, the joint
is acted upon by only two forces, which do not have the same line of
action; therfore, in order to satisfy $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for such a
joint, both $F_{ab}$ and $F_{ac}$ must be equal to zero.

It is also interesting to note that, once the reactions have been deter-

mined on a *simple* truss, then all the bar stresses can be determined by using only the method of joints and never resorting to the method of sections.   It is apparent that this is so, since there are only two unknown bar stresses acting on the joint that was located *last* in arranging the lay-out of the bars of the truss.   After these two bar stresses have been determined by isolating this joint, then it will be found that the joints at the far ends of these two members are acted upon by only two unknown bar stresses.   Thus, by isolating the joints in the reverse order from that in which they were established in laying out the truss, the method of joints may be used to determine all the bar stresses.   This is the explanation of why it is possible to solve Example 4·6 in this manner.   It should be noted, however, that in many cases of simple trusses the calculation of all the bar stresses may be expedited by combining the use of both the method of sections and the method of joints, as is illustrated by some of the examples in Art. 4·11.

When applying the method of sections, if the isolated portion of the truss is acted upon by three unknown bar stresses that are neither parallel nor concurrent, then all three unknown stresses may be determined from the three equilibrium equations available for the isolated portion. It is presumed, of course, that the reactions acting on any such portion have been determined previously.   Of course, if there are only one or two unknown bar stresses, these may be determined by using a like number of the available equations.   The remainder of the equations that must be satisfied by the system of forces acting on the isolated portion may then be used simply as a check on the calculations up to this point.

It is sometimes possible to find some of the unknown stresses by the method of sections even when there are more than three unknowns acting on the isolated portion.   For example, suppose that the lines of action of all but one of the unknown stresses intersect at a point *a*.   Then, the stress in this one remaining bar can be determined from the equation $\Sigma M_a = 0$, that is, by summing up the moments of the forces about point *a*.   Another similar case would be one where all the unknown bar stresses except one are parallel.   The stress in this remaining bar can then be determined by summing up all the force components that are perpendicular to the direction of the other unknown bar stresses.   In each of the above cases, the remaining two equilibrium equations for the isolated portion will involve more unknowns than there are equations, and hence no immediate solution for these remaining unknowns is possible.

In applying either the method of joints or the method of sections, it is important to realize that it makes no difference how many bars have been cut in which the bar stresses are *known*.   Only the number of unknown bar stresses is important.

**4·8  Statical Stability and Statical Determinancy of Truss Structures.** Up to this point the emphasis has been placed on the methods of computing the bar stresses in trusses. For this purpose, all the examples that have been used have been statically determinate and stable. With these ideas as a background, it is now possible to discuss the question of statical stability and determinancy of trusses from a general standpoint.

In discussing the arrangement of the members of a simple truss, it was shown that a rigid truss is formed by using three bars to connect three joints together in the form of a triangle and then using two bars to connect each additional joint to the framework already constructed. Thus, to form a rigid simple truss of $n$ joints, it is necessary to use the three bars of the original triangle plus two additional bars for each of the remaining $(n - 3)$ joints. If $b$ denotes the total number of bars required. then

$$b = 3 + 2(n - 3) = 2n - 3$$

This is the minimum number of bars that can be used to form a rigid simple truss. To use more is unnecessary and to use fewer results in a nonrigid or unstable truss. If a simple truss having $n$ joints and $(2n - 3)$ bars is supported in a manner that is equivalent to three links that are neither parallel nor concurrent, then the structure is stable under a general condition of loading and the reactions are statically determinate. In the previous article, it is pointed out that, once the reactions are found, all the bar stresses of a simple truss can be computed by the method of joints.

It may be concluded, therefore, that a stable simple truss having three independent reaction elements and $(2n - 3)$ bars is statically determinate with respect to both reactions and bar stresses. If there are more than three reaction elements, the structure is statically indeterminate with respect to its reactions; if there are more than $(2n - 3)$ bars but only three reaction elements, it is indeterminate with respect to the bar stresses; and if there is an excess of both bars and reaction elements, the structure is indeterminate with respect to both reactions and bar stresses.

The same discussion and conclusions apply equally well to a compound truss. Suppose a compound truss is formed by connecting two simple trusses together by means of three additional bars that are neither parallel nor concurrent. If the two simple trusses have $n_1$ and $n_2$ joints, respectively, the total number of bars $b$ in the compound truss is

$$b = (2n_1 - 3) + (2n_2 - 3) + 3 = 2(n_1 + n_2) - 3$$

or if $n$ denotes the total number of joints in the compound truss, *i.e.*,

if $n = n_1 + n_2$, then

$$b = 2n - 3$$

Thus, the minimum number of bars that can be used to form a rigid com-
pound truss is the same as in the case of a simple truss. If the remainder
of the discussion were carried out in a similar manner for a compound
truss, it would be found that the conclusions of the previous paragraph
apply equally well to both simple and compound trusses.

It is desirable to discuss this question of determinancy and stability
from a more general standpoint. Suppose a truss structure has $r$ inde-
pendent reaction elements, $b$ bars, and $n$ joints. If the truss as a whole
is in equilibrium, then every isolated portion must likewise be in equilib-
rium. To isolate an entire bar or some portion of it would produce no
new information since the equilibrium conditions of the bars were con-
sidered during the establishing of the definition of the term bar stress.
However, it is possible to isolate each of the $n$ joints in turn and to write
for each of these joints two new and independent equations of static
equilibrium, $\Sigma F_x = 0$ and $\Sigma F_y = 0$. In this manner, a total of $2n$
independent equations would be obtained, involving as unknowns the $r$
reaction elements and the $b$ bar stresses, a total of $(r + b)$ unknowns.
These $2n$ equations must be satisfied simultaneously by the $(r + b)$
unknowns. By comparing the number of unknowns with the number of
independent equations, it is possible to decide whether a truss structure is
unstable, statically determinate, or indeterminate. The reasoning
involved is similar to that used in Art. 2·5. If $r + b$ is less than $2n$,
there are not enough unknowns available to satisfy the $2n$ equations
simultaneously and therefore the structure is said to be *statically unstable*.
If $r + b$ is equal to $2n$, the unknowns can then be obtained from the
simultaneous solution of the $2n$ equations and therefore the structure is
said to be *statically determinate*. If $r + b$ is greater than $2n$, there are
too many unknowns to be determined from these $2n$ equations alone and
therefore the structure is said to be *statically indeterminate*. The cri-
terion establishes the combined degree of indeterminancy with respect
to both reactions and bar stresses. It is apparent that these conclusions
agree with the foregoing discussion of simple and compound trusses.

At first glance, it might seem that the total number of independent equations
of static equilibrium in a truss structure should include not only the $2n$ equations
noted in the previous paragraph but also the three equations $\Sigma F_x = 0$, $\Sigma F_y = 0$,
and $\Sigma M = 0$ applied to the entire structure as a free body. However, the follow-
ing demonstration will prove that this offhand opinion is not so and that there
are only $2n$ independent equations: Consider any truss as a free body acted upon
by its reactions and applied loads such as the truss shown in Fig. 4·7A. Suppose

the system of forces shown in Fig. 4·7B is superimposed on the system in Fig. 4·7A; the combination of these two loading systems will then be as shown in Fig. 4·7C. The load system in B is a special system consisting of several pairs of equal and opposite forces, one pair for each member of the truss. For any particular member, both forces of its pair act along the member, one acting on the joint at one end of the member and the other on the joint at the other end. Each of the forces is numerically equal to *the stress in that member produced by the forces in system A* and acts in a sense that is the same as the action of this bar stress on the joint. Each pair of forces is in equilibrium, of course, and hence all the pairs acting together form a system that is in equilibrium.

Considering now the combined system in C, it is found that the forces acting at each joint of the truss are the same forces that would be acting on that joint if, under the loading of A, it were isolated by itself as a free body. If, however, the applied forces, reactions, and bar stresses satisfy simultaneously the 2n equations of statics obtained by isolating the n joints and writing $\Sigma F_x = 0$ and
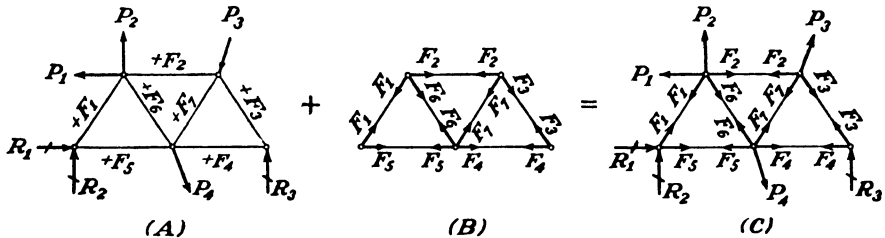


Fig. 4·7

$\Sigma F_y = 0$ for each joint, the forces acting on *each* joint in C form a concurrent system of forces that are in equilibrium. Since at each joint the forces are in equilibrium, the combined system in C of all joints is in equilibrium and satisfies the equations $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$ for the entire truss. Since the combined system in C is in equilibrium and the portion of this system shown in sketch B is also in equilibrium by itself, then the remaining portion of the system as shown in sketch A must be in equilibrium and therefore must satisfy the equations $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$ for the entire truss. It may be concluded, therefore, that, if the reactions, bar stresses, and applied loads satisfy the 2n equations of equilibrium obtained by isolating the joints of the truss, then the reactions and applied loads will automatically satisfy the three equations of equilibrium for the truss as a whole and that thus there are only 2n independent equations of static equilibrium involved in a truss.

It should be noted that comparing the count of the unknowns and the independent equations establishes a criterion which is *necessary* but not always *sufficient* to decide whether a truss is stable or not. If $b + r$ is less than 2n, then this comparison is sufficient for deciding that the truss is *statically unstable*. If, however, $b + r$ is equal to or greater than 2n, it does not automatically follow that the truss is stable. This statement

may be verified by considering the examples shown in Fig. 4·8. In all
four of these cases, the structures are unstable, whereas the count by
itself indicates that $A$ and $B$ are statically determinate and $C$ and $D$ are
indeterminate to the first degree. $A$ and $D$ are unstable under a general
condition of loading because in each case the reactions are equivalent to
parallel links. $B$ and $C$ are unstable, not because of the arrangement of
the reactions, but because of the arrangement of the bars. In $B$, for
example, the reactions are statically determinate, but the truss is unstable
and would collapse because there is nothing to carry the shear in the
second panel from the right end.

These and other considerations lead to the conclusion that, even
though the count indicates that the structure is either statically determi-



$b = 13$   $r = 3$   $n = 8$
$b + r = 16$    $2n = 16$

$(A)$

$b = 13$   $r = 3$   $n = 8$
$b + r = 16$    $2n = 16$

$(B)$

$b = 17$   $r = 4$   $n = 10$
$b + r = 21$    $2n = 20$

$(C)$

$b = 25$   $r = 4$   $n = 14$
$b + r = 29$    $2n = 28$

$(D)$
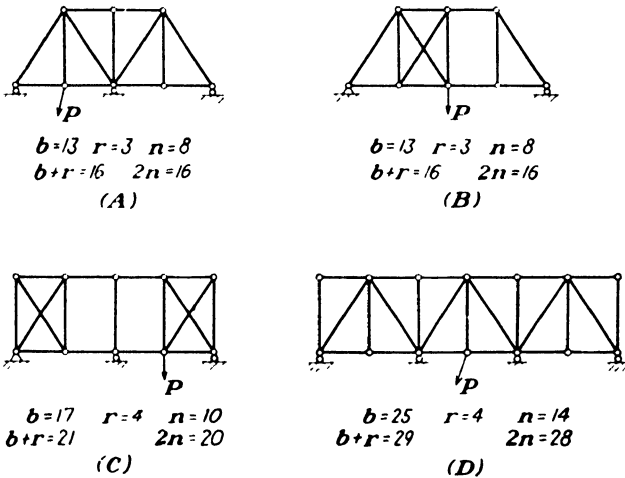
FIG. 4·8

nate or indeterminate, for it to be stable also, it is necessary that the follow-
ing conditions shall likewise be satisfied: (1) The reactions must be
equivalent to three or more links that are neither parallel nor concurrent.
(2) The bars of the truss must also be arranged in an adequate manner.
It is sometimes difficult to determine whether or not the arrangement of
the bars is adequate. In such cases, if the arrangement is inadequate it
will become apparent; for when a stress analysis is attempted, it will yield
results that are inconsistent, infinite, or indeterminate.

**4·9   Examples Illustrating the Determination of Stability and
Determinancy.** It is easy to investigate the stability and deter-
minancy of a truss structure that is formed by supporting in some manner
a truss that is in itself a rigid body. The truss itself may be merely a
simple or compound truss or, in some cases, a simple or compound truss

modified by adding more than the necessary number of bars. In either case, the bars, reactions, and joints can be counted and the criteria of the last article applied to decide whether the structure is unstable, statically determinate, or statically indeterminate. This count, of course, enables one to classify the structure with respect to both bar stresses and reactions. If the count shows that the structure is statically determinate or indeterminate, the question of stability must still be decided, for the count by itself is not sufficient to prove that the structure is stable.
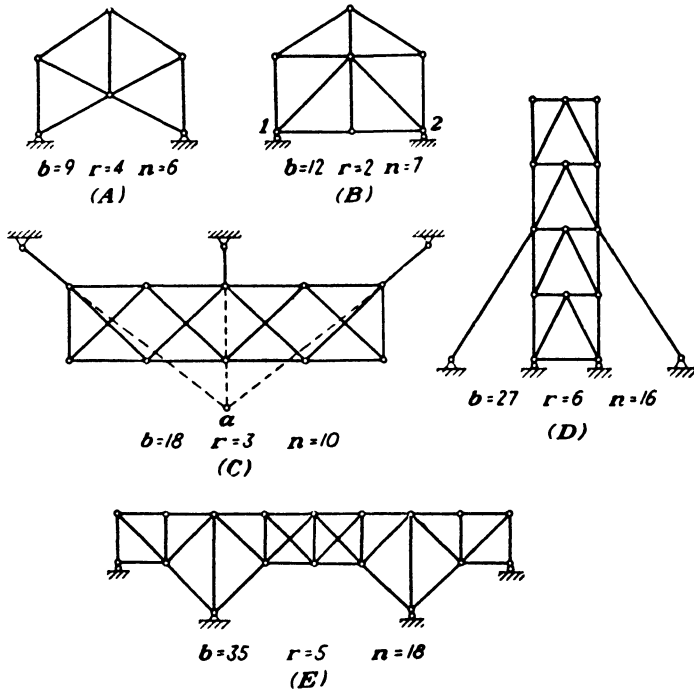


Fig. 4·9

It is also easy to classify this type of structure with respect to its reactions only. If there are less than three independent reaction elements, the structure is statically unstable under a general condition of loading regardless of how the bars of the truss are arranged. If there are three or more independent reaction elements and they are arranged so as to be equivalent to three or more links that are neither parallel nor concurrent, the structure is stable with respect to its reactions. For a stable structure, if there are exactly three reaction elements, these elements are statically determinate; if there are more than three reaction elements, the structure is statically indeterminate with respect to its reactions alone to a degree that is equal to the number of reaction ele-

ments in excess of three.    Structures in this general category are shown in
Fig. 4·9.

The count of the bars, joints, and reaction elements is shown in each
of the sketches in Fig. 4·9.    Considering only the reactions, structure
$A$ is stable and statically indeterminate to the first degree.    Since
$b + r = 13$ and $2n = 12$, it is also indeterminate to the first degree,
considering both reactions and bar stresses.    The count of structure $B$
indicates that it is statically determinate since $b + r$ and $2n$ are both
equal to 14.    A consideration of the reactions, however, discloses that this
structure is actually unstable.    Likewise, the count indicates that
structure $C$ is statically indeterminate to the first degree, but considera-
tion of the reactions shows that it is unstable.    Both the count and the
consideration of the reactions indicate that structure $D$ is indeterminate
to the first degree.    With respect to the reactions only, structure $E$ is
indeterminate to the second degree, but a count of both bars and reactions
discloses that it is actually indeterminate to the fourth degree.

There is another important type of truss structure that is built up out
of more than just one rigid truss.    In this type, the structure is composed
of several rigid trusses connected together in some manner and then the
whole assemblage mounted on a certain number of supports.    In such
cases, the supports are usually arranged so as to provide more than three
independent reaction elements.    The connections between the several
rigid trusses are, however, not completely rigid, so that certain equations
of condition (or construction) are introduced so as to reduce the degree
of indeterminancy or perhaps even to make the reactions statically deter-
minate.    This type of structure is the hardest to analyze from a stability
or determinancy standpoint.    However, some of the most important
trussed structures—for example, cantilever and three-hinged arch bridges
—belong in this category, and therefore it is important for the student to
master the methods of investigating this type.    Structure of this general
type are illustrated in Fig. 4·10.

The stability and determinancy of structures of the type shown in
Fig. 4·10 may be investigated by comparing the count of the bars and
reaction elements with the count of the joints.    With this criterion, it
will be concluded that structures $A$, $B$, $D$, $E$, and $F$ are statically deter-
minate and structure $C$ is indeterminate to the first degree.    In structures
of this type, it is also important to consider whether or not the structure
is statically determinate with respect to its reactions alone.    This may
be done by comparing the count of the unknown reaction elements with
the number of available equations in the same manner as discussed
previously in Arts. 2·5 and 2·6.    In these cases, the available equations
include the three equations of static equilibrium for the structure as a

whole plus any equations of condition which may be introduced by the
manner in which the several rigid trusses are connected together.

If two trusses are hinged together at a common joint, such as the
joints marked $a$ in Figs. 4·10 $A$ to $E$, one equation of condition is intro-



$b = 16$   $r = 4$   $n = 10$

(A)

$b = 43$   $r = 5$   $n = 24$

(B)

$b = 42$   $r = 5$   $n = 23$

(C)

$b = 71$   $r = 7$   $n = 39$

(D)

9 panels @ 25'

$b = 34$   $r = 6$   $n = 20$

(E)

8 panels @ 25'

$b = 31$   $r = 5$   $n = 18$

(F)
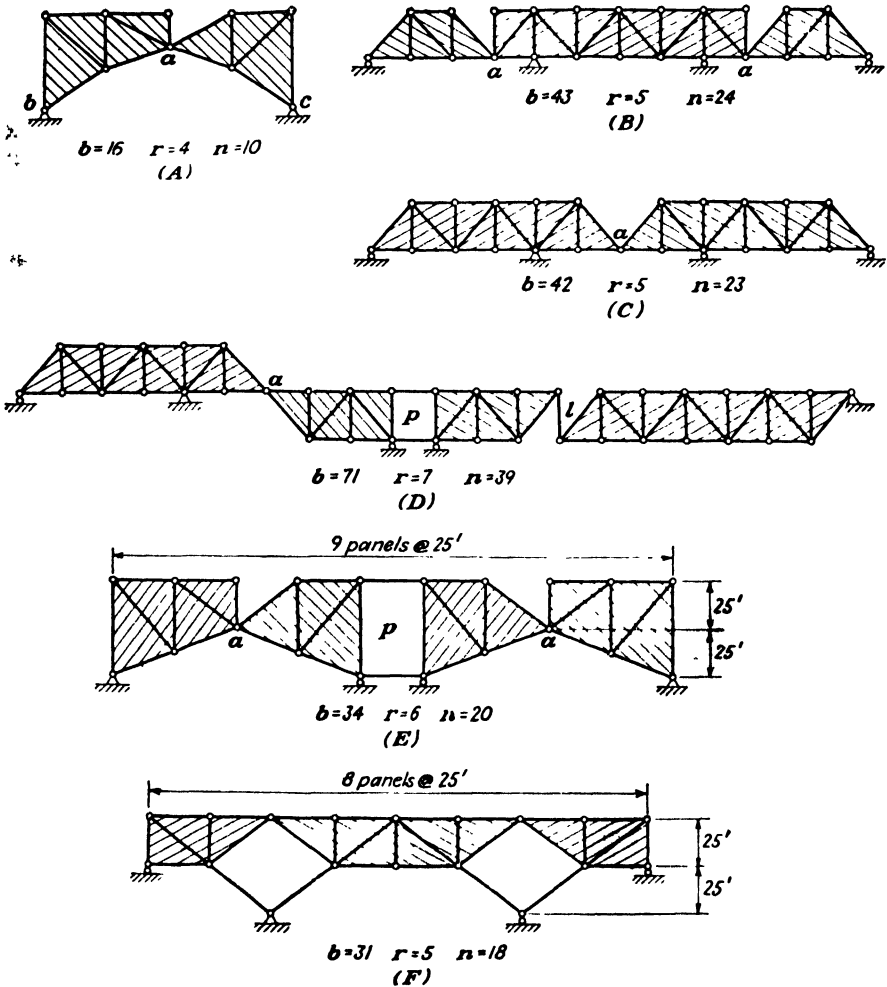
Fig. 4·10

duced, viz., that the bending moment about that point must be zero since
the hinge cannot transmit a couple from one truss to the other. If two
trusses are connected together by a link or roller, such as the link marked $l$
in structure $D$, two equations of condition are introduced since then both
the direction and point of application of the interacting force are known.

This means, therefore, (1) that the bending moment about *either* end of the link must be zero and (2) that the interacting force between the two trusses cannot have a component perpendicular to the link. If two trusses are connected together by two parallel bars, as is done in panels *p* of structures *D* and *E*, one equation of condition is introduced, *viz.*, that the interaction between the two trusses cannot involve a force perpendicular to the two bars. In the case of structures *D* and *E*, this means that the shear acting on panel *p* must be zero.

From this discussion, it is apparent that one equation has been introduced in structure *A*, two in *B*, one in *C*, four in *D*, and three in *E*. It will be concluded, therefore, that, with respect to reactions only, structures *A*, *B*, *D*, and *E* are statically determinate and structure *C* is indeterminate to the first degree. Structure *F* is a special type of structure, called a Wichert[1] truss in this country, that can be counted only by considering the bars, joints, and reactions.

There is no obvious instability in any of the structures of Fig. 4·10. If, however, one attempts to compute the reactions and bar stresses for either structures *E* or *F*, the results will be inconsistent, infinite, or indeterminate; therefore, these structures are actually unstable. In both cases, by changing only the geometry of the structure, it is possible to make the structure stable. Structures *E* and *F* are therefore said to be *geometrically unstable*. This type of instability may arise whenever equations of condition are introduced by the arrangement of the structure. Sometimes the instability is obvious, but usually it does not become apparent until one attempts to compute the reactions, etc.[2]

**4·10 Conventional Types of Bridge and Roof Trusses.** The members of a truss may be arranged in an almost unlimited number of ways, but the vast majority of trusses encountered in bridge or building work belong to one of the common types shown in Figs. 4·11 and 4·12. Since they are encountered so frequently, the student should be familiar with the names of these conventional types.

Trusses *A*, *B*, *C*, *D*, and *E* of Fig. 4·11 are simple trusses, while the remaining trusses are compound trusses built up out of the simple trusses (shaded). In order to achieve economical design of single-span steel-truss bridges, it is essential for the ratio of depth of truss to length of span to be between $\frac{1}{5}$ and $\frac{1}{10}$, for the diagonals to slope at approximately 45° to the horizontal, and for the panel lengths not to exceed 30 to 40

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[1] STEINMAN, D. B., "The Wichert Truss," D. Van Nostrand Company, Inc., New York, 1932.

[2] For a more complete discussion see W. M. FIFE and J. B. WILBUR, "Theory of Statically Indeterminate Structures," McGraw-Hill Book Company, Inc., New York, 1937.

ft.   Trusses $A$, $B$, $C$, and $D$ can meet these requirements if the span is not too long.   For long-span bridges, however, it becomes necessary to use one of the subdivided types such as $F$, $G$, or $H$.

All of the roof trusses shown in Fig. 4·12 are simple trusses with the exception of the Fink truss.   This is a compound truss.



(A)
Howe Truss

(B)
Pratt Truss

(C)
Warren Truss

(D)
Warren Truss with verticals

(E)
K Truss

(F)
Sub-divided Warren Truss

(G)
Sub-divided Pratt Truss
or Baltimore Truss

(H)
Baltimore Truss with inclined
chord or Petit Truss
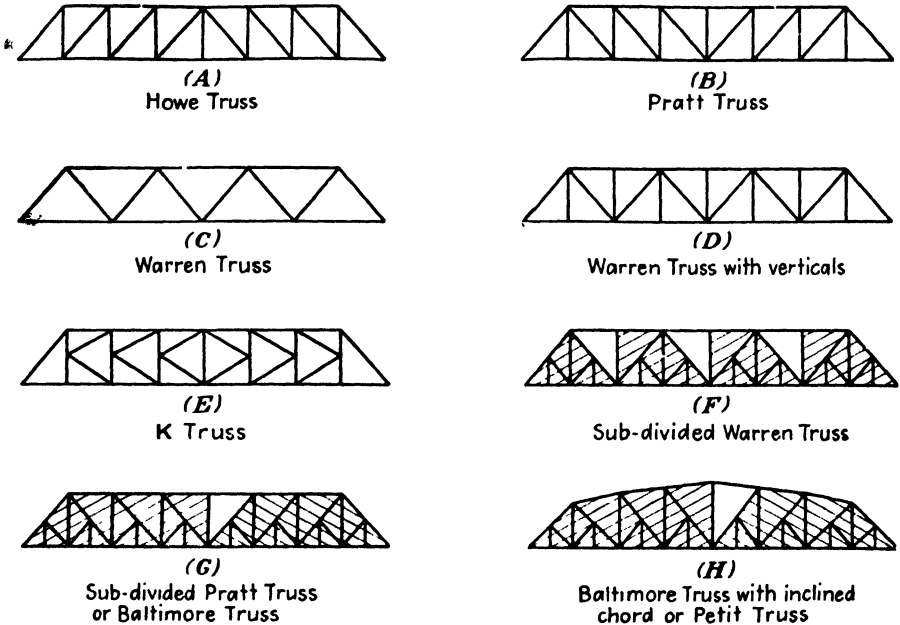
BRIDGE TRUSSES

FIG. 4·11

(A)
Howe Truss

(B)
Pratt Truss

(C)
Fan Truss

(D)
Fink Truss
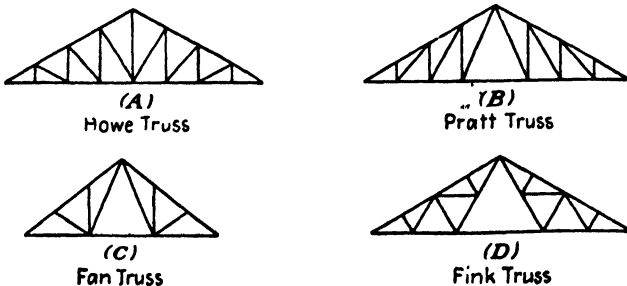
ROOF TRUSSES

FIG. 4·12

**4·11   Illustrative Examples of Stress Analysis of Determinate Trusses.**   The following examples illustrate the application of the previous discussions to the stress analysis of several conventional types of trusses.   The analysis of such trusses is discussed further in Chap. 7.

**Example 4·7**  *Determine the bar stresses in all members of this Pratt truss with a curved top chord:*
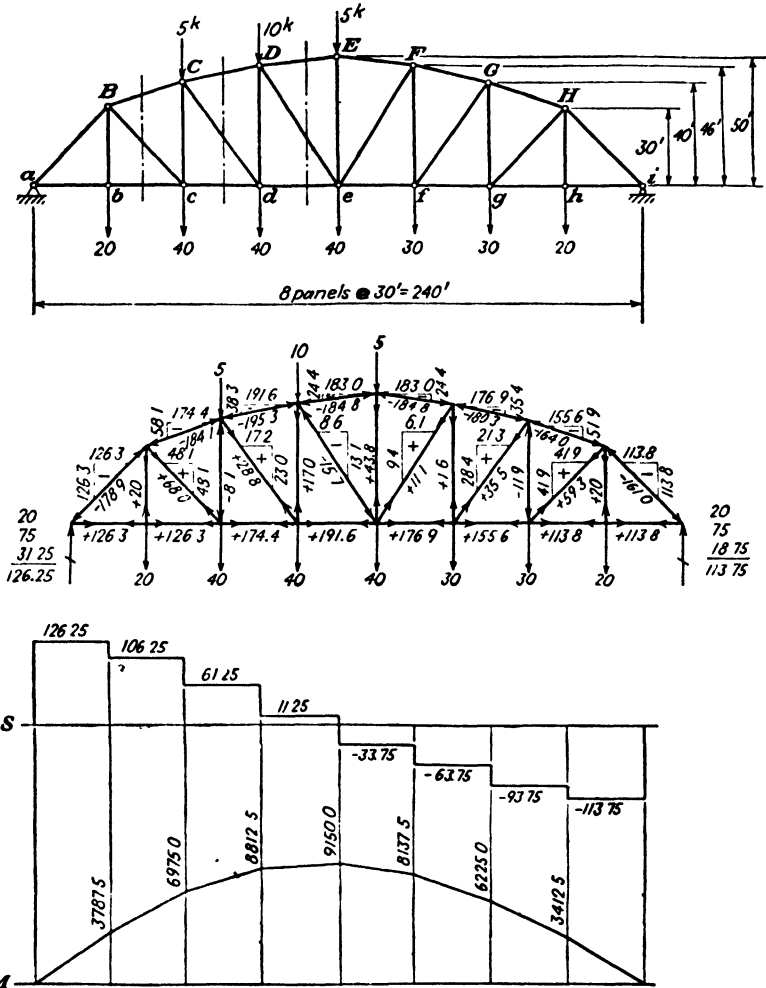
$$0 = M_a$$

$$126.25 \times 30 = \underline{3,787.5}$$
$$-20 \qquad\qquad 3,787.5 = M_b$$
$$\overline{106.25} \times 30 = \underline{3,187.5}$$
$$-45 \qquad\qquad 6,975.0 = M_c$$
$$\overline{61.25} \times 30 = \underline{1,837.5}$$
$$-50 \qquad\qquad 8,812.5 = M_d$$
$$\overline{11.25} \times 30 = \underline{337.5}$$
$$-45 \qquad\qquad 9,150.0 = M_e$$
$$\overline{-33.75} \times 30 = \underline{-1,012.5}$$
$$-30 \qquad\qquad 8.137.5 = M_f$$
$$\overline{-63.75} \times 30 = \underline{-1,912\,5}$$
$$-30 \qquad\qquad 6,225.0 = M_g$$
$$\overline{-93.75} \times 30 = \underline{-2,812.5}$$
$$-20 \qquad\qquad 3,412.5 = M_h$$
$$\overline{-113.75} \times 30 = \underline{-3,412.5}$$
$$0 \quad = M_i$$

**Discussion:**

This Pratt truss is a simple truss and can therefore be analyzed by using simply the method of joints. This procedure, however, is not particularly efficient in a case where the two chords are not parallel. Probably the best procedure is first to find the horizontal components in the members of the curved chord. These may be computed by passing a vertical section through a panel and taking moments about the appropriate joint on the bottom chord. These computations are facilitated if the bending moments are known at the various joints along the bottom chord.

The bending moments at the bottom-chord joints may be computed very conveniently by drawing the shear and bending-moment diagrams as shown. In this case where all the loads and reactions are vertical, the bending moment about any top-chord joint is the same as that about the bottom-chord joint directly under it. Of course, if there are horizontal loads, this relationship is not necessarily true.

The horizontal components in the top chord being known, the remainder of the stress analysis can be accomplished by the method of joints. It should also be noted that it is easy to compute the vertical components in the diagonals once the shears in the panels and the vertical components in the top chords are known.

**Example 4·8**    *Determine the bar stresses in all members of this Fink roof truss:*
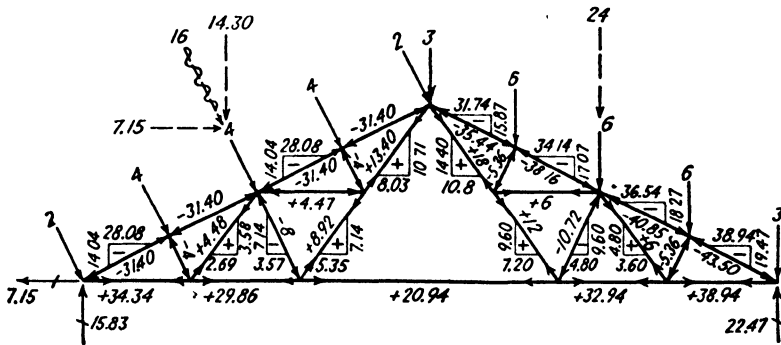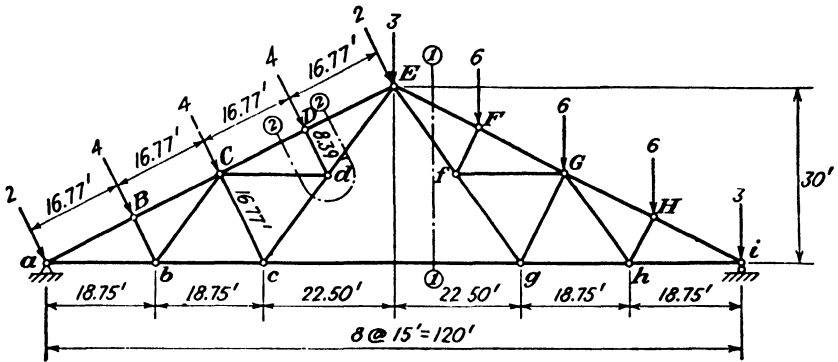


$$14.3 \times \frac{82.5}{120} = 9.83 \qquad 16 \times \frac{33.54}{120} = 4.47$$

$$24 \times \frac{30}{120} = 6 \qquad\qquad 24 \times \frac{90}{120} = 18$$

$$\overline{15.83} \qquad\qquad\qquad \frac{\overline{22.47}\quad 24}{15.83\quad 14.3}$$

$$F_{cg}: \qquad 22.47 \times \frac{60}{30} = 44.94 \qquad\qquad 38.30 \quad 38.3$$

$$-(18 \times 30 + 3 \times 60)\frac{1}{30} = -24 \qquad\qquad O.K.$$

$$F_{cg} = +\overline{20.94}$$

**Discussion:**

Although this Fink truss is a compound truss, it can be analyzed by using only the method of joints. For example, after the reactions have been computed, the method of joints can be applied successively at joints i, H, and h. Since there are more than two unknown bar stresses at each of joints g and G, it is not possible to consider these joints as the next step in the analysis. One can, however, determine the stress in bar Ff by isolating joint F and the stress in bar fG by isolating joint f. Then the stress analysis can be completed, still by means of the method of joints. This procedure is possible, however, only because joints E, F, and G and likewise joints E, f, and g lie on straight lines.

However, *it is usually preferable to proceed as follows: After applying the method of joints at joints i, H, and h, the stress in bar cg can be obtained by isolating the portion to the right of section 1-1 and taking moments about point E; the rest of the analysis can then be completed by the method of joints.*

*Note also that the stress analysis can be expedited by finding the stresses in bars Cd, Cb, Gf, and Gh, using sections similar to section 2-2, and taking moments about points such as point E in the case of section 2-2.*

*The main point to remember is to consider the application of both the method of joints and the method of sections and combine the two approaches in such a way as to expedite the calculations.*

*Note that the geometry of the truss is rather complicated. This is often true in the case of roof trusses. In such cases it is often easier to accomplish the stress analysis of the truss by graphical methods.*

**Example 4·9** *Determine the stresses in all the members of this Howe truss:*
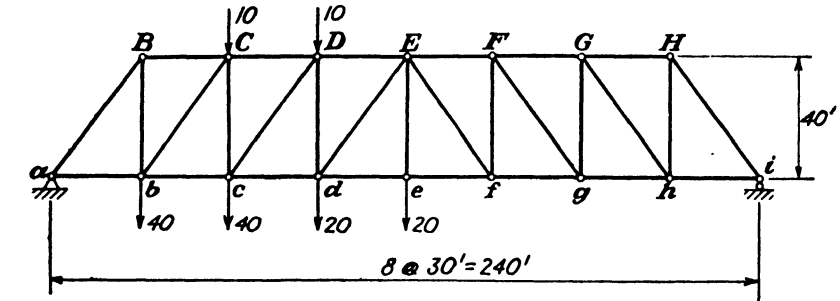


Index stresses:
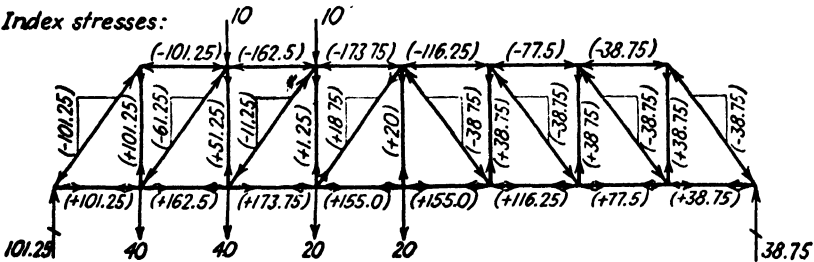


$$80 \times \frac{6.5}{8} = 65 \qquad 80 \times \frac{1.5}{8} = 15$$

$$40 \times \frac{4.5}{8} = 22.5 \qquad 40 \times \frac{3.5}{8} = 17.5$$

$$20 \times \frac{5.5}{8} = \underline{13.75} \qquad 20 \times \frac{2.5}{8} = \underline{\quad 6.25}$$

$$\overline{101.25} \Updownarrow \qquad\qquad\qquad \underline{38.75} \Updownarrow$$

$$\overline{101.25}$$

$$\overline{140.0} \qquad \therefore O.K.$$

*Bar stresses:*
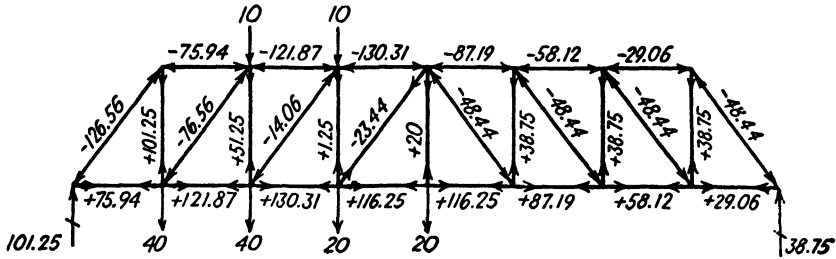


$$\text{For verticals, bar stress} = 1 \times (\textit{index stress})$$
$$\text{For diagonals, bar stress} = {}^{50}\!/_{40} \times (\textit{index stress})$$
$$\text{For chords, bar stress} = {}^{30}\!/_{40} \times (\textit{index stress})$$

Check:                  $F_{ef} = 38.75 \times {}^{120}\!/_{40} = +116.25$

**Discussion:**

*After the reactions have been computed, it is possible to compute the vertical components in all the web members by working from one end of the truss to the other, using either the method of joints or the method of sections. Then it is possible to compute the horizontal components in the diagonals and apply the method of joints to obtain the chord stresses. In this case, however, the computation of the chord stresses depends only on the horizontal components in the various diagonals. Since all the diagonals have the same slope, the ratio between the vertical and horizontal components is the same for all of them—in this case as 40 is to 30. In applying $\Sigma F_x = 0$ at the various joints to obtain the chord stresses, it is therefore permissible to use the values of vertical components temporarily in place of the horizontal components in the diagonals. In this manner, the values obtained for the chord stresses are not equal to the true bar stresses in the chords, but the ratio between these values and the true values is constant and equal to the ratio between the vertical and horizontal components in the diagonals.*

*These values of the chord stresses which must be multiplied by a constant factor to obtain the true bar stress are called index stresses for the chords. Likewise, the vertical components in the web members may be referred to as index stresses for the webs. These index stresses may be written down easily, as shown by the numbers in parentheses in the first stress diagram. Then, the true bar stresses may be obtained by multiplying the index stresses by certain factors, as indicated in the second stress diagram.*

*The use of index stresses is helpful in analyzing parallel-chord trusses that have equal panels and are acted upon by transverse loads. In other cases, the index-stress method becomes involved and is usually inferior to the other methods already discussed.*

**4·12  Exceptional Cases.**  Occasionally one encounters certain trusses that cannot be classified as either simple or compound.  Such a truss is shown in Fig. 4·13a.  In these cases, it is usually difficult to tell by inspection whether the truss is rigid or not and whether it is statically determinate or indeterminate. In this particular case, a count of the structure shows that there are nine bars and six joints, which indicates that the structure is statically determinate.  Whether or not the truss is stable is not apparent, but one way of finding out is to attempt a stress analysis and discover whether the results are consistent or not.

After computing the reactions, it is found that there is no joint at which there

are only 2 unknown bar stresses. Applying the method of joints will therefore not yield immediate solutions for the bar stresses as in the case of simple trusses. It is also found that the method of sections likewise will not yield an immediate solution for any of the bar stresses. Of course, it is possible to set up and solve 9 simultaneous equations involving the 9 unknown bar stresses by using 9 of the 12 equations that result from applying the method of joints to the 6 joints of the structure. If the 3 reactions have already been computed, the 3 remaining equations may be used for checking the results obtained for the 9 bar stresses.

Setting up nine equations in this manner is a poor way to solve this problem, however. Several other approaches are much superior, one of which is to proceed as follows: After computing the reactions, assume that the horizontal component of the bar stress in member $FE$ is a tension of $H$. Then, from joint $F$, the horizontal component in $FA$ must also be $+H$, and the bar stress in $FC$ is $-0.5H$. By isolating joint $C$, it is then found that the horizontal and vertical components in both bars $BC$ and $CD$ are $+0.5H$ and $+0.25H$, respectively. The stresses in these five bars having been found in terms of $H$, it is now possible to pass section 1-1 through the truss, thus isolating the portion to the right of this section. Summing up the moments about point $a$, an equation is obtained involving $H$ as the only unknown.
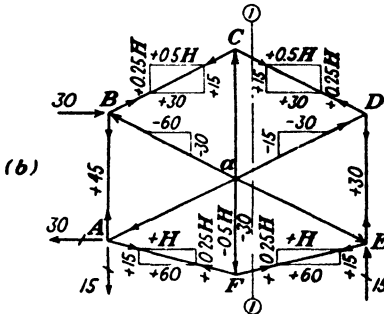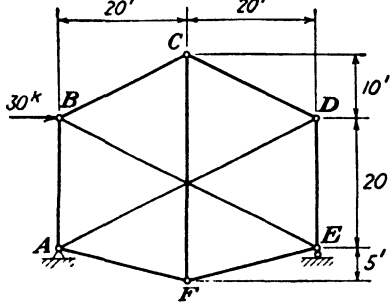


FIG. 4·13

$$\sum_{}^{n} M_a = 0, \ \widehat{+}, \ 15H - (20)(0.5H) - (15)(20) = 0$$

whence $H = +60$. With $H$ known, all the other bar stresses may be found by the method of joints as shown in Fig. 4·13c. Since, in this manner, it is possible to make a consistent stress analysis of this truss under any condition of loading, it may be concluded that it is statically determinate and stable.

Trusses of this type, which cannot be classified as either simple or compound, may be called *complex* trusses. Prof. S. Timoshenko uses this terminology.[1] In his excellent discussion of complex trusses, Timoshenko describes a general method of analysis of complex trusses called Henneberg's method.[2]

While the student should know how to recognize a complex truss and some-

[1] TIMOSHENKO, S., and D. H. YOUNG, "Engineering Mechanics—Statics," McGraw-Hill Book Company, Inc., New York, 1937.
[2] This method was developed by L. Henneberg in his "Statik der Starren Systeme," Darmstadt, 1886.

thing about investigating its stability and stress analysis, he will not encounter this type often enough to warrant devoting more space to the subject here. If additional information is required, the student is referred to Timoshenko's book.[1] Several problems at the end of this chapter emphasize the fact that complex trusses may often be arranged so as to be geometrically unstable. Cases of this type are not always obvious and may not become apparent until the stress analysis is attempted and is found to lead to inconsistent results.

**4·13   Rigid Frames.**   Before closing this chapter on truss structures, it is important to call to the attention of the student the difference between an ideal truss and a so-called "rigid frame." The members of a rigid frame are usually connected together by moment-resisting (rigid) joints instead of being hinged together as in an ideal truss. Thus, a rigid frame may be defined as a structure composed of a number of members all lying in one plane and connected together so as to form a rigid framework by joints, some or all of which are moment-resisting (rigid) instead of hinged.

A moment-resisting joint is capable of transmitting both a force and a couple from one member to the other members connected by the joint. Such a joint can be formed by riveting or welding all the members to gusset plates. The detail of such joints is such that the angles between the ends of the various members at a joint remain essentially unchanged as the frame distorts under load. For this reason, moment-resisting joints are usually referred to as rigid joints.

By a strict interpretation of these definitions, a modern truss with riveted or welded joints should actually be classified as a rigid frame. However, since a satisfactory stress analysis may usually be obtained by assuming that such a truss acts as if it were pin-jointed, such structures are called trusses. The term rigid frame is reserved to designate structures of the type shown in all of Fig. 4·14 except sketch b. When rigid frames are represented by line diagrams as is done in this figure, moment-resisting joints are designated by indicating little fillets between the members meeting at a joint. Any pin joints are represented in the usual manner.

The stability and determinancy of rigid frames may be investigated by methods similar to those used for trusses. For this purpose, a criterion may be established comparing the number of unknown stress components and reaction elements with the number of independent equations of static equilibrium available for their solution. As in the case of trusses, the number of unknowns and equations may be expressed in terms of the number of members, joints, and reaction elements.

The total number of independent unknowns is equal to the sum of

[1] TIMOSHENKO S., and D. H. YOUNG, "Engineering Mechanics—Statics," McGraw-Hill Book Company, Inc., New York, 1937.

the number of unknown reaction elements plus the number of independent unknown stress components in the members. In a frame with rigid joints, the action of a joint on a member may consist of a couple as well as a force. Likewise, this force may have both axial and transverse components. As a result, the cross sections of a member may be subjected to an axial force, shear, and bending moment. However, if the axial force, shear, and bending moment are known at one end of a member, then similar quantities may be found for any other cross section of the member. There are therefore only three independent stress components
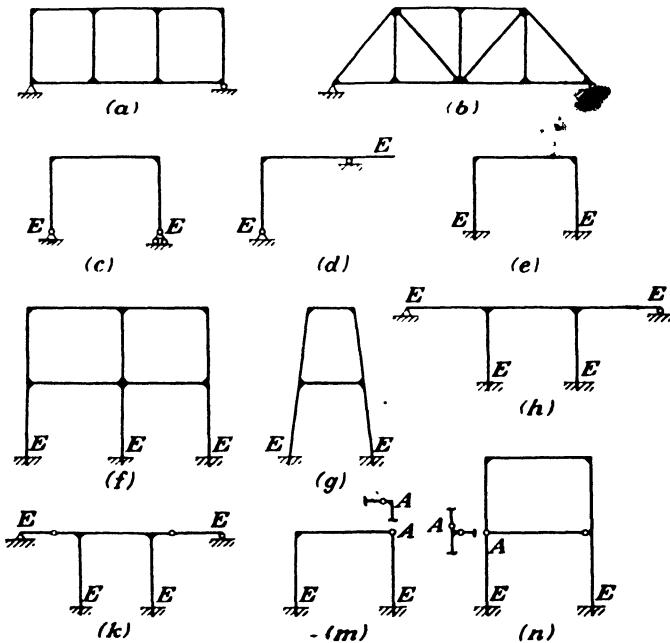


FIG. 4·14

for each member of the frame. If the number of reaction elements is $r$ and the number of members is $b$, the total number of independent unknowns in a rigid frame is equal to $3b + r$.

If a rigid joint is isolated as a free body, it will be acted upon by a system of forces and couples. For equilibrium of such a joint, this system, therefore, must satisfy three equations of static equilibrium, $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$. If the entire frame is in equilibrium, then each of its joints must be in equilibrium. If there are $n$ rigid joints in the frame, each of these joints can be isolated as a free-body and a total of $3n$ equations of static equilibrium obtained. As in the discussion of trusses, it may be shown that the three equations of equilibrium of the

entire structure are not independent of these equations, and therefore it may be concluded that there are only $3n$ equations of static equilibrium for the entire rigid frame.

Occasionally hinges or some other special conditions of construction are introduced into the structure. If in this manner $s$ special equations of condition are introduced, the total number of equations available for the solution of the unknowns will equal $3n + s$. The criterion for stability and determinancy of the rigid frame is obtained by comparing the number of unknowns, $3b + r$, with the number of independent equations, $3n + s$. As before, it may, therefore, be concluded that

If $3n + s > 3b + r$, the frame is *unstable*.

If $3n + s = 3b + r$, the frame is *statically determinate*.

If $3n + s < 3b + r$, the frame is *statically indeterminate*.

If the criterion indicates that the frame is statically determinate or indeterminate, it should be remembered from the similar discussion in Art. 4·8 that the count alone does not prove absolutely that the structure is stable.

This criterion establishes the combined degree of determinancy with respect to both reactions and stress components. The degree of determinancy with respect to reactions only may be established in the same manner as discussed in Art. 4·9 for truss structures and also in Arts. 2·5 and 2·9.

Table 4·1 shows the application of the above criterion to the frames in Fig. 4·14:

Table 4·1

| Frame | $n$ | $s$ | $b$ | $r$ | $3n + s$ | $3b + r$ | Classification |
|-------|-----|-----|-----|-----|----------|----------|----------------|
| $a$ | 8 | 0 | 10 | 3 | 24 | 33 | Indeterminate—9th degree |
| $b$ | 8 | 0 | 13 | 3 | 24 | 42 | Indeterminate—18th degree |
| $c$ | 4 | 0 | 3 | 3 | 12 | 12 | Determinate |
| $d$ | 4 | 0 | 3 | 3 | 12 | 12 | Determinate |
| $e$ | 4 | 0 | 3 | 6 | 12 | 15 | Indeterminate—3d degree |
| $f$ | 9 | 0 | 10 | 9 | 27 | 39 | Indeterminate—12th degree |
| $g$ | 6 | 0 | 6 | 6 | 18 | 24 | Indeterminate—6th degree |
| $h$ | 6 | 0 | 5 | 9 | 18 | 24 | Indeterminate—6th degree |
| $k$ | 6 | 2 | 5 | 9 | 20 | 24 | Indeterminate—4th degree |
| $m$ | 4 | 1 | 3 | 6 | 13 | 15 | Indeterminate—2nd degree |
| $n$ | 6 | 3 | 6 | 6 | 21 | 24 | Indeterminate—3d degree |

In applying this criterion, any extremity of the frame, such as those marked $E$ in Fig. 4·14, should be counted as a rigid joint even though just one member is connected to it. Sometimes the count of the $s$ special equations of condition is rather difficult to make. It is quite obvious in

structure $k$ of Fig. 4·14 that the insertion of the two hinges has introduced
two equations of condition.   The insertion of the hinge joint $A$ in struc-
ture $m$ introduces one equation of condition, but the insertion of a similar
joint $A$ as shown in structure $n$ introduces two equations of condition.
In each case, the validity of these counts is more apparent if one considers
the auxiliary sketches of the joints shown in each case.   The auxiliary
sketches show the manner in which the same structural action may be
obtained at these joints by insertion of hinges in the ends of the members
meeting at the joint.   These alternate arrangements require one hinge
for structure $m$ and two hinges for the joint $A$ in structure $n$.   To general-
ize, it may be stated that the number of special condition equations
introduced by the insertion of a hinge joint in a rigid frame is equal to
the number of bars meeting at that joint minus one.   If the $s$ special
equations are counted in this manner, the criterion yields the correct
results.

    After reading this last paragraph, the reader will no doubt appreciate
the truth of the statement that it is almost impossible to count some
structures properly without first knowing the answer.   Because of the
difficulties encountered in counting some structures, the authors feel
that, while criteria such as the above are sometimes very useful, the
stress analyst should rely on a more fundamental approach to determine
the degree of indeterminancy of an indeterminate structure.   The most
fundamental approach is to remove supports and/or to cut members
until the structure has been reduced to a statically determinate and stable
structure.   The number of restraints that must be removed to accomplish
this result is equal to the degree of indeterminancy of the actual structure.

## 4·14   Problems for Solution.

   **Problem 4·1**   Classify the trusses of Fig. 4·15 as being simple, compound,
or complex.
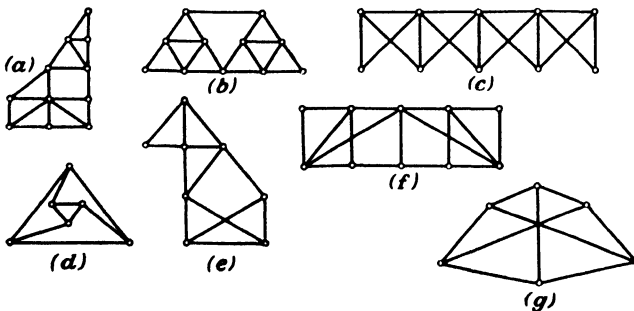


FIG. 4·15

**Problem 4·2**  Classify the truss structures of Fig. 4·16 as being statically determinate or indeterminate, stable or unstable.  If the structure is indeterminate, state the degree of indeterminancy both with respect to reactions and bar stresses and with respect to reactions only.  If the structure is unstable, state the reason for the instability.
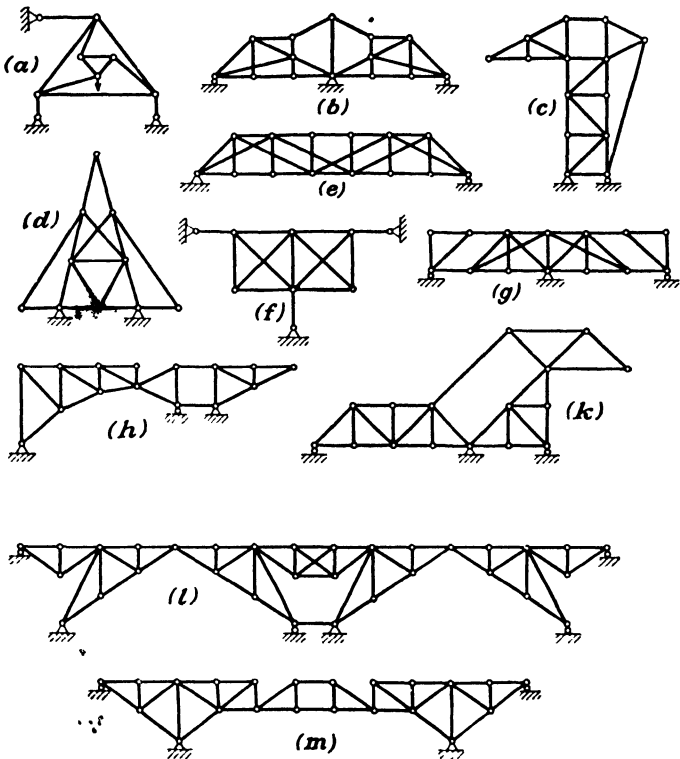


FIG. 4·16

**Problem 4·3**  Compute the bar stresses in the lettered bars of the trusses of Fig. 4·17 due to the loads shown.
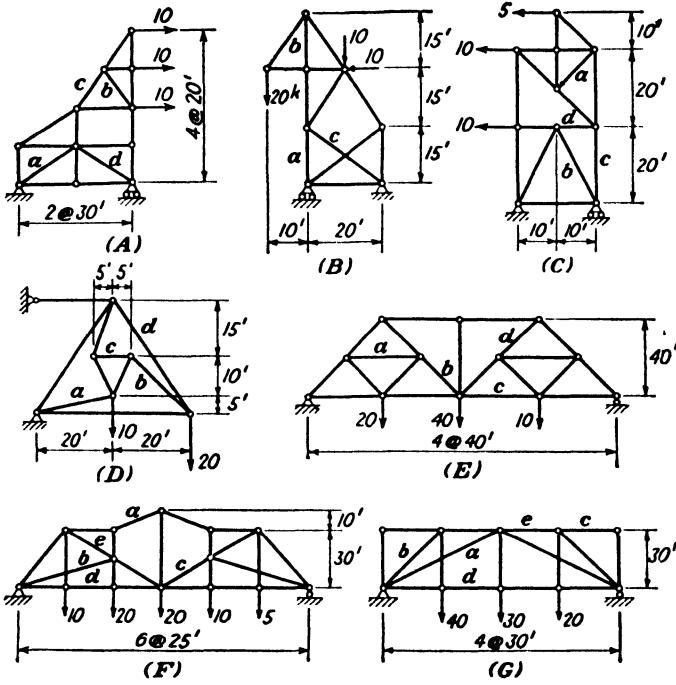


Fig. 4·17

**Problem 4·4**   Compute all the bar stresses in the trusses of Fig. 4·18 due to the loads shown.
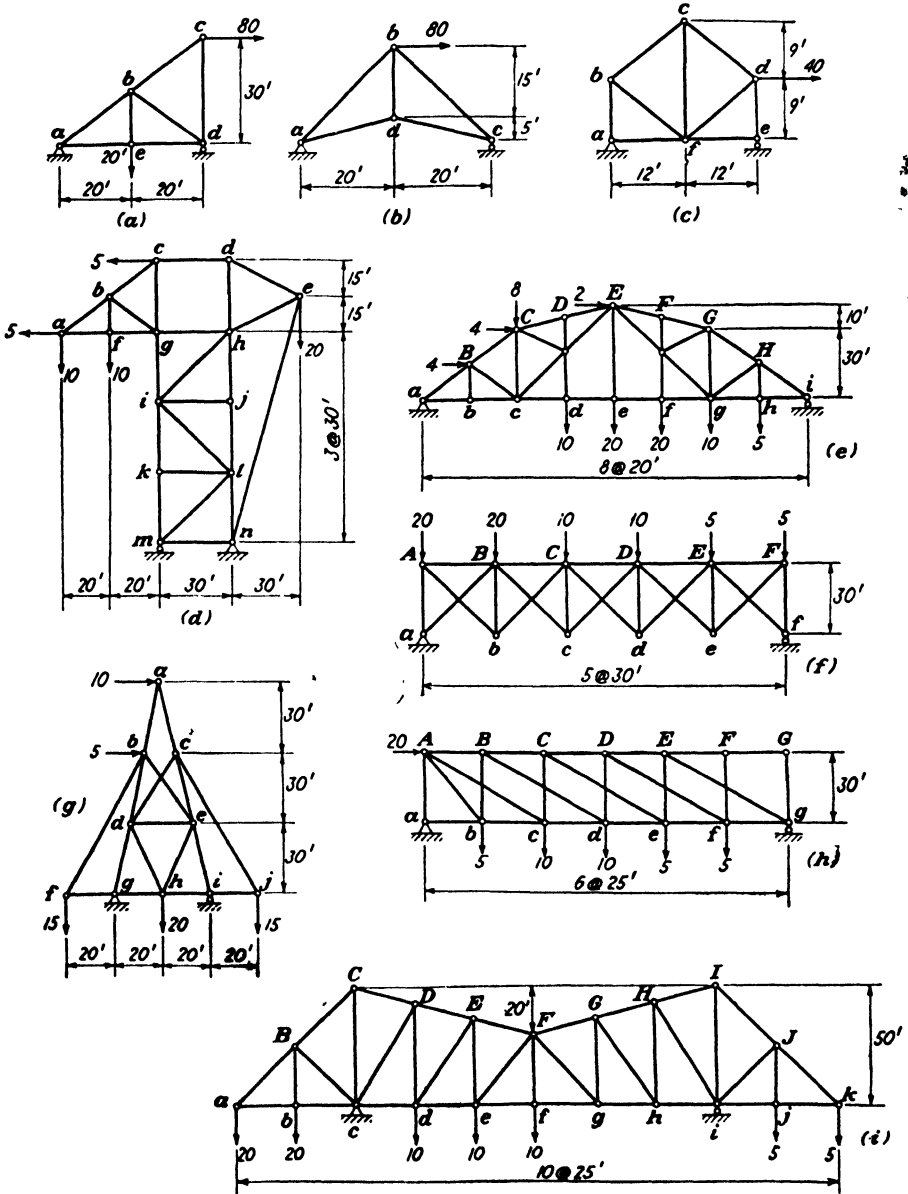


FIG. 4·18
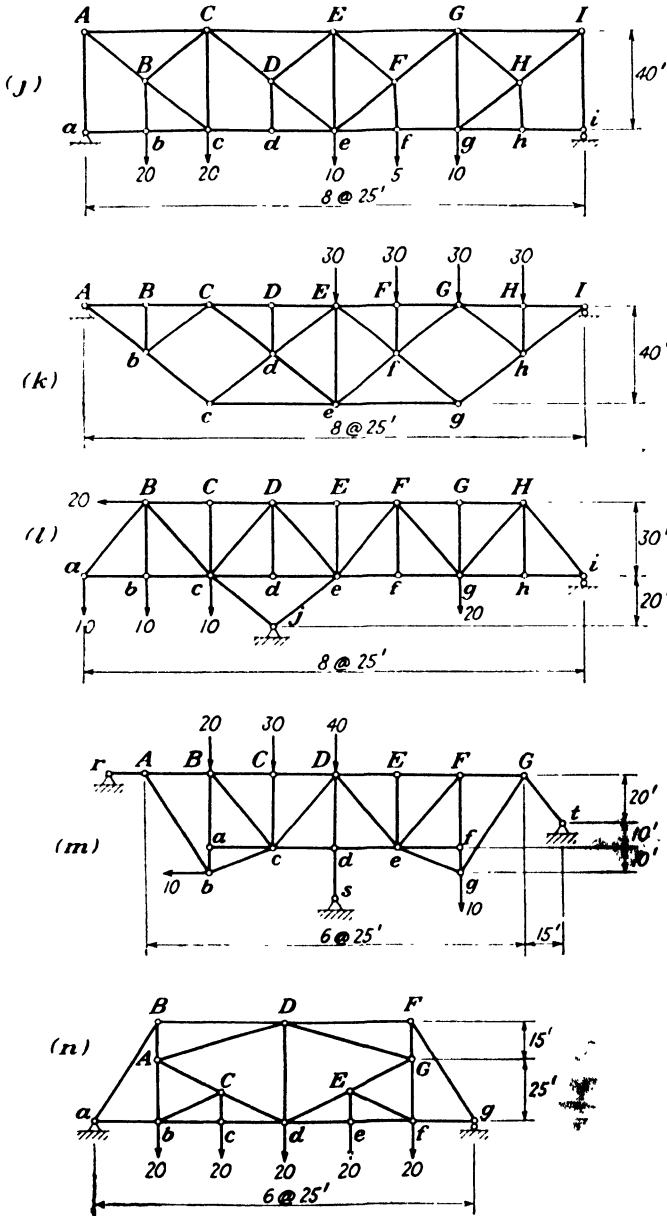
FIG. 4·18 (Cont.)
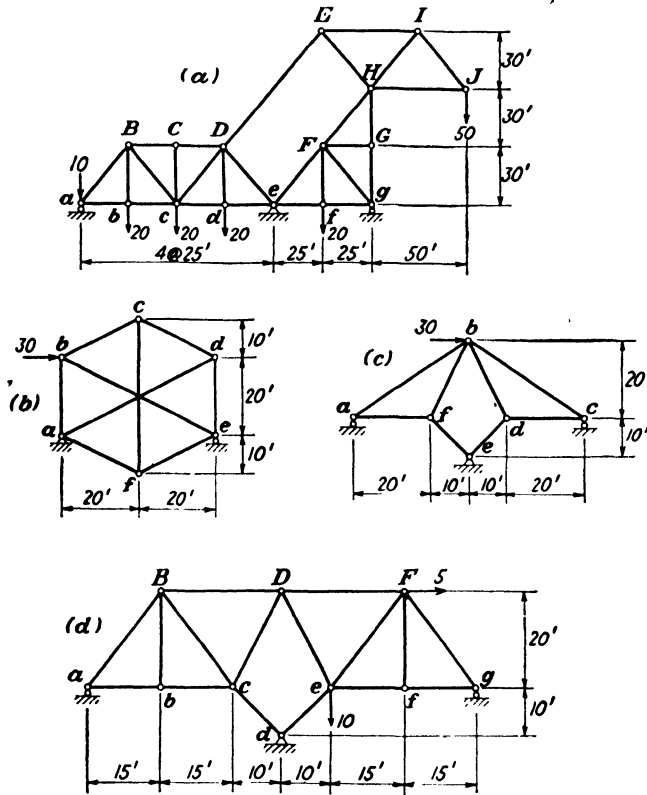
FIG. 4·19

**Problem 4·5** Compute the bar stresses in the structures of Fig. 4·19. (*Hint:* Remember that structures of this type may be geometrically unstable.)

**Problem 4·6** Compute the bar stresses in the structures of Fig. 4·20. Also draw the shear and bending-moment curves for those members in which such stress conditions exist.
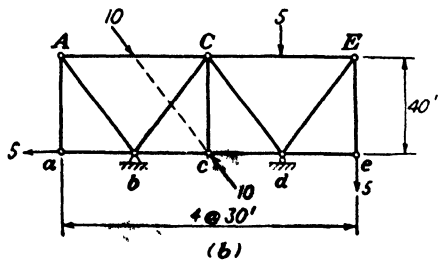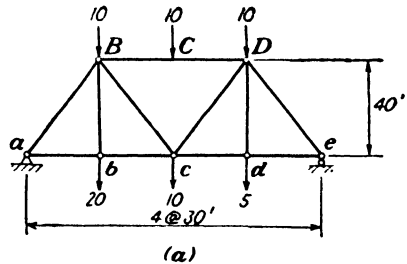
# CHAPTER 5

## GRAPHIC STATICS

**5·1 Introduction.** Graphic statics is that branch of mechanics which deals with graphical rather than algebraic solutions of problems of statics. In this country, there seems to be an aversion toward graphical solutions among students and engineers. There are some problems, however, where the graphical solution is distinctly superior to the algebraic approach. There are other problems where just the reverse is true. In the middle ground between these two extreme situations, which method the engineer decides to use usually depends on his personal preference and background.

Among the problems that may be solved advantageously by graphical methods are the following:

1. Determination of the bar stresses in a truss that has a complex configuration and that is to be analyzed for a limited number of loading conditions, for example, the analysis of a roof truss or the determination of erection stresses in certain types of structures, such as cantilever trusses
2. Cases where the true resultant deflection is required for every joint of a truss

The student will find not only that a knowledge of graphical methods is useful in the solution of problems such as these but also that there are certain educational advantages which result from a study of the basic principles of graphical methods. He will find that these ideas aid him in visualizing and representing physical phenomena and often assist him in thinking about the algebraic solution of certain problems.

In this book, the discussion of graphic statics will be limited to the solution of two-dimensional, or planar, structures. Graphical methods can be extended, of course, to the more general three-dimensional problems, but in most cases the complexities introduced by the third dimension are greater for the graphical methods than for the algebraic methods.

**5·2 Definitions.** Before the fundamental principles of graphic statics are discussed, it is first necessary to emphasize certain ideas and definitions concerning forces and force systems. A *force* may be defined as any action which tends to change the state of motion (or rest) of the body to which it is applied. The forces acting on a body may be classified as either outer forces or inner forces (stresses). The outer forces

may be further subdivided into the loads (active forces) applied to the structure from without and the reactions (resisting forces) balancing or restraining the effects of the loads. The inner forces, or stresses, are usually developed between the particles of the body by the outer forces acting on it.

A force may be completely identified by the following specifications: (1) its point of application; (2) its direction; (3) its magnitude. According to this terminology, the "direction of a force" is intended to define the slope of its line of action while the "magnitude" indicates not only its numerical size but also the sense in which the force acts along this line of action, i.e., whether toward or away from a body. A force is there-fore a vector quantity since it has both magnitude and direction. Hence, a force may be represented graphically by a line drawn toward or from the point of application and having a length that indicates the numerical size of the force to a certain scale. The slope of this line indicates the direction of the force and an arrowhead the sense in which the force

FIG. 5·1
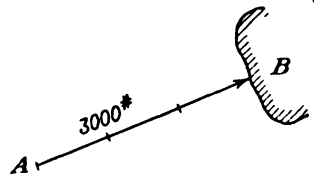
acts along this line. A 3,000-lb. force is represented in this manner by the vector $\overline{AB}$ in Fig. 5·1. When this vector notation is used, the order of the letters indicates the sense of the force. Thus, $\overline{AB}$ means that the force acts from $A$ toward $B$.

The use of the term "point of application" of a force implies that it is possible to concentrate a force at a point. Physically, of course, this is impossible since a finite load applied at a point, i.e., applied to a zero area, would develop infinite contact stresses intensities in the material. No material can withstand such stresses, since it will deform at the point of contact until a small finite contact area is developed over which the load is distributed at finite stress intensities. However, as far as the equilibrium condition of the body as a whole is concerned, it is legitimate to replace the actual load distributed over a small area by the equivalent total load concentrated at a point.

As previously explained in Art. 2·3, it is usually permissible to assume structures as being rigid (nondeformable) bodies in investigating their equilibrium conditions. Thus, in most problems of graphic statics, it will be assumed that the structure is a rigid body and hence that the geometry after the application of the loads is essentially the same as before.

**5·3 Composition and Resolution of Forces.** It is sometimes desirable to replace two forces by a single force that exerts the same effect. This single force that would have the same effect in producing motion is

called the *resultant* of these forces. It may be demonstrated that the magnitude and direction of the resultant of two concurrent forces may be obtained by drawing the diagonal of a parallelogram which has been constructed with the vectors representing these two forces drawn as



sides. Thus, to determine the magnitude and direction of the resultant of the forces $F_1$ and $F_2$ shown in Fig. 5·2a, a parallelogram is constructed as shown in Fig. 5·2b. This parallelogram is constructed on the basis of two sides $OA$ and $OB$ obtained by drawing through point $O$ two vectors $\overline{OA}$ and $\overline{OB}$ representing the forces $F_1$ and $F_2$, respectively. The magnitude and direction of the resultant $R_{12}$ is given by the vector $\overline{OC}$, which is the diagonal of this parallelogram.

From Fig. 5·2d, it is apparent that the same results would be obtained from a parallelogram constructed by drawing vectors $\overline{AO}$ and $\overline{BO}$ both running into point $O$ rather than away from this point. Likewise, from Fig. 5·2c, it is apparent that the same results could also be obtained by drawing either of the vector triangles $OAC$ or $OBC$ instead of the parallelogram. In constructing these triangles, either force may be drawn first and then the other force laid out from the end of the first vector. The resultant is then obtained in magnitude and direction from the closing vector of the triangle drawn from the beginning of the first vector to the end of the second.
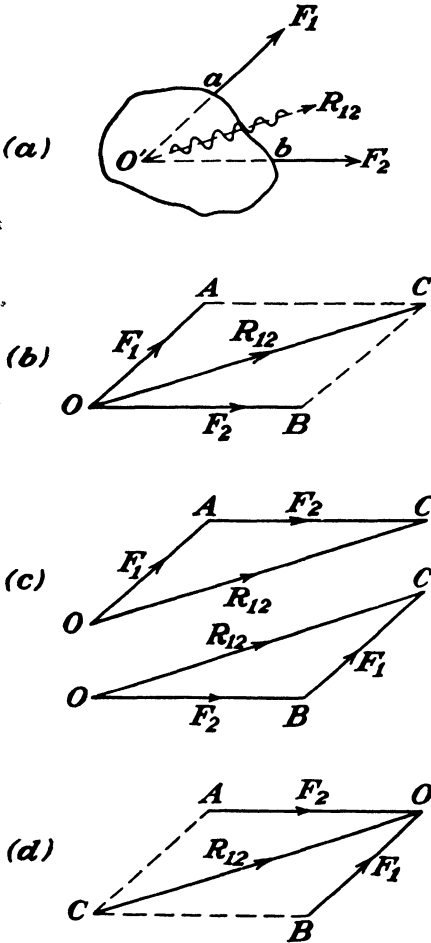
The magnitude and direction of the resultant $R_{12}$ having been determined in one of these ways, its point of application may be considered to be located at any point along its line of action. The line of action of the resultant must pass through the point of intersection of the two forces

$F_1$ and $F_2$, or through point $O'$ in Fig. 5·2a.   If this were not so, the resultant would not exert the same effect as the two forces that it replaces since the moment of the resultant about an axis through any point in the plane would not be the same as the sum of the moments of the two forces about that axis.   For example, the moments of both $F_1$ and $F_2$ about an axis through $O'$ are zero, but the moment of the resultant $R_{12}$ will also be zero only if the line of action of $R_{12}$ also passes through point $O'$.

The validity of this parallelogram construction for determining the magnitude and direction of the resultant of two concurrent forces may be demonstrated in the following manner: Consider the two forces $F_1$ and $F_2$ acting at point $O$ on the body shown in Fig. 5·3. The resultant $R_{12}$ of these forces also acts at point $O$ along some line of action specified by the unknown angle $\alpha$. The unknown magnitude and direction of this resultant may be determined in the following manner: If this body is moved so that point $O$ is given some arbitrary movement $\delta$, the forces $F_1$ and $F_2$ will perform a certain amount of work. If
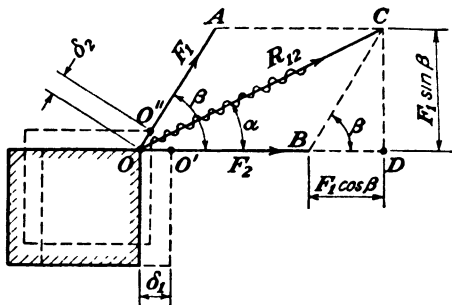


Fig. 5·3

the resultant $R_{12}$ is to exert the same effect as the two forces $F_1$ and $F_2$, then it must perform the same amount of work during the movement $\delta$.   Suppose the body is given an arbitrary translation so that point $O$ moves to point $O'$.   Upon equating the work done by $F_1$ and $F_2$ to the work done by $R_{12}$, Eq. ($a$) is obtained,

$$(R_{12} \cos \alpha)(\delta_1) = (F_2)(\delta_1) + (F_1 \cos \beta)(\delta_1)$$

or

$$R_{12} = \frac{F_2 + F_1 \cos \beta}{\cos \alpha} \qquad (a)$$

In the same manner, translating the body so that point $O$ moves $O''$,

$$[R_{12} \cos (\beta - \alpha)](\delta_2) = (F_1)(\delta_2) + (F_2 \cos \beta)(\delta_2)$$

$$R_{12} = \frac{F_1 + F_2 \cos \beta}{\cos (\beta - \alpha)} \qquad (b)$$

These two equations may now be solved for $R_{12}$ and $\alpha$.   Equating the right-hand sides of these equations leads to the following expression for $\alpha$:

$$\tan \alpha = \frac{F_1 \sin \beta}{F_2 + F_1 \cos \beta} \qquad (c)$$

Therefore,

$$\cos \alpha = \frac{F_2 + F_1 \cos \beta}{\sqrt{(F_1 \sin \beta)^2 + (F_2 + F_1 \cos \beta)^2}} \qquad (d)$$

Substituting in Eq. (a) from Eq. (d) leads to

$$R_{12} = \sqrt{(F_1 \sin \beta)^2 + (F_2 + F_1 \cos \beta)^2} \qquad (e)$$

By use of Eq. (c), the angle $\alpha$ may be laid off graphically as indicated in Fig. 5·3, while the magnitude of the vector $R_{12}$ is shown by Eq. (e) to be given by the length of the hypotenuse of the right triangle $ODC$. It is obvious, therefore, that the vector $\overline{OC}$ representing the resultant is likewise the diagonal of the parallelogram of forces $OACB$, which justifies the parallelogram construction described above.

This process of replacing forces $F_1$ and $F_2$ by a single resultant force $R_{12}$ is known as the *composition of forces* $F_1$ and $F_2$. The reverse of this process, that of replacing the effect of a single force $R$ by two equivalent forces (called two *components*) $F_1$ and $F_2$, is called the *resolution of force R*. In the latter process, the direction of the two components might be given and their magnitude obtained from the force triangle or parallelogram; or the direction and magnitude of one component might be given and the direction and magnitude of the other component determined in a similar manner. The magnitude and direction of the two components $F_1$ and $F_2$ having been determined, they may both be applied at the point of application of the force $R$. Of course, it is permissible to apply both the components $F_1$ and $F_2$ at any point along the line of action of the force $R$.

**5·4  Resultant of Several Forces in a Plane—Force Polygon.** Consider a body subjected to a coplanar system of forces $F_1$, $F_2$, $F_3$, and $F_4$ as shown in Fig. 5·4a. Suppose that it is required to find the resultant of these forces graphically. As described in the previous article, the resultant $R_{12}$ of the forces $F_1$ and $F_2$ may be obtained in magnitude and direction from the force triangle 012. The line of action of this resultant is drawn parallel to the vector $\overline{O2}$ and through the intersection of the lines of action of the forces $F_1$ and $F_2$. In the same manner, the resultant $R_{123}$ of the forces $R_{12}$ and $F_3$ may be obtained; and then the resultant $R_{1234}$ of the forces $R_{123}$ and $F_4$. The last resultant $R_{1234}$ is of course the resultant of all four forces $F_1$, $F_2$, $F_3$, and $F_4$.

The figure obtained by combining the force triangles 012, 023, and 034 and then omitting the dashed lines 02 and 03 is called the *force polygon* for the forces $F_1$, $F_2$, $F_3$, and $F_4$. From this force polygon, the resultant $R_{1234}$ of the entire system may be found directly without completing the intermediate force triangles. The magnitude and direction of this resultant are given by the vector drawn from the initial to the final point of the force polygon—in this case by the vector $\overline{O4}$. To establish the line of action of this resultant in the space diagram, the lines of action of the intermediate resultants $R_{12}$ and $R_{123}$ must be established as described above.

This method of obtaining the magnitude, direction, and line of action of the resultant is applicable as long as the lines of action of the forces are not parallel and intersect within the limits of the drawing. When this method is not applicable, it is necessary to resort to the use of a so-called "equilibrium (funicular) polygon," which will be described in Art. 5·7.

It should be noted that the order of drawing the forces in the force polygon is immaterial and that they are usually considered in a clockwise order simply as a matter of convenience.



(a)
Space Diagram

(b)
Force Polygon
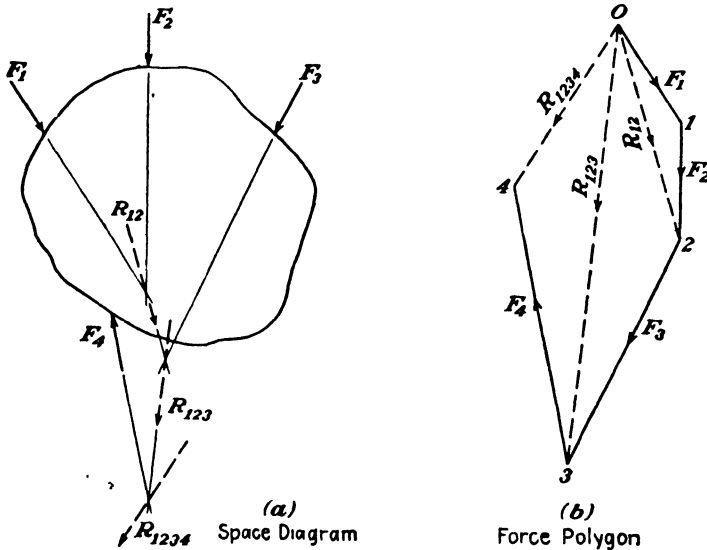
Fig. 5·4

**5·5  Equilibrium Conditions for Concurrent and Nonconcurrent Coplanar Force Systems.**  Suppose that the force $F_5$ is added to the nonconcurrent force system already shown in Fig. 5·4a. The new system $F_1$, $F_2$, $F_3$, $F_4$, and $F_5$ is shown in Fig. 5·5. Let the force $F_5$ have the same line of action as the resultant $R_{1234}$, and further let it be numerically equal to this resultant but acting in the opposite sense. Then in this case it will be found that the force polygon for all five forces will close back on the original starting point $O$. Closure of the force polygon indicates that the equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$ are satisfied by the five forces themselves and that therefore their resultant effect cannot be a resultant force. The fact that, in the space diagram, $F_5$ and $R_{1234}$ have the same lines of action and are also numerically equal, but opposite in sense, indicates that $F_5$ in effect holds the other four forces in

equilibrium.   In such a case, the force $F_5$ is called the *equilibrant* of the other four forces.

Suppose, however, that force $F_5$, instead of having the same line of action as $R_{1234}$, is displaced laterally a distance $a$ as indicated by the dotted force $F'_5$ in Fig. 5·5.   Now, although $F'_5$ closes the force polygon, thus satisfying equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$, in the space diagram the equal and opposite forces $F'_5$ and $R_{1234}$ are parallel, but their lines of action are displaced by the distance $a$.   Obviously, therefore, the resultant of the new system $F_1$, $F_2$, $F_3$, $F_4$, and $F'_5$ is a couple equal in magnitude to $F'_5a$, and the system is not in equilibrium since $\Sigma M \neq 0$.
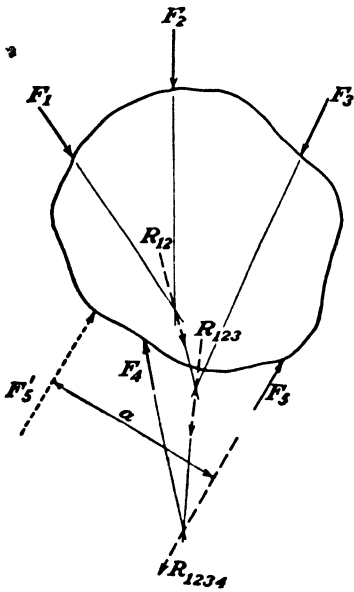
In the case of a nonconcurrent force system, it is therefore evident that closure of the force polygon is a necessary but not a sufficient condition to show that the system is in equilibrium.   In addition to this condition, it is necessary to show in the space diagram that the system is not equivalent to a couple, *i.e.*, that one force has the same line of action but is opposite in sense to the resultant of the remaining forces of the system.

Of course, if the force system is a concurrent system with the lines of action of all the forces intersecting in a common point, then it is impossible for the resultant effect of the system to be a couple.   In such a case, closure of the force polygon indicating that the resultant effect of the system is not a resultant force is then sufficient to prove that this concurrent force system is in equilibrium.



FIG. 5·5

**5·6   Determination of Reactions by the Three-force Method.**
If only three nonparallel forces act on a body, it is easy to show that they must be concurrent in order to be in equilibrium.   Consider any two of these three forces.   The line of action of the resultant of these two forces must pass through their point of intersection.   Then, in order for the remaining force to be the equilibrant of the other two, the line of action of this third force must coincide with that of the resultant of the other two.   It may therefore be concluded that the lines of action of all three forces must intersect in a common point if the system is to be in equilibrium.

This conclusion furnishes the basis for the so-called "three-force method" of determining the reactions of a statically determinate structure.

Consider, for example, the beam shown in Fig. 5·6. Suppose that it is desired to find the reactions which are required to keep this structure in static equilibrium. First determine the magnitude and line of action of the resultant of the applied loads, using the force polygon and the space diagram. Then the structure may be considered to be acted upon by three forces—the resultant of the applied loads ($R_{12}$ in this case) and the two reactions ($R_a$ and $R_b$). For equilibrium of the structure, it is necessary for these three forces to be concurrent. In this case, the magnitude of both reactions and the direction of $R_a$ are unknown, but the point of application of $R_a$ and the line of action of $R_b$ are both known. The line of action of $R_b$ is known to be a vertical line passing through point $b$. Point $o$, the point of intersection of $R_{12}$ and $R_b$, must therefore be the point of concurrency of the three forces. The line of action of $R_a$ must therefore be along the line $oa$. Now, the directions of both reactions being known, their magnitudes may easily be determined since it is known that the vectors representing these reactions must close the force polygon. In this case, upon drawing a line through point 2 parallel to $R_a$ and through point $O$, a line parallel to $R_b$ establishes the intersection point 3, which determines the length of vectors $\overline{23}$ and $\overline{30}$ representing the reactions $R_a$ and $R_b$, respectively. Of course, it is immaterial whether the force polygon is closed in this manner or by drawing the line parallel to $R_a$ through point $O$ and the line parallel to $R_b$ through point 2.



FIG. 5·6

It should be noted that this three-force method is not a completely general method of determining the reactions of a statically determinate structure. It may be used only when the line of action of the resultant of the applied loads intersects the known line of action of one of the reactions.
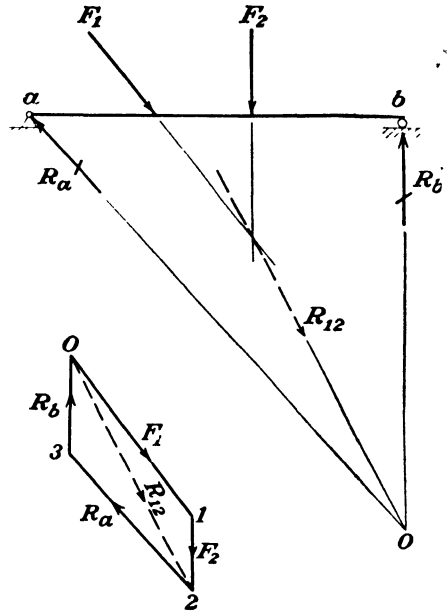
**5·7  Funicular (Equilibrium) Polygon.**  A method is discussed in Art. 5·4 by which the resultant of several coplanar forces may be determined.  This method fails when any point of intersection on the space diagram falls outside of the paper and is not applicable to a system of parallel forces.  However, a general method using the *funicular (equilibrium) polygon* is applicable to any coplanar force system.

Suppose that the resultant of the forces $F_1$, $F_2$, and $F_3$ shown in Fig. 5·7a is required.  The magnitude and direction of the resultant $R_{123}$ is obtained from vector $\overline{O3}$ in the force polygon $O123$.  The line of action of this resultant on the space diagram may be determined as follows: Suppose, by using the force triangle $OP1$, that force $F_1$ is resolved into any two components $P1$ and $OP$ at some point on its line of action.



(a)

(b)
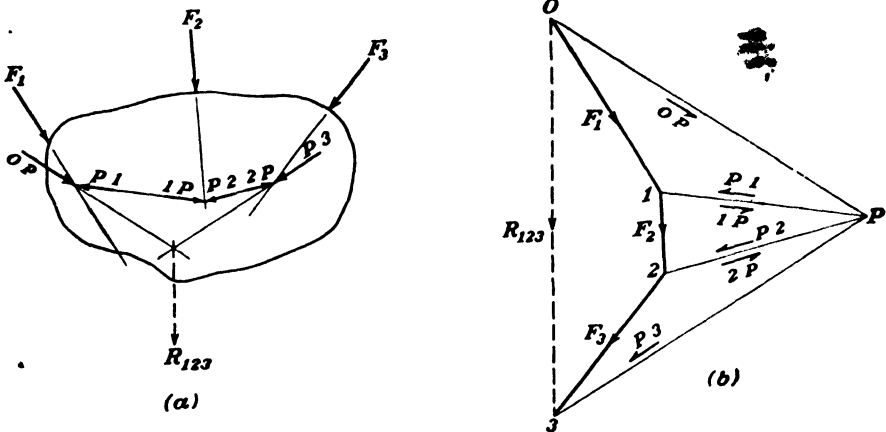
Fɪɢ. 5·7

Suppose that the line of action of the component $P1$ is extended until it intersects the line of action of $F_2$.  At this point, resolve $F_2$ into components one of which, $1P$, is collinear with, but equal and opposite to, $P1$, and the other is equal to $P2$ as obtained from the force triangle $1P2$.  In the same manner, resolve $F_3$ into components $2P$ and $P3$ as shown.  Now the original force system of $F_1$, $F_2$, and $F_3$ has been replaced by six components, $OP$ and $P1$, $1P$ and $P2$, and $2P$ and $P3$.  Of these six components, the pairs $P1$ and $1P$ and $P2$ and $2P$ are collinear but equal and opposite, and therefore each of these pairs is in equilibrium. The resultant of the six components and therefore of the original force system is the resultant of the two remaining components $OP$ and $P3$ and acts through their point of intersection.

The construction thus drawn between the lines of action of the forces in the space diagram is called the equilibrium, or funicular, polygon.  The

sides of this polygon drawn between the forces are called *strings*. The
point $P$ on the force polygon through which all the components are
directed is called the *pole*. The lines drawn from the vertices of the force
polygon to the pole $P$ are called *rays*.

In an actual problem, the line of action of the resultant is located by
constructing the funicular polygon in a slightly different manner from that
described above. First a convenient pole $P$ is selected, and rays are
drawn from this pole to the vertices of the force polygon. Then the
strings of the funicular polygon are drawn on the space diagram parallel
to the corresponding rays of the force polygon. Note that a string is
drawn between the lines of action of two forces which are adjacent to
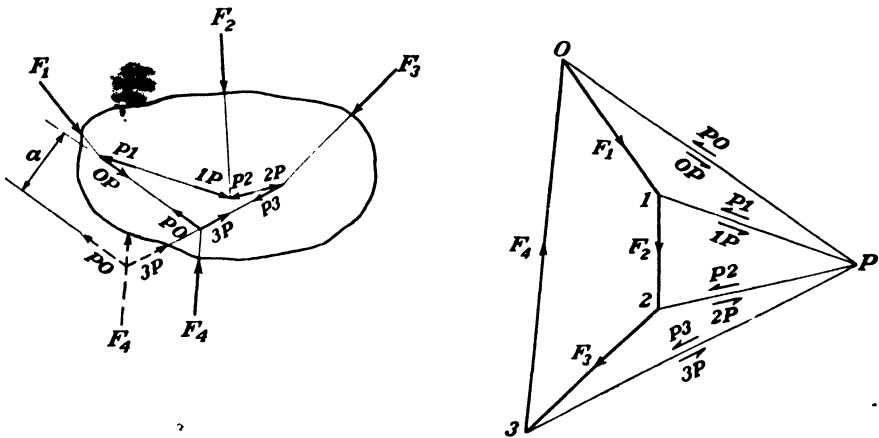


Fig. 5·8

each other in the force polygon and further that this string is drawn
parallel to the ray directed through the intersection of these two adjacent
force vectors. While it is not necessary to do so, usually the starting point
for the funicular polygon is selected as some point on the line of action of
the first force to have been laid out in the force polygon. Then, the
intersection of the first and last string of the funicular polygon (such as
the string between $F_1$ and $R_{123}$ and that between $F_3$ and $R_{123}$, respectively)
is a point on the line of action of the resultant of the system, which has
been determined previously in magnitude and direction from the force
polygon.

Suppose that a fourth force $F_4$ is added to the system in Fig. 5·7.
Suppose further that $F_4$ is collinear with, but equal and opposite to,
$R_{123}$. The new system will then be as shown in Fig. 5·8.

In such a case, the force polygon will close, indicating that $\Sigma F_x = 0$
and $\Sigma F_y = 0$. Likewise, when the funicular polygon is drawn, it is

found that the first and last strings drawn from forces $F_1$ and $F_4$, respectively, are actually collinear.  Of course, drawing the funicular polygon actually is just a means of replacing the four forces by the eight components shown.  It is apparent that these eight components may be considered as four pairs, each of which is in equilibrium, and therefore the original force system must be in equilibrium.

Suppose that, instead of $F_4$ being collinear with $R_{123}$, its line of action is displaced to the dashed-line position, still being parallel and equal and opposite to $R_{123}$.  Then, the force polygon will still close, but the first and last strings will no longer be collinear.  They now will be parallel but displaced a distance $a$ apart.  In this case, the eight components represented by the funicular polygon will consist of three equal and opposite pairs in equilibrium, but the fourth pair $OP$ and $PO$ will be parallel, equal, and opposite and will be equivalent to a couple equal to $(OP)(a)$.  The original force system will now be equivalent to a couple of $(OP)(a)$ and will no longer be in equilibrium.

It may therefore be concluded that, for a system of nonconcurrent forces to be in equilibrium, it is necessary not only for the force polygon to be a closed figure but also for the funicular polygon to be a closed figure, *i.e.*, the first and last strings of the funicular polygon must be coincident.  If the force polygon closes but the funicular polygon does not, the force system will be equivalent to a couple.

There is another principle associated with a funicular polygon, which may often be used to advantage.  The strings of a funicular polygon may be considered to represent links connected together at the vertices of the polygon by frictionless pins.  If one considers the fundamental principles involved in the construction of a funicular polygon, it is evident that a linkage of this shape will support the system of loads applied to its joints.  Of course, in cases where the funicular polygon is not a closed figure, it is also necessary to provide the proper reactions for the linkage acting along the directions of the first and last strings of the polygon. The magnitude of these two reactions are given by measuring the length of the first and last rays of the force polygon.

**5·8  Use of Funicular Polygon to Determine Reactions.**  The use of the funicular polygon in determining the reactions of a statically determinate structure may be explained by considering the beam and loading shown in Fig. 5·9.  In this case, the point of application and direction of the right reaction and the point of application of the left reaction are known, leaving as unknowns the magnitude of both reactions and the direction of the left one.  These three unknowns may be found by knowing that both the force and funicular polygons must close if the combined system of loads and reactions is to be in equilibrium.

A portion of the force polygon may be drawn immediately, *viz.*, vectors $\overline{01}$ and $\overline{12}$ representing the applied loads. Now select a pole $P$, and draw the rays to points $0$, $1$, and $2$. Draw a string parallel to ray $PO$ between the unknown line of action of $R_L$ and the force $F_1$. While the direction of $R_L$ is unknown, it is known that point $a$ is one point on its line of action and therefore this string may be drawn from point $a$ to an intersection with $F_1$. Likewise, draw successively the strings between $F_1$ and $F_2$ and then between $F_2$ and the known line of action of $R_R$. This last string intersects $R_R$ at point $b$. The closing string of the funicular polygon will be the string $ab$. Now the corresponding ray may be drawn in the force polygon parallel to this closing string. This ray must go through the vertex that is the intersection of the vectors, representing $R_R$ and $R_L$. Since string 3 was drawn between $F_2$ and $R_R$ parallel
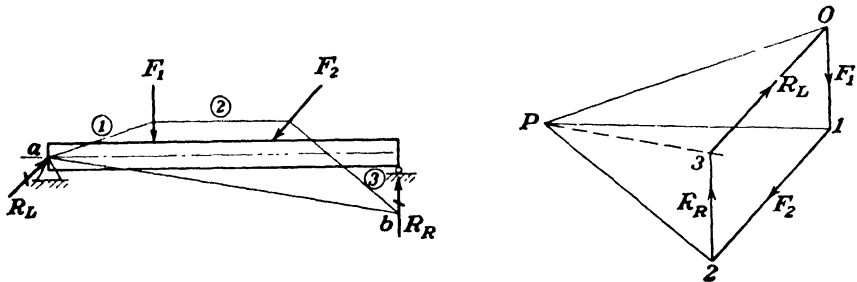


Fig. 5·9

to ray $P2$, one end of the vector representing $R_R$ must be at point 2 on the force polygon. It is known further that this reaction is vertical. Through point 2, therefore, draw a vertical vector representing $R_R$. The other end of this vector must lie on the ray parallel to the closing string, and thus vertex 3 of the polygon is located. Vector $\overline{23}$ gives the magnitude of $R_R$, and vector $\overline{30}$, the closing vector of the force polygon, gives the direction and magnitude of $R_L$.

This procedure is straightforward, but sometimes students become confused as to whether to draw the vector for the reaction with the known direction through the first or last vertex of the force, *i.e.*, in this case whether through point $0$ or 2. Such confusion is not necessary if one remembers that a given string drawn between the lines of action of two forces is parallel to the ray which passes through the intersection of the two vectors representing these forces in the force polygon. In this case, string 1 is drawn parallel to ray $PO$ and string 3 parallel to ray $P2$. This implies that one end of the vector representing $R_L$ is at point $0$ on the force polygon and one end of the vector for $R_R$ at point 2.

This method of determining reactions is general and is not limited to special cases as is the three-force method.

**5·9 Funicular Polygons Drawn through One, Two, or Three Specified Points.** When one reconsiders the procedure for drawing a funicular polygon for a given set of forces, it is apparent that it is possible to draw an infinite number of funicular polygons for that particular set of forces, for any one of an infinite number of points can be selected as the pole of the force polygon. Sometimes, however, it is necessary to draw the funicular polygon so that it will pass through certain specific points in the space diagram. Under such conditions, it becomes necessary to limit the selection of the pole to certain specific points.

First consider the case where it is necessary to pass the funicular polygon through *one* specific point in the space diagram, such as point $a$ in Fig. 5·10. This may be done by drawing between forces $F_2$ and $F_3$
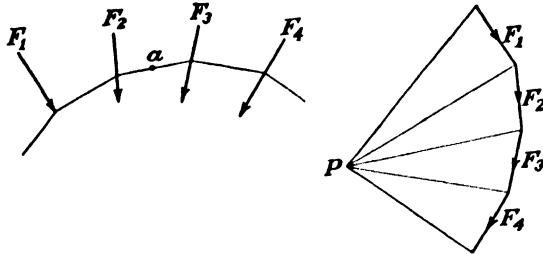


FIG. 5·10

any desired string that also passes through point $a$. The ray corresponding to this string may now be drawn parallel to it and passing through the intersection of the vectors for $F_2$ and $F_3$ in the force polygon. The pole $P$ may now be selected as any point along this ray and the remainder of the corresponding funicular polygon completed as shown. Obviously, any one of an infinite number of poles $P$ can be selected in this manner, and hence an infinite number of polygons can be drawn passing through the single point $a$.

When it becomes necessary to pass the funicular polygon through two particular points such as $a$ and $b$ in Fig. 5·11, the procedure must be altered somewhat. Suppose temporarily that these forces are imagined to be applied to a structure supported by a hinge support at point $a$ and a roller support supplying a vertical reaction at point $b$. Then, proceeding in the usual manner, these two imaginary reactions can be obtained, using the funicular polygon labeled with the strings 1, 2, 3, and 4 and the closing string $ab'$. This funicular polygon is drawn using the pole $P$. Of course, the value so determined for the reactions $R_R$ and $R_L$ will be the same regardless of what point is chosen for the pole $P$, and

therefore the position of vertex $v$ in the force polygon is unique. If the pole $P$ has been selected as desired, the resulting funicular polygon will pass through points $a$ and $b$ and the closing string of this polygon will be the line $ab$. The corresponding ray for this string will be parallel to $ab$ and will also pass through vertex $v$ of the force polygon. This ray has been so drawn and labeled $5'$. Any point along this ray may now be selected as the pole $P'$, which will result in a funicular polygon with strings $1'$, $2'$, $3'$, $4'$, and $5'$, which pass through the two specified points $a$ and $b$ in the space diagram. Again, it is apparent that any one of an infinite number of poles $P'$ can be selected in this manner and hence an infinite
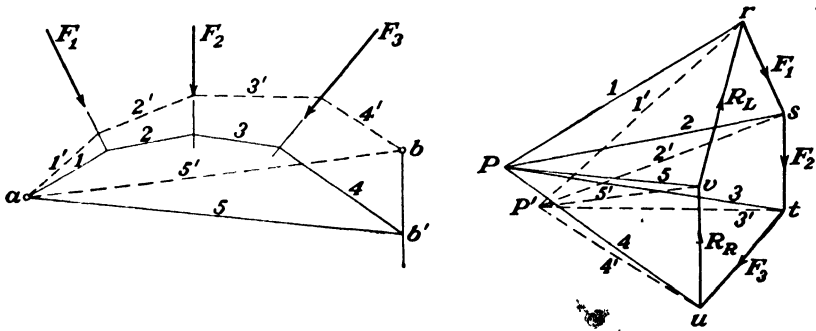


FIG. 5·11

number of funicular polygons can be drawn passing through two specified points $a$ and $b$.

Now consider a case where it is necessary to pass the funicular polygon through three specified points $a$, $b$, and $c$, as shown in Fig. 5·12. As before, assume temporarily that these forces are acting on a structure supported by a hinge support at $a$ and a roller support supplying a vertical reaction at $c$. Continue as before, and determine these reactions $R_L$ and $R_R$ by drawing the funicular polygon with strings 1, 2, 3, 4, and 5 and thus locating vertex $v$ in the force polygon. As demonstrated in the preceding paragraph, the pole of a funicular polygon passing through points $a$ and $c$ must lie somewhere along the line $vx$ parallel to line $ac$. Now consider only the forces lying between points $a$ and $b$, and assume these forces to be supported by a hinge support at $a$ and a roller support supplying a vertical reaction at point $b$. These reactions $R'_L$ and $R'_R$ may be determined by using the funicular polygon with strings 1, 2, 3, and 6 and thus locating vertex $w$ in the force polygon. Likewise, the pole of a funicular polygon passing through points $a$ and $b$ must lie somewhere along the line $wy$ parallel to line $ab$. Hence, for the polygon to pass through all three points $a$, $b$, and $c$, the pole $P'$ must be located at the intersection of the lines $vx$ and $wy$. In this case, only one pole $P'$ may

be located in this manner, and hence only one funicular polygon may be passed through three specified points. Further, a funicular polygon cannot be made to pass through more than three specified points.
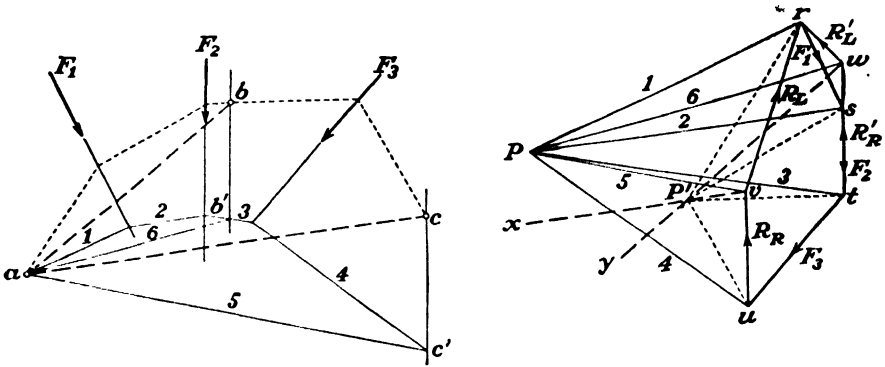


FIG. 5·12

## 5·10 Graphical Determination of Shear and Bending Moment.

After finding the reactions of a beam graphically, shear and bending moment may also readily be found by graphical methods. Shear, being the transverse component of the resultant of the forces applied either to the left or to the right of a section, may easily be found from the force polygon. Graphical determination of bending moment, however, requires additional considerations beyond the techniques already discussed.

The bending moment at a section is equal to the moment of the resultant of the forces applied either to the left or to the right of the section. Of course, the magnitude and direction of such a resultant may be determined from the force polygon, while a point on its line of action is located by the intersection of the appropriate strings of the funicular polygon. By using this information and scaling the lever arm of the resultant, it is possible to compute the desired moment. It is simpler, however, to compute the moment by using the procedure developed by the following considerations:

Consider the force system $F_1$, $F_2$, and $F_3$ shown in Fig. 5·13, and suppose that it is desired to compute the sum of the moments of these three forces about point $a$. The sum of the moments of these three forces is equal to the moment of their resultant about point $a$, which will be called $M_a$. Then,

$$M_a = (R_{123})(m) = (\overline{OT})(m) \tag{a}$$

where $\overline{OT}$ is measured to the force scale and $m$ to the distance scale. If

the line $de$ is drawn parallel to $OT$, then the triangles $OPT$ and $cde$ are
similar. Therefore, drawing $H$ perpendicular to $OT$,

$$\frac{de}{m} = \frac{OT}{H} \qquad \text{or} \qquad Hde = OTm \qquad (b)$$

Hence,

$$M_a = Hde \qquad (c)$$

where $de$ is measured to the distance scale and $H$ to the force scale. $H$ is
called the pole distance.

In general, determining the product of $Hde$ furnishes a convenient
way of evaluating the moment of the resultant (and hence of the three
forces) about point $a$. The following procedure summarizes this graph-
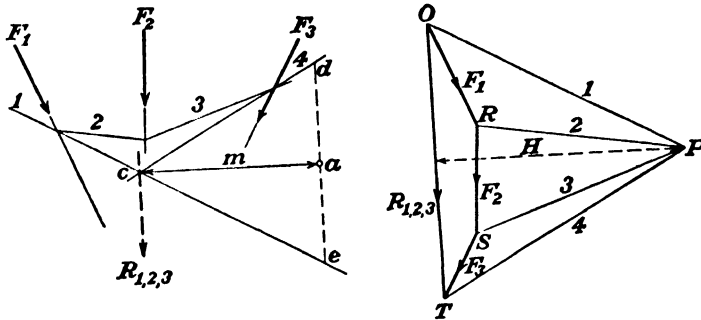


FIG. 5·13

ical method of determining the moment of a system of forces about a
given point $a$:

1. Construct the force polygon for this system. Select a pole $P$,
and draw the corresponding funicular polygon.

2. Draw through point $a$ in the space diagram a line parallel to the
direction of the resultant of the system as determined in the force polygon.

3. Measure to the distance scale the intercept of this line between the
strings of the funicular polygon, the intersection of which determines a
point on the line of action of the resultant.

4. Also measure to the force scale the pole distance $H$ that is the per-
pendicular distance from the pole $P$ to the resultant vector in the force
polygon.

5. The moment of the force system about point $a$ is then equal to
the product of the intercept, from step 3, and the pole distance, from
step 4.

The following examples illustrate how conveniently this procedure
may be applied to the computation of the bending moment at various
points in a beam.

It is often necessary to consider cases where the load is distributed instead of concentrated. In such cases, the portion of the beam so loaded must be divided into a series of short sections. The total load acting on each section is then assumed to be concentrated at the center of gravity of the load for that section. The graphical solution for reactions, shear, and bending moment is then carried out as usual, considering the distributed load to be replaced by this series of concentrated loads. The values of the reactions found in this manner are not in error, but the values of shear and bending moment are exact only at the ends of the various short sections into which the distributed load is divided. The ordinates of the shear and bending-moment curves at intermediate points are not appreciably in error, however, provided that the lengths of the sections are reasonably small.

**Example 5·1**    *Draw the shear and bending-moment diagrams for this beam:*



$$M_c = 31.0 \times 2.30 = 71.3^{k'}$$
$$M_d = 31.0 \times 3.80 = 117.9$$
$$M_e = 31.0 \times 1.45 = 44.9$$
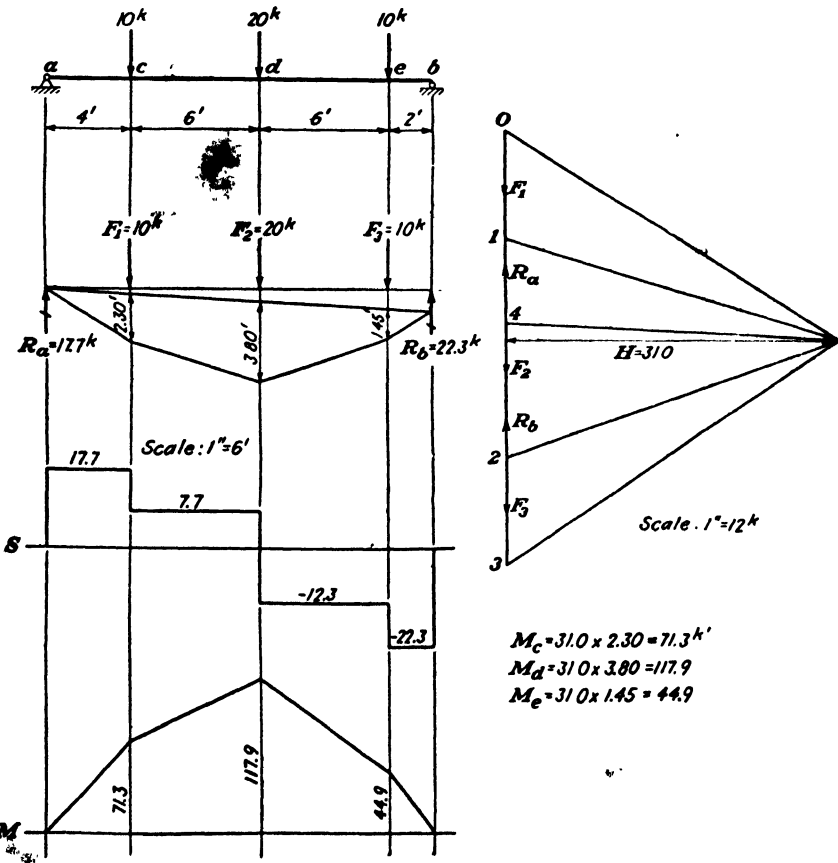
$$M_c = 31.0 \times 2.30 = 71.3^{k'}$$
$$M_d = 31.0 \times 3.80 = 117.9^{k'}$$
$$M_e = 31.0 \times 1.45 = 44.9^{k'}$$

**Discussion:**

*In this example, all the loads are vertical and develop vertical reactions. As a result, at any point along the beam, the resultant of the loads either to the right or to the left of that section is a vertical force. The computation of shear and, particularly, bending moment is therefore considerably simplified. For example, in computing the bending moments at various points, the pole distance $H$ will be constant for every point, and the intercepts measured in the funicular polygon will all be along vertical lines.*

**Example 5·2**   *Draw the shear and bending-moment diagrams for this beam:*
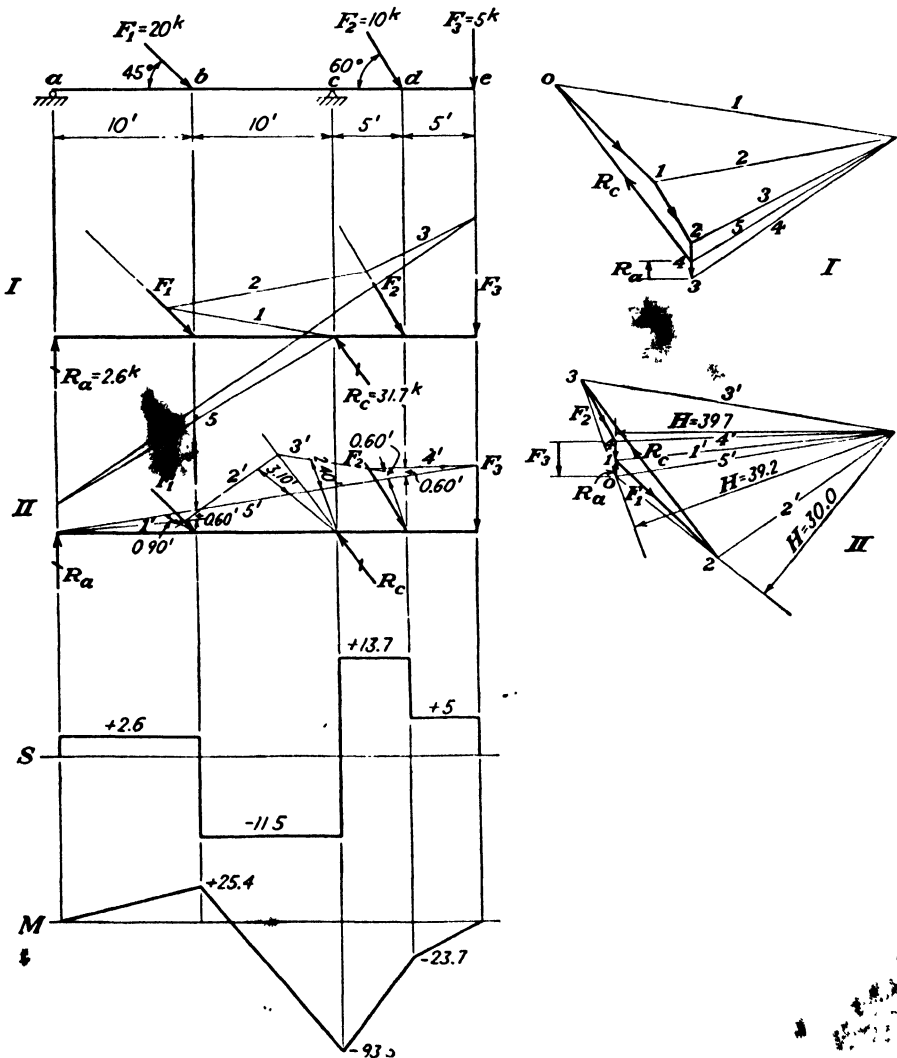
<center>*Computed by using forces on*</center>

| *Left* | | *Right* |
|---|---|---|
| $M_b = 39.7 \times 0.60 = 23.8$ | or | $30.0 \times 0.90 = 27.0$ |
| $M_c = 30.0 \times 3.10 = 93.0$ | or | $39.2 \times 2.40 = 94.0$ |
| $M_d = 39.2 \times 0.60 = 23.6$ | or | $39.7 \times 0.60 = 23.8$ |

<center>Average $M_b = 25.4^{k'}$</center>
<center>Average $M_c = 93.5^{k'}$</center>
<center>Average $M_d = 23.7^{k'}$</center>

**Discussion:**

In simple problems such as Example 5·1, the graphical method may be applied without difficulty. As soon as the loads are inclined or the beam is no longer supported in a simple end-supported manner, the method becomes considerably more complex.

In this case, after finding the reactions using force and funicular polygons I, it is necessary to redraw the force polygon so that the forces are laid out in the same order as they are encountered in traversing the member from one end to the other. After force polygon II and its corresponding funicular polygon have been laid out in this manner, it is possible to compute the shear and bending moment at all the various points along the beam. Normally, the second funicular polygon may be superimposed on the first. Here, however, two separate space diagrams have been drawn to avoid unnecessary confusion.

Note that the bending moments have been computed from the forces on each side of the section to obtain a check.

## 5·11 Stresses in Trusses—Maxwell Diagram—Bow's Notation.

The graphical method of joints is a convenient method for determining the bar stresses of certain statically determinate trusses. Assuming that the reactions have been determined previously by either graphical or algebraic methods, the bar stresses can then be determined by drawing a series of force polygons, one for each joint. It will be found convenient, however, to combine all these polygons into one composite figure called a *Maxwell diagram* after its originator, Clerk Maxwell.

At any joint, the bar stresses and external forces form a concurrent coplanar force system, which to be in equilibrium must produce a closed force polygon. Closure of the force polygon is equivalent to satisfying the two algebraic conditions $\Sigma F_x = 0$ and $\Sigma F_y = 0$. Since the directions of all forces acting on a joint are known, the *magnitude* of *two* unknown bar stresses may be determined, therefore, by making the force polygon for the joint close. It is thus possible to determine all the bar stresses in a *simple truss* by starting at a joint where there are only two unknown stresses and then considering each of the other joints in turn, always working with a joint where there are not more than two unknowns.

In applying the graphical method of joints, the external and internal forces may be identified by using *Bow's notation*. To apply this notation, the spaces between external forces are designated by Roman numerals and those between bars by Arabic numerals as illustrated in Fig. 5·14. Then, an external force may be identified by reading the numbers on each

side of it in a clockwise order; for example, the force acting at joint $B$ is
called force I-II.   Likewise, the internal force with which a member
acts on a joint is identified by reading the numbers on each side of that
member in a clockwise order about that joint; for example, member $Bc$
acts on joint $B$ with a force 32.

After the space diagram has been labeled according to Bow's notation,
the reactions of a truss such as the one in Fig. 5·14 may be computed by
either graphical or algebraic methods, whichever is more convenient.
Then it is possible to isolate joint $a$ where two bar stresses are unknown
and to determine these unknowns by drawing the force polygon shown,
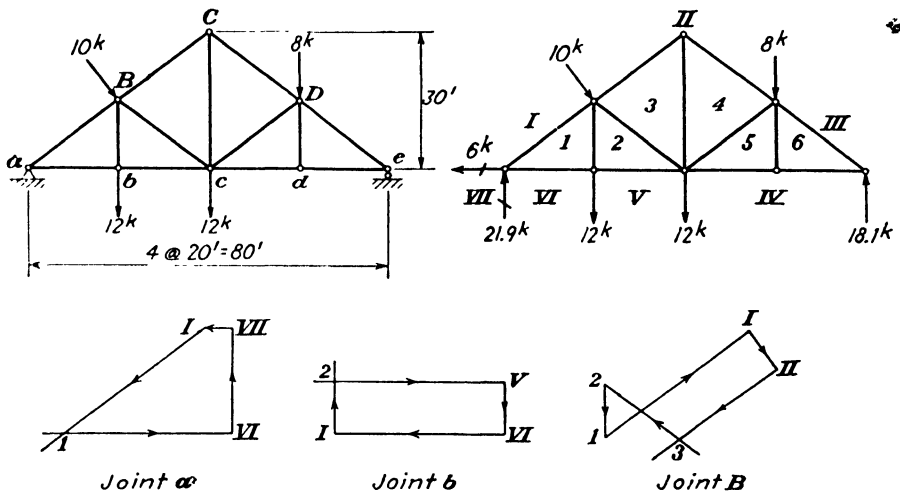


FIG. 5·14

The vectors of this polygon should be laid out in the clockwise order
of the forces around the joint.   The ends of a vector should be identified by
the same numbers that lie on each side of the corresponding force in the
space diagram, arranged so that reading the number first at the rear and
then at the front end of a vector places them in the same order as that in
which they are encountered in going clockwise around the joint.   If
this procedure is followed, then the numbers of the vertices from start to
finish of the force polygon read in the same order as that in which the
numbers are encountered in going clockwise around the joint.   In this
way, it is found from the force polygon for joint $a$ that the forces with
which bars $aB$ and $ab$ act on joint $a$ are measured by the vectors $\overline{I\text{-}1}$
and $\overline{1\text{-}VI}$, respectively, which indicate that the character of stress in $aB$
is compression and that in $ab$ is tension.

Having the stress in $ab$, it is now possible to proceed to joint $b$ and
construct a force polygon from which the stresses in bars $Bb$ and $bc$ may

be determined.   Then proceeding to joint $B$, there are only two unknown stresses, in bars $Bc$ and $BC$, which may be determined from the force polygon for this joint.   Considering the remaining joints in turn enables one to complete the stress analysis of the truss.   Instead of constructing a separate polygon for each joint, however, it is more convenient to draw one Maxwell diagram that in effect combines all the separate joint polygons.

To construct a Maxwell diagram, first draw a force polygon for all the external forces, laying out the vectors in the same order as the forces are encountered in going clockwise around the structure.   If the reactions
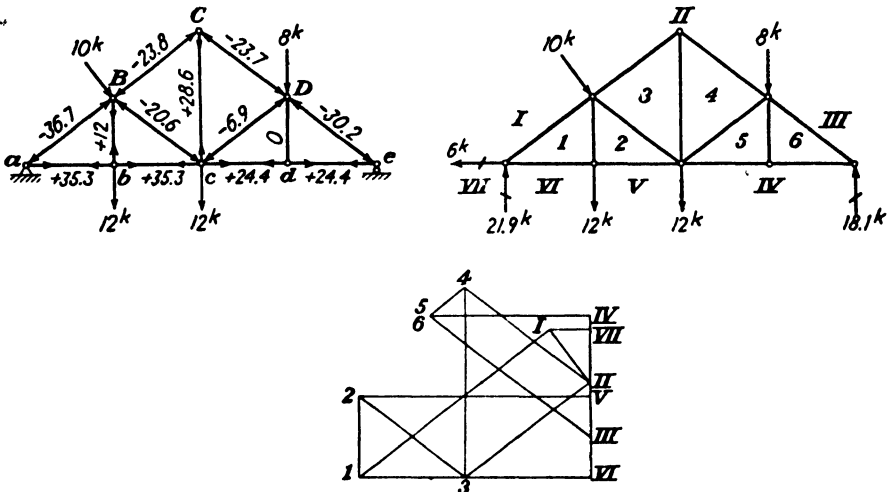


Fig. 5·15

have been determined graphically and the resulting force polygon has not been laid out in this manner, then a new polygon must be drawn with the vectors so arranged.   The vertices of this polygon should be labeled in the same manner as described above for the joint force polygon. The Maxwell diagram for the truss of Fig. 5·14 was started in this manner and is shown in Fig. 5·15.   Now consider a joint such as $a$ where there are only two unknown bar stresses, and note the numbers of the spaces surrounding this joint.   All but one, 1, have corresponding vertices in the part of the Maxwell diagram drawn so far.   The missing vertex, number 1, may be located by drawing through the two adjacent vertices, I and VI, lines parallel to the intervening bars, $aB$ and $ab$, respectively. Now, considering joint $b$ in the same way, vertex 2 is the only one missing and is located by drawing through vertices 1 and V lines parallel to bars $bB$ and $bc$, respectively.   The remaining missing vertices 3, 4, 5, and 6

may be located in turn by considering successively joints $B$, $C$, $c$, and $d$ (or $D$). Note that, at the time when each of these joints is considered, there is only one missing vertex associated with that joint.

The construction of the Maxwell diagram having been completed, it is a simple matter to determine the magnitude and sense of the force with which a bar acts on a given joint. *Read the numbers on each side of a bar in a clockwise order around a joint. The magnitude and sense with which that bar acts on that joint are given by the vector measured from the vertex of the first number to the vertex of the second number.* The bar stresses so determined in this example are recorded on the line diagram of the truss.

After comparing the Maxwell diagram with the force polygon for the separate joints, it is evident that the diagram is simply a composite figure in which all the joint polygons have been superimposed. It is also evident that the use of the clockwise direction throughout this discussion is arbitrary: the whole system would have worked equally well if everything were reversed and taken in a counterclockwise order throughout.

**5·12  Certain Ambiguous Cases—Fink Roof Truss.** The Maxwell diagram described in the previous article may be drawn without difficulty for any *simple truss.* When the ideas are applied to a *compound truss*, the diagram may be drawn up to a certain point; then one discovers that at each of the remaining joints there are more than two unknown bar stresses and therefore more than one missing vertex.

Consider a compound truss such as the Fink roof truss shown in Fig. 5·16. After finding the reactions, a force polygon for the external forces may be laid out and the Maxwell diagram started in the conventional manner by considering first joint $a$ and then proceeding in turn to joints $B$ and $b$. Considering now either joint $C$ or $c$, there are three unknown bar stresses and therefore two unknown vertices at either of these joints; it is thus impossible to continue with the Maxwell diagram. Of course, it is possible to go across to joint $i$ and work back successfully at joints $h$ and $H$, but then the same dilemma is encountered at either of joints $G$ or $g$  One of the several alternative methods available to circumvent this difficulty is discussed below:

Suppose temporarily that we replace bars $Cd$ and $dD$ by the bar $Dc$ as indicated by the dashed line. We shall call the space enclosed by triangle $cDE$ by number 6' and the space enclosed by triangle $cCD$ by number 4'. Such a replacement does not alter the stresses in bars $aB$, $ab$, $bB$, $bc$, $bC$, $BC$, $DE$, $Ed$, or $cg$ of the original truss. This is evident when one considers the computation of the stresses in these members using the sections indicated. The locations of vertices 1, 2, and 3 of the Maxwell diagram, therefore, remain the same for either the original or the

altered truss. It is thus possible to locate vertex 4′ of the altered truss
by considering joint $C$ and then to proceed to joint $D$ to locate vertex 6′.
The location of vertex 6′ so determined for the altered truss coincides with
6 for the original truss, since in either case the stress in bars $DE$ and
$dE$ is the same. It is now possible to return to the original truss and,
by considering in turn joints $D$ and $C$, to locate the correct positions
of vertices 5 and 4, respectively. Now it is easy to proceed in the
conventional manner and locate the remaining vertices 7 to 13.
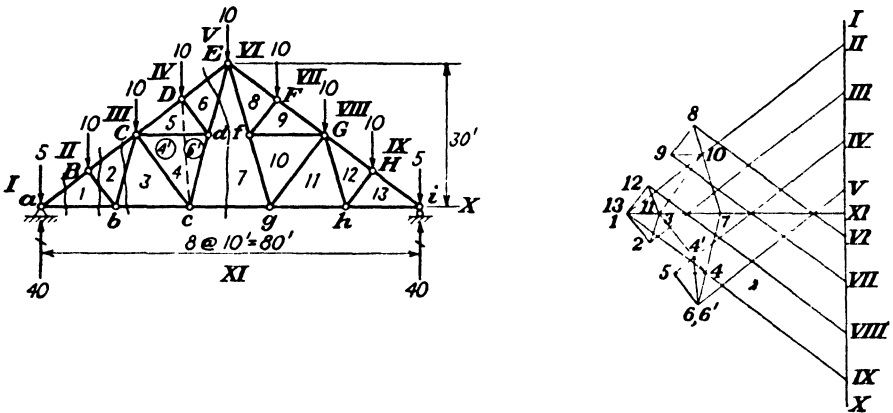


Fig. 5·16

## 5·13 Reactions and Bar Stresses of Three-hinged Arches.
Once the reactions of a three-hinged arch have been determined, there
is no difficulty encountered in drawing a Maxwell diagram to find the
bar stresses. The reactions may be determined, of course, either ana-
lytically or graphically. The graphical solution for the reactions of
three-hinged arches requires some additional considerations, however.

One graphical method of finding the reactions utilizes an important
characteristic of a three-hinged arch. Consider the arch shown in Fig.
5·17. The reactions of the structure may be computed by superimposing
the separate effects of (1) the loads applied to the left half acting by them-
selves and (2) those applied to the right half. It is easy to find the
separate effects, for in each case one half of the arch is not acted upon by
any external loads. In such cases, the reaction acting on the unloaded
half must be directed through the center of the crown hinge at point $b$
so that the bending moment about the hinge will be zero. In both cases
I and II, therefore, the graphical solution is the same as the case of an
end-supported beam where the magnitude of both reactions but the
direction of only one of them is unknown. The reactions in cases I and II

are obtained by selecting the poles $P_1$ and $P_2$ and drawing the funicular polygons shown.   If the two force polygons are plotted together as shown, it is then a simple matter to superimpose the two cases graphically and find the resultant reactions $R_a$ and $R_c$ developed by the combined system of loads.

A second graphical method of finding the reactions involves passing the funicular polygon of the external loads through the three hinges $a$, $b$, and $c$.   In Art. 5·7, it is pointed out that a system of external forces can be supported by a linkage system which has the same shape as the funicular polygon for those forces.   Considering one half of the arch at a *
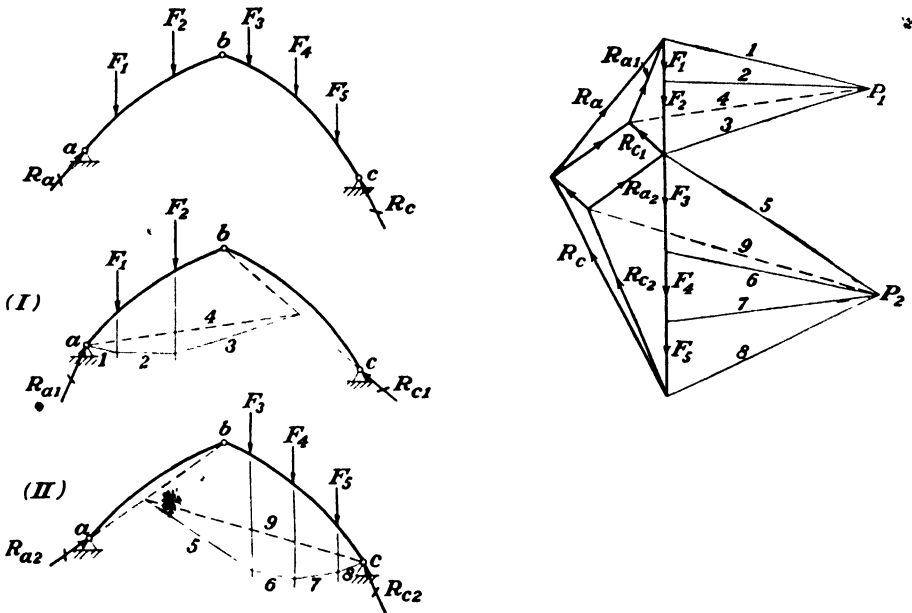


FIG. 5·17

time, the external loads acting on each half can be supported on a linkage with end reactions corresponding to a funicular polygon whose end strings pass through the hinges at the support and crown.   Since the action of the left half on the right half must be equal and opposite to the action of the right on the left, the end strings at the crown hinge of the funicular polygon of each half must be collinear.   This means that the two separate polygons must be capable of being combined into one continuous polygon for the entire arch, which passes through all three hinges $a$, $b$, and $c$. If a pole is found so that the funicular polygon for all the external loads passes through these three points, the end reactions may then be obtained by measuring the first and last rays on the force polygon.

**5·14  Problems for Solution.**

**Problem 5·1**  Find graphically the resultant of the forces shown in Fig. 5 18. Indicate its magnitude.  Show its direction by indicating its horizontal and vertical components.  Use scale 1 in. = 1 ton.
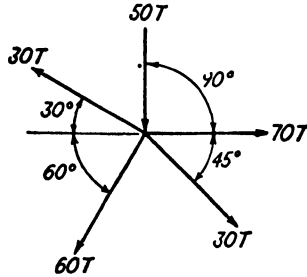


FIG. 5·18

**Problem 5·2**  Find graphically the forces necessary to hold each of the frames of Fig. 5 19 in equilibrium.  Indicate the magnitude of both the force and its horizontal and vertical components.  Locate the line of action of the resultant force with reference to horizontal and vertical axes drawn through the center of each frame.
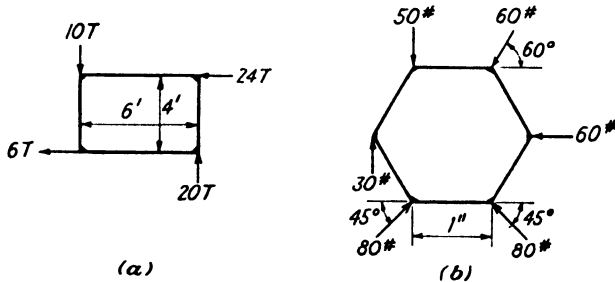


FIG. 5·19

**Problem 5·3**  Find the resultant of the forces shown in Fig. 5 20 by use of the funicular polygon.  Indicate its magnitude and direction, and locate the intersection of its line of action with the horizontal base line.  Scales 1 in. = 5 ft, and 1 in. = 50 lb.
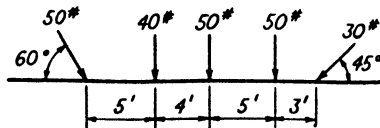


FIG. 5·20

**Problem 5·4**  Find the horizontal and vertical components of the reactions of the structures of Fig. 5·21, using the three-force method.
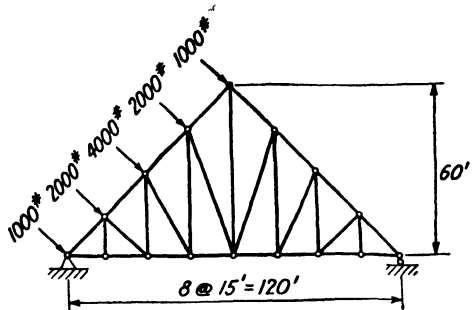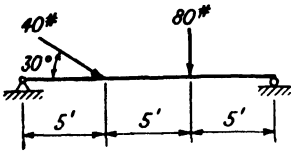
FIG. 5·21

**Problem 5·5**   Find the horizontal and vertical components of the reactions of the structures of Fig. 5·22, using the funicular polygon.
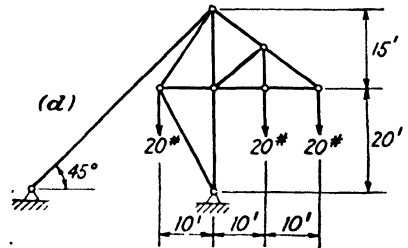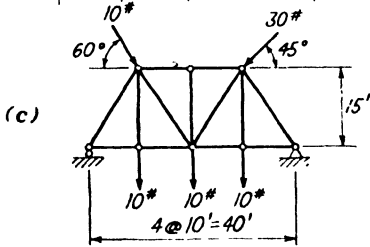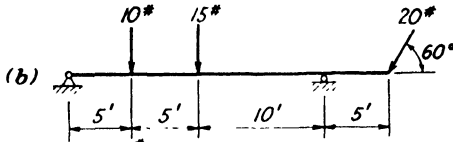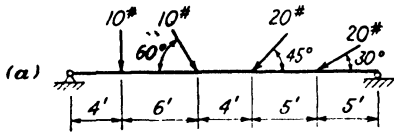


FIG. 5·22

**Problem 5·6**   Find graphically the bar stresses in the members of the Pratt roof truss of Fig. 5·23.
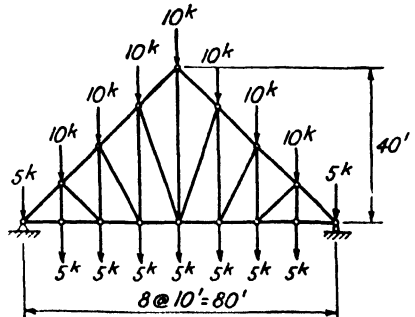


FIG. 5·23

**Problem 5·7**   Find graphically the bar stresses in the members of the Fink roof truss of Fig. 5·24.
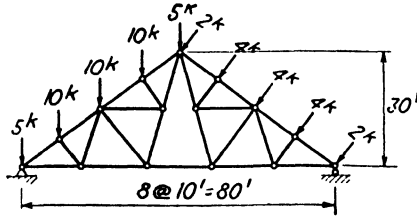


FIG. 5·24
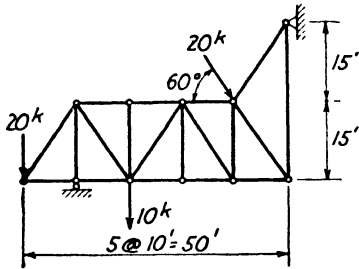


FIG. 5·25

**Problem 5·8**   Find the bar stresses in the members of the truss of Fig. 5·25, using graphical methods.

**Problem 5·9**   Draw the curves of shear and bending moment for the beams of Fig. 5·26, using graphical methods.



(a)

(b)

(c)

(d)

(e)
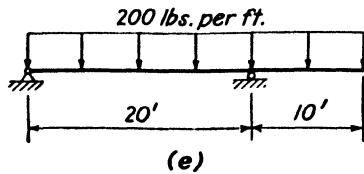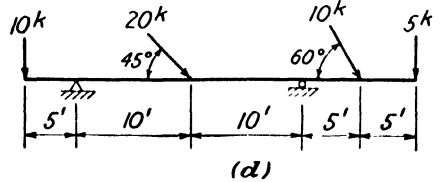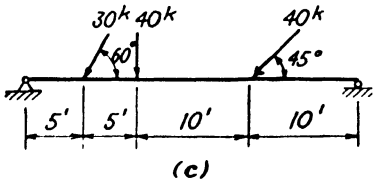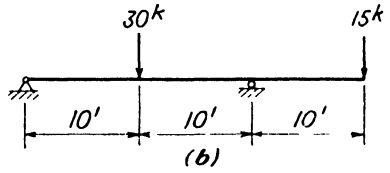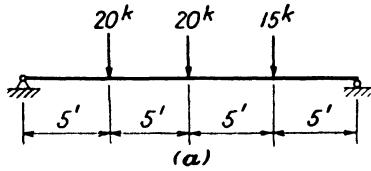
FIG. 5·26

**Problem 5·10** Draw the funicular polygon for the beam and loading of Fig. 5·27, so that it passes through the points of support and also through a point 20 ft below the center of the span.
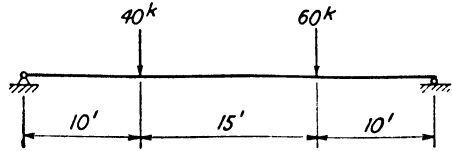


FIG. 5·27

**Problem 5·11** A cable is suspended from two points at the same elevation and 20 ft apart. The cable supports nine weights of 100 lb each, spaced 2 ft apart, the distance between each support and the nearest weight being 2 ft. The lowest point on the cable is 5 ft below the line joining the supports. What is the length of the cable, and what is the maximum tension in it? Use graphical methods.

**Problem 5·12** Find the reactions and bar stresses of the trusses of Fig. 5 28, using graphical methods.
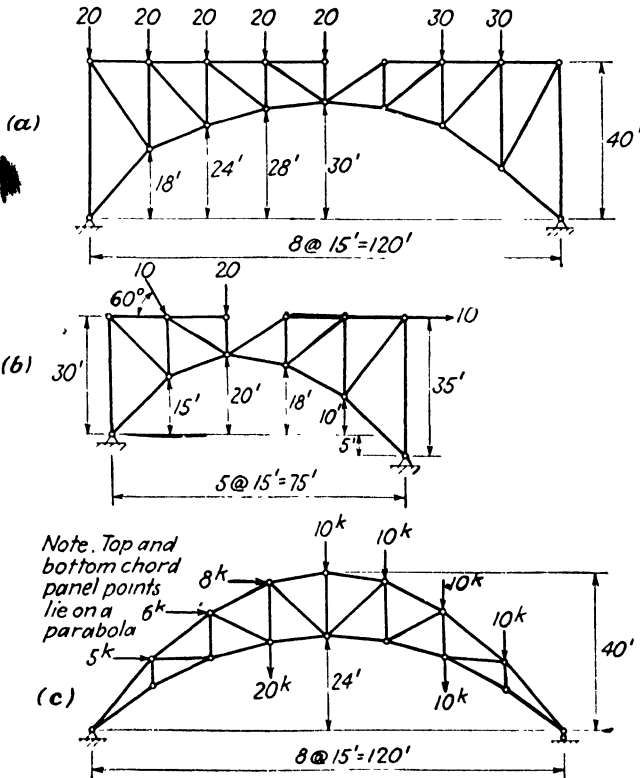


FIG. 5·28

**Problem 5·13** Using graphical methods, find the reactions and bar stresses of the truss shown in Example 4·5.

# CHAPTER 6

## INFLUENCE LINES

**6·1 Introduction.** Chapters 2 to 5 are devoted to a consideration of the basic ideas involved in the computation of reactions, shears, bending moments, and bar stresses for statically determinate structures. Before any of these functions can be computed, it is of course necessary to establish the condition of loading for which the analysis is to be made. In Chap. 1, distinction is drawn between dead loads, such as the weight of the structure itself, which remain stationary, and live loads, which may vary in position on a structure.

When one is designing any specific part of a structure, it is necessary to proportion the part under consideration so that it has sufficient strength to withstand the greatest stress to which it may be subjected during the life of the structure. In order to design such a part, the greatest contribution of the live load to the total design stress is one of the items that must be determined. The stress produced in a given part by the live load varies with the position of the load on the structure. There is always one position of the live loads on a structure that will cause the maximum live stress in any particular part of the structure. The part of the structure and the type of stress involved may be, for example, the reaction at a support; the bending moment or shear at a section in a beam or girder; the tension or compression in a truss member; or the load carried by a particular rivet. The proper design of the various parts will, in general, depend on different live-load positions.

It should therefore be clear that it is essential for the structural analyst to understand clearly the methods by which the position of live load which causes the maximum stress at any point may be determined.

**6·2 Illustration of Variation in Stress with Position of Load.** Suppose that a downward load of unity be placed at point $A$ on the beam $AB$ of Fig. 6·1. By taking moments about $B$, the reaction $R_{Ay}$ is found to act upward and to equal 1. At $A'$ on the base line $A'B'$, $A'$ being directly beneath $A$, let the distance $+1$ be plotted vertically. Let the applied unit load now travel to $C$; upon taking moments about $B$, $R_{Ay}$ is found to equal $+\frac{9}{10}$. Plot this ordinate $+\frac{9}{10}$ at $C'$ on the base line $A'B'$, $C'$ being directly below the point of application of the unit load. Let the unit load now travel to $D$; $R_{Ay}$ becomes $+\frac{8}{10}$; plot $+\frac{8}{10}$ vertically at $D'$, which is directly below $D$.

Repeat the procedure for all positions of the unit load between $A$

and $B$.  The resultant reaction values, plotted in each instance from the base line $A'B'$, and directly below the particular load position, all lie on a straight line.  This might have been foreseen, since, for the unit load at any section at distance $x$ from $B$, $R_{Ay}$ equals $+x/20$.  The ordinate $+x/20$ is plotted at distance $x$ from $B'$, and the plot of $x/20$ against $x$ is linear.

Because of the manner of plotting this curve from the base line $A'B'$, a number of important conclusions can be drawn.

1. The ordinate at any point on this curve equals the value of $R_{Ay}$ if a unit load is applied at that section.  (Note that all ordinates refer to the reaction at point $A$, and that it is the position of the unit load causing this reaction which is the variable in constructing the curve.)



FIG. 6·1

2. As the unit load travels from $B$ to $A$, the reaction at $A$ increases linearly.  The maximum value of $R_{Ay}$ is seen to occur when the load is applied at $A$.

3. Since all the ordinates to this curve are positive, it may be concluded that a unit load applied at any point along the span $AB$ causes an upward reaction at $A$.  Hence, if this structure were to be loaded with a uniform live load, the live load should extend over the entire span $AB$ in order to give a maximum value to $R_{Ay}$.
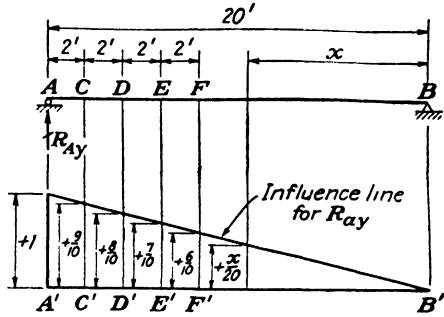
**6·3   The Influence Line—Definition.**   The curve drawn in Fig. 6·1 is called an *influence line* because it shows the influence on a certain function of a unit load as it travels across the structure.   In this particular case the function under consideration is the vertical reaction at $A$. The function may, however, be anything that varies as the load moves across the span, such as moment or shear at a given section in a girder or beam, or stress in a particular truss member, or deflection of a given point on a structure.

An influence line may be defined as follows: *An influence line is a curve the ordinate to which at any point equals the value of some particular function due to a unit load acting at that point.*

Curve $b$ of Fig. 6·2 shows the influence line for moment at $C$, the center of an end-supported beam.   That this curve satisfies the definition of an influence line may be verified by checking the ordinate at any point. If, for example, a unit load is applied at $D$, the moment at $C$ equals $\frac{1}{4} \times 10 = +2.5$.   This is the ordinate to the influence line at point $D$.

Curve *c* of this figure shows the influence line for shear at *D*, the left quarter point. That this curve satisfies the definition of an influence line may also be seen by checking the ordinate at any point. If, for example, the unit load is applied just to the right of *D*, the shear at *D* equals $+\frac{3}{4}$.

By definition, an influence line shows the effect of a unit load as it travels across the span. It should be apparent that such a curve is closely related to a live load moving across a bridge. The usefulness of influence lines is not, however, limited to bridge structures, since they
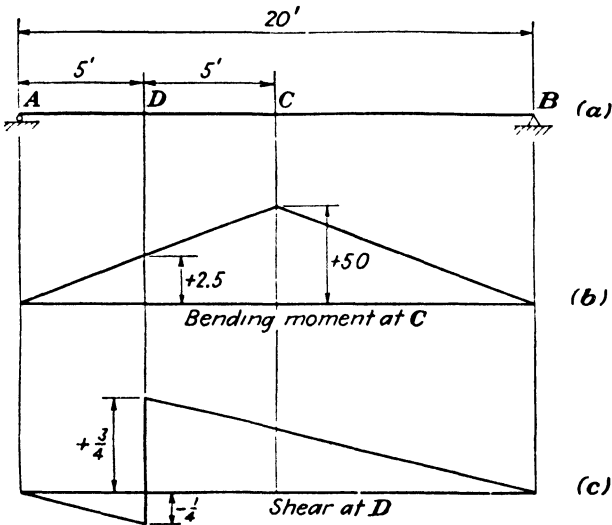


FIG. 6·2

are of importance in the determination of maximum stresses in any structure subject to the action of live loads. These live loads may be the movable loads in an office building, or the aerodynamic loads on the wing of an airplane, or the hydrostatic support caused by the displacement of waves on the hull of a ship.

**6·4  Construction of Influence Lines for Beams.** Consider the beam shown in Fig. 6·3a. To illustrate the method of constructing influence lines, an influence line for the shear just to the left of point *A* will first be constructed, as shown in Fig. 6·3b. When a unit load is applied at any position to the left of this section, the shear just to the left of *A* equals the unit load and is negative. Hence the influence line has the ordinate −1 from *C* to *A*. When a unit load is applied in any position between *A* and *B*, the shear just to the left of *A* is zero. Hence the ordinate to the influence line is zero in this portion of the beam.

The influence line for the shear at $D$ will now be constructed, as shown in Fig. 6·3c.  If a unit load is applied at $C$, the shear at $D$ may be computed from the reaction at $B$ and is seen to equal $+\frac{1}{2}$.  As the unit load travels from $C$ to $A$, the reaction at $B$ and hence the shear at $D$ decreases to zero.  Hence the influence line for the shear at $D$ varies from $+\frac{1}{2}$ at $C$ to zero at $A$.  Actually, this variation between $C$ and $A$ is linear.  That
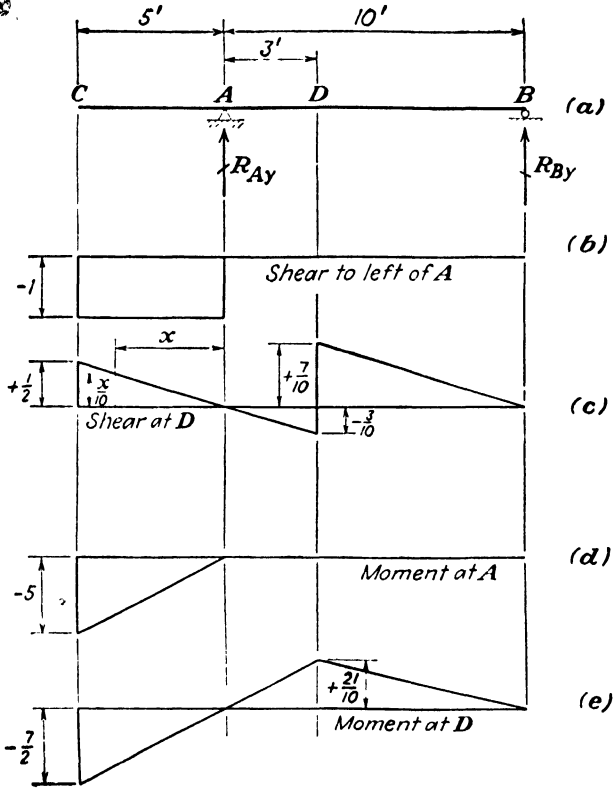


Fig. 6·3

the influence line between $C$ and $A$ is a straight line may be seen by either of the following two methods:

1. The unit load may be applied at any point between $C$ and $A$ and the shear at $D$ computed.  When this value is plotted at the point of application of the load, it will be found to lie on a straight line.

2. If the distance from $A$ to the load is denoted by $x$, the vertical reaction at $B$ acts downward and has the value $x/10$.  The shear at $D$ therefore equals $+x/10$.  The plot of $+x/10$ against $x$ is a straight line.

As the unit load travels from $A$ to a point just to the left of $D$, the reaction at $B$ increases from zero to $+\frac{3}{10}$.  Hence the shear at $D$ goes

from zero to $-\frac{3}{10}$. The ordinate to the influence line at a point just to the left of $D$ is therefore $-\frac{3}{10}$. That the influence line is a straight line varying from zero at $A$ to $-\frac{3}{10}$ at $D$ may be seen on the basis of either of the arguments which led to the conclusion that it was straight between $C$ and $A$.

Now consider the unit load placed just to the right of $D$. If the shear at $D$ is computed from the forces to the right of $D$, as has been done previously, it is necessary to consider two forces, the reaction at $B$ and the unit load itself. If, on the other hand, the shear at $D$ is computed from the forces to the left of $D$, it is necessary to consider only the reaction at $A$.

It is often preferable, in computing ordinates to influence lines, to work from the forces on the side of the section that is away from the unit load. For the case under consideration, $R_{Ay} = +\frac{7}{10}$, so that the shear at $D$ is $+\frac{7}{10}$. This is the ordinate to the influence line just to the right of $D$. It is to be noted that as the unit load passes $D$, while moving from left to right, the shear at $D$ increases suddenly from $-\frac{3}{10}$ to $+\frac{7}{10}$. As the unit load travels from just to the right of $D$ to $B$, the reaction at $A$, hence the shear at $D$, decreases linearly from $+\frac{7}{10}$ to zero. Hence the influence line is a straight line running from $+\frac{7}{10}$ at $D$ to zero at $A$.

The influence line for the moment at $A$ will now be constructed, as shown in Fig. 6·3$d$. When a unit load is placed at $C$, the moment at $A$ equals $-5$. As the load travels from $C$ to $A$, the moment at $A$ decreases linearly to zero. With the load at any position between $A$ and $B$, the moment at $A$ equals zero, as can be seen from a consideration of the forces to the left of $A$.

To construct the influence line for the moment at $D$, as shown in Fig. 6·3$e$, one may proceed as follows: Owing to a unit load at $C$, the moment at $D$ equals $-\frac{7}{2}$, as may easily be computed from the reaction at $B$. As the load goes from $C$ to $A$, the moment at $D$ decreases linearly to zero. Hence the influence line is a straight line from $-\frac{7}{2}$ at $C$ to zero at $A$. As the load goes from $A$ to $D$, the reaction at $B$ increases linearly from zero to $+\frac{3}{10}$; the moment at $D$, computed from this reaction, increases linearly from zero to $+\frac{3}{10} \times 7 = +\frac{21}{10}$; hence the influence line is a straight line from zero at $A$ to $+\frac{21}{10}$ at $D$. As the load goes from $D$ to $B$, the reaction at $A$ decreases linearly from $+\frac{7}{10}$ to zero; the moment at $D$, computed from this reaction, decreases linearly from $+\frac{7}{10} \times 3 = \frac{21}{10}$ to zero at $B$.

**6·5 Properties of the Influence Line.** Influence lines may be used for two very important purposes: (1) to determine what position of live loads will lead to a maximum value of the particular function for which an influence line has been constructed; (2) to compute the value

of that function with the loads so placed or, in fact, for any loading condition.

Since the ordinate to an influence line equals the value of a particular function due to a unit load acting at the point where the ordinate is measured, the following two theorems hold:

1. *To obtain the maximum value of a function due to a single concentrated live load, the load should be placed at the point where the ordinate to the influence line for that function is a maximum.* It is obvious that if the maximum positive value of a function is desired, the load should be placed at the point where the ordinate to the influence line has its maximum positive value, while, if the maximum negative value is to be obtained, the position of the load is determined by the maximum negative ordinate.

2. *The value of a function due to the action of a single concentrated live load equals the product of the magnitude of the load and the ordinate to the influence line for that function, measured at the point of application of the load.* This follows from the principle of superposition. Further, the total value of a function due to more than one concentrated load can be obtained by superimposing the separate effects of each concentrated load, as determined by theorem 2.

To illustrate the application of these two theorems, suppose a concentrated live load of 10,000 lb is applied to the beam of Fig. 6·3a. If the influence line of Fig. 6·3c is used, the maximum positive shear that this load can cause at $D$ occurs with the load just to the right of $D$ and equals $10,000(+\frac{7}{10}) = +7,000$ lb. The maximum negative shear at the same section occurs with the load just to the left of $D$ and equals $10,000 \ (-\frac{3}{10}) = -3,000$ lb. From Fig. 6·3e, the maximum positive moment at $D$ occurs with the load at $D$ and equals

$$10,000(+\tfrac{21}{10}) = +21,000 \text{ ft-lb}$$

From the definition of an influence line, the following theorem, dealing with uniformly distributed live loads, is apparent:

3. *To obtain the maximum value of a function due to a uniformly distributed live load, the load should be placed over all those portions of the structure for which the ordinates to the influence line for that function have the sign of the character of the function desired.*

To compute, from the influence line, the actual value of the function due to a uniformly distributed live load, the following theorem should be used:

4. *The value of a function due to uniformly distributed live load is equal to the product of the intensity of the loading and the net area under that portion of the influence line, for the function under consideration, which corresponds to the portion of the structure loaded.*

That the foregoing theorem is correct may be seen from the following: Let $AB$ be the influence line for a given function $F$, for a portion of a structure, as shown in Fig. 6·4, that is subjected to a uniformly distributed load of $w$ lb per ft, applied continuously to the structure between two points $M$ and $N$. That portion of the uniform load applied in distance $dx$ may be treated as a concentrated load equal to $w\,dx$. By theorem 2, the value of the function $F$ due to this differential load is given by $dF = w\,dx\,y$. The total value of $F$ due to the load between $M$ and $N$ is obtained by integrating $dF$ between $x = 0$ and $x = a$, or



FIG. 6·4

$F = \int_0^a wy\,dx = w\int_0^a y\,dx = w$ multiplied by the area under that portion of the influence line which corresponds to the portion of the structure loaded.
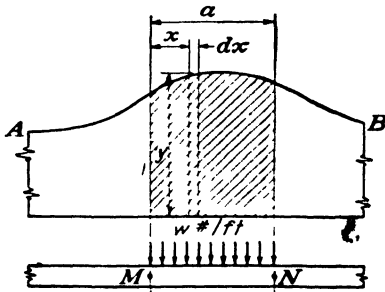
To illustrate the application of theorems 3 and 4, suppose a uniform live load of 1,000 lb per ft is applied to the beam of Fig. 6·3a. To obtain the maximum positive shear at $D$ (Fig. 6 3c), the uniform load should extend from $C$ to $A$ and from $D$ to $B$. The value of this maximum positive shear at $A$ is given by

$$1,000[\tfrac{1}{2}(5)(+\tfrac{1}{2}) + \tfrac{1}{2}(7)(+\tfrac{7}{10})] = +3,700 \text{ lb}$$

For maximum negative shear at $D$, the structure should be loaded from $A$ to $D$, leading to a resultant shear at $D$ equal to

$$1,000[\tfrac{1}{2}(3)(-\tfrac{3}{10})] = -450 \text{ lb}$$

For maximum positive moment at $D$, refer to Fig. 6·3e. The structure should be loaded from $A$ to $B$; the resultant moment equals

$$1,000[\tfrac{1}{2}(10)(+2\tfrac{1}{10})] = +10,500 \text{ ft-lb}$$

For maximum values of functions due to a concentrated live load and uniformly distributed live load acting simultaneously, the maximum function due to each acting separately should be computed by the methods already given and the results superimposed. For example, to obtain the maximum negative moment at $A$, in the beam of Fig. 6·3a, due to a uniform load of 1,000 lb per ft and a single concentrated load of 10,000 lb, it is found by reference to Fig. 6·3d that the uniform load should extend from $C$ to $A$, and the concentrated load should be placed at $C$. The maximum negative moment at $A$ is then given by

$$1,000[\tfrac{1}{2}(5)(-5)] + 10,000(-5) = -62,500 \text{ ft-lb}$$

Suppose that a uniform load of 1,000 lb per ft extends over the entire length of the beam of Fig. 6·3a.  Functions are then computed on the basis of the algebraic sum of the component areas that comprise the entire influence line.  From Fig. 6·3e, the resultant moment at $D$ would, for example, be given by $1,000[\frac{1}{2}(5)(-\frac{1}{2}) + \frac{1}{2}(10)(+2\frac{1}{20})] = -3,500$ ft-lb.

**6·6   Influence Lines for Girders with Floor Systems.**   The structural action of floor systems is discussed in Art. 3·9.  The construction of influence lines for girders with floor systems may be illustrated by a consideration of Fig. 6·5.  An influence line for the shear in panel $BC$ will first be drawn, as shown in Fig. 6·5b.  It should be noted
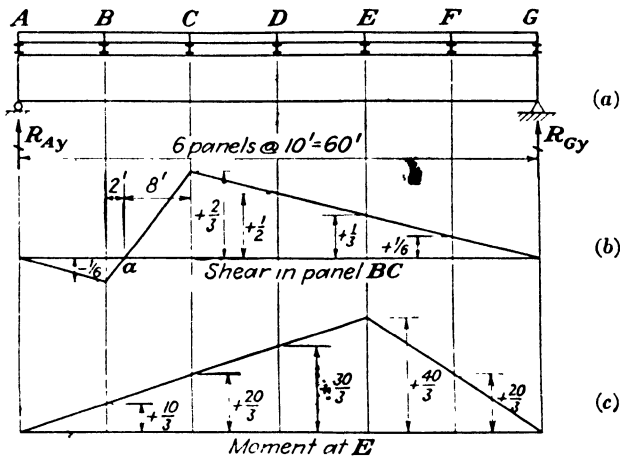


Fig. 6·5

that, since live loads can be applied to the girder only by the floor beams which are located at panel points $A$, $B$, . . . , $G$, the live shear has the same value at any section in a given panel of the girder.

When a unit load is placed at $A$, $R_{Gy} = 0$.  The floor-beam reactions to the right of panel $BC$, that is, the forces applied to the girder by the floor beams at $C$, $D$, $E$, $F$, and $G$, are also zero.  Hence, by computing the shear in panel $BC$ from the forces acting on the girder to the right of the panel, the shear equals zero.  When the unit load is placed at $B$, $R_{Gy} = +\frac{1}{6}$; the floor-beam reactions at $C$, $D$, . . . , $G$ still equal zero; hence the shear in panel $BC$ equals $-\frac{1}{6}$.

As the unit load travels along a stringer from one panel point to another, the influence line will be a straight line for the panel under consideration, provided that the stringers act as end-supported beams spanning the adjacent floor beams.  That this is so may be seen from the following: As the unit load travels from one panel point to another, the

reactions on the stringer, which are also the forces applied to the girder by the floor beams, vary linearly; hence any stress function for the girder, such as shear in a given panel, will also vary linearly. The influence line for shear in panel $BC$ is therefore a straight line from zero at $A$ to $-\frac{1}{6}$ at $B$.

When a unit load is placed at $C$, $R_{Av} = +\frac{2}{3}$, while the floor-beam reactions at $A$ and $B$ are zero. Then the shear in panel $BC$ equals $+\frac{2}{3}$, as is easily computed from the forces to the left of the panel. The influence line is a straight line from $B$ to $C$.

The ordinates to the influence line at panel points $D$, $E$, $F$, and $G$ may be computed by a procedure similar to that used in determining the ordinate at $C$, and the influence line will in each case be a straight line between panel points. It will be found that the influence line is a straight line from $C$ to $G$. These computations can be eliminated by the following reasoning: As the load travels from $C$ to $G$, $R_{Av}$ decreases linearly from $+\frac{2}{3}$ to zero, as is easily seen from a consideration of the *external* forces acting on the structure; since the floor-beam reactions at $A$ and $B$ remain zero, the shear in panel $BC$ decreases linearly from $+\frac{2}{3}$ to zero.

It should be pointed out that an influence line may always be constructed by computing the value of the function under consideration for successive positions of the unit load and taking care that one includes all points where the slope of the influence may change. Panel points constitute such points; but as will be seen later, it is possible to arrange a structure so that other points may also be critical. Experience in constructing influence lines makes it possible for one to recognize the fact that certain portions of the line are linear. This leads to a saving in computations, but it is not a necessary procedure.

Consider now the influence line for the moment at panel point $E$ of the girder of Fig. 6·5a, which is given in Fig. 6·5c. As a unit load travels from $A$ to $E$, $R_{Gv}$ increases linearly from zero to $+\frac{2}{3}$, as may be seen from a consideration of the external forces, while the floor-beam reactions at $F$ and $G$ are zero. Hence the moment at panel point $E$ increases linearly from zero at $A$ to a value equal to $+\frac{2}{3} \times 20 = +\frac{40}{3}$ at $E$, and the influence line is a straight line from zero at $A$ to $+\frac{40}{3}$ at $E$. As the unit load travels from $E$ to $G$, $R_{Av}$ decreases linearly from $+\frac{1}{3}$ to zero; the floor-beam reactions at $A$, $B$, $C$, and $D$ are zero; hence the influence line is a straight line from $+\frac{1}{3} \times 40 = +\frac{40}{3}$ at $E$ to zero at $G$.

It is not necessary for the stringers in every panel to be simply supported by the adjacent floor beams. Figure 6·6a illustrates a case where the stringers are cantilevered in panel $BE$ and where the end stringer in panel $EF$ cantilevers to point $G$. The construction of an influence line for such a structure will be illustrated by considering the moment at

panel point $E$ in the girder, as shown in Fig. 6·6b.   As the unit load goes from $A$ to $C$, consideration of the forces acting on the free body consisting of the girder plus stringer $AC$ and the floor beams connecting stringer $AC$ to the girder shows that $R_{Fy}$ increases linearly from zero to $+\frac{1}{2}$.   Since the floor-beam reactions at $E$ and $F$ are zero, the moment in the girder at $E$ increases linearly from zero at $A$ to $+\frac{1}{2} \times 10 = +5$ at $C$.   With the unit load at $D$, $R_{Ay} = +\frac{1}{2}$; the floor-beam reactions at $A$ and $B$ are zero; hence the moment at $E$ equals $+\frac{1}{2} \times 20 = +10$.   As the unit load travels from $D$ to $G$, $R_{Ay}$ varies linearly from $+\frac{1}{2}$ to $-\frac{1}{6}$; the floor-beam reactions at $A$ and $B$ remain at zero; hence the moment at $E$ varies linearly from $+10$ at $D$ to $-\frac{1}{6} \times 20 = -3.33$ at $G$.



(a)

(b)

Moment at $E$

FIG. 6·6

**6·7  Interpretation of Influence Lines for Girders with Floor Systems.**  The four theorems of Art. 6·5 dealing with the use of influence lines are per-fectly general and are applicable to influence lines for girders with floor systems.  Suppose that live loads consisting of a uniform load of 1,000 lb per ft and a single concentrated load of 10,000 lb are applied to the structure of Fig. 6·5a.  To obtain the maximum live shear in panel $BC$, refer to the influence line of Fig. 6·5b.  It is first necessary to locate point $a$ at which this influence line crosses the base line.  Such a point is called a neutral point, since a load applied at this point has no effect on the function under consideration.  This point may be located by similar triangles; its distance from $B$ will be found to be 2 ft.  The maximum positive live shear in panel $BC$ occurs when the uniform load extends from the neutral point to $G$ and when the concen-trated load is at $C$; it is equal to

$$1,000[\tfrac{1}{2}(+\tfrac{2}{3})(48)] + 10,000(+\tfrac{2}{3}) = 22,667 \text{ lb}$$

Maximum negative live shear in this panel occurs when the uniform load extends from $A$ to the neutral point and the concentrated load is at $B$; it has a value equal to $1,000[\tfrac{1}{2}(-\tfrac{1}{6})(12)] + 10,000(-\tfrac{1}{6}) = -2,667$ lb. In Fig. 6·5c, the maximum positive live moment at panel point $E$, due to the same live load, occurs with the uniform load extending over the entire span and with the concentrated load at $E$.  Its value is equal to $1,000[\tfrac{1}{2}(+\tfrac{40}{3})(60)] + 10,000(+\tfrac{40}{3}) = +533,333$ ft-lb.

The foregoing method of computing maximum live shears and
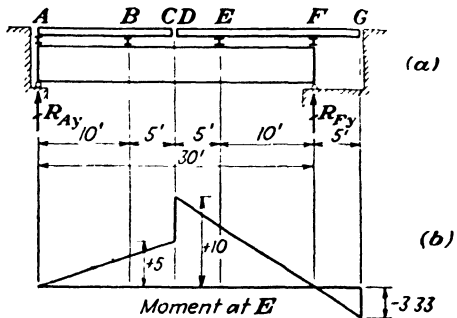
moments, based on locating neutral points and using exact areas under influence lines, is *exact*. The following *approximate* method is of importance, since it often involves less computation and is well suited to efficient organization of computations for complicated structures. In the approximate method, it is assumed that for uniform live load there is acting at each panel point either a full panel load or no panel load whatever, depending on whether the ordinate to the influence line indicates that a load at that panel point increases or decreases the value of the function for which a maximum value is desired.

A full panel load is the maximum possible load that can be applied to a girder by a floor beam. It can occur only when the stringers adjacent to the panel are fully loaded, and it is equal (for panels of equal length) to $wl$, where $w$ is the intensity of the uniform load and $l$ is the length of the panel.

Consider again the structure of Fig. 6·5a acted upon by live loads consisting of a uniform load of 1,000 lb per ft and a single concentrated load of 10,000 lb. For the uniform live load, the full panel load equals $(1,000)(10) = 10,000$ lb. To compute the maximum positive live shear in panel $BC$ by the approximate method, this full panel load is placed at $C$, $D$, $E$, and $F$, since the influence line of Fig. 6·5b has positive ordinates at these panel points. No panel load will be placed at $B$, where the ordinate to the influence line is negative. The concentrated load will, as in the exact method, be placed at $C$. The resultant maximum positive live shear in panel $BC$ is equal to

$$10,000(\tfrac{2}{3} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{6}) + 10,000(\tfrac{2}{3}) = 23,333 \text{ lb}$$

The corresponding value was 22,667 lb by the exact method, so that the result by the approximate method is seen to be slightly on the safe side, *i.e.*, slightly larger than the exact value. The approximate method assumes a full panel load acting at $C$, which could not occur without completely loading stringer $BC$; loading stringer $BC$ would cause a floorbeam reaction at $B$ equal to half a full panel load, which by itself would cause negative shear in panel $BC$. Because the negative shear due to the partial panel load applied at $B$ is neglected in the approximate method, the resultant positive shear computed is necessarily on the safe side. The approximate method of computing maximum values of functions never gives smaller values than does the exact method.

To find the maximum positive live moment at $E$ for the same structure and loading by the approximate method, refer to the influence line of Fig. 6·5c. For the uniform load, a full panel load of 10,000 lb is applied at all intermediate panel points, since all the corresponding ordinates to the influence line are positive. The concentrated load is

placed at $E$.   The maximum positive live moment is given by

$$10{,}000(+{}^{10}\!/_3 + {}^{20}\!/_3 + {}^{30}\!/_3 + {}^{40}\!/_3 + {}^{20}\!/_3) + 10{,}000(+{}^{40}\!/_3)$$
$$= +533{,}333 \text{ ft-lb}$$

This is the same value as that obtained by the exact method.

**6·8   Series of Concentrated Live Loads—Use of Moment Diagram.**   The methods of using the influence line as previously presented apply to uniformly distributed live loads and to single concentrated live loads.   They cannot, however, be used directly when the live load consists of a series of concentrated loads of given magnitude and spacing, such as are actually applied by the wheels of a locomotive or of a series of trucks.   When there is more than one concentrated load, it is not possible, in general, to tell by inspection which of the concentrated

| 10 | | 20 | 20 | 20 | | 20 | 20 | 20 | Loads in kips per wheel |
|---|---|---|---|---|---|---|---|---|---|
| ① | | (2) | (3) | (4) | | ⑤ | ⑥ | ⑦ | Number of load |
| | 10' | | 5' | 5' | 10' | | 5' | 5' | Wheel spacing |
| | 10' | | 15' | 20' | 30' | | 35' | 40' | Distance from load **1** |
| 10 | | 30 | 50 | 70 | | 90 | 110 | 130 | Sum of loads from left |
| 400 | | 1000 | 1500 | 1900 | | 2100 | 2200 | | Sum of moments about load **7** |
| 350 | | 850 | 1250 | 1550 | | 1650 | | | Sum of moments about load **6** |
| 300 | | 700 | 1000 | 1200 | | | | | Sum of moments about load **5** |
| 200 | | 400 | 500 | | | | | | Sum of moments about load **4** |
| 150 | | 250 | | | | | | | Sum of moments about load **3** |
| 100 | | | | | | | | | Sum of moments about load **2** |

FIG. 6·7

loads should be placed at the maximum ordinate of the influence line in order to make the given function a maximum.

The method that should be followed for such a live load is essentially one of trial.   In order to expedite the various trial solutions, it is desirable to organize such an analysis carefully, so as to minimize computations. For a series of concentrated loads, a moment diagram, such as is shown in Fig. 6·7, can be used to advantage.   This particular moment diagram is computed for the seven concentrated loads spaced as shown.   The diagram is practically self-explanatory.   The numbers in the six bottom rows may be explained by a single illustration: The number 1,900 under load 4 and in the horizontal line labeled Sum of moments about load 7 represents the moment about load 7 of loads 1 to 4; thus

$$10(40) + 20(30) + 20(25) + 20(20) = 1{,}900$$

To illustrate the use of the moment diagram, suppose that it is desired to compute the moment at load 3 in the beam of Fig. 6·8, due to the

loading of Fig. 6·7 located as shown in Fig. 6·8. The moment about $B$ of the applied loads is equal to 1,650 (the moment of loads 1 to 5 about load 6) plus 110 (the sum of the loads from 1 to 6), multiplied by 2 (the distance from load 6 to point $B$), whence $1,650 + 110(2) = 1,870$ kip-ft. Dividing this moment by the span of the beam, $R_{Ay}$ is found to equal $1,870/50 = +37.4$ kips. Hence the moment at load 3, working from the forces to the left, is given by $+37.4(28) - 250 = +798$ kip-ft. It is to be noted that the moment of 250 kip-ft which was subtracted is the moment of loads 1 and 2 about load 3.
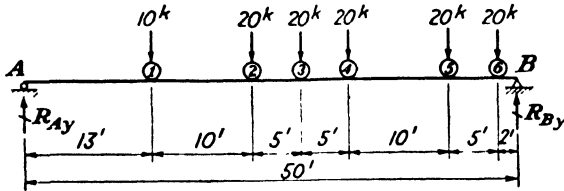


Fig. 6 8

As a second example of the use of the moment diagram, the shear in panel $BC$ of the girder of Fig. 6·9 will be computed for the loads shown acting, these loads being a part of the loading of Fig. 6·7. For the loads so placed, load 1 is not on the span. The girder reaction at $A$ is given by

$$R_{Ay} = \frac{(1,650 - 350) + (110 - 10)3}{36} = +44.5 \text{ kips}$$

The sum of the floor-beam reactions at $A$ and $B$ is equal to

$$20 + \tfrac{5}{9}(20) = 31.1$$

Hence the shear in panel $BC$ equals $+44.5 - 31.1 = +13.4$ kips.
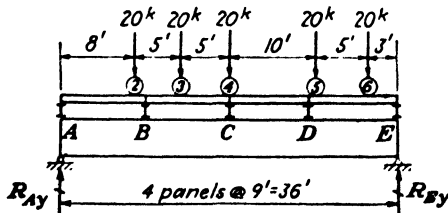


Fig. 6·9

In using the moment diagram it is usually convenient to reproduce it to scale on cardboard and place it in the proper position on a drawing of the structure to be analyzed, which is drawn to the same scale.

**6·9 Series of Concentrated Live Loads—Computation of Maximum Moment.** The computation of maximum moment at a given section in a girder will be illustrated by computing the maximum moment at $C$ of the girder of Fig. 6·10a, due to the live loading corresponding to the moment diagram of Fig. 6·7. The influence line for the

moment at $C$ is first constructed, as shown in Fig. 6·10$b$. The maximum moment at $C$ will occur when one of the concentrated loads is at $C$; the first part of the problem consists in finding out which load should be at $C$ in order to cause this maximum moment.

Before attempting this trial solution, the slope of each portion of the influence line, going from right to left, is first computed. For example, the portion of the influence line for the moment at $C$, which runs from $F$ to $C$, has a slope of $+12\frac{2}{30} = +\frac{2}{5}$.*

Using the moment diagram of Fig. 6·7, place load 1 at $C$. This causes a certain moment at $C$, which, however, will not be computed at this
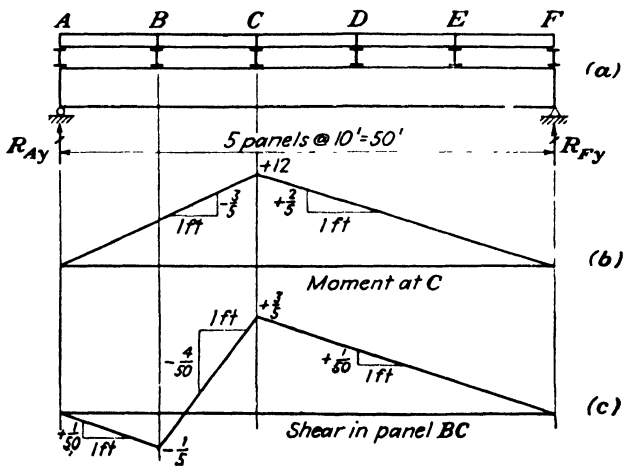


FIG. 6·10

stage of the analysis. Instead, the entire system of loads will be moved to the left until load 2 is at $C$, and computations will be carried out to determine whether the moment at $C$ has been increased or decreased by this change in the position of the loads. To see whether the moment has become larger or smaller, it is convenient to divide the loads under consideration into three groups: (1) those loads which were on the structure before the loads were moved and which remain on the structure after the loads are moved; (2) those loads which were on the structure before the loads were moved but which have passed off the structure after the loads are moved; (3) those loads which were not on the structure before the loads were moved but which are on the structure after the loads

* The ordinates to an influence line for moment may be interpreted to give the moment per pound of applied load and hence are in units of foot-pounds per pound = feet. The increment in ordinates per ft. are therefore in units of ft. per foot: that is, they are nondimensional

are moved. For convenience we shall refer to these three load groups as load group 1, load group 2, and load group 3, respectively.

The following computations determine whether this load move has increased or decreased the moment at $C$. It should be noted that, if a load $P$ moves a distance $d$ and if the slope of the influence line is $m$, the corresponding change in moment equals $Pdm$.

| Load 1 at section; move up load 2 | Increase in moment | Decrease in moment |
|---|---|---|
| Load group 1.... .. Loads 1 to 5 | $80(10)(+\frac{2}{5}) = +320$ | $10(10)(-\frac{3}{5}) = -60$ |
| Load group 2 .. . ... None | 0 | 0 |
| Load group 3 ... .... Load 6 | $20(5)(+\frac{2}{5}) = +10$ | 0 |
| All loads combined ............... | $+360$ | $-60$ |

The net change in moment is $+360 - 60 = +300$ kip-ft, so that a larger moment at $C$ occurs with load 2 at the section (*i.e.*, at $C$) than with load 1. However, it may be that a still larger moment occurs with load 3 at the section. The loads will now be moved to the left until load 3 is at $C$, and computations will be made to determine whether this new movement of loads has increased or decreased the moment at $C$.

| Load 2 at section; move up load 3 | Increase in moment | Decrease in moment |
|---|---|---|
| Load group 1..............All loads | $100(5)(+\frac{2}{5}) = +200$ | $30(5)(-\frac{3}{5}) = -90$ |
| Load group 2. ......... . None | 0 | 0 |
| Load group 3 .. ...... ...None | 0 | 0 |
| All loads combined.... .......... | $+200$ | $-90$ |

Since 200 is greater than 90, the moment has again increased. We shall now find out whether there will be still a further increase if load 4 is moved to the section.

| Load 3 at section; move up load 4 | Increase in moment | Decrease in moment |
|---|---|---|
| Load group 1.. ........ . ...All loads | $80(5)(+\frac{2}{5}) = +160$ | $50(5)(-\frac{3}{5}) = -150$ |
| Load group 2 . .......... None | 0 | 0 |
| Load group 3 ............None | 0 | 0 |
| All loads combined....... ...... ... | $+160$ | $-150$ |

Again, the moment has increased. We shall now move up load 5.

| Load 4 at section; move up load 5 | Increase in moment | Decrease in moment |
|---|---|---|
| Load group 1........Loads 2 to 7 | $60(10)(+\frac{2}{5}) = +240$ | $60(10)(-\frac{3}{5}) = -360$ |
| Load group 2........ Load 1 | 0 | 0 |
| Load group 3........ None | 0 | 0 |
| All loads combined............... | $+240$ | $-360$ |

Note that, although load 1 was considered as on the structure with load 4 at the section, it caused no moment at $C$, so that no change occurred when it passed off the structure. Since 240 is less than 360, moving up load 5 caused a decrease in the moment at $C$. Hence the maximum moment at $C$ occurs with load 4 at $C$. With some experience in moving up loads, one might have foreseen that the maximum moment at $C$ would not occur with load 1 at the section and that it probably would not occur with load 2 at the section. This would have eliminated a portion of the foregoing computations.

With the position of loads causing maximum moment at $C$ known, the value of this moment may now be computed, either from the ordinates of the influence line directly or by using the moment diagram. By the latter procedure,

$$R_{Ay} = \frac{2,200 + 130(10)}{50} = +70 \text{ kips}$$

The moment of the floor-beam reactions at $A$ and $B$ about $C$ is equal to the moment of loads 1, 2, and 3 about load 4, which is 500 kip-ft. Hence the maximum positive live moment at $C$ equals

$$+70(20) - 500 = +900 \text{ kip-ft}$$

**6·10   Series of Concentrated Live Loads—Computation of Maximum Shear.** The foregoing method of moving up loads, based on the use of the influence line, is perfectly general and may be used for any influence line. As a second illustration of its application, the maximum positive shear in panel $BC$ of the structure of Fig. 6·10a, due to the live loading of Fig. 6·7, will be computed. The solution might start by placing load 1 at $C$ (the maximum positive ordinate to the influence line) and moving up load 2 to find out whether the shear in panel $BC$ increases or decreases. This step is scarcely necessary, however, since an examination of the loading and the influence line of Fig. 6·10c will lead to the conclusion, without computations, that this movement will increase the shear in panel $BC$.

| Load 2 at section; move up load 3 | Increase in shear | Decrease in shear |
|---|---|---|
| Load group 1    ..   .   All loads | $100(5)(+\frac{1}{50}) + 10*(5)(+\frac{1}{50}) = +11\ 0$ | $20(5)(-\frac{4}{50}) = -8\ 0$ |
| Load group 2             None | 0 | 0 |
| Load group 3    .    .   None | 0 | 0 |
| All loads combined   .    . | $+11.0$ | $-8\ 0$ |

* This term is for load 1. The negative shear due to this load has decreased, leading to an increase in the positive shear in panel $BC$.

This shows an increase in the positive shear in panel *BC*. Hence we shall move load 4 up to the section.

| Load 3 at section; move up load 4 | Increase in shear | Decrease in shear |
|---|---|---|
| Load group 1.......... All loads | 30(5)(+$\frac{1}{60}$) + 10(5) (+$\frac{1}{60}$) = +9.0 | 40(5)(−$\frac{1}{60}$) = −16.0 |
| Load group 2............None | 0 | 0 |
| Load group 3..... ......None | 0 | 0 |
| All loads combined. ... ........ | +9.0 | −16.0 |

This shows a decrease in the positive shear in panel *BC*. Hence the maximum shear in panel *BC* occurs when load 3 is at *C*. The value of this maximum shear may be obtained as follows, using the moment diagram of Fig. 6·7:

$$R_{Ay} = \frac{2,200 + 130(5)}{50} = +57.0 \text{ kips}$$

The sum of the floor-beam reactions at *A* and *B* equals $10 + {}^{20}\!/_2 = 20$ kips; hence the maximum positive live shear in panel *BC* equals

$$+57.0 - 20.0 = +37.0 \text{ kips}$$

**6·11 Absolute Maximum Live Shear.** The methods which have been given for the computation of maximum shear due to live loads assume that the section or panel in which the shear is to be computed is known. It is often desirable to compute the absolute maximum live shear in a member, *i.e.*, the maximum live shear that can occur at any section in the member. For a simple end-supported beam or girder, absolute maximum live shear will occur at a section immediately adjacent to one of the end reactions. If the beam or girder is not a simple end-supported member, the absolute maximum live shear will occur on one side of one of the reactions. The true value of absolute maximum live shear can be determined only by computing the maximum live shear at each such section.

**6·12 Absolute Maximum Live Moment.** Similarly, the methods which have been given for the computation of maximum moment due to live loads assume that the section at which maximum live moment is to be computed is known. It is often necessary to compute the absolute maximum live moment for a beam or girder. For a simple end-supported beam, this occurs at mid-span for either a uniform live load or a single concentrated live load. For a simple end-supported girder with a floor system, absolute maximum live moment occurs at the panel point nearest the center of the span. For a girder wholly or partly cantilevered, abso-

lute maximum live moment is likely to occur at a reaction.   If the section where absolute maximum live moment occurs cannot be definitely identified by inspection, it is necessary to compare maximum moments computed for various sections where absolute maximum live moment is likely to occur.

A special case of importance consists in determining the absolute maximum live moment due to the action of a series of concentrated live loads on an end-supported beam, as shown in Fig. 6·11.   The moment curve for a series of concentrated loads is a series of straight lines intersecting at the positions of the loads, so that absolute maximum live moment must occur directly beneath one of the loads. Two questions must be answered: (1) Under which load does absolute maximum live moment occur?   (2) What is the position of this load when absolute maximum live moment occurs?
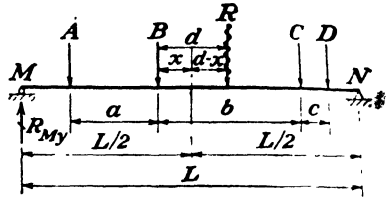


FIG. 6·11

The answer to the first question must often be determined by trial, but the second question is subject to direct analysis.   Assume that in Fig. 6·11 the absolute maximum live moment will occur under load $B$. Let the distance from the center of the span to load $B$ be denoted by $x$ and the distance from load $B$ to the resultant $R$ of all the loads $A$, $B$, $C$, and $D$ be denoted by $d$.   We wish to determine the value of $x$ that will make the moment at load $B$ a maximum.   The value of $R_{My}$ may be determined by taking moments about $N$ and considering the resultant force $R$ rather than the actual loads $A$, $B$, $C$, and $D$.   Thus

$$R_{My} = \frac{R\left(\frac{L}{2} + x - d\right)}{L} = \frac{R}{2} + \frac{Rx}{L} - \frac{Rd}{L}$$

Denoting by $M_B$ the moment under load $B$,

$$M_B = R_{My}\left(\frac{L}{2} - x\right) - Aa = \left(\frac{R}{2} + \frac{Rx}{L} - \frac{Rd}{L}\right)\left(\frac{L}{2} - x\right) - Aa$$

$$= \frac{RL}{4} - \frac{Rd}{2} - \frac{Rx^2}{L} + \frac{R\,xd}{L} - Aa$$

For a maximum value of $M_B$,

$$\frac{dM_B}{dx} = -\frac{2Rx}{L} + \frac{Rd}{L} = 0$$

whence $x = d/2$.

We may therefore conclude that *the maximum moment directly beneath one of a series of concentrated live loads that are applied to a simple end-supported beam occurs when the center of the span is halfway between that particular load and the resultant of all the loads on the span.*

If there are only two concentrated loads to consider, the absolute maximum live moment will occur under the heavier of the two loads. Such a case is illustrated in Fig. 6·12, where the distance from the 10-kip load to the resultant $R$ of the two loads equals $(5 \times 12)/15 = 4$ ft. For absolute maximum moment, the 10-kip load is placed 2 ft from the center of the span, and thus the resultant $R$ is 2 ft on the other side of the span center. At this point one should check to see whether or not both loads are on the span. If not, the absolute maximum moment occurs at mid-span when the heavy load is at the center of the span. For this case, both loads are on the span. The absolute maximum live moment occurs directly beneath the 10-kip load and is given by
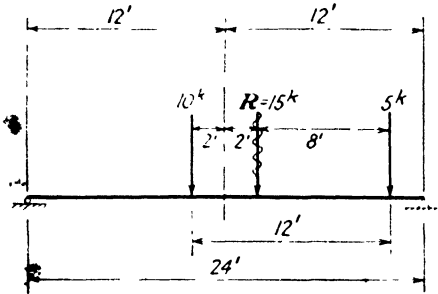


FIG. 6·12

$$M = \frac{15(12 - 2)^2}{24} = +62.5 \text{ kip-ft}$$

If there are more than two concentrated loads, it may not be possible to tell by inspection under which load the absolute maximum live moment will occur. It will usually occur under a large load near the center of the group of loads. The maximum moment that can occur under each of the loads may be determined by the foregoing method, and the largest of these moments will be the absolute maximum live moment.

**6·13 Influence Lines for Trusses—General.** Influence lines may be constructed for the stresses in truss members and are important in determining the location of live loads leading to maximum stress in truss members, as well as for computing the actual values of these maximum stresses. The same general procedure as that used for constructing influence lines for beams and girders is applicable to trusses. It is always possible to compute the ordinate to the influence line for a unit load at each panel point of the truss. Usually the stringers act as end-supported beams between the floor beams, so that the influence line is a straight line between panel points. As was the case with beams and girders, it is often possible to reduce the amount of computation by recognizing

the fact that the influence line is a straight line for several successive panels.

Once the influence line has been constructed for the stress in a given truss member, the interpretation of the curve with respect to loading criteria and stress analysis is identical with that for beams and girders.

Influence lines for trusses are drawn to correspond to a unit load traveling across the *loaded chord*, i.e., the chord containing the panel points at which the live load is applied.

**6·14   Influence Lines for a Pratt Truss.**   The construction and use of influence lines for trusses will be illustrated by a consideration of
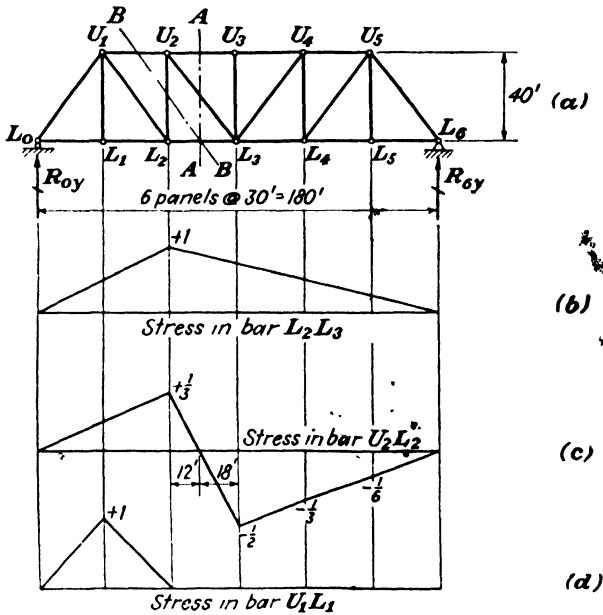


Fig. 6·13

the Pratt truss of Fig. 6·13a.   To construct the influence line for a chord member, such as bar $L_2L_3$, take moments about $U_2$ of the forces acting on one side of section $A$-$A$.   With the unit load to the left of the section, the tension in $L_2L_3$ equals $R_{6y}$ multiplied by 120 and divided by the truss height of 40 ft and hence is directly proportional to $R_{6y}$.   Since $R_{6y}$ varies linearly as a unit load travels from $L_0$ to $L_2$, the influence line is a straight line from zero at $L_0$ to $+\frac{1}{3}(120_{40}) = +1$ at $L_2$.   Had this linearity not been recognized, the value of the ordinate to the influence line at $L_1$ might have been computed independently and would have been found to equal $+\frac{1}{6}(120_{40}) = +\frac{1}{2}$.   With the unit load at $L_2$ or at any point to the right of section $A$-$A$, the tension in $L_2L_3$ equals $R_{0y}$ multiplied by

60 and divided by 40. Since $R_{0y}$ varies linearly as a unit load travels from $L_2$ to $L_6$, the influence line is a straight line from $+\frac{2}{3}(^{60}\!/_{40}) = +1$ at $L_2$ to zero at $L_6$. This influence line is shown in Fig. 6·13b, where tension is plotted above the base line.

The construction of an influence line for stress in a web member will be illustrated by Fig. 6·13c, where the vertical bar $U_2L_2$ is considered. When a unit load is to the left of section B-B, the tension in this member equals the reaction $R_{6y}$. Hence the influence line is a straight line from zero at $L_0$ to $+\frac{1}{3}$ at $L_2$. When the unit load is to the right of section B-B, the compression in $U_2L_2$ equals the reaction $R_{0y}$. Hence the influence line is a straight line from $-\frac{1}{2}$ at $L_3$ to zero at $L_6$, negative stress values being plotted below the base line and indicating compression. Between panel points $L_2$ and $L_3$, the influence line is a straight line, assuming that the stringers are constructed so that they act as end-supported beams between the panel points $L_2$ and $L_3$.

For both the bars that have been considered, the influence lines extend over the entire length of the truss. Such bars are called *primary* truss members. Consider now the vertical member $U_1L_1$, the influence line for which is shown in Fig. 6·13d. Upon applying the method of joints to $L_1$, it is seen that the stress in this bar is zero if the unit load is applied to any panel point other than $L_1$, in which case it equals $+1$. Such a member of the unit load, which is stressed for certain positions only, is called a *secondary* truss member.

To obtain maximum live stresses in truss members by use of the influence line, no new principles are involved. For example, suppose that it is desired to determine the maximum compression in the vertical $U_2L_2$ due to a uniform live load of 2,000 lb per ft and a single concentrated live load of 15,000 lb.

By the exact method, the neutral point of the influence line of Fig. 6·13c is determined by similar triangles as being located 12 ft to the right of $L_2$. For maximum compression in $U_2L_2$, the uniform load should extend from the neutral point to $L_6$, while the concentrated load should be placed at $L_3$. The value of this maximum compression is given by

$$2,000(\tfrac{1}{2})(108)(-\tfrac{1}{2}) + 15,000(-\tfrac{1}{2}) = -61,500 \text{ lb}$$

By the approximate method, the panel loads for the uniform load equal $2,000 \times 30 = 60,000$ lb and are placed at $L_3$, $L_4$, and $L_5$. The concentrated load is placed at $L_3$. The maximum live compression is given by

$$60,000(-\tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{6}) + 15,000(-\tfrac{1}{2}) = -67,500 \text{ lb}$$

**6·15 Influence Lines for Truss with K Diagonals.** For simple cases, such as that of the Pratt truss of Art. 6·14, it is relatively simple to

eliminate as many computations as possible in the construction of influence lines by recognizing the fact that certain portions of the influence line are linear over several successive panels.   For more complicated trusses, it is often necessary either (1) to compute ordinates for each successive panel point or (2) first to construct influence lines for members other than that under consideration and use the data thus obtained in constructing the influence line actually desired.   This latter procedure may be illustrated by considering the diagonal $U_2M_3$ of the truss of
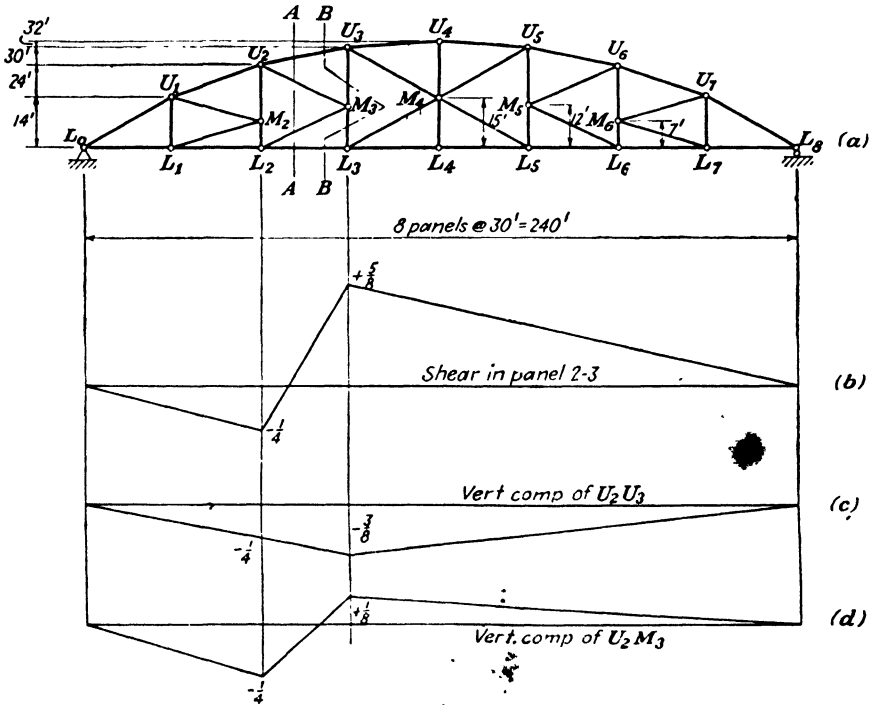


Fig. 6·14

Fig. 6·14a, which·has K-type diagonals and for which the top–chord panel points lie on a parabola.

A consideration of joint $M_3$ shows that the horizontal components of the stresses in bars $U_2M_3$ and $L_2M_3$ are always equal in magnitude but opposite in character.   Since the slopes of these two bars are the same, the vertical components of these stresses are likewise equal in magnitude and opposite in character: thus they act in the same direction when holding in equilibrium the vertical forces applied to that portion of the structure on one side of section $A$-$A$.   The top chord $U_2U_3$ has a vertical

component of stress that must also be considered in the foregoing condition of equilibrium.

The influence line for the total shear in panel 2-3 is first drawn, as shown in Fig. 6·14b. Next, the influence line for the vertical component of the stress in bar $U_2U_3$ is constructed, with stresses in that member determined by taking moments about $L_3$ of the forces acting on one side of section B-B. This influence line is found to be a triangle with its apex at panel point 3, where the ordinate is equal to $-(\frac{3}{8})(^{150}\!\!\!/_{30})(\frac{6}{30}) = -\frac{3}{8}$.

Upon applying $\Sigma F_y = 0$ to that portion of the truss to the left of section A-A, the tensile vertical component of the stress in $U_2M_3$ equals half the sum of the positive shear in panel 2-3 and the tensile vertical component in $U_2U_3$. Thus each ordinate to the influence line for the vertical component of stress in $U_2M_3$ equals half the algebraic sum of the ordinates, at the same section, to the influence lines of Figs. 6·14b and c. Since these two influence lines change directions at $L_2$ and $L_3$ only, the influence line for the vertical component of stress in $U_2M_3$ will change direction at those panel points only. The critical ordinates, i.e., the ordinates where the direction of the resultant influence line changes, will be as follows:

At $L_2$, $[-\frac{1}{4} + (-\frac{1}{4})]\frac{1}{2} = -\frac{1}{4}$     At $L_3$, $[+\frac{5}{8} + (-\frac{3}{8})]\frac{1}{2} = +\frac{1}{8}$

The resultant influence line for the vertical component of the stress in bar $U_2M_3$ is shown in Fig. 6·14d.

**6·16   Maximum Stress in Truss Member Due to Series of Concentrated Live Loads.**   Once the influence line has been constructed
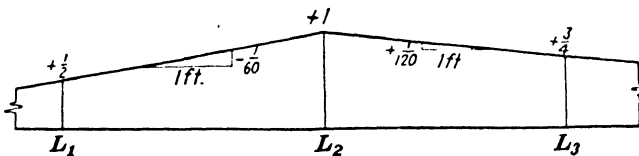


Fig. 6·15

for stress in a truss member, the position in which a series of concentrated live loads should be placed, in order to make a given character of stress in that member have a maximum value, may be determined by moving up loads in the manner already described for girders. To illustrate, the position of the loads of Fig. 6·7 that gives maximum tension in bar $L_2L_3$ of the truss of Fig. 6·13a will be determined. The necessary portion of the influence line for this member (see Fig. 6·13b for the complete influence line) is reproduced in Fig. 6·15.

The computations are as follows (load 1 at $L_2$ does not gives maximum, by inspection):

                                    Increase in tension      Decrease in tension

Load 2 at $L_2$; move up load 3

   Load group 1—All loads   $100(5)(+\frac{1}{4}20) = +5\frac{9}{12} > 30(5)(-\frac{1}{60}) = -\frac{5}{6}$

Load 3 at $L_2$; move up load 4

   Load group 1—All loads   $80(5)(+\frac{1}{4}20) = +1\frac{9}{3} < 50(5)(-\frac{1}{60}) = -2\frac{5}{6}$  ·
                        $\therefore$ Maximum tension in $L_2L_3$ occurs with load 3 at $L_2$

   To compute the value of this maximum tension, two different pro-
cedures are suggested.
   *Method* 1 (based on moment diagram of Fig. 6·7):

$$R_{0y} = \frac{2,200 + 130(95)}{180} = 80.8 \text{ kips}$$

Hence the stress in $L_2L_3$ is given by

$$\frac{+80.8(60) - 250}{40} = 115.0 \text{ kips}$$

   *Method* 2 (based on computing ordinates to the influence line at each
panel point):

| Panel point | Floor-beam reaction | | Influence-line ordinate | Increment of stress in $L_2L_3$ |
|---|---|---|---|---|
| 1 | $10(\frac{3}{6}) + 20(\frac{1}{6})$ | $= +8\ 3$ | $+\frac{1}{2}$ | $+4\ 2$ |
| 2 | $10(\frac{3}{6}) + 20(\frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{3}{6} + \frac{1}{6})$ | $= +78,3$ | $+1$ | $+78.3$ |
| 3 | $20(\frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{5}{6})$ | $= +43\ 3$ | $+\frac{3}{4}$ | $+32\ 5$ |
| | | | Total stress in $L_2L_3 = \Sigma = +115\ 0$ kips | |

   **6·17   Influence Tables.**   It is often advantageous to express
influence data in the form of influence tables rather than in the form of
curves.   The influence table (Table 6 1) refers to the truss of Fig.
6·13a.   It gives the stress in each bar of the truss due to a unit load at
each panel point.   Stresses in bars $L_2L_3$, $U_2L_2$, and $U_1L_1$ were taken
directly from Figs. 6·13b, c, and d, respectively.   Stresses for other bars
may be checked by the student.
   In utilizing an influence table to compute maximum live stresses by
the approximate method, it is convenient to prepare a second table that
is a summary of the influence table, as illustrated by Table 6·2.
   In the summary of the influence table, the sum of the positive ordi-
nates for a member is obtained by adding up, for that member, all the
positive values from the influence table.   The product of this sum and

the panel load for uniform live load equals the maximum tension in that member due to uniform live load.

Table 6·1    Influence Table for Truss of Fig. 6·13a

| Bar | Stress due to unit load at | | | | | | |
|---|---|---|---|---|---|---|---|
| | $L_0$ | $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | $L_6$ |
| $L_0L_1$ | 0.000 | +0.625 | +0.500 | +0 375 | +0 250 | +0.125 | 0.000 |
| $L_1L_2$ | 0 000 | +0.625 | +0.500 | +0 375 | +0.250 | +0.125 | 0.000 |
| $L_2L_3$ | 0.000 | +0.500 | +1.000 | +0 750 | +0.500 | +0.250 | 0.000 |
| $L_0U_1$ | 0.000 | −1.041 | −0.833 | −0 625 | −0.417 | −0 208 | 0.000 |
| $U_1U_2$ | 0.000 | −0.500 | −1.000 | −0.750 | −0.500 | −0.250 | 0.000 |
| $U_2U_3$ | 0 000 | −0.375 | −0.750 | −1 125 | −0.750 | −0.375 | 0.000 |
| $U_1L_2$ | 0.000 | −0.208 | +0.833 | +0 625 | +0.417 | +0.208 | 0.000 |
| $U_2L_3$ | 0.000 | −0.208 | −0.417 | +0 625 | +0.417 | +0.208 | 0.000 |
| $U_1L_1$ | 0.000 | +1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $U_2L_2$ | 0.000 | +0.167 | +0.333 | −0 500 | −0.333 | −0.167 | 0.000 |
| $U_3L_3$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

The sum of the negative ordinates for a member is obtained by adding up, for that member, all the negative ordinates from the influence table. The product of this sum and the panel load for uniform live load equals the maximum compression in that member due to uniform live load.

The sum of all the ordinates for a member is obtained by adding algebraically, for that member, the sum of the positive ordinates and the sum of the negative ordinates. If the dead panel loads are equal, the product

Table 6·2    Summary of Influence Table for Truss of Fig. 6·13a

| Bar | Sum of ordinates | | | Max. ordinates | | Loaded length for | |
|---|---|---|---|---|---|---|---|
| | Positive | Negative | All | Positive | Negative | Tension | Compression |
| $L_0L_1$ | +1.875 | 0.000 | +1.875 | +0.625 | 0.000 | 180 | 0 |
| $L_1L_2$ | +1.875 | 0.000 | +1.875 | +0.625 | 0.000 | 180 | 0 |
| $L_2L_3$ | +3.000 | 0.000 | +3.000 | +1.000 | 0.000 | 180 | 0 |
| $L_0U_1$ | 0.000 | −3.124 | −3.124 | 0.000 | −1.041 | 0 | 180 |
| $U_1U_2$ | 0.000 | −3.000 | −3.000 | 0.000 | −1.000 | 0 | 180 |
| $U_2U_3$ | 0.000 | −3.375 | −3.375 | 0.000 | −1.125 | 0 | 180 |
| $U_1L_2$ | +2.083 | −0.208 | +1.875 | +0 833 | −0.208 | 144 | 36 |
| $U_2L_3$ | +1.250 | −0.625 | +0.625 | +0.625 | −0.417 | 108 | 72 |
| $U_1L_1$ | +1.000 | 0.000 | +1.000 | +1.000 | 0.000 | 60 | 0 |
| $U_2L_2$ | +0.500 | −1.000 | −0.500 | +0.333 | −0.500 | 72 | 108 |
| $U_3L_3$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0 | 0 |

of this sum and the dead panel load equals the dead stress for any member except the verticals. For verticals, this product must be corrected for that portion of the dead load applied at the top-chord panel point.

The maximum positive ordinate for a member is obtained by choosing, for that member, the maximum positive value from the influence table. The product of this value and the concentrated live load equals the maximum tension in that member due to the concentrated live load.

The maximum negative ordinate for a member is obtained by choosing, for that member, the maximum negative value from the influence table. The product of this value and the concentrated live load equals the maximum compression in that member due to the concentrated live load.

The procedure for determining the total dead plus live stress in any member from the summary of the influence table will be illustrated by considering bar $U_2L_3$ and the following loads:

Dead load = 2,000 lb per ft　　　Uniform live load = 1,000 lb per ft
Concentrated live load = 10,000 lb

|  | Max. tension, kips | Max. compression, kips |
|---|---|---|
| Dead ........................... | $60(+0.625) = +37.5$ | $+37.5$ |
| Live: |  |  |
|   Uniform.... ................... | $30(+1.250) = +37.5$ | $30(-0.625) = -18.8$ |
|   Concentrated..................... | $10(+0.625) = + 6 3$ | $10(-0.417) = - 4.2$ |
|     Total dead plus live stress... . | $+81 3$ | $+14.5$ |

Hence there is no stress reversal in this case.

**6·18 Loaded Length.** The loaded length is the length of that portion of a structure loaded with uniform live load to produce maximum live stress of a given character. This loaded length may be determined from the influence line for the member under consideration. For example, consider the girder of Fig. 6·5. The loaded length for positive moment at $E$ is 60 ft. The loaded length for positive shear in panel *BC* is 48 ft; for negative shear in the same panel it is 12 ft. Loaded length is the controlling parameter in many formulas for impact.

When equivalent live loads are used in place of a series of concentrated live loads, the concentrated live load is not physically dissociated from the uniform live load. Hence it is correct to use the same loaded length for the concentrated live load that is used for the uniform live load, *i.e.*, the impact factor for both components of the equivalent live load is based on the loaded length corresponding to the uniform load.

In computing loaded length, it is often permissible to estimate the location of the neutral point to the nearest half panel. This saves computations and does not introduce important errors into total computed stresses (dead + live + impact).

If desired, two columns may be added to the influence-table summary (Table 6·2) in which loaded lengths for tension and compression are given. The inclusion of these data in the summary of the influence table is helpful for computing impact stresses in the various members.

**6·19 Alternate Approach for Determination of Influence Lines.** An alternate approach to the construction of influence lines may be made by introducing an imaginary distortion into the truss member or girder section under consideration. This method is of more interest than use in connection with statically determinate structures, but it is of
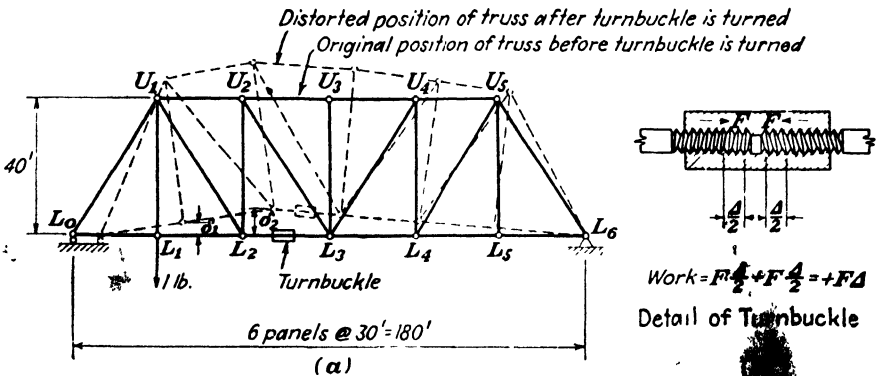


FIG. 6·16

importance in constructing influence lines for statically indeterminate structures, by both analytical and structural-model procedures.

The method will be illustrated by a consideration of the truss of Fig. 6·16a, in which it is desired to construct the influence line for member $L_2L_3$. Suppose a turnbuckle is imagined as inserted in member $L_2L_3$. If this turnbuckle were turned so that bar $L_2L_3$ were shortened by a small amount $\Delta$, the structure would take the shape indicated by the dotted position of the truss. Since the truss is statically determinate, no elastic restraint would be encountered as the turnbuckle is turned; and if there were no loads on the structure, there would be no stress in any of the members. Suppose, however, that a unit load is imagined as being applied at any particular panel point such as $L_1$. Let $F$ be the tension in $L_2L_3$ due to this unit load. Then, when the turnbuckle is taken up, it would be necessary to do work on the structure, to an amount equal to $+F\Delta$, since the turnbuckle would exert a tensile force equal to $+F$ on

bar $L_2L_3$ at each end of the turnbuckle, and the total distance traveled by the two forces $F$, as shown in Fig. 6·16b, would equal $\Delta$. The unit load at $L_1$ would move vertically through a distance $\delta_1$ and while so moving would do work on the structure equal to $(-1)(\delta_1)$. The negative sign is introduced because the movement $\delta_1$ is in a direction opposite to that of the unit load.

The stresses in the structure will not change during this distortion, so that the elastic strain energy stored in the members of the truss due to the fact that they are stressed will remain constant. Since the strain energy remains constant, there has been no net work done on the truss during the distortion, when all the forces are considered. Expressed in symbols,

$$+F(\Delta) - (1)(\delta_1) = 0$$

where

$$F = +\frac{\delta_1}{\Delta}$$

Had the unit load been placed at any other panel point $L_n$, similar considerations would lead to the conclusion that

$$F = +\frac{\delta_n}{\Delta}$$

But $F$ is the stress in bar $L_2L_3$ due to a unit load at $L_n$; hence it is the ordinate at panel point $L_n$ to the influence line for the stress in bar $L_2L_3$. The value of $\Delta$ is independent of the panel point considered. We may therefore conclude that the bottom chord of the truss as shown by the dotted lines of Fig. 6·16a has the *shape* of the influence line for tension in $L_2L_3$. The *scale* of the influence line is determined by dividing the deflections $\delta_n$ by the imposed distortion $\Delta$. If $\Delta$ is made equal to unity, the values $\delta_n$ are themselves numerically equal to the ordinates to the influence line.

To use fully this method of constructing influence lines requires either a model of the structure or a knowledge of the methods for computing deflections. However, even without computing deflections, it is often possible to visualize the shape that a structure would take and thus arrive at the shape of the influence line. In this manner, the critical points at which influence-line ordinates must be computed may be located. The position of live loads to cause maximum live stresses can often be told from the shape of the influence line, without actually computing the values of the critical ordinates.

For example, the dotted lines in Fig. 6·17 show the position the truss would take if the diagonal $U_2L_3$ were shortened. Since the bottom chord

would change slope only at $L_2$ and $L_3$, ordinates to the influence line for stress in $U_2L_3$ need be computed at these two panel points only. It may be concluded, moreover, without actually computing these ordinates, that by the approximate method the maximum live tension in $U_2L_3$ will occur with panel loads for uniform live load at $L_3$, $L_4$, and $L_5$ and with the concentrated live load at $L_3$. The actual value of this maximum live stress can be computed by the equations of statics, with the loads in the foregoing positions.
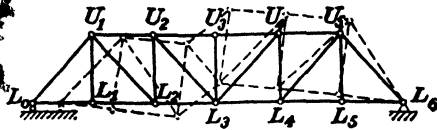


FIG. 6·17

**6·20   Problems for Solution.**

**Problem 6·1**   Referring to Fig. 6·18, construct the influence lines for (a) shear at $a$; (b) moment at $a$; (c) reaction at $b$.

**Problem 6·2**   Referring to Fig. 6·18 and using a live load consisting of a uniform load of 500 lb per ft and a concentrated load of 5,000 lb, compute (a) the maximum upward reaction at $b$; (b) the maximum positive and negative moments at $a$; (c) the maximum positive and negative shears at a section just to the right of the support at $b$.   (d) If the dead load is 1,000 lb per ft, compute, from the influence line, the maximum moment at $a$ due to dead plus live load.
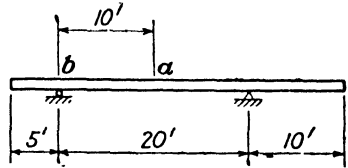


FIG. 6·18

**Problem 6·3**   For the structure of Fig. 6·19, construct influence lines for (a) shear in panel $AB$; (b) moment at panel point $C$.
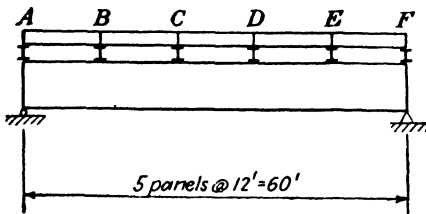
**Problem 6·4**   Compute, by both the exact and the approximate methods, the maximum live shear in panel $AB$ and the maximum live moment at panel point $C$ of the structure of Fig. 6·19, due to a uniform live load of 1,200 lb per ft and a single concentrated live load of 18,000 lb.



5 panels @ 12'=60'

FIG. 6·19

**Problem 6·5**   For the structure of Fig. 6·20, construct influence lines for (a) shear in panel $DE$; (b) moment at panel point $E$.
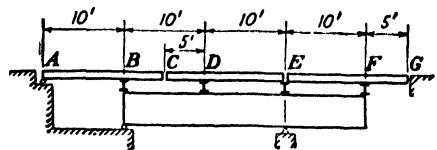
**Problem 6·6**   Compute, by the approximate method, the maximum live shear in panel $DE$ and the maximum live moment at panel point $E$ of Fig. 6·20, due to the live load of Prob. 6·4.



FIG. 6·20

**Problem 6·7**  Construct a moment diagram for the loading of Fig. 6·21.

**Problem 6·8**  Using the moment diagram for the loading of Fig. 6 21 and the structure of Fig. 6·19, determine (a) the left girder reaction when load 3 is at C; (b) the left girder reaction when load 5 is at C; (c) the shear in panel AB when load 3 is at E; (d) the moment at panel point D when load 2 is at D.

**Problem 6·9**  For the loading of Fig. 6·21, compute the maximum live moment at panel point B of the girder of Fig. 6·10a.

**Problem 6·10**  For the loading of Fig. 6·21, compute the maximum positive shear in panel CD of the girder of Fig. 6·10a.



(All loads are in kips)

FIG. 6·21

**Problem 6·11**  Compute the absolute maximum shear in the beam of Fig. 6·3a, due to a live load of 1,000 lb per ft, plus a single concentrated live load of 5,000 lb.

**Problem 6·12**  Compute the absolute maximum moment due to two concentrated live loads of 10 kips each, spaced at 10 ft and acting on an end-supported beam with a span of 20 ft.

**Problem 6·13**  Compute the absolute maximum moment due to four concentrated live loads of 12 kips each, spaced at 6 ft each and acting on an end supported beam with a span of 25 ft.

**Problem 6·14**  Compute the maximum live stress in bars a and b of the truss of Fig. 6·22, due to a uniform live load of 750 lb per ft. Consider both tension and compression.
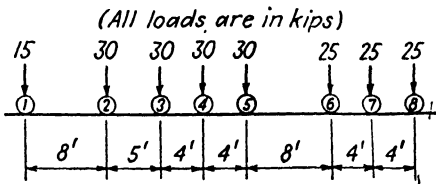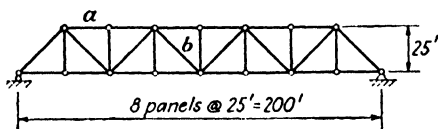


FIG. 6·22

**Problem 6·15**  The top-chord panel points of the truss of Fig. 6·23 lie on a parabola.   Draw influence lines for (a) horizontal component of the stress in bar a; (b) stress in bar b; (c) stress in bar c; (d) stress in bar d.

**Problem 6·16**  Construct an influence line for the stress in bar $U_2M_2$ of the truss of Fig. 6·14a.

**Problem 6·17**  Compute the maximum stress in bar a of the truss in Fig. 6·23, due to the loading of Fig. 6·21.
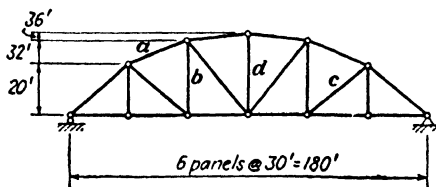


FIG. 6·23

**Problem 6·18**  Prepare an influence-table summary of bars a, b, c, and d of the truss of Fig. 6·23.   Include loaded lengths in the summary.

# CHAPTER 7

## BRIDGE AND ROOF TRUSSES

**7·1 Introduction.** Loadings for trusses are discussed in Chap. 1; stress analysis for trusses is treated in Chaps. 4 and 5, by mathematical and graphical procedures, respectively; influence lines for trusses and the determination of maximum live stresses are considered in Chap. 6. In this chapter, we shall bring together the ideas contained in the earlier chapters and apply these concepts to the broader aspects of truss analysis. In addition to considering the general analysis of a typical roof truss and a typical bridge truss, we shall consider some further specific matters of importance, such as stress reversal and the effect of counters. Attention will then be given to the effect of skewing a bridge with respect to its
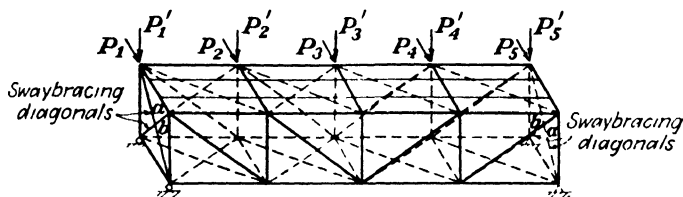


Fig. 7·1

abutments. Finally, a brief consideration will be given to movable bridges.

The trusses considered will be analyzed as planar structures, but it should be understood that actually they are portions of three-dimensional frameworks. To see why this procedure is permissible, refer to the deck bridge of Fig. 7·1, in which only the lateral loads $P_1$, $P_2$, . . . , $P_5$ will be assumed as acting, these loads lying in the plane of the top-chord lateral system. Suppose that a horizontal plane is passed through the structure at some elevation between the top and bottom chords. Considering the isolated part of the structure above this plane, it will be seen that the horizontal components of the stresses in the sway-bracing diagonals at the ends of the structure must hold the top-chord lateral system, acted upon by the lateral loads, in equilibrium.[1] These horizontal components provide the end reactions on the top-chord lateral system,

---

[1] This statement is not completely correct, because under an unsymmetrical system of lateral loads, the horizontal components of the stresses in the diagonals of the main trusses would likewise assist in furnishing the reactions to the top-chord lateral system.

which may therefore be analyzed as a planar truss. The vertical components of the stresses in the sway-bracing diagonals must be held in equilibrium by the stresses in the end verticals of the main vertical trusses. Thus the sway bracing in each end acts as a planar truss that transfers a reaction on the top-chord lateral system to the foundation.

Next, consider the same structure, with only the vertical loads $P'_1, P'_2, \ldots, P'_5$ assumed to be acting, these loads lying in the plane of the rear vertical truss. Suppose that a vertical plane, parallel to the main vertical trusses, is passed through the structure between these trusses. The sway-bracing diagonals at each end will be the only members cut that are capable of carrying vertical components of stress. If the diagonals a are capable of carrying compression, they will be stressed, and this will affect the stresses in the end verticals of the rear vertical truss. This effect is of secondary importance, however, so that it is permissible and on the safe side to analyze the rear vertical truss as a planar truss acted upon by the loads $P'_1, P'_2, \ldots, P'_5$.

If additional sway bracing is introduced at intermediate panel points, the situation becomes more complicated. As the panel points of the loaded vertical truss deflect vertically, the sway bracing at each intermediate panel point acts in a manner such that the corresponding panel point on the unloaded vertical truss must also undergo a certain amount of vertical movement. This produces what are called *participating stresses* in the unloaded vertical truss. However, if both vertical trusses have the same loading, there would, because of symmetry, be no participating stresses of this type. In an actual bridge, the vertical loads on the two main vertical trusses will not always be the same, but they are usually so nearly the same that the stresses in the diagonals of sway bracing at intermediate panel points may be assumed as zero. Each main vertical truss can then be analyzed separately as a planar structure.

**7·2  General Analysis of a Roof Truss.** The general analysis of a roof truss includes, not only the computation of stresses in each member due to the various types of load that must be supported by the truss, but the combination, for each truss member, of the stresses due to each type of loading, in a manner such that the maximum stress that can result from the combined effects of the different types of loading is obtained. To illustrate this procedure, consider the wall-supported roof truss of Fig. 7·2, which will be taken as an intermediate truss of a series of trusses spaced at 20 ft center to center. It will be assumed that the framing of the roof is such that loads from the roof are applied to the truss at the top-chord panel points only. The roofing, including the purlins that support the roofing between trusses, will be assumed to weigh 4.3 lb per sq ft of roof area; the truss itself will be assumed

to weigh 75 lb per horizontal foot, with the weight equally divided between the top- and bottom-chord panel points. The snow load will be taken as 20 lb per sq ft projected horizontal area of the roof. The ice load will be taken as 10 lb per sq ft projected horizontal area of the roof. Wind loads shall be in accordance with the recommendations of the ASCE report[1] discussed in Art. 1·11 and shall be based on a maximum
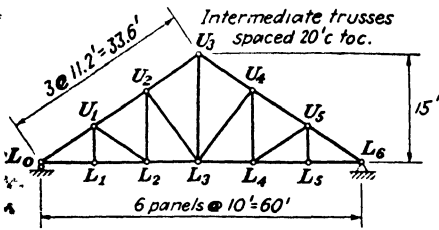


FIG. 7·2

wind velocity of 100 mph. The analysis will be limited to the following combinations of loads: (1) dead plus snow over the entire roof; (2) dead plus wind plus snow on leeward side;[2] (3) dead plus ice over the entire roof plus wind. It is to be noted that the wind can blow from either the right or the left.

The dead stresses will first be computed. For the bottom chord, the dead panel loads are given by $7\frac{1}{2}(10) = 375$ lb; for the top chord, the dead panel loads are given by $375 + 4.3(11.2)(20) = 1,345$ lb. These panel loads, together with the dead stresses they produce, are shown in Fig. 7·3. Since the dead stresses are symmetrical about the center line of the truss, only half the truss is shown. The computations leading to the determination of the dead stresses are omitted, since they involve no new procedures.

Loading condition 1 calls for snow over the entire roof; loading condition 2 calls for snow on the leeward side only. We shall first compute the truss stresses for snow on the leeward side only; because the truss is symmetrical, we may then determine the stresses due to snow over the entire roof by the principle of superposition, thus avoiding a complete second stress analysis. The snow panel load is given by

$$20 \times 10 \times 20 = 4,000 \text{ lb}$$

[1] Wind Bracing in Steel Buildings, *Proc. ASCE*, March, 1936, p. 397.

[2] This combination of loads is considered in the present discussion because it is one frequently investigated. It corresponds to a reasonable condition if wind loads are computed by a formula such as that by Duchemin (see footnote, Art. 1·11), since the snow might be likely to be blown from the roof by the pressure on the windward slope but remain on the leeward slope where there is neither pressure nor suction. When the recommendations of the ASCE report are followed, however, this particular load combination does not appear to be so likely to occur, since one may encounter, as in this particular problem, suction on both slopes, with the greater suction on the leeward slope. With this condition of wind load, the snow, if it remained on the roof at all, would appear more likely to remain on the windward side. However, it is always possible that snow, having fallen previous to the action of the wind, would have crusted over in a manner such as to act as assumed for loading condition 2.

With the snow on the leeward side only, the snow panel load at $U_3$ equals only half this value. These panel loads, together with the stresses they produce, are shown in Fig. 7·4. Since this is an unsymmetrical loading condition, stresses will be computed for the entire truss.



STRESSES DUE TO DEAD LOAD
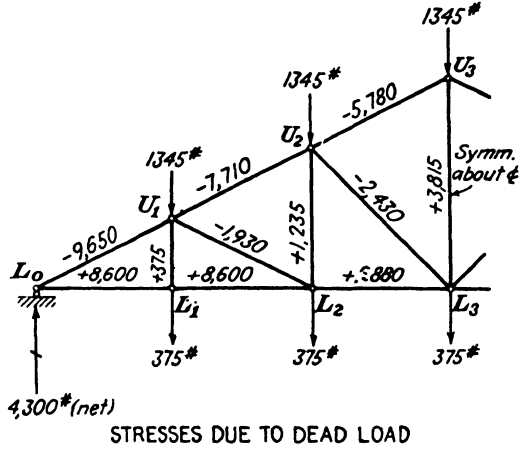
Fig. 7·3

To obtain the stresses due to full snow load, we may use the principle of superposition, as indicated by Fig. 7 5, since the stresses **due** to loading $a$ (as already computed in Fig. 7·4), superimposed upon the stresses due to loading $b$ (which can be obtained from Fig. 7·4 by symmetry), equal



STRESSES DUE TO SNOW ON LEEWARD SIDE
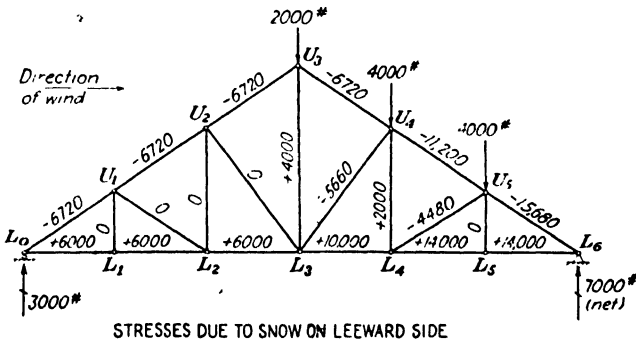
Fig. 7·4

the stresses due to loading $c$, which corresponds to full snow load. For example, the stress in bar $L_0U_1$ due to full snow load equals

$$-6,720 - 15,680 = -22,400$$

for bar $U_2L_3$, it equals $0 - 5,660 = -5,660$; etc. Since this is a symmetrical loading condition, only half the truss is shown. The stresses in

Fig. 7·6 could, of course, have been obtained by a separate stress analysis for the full snow-load condition.

Loading condition 3 calls for ice over the entire roof. Since the intensity of the ice load is half the intensity of the snow load, the stresses due to full ice load may be obtained from Fig. 7·6 by direct proportion, equaling half the stresses due to full snow load. Stresses due to full ice load are shown in Fig. 7·7.
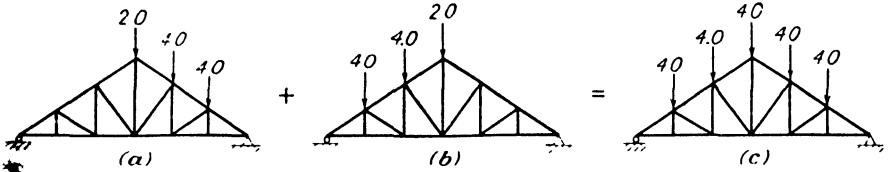


FIG. 7·5
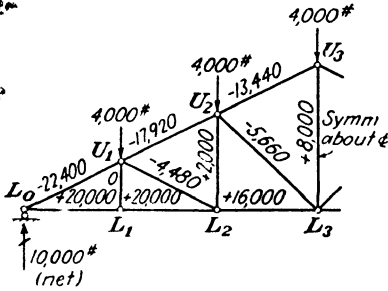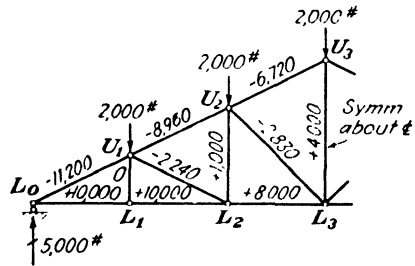


STRESSES DUE TO FULL SNOW LOAD      STRESSES DUE TO FULL ICE LOAD

FIG. 7·6                  FIG. 7·7

For a wind velocity of 100 mph, by Eq. (1·3),

$$q = 0.002558(100)^2 = 25.6 \text{ lb per sq ft}$$

For the roof, $\tan \alpha = {}^{15}\!/_{30} = 0.5$; $\alpha = 26.6°$, whence, by Eq. 1·4, the windward slope is subjected to a suction of

$$p = [0.07(26.6) - 2.10](25.6) = 6.4 \text{ lb per sq ft}$$

The suction on the leeward slope equals $0.6(25.6) = 15.4$ lb per sq ft. Thus, for the windward side, the wind panel load equals

$$6.4(11.2)(20) = 1,435 \text{ lb}$$

For purposes of analysis it is more convenient to treat the vertical and horizontal components of the wind panel load, which equal 1,280 lb upward and 640 lb upwind, respectively. For the leeward side, the wind panel load equals $15.4(11.2)(20) = 3,450$ lb. This panel load has an upward vertical component of 3,080 lb and a down-wind horizontal

component of 1,540 lb.   Figure 7·8 shows the truss acted upon by the
vertical and horizontal components of the wind panel loads and the
stresses for that condition of loading.   The half panel loads at $L_0$ and
$L_6$ were included in the analysis since the horizontal component of load
applied at $L_0$ causes stresses in the truss.   Since the wind loading is
unsymmetrical, stresses are computed for the entire truss.

<div align="center">

**Table 7·1   Stress Table for Roof Truss of Fig. 7·2**

(All stresses in lb)

</div>

| Bar | $D$ | $S$ | $S_L$ | $I$ | $W$ | (1) $D + S$ | (2) $D + W + S_L$ | (3) $D + W + I$ | Max. stress | Load combinations |
|---|---|---|---|---|---|---|---|---|---|---|
| $L_0L_1$ | +8,600 | +20,000 | + 6,000 / +11,000 | +10,000 | − 9,460 / −13,950 | +28,600 | + 5,110 / + 8,650 | +9,140 / +4,650 | +28,600 | |
| $L_1L_2$ | +8,600 | +20,000 | + 6,000 / +11,000 | +10,000 | − 9,460 / −13,950 | +28,600 | + 5,110 / + 8,650 | +9,140 / +4,650 | +28,600 | 1 |
| $L_2L_3$ | +6,880 | +16,000 | + 6,000 / +10,000 | + 8,000 | − 7,860 / −10,100 | +22,880 | + 5,020 / + 6,780 | +7,020 / +1,780 | +22,880 | 1 |
| $L_0U_1$ | −9,650 | −22,400 | − 6,720 / −15,680 | −11,200 | +10,950 / +13,470 | −32,050 | − 5,420 / −11,860 | −9,900 / −7,380 | −32,050 | 1 |
| $U_1U_2$ | −7,710 | −17,920 | − 6,720 / −11,200 | − 8,960 | + 9,880 / +10,880 | −25,630 | − 4,550 / − 8,030 | −6,790 / −5,790 | −25,630 | 1 |
| $U_2U_3$ | −5,780 | −13,440 | − 6,720 / − 6,720 | − 6,720 | + 8,800 / + 8,300 | −19,220 | − 3,700 / − 4,200 | −3,700 / −4,200 | −19,220 | 1 |
| $U_1L_1$ | −1,930 | − 4,480 | 0 / − 4,480 | − 2,240 | + 1,790 / + 1,320 | − 6,410 | − 140 / − 2,090 | −2,380 / + 150 | + 150 / − 6,410 | 3 / 1 |
| $U_2L_3$ | −2,430 | − 5,660 | 0 / − 5,660 | − 2,830 | + 2,260 / + 5,450 | − 8,090 | − 170 / − 2,640 | −3,000 / + 190 | + 190 / − 8,090 | 3 / 1 |
| $U_1L_2$ | + 375 | 0 | 0 / 0 | 0 | 0 / 0 | + 375 | + 375 | + 375 | + 375 | 1, 2, 3 |
| $U_2L_2$ | +1,235 | + 2,000 | 0 / + 2,000 | + 1,000 | − 800 / − 1,925 | + 3,235 | + 435 / + 1,310 | +1,435 / + 310 | + 3,235 | 1 |
| $U_2L_3$ | +3,815 | + 8,000 | + 4,000 | + 4,000 | − 5,150 | +11,815 | + 2,365 | +2,365 | +11,815 | 1 |

In Table 7·1, we shall list the stresses as given in Figs. 7·3, 7·4, and
7·6 to 7·8.   Then we shall combine these stresses to give the total stress
in each member corresponding to the three specified loading combina-
tions.   Finally we shall choose, for each member, the maximum stress of
each character which occurs in any of the three loading combinations,
these maximum stresses being those which would control the design of

the members. The abbreviations used in the column headings are as follows: $D$ = dead stress; $S$ = stress due to snow load on both sides; $S_L$ = stress due to snow on the leeward side only; $I$ = stress due to ice on both sides; $W$ = stress due to wind. We shall list in the table only the bars in half the truss; but, in entering stresses for the columns headed $S_L$ and $W$, we shall enter first the stress in the member itself, due to the loading under consideration and second the stress in the symmetrically placed member, due to the same loading.[1] This second entry covers the case which would exist were the direction of the wind to reverse, a condition for which the design must, of course, provide. In carrying forward the totals that include either $S_L$ or $W$, the sums corresponding to both entries are computed.
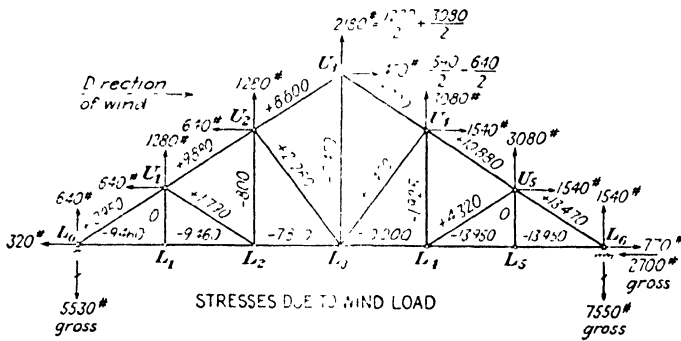


Fig. 7·8

**7·3  Permissible Fiber Stresses for Members Stressed by Wind.** When members carry stresses due to wind loads, it is usual to permit somewhat higher fiber stresses in design than would otherwise be the case. For example, the specifications of the AISC state that members subject to stresses produced by a combination of wind and other loads may be proportioned for unit stresses $33\frac{1}{3}$ per cent greater than those specified in Art. 1·22, provided that the section thus required is not less than that required for the combination of other loads, without including wind. They state further that this increase of $33\frac{1}{3}$ per cent is also permissible for members subject only to stresses produced by wind loads.

Instead of designing members carrying wind loads on the basis of increased fiber stresses equal to $\frac{4}{3}$ of the usual permissible fiber stresses, it is obvious that the same results will be obtained by designing on the basis of the usual permissible fiber stresses, but allowing for only $\frac{3}{4}$ of computed total stresses in such members, since $F/\frac{4}{3}f = \frac{3}{4}F/f$. Hence, in preparing a stress table such as Table 7·1, one might proceed by entering in the column headed Max. stress the largest of the following: $D + S$;

[1] This procedure was not followed for the lower-chord members. Why?

$\frac{3}{4}(D + W + S_L)$; $\frac{3}{4}(D + W + I)$. The member would then be designed for this maximum stress, with the usual permissible fiber stresses.

**7·4   General Analysis of a Bridge Truss.** The general analysis of a bridge truss consists in computing the stress in each member, due to each type of loading, and the combination, for each truss member, of these stresses into the maximum total stress that will control design.
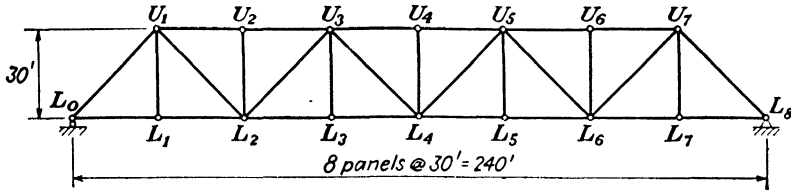


FIG. 7·9

To illustrate this procedure, we shall consider the Warren type of highway truss of Fig. 7·9, which will be analyzed for dead, live, and impact loads. It will be assumed that the truss, including details and secondary bracing, weighs 0.500 kip per ft, with this dead weight divided equally

TABLE 7·2   Influence Table for Truss of Fig. 7·9

| Bar | Stress due to unit load at | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $L_0$ | $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | $L_6$ | $L_7$ | $L_8$ |
| $L_0L_2$* | 0 | +0.875 | +0 750 | +0 625 | +0 500 | +0 375 | +0.250 | +0.125 | 0 |
| $L_2L_4$ | 0 | +0.625 | +1 250 | +1.875 | +1.500 | +1 125 | +0.750 | +0.375 | 0 |
| $L_0U_1$ | 0 | −1 238 | −1.060 | −0.885 | −0 708 | −0.530 | −0.351 | −0.177 | 0 |
| $U_1U_3$ | 0 | −0 750 | −1.500 | −1.250 | −1 000 | −0.750 | −0 500 | −0 250 | 0 |
| $U_3U_4$ | 0 | −0 500 | −1.000 | −1.500 | −2 000 | −1.500 | −1 000 | −0.500 | 0 |
| $U_1L_2$ | 0 | −0 177 | +1.060 | +0 885 | +0 708 | +0.530 | +0 351 | +0 177 | 0 |
| $L_2U_3$ | 0 | +0 177 | +0.351 | −0.885 | −0.708 | −0.530 | −0 351 | −0 177 | 0 |
| $U_3L_4$ | 0 | −0 177 | −0.351 | −0.530 | +0.708 | +0 530 | +0.351 | +0.177 | 0 |
| $U_1L_1$ | 0 | +1 000 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $U_2L_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $U_3L_3$ | 0 | 0 | 0 | +1.000 | 0 | 0 | 0 | 0 | 0 |
| $U_4L_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

* By the method of joints, the stress in $L_0L_1$ is equal to the stress in $L_1L_2$ as long as only vertical loads are applied at joint $L_1$. Hence to save space in this and the following tables, these two members will be treated as a single member.

between the top- and bottom-chord panel points. The weight per foot of that portion of the floor system carried by the truss will be taken as 0.800 kip, acting at the bottom-chord panel points. For live load, an equivalent live-load system will be used, consisting of a uniform load of 0.650 kip per ft and a single concentrated load of 20.0 kips. Impact will

be computed in accordance with Eq. (1·1). Maximum live stresses are to be computed by the approximate method. For each member of the left half of the truss, the maximum total stress of each character is to be obtained.

We shall first compute the ordinates to the influence line for each member of the left half of the truss, considering the effect of a unit load at each bottom-chord panel point of the entire truss. These ordinates are summarized in the influence table (Table 7·2). Since no unusual conditions are encountered in the computations of these values, the details of the computations will be omitted.

A summary of the influence table (Table 7·2) will next be prepared (Table 7·3).

Preparatory to the construction of a stress table for the members under consideration, we shall compute the panel loads for dead and live

Table 7·3   Summary of Influence Table for Truss of Fig. 7·9

| Bar | Sum of ordinates | | | Maximum ordinates | | Loaded length ft. for | |
|---|---|---|---|---|---|---|---|
| | Positive | Negative | All | Positive | Negative | Tension | Compression |
| $L_0L_2$ | +3.500 | 0 | +3 500 | +0.875 | 0 | 210 | 0 |
| $L_2L_4$ | +7 500 | 0 | +7 500 | +1.875 | 0 | 210 | 0 |
| $L_0U_1$ | 0 | −4 952 | −4 952 | 0 | −1 238 | 0 | 240 |
| $U_1U_3$ | 0 | −6 000 | −6 000 | 0 | −1.500 | 0 | 240 |
| $U_3U_4$ | 0 | −8 000 | −8 000 | 0 | −2 000 | 0 | 210 |
| $U_1L_2$ | +3 714 | −0 177 | +3 537 | +1.060 | −0.177 | 206 | 34 |
| $L_2U_3$ | +0 531 | −2 654 | −2 123 | +0.354 | −0.885 | 69 | 171 |
| $U_3L_4$ | +1 769 | −1.061 | +0 708 | +0.708 | −0.530 | 137 | 103 |
| $U_1L_1$ | +1 000 | 0 | +1 000 | +1 000 | 0 | 60 | 0 |
| $U_2L_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $U_3L_3$ | +1 000 | 0 | +1 000 | +1.000 | 0 | 60 | 0 |
| $U_4L_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

load. For dead load, the top-chord panel load equals 30(0.250) = 7.5 kips; the bottom-chord panel load equals 7.5 + 30(0.800) = 31.5 kips. For uniform live load, the panel load equals 30(0.650) = 19.5 kips; the concentrated live load, which is already expressed as a panel load, equals 20.0 kips.

In the stress table (Table 7·4), the dead stresses are computed by the principle of superposition, as shown in Fig. 7·10. The stresses due to the loading of Fig. 7·10a are first computed. This loading assumes that the total dead load is applied at the bottom chord, so that the dead panel load for bottom-chord panel points equals 31.5 + 7.5 = 39.0. Since

### Table 7·4  Stress Table for Truss of Fig. 7·9
(All stresses in kips)

| Bar | D = dead stress | L = live stress | | | Impact fraction | I = impact stress | L + I | Total = D + L + I |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| | | Uniform | Concentrated | Total | | | | |
| $L_0L_2$ | +136.5 | + 68.3 | +17.5 | + 85 8 | 0.137 | +11.8 | + 97 6 | +234.1 |
| | ...... | ...... | ...... | ...... | ..... | ..... | ....... | ....... |
| $L_2L_4$ | +292.5 | +116 2 | +37.5 | +183 7 | 0.137 | +25.2 | +208.9 | +501.4 |
| | ....... | ... . | ..... | ...... | ..... | ..... | ......... | ...... |
| $L_0U_1$ | .... .. | . . | . . | ..... . | .. | . . | ...... | ...... |
| | −193.0 | − 96 5 | −24.7 | −121 2 | 0 137 | −16.6 | −137.8 | −330.8 |
| $U_1U_3$ | −231.0 | −117 0 | −30.0 | −117 0 | 0 137 | −20 2 | −167.2 | −101.2 |
| $U_3U_4$ | ...... | | | .... | | | ..... . | . ... |
| | −312 0 | −156 0 | − 10 0 | −196 0 | 0.137 | −26 8 | −222 8 | −534.8 |
| $U_1L_2$ | +138 0 | + 72 5 | +21 2 | + 93 7 | 0 151 | +11 2 | +107.9 | +215 9 |
| | . ... . | − 3 5 | − 3 5 | − 7 0 | 0 300 | − 2 1 | − 9 1 | .... |
| $L_2U_3$ | ..... . | + 10.1 | + 7.1 | + 17.5 | 0 258 | + 4 5 | + 22 0 | |
| | − 83.0 | − 51 8 | −17.7 | − 69 5 | 0 169 | −11.8 | − 81 3 | −161 3 |
| $U_3L_4$ | +· 27.6 | + 31 5 | +11 2 | + 18 7 | 0 191 | + 9.3 | + 58 0 | + 85 6 |
| | .. | − 20.7 | −10 6 | − 31.3 | 0 219 | − 6.9 | − 38 2 | − 10 6 |
| $U_1L_1$ | + 31.5 | + 19 5 | +20 0 | + 39.5 | 0.270 | +10.7 | + 50 2 | + 81.7 |
| | ....... | ....... | ..... | ..... | ..... | ...... | ....... | ....... |
| $U_2L_2$ | ....... | ........ | ........ | ........ | ..... | ....... | ....... | ...... |
| | − 7.5 | ........ | ..... | ...... | ..... | ..... | ....... | − 7.5 |
| $U_3L_3$ | + 31.5 | + 19.5 | +20.0 | + 39 5 | 0.270 | +10.7 | + 50.2 | + 81 7 |
| | ....... | ....... | ...... | ....... | ..... | ...... | ....... | ....... |
| $U_4L_4$ | ....... | ........ | ........ | ........ | ...... | ....... | ....... | ....... |
| | − 7.5 | ........ | ..... | ........ | ..... | ....... | ........ | − 7.5 |

the influence table is based on the application of loads to the bottom-chord panel points, the stresses for this condition of loading are obtained by multiplying, for each member, the sum of all the ordinates to the influence line by the total dead panel load of 39.0.  The stresses due to

the loading of Fig. 7·10b are next computed, the value 7.5 being that of the top-chord dead panel load. This stress analysis is exceedingly simple, and results, as shown in the figure, in a stress of −7.5 in all verticals and zero stress in all other members. If the loadings of Figs. 7·10a and b are superimposed, the loading of Fig. 7·10c, which corresponds to the actual dead loading, is obtained. Hence the dead stresses may be obtained by superimposing the stresses computed for the loadings of Figs. 7·10a and b. To summarize, the dead stresses for a given member may be computed as follows: (1) Multiply the *total* dead panel load by the sum of all the ordinates to the influence line. (2) To the stress from
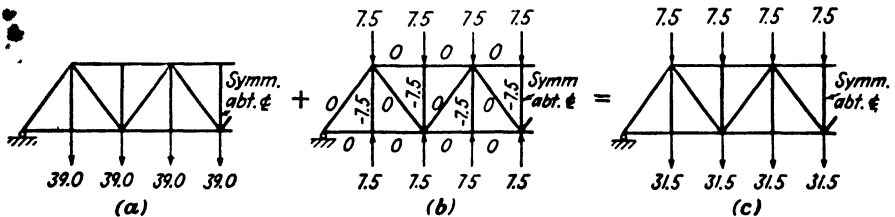


Fig. 7·10

step 1, subtract, *for the verticals only*, the magnitude of the top-chord dead panel load.

Had the dead load not been uniformly distributed, the stresses corresponding to the loading of Fig. 7·10a would require recourse to the influence table, with stresses obtained by summing the cross products of the individual panel loads and corresponding ordinates from the influence table. It is, of course, possible, and sometimes desirable, to obtain dead stresses by a separate analysis for dead loads, without making use of influence data.

Maximum tension and compression due to uniform live load are entered in the next column. These values are obtained by multiplying the uniform live panel load (19.5) by the sum of the positive and negative ordinates, respectively, to the influence line. Maximum tension and compression due to concentrated live load are entered in the following column. These values are obtained by multiplying the concentrated live panel load (20.0) by the maximum positive and negative ordinates, respectively, to the influence line. Total live stress of each character is then obtained by summing the effects of the uniform and concentrated live loads.

Impact fractions, computed in accordance with Eq. (1·1), are then entered; these impact fractions multiplied by the *total* live stresses give the impact stresses, which are then added to the total live stresses, leading to the total stresses due to live load and impact. Total stresses due

to dead load plus live load plus impact are entered in the final column. It is to be noted that live and impact stresses of both characters are carried forward in the stress table, until it is definitely ascertained that no stress reversal (see Art. 7·5) can occur. In the actual design of a truss, stresses due to other causes, such as wind loads, would be considered, such stresses being combined with dead, live, and impact stresses in additional columns in the stress table. In this example, stresses due to wind have been omitted because the analysis of lateral systems and portals has been reserved for discussion in a later chapter.

7·5  **Stress Reversal.**  In the highway truss analyzed in Art. 7·4, all the members were assumed to be capable of carrying either tension or compression, and the maximum stress of each character was obtained. In preparing the stress table of Table 7·4, the total stress due to dead plus live plus impact was obtained by simply adding the $L + I$ stresses algebraically to the dead stresses. In the case of only one member, $U_3L_4$, did the live plus impact total exceed the dead stress, when the live plus impact total was of opposite character from the dead stress.

For $U_3L_4$, the interpretation of this condition is as follows: Depending upon the position of the live load on the structure, the total stress due to dead + live + impact may be either tension or compression. This alternation of stresses as live load travels across the span is called stress reversal. When members are subjected to stress reversal, it is usual to follow a more conservative design procedure than would result if the usual permissible fiber stresses were used in conjunction with the $D + L + I$ totals of Table 7·4. For example, the specifications of the AASHO state that "if the alternate stresses (stress reversals) occur in succession during one passage of the live load, each shall be increased by 50 per cent of the smaller. . . . If the live-load and dead-load stresses are of opposite sign, only 70 per cent of the dead-load stress shall be considered as effective in counteracting the live-load stress." The following of such a specification results in a build-up of total stress for which a member subject to stress reversal shall be designed. Since the usual fiber stresses are used in conjunction with this increased total, a more conservative design results than would otherwise be the case.

To illustrate the application of the foregoing specification, consider $U_4L_4$:

| | | |
|---|---|---|
| Dead | +27.6 | 0.7(+27.6) = +19.3 |
| Live + impact | +58.0 | −38.2 |
| | +85.6 | −18.9 |
| | + 9.5 = 18.9(0.50) | − 9.5 = 18.9(0.50) |
| | +95.1 kips | −28.4 kips |

The member should be designed to withstand both these stresses.

7·6 **Counters.** A member with a large slenderness ratio (length divided by radius of gyration) will buckle when subjected to relatively small compressive forces. Such a member can carry tension satisfactorily but can carry only a negligible amount of compression. The diagonals of a truss may be designed so that they act in such a manner, in which case they are called tension diagonals. If a truss were to carry dead loads only, as shown in Fig. 7·11a, single tension diagonals in each panel would be satisfactory, since their slopes could be chosen in a manner such that the dead shear in each panel would produce tension in the diagonals. When live loads are considered, it is always possible that, for some position of the live loads, the maximum shear in a given panel due to live loads plus impact, of a character opposite to that of the dead shear, may exceed the dead-load shear. This would tend to produce compression in the
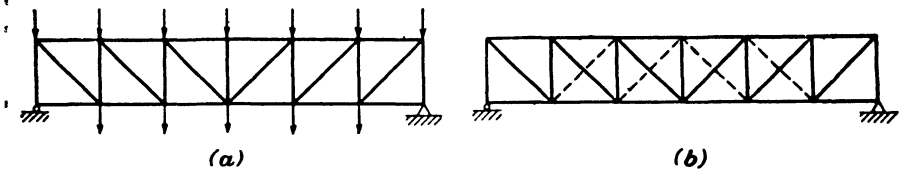


(a)                                       (b)

Fig. 7·11

tension diagonal, a condition that must be avoided, unless a second tension diagonal, called a *counter*, is added to the panel, the counter having a slope opposite to that of the main tension diagonal. The counter will have no stress due to dead loads only, since it buckles slightly under the action of the dead shear. When the shear in the panel changes sign owing to live loads and impact, however, the main tension diagonal buckles and has zero stress and the counter comes into action, carrying the resultant shear due to dead plus live plus impact as a tension member.

In designing a truss with tension diagonals, one should first determine the panels where counters are required. Counters are most likely to be necessary in the panels nearest the center of the truss, since the dead shear in such panels is smaller than in the end panels, while the live shear of opposite character to that of the dead shear is larger than in the end panels and hence more likely to exceed the dead shear. A typical truss with counters is shown in Fig. 7·11b, in which the counters are indicated by the dotted diagonals of the four panels nearest the center of the truss.

To illustrate the determination of maximum stresses in a member of a truss with counters, consider the computation of the maximum compression in member $U_3L_3$ of the structure of Fig. 7·12a. An influence line can be constructed that will show how the stress in $U_3L_3$ will vary as a unit load moved across the span. Such an influence line cannot be used

to determine how to place live loads for maximum compression in $U_3L_3$, however, for the following reason: The computation of each ordinate will depend on whether the main diagonal or the counter is in action in each of the two center panels. This condition will vary for different positions of the unit load. For example, with the unit load at $L_2$, $L_2U_3$ and $L_3U_4$ will be in action and $U_2L_3$ and $U_3L_4$ will have zero stress, while, with the
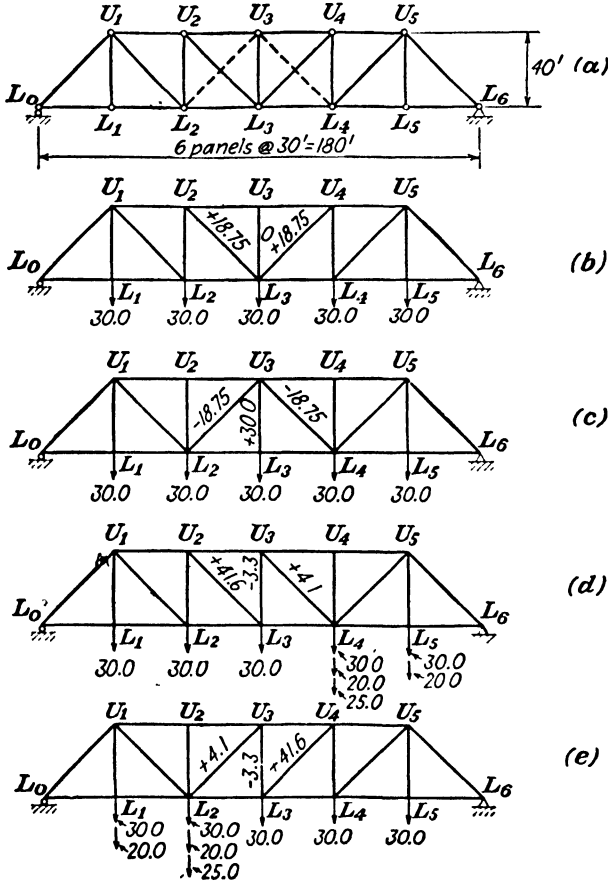


Fig. 7·12

unit load at $L_3$, $U_2L_3$ and $L_3U_4$ will be in action and $L_2U_3$ and $U_3L_4$ will have zero stress. Hence, in effect, *the different ordinates to the influence line will correspond to the action of different structures, and under such circumstances the principle of superposition does not hold.* The maximum compression in $U_3L_3$ due to dead plus live plus impact loads depends on which diagonals are in action under the *total* loading leading to that maximum compression.

To simplify this illustration, let us assume that the dead panel loads are applied to the bottom chord only and equal 30.0 kips; that the uniform live panel is 20.0 kips and the concentrated live load is 25.0 kips; and that impact is to be neglected.

The four possible ways in which the diagonals of the two center panels can act are shown in Figs. 7·12b to e. If they act as shown in Fig. 7·12b, $U_3L_3$ cannot be stressed, as may be seen by applying the method of joints to joint $U_3$. If they act as shown in Fig. 7·12c, the live load cannot be placed so as to produce compression in $U_3L_3$, as may be seen by applying the method of joints to $L_3$. Even under dead loads this represents an impossible condition, since it is necessary for $L_2U_3$ and $U_3L_4$ to carry compression. If the diagonals act as shown in Fig. 7·12d, the live loads will be placed as shown to produce maximum compression in $U_3L_3$. This is a possible condition since $U_2L_3$ and $U_3L_4$ are both in tension and leads to a resultant compression of $-3.3$ kips in $U_3L_3$. If the diagonals act as shown in Fig. 7 12e, there results the same compression of $-3.3$ kips in $U_3L_3$, this also being a possible condition, since $L_2U_3$ and $L_3U_4$ are both in tension.

Hence we may conclude that the maximum compression which can occur in $U_3L_3$ is $-3.3$ kips.

When a truss has counters, the determination of maximum stresses in members influenced by the action of the counters must be approached by trial. Each position of live load that may reasonably lead to the maximum stress desired must be investigated. If diagonals capable of carrying tension only have been assumed to be in action, the results of any investigation are invalidated unless the computed stresses in such members indicate that they actually carry tensile stresses.

**7·7 Movable Bridges—General.** When the topography of a bridge site is such that it is desirable to have the roadway close to the surface of the body of water crossed by the bridge, the vertical under-clearance requirements of the navigation passing beneath the bridge may require a movable bridge. A movable bridge is one that may be moved to permit the passage of navigation. The three most important types of movable bridges are (1) bascule bridges, (2) vertical-lift bridges, and (3) horizontal-swing bridges. The type to be used depends largely upon the horizontal and vertical clearance requirements of the navigation. Whether a low-level movable bridge or a high-level fixed bridge should be used in a given site can usually be determined only by a careful economic study.

**7·8 Bascule Bridges.** A bascule bridge may prove economical where horizontal navigation requirements do not necessitate too long a span and where a high vertical clearance is required. A typical bascule

bridge is shown in Fig. 7·13.   Motive power drives a pinion at $D$, which engages the rack $E$, thus opening or closing the span.   The required motive power is reduced by the action of the counterweight $C$.

The dead-load stresses in a bascule span change as the bridge is opened or closed, and it is possible that the dead stresses in certain members during such an operation may exceed the total stresses with the bridge closed and subjected to traffic.

To find the maximum dead stresses that occur while the span is being raised or lowered,[1] let $F_H$ be the dead stress in any member, with the span
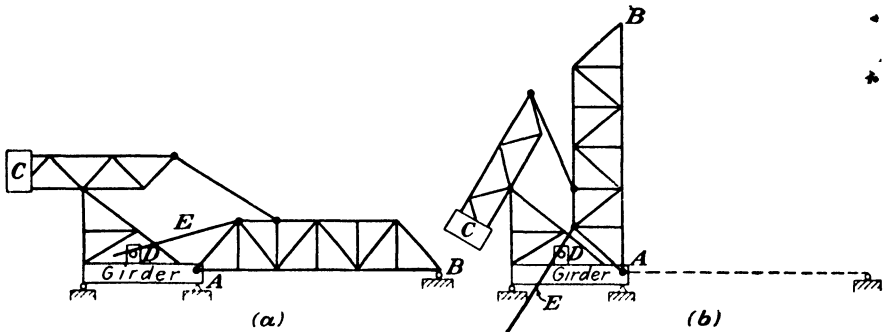


FIG. 7·13

horizontal, and $F_V$ be the dead stress in the same member after the span is vertical (*i.e.*, after having rotated through 90° from its closed position), both these values being easily computed by the usual methods of analysis. With the bridge partly opened and the bottom chord making an angle $\alpha$ with the horizontal, as shown in Fig. 7·14, each dead panel load may be resolved into two components, one perpendicular and one parallel to the bottom chord.   The components of dead load that are perpendicular to the bottom chord will cause stresses equal to $F_H \cos \alpha$, while the components of dead load that are parallel to the bottom chord will cause stresses equal to $F_V \sin \alpha$.   Hence, for any angle $\alpha$, the total dead stress $F_D$ in any member is given by

$$F_D = F_V \sin \alpha + F_H \cos \alpha \tag{a}$$

Placing the derivative of $F_D$ with respect to $\alpha$ equal to zero,

$$\frac{dF_D}{d\alpha} = F_V \cos \alpha - F_H \sin \alpha = 0$$

whence $\tan \alpha = F_V/F_H$.   Substituting this value of $\alpha$ into Eq. *a*,

$$\text{Max } F_D = F_V \frac{F_V}{\sqrt{F_V^2 + F_H^2}} + F_H \frac{F_H}{\sqrt{F_V^2 + F_H^2}} = \sqrt{F_V^2 + F_H^2} \tag{7·1}$$

[1] See HOVEY. O. E., "Movable Bridges," Vol. I, p. 219, John Wiley & Sons, Inc., New York, 1926

With the bridge closed, the dead-load reaction at the free end will be zero, since the counterweight holds the dead loads in equilibrium, but live loads produce reactions at each end, in the same manner as for an end-supported span.
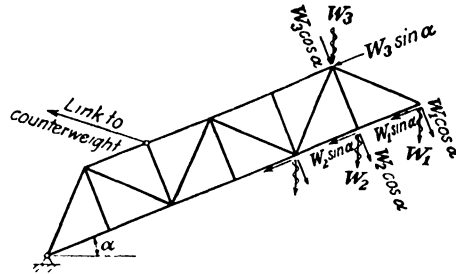


Fig. 7·14

**7·9 Vertical-lift Bridges.** When the horizontal-clearance requirement is greater than the vertical-clearance requirement for navigation, a vertical-lift bridge is likely to prove economical. A typical vertical-lift bridge is shown in Fig. 7·15. The span $AB$ is raised or lowered vertically by cables running over sheaves at $D$ that are supported at the tower tops. The motive power required for this motion is reduced by the counterweights $C$. These counterweights are usually designed to balance the entire dead load of the movable span, so that the dead-load
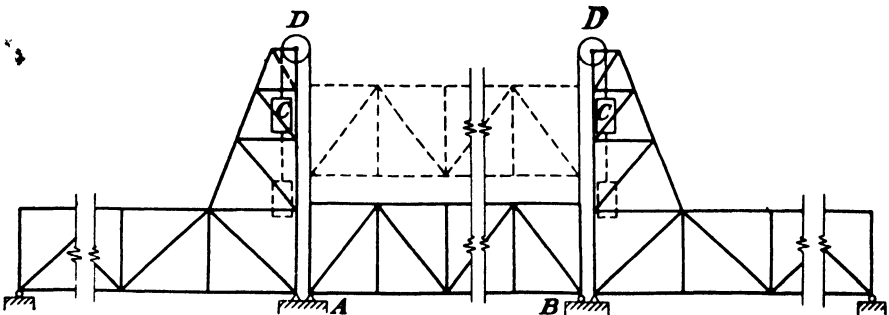


Fig. 7·15

reactions are taken by the cables. Live loads on the movable span produce reactions on the piers at $A$ and $B$, however.

**7·10 Horizontal-swing Bridges.** Horizontal-swing bridges give unlimited vertical clearance, but the center pier constitutes an obstruction to traffic. A large horizontal area is required for this type of movable bridge. Horizontal-swing bridges may be of either the center-bearing type, as shown in Fig. 7·16a, or the rim-bearing type, as shown in Fig.

7·16b.   In either case, the bridge is opened by swinging it horizontally about the vertical center line.   When the bridge is open, the two spans cantilever from the center pier and are statically determinate.   When the bridge is closed, the trusses are continuous and hence statically indeterminate.   Stress analysis for the closed condition depends on principles discussed in the portion of this book dealing with statically indeterminate structures.

When a swing bridge is closed, the dead reactions developed at the outer ends of the structure depend on the design.   If these ends just
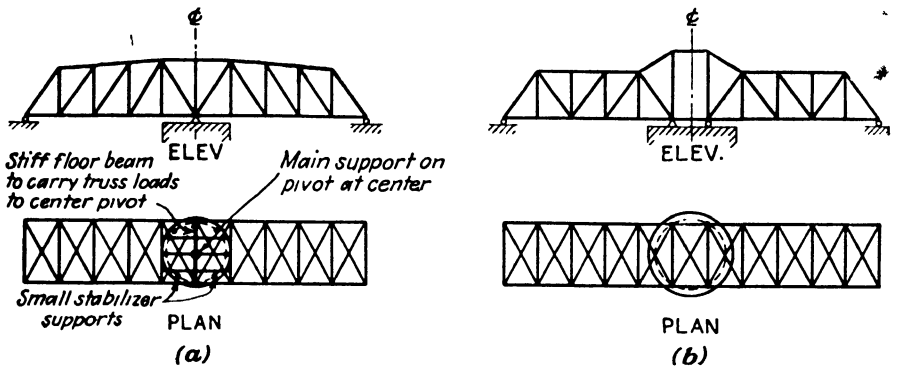


Fig. 7·16

touched their supports, any live load on one span would cause uplift at the far end of the other span.   This condition is usually avoided by lifting the ends a slight amount when they are closed.   The desired dead-load reaction can be computed so that it will exceed the maximum live-load reaction of opposite character.

**7·11   Skew Bridges.**   If the abutments are not perpendicular to the longitudinal axis of a bridge, the bridge is said to be skewed.   Figure 7·17b shows the plan view of such a structure.   In order to keep the connections from becoming complicated, the floor beams are usually kept perpendicular to the main trusses.   This is likely to make the main trusses unsymmetrical.   The two end posts of a truss should have the same slope so that the end portals will lie in a plane.   This may lead to an inclined hanger as shown by bar $a$, Fig. 7·17a.
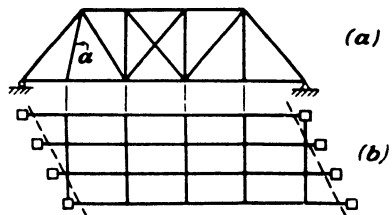


Fig. 7·17

When analyzing stresses in the members of a truss of a skewed bridge, one may proceed by the same general approach followed for a bridge that

is perpendicular to the abutments.   However, in dealing with live loads, one should take into consideration the irregularity of the floor system. This will alter the details of the computation of live panel loads, which may not be the same for all panel points.

### 7·12   Problems for Solution

Problem 7·1   For the roof truss and loadings of Art. 7·2:

*a.* Compute the maximum stress of each character for each member of the left half of the truss, due to dead plus wind loads combined.

*b.* Comparing the stresses computed in part *a* to the maximum stresses given in the stress table of Table 7 1, for what members if any would the loading condition of part *a* control design?   (NOTE: In part *b* do not consider permissible increases in fiber stresses for members stressed by wind.)

*c.* Is it necessary for this truss to be anchored at its supports to prevent uplift?   If so, determine the uplift force for which each anchorage should be designed, allowing a factor of safety of 50 per cent in this force.
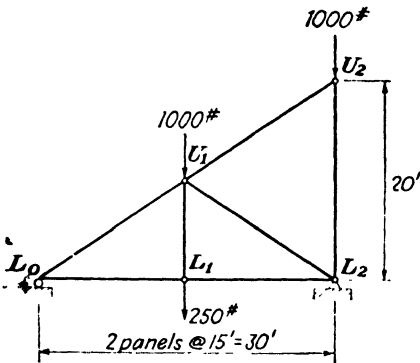


FIG. 7·18

Problem 7·2   The roof truss of Fig. 7·18 is an intermediate truss of a series of trusses spaced on 15-ft centers. The dead loads acting on this truss are as shown in the figure.   The snow load is 20 lb per sq ft horizontal projected area.   Wind loads are to be computed by the Duchemin formula and based on a wind-pressure intensity of 20 lb per sq ft on a vertical surface.   Compute the maximum stress of each character in each member due to each of the following load combinations: (1) dead plus snow; (2) dead plus wind (note that wind may blow from either direction).   What stress will control the design of each member?

Problem 7·3   The dead panel loads in kips are shown on the bridge truss of Fig. 7·19.   The truss is subjected to a uniform live load of 1.0 kip per ft and a concentrated live load of 15.0 kips.   Impact is to be computed in accordance with Eq. (1·1).   Prepare a stress table for the members of the left half of this truss, in which the following columns are included: dead; uniform live; concentrated live; total live; impact; live plus impact; total dead plus live plus impact.   For live stresses, use the approximate method of analysis.   Give stresses of both characters where they exist.   Combine dead
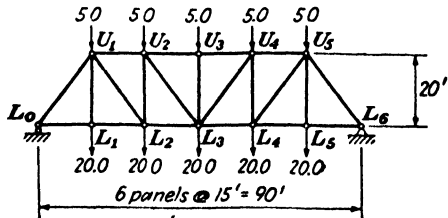


FIG. 7·19

stresses with live plus impact stresses, (a) by algebraic addition; (b) in accordance with the specifications of the AASHO for members subject to stress reversal.

**Problem 7·4**  Assuming that the diagonals of the truss of Fig. 7·19 can carry tension only, at what panels are counters necessary?

**Problem 7·5**  The loads shown acting on the truss of Fig. 7·20 are the dead panel loads in kips. The diagonals of this truss can carry tension only. The truss is acted on by the following live load: uniform, 0.700 kip per ft; concentrated, 20.0 kips. Impact is to be computed in accordance with Eq. (1 1). Compute the maximum stress due to dead plus live plus impact in (a) $L_3U_4$; (b) $U_3U_4$.
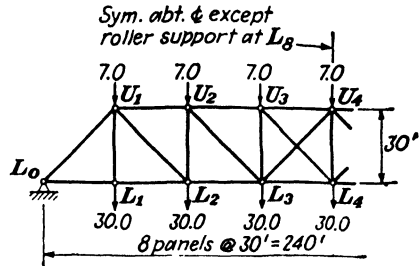


Sym. abt. ₵ except roller support at $L_8$

FIG. 7·20

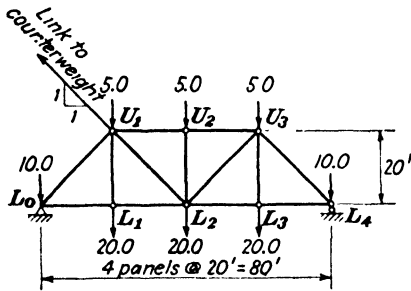**Problem 7·6**  Figure 7·21 shows a bascule span acted upon by its dead panel loads in kips.

a. Compute the dead stresses, including the tension in the link to the counter-weight, with the span in the closed position as shown.

b. Compute the maximum dead stress that will occur as the span is raised through an angle of 90°, for all members except $L_0U_1$.

c. Is the method of analysis used in part b applicable to the determination of the maximum dead stress in $L_0U_1$?



FIG. 7·21

**Problem 7·7**  Figure 7 22 shows a horizontal-swing bridge truss acted upon by its dead panel loads in kips. Dashed line members carry tension only.
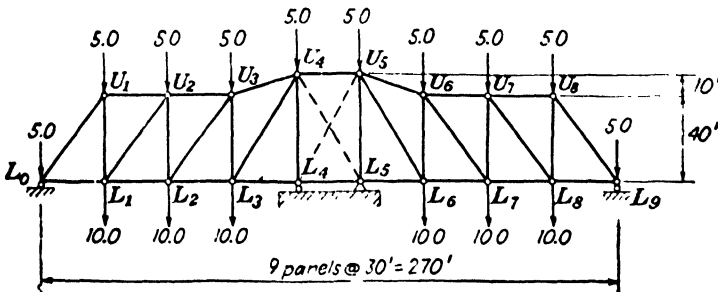


FIG. 7·22

a. Compute the dead stresses when the bridge is swung into its open position.

b. When the bridge is closed, each end of the structure is raised 1 in. above the elevation it has when the bridge is open. If a force of 10 kips applied upward

at $L_o$ when the bridge is open would raise $L_o$ by 1 in. and lower $L_9$ by ¼ in., compute the dead reaction at $L_o$ and $L_9$ with the bridge in the closed position.

  *c.* Compute the dead stresses in the truss with the bridge in the closed position, based on the dead reactions corresponding to **part** *b.*

  **Problem 7·8**   For the single-track railroad skew bridge for which the plan view is shown in Fig. 7·23, the weight of the track and ties is 500 lb per ft of
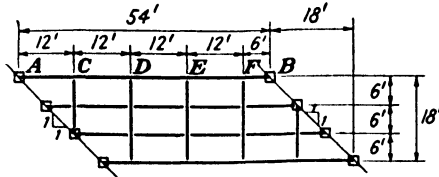


FIG. 7·23

track; the weight of each stringer, including details, is 125 lb per ft; the weight of each floor beam, including details, is 175 lb per ft.

  *a.* Compute those portions of the dead panel loads acting on girder *AB* that are applied by the floor beams to the girder at panel points *C, D, E*, and *F.*

  *b.* If the track is subjected to a uniform live load of 5,000 lb per ft, extending over the entire structure, what are the live panel loads acting on the girder *AB* that are applied by the floor beams to the girder at panel points *C, D, E,* and *F?*

# CHAPTER 8

## LONG-SPAN STRUCTURES

**8·1 Introduction.** As the span of a structure becomes larger, the bending moments to which the structure is subjected increase rapidly if simple end-supported structures are used. Even if the load per foot to be carried by the structure did not increase with the span, the moment due to distributed loads would vary with the square of the span. Actually, the dead weight increases with the span, so that bending moment increases at a rate greater than the square of the span. Since the chord stresses in trusses depend on bending moments to be carried by the truss, these considerations are of importance in the design of trusses as well as in the design of beams and girders.

For an economical structure, it is desirable, for the case of long spans, to adopt some means of construction that will reduce the bending moments to values less than would occur for simple end-supported structures. There are a number of methods by which this can be accomplished. In this chapter, some of these methods will be illustrated by considering the analysis of several types of long-span structures.

**8·2 Cantilever Structures — General.** In a cantilever structure, bending moments are reduced by shortening the span in which positive bending occurs, by supporting an end-supported beam, of a length shorter than the total span, on cantilevered arms that act in negative bending. The structure of Fig. 8 1 shows cantilever construction and is statically determinate. The maximum moment in beam $BC$ equals
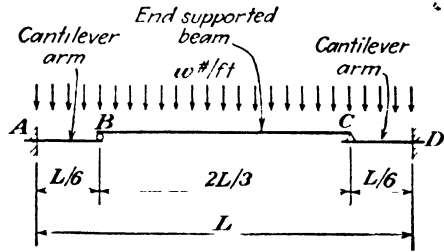
Fig. 8·1

$$\tfrac{1}{8}w\left(\frac{2L}{3}\right)^2 = \frac{wL^2}{18}$$

The maximum moment in the cantilever arm $AB$ occurs at $A$ and equals

$$-\frac{w}{2}\left(\frac{2L}{3}\right)\left(\frac{L}{6}\right) - w\left(\frac{L}{6}\right)\left(\frac{L}{12}\right) = -\frac{5wL^2}{72}.$$ Had a simple span of length $L$ been used, the maximum moment would have been $+wL^2/8$. Hence, in this particular case, the reduction in maximum moment resulting from

cantilever construction is from $wL^2/8$ to $5wL^2/72$, or about 45 per cent. It should be pointed out that maximum moment is not the only criterion by which the relative merit of alternate types of construction should be judged, but it is an important factor in determining the desirability of using a particular type of structure for a given span. Moments and shears along the entire length of the structure, computed for all types of loads including live loads, must be considered in the actual design.

To obtain the fixity of supports at $A$ and $D$ for the cantilevers $AB$ and $CD$ of Fig. 8·1 would be difficult under many conditions. The same principle of cantilever construction is present, however, in the structure of Fig. 8·2, where this difficulty has been eliminated. The moments in this structure, between $A$ and $D$, are identical with the
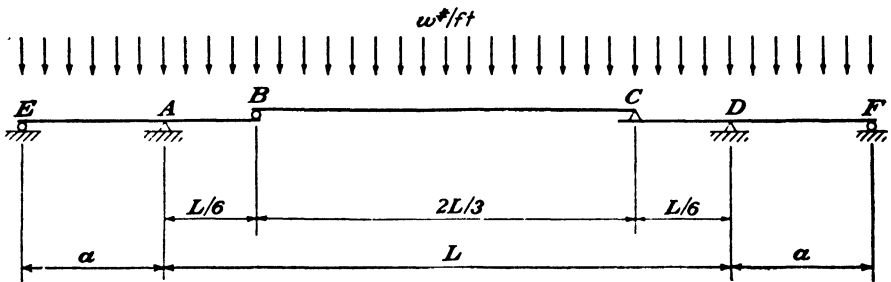


Fig. 8·2

moments in the structure of Fig. 8 1. The moments at $A$ and $D$ however, are resisted by the flanking spans $AE$ and $DF$. The moment at $A$, for example, is held in equilibrium by the reaction at $E$ and the load applied between $E$ and $A$.

**8·3 Statical Condition of Cantilever Structures.** For a cantilever structure to be statically determinate with respect to its outer forces, there must be as many independent equations available for the determination of these outer forces as there are independent reactions for the structure. Except for a simple cantilever beam, there are always more than three independent reactions for a cantilever structure, but there are only three independent equations of statics that can be applied to the entire structure. To make a cantilever structure statically determinate, it is therefore necessary to introduce certain construction features that make it possible to apply the equations of statics to certain portions of the structure, thus obtaining additional independent equations of condition. Some of these construction features are inherent to cantilever construction; others must be specially provided.

Such construction features are illustrated by the structure of Fig. 8·3. The hinge at $a$ constitutes one such construction feature, since

$\Sigma M_a = 0$ may be applied to those forces acting on that portion of the structure lying on *either* side of joint $a$.   The hinge at $b$ is similar in its effect, since it permits one to apply $\Sigma M_b = 0$ to those forces acting on that portion of the structure lying on either side of joint $b$.

The hinge at $c$ permits the application of $\Sigma M_c = 0$ to those forces acting on that portion of the structure lying on either side of joint $c$. Instead of utilizing the hinge at $c$ in this manner, the following alternate interpretation is usually advantageous: Because there are hinges at both $b$ and $c$, the hanger $bc$ carries direct stress only and hence has no component of stress perpendicular to $bc$, that is, $bc$ is a link.   One may therefore conclude that, if a section is taken through $bc$, the sum of the forces perpendicular to $bc$ which act on either side of the section equals zero. For the case under consideration, this means that the equation $\Sigma F_x = 0$ may be applied to those forces acting on that portion of the structure lying on one side of section $A$-$A$.
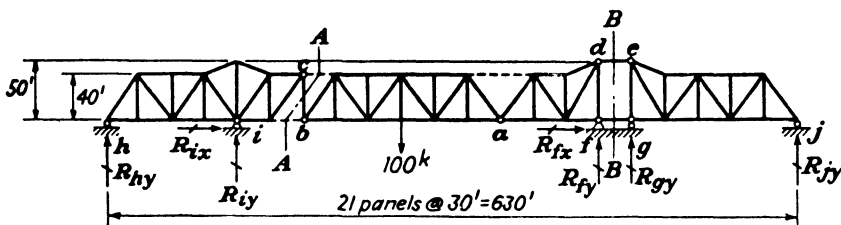


·Fɪɢ. 8·3

The omission of the diagonal in panel $defg$ constitutes another construction feature that permits an additional independent equation of statics with respect to external forces, since, as a result, no shear can be carried by this panel.   Hence the equation $\Sigma F_y = 0$ may be applied to the forces acting on one side of section $B$-$B$.

There are, therefore, the following seven independent equations available for determining the reactions of this structure:

(1) $\Sigma M = 0$   for all forces acting on the structure
(2) $\Sigma F_x = 0$   for all forces acting on the structure
(3) $\Sigma F_y = 0$   for all forces acting on the structure
(4) $\Sigma M_a = 0$   for all forces acting on one side of hinge $a$
(5) $\Sigma M_b = 0$   for all forces acting on one side of hinge $b$
(6) $\Sigma F_x = 0$   for all forces acting on one side of section $A$-$A$
(7) $\Sigma F_y = 0$   for all forces acting on one side of section $B$-$B$

There are also seven independent reactions, $R_{hy}$, $R_{iy}$, $R_{ix}$, $R_{fy}$, $R_{fx}$, $R_{gy}$, and $R_{jy}$.   Therefore this structure is statically determinate with respect to its outer forces.   That the structure is also statically determi-

nate with respect to both outer and inner forces may be verified from the fact that the total number of bars (75) plus the total number of reactions (7) equals twice the number of joints $(2 \times 41) = 82$.

**8·4   Stress Analysis for Cantilever Trusses.**   To determine the reactions for the structure of Fig. 8·3 due to the load acting as shown, one may proceed as follows: Assume all reactions to act in the directions shown in the figure.   Apply the equation $\Sigma F_x = 0$ to that portion of the structure to the left of section $A$-$A$; this shows that $R_{ix} = 0$.

Apply $\Sigma F_x = 0$ to the entire structure; this shows that $R_{fx} = 0$.

Apply $\Sigma M_b = 0$ to that portion of the structure to the left of hinge $b$:
$$+ R_{hy}(6)(30) + R_{iy}(2)(30) = 0; \qquad R_{iy} = -3R_{hy}$$

Apply $\Sigma M_a = 0$ to that portion of the structure to the left of hinge $a$:
$$- (100)(3)(30) + R_{hy}(12)(30) + (-3R_{hy})(8)(30) = 0; \qquad R_{hy} = -25;$$
$$R_{iy} = -3(-25) = +75$$

Apply $\Sigma F_y = 0$ to that portion of the structure to the left of section $B$-$B$:
$$-25 + 75 - 100 + R_{fy} = 0; \qquad R_{fy} = +50$$

Apply $\Sigma F_y = 0$ to that portion of the structure to the right of section $B$-$B$:
$$+ R_{gy} + R_{iy} = 0; \qquad R_{gy} = -R_{iy}$$

Finally, applying $\Sigma M_a = 0$ to that portion of the structure to the right of hinge $a$:
$$-50(3)(30) - (-R_{iy})(4)(30) - R_{iy}(9)(30) = 0; \qquad R_{iy} = -30;$$
$$R_{gy} = +30$$

The available equations can often be applied in different sequences; but if the structure is stable, the same results will be obtained, regardless of the sequence followed.

Once the reactions are known, the bar stresses can be computed by the usual methods of analysis for statically determinate trusses.   Since the analysis can be carried out for a unit load at any panel point, the construction of influence lines for reactions or bar stresses involves no special difficulties, although it is often advantageous first to construct influence lines for reactions or bar stresses other than the one under consideration and use the data thus obtained in constructing the influence line actually desired.

The foregoing procedure is illustrated by constructing the influence line for bar $a$ of the structure of Fig. 8·4a.   The influence line for the stress in bar $FE$ will first be constructed, with stress computed by taking moments about $D$ of the forces acting on that part of the structure between $D$ and section $M$-$M$.   The influence line for the stress in bar $a$ is then drawn, with stresses computed by taking moments about $G$ of the forces acting on that part of the structure between sections $M$-$M$

and *N-N.* The forces entering into the resultant equations consist of the stress in the hanger *FE*, the applied unit load, and the stress in bar *a* itself.
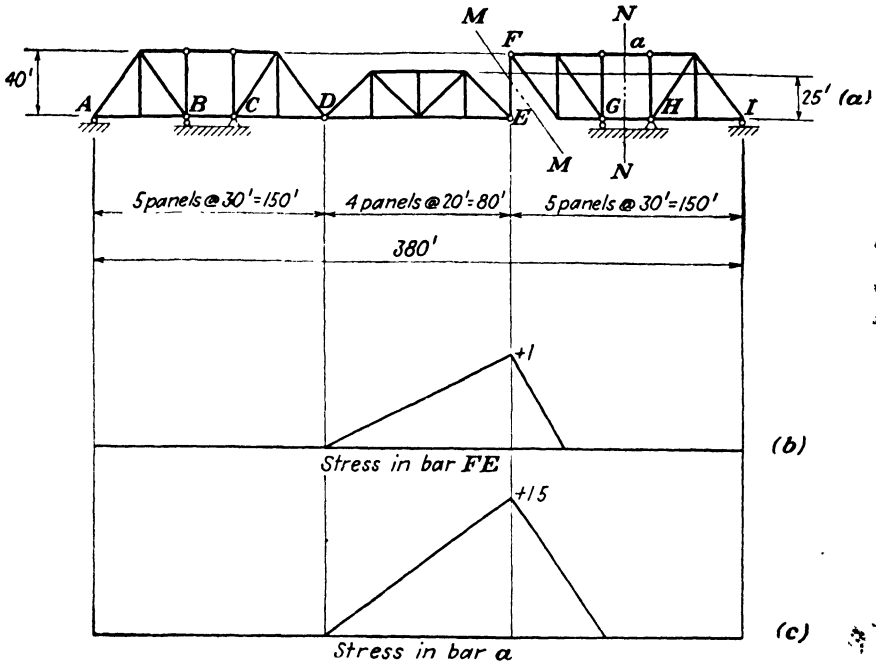


Fig. 8·4

**8·5 Continuous Structures.** The reduction in maximum moment that results from the shortening of the span effective in producing positive bending in cantilever construction may be obtained in a somewhat similar manner by continuity in a structure, although continuous structures are usually statically indeterminate. The fixed end beam shown in Fig. 8·5*a* will deflect under a uniformly distributed load as shown by the dotted line *ABCD*. At points *B* and *C*, the bending changes from one character to another, so that the curvature of the deflected beam reverses. At such points (called *points of inflection*) the bending moment is zero, so that the curve of moments for the beam *AD* is just as it would be if there were hinges at these inflection points. Because of construction difficulties in developing the fixed end moments at *A* and *D* for long-span structures, partial restraint against changes of slope at these points may be obtained by the addition of flanking spans, as shown in Fig. 8·5*b*. The location of the points of inflection *B'* and *C'* will depend on the span ratio *a*. The continuous truss shown in Fig.

8·5c derives the same advantages from continuity as does the continuous beam of Fig. 8·5b. With five independent reactions, this continuous truss is statically indeterminate to the second degree, and its analysis
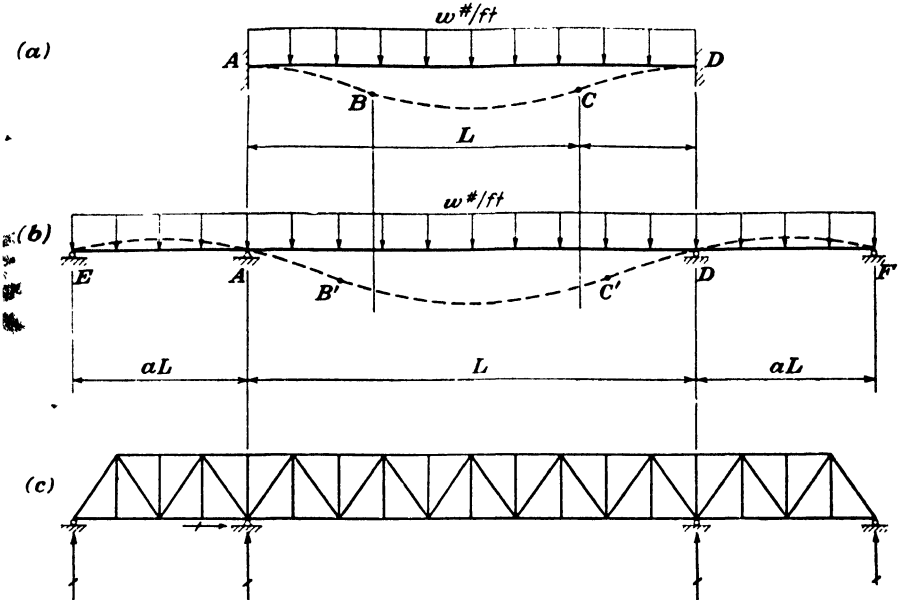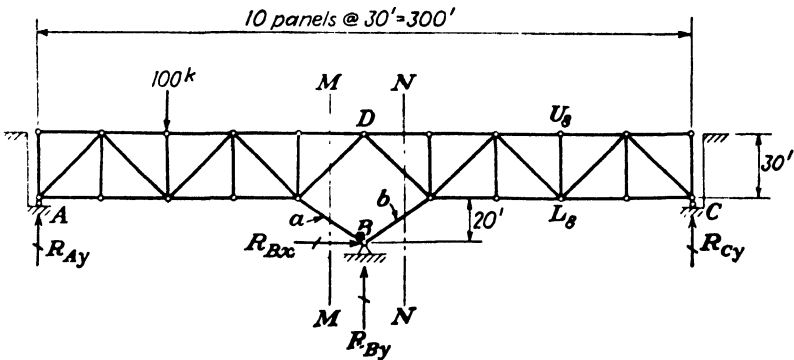
(a)

(b)

(c)



FIG. 8·5



FIG. 8·6

depends on methods that take into consideration the elastic properties of the structure.

It is possible, however, by omitting certain bars, to make a continuous truss statically determinate. This is accomplished in the Wichert[1]

[1] See STEINMAN, D. B., "The Wichert Truss," Van Nostrand Co., Inc., New York, 1932.

truss, invented and patented by E. M. Wichert of Pittsburgh, Pa., by omitting the vertical members over intermediate points of support. Such a structure is shown in Fig. 8·6. This structure has 40 bars and 4 reactions, or a total of 44 unknowns; there are 22 joints, hence 44 equations of statics available for the determination of inner and outer forces; the structure therefore has the correct count for statical determination and is actually statically determinate unless certain slopes are given to bars $a$ and $b$, in which case it is possible for the structure to become geometrically unstable, a condition that can be recognized by inconsistent results from the application of the equations of statics.

For the structure of Fig. 8·6, acted upon by the load shown, the application of $\Sigma F_x = 0$ to the entire structure shows that $R_{Bx} = 0$. To determine the vertical reactions, one may proceed as follows: Taking moments about $D$ of the forces to the right of section $N$-$N$ and assuming all vertical reactions to act upward,

$$-R_{cy}(5)(30) + X_b(50) = 0; \qquad X_b = +3R_{cy};$$
$$Y_b = +\tfrac{2}{3}X_b = +2R_{cy}$$

Taking moments about $D$ of the forces to the left of section $M$-$M$,

$$+R_{Ay}(5)(30) - 100(3)(30) - X_a(50) = 0; \qquad X_a = +3R_{Ay} - 180$$
$$Y_a = +\tfrac{2}{3}X_a = +2R_{Ay} - 120$$

Applying $\Sigma F_y = 0$ at joint $B$,

$$+R_{By} + 2R_{cy} + 2R_{Ay} - 120 = 0$$

whence

$$R_{By} = 120 - 2R_{Ay} - 2R_{cy}$$

Since the foregoing equation expresses the center reaction in terms of the two end reactions, these end reactions may now be determined by applying $\Sigma M = 0$ and $\Sigma F_y = 0$ to the entire structure. For $\Sigma M_A = 0$,

$$+ 100(2)(30) - (120 - 2R_{Ay} - 2R_{cy})(5)(30) - R_{cy}(10)(30) = 0$$

For $\Sigma F_y = 0$,

$$R_{Ay} - 100 + (120 - 2R_{Ay} - 2R_{cy}) + R_{cy} = 0$$

The solution of these two equations leads to $R_{Ay} = +40$ and $R_{cy} = -20$, whence $R_{By} = 120 - 2(+40) - 2(-20) = +80$.

With the reactions known, the bar stress analysis presents no further difficulty. Since an analysis can be carried out in the foregoing manner for a unit load at any panel point, influence lines for reactions and bar stresses can be constructed.

**8·6   Arches—General.**   Another method of reducing maximum moments in long-span structures consists in adopting a structural layout in which applied vertical loads produce horizontal reactions that act in a manner such that the moments due to these horizontal reactions tend to reduce the moments that would otherwise exist.   Figure 8·7 shows an arch, which is a structure that develops horizontal *thrust* reactions under the action of vertical loads.   This particular arch is of the two-hinged type.   The vertical reactions on this structure may be determined by
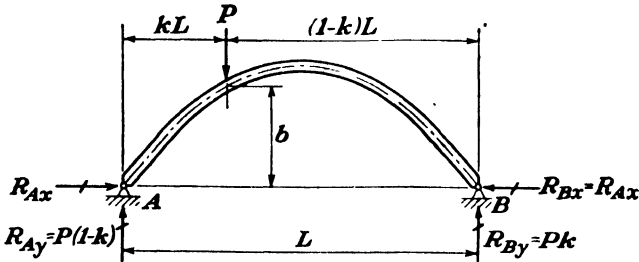


Fig. 8·7

statics by taking moments about an end hinge of all the forces acting on the arch; for the load $P$ acting as shown, they have the values given on the figure.   The relation between the horizontal reactions $R_{Ax}$ and $R_{Bx}$ can be determined by statics ($\Sigma F_x = 0$), but the actual values of these reactions can be obtained only on the basis of an elastic analysis, since



Fig. 8·8

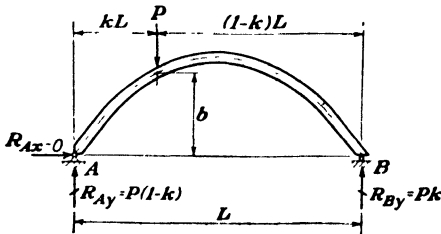with four independent reactions this two-hinged arch is statically indeterminate to the first degree.

If the hinge at one end were replaced with a roller, as shown in Fig. 8-8, the structure would be, not an arch, but a statically determinate curved beam, and the moment at the point of application of the load would equal $P(1 - k)kL$.   For the two-hinged arch of Fig. 8·7, however, this moment is reduced by the moment $R_{Ax}b$.

An arch may be made statically determinate by building in a third hinge at some internal point, such as at the crown, in addition to the end hinges.   Such a structure is shown in Fig. 8·9 and is called a three-hinged arch.   This structure has four independent reactions; three equations of statics can be applied to the structure as a whole, and one equation of condition can be obtained by taking moments about the hinge at $C$ of the forces acting on either side of the hinge.   Then taking moments about $A$ of all the forces acting on the structure leads to

$$R_{By} = +100(^{20}\!\!/_{100}) = +20$$

similarly, taking moments about hinge $B$, $R_{Ay} = +100(^{80}\!\!/_{100}) = +80$. Now, taking moments about the hinge $C$, of the forces acting on that portion of the structure to the right of the hinge,

$$+R_{Bx}(30) - 20(50) = 0; \qquad R_{Bx} = +33.3$$

Applying $\Sigma F_x = 0$ to the entire structure, $R_{Ax} = +33.3$.

The moment at the point of application of the load is given by

$$M_D = +80(20) - 33.3(25) = +767 \text{ kip-ft}$$

For a simple end-supported beam of the same span and loading, the moment at the load would equal $+80(20) = +1,600$ kip-ft.  Hence the arch construction has reduced this moment by 52 per cent.  The arch ribs must, however, carry compression that is not present in the end-supported beam.  For example, the compression at the crown $C$ in the arch of Fig. 8·9, where the rib is horizontal, equals the horizontal reaction and therefore has a value of $-33.3$ kips



FIG. 8·9

for the load considered.  It is, however, usually more economical to carry loads in direct stress than in bending, although, if the direct stress is compression, one must provide stability against elastic buckling.
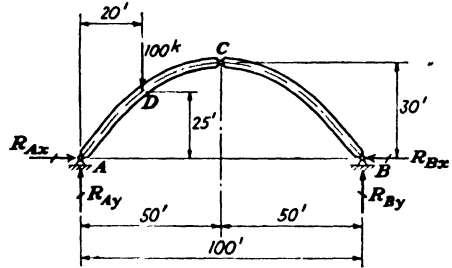
**8·7   Analysis of Three-hinged Trussed Arch.**   The arch ribs $AC$ and $BC$ of the structure of Fig. 8·9 may be replaced by trusses as shown in Fig. 8·10.  Since there are four reactions on this structure, it would be statically indeterminate were it not for the hinge at $e$, which is effective since the bar $EF$, shown dotted, is connected at its ends in such a manner that it can carry no direct stress.

The reactions for this structure may be computed as follows: Taking moments about $a$, of all the forces acting on the structure,

$$+100(30) + 200(2)(30) + 300(6)(30) - R_{iy}(8)(30) = 0;$$
$$R_{iy} = +287.5 \text{ kips}$$

Taking $\Sigma F_y = 0$ for all forces acting on the structure,

$$100 - 200 - \ldots + 287.5 = 0; \qquad R_{ay} = +312.5 \text{ kips}$$

To obtain $R_{ix}$, apply $\Sigma M = 0$ about the hinge at $e$, considering the forces acting on the part of the structure to the right of the hinge.

$$+300(2)(30) + R_{\cdot x}(48) - 287.5(4)(30) = 0; \qquad R_{ix} = +344 \text{ kips}$$

Since $\Sigma F_x = 0$ for the entire structure, $R_{ax}$ also equals $+344$ kips. $R_{ax}$ might have been computed by taking moments about the hinge at $e$
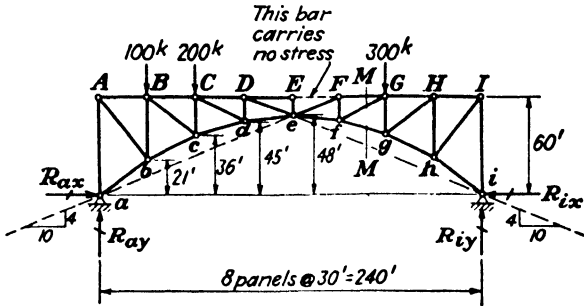


Fig 8·10

of the forces to the left of the hinge, leading to

$$+312.5(1)(30) - R_{ax}(48) - 100(3)(30) - 200(2)(30) = 0;$$
$$R_{ax} = +344 \text{ kips}$$

In computing bar stresses, the effect of the horizontal reactions must not be overlooked. To compute the stress in $FG$, for example, taking moments about $f$ of the forces to the right of section $M$-$M$,

$$+300(30) + 344(15) - 287.5(90) - F_{FG}(15) = 0; \qquad F_{FG} = -93 \text{ kips}$$

If there is no load between the center hinge and one end of the truss, the resultant reaction at that end of the truss must have a direction such that it passes through the center hinge, because the moment about the hinge of the forces acting on that side of the hinge must equal zero. Thus, if we consider the action of a unit vertical load at $B$ on the structure of Fig. 8·10, $R_{iy} = +\frac{1}{8}$; since the resultant reaction at $i$ lies along the dotted line drawn through $i$ and $e$, we conclude immediately that

$$R_{ix} = +\frac{1}{8}(1\frac{1}{4}) = +\frac{5}{16}$$

This fact is often convenient in analysis, particularly in constructing influence lines.

If the three-hinged arch of Fig. 8·10 were subjected to equal vertical panel loads at each top-chord panel point (or at each bottom-chord panel point), the following stress condition would result: (1) The stress in each top chord would be zero.  (2) The stress in each diagonal would

be zero.   (3) The stress in each vertical would equal the top-chord panel load.   (4) The horizontal component of stress in each bottom chord would be the same and equal to the horizontal reactions.   These facts may be verified by the student and would be useful, for example, in computing dead stresses for a uniformly distributed dead load.   Such conditions exist because the bottom-chord panel points of this structure lie on a parabola.   If a funicular polygon were drawn for the loads under consideration, so that it passed through the three hinges, the polygon would coincide with the location of the bottom chords.

**8·8   Influence Lines for Three-hinged Trussed Arch.**   Influence lines for a three-hinged trussed arch may be constructed by considering successive positions of the unit load, but a method similar to the following will often prove advantageous: We shall construct the influence line for bar $FG$ of the structure of Fig. 8·10.   We shall first construct the influence line for that portion of the stress in bar $FG$ due to the unit load and the vertical reactions only.   Since these reactions have the same values they would have for an end-supported beam, the influence line for this portion of the stress is a triangle with its maximum value occurring when the load is directly over the center of moments $f$,
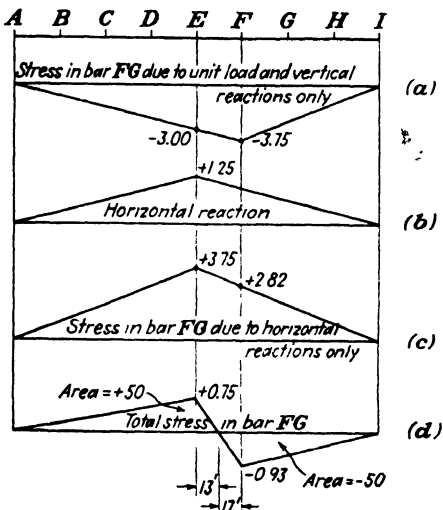


Fig. 8·11

where the ordinate equals $-\frac{5}{8}(\frac{90}{15}) = -3.75$, as shown in Fig. 8·11a. We shall next construct an influence line for $R_{ax} = R_{ix}$.   As the unit load travels from $A$ to $E$, $R_{iy}$, hence $R_{ix}$ (which equals $\frac{5}{2}R_{iy}$) increases linearly. With the load at $E$, $R_{ix}$ equals $\frac{5}{2}(+\frac{1}{2}) = +1.25$.   Hence the influence line for $R_{ix}$ is a straight line from zero at $A$ to $+1.25$ at $E$.   By similar reasoning, considering the reactions at $a$, the influence line for $R_{ax} = R_{ix}$ is a straight line from $+1.25$ at $E$ to zero at $I$.   The influence line for the magnitude of this horizontal reaction is shown in Fig. 8·11b.   The stress in bar $FG$ due to the horizontal reactions equals

$$+R_{ix}(\frac{45}{15}) = +3R_{ix}$$

hence the influence line for this portion of the stress in bar $FG$ is a triangle with its apex at $E$, where the ordinate equals $+3(+1.25) = +3.75$, as shown in Fig. 8·11c.   The influence line for the total stress in bar $FG$

is now obtained by superimposing the influence lines of Figs. 8·11a and *c*, leading to the influence line of Fig. 8·11*d*.

It will be noted that the net area under this influence line is zero, as should be the case, since a uniform load extending over the entire structure would cause no stress in the top chords.

**8·9   Three-hinged Trussed Arches with Supports at Different Elevations.**   The points of support of a three-hinged trussed arch may be at different elevations, as shown in Fig. 8·12.   The vertical reactions will then differ from the values they would have for an end-supported
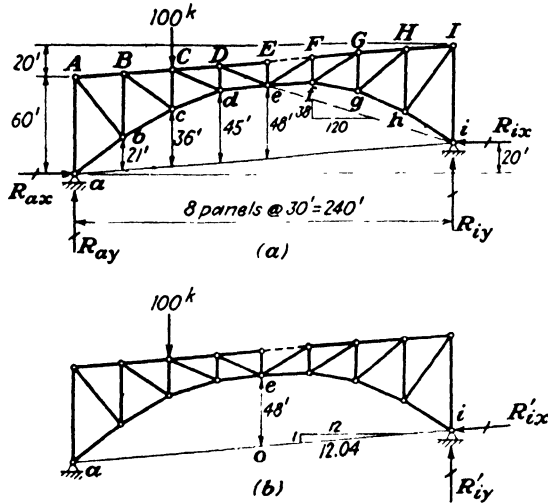


Fig. 8·12

truss, since, when moments are taken about one point of support, the horizontal reaction at the far end enters into the equation.   The reactions can, however, be obtained by statics.   Referring to Fig. 8 12*a* and taking moments about *a* of all the forces acting on the structure,

$$+100(60) - R_{ix}(20) - R_{iy}(240) = 0$$

whence $R_{ix} = +300 - 12R_{iy}$.   Now, taking moments about the hinge at *e* of the forces acting to the right of the hinge,

$$+R_{ix}(38) - R_{iy}(120) = 0;   R_{iy} = +\tfrac{38}{120}R_{ix}$$
$$= +\tfrac{38}{120}(+300 - 12R_{iy})$$

whence

$$R_{iy} = +19.8 \text{ kips and } R_{ix} = +300 - 12(19.8) = +62.5 \text{ kips}.$$

The foregoing solution required the solution of two simultaneous equations, which might have been avoided by taking the reactions as

shown in Fig. 8·12b, where $R'_{ix}$ is taken as acting along a line passing through the two points of support and $R'_{iy}$, while a vertical reaction, differs from $R_{iy}$, since $R'_{ix}$ has a vertical component.   Taking moments about $a$,

$$+100(60) - R'_{iy}(240) = 0; \qquad R'_{iy} = +25.0$$

Taking moments about $e$ and considering $R'_{ix}$ to act at point $o$,

$$+ \frac{12.00}{12.01} R'_{ix} (48) - 25.0(120) = 0$$

whence $R'_{ix} = +62.7$ kips. Hence the actual horizontal reaction at $i$ is given by

$$R_{ix} = +62.7 \left( \frac{12.00}{12.01} \right) = +62.5 \text{ kips}$$

while the actual vertical reaction at $i$ is given by

$$R_{iy} = +25.0 - \frac{1}{12.01} (62.7) = +19.8 \text{ kips}$$

For the particular case under consideration, since there are no loads applied to the right of the hinge at $e$, we might have concluded immediately that the resultant reaction at $i$ passed through $e$, whence

$$R_{ix} = {}^{120}\!/_{38} R_{iy} = +3.16 R_{iy}$$

Then, taking moments about $a$,

$$+100(60) - 3.16 R_{iy}(20) - R_{iy}(240) = 0$$

whence

$$R_{iy} = +19.8 \text{ kips and } R_{ix} = +3.16(+19.8) = +62.5 \text{ kips}$$

**8·10  Suspension Bridges.**  An important method of reducing bending moments in long-span structures consists in providing partial support at points along the span by means of a system of cables, as in a suspension bridge. Referring to Fig. 8·13, a suspension bridge is usually erected in a manner such that all the dead load is carried by the cable. When live load is applied to such a structure, tension in the hangers transfers a large portion of the live load to the cable.   Hence, the stiffening truss $AB$ is subjected to no dead moments, and the live moments it must carry are substantially reduced.   For long-span structures, this is of particular importance, since so much of the load is carried by the cable in tension, which is a highly efficient manner of carrying loads.

A suspension bridge such as that of Fig. 8·13 is statically indeterminate.  By the introduction of certain features of construction, it may, however, be made statically determinate.  The analysis of statically determinate suspension bridges is treated in Art. 11·10.
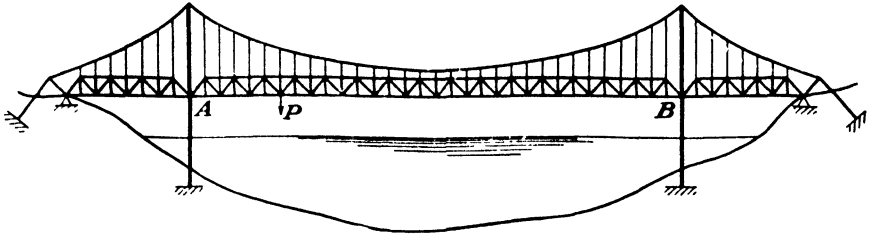
FIG. 8·13

## 8·11 Problems for Solution.

**Problem 8·1**   *a.* Construct an influence line for the vertical reaction at *A* of the structure of Fig. 8 2, taking $a = L/2$.

*b.* How does the maximum reaction at *A* due to a uniform live load of 1,000 lb per ft compare with the maximum pier load that would result at *A* if *EA*, *AD*, and *DF* were simple end-supported spans?

**Problem 8·2**   Referring to the cantilever bridge of Fig. 8 3, construct an influence line for (*a*) the stress in the hanger *bc*; (*b*) the vertical reaction at *i*; (*c*) the stress in the top chord *de*.   (*d*) Compute the maximum stress in bar *de* due to the following loading: dead load, 2,000 lb per ft; uniform live load, 1,000 lb per ft; concentrated live load, 10,000 lb.

**Problem 8·3**   Determine the stresses in all the bars of the Wichert truss of Fig. 8 6 due to a load of 100 kips acting at *D.*

**Problem 8·4**   Construct an influence line for the reaction at *A* of the Wichert truss of Fig. 8 6.

**Problem 8·5**   A three-span Wichert truss is the same on each side of an axis of symmetry as that portion of Fig. 8 6 to the left of $U_8L_8$, except for the fact that the pier symmetrical to *B* has a roller support.   Compute all the reactions on this structure due to a uniform load of *w* lb per ft extending over the entire structure, applied to the top-chord panel points.

**Problem 8·6**   Compute the shear and direct stress at point *D* of the structure of Fig. 8 9 due to the load shown, assuming that the slope of the arch rib at *D* is 30° from the horizontal.

**Problem 8·7**   Compute the stresses in all the members of the three-hinged arch of Fig. 8 10 due to a uniform load of 1,000 lb per ft applied along the entire length of the structure.   Panel loads are to be applied at the top-chord panel points.

**Problem 8·8**   Construct an influence line for the stress in bar *bc* of the structure of Fig. 8 10.

**Problem 8·9**   What is the maximum stress in bar *Bc* of the structure of Fig. 8 10 due to the following loading: Dead load, 1,000 lb per ft; uniform live load, 500 lb per ft; concentrated live load, 5,000 lb?

**Problem 8·10**   Solve Prob. 8 8, using Fig. 8·12*a.*

**Problem 8·11**   Solve Prob. 8 9, using Fig. 8·12*a.*

# CHAPTER 9

## THREE-DIMENSIONAL FRAMEWORKS

**9·1 Introduction.** Although most engineering structures are three-dimensional, it is often permissible to break a three-dimensional structure down into component planar structures and to analyze each planar structure for loads lying in its plane. Consider, for example, a typical through-parallel-chord truss highway bridge. Such a structure is three-dimensional, but it can be broken down into six component structures, each of which is planar, the two main vertical trusses, the top-chord lateral system, the bottom-chord lateral system, and the two end portals. Often a given member must be considered as part of more than one component planar structure: a bottom chord of a vertical truss, for example, is also likely to be a chord of the bottom chord lateral system. This introduces no difficulty, since for such members the stresses can be computed for each component planar structure in which it participates and the resultant stresses superimposed to give the total stress in the member.

In some three-dimensional structures, however, the stresses are interrelated between members not lying in a plane, in a manner such that the analysis cannot be carried out on the basis of component planar structures. For such structures, a special consideration of the analysis of three-dimensional structures is necessary.

Structures that may fall into this latter classification include towers, guyed masts, derricks, framing for domes, and framing for aircraft, to mention only a few. Such structures may be either statically determinate or statically indeterminate. In this chapter, consideration will be given to statically determinate three-dimensional structures, but the methods for analyzing statically indeterminate structures, given elsewhere in this book, are applicable in principle to statically indeterminate three-dimensional structures.

In this treatment of three-dimensional structures, analysis will be made with reference to three coordinate axes. *OX* and *OY* will be used as in the case of planar structures, i.e., with *OX* horizontal and *OY* vertical; the third axis *OZ* is horizontal and perpendicular to the plane *XOY*.

It should be pointed out that the basic approach to the analysis of three-dimensional structures is the same as for planar structures. Any equation of statics may be applied to the structure as a whole or to any portion of the structure. There are, however, more equations of statics

available, since forces may be summed up along a new coordinate axis
and moments may be taken about two new coordinate axes.

**9·2   Statical Condition.**   It is usually assumed that the members
of a three-dimensional framework are pin-connected in a manner such
that the members carry axial stress only.   Hence there is only one
independent component of stress for each member of the framework.
Although each member can have three components of stress, one parallel
to each of the three coordinate axes, the relations between these three
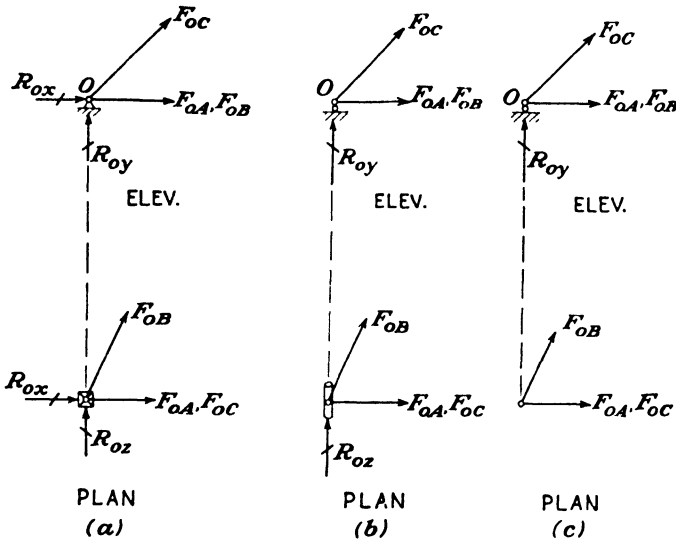components can be computed from the projections of the member.



Fɪɢ. 9·1

At a point of support of a space framework, it is possible to have three
independent components of force reactions, although the structure may
be designed at a point of support so that one or more of these reaction
components equals zero.   In Fig. 9·1a, the hinge support shown, which
is actually a universal joint, can develop three independent reactions,
$R_{ox}$, $R_{oy}$, and $R_{oz}$.   Suppose that the hinge is replaced with a roller, as
shown in Fig. 9·1b.   Since this resists horizontal movement in the Z
direction only, the reactions $R_{oy}$ and $R_{oz}$ can be developed but $R_{ox}$ equals
zero.   If the roller is replaced by a spherical ball, there can be no hori-
zontal reaction whatever and only the vertical reaction $R_{oy}$ can be
developed.   This condition is shown in Fig. 9·1c.

To represent these three types of support in plan view, heavy dotted
lines will be drawn along the lines of action where horizontal reactions
can exist.   This is illustrated in Fig. 9·2, where a represents a hinge-

type support in which horizontal reactions can be developed in both the $X$ and $Z$ direction; $b$ represents a roller-type support, with the roller so placed that a horizontal reaction can be developed in the $Z$ direction only; $c$ represents a roller-type support, with the roller so placed that a horizontal reaction can be developed in the $X$ direction only; and $d$ represents a ball-type support, in which there can be no horizontal reaction.

Thus the total number of independent unknown stress elements present in the analysis of a three-dimensional framework equals the number of bars plus the number of independent reaction components, there being one, two, or three of the latter at each point of support, depending on the type of construction used at the reaction point.

For a three-dimensional framework, six independent equations of statics may be written regarding the equilibrium of the external loads
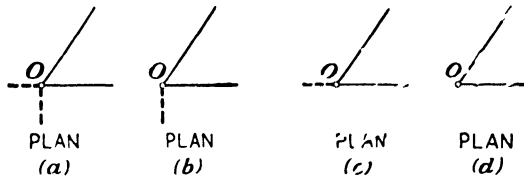


PLAN　　　PLAN　　　PLAN　　　PLAN
(a)　　　　(b)　　　　(c)　　　　(d)

Fig. 9 2

and reactions acting on the entire structure. If $OX$, $OY$, and $OZ$ represent the three coordinate axes, these equations are $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_z = 0$, $\Sigma M_x = 0$, $\Sigma M_y = 0$, and $\Sigma M_z = 0$. $\Sigma M_x$ denotes the sum of the moments about the $OX$ axis of all the forces acting on the structure, etc.

*It may therefore be concluded that a necessary (although not sufficient) condition for statical determination of a three-dimensional framework with respect to its outer forces is that the total number of independent reactions shall equal six.*

If we now consider both the internal and the external forces, three independent equations of statics can be written for each joint, *viz.*, $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma F_z = 0$. Equations of statics applied to the structure as a whole will not furnish further independent equations. *It may therefore be concluded that a necessary (although not sufficient) condition for statical determination of a three dimensional framework with respect to both inner and outer forces is that the total number of bars plus the total number of independent reactions shall equal three times the number of joints.*

The application of these principles may be illustrated by considering Fig. 9·3d. Considering first only the external forces, if the horizontal

reactions are arranged as shown in the plan view, there is a total of 9 independent reactions, so that the structure is statically indeterminate to the $9 - 6 =$ third degree. If rollers are substituted for hinges, so that the horizontal reactions act as shown in Fig. 9·3b, the number of independent reactions is six and the structure is statically determinate. Suppose, however, that the rollers are placed so that the horizontal reactions have the directions indicated in Fig. 9·3c. Then, while the numerical count indicates that this structure is statically determinate with respect to its outer forces, it is actually unstable. The $Z$ reaction at $b$ for example, will, apparently have two values, depending on whether it is determined by applying $\Sigma F_z = 0$ to the entire structure (in which case it equals zero) or whether it is computed by applying the equation $\Sigma M_y = 0$ about a vertical axis through $a$ (in which case it must have a value). This shows that the numerical count, while a necessary condition for statical determination, is not a sufficient criterion. The reactions must be placed so that they can resist translation along and rotation about each of the three coordinate axes, if a three-dimensional



FIG. 9·3

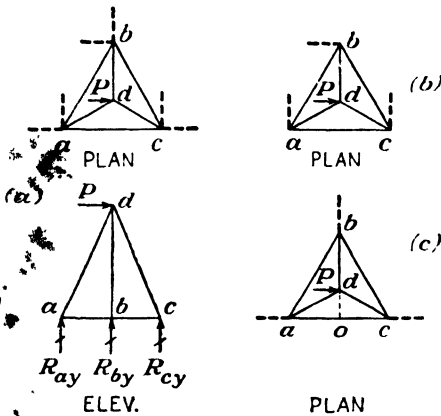structure is to be stable. The reactions shown in Fig. 9·3c all pass through point $o$; they cannot resist rotation about a vertical axis passing through that point.

Considering now both internal and external forces, in Fig. 9·3a, there are present 15 independent stress unknowns: 6 bar stresses and 9 reactions. There are 4 joints and hence $4 \times 3 = 12$ independent equations of statics. Therefore the structure is statically indeterminate to the $15 - 12 =$ third degree. With the horizontal reactions arranged as shown in Fig. 9·3b, there are only 12 independent stress unknowns: 6 bar stresses and 6 reactions. There are still 12 independent equations of statics. Hence the structure is statically determinate with respect to both inner and outer forces.

**9·3 Determination of Reactions.** If a three-dimensional framework is statically determinate with respect to its outer forces and if it is supported at only three points, its reactions can be readily determined by applying the equations of statics to the structure as a whole. When there are more than three points of support, it is usually necessary to determine

some or all of the bar stresses before the reactions can be evaluated.    In
this article, consideration will be given to structures where the reactions
can be determined directly.

In Fig. 9·4, the vertical reactions will first be determined.  If mo-
ments are taken about a horizontal axis passing through any two points
of support, the vertical reaction at the third point of support will be the
only unknown occurring in the resulting equation.  Upon applying
$\Sigma M_x = 0$ about $ac$ as an axis, $R_{by}$ is the only external force that could have
a moment; hence, $R_{by} = 0$.  Upon
applying $\Sigma M_z = 0$ about the line of
action of $R_{az}$,

$$+10{,}000(20) - R_{cy}(20) = 0$$

from which $R_{cy} = +10{,}000$ lb.  Apply-
ing $\Sigma F_y = 0$ to the entire structure,

$$R_{ay} + 10{,}000 = 0$$

from which $R_{ay} = -10{,}000$ lb.

To determine the horizontal reac-
tions, if $\Sigma M_y = 0$ is applied about a
vertical axis passing through the inter-
section of the lines of action of any two
horizontal reactions, the third hori-
zontal reaction will be the only
unknown occurring in the resulting
equation.  Taking moments, for ex-
ample, about a vertical axis through



FIG. 9·4

point $o$ and assuming $R_{cz}$ to act toward the rear of the structure,

$$-10{,}000(5) - R_{cz}(20) = 0; \qquad R_{cz} = -2{,}500 \text{ lb}$$

Now, applying $\Sigma F_z = 0$ to the entire structure and assuming $R_{az}$ to act
toward the rear of the structure,

$$R_{az} - 2{,}500 = 0; \qquad R_{az} = +2{,}500 \text{ lb}$$

Finally, applying $\Sigma F_x = 0$ to the entire structure and assuming $R_{bx}$ to act
to the left,

$$+10{,}000 - R_{bx} = 0; \qquad R_{bx} = +10{,}000 \text{ lb}$$

As was the case with planar structures, the particular equations of
statics used and the order in which they are applied may be varied in
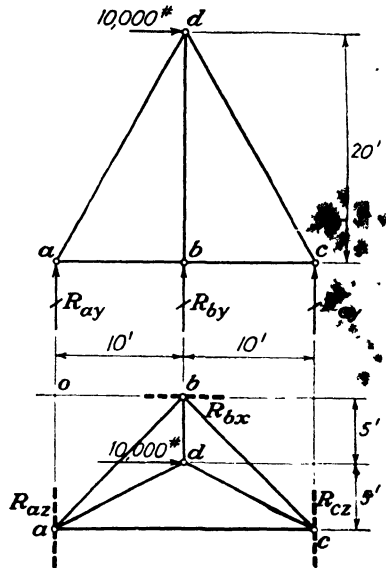accordance with the ingenuity of the analyst.

**9·4  Determination of Bar Stresses.**  A bar of a three-dimensional framework may have projections on each of three coordinate axes. This is illustrated in Fig. 9·5, where the bar $ab$ has the projections $ax$, $ay$, and $az$ in the directions of the $OX$, $OY$, and $OZ$ axes, respectively.
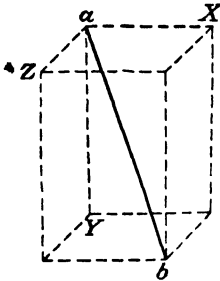
In terms of these projected lengths, the length of the bar $ab$ is given by $ab = [(ax)^2 + (ay)^2 + (az)^2]^{1/2}$. Since the stress $F_{ab}$ is axial, the components of $F_{ab}$ parallel to the coordinate axes are

$$X_{ab} = F_{ab}\frac{ax}{ab}; \qquad Y_{ab} = F_{ab}\frac{ay}{ab}; \qquad Z_{ab} = F_{ab}\frac{az}{ab}$$

By combining these relations, it is easy to express one component of stress in terms of either of the other stress components.  For example, $X_{ab} = Y_{ab}(ax/ay)$, etc.



FIG. 9·5

any joint where the converging bars do not lie in a plane, three equations of statics are available for bar stress determination.  Hence, if not more than three bars with unknown stresses meet at such a joint, these stresses may be determined.  This general method of procedure, which is the method of joints expanded to three dimensions, may be illustrated by its application to joint $d$ of the structure of Fig. 9·4.[1]  The following table of dimensions will first be prepared:

Table 9·1

| Member | Length | Projections | | |
|---|---|---|---|---|
| | | X | Y | Z |
| $ad$ | 22.9 | 10 | 20 | 5 |
| $bd$ | 20.6 | 0 | 20 | 5 |
| $cd$ | 22.9 | 10 | 20 | 5 |
| $ab$ | 14.14 | 10 | 0 | 10 |
| $bc$ | 14 14 | 10 | 0 | 10 |
| $ac$ | 20 0 | 20 | 0 | 0 |

At joint $d$, assuming all bars to be in tension, the following equations may be written:

$$\Sigma F_x = 0, \qquad +10{,}000 - X_{ad} + X_{cd} = 0$$
$$\Sigma F_y = 0, \qquad -Y_{ad} - Y_{bd} - Y_{cd} = 0$$
$$\Sigma F_z = 0, \qquad +Z_{ad} - Z_{bd} + Z_{cd} = 0$$

[1] The student should note that this particular application is given solely for the purpose of illustrating a procedure of general importance.  The solution of this joint could be substantially simplified by the application of theorem I, Art. 9·6.

The nine components of stress involved in these three equations may be expressed in terms of the three independent bar stresses $F_{ad}$, $F_{bd}$, and $F_{cd}$ as follows:

$$+10,000 - \frac{10}{22.9} F_{ad} + \frac{10}{22.9} F_{cd} = 0$$

$$-\frac{20}{22.9} F_{ad} - \frac{20}{20.6} F_{bd} - \frac{20}{22.9} F_{cd} = 0$$

$$+\frac{5}{22.9} F_{ad} - \frac{5}{20.6} F_{bd} + \frac{5}{22.9} F_{cd} = 0$$

The simultaneous solution of these three equations leads to

$$F_{ad} = +11,150 \text{ lb}; \qquad F_{bd} = 0; \qquad F_{cd} = -11,450 \text{ lb}$$

In this particular structure, the vertical reactions can first be determined, as is shown in Art. 9·3, and the stresses in bars $ad$, $bd$, and $cd$ can be computed more easily by taking advantage of that fact. At joint $a$, for example, the vertical reaction acts downward and equals 10,000 lb. Applying $\Sigma F_y = 0$ at joint $a$,

$$-10,000 + Y_{ad} = 0; \qquad Y_{ad} = +10,000;$$

$$F_{ad} = +10,000 \left(\frac{22.9}{20.0}\right) = +11,150*$$

$F_{bd}$ and $F_{cd}$ can be similarly obtained from $\Sigma F_y = 0$ at joints $b$ and $c$, respectively. However, it is important for the general approach based on writing three simultaneous equations at joint $d$ to be understood, since in more complicated three-dimensional structures it may be the only procedure that can be used.

The analysis of the bars connecting the points of support of Fig. 9·4 will now be carried out, advantage being taken of the fact that the horizontal reactions are determined in Art. 9·3. The values of these reactions are shown in Fig. 9·6. As an illustration, joint $a$ will be considered and the equation $\Sigma F_z = 0$ written. Noting that $Z_{ad} = +11,450(5/22.9)$,
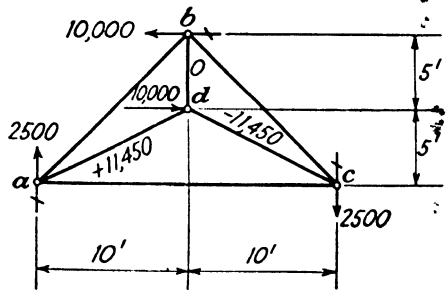


**PLAN**

Fig. 9·6

* Note that this equation may be written directly as

$$-10,000 + \frac{20.0}{22.9} F_{ad} = 0; \qquad F_{ad} = +11,150$$

$$+2,500 + 11,450 \left(\frac{5}{22.9}\right) + F_{ab}\left(\frac{10}{14.14}\right) = 0; \qquad F_{ab} = -7,070 \text{ lb}$$

The stress in bar $ac$ can now be found by a similar procedure, by applying $\Sigma F_x = 0$ to joint $a$. $F_{bc}$ can be obtained by applying $\Sigma F_z = 0$ at joint $c$.

## 9·5 Case Where Reactions Cannot Be Computed without Considering Bar Stresses.

The structure of Fig. 9·7 is statically



FIG. 9·7

determinate with respect to its inner and outer forces combined. Since there are only three horizontal reactions, these may be determined by a consideration of the external forces only but the vertical reactions cannot be computed without taking the bar stresses into consideration. If the stresses in $ab$, $ac$, $ad$, and $ae$ can be computed, then the vertical reactions may be obtained from the vertical components of these bar stresses.

At joint $a$, 4 unknown bar stresses are present, so that, with only 3 equations of statics available, a direct solution for these bar stresses cannot be made. A stress analysis of this structure, on the basis of statics only, should be possible, however, since with 5 joints there are 15 independent equations of statics, and there are only 15 independent unknown stresses—8 bars, 4 vertical reactions, and 3 horizontal reactions.
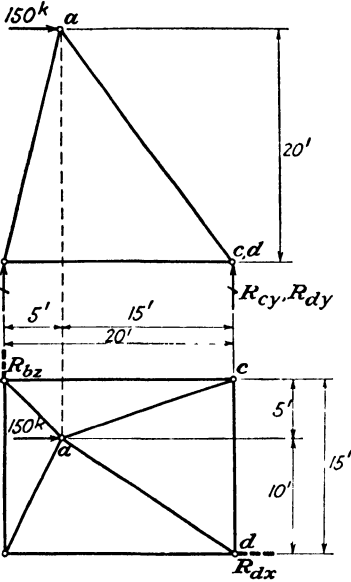
Table 9·2

| Bar | Projection | | | Length |
|---|---|---|---|---|
| | X | Y | Z | |
| ab | 5 | 20 | 5 | 21.2 |
| ac | 15 | 20 | 5 | 25.5 |
| ad | 15 | 20 | 10 | 26.9 |
| ae | 5 | 20 | 10 | 22.9 |
| bc | 20 | 0 | 0 | 20.0 |
| cd | 0 | 0 | 15 | 15.0 |
| de | 20 | 0 | 0 | 20.0 |
| eb | 0 | 0 | 15 | 15.C |

For this structure, however, the equations of statics are simultaneous in character. A convenient approach under this condition is to adopt, as a temporary unknown, one of the bar stresses. Other bar stresses and reactions can then be expressed in terms of this temporary unknown, and eventually one of the equations of statics will permit its evaluation. To illustrate this procedure, take $F_{cd}$ as the temporary unknown. At joint $c$, apply $\Sigma F_z = 0$.

$$F_{cd} + \frac{5}{25.5} F_{ac} = 0; \qquad F_{ac} = -5.10 F_{cd}$$

At joint $d$, apply $\Sigma F_z = 0$.

$$F_{cd} + \frac{10}{26.9} F_{ad} = 0; \qquad F_{ad} = -2.69 F_{cd}$$

Assume that the vertical reactions at $c$ and $d$ act up. Then at joint $c$, applying $\Sigma F_y = 0$,

$$R_{cy} + \frac{20}{25.5} F_{ac} = 0; \qquad R_{cy} = -0.784 F_{ac} = -0.784(-5.10 F_{cd})$$
$$= +4.00 F_{cd}$$

At joint $d$, applying $\Sigma F_y = 0$,

$$R_{dy} + \frac{20}{26.9} F_{ad} = 0; \qquad R_{dy} = -0.744 F_{ad} = -0.744(-2.69 F_{cd})$$
$$= +2.00 F_{cd}$$

Now, taking moments about $be$ of all the forces acting on the structure

$$+150(20) - R_{cy}(20) - R_{dy}(20) = 0$$

whence

$$+150(20) - 4.00 F_{cd}(20) - 2.00 F_{cd}(20) = 0_d; \qquad F_{cd} = +25 \text{ kips}$$

Since $R_{cy}$, $R_{dy}$, $F_{ac}$, and $F_{ad}$ have already been expressed in terms of $F_{cd}$, they may now be evaluated. With these stresses known, the remainder of the structure can be analyzed without difficulty.
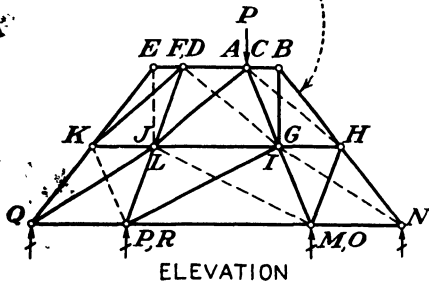
**9·6  Special Theorems.** While three-dimensional frameworks can be analyzed by the methods that have been presented, the following theorems are of importance because they often result in an appreciable saving in computations:

I. *If all the bars meeting at a joint, with the exception of one bar n, lie in a plane, the component normal to that plane of the stress in bar n is equal to the component normal to that plane of any external load or loads applied at that joint.* That this theorem is correct may be seen from a consideration of the static equilibrium of the joint, summing up all the forces

normal to the plane that contains all bars except *n*. In the structure of Fig. 9·4, for example, suppose that this theorem is applied to joint *d*. Bars *ad* and *dc* lie in the plane *adc;* the component of stress in bar *bd*, normal to plane *adc*, must equal the component of the applied load normal to the same plane. For this particular case, the applied load also lies in the plane *adc* and hence has no component normal to that plane. We can conclude, then, that the stress in bar *bd* is zero. Recognition of this fact would simplify the analysis of joint *d* carried out in Art. 9·4 by means of three simultaneous equations, since only two equations would be necessary.

On the basis of theorem I, two corollary theorems may be stated:

II. *If all the bars meeting at a joint, with the exception of one bar n, lie in a plane and if no external load is applied at that joint, the stress in bar n is zero.*

III. *If all but two bars at a joint have no stress and these two are not colinear, and if no external load acts at that joint, the stress in each of these two bars is zero.*
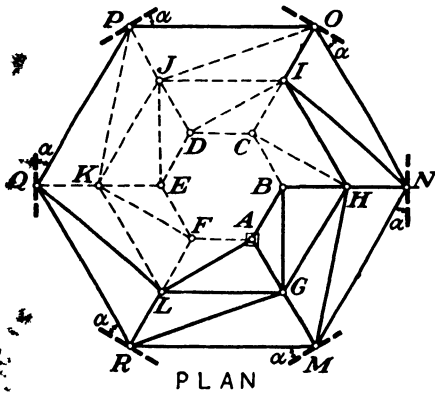
**9·7 Application of Special Theorems—Schwedler Dome.** The importance of these three theorems in the analysis of three-dimensional frameworks may be illustrated by considering the Schwedler dome shown in Fig. 9 8, acted upon by a vertical load *P* applied at joint *A*. The bars shown by dotted lines in the plan view have no stress, as may be concluded from the foregoing theorems, applied as follows:



*B, H, and N do not necessarily lie on a straight line, etc.*

ELEVATION

PLAN

Fig. 9·8

At joint *F*, bars *EF*, *KF*, and *LF* all lie in a plane, but *AF* does not. Since no load is applied at joint *F*, the stress in bar *AF* is zero, in accordance with theorem II. A similar consideration of joints *E*, *D*, *C*, and *B* leads to the conclusion that bars *EF*, *DE*, *CD*, and *BC*, respectively, all have zero stress.

Considering joint *F* again, since the stresses in *FA* and *FE* are zero, bars *KF* and *LF* comprise two bars meeting at a joint where no load is

applied. Hence the stresses in $KF$ and $LF$ are zero, on the basis of theorem III. A similar consideration of joints $E$, $D$, and $C$ leads to the conclusion that bars $KE$ and $JE$, $JD$ and $ID$, and $IC$ and $HC$, respectively, all have zero stress.

Now, considering point $K$, since $KE$ and $KF$ have zero stress, bar $KL$ is a single bar lying outside of the plane of bars $JK$, $PK$, and $QK$. Hence bar $KL$ has zero stress. In a similar manner, a consideration of joints $J$ and $I$ shows that bars $JK$ and $IJ$ have zero stress.

Considering joint $K$ again, since bars $JK$, $EK$, $FK$, and $LK$ have zero stress, bars $QK$ and $PK$ comprise two bars meeting at a joint where no external load is applied. The stresses in these two bars are therefore zero. A similar consideration of joint $J$ shows that bars $PJ$ and $OJ$ also have zero stress.

Hence, when this framed dome is acted upon by a vertical load at $A$, only the bars shown by solid lines in the plan view of Fig. 9·8 carry stress. To complete the analysis of this structure, the method of joints may now be applied successively to joints $A$, $B$, $L$, $G$, $H$, and $I$, and the stresses in all the bars except those of the base ring thus determined. The vertical reactions can then be determined by applying $\Sigma F_y = 0$ at each point of support.

To determine the horizontal reactions and the stresses in the base-ring bars, all of which lie in a plane, it is necessary to take the stress in one of the base-ring bars as a temporary unknown. Suppose $F_{RM}$ is taken as the temporary unknown. At joint $R$, the application of $\Sigma F_x = 0$ and $\Sigma F_z = 0$ permits one to express $F_{QR}$ and the horizontal reaction at $R$ in terms of $F_{RM}$. (Note that joint $R$ is also acted upon by the $X$ and $Z$ components of stress in bars $LR$ and $GR$.) Proceeding clockwise around the base ring and successively writing similar equations at joints $Q$, $P$, $O$, and $N$ enable one to express the stresses in all the base-ring bars, and all the horizontal reactions except that at $M$, in terms of $F_{RM}$. If the equations $\Sigma F_x = 0$ and $\Sigma F_z = 0$ are now applied at joint $M$, the values of $F_{RM}$ and the horizontal reaction at $M$ may be obtained. Since all the other stresses and horizontal reactions have been previously expressed in terms of $F_{RM}$, their values may now be determined.

It is possible for a Schwedler dome to be geometrically unstable, even though the statical count holds, if the angles between the horizontal reactions and the base ring bars have certain values.[1]

**9·8  Towers.**  Unless the legs of a framed tower have a constant batter throughout their length, the structure should be analyzed on the basis of three-dimensional considerations. Figure 9·9a shows the side elevation of a tower of triangular cross section, in which the batter of

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[1] SPOFFORD, G. M., "Theory of Structures," 4th ed., Chap. XVI, Space Frameworks, McGraw-Hill Book Company, Inc., New York, 1939.

the legs is not constant. Figure 9·9b shows the arrangement of horizontal reactions. This structure is statically determinate, as may be verified, and it may be analyzed panel by panel, beginning with the top panel and working down. Whenever, as in this case, adjacent legs of the top panel lie in a plane, the stress in any of the top ring bars may be computed most easily by utilizing the three theorems of Art. 9·6. For example, $F_{ab}$ may be computed by noting that its component normal to the face $acfd$ must balance the component of the external load at joint $a$, which is also normal to this plane. Once the top ring bar stresses are known, the stresses in the legs and diagonals of the top panel may be computed. The stresses in these bars together with any external loads at joints $d$, $e$, and $f$ comprise the loading on the second panel which may now be analyzed similarly, etc.

If, however, adjacent legs of a panel do not lie in a plane, a more general approach must be used to obtain the ring bar stresses. Starting as before with the top panel, the stress in any top ring bar may be taken as a temporary unknown. If $F_{ab}$ is chosen for this purpose, application of the method of joints at joint $b$ permits one to express $F_{bc}$ in terms of $F_{ab}$. Similar treatment at $c$ gives $F_{ca}$ in terms of $F_{ab}$. Finally, $F_{ab}$ may be evaluated by applying the method of joints at joint $a$.

It is customary to have secondary bracing members in each horizontal plane where the batter of the tower legs changes and often at panel points where no change in batter occurs. In a tower with a rectangular cross section, for example, this bracing may consist of horizontal diagonals connecting diagonally opposite panel points. The presence of such members may make a tower statically indeterminate. It is common practice, however, to assume for the purpose of analyzing the main members of the tower that the stresses in these secondary bracing members are zero, thus permitting such analyses to be carried out by the principles of statics.



FIG. 9·9

**9·9 Tower with Straight Legs.** If the batter of the tower legs is constant over the entire height, the tower may be analyzed on the basis of component planar trusses. Such a structure is shown in Fig. 9·10.
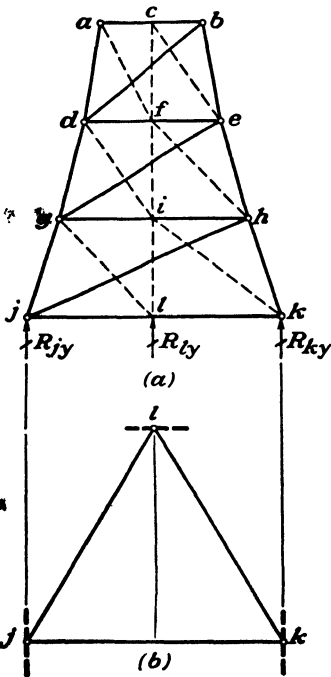
A load $P$, applied at any joint, may be resolved into three components: $C_1$, parallel to the tower leg; $C_2$, horizontal and lying in the plane of one adjacent face of the tower; and $C_3$, horizontal and lying in the plane of the other adjacent face of the tower.

It is easy to show by the theorems of Art. 9·6 that $C_1$ causes stresses in the bars of leg $GC$ only, $C_2$ causes stresses in the bars of tower side $CDGH$ only, and $C_3$ causes stresses in the bars of tower side $ACEG$ only.

Thus the stresses due to each of the components $C_1$, $C_2$, and $C_3$ can be obtained by carrying out a separate planar analysis, and the total stress in any bar due to load $P$ can be obtained by superposition of the effects of its three components. Since each panel load can be handled in the foregoing manner, this constitutes a general procedure for analysis.

If all the faces of such a tower are identical, influence data can be prepared, giving the stresses in each bar of one of the faces due to (1) a unit horizontal load, applied successively at each joint of that face; and (2) a unit load parallel to the tower leg, applied successively at each joint of that face. Such influence data, prepared for one face of the tower, will be applicable to all faces of the tower. By resolving panel loads into com-



FIG. 9·10

ponents as previously outlined and by using these influence data, stresses in any member, due to any condition of external loading, may be obtained by superposition.
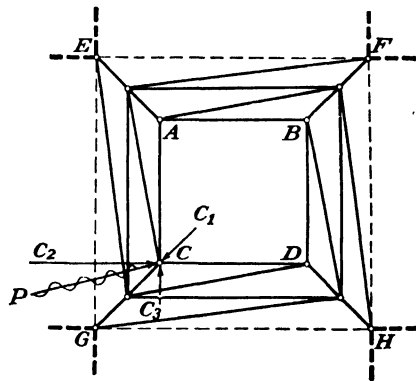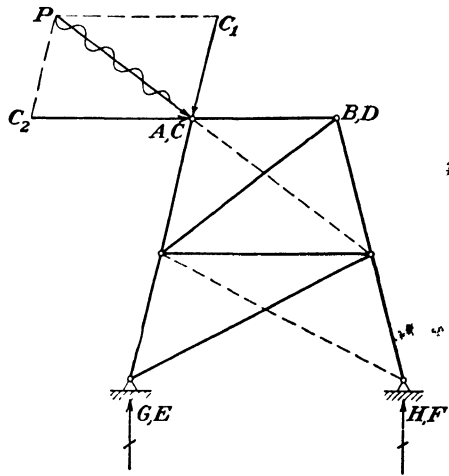
## 9·10 Problems for Solution.

**Problem 9·1**   Show that the tower of Fig. 9·10 is statically determinate.

**Problem 9·2**   Find the reactions on the structure of Fig. 9 11 due to the load of 1,000 lb acting as shown.

**Problem 9·3** Find the bar stresses in the structure of Fig. 9·11 due to the load of 1,000 lb acting as shown.

**Problem 9·4** Find the reactions and bar stresses of the structure of Fig 9·11 if the load of 1,000 lb is applied at joint $d$ but with a direction such that it passes through point $g$, which lies at the center of the equilateral triangle $def$.
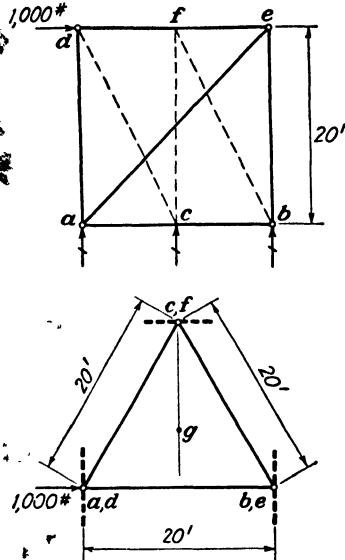
**Problem 9·5** *a.* Show that the structure of Fig. 9·12 is statically indeterminate to the first degree.

*b.* Find the reactions on the structure of Fig. 9·12 and the stresses in all the bars, for the load of 10,000 lb acting as shown, assum-
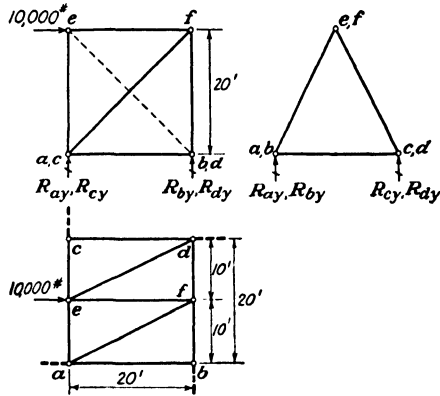
FIG. 9·11

FIG. 9·12

ing that bar $ef$ is in compression and has a stress equal to half the applied load.

**Problem 9·6** If the angle $\alpha$ between the horizontal reaction and the base-ring bar has the same value at each point of support of a Schwedler dome, as shown in Fig. 9 8, prove that, under the action of any vertical load on the dome, the algebraic sum of the horizontal reactions is zero.

**Problem 9·7** A tower similar to that of Fig. 9 9 has 10 panels with a height of 10 ft each; $ab = bc = ca = 10$ ft; $de = ef = fd = 12$ ft; at the tower base, $jk = kl = lj = 40$ ft. At joint $a$, the following external loads are applied: a horizontal load of 10,000 lb, acting to the right, in a direction parallel to the $OX$ axis; a horizontal load of 5,000 lb, acting to the rear, in a direction parallel to the $OZ$ axis; and a vertical load of 20,000 lb, acting downward.

*a* What are the stresses in bars $ab$, $bc$, and $ca$?

*b.* What are the resultant $X$, $Y$, and $Z$ components of the forces applied at joints $d$, $e$, and $f$ by the legs and diagonals of the top panel?

*c.* What are the reactions on the tower?

**Problem 9·8** The tower of Prob. 9·7 weighs 50,000 lb. The maximum wind load acting on the tower exerts a lateral pressure equal to 400 lb per ft of tower height. Against what uplift must a point of tower support be designed if the specifications state that the tower supports shall be designed for 150 per cent of the maximum net uplift?

# CHAPTER 10

# GRAVITY STRUCTURES

**10·1  Introduction.**  A gravity structure is one in which the weight of the structure itself plays an important part in holding in equilibrium the forces to which the structure is subjected.  In a gravity dam, for example, the tendency of the dam to overturn by rotating about its downstream toe, because of the hydrostatic pressure on the upstream face of the dam, is resisted by the moment, about the down stream toe, of the weight of the dam.  Similarly, the tendency of the dam to slide horizontally in a downstream direction is resisted by the friction of the dam on its foundation, which in turn is a function of the weight of the dam.  A retaining wall, built to resist earth pressure, is another example of a gravity structure.  Such a wall may be designed with a thickness such that on a horizontal plane the vertical stresses are all compressive; for a thinner wall, the presence of tensile stresses on the side to which earth pressure is applied may require vertical reinforcing rods.  In either event, however, the weight of the wall is an important factor in the stability of the structure, so that retaining walls are classified as gravity structures.

**10·2  Stresses in a Gravity Dam.**  Let us consider the dam shown in Fig. 10·1a with the object of investigating the distribution of normal
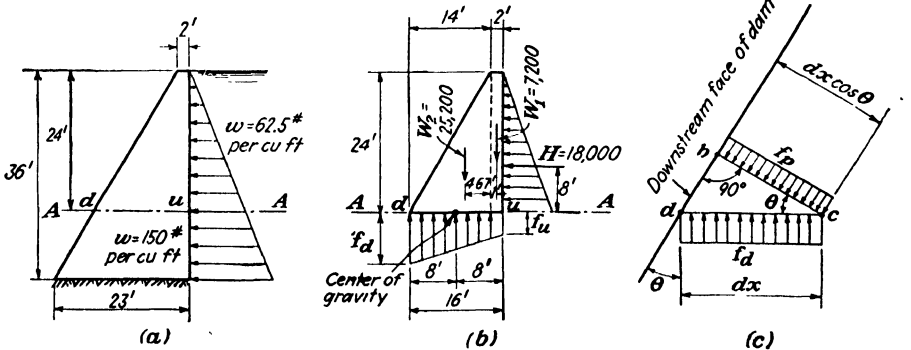


FIG. 10·1

stresses along section $A$-$A$.  We shall consider a strip of the dam 1 ft in thickness, measured along the length of the dam, and assume that this strip, under the external loads to which it is subjected, acts independently of the adjacent strips.  There will be applied to this strip above section

$A$-$A$ the resultant hydrostatic force $H$ and the forces $W_1$ and $W_2$, which comprise the weight of the dam, as shown in Fig. 10·1b. The intensity of hydrostatic pressure at elev. $A$-$A$ is 62.5(24) = 1,500 lb per sq ft. Since the distribution of this pressure is triangular, the resultant pressure $H$ is given by $\frac{1}{2}(1,500)(24) = 18,000$ lb and acts 8 ft above section $A$-$A$ as shown. The forces $W_1$ and $W_2$ are computed as follows:

$$W_1 = 150(2)(24) = 7,200 \text{ lb}, \qquad W_2 = 150(\tfrac{1}{2})(14)(24) = 25,200 \text{ lb}$$

They act as shown in the figure.

If we assume the normal fiber stresses to be distributed linearly along section $A$-$A$, the following relation may be used:

$$f = \frac{P}{A} + \frac{Mx}{I} \tag{10·1}$$

where $f$ = compressive stress[1] developed, with a positive value indicating compression[2]

$\quad P$ = resultant vertical force applied above the section under consideration, with downward forces taken as positive

$\quad A$ = cross-sectional area at the section under consideration;

$\quad M$ = resultant moment about the center of gravity of $A$ of all forces applied above the section under consideration, with moments causing compression in the downstream face taken as positive

$\quad x$ = horizontal distance, measured from the center of gravity of $A$ to the point where $f$ is to be determined, with this distance taken as positive for points downstream from the center of gravity

$\quad I$ = moment of inertia, about the center of gravity of $A$ of the cross-sectional area $A$ itself

For the problem under consideration

$P = 7,200 + 25,200 = +32,400$ lb
$A = 16(1) = 16$ sq ft
$M = +18,000(8) - 7,200(7) - 25,200(1.33) = +60,000$ ft-lb
$x_d$ (to downstream face) = $+8$ ft,    $x_u$ (to upstream face) = $-8$ ft
$I = \frac{1}{12}(1)(16)^3 = 341$ ft$^4$

Hence, at the downstream face,

$$f_d = \frac{32,400}{16} + \frac{60,000(+8)}{341} = 2,030 + 1,410$$
$$= +3,440 \text{ lb per sq ft (compression)}$$

---

[1] In this chapter, for the sake of brevity, the term *stress* instead of *stress intensity* is used where the meaning is obvious.

[2] This is opposite to the customary sign convention for stresses and is used in this instance because of the predominance of compressive stresses in dam analysis.

while, at the upstream face,

$$f_u = \frac{32,400}{16} + \frac{60,000(-8)}{341} = 2,030 - 1,410$$

$$= +620 \text{ lb per sq ft (compression)}$$

From $d$ to $u$, the stress intensity varies linearly, giving a trapezoidal distribution.

It should be noted that the foregoing intensities are for stresses perpendicular to section $A$-$A$. For the upstream face, the value computed in this particular case represents the principal stress since the upstream face is vertical. Owing to the slope of the downstream face, however, the value of stress computed at $d$ is not a principal stress. In Fig. 10·1c, since $bd$ is a free face, the shear on this plane is zero. Since the shear on any two mutually perpendicular planes has the same value at a given point, the shear along $bc$ is also zero. This identifies $bc$ as the plane on which the principal stress $f_p$ occurs. To evaluate $f_p$, take moments about $c$ of the forces acting on the differential wedge element $bcd$.

$$f_p \frac{(dx \cos \theta)^2}{2} = f_d \frac{(dx)^2}{2}$$

whence $f_p = f_d \sec^2 \theta$. For this problem, $\sec \theta = 1.16$, whence, at $d$,
$$f_p = +3,440(1.16)^2 = +4,640 \text{ lb per sq ft (compression)}$$

**10·3   Location of Resultant Force.**   The forces applied to a dam above a given elevation can be represented by a single resultant force, and the axial force and bending moment at that elevation can be computed from this resultant. The resultant force $R$ applied above section $A$-$A$ of the dam of Fig. 10·1a is shown in Fig. 10·2. The horizontal component of this resultant must equal 18,000 lb; the vertical component must equal $7,200 + 25,200 = 32,400$ lb.

The moment of the resultant $R$ about any point such as $a$ must equal the moment of the forces that have been combined, whence
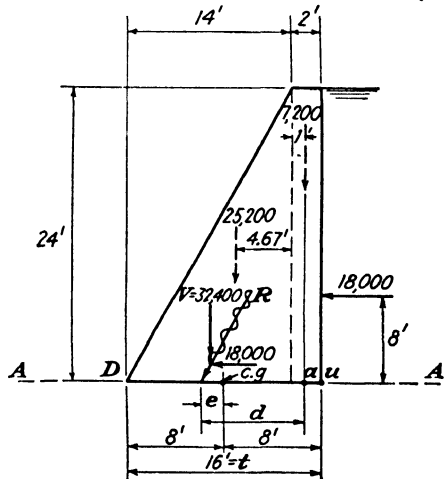


FIG. 10·2

$$+18,000(8) + 25,200(5.67) = 32,400\, d$$

from which $d = 8.85$ ft. The resultant therefore intersects plane $A$-$A$ at a distance of $8.85 - 7.00 = 1.85$ ft downstream from the center of

gravity of the cross section.   Now, working directly from the resultant,

$$P = +32,400 \text{ lb and } M = 32,400(+1.85) = +60,000 \text{ ft-lb}$$

These values check those previously obtained.

For the general case, assume that the resultant intersects a given horizontal section at distance $e$ downstream from the center of gravity of the cross section.   Denoting by $t$ the width of the dam at the section under consideration and by $V$ the vertical component of the resultant $R$, and then referring to Eq. (10·1), $P = +V$; $A = t$; $M = +Ve$; $x_d$ (to downstream face) $= +t/2$; $x_u$ (to upstream face) $= -t/2$; $I = t^3/12$, whence, at the downstream face,

$$f_d = +\frac{V}{t} + \frac{(+Ve)(+t/2)}{t^3/12} = \frac{V}{t}\left(1 + 6\,\frac{e}{t}\right) \tag{a}$$

while, at the upstream face,

$$f_u = +\frac{V}{t} + \frac{(+Ve)(-t/2)}{t^3/12} = \frac{V}{t}\left(1 - 6\,\frac{e}{t}\right) \tag{b}$$

From Eqs. (a) and (b), it may be seen that, if the resultant acts within the middle third of the cross section, i.e., if $-t/6 < e < +t/6$, the entire cross section will be in compression.   If the resultant is at the downstream limit of the middle third, i.e., $e = +t/6$, $f_d = 2V/t$ and $f_u = 0$.   Under this condition, the distribution of the normal fiber stresses along the section is triangular, with the intensity at the downstream face equal to twice the value it would have if the vertical component of the resultant were uniformly distributed over the cross-sectional area $A$.

**10·4   Resultant Outside of Middle Third.**   If, however, the resultant acts outside of the middle third of the cross section, tension will be developed at one face.   Plain masonry has little tensile strength, and for design purposes it is usually assumed to have none.   Unless the dam is adequately reinforced with steel so that it can develop the necessary tensile strength, the previously given method of analysis is not wholly applicable, since it is based on the idea that stresses vary linearly across the entire section.

Consider the stress distribution along section $A$-$A$ of the dam of Fig. 10·1a if, by means of flashboards, the elevation of the water surface is raised to a height of 3 ft above the top of the dam, as shown in Fig. 10·3. The resultant horizontal water pressure equals $62.5(27)(1\frac{1}{2})(27) = 22,700$ lb and acts 9 ft above section $A$-$A$.   To determine the location of the resultant, take moments about $a$:

$$+\ 22,700(9) + 25,200(5.67) = +32,400\ d;\ d = 10.7 \text{ ft}$$

Hence $e = 10.7 - 7.0 = 3.7$ ft. Since this is greater than $16\frac{2}{6} = 2.67$ ft, the resultant lies outside the middle third of the section. Application of Eq. (10·1) would lead to tensile stresses on the upstream face of the dam. Since the dam could not resist linearly distributed tensile stresses, the stresses computed by Eq. (10·1) would not be valid.

To determine the stress distribution along section $A$-$A$, static equilibrium of the portion of the dam above this section requires that the resultant of the normal stresses along the section shall be equal and opposite to the vertical component of $R$ and applied at the point where $R$ intersects section $A$-$A$. Hence, the resultant of these normal stresses is an upward force of 32,100 lb applied at a distance of $8.00 - 3.7 = 4.3$ ft from $d$, the downstream edge of section $A$-$A$.

Consistent with a linear stress distribution involving compressive stresses only, the stresses will be distributed triangularly. The center of gravity of this triangle will coincide with the resultant of the normal stresses on the section. Hence the length of the triangle along section $A$-$A$ will equal $3(4.3) = 12.9$ ft, as shown in Fig. 10·3.

Moreover, because of the triangular distribution, the intensity



FIG. 10·3

of normal stresses at the downstream face equals twice the value that would occur if the vertical component of the resultant were distributed uniformly over the entire area under compression. Hence

$$f_d = \frac{2(32,100)}{12.9} = 5,020 \text{ lb per sq ft}$$

The principal stress at $d$ is given by $f_p = 5,020(1.16)^2 = 6,750$ lb per sq ft.

**10·5 Determination of Gravity-dam Profile.** The profile of a gravity dam is usually laid out so that for a horizontal section at any elevation the resultant lies within the middle third. For a given profile, the location of the resultant will depend upon the loading condition considered. For reasons of economy, it is desirable, under what proves to
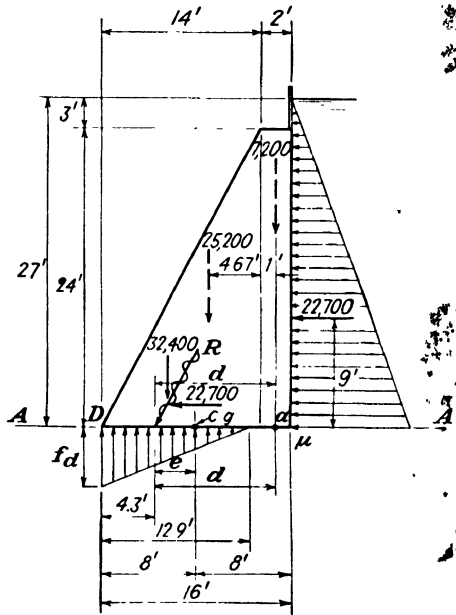
be a critical loading condition for a given elevation, for the resultant at that elevation to be as near as is practicable to an outer edge of the middle third.   Thus, for the maximum overturning condition, *i.e.*, water at highest elevation, plus ice pressure, etc., the resultant should be near the downstream edge of the middle third, while, with the dam empty, the resultant should be near the upstream edge of the middle third.   The foregoing criteria can rarely be completely met in an actual case, but they serve as a guide for profile determination.   For the following procedure, either analytical or graphical methods, or a combination of the two, may be used:



FIG. 10·4

The width of the dam at the top will be determined from practical considerations, such as minimum width for durability, required width for equipment, roadway, etc.   Through the upstream edge of the top of the dam, construct a vertical reference line, called the dam axis, as shown in Fig. 10·4.   At any elevation, the horizontal distance $u$ from the dam axis to the upstream face is called the upstream projection; the corresponding distance to the downstream face is called the downstream projection.

Divide the height of the dam by horizontal lines at representative elevations such as elev. (1) or elev. (2) at which the upstream and downstream projections are to be determined.   Compute the resultant horizontal water pressure applied to each vertical portion of the dam ($H_1$, $H_2$, etc.).

At elev. (1), assume tentative values of $u_1$ and $d_1$.   On the basis of these tentative values, evaluate and locate $W_1$, the weight of the dam above elev. (1), and, provided that $u_1 \neq 0$, $V_1$ the vertical component of water pressure applied above elev. (1).   Evaluate and locate $R_1$, the resultant of the forces $W_1$, $H_1$, and $V_1$.   On the basis of the tentative values of $u_1$ and $d_1$, investigate in the foregoing manner each loading condition that may affect the profile.   If, considering all loading conditions, the tentative values of $u_1$ and $d_1$ are not satisfactory, successively investigate new values of $u_1$ and $d_1$ until acceptable values are obtained.
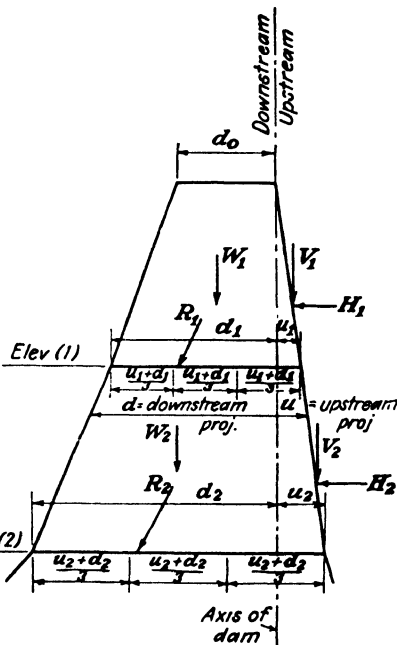
Then, at elev. (2), assume tentative values of $u_2$ and $d_2$. On the basis of these tentative values, evaluate and locate $W_2$, the weight of the dam between elevs. (1) and (2), and, provided that $u_2 - u_1 \neq 0$, $V_2$ the vertical component of water pressure applied between elevs. (1) and (2). Evaluate and locate $R_2$, the resultant of $R_1$, $W_2$, $H_2$, and $V_2$. Still using the tentative values of $u_2$ and $d_2$, investigate each loading condition in the foregoing manner. By successive trials, establish acceptable values of $u_2$ and $d_2$.

Working down the dam, section by section, in a similar manner, establish $u$ and $d$ at each representative elevation. With the profile determined as described, no tensile stresses will occur in the dam. This does not, however, ensure that the compressive stresses in the dam will be within allowable limits. The actual stresses at each elevation should be computed, and this would often influence the layout of the profile.

**10·6   Stresses at Base of Gravity Dam.** If an impervious dam is bonded at its base to an impervious foundation, so that no upward water
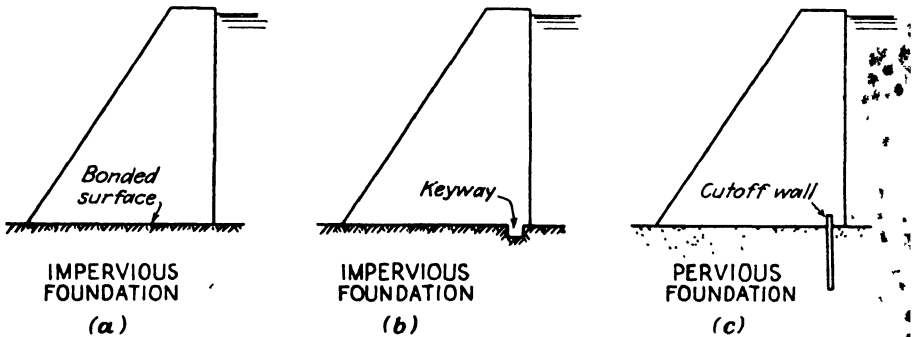


Fig. 10·5

pressure can act on the base of the dam, the determination of the stress distribution along the base of the dam can be carried out by following the same procedures as those used for any other horizontal section through the dam. Such a dam is shown in Fig. 10·5a. Actually, it would be unlikely that the dam could be bonded to the foundation so securely as to ensure that there would be no upward water pressure under the dam. To prevent such pressure a keyway embedded near the upstream side of an impervious foundation might be used, as shown in Fig. 10·5b. For a pervious foundation, a deep cutoff wall near the upstream side of the foundation, as shown in Fig. 10·5c, may be effective.

If the foundation is such that upward water pressure cannot act on the base of the dam, stresses along the base may be determined by Eq. (10·1), provided that the resultant acts within the middle third of the base, so that no tension occurs. If the resultant lies outside of the middle

third but within the base of the dam, the compression on the base of the dam will be distributed triangularly, with the center of gravity of the triangle at the point of application of the resultant force to the foundation. The maximum foundation stress will then occur at the edge of the foundation and will equal twice the value obtained if the vertical component of the resultant were uniformly distributed over the area in compression.

Should the resultant lie outside of the base of the dam, the dam would fail by overturning. A reasonable factor of safety against overturning should be provided.

If the angle made by the resultant with the vertical exceeds the friction angle of the dam on its foundation, the dam will fail by sliding. A liberal factor of safety should be observed in this connection, since the friction angle cannot as a rule be predicted with precision.

**10·7 Upward Water Pressure.** Consider the dam shown in Fig. 10·6a in which the water is at the same elevation on both the upstream
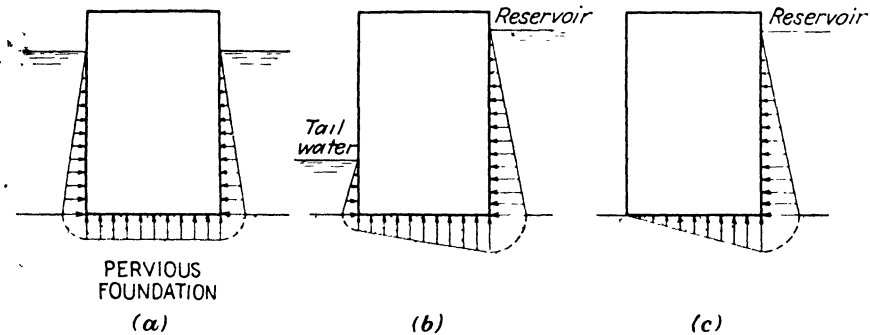


FIG. 10·6

and downstream sides. At the base of each side, let the hydrostatic pressure be $p$. Since the foundation is pervious, there will be water under the dam. No flow of water occurs, however, and thus there is an upward water pressure of intensity $p$ over the entire base of the dam. In the dam of Fig. 10·6b, the tail water is at a lower elevation than the water in the reservoir. Let the hydrostatic pressure at the base of the upstream face be $p_u$ and at the base of the downstream face be $p_d$. Since $p_u > p_d$, water will flow downstream under the dam, losing pressure as it flows. If this loss of pressure is assumed to be linear, the upward water pressure on the base of the dam will be distributed trapezoidally, having intensities $p_u$ and $p_d$ at the upstream and downstream faces, respectively. If there is no water on the downstream edge of the base, as shown in Fig. 10·6c, $p_d = 0$ and the distribution of upward water pressure is triangular.

It is often assumed that the upward water pressure does not come in contact with the entire area of the base of the dam. It might be specified, for the dam of Fig. 10·6c, for example, that the structure should be designed for an upward water pressure varying linearly from half the hydrostatic pressure at the upstream edge of the base to zero at the downstream edge.

As pointed out in the preceding article, the amount and distribution of upward water pressure can be controlled by such means as keyways and cutoff walls. The use of drains to carry away seepage water provides another means of control.

While we have discussed upward water pressure as a phenomenon that occurs along the base of the dam, it is evident that it will occur along any pervious horizontal section through the dam itself. In the following article, the effect of upward water pressure on stress computations along the base of the dam is considered. The methods given are equally applicable when one considers the effect of upward water pressures on other horizontal sections.

**10·8  Effect of Upward Water Pressure on Stresses at Base of Dam.** Let us consider the stresses along the base of the dam of Fig. 10 1, assuming that the water is 6 ft below the top of the dam and that the base of the dam is subjected to upward water pressure varying uniformly in intensity from half the hydrostatic pressure at the upstream edge of the base to zero at the downstream edge of the base, as shown in Fig. 10 7. As in previous examples, $H$, $W_1$, and $W_2$ are first evaluated. On the base of the dam, there will act upward, first, the upward water pressure and, second, the soil pressure. The resultant $U$ of the upward water pressure equals $937(1\frac{1}{2})(23) = 10,800$ lb and acts at the third point of the base as shown.

To evaluate $V$, the vertical component of the resultant of the soil pressure on the base of the dam, apply $\Sigma F_y = 0$ to all the forces acting on the dam.

$$+56,700 + 10,800 - 10,800 - V = 0$$

whence $V = +56,700$ lb. To locate $V$, take moments about any point such as $a$ of all the forces acting on the dam.

$$+56,700d + 10,800(6.67) - 56,700(8) - 28,100(10) = 0$$

whence $d = +11.63$ ft and $e = 12.63 - 11.50 = 1.13$ ft. Since this is less than $23/6 = 3.81$ ft, the resultant soil pressure lies within the middle third and the soil is in compression along the entire width of the base. The intensities of soil pressures may therefore be computed by Eq. (10·1), in which $P = V = +56,700$ lb and

$$M = +Ve = +56,700(1.13) = +64,000 \text{ ft-lb}$$

This leads to soil pressures of 3,191 lb per sq ft and 1,739 lb per sq ft at the downstream and upstream edges, respectively, as shown in Fig. 10·7. The total pressure on the base of the dam is obtained by superimposing the soil pressure and the upward water pressure and is also shown in the figure.

Had the resultant soil pressure been applied outside the middle third, the distribution of soil pressure would have been triangular, with the
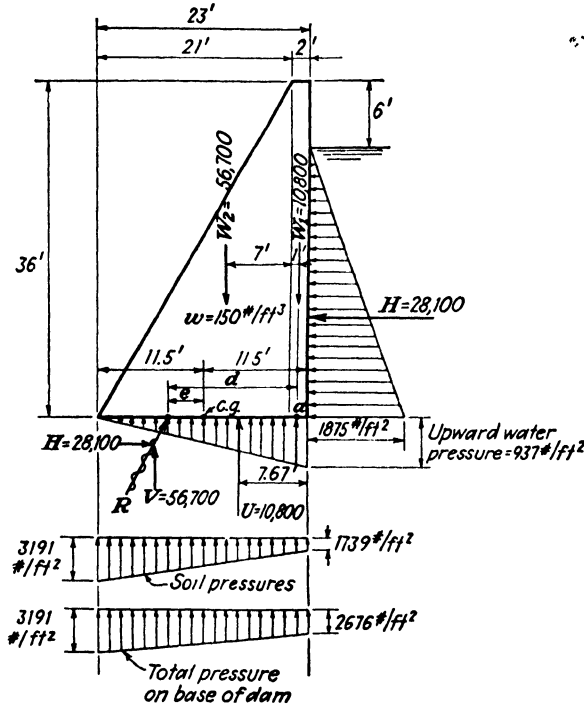


FIG. 10·7

center of gravity of the triangle coinciding with the resultant soil pressure and with a maximum intensity of soil pressure equal to twice the value obtained if the vertical component of the resultant soil pressure were uniformly distributed over the area acted upon by the soil pressure. The total pressure on the base of the dam would still be obtained by superimposing the soil and upward water pressures, the resulting pressure curve being discontinuous at the point where the soil pressure equals zero.

It is to be noted that the presence of upward water pressure increases the tendency of the dam to fail by both overturning and sliding.

**10·9  Gravity Retaining Walls.**  While the weight of all retaining walls plays an important part in enabling the wall to resist the earth

pressure to which it is subjected, we shall distinguish between gravity retaining walls, in which the resultant is kept within the middle third at all elevations, and cantilever retaining walls, in which the wall acts as a vertical cantilever beam, carrying its lateral load in bending and requiring steel reinforcing.

The determination of soil pressures that act on the back of retaining walls is discussed in Art. 1·12, where it is pointed out that the lateral pressure caused by soils varies when the wall yields, reaching a minimum value called the active pressure after the wall has yielded a small amount. Although a gravity retaining wall is a relatively rigid structure, it is nevertheless customary to assume that it will yield sufficiently to permit its design on the basis of active soil pressures, unless the top of the wall is actually restrained against lateral movement by a rigid connection to a relatively immovable support.

In Art. 1·12, the general equation for the resultant of the active soil pressure on the back of a retaining wall, based on the theory developed by Coulomb, is given in Eq. (1·6). It is further shown that, where the surface of the soil retained is horizontal ($i = 0$), where the back of the retaining wall is vertical ($\theta = 0$), and where $\phi$, the angle of internal friction of the soil, equals $\phi'$, the friction angle of the soil on masonry, Eq. (1·6) reduces to Eq. (1·7),

$$P = \frac{1}{2}\gamma H^2 \left[ \frac{\cos\phi}{(1 + \sqrt{2}\sin\phi)^2} \right] \qquad (1\cdot7)$$

where $P$ is the resultant soil pressure for a strip of wall 1 ft long and acts as shown in Fig. 1·3; $\gamma$ is the weight of earth per unit volume, and $H$ is the vertical depth of the soil above the base of the wall.

The analysis of retaining walls follows the same general procedure used for gravity dams. Consider the gravity wall shown in Fig. 10·8 with the object of determining the soil pressures developed along the base of the wall. For this case, Eq. (1·7) is applicable, whence, considering a strip of wall 1 ft in length,

$$P = \frac{1}{2}(100)(20)^2 \left[ \frac{0.867}{(1 + 0.500\sqrt{2})^2} \right] = 5{,}980 \text{ lb}$$

This resultant earth pressure acts one-third of the way up from the base of the wall, making an angle of 30° with the normal to the back of the wall as shown. The horizontal and vertical components of $P$ are also shown on the figure. For the wall,

$$W_1 = 150(1)(20) = 3{,}000 \text{ lb and } W_2 = 150(\tfrac{1}{2})(20)(6) = 9{,}000 \text{ lb}$$

The vertical component $V$ of the resultant pressure on the base of the wall is obtained by applying $\Sigma F_y = 0$ to all the forces acting on the wall.

$$+9,000 + 3,000 + 2,990 - V = 0$$

whence $V = +14,990$ lb. To locate the resultant, taking moments about $a$ of all the forces acting on the wall,

$$+14,990d + 2,990(0.5) - 5,180(6.67) - 9,000(2.50) = 0$$

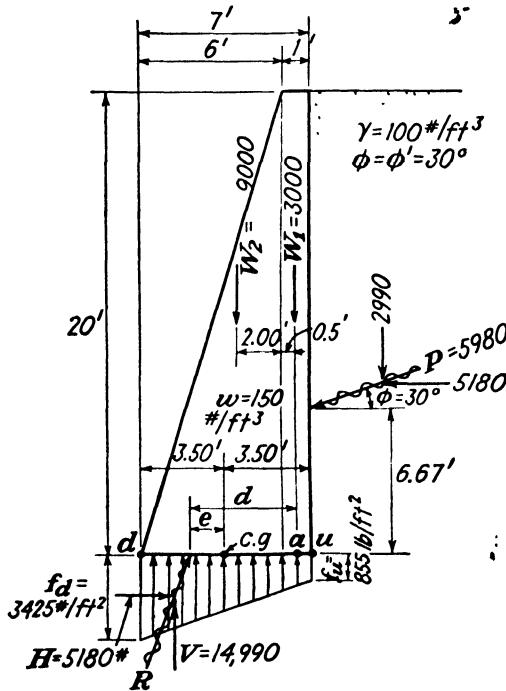whence $d = 3.70$ ft and $e = 3.70 + 0.50 - 3.50 = +0.70$ ft, which is



Fig. 10·8

less than $\frac{7}{6} = 1.17$ ft. Hence to determine soil pressures, Eq. (10·1) is applicable. Upon using $P = V = 14,990$ lb and

$$M = +14,990(0.70) = +10,500 \text{ ft-lb}$$

this leads to the values shown at the base of the wall in Fig. 10·8.

**10·10 Cantilever Retaining Walls.** A typical cantilever retaining wall is shown in Fig. 10·9. The determination of stresses on any horizontal section such as $A$-$A$ involves the same procedures as would be followed for a gravity retaining wall, although the resultant would

doubtless lie outside of the middle third or even outside of the section itself. This, however, would lead to no serious difficulty, since with proper steel reinforcing the section could be designed to withstand the axial stress and bending caused by the resultant.

In determining the distribution of soil pressures on the base of the footing of the wall, we must give careful consideration to the forces acting on the wall. There will be acting on this wall $P$, the resultant soil pressure acting on the vertical line $BB$; $W_1$, the weight of the earth directly over the footing of the wall; $W_2$, the weight of the wall itself; and $R$, the resultant soil pressure acting on the base of the footing.

To determine $P$, note that $\phi'$ should be replaced by $\phi$, since the angle which $P$ makes with the normal to $BB$ is determined by the angle of internal friction of the soil rather than by the friction angle of soil on masonry. The application of Eq. (1·7) leads to

$$P = \frac{1}{2}(100)(20)^2$$

$$\left[\frac{0.867}{(1 + 0.500\sqrt{2})^2}\right] = 5,980 \text{ lb}$$

$W_1 = 100(3)(18) = 5,400 \text{ lb}; \ W_2 = 150[2(18) + 2(8)] = 7,800 \text{ lb}$



Fig. 10·9

To evaluate $V$, the vertical component of $R$,

$$7,800 + 5,400 + 2,990 - V = 0; \quad V = +16,190 \text{ lb}$$

To locate $R$,

$$+16,190d + 2,990(1.5) - 7,800(2.5) - 5,180(6.67) = 0;$$
$$d = 3.05 \text{ ft;} \quad e = 3.05 + 1.50 - 4.00 = +0.55 \text{ ft}$$

Since the resultant lies within the middle third of the base of the footing, Eq. (10·1) is applicable for determining soil pressures on the base. Upon using $P = V = +16,190$ lb and $M = +16,190(0.55) = +8,900$ ft-lb, this leads to the values shown on the figure.

An alternate approach to the determination of $P$, the resultant earth pressure acting on the side of the wall, which is considered preferable by some engineers, consists in computing the resultant pressure along line
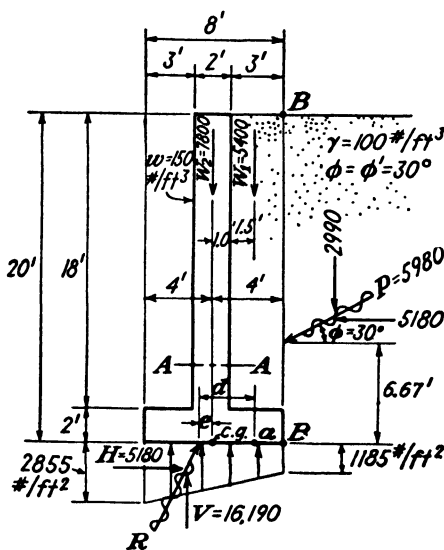
$CC$ as shown in Fig. 10·10. This necessitates the use of Eq. (1·6), since $\theta$ no longer equals zero. In computing $W_1$ to correspond to this method of determining $P$, only the weight of the triangular wedge of earth is included. The line $CC$ corresponds more closely to the failure plane of the earth than does line $BB$ of Fig. 10·9. Ordinarily, the two methods of determining $P$ lead to much the same values of soil pressure along the base of the footing.
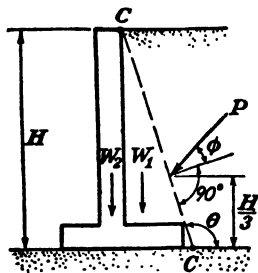


FIG. 10·10

### 10·11 Problems for Solution.

**Problem 10·1** Find the distribution of normal stresses along section $A$-$A$ of the dam of Fig. 10·11, which is not reinforced so as to be capable of carrying tension.

**Problem 10·2** Find the maximum elevation to which the water behind the dam of Fig. 10·11 can rise (using flashboards if necessary), the entire section $A$-$A$ remaining in compression.

**Problem 10·3** Find the distribution of soil pressure along the base of the dam of Fig. 10·11, assuming no upward water pressure on the base of the dam. The water level is to be assumed at the top of the dam.

**Problem 10·4** Referring to the dam of Fig. 10·11 and with the water level at the top of the dam, let the base of the dam be subjected to upward water pressure varying linearly from one-third of the hydrostatic pressure at the upstream edge of the base to zero at the downstream edge of the base. Determine (a) the distribution of soil pressure along the base of the dam; (b) the distribution of total pressure along the base of the dam.

**Problem 10·5** Consider that the structure of Fig. 10·11 is a gravity retaining wall, retaining soil that is level with the top of the wall. The earth weighs 100 lb per cu ft; it has an angle of internal friction of 30° and a friction angle on masonry of 30°. Determine the distribution of soil pressure along the base of the wall.



FIG. 10·11

**Problem 10·6** Referring to Prob. 10·5, what is the minimum width of the base of the wall that can be used if the soil is to be in compression along the entire base of the wall? The width of the base is to be changed by varying the dimension of 24 ft.

**Problem 10·7** Determine the distribution of soil pressure along the base of the cantilever retaining wall of Fig. 10·9, using the soil pressure on the back of the wall in accordance with the method described in the discussion of Fig. 10·10.
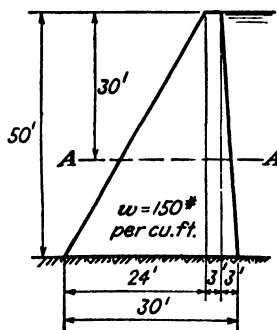
# CHAPTER 11

## CABLES

**11·1 Introduction.** Cables are used in many important types of engineering structures. They form the main load-carrying elements for suspension bridges and cable-car systems. They are used extensively for permanent guys on structures such as derricks and radio towers. They are also used for temporary guys during erection. Although exact cable analyses may require mathematical procedures beyond the scope of this book, a knowledge of certain fundamental relationships for cables is important in structural engineering.

When a cable supports a load that is uniform per unit length of the cable itself, such as its own weight, it takes the form of a catenary; but unless the sag of the cable is large in proportion to its length, the shape taken may often be assumed to be parabolic, the analysis being thus greatly simplified.

**11·2 General Cable Theorem.** Consider the general case of a cable supported at two points $a$ and $b$, which are not necessarily at the
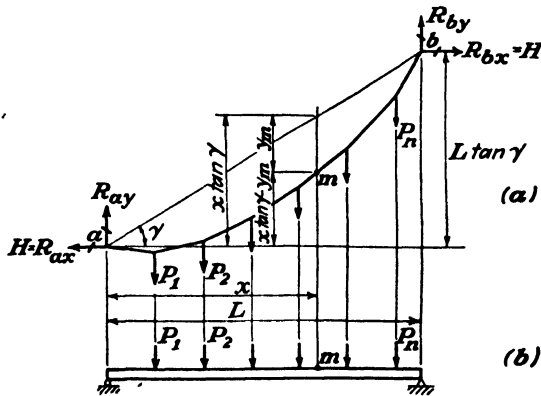


FIG. 11·1

same elevation, and acted upon by any system of vertical loads $P_1$, $P_2$, . . . , $P_n$ as shown in Fig. 11 1$a$. The cable is assumed to be perfectly flexible, so that the bending moment at any point on the cable must be zero. Since all the loads are vertical, the horizontal component of cable stress, which will be denoted by $H$ has the same value at any point on the cable and the horizontal reactions are each equal to $H$.

Let $\Sigma M_b$ = the sum of the moments about $b$ of all the loads $P_1$, $P_2$, . . . , $P_n$

$\Sigma M_m$ = the sum of the moments about any point $m$ on the cable of those of the loads $P_1$, $P_2$, . . . , $P_n$ that act on the cable to the left of $m$

Taking moments about $b$, of all the forces acting on the cable,

$$+H(L \tan \gamma) + R_{ay}L - \Sigma M_b = 0$$

from which

$$R_{ay} = \frac{\Sigma M_b}{L} - H \tan \gamma \qquad (a)$$

Taking moments about $m$ of those forces acting on that portion of the cable to the left of $m$,

$$+H(x \tan \gamma - y_m) + R_{ay}x - \Sigma M_m = 0$$

Substituting $R_{ay}$ from Eq. $(a)$ and simplifying,

$$H y_m = \frac{x}{L} \Sigma M_b - \Sigma M_m \qquad (b)$$

In interpreting Eq. $(b)$, it should be noted that $y_m$ is the vertical distance from the cable at point $m$ to the cable chord $ab$ which joins the points of cable support. The right side of Eq. $(b)$ may be seen to equal the bending moment that would occur at point $m$ (see Fig. 11·1b) if the loads $P_1$, $P_2$, . . . , $P_n$ were applied to an end-supported beam of span $L$ and $m$ were a point on this imaginary beam, located at distance $x$ from the left support.

From Eq. $(b)$, we may therefore state the following general cable theorem: *At any point on a cable acted upon by vertical loads, the product of the horizontal component of cable stress and the vertical distance from that point to the cable chord equals the moment which would occur at that section if the loads carried by the cable were acting on an end-supported beam of the same span as that of the cable.*

It is to be emphasized that this theorem is applicable to any set of vertical loads and holds true whether the cable chord is horizontal or inclined.

**11·3 Application of General Cable Theorem.** Suppose that the loading on a cable is defined and that the distance from the cable to the cable chord is known at one point, as is the case in Fig. 11·2. Neglecting the weight of the cable itself, the bending moment at point $d$ on an imaginary beam of equal span is equal to

$$2{,}330(20) - 1{,}000(10) = 36{.}600 \text{ ft-lb}$$

Hence, by the general cable theorem, $10H = 36,600$, or $H = 3,660$ lb. To determine the distance of any other point such as $c$ from the cable chord, the general cable theorem is applied at section $c$, leading to $3,660y_c = 2,330(10)$, from which $y_c = 6.38$ ft. The segment of the cable between $a$ and $c$ lies along a straight line, since the weight of the cable has been neglected, and has a length equal to $\sqrt{(10)^2 + (6.38)^2} = 11.85$ ft. Since the horizontal component of cable stress equals 3,660 lb, the actual cable stress between $a$ and $c$ equals $3,660(11.85/10) = 4,340$ lb. The left vertical reaction on the cable equals the vertical component of cable stress in segment $ac$ and is given by $3,660(6.38/10) = 2,330$ lb.

For this particular case, this value equals the left vertical reaction on the imaginary end-supported beam. Had the cable chord been inclined, however, these two vertical reactions would have had different values.



FIG. 11·2

**11·4  Shape of Uniformly Loaded Cable.** The case of a loading that is uniform per horizontal foot and applied over the entire span of the cable is important not only because it is substantially the type of cable loading occurring in suspension bridges, but because a cable carrying only its dead weight can be treated approximately on the assumption that the dead weight is uniform per horizontal foot. In Fig. 11·3, the general cable theorem leads to
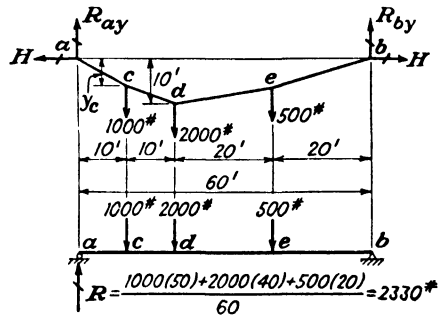
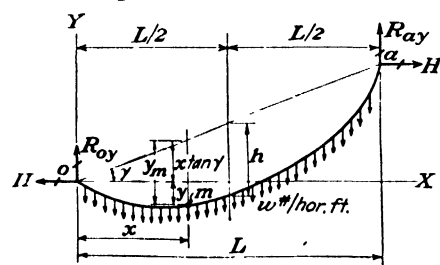

FIG. 11·3

$$Hy_m = \frac{wLx}{2} - \frac{wx^2}{2} \qquad (a)$$

Let the particular value of $y_m$ at mid-span be denoted by $h$. The distance $h$ is called the cable sag and is measured vertically in all cases. For the mid-span, where $x = L/2$ and $y_m = h$, the foregoing equation reduces to $Hh = wL^2/8$, whence

$$H = \frac{wL^2}{8h} \qquad (11\cdot1)$$

This relation for $H$ is of primary importance. Note that it holds whether the cable chord is inclined or horizontal. Substituting this value of $H$

into Eq. (a) and solving for $y_m$ lead to

$$y_m = \frac{4hx}{L^2}(L - x) \qquad\qquad (11 \cdot 2)$$

Equation (11·2) defines the shape of the cable, located with respect to the cable chord, and in terms of the cable sag. It is often desirable to define the shape of the cable with respect to a horizontal axis. If the origin $o$ of the axes is taken at the left end of the cable, as shown in Fig. 11·3, the relation $y = +x \tan \gamma - y_m$ may be used. Substituting $y_m$ as expressed by Eq. (11·2) leads to

$$y = \frac{4hx}{L^2}(x - L) + x \tan \gamma \qquad\qquad (11 \cdot 3)$$

If the cable chord is horizontal, $\tan \gamma = 0$, whence

$$y = \frac{4hx}{L^2}(x - L) \qquad\qquad (11 \cdot 4)$$

If the cable chord is horizontal and if it is desired to define the cable curve with respect to axes with their origin at $c$, the low point on the cable,



FIG. 11·4

which is at mid-span, reference to Fig. 11·4 shows that, since

$$x = \left(\frac{L}{2}\right) + x_c \text{ and } y = -h + y_c$$

these relations may be substituted into Eq. (11·4), leading to
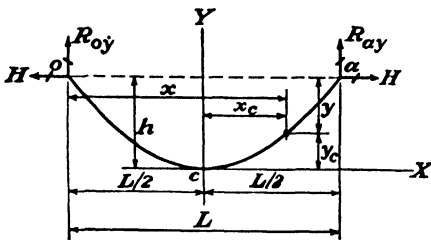
$$y_c = \frac{4h}{L^2} x_c^2 \qquad (11 \cdot 5)$$

**11·5 Stresses in Uniformly Loaded Cable.** The stress at any point in a cable is axial. For a uniformly loaded cable, the horizontal component of cable stress can be computed by means of Eq. (11·1). Consider a differential element of cable, of length $ds$ and horizontal projection $dx$. Then $T_x$, the tension in the cable at any distance $x$ from the origin, is given by $H\, ds/dx$. For the case of the inclined cable chord, as shown in Fig. 11·3, differentiation of Eq. (11·3) leads to

$$\frac{dy}{dx} = \frac{8hx}{L^2} - \frac{4h}{L} + \tan \gamma$$

$$= \frac{8\theta x}{L} - 4\theta + \tan \gamma$$

where $\theta = h/L$ and is called the sag ratio.

Further, since $ds = [1 + (dy/dx)^2]^{1/2} dx$, $T_x = H[1 + (dy/dx)^2]^{1/2}$, or

$$T_x = H\left(1 + \frac{64\theta^2 x^2}{L^2} + 16\theta^2 + \tan^2 \gamma - \frac{64\theta^2 x}{L}\right.$$
$$\left. + \frac{16\theta x}{L} \tan \gamma - 8\theta \tan \gamma\right)^{1/2} \quad (a)$$

The maximum stress occurs at one end of the cable.

For $x = 0$,    $T_{max} = H(1 + 16\theta^2 + \tan^2 \gamma - 8\theta \tan \gamma)^{1/2}$
For $x = L$,    $T_{max} = H(1 + 16\theta^2 + \tan^2 \gamma + 8\theta \tan \gamma)^{1/2}$  $\Bigg\}$    (11·6)

If the cable chord is horizontal, $\tan \gamma = 0$, whence at either end of the cable

$$T_{max} = H(1 + 16\theta^2)^{1/2} \quad (11\ 7)$$

For the special case of the horizontal cable chord, Eq. (11·7) may also be derived as follows: The maximum cable stress occurs at the end of the cable and equals the resultant reaction on the cable.   For this reaction, the horizontal component is $H$, while the vertical component equals half the total load on the cable, or $wL/2$, which may also be expressed as $4Hh/L = 4H\theta$, since $H = wL^2/8h$; hence

$$T_{max} = (H^2 + R_v^2)^{1/2}$$
$$= (H^2 + 16H^2\theta^2)^{1/2}$$
$$= H(1 + 16\theta^2)^{1/2}$$

### 11·6   Illustrative Examples.

**Example 11·1**   *A cable supporting a uniform load of 1 kip per horizontal foot is suspended between two points of equal elevation, which are 2,000 ft apart, with a sag ratio such that the horizontal component of cable stress is 2,500 kips.   What is the maximum stress in the cable?*

**Solution:** *Since* $H = wL^2/8h$, $2,500 = 1(2,000)^2/8h$; $h = 200\ ft$;

$$\theta = \frac{h}{L} = \frac{200}{2,000} = \frac{1}{10}$$
$$T_{max} = H(1 + 16\theta^2)^{1/2} = 2,500(1 + {}^{16}\!/_{100})^{1/2} = 2,690\ kips$$

**Example 11·2**   *A cable supporting a uniform load of 100 lb per horizontal foot is suspended between two points 200 ft apart horizontally, with one point of support 50 ft higher than the other.   The tension in the cable is adjusted until the sag at mid-span (i.e., the vertical distance from the cable to the cable chord at mid-span) equals 12.5 ft.   What is the maximum stress in the cable?*

**Solution:** $H = \dfrac{wL^2}{8h} = \dfrac{100(200)^2}{8(12.5)} = 40,000\ lb$

$$\theta = \frac{h}{L} = \frac{12.5}{200} = 0.0625; \qquad \tan \gamma = \frac{50}{200} = 0.25$$

*The maximum cable stress occurs at the high point of support and is given by*

$$T_{max} = H(1 + 16\theta^2 + \tan^2 \gamma + 8\theta \tan \gamma)^{\frac{1}{2}}$$
$$= 40,000[1 + 16(0.0625)^2 + (0.25)^2 + 8(0.0625)(0.25)]^{\frac{1}{2}}$$
$$= 44,700 \ lb$$

**11·7 Length of Uniformly Loaded Cable.** If $s_o$ is the total length of a cable, then

$$s_o = \int_0^L ds = \int_0^L \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx \qquad (a)$$

For the case of a horizontal cable chord, using Eq. (11·5), which is based on the origin being located at the low point on the cable, which is at mid-span,

$$\frac{dy}{dx} = \frac{8hx}{L^2}$$

whence

$$s_o = 2 \int_0^{L/2} \left[ 1 + \frac{64h^2x^2}{L^4} \right]^{\frac{1}{2}} dx \qquad (b)$$

Integration of this relation leads to the following exact expression:

$$s_o = \frac{L}{2} (1 + 16\theta^2)^{\frac{1}{2}} + \frac{L}{8\theta} \ln [4\theta + (1 + 16\theta^2)^{\frac{1}{2}}] \qquad (11·8)$$

The use of Eq. (11·8) requires the use of natural logarithms. A very useful approximate expression for determining cable length when the cable chord is horizontal may be obtained by expanding the term

$$\left[ 1 + \left( \frac{64h^2x^2}{L^4} \right) \right]^{\frac{1}{2}}$$

occurring in Eq. (b) into an infinite series by the binomial theorem and considering only the first few terms of the resultant series. This leads to

$$s_o = 2 \int_0^{L/2} \left[ (1)^{\frac{1}{2}} + \frac{1}{2} (1)^{-\frac{1}{2}} (64) \frac{h^2x^2}{L^4} \right.$$
$$\left. + \frac{1}{2} \left( -\frac{1}{2} \right) (1)^{-\frac{3}{2}} (64)^2 \frac{h^4x^4}{L^8} + \cdots \right] dx$$

which simplifies into

$$s_o = L \left( 1 + \frac{8\theta^2}{3} - \frac{32\theta^4}{5} + \cdots \right) \qquad (11·9)$$

This equation converges rapidly, the first three terms giving sufficient accuracy for many purposes.

If the cable chord is inclined, an exact equation for $s_o$ could be obtained by substituting $dy/dx$ as obtained from Eq. (11·3) into Eq. (a). This leads, however, to an extremely cumbersome result. For many purposes the following approximate treatment gives results of sufficient accuracy:

Assume that the length of such a cable is the same as it would be for a cable with a horizontal cable chord, where the span equals the length of the inclined cable chord, that is, $L \sec \gamma$, and the sag equals $h \cos \gamma$, which is an approximate expression for the maximum perpendicular distance from the cable to the cable chord. With these assumptions,

$$\theta' = \frac{(h \cos \gamma)}{(L \sec \gamma)} = \frac{\theta}{\sec^2 \gamma}$$

Now, applying Eq. (11·9) to this hypothetical cable and using only the first two terms,

$$s_o = L \sec \gamma \left(1 + \frac{8}{3} \frac{\theta^2}{\sec^4 \gamma}\right)$$

$$= L \left(\sec \gamma + \frac{8}{3} \frac{\theta^2}{\sec^3 \gamma}\right) \qquad (11·10)$$

To find the approximate length of the cable of Example 11·1, Art. (11·6), use Eq. (11·9) ($\theta = \frac{1}{10}$).

$$s_o = 2,000 \left[1 + \frac{8}{3}\left(\frac{1}{100}\right) - \frac{32}{5}\left(\frac{1}{10,000}\right)\right]$$
$$= 2,000(1.000 + 0.0267 - 0.0006) = 2,052 \text{ ft}$$

To find the approximate length of the cable of Example 11·2, Art. (11·6), use Eq. (11·10).

$$\theta = 0.0625; \qquad \sec^2 \gamma = 1 + \tan^2 \gamma = 1 + (0.25)^2 = 1.0625;$$
$$\sec \gamma = 1.031$$

$$s_o = 200 \left[1.031 + \frac{8}{3} \frac{(0.0625)^2}{(1.031)^3}\right] = 208 \text{ ft}$$

**11·8  Elastic Stretch of Cables.** When a cable supports a load, it undergoes an elastic stretch, which is often of importance in determining cable sags and for other purposes. By the definition of the modulus of elasticity,

$$E = \frac{F/A}{\Delta L/L}, \qquad \text{whence } \Delta L = \frac{FL}{AE}$$

An element of a cable of length $ds$ is subject to a tension $T_x$. A convenient method of determining the elastic stretch of a cable consists in first determining $T_{av}$, which by definition will be taken as that average tension which if applied throughout the length of the cable will cause

the same total elastic change of length as actually occurs. Stated mathematically,

$$\frac{T_{av}s_o}{AE} = \int_0^{s_o} \frac{T_x \, ds}{AE}$$

where $A$ and $E$ are assumed constant. Hence

$$T_{av} = \frac{1}{s_o} \int_0^{s_o} T_x \, ds = \frac{1}{s_o} \int_0^{s_o} H \frac{ds}{dx} ds = \frac{H}{s_o} \int_0^L \left[1 + \left(\frac{dy}{dx}\right)^2\right] dx \quad (a$$

Considering a loading that is uniform per horizontal foot, and an inclined cable chord, $dy/dx$ may be obtained from Eq. (11·3), and equals $(8hx/L^2) - (4h/L) + \tan \gamma$. Substituting this value of $dy/dx$ into Eq. (a), and integrating,

$$T_{av} = \frac{HL}{s_o} \left(1 + \frac{16}{3} \theta^2 + \tan^2 \gamma\right) \tag{b}$$

Using Eq. (11·10) to express $s_o$,

$$T_{av} = H \frac{1 + \dfrac{16\theta^2}{3} + \tan^2 \gamma}{\sec \gamma + \dfrac{8}{3} \dfrac{\theta^2}{\sec^3 \gamma}} \tag{11·11}$$

If the cable chord is horizontal, $\gamma = 0$, whence

$$T_{av} = H \frac{1 + \dfrac{16\theta^2}{3}}{1 + \dfrac{8\theta^2}{3}} \tag{11·12}$$

Suppose, for example, that it is required to find the elastic stretch of the cable of Example 11·1, Art. (11·6), for which $s_o$ was found to equal 2,052 ft in Art. (11·7). Taking $E = 27,600,000$ psi and $A = 50$ sq in.,

$$T_{av} = 2,500 \frac{1 + \frac{16}{3}(\frac{1}{100})}{1 + \frac{8}{3}(\frac{1}{100})} = 2,570 \text{ kips}$$

$$\text{Elastic stretch} = \frac{T_{av}s_o}{AE} = \frac{2,570(2,052)}{\frac{50}{144}(27,000)(144)} = 3.91 \text{ ft}$$

To the nearest foot, the *unstressed* length of this cable would be

$$2,052 - 4 = 2,048 \text{ ft}$$

**11·9 Guyed Structures.** The application of the various relations for cables to guys can be shown by a consideration of the structure shown in Fig. 11·5. Suppose that a cable weighing 4.16 lb per ft is to be used as a guy and that it is required that this guy hold the mast *BC* vertically

when the load of 100,000 lb is applied. Taking moments about $C$ of the forces acting on the mast and noting that $H$, the horizontal component of cable stress, acts to the left on the mast,

$$100H = 100,000(10); \qquad H = 10,000 \text{ lb}$$

This value of $H$ must be developed to hold the mast vertically; thus the cable sag must conform to the following value [in this approximate solution the cable is assumed to be parabolic and acted upon by a uniform load of

$w = 4.16(180.2/150)$
$\quad = 5.00$ lb per horizontal foot]:

$$10,000 = \frac{wL^2}{8h} = \frac{5.00(150)^2}{8h};$$

$$h = 1.41 \text{ ft}$$
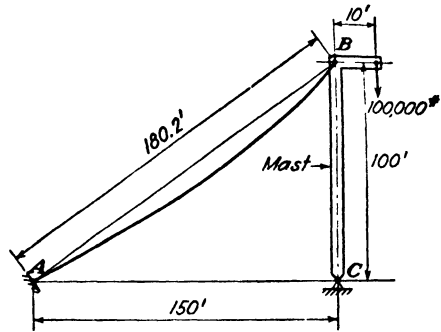


FIG 11·5

The maximum tension in the guy would occur at $B$ and by using Eq. (11·6) equals

$$T_{max} = 10,000\left[1.000 + 16\left(\frac{1.41}{150}\right)^2 + \left(\frac{100}{150}\right)^2 + 8\left(\frac{1.41}{150}\right)\left(\frac{100}{150}\right)\right]^{\frac{1}{2}}$$
$$= 12,240 \text{ lb}$$

It should be noted that a close approximation to the foregoing solution can be obtained by considering the cable to act as a straight tie rod lying along the chord $AB$. The length of this chord is 180.2 ft; the tension would equal

$$10,000\left(\frac{180.2}{150}\right) = 12,000 \text{ lb}$$

The only error introduced in this approximate solution arises from the fact that the slope of the cable at $B$ is actually steeper than that of the chord. Unless the sag ratio of the cable is large, this difference in slope is not likely to be of importance.

It is usually necessary to have more than one guy on a mast or other guyed structure, in order that the mast may be held against overturning in more than one direction. Under these conditions, the guys are adjusted so that they have a certain initial tension; an initial tension equal to about half the maximum tension that will occur under load is commonly used. In Fig. 11·6, suppose that the sags of the guys $AB$ and $AC$ have been adjusted so that, with no horizontal load applied to

the mast, the horizontal component of cable tension in each guy is 6,000 lb. Under these circumstances, the maximum initial stress in each guy, following the approximate method of analysis suggested in the discussion of Fig. 11·5, will equal $+6,000 \times 1.414 = +8,500$ lb.

When the load of 5,000 lb is applied at the top of the mast, a resultant horizontal force of 10,000 lb, acting to the left, must be applied at $A$ by the two guys to maintain equilibrium. By an approximate method of analysis that leads to reasonably accurate results, it may be assumed that this load of 10,000 lb is applied by increasing the horizontal component of tension in guy $AB$ by 5,000 lb and decreasing the horizontal component of tension in guy $AC$ by the same amount, leading to $H_{AB} = +11,000$ lb and $H_{AC} = +1,000$ lb. The maximum guy stress is then equal to

$$+11,000 \times 1.414 = +15,500 \text{ lb}$$
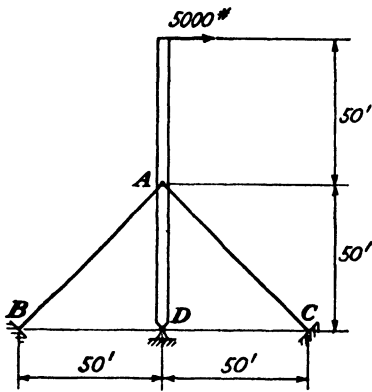
Actually, when the load of 5,000 lb is applied at the top of the mast, point $A$ moves to the right. Guy $AB$ permits this movement since it undergoes additional elastic stretch and since its sag decreases. The sag in guy $AC$ increases, and the length of that guy shortens elastically. An exact analysis,



Fig 11·6

taking into account elastic changes of length and sag changes, can be made, but it is relatively complicated. The foregoing approximate solution is more practical for usual design problems.

**11·10 Statically Determinate Suspension Bridges.** A suspension bridge is usually constructed in a manner such that the dead loads are carried entirely by the cables. A large portion of the dead load comes from the roadway and is uniform. It is commonly assumed that the entire dead load is uniform per horizontal foot. On the basis of this assumption, the cables are parabolic under dead load only. When live load is applied, with partial loadings so as to give maximum stresses in members, the cables tend to change their shape. In order to prevent local changes of slope in the roadway, due to live load, from being too large, the floor beams of the floor system are usually framed into stiffening trusses, which in turn are supported by hangers running to the cables. These stiffening trusses distribute the live load to the various hangers, in a manner such that even under live loads the cable may be assumed to remain essentially parabolic. As long as the cable remains parabolic, it must be acted upon by a load that is uniform per horizontal foot.

Since the hangers are equally spaced, this is equivalent to stating that the hanger stresses in a given span must be equal. In the "elastic" theory of suspension bridges, all the hangers in a given span are assumed to have equal stresses. That this assumption is not strictly correct may be shown by the more accurate and more complicated "deflection" theory of suspension bridges. However, unless a suspension bridge is long and flexible, the elastic theory may lead to results that are not greatly in error.

If the stiffening trusses of a suspension bridge are arranged and supported as shown in Fig. 11·7a, with a hinge at some intermediate point
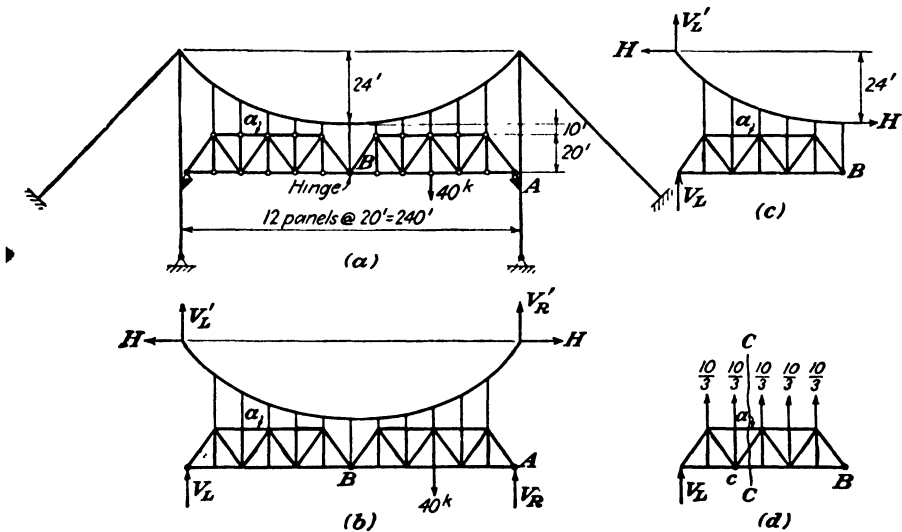


FIG. 11·7

in the main-span stiffening truss, the structure is statically determinate, provided that it is assumed that all the hangers in a given span have equal stresses. The application of the various relations for cables to a statically determinate suspension bridge will be illustrated by an analysis of the structure shown in that figure.

Suppose this bridge is subjected to a live load of 40 kips acting as shown in the figure. Consider the equilibrium of all the forces acting on that portion of the structure shown in Fig. 11·7b. The horizontal components of cable reaction at each end of the cable are equal and have the same line of action. Taking moments about point $A$,

$$(V_L + V'_L)(240) - 40(60) = 0; \qquad V_L + V'_L = +10 \text{ (upward)}$$

Now consider the equilibrium of all the forces acting on that portion

of the structure shown in Fig. 11·7c, taking moments of these forces about the hinge at B.

$$(V_L + V'_L = 10)(120) + H(30) - H(54) = 0; \qquad H = 50.0 \text{ kips}$$

The maximum cable stress in the main span occurs at the ends of the cable and by Eq. (11·7) equals $50.0(1 + {}^{16}\!/_{100})^{\frac{1}{2}} = 53.9$ kips.

Let $X$ equal the stress in each hanger. The equivalent uniform load of the hangers equals $X/20$ kips per ft. To evaluate $X$, use the relation $H = wL^2/8h$, where $w = X/20$.

$$50.0 = \frac{X}{20} \frac{(240)^2}{8(24)}; \qquad X = + \frac{10}{3} \text{ kips}$$

Once the hanger stresses have been determined, the stresses in the bars of the stiffening truss are readily evaluated. For example, to find the stress in bar $a$, first find $V_L$ by taking moments about $B$ of the forces acting on the portion of the structure shown in Fig. 11·7d.

$$+120V_L + {}^{10}\!/_3(20 + 40 + 60 + 80 + 100) = 0; \qquad V_L = -{}^{25}\!/_3 \text{ (down)}$$

Now taking moments about $c$ of the forces acting on the portion of the structure to the left of section $C$-$C$ (Fig. 11·7d),

$$-{}^{25}\!/_3(10) + {}^{10}\!/_3(20) + F_a(20) = 0; \qquad F_a = +13.33 \text{ kips}$$

For this particular structure, the side spans are not suspended. The cables of the side spans act as guys to the towers. The horizontal component of cable stress is the same for side and center spans, as can be seen by taking moments about the hinge at a tower base of the forces acting on a tower. $V_L$ is assumed to act on the center line of the tower.

### 11·11   Problems for Solution.

**Problem 11·1**   A cable with a span of 1,000 ft carries concentrated loads spaced at horizontal intervals of 200 ft. The magnitudes of these loads, from left to right, in kips, are 100, 50, 200, and 300. The right end of the cable is 100 ft higher than the left end. The maximum distance, measured vertically from the cable to the cable chord, is 50 ft. Neglecting the weight of the cable itself;

*a.* What is the vertical distance from the cable chord to the point of application of each load?

*b.* What is the length of the cable?

*c.* What is the maximum tension in the cable?

**Problem 11·2**   A side-span suspension-bridge cable has a span of 500 ft and a sag ratio of $\frac{1}{10}$. The slope of the cable chord is defined by tan $\gamma = 0.7$. The load on the cable is 1,000 lb per horizontal foot; $E = 27,000,000$ psi; the area of the cable is 50 sq in.

a. What is the maximum slope of the cable?

b. What is the maximum stress in the cable?

c. Compute, to the nearest foot, the length of this cable.

d. Compute, to the nearest foot, the unstressed length of this cable.

**Problem 11·3** Prepare a set of curves from which the ratio $s_o/L$ can be read, for values of $\theta$ ranging from 0 to 0.25 and values of $\tan \gamma$ ranging from 0 to 1, for cables where the loading is uniform per horizontal foot.

**Problem 11·4** A cable with a span of 1,000 ft and a horizontal cable chord carries 1,500 lb per horizontal foot. The tension is adjusted until the maximum cable stress is 2,000,000 lb. The temperature is $+50°F$; $E = 27,000$ kips per sq in.; $A = 40$ sq in.

a. What is the corresponding sag at mid-span?

b. What is the unstressed length of this cable at $+100°F$?

**Problem 11·5** The top of a derrick mast is guyed with 12 guys, spaced 30° apart as seen in plan view. Each guy has a span of 400 ft and a vertical rise of 150 ft. The guy cable weighs 5 lb per ft of cable; the sag of each guy is 4 ft.

a. What compression is exerted on the derrick mast by the sum of the vertical cable reactions?

b. What is the maximum stress in each guy?

c. The mast of the derrick is 150 ft high. If the boom line exerts a horizontal force of 100,000 lb at the top of the mast, what is the approximate value of the maximum stress occurring in any guy? (Assume that each guy participates in resisting this force, in proportion to the cosine of the angle between the guy and the force, as seen in plan view.)

**Problem 11·6** A suspension bridge is similar to that shown in Fig. 11·7a, having 20 panels of 20 ft each and a cable sag ratio of $\frac{1}{10}$. The stiffening truss is 20 ft deep.

a. If a live load of 1,000 lb per ft acts over the entire bottom chord, compute the maximum cable stress in the suspended span and the maximum bottom-chord stress in the stiffening truss.

b. Construct influence lines for (1) horizontal component of cable stress; (2) hanger stress; (3) stress in the stiffening-truss diagonal of the second panel from the left tower.

c. The dead load on this structure is 5 kips per ft. The live load consists of a uniform load of 2 kips per ft and a single concentrated load of 20 kips. Neglecting impact, what cross-sectional area is required for each hanger, using a working stress of 50 kips per sq in. for cables in tension?

# CHAPTER 12

# APPROXIMATE ANALYSIS
# OF STATICALLY INDETERMINATE STRUCTURES

**12·1 Introduction.** From a broad viewpoint, the analysis of every structure is approximate, for it is necessary to make certain assumptions in order to carry out the analysis. For example, in computing the stresses in a pin-connected truss, it is assumed that the pins are frictionless, so that the truss members carry axial stress only. It is, of course, impossible to build a pin connection that is frictionless, and as a result the stress analysis of a pin-connected truss is approximate. It may therefore be said that there is no such thing as an "exact" analysis.

However, if proper judgment is exercised in making the assumptions upon which the analysis of a given structure is based, the resultant errors will be small. A stress analysis based on the usual assumptions that underlie structural theory is often called "exact," although it may be seen that, strictly speaking, this term is not used correctly. It is, however, a convenient term to use, for it is desirable to distinguish between analyses based on the usual assumptions and which are relatively exact, and analyses based on further assumptions which introduce further errors and which are therefore frankly approximate.

When one speaks of an approximate analysis for a given structural problem, he does not necessarily refer to any particular set of assumptions and resultant approximations. The particular approximate method to be used under any given circumstances will depend on the time available for the analysis and the degree of accuracy considered necessary.

For structural types that occur commonly in structural analysis, one may take advantage of approximate methods of analysis worked out by others and investigated as to their accuracy so that they can be used with a fair degree of confidence. The approximate methods described in engineering literature do not, however, cover all cases. A good stress analyst should be familiar enough with the action of statically indeterminate structures to be able to set up his own assumptions when he encounters circumstances not covered by the literature.

In this chapter, a number of approximate solutions for common types of statically indeterminate structures are given. A knowledge of these methods is of importance, but of perhaps greater importance is the fact that the procedures here outlined will serve as a basis for making intelligent assumptions that will permit simplified approximate analyses of other types of statically indeterminate structures.

12·2   Importance of Approximate Methods in Analyzing Statically Indeterminate Structures.   The analysis of a statically determinate structure does not depend on the elastic properties of its members. Because of this, relatively simple "exact" stress analyses can be carried out for such structures.

In a statically indeterminate structure, however, stress analysis depends on the elastic properties of members.   These elastic properties include modulus of elasticity, cross-sectional area, cross-sectional moment of inertia, and length of member.   That this is so may be visualized by reference to Fig. 12·1.   Suppose that the stiffness of beam $AB$ is made very small in comparison with the stiffness of beam $CD$.   This may be accomplished by making $E_1I_1/L_1^3$ very small in comparison with $E_2I_2/L_2^3$.   Then beam $CD$ will carry a greater portion of load $P$ than will beam $AB$.   Suppose further that the tension tie $EF$ that connects the two beams is given a very small stiffness by making $E_3A/h$ small.   This might be accomplished by constructing the tie $EF$ of rubber, which has a very low value of $E$.   Under this condition the beam $CD$ would carry still more of the load $P$.
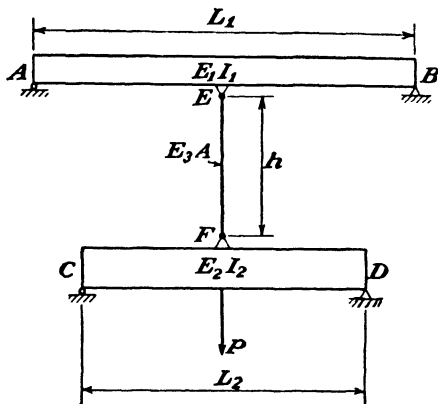


Fig. 12·1

If the quantities determining the stiffnesses of members in a statically indeterminate structure are known, a so-called "exact" analysis may be carried out that will yield results of the same order of accuracy as can be obtained for statically determinate structures.   In actual practice, however, the following three factors may prevent an exact analysis:

1. The stress analyst may lack the knowledge necessary to carry out the statically indeterminate analysis.

2. The time required to carry out a statically indeterminate analysis may be so great that an exact solution must be abandoned.   In some cases, the need of meeting time schedules may be the controlling factor. In other instances, economic considerations may make it desirable to use an approximate method of analysis.   It may be less expensive to use more material, as a result of basing design on approximate stresses and on a higher apparent factor of safety with respect to the computed stresses, than to save material by basing design on exact stresses and a lower apparent factor of safety.   This attitude may sometimes be prop-

erly taken in designing relatively unimportant structures or secondary
portions of important structures.   The analyst will also be influenced
in this connection by his judgment as to the magnitude of the errors likely
to be introduced by the approximate method he proposes to use.

3. When the design of a statically indeterminate structure is first
begun, the areas and moments of inertia of its members are not known.
It is therefore necessary to carry out an approximate analysis for stresses
in the structure, so as to obtain some idea as to the required sizes of these
members.   Once these tentative sizes have been assigned, an elastic
analysis may be carried out.   In general, the first elastic analysis will
show that the actual fiber stresses in the structure are not satisfactory,
and it is only by successive designs that a satisfactory final result can be
obtained.   Approximate analyses of statically indeterminate structures
are therefore important in preliminary design stages.

**12·3   Number of Assumptions Required.**   It has previously
been pointed out that, for an analysis of a structure to be possible
on the basis of the equations of statics only, there must be available
as many independent equations of statics as there are independent
components of stress in the structure.   If there are $n$ more inde-
pendent components of stress than there are independent equations
of statics, the structure is statically indeterminate to the $n$th degree.
It will then be necessary to make $n$ independent assumptions, each of
which supplies an independent equation of statics, in order that an
approximate solution can be worked out on the basis of statics only.

If fewer than $n$ assumptions are made, a solution based on statics
only will not be possible.   If more than $n$ assumptions are made, the
assumptions will not in general be consistent with each other and the
application of the equations of statics will lead to inconsistent results,
depending on which equations are
used and the order in which they
are used.   The first step in the
approximate analysis of a stati-
cally indeterminate structure is to
find the degree to which the struc-
ture is indeterminate and hence
the number of assumptions to be
made.
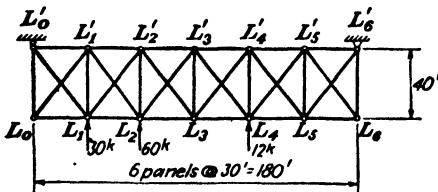


FIG. 12·2

**12·4   Parallel-chord Trusses with Two Diagonals in Each
Panel.**   Trusses of this type occur frequently in structural engineering,
as, for example, in the top- and bottom-chord lateral systems of a bridge,
as described in Art. 1·21.   The approximate analysis of such a truss will
be illustrated by considering the truss of Fig. 12·2, in which it is assumed

that all members are capable of carrying either compression or tension. It should first be noted that this truss is statically indeterminate to the sixth degree; that this is true may readily be seen from the fact that, if one diagonal were removed from each panel, the remaining members would form a statically determinate truss. It is therefore necessary to make six independent assumptions as to stress conditions. *It will be assumed that in each panel the shear is equally divided between the two diagonals;* since there are six panels, this amounts to six independent assumptions. It is now easy to complete the analysis of the structure, by using only relations of statics. The solution is shown in Fig. 12·3, the method of index stresses having been employed. The shear in each panel is first computed from the external forces. As an illustration, the shear in panel (1-2) is −39 kips. This shear is equally divided between
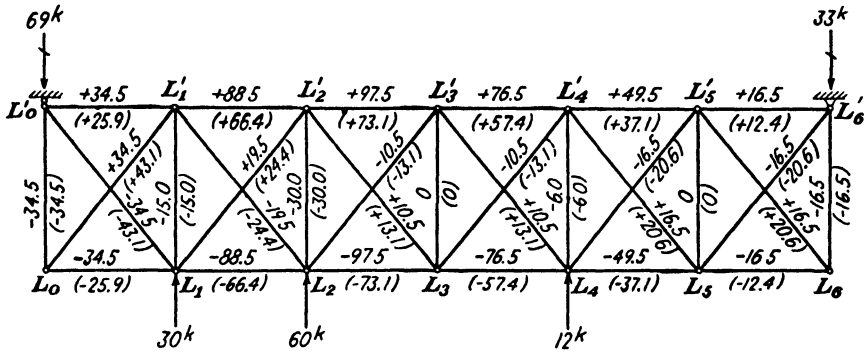


FIG. 12·3

$L_1L'_2$ and $L'_1L_2$, so that the index stresses in these two bars are +19.5 and −19.5, respectively. Index stresses for all diagonals are determined in a similar manner. From these index stresses, the index stresses in all other members may be determined in the usual manner. In Fig. 12·3 the actual stresses are given in parentheses for each member.

In trusses of this type, the diagonals are often designed as tension members only, by making their slenderness ratios (unsupported length divided by radius of gyration) large, since then, when a diagonal is subjected to compression, it will buckle slightly and carry only a negligible load. When the diagonals are designed in this manner, the total shear in each panel is carried in tension by a single diagonal. A consideration of the total shear on each panel enables one to tell which diagonal has a tendency to buckle and therefore carries no load. In Fig. 12·3, if it is assumed that the diagonals were designed so that they could carry tension only, all the diagonals that carried compression in the previous analysis would now have zero stress. This, in effect, makes the truss statically

determinate, and hence no difficulty is encountered in completing the analysis.

**12·5  Multiple-system Trusses.**  Trusses with redundant web members can, under some circumstances, be analyzed approximately by imagining that the indeterminate truss is equivalent to two or more statically determinate trusses superimposed on each other.  The truss of Fig. 12·2, for example, might be built up by combining the two trusses
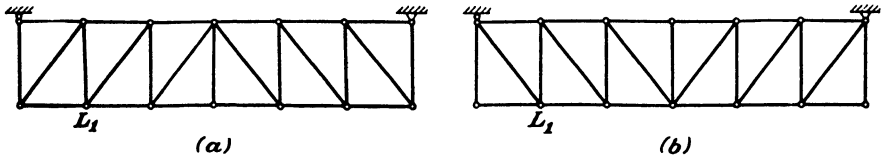


FIG. 12·4

shown in Fig. 12·4.  For this particular type of truss, however, a vertical load applied at any panel point such as $L_1$ could be carried by either of the trusses shown in Fig. 12·4.  Hence, it is not possible to assign such a load to one component truss only.  Consider next the truss shown in Fig. 12·5a, which is statically indeterminate to the first degree.  The truss might be built up by superimposing the two trusses shown in Figs. 12·5b and c.  If a vertical load is applied at any panel point for this truss, it can be assumed to be carried by one component truss only.  Suppose, for example, a vertical load is applied at $L_1$, $L_3$ or $L_5$; it can then be
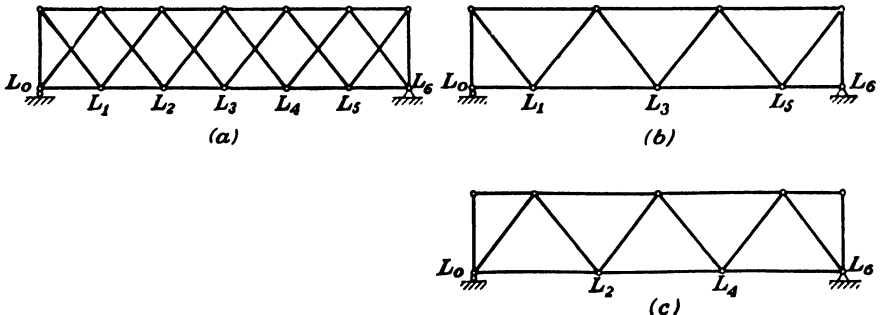


FIG. 12·5

assumed to be carried entirely by the truss of Fig. 12·5b, since the diagonals of that truss can hold the vertical load in equilibrium.  Loads applied at these panel points on the truss of Fig. 12·5c could not be carried unless the bottom chords carried them in bending.  Since the truss members are designed to carry axial stress only, this is not possible.  Similarly, vertical loads at $L_2$ or $L_4$ could be assumed to be carried entirely by the truss of Fig. 12·5c.  The loads applied to the truss of Fig. 12·5a

can therefore be assigned definitely to one or the other of the component trusses. Each of these component trusses is statically determinate and can easily be analyzed for the loading it is assumed to carry. Stresses in the actual truss can then be determined by superimposing the stresses from the two component trusses. The stress in any diagonal is obtained from the component truss in which that diagonal occurs, but the stress in any chord or end post is obtained by adding up the stresses for that chord or end post, as obtained for each component truss.
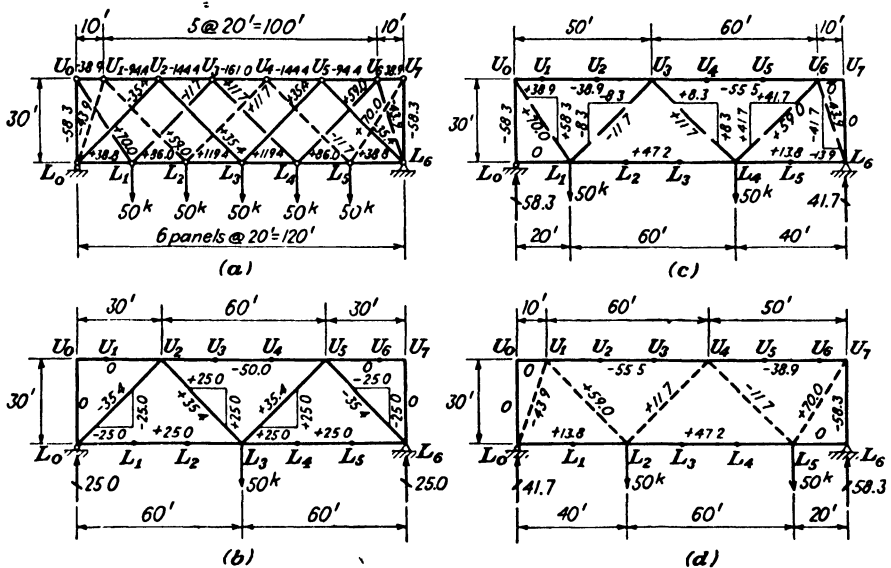


FIG. 12·6

Since the truss of Fig. 12·5a is statically indeterminate to the first degree, the approximate analysis here outlined is based on a single assumption, that, in any *one* panel, the total external shear is divided between the two diagonals of that panel in the manner which would occur if the two component trusses could operate independently under the loadings applied at their respective panel points. This assumption having been made for the division of shears in *one* panel, the division of shears on the same basis in other panels follows directly from the equations of statics, so that no further assumptions are involved.

To illustrate this approximate method of analyzing trusses with multiple web systems, consider the truss shown in Fig. 12·6a. This truss may be considered as composed of three component trusses, as indicated by the full, dash, and dotted lines used in drawing the diagonals. The stresses for each of the three component trusses are given in Figs.

12·6b to d. These component stresses are then superimposed and the total stresses written on Fig. 12·6a. For example, the total stress in $U_2L_3$ is taken directly from Fig. 12·6b (+35.4), while the total stress in $U_2U_3$ is obtained by adding up the stresses for this member as obtained from the component trusses, $-50.00 - 38.9 - 55.5 = -144.4$.

The same approach may be followed if it is desired to construct influence lines for stresses in bars of multiple-system trusses. In Figs. 12·7a and b, influence lines for bars $U_2L_3$ and $U_2U_3$ are drawn. Since $U_2L_3$ is a member of the component truss of Fig. 12·6b, the influence line is first drawn by considering that truss only. The ordinate at $L_3$ is the only significant ordinate to this influence line, however, since, when the unit load is at any other bottom-chord panel point, it is carried by one of the other component trusses, and the stress in $U_2L_3$ is zero. Since the
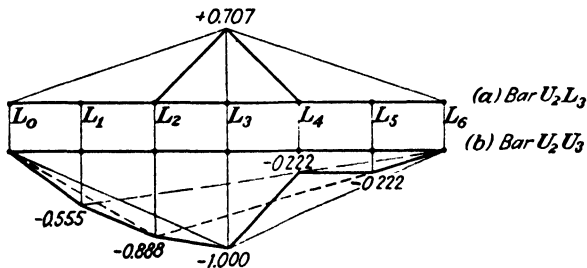


FIG. 12·7

influence line is a straight line between panel points, the resultant influence line for this member is as shown by the heavy line in Fig. 12·7a.

Bar $U_2U_3$ is a member of each of the three component trusses. Considering it in turn as a member of each of the trusses shown in Figs. 12·6b to d, the influence line as a member of each component truss is constructed; these three influence lines are shown by the full, dash, and dotted lines, respectively, in Fig. 12·7b. The ordinate at $L_3$ is the only one that has any significance for the influence line drawn on the basis of Fig. 12·6b, the ordinates at $L_1$ and $L_4$ are the only ones that have any significance for the influence line drawn on the basis of Fig. 12·6c, and the ordinates at $L_2$ and $L_5$ are the only ones that have any significance for the influence line drawn on the basis of Fig. 12·6d. These significant points are connected by the heavy solid line shown in Fig. 12·7b, giving the resultant influence line for stress in bar $U_2U_3$.

**12·6 Portals.** Portal structures, similar to the end portals of the bridge described in Art. 1·21, have as their primary purpose the transfer of horizontal loads, applied at their tops, to their foundations. Clearance requirements usually lead to the use of statically indeterminate structural layouts for portals, and approximate solutions are often used in their

analyses. . Consider the portal shown in Fig. 12·8a, all the members of which are capable of carrying bending and shear as well as axial stress. The legs are hinged at their bases and rigidly connected to the cross girder at the top. . This structure is statically indeterminate to the first degree: hence, one assumption must be made. Solutions of this type of structure, based on elastic considerations, show that the total horizontal shear on the portal will be divided almost equally between the two legs; it will therefore be assumed that the horizontal reactions for the two legs are equal to each other and therefore equal to $P/2$. The remainder of the analysis may now be carried out by statics. The vertical reaction on the right leg can be obtained by taking moments about the hinge at the
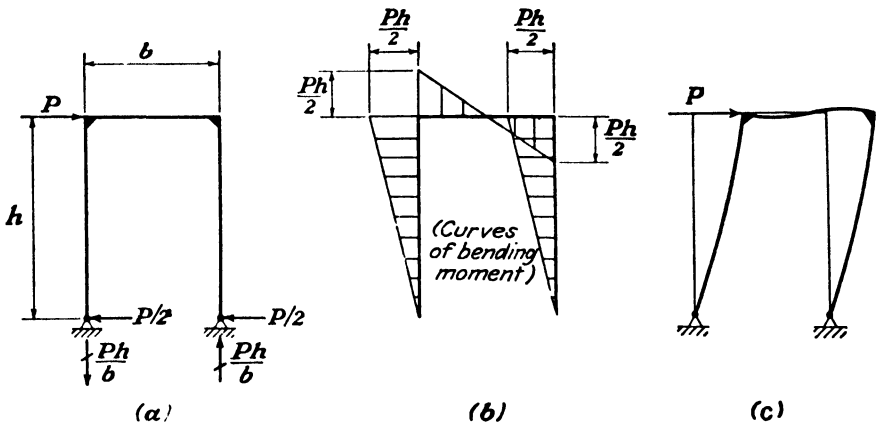


FIG. 12·8

base of the left leg. The vertical reaction on the left leg can then be found by applying $\Sigma F_y = 0$ to the entire structure. Once the reactions are known, the curves of bending moment and shear are easily computed, leading to values for bending moment as given in Fig. 12·8b. It is well to visualize the distorted shape of the portal under the action of the applied load. This is shown, to an exaggerated scale, in Fig. 12·8c.

Consider now a portal similar in some ways to that of Fig. 12·8a but with the bases of the legs fixed, as shown in Fig. 12·9a. This structure is statically indeterminate to the third degree, so that three assumptions must be made. As was the case when the legs were hinged at their bases, it will again be assumed that the horizontal reactions for the two legs are equal and hence equal to P/2. Figure 12·9c shows the distorted shape of the portal under the action of the applied load. It will be noted that near the center of each leg where is a point of reversal of curvature. These are points of inflection where the bending moment is changing sign and hence has zero value. It will therefore be assumed that there is

a point of inflection at the center of each leg: this is structurally equivalent to assuming that hinges exist at points $a$ and $a'$, as shown in Fig. 12·9c. The vertical reactions on this portal equal the axial stresses in the portal legs and may be determined by successively taking moments about $a$ and $a'$ of all the forces acting on that portion of the structure
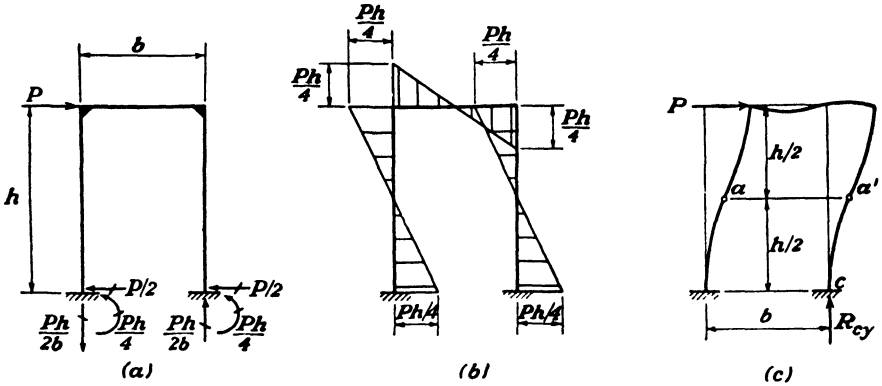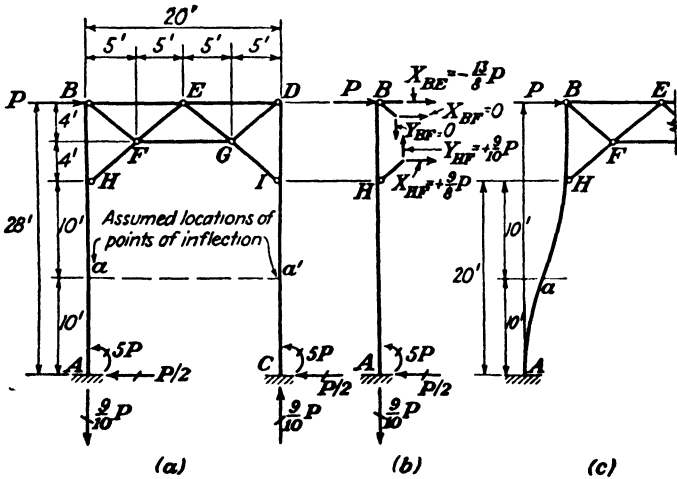


FIG. 12·9



FIG. 12·10

above $a$ and $a'$. For example, taking moments about $a$,

$$+P\frac{h}{2} - R_{cv}b = 0; \qquad R_{cv} = +\frac{Ph}{2b}$$

The moment reaction at the base of each leg equals the shear at the point of inflection in the leg multiplied by the distance from the point of inflection to the base of the leg and therefore equals $(P/2)(h/2) = Ph/4$.

Once the reactions are known, the curves of shear and bending moment for the members of the portal are easily determined by statics. The curves of bending moment for this structure and loading are given in Fig. 12·9b.

Portals for bridges are often arranged in a manner similar to that shown in Fig. 12·10a. In such a portal, the legs $AB$ and $CD$ are continuous from $A$ to $B$ and $C$ to $D$, respectively, and are designed so as to be capable of carrying bending moment and shear as well as axial stress. The other members that comprise the truss at the top of the portal are considered as pin-connected and carrying axial stress only. Such a structure is statically indeterminate to the third degree; the following three assumptions will be made:

1. The horizontal reactions are equal.

2. A point of inflection occurs midway between the base $A$ of the leg $AB$ and the end $H$ of the knee brace for leg $AB$.

3. A point of inflection occurs midway between the base $C$ of the leg $CD$ and the end $I$ of the knee brace for leg $CD$.

The horizontal reactions therefore each equals $P/2$. The moment at the base of each leg equals the shear in the leg multiplied by the distance from the point of inflection to the base of the leg and has a value of $(P/2)(10) = 5P$. Vertical reactions can be obtained by successively taking moments about the points of inflection $a$ and $a'$ of the forces acting on that portion of the structure above $a$ and $a'$. These equations show the vertical reactions each equal to $9P/10$ and to act in the directions shown in the figure.

To find the stresses in the bars connected to the legs, one may proceed as follows: Considering the leg $AB$ as a rigid body and taking moments about $B$ of the forces acting on the leg,

$$+\frac{P}{2}(28) - 5P - X_{HF}(8) = 0; \qquad X_{HF} = +\frac{9P}{8}$$

$$\therefore Y_{HF} = +\frac{9P}{8}\left(\frac{4}{5}\right) = +\frac{9P}{10}$$

To find the stress in bar $BF$, apply $\Sigma F_y = 0$ to all the forces acting on leg $AB$.

$$+Y_{BF} - \frac{9P}{10} + \frac{9P}{10} = 0 \qquad \therefore Y_{BF} = 0 \qquad \therefore X_{BF} = 0$$

To find the stress in bar $BE$, apply $\Sigma F_x = 0$ to all the forces acting on leg $AB$.

$$+X_{BE} + 0 + \frac{9P}{8} + P - \frac{P}{2} = 0 \qquad \therefore X_{BE} = -\frac{13P}{8}$$

With all the forces acting on leg $AB$ known, as shown in Fig. 12·10*b*, the axial stress on any section of the leg may be computed, and curves of shear and moment for the leg may be constructed. Leg $CD$ can be analyzed in a similar manner. The remaining bar stresses may then be computed without difficulty.

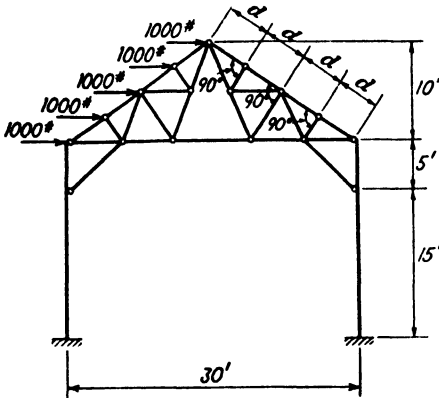**12·7 Mill Bents.** Bents of mill buildings are often constructed as shown in Fig. 12·11. An approximate analysis of stresses in such a bent when it is acted on by lateral loads may be carried out on the basis of assumptions identical with those made for the portal of Fig. 12·10, *viz.*, the horizontal reactions at the bases of the legs are equal, and a point of inflection occurs in each leg at an elevation of 7.5 ft above the base of the leg. The application of the equations of statics to carry out the analysis after these assumptions have been made follows the same general procedure as that employed for the portal.



Fig. 12·11

**12·8 Towers with Straight Legs.** If the legs of a statically determinate tower have a constant batter throughout their length, each face of the tower lies in a plane. For such a tower, stress analysis for lateral loads can be carried out by resolving all lateral loads into components lying in the planes of the faces of the tower that are adjacent to the joints where the lateral loads are applied. Each face of the tower may then be analyzed as a planar truss, acted upon by forces lying in the plane of that truss. Such an analysis leads directly to the stresses in the web members of each face, while the stresses in the legs are obtained by superimposing the stresses resulting from the analyses of the adjacent faces.

The planar truss of each face however, may be statically indeterminate. For such a tower, stress analysis may still be carried out on the basis of planar trusses, but it is necessary to make certain assumptions if the analysis is to be based on statics only. In Fig. 12·12*a*, let it be required to determine the stresses in the diagonals of panel *abcd*. Passing section *M-M* through this panel, at the elevation of the intersection of the diagonals, take moments about *o*, which is at the intersection of the extended legs, of all the forces acting on that portion of the truss above section *M-M*. Since the legs extended pass through the origin of the moments, the moments of forces $P_1$, $P_2$, and $P_3$ are held in equilibrium by the moments of the horizontal components in the diagonals *ad* and *bc*, which act with the lever arm *h*.

If the diagonals can carry tension only, diagonal *ad* will have zero stress, so

that only one unknown appears in the foregoing equation, and the horizontal component of stress in bar $bc$ is obtained directly. If, however, the diagonals can carry compression as well as tension, it may be assumed that the horizontal components of stress in the two diagonals are numerically equal but opposite in sign. This reduces the number of independent unknowns in the foregoing equation to one, and the horizontal components of the diagonal stresses may thus be determined.

If the horizontal bars at the intermediate panel points are omitted, as shown in Fig. 12·12b, the tower truss may be considered as being composed of two component trusses as shown in Figs. 12·12c and d and each truss may be analyzed by statically determinate procedures for the loads assigned to it. Resultant
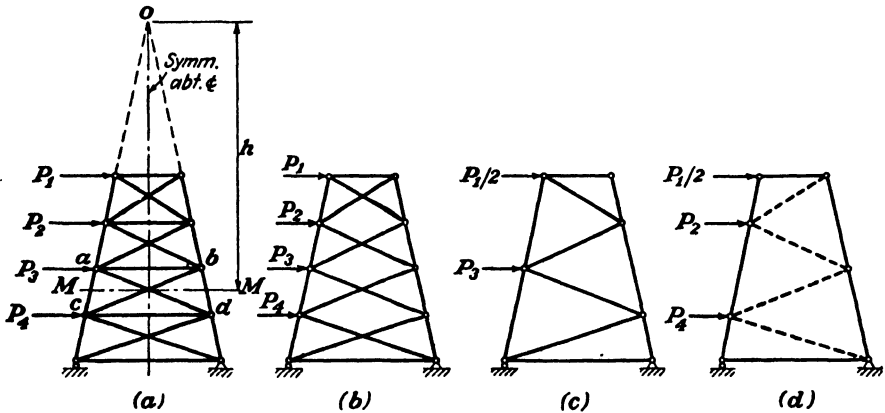


FIG. 12·12

stresses in the actual truss may then be determined by superimposing the stresses from the two component trusses.

It is therefore seen that the same general procedures as those used for the approximate analyses of end-supported indeterminate trusses may be applied in the approximate analyses of tower sides which are actually statically indeterminate planar cantilever trusses.

**12·9  Stresses in Building Frames Due to Vertical Loads.** A building frame consists primarily of girders, which carry vertical loads to columns, and of the columns themselves. While such a frame might be built like that shown in Fig. 12·13a, which is statically determinate, it would have little resistance against horizontal forces, such as wind loads, which it must also carry. It is therefore actually built as shown in Fig. 12·13b, in which the girders are rigidly connected to the columns so that all the members can carry bending moment, shear, and axial stress. Such a frame is called a rigid frame; it is also referred to as a building bent. Because of the rigid construction, a building frame is highly indeterminate. The degree to which it is indeterminate can be

investigated by an examination of Fig. 12·13c. Suppose that each girder is cut near mid-span as shown. The resulting structure will be statically determinate, since each column, together with its girder stubs, acts as a cantilever. To arrive at this condition, however, it is necessary to remove the bending moment, shear, and axial stress in each girder where it is cut. If $n$ is the number of girders in the bent, it is necessary to remove $3n$ redundant stresses to make the bent statically determinate:
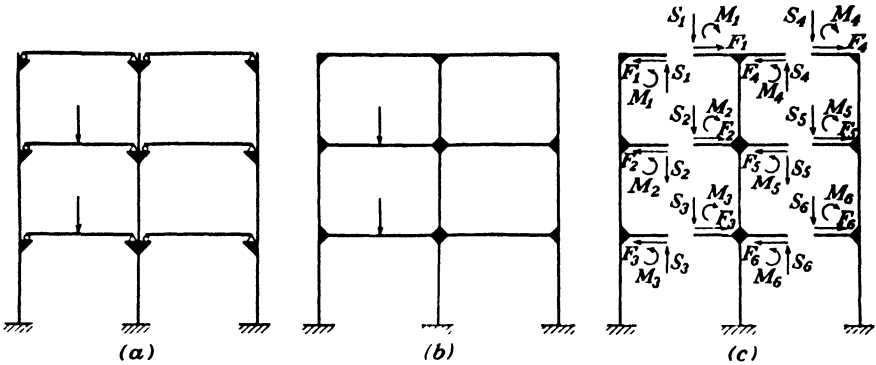


Fig. 12·13

hence the bent is indeterminate to the $3n$th degree. The bent of Fig. 12·13b is therefore statically indeterminate to the 18th degree. A bent with 100 stories and 8 stacks of columns would include 700 girders and be statically indeterminate to the 2,100th degree. It is therefore highly advantageous to have approximate methods available for the analysis of such structures.

Since a building frame of the type just considered is statically indeterminate to a degree equal to three times the number of girders, it will
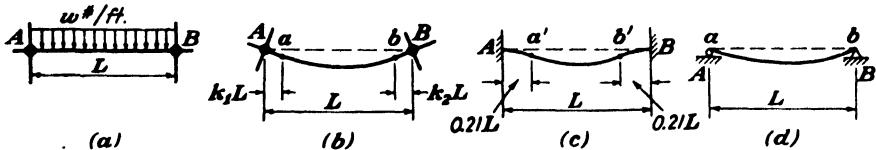


Fig. 12·14

be necessary to make three stress assumptions for each girder in the bent if an analysis is to be carried out on the basis of statics only. In Fig. 12·14a, if a girder is subjected to a load of $w$ lb per ft, extending over the entire span, both of the joints $A$ and $B$ will rotate as shown in Fig. 12·14b, since, while they are partly restrained against rotation, the restraint is not complete. Had the supports at $A$ and $B$ been completely fixed against rotation, as shown in Fig. 12·14c, it can easily be shown

from a consideration of stresses in a fixed-end beam that the points of inflection would be located at a distance of 0.21L from each end.   If the supports at A and B were hinged, as shown in Fig. 12·14d, the points of zero moment would be at the end of the beam.   For the actual case of partial fixity, the points of inflection may be assumed to lie somewhere between the two extremes of 0.21L and 0.00L from the ends of the beam. If they are assumed to be located at one-tenth of the span length from each end joint, a reasonable approximation has been made.

Solutions of building bents based on elastic action show that under vertical loads the axial stress in the girders is usually very small.

The following three assumptions will therefore be made for each girder, in analyzing a building bent acted upon by vertical loads:

1. The axial stress in the girder is zero.

2. A point of inflection occurs at the one-tenth point measured along the span from the left support.

3. A point of inflection occurs at the one-tenth point measured along the span from the right support.
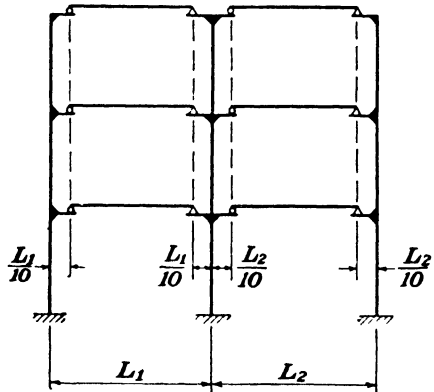
This is equivalent to assuming



FIG 12·15

that the bent acts structurally in the same manner as does the statically determinate bent of Fig. 12·15.   Girders may then be analyzed by statics, as will be illustrated by considering Fig. 12·16.   The maximum positive moment occurs at span center and is given by

$$M = +\tfrac{1}{8}(1.0)(16)^2 = +32.0 \text{ kip-ft}$$

The maximum negative moment occurs at either end of the span, and is given by

$$M = -8.0(2) - 1.0(2)(1) = -18.0 \text{ kip-ft}$$

The maximum shear occurs at each end of the span and is given by

$$S = 8.0(1) + 2.0(1) = 10 \text{ kips}$$

Since the end shears acting on the girders are equal to the vertical forces applied to the columns by the girders, the axial stresses in the columns are easily found by summing up the girder shears from the top of the column down to the column section under consideration.

To produce maximum compression in a column, bays on both sides of

the column are loaded.   For interior columns, moments applied at a given floor by the two girders oppose each other; hence the column moments are small and are often neglected in design.   For exterior columns, however, girders apply moments to only one side of the columns; hence the column

moments are larger and must be considered in design. In computing column moments, however, assumption 1 is invalid.   Girder moments should be divided between columns in proportion to their stiffnesses.
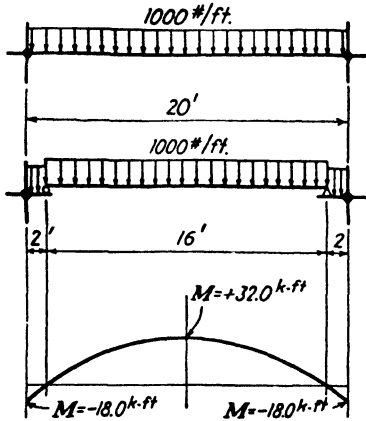


FIG. 12·16

**12·10   Stresses in Building Frames Due to Lateral Loads— General.**   In Art. 12·9, it is pointed out that approximate methods of analyzing bents are of importance because of the fact that such structures are highly indeterminate.   It is shown that the degree of statical indetermination for a bent such as that of Fig. 12·17a equals three times the number of girders in the bent.   The number of assumptions that must be made to permit an analysis by statics depends on the structure itself so that for this bent one must make three times as many assumptions as there are girders, regardless of the type of loading considered.   The assumptions made in analyzing building bents acted upon by vertical
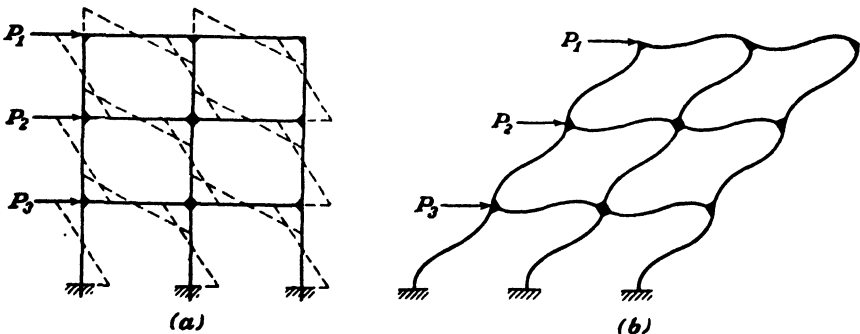


FIG. 12·17

loads will not, however, be suitable for lateral-load analysis, for the structural action of bents is entirely different when lateral loads are considered.   This may be seen by a consideration of Fig. 12·17b, which illustrates, to an exaggerated scale, the shape that a building frame takes

under the action of lateral loads.   It will be noted that, while points of inflection occur in the members, they do not occur in the same manner as under the action of vertical loads.   Actually, when a building bent is acted upon by lateral loads, there will be, as shown in Fig. 12·17b, a point of inflection near the center of each girder and each column.   The assumption that points of inflection occur at the mid-points of all members is therefore a reasonable one and is often among those made to carry out by statics an approximate analysis of building bents under lateral loads.   The moment curves for the structure of Fig. 12·17 are of the type shown by the dotted lines of Fig. 12·17a.
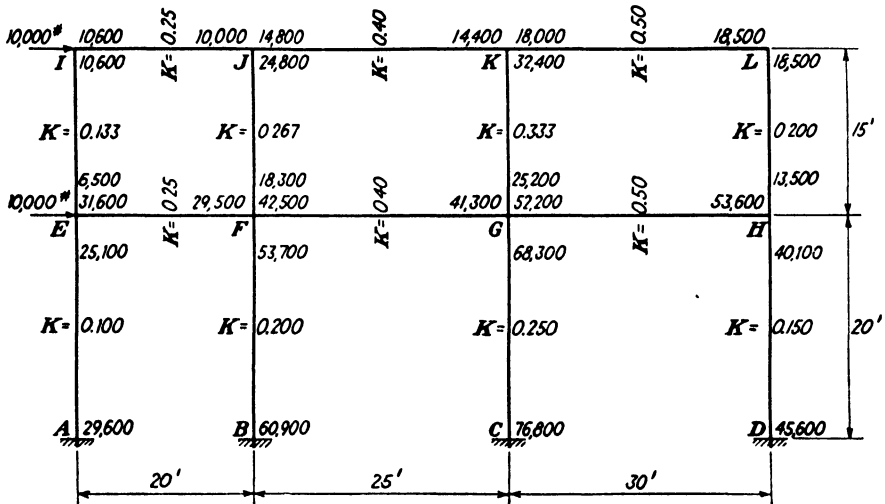


FIG. 12·18

In this treatment, three approximate methods for analyzing building frames acted upon by lateral loads will be given.   These methods are:

1. The portal method
2. The cantilever method
3. The factor method

In order that the relative accuracies of these methods may be considered, all three methods are applied to the same bent.   This bent and its loading are shown in Fig. 12·18.   It is possible to make a so-called "exact" solution for this bent and loading, using, for example, the "slope-deflection" method of analyzing statically indeterminate rigid frames, as treated in Chap. 14.   Such a solution is of interest for the comparison of the results of the approximate methods and has led to the values of end moments in foot-pounds in the girders and columns of the bent of Fig. 12·18 that are shown by the numbers at the ends of the members.

For the slope-deflection solution and also for the factor-method solution, it is necessary to know the relative stiffness of each member. Relative stiffness is denoted by $K$ and for a given member is obtained by dividing its moment of inertia by its length. Values of $K$ are shown for each member of the bent of Fig. 12·18.

**12·11  The Portal Method.** In the portal method, the following assumptions are made:

**1.** There is a point of inflection at the center of each girder.

**2.** There is a point of inflection at the center of each column.

**3.** The total horizontal shear on each story is divided between the columns of that story in a manner such that each interior column carries twice as much shear as each exterior column.

This last assumption is arrived at by considering each story to be made up of a series of portals, as shown in Fig. 12·19. Thus, while an
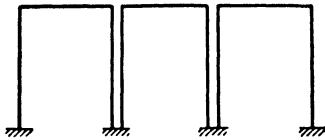


FIG. 12·19

exterior column corresponds to a single portal leg, an interior column corresponds to two portal legs, so that it becomes reasonable to assume interior columns to carry twice the shear of exterior columns. If there are $m$ columns in a story, assumption 3 is equivalent to making $(m-1)$ assumptions per story, regarding column-shear relations.

With reference to the bent of Fig. 12·18, application of the portal method results in the making of the following number of assumptions:

| | |
|---|---|
| Inflection points in girders . . . . . . . . . . . . . . . | $2 \times 3 = 6$ |
| Inflection points in columns. . . . . . . . . . . . . . | $4 \times 2 = 8$ |
| Column shear relations. . . . . . . . . . . . . . . . . | $2 \times 3 = 6$ |
| Total. . . . . . . . . . . . . . . . . . . . . . . . | 20 |

Since there are six girders in this bent, the structure is indeterminate to the eighteenth degree. Hence, the portal method makes more assumptions than are necessary. However, it so happens that the additional assumptions are consistent with the necessary assumptions, and no inconsistency of stresses, as computed by statics, results.

To illustrate the application of the portal method, it will now be applied to the bent of Fig. 12·18. The following discussion refers to Fig. 12·20, where the results of the portal-method analysis are given.

*Column shears:* In accordance with assumption 3, let $x$ = shear in each exterior column of a given story; then $2x$ = shear in each interior column of the same story. For the first story,

$$x + 2x + 2x + x = 6x = 10,000 + 10,000 = 20,000; \quad x = 3,333;$$
$$2x = 6,667$$

For the second story, $6x = 10,000; x = 1.667; 2x = 3,333.$

*Column moments:* In accordance with assumption 2, the moment at the center of each column is zero. Hence, each end moment for a given column equals the shear on that column multiplied by half the length of that column. For example, $M_{AE}$, the moment at the $A$ end of column $AE$, equals $3{,}333 \times 10 = 33{,}333$ ft-lb; $M_{FJ} = 3{,}333 \times 7.5 = 25{,}000$ ft-lb; etc.
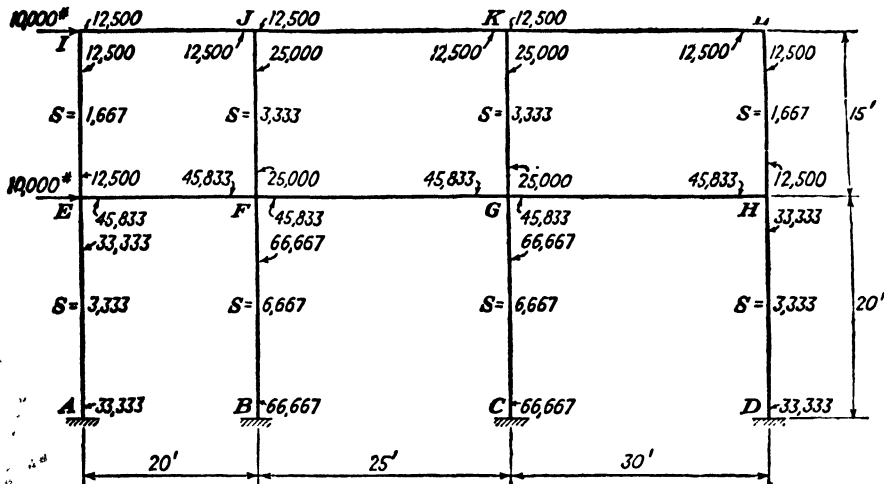


FIG. 12·20

*Girder moments:* Reference to Fig. 12·17b, which shows the type of deformation occurring in a building frame acted upon by lateral forces, indicates that girder and column moments act in opposite directions on a joint. This fact is further clarified in Fig. 12·21a, from which the following equation may be written: $M_{c1} + M_{c2} = M_{g1} + M_{g2}$. Hence we
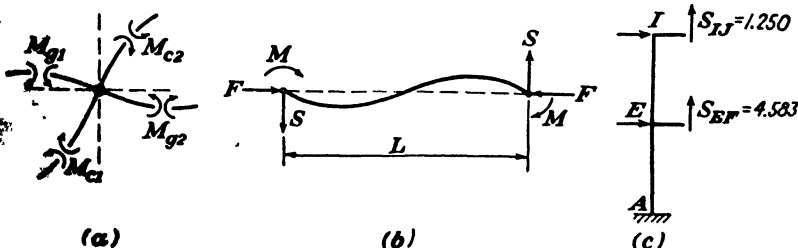


FIG. 12·21

conclude that for any joint the sum of the column end moments equals the sum of the girder end moments. This relation may be used to determine girder end moments, since the column end moments have already been evaluated. At joint $E$, for example,

$$M_{EF} = 33{,}333 + 12{,}500 = 45{,}833 \text{ ft-lb}$$

Since by assumption 1 there is a point of inflection at the center of girder *EF*, $M_{FE}$ also equals 45,833 ft-lb. Equating girder moments to column moments at joint *F* gives $M_{FG} + 45,833 = 66,667 + 25,000$, whence $M_{FG}$ also equals 45,833 ft-lb. Continuing across the girders of the first floor in this manner, all the end moments in the girders of the first floor equal 45,833 ft-lb. Girder end moments in the roof may be determined in a similar manner; each will be found to equal 12,500 ft-lb.

*Girder shears:* In Fig. 12·21b, if $\Sigma M = 0$ is written for the forces acting on a girder, taking moments about one end of the girder, $SL = 2M$, whence $S = 2M/L$. Hence the shear in girder *EF* is given by

$$S_{EF} = \frac{2 \times 45,833}{20} = 4,583 \text{ lb}; \quad S_{IJ} = \frac{2 \times 12,500}{20} = 1,250 \text{ lb}; \text{ etc.}$$

*Column axial stresses:* In Fig. 12·21c, the axial stresses in the columns may be obtained by summing up, from the top of the column, the shears applied to the column by the girders. Thus

$$F_{EI} = +1,250 \text{ lb}; \qquad F_{AE} = +1,250 + 4,583 = +5,833 \text{ lb}; \text{ etc.}$$

Girder axial stresses, while not usually important in design, may be obtained in a similar manner, by summing up, from one end of the girder, the shears applied to the girder by the columns; one would, of course, include the effects of the lateral loads themselves in such a summation.

**12·12 The Cantilever Method.** In the cantilever method, the following assumptions are made:

1. There is a point of inflection at the center of each girder.
2. There is a point of inflection at the center of each column.
3. The intensity of axial stress in each column of a story is proportional to the horizontal distance of that column from the center of gravity of all the columns of the story under consideration.

This last assumption is arrived at by considering that the column axial stress intensities can be obtained by a method analogous to that used for determining the distribution of normal stress intensities on a transverse section of a cantilever beam.

·If there are *m* columns in a story, assumption 3 is equivalent to making $(m - 1)$ assumptions regarding column axial-stress relations, for each story. Hence, as with the portal method, the cantilever method makes more assumptions than are necessary, but again the additional assumptions prove to be consistent with the necessary assumptions.

To illustrate the application of the cantilever method, it will now be applied to the building frame of Fig. 12·18. The following discussion refers to Fig. 12·22b, where the results of the cantilever-method analysis are given.

*Column axial stresses:* Assuming that all columns have the same cross-sectional area, the center of gravity of the columns in each story is found by the following equation:

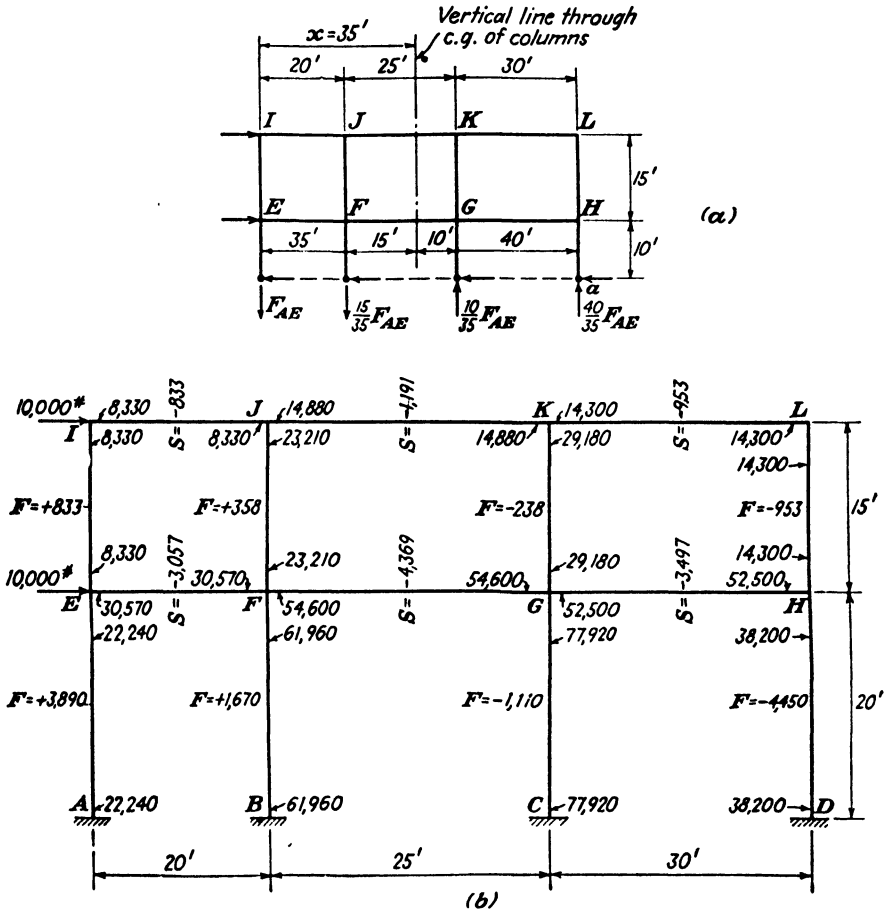$$x = \frac{20 + 45 + 75}{4} = 35.0 \text{ ft from } AEI$$



(a)



(b)

Fɪɢ. 12·22

For the first story, refer to Fig. 12·22a.　If the axial stress in $AE$ is denoted by $+F_{AE}$, then, by assumption 3, the axial stresses in $BF$, $CG$, and $DH$ will be $+^{15}\!/_{35}F_{AE}$, $-^{10}\!/_{35}F_{AE}$, and $-^{40}\!/_{35}F_{AE}$, respectively. Taking moments about $a$, the point of inflection in column $DH$, of all the forces acting on that part of the bent lying above the horizontal plane

passing through the points of inflection of the columns of the first story,

$$+10{,}000(25) + 10{,}000(10) - F_{AE}(75) - {}^{15}\!/_{35}F_{AE}(55)$$
$$+ {}^{10}\!/_{35}F_{AE}(30) = 0$$

whence $F_{AE} = +3{,}890$; $F_{BF} = +{}^{15}\!/_{35}(+3{,}890) = +1{,}670$; etc.

For the second story, the column axial stresses are in the same ratio to each other as they are in the first story. They would be evaluated in a similar manner, taking moments about the point of inflection in $HL$ of all the forces acting on that portion of the bent lying above a horizontal plane passing through the points of inflection of the columns of the second story. Column axial stresses are shown at the center of each column of Fig. 12·22$b$.

*Girder shears:* The girder shears may be obtained from the column axial stresses at the various joints. For example, at joint $E$,

$$S_{EF} = +833 - 3{,}890 = -3{,}057$$

at joint $F$, $S_{FG} = -3{,}057 + 358 - 1{,}670 = -4{,}369$; etc. Girder shears are shown at the center of each girder of Fig. 12·22$b$.

*Girder moments:* Since the moment at the center of each girder is zero, the moment at each end of a given girder equals the shear in that girder multiplied by half of the length of that girder. For example,

$$M_{EF} = 3{,}057 \times 10 = 30{,}570 \text{ ft-lb}; M_{KJ} = 1{,}191 \times 12.5 = 14{,}880 \text{ ft-lb};$$

etc.

*Column moments:* Column moments are determined by beginning at the top of each column stack and working progressively toward its base, as shown in the following illustration: At joint $J$, the column moment equals the sum of the girder moments, whence

$$M_{JF} = 8{,}330 + 14{,}880 = 23{,}210 \text{ ft-lb}$$

Since there is a point of inflection at the center of $FJ$, $M_{FJ}$ also equals 23,210 ft-lb. At joint $F$, $M_{FB} + 23{,}210 = 30{,}570 + 54{,}600$, whence $M_{FB} = 61{,}960$ ft-lb. $M_{BF}$ also equals 61,960 ft-lb, since a point of inflection is assumed midway between $B$ and $F$.

**12·13  The Factor Method.** The factor method of analyzing building frames acted upon by lateral loads is more accurate than either the portal or the cantilever method. Whereas the portal and cantilever methods depend on certain stress assumptions that make possible a stress analysis based on the equations of statics, the factor method depends on certain assumptions regarding the elastic action of the structure, which makes possible an approximate slope-deflection analysis of the bent. While based upon the slope-deflection method of analysis, it is possible to formulate a relatively simple set of rules by which the method can be

applied without knowledge of the elastic principles upon which it is based.

Before applying the factor method, it is necessary to compute the value of $K = I/L$ for each girder and each column. It is not necessary to use absolute values of $K$, since the stresses depend upon the relative stiffnesses of the members of the bent. It is, however, necessary for the $K$ values for the various members to be in the correct ratio to each other.

The factor method may be applied by carrying out the following six steps:

1. For each joint, compute the girder factor $g$ by the following relation: $g = \Sigma K_c / \Sigma K$, where $\Sigma K_c$ denotes the sum of the $K$ values for the columns meeting at that joint and $\Sigma K$ denotes the sum of the $K$ values for all the members of that joint. Write each value of $g$ thus obtained at the near end of each girder meeting at the joint where it is computed.

2. For each joint compute the column factor $c$ by the following relation: $c = 1 - g$, where $g$ is the girder factor for that joint as computed in step 1. Write each value of $c$ thus obtained at the near end of each column meeting at the joint where it is computed. For the fixed column bases of the first story, take $c = 1$.

3. From steps 1 and 2, there is a number at each end of each member of the bent. To each of these numbers, add half of the number at the other end of the member.

4. Multiply each sum obtained from step 3 by the $K$ value for the member in which the sum occurs. For columns, call this product the column moment factor $C$; for girders, call this product the girder moment factor $G$.

5. The column moment factors $C$ from step 4 are actually the approximate relative values for column end moments for the story in which they occur. The sum of the column end moments in a given story may be shown by statics to equal the total horizontal shear on that story multiplied by the story height. Hence, the column moment factors $C$ may be converted into column end moments, by direct proportion, for each story.

6. The girder moment factors $G$ from step 4 are actually approximate relative values for girder end moments for each joint. The sum of the girder end moments at each joint is equal, by statics, to the sum of the column end moments at that joint, which can be obtained from step 5. Hence, the girder moment factors $G$ may be converted to girder end moments, by direct proportion, for each joint.

The factor method will now be illustrated by applying it to the bent of Fig. 12·18. The following discussion refers to Fig. 12·23, where computations for the factor method and the results obtained by it are
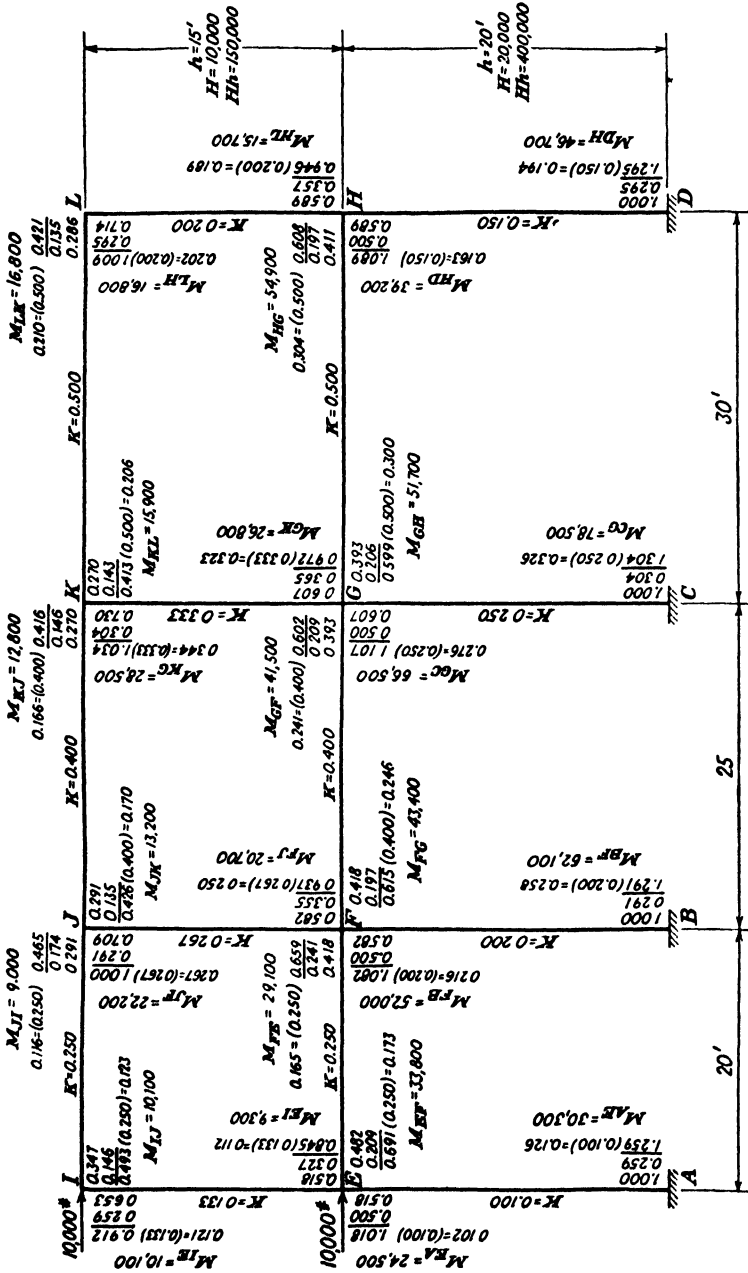
Fig. 12·23

shown.  Values of $K$ are written on the various members as part of the given data.  For each story, on the right side of the figure, values of $H$, the total horizontal shear on the story, and $Hh$, the product of $H$ and the story height $h$, are first worked out.  Illustrative details of the solution follow:

*Step* 1: *Computation of girder factors:*

For joint $E$: $g_E = \dfrac{0.133 + 0.100}{0.133 + 0.100 + 0.250} = 0.482$

This number is written at the left end of girder $EF$.

For joint $F$: $g_F = \dfrac{0.267 + 0.200}{0.267 + 0.200 + 0.250 + 0.400} = 0.418$

This number is written at the left end of girder $FG$ and at the right end of girder $EF$.

For joint $I$: $g_I = \dfrac{0.133}{0.133 + 0.250} = 0.347$

This number is written at the left end of girder $IJ$.

Girder factors for all other joints are computed in a similar manner and written at the near end of each girder meeting at the joint where the girder factor is computed.

*Step* 2: *Computation of column factors:*

For joint $E$: $c_E = 1 - g_E = 1.000 - 0.482 = 0.518$.  This number is written at the top of column $AE$ and at the bottom of column $EI$.

For joint $J$: $c_J = 1.000 - 0.291 = 0.709$.  This number is written at the top of column $FJ$.

For joint $A$: $c_A = 1.000$, since this is a fixed column base of the first story.  This number is written at the bottom of column $AE$.

Column factors for all other joints are computed by similar procedures and written at the near end of each column meeting at the joint where the column factor is computed.

*Step* 3: *Increasing the number at each end of each member by half of the number at the other end of that member:*

For joint $A$: Member $AE$: $1.000 + 0.5(0.518) = 1.259$
For joint $E$: Member $EI$: $0.518 + 0.5(0.653) = 0.845$
　　　　　　 Member $EF$: $0.482 + 0.5(0.418) = 0.691$
　　　　　　 Member $EA$: $0.518 + 0.5(1.000) = 1.018$

Similar computations for all joints are made directly on Fig. 12·23.

*Step* 4: *Computation of column moment factors and girder moment factors:*

For joint $A$: Member $AE$: $C_{AE} = 1.259(0.100) = 0.126$
For joint $E$: Member $EI$: $C_{EI} = 0.845(0.133) = 0.112$
Member $EF$: $G_{EF} = 0.691(0.250) = 0.173$
Member $EA$: $C_{EA} = 1.018(0.100) = 0.102$

Similar computations for all joints are made directly on Fig. 12·23.

*Step* 5: *Determination of column moments:*

Since the column moment factors are relative values of column end moments for each story of the bent, this is another way of saying that

$$M_{AE} = AC_{AE}; \qquad M_{EA} = AC_{EA}; \qquad M_{BF} = AC_{BF}, \text{ etc.}$$

where $M_{AE}$, $M_{EA}$, $M_{BF}$, etc., are the actual moments at the end of the columns and $A$ has the same value for all the columns of a given story.
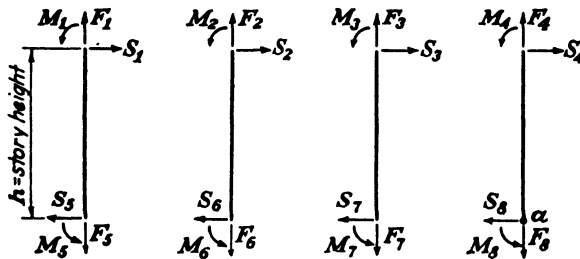


FIG. 12·24

The sum of the end moments may therefore be expressed by

$\Sigma$ column end moments

$$= A(C_{AE} + C_{EA} + C_{BF} + C_{FB} + C_{CG} + C_{GC} + C_{DH} + C_{HD})$$
$$= A\Sigma C \text{ for story} \qquad (a)$$

Consider the static equilibrium of all the forces acting on all the columns of a given story. Refer to Fig. 12·24, and take moments about the base of the right-hand column, at point $a$.

$$(S_1 + S_2 + S_3 + S_4)h$$
$$= M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7 + M_8$$

The sum $S_1 + S_2 + S_3 + S_4$ equals $H$, the total horizontal shear on the story. The sum $M_1 + M_2 + \cdots + M_8$ equals the sum of the column end moments for the story. Hence

$$\Sigma \text{ column end moments } = Hh \qquad (b)$$

From Eqs. $(a)$ and $(b)$,

$$A = \frac{Hh}{\Sigma C \text{ for story}} \qquad (c)$$

For each story, $A$ may be determined by Eq. $(c)$. The end moment for each column of that story may then be obtained by multiplying the respective column moment factor by $A$.

This procedure is illustrated by its application to the first story of the bent of Fig. 12·23.

$$A_1 = \frac{400,000}{0.126 + 0.102 + 0.258 + 0.216 + 0.326 + 0.276 + 0.194 + 0.163}$$
$$= 241,000$$

$$M_{AE} = 0.126(241,000) = 30,300 \text{ ft-lb}$$
$$M_{EA} = 0.102(241,000) = 24,500 \text{ ft-lb}$$
$$M_{BF} = 0.258(241,000) = 62,100 \text{ ft-lb}$$

Moments in the other column ends for the first story are similarly obtained, by using $A_1 = 241,000$. Moments in the column ends of the second story are obtained from $A_2$, which is computed by applying Eq. $(c)$ to the second story, leading to $A_2 = 83,000$.

*Step 6: Determination of girder moments:*

Since the girder moment factors are relative values of girder end moments for a given joint, this is another way of saying, with reference to Fig. 12·25, that $M_{AB} = B_A G_{AB}$ and $M_{AC} = B_A G_{AC}$, where $B_A$ has the same value in each of the foregoing relations. Moreover, since at any joint, the sum of the girder moments equals the sum of the column moments, $B_A$ may be evaluated from the following relation:
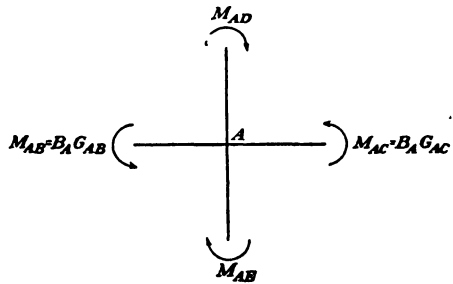


FIG. 12·25

$$B_A G_{AB} + B_A G_{AC} = M_{AE} + M_{AD}$$

whence, at any joint $N$,

$$B_N = \frac{\text{sum of column moments at joint } N}{\text{sum of girder moment factors at joint } N} \qquad (d)$$

For each joint, $B_N$ may be evaluated by Eq. $(d)$. The end moment for each girder at that joint may then be obtained by multiplying the

respective girder moment factor by $B_N$.   This procedure is illustrated by its application to joint $F$ of the bent of Fig. 12·23.

$$B_F = \frac{52{,}000 + 20{,}700}{0.165 + 0.246} = 176{,}500$$

$$M_{FE} = 0.165(176{,}500) = 29{,}100 \text{ ft-lb}$$

$$M_{FG} = 0.246(176{,}500) = 43{,}400 \text{ ft-lb}$$

It will be noted that the application of this procedure for step 6 to exterior joints of a bent results in discovery that girder end moments at these joints equal the sum of the column end moments, as they should by statics.   Thus girder end moments at exterior columns may be obtained directly from column end moments by statics, and the computation of girder moment factors at these joints is not necessary.

The shears and axial stresses in the columns and girders may be computed by the equations of statics, once the end moments are known.

### 12·14   Problems for Solution.

**Problem 12·1**   Owing to a uniform live load of 500 lb per ft, compute the maximum stress of each character in the following bars of the truss of Fig. 12·2: (a) $L_3L_4$; (b) $L'_3L_4$.   (Diagonals can carry compression.)

**Problem 12·2**   Owing to a uniform live load of 1,000 lb per ft and a single concentrated live load of 10,000 lb, compute the maximum stress in the following bars of the truss of Fig. 12·6a: (a) $U_5U_6$; (b) $L_4U_6$; (c) $U_0L_0$.

**Problem 12·3**   The portal of Fig. 12·8a is acted upon by a uniformly distributed wind load of 200 lb per ft along the entire length of the left column. If $h = 40$ ft and $b = 30$ ft, construct the moment curves for all the members of the portal.

**Problem 12·4**   Solve Prob. 12·3, using the portal of Fig. 12·9a.

**Problem 12·5**   Solve Prob. 12·3, using the portal of Fig. 12·10a, with dimensions as shown on the figure.

**Problem 12·6**   For the mill bent of Fig. 12·11,

*a.* Draw the shear and moment curves for the left supporting column.

*b.* What are the forces applied to the roof truss by the columns and knee braces?

**Problem 12·7**   A tower of rectangular cross section, with sides similar to that of Fig. 12·12a, has five vertical panels, the height of each being 10 ft.   The width of each side of the tower is 15 ft at the base and 7.5 ft at the top.   Each panel point on the left side of one face of the tower is subjected to a horizontal panel load of 1,000 lb, acting to the right and lying in the plane of the face under consideration.   Determine the stresses in all the members of this tower face, assuming that the diagonals of each panel have equal stresses but opposite character of stress.

**Problem 12·8**   A building bent has three equal bays of 20 ft each and three stories of 12 ft each.   The columns of the first story are fixed at their bases.

For each girder the dead load is 500 lb per ft and the live load is 300 lb per ft. Determine (a) the maximum positive girder moment occurring in the bent; (b) the maximum negative girder moment occurring in the bent; (c) the maximum girder shear occurring in the bent; (d) the maximum exterior column compression occurring in the bent; (e) the maximum interior column compression occurring in the bent; (f) the maximum exterior column moment occurring in the bent; (g) the maximum interior column moment occurring in the bent.

**Problem 12·9**  The building bent of Prob. 12·8 is acted upon by a horizontal force of 5,000 lb applied at each girder elevation on the left exterior columns.  Determine the bending moment at each end of each member by the portal method.

**Problem 12·10**  Solve Prob. 12·9 by the cantilever method.

**Problem 12·11**  Solve Prob. 12·9 by the factor method, if the moment of inertia of each girder is three times as great as the moment of inertia of each column.

# CHAPTER 13

## DEFLECTIONS OF STRUCTURES

**13·1 Introduction.** Engineering structures are constructed from materials that deform slightly when subjected to stress or a change in temperature. As a result of this deformation, points on the structure undergo certain movements called deflections. Provided that the elastic limit of the material is not exceeded, this deformation and the resulting deflection disappear when the stress is removed and the temperature returns to its original value. This type of deformation or deflection is called *elastic* and may be caused either by loads acting on the structure or by a change in temperature.

Sometimes, the deflection of the structure is the result of settlement of the supports, play in pin joints, shrinkage of concrete, or some other such cause. In such cases, the cause of the deflection remains in action permanently, and therefore the resulting deflections never disappear. This type of deflection may be called *nonelastic* to distinguish it from the elastic type mentioned above. In either type, it should be noted, at this time, that deformation and deflection may occur with or without stress in the structure. This will be discussed in detail later.

The structural engineer often finds it necessary to compute deflections. For example, in the erection of cantilever or continuous bridges, or in designing the lifting devices for swing bridges, it is imperative to compute the deflection of various points on the structure. In certain cases, computations must be made to see that the deflection of a structure does not exceed certain specified limits. For example, the deflection of the floor of a building must be limited to minimize the cracking of plaster ceilings, and the deflection of a shaft must be limited to ensure the proper functioning of its bearings. Perhaps the most important reason for the structural engineer's interest in deflection computations is that the stress analysis of statically indeterminate structures is based largely on an evaluation of their deflection under load.

Numerous methods have been presented in the literature for computing deflections. Of the various methods the following are considered the most fundamental and useful and will therefore be discussed in this chapter:

1. Methods to compute one particular deflection component at a time:

   *a.* Method of virtual work (applicable to any type of structure)

   b. Castigliano's second theorem (applicable to any type of structure)·

2. Methods to compute several deflection components simultaneously:
   a. Williot-Mohr method (applicable to trusses only)
   b. Bar-chain method (applicable to trusses only)
   c. Moment-area method (elastic-load method or conjugate-beam method) (applicable to beams and frames)

**13·2  Nature of the Deflection Problem.**  The computation of the deflection of a structure is essentially a problem in geometry or trigonometry.  Of course, it is first necessary to define the deformation of the particles or elements of the structure, but once this is done the deflections may be computed by using geometrical or trigonometrical principles.

This is particularly evident in the case of a simple truss, which is usually composed of triangles.  The configuration of these triangles may be determined if the lengths of their three sides are known.  Thus, if the lengths of the members before and after deformation are known, the position of the joints before and after may be calculated by trigonometry. From the difference of the two positions of any joint, its deflection may then readily be determined.  This procedure, though simple in theory, is a laborious one and therefore not suitable for practical application.

In the case of a truss, it is also possible to solve the deflection problem graphically simply by superimposing the layouts of the deformed and undeformed truss.  This procedure is obvious and simple in theory, but in order to achieve any accuracy such a large scale would have to be used that it would be physically impossible for a draftsman to make the drawing.

The so-called "method of rotation" is another means of computing truss deflections that is simple in theory but impractical in application.  This method gives us some useful ideas concerning the kinematics of truss deflection, however. In applying this method, the deflection of any joint of a simple truss due to a change in length of any one member may be determined by investigating the resulting rotation of one portion of the truss with respect to the other, the latter being assumed as fixed in position.  By considering the effect of each member separately and summing up the results, the total deflection of any joint due to the change in length of all members may be determined.

To illustrate this procedure, consider first the effect of a change in length of the upper chord $U_2U_3$ of the truss of Fig. 13·1.  Since all deformations are small, it is permissible to assume that the rotations of members are so small that

$$\alpha = \sin \alpha = \tan \alpha$$

It is further permissible to consider that the arc, along which a point actually

travels as a body is rotated through a small angle, coincides with its tangent for all practical purposes. To consider the effect of the change in length of $U_2U_3$ on the deflections, first remove the pin at joint $U_3$ and then allow this change in length, $\Delta L$, to take place. If the left-hand portion of the truss is considered to be held fixed in position, it then is necessary to rotate member $U_2U_3$ about $U_2$ and the dotted portion of the truss about $L_3$ until the points $U_3$ coincide again. During these rotations, point $U_3$ on member $U_2U_3$ may be considered to move vertically and on the dotted portion horizontally. In such a case, the intersection of these two paths will fall along the original position of $U_2U_3$, and the final position of $U_3$ is located as shown. Then,
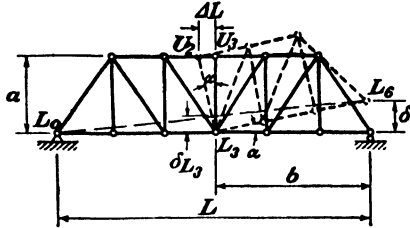


FIG. 13·1

$$\delta = \alpha b = \frac{\Delta L}{a} b = \frac{b}{a} \Delta L$$

Now if the supports do not settle, joint $L_6$ must not actually move vertically so the entire truss must be rotated clockwise about $L_0$ until $L_6$ is back on its support. The dashed line joining $L_0$ and $L_6$ will therefore correspond to the line of zero deflection, and the downward vertical deflection of joint $L_3$ will be, by proportion,



FIG. 13·2

$$\delta_{L_3} = \frac{L-b}{L} \delta = \frac{L-b}{L} \frac{b}{a} \Delta L$$

In a similar manner, the deflection caused by the change in length of a typical diagonal may be evaluated by proceeding as indicated in Fig. 13·2. These considerations demonstrate the impracticality of this method, but the ideas involved are directly applicable to the Williot-Mohr method, which is discussed in Art. 13·11.
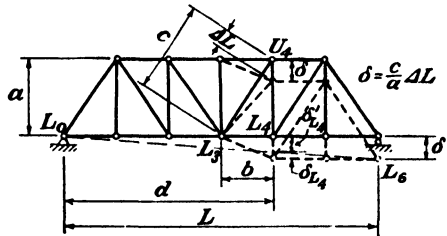
· While the various methods here discussed are impractical, it is important to recognize their existence, for it gives us confidence to know that the deflection problem can be solved using simple "everyday" ideas. Further, it is now obvious that some refinement must be introduced in theory to reduce the labor in the practical solution of such problems.

**13·3 Principle of Virtual Displacements.** Perhaps the most general, direct, and foolproof method for computing the deflections of structures is the *method of virtual work*. This method is based on an application of the *principle of virtual displacements*, which was originally

formulated by John Bernoulli in 1717.   This principle may be developed by the following considerations:

Consider a truly rigid body which is in static equilibrium under a system of forces $Q$.   In this sense, a rigid body is intended to mean an undeformable body in which there can be no relative movement of any of its particles.   Suppose first that, as shown in Fig. 13·3, this rigid body is translated without rotation a small amount by some other cause which is separate from and independent of the Q-force system.   Upon selecting an origin $o$ and two coordinate reference axes $x$ and $y$, this translation may be defined by $\delta_o$, the actual translation of the origin $o$, or by the two components $\delta_{ox}$ and $\delta_{oy}$, in the $x$ and $y$ directions, both assumed to be plus when in the sense shown.   Since the body is rigid, every point on the body will be translated through exactly the same distance as point $o$.
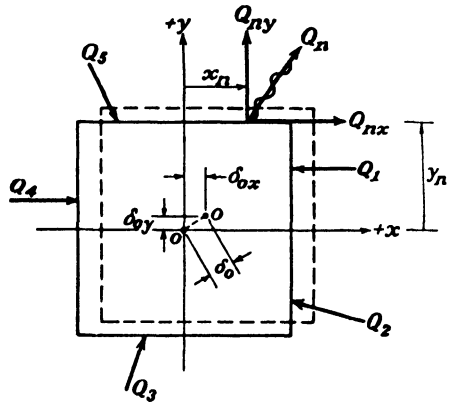


Fig. 13·3

All the $Q$ forces may be resolved into $x$ and $y$ components, designated as $Q_{nx}$ and $Q_{ny}$ for any particular force $Q_n$ and assumed to be plus when in the same sense as the plus sense of the corresponding coordinates.   Since these $Q$ forces are in static equilibrium, the following equations are satisfied by the components of these forces:

$$\left. \begin{array}{c} \Sigma Q_{nx} = 0 \\ \Sigma Q_{ny} = 0 \\ \Sigma(Q_{nx}y_n - Q_{ny}x_n) = 0 \end{array} \right\} \tag{a}$$

Consider now the work $W_Q$ done by only these $Q$ forces as they "ride along" when the rigid body is translated a small amount $\delta_o$ by some *other* cause.   Since this translation is small, all the $Q$ forces may be assumed to maintain the same position relative to the body and, hence, to remain in equilibrium during the translation.   Then, we may write that,

$$W_Q = \Sigma(Q_{nx}\delta_{ox} + Q_{ny}\delta_{oy}) = \delta_{ox}\Sigma Q_{nx} + \delta_{oy}\Sigma Q_{ny}$$

and, therefore, in view of Eqs. (a), the total work done by the $Q$ forces in such a case is equal to zero.

In a similar manner, we may consider the work done by the $Q$ forces during a small rotation $\alpha_o$ of the rigid body about point $o$.   During a small angular rotation, a point may be assumed to move along the normal to the radius drawn from the center of rotation to that point, *i.e.*, along the tangent rather than the arc.   Hence, the $x$ and $y$ components of the displacement of any point $n$ may be computed to be as shown in Fig. 13·4.   Since the angular rotation is small, the $Q$

forces again may be assumed to maintain their same position relative to the body and hence to remain in equilibrium during the small rotation. Then, we may write that

$$W_Q = \Sigma(Q_{nx}\alpha_o y_n - Q_{ny}\alpha_o x_n) = \alpha_o \Sigma(Q_{nx}y_n - Q_{ny}x_n)$$

and, therefore, in view of Eq. (a), the total work done by the $Q$ forces during the rotation of a rigid body is also equal to zero.

After a little thought, it is evident that *any* small displacement of a rigid body may be broken down into a translation of a given point on the body plus a rotation of the rigid body about that point. Since, in the case of either translation or rotation, the work done by the $Q$ system



Fig. 13·4

(which is a system in equilibrium) has been shown to equal zero, the following principle is obviously true in the general case where a rigid body may be given *any* type of small displacement:

*Bernoulli's principle of virtual displacements:* If a system of forces $Q$ acting on a *rigid* body is in *equilibrium* and remains in equilibrium as the body is given *any small virtual* displacement, the *virtual* work done by the $Q$-force system is equal to zero.
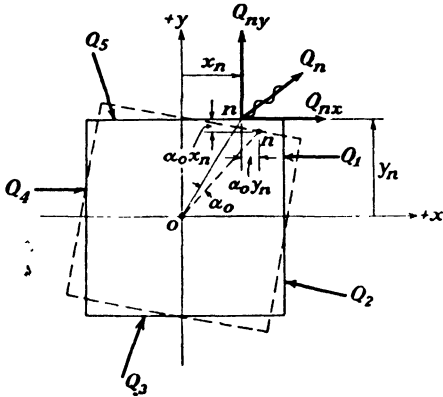
In this statement, the term *virtual* has been used to indicate that the action producing the displacement is separate from and independent of the $Q$-force system. The work done by the $Q$-force system as it "rides along" during such a virtual displacement would be called *virtual work*.

**13·4   Fundamentals of Method of Virtual Work.**   The principle of virtual displacements may now be used to develop the basis for the method of virtual work for computing the deflection of structures. This method is applicable to any type of structure—beam, truss, or frame; planar or space frameworks. For simplicity, however, consider any planar structure such as that shown in Fig. 13·5. Suppose that this structure is in static equilibrium under the external loads and reactions of the $Q$-force system shown.

Since the body as a whole is in equilibrium, any particular particle such as the crosshatched one may be isolated and will also be in equilibrium under the internal $Q$ stresses developed by the external $Q$ forces. If this particle and its adjoining particles are isolated, it will be acted upon by internal $Q$ stresses on any internal boundaries with adjacent

particles but by external $Q$ forces on any external boundaries. On the adjacent internal boundaries of any two adjoining particles, the internal stresses will·be numerically equal but opposite in sense.

Now suppose that the body is subjected to a small change in shape caused by some source *other* than the $Q$-force system. Such a change in shape would give the $Q$-force system a ride and would be called a *virtual* change in shape or a *virtual* distortion. Owing to this change in shape, any particle such as the crosshatched one might be deformed as well as translated and rotated as a rigid particle. Hence, the boundaries of each particle may be displaced, and therefore the $Q$ stresses acting on such
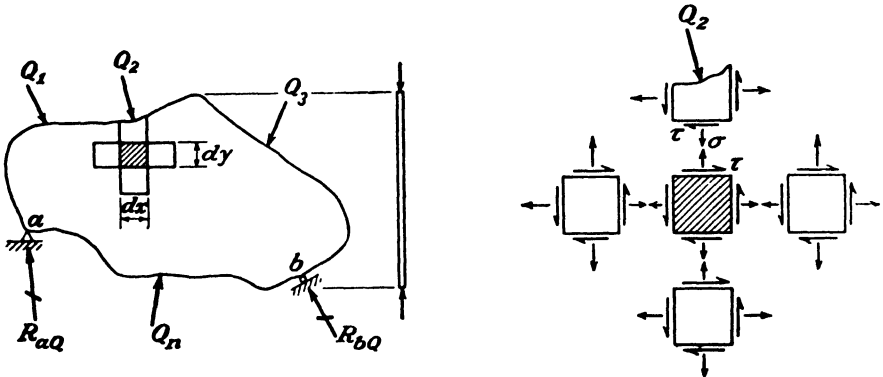


FIG. 13·5

boundaries would move and, hence, do virtual work. Let the virtual work done by the $Q$ stresses on the boundaries of the differential particle be designated by $dW_s$. Part of this virtual work will be done because of the movements of the boundaries of the particle caused by the deformation of the particle itself; this part will be called $dW_d$. The remaining part of $dW_s$ will be the virtual work done by the $Q$ stresses during the remaining part of the displacement of the boundaries and will be equal to $dW_s - dW_d$. However, this remaining displacement is caused by the translation and rotation of the particle as a *rigid body*, and, according to the principle of virtual displacements, the virtual work done in such a case is equal to zero. Hence,

$$dW_s - dW_d = 0$$

or

$$dW_s = dW_d$$

If the virtual work done by the $Q$ stresses on all particles of the body is now added up, this equation becomes,

$$W_s = W_d \qquad (13·1)$$

To evaluate first $W_s$, we recognize that this term represents the total virtual work done by the $Q$ stresses and forces on all the boundaries of all particles.     However, for every *internal* boundary of a particle, there is an adjoining particle whose adjacent boundary is actually the same line on the body as a whole, and therefore these adjacent boundaries are displaced exactly the same amount.     Since the forces acting on the two adjacent internal boundaries are numerically equal but opposite in sense, the total virtual work done on the pair of adjoining internal boundaries is zero.     Hence, since all the internal boundaries occur in pairs of adjoining boundaries, there is no net virtual work done by the forces on all the *internal* boundaries.     $W_s$, therefore, consists only of the work done by the *external* $Q$ forces on the *external* boundaries of the particles.     Equation (13·1) may therefore be interpreted in the following manner:

*Law of virtual work: If a deformable body is in equilibrium under a Q-force system and remains in equilibrium while it is subjected to a small virtual distortion, the external virtual work done by the external Q forces acting on the body is equal to the internal virtual work of distortion done by the internal Q stresses.*

This law of virtual work is the basis of the method of virtual work used to compute deflections.     Before such computations can be made, suitable expressions must be developed so that the external virtual work and internal virtual work of distortion may be evaluated.     In addition, certain "tricks" must be used in selecting a suitable $Q$ system so that the desired deflection components may be computed.     All this will be explained in the following articles.

It is important to emphasize the assumptions and limitations of this development in order to appreciate the flexibility and generality of the method of virtual work.

1. The only requirement of the external $Q$ forces and the internal $Q$ stresses is that they shall form a system of forces which are in equilibrium and remain in equilibrium throughout the virtual distortion.     This requirement will not be satisfied if the virtual distortion has varied the geometry of the structure appreciably.

2. The relations that have been derived are independent of the cause or type of distortion—they are true whether the distortion is due to loads, temperature, errors in lengths of members, or other causes or whether the material follows Hooke's law or not.

**13·5   Expressions for External and Internal Virtual Work.**   It is easy to evaluate $W_s$, the external virtual work done by a system of $Q$ forces acting on any structure.   Let $\delta$ denote the displacement of the point of application of a $Q$ force during the virtual distortion of the structure.   This displacement $\delta$ is to be measured along the same direc-

tion as the line of action of its corresponding $Q$ force. The virtual work done by the force $Q_1$ will then be $Q_1$ times $\delta_1$, and therefore the total external virtual work done by all the $Q$ forces including both loads and reactions will be

$$W_e = Q_1\delta_1 + Q_2\delta_2 + Q_3\delta_3 + \cdots$$

This expression may be represented as follows:

$$W_e = \Sigma Q\delta \qquad (13\cdot2)$$

which simply means that the product of $Q$ times $\delta$ must be evaluated for each $Q$ force and summed up for all the $Q$ forces including both loads and reactions. Note that having assumed the work to be positive in this formula implies that $\delta$ should be considered plus when in the same sense as its corresponding $Q$ force.

It is also relatively easy to evaluate $W_d$, the internal virtual work of distortion. In this instance, however, expressions for $W_d$ must be developed for different types of distortion i.e., for axial change in length, shear



FIG. 13·6

distortion, bending distortion, etc. Consider first the simple case of a little element having an undistorted length of $s$ and a cross-sectional area of $a$ as shown in Fig. 13·6. Suppose that the $Q$ stress intensities $f_Q$ are uniformly distributed over the cross section $a$ so that the axial stress is $(f_Q)(a)$. If the virtual distortion of this element is simply a uniform axial strain $e$, the axial change in length $\Delta s$ will be equal to $(e)(s)$. The internal virtual work of distortion done by the $Q$ stresses in this case will simply be
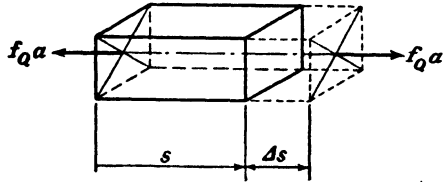
$$W_d = f_Q a\ \Delta s = f_Q a e s \qquad (13\cdot3)$$

This expression may now be used to evaluate the internal virtual work of distortion for a beam or a member of a truss or frame.

Consider the case of a member distorted by a two-dimensional system of $P$ loads, or by a change in temperature. Suppose that the centroidal axis of the member is straight and that all the cross sections of the member have axes of symmetry lying in the plane of the $P$ loads. As a result, all the internal $P$ stresses will also be parallel to this same plane. The axial stress $F_P$ at any cross section will produce an axial strain that is uniform for all the fibers at this section. Suppose that the strain produced by the change in temperature is also uniform at this cross section. Let $e_o$ denote the uniform axial strain at any particular cross section produced by these two effects. The $P$-load system may also produce a

shear and bending moment at this cross section. Because of these two effects, a longitudinal element such as that of Fig. 13·7 will be subjected to shear and normal stresses as shown. More detailed analysis reveals that, unless a member is very deep in comparison with its length, the effect of the shear stress is small in comparison with the deflection caused by the elongation and contraction of these elements under the longitu-dinal normal stresses. In this book, therefore, the effect of shear-ing stresses will be neglected.
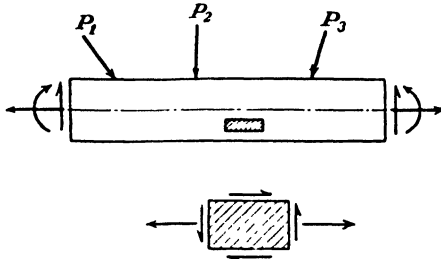
Neglecting the effect of shear distortion, it is then easy to evaluate the virtual work of dis-tortion done simply as a result of the elongation and contraction of the longitudinal elements. Sup-pose that the $Q$-force system lies in the same plane as the $P$ forces of Fig. 13·7 and exerts some axial force, shear, and bending moment on the cross sections of the member. We now want to develop an expression for the internal virtual work of distortion done by the resulting $Q$ stresses as the member is sub-jected to virtual distortion of the type described in the previous para-graph. We select the $s$ and $y$ axes so that the $s$ axis coincides with the centroidal axis of the member. If we now consider a longitudinal ele-ment located at the position $(s,y)$, in this member, as shown in Fig. 13·8, it is apparent that, under the conditions described, this ele-ment is precisely similar to the one shown in Fig. 13·6. The virtual work for a longitudinal element may therefore be evaluated by means of Eq. (13·3). Doing this
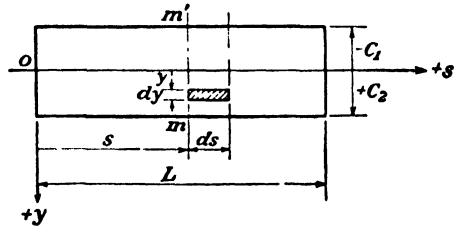


Fig. 13·7



Fig. 13·8

for all the elements of the member and summing them up will give us the virtual work of distortion for the entire member.

Assuming that the normal fiber stresses may be found by elementary beam theory, let

$M_P$ = bending moment on section $mm'$ due to $P$ loads (which cause vir-tual distortion)

$M_Q$ = bending moment on section $mm'$ due to $Q$ loads

$F_P$ = axial force on section $mm'$ due to $P$ loads

$F_Q$ = axial force on section $mm'$ due to $Q$ loads

$f_P$ = normal stress at point $(s,y)$ due to $P$ loads = $(F_P/A) + (M_Py/I)$

$f'_P$ = normal stress at point $(s,y)$ due to $F_P = F_P/A$

$f''_P$ = normal stress at point $(s,y)$ due to $M_P = M_P y/I$

$f_Q$ = normal stress at point $(s,y)$ due to $Q$ loads = $(F_Q/A) + (M_Q y/I)$

$e$ = axial strain of a longitudinal element at a point $(s,y)$

$e_o$ = axial strain of a longitudinal element at a point on the centroidal axis

$I$ = moment of inertia of cross section $mm'$ about its centroidal axis

$A$ = cross-sectional area of cross section $mm'$

$E$ = modulus of elasticity of the material

$b$ = width of cross section $mm'$ at fiber $y$

Considering now the longitudinal element of the member at point $(s,y)$, the length of which is $ds$, the width $b$, and the depth $dy$, then,

$$e = e_o + \frac{f''_P}{E} = e_o + \frac{M_P y}{EI}$$

and the virtual work of distortion for this element is

$$(f_Q b \, dy)(e \, ds) = \left(\frac{F_Q}{A} + \frac{M_Q y}{I}\right)(b \, dy)\left(e_o + \frac{M_P y}{EI}\right) ds$$

The total virtual work of distortion for the entire member is therefore

$$W_d = \int_0^L \int_{-C_1}^{+C_2} \left(\frac{F_Q}{A} + \frac{M_Q y}{I}\right)(b \, dy)\left(e_o + \frac{M_P y}{EI}\right) ds$$

$$= \int_0^L \int_{-C_1}^{+C_2} \left(\frac{F_Q e_o}{A} + \frac{M_Q e_o}{I}y + \frac{F_Q M_P}{EIA}y + \frac{M_Q M_P}{EI^2}y^2\right) b \, dy \, ds$$

Noting that $\int_{-C_1}^{C_2} b \, dy = A$, $\int_{-C_1}^{C_2} yb \, dy = 0$, and $\int_{-C_1}^{C_2} y^2 b \, dy = I$, this expression simplifies to

$$W_d = \int_0^L F_Q e_o \, ds + \int_0^L \frac{M_Q M_P}{EI} ds \qquad (13\cdot4)$$

This expression may now be applied and further simplified for the common types of members encountered in planar structures, such as beams and members of trusses and frames.

**13·6 Deflection of Trusses, Using Method of Virtual Work.** An expression for the law of virtual work as applied specifically to trusses may be obtained by substituting in Eq. (13·1) from Eqs. (13·2) and (13·4). Consider first the case of an ideal pin-jointed truss where both the distorting $P$ loads and the $Q$ loads are applied only at the joints of the truss. In such a case, the individual members will be subjected only to axial forces with no shear or bending moment involved, and the second

term in Eq. (13·4) will disappear. Furthermore, $F_Q$ will be constant throughout the length of a given member and, since

$$\int_0^L e_o \, ds = \text{axial change in length of a member} = \Delta L$$

the virtual work of distortion for one particular truss member becomes

$$W_d = F_Q \int_0^L e_o \, ds = F_Q \, \Delta L$$

Summing such products up for all the members of the truss, the internal virtual work of distortion for the entire truss may be represented as

$$W_d = \Sigma F_Q \, \Delta L$$

and, therefore, the law of virtual work as applied to an ideal pin-jointed truss becomes

$$\Sigma Q \delta = \Sigma F_Q \, \Delta L \tag{13·5}$$

Suitable expressions for $\Delta L$ may easily be developed depending on whether the change in length is produced by the $P$ loads, by a change in temperature, or by some other cause. For a prismatic member having a constant cross-sectional area $A$ and a constant modulus of elasticity $E$, *If the distortion is due to joint loads P on the truss,*

$$\Delta L = (e_o)(L) = \left(\frac{f_P'}{E}\right)(L) = \left(\frac{F_P}{A}\right)\left(\frac{L}{E}\right) = \frac{F_P L}{A E} \tag{13·5a}$$

*If the distortion is due to a uniform change in temperature t,*

$$\Delta L = (e_o)(L) = (\epsilon t)(L) = \epsilon t L \tag{13·5b}$$

*If the distortion is caused by both these effects acting simultaneously,*

$$\Delta L = \frac{F_P L}{A E} + \epsilon t L \tag{13·5c}$$

where, in addition to the notation introduced previously,

$F_P$ = bar stress in the member due to distorting loads $P$
$F_Q$ = bar stress in the member due to loads $Q$
$\epsilon$ = coefficient of thermal expansion of the material
$L$ = length of the member

Equation (13·5) is the basis for the method of virtual work for computing the deflection of ideal pin-jointed trusses, but as yet we do not know how to use it to do this job for us. Suppose, for example, that we wish to compute the vertical component of the deflection of joint $c$ caused by the $P$ loads shown in Fig. 13·9a. Suppose that we select as the $Q$-load system a unit vertical load at joint $c$ together with its reactions. If we imagine that we first apply this $Q$ system to the structure, then,

when we apply the actual distorting loads $P$, the $Q$ loads will be given a ride and will do a certain amount of external virtual work. According to the law of virtual work, the internal $Q$ stresses will do an equal amount of internal virtual work as the members change length owing to the $F_P$ stresses. Applying Eq. (13·5),

$$(1)(\delta_c^{\downarrow}) + W_R = \sum F_Q \frac{F_P L}{AE}$$

where $W_R$ represents the virtual work done by the $Q$ reactions if the support points move and could be evaluated numerically if such movements were known. If the supports are unyielding, $W_R = 0$ and

$$(1)(\delta_c^{\downarrow}) = \sum F_Q F_P \frac{L}{AE}$$

The bar stresses $F_Q$ and $F_P$ due to the $Q$- and $P$-load systems, respectively, may easily be computed. These data combined with the given values of $L$, $A$, and $E$ give us enough information to evaluate the right-hand side of the above equation and, therefore, to solve for the unknown value of $\delta_c$.



FIG. 13·9

Figure 13·10 shows how to select suitable $Q$ systems for use in the computation of other deflection components that may be required. Note that the "trick" is simply to select the $Q$-force system in such a way that the desired deflection is the only unknown $\delta$ appearing on the left-hand side of the equation. Some students worry about the deflection produced by the $Q$ system. We do not care what this is. We want to find the deflection produced by the given cause of distortion. The $Q$ system is purely a system that rides along and does virtual work and, thereby, enables us to compute the desired deflection. Note also that in these computations we never have to worry about or compute the *real* work done by the $P$ loads as they deflect the structure.

The following illustrative examples show how to organize these computations in certain typical problems. In using the law of virtual work, it is particularly important to note the sign convention involved. In setting up the formulas for external and internal virtual work done by the $Q$ system, it was assumed that the work was positive. This implies first that $\delta$ is to be considered positive when it is in the same direction as its corresponding $Q$ force. Furthermore, this implies that both $F_Q$ and $\Delta L$ are to be considered positive when in the same sense. If $F_Q$ is considered plus when tension, then $\Delta L$ is plus when an elongation
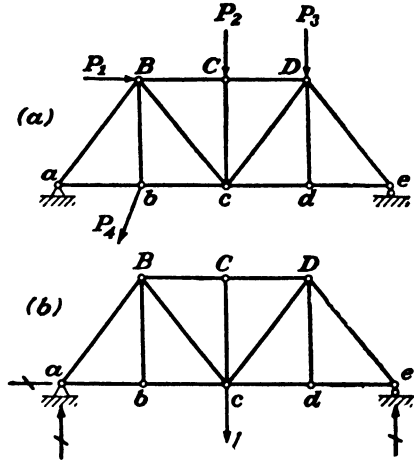
and, therefore, $F_P$ is plus when tension and $t$ plus when an increase in temperature.

After studying the following examples, it will be evident that there are two principal sources of difficulties—units and signs. The novice will probably have less difficulty with units if force units are assigned to the $Q$ forces and stresses, although some authorities treat these forces as dimensionless. Ordinarily it is advisable to use the same length units
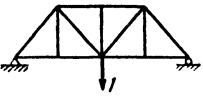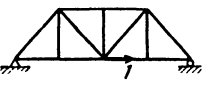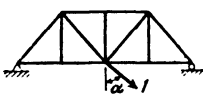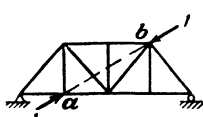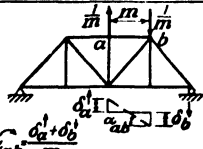
| Deflection Component | Q-System | $\Sigma Q\delta = \Sigma F_Q \,\Delta L$ | Remarks |
|---|---|---|---|
| 1. Vertical Component of Deflection of Joint | | $(1)(\delta_V^{\downarrow}) + W_R = \Sigma F_Q \,\Delta L$<br><br>$W_R$ = external virtual work done by $Q$ reactions | δ plus when in same direction as corresponding Q force; plus direction indicated by arrows; therefore, minus direction opposite to arrows; $F_Q$ plus when tension; $\Delta L$ plus when elongation; $F_P$ plus when ten direction.$F_P$ plus when tension; if supports are unyielding; $W_R = 0$ if supports are unyielding; $W_R$ easily evaluated if support settlements known. |
| 2. Horizontal Comp. of Deflection of Joint | | $(1)(\delta_H^{\rightarrow}) + W_R = \Sigma F_Q \,\Delta L$ | |
| 3. Any Component of Deflection of Joint | | $(1)(\delta_\alpha^{\searrow}) + W_R = \Sigma F_Q \,\Delta L$ | |
| 4. Relative Deflection of two joints along line joining them | | $(1)(\delta_a^{\nearrow}) + (1)(\delta_b^{\nearrow}) + W_R = \Sigma F_Q \,\Delta L$<br>$(1)(\delta_a^{\nearrow} + \delta_b^{\nearrow}) + W_R =$<br>$(1)(\delta_{a-b}^{\nearrow}) + W_R = \Sigma F_Q \,\Delta L$<br>$\delta_{a-b}$ = rel mov $a$ and $b$ together | |
| 5. Rotation of a truss bar | | $(\tfrac{1}{m})(\delta_a^{\downarrow}) + (\tfrac{1}{m})(\delta_b^{\downarrow}) + W_R = \Sigma F_Q \,\Delta L$<br>$(\tfrac{1}{m})(\delta_a^{\downarrow} + \delta_b^{\downarrow}) + W_R =$<br>$(1)(\alpha_{a-b}^{\frown}) + W_R = \Sigma F_Q \,\Delta L$ | |

FIG. 13·10

throughout a problem; however, in some cases it is desirable to mix such units in order to obtain more convenient numbers. For instance, in these examples $A$ and $E$ are used in inch units, while $L$ is used in feet. What is done in this respect is largely a matter of personal preference, but everyone should follow one rule: Be sure the units are consistent. As for signs, no difficulty should be encountered if one is careful and follows the convention noted above. Be sure to check the signs of all products, however. Note also that $F_Q$ and $F_P$ are actual bar stresses, not the horizontal or vertical components.

**Example 13·1**  (a)· *Compute vertical component of deflection of joint c due to 100ᵏ load shown.  E = 30 × 10³ kips per sq in.*

(b) *Compute vertical component of deflection of joint c due to decrease of temperature of 50°F in bottom chord only.  $\epsilon$ = 1/150,000 per °F.*

(a) $\sum Q\delta = \sum F_Q \,\Delta L = \sum F_Q F_P \dfrac{L}{AE}$

$(1^k)(\delta\frac{1}{c}) = \dfrac{1}{E}\sum F_Q F_P \dfrac{L}{A}$

$= \dfrac{+325.01^{k^2/\prime\prime\prime^2}}{30 \times 10^{3\,k/\prime\prime^2}}$

$\therefore \delta_c = \underline{+0.01083\,ft}$  $\therefore$ down



*Cross-sectional areas in sq. ins., shown in parentheses*
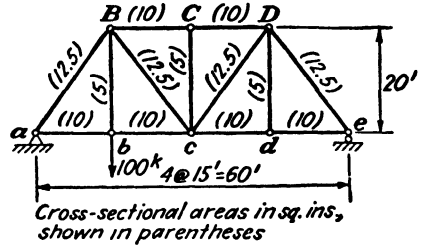
(b) $\Sigma Q\delta = \Sigma F_Q \,\Delta L = \Sigma F_Q \epsilon t L$

$(1^k)(\delta\frac{1}{c}) = \epsilon\Sigma F_Q t L$

$= \left(\dfrac{1}{150,000}\,per\,°F\right)(-1,125^{k°F'})$

$\therefore \delta_c = \underline{-0.0075\,ft}$  $\therefore$ up
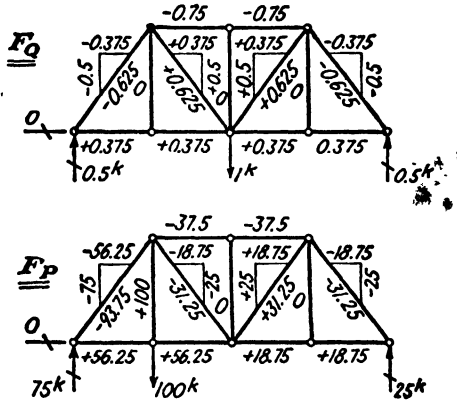




| Bar | L | A | $\dfrac{L}{A}$ | $F_Q$ | $F_P$ | $F_Q F_P \dfrac{L}{A}$ | t | $F_Q t L$ |
|---|---|---|---|---|---|---|---|---|
| Units | ′ | ′′² | ′/′′² | k | k | k²/′′² | °F | k°F′ |
| ab | 15 | 10 | 1.5 | +0.375 | + 56.25 | + 31.64 | −50 | − 281 25 |
| bc | 15 | 10 | 1.5 | +0.375 | + 56.25 | + 31.64 | −50 | − 281 25 |
| cd | 15 | 10 | 1.5 | +0 375 | + 18.75 | + 10.55 | −50 | − 281.25 |
| de | 15 | 10 | 1.5 | +0.375 | + 18.75 | + 10.55 | −50 | − 281.25 |
| BC | 15 | 10 | 1.5 | −0.75 | − 37.5 | + 42.19 | 0 | 0 |
| CD | 15 | 10 | 1.5 | −0.75 | − 37.5 | + 42.19 | 0 | 0 |
| aB | 25 | 12 5 | 2 | −0.625 | − 93.75 | +117.19 | 0 | 0 |
| Bc | 25 | 12.5 | 2 | +0.625 | − 31.25 | − 39.06 | 0 | 0 |
| cD | 25 | 12.5 | 2 | +0.625 | + 31.25 | + 39.06 | 0 | 0 |
| De | 25 | 12.5 | 2 | −0.625 | − 31.25 | + 39.06 | 0 | 0 |
| bB | 20 | 5 | 4 | 0 | +100 | 0 | 0 | 0 |
| cC | 20 | 5 | 4 | 0 | 0 | 0 | 0 | 0 |
| dD | 20 | 5 | 4 | 0 | 0 | 0 | 0 | 0 |
| Σ | .. | .... | ... | ........ | ........ | +325 01 | .... | −1,125 |

**Example 13·2** *Compute the horizontal component of the deflection of joint E, due to the load shown.* $E = 30 \times 10^3$ *kips per sq in.*



Cross-sectional areas in sq. ins., shown in parenthesis
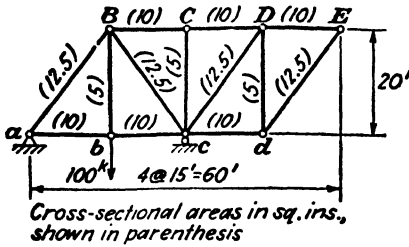
$$\sum Q\delta = \sum F_Q \,\Delta L = \sum F_Q F_P \frac{L}{AE}$$

$$(1^k)(\delta_E^{\rightarrow}) = \frac{1}{E}\sum F_Q F_P \frac{L}{A} = \frac{+56.25^{k^2/''^2}}{30 \times 10^{3\,k/''^2}}$$

$$\therefore \delta_E = +0.001875 \, ft \qquad \therefore \text{ to right}$$
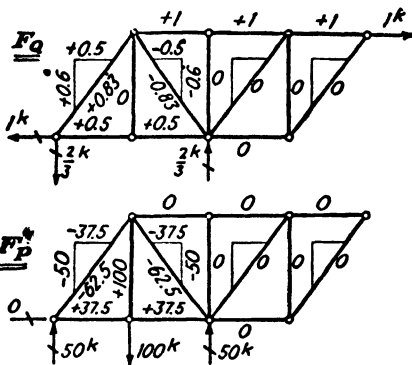
| Bar | L | A | $\dfrac{l}{A}$ | $F_Q$ | $F_P$ | $F_Q F_P \dfrac{L}{A}$ |
|---|---|---|---|---|---|---|
| Units | ' | ''² | '/''² | k | k | $k^2/''^2$ |
| ab | 15 | 10 | 1 5 | +0 5 | +37 5 | + 28 13 |
| bc | 15 | 10 | 1.5 | +0.5 | +37 5 | + 28 .13 |
| aB | 25 | 12 5 | 2 | +0 83 | −62 5 | − 104 17 |
| Bc | 25 | 12 5 | 2 | −0 83 | −62 5 | +104 17 |
| Σ | . | .... | ... | .... | ..... | + 56 25 |

**Discussion:**

· *Note that any bar in which either $F_Q$ or $F_P$ is zero may be omitted from the tabulation since the product of $F_Q F_P (L/A)$ would be zero for such a bar.*

**Example 13·3** *For the truss in Example 13·2, compute the horizontal component of the deflection of joint E, due to the following movements of the supports:*

$$At \ a, \ horizontal = 0.5'', \ to \ left$$
$$At \ a, \ vertical \quad = 0.75'', \ down$$
$$At \ c, \ vertical \quad = 0.25'', \ down$$

*Use the stress analysis for the Q system from the previous problem. In this example, the deflection is caused simply by support movements. There are no changes in length of the members, that is, $\Delta L = 0$ for all members.*

$$\Sigma Q\delta = \Sigma F_Q \,\Delta L = 0$$
$$(1^k)(\delta_E^{\rightarrow}) + (1^k)(0.5'') + (\tfrac{2}{3}^k)(0.75'') - (\tfrac{2}{3}^k)(0.25'') = 0$$
$$\therefore \delta_E = -0.5 - 0.5 + 0.167 = -0.833 \ in. \qquad \therefore \text{ to left}$$

**Discussion:**

*In evaluating the external virtual work done by the Q reactions, be careful to include the proper sign for the work. Any particular reaction does plus or minus virtual work*

*depending on whether its point of application moves in the same or the opposite sense as the reaction, respectively.*

*Deflections due to support settlement may also be evaluated simply by a consideration of the kinematics of the problem.    In sketch a, the solid line shows the outline of the truss in its original position.    The truss may then be translated as a rigid body until the support at a is in its final position—the translated position is shown by the dashed line.    The truss must then be rotated counterclockwise about a until the support at c is in its proper position. The final position of the truss is shown by the dotted line.    In these sketches, the movements have been exaggerated tremendously to clarify the mechanics of the problem.    The horizontal movement of point E may therefore be computed as*

Due to translation . . . . . . . . . . . . .  *0.5" to left*

Due to rotation about a . . . . . . . . . .  $\dfrac{0.5''}{30'} \times 20' = 0.\dot{3}''$ *to left*

Total horizontal movement of point E . . . .  $0.5'' + 0.\dot{3}'' = 0.8\dot{3}''$ *to left*



(a)                                    (b)
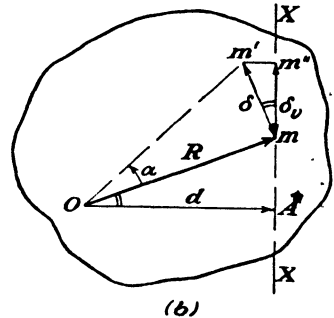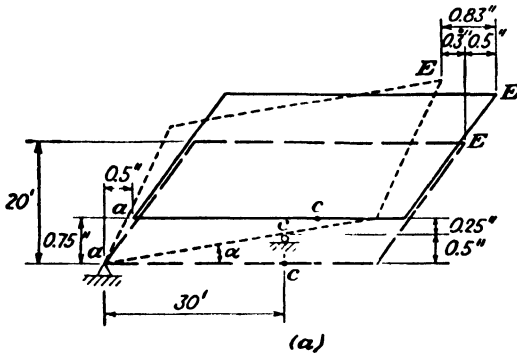
*The computation of the movement during rotation requires some explanation, and it likewise illustrates the application of a useful theorem.    Consider the movement of a point m as a rigid body is rotated about some center O through some small angle α.    In sketch·b, this angle has been exaggerated tremendously so that the geometry is clearer.    Actually, the angle α is so small that the angle (in radians), its sine, and its tangent are all essentially equal to each other.    This means that it is legitimate to consider that m moves to its rotated position m' along the tangent shown rather than along the actual arc.    Suppose that it is desired to obtain the component of the displacement mm' along some given direction through m such as XX.    Drop a perpendicular OA from point O to this direction XX. From the sketch it is apparent that triangles mm'm'' and OmA are similar.    Then,*

$$\frac{\delta_v}{\delta} = \frac{d}{R}, \qquad \delta_v = \frac{\delta}{R} d = \alpha d$$

*since α ≈ δ/R.    Therefore, the following theorem may be stated:*

*If a rigid body is rotated about some center O through some small angle α, the component of the displacement of a point m along some direction XX through that point is equal to the angle α times the perpendicular distance from O to the line XX.*

*Applied to the above truss,*

$$\alpha = \frac{0.5''}{30'}$$

*Therefore, the horizontal movement of E during rotation about O is*

$$\frac{0.5''}{30'} (20') = 0.\dot{3}''$$

**Example 13·4**  *Compute the relative deflections of joints b and D along the line joining them, due to the following causes:*

(a)  *The loads shown.  $E = 30 \times 10^3$ kips per sq in.*

(b)  *An increase in temperature of 80°F in top chord; decrease of 20°F in bottom chord.*  $\epsilon = 1/150,000$ per °F.
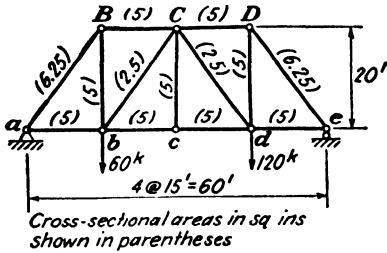


Cross-sectional areas in sq ins shown in parentheses
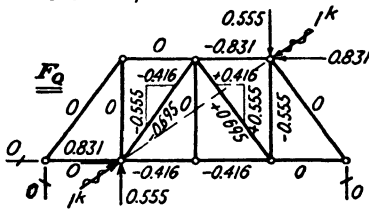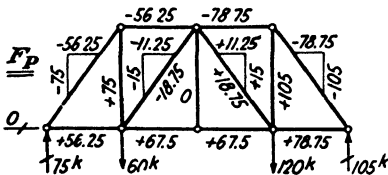
(a)  $\sum Q\delta = \sum F_Q \, \Delta L = \sum F_Q F_P \dfrac{L}{AE}$

$(1^k)(\delta_b^{'}) + (1^k)(\delta_D^{'})$

$= \dfrac{1}{E} \sum F_Q F_P \dfrac{L}{A}$

$(1^k)(\delta_{b-D}) = \dfrac{55.0^{k^2/''^2}}{30 \times 10^{3 \, k/''^2}}$

$\therefore \delta_{b-D} = +0.00183 \, ft \quad \therefore$ together

(a)

(b)  $\Sigma Q\delta = \Sigma F_Q \, \Delta L = \Sigma F_Q \epsilon t L = \epsilon \Sigma F_Q t L$

$(1^k)(\delta_{b-D})$

$= \left(\dfrac{1}{150,000} \, per \, °F\right) (-747.2^{k°F'})$

$\therefore \delta_{b-D} = -0.00499 \, ft \qquad \therefore$ apart

(b)

| Bar | L | A | $\dfrac{L}{A}$ | $F_Q$ | $F_P$ | $F_Q F_P \dfrac{L}{A}$ | $t$ | $F_Q t L$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Units | $'$ | $''^2$ | $'/''^2$ | k | k | $k^2/''^2$ | °F | $k°F'$ |
| bc | 15 | 5 | 3 | $-0.416$ | $+67.5$ | $-84.5$ | $-20$ | $+125$ |
| cd | 15 | 5 | 3 | $-0.416$ | $+67.5$ | $-84.5$ | $-20$ | $+125$ |
| CD | 15 | 5 | 3 | $-0.831$ | $-78.75$ | $+197.0$ | $+80$ | $-997.2$ |
| bC | 25 | 2.5 | 10 | $-0.695$ | $-18.75$ | $+130$ | 0 | 0 |
| Cd | 25 | 2.5 | 10 | $+0.695$ | $+18.75$ | $+130$ | 0 | 0 |
| dD | 20 | 5 | 4 | $-0.555$ | $+105$ | $-233$ | 0 | 0 |
| $\Sigma$ | .. | ... | .. | ....... | ........ | $+55.0$ | ... | $-747.2$ |

**Discussion:**

*Many students are confused about temperature or settlement deflection problems dealing with statically determinate trusses.  They "feel" that stresses must be developed*

*in the members in such cases. However, no reactions can be developed on a statically determinate truss unless there are loads acting on the structure. This can be proved by applying equations of statics. If there are no reactions or external loads, then there can be no internal bar stresses. From a physical standpoint, the distortion of such trusses that is the result of support settlements or changes in length due to temperature may take place without encountering resistance, and therefore no reactions and bar stresses can be developed.*

All the trusses discussed so far have been ideal pin-connected trusses acted upon by $Q$ or $P$ loads, which were always applied at the joints. If a pin-connected truss is distorted by $P$ loads, some of which are applied to certain members between the joints, such members are subjected to $M_P$ bending moments. If, however, only joint deflections are desired, the $Q$ system consists simply of certain joint loads, which cause no $M_Q$ bending moments. The second term on the right side of Eq. (13·4) disappears, therefore, and the law of virtual work for such a case is the same as Eq. (13·5). The same thing is likewise true in the cases where, in order to compute the deflection desired, it is necessary to apply $Q$ loads between the joints but the $P$ loads causing distortion are applied only at the joints so that there are no $M_P$ bending moments.

The case of a pin-connected truss having both $P$ and $Q$ loads applied between the joints is handled like that of a rigid frame structure, which is discussed in the next article. The case of riveted trusses is also discussed there.

**13·7  Deflection of Beams and Frames, Using Method of Virtual Work.** An expression for the law of virtual work as applied to beams or frames may likewise be obtained by substituting in Eq. (13·1) from Eqs. (13·2) and (13·4). Consider first the case of a beam distorted by transverse loads. In such a case, if the reactions have no axial components, the cross sections of the beam are subjected only to shear and bending moment with no axial force. The first term of Eq. (13·4) will disappear, and the law of virtual work as applied to this case will be simply
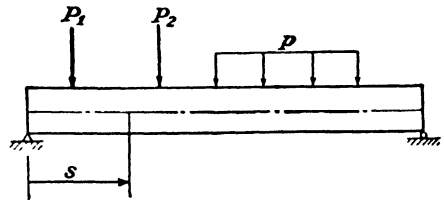


Fig. 13·11

$$\sum Q\delta = \int M_Q M_P \frac{ds}{EI} \qquad (13\cdot6)$$

Remember that the effect of shearing distortion has been neglected in the development of Eq. (13·4). Such a case is shown in Fig. 13·11. To find the vertical, horizontal, or, in fact, any component of the deflec-

tion of a point on such a beam, a unit load is applied in the appropriate direction. This unit load together with its reactions constitutes the $Q$ system, which rides along during the distortion of the beam. The solution of a beam-deflection problem is basically similar to the solution of a truss-deflection problem but differs in the detail of evaluating the right-hand side of Eq. (13·6).

Before the integration of the right-hand side can be accomplished, both $M_Q$ and $M_P$ must first be expressed as functions of $s$. It is usually necessary to separate the integration for the entire beam into the sum of several integrals, one for each of several portions of the beam. The integration must be broken at points where there is a change in the functions representing $M_Q$, $M_P$, or $I$ in terms of $s$. The integration process can often be simplified by selecting different origins for the measurement



FIG. 13·12

of $s$ for these various portions of the beam. The technique of organizing such computations will be illustrated in the following examples.
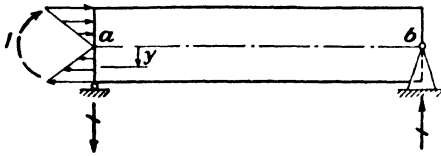
Note particularly the sign convention used for the various terms in Eq. (13·6). Any suitable convention may be used for $M_Q$ and $M_P$ as long as the same convention is used for both. Usually, the ordinary beam convention is the most satisfactory. $\delta$ is, of course, plus when in the same sense as its corresponding $Q$ force.

Often it is necessary to find the change in slope of some cross section of a beam. To do this, select as the $Q$ system a distributed load, such as that shown in Fig. 13·12, and its reactions. This load is distributed across the cross section in a manner such as to be equivalent to a unit couple. Let the intensity of this load at a distance $y$ from the centroidal axis be $q_y$. Considering only the effect of bending distortion, a cross section that was plane before bending remains plane and normal to the elastic curve after bending. If a cross section is rotated through a small angle $\alpha$, a point at distance $y$ from the centroidal axis would move $(\alpha)(y)$. Then, the external virtual work done by the distributed $q$ load during the rotation of the cross section caused by the $P$ system would be

$$\int (q_y b\, dy)(\alpha y) = \alpha \int q_y by\, dy = (1)(\alpha)$$

since the moment of the $q$ load about the centroidal axis $= \int q_y by\, dy = 1$. Having recognized that the external virtual work done by such a distributed $q$ load is simply its resultant unit couple times $\alpha$, we may henceforth consider that we apply a unit couple to the cross section and not

trouble about showing the distributed load in detail. Applying Eq. (13·6) in this case, therefore, yields the following:

$$(1) \quad (\alpha_a^{\curvearrowright}) + W_R = \int M_Q M_P \frac{ds}{EI}$$

From this point on, the rest of the solution is similar to that for a vertical deflection.

---

**Example 13·5**  *Compute the vertical deflection at A, due to load P applied at point B.*

$$\sum Q\delta = \int M_Q M_P \frac{ds}{EI}$$

$$(1^k)(\delta_A \downarrow) = \int_A^B M_Q M_P \frac{dx}{EI}$$

$$+ \int_B^C M_Q M_P \frac{dx}{EI}$$

From A to B,
$$0 < x < a, \qquad M_Q = -x, \qquad M_P = 0$$

From B to C,
$$0 < x < (L - a), \qquad M_Q = -(a + x),$$
$$M_P = -Px$$

$$(1^k)(\delta_A \downarrow) = 0 + \frac{1}{EI} \int_0^{L-a} [-(a + x)]$$
$$[-Px] \, dx$$

$$= \frac{P}{EI} \left[ \frac{ax^2}{2} + \frac{x^3}{3} \right]_0^{L-a}$$

$$= \frac{P}{EI} \left[ \frac{a(L - a)^2}{2} + \frac{(L - a)^3}{3} \right]$$

$$\delta_A^\downarrow = \frac{P}{6EI} (L - a)^2 (2L + a)$$

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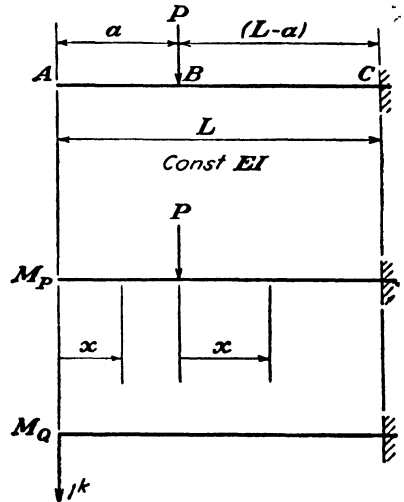
**Example 13·6**  *Compute the vertical deflection of a, due to the load shown.*

$$E = 30 \times 10^3 \text{ kips per sq in.} \quad I = 200 \text{ in.}^4$$
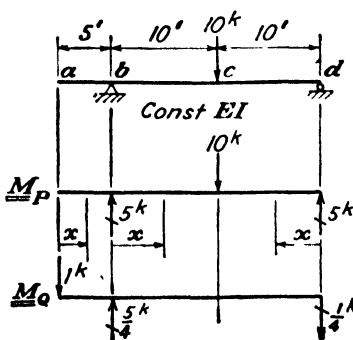
$$\sum Q\delta = \int M_Q M_P \frac{ds}{EI}$$

$$(1^k)(\delta_a \downarrow) = \int_a^b M_Q M_P \frac{dx}{EI} + \int_b^c + \int_d^c$$

From a to b,
$$M_P = 0$$

From b to c,
$$0 < x < 10, \qquad M_Q = \frac{x}{4} - 5, \qquad M_P = 5x$$

From d to c,
$$0 < x < 10, \qquad M_Q = -\frac{x}{4}, \qquad M_P = 5x$$

$$(1^k)(\delta_a^{\downarrow}) = \frac{1}{EI}\left[0 + \int^{10}\left(\frac{x}{4} - 5\right)(5x)\,dx + \int_0^{10}\left(\frac{x}{4}\right)(5x)\,dx\right]$$

$$= \frac{1}{EI}\left\{\left[\frac{5}{12}x^3 - \frac{25}{2}x^2\right]_0^{10} + \left[-\frac{5}{12}x^3\right]_0^{10}\right\} = -\frac{1,250^{k'^2}}{EI}$$

$$\therefore\ \delta_a = \frac{-1250^{k'^3}}{(30 \times 10^3 \times 144)^{k/'^2}\left(\frac{200}{144 \times 144}\right)^{'^4}} = \underline{-0.030\ ft} \quad \therefore\ up$$

**Discussion:**

*The origin for measuring $x$ in any given portion may be selected at will, but note that the same origin must be used for $x$ in the expressions for both $M_Q$ and $M_P$ for a given portion. An origin should be selected that minimizes the number of terms in the expressions for $M_Q$ and $M_P$ and that also reduces the labor in substituting the limits of integration.*

**Example 13·7**   *Compute the change in slope of the cross section at point a caused by the load shown.* $E = 30 \times 10^3$ *kips per sq in.* $I_1 = 150$ *in.*[4]   $I_2 = 200$ *in.*[4]



$$\sum Q\delta = M_Q M_P \frac{ds}{EI}$$

$$(1^{k'})(\alpha_a\frown) = \int_c^b M_Q M_P \frac{dx}{EI}$$

$$+ \int_b^c + \int_e^d + \int_d^c$$
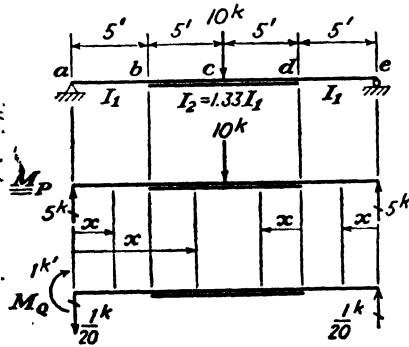
From a to b,

$$0 < x < 5, \quad I_1, \quad M_Q = 1 - \frac{x}{20}, \quad M_P = 5x$$

From b to c,

$$5 < x < 10, \quad 1.33 I_1, \quad M_Q = 1 - \frac{x}{20}, \quad M_P = 5x$$

From e to d,

$$0 < x < 5, \quad I_1, \quad M_Q = \frac{x}{20}, \quad M_P = 5x$$

From d to c,

$$0 < x < 5, \quad 1.33 I_1, \quad M_Q = \frac{1}{4} + \frac{x}{20}, \quad M_P = 25 + 5x$$

$$(1^{k'})(\alpha_a\frown) = \frac{1}{EI_1}\left[\int_0^5\left(1 - \frac{x}{20}\right)(5x)\,dx + \int_5^{10}\left(1 - \frac{x}{20}\right)(5x)\frac{dx}{1.33}\right.$$

$$\left. + \int_0^5\left(\frac{x}{20}\right)(5x)\,dx + \int_0^5\left(\frac{1}{4} + \frac{x}{20}\right)(25 + 5x)\frac{dx}{1.33}\right] = \frac{1}{EI}\left\{\left[\frac{5x^2}{2} - \frac{x^3}{12}\right]_0^5\right.$$

$$\left. + \frac{1}{1.33}\left[\frac{5x^2}{2} - \frac{x^3}{12}\right]_5^{10} + \left[\frac{x^3}{12}\right]_0^5 + \frac{1}{1.33}\left[\frac{25x}{4} + \frac{5}{4}x^2 + \frac{x^3}{12}\right]_0^5\right\}$$

$$25)\ +\frac{1}{1.33}\left[\frac{5}{2}(100-25)-\left(\frac{1,600-125}{12}\right)\right]$$

$$+\frac{1}{1.33}\left[\frac{25}{4}(5)+\frac{5}{4}(25)+\frac{125}{12}\right]\Bigg\}$$

$$=\frac{1}{EI_1}\left[62.5+\frac{1}{1.33}(187.5-62.5+62.5)\right]=\frac{1}{EI}(62.5+140.6)=\frac{203.1^{k^2/^2}}{EI_1}$$

$$\therefore \alpha_a=\frac{(203.1)^{k/^2}}{(30\times10^3\times144)^{k/^2}\left(\frac{150}{144\times144}\right)^{/^4}}=\underline{+0.0065\ radian}\qquad \therefore\ clockwise$$

**Discussion:**

*The selection of the origins for measuring $x$ is not necessarily the best in this solution but was intended to illustrate several of the possible ways of handling the problem. Whenever cancellations are made before integration or before substitution of limits, be very sure that the cancellation is legitimate—check to see that both the terms and the limits are the same.*

---

**Example 13·8** *Compute the change in slope of the cross section of point B. E and I constant.*

---

$$\sum Q\delta=\int M_Q M_P\frac{ds}{EI}$$

$$(1^{k\prime})(\alpha_B\frown)=\int_A^B M_Q M_P\frac{dx}{EI}$$

*From A to B,* $\qquad 0<x<L$

$$M_Q=\frac{x}{L},\qquad M_P=\frac{wL}{2}x-\frac{wx^2}{2}$$

$$(1^{k\prime})(\alpha_B\frown)=\int_0^L\left(\frac{x}{L}\right)\left(\frac{wLx}{2}-\frac{wx^2}{2}\right)\frac{dx}{EI}$$

$$=\frac{w}{EI}\left[\frac{x^3}{6}-\frac{x^4}{8L}\right]_0^L=\frac{wL^3}{24EI}$$

$$\therefore \alpha_B=\frac{wL^3}{24EI}\qquad \therefore\ counterclockwise$$
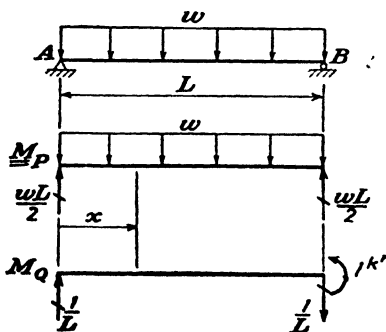


Consider now the more general case of a beam or rigid frame where the cross sections of the members are subjected to an axial force as well as shear and bending moment. Several cases of this type are shown in Fig. 13·13. The deflection of such structures may also be caused by temperature changes as well as by the axial forces and bending moments developed by the $P$ loads. As a result, both terms in Eq. 13·4) must be considered in evaluating the internal virtual work. The law of virtual work as applied to such cases may be expressed as follows:
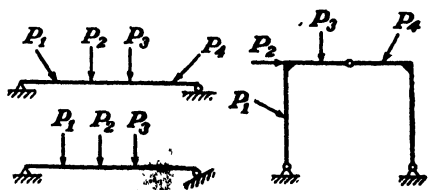
$$\sum Q\delta = \int F_Q e_0 \, ds + \int M_Q M_P \frac{ds}{EI} \qquad (13\cdot7)$$

where the axial strain $e_0$, including the effect of both axial force and temperature, is

$$e_0 = \epsilon t + \frac{F_P}{AE}$$

Usually the entire structure may be split up into several portions over each of which $F_Q$, $F_P$, $A$, and $t$ are constant. Likewise, to evaluate the second term on the right side of Eq. (13·7), the integration may be split up into several parts, each of which covers a portion over which the functions representing $M_Q$, $M_P$, and $I$ remain of the same form. The portions used in evaluating the first term in Eq. (13·7) are not necessarily the same portions as those used to evaluate the second term. In any event: Eq. (13·7) may be represented more conveniently in the following form

$$\sum Q\delta = \sum F_Q \, \Delta L + \sum \int M_Q M_P \frac{ds}{EI} \qquad (13\cdot8)$$

where

$$\Delta L = \int e_0 \, ds = \epsilon t L + \frac{F_P L}{AE}$$

The summation signs on the right side of Eq. (13·8) indicate that such terms must be summed up for all the separate portions of all members of the structure.
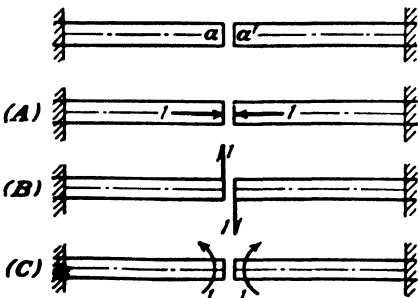


FIG. 13·14

The application of Eq. (13·8) to a specific problem involves the techniques already illustrated in Examples 13·1 to 13·8. Example 13·9 will show how to arrange the computations. Most beam or frame deflection problems involve finding some component of the deflection of a point or the change in slope of a cross section. Sometimes, however, it is necessary to find the relative deflections of two adjacent cross sections such as $a$ and $a'$ in Fig. 13·14. The relative horizontal, vertical, or angular displacements at points $a$ and $a'$ may be found by selecting the $Q$ systems shown in sketches $A$, $B$, or $C$, respectively.

**Example 13·9**  *Compute the change in slope of the cross section on the left side of the hinge at C due to the load shown.*

---

$$\sum Q\delta = \sum F_Q F_P \frac{L}{AE} + \sum \int M_Q M_P \frac{ds}{EI}$$

$$(1^{k'})(\widehat{\alpha_{CL}}) = \int_A^B M_Q M_P \frac{dy}{EI}$$

$$+ \int_B^C M_Q M_P \frac{dx}{EI}$$

$$+ \int_E^D + \int_D^C + \sum F_Q F_P \frac{L}{AE}$$

NOTE: $F_Q F_P \dfrac{L}{AE}$ may be evaluated for same portions $AB$, $BC$, $ED$, and $DC$. Dashed line indicates lower fibers for applying beam convention to signs of $M_Q$ and $M_P$.
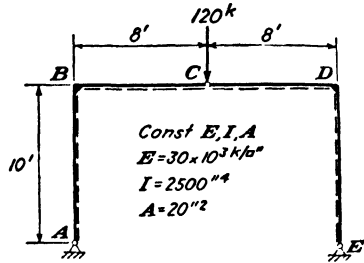
*From A to B,*

$$L = 10' \qquad 0 \to y \to 10$$
$$F_Q = +\frac{1}{16} \qquad M_Q = -\frac{y}{20}$$
$$F_P = -60 \qquad M_P = -48y$$

*From B to C,*

$$L = 8' \qquad 0 \to x \to 8$$
$$F_Q = -\frac{1}{20} \qquad M_Q = -\frac{1}{2} - \frac{x}{16}$$
$$F_P = -48 \qquad M_P = -480 + 60x$$

*From E to D,*

$$L = 10 \qquad 0 \to y \to 10$$
$$F_Q = -\frac{1}{16} \qquad M_Q = -\frac{y}{20}$$
$$F_P = -60 \qquad M_P = -48y$$

*From D to C,*

$$L = 8 \qquad 0 \to x \to 8$$
$$F_Q = -\frac{1}{20} \qquad M_Q = -\frac{1}{2} + \frac{x}{16}$$
$$F_P = -48 \qquad M_P = -480 + 60x$$



$Const\ E, I, A$
$E = 30 \times 10^3\ k/b''$
$I = 2500''^4$
$A = 20''^2$



$M_P, F_P$



$M_Q, F_Q$
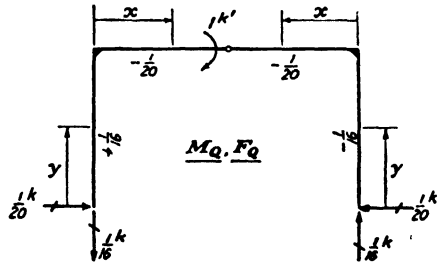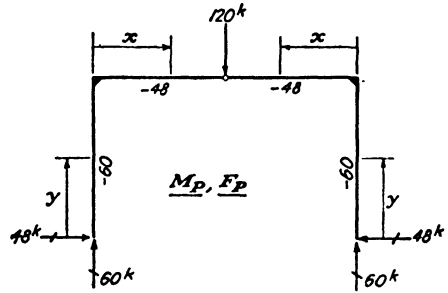
$$(1^{k'})(\widehat{\alpha_{CL}}) = \frac{1}{EI}\left[ 2\int_0^{10}\left(\frac{-y}{20}\right)(-48y)\,dy + \int_0^8\left(-\frac{1}{2} - \frac{x}{16}\right)(-480 + 60x)\,dx \right.$$

$$\left. + \int_0^8\left(-\frac{1}{2} + \frac{x}{16}\right)(-480 + 60x)\,dx \right] + \frac{2}{AE}\left(-\frac{4}{20}\right)(-48)(8)$$

$$= \frac{1}{EI}\left[ (1.6y^3)\Big|_0^{10} + 2(240x - 15x^2)\Big|_0^8 \right] + \frac{38.4}{AE}$$

$$(1^{k'})(\widehat{\alpha_{CL}}) = \frac{3,520}{EI} + \frac{38.4}{AE} = \frac{(3,520)^{k'/'}}{(30 \times 10^3 \times 144)^{k''/'^2} \left(\dfrac{2,500}{144 \times 144}\right)^{/4}}$$

$$+ \frac{(38.4)^{k'/'}}{(30 \times 10^3 \times 144)^{k''/'^2} \left(\dfrac{20}{144}\right)^{/2}}$$

$$\therefore \alpha_{CL} = +0.00676 + 0.000064 = \underline{+0.006824 \ radian}, \qquad \therefore clockwise$$

**Discussion:**

*In problems involving both bending and axial distortion, be particularly careful of units. Note that the contribution of the axial-distortion term is only about 1 per cent of that due to bending distortion. This is more or less typical of the relative size of these two effects in frame-deflection problems. It is therefore usually permissible to neglect the effect of axial distortion in such cases.*

---

A truss with riveted joints is essentially a rigid frame. In the discussion in Art. 4·2, it will be recalled that the members of such trusses are subjected to shear and bending moment as well as axial forces even when the loads are applied at the joints. However, as far as the stresses and strains in the members are concerned, a given riveted truss may be considered to be equivalent to a corresponding pin-connected truss as shown in Fig. 13·15. This equivalent truss, however, is loaded not only with the given joint loads but also by couples on the ends of each bar, which are equal to the moments at the ends of the corresponding members of the riveted truss. More detailed analysis shows that, in most cases, these end couples by themselves produce very small bar stresses in the members. In other words, the bar stresses in the equivalent pin-connected truss are produced almost entirely by the joint loads.
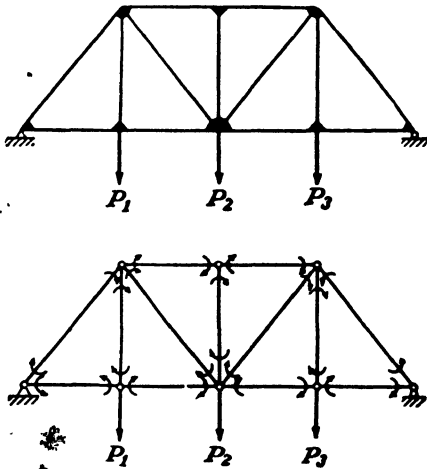
FIG 13·15

In the previous article, it is pointed out that the joint deflections of a pin-connected truss are a function only of the axial change in length of the members and do not depend on the bending distortion of such members. The deflection of the joints of the equivalent pin-connected truss under both the joint loads and the end couples is therefore essentially the same as that of an ideal pin-connected truss acted upon by only the joint loads. Hence, a riveted truss is assumed to be an ideal pin-connected truss when the deflection of the joints are computed. It should be realized, of course, that the bending of the members of riveted trusses does affect the deflection of points other than the joints.

At no place in the previous discussion has the case of a beam or a frame with a curved axis and varying cross sections been considered. The detailed consideration of such structures is beyond the scope of this book. If the curvature and variation of the cross sections are not great, the normal stresses in such structures may be assumed to be distributed linearly and the deflection of the structure may therefore be computed by applying Eq. (13·7). In such cases, the integrals on the right-hand side of this equation can seldom be evaluated exactly, and their values must be approximated by a summation process. For this purpose, the axis of the structure is divided into a number of short portions of equal length $\Delta s$. The values of $F_Q$, $e_o$, $M_Q$, $M_P$, and $I$ are computed for the cross section at the center of each of these short portions. The products of $F_Q e_o \, \Delta s$ and $M_Q M_P \, \Delta s/EI$ may then be evaluated for each portion. The sum of these products for all of the portions approximates the value of the right-hand side of Eq. (13·7).

The accuracy of this summation process increases as the length of the individual portions is decreased.

**13·8 Development of Moment-area Theorems.** The moment-area theorems may often be used more conveniently than the method of virtual work in the computation of slopes and deflections of beams and frames, particularly when the distortion is caused by concentrated rather than distributed loads. These theorems are based on a consideration



Fig. 13·16

of the geometry of the elastic curve of the beam and the relation between the rate of change of slope and the bending moment at a point on the elastic curve.
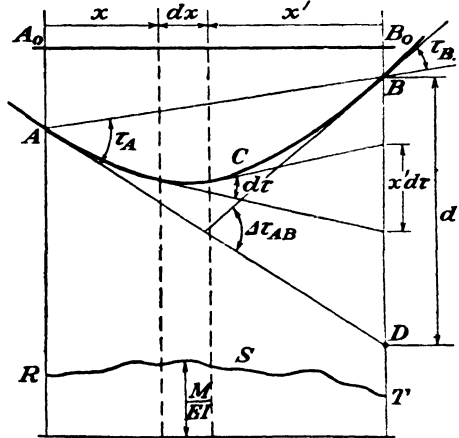
Referring to Fig. 13·16, consider a portion $ACB$ of the elastic curve of a beam that was *initially straight* and in the position $A_oB_o$ in its unstressed condition. Draw the tangents to the elastic curve at points $A$ and $B$. The tangent at $A$ intersects the vertical through $B$ at $D$. The angle $\Delta \tau_{AB}$ is the change in slope between the tangents at points $A$ and $B$. Recognize that the deflection and curvature have been exaggerated tremendously in this sketch. Actually, the inclination of any tangent to the elastic curve is so small that an angle such as $\tau_A$ is approximately equal to its sine and its tangent, and its cosine is approximately equal to unity.

Consider a differential element of this curve having a horizontal projection $dx$, and draw the tangents to the elastic curve at each end of this element. The change in slope between these tangents is the angle $d\tau$, and its value may be obtained by considering Fig. 13·17.

$$d\tau = \frac{(f_c\, dx)/E}{c} = \frac{Mc}{EI}\frac{dx}{c} = \frac{M}{EI}\, dx$$

It is then evident that the total change in slope between the tangents at $A$ and $B$ is the sum of all the angles $d\tau$ for all the elements $dx$ along the elastic curve $ACB$, or



Fig. 13·17
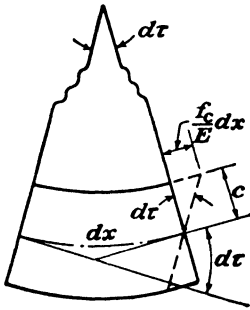
$$\Delta\tau_{AB} = \int_A^B d\tau = \int_A^B \frac{M}{EI}\, dx \qquad (13\cdot9)$$

Let $RST$ be the bending-moment curve for the portion $AB$, every ordinate of which has been divided by the $EI$ of the beam at that point. Such a curve is called the $M/EI$ curve. It is evident that the integral in Eq. (13·9) may be interpreted as the area under the $M/EI$ curve between $A$ and $B$. Hence, from Eq. (13·9), the *first moment-area theorem* may be stated:

*The change in slope of the tangents of the elastic curve between two points A and B is equal to the area under the M/EI curve between these two points.*

In view of the fact that the distortion and slopes are actually small, it is evident that the intercept on the line $BD$ (drawn normal to the unstrained position of the beam) between the tangents at the ends of the element $dx$ may be written as $x'\, d\tau$, and consequently
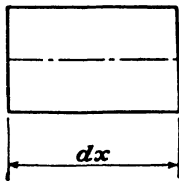
$$d = \int_A^B x'\, d\tau = \int_A^B \frac{M}{EI}\, x'\, dx \qquad (13\cdot10)$$

The last integral may be interpreted as the static moment, about an axis through point $B$, of the area under the $M/EI$ curve between points $A$ and $B$. Hence, from Eq. (13·10), the *second moment-area theorem* may be stated:

*The deflection of point B on the elastic curve from the tangent to this curve at point A is equal to the static moment about an axis through B of the area under the M/EI curve between points A and B.*

Note that this deflection is measured in a direction normal to the original position of the beam.

The two theorems may be used directly to find the slopes and deflections of beams simply by drawing the moment curve for the loads causing distortion and then computing the area and static moments of all or part of the corresponding $M/EI$ curve. This procedure is illustrated in Examples 13·10 to 13·12. It will become apparent from these examples that these computations may be facilitated by introducing some new ideas. The analogy based on these ideas is discussed in the next article and will be called the *elastic-load method.*

These theorems may be extended without difficulty to members that were not initially straight. The consideration of such cases is beyond the scope of this book.

**Example 13·10** *Using the moment-area theorems, compute the deflection at points b and c and the slope of the elastic curve at point b. E and I are constant.*

*Applying second moment-area theorem,*

$$\delta_b = \left(\frac{180}{EI}\right)\left(\frac{6}{2}\right)(4) = \frac{2,160}{EI} \text{ (down)}$$

$$\delta_c = \left(\frac{180}{EI}\right)\left(\frac{6}{2}\right)(4+6) = \frac{5,400}{EI} \text{ (down)}$$

*Applying first moment-area theorem,*

$$\Delta\tau = \left(\frac{180}{EI}\right)\left(\frac{6}{2}\right) = \frac{540}{EI}$$

$$\therefore \tau_b = \frac{540}{EI}$$
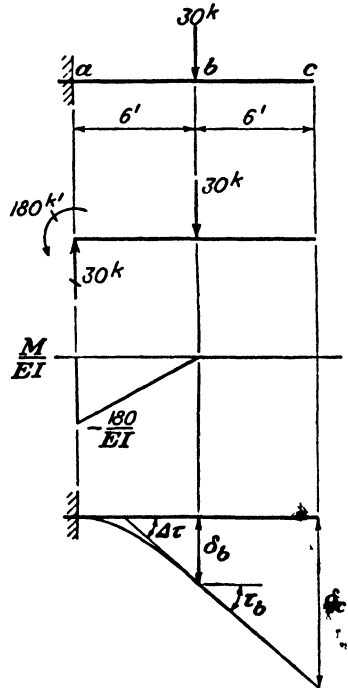


A sign convention could be formulated for the application of the moment-area theorems. This has not been done herein for fear that it might be confused with the convention proposed in the next article for the elastic-load method. When applying the moment-area theorems, signs

**Example 13·11** *Using the moment-area theorems, compute the slope of the elastic curve at points a and m, and the deflection at m.*
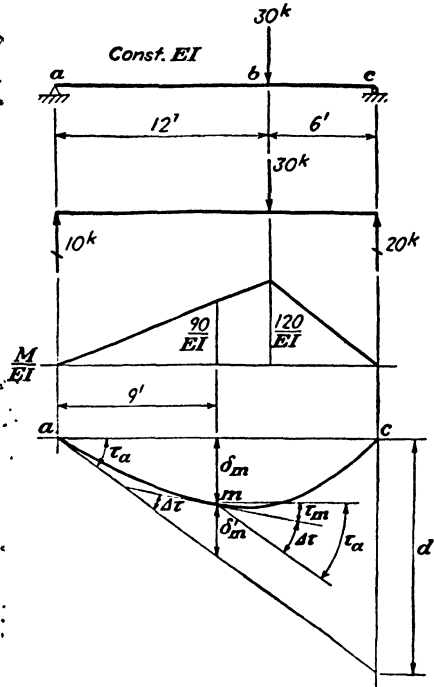


*Since slopes are small,* $\tan \tau_a = \tau_a$

$$\therefore \tau_a = \frac{d}{18}$$

*Applying second moment-area theorem,*

$$d = \left(\frac{120}{EI}\right)\left[\left(\frac{12}{2}\right)(6+4) + \left(\frac{6}{2}\right)(4)\right]$$

$$= \frac{8,640}{EI} \qquad \therefore \tau_a = \frac{480}{EI}$$

*From sketch of elastic curve,* $\tau_m = \tau_a - \Delta\tau$
*But by first moment-area theorem*

$$\Delta\tau = \left(\frac{90}{EI}\right)\left(\frac{9}{2}\right) = \frac{405}{EI}$$

$$\therefore \tau_m = \frac{480 - 405}{EI} = \frac{75}{EI}$$

*Also, from sketch* $\delta_m = 9\tau_a - \delta'_m$
*Applying second moment-area theorem,*

$$\delta'_m = \left(\frac{90}{EI}\right)\left(\frac{9}{2}\right)(3) = \frac{1,215}{EI}$$

$$\therefore \delta_m = (9)\left(\frac{480}{EI}\right) - \frac{1,215}{EI} = \frac{3,105}{EI}$$

may be handled as shown in Examples 13·10 to 13·12. In these examples, the elastic curve is first sketched roughly showing the proper curvature as indicated by the signs of the bending-moment diagram. In the computation of a deflection (or angle) shown on this sketch, the contribution of a particular portion of the *M EI* diagram (regardless of the sign of *M* in that portion) is considered as plus or minus depending on whether it tended to increase or decrease, respectively, the magnitude of that deflection as shown on the assumed sketch. Thus, a positive final result confirms the sense shown for the deflection.

**Example 13·12**   *Using the moment-area theorems, compute the deflections at points b and e, and the slope at point c.   Constant E amd I.*

$\tau_d = \dfrac{d_1}{20},$

$$d_1 = \frac{1}{EI}\left[(37.5)\left(\frac{20}{2}\right)(10)\right.$$

$$\left. -(25)\left(\frac{10}{2}\right)\left(10+\frac{2}{3}\times 10\right)\right] = \frac{5{,}000}{3EI}$$

$$\therefore \tau_d = \frac{250}{3EI}$$

$\delta_b = 10\tau_d + \delta'_b,$

$$\delta'_b = \frac{1}{EI}\left[(25)\left(\frac{10}{2}\right)\left(\frac{20}{3}\right)\right.$$

$$\left. -(37.5)\left(\frac{10}{2}\right)\left(\frac{10}{3}\right)\right] = \frac{625}{3EI}$$

$$\therefore \delta_b = \frac{2{,}500}{3EI} + \frac{625}{3EI} = \frac{3{,}125}{3EI}\ (down)$$

$\delta_e = 5\tau_d - \delta'_e,$

$$\delta'_e = \frac{1}{EI}\left[(25)\left(\frac{5}{2}\right)\left(\frac{10}{3}\right)\right] = \frac{625}{3EI}$$

$$\therefore \delta_e = \frac{1{,}250}{3EI} - \frac{625}{3EI} = \frac{625}{3EI}\ (up)$$

$\tau_c = \tau_d + \Delta\tau,$

$$\Delta\tau = \frac{(25+12.5)(5)}{2EI} - \frac{(18.75)(5)}{2EI} = \frac{93.75}{2EI}$$

$$\therefore \tau_c = \frac{250}{3EI} + \frac{93.75}{2EI} = \frac{781.25}{6EI}$$
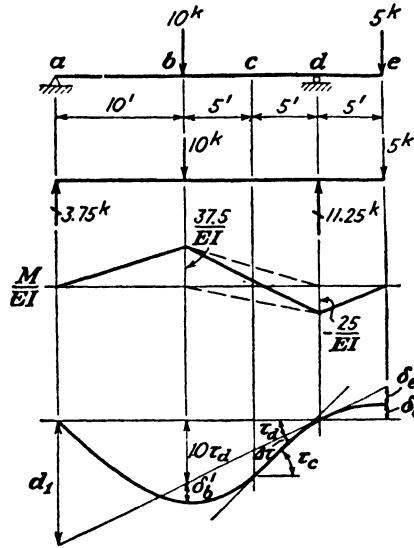


**Discussion:**

Note that it becomes more difficult to handle signs properly when a beam is bent by both plus and minus bending moment.   To minimize these difficulties, it is advisable to sketch the elastic curve as accurately as possible.   The curvature can be sketched correctly by simply following the bending-moment diagram—plus bending moment makes the elastic curve concave upward and minus bending moment makes it concave downward. It will be found advantageous to handle these more difficult problems by the elastic-load method, for thereby the sign difficulty may be handled more or less automatically.

Note also that, in a portion such as bd where a straight-line portion of the moment curve goes from plus to minus, it is often advantageous for computation purposes to replace the actual moment curve by the dashed lines shown.   In this case, the plus and minus triangles under the actual moment curve are replaced by the plus triangle with an altitude of 37.5 and a base of 10 and the minus triangle with an altitude of 25 and also a base of 10. A little thought will verify that this procedure is legitimate for the computation of the net area or of the net static moment of the area about any vertical axis.

**13·9 Elastic-load Method.** The ideas involved in the elastic-load method may be developed by considering the elastic curve $ACB$ of a member $AB$, which was originally straight and has been bent as shown in Fig. 13·18. Let $RST$ be the $M/EI$ curve. Applying the second moment-area theorem,

$$d = \int_A^B x' \frac{M}{EI} \, dx$$

then,

$$\tau_A = \frac{d}{L} = \frac{1}{L} \int_A^B x' \frac{M}{EI} \, dx$$

This equation simply states that $\tau_A$ is equal to the static moment, about an axis through $B$ of the area under the $M/EI$ curve between $A$ and B, divided by $L$. It is interesting to note, however, that the form of this computation for $\tau_A$ seems very familiar to us. Suppose that we imagine that the $M/EI$ diagram, $RST$, represents a distributed vertical load applied to a simple beam supported at points $A$ and $B$ as shown by the dashed lines in Fig. 13·18b. Then the computation for the vertical reaction at point $A$ would involve taking moments about an axis through point $B$, and, as a matter of fact, this vertical reaction at point $A$ would be exactly equal to the value of $\tau_A$ as computed above.



FIG. 13·18
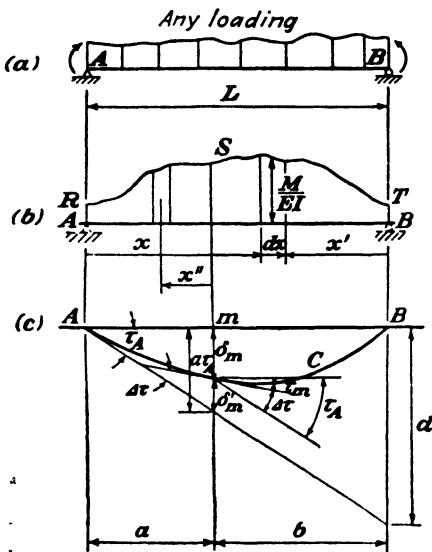
This analogy may be carried still further if we consider the form of the computation for $\tau_m$ and $\delta_m$. Note that $\tau_m$ gives the slope of the tangent with reference to the direction of the chord $AB$ of the elastic curve and that $\delta_m$ gives the deflection of point $m$ from the same chord $AB$. Consider first $\tau_m$; then

$$\tau_m = \tau_A - \Delta\tau$$

But, according to the first moment-area theorem, $\Delta\tau = \int_A^m M/EI \, dx$, and hence

$$\tau_m = \tau_A - \int_A^m \frac{M}{EI} \, dx$$

Continuing to consider the $M/EI$ diagram as the loading and $\tau_A$ as the reaction at point $A$ on the imaginary beam, we see from this equation that $\tau_m$ is equal to the reaction $\tau_A$ minus the load applied between $A$ and $m$. In other words, $\tau_m$ may be interpreted as being the shear of point $m$ of this imaginary beam. Likewise,

$$\delta_m = (a)(\tau_A) - \delta'_m$$

But, from the second moment-area theorem, $\delta'_m = \int_A^m x''(M/EI)\ dx,$ and hence

$$\delta_m = (\tau_A)(a) - \int_A^m x'' \frac{M}{EI}\ dx$$

To continue the analogy, from this equation it is apparent that $\delta_m$ may be interpreted as the bending moment at point $m$ of the imaginary beam.

From these considerations, it may be concluded that the elastic curve of a beam $AB$ is exactly the same as the bending-moment curve for an imaginary simple end-supported beam of the same span $AB$, which is loaded with a distributed transverse load equal to the $M/EI$ diagram of actual beam $AB$. Further, the slope of the tangent to the elastic curve at any point is equal to the corresponding ordinate of the shear diagram for the imaginary beam $AB$ loaded with the $M/EI$ diagram. When used in this manner, the $M/EI$ diagram is referred to as the "elastic load," and therefore this procedure for computing the deflection and slope of a beam is called the *elastic-load method*.

By using this analogy, the problem of computing deflections and slopes of a beam is reduced to a procedure well known to a structural engineer. All he has to do is to compute the reactions, shear, and bending moment on an imaginary end-supported beam loaded with a distributed transverse load. The following examples illustrate the convenience of the elastic-load method.

The example used in developing these ideas has unyielding supports at points $A$ and $B$. As a result, these points do not deflect, and therefore the chord of the elastic curve joining points $A$ and $B$ remains horizontal and coincident with the undistorted position of the axis of the beam. The slope $\tau_m$ and the deflection $\delta_m$ give, therefore, the true slope and deflection with reference to the original position of the beam. The elastic-load method may be applied, of course, to any portion $AB$ of a beam whether the chord $AB$ of the elastic curve remains fixed in position or not. Remember, however, that the shear and bending moment obtained by the elastic-load method give the slope and deflection *measured with reference to the chord AB*. If the chord has moved, the quantities so obtained do not give the true slope and deflection measured with refer-

ence to the original position of the beam until they are corrected for the effect of the chord movement.

In the general case where the chord may or may not move, the procedure of applying the elastic-load method may be summarized in the following statement:

*The slopes and deflections of an elastic curve measured with reference to one of its chords AB are equal, respectively, to the shears and moments of an imaginary end-supported beam of span AB loaded with a distributed load consisting of the M/EI diagram for that portion AB.*

In order to take full advantage of the elastic-load method, it is desirable to follow the same sign convention and principles as those used in drawing regular load, shear, and bending-moment curves. Since upward loads are considered as positive in such computations, plus $M/EI$ ordinates indicate upward loads. Plotting the shear and bending-moment curves for the imaginary beam according to the usual beam convention, plus bending moment would be plotted above the axis and minus values below. Plus bending moment on the imaginary beam therefore indicates deflections above the chord, and minus values indicate deflections below the chord. Likewise, plus shear on the imaginary beam indicates that the elastic curve slopes upward, proceeding from left to right, and minus shear that it slopes downward.

**Example 13·13**   *Compute slopes and deflections of this beam, using the elastic-load method.*

$\Sigma M_d = 0$

| | | | | |
|---|---|---|---|---|
| (1) | $120 \times 3 =$ | $360 \times 14 =$ | $5{,}040$ |
| (2) | $120 \times 6 =$ | $720 \times 9 =$ | $6{,}480$ |
| (3) | $30 \times 3 =$ | $90 \times 8 =$ | $720$ |
| (4) | $150 \times 3 =$ | $450 \times 4 =$ | $1{,}800$ |

$$\frac{1{,}620 \qquad \overline{14{,}040}}{18}$$

$$780 \;\ddagger$$

$\Sigma M_a = 0$

$$360 \times 4 = 1{,}440$$
$$720 \times 9 = 6{,}480$$
$$90 \times 10 = 900$$
$$450 \times 14 = 6{,}300$$
$$\frac{\overline{15{,}120}}{18} = 840 \;\ddagger$$

$\tau_a = -780$
$\qquad +360$
$\tau_b = -\overline{420}$
$\qquad +720$
$\qquad +\;90$
$\tau_c = +\overline{390}$
$\qquad +450$
$\tau_d = +\overline{840}$

$$0 = \delta_a$$
$$-780 \times 6 = -4{,}680$$
$$+360 \times 2 = \qquad 720$$
$$\overline{\qquad -3{,}960} = \delta_b$$
$$-420 \times 6 = -2{,}520$$
$$+\;90 \times 2 = +\;180$$
$$+720 \times 3 = +2{,}160$$
$$\overline{\qquad -4{,}140} = \delta_c$$
$$+390 \times 6 = +2{,}340$$
$$+450 \times 4 = +1{,}800$$
$$\overline{\qquad 0} = \delta_d$$

*Maximum $\delta$ occurs where $\tau = 0$, that is, at the point where shear is zero on imaginary beam. This is between b and c where*

$$EI\tau = -420 + 120x_1 + \left(\frac{30}{6}\right)\frac{x_1^2}{2}$$
$$= -420 + 120x_1 + 2.5x_1^2 = 0$$
$$\therefore \; x_1^2 + 48x_1 = 168$$
$$(x_1 + 24)^2 = 168 + (24)^2 = 744$$
$$x_1 + 24 = +27.28 \qquad \therefore \; x_1 = 3.28'$$

*At this point,*

$$EI\delta_{max} = -3{,}960 - (420)(3.28) + (120)(3.28)\left(\frac{3.28}{2}\right) + (5)(3.28)\left(\frac{3.28}{2}\right)\left(\frac{3.28}{3}\right)$$
$$= -3{,}960 - 1{,}378 + 646 + 29 = -4{,}663 \; kip\text{-}ft^3$$

**Discussion:**

    *Note the units of these various numbers. M is in kip-feet. Therefore, the area under the M curve is in kip-feet², and the static moment of this area is in kip-feet³. Thus, if E and I are also substituted in kip and foot units, values of $\delta$ will be in feet and of $\tau$ in radians.*
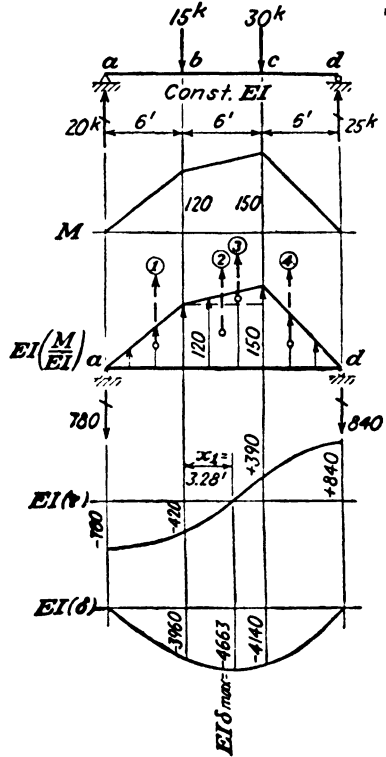
**Example 13·14**  *Compute slopes and deflections of this beam.*

$$E = 30 \times 10^3 \; k/''^2. \quad I_1 = 600 \; in.^4 \quad I_2 = 300 \; in.^4$$
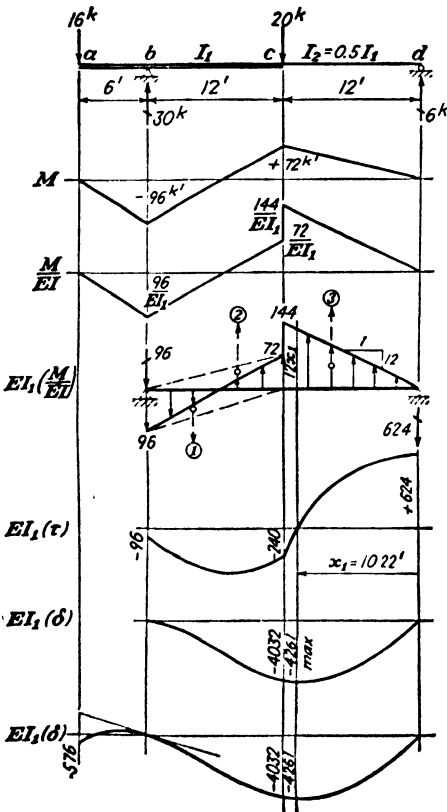
The chord $bd$ of this beam remains horizontal. Slopes and deflections with reference to this chord are therefore the actual slopes and deflections. Use the elastic-load method to obtain these, using an imaginary beam supported at $b$ and $d$. The reaction at $b$ on this beam gives the true slope of the tangent to the elastic curve at point $b$. With this slope known, it is easy to apply the moment-area theorems directly to obtain slopes and deflections in the contilever portion $ab$.

$$\curvearrowleft^+$$
$$\Sigma M_b = 0$$

(1) $\quad 96 \times 6 = -576 \times \; 4 = +\; 2{,}304$
(2) $\quad 72 \times 6 = \quad 432 \times \; 8 = - \; 3{,}456$
(3) $\quad 144 \times 6 = \quad 864 \times 16 = -13{,}824$
$$\qquad\qquad +720 \qquad\qquad \overline{-14{,}976}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \overline{24}$$
$$\qquad\qquad\qquad\qquad\qquad = \underline{\underline{624}} \; \ddagger$$

$$\curvearrowright^+$$
$$\Sigma M_d = 0$$
$$576 \times 20 = -11{,}520$$
$$432 \times 16 = \quad 6{,}912$$
$$864 \times \; 8 = \quad 6{,}912$$
$$\qquad\qquad \overline{\dfrac{2{,}304}{24}} = \underline{\underline{96}} \; \ddagger$$

$$\tau_b = \quad -96$$
$$\qquad\quad -576$$
$$\qquad\quad +432$$
$$\tau_c = \quad -240$$
$$\qquad\quad +864$$
$$\tau_d = \; +624$$

$$\qquad\qquad\qquad 0 = \delta_b$$
$$-96 \times 12 = -1{,}152$$
$$-576 \times \; 8 = -4{,}608$$
$$+432 \times \; 4 = +1{,}728$$
$$\qquad\qquad \overline{-4{,}032} = \delta_c$$
$$-240 \times 12 = -2{,}880$$
$$+864 \times \; 8 = +6{,}912$$
$$\qquad\qquad\qquad 0 = \delta_d$$

Then at point $a$,
$$EI_1\delta_a = 96 \times 6 - 96 \times 3 \times 4$$
$$= \underline{-576 \; (down)}$$

*Maximum occurs just to right of point c, where τ = 0.*

$$EI_1\tau = +624 - (12x_1)\left(\frac{x_1}{2}\right) = 0 \qquad \therefore x_1{}^2 = 104, \qquad x_1 = \underline{10.22'}$$

$$EI_1\delta_{max} = -(624)(10.22) + \frac{(12)(10.22)^3}{(2)(3)} = -6{,}385 + 2{,}124 = \underline{-4{,}261} \ (down)$$

*Substituting now for E and $I_1$,*

$$\delta_{max} = \frac{-(4{,}261)^{k'^3}}{(30 \times 10^3 \times 144)^{k''^2}\left(\frac{600}{144^2}\right)^{'4}} = \underline{-0.0341 \ ft} \ (down)$$

## 13·10 Application of Moment-area Theorems and Elastic-load Method to Deflection of Beams and Frames.

To utilize the moment-area approach at its maximum efficiency, it is often desirable to combine the direct application of the two theorems with the use of the elastic-load procedure. This has already been illustrated in Example 13·14 and will be further illustrated in the examples that follow. To plan the method of attack in a given problem, it is advisable first to sketch the elastic curve of the structure. A certain amount of practice is necessary to develop proficiency in sketching such curves. However, even the novice can get the curvature correct by simply following the bending-moment diagram of the member and noting whether the bending is plus or minus.

Once the elastic curve has been sketched approximately, it then is easy to plan the solution. To obtain deflections with respect to a tangent, use the moment-area theorems directly. To obtain deflections with respect to a chord, use the elastic-load method. In applying the elastic-load method as described here, the *portion of the elastic curve of the actual beam corresponding to the span of the imaginary beam must never include an intermediate hinge.* At such hinges, there may be a sharp change in the slope of the elastic curve. This sudden change is not included in an elastic load that consists simply of the $M/EI$ diagram. Such cases may be handled by an extension of the elastic-load method called the conjugate-beam method or by the combined use of the moment-area theorems and the elastic-load method as illustrated in the following examples.

For the reasons just noted, it is likewise true that the moment-area theorems cannot be applied between two points on the elastic curve if there is a hinge within that portion of the beam.

**Example 13·15**   *Compute the maximum deflection of this beam.*
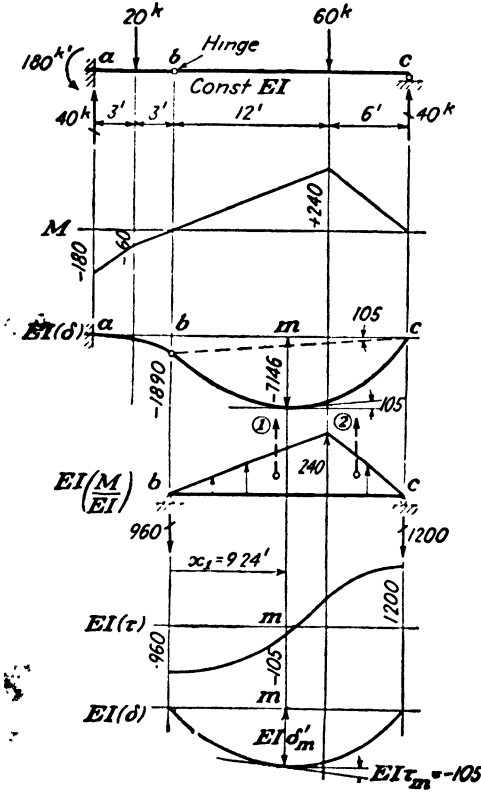


*After studying the sketch of the elastic curve, it is apparent that $\delta_b$ may be computed by applying the second moment-area theorem to the portion ab. This deflection establishes the position of the chord bc. The deflections and slopes with reference to this chord may be obtained using the elastic-load method applied to an imaginary beam of span bc.*

$$EI\delta_b = [(180)(\tfrac{3}{2})(5) + (60)(\tfrac{6}{2})(3)]$$
$$= 1,890(\downarrow)$$

*Rotation of chord bc* $= \dfrac{1,890}{18} = \underline{105}$

(1) $240 \times 6 = 1,440 \times 10 = 14,400$
(2) $240 \times 3 = \phantom{1,44}720 \times \phantom{1}4 = \phantom{1}2,880$
$$\overline{\phantom{1}17,280}$$
$$\overline{\phantom{1}18}$$
$$= 960 \updownarrow$$
$$1,440 \times 8 = 11,520$$
$$720 \times 14 = 10,080$$
$$\overline{21,600}$$
$$\frac{21,600}{18} = \underline{1.200}$$

*The point of maximum $\delta$ occurs where the tangent to the elastic curve is horizontal, i.e., where the tangent slopes down to the right with respect to the chord bc, or $EI\tau_m = -105$.*

$$EI\tau_m = -960 + (20x_1)\left(\frac{x_1}{2}\right)$$
$$= -105 \qquad x_1^2 = 85.5$$
$$x_1 = 9.24'$$
$$EI\delta'_m = -(960)(9.24) + \frac{(20)(9.24)^3}{6}$$
$$= \underline{-6,225}$$
$$EI\delta_m = -6,225 - (105)(18 - 9.24)$$
$$= \underline{-7,146}$$

**Discussion:**

   Note that it is not permissible to apply the elastic-load method to an imaginary beam of span ac because of the presence of the hinge at b.

**Example 13·16**    *Compute the slopes and deflections of this beam.    Constant EI.*
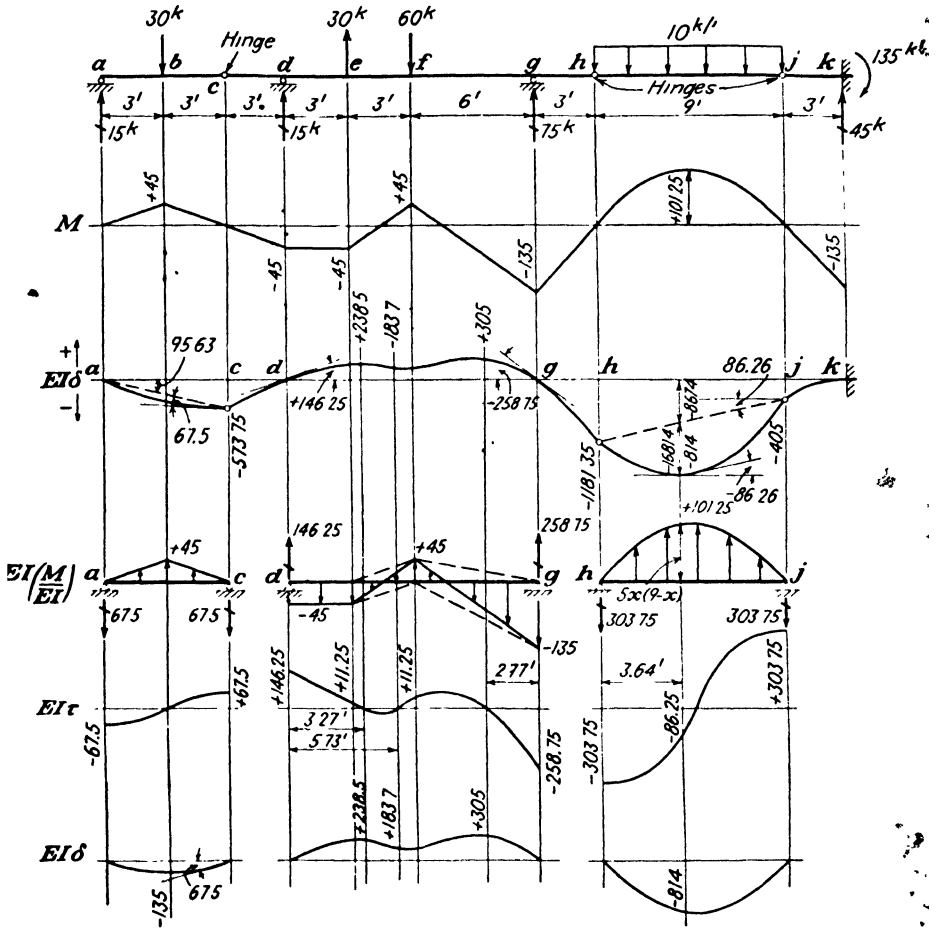


**Discussion:**

   The chord *dg* of the elastic curve does not move, and therefore slopes and deflections determined with reference to it by applying the elastic-load method to the imaginary beam *dg* are the true slopes and deflections of the beam in this portion.    These computations are straightforward and produce the results shown for the location and magnitude of the maximum deflections in the portion *dg*.

   The reactions of this imaginary beam establish the slope of the tangent of the elastic curve at *d* and *g*.    Is is then easy to use the second moment-area theorem to compute the additional deflection of the hinges at *c* and *h* from these tangents.    The total deflections of these hinges from the original position of the beam are shown.

   It is also easy to apply the second moment-area theorem to the cantilever portion *jk* and thus compute the deflection of the hinge at *j*.

*In this manner, the position of the chords ac and hj is established, and it is now possible to compute the true slopes and deflections in these portions. The position and magnitude of the maximum deflections in these portions are shown. The slopes and deflections with respect to the chords ac and hj may be obtained by applying the elastic-load method to the imaginary beams of the same spans.*

*Note that there can be no point in the portion ac which deflects more than c, for the slope of the chord is greater than the slope of the tangent at c with respect to the chord.*

The moment-area theorems and the elastic-load method may also be used advantageously in the computation of frame deflections. The frame deflections computed in this manner, however, do not include the effect of axial changes in length of the members. Fortunately, it is usually permissible to neglect the effect of axial distortion in most frame-deflection problems (see Example 13 9).

**Example 13·17** *Compute the deflections of this frame:*
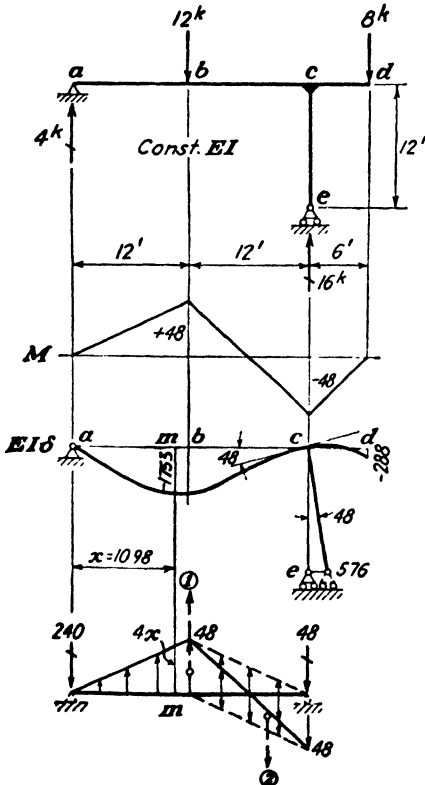


(1) $48 \times 12 = 576 \times 1\frac{2}{24} = 288 \downarrow$

(2) $48 \times 6 = 288 \times \frac{1}{24} = 48 \uparrow$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad 240 \downarrow$

(1) $\qquad\qquad 576 \times 1\frac{2}{24} = 288 \downarrow$

(2) $\qquad\qquad 288 \times 2\frac{0}{24} = 240 \uparrow$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad 48 \downarrow$

*To find point of maximum deflection,*

$$EI\tau_m = 0 = -240 + (4x)\left(\frac{x}{2}\right)$$

$$x^2 = 120, \qquad x = 10.98$$

$$EI\delta_m = -(240)(10.98) + (4)\frac{(10.98)^3}{6}$$

$$= -1.753 \,(\downarrow)$$

$$Ei\delta_d = (48)(6) - (48)(3)(4)$$

$$= -288 \,(\downarrow)$$

$$EI\delta_e = (48)(12) = 576 \,(\rightarrow)$$

*Since the joint at c is rigid, the tangents to the elastic curves of all members meeting at that joint rotate through the same angle. Since there is no bending moment in the column, the elastic curve is a straight line inclined at the same angle as the tangent of the elastic curve of the beam.*

**Example 13·18**   *Compute the deflection at point e on this frame:*

$EI\tau_c = (360)(12)(1\%_{24}) = 2,880$

$EI\delta_e = (2,880)(12) + (360)(6)(8)$

$= \underline{51,840}$

$(\leftarrow)$
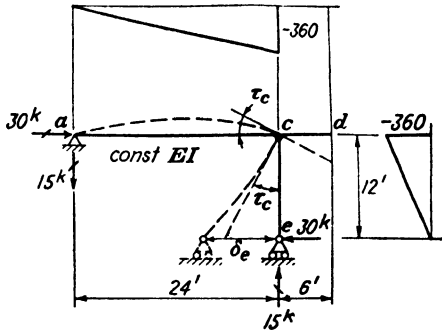


It should be apparent from these examples and the problems at the end of the chapter that the moment-area methods may be applied most advantageously when the moment diagram is composed of a series of straight lines, *i.e.*, when the beam is acted upon by concentrated loads. In such cases, the area under the $M/EI$ curve may be broken up into rectangles and triangles, and the necessary computations are quite simple. When the load is distributed, however, and the moment curve becomes a curved line, the computations become more difficult. When the load is uniformly distributed, the moment curve is parabolic, and the $M/EI$ diagram may be broken up into triangles, rectangles, and parabolic segments. In the case of more complicated distributions, however, it is usually necessary to divide the $M/EI$ diagram into a series of short portions each of which may be broken up into essentially triangles and rectangles. Sometimes it is desirable to use Simpson's rule to improve the results of approximate summation.

Sometimes it is advisable to use graphical methods in applying the elastic-load method. Suppose, for example, that the $M/EI$ diagram is very irregular in shape owing either to a complicated loading of the structure or to variation in the moment of inertia of the beam. In such cases, the shear and moment diagrams produced by the irregular elastic load acting on the imaginary beam may be obtained graphically in the manner discussed in Art. 5·10.

**13·11   Williot-Mohr Method.**   When the method of virtual work is used to compute truss deflections, only one component of the deflection of a joint can be calculated at a time. To obtain the magnitude and direction of the true absolute movement of a joint, both horizontal and vertical components of its movement must be determined. Therefore, two separate applications of the method of virtual work are usually

required in order to determine the resultant deflection of *each* truss joint. *One* graphical solution using the Williot-Mohr method, however, will determine the resultant deflection of *all* joints of the truss, and therefore this method obviously has a very important advantage in the solution of certain deflection problems.

The fundamentals of the Williot-Mohr method may be developed by considering the simple truss shown in Fig. 13·19. Assume that the changes of length of the members, $\Delta L$, have been computed for the given condition of distortion, using Eq. (13·5a), (13·5b), or (13·5c). The deflected position of the truss may then be determined in the following manner as indicated by the dashed lines in Fig. 13·19a. First, remove the pin at joint $D$, and allow the change in length of member $AB$ to take place. This will cause member $DB$ to move parallel to itself as shown, and all points on this member will move horizontally to the right an amount
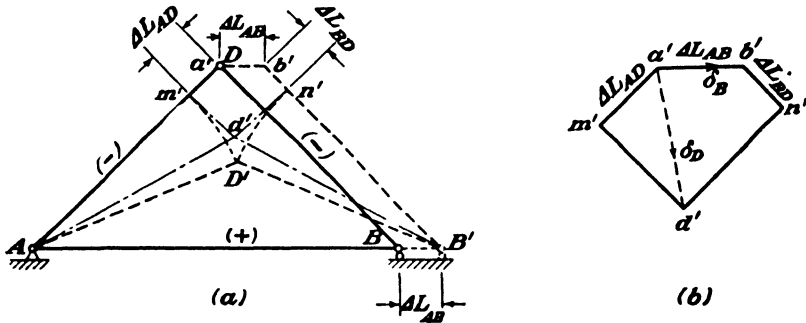


Fig. 13·19

equal to $\Delta L_{AB}$. Now, upon allowing members $AD$ and $BD$ to change length, the $D$ ends of each will move as shown if they remain connected to member $AB$ at points $A$ and $B'$, respectively. Before pin $D$ can be replaced and the truss connected together again, it is necessary to make the $D$ ends of members $AD$ and $BD$ coincide again. This may be done by rotating $AD$ about $A$ and $BD$ about $B'$ until the arcs intersect. The deflection of any joint can then be determined from the original and deflected position of the joint.

This procedure is straightforward but difficult to apply in practice because the deflections and changes in length are actually very much smaller than have been indicated in this sketch, so that a very large scale drawing is necessary in order to obtain any accuracy. Since the distortions are small, however, the angular rotation of any member is also small, in fact so small that it is permissible to assume that, during the rotation of a member, a point moves along the tangent drawn normal to the original direction of the member rather than along the true arc,

as shown by the dot-dash lines in Fig. 13·19a.  If $\Delta L$ had not been tremendously exaggerated in this sketch, the dashed and dot-dash lines would coincide for all practical purposes.  Introducing this simplification makes it possible to obtain the joint deflections without drawing the entire lengths of the members because it is no longer necessary to draw the arcs about the centers of rotation.

The simplified diagram shown in Fig. 13·19b is similar to the portion of Fig. 13·19a marked with the same letters.  It involves only the changes in length of the members and the tangents to the arcs of rotation and enables one to find the relative movements of the various joints. Such a diagram is called a *Williot diagram* after the French engineer who suggested it.  As before, imagine that the pin at $D$ is temporarily removed, and allow the changes in length to take place one at a time. Upon selecting a suitable scale for $\Delta L$ and designating the points in this diagram by the lower-case letters of the corresponding truss joints, the diagram is started by locating point $a'$.  In this simple truss, it is known that joint $A$ remains fixed in position and member $AB$ remains horizontal. Joint $B$ therefore moves horizontally to the right relative to $A$ an amount equal to $\Delta L_{AB}$ as indicated by the relative positions of points $a'$ and $b'$. Owing to the shortening, $\Delta L_{AD}$, the $D$ end of member $AD$, moves downward to the left parallel to $AD$ with reference to joint $A$, as represented [1] by the vector $\overline{a'm'}$; similarly, owing to the shortening, $\Delta L_{BD}$, the $D$ end of member $BD$, moves downward to the right parallel to $BD$ with reference to joint $B$, as represented by the vector $\overline{b'n'}$.  In order to make the $D$ ends coincide, member $AD$ must be rotated about $A$ and member $BD$ about $B$.  During these rotations, the $D$ ends are assumed to move along the tangents as represented by the vectors $\overline{m'd'}$ and $\overline{n'd'}$, respectively. Note that these tangents are perpendicular to members $AD$ and $BD$, respectively.  The vectors $\overline{a'd'}$ and $\overline{a'b'}$ represent the deflections of joints $D$ and $B$, respectively, with respect to joint $A$.  In this case, since joint $A$ is actually fixed in position, these vectors represent the true absolute movements of these joints.  Of course, the lengths of these vectors are measured to the same scale as that used in plotting $\Delta L$.

The construction of the Williot diagram for a more elaborate truss is carried out in essentially the same manner as described above.  Temporarily, all members are assumed to be disconnected.  Then, after the members are imagined to undergo their changes in length, they are reassembled one at a time, and the Williot diagram is constructed to show the resulting joint displacements.  In such cases, the true relative movement of the two ends of a bar are not known as was true of joints $A$ and $B$ in the above simple case.  The Williot diagram may be drawn, however, on the arbitrary assumption that some bar remains fixed in direction,

*i.e.,* that the relative movement of the joints at the end of this bar is parallel to the bar and equal to the change in its length. When these two points are located, a third point corresponding to the third joint of a truss triangle formed by these three joints may be located in the manner described above. The remainder of the construction may be carried out proceeding from joint to joint, always working with a triangle two joints of which have already been located, and locating the new joint from these two points. If the assumed orientation is correct, the vectors corresponding to certain known deflection conditions will be oriented correctly. If these vectors are not consistent with the known conditions. the Mohr correction diagram must be added to the Williot diagram.

The Williot diagram for the truss of Fig. 13·20 has been drawn on the assumption that joint $a$ is fixed in position and member $ab$ fixed in direction. If this orientation is correct, the vectors drawn from point $a'$ to points $b'$, $c'$, $d'$, etc., will be the true deflections of joints $b$, $c$, $d$, etc., respectively. Joint $c$ on the truss, being a roller support, can have no vertical deflection and is constrained to move horizontally. On the basis of the assumed orientation, however, joint $c$ moves upward to the right as shown by the vector $\overline{a'c'}$, and the other joints have moved as indicated to an exaggerated scale by the dashed lines on the line diagram of the truss. This means that the assumption that member $ab$ remains fixed in direction is in error, and it is now necessary to rotate the truss, as a whole, clockwise about $a$ to bring joint $c$ back down on the support. The amount of rotation required may be determined by knowing that the true deflection of $c$ must be a horizontal vector. This true deflection is the resultant of vector $\overline{a'c'}$ and the vector representing the movement of $c$ during the rotation of the truss about $a$. During a small angular rotation of the truss about $a$, joint $c$ may be assumed to move along the tangent to the true arc and hence normal to the line $ac$ on the truss, or, in this case, vertically. If this vector, giving the movement of $c$ with reference to $a$ during rotation, is added on the diagram as the vertical vector $\overline{c''a'}$ drawn through $a'$, then the resultant of the vector $\overline{c''a'}$ ($\downarrow$) and the vector $\overline{a'c'}$ ($\nearrow$) must be the horizontal vector $\overline{c''c'}$ ($\rightarrow$), which is the true deflection of joint $c$, and thus point $c''$ is located.

During the rotation of the truss about $a$, not only joint $c$ but all other joints may be assumed to move normal to the radius from the joint to the center of rotation $a$ an amount equal to that radius times the angle of rotation. In Fig. 13·21a, the arrows on the joints indicate the direction of the joint movements as the truss is rotated clockwise about $a$ through a small angle $\alpha$. If all these vectors are drawn to scale and acting toward point $a''$, as shown in Fig. 13·21b, the movement of joint $b$ during rotation is represented by vector $\overline{b''a''}$, of joint $c$ by $\overline{c''a''}$, of joint $d$ by $\overline{d''a''}$,

etc.   After these double-prime points have been connected as shown, consider any corresponding portions of these two figures such as triangles $dac$ and $d''a''c''$.   Since $c''a''$ is perpendicular to $ca$ and $d''a''$ is perpendicular to $da$, therefore angle $dac$ = angle $d''a''c''$.   Since $c''a'' = \alpha ca$
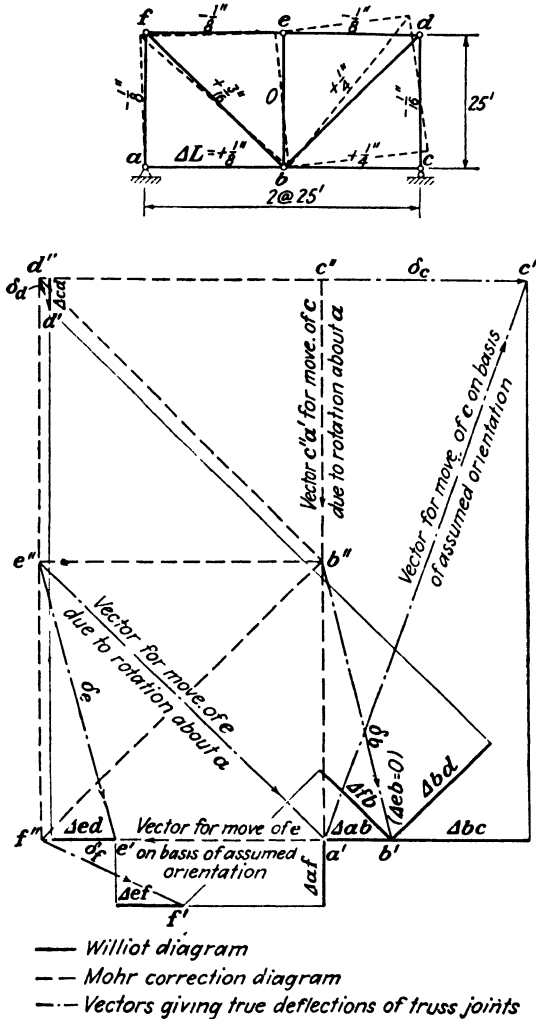


Fig. 13·20

and $d''a'' = \alpha da$, then $c''a''/d''a'' = ca/da$.   Therefore, triangles $dac$ and $d''a''c''$ are similar.   In this manner, it may be shown that Fig. 13·21a is similar to Fig. 13·21b.   Such considerations lead to the following conclusion:

*If a rigid body is rotated about some center of rotation O through a small angle, and if the movements of two points i and k are plotted as vectors O″i″ and O″k″ from or to a point O″, and if the shape of the body is then plotted to a scale with points i″ and k″ as a base, each line on this scale sketch will be perpendicular to the corresponding line on the actual body and the line drawn from any point on this scale sketch to the point O″ will represent, as a vector, the movement of the corresponding point on the real body during the rotation about O.*

This principle is the basis for the Mohr correction diagram.

In the example in Fig. 13·20, the Mohr correction diagram $a'b''c''$-$d''e''f'''$ is superimposed on the Williot diagram, using the vector $c''a'$ as a



(a)    (b)

Fig. 13·21

basis, every line on it being at 90° to the corresponding line of the line diagram of the truss. The vectors from the double-prime points to the pole $a'$ give the movements of the corresponding joints during the rotation of the truss about joint $a$. The true deflection of any joint may now be determined as the resultant of the vector from the corresponding double-prime point to $a'$ and the vector from $a'$ to the corresponding single-prime point, *i.e.*, the true deflection of any joint may be found in magnitude and direction from the vector drawn *from* the *double-prime point* on the Mohr correction diagram *to* the *single-prime point* on the Williot diagram.
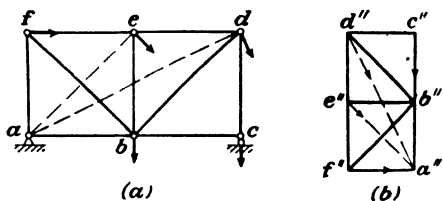
**13·12 Application of Williot-Mohr Method.** The Williot diagram will be more compact and will require a smaller Mohr correction diagram if the orientation that is assumed to start the Williot diagram is reasonably correct. In many cases, it is quite apparent that certain bars do not change direction very much. For example, in the case of an end-supported truss the center vertical remains essentially vertical and is therefore a good member to assume fixed in direction in starting the Williot diagram. This is illustrated in Example 13·19.

It is also interesting to note that the Mohr correction diagram may be eliminated entirely if the relative movements of the two ends of a bar are computed first by the method of virtual work. For example, consider the truss shown in Fig. 13·22. Joint $a$ in this case does not move. Consider one of the bars that meet at joint $a$ such as bar $ab$. The relative movement of the two ends of this bar is the resultant of the relative movements parallel and perpendicular to the bar. The relative movement of the two ends parallel to the bar is simply equal to its change

in length, while the relative movement normal to the bar may be computed by the method of virtual work.    In this case, since point $a$ does not move, all that is necessary is to compute the movement of point $b$ normal to the bar, or vertically.    Once this has been done, these two vectors may be laid out as shown, and the vector $\overline{a'b'}$ then gives the true relative movements of joints $a$ and $b$..    The Williot diagram may now be drawn on the basis of the positions of points $a'$ and $b'$, locating first point $B'$ from the triangle $abB$, etc.    Since this diagram has been oriented correctly, the absolute movements of all the joints may be obtained from it directly without adding the Mohr correction diagram. This procedure is illustrated in Example 13·20.
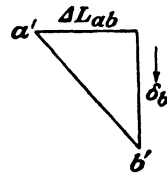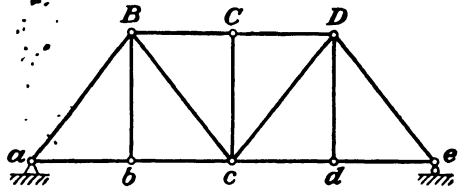


The Williot diagram may be drawn without difficulty in the manner described in Art. 13·11 for any simple truss.    It should be noted, however, that certain difficulties are encountered in applying these ideas to a compound truss such as that shown in Fig. 13·23.    Suppose that the Williot



FIG. 13·22

diagram is started in this instance by assuming joint $g$ fixed in position and bar $gG$ in direction.    Points $F'$, $f'$, $e'$, and $E'$ can be located without difficulty on the left side and similarly points $H'$, $h'$, $i'$, and $I'$ on the right. It is impossible to proceed beyond these points, however, by using the previously discussed procedures.    One way of overcoming the difficulty is to recognize that the relative movement of joints $C$ and $e$ parallel to the line $CDe$ is simply equal to the sum of the changes in length of bars $CD$
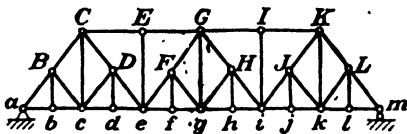


FIG. 13·23

and $De$ and, similarly, that the relative movement of joints $c$ and $e$ along line $cde$ is equal to the sum of the changes in length of $cd$ and $de$.    On this basis, temporarily consider that there are no joints at $D$ and $d$ and that bars $cD$ and $Dd$ have been omitted, and proceed as usual to locate points $C'$ and $c'$.    Having located these points, reinsert the omitted bars, and backtrack to locate points $D'$ and $d'$.    If the work has been done correctly, it is now possible to locate $e'$ from $D'$ and $d'$ and see whether or not it coincides with the existing location of point $e'$.    Points $B'$, $b'$, and

$a'$ can now be located without difficulty, and the remaining points to the right of $iI$ may be located in the manner just described for the points to the left of $Ee$.

The application of the Williot-Mohr method to the case of a three-hinged arch requires a somewhat different technique, which is illustrated by Example 13·21.

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**Example 13·19**   *Draw the Williot-Mohr diagram for this truss:*

---
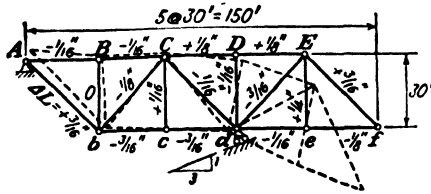


Assume point *c* fixed in position.
Assume bar *cC* fixed in direction.
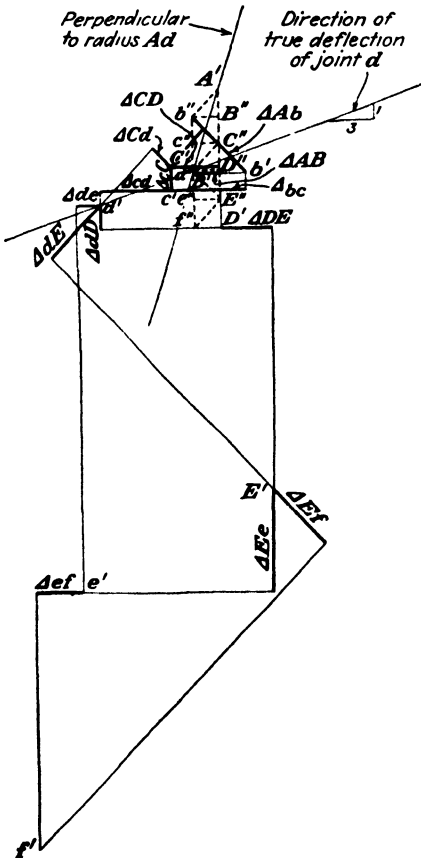
| Point | Deflection | |
|---|---|---|
| | *Horizontal* | *Vertical* |
| A | 0 | 0 |
| B | 0.063″ ← | 0.130″ ↓ |
| C | 0.125″ ← | 0.050″ ↓ |
| D | 0 | 0.125″ ↓ |
| E | 0.125″ → | 0.700″ ↓ |
| b | 0.125″ → | 0.130″ ↓ |
| c | 0.063″ ← | 0.112″ ↓ |
| d | 0.250″ ← | 0.083″ ↓ |
| e | 0.313″ ← | 0.962″ ↓ |
| f | 0.438″ ← | 1.513″ ↓ |

**Discussion:**

The vectors on the Williot diagram drawn from $c'$ to $A'$, $B'$, $b'$, $C'$, etc., give the movements of joints $A$, $B$, $b$, $C$, etc., it being assumed that joint $c$ remains fixed in position and bar $cC$ remains fixed in direction.  In such a case, the resulting joint deflections would be as shown to an exaggerated scale by the dashed lines on the line diagram.  Obviously, the assumed orientation is incorrect; for joint $A$ has moved off of the hinge support, and the movement of joint $d$ is not parallel to the supporting surface.  Note, however, that the assumed orientation is not badly in error.

Joint $A$ may be brought back to the hinge support by translating the truss as a rigid body parallel to the vector $\overline{A'c'}$ a distance equal to the scaled length of this vector.  During such a translation, every joint would move the distance indicated by the vector $\overline{A'c'}$.  After such translation, the resultant movement of any joint, such as $E$, would be the vector sum of the vector $\overline{A'c'}$ and the vector $\overline{c'E'}$, and, therefore, the vector $\overline{A'E'}$.  Therefore, it is apparent that the resultant movements of the various joints after joint $A$ has been restored to its correct position are given by the vectors drawn from point $A'$ to $B'$, $b'$, $C'$, $c'$ $D'$, $d'$, etc.  Obviously, the truss is still not oriented correctly, for the vector $\overline{A'd'}$ does not indicate the proper direction for the movement of joint $d$.  The truss must therefore be rotated counterclockwise about $A$ so that $d$ moves the amount given by the vector $\overline{d''A'}$.  Then, the total movement of joint $d$ is the vector sum of vectors $\overline{d''A'}$ and $\overline{A'd'}$, that is, the vector $\overline{d''d'}$, which is in the proper direction.

The Mohr correction diagram may then be completed as shown by the dotted lines, the line $d''A'$ being used as a base line.  The final true resultant deflection of any joint is then given by the vector drawn from the corresponding double-prime point to the corresponding single-prime point.

---

**Example 13·20**  Draw the Williot diagram for the same truss as shown in Fig. 13·20.  In this instance, however, first compute the true relative position of the two ends of a bar so that the Williot diagram may be drawn with the correct orientation and the Mohr correction diagram will thereby be eliminated.

---

In this case joint $a$ is truly fixed in position.  Computing simply the vertical deflection of joint $b$ gives us the only additional information necessary to plot the true relative position of joints $a$ and $b$.
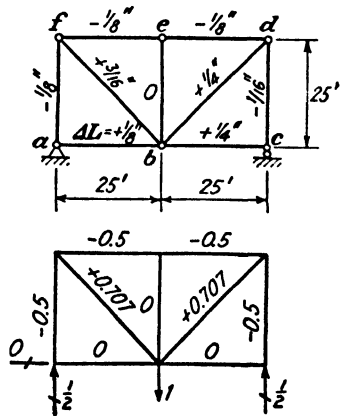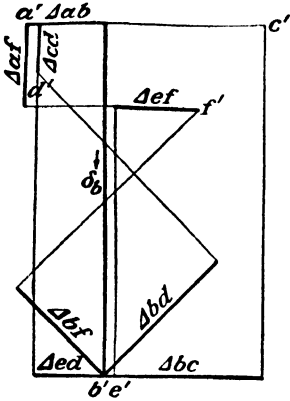
| Bar | $F_Q$ | $\Delta L$ | $F_Q \Delta L$ |
|-----|-------|-----------|---------------|
| fe | −0.5 | −0.125 | +0.063 |
| ed | −0.5 | −0.125 | +0.063 |
| af | −0 5 | −0.125 | +0.063 |
| cd | −0 5 | −0.063 | +9.031 |
| fb | +0 707 | +0.188 | +0 133 |
| bd | +0 707 | +0.250 | +0.177 |
| Σ | . . | . . . . . . | +0 530 |

$$(1^k)(\delta_b^{\downarrow}) = \Sigma F_Q \Delta L = +0.530^{k''}$$
$$\therefore \delta_b = 0.530 \text{ in. } \downarrow$$

*Note the much reduced size of this diagram*

**Example 13·21**  *Find the deflections of this three-hinge arch, using the Williot-Mohr method:*

**For left half**

   *Assume a fixed in position.*
   *Assume aA fixed in direction.*

**For right half**

   *Assume e fixed in position.*
   *Assume eE fixed in direction.*
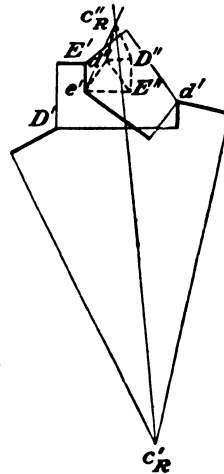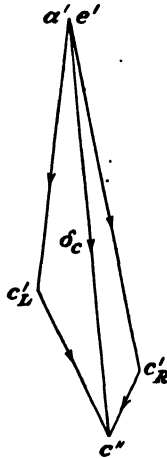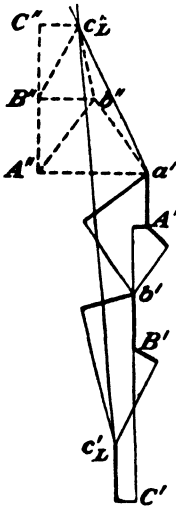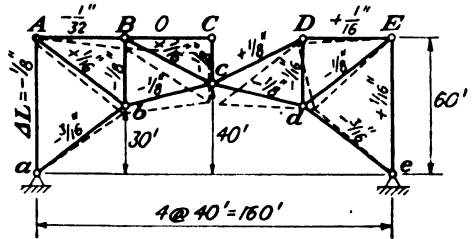
**Discussion:**

   *The presence of the hinge at point c introduces certain complications in applying the Williot-Mohr method to this problem.   After making the assumptions indicated, draw the Williot diagrams in the usual manner for the two separate halves.   On this basis, the vector $\overline{a'c'_L}$ on the left diagram indicates a certain deflection for joint c.   Similarly, the vector $\overline{e'c'_R}$ on the right diagram indicates a different deflection for joint c.   If the assumed orientations are correct, the deflection of joint c will be the same whether determined from the left or the right diagram.   Since these vectors are not equivalent, the left half must be rotated about a, causing joint c to move normal to the radius ac.   Similarly, the right half is rotated about e, causing joint c to move normal to the radius ec.   These rotations must be such as to produce the same resultant displacement of joint c for either half.   They are determined as indicated in the center vector diagram being given by the vectors $\overline{c'_L c''}$ for the left half and $\overline{c'_R c''}$ for the right half.   These vectors, therefore, establish the bases $\overline{c''_L a'}$ and $\overline{c''_R e'}$ for the Mohr correction diagrams for the left and right halves, respectively.   The true deflections of the various joints may now be obtained from the vectors drawn from the double- to the single-prime points.*

---

**13·13   Bar-chain Method.**   The  bar-chain method  is  similar  to  the Williot-Mohr method in that by one application of the method we are able to compute the deflection of several joints of a truss simultaneously.   This method was first suggested by H. Müller-Breslau and is essentially an adaptation of the elastic-load method, applied to trusses instead of beams.

   To develop the fundamentals of this procedure, consider the case of the simple truss shown in Fig. 13·24a, where it is desired to compute the vertical component of the deflection of the lower-chord panel points.   For the given cause of distortion, the changes in length of the members may be computed from Eqs. (13·5a), (13·5b), or (13·5c). As a result of these changes in length, the angles of the truss triangles will change by certain small amounts, and the truss may change shape as indicated to an exaggerated scale by the dashed lines in Fig. 13·24b.   The changes in the angles of the triangles may be com-



FIG. 13·24

puted rather easily by means of the expressions developed below.   The bottom chord of this truss may be isolated and its deflected position drawn as shown in Fig. 13·24c.   This suggests to us immediately that the problem of computing the deflected shape of this bottom chord resembles that of computing the elastic curve of a straight beam by the moment-area or elastic-load method.   In the case of a
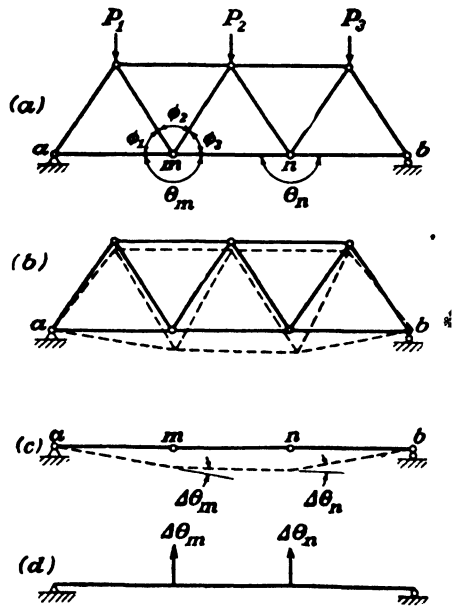
beam, however, the elastic curve is a smooth, continuous curve the slope of which is continually changing, whereas the deflection curve of this bottom chord is composed of a series of straight lines which change slope only at the truss joints. This series of straight lines is called a *bar chain*, and from this terminology the method gets its name.

In a beam, the change in slope between the tangents at the ends of a differential element of the elastic curve, $dx$, is equal to $M/EI\ dx$, which is the differential area under the $M/EI$ diagram at that element. At any joint $m$ of the bar chain, however, the change in slope between the adjacent bars of the chain is equal to the change in the angle $\theta_m$, that is, $\Delta\theta_m$, and between joints there is no change in slope. When the moment-area method is applied to a bar chain, the $M/EI$ diagram should be replaced, therefore, by a series of ordinates, one at each intermediate joint equal to $\Delta\theta$ at that joint. Hence, the elastic load used for the bar chain consists of a series of concentrated loads such as those shown in Fig. 13·24d. The technique of applying the moment-area or the elastic-load method to obtain the slope and deflection of the bar chain, however, is exactly the same as in the case of beams. The angle $\theta$ should be considered as the angle included on the lower side between two adjacent bars of the chain. $\Delta\theta$ should be considered plus (and therefore an upward elastic load) when $\theta$ increases.

It is easy to compute the change in $\theta$ if the angle changes of the truss triangles are known. For example, at
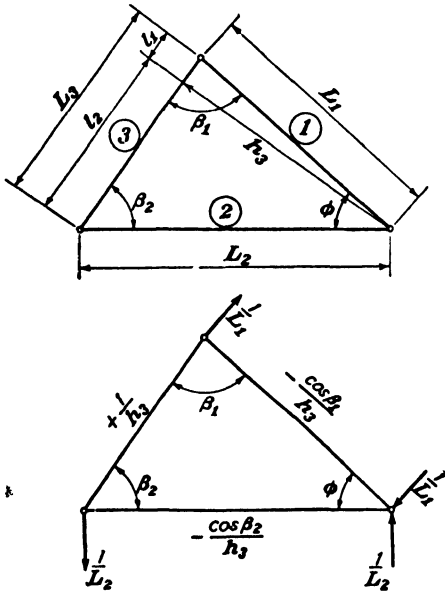


FIG. 13·25

joint $m$ in this case, $\Delta\theta_m$ is numerically equal but opposite in sense to the algebraic sum of the angle changes in angles $\phi_1$, $\phi_2$, and $\phi_3$. An expression to be used for computing such changes in the angles of the truss triangles will now be developed.

Consider the triangle shown in Fig. 13·25. Suppose that we want to compute the change in angle $\phi$ caused by these bars increasing in length by $\Delta L_1$, $\Delta L_2$, and $\Delta L_3$. The increase $\Delta\phi$ in angle $\phi$ may be computed by the method of virtual work, using the $Q$ system shown. Let $\alpha_1$ represent the clockwise rotation of bar (1) and $\alpha_2$ the counterclockwise rotation of bar (2). Then, evaluating the external virtual work done by this $Q$ system,

$$\Sigma Q\delta = (1)(\alpha_1^{\curvearrowright}) + (1)(\alpha_2^{\curvearrowright}) = (1)(\alpha_1^{\curvearrowright} + \alpha_2^{\curvearrowright}) = (1)\,\Delta\phi$$

since

$$\alpha_1^{\curvearrowright} + \alpha_2^{\curvearrowright} = \text{increase in } \angle\phi = \Delta\phi$$

Applying now the law of virtual work,

$$\Sigma Q\delta = \Sigma F_Q \, \Delta L$$

$$(1) \quad \Delta\phi = \left(\frac{1}{h_3}\right)\Delta L_3 + \left(-\frac{\cos\beta_2}{h_3}\right)\Delta L_2 + \left(-\frac{\cos\beta_1}{h_3}\right)\Delta L_1 \qquad (a)$$

From the geometry of the triangle, the following relations exist:

$$h_3 = L_2 \sin\beta_2 \quad \text{and} \quad h_3 = L_1 \sin\beta_1 \qquad (b)$$

Likewise,

$$L_3 = l_1 + l_2 = h_3 \cot\beta_1 + h_3 \cot\beta_2 = h_3(\cot\beta_1 + \cot\beta_2)$$

$$\frac{1}{h_3} = \frac{\cot\beta_1 + \cot\beta_2}{L_3} \qquad (c)$$

Substituting from Eqs. (b) and (c) in Eq. (a),

$$\Delta\phi = \frac{\Delta L_3}{L_3}(\cot\beta_1 + \cot\beta_2) - \frac{\Delta L_2}{L_2}\cot\beta_2 - \frac{\Delta L_1}{L_1}\cot\beta_1$$

Or this may be written in the following convenient form, noting that $\Delta L = eL$:

$$\Delta\phi = (e_3 - e_1)\cot\beta_1 + (e_3 - e_2)\cot\beta_2 \qquad (13 \cdot 11a)$$

Thus, the angle change $\Delta\phi$ is equal to the difference in the strain of the side opposite and one adjacent side multiplied by the cotangent of the angle between these sides, plus the difference in the strain of the side opposite and the other adjacent side of the triangle multiplied by the cotangent of the angle between these two sides. When the strain is due to an axial-stress intensity $f$, this equation may be expressed more conveniently as,

$$E \, \Delta\phi = (f_3 - f_1)\cot\beta_1 + (f_3 - f_2)\cot\beta_2 \qquad (13 \cdot 11b)$$

A plus value of $\Delta\phi$ indicates an increase in angle $\phi$ and minus values a decrease.

After the angle changes of a simple truss have been computed by using either Eq. (13·11a or 13·11b), $\Delta\theta$ may be computed at any joint of the bar chain. The moment-area method or the elastic-load procedure may then be applied to find the vertical components of the deflection of any bar chain that is *initially straight and horizontal.* This is illustrated in Examples 13·22 and 13·23. It should be recalled that, in applying the elastic-load method, the shears and moments of the imaginary beam give the slopes and deflections of the deflection curve, measured with reference to the chord corresponding to the support points of this imaginary beam. Thus, in Example 13·23, where the chord *af* rotates, the true deflections may be obtained by drawing the known line of zero deflections through points *a* and *d* and correcting the deflections measured from the chord *af* in the manner indicated.

**Example 13·22** *Using the bar chain method, compute the vertical component of the deflection of joint b of the truss in Fig. 13·20.*
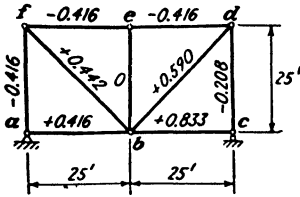


*The figures shown on the line diagram are* $e \times 10^3$. *The strains have all been multiplied by 1,000 to obtain more convenient numbers. The final results will then be divided by 1,000 to obtain the true answers.*
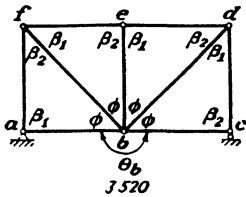
| ⊀ | $e_1 - e_1$ | col $\beta_1$ | $e_2 - e_2$ | col $\beta_2$ | $\Delta\phi$ |
|---|---|---|---|---|---|
| abf | -0 416 - 0 416 | 0 | -0 416 - 0 442 | 1 | -0 858 |
| fbe | -0 416 - 0 442 | 1 | -0 416 - 0 | 0 | -0 858 |
| ebd | -0 416 - 0 | 0 | -0 416 - 0 590 | 1 | -1 006 |
| dbc | -0 208 -0 590 | 1 | -0 208 -0 833 | 0 | -0 798 |
| ΣΔφ | . | | | | -3 520 |
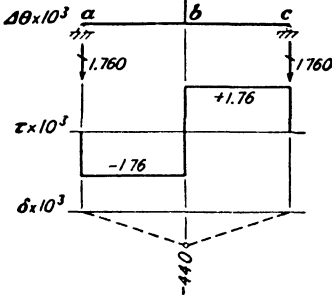
*Since, in this case, $\Delta\theta$ is equal in magnitude but opposite in sense to $\Sigma \Delta\phi$ at joint b,*

$$\therefore \Delta\theta_b = +3.520$$

*and, therefore, an upward elastic load.*
$$1.760 \times 25 = -44.0$$

$$\therefore \delta_b = 0.044 \ ft \ (down)$$
$$= 0.528 \ in.$$

**Example 13·23**   *Using the bar-chain method, compute the vertical components of the deflection of the bottom-chord panel points.*
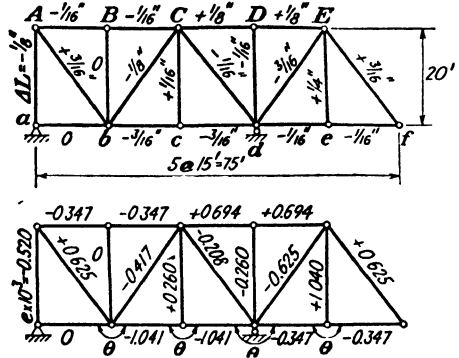
For convenience, multiply e by $10^3$, and divide final results by $10^3$ to obtain true answers.

$A -\frac{1}{16}''$   $B -\frac{1}{16}''$   $C +\frac{1}{8}''$   $D +\frac{1}{8}''$   $E$

$20'$

$0$   $b -\frac{3}{16}''$   $c -\frac{3}{16}''$   $d -\frac{1}{16}''$   $e -\frac{1}{16}''$   $f$

$5@15\frac{3}{4}'-75'$

$-0347$   $-0347$   $+0694$   $+0694$

$e_1 \times 10^3 = 0520$

$0$   $\theta$ $-1.041$   $\theta$ $-1.041$   $\theta$ $-0.347$   $\theta$ $-0.347$

| ∠ | $(e_1-e_1)\times 10^3$ (1) | col $\beta_1$ (2) | (1)× (2) | $(e_1-e_2)\times 10^3$ (3) | col $\beta_2$ (4) | (3)× (4) | $\Delta\phi\times 10^3$ | $\Delta\theta\times 10^3$ |
|---|---|---|---|---|---|---|---|---|
| abA | $-0.520-0 = -0\ 520$ | 0 | 0 | $-0\ 520-0.625 = -1\ 145$ | 1.3 | $-1\ 527$ | $-1.527$ | |
| AbB | $-0\ 347-0.625 = -0.972$ | 0.75 | $-0.729$ | $-0.347-0 = -0\ 347$ | 0 | 0 | $-0.729$ | $+1.301$ |
| BbC | $-0\ 347-0 = -0.347$ | 0 | 0 | $-0\ 347+0.417 = +0\ 070$ | 0 75 | $+0.052$ | $+0.052$ | |
| Cbc | $+0\ 260+0\ 417 = +0\ 677$ | 1.3 | $+0.903$ | $+0\ 260+1.041 = +1.301$ | 0 | 0 | $+0.903$ | |
| bcC | $-0.417+1\ 041 = +0.624$ | 0 75 | $+0\ 468$ | $-0.417-0.260 = -0.677$ | 1.3 | $-0.903$ | $-0.435$ | $+0.435$ |
| Ccd | $-0\ 208-0\ 260 = -0\ 468$ | 1.3 | $-0.624$ | $-0\ 208+1.041 = +0\ 833$ | 0.75 | $+0.625$ | 0 | |
| cdC | $+0\ 260+1\ 041 = +1\ 301$ | 0 | 0 | $+0\ 260+0.208 = +0\ 468$ | 1.3 | $+0.624$ | $+0.624$ | |
| CdD | $+0.694+0.208 = +0\ 902$ | 0.75 | $+0.676$ | $+0\ 694+0.260 = +0.954$ | 0 | 0 | $+0.676$ | $-4.509$ |
| DdE | $+0.694+0.260 = +0\ 954$ | 0 | 0 | $+0.694+0.625 = +1.319$ | 0.75 | $+0.989$ | $+0.989$ | |
| Ede | $+1.040+0.625 = +1.665$ | 1.3 | $+2.220$ | $+1.040+0.347 = +1.387$ | 0 | 0 | $+2.220$ | |
| deE | $-0.625+0\ 347 = -0.278$ | 0.75 | $-0.208$ | $-0.625-1.040 = -1.665$ | 1.3 | $-2.220$ | $-2.428$ | $+2.252$ |
| Eef | $+0.625-1.040 = -0.415$ | 1 3 | $-0.553$ | $+0.625+0.347 = +0.972$ | 0.75 | $+0.729$ | $+0.176$ | |

1.301　0.435　4.509　2.252

$\Delta\theta \times 10^3$  a  |b  |c  |d  |e  f

{0.051  
+1.787　　　　　　　　　0.470↑  
+1.352  
$\tau \times 10^3$  +0.051  
　　　　　　　　　　　　　−0.470  
−2.722  

Net δ / 0  
0.182" down　0.130" down　0　0.681"　0.957"  
(Line of 0 defl.)  
$\delta \times 10^3$  
0　+0.77　+21.05　+47.86　+7.03　0
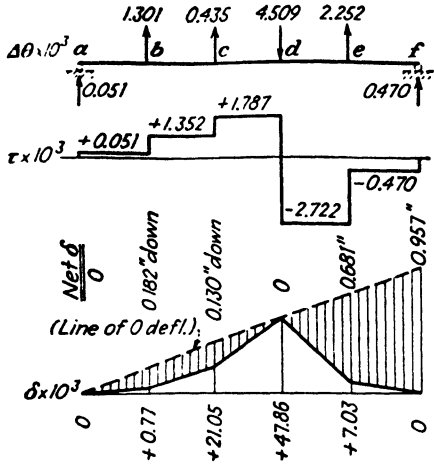
*Using imaginary beam of span af,*

$\Sigma M_f$ ↷+  
1.301 × 4 = 　5.204  
0.435 × 3 = +1.305  
−4.509 × 2 = −9.018  
2.252 × 1 = +2.252  
　　　　　5|−0.257  
$\overline{0.521\downarrow}$　　$0.051\uparrow$

$\Sigma M_a$ ↶+  
1.301 × 1 = + 1.301  
0.435 × 2 = + 0 870  
−4.509 × 3 = −13.527  
2.252 × 4 = + 9 008  
　　　　　5|− 2.348  
　　　　　$\overline{0.470\uparrow}$

| $\tau \times 10^3$ | | |
|---|---|---|
| +0 051 | | $0 = \delta_a \times 10^3$ |
| +1.301 | +0.051 × 15 = + 0 77 | |
| +1 352 | | $+ 0.77 = \delta_b \times 10^2$ |
| +0 435 | +1.352 × 15 = + 20 28 | |
| +1.787 | | $+21 05 = \delta_c \times 10^3$ |
| −4.509 | +1.787 × 15 = +26.81 | |
| −2.722 | | $47 86 = \delta_d \times 10^3$ |
| +2.252 | −2.722 × 15 = −40 83 | |
| −0.470 | | $+ 7.03 = \delta_e \times 10^3$ |
| +0.470 | −0.470 × 15 = − 7.05 | |
| 0 | | $− 0.02 = \delta_f \times 10^2$ |

Certain modifications must be introduced into this approach before it can be applied to a bar chain that is not initially straight and horizontal. Suppose that we wish to compute the vertical components of the deflection of the lower-chord bar chain of the truss shown in Fig. 13·26a. The effect of the changes in the $\theta$ angles on the deflection of this bar chain can be computed in the same manner as if the bar chain were initially horizontal. This is evident when one considers the lower-chord bar chain shown in Fig. 13·26b. Imagine that we allow only the angle $\theta_b$ to change and that at the same time we hold the portion of the bar chain to the left of joint b fixed in position. The vertical component of the movement of point E would then be equal to $x'\,\Delta\theta_b$, the same value as it would have been if the bar chain were initially straight. Continuing in this manner, we may show that the vertical deflection of this lower-chord bar chain produced simply by the angle changes $\Delta\theta$ may be computed in the same manner as if the bar chain were initially horizontal, using the imaginary beam shown in Fig. 13·26c.

The total deflection of this lower-chord bar chain is produced, however, not only by the angle changes $\Delta\theta$ but also by the changes in length of the members

of the bar chain themselves. For example, consider the effect of an elongation of bar $cd$. If all the joints of the bar chain were locked so that no angle changes could occur and if the bars to the left of $c$ were held fixed in position, the elongation of bar $cd$ would cause all joints to the right of $c$ to move vertically an amount equal to $\Delta L_{cd} \sin \alpha$, as indicated by the ordinates to the solid line in Fig. 13·26d. In order to return point $E$ to its proper position, the entire bar chain must be rotated about point $A$ as a rigid body, causing the vertical displacements shown by the dashed line in Fig. 13·26d. The net effect of the two displacements shown in Fig. 13·26d is shown in Fig. 13·26e. This displacement diagram would be the bending-moment diagram for the loads shown in Fig. 13·26f. In the same manner, it could be shown that the contribution of the change in length of any bar of the chain to the vertical deflection would be equal to the bending moment produced by two similar equal and opposite loads acting at the joints at the ends of that bar.

These considerations lead us to the following conclusion: To compute the total deflections of a bar chain that is not initially straight, the bending moments may be computed on an imaginary beam that is acted upon by a modified elastic loading as indicated in Fig. 13·26g. At any intermediate joint $m$ of the bar chain, this loading is equal to the angle change $\Delta\theta_m$ modified by certain contributions proportional to the strains in the adjacent members of the bar chain. Let this modified elastic load at joint $m$ be called $W_m$. Then,

$$W_m = \Delta\theta_m - e_L \tan \alpha_L + e_R \tan \alpha_R \qquad (13·12)$$

in which $e_L$ and $\alpha_L$ represent the strain and initial slope with the horizontal of the adjacent member of the bar chain on the *left* of joint $m$

$e_R$ and $\alpha_R$ represent similar quantities for the adjacent member on the *right* of joint $m$

$\Delta\theta_m$ represents the change in angle $\theta_m$, which, as defined above, is the angle included on the lower side between the two adjacent members of the chain
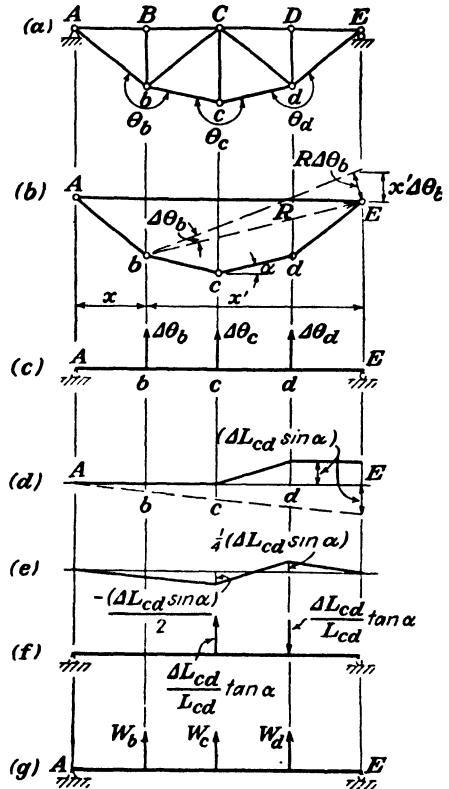


FIG. 13·26

It is important for the proper sign convention to be adhered to in using this equation.

$W_m$ is plus when indicating an upward elastic load
$\Delta\theta_m$ is plus when $\theta_m$ increases
   $e$ is plus when the member elongates
   $\alpha$ is plus when the initial slope of the member is up to the right

**Example 13·24**  *Using the bar-chain method, compute the vertical components of the deflection of the lower-chord panel points.*
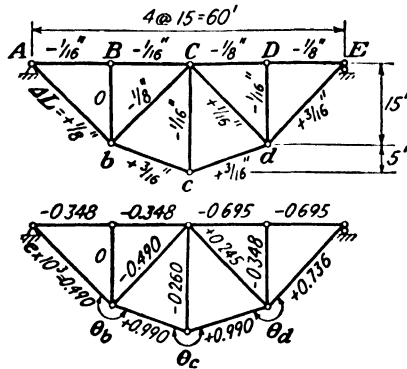


| ₄ | $(e_1-e_1) \times 10^2$ (1) | col $\beta_1$ (2) | (1) × (2) | $(e_1-e_2) \times 10^2$ (3) | col $\beta_2$ (4) | (3) × (4) | $\Delta\phi \times 10^3$ | $\Delta\theta \times 10^3$ |
|---|---|---|---|---|---|---|---|---|
| AbB | $-0.348 - 0.490 = -0.838$ | 1.0 | $-0.838$ | $-0.348 - 0 = -0.348$ | 0 | 0 | $-0.838$ | |
| BbC | $-0.348 - 0 = -0.348$ | 0 | 0 | $-0.348 + 0.490 = +0.142$ | 1.0 | $+0.142$ | $+0.142$ | $+0.883$ |
| Cbc | $-0.260 + 0.490 = +0.230$ | 1.0 | $+0.230$ | $-0.260 - 0.990 = -1.250$ | 0.3 | $-0.417$ | $-0.187$ | |
| bcC | $-0.490 - 0.990 = -1.480$ | 0.500 | $-0.740$ | $-0.490 + 0.260 = -0.230$ | 1.0 | $-0.230$ | $-0.970$ | $+0.838$ |
| Ccd | $+0.245 + 0.260 = +0.505$ | 1.0 | $+0.505$ | $+0.245 - 0.990 = -0.745$ | 0.500 | $-0.373$ | $+0.132$ | |
| cdC | $-0.260 - 0.990 = -1.250$ | 0.3 | $-0.417$ | $-0.260 - 0.245 = -0.505$ | 1.0 | $-0.505$ | $-0.922$ | |
| CdD | $-0.695 - 0.245 = -0.940$ | 1.0 | $-0.940$ | $-0.695 + 0.348 = -0.347$ | 0 | 0 | $-0.940$ | $+3.293$ |
| DdE | $-0.695 + 0.348 = -0.347$ | 0 | 0 | $-0.695 - 0.736 = -1.431$ | 1.0 | $-1.431$ | $-1.431$ | |

Then, $W_m = \Delta\theta_m - e_L \tan \alpha_L + e_R \tan \alpha_R$

$$W_b \times 10^2 = +0.883 - (0.490)(-1.0) + (0.990)(-0.3) = +1.043$$
$$W_c \times 10^2 = +0.838 - (0.990)(-0.3) + (0.990)(0.3) = +1.498$$
$$W_d \times 10^2 = +3.293 - (0.990)(0.3) + (0.736)(1.0) = +3.699$$

› *Finding the reactions on the imaginary beam*
*AE,*

$\Sigma M_E \qquad \curvearrowleft +$

$1.043 \times 3 = 3.129$
$1.498 \times 2 = 2.996$
$3.699 \times 1 = 3.699$
$\qquad\qquad 4\overline{)9.824}$

$\overline{6.240}\uparrow \qquad 2.456\downarrow$

$\Sigma M_A \qquad \curvearrowleft +$

$1.043 \times 1 = 1.043$
$1.498 \times 2 = 2.996$
$3.699 \times 3 = 11.097$
$\qquad\qquad 4\overline{)15.136}$

$\qquad\qquad \overline{3.784}\downarrow$

$\qquad\qquad 0 = \delta_A \times 10^3$

$-2.456 \qquad -2.456 \times 15 = -36.84$
$+1.043 \qquad\qquad \overline{-36.84} = \delta_b \times 10^3$
$-1.413 \qquad -1.413 \times 15 = -21.20$
$+1.498 \qquad\qquad \overline{-58.04} = \delta_c \times 10^3$
$+0.085 \qquad +0.085 \times 15 = +1.28$
$+3.699 \qquad\qquad \overline{-56.76} = \delta_d \times 10^3$
$+3.784 \qquad +3.784 \times 15 = +56.76$
$-3.784 \qquad\qquad \overline{0} = \delta_E \times 10^3$
$\overline{0}$
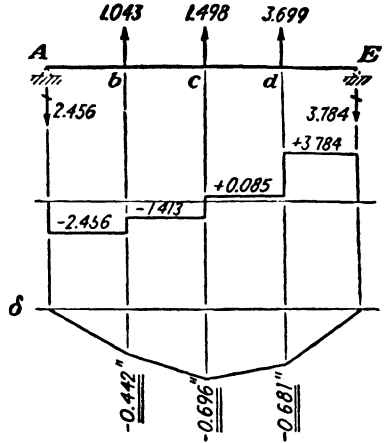
Example 13·24 illustrates the application of this method to a bar chain that is not initially straight.  It is important to note that the shear produced on the imaginary beam by the elastic load has no significance in such cases.  Of course, in cases where the bar chain is initially straight and horizontal, such shears give the slope of the members of the deflected bar chain.  The elastic load for the general case of a polygonal bar chain as expressed in Eq. (13·12), however, was determined so that the bending moment on the imaginary beam gave the deflection of the bar chain, but no significance was attached to the shear produced by this load.  These statements may be verified by the computations in Example· 13·24.  In this case, since bar *Bb* does not change length, the vertical deflections of joints *B* and *b* are the same.  The rotation of bar *AB* is therefore 0.002456 radian clockwise.  Angle *BAb* is reduced by 0.000490 radian.  Bar *Ab*, therefore, is found to rotate clockwise 0.001966 radian, but this value is not equal to the shear of 0.002456.  Likewise, it will be found that none of the other rotations of the members of the bar chain are equal to the corresponding shears.  Of course, the rotation of bar *Ab* being known, the rotation of the remaining bars of the chain may be found by working from joint to joint along the chain and using the $\Delta\theta$ values already computed.  In this manner, bar *bc* is found to rotate 0.001083 radian clockwise; *cd*, 0.000245 radian clockwise; and *dE*, 0.003048 radian counterclockwise.

The bar-chain method may be easily applied in a similar manner to any simple truss. A suitable bar chain is selected joining those joints the vertical deflection of which is desired. For any such bar chain, the elastic loads may be computed from Eq. (13·12). In the special cases where the bar chain is initially straight and horizontal (as illustrated in Examples 13·22 and 13·23), $\alpha$ will be zero for all bars and $W_m$ equals simply $\Delta\theta_m$. Note that Eq. (13·12) cannot be used if the bar chain includes a vertical member since, then, $\alpha = 90°$ and $\tan \alpha = \infty$. It is never necessary to include such a member, however, since the difference in the vertical deflection of the ends of such a vertical is simply equal to the change in length of that member.

This method may also be applied to compound trusses such as that shown in Fig. 13·23. In such cases, however, it is necessary to insert imaginary bars between such joints as $D$ and $E$, $E$ and $F$, etc., so as to divide the truss into triangles for the computation of the angle changes. The changes in length of any one of these bars may be obtained by computing the relative deflection of the joints at the end of the bar by the method of virtual work. The method may likewise be applied to three-hinged arches such as that of Example 13·21. Here again, however, it is necessary to insert an imaginary bar between $C$ and $D$ before the angle changes may be computed. The change in length of this imaginary bar is equal to the relative deflection of joints $C$ and $D$, which may likewise be computed by the method of virtual work.

**13·14 Castigliano's Second Theorem.** In 1879, Castigliano published the results of an elaborate research on statically indeterminate structures in which he used two theorems which bear his name. *Castigliano's second theorem* may be stated as follows:

*In any structure the material of which is elastic and follows Hooke's law and in which the temperature is constant and the supports unyielding,*



FIG. 13·27

*the first partial derivative of the strain energy with respect to any particular force is equal to the displacement of the point of application of that force in the direction of its line of action.*
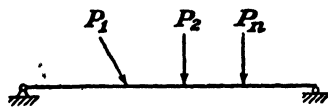
In this statement, the words *force* and *displacement* should be interpreted also to mean *couple* and *angular rotation*, respectively. Further, it also is implied that, during the distortion of the structure, there is no appreciable change in its geometry. The application of this theorem is therefore limited to cases where it is legitimate to superimpose deflections.

To derive this theorem, consider any structure that satisfies the stated conditions, such as the beam shown in Fig. 13·27. Suppose that this beam is loaded gradually by the forces $P_1$, $P_2$, . . . , $P_n$. Then, the external work done by these forces (let us call this $W_E$) is some function of these forces. According to the principle of the conservation of energy, we know that, in any elastic structure at rest and in equilibrium under

a system of loads, the internal work or strain energy stored in the struc-
ture is equal to the external work done by these loads during their gradual
application. Designating the internal work or strain energy by $W_I$, we
may therefore write

$$W_I = W_E = f(P_1, P_2, \cdots, P_n) \tag{a}$$

Suppose now that the force $P_n$ is increased by a small amount $dP_n$; the
internal work will be increased, and the new amount will be

$$W'_I = W_I + \frac{\partial W_I}{\partial P_n} dP_n \tag{b}$$

The magnitude of the total internal work, however, does not depend
upon the order in which the forces are applied—it depends only on the
final value of these forces. Further, if the material follows Hooke's
law, the distortion and deflection caused by the forces $P_1, P_2, \ldots, P_n$
and hence the work done by them are the same whether these forces are
applied to a structure already acted upon by other forces or not, as long
as the total fiber stresses due to all causes remain within the elastic limit.
If, therefore, the infinitesimal force $dP_n$ is applied first and the forces
$P_1, P_2, \ldots, P_n$ are applied later, the final amount of internal work will
still be the same amount as that given by Eq. (b).

The force $dP_n$ applied first produces an infinitesimal displacement
$d\delta_n$, so that the corresponding external work done during the application
of $dP_n$ is a small quantity of the second order and can be neglected. If
the forces $P_1, P_2, \ldots, P_n$ are now applied, the external work done just
by them will not be modified owing to the presence of $dP_n$ and hence will
be equal to the value of $W_E$ given by Eq. (a). However, during the
application of these forces, the point of application of $P_n$ is displaced an
amount $\delta_n$, and therefore $dP_n$ does external work during this displace-
ment equal to $(dP_n)(\delta_n)$. Let the total amount of external work done by
the entire system during this loading sequence be $W''_E$. Then,

$$W''_E = W_E + dP_n\delta_n \tag{c}$$

But, according to the principle of the conservation of energy, $W'_E$ equals
$W'_I$, and therefore

$$W_E + dP_n\delta_n = W_I + \frac{\partial W_I}{\partial P_n} dP_n \tag{d}$$

However, since $W_E$ is equal to $W_I$, Eq. (d) reduces to simply

$$\frac{\partial W_I}{\partial P_n} = \delta_n \tag{13·13}$$

This latter equation is the mathematical statement of Castigliano's second theorem.

In order to use Castigliano's theorem, it is first necessary to develop suitable expressions for the strain energy stored, or the internal work done, by the stresses in a member. Consider first the case of the strain energy stored in a bar by an *axial stress F* as this stress gradually increases from zero to its final value. Consider a differential element of such a bar bounded by two adjacent cross sections as shown in Fig. 13·28. Suppose that this element is acted upon by a stress $F_t$, some intermediate value between zero and the final value of the stress $F$. Suppose this stress is now increased by an amount $dF_t$, causing the element to change in length by an amount $\Delta(dL)_t$, where
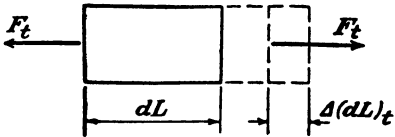
Fig. 13·28

$$\Delta(dL)_t = dF_t \frac{dL}{AE} \qquad (e)$$

Neglecting quantities of the second order, the internal work done during the application of $dF_t$ is equal to $(F_t)[\Delta(dL)_t]$, and therefore the total internal work $dW_I$ done in this element during the increase of the stress $F$ from zero to its final value is

$$dW_I = \int_0^F F_t\, \Delta(dL)_t = \int_0^F F_t \frac{dL}{AE}\, dF_t = \frac{F^2}{2AE}\, dL \qquad (f)$$

For the entire member, the internal work will be the sum of the terms $dW_I$ for all the elements $dL$, or

$$W_I = \int_0^L \frac{F^2}{2AE}\, dL = \frac{F^2 L}{2AE} \qquad (g)$$

For all the members of the structure, the internal work will be the sum of such terms for every bar of the structure, or
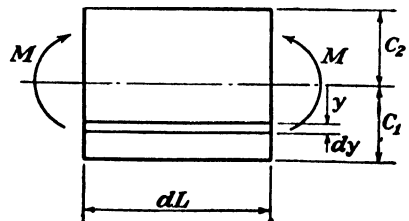
Fig. 13·29

$$\text{\textit{Strain energy stored by axial stresses,}}\ W_I = \sum \frac{F^2 L}{2AE} \qquad (13·14)$$

This equation may now be used to develop an expression for the strain energy stored in a beam by the fiber stresses produced by a *bending moment M*. Consider a differential element of a beam having a length $dL$ as shown in Fig. 13·29. This element of the beam may be considered to be a bundle of little fibers each having a length $dL$, a depth $dy$, and a

width normal to the plane of the paper of $b$.  The axial stress on such a fiber would be

$$F = fb \, dy = \frac{My}{I} b \, dy \tag{h}$$

The strain energy stored in all such fibers of the beam may be evaluated by applying Eq. (13·14), summing up the contributions of all the fibers across the element $dL$, and then summing up these quantities for all the elements in the length of the beam, or

$$W_I = \int_0^L \int_{-C_2}^{C_1} \left(\frac{My}{I} b \, dy\right)^2 \frac{dL}{2(b \, dy)E} = \int_0^L \frac{M^2}{2EI} \int_{-C_2}^{C_1} \frac{y^2 b \, dy}{I} dL$$

$$= \int_0^L \frac{M^2}{2EI} dL \tag{i}$$

since $\int_{-C_2}^{C_1} y^2 b \, dy = I$.  Hence, for all the beam elements of the structure,

$$\textit{Strain energy stored by bending moment, } W_I = \sum \int \frac{M^2}{2EI} dL \tag{13·15}$$

As discussed previously in Art. 13·5, it is usually permissible to neglect the strain energy stored by the shear stresses in a beam.

**13·15   Computation of Deflections, Using Castigliano's Second Theorem.**   Castigliano's second theorem is used principally in the analysis of statically indeterminate structures, although it is sometimes used to solve deflection problems.  The technique of applying this method in the latter case is essentially the same as that of solving such problems by the method of virtual work.  In fact, the following illustrative problems will demonstrate that the actual numerical computations involved in either of these methods are almost identical.

Example 13 25 illustrates how to apply this method when the deflection of the point of application of one of the loads is required.   If this load has some numerical value, it should temporarily be denoted by a symbol, instead.   Then, after the partial differentiation of the bending-moment expression has been performed, the symbol may be replaced by the given numerical value.

Sometimes we wish to compute the deflection at a point where no load is acting.   In such cases, we may temporarily introduce an imaginary force (or couple) acting at the point and in the direction of the desired deflection component.   Then, after the partial differentiation of the strain energy has been performed, we may let the introduced force equal zero and proceed with the numerical calculations.   In this way, we obtain the desired deflection produced by only the given loads.   Example 13·26 illustrates this procedure.

By introducing suitable imaginary loads in this manner, it is possible to compute any desired deflection component.   The technique of selecting the

appropriate imaginary loads in such cases is exactly the same as selecting the proper $Q$-force system in applying the method of virtual work. Example 13·27 illustrates this point and also demonstrates the close similarity between the solution by Castigliano's theorem and that by the method of virtual work.

It should be noted that Castigliano's theorem may be applied to any type of structure, whether beam, truss, or frame. Its use is restricted, however, to cases where the deflection is caused by loads. This theorem is not applicable to the computation of deflections produced by support settlements or temperature changes.

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**Example 13·25**   *Compute the vertical deflection of point b due to the load shown.*
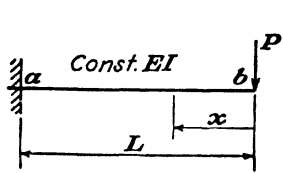
$$W_I = \sum \int \frac{M^2\,dx}{2EI}, \qquad \text{but}$$

$$\frac{\partial W_I}{\partial P} = \delta_b^{\downarrow} = \sum \int M \frac{\partial M}{\partial P} \frac{dx}{EI}$$

*From b to a,*

$$0 < x < L, \qquad M = -Px, \qquad \frac{\partial M}{\partial P} = -x$$

*Therefore,*

$$\delta_b^{\downarrow} = \int_0^L (-Px)(-x)\,\frac{dx}{EI} = \left[\frac{Px^3}{3EI}\right]_0^L$$

*Whence*

$$\delta_b = \frac{PL^3}{3EI}$$

---

**Example 13·26**   *Compute the change in slope of the cross section at a due to the loads shown.*

*Suppose that a couple $M_1$ was applied temporarily at a. Considering this as part of the load system, we may proceed as follows:*

$$W_I = \sum \int \frac{M^2\,dx}{2EI} \qquad \text{but}$$

$$\frac{\partial W_I}{\partial M_1} = \alpha_a = \sum \int M \frac{\partial M}{\partial M_1} \frac{dx}{EI}$$
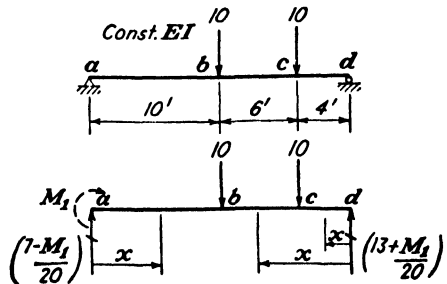
*From a to b,*   $0 < x < 10,$   $M = M_1 + \left(7 - \dfrac{M_1}{20}\right)x,$   $\dfrac{\partial M}{\partial M_1} = \left(1 - \dfrac{x}{20}\right)$

*From d to c,*   $0 < x < 4,$   $M = \left(13 + \dfrac{M_1}{20}\right)x,$   $\dfrac{\partial M}{\partial M_1} = \dfrac{x}{20}$

*From c to b,*   $4 < x < 10,$   $M = \left(13 + \dfrac{M_1}{20}\right)x - 10(x - 4),$   $\dfrac{\partial M}{\partial M_1} = \dfrac{x}{20}$

Then,

$$\frac{\partial W}{\partial M_1} = EI\alpha_a\overset{\frown}{\phantom{a}} = \int_0^{10} \left[ M_1 + \left( 7 - \frac{M_1}{20} \right) x \right] \left( 1 - \frac{x}{20} \right) dx$$

$$+ \int_0^4 \left[ \left( 13 + \frac{M_1}{20} \right) x \right] \left( \frac{x}{20} \right) dx + \int_4^{10} \left[ \left( 13 + \frac{M_1}{20} \right) x - 10(x - 4) \right] \left( \frac{x}{20} \right) dx$$

*In this equation, we may now let $M_1 = 0$ since it is an imaginary load.*
*Then,*

$$EI\alpha_a\overset{\frown}{\phantom{a}} = \int_0^{10} (7x) \left( 1 - \frac{x}{20} \right) dx + \int_0^4 (13x) \left( \frac{x}{20} \right) dx + \int_4^{10} (3x + 40) \left( \frac{x}{20} \right) dx$$

$$= \left[ \frac{7x^2}{2} - \frac{7x^3}{60} \right]_0^{10} + \left[ \frac{13x^3}{60} \right]_0^4 + \left[ \frac{3x^3}{60} + x^2 \right]_4^{10} = \left( 350 - \frac{700}{6} \right) + \left( \frac{208}{15} \right)$$

$$+ \left[ \left( \frac{1,000 - 64}{20} \right) + (100 - 16) \right]$$

*or,*

$$\alpha_a = \frac{378}{EI}$$

*If $E = 30 \times 10^3$ k/in² and $I = 200$ in.⁴,*

$$\alpha_a = \frac{(378)^{k/²}}{(30 \times 10^3 \times 144)^{k/²} (200/144^2)^{/4}} = 0.00907 \ radian$$

---

**Example 13·27**    *Solve part a of Example 13·4 using Castigliano's second theorem.*

---

*To solve for the relative deflection of joints b and
D, add to the given load system the loads P acting
as shown.*
*Then*

$$W_I = \sum \frac{F^2 L}{2AE} \quad and$$

$$\frac{\partial W_I}{\partial P} = \delta_{b-D} = \sum \frac{FL}{AE} \frac{\partial F}{\partial P}$$
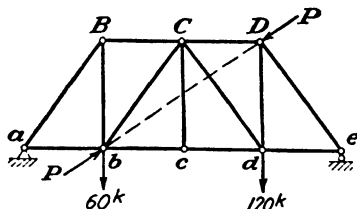


*Using the bar stresses computed in Example 13·4 for the 60- and 120-kip loads and noting
that the stresses due to the loads P are P times the stresses caused by the unit loads in
that problem, we may now proceed as follows:*

| Bar | L | A | $\frac{L}{A}$ | F | $\frac{\partial F}{\partial P}$ | $F \frac{L}{A} \frac{\partial F}{\partial P}$ |
|---|---|---|---|---|---|---|
| Units | ' | '' ² | '/'' ² | k | k/k | k''/'' ² |
| bc | 15 | 5 | 3 | + 67.5 − 0 416P | −0 416 | − 84.5 |
| cd | 15 | 5 | 3 | + 67.5 − 0.416P | −0 416 | − 84.5 |
| CD | 15 | 5 | 3 | − 78.75 − 0 831P | −0 831 | +197.0 |
| bC | 25 | 2 5 | 10 | − 18 7 − 0.695P | −0 695 | +130 |
| Cd | 25 | 2.5 | 10 | + 18.7 + 0.695P | +0 695 | +130 |
| dD | 20 | 5 | 4 | +105 − 0 555P | −0 555 | −233. |
| Σ | .. | ... | .. | ................. | ....... | + 55.0 |

*Before evaluating the product in the last column of this tabulation, set P equal to zero since only the contribution of the constant portion of F need be included in the product. Thus,*

$$\delta_{b-D} = \frac{+55}{E} = \frac{(55)^{k'/''^2}}{(30 \times 10^3)^{k'/''^2}} = +0.00183 \; ft \; (together)$$

*Note that those bars which have zero stresses due to either the imaginary loads P or the given applied loads do not contribute to the products in the last column and therefore do not need to be included in the tabulation.*

**13·16   Deflection of Space Frameworks.**   The deflections of the joints of a space framework may be computed without difficulty by either the method of virtual work or Castigliano's second theorem. The expressions developed previously in order to apply these methods to planar trusses may likewise be applied to space frameworks (which are simply three-dimensional "trusses"), as illustrated in Example 13·28.

**Example 13·28**   *Compute the z component of the deflection of joint d due to the load shown. The cross-sectional area of all members is 2 sq in. E = 30 × 10³ kips per sq in.*
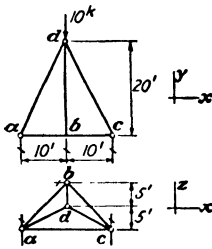


| Bar | Projection | | | L |
|-----|-----|-----|-----|-----|
| | x | y | z | |
| ab | 10 | 0 | 10 | 14 14 |
| bc | 10 | 0 | 10 | 14 14 |
| ac | 20 | 0 | 0 | 20 |
| ad | 10 | 20 | 5 | 22 91 |
| bd | 0 | 20 | 5 | 20 62 |
| cd | 10 | 20 | 5 | 22 91 |

*Using the method of virtual work,*

$$\sum Q\delta = \sum F_Q \Delta L = \sum F_Q F_P \frac{L}{AE}$$

*or*

$$(1^k)(\delta_{ds}) = \frac{1}{AE}\sum F_Q F_P L$$



$F_Q$:
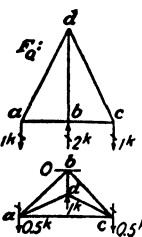
| Bar | Components | | | $F_Q$ |
|-----|-----|-----|-----|-----|
| | X | Y | Z | |
| ab | +0 25 | 0 | +0 25 | +0 354 |
| bc | +0 25 | 0 | +0 25 | +0 354 |
| ac | -0 75 | 0 | 0 | -0 75 |
| ad | +0 5 | +1 | +0 25 | +1 145 |
| bd | 0 | -2 | -0 5 | -2 062 |
| cd | +0 5 | +1 | +0 25 | +1 145 |

$F_P$:
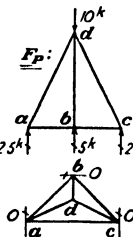
| Bar | Components | | | $F_P$ |
|-----|-----|-----|-----|-----|
| | X | Y | Z | |
| ab | +0 625 | 0 | +0 625 | +0 884 |
| bc | +0 625 | 0 | +0 625 | +0 884 |
| ac | +0 625 | 0 | 0 | +0 625 |
| ad | -1 25 | -2 5 | -0 625 | -2 863 |
| bd | 0 | -5 | -1 25 | -5 154 |
| cd | -1 25 | -2 5 | -0 625 | -2 863 |

| Bar | L | $F_Q$ | $F_P$ | $F_Q F_P L$ |
|-----|---|-------|-------|-------------|
|     | ' | k | k | $k^{2}{}'$ |
| ab | 14.14 | +0.354 | +0.884 | + 4.4 |
| bc | 14.14 | +0.354 | +0.884 | + 4.4 |
| ac | 20 | −0.75 | +0.625 | − 9.4 |
| ad | 22.91 | +1.145 | −2.863 | − 75.0 |
| bd | 20.62 | −2.062 | −5.154 | +219.4 |
| cd | 22.91 | +1.145 | −2.863 | − 75.0 |
| Σ | . ... | ....... | ....... | + 68.8 |

*whence*

$$(1^k)(\delta_{dz}) = \frac{+68.8^{k^2}{}'}{(2''^2)(30 \times 10^3 \ k/''^2)}$$

$$\delta_{d\bar{z}} = +\ 0.00115\ ft$$

**13·17　Other Deflection Problems.** All the examples in this chapter have illustrated the computation of the deflection of statically determinate structures. All the methods presented may be applied, however, to both statically determinate and indeterminate structures. Of course, the stress analysis of an indeterminate structure must be completed before the deformation of the elements of the structure may be defined. Once this has been done, however, the computations of the resulting deflections of the structure are essentially the same as if it were statically determinate. Several examples illustrating such computations are included in Art. 14·17.

In this chapter, the discussion of beam deflections has been limited to cases where the centroidal axis of the beam is straight and where all the cross sections have axes of symmetry that lie in the same plane as the loads. Further, no discussion has been included concerning the deflections produced by shear. All such cases are beyond the scope of this book and are included in more advanced treatments of this subject. It should be noted, however, that either the method of virtual work or Castigliano's second theorem may be extended without difficulty to cover such cases. It is also possible to apply these methods to members that involve torsion.

**13·18　Cambering of Structures.** Cambering a structure consists in varying the unstressed shape of the members of the structure in a manner such that under some specified condition of loading the structure attains its theoretical shape. The purpose of such a procedure is twofold. (1) It improves the appearance of the loaded structure. (2) It ensures that the geometry of the loaded structure corresponds to the theoretical shape used in the stress analysis.

To illustrate this procedure, consider the problem of cambering a truss. In this case, the truss members are fabricated so that they are either longer or shorter than their theoretical lengths. Since the members receive their maxi-

mum stress under different positions of the live load, trusses cannot be cambered so as to assume their theoretical shape when each member receives its maximum stress. As a practical compromise, trusses are usually cambered so that they attain their theoretical shape under dead load or under dead load plus some fraction of full live load over the entire structure.

To camber a truss exactly, we compute the change of length of each member under the stresses produced by the cambering load and then fabricate the compression members the corresponding amounts too long and the tension members the corresponding amounts too short. Then, when the truss is erected and subjected to its cambering load, it will deflect into its theoretical shape. The advantage of this exact method is that any truss so cambered can be assembled free of initial stress. The disadvantage is that most members are affected and the required changes in length are sometimes small and difficult to obtain.

The practical method of cambering trusses, therefore, is to alter only the lengths of the chord members. For example, if each top-chord member of an end-supported truss is lengthened $3/16$ in. for every 10 ft of its horizontal projection, this is equivalent to changing both top and bottom chords by one-half of this amount. This change in length is the same as that produced by a stress intensity of $29{,}000{,}000 \times 3/32 \times 1/120 = 22{,}600$ psi. Since only the chords are being corrected, they must be overcorrected to allow for the contribution of the web members to the deflection. Assuming that the chords contribute 80 per cent of the deflection, the change in length specified above corresponds to that caused by a load which would produce a chord stress intensity of

$$0.8 \times 22{,}600 = 18{,}000 \text{ psi}$$

In other words, the rule of thumb suggested above corresponds to a cambering load essentially equal to dead load plus full live and impact over the entire structure.

This approximate method of cambering may be applied without difficulty to statically determinate trusses. It must be applied with caution, however, to statically indeterminate trusses; otherwise, the assembled truss may have to be forced together, initial stresses thus being introduced into the truss.
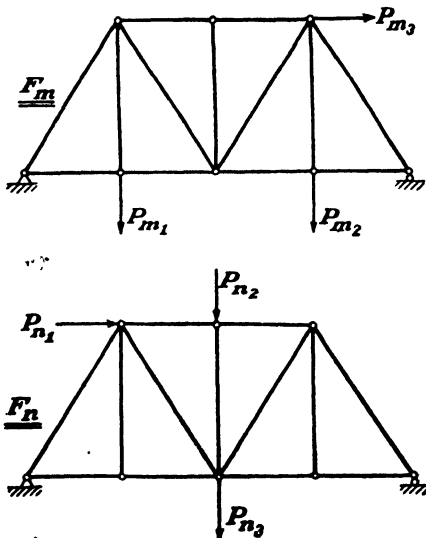


FIG. 13·30

## 13·19  Maxwell's Law of Reciprocal Deflections. Betti's Law.
Maxwell's law is a special case of the more general Betti's law. Both these laws are applicable to any type of structure, whether beam, truss, or frame. To simplify this discussion, however, these ideas will be

developed by considering the simple truss shown in Fig. 13·30.   Sup-
pose that this truss is subjected to two separate and independent systems
of forces, the system of forces $P_m$ and the system of forces $P_n$.   The $P_m$
system develops the bar stresses $F_m$ in the various members of the truss,
while the $P_n$ system develops the bar stresses $F_n$.   Let us imagine two
situations.   First, suppose that the $P_m$ system is at rest on the truss and
that we then further distort the truss by applying the $P_n$ system.   As a
second situation, suppose that just the reverse is true, *i.e.*, that the $P_n$
system is acting on the truss and that then we further distort the truss
by applying the $P_m$ system.   In both situations, we may apply the law
of virtual work and thereby come to a very useful conclusion known as
Betti's law.

   For purposes of this derivation, we shall assume that the supports of
the structure are unyielding and that the temperature is constant.
Also, let

$\delta_{mn}$ = the deflection of the point of application of one of the forces
        $P_m$ (in the direction and sense of this force) caused by the
        application of the $P_n$ force system

$\delta_{nm}$ = the deflection of the point of application of one of the forces
        $P_n$ caused by the application of the $P_m$ force system

Consider now the application of the law of virtual work to the first situa-
tion.   In this case, the $P_m$ force system is in the role of the $Q$ forces and
will be given a ride as a result of the distortion caused by the $P_n$ system.
Thus, applying Eq. (13·5),

$$\Sigma P_m \delta_{mn} = \Sigma F_m \Delta L$$

where $\Delta L = F_n L / AE$, and thus

$$\sum P_m \delta_{mn} = \sum F_m F_n \frac{L}{AE} \tag{a}$$

In the second situation, however, the $P_n$ force system will now be in
the role of the $Q$ forces and will be given a ride as a result of the distor-
tion caused by the $P_m$ system.   Then, applying Eq. 13·15,

$$\Sigma P_n \delta_{nm} = \Sigma F_n \Delta L$$

where $\Delta L = F_m L / AE$, and thus

$$\sum P_n \delta_{nm} = \sum F_n F_m \frac{L}{AE} \tag{b}$$

From Eqs. (*a*) and (*b*), it may be concluded that

$$\Sigma P_m \delta_{mn} = \Sigma P_n \delta_{nm} \qquad (13 \cdot 16)$$

which when stated in words is called *Betti's law.*

*In any structure the material of which is elastic and follows Hooke's law and in which the supports are unyielding and the temperature constant, the external virtual work done by a system of forces $P_m$ during the distortion caused by a system of forces $P_n$ is equal to the external virtual work done by the $P_n$ system during the distortion caused by the $P_m$ system.*
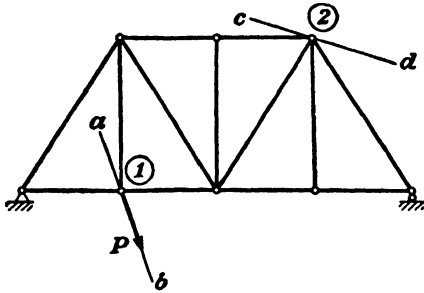
Betti's law is a very useful principle and is sometimes called the generalized Maxwell's law. This suggests that Maxwell's law of reciprocal deflections may be derived directly from Betti's law.
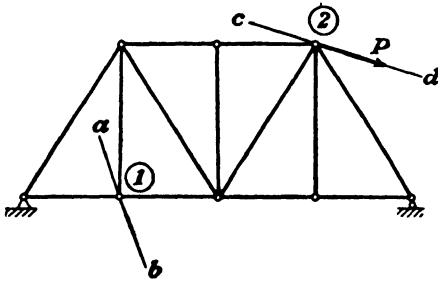
Consider any structure such as the truss shown in Fig. 13 31. Suppose that the truss is acted upon first by a load $P$ at point 1; then suppose that the truss is acted upon by a load of the same magnitude $P$ but applied now at point 2. Let

$\delta_{12}$ = deflection of point 1 in direction $ab$ due to a load $P$ acting at point 2 in direction $cd$

$\delta_{21}$ = deflection of point 2 in direction $cd$ due to a load $P$ acting at point 1 in direction $ab$



FIG. 13·31

Applying Betti's law to this situation,

$$(P)(\delta_{12}) = (P)(\delta_{21})$$

and therefore

$$\delta_{12} = \delta_{21} \qquad (13 \cdot 17)$$

which, when stated in words, is called *Maxwell's law of reciprocal deflections:*

*In any structure the material of which is elastic and follows Hooke's law and in which the supports are unyielding and the temperature constant, the deflection of point 1 in the direction ab due to a load P at point 2 acting in a*

*direction cd is numerically equal to the deflection of point 2 in the direction cd due to a load P at point 1 acting in a direction ab.*

Maxwell's law is perfectly general and is applicable to any structure. This reciprocal relationship exists likewise between the rotations pro-
duced by two couples and also between the deflection produced by a couple of $P$ and the rotation produced by a force $P$. This generality is illustrated by the beam in Fig. 13·32. That $\delta_{cb} = \delta_{bc}$ is a straight-forward application of Maxwell's law. Note also that the rotation $\alpha_{ac}$ in radians produced by a force of $P$ lb is numerically equal to the deflection $\delta_{ca}$ in feet produced by a couple of $P$ ft-lb. In the latter case, be careful of the units.
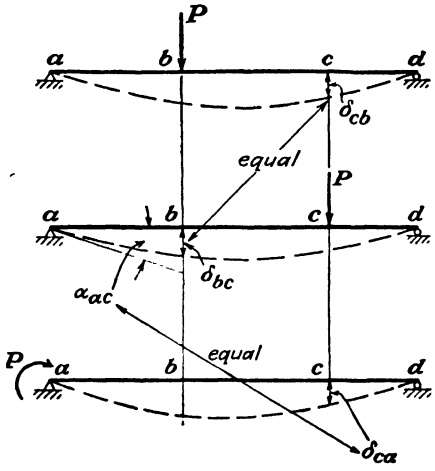
It is important to become familiar with the subscript notations used in the above discussion to



FIG. 13·32

denote the deflections. The first subscript denotes the point where the deflection is measured and the second subscript the point where the load causing the deflection is applied.

**13·20   Influence Lines for Deflections.**   Suppose that we wish to draw the influence line for the vertical deflection at point $a$ on the beam in Fig. 13·33. The ordinates of such an influence line may be computed and plotted by placing a unit vertical load successively at various points along the beam and in each case computing the resulting vertical deflection of point $a$. In this manner, when the unit load is placed at any point $m$ it produces a deflection $\delta_{am}$ at point $a$; or when placed at some other point $n$, it produces a deflection $\delta_{an}$ at point $a$. Note, however, the advantage of applying Maxwell's law to this problem. If we simply place a unit vertical load at point $a$, the deflections $\delta_{ma}$ and $\delta_{na}$ at points $m$ and $n$ will be equal, respectively, to $\delta_{am}$ and $\delta_{an}$ according to Maxwell's law. In other
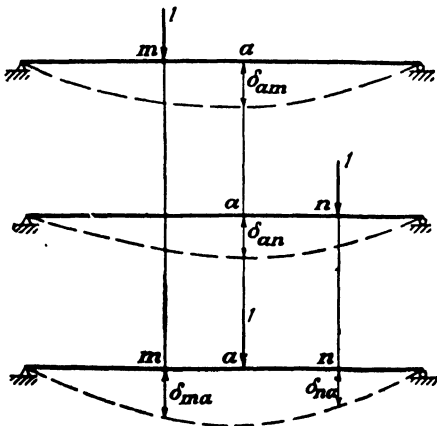


FIG .13·33

words, the elastic curve of the beam when the unit load is placed at point *a* is the influence line for the vertical deflection at point *a*. Thus, by computing the ordinates of the elastic curve for one simple problem we obtain the ordinates of the desired influence line.

This idea may be used to obtain the influence line for the deflection of any point on any structure. To obtain the influence line for the deflection of a certain point, simply place a unit load at that point, and compute the resulting elastic curve of the structure.

**13·21   Problems for Solution.**

**Problem 13·1**   Using the method of virtual work, compute the vertical component of the deflection of joint *d* due to the load shown, for the structure of **Fig. 13·34.** Bar areas in square inches are shown in parentheses.

$$E = 30 \times 10^3 \text{ kips per sq in.}$$
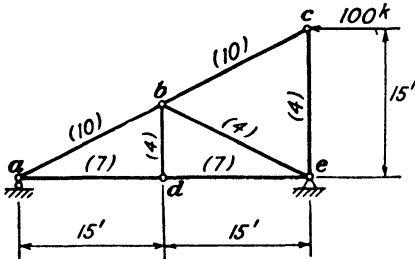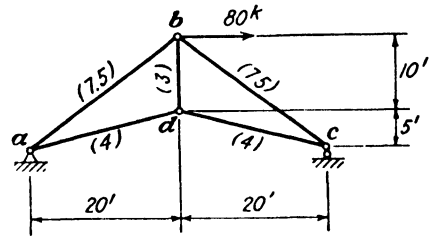


FIG. 13·34          FIG. 13·35

**Problem 13·2**   Using the method of virtual work, compute the horizontal component of the deflection of joint *c* due to the load shown, for the structure of **Fig. 13·35.** Bar areas in square inches are shown in parentheses.

$$E = 30 \times 10^3 \text{ kips per sq in.}$$

**Problem 13·3**   For the structure of Fig. 13·36, find the horizontal component of the deflection of joint *d* if bar *cf* is shortened 1 in.
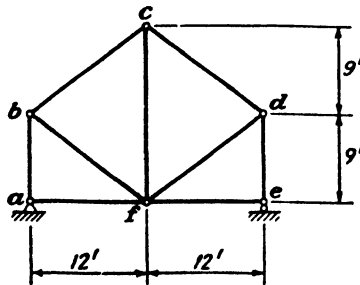


FIG. 13·36

**Problem 13·4**  For the structure of Fig. 13·37, using the method of virtual work, determine the relative movement of joints $M_3$ and $U_4$ along the line $M_3U_4$, (a) due to the loading shown; (b) due to a uniform increase in temperature of 50°F in the bottom chord.  Bar areas in square inches are shown in parentheses. $E = 30 \times 10^3$ kips per sq in.  $\epsilon = 1/150,000$ per °F.
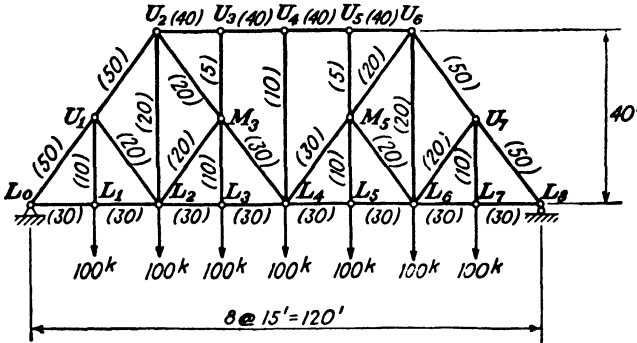


FIG. 13·37

**Problem 13·5**  For the structure of Fig. 13 38, using the method of virtual work, find the vertical deflection of point $a$ due to the load shown.  Cross-sectional areas of members in square inches are shown in parentheses.

$$E = 30 \times 10^3 \text{ kips per sq in.}$$
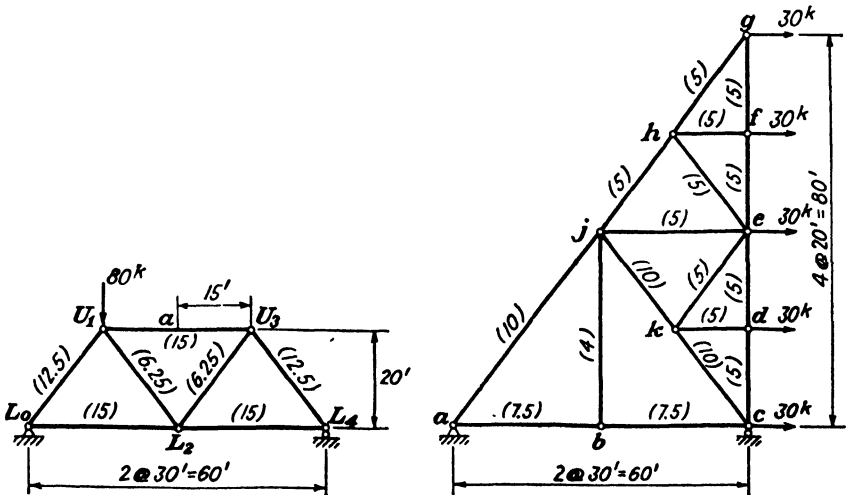


FIG. 13·38



FIG. 13·39

**Problem 13·6**  For the structure of Fig. 13·39, cross-sectional areas of members in square inches are shown in parentheses.  $E = 30 \times 10^3$ kips per sq in  Compute the angular rotation of member $bj$ due to the loads shown.

**Problem 13·7**  For the structure of Fig. 13·40, $E_1$ of member $gf = 20 \times 10^3$ kips per sq in., and $E_2$ of all the other members $= 30 \times 10^3$ kips per sq in. Cross-sectional areas in square inches are shown in parentheses. Using the method of virtual work,

*a.* Compute the vertical component of the deflection of point $c$ due to the load shown.

*b.* If a turnbuckle in member $gf$ were adjusted so as to shorten the member 0.5 in., what would be the vertical and horizontal components of the movement of point $c$ due only to this adjustment?
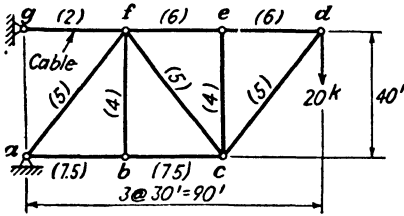
FIG. 13·40
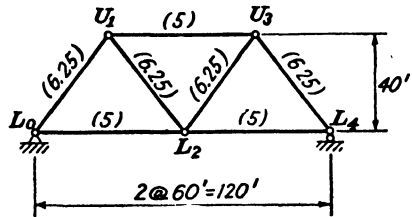
FIG. 13·41

**Problem 13·8**  Refer to Fig. 13·41.  During repairs to the right abutment of this truss, it was necessary to support the truss temporarily at joint $L_2$ on a hydraulic jack.  If the gross dead reaction at $L_4$ is 50 kips, compute the distance the jack must raise the truss at joint $L_2$ in order to free the support at $L_4$ and lift it 2 in. above its normal position.  Cross-sectional areas of bars in square inches are shown in parentheses.  $E = 30 \times 10^3$ kips per sq in.
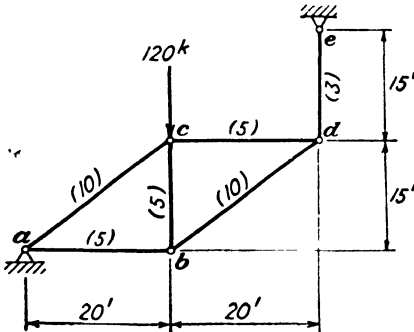
FIG. 13·42
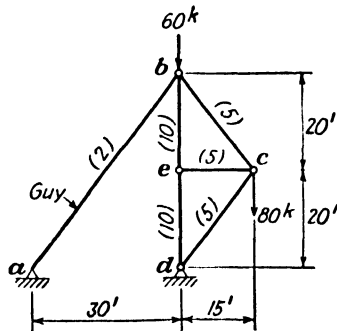
FIG. 13·43

**Problem 13·9**  Refer to Fig. 13·42.  Cross-sectional areas of members in square inches are shown in parentheses.  $E$ of member $ed = 20 \times 10^3$ kips per sq in. $= E_1$, and $E$ of all other members $= 30 \times 10^3$ kips per sq in. $= E_2$. Using the method of virtual work, determine the direction and magnitude of the resultant deflection of joint $d$ due to load shown.

**Problem 13·10**  For the structure of Fig. 13·43, the cross-sectional areas of members in square inches are shown in parentheses.  $E$ of guy $= 20 \times 10^3$

kips per sq in. = $E_1$, and $E$ of all other members = $30 \times 10^3$ kips per sq in. = $E_2$.

*a.* Compute the vertical component of the deflection of joint *c* due to the loads shown.

*b.* How much would the length of the guy *ab* have to be changed, by adjusting a turnbuckle in the guy, to return joint *c* to its undeflected vertical elevation?

**Problem 13·11**  Using the method of virtual work, determine the vertical deflection and the change in slope of the cross section at point *a* of the beam of Fig. 13·44.  *E* and *I* are both constant.
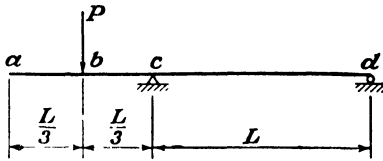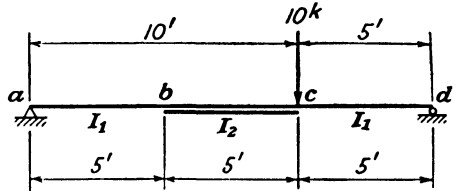


FIG. 13·44                    FIG. 13·45

**Problem 13·12**  For the girder of Fig. 13·45, using the method of virtual work,

*a.* Compute the vertical deflection of point *b* due to the load shown.

*b.* Compute the change in slope of the cross section at point *d* due to a uniformly distributed load of 2 kips per ft applied over the entire span.

$$E = 30 \times 10^3 \text{ kips per sq in., } I_1 = 300 \text{ in.}^4, I_2 = 500 \text{ in.}^4$$
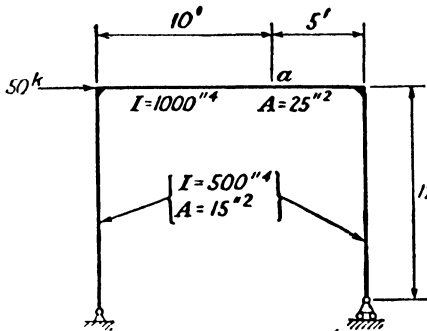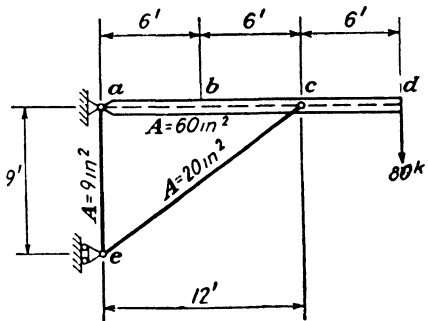


FIG. 13·46                    FIG. 13·47

**Problem 13·13**  Compute the vertical deflection of point *a* of the frame of Fig. 13·46, considering the effect of distortion due to both direct stress and bending.  $E = 30 \times 10^3$ kips per sq in.

**Problem 13·14**  For the structure of Fig. 13·47, *I* of member *ad* = 3,456 in.⁴, and $E = 30 \times 10^3$ kips per sq in.  Using the method of virtual work, compute the vertical component of the deflection of point *b*.

**Problem 13·15**  For the truss of Fig. 13·48, the cross-sectional areas of members in square inches are shown in parentheses.  $E = 30 \times 10^3$ kips per

sq in. Compute the relative deflection of point $a$ and joint $L_3$ along the line joining them, due to the loads shown.
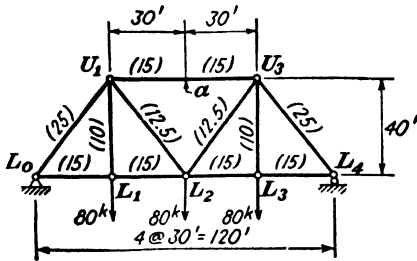


FIG. 13·48
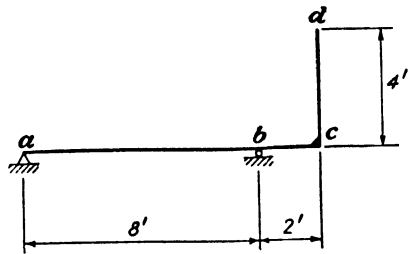


FIG. 13·49

**Problem 13·16** Refer to Fig. 13·49. Using the method of virtual work, compute the horizontal component of the deflection of point $d$ due to the following support movements:

*Point a:* Horizontal = 0.36 in. (to left)
    Vertical  = 0.48 in. (down)
*Point b:* Vertical  = 0.96 in. (down)

**Problem 13·17** Refer to Fig. 13·50. Using the moment-area method,

 *a.* Compute the vertical deflection of points $a$, $c$, and $d$ and the slope of the elastic curve at points $b$ and $c$ in terms of $E$ and $I$.

 *b.* If $E = 30 \times 10^3$ kips per sq in. and $I = 300$ in.⁴, compute from part $a$ the vertical deflection of $a$ in inches and the slope at $b$ in radians.
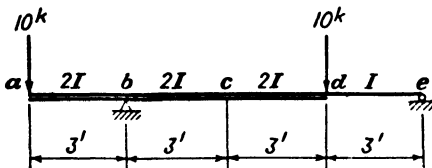


FIG. 13·50
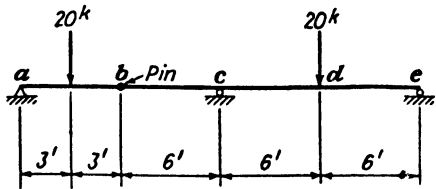


FIG. 13·51

**Problem 13·18** Refer to Fig. 13·51. Using the elastic-load procedure and the moment-area theorems,

 *a.* Find the vertical deflection at 3-ft intervals along the beam in terms of $E$ and $I$.

 *b.* Find the position and magnitude of the maximum vertical deflection in span $ce$.

$E$ and $I$ are constant.

**Problem 13·19** Refer to Fig. 13·52. Using the moment-area method, find, due to the load shown, the position and magnitude of the maximum vertical deflection in member $bc$. $E = 30 \times 10^3$ kips per sq in.
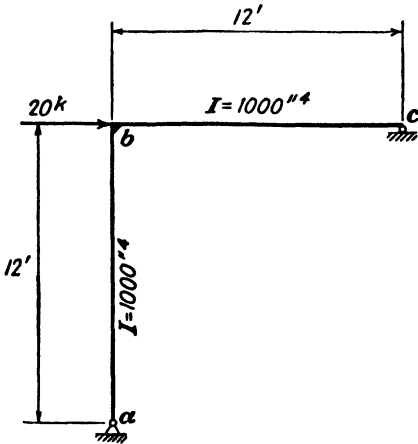
FIG. 13·52
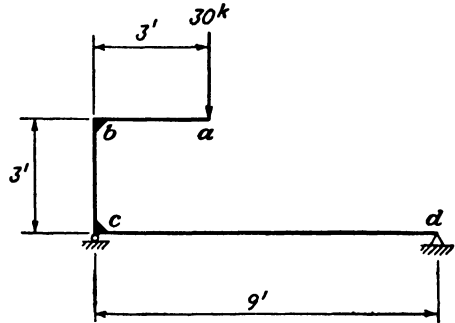


FIG. 13·53

**Problem 13·20**   Refer to Fig. 13·53. $E$ and $I$ are constant throughout, $E = 30 \times 10^3$ kips per sq in., and $I = 432$ in.⁴ Due to the load shown, compute the vertical component of the deflection of point $a$ on the bracket attached to this beam.

**Problem 13·21**   Refer to Fig. 13 54. $E$ and $I$ are constant, $E = 30 \times 10^3$ kips per sq in., and $I = 192$ in.⁴ Compute the location and magnitude of the maximum vertical deflection of the beam $ab$.
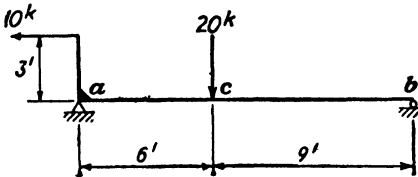


FIG. 13·54
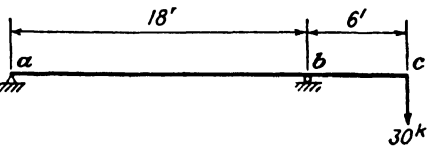


FIG. 13·55

**Problem 13·22**   Refer to Fig. 13·55. $E = 30 \times 10^3$ kips per sq in., and $I = 1,440$ in.⁴ If the support at $b$ settled downward ⅛ in., determine the location and magnitude of the maximum upward deflection in the portion of the beam $ab$ resulting from both the load shown and the settlement.
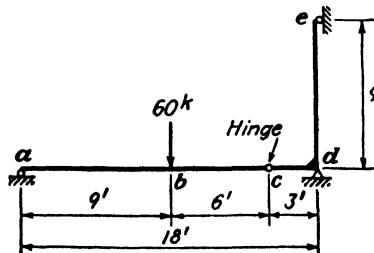


FIG. 13·56

**Problem 13·23**   Refer to Fig. 13·56.   Compute the position and magnitude of the maximum vertical deflection of this structure, using the moment-area theorems or their elastic-load adaptation.   $E$ and $I$ are constant, $E = 30 \times 10^3$ kips per sq in., $I = 1,200$ in.[4]

**Problem 13·24**   Refer to Fig. 13 57.   $E = 30 \times 10^3$ kips per sq in., and $I = 576$ in.[4]   Compute the maximum vertical deflection of this beam, using the moment-area theorems or their elastic-load adaptation.
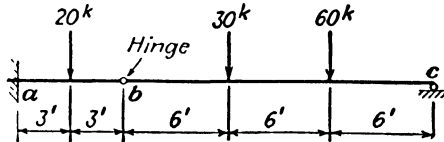


FIG. 13·57

**Problem 13·25**   Refer to Fig. 13 58.   Compute the position and magnitude of the maximum vertical deflection in this beam.   $E$ and $I$ are constant, $E = 30 \times 10^3$ kips per sq in., $I = 432$ in.[4]
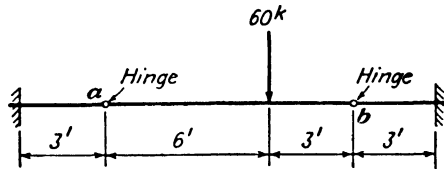


FIG. 13·58

**Problem 13·26**   Refer to Fig. 13·59.   Using the moment-area theorems or their elastic-load adaptation, compute the position and magnitude of .the maximum vertical deflection in the portion *bc* of this structure.   $E$ and $I$ are constant, $E = 30 \times 10^3$ kips per sq in., $I = 1,440$ in.[4]
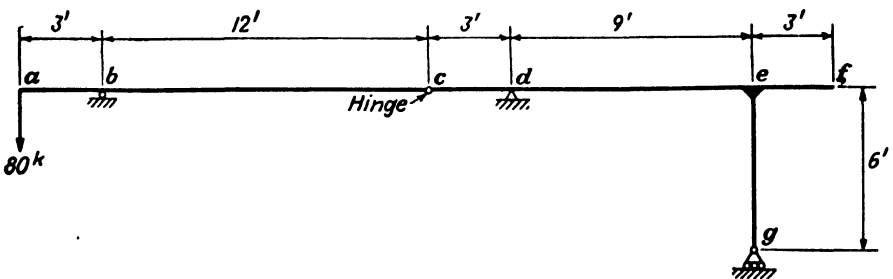


FIG. 13·59

**Problem 13·27**   Refer to Fig. 13·60.   Assuming that the couple of 120 kip-ft applied at *b* causes the moment reaction of 60 kip-ft acting as shown at
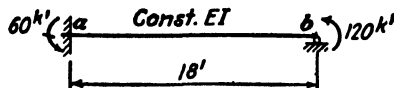


FIG. 13·60

support $a$, compute the position and magnitude of the maximum vertical deflection of this beam.  $E = 30 \times 10^3$ kips per sq in., and $I = 576$ in.[4]
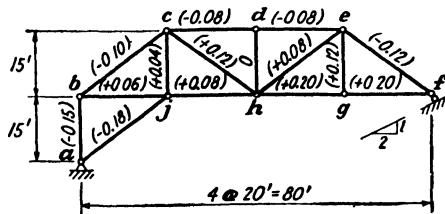


FIG. 13·61

**Problem 13·28** Refer to Fig. 13·61. Find the horizontal and vertical components of the deflection of the joints of this truss, using the Williot-Mohr method. Construct the Williot diagram, assuming point $h$ fixed in position and bar $hd$ in direction. Record the results in inch units. Use a scale of 1 in. = 0.20 in. and place point $h'$ in the lower right-hand corner of the page about 1 in. from the bottom and 3 in. from the right-hand edge. $\Delta L$'s of bars in inches are shown in parentheses.

**Problem 13·29** Using the Williot-Mohr method, find the horizontal and vertical components of all the joints of the trusses in the following problems: (a) Prob. 13 6; (b) Prob. 13 7; (c) Prob. 13·4; (d) Prob. 13·9; (e) Prob. 13 10.

**Problem 13·30** Using the bar-chain method, compute (a) the vertical deflection of the bottom-chord panel points of the truss of Prob. 13 15; (b) the vertical deflection of the top-chord panel points of the truss of Prob. 13·7; (c) the horizontal deflection of panel points $c$, $d$, $e$, $f$, and $g$ of the truss of Prob. 13 6; (d) the horizontal deflection of panel points $a$, $j$, $h$, and $g$ of the truss of Prob 13 6; (e) the vertical deflection of the joints of the bar chain $L_0 U_2 U_6 L_8$ of the truss of Prob. 13 4; (f) the vertical deflection of the lower-chord panel points of the three-hinged arch of Example 13·21.

# CHAPTER 14

## STRESS ANALYSIS
## OF INDETERMINATE STRUCTURES

**14·1 Introduction.** In the last 25 years, statically indeterminate structures have been used more and more extensively. This is no doubt due to their economy and increased rigidity under moving or movable loads. The details of reinforced-concrete and welded construction are such that structures of these types are usually wholly or partly continuous in their structural action and are therefore usually statically indeterminate. A knowledge of the analysis of indeterminate structures has thus become increasingly important as the use of these types of construction has become more extensive. Typical examples of indeterminate structures are continuous beams and trusses, two-hinged and hingeless arches, rigid-frame bridges, suspension bridges, and building frames.

Statically indeterminate structures differ from statically determinate ones in several important respects, *viz.:*

1. Their stress analysis involves not only their geometry but also their elastic properties such as modulus of elasticity, cross-sectional area, and moment of inertia. Thus, to arrive at the final design of an indeterminate structure involves assuming preliminary sizes for the members, making a stress analysis of this design, redesigning the members for these stresses, making a new stress analysis of the revised design, redesigning again, reanalyzing, etc., until one converges on the final design.

2. In general, stresses are developed in indeterminate structures, not only by loads, but also by temperature changes, support settlements, fabrication errors, etc.

In order to understand the stress analysis of indeterminate structures, it is imperative to understand first the fundamental difference between an unstable, a statically determinate, and a statically indeterminate structure. For this reason, a short review of certain fundamentals is justified. Suppose that bar *AB* is supported only by a hinge support at *A* as shown in Fig. 14·1a. If the 10-kip load shown is then applied to the bar, it will obviously rotate freely about the hinge at *A*. A bar supported in this manner will therefore be an *unstable structure*. By means of an additional roller support at *B*, however, as shown in Fig. 14·1b, we obviously prevent rotation about *A* and therefore have arranged the structure so that it is *stable;* by inspection it is also a *statically determinate structure.*

If we add still another roller support at $C$, as shown in Fig. 14·1c, we then have more reactions than the minimum required for static equilibrium and no longer have a statically determinate structure but, instead, a *statically indeterminate structure*.  With this new arrangement, not only do we prevent the bar from translating or rotating as a rigid body, but also by the addition of the roller support at $C$ we prevent this point from deflecting vertically.



(a)

(b)

(c)

FIG. 14·1

This observation, however, immediately suggests one method by which we compute the vertical reaction at $C$.  If the support at $C$ is temporarily removed, suppose that the 10-kip load acting on the beam supported simply by the remaining supports causes a downward deflection of 3 in. at $C$, as shown in Fig. 14·2a.  If the support at $C$ is now reestablished, it will have to provide an upward reaction in order to return point $C$ to a position of zero deflection.  Just how much this upward reaction will have to be may be determined by first finding how much a 1-kip load will deflect point $C$.  If a 1-kip load acting at point $C$ on the beam supported at $A$ and $B$ deflected this point $\frac{1}{2}$ in. as shown in Fig. 14·2b,



(a)

(b)

(c)
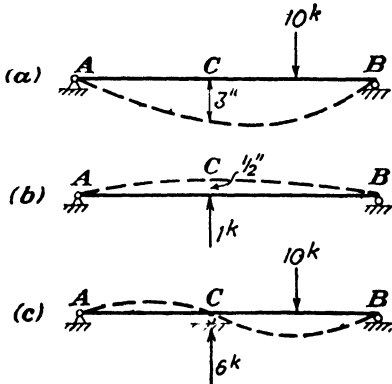
FIG. 14·2

then an upward force of 6 kips at point $C$, together with the 10-kip load, will produce a net deflection of zero at this point as shown in Fig. 14·2c.  The vertical reaction at point $C$ of the beam shown in Fig. 14·1c must therefore be 6 kips acting upward.  Once this reaction has been determined, the remaining reactions at $A$ and $B$ may easily be computed by statics.

**14·2  Use of Superposition Equations in Analysis of Indeterminate Structures.**  Many simple indeterminate structures may be analyzed in the manner just discussed.  Such an informal approach is confusing, however, in the case of more complicated structures.  It is therefore desirable to develop a more formal and orderly procedure to facilitate the solution of the more complicated problems and also to handle the simple problems with maximum efficiency.

The ideas and philosophy behind a more orderly approach may be
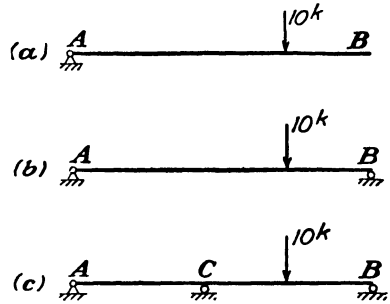
developed by considering a specific example such as the indeterminate beam shown in Fig. 14·3, the supports of which are unyielding. This beam is statically indeterminate to the first degree, *i.e.*, there is one more reaction component than the minimum necessary for static equilibrium. One of the reaction components may be considered as being extra, or *redundant.* In this case, consider the vertical reaction at $b$ as being the redundant reaction.

Suppose that we remove the vertical support at $b$ and replace it by the force $X_b$ which it supplies to the actual structure. We then have left a cantilever beam acted upon by the applied load and the unknown redundant force $X_b$. This statically determinate and stable cantilever beam that remains after the removal of the redundant support is called the *primary structure.*
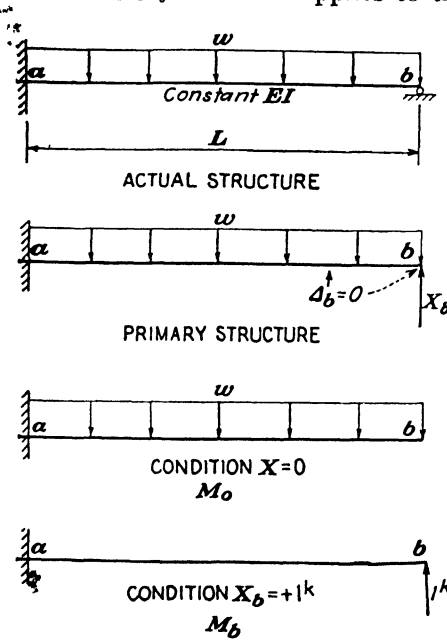


ACTUAL STRUCTURE

PRIMARY STRUCTURE

CONDITION $X=0$
$M_0$

CONDITION $X_b = +1^k$
$M_b$

Fig. 14·3

If the redundant force $X_b$ acting on the primary structure has the same value as the vertical reaction at $b$ on the actual structure, then the shear and bending moment at any point and the reactions at point $a$ are the same for the two structures. If the conditions of stress in the actual structure and the primary structure are the same, then the conditions of distortion of the two structures must also be exactly the same. If the conditions of distortion of the two structures are the same and the deflections of the supports at point $a$ on each are also identical, then the deflections at any other corresponding points must be the same. Therefore, since there is no vertical deflection of the support at $b$ on the actual structure, the vertical deflection of point $b$ on the primary structure, due to the combined action of the applied load and $X_b$, must also be equal to zero.

It is possible, however, to express this latter statement mathematically and thus obtain an equation from which the value of the unknown redundant $X_b$ may be determined. Assuming the positive direction of the redundant $X_b$ to be upward, let us introduce the following notation:

Let $\Delta_b$ = upward deflection of point $b$ on the primary structure, due to all causes

$\quad\Delta_{bo}$ = upward deflection of point $b$ on the primary structure, due only to the applied load with the redundant removed, hereafter called "condition $X = 0$" (see Fig. 14·3)

$\quad\Delta_{bb}$ = upward deflection of point $b$ on the primary structure, due only to redundant $X_b$

It is impossible to compute $\Delta_{bb}$ until the magnitude of $X_b$ is known. If, however, we also let

$\quad\delta_{bb}$ = upward deflection of point $b$ on the primary structure, due only to a unit upward load at $b$, hereafter called "condition $X_b = +1$ kip" (see Fig. 14·3)

then we can say, as long as the principle of superposition is valid, that $X_b$ will cause a deflection which is $X_b$ times that produced by a unit value of $X_b$, or

$$\Delta_{bb} = X_b\delta_{bb} \qquad\qquad (a)$$

It is a physical fact, however, that $\Delta_b$, the total deflection due to all causes, is equal to the superposition of the contributions of the separate effects, *viz.*, the applied load and the redundant $X_b$. Thus,

$$\Delta_b = \Delta_{bo} + \Delta_{bb} \qquad\qquad (b)$$

or, substituting from Eq. $(a)$,

$$\Delta_b = \Delta_{bo} + X_b\delta_{bb} \qquad\qquad (14\cdot1)$$

This equation is called a *superposition equation* for the deflection of point $b$ on the primary structure.

Since $\Delta_b$ must be zero, we may solve Eq. (14·1) for $X_b$ and obtain

$$X_b = -\frac{\Delta_{bo}}{\delta_{bb}} \qquad\qquad (14\cdot2)$$

It is a simple matter, of course, to compute the numerical value of $\Delta_{bo}$ and $\delta_{bb}$ by any of the methods available for computing beam deflections. In substituting these values in Eq. (14·2), they should be considered plus when the deflections are upward as specified above. A plus value for $X_b$ will then indicate that it acts upward; a minus value, that it acts downward. The organization of the numerical computations in problems such as this is illustrated by the examples in Art. 14·4.

### 14·3   General Discussion of Use of Superposition Equations in Analysis of Indeterminate Structures.   The method of attack sug-

gested in Art. 14·2 is the most general available for the stress analysis of indeterminate structures.   There are other methods, of course, that are definitely superior for certain specific types of structures, but there is no other one method that is as flexible and general as the method based on using superposition equations.   It is applicable to any structure, whether beam, frame, or truss or any combination of these simple types. It is applicable whether the structure is being analyzed for the effect of loads, temperature changes, support settlements, fabrication errors, or any other cause.

There is only one restriction on the use of this approach, *viz.*, that the principle of superposition is valid.   From the discussion in Art. 2·12, it will be recalled that the principle of superposition is valid unless the geometry of the structure changes an essential amount during the application of the loads or the material does not follow Hooke's law. All the other methods discussed here are likewise subject to this same limitation, however.

Before presenting numerical examples illustrating the application of this method to certain typical problems, it is desirable to emphasize and extend the ideas suggested in Art. 14·2.   The student may find that he has difficulty understanding the somewhat generalized discussion that follows.   If he will read this article, however, absorbing what he can, and then study the examples in Arts. 14·4 and 14·5, he should then be able to understand the remainder of this article when he rereads it.

Suppose that we wish to analyze some particular indeterminate structure for any or all of the various causes which may produce stress in the structure.   The structure may be of any type or statically indeterminate to any degree.   In any case, the very first step in the analysis is to determine the degree of indeterminacy.   Suppose the structure is indeterminate to the $n$th degree.   We then select $n$ redundant restraints; remove them from the structure; and replace them by the $n$ redundant stress components that they supply to the structure.   All these $n$ redundants $X_a, X_b, \ldots, X_n$ will then be acting together with the applied loads on the *primary structure* that remains after the removal of the $n$ restraints.

A wide choice of redundants is available in most problems, but it is usually possible to minimize the numerical calculations by choosing them judiciously.   Some of the principles to follow in selecting the redundants are discussed in Art. 14·6.   Suffice it to say for the present that the redundants must be selected so that the resulting primary structure is stable and statically determinate.[1]

---

[1] Occasionally it is advantageous to select a statically indeterminate and stable primary structure (see Art. 15·3).

Having selected the redundants, we then may reason that, if the redundant forces acting on the primary structure have the same values as the forces supplied by the corresponding restraints on the actual structure, then the entire conditions of stress in the actual and primary structure are the same.  As a result, the conditions of distortion of the two structures are the same.  If, in addition, the deflections of the supports of the primary structure are the same as the deflections of the corresponding supports of the actual structure, then the deflections of any other corresponding points on the two structures must be identical. It may be concluded, therefore, that the deflections of the points of application of the $n$ redundants on the primary structure must be equal to the specified deflections of the corresponding points on the actual structure.

If $n$ superposition equations are written for these $n$ deflections, they will involve among them all the $n$ redundants as unknowns.  Since all these $n$ deflection equations must be satisfied simultaneously by the $n$ unknowns, simultaneous solution of these equations leads to the values of the $n$ unknown redundants.

In applying this method, it is convenient to have a definite notation for the various deflection terms involved in the solution.  The total deflection of a point $m$ on the primary structure, due to all causes, is denoted by $\Delta_m$.  The various supplementary deflections contributing to this total are identified, however, by two subscripts, the first of which denotes the point at which the deflection occurs and the second of which denotes the loading condition which causes that particular contribution. Thus, the deflection of a point $m$ produced by certain typical causes would be denoted as follows:

$\Delta_m$ = total deflection of point $m$, due to all causes
$\Delta_{mo}$ = deflection of point $m$, due to condition $X = 0$
$\Delta_{mT}$ = deflection of point $m$, due to change in temperature
$\Delta_{mS}$ = deflection of point $m$, due to settlement of supports of primary structure
$\Delta_{mE}$ = deflection of point $m$, due to fabrication errors
$\delta_{ma}$ = deflection of point $m$, due to condition $X_a = +1$ kip
$\delta_{mb}$ = deflection of point $m$, due to condition $X_b = +1$ kip
$\delta_{mm}$ = deflection of point $m$, due to condition $X_m = +1$ kip
Etc.

Note that the deflections produced by unit values of the various redundants are denoted by a small delta ($\delta$).

Any redundant may be arbitrarily assumed to act in a certain sense, thus establishing the positive sense of that redundant.  Any deflection

of the point of application of that redundant should then be measured along its line of action and should be considered positive when in the same sense as that assumed for the redundant.

Thus, using the above notation and sign convention, we have the following form for the $n$ superposition equations involving the $n$ redundants—one equation for the known total deflection of the point of application of each of the $n$ redundants:

$$\left.\begin{aligned}
\Delta_a &= \Delta_{ao} + \Delta_{aT} + \Delta_{aS} + \Delta_{aE} + X_a \delta_{aa} + X_b \delta_{ab} + \cdots + X_n \delta_{an} \\
\Delta_b &= \Delta_{bo} + \Delta_{bT} + \Delta_{bS} + \Delta_{bE} + X_a \delta_{ba} + X_b \delta_{bb} + \cdots + X_n \delta_{bn} \\
&\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
\Delta_n &= \Delta_{no} + \Delta_{nT} + \Delta_{nS} + \Delta_{nE} + X_a \delta_{na} + X_b \delta_{nb} + \cdots + X_r \delta_{nn}
\end{aligned}\right\} \quad (14 \cdot 3)$$

If the known values of $\Delta_a$, $\Delta_b$, . . . , $\Delta_n$ are substituted and the various deflection terms on the right-hand side of these equations are evaluated by *any* suitable method, then the $n$ equations may be solved simultaneously[1] for the value of the redundants $X_a$, $X_b$, etc.

Note that the values of $X_a$, $X_b$, etc., obtained from this solution are dimensionless. This is obviously so, since $X_a$, $X_b$, etc., must be pure numbers in order that all the terms in each of Eqs. (14·3) may have the same deflection units. In order to assign units to these values for $X_a$, $X_b$, etc., it is necessary to note the units of the unit values of these redundants that were used in the computation of $\delta_{aa}$, $\delta_{ab}$, etc.

Note further that the redundants selected may be forces or couples. The deflections designated by $\Delta$ or $\delta$ may therefore represent a linear or angular deflection depending on whether they represent the deflection of the point of application of a force or a couple. In the examples that follow, the positive direction of any particular deflection is indicated by the small arrow appended to the symbol, as $\Delta_a^{\uparrow}$, $\Delta_b^{\downarrow}$, $\Delta_c^{(\rightarrow \leftarrow)}$, etc.

**14·4  Examples Illustrating Stress Analysis Using Superposition Equations.** The following examples illustrate the application of the superposition-equation approach to the stress analysis of typical indeterminate structures acted upon by specified external loads.

After determining the degree of indeterminacy, one has considerable latitude in selecting the redundants. However, never make the mistake of selecting a statically determinate reaction component, bar stress, shear, or moment, as a redundant. If such a quantity is statically determinate, it is necessary for the stability of the structure; to remove it would leave an unstable primary structure. Such an error will never go undetected, however, since an attempted stress analysis of the primary structure would lead to impossible or inconsistent results.

[1] A convenient tabular method for solving simultaneous equations is discussed in connection with Example 14·22.

The reader should consider alternate selections of the redundants in the following examples and try to decide whether in each case there is some other selection involving less numerical calculation than those used. The question of selection of redundants is discussed further in Art. 14·6.

The positive sense of a redundant may be chosen arbitrarily. The deflection of the point of application of this redundant should likewise be considered positive when in the same sense. Consideration of any of these examples should make it apparent that the final interpretation of the direction of the redundants will not be affected by the initial choice of its positive sense.

In applying the superposition approach, any suitable method may be used to compute the various deflections involved. Of course, as in any deflection problem, care must be exercised in handling units and signs. Note again that the values of $X_a$, $X_b$, etc., obtained from these solutions are simply numerical values, $i.e.$, that they are dimensionless. Such numerical values indicate, for example, that $X_a$ is, say, 10.5 times larger than the unit load in condition $X_a = +1$ kip. Thus, if this is a 1-kip unit load, $X_a$ should be considered as 10.5 kips in stating the final results or in using it in subsequent computations.

The units will always be consistent if the same force and distance units are used throughout a given problem. If, however, we mix up feet, inches, pounds, and kips in problems involving more than one type of distortion, we may run into difficulty. Similar difficulty will be encountered if units are mixed in dealing with problems involving temperature changes, support settlements, etc., such as are discussed in Art. 14·5.

It should be noted that $\Delta_n$, the total deflection of the point of application of a redundant $X_n$ due to all causes, is almost always equal to zero. In fact, the only case in which $\Delta_n$ would have a value other than zero would be when $X_n$ represented a redundant reactive force or couple at a support on the actual structure, which underwent a movement in the direction of $X_n$.

**Example 14·1**   *Compute the reactions of this beam.   E and I constant.*

Structure is indeterminate to first degree.
Select $X_b$ as redundant.
Then

$$\Delta_b^\uparrow = \Delta_{bo} + X_b\delta_{bb} = 0$$

*Evaluate $\Delta_{bo}$ and $\delta_{bb}$ by method of virtual work.*

$$\sum Q\delta = \sum \int M_Q M_P \frac{ds}{EI}$$

$\Delta_{bo}$:

$$M_Q = M_b \qquad M_P = M_o,$$

$$(1^k)(\Delta_{bo}^\uparrow) = \int M_b M_o \frac{ds}{EI}$$

*From b to a,*

$$M_b = x, \qquad M_o = -x^2$$

$$(1^k)(\Delta_{bo}^\uparrow) = \int_0^{20} (x)(-x^2) \frac{dx}{EI},$$

$$\Delta_{bo} = \frac{-40,000^{k'^3}}{EI}$$

$\delta_{bb}$:

$$M_Q = M_b, \qquad M_P = M_b,$$

$$(1^k)(\delta_{bb}^\uparrow) = \int M_b^2 \frac{dx}{EI}$$

$$(1^k)(\delta_{bb}^\uparrow) = \int_0^{20} (x)^2 \frac{dx}{EI}, \qquad \delta_{bb} = \frac{2,667^{k'^3}}{EI}$$

*Then*

$$\frac{-40,000}{EI} + \frac{2,667}{EI} X_b = 0$$

$$\therefore X_b = +15 \qquad \therefore upward$$

*Then, by statics, the reactions at a are found
to be as shown.*
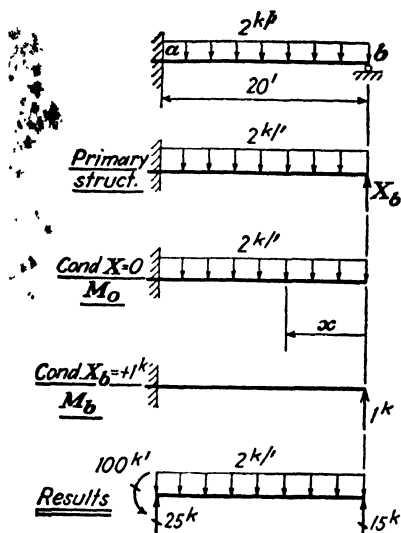
**Example 14·2** *Compute the reactions and draw bending-moment diagram for this beam. E and I constant.*

Structure is indeterminate to first degree. Select $X_c$ as redundant.
Then,

$$\Delta_c^{\uparrow} = \Delta_{co} + X_c\delta_{cc} = 0$$

Evaluate $\Delta_{co}$ and $\delta_{cc}$ by the elastic-load method.

$\Delta_{co}$: Compute first the reaction at $b$ on imaginary beam.

$$216 \times 3 = 648 \times 4 = 2,592$$
$$216 \times 4.5 = 972 \times 9 = 8,748$$
$$\frac{11,340}{15} = 756$$

Then,

$$EI\Delta_{co} = 756 \times 18 = 13,608\uparrow$$

$$\therefore \Delta_{co} = \frac{+13,608''}{EI}$$

$\delta_{cc}$: Compute first the reaction at $b$ on imaginary beam.

$$\tfrac{2}{3} \times 18 \times 7.5 = 90$$

Then,

$$EI\delta_{cc} = 90 \times 18 + 18 \times 9 \times 12$$
$$= 3,564\uparrow$$

$$\therefore \delta_{cc} = \frac{+3564^{k'^3}}{EI}$$

Then,

$$\frac{13,608}{EI} + \frac{3,564}{EI}X_c = 0$$

$$\therefore X_c = -3.82 \qquad \therefore down$$

The reaction at $a$ and $b$ can now be computed easily by statics. Likewise, the BM diagram may be constructed without difficulty.
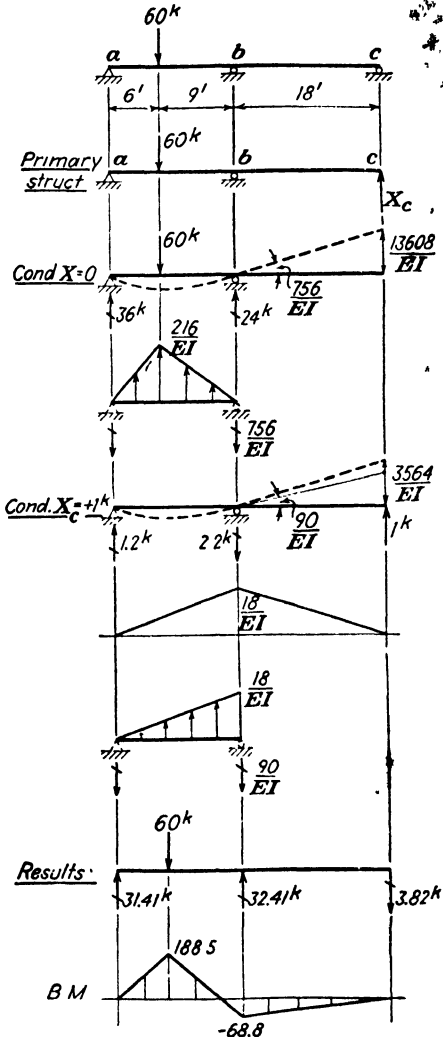
**Example 14·3**   *Compute the reactions and bar stresses for this truss. Cross-sectional areas in square inches are shown in parentheses.* $E = 30 \times 10^3$ *kips per sq in.*



*Indeterminate to first degree. Select $X_b$ as redundant. Then,*

$$\Delta_b^{\uparrow} = \Delta_{bo} + X_b \delta_{bb} = 0$$

*Evaluate $\Delta_{bo}$ and $\delta_{bb}$ by the method of virtual work.*

$$\sum Q\delta = \sum F_Q \, \Delta L = \sum F_Q F_P \frac{L}{AE}$$

$\Delta_{bo}$: $\quad F_Q = F_b, \qquad F_P = F_o, \qquad (1^k)(\Delta_{bo}^{\uparrow}) = \frac{1}{E}\sum F_b F_o \frac{L}{A}$

$\delta_{bb}$: $\quad F_Q = F_b, \qquad F_P = F_b, \qquad (1^k)(\delta_{bb}^{\uparrow}) = \frac{1}{E}\sum F_b^2 \frac{L}{A}$
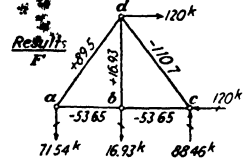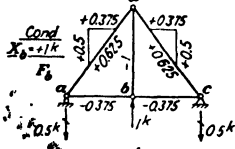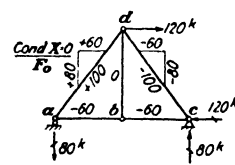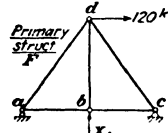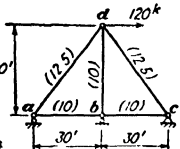
| Bar | $L$ | $A$ | $\dfrac{L}{A}$ | $F_o$ | $F_b$ | $F_b F_o \dfrac{L}{A}$ | $F_b^2 \dfrac{L}{A}$ | $X_b F_b$ | $F$ |
|---|---|---|---|---|---|---|---|---|---|
| Units | $'$ | $''^2$ | $'/''^2$ | $k$ | $k$ | $k^2'/''^2$ | $k^2'/''^2$ | $k$ | $k$ |
| ab | 30 | 10 | 3 | $-60$ | $-0.375$ | $+67.5$ | $+0.422$ | $+6.35$ | $-53.65$ |
| bc | 30 | 10 | 3 | $-60$ | $-0.375$ | $+67.5$ | $+0.422$ | $+6.35$ | $-53.65$ |
| ad | 50 | 12.5 | 4 | $+100$ | $+0.625$ | $+250$ | $+1.563$ | $-10.6$ | $+89.4$ |
| dc | 50 | 12.5 | 4 | $-100$ | $+0.625$ | $-250$ | $+1.563$ | $-10.6$ | $-110.6$ |
| bd | 40 | 10 | 4 | 0 | $-1.0$ | 0 | $+4.0$ | $+16.93$ | $+16.93$ |
| $\Sigma$ | | | | . . | . . | $+135$ | $+7.970$ | | |

$$(1^k)(\Delta_{bo}) = \frac{+135^{k^2'/''^2}}{E}, \qquad \Delta_{bo} = \frac{135^{k'/''^2}}{E}$$

$$(1^k)(\delta_{bb}) = \frac{+7.97^{k^2'/''^2}}{E}, \qquad \delta_{bb} = \frac{7.97^{k'/''^2}}{E}$$

$$\therefore \frac{135}{E} + \frac{7.97}{E} X_b = 0, \qquad X_b = -16.93 \qquad \therefore \text{ down}$$

*The remaining reactions and the bar stresses can now be computed by statics. The bar stresses could likewise be computed easily in a tabular manner by noting that by superposition*

$$F = F_o + X_b F_b$$

**Example 14·4**  *Compute the reactions of this truss.   Cross-sectional areas in square inches are shown in parentheses.*   $E = 30 \times 10^3$ *kips per sq in.*
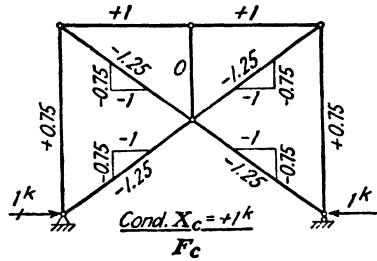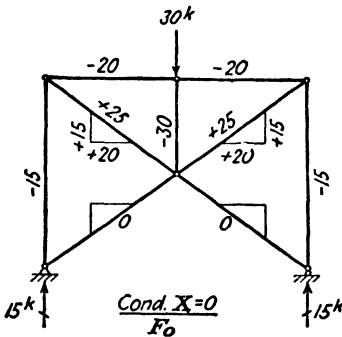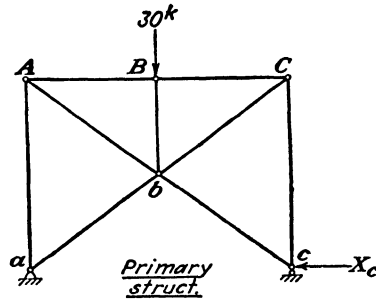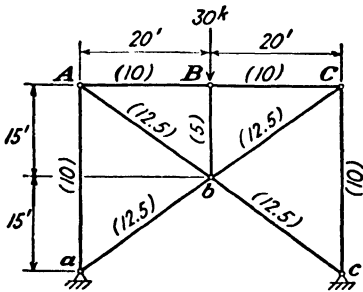


Indeterminate to first degree.   Select $X_c$ as redundant.

$$\Delta_c^{\leftarrow} = \Delta_{co} + X_c \delta_{cc} = 0$$

Evaluate $\Delta_{co}$ and $\delta_{cc}$ by the method of virtual work.

$\Delta_{co}$:   $\qquad F_Q = F_c, \qquad F_P = F_o, \qquad (1^k)(\Delta_{co}^{\leftarrow}) = \dfrac{1}{E}\sum F_c F_o \dfrac{L}{A}$

$\delta_{cc}$:   $\qquad F_Q = F_c, \qquad F_P = F_c, \qquad (1^k)(\delta_{cc}^{\leftarrow}) = \dfrac{1}{E}\sum F_c^2 \dfrac{L}{A}$

Owing to symmetry of the load, structure, and redundant, only one-half of the bars have to be included in the tabulation.

| Bar | L | A | $\dfrac{L}{A}$ | $F_o$ | $F_c$ | $F_oF_c\dfrac{L}{A}$ | $F_c^2\dfrac{L}{A}$ |
|------|------|------|------|------|------|------|------|
| Units | $'$ | $''^2$ | $'/''^2$ | $k$ | $k$ | $k^2 '/''^2$ | $k^2 '/''^2$ |
| $AB$ | 20 | 10 | 2 | $-20$ | $+1.0$ | $-40$ | $+2.0$ |
| $ab$ | 25 | 12 5 | 2 | 0 | $-1\ 25$ | 0 | $+3.125$ |
| $Aa$ | 30 | 10 | 3 | $-15$ | $+0.75$ | $-33\ 75$ | $+1.688$ |
| $Ab$ | 25 | 12 5 | 2 | $+25$ | $-1.25$ | $-62.5$ | $+3.125$ |
| $\tfrac{1}{2}Bb$ | 7.5 | 5 | 1 5 | $-30$ | 0 | 0 | 0 |
| $\tfrac{1}{2}\Sigma$ | .... | .... | .... | .... | ..... | $-136.25$ | $+9.938$ |

$$\therefore \Delta_{co} = \frac{-272.5^{k'/''^2}}{E}, \qquad \delta_{cc} = \frac{19.875^{k'/''^2}}{E}$$

$$-\frac{272.5}{E} + \frac{19.875}{E} X_c = 0$$

$$\therefore X_c = +13.7 \qquad \therefore \leftarrow\!\!\rightarrow$$

Results:
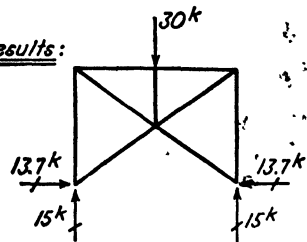
**Example 14·5** *Compute the bar stresses in the members of this truss. E and A are constant for all members.*
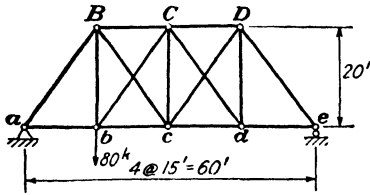


This truss is statically determinate externally, but indeterminate to the second degree with respect to its bar stresses. Cut bars $bC$ and $Cd$, and select their bar stresses as the redundants. Then

$$\Delta_a = \Delta_{ao} + X_a\delta_{aa} + X_b\delta_{ab} = 0 \quad (1)$$

$$\Delta_b = \Delta_{bo} + X_a\delta_{ba} + X_b\delta_{bb} = 0 \quad (2)$$
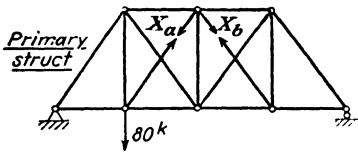
By Maxwell's law, $\delta_{ab} = \delta_{ba}$, and by symmetry of the structure and in the selection of the redundants $\delta_{aa} = \delta_{bb}$. Therefore, only four deflections need to be computed: $\Delta_{ao}$, $\Delta_{bo}$, $\delta_{ab}$, and $\delta_{aa}$. Using the method of virtual work, we find:
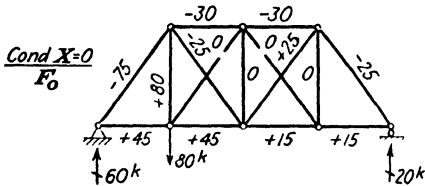
$$(1^k)(EA\Delta_{ao}) = \Sigma F_a F_o L = -2040^{k^2\prime}$$
$$(1^k)(EA\Delta_{bo}) = \Sigma F_b F_o L = +760^{k^2\prime}$$
$$(1^k)(EA\delta_{ab}) = \Sigma F_a F_b L = +12.8^{k^2\prime}$$
$$(1^k)(EA\delta_{aa}) = \Sigma F_a^2 L = 86.4^{k^2\prime}$$

Substituting these values in (1) and (2) and canceling out $EA$,

$$-2,040 + 86.4X_a + 12.8X_b = 0$$
$$+760 + 12.8X_a + 86.4X_b = 0$$

Solving these simultaneously,

$$X_a = +25.5, \qquad X_b = -12.6$$

In any bar,

$$F = F_o + X_a F_a + X_b F_b$$
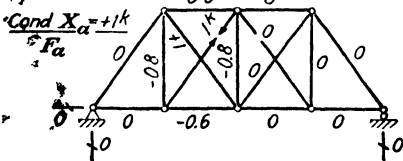
| Bar | L | $F_o$ | $F_a$ | $F_b$ | $F_oF_aL$ | $F_oF_bL$ | $F_a^2L$ | $F_aF_bL$ | $F_aX_a$ | $F_bX_b$ | $F$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Units | $'$ | $k$ | $k$ | $k$ | $k^{2\prime}$ | $k^{2\prime}$ | $k^{2\prime}$ | $k^{2\prime}$ | $k$ | $k$ | $k$ |
| bc | 15 | $+45$ | $-0\ 6$ | $0$ | $-405$ | $0$ | $+\ 5.4$ | $0$ | $-15\ 3$ | $0$ | $+29.7$ |
| cd | 15 | $+15$ | $0$ | $-0.6$ | $0$ | $-135$ | $0$ | $0$ | $0$ | $+\ 7\ 6$ | $+22.6$ |
| BC | 15 | $-30$ | $-0\ 6$ | $0$ | $+270$ | $0$ | $+\ 5\ 4$ | $0$ | $-15.3$ | $0$ | $-45.3$ |
| CD | 15 | $-30$ | $0$ | $-0\ 6$ | $0$ | $+270$ | $0$ | $0$ | $0$ | $+\ 7\ 6$ | $-22.4$ |
| bB | 20 | $+80$ | $-0\ 8$ | $0$ | $-1,280$ | $0$ | $+12\ 8$ | $0$ | $-20.4$ | $0$ | $+59.6$ |
| cC | 20 | $0$ | $-0\ 8$ | $-0\ 8$ | $0$ | $0$ | $+12.8$ | $+12.8$ | $-20.4$ | $+10\ 1$ | $-10.3$ |
| dD | 20 | $0$ | $0$ | $-0.8$ | $0$ | $0$ | $0$ | $0$ | $0$ | $+10\ 1$ | $+10$ |
| Bc | 25 | $-25$ | $+1$ | $0$ | $-625$ | $0$ | $+25\ 0$ | $0$ | $+25.5$ | $0$ | $+\ 0.5$ |
| bC | 25 | $0$ | $+1$ | $0$ | $0$ | $0$ | $+25.0$ | $0$ | $+25.5$ | $0$ | $+25.5$ |
| cD | 25 | $+25$ | $0$ | $+1$ | $0$ | $+625$ | $0$ | $0$ | $0$ | $-12\ 6$ | $+12\ 4$ |
| Cd | 25 | $0$ | $0$ | $+1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-12.6$ | $-12\ 6$ |
| $\Sigma$ | | $\cdot$ | ...... | ... | $-2,040$ | $+760$ | $+86\ 4$ | $+12\ 8$ | | | |

## Discussion:

*In tabulating the computations in such a truss, it is unnecessary to include bars having neither an $F_a$ nor an $F_b$ stress. Be sure to include the redundant bars, however. Why?*

*There may be some question about cutting truss bars as was done in this problem. Students sometimes worry about the two ends of a cut bar being unstable, since as shown in the above line diagrams there is nothing to prevent these two ends from rotating about the pin joints. To be strictly correct, we should show the cut in such cases as shown in Fig. 14·4. Since the shear and moment in a straight bar hinged at both ends are statically*
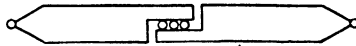


FIG. 11·4

*determinate, the restraints of this type cannot be removed. Thus, when we say that we cut such a bar, we mean that we remove its capacity to carry axial stress but retain its ability to carry shear and moment. This could be accomplished by cutting through the member and then inserting a nest of rollers in the manner shown.*

*It is impractical, however, to show such a detail every time we remove the capacity to carry axial stress from a bar that is hinged at its ends. In such cases, therefore, we shall imply such a detail but show the cut as has been done in the line diagrams in this example.*

*These remarks are likewise applicable to the tie rod in Example 14·8.*

**Example 14·6** *Compute the reactions and draw the bending-moment diagram for this beam.* $E = 30 \times 10^{1}$ *kips per sq in.*
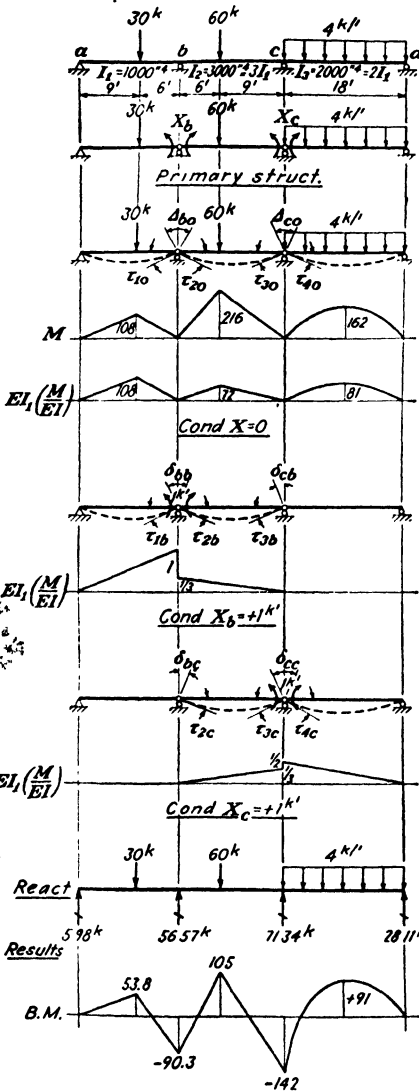
This beam is statically indeterminate to the second degree. Select moments at supports as redundants. Then,

$$\Delta_b^{\curvearrowright} = \Delta_{bo} + X_b\delta_{bb} + X_c\delta_{bc} = 0 \quad (1)$$

$$\Delta_c^{\curvearrowright} = \Delta_{co} + X_b\delta_{cb} + X_c\delta_{cc} = 0 \quad (2)$$

Note $\delta_{bc} = \delta_{cb}$ by Maxwell's law. Use moment-area theorems to compute $\Delta_{bo}$, $\Delta_{co}$, $\delta_{bb}$, $\delta_{cb}$, and $\delta_{cc}$.

$$EI_1\tau_{10} =$$
$$\frac{108}{15}\left[\frac{(9)(6)}{2}\right.$$
$$\left. + \frac{(6)(11)}{2}\right] = 432$$

$$EI_1\tau_{20} =$$
$$\frac{72}{15}\left[\frac{(9)(6)}{2}\right.$$
$$\left. + \frac{(6)(11)}{2}\right] = 288$$

$\left.\begin{array}{}\\ \\ \\ \\ \\ \\ \end{array}\right\}$ $EI_1\Delta_{bo} = +720^{k'^2}$

$$EI_1\tau_{30} =$$
$$\frac{72}{15}\left[\frac{(6)(4)}{2}\right.$$
$$\left. + \frac{(9)(9)}{2}\right] = 952$$

$$EI_1\tau_{40} =$$
$$\left(\frac{2}{3}\right)\frac{(81)(18)}{2} = 486$$

$\left.\begin{array}{}\\ \\ \\ \\ \end{array}\right\}$ $EI_1\Delta_{co} = +738^{k'^2}$

$$EI_1\tau_{1b} =$$
$$\left(\frac{1}{15}\right)\frac{(15)(10)}{2} = 5$$

$$EI_1\tau_{2b} =$$
$$\left(\frac{1}{3}\right)\frac{(15)(10)}{(15)(2)} = \frac{5}{3}$$

$\left.\begin{array}{}\\ \\ \\ \\ \end{array}\right\}$ $EI_1\delta_{bb} = +6.67^{k'^2}$

$$EI_1\tau_{3b} = \left(\frac{1}{3}\right)\frac{(15)(5)}{(15)(2)} = \frac{5}{6}$$
$$\therefore EI_1\delta_{cb} = +0.83^{k'^2}$$

$$EI_1\tau_{3c} =$$
$$\left(\frac{1}{3}\right)\frac{(15)(10)}{(15)(2)} = \frac{5}{3}$$

$$EI_1\tau_{4c} =$$
$$\left(\frac{1}{2}\right)\frac{(18)(12)}{(18)(2)} = 3$$

$\left.\begin{array}{}\\ \\ \\ \end{array}\right\}$ $EI_1\delta_{cc} = +4.67^{k'^2}$
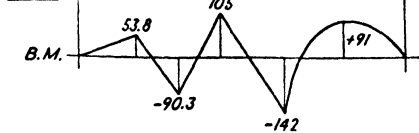
*Substituting these values in (1) and (2) and canceling out $EI_1$,*

$$720 + 6.67X_b + 0.83X_c = 0$$
$$738 + 0.83X_b + 4.67X_c = 0$$

*Solving these simultaneously,*

$$X_b = -90.3, \qquad X_c = -142.0$$

*With these moments known it is easy to isolate various portions of the beam and, by using statics, to compute shears, reactions, and bending moments, leading to the results shown.*

### Discussion:

*When we select the support moments at b and c as the redundants, we remove the moment restraints from the actual structure at these points, i.e., we insert hinges at these points.*

*Note the manner in which the varying I is handled in this problem. One I is selected as a base, and all other I's are expressed in terms of the standard I. Once this is done, the actual values of the various I's need never be substituted in the solution of this problem.*

**Example 14·7** *Compute the reactions and draw the bending-moment diagram for this frame. E and I are constant. Consider only bending distortion.*



Actual struct.

Primary struct.

$\dfrac{720}{EI}$

Cond. X=0

Cond. $X_c$=+1$^k$

Reactions

B.M. Diagram
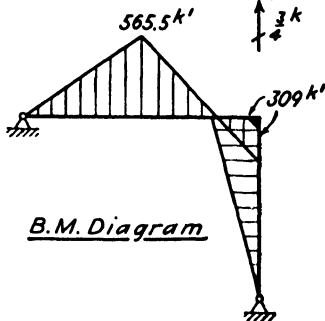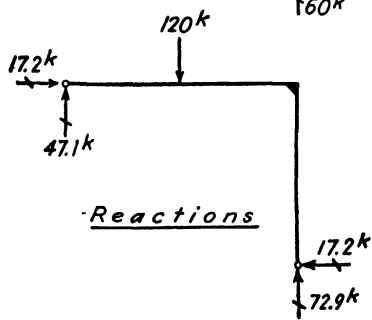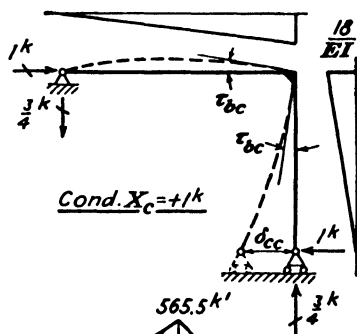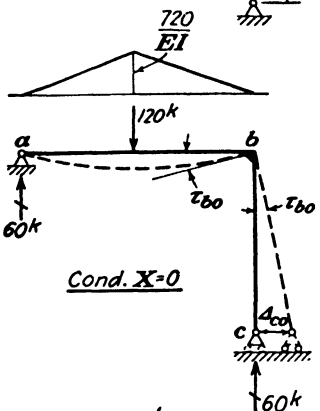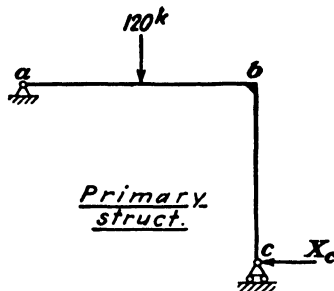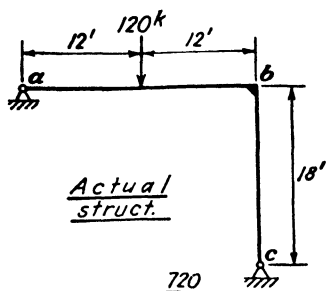
This frame is indeterminate to the first degree. Select $X_c$ as the redundant. Then,
$\Delta_c = \Delta_{co} + X_c \delta_{cc} = 0$. Compute $\Delta_{co}$ and $\delta_{cc}$ by the moment-area theorems.

$$EI\tau_{bo} = (720)(12)\left(\frac{12}{24}\right) = 4,320 \qquad \therefore \Delta_{co} = -\left(\frac{4,320}{EI}\right)(18) = \frac{-77,760^{k''}}{EI}$$

$$EI\tau_{be} = (18)(12)\left(\frac{16}{24}\right) = 144 \qquad \therefore \delta_{cc} = \left(\frac{144}{EI}\right)(18) + \left(\frac{18}{EI}\right)(9)(12) = +\frac{4,536^{k^3}}{EI}$$

Hence,

$$\frac{-77,760}{EI} + \frac{4,536}{EI} X_c = 0, \qquad X_c = \underline{+17.16} \qquad \therefore \leftarrow\!\!\uparrow$$

tations may be easily done by using only statics.

**Example 14·8**   *Computing the stress in the tie rod of this structure.*

*Structure is indeterminate to the first degree. Cut the tie rod, and select its bar stress as the redundant.   Then,*

$$\Delta_a = \Delta_{ao} + X_a \delta_{aa} = 0$$

*Evaluate $\Delta_{ao}$ and $\delta_{aa}$ by the method of virtual work including distortion due to both bending and axial stress.*

$$\sum Q\delta = \sum F_Q F_P \frac{L}{AE} + \sum \int M_Q M_P \frac{ds}{EI}$$

*From A to B,*

$$0 < x < 12 \qquad L = 12$$
$$M_o = 5x \qquad F_o = 0$$
$$M_a = -0.4x \qquad F_a = -0.6^k$$

*From C to E,*

$$0 < x < 6$$
$$M_o = 15x \qquad F_o = 0$$
$$M_a = -0.4x \qquad F_a = 0$$

*From E to B,*

$$6 < x < 12$$
$$M_o = 120 - 5x \qquad F_o = 0$$
$$M_a = -0.4x \qquad F_a = 0$$

*From D to B,*

$$L = 10$$
$$M_o = 0 \qquad F_o = 0$$
$$M_a = 0 \qquad F_a = +1^k$$



*Primary struct*

*Cond. X=0*

*Cond. $X_a = +1^k$*

$$(1^k)(\Delta_{ao}) = \sum F_a F_o \frac{L}{AE} + \sum \int M_a M_o \frac{ds}{EI} = 0 + \int_0^{12} (5x)(-0.4x)\frac{dx}{EI_2}$$
$$+ \int_0^6 (15x)(-0.4x)\frac{dx}{EI_2} + \int_6^{12} (120 - 5x)(-0.4x)\frac{dx}{EI_2}$$
$$\therefore EI_2 \Delta_{ao} = -3{,}168^{k'^3}$$

$$(1^k)(\delta_{aa}) = \sum F_a^2 \frac{L}{AE} + \sum \int M_a^2 \frac{ds}{EI} = \frac{(-0.6)^2(12)}{EA_2} + \frac{(1^2)(10)}{EA_1}$$
$$+ 2\int_0^{12} (-0.4x)^2 \frac{dx}{EI_2} \qquad \therefore EI_2 \delta_{aa} = +206.05^{k'^3}$$

*Since*

$$\frac{I_2}{A_2} = \left(\frac{600}{144^2}\right)\left(\frac{144}{20}\right) = \left(\frac{30}{144}\right)^{'2} \qquad and \qquad \frac{I_2}{A_1} = \left(\frac{600}{144^2}\right)\left(\frac{144}{2}\right) = \left(\frac{300}{144}\right)^{'2}$$

*then,*

$$\frac{-3{,}168}{EI_2} + \frac{206.05}{EI_2} X_a = 0 \qquad \therefore X_a = +15.38 \qquad \therefore tension$$

*If the distortion due to axial stress had been neglected, then*

$$X_a = \frac{3168}{184.32} = +17.2$$
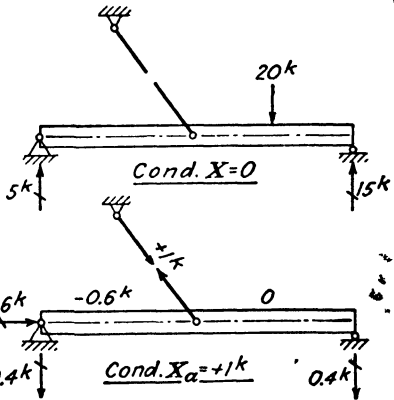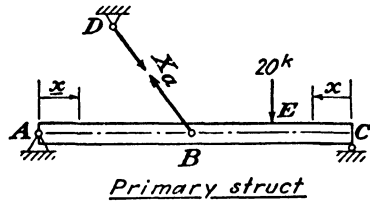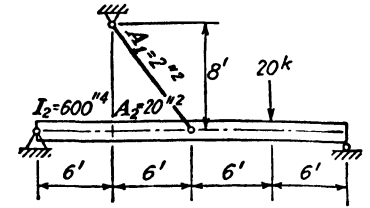
**14·5  Additional Examples Involving Temperature, Settlement, etc.**   The following examples illustrate the application of the superposition-equation approach to the stress analysis of typical indeterminate structures subjected to temperature changes, support movements, fabrication errors, etc.

Fundamentally, these problems are no more difficult to handle than those involving simply the effect of loads.  Always remember, however, that the deflections $\Delta_{aT}$, $\Delta_{aS}$, etc., refer to deflections of points on the *primary* structure, due to a change in temperature of the *primary* structure, due to the settlement of the supports of the *primary* structure, etc.  These deflections when superimposed correctly must add up so that the total deflection of point $a$ on the *primary* structure is equal to the known deflection of the corresponding point on the *actual* structure.

---

**Example 14·9**   *Compute bar stresses due to an increase of 60°F in the temperature at bars aB, BC, and Cd.   No change in the temperature of any other bars.   $\epsilon = 1/150{,}000$ per °F.   $E = 30 \times 10^3$ kips per sq in.   Cross-sectional areas in square inches are shown in parentheses.*

---

*This truss is indeterminate to the first degree. Cut bar bC, and select its bar stress as the redundant. Then,*
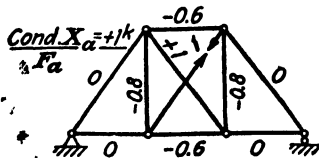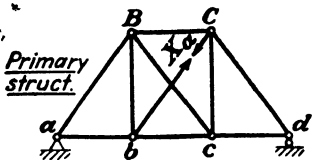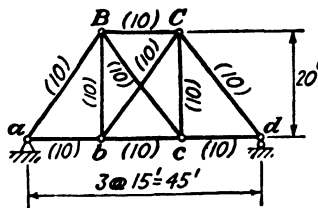
$$\Delta_a' = \Delta_{aT} + X_a \delta_{aa} = 0$$

*Using the method of virtual work,*

$$\Delta_{aT}:\quad F_Q = F_a \qquad (1^k)(\Delta_{aT}')$$
$$= \sum F_a \, \epsilon t L = \epsilon \sum F_a t L$$

$$\delta_{aa}:\quad F_Q = F_P = F_a \qquad (1^k)(\delta_{aa}') = \frac{1}{E}\sum F_a^2 \frac{L}{A}$$

| Bar | L | A | $\frac{L}{A}$ | $F_a$ | $F_a^2\frac{L}{A}$ | $t$ | $F_a t L$ | $X_a F_a$ |
|---|---|---|---|---|---|---|---|---|
| Units | $'$ | $''^2$ | $'/''^2$ | $k$ | $k^2'/''^2$ | °F | $k$ °F$'$ | $k$ |
| BC | 15 | 10 | 1 5 | −0.6 | +0 54 | +60 | −540 | − 7.5 |
| bc | 15 | 10 | 1 5 | −0.6 | +0 54 | 0 | 0 | − 7.5 |
| Bb | 20 | 10 | 2 | −0 8 | +1 28 | 0 | 0 | −10.0 |
| Cc | 20 | 10 | 2 | −0 8 | +1.28 | 0 | 0 | −10.0 |
| Bc | 25 | 10 | 2 5 | +1 0 | +2.5 | 0 | 0 | +12 5 |
| bC | 25 | 10 | 2 5 | +1.0 | +2 5 | 0 | 0 | +12 5 |
| Σ | .. | . | . | .... | +8 64 | | −540 | |

$$\Delta_{aT} = -(540)\,°\mathrm{F}' \left(\frac{1}{150{,}000}\right)^{/°\mathrm{F}} = \underline{-0.0036\,ft}$$

$$\delta_{aa} = +\frac{8.64^{k'/''^2}}{30 \times 10^3 \,{}^{k/''^2}} = \underline{+0.000288\,ft}$$

$$-0.0036 + 0.000288 X_a = 0, \qquad X_a = \underline{+12.5}$$

$$\therefore tension$$


B (10) C

(10) (10)(10) (10) (10)

a                           d
(10) b (10) c (10)

3 @ 15' = 45'

20'


Primary struct.

B        C

a              d
b        c


Cond. $X_a = +1^k$

$F_a$

−0.6

0        0

−0.8        −0.8

0   −0.6   0

**Example 14·10** *Compute the bar stresses of the two-hinged trussed arch in Example 14·4 caused by forcing member AB into place even though it had been fabricated ⅛" too short.*

*In this case,*

$$\Delta_c^{\leftarrow} = \Delta_{cE} + X_c \delta_{cc} = 0$$

*Using the method of virtual work, $\Delta_{cE}$ may be evaluated as follows, using information from Example 14·4:*

$$(1^k)(\Delta_{cE}^{\leftarrow}) = \Sigma F_c \, \Delta L_E = (+1^k)(-\tfrac{1}{96}), \qquad \Delta_{cE} = -0.0104'$$

*From Example 14·4,*

$$\delta_{cc} = + \frac{19.875^{k'/''^2}}{E} = + \frac{19.875^{k'/''^2}}{30 \times 10^3 \, {}^{k/''^2}} = +0.0006625'$$

$$\therefore -0.0104' + 0.0006625' X_c = 0 \qquad \therefore X_c = \underline{+15.7} \qquad \therefore \dashv\vdash$$

*The bar stress in any member can then be computed, since*

$$F = X_c F_c = 15.7 F_c$$

**Example 14·11**  *Compute the bar stresses in this truss due to the following support movements:*

> *Support at a, 0.24 in. down.*
> *Support at c, 0.48 in. down.*
> *Support at e, 0.36 in. down.*

*Cross-sectional area in square inches shown in parentheses.*

$$E = 30 \times 10^3 \text{ kips per sq in.}$$



Primary struct.



Cond. $X_C = +1^k$



Settlement cond.
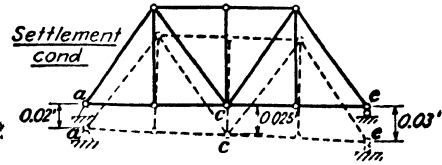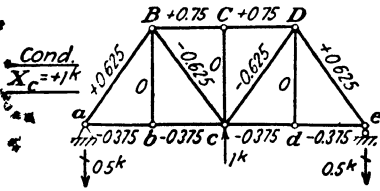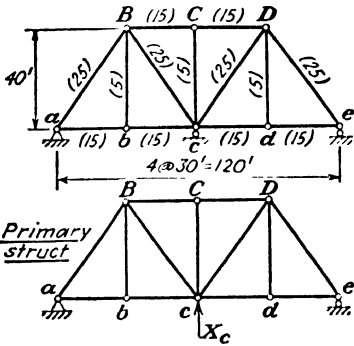


*Selecting the vertical reaction at c as the redundant,*

$$\Delta_c^{\uparrow} = \Delta_{cS} + X_c\delta_{cc} = -0.04'$$

*Using the method of virtual work,*

$\Delta_{cs}$:  
$$\Sigma Q\delta = 0$$
$$(1^k)(\Delta_{cS}^{\uparrow}) + (0.5^k)(0.02')$$
$$+ (0.5^k)(0.03') = 0$$
$$\therefore \Delta_{cS} = \underline{-0.025'}$$

*which checks value obtained by geometry as shown on sketch.*

$\delta_{cc}$:  
$$(1^k)(\delta_{cc}^{\uparrow}) = \frac{1}{E}\sum F_c^2 \frac{L}{A}$$

*Because of symmetry only one-half of the bars of the truss need be listed in the tabulation.*

| Bar | L | A | $\frac{L}{A}$ | $F_c$ | $F_c^2\frac{L}{A}$ | $X_cF_c$ |
|-----|---|---|------|-------|----------|-------|
| Units | ' | $''^2$ | $'/''^2$ | k | $k^2'/''^2$ | k |
| ab | 30 | 15 | 2 | $-0.375$ | $+0.281$ | $+26.0$ |
| bc | 30 | 15 | 2 | $-0.375$ | $+0.281$ | $+26.0$ |
| BC | 30 | 15 | 2 | $+0.75$ | $+1.125$ | $-52.1$ |
| aB | 50 | 25 | 2 | $+0.625$ | $+0.781$ | $-44.1$ |
| Bc | 50 | 25 | 2 | $-0.625$ | $+0.781$ | $+44.1$ |
| $\frac{1}{2}\Sigma$ | .. | .. | . | ........ | $+3.25$ | |

$$\delta_{cc} = \frac{+6.50^{k'/''^2}}{30 \times 10^3 {}^{k/''^2}} = \underline{+0.000216'}$$

$$\therefore -0.025 + 0.000216X_c = -0.04'$$

$$X_c = \frac{-0.015}{0.000216} = \underline{-69.5} \qquad \therefore \text{ down}$$

**Example 14·12**  *Compute the reactions and draw the bending-moment diagram for the beam in Example 14·6, due to the following support movements:*

$$Pt.\ a:\ 0.02\ ft\ down$$
$$Pt.\ b:\ 0.04\ ft\ down$$
$$Pt.\ c:\ 0.05\ ft\ down$$
$$Pt.\ d:\ 0$$

$$E = 30 \times 10^3\ kips\ per\ sq\ in.$$

*In this case,*

$$\Delta_b = \Delta_{bs} + X_b\delta_{bb} + X_c\delta_{bc} = 0 \quad (1)$$

$$\Delta_c = \Delta_{cs} + X_b\delta_{cb} + X_c\delta_{cc} = 0 \quad (2)$$

*Using either the method of virtual work or the geometry of the adjacent sketch,*

$$\Delta_{bs} = -0.000667\ radian$$
$$\Delta_{cs} = -0.003444\ radian$$

*From Example 14·6,*

$$\delta_{bb} = \frac{6.667^{k'^2}}{EI_1} \qquad \delta_{cc} = \frac{4.667^{k'^2}}{EI_1}$$

$$\delta_{bc} = \delta_{cb} = \frac{0.833^{k'^2}}{EI_1}$$
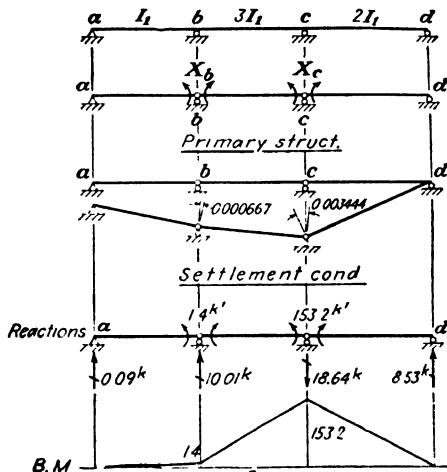
*Upon substituting these values Eqs. (1) and (2) become*

$$6.667X_b + 0.833X_c = 0.000667EI_1$$
$$0.833X_b + 4.667X_c = 0.003444EI_1$$

*from which, since*

$$EI = (30 \times 10^3 \times 144)^{k'^2} \left(\frac{1,000}{144^2}\right)^{'4} = 0.208 \times 10^6\ ^{k'^2}$$

$$X_b = +0.00000685EI_1 = +1.4$$
$$X_c = +0.000736EI_1 = +153.2$$

*The reactions and BM diagram may then easily be computed.*

## 14·6  General Remarks Concerning Selection of Redundants.

From the previous discussion, we recognize that there is considerable latitude in selecting redundants, the only restriction being that they shall be selected so that a *stable* primary structure remains.   By proper selection of the redundants, however, we may minimize the numerical computations.   This objective may be achieved by adhering to the two following policies:

1. Take advantage of any symmetry of the structure.

2. Select the primary structure so that the effect of any of the various loading conditions is localized as much as possible.

Consideration of several alternate selections of the redundants for the continuous truss shown in Fig. 14·5 will illustrate the validity of these statements. This structure is indeterminate to the second degree. Any selection of the redundants will involve two equations of the following form:

$$\left.\begin{array}{l} \Delta_a = \Delta_{ao} + X_a \delta_{aa} + X_b \delta_{ab} = 0 \\ \Delta_b = \Delta_{bo} + X_a \delta_{ba} + X_b \delta_{bb} = 0 \end{array}\right\} \qquad (a)$$

Only the following five different deflection terms are involved, since $\delta_{ab}$ equals $\delta_{ba}$:

$$\left.\begin{array}{llll} (1) \ (\Delta_{ao}) = \sum F_a F_o \dfrac{L}{AE}; & (1) \ (\delta_{aa}) = \sum F_a^2 \dfrac{L}{AE}; \\ & (1) \ (\delta_{bb}) = \sum F_b^2 \dfrac{L}{AE} \\ (1) \ (\Delta_{bo}) = \sum F_b F_o \dfrac{L}{AE}; & (1) \ (\delta_{ab}) = \sum F_a F_b \dfrac{L}{AE} \end{array}\right\} \qquad (b)$$

Before these terms can be evaluated, $F_o$, $F_a$, and $F_b$ stresses must be computed.



ACTUAL STRUCTURE

PRIMARY STRUCTURE NO.1

PRIMARY STRUCTURE NO.2

PRIMARY STRUCTURE NO.3

FIG. 14·5

If the structure is symmetrical and if symmetrical redundants are selected, then the $F_b$ stresses can be obtained from the $F_a$ stresses by symmetry. Further, $\delta_{bb}$ will be equal to $\delta_{aa}$ in such a case, leaving only four deflection terms to be evaluated. The evaluation of these terms will involve less computation if the redundants are selected so as to restrict the effect of the various loading conditions to as few bars as possible. The latter will be true whether or not the structure is symmetrical.

All the three alternate selections of the redundants shown in Fig. 14·5 take advantage of symmetry. The various loading conditions affect the portions of the structure indicated in each case. Comparison of these
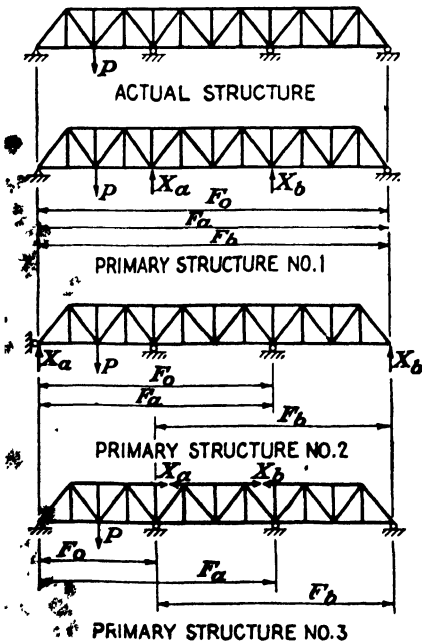
primary structures shows clearly that selection 3 is the best since it is most effective in localizing the effects of the various loading conditions.

There are several other items to be noted at this time. In the discussion of Example 14·5, it was pointed out that the shear and moment are statically determinate in a straight bar which is hinged at both ends. If such a bar is cut, only its axial stress may be statically indeterminate and, therefore, considered as a redundant. On the other hand, the shear, moment, and axial stress are often all statically indeterminate in a member that is rigidly connected at its ends to the rest of the structure. If such a member is cut, the shear, moment, and axial stress may all be considered as redundants, provided that a stable structure remains when the restraints corresponding to all three of these elements are removed (as an illustration, see Example 14·14).

**14·7 Analysis of Indeterminate Structures, Using Castigliano's Theorem; Theorem of Least Work.** The previous approach to the analysis of an indeterminate structure involves writing superposition equations for the deflections of the points of application of the redundants. Instead of doing this, however, expressions for these deflections may be set up, using Castigliano's second theorem. The latter approach is actually very similar to the former. It is a somewhat more automatic procedure, however, and is therefore preferred by some students and engineers. Since Castigliano's theorem should really be limited to the computation of the deflections produced simply by loads on the structure, this method lacks the generality of the superposition-equation approach.

Consider, for example, the indeterminate beam shown in Fig. 14·8. After the degree of indeterminacy has been established and the redundant and the resulting primary structure have been selected, the deflection of the point of application of redundant $X_b$ may then be evaluated by using Castigliano's second theorem. In this particular case, only bending distortion is involved; therefore,

$$W_I' = \sum \int M^2 \frac{ds}{2EI}$$

but

$$\frac{\partial W_I}{\partial X_b} = \Delta_b \uparrow \qquad\qquad (a)$$

Since point $b$ on the actual structure does not deflect, $\Delta_b$ on the primary structure must equal zero. As a result,

$$\frac{\partial W_I}{\partial X_b} = \sum \int M \frac{\partial M}{\partial X_b} \frac{ds}{EI} = 0 \qquad\qquad (b)$$

However, $M$, being equal to the total bending moment on the primary structure due to all causes, may be expressed as being the superposition of the contribution of the applied load only and the contribution of the redundant $X_b$. Thus,

$$\left. \begin{array}{l} M = M_o + X_b M_b \\[4pt] \dfrac{\partial M}{\partial X_b} = M_b \end{array} \right\} \qquad (c)$$

Equation ($b$), therefore, becomes,

$$\sum \int M_o M_b \frac{ds}{EI} + X_b \sum \int M_b^2 \frac{ds}{EI} = 0 \qquad (d)$$

It is easy to evaluate these integrals for the primary structure and then solve for $X_b$.

If the superposition approach is used in this example, $\Delta_{bo}$ and $\delta_{bb}$ may be evaluated by the method of virtual work and found to be

$$\left. \begin{array}{l} (1)\ (\Delta_{bo}) = \displaystyle\sum \int M_o M_b \frac{ds}{EI} \\[10pt] (1)\ (\delta_{bb}) = \displaystyle\sum \int M_b^2 \frac{ds}{EI} \end{array} \right\} \qquad (e)$$

From Eqs. ($e$) it is immediately apparent that Eq. ($d$) is actually a statement that

$$\Delta_{bo} + X_b \delta_{bb} = 0 \qquad (f)$$

Thus, if the method of virtual work is used as a basis for evaluating $\Delta_{bo}$ and $\delta_{bb}$ in the superposition-equation approach, the two methods are essentially identical.

The above illustration involves a structure that is statically indeterminate to only the first degree. In more highly indeterminate structures, the procedure is essentially the same. After selecting the $n$ redundants and the resulting primary structure, express the displacement of the point of application of each redundant by $n$ separate applications of Castigliano's second theorem. This will result in $n$ simultaneous equations involving the $n$ redundants, the value of which may then be obtained by simultaneous solution of the equations. The procedure for analyzing a multiply redundant frame in this manner is illustrated by Example 14·14.

In the examples of Art. 14·8, equations comparable with Eq. ($b$) are evaluated in a slightly different manner. Thus it is possible to use Castigliano's theorem somewhat more automatically and effectively in certain problems. The procedure suggested in the above illustration

may likewise be used to advantage in certain other problems. In cases of the latter type, however, there is no advantage to using the Castigliano approach instead of superposition equations.

If in the analysis of indeterminate structures the deflection of the point of application of a redundant is zero, then applying Castigliano's theorem as in Eq. (a) reduces to the statement that the first partial derivative of the strain energy with respect to that redundant is equal to zero. This is equivalent to stating that the value of the redundant must be such as to minimize the strain energy. This special case of Castigliano's second theorem is often called the *theorem of least work* and may be stated as follows:

*In a statically indeterminate structure, if there are no support move-ments and no change of temperature, the redundants must be such as to make the strain energy a minimum.*

**14·8 Examples Illustrating Stress Analysis Using Castigliano's Theorem.** The examples that follow have been chosen primarily to illustrate the use of Castigliano's second theorem in the stress analysis of indeterminate structures. If, in each case, we solved the problem by the superposition-equation approach, using the method of virtual work to evaluate the various deflection terms, we should find that the actual computations would be essentially the same as those involved in the Castigliano solution. The only difference in the two approaches lies in the somewhat more automatic manner of setting up the solution when the Castigliano approach is used.

There is a difference between the two approaches that is worthy of note. In a Castigliano solution, the redundants carry their own units throughout a solution. For example, in Eq. (a), $X_b$ must be in kips if $W_i$ is in kip-feet in order that the change in $W_i$ divided by the change in $X_b$ shall equal $\Delta_b$ in feet. As a result, if $X_b$ is in kips, $M_b$ must have units of kip-feet per kip if the units in Eqs. (c) are to be consistent. If it is recognized, therefore, that the redundants in a Castigliano solution carry their own units, dimensional checks should be consistent at all times.

Strictly speaking, Castigliano's theorem is applicable only when the deflection of the structure is caused by loads. It is possible, however, to handle the stress analysis of an indeterminate structure for the effect of temperature change, settlement of supports, etc., by proceeding as follows: Select the primary structure, and temporarily remove all the redundants. Now allow the temperature change or settlement to take place on the primary structure. Compute the resulting displacements of the points of application of the redundants on this primary structure. Such computations may be performed by means of the method of virtual

work or some other suitable method? Now apply the redundants. As they are applied, they must restore their points of application to their correct positions. Castigliano's theorem may be used to evaluate the deflections produced by the redundants. Substituting the previously computed values for the restoring deflections, we thereby obtain equations containing only the redundants as the unknowns, which may then be obtained by simultaneous solution of these equations. While this procedure is often not as straightforward as that of the superposition-equation approach, there are nevertheless instances where it may be used advantageously.

---

· **Example 14·13** *Compute the bar stresses in the members of this truss. Cross-sectional areas in square inches shown in parentheses.* $E = 30 \times 10^3$ *kips per sq in.*

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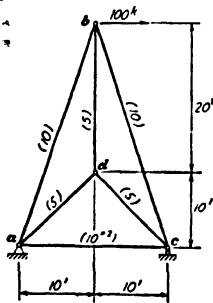
Truss is indeterminate to first degree. Select stress in bar bd as the redundant.

$$W_I = \sum \frac{F^2 L}{2AE}, \qquad \frac{\partial W_I}{\partial X_1} = \Delta_1 \begin{smallmatrix}\downarrow\\\uparrow\end{smallmatrix} = 0$$

$$\therefore \frac{\partial W_I}{\partial X_1} = \frac{1}{E} \sum \frac{FL}{A} \frac{\partial F}{\partial X_1} = 0$$

| Bar | $L$ | $A$ | $\dfrac{L}{A}$ | $F$ | $\dfrac{\partial F}{\partial X_1}$ | $F \dfrac{L}{A} \dfrac{\partial F}{\partial X_1}$ |
|---|---|---|---|---|---|---|
| Units | $\dfrac{L}{\prime}$ | $\prime\prime^2$ | $\prime/\prime\prime^2$ | $k$ | $k/k$ | $k'/\prime\prime^2$ |
| ab | 31 7 | 10 | 3 17 | $+158\ 5 - 0\ 529X_1$ | $-0\ 529$ | $-265\ + 0\ 886X_1$ |
| bc | 31 7 | 10 | 3 17 | $-158\ 5 - 0\ 529X_1$ | $-0\ 529$ | $+265\ + 0\ 886X_1$ |
| ca | 20 | 10 | 2 | $+50\ \ \ - 0\ 333X_1$ | $-0\ 333$ | $-33\ 3 + 0\ 222X_1$ |
| ad | 14 1 | 5 | 2 83 | $+0\ 707X_1$ | $+0\ 707$ | $+1\ 414X_1$ |
| bd | 20 | 5 | 4 | $+X_1$ | $+1\ 0$ | $+4\ 000X_1$ |
| cd | 14 1 | 5 | 2 83 | $+0\ 707X_1$ | $+0\ 707$ | $+1\ 414X_1$ |
| $\Sigma$ | | | .. | .... | . | $-33\ 3 + 8.822X_1$ |

$$\therefore -33.3 + 8.822X_1 = 0, \qquad X_1 = +3.78$$

The remaining bar stresses may be found easily by statics or by extending the above tabulation.



Primary Structure
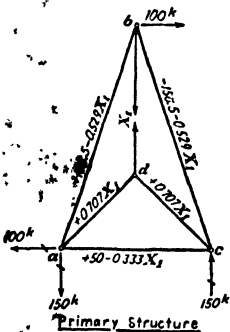
**Example 14·14**  *Solve this frame using Castigliano's theorem.    Include the effect of distortion due to both axial stress and bending.*

This frame is indeterminate to the third degree.  Cut the girder at midspan and select the moment, axial stress, and shear as the three redundants $X_a$, $X_b$ and $X_c$.

$$W_I = \sum \int M^2 \frac{ds}{2EI} + \sum \frac{F^2 L}{2AE}$$

But,

$$\frac{\partial W_I}{\partial X_a} = \Delta_a \,\rangle \,\zeta = 0$$

$$\frac{\partial W_I}{X_b} = \Delta_b^{\rightarrow\cdots\leftarrow} = 0$$

$$\frac{\partial W_I}{\partial X_c} = \Delta_c^{\,\Updownarrow} = 0$$

Therefore, differentiating and canceling out $E$,

$$\sum \int M \frac{\partial M}{\partial X_a} \frac{ds}{I} + \sum F \frac{L}{A} \frac{\partial F}{\partial X_a} = 0 \quad (1)$$

$$\sum \int M \frac{\partial M}{\partial X_b} \frac{ds}{I} + \sum F \frac{L}{A} \frac{\partial F}{\partial X_b} = 0 \quad (2)$$

$$\sum \int M \frac{\partial M}{\partial X_c} \frac{ds}{I} + \sum F \frac{L}{A} \frac{\partial F}{\partial X_c} = 0 \quad (3)$$



Primary structure

From F to C,

$0 < x < 2$,    $M = X_a - xX_c$,    $\frac{\partial M}{\partial X_a} = 1$    $\frac{\partial M}{\partial X_b} = 0$    $\frac{\partial M}{\partial X_c} = -x$

$L = 2'$,    $F = X_b$,    $\frac{\partial F}{\partial X_a} = 0$    $\frac{\partial F}{\partial X_b} = 1$    $\frac{\partial F}{\partial X_c} = 0$

From C to B,

$2 < x < 10$,    $M = X_a - xX_c - 100(x - 2)$,    $\frac{\partial M}{\partial X_a} = 1$    $\frac{\partial M}{\partial X_b} = 0$    $\frac{\partial M}{\partial X_c} = -x$

$L = 8'$,    $F = X_b$,    $\frac{\partial F}{\partial X_a} = \frac{\partial F}{\partial X_c} = 0$    $\frac{\partial F}{\partial X_b} = 1$

From B to A,

$0 < y < 15$,    $M = X_a - yX_b - 10X_c - 800$,    $\frac{\partial M}{\partial X_a} = 1$    $\frac{\partial M}{\partial X_b} = -y$    $\frac{\partial M}{\partial X_c} = -10$

$L = 15'$,    $F = -100 - X_c$,    $\frac{\partial F}{\partial X_a} = \frac{\partial F}{\partial X_b} = 0$    $\frac{\partial F}{\partial X_c} = -1$

From F to D,

$0 < x < 10$,    $M = X_a + xX_c$,    $\frac{\partial M}{\partial X_a} = 1$    $\frac{\partial M}{\partial X_b} = 0$    $\frac{\partial M}{\partial X_c} = x$

$L = 10'$,    $F = X_b$,    $\frac{\partial F}{\partial X_a} = \frac{\partial F}{\partial X_c} = 0$    $\frac{\partial F}{\partial X_b} = 1$
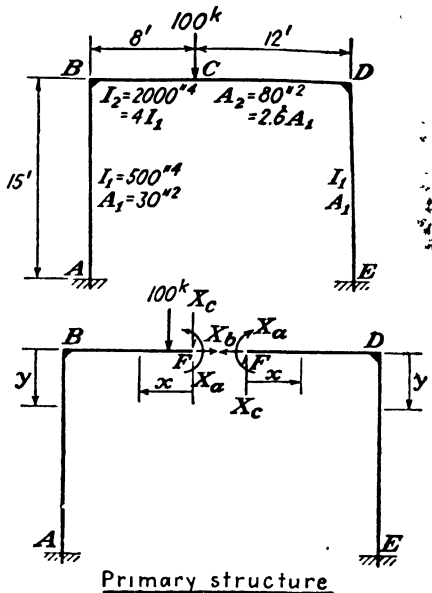
*From D to E,*

$0 < y < 15,$     $M = X_a - yX_b + 10X_c,$     $\dfrac{\partial M}{\partial X_a} = 1$     $\dfrac{\partial M}{\partial X_b} = -y$

$$\dfrac{\partial M}{\partial X_c} = 10$$

$L = 15',$          $F = X_c,$          $\dfrac{\partial F}{\partial X_a} = \dfrac{\partial F}{\partial X_b} = 0$     $\dfrac{\partial F}{\partial X_c} = 1$

*Setting up Eq. (1),*

$$\int_0^2 (X_a - xX_c)(1)\frac{dx}{4I_1} + \int_2^{10} (X_a - xX_c - 100x + 200)(1)\frac{dx}{4I_1}$$

$$+ \int_0^{15} (X_a - yX_b - 10X_c - 800)(1)\frac{dy}{I_1} + \int_0^{10} (X_a + xX_c)(1)\frac{dx}{4I_1}$$

$$+ \int_0^{15} (X_a + 10X_c - yX_b)(1)\frac{dy}{I_1} = 0$$

*Combining, canceling, and integrating, we obtain*

$$35X_a - 225X_b = 12,800 \tag{1}$$

*Now setting up Eq. (2)*

$$\int_0^{15} (X_a - yX_b - 10X_c - 800)(-y)\frac{dy}{I_1} + \int_0^{15} (X_a - yX_b + 10X_c)(-y)\frac{dy}{I}$$

$$+ X_b \frac{(1)(10)}{2.6A_1} + X_b \frac{(1)(10)}{2.6A_1} = 0$$

*Again combining and canceling before integrating, we obtain*

$$(-225)X_a + \left(2,250 + 7.5\frac{I_1}{A_1}\right)X_b = -90,000$$

*However,*

$$\frac{I_1}{A_1} = \left(\frac{500}{144^2}\right)\left(\frac{144}{30}\right) = \frac{50}{432},$$

*and therefore*

$$-225X_a + 2,250.87X_b = -90,000 \tag{2}$$

*In a similar manner, Eq. (3) reduces to*

$$\left(3,166.7 + 30\frac{I_1}{A_1}\right)X_c = -\left(125,866.7 + 1,500\frac{I_1}{A_1}\right),$$

$$3,170.14X_c = -126,040.2 \tag{3}$$

*Solving Eqs. (1), (2), and (3),*

$$X_a = +304.2^{k'}$$
$$X_b = -9.56^{k}$$
$$X_c = -39.76^{k}$$

*If the effect of axial distortion were neglected, all the terms containing A would be omitted. Then, we should find*

$$X_a = +304.0^{k'}$$
$$X_b = -9.60^{k}$$
$$X_c = -39.74^{k}$$

**Discussion:**

These results indicate that the effect of axial distortion may be neglected in comparison with the effect of bending in the rigid-frame type of structure.

**Example 14·15**   *Compute the stress in the tie rod.   Include the effect of both axial stress and bending.*

*This structure is indeterminate to the first degree.   Select the tie-rod stress as the redundant.   Then*

$$\frac{\partial W_I}{\partial X_1} = \Delta_1 = 0$$

$$\frac{\partial W_I}{\partial X_1} = \sum \int M \frac{\partial M}{\partial X_1} \frac{dx}{EI}$$

$$+ \sum F \frac{\partial F}{\partial X_1} \frac{L}{AE} = 0 \quad (1)$$

*From B to C,*

$$M = 0, \quad F = X_1, \quad \frac{\partial F}{\partial X_1} = 1,$$
$$L = 20'$$

*From B to D,*

$$M = 0.6xX_1 \qquad F = -0.8X_1$$
$$\frac{\partial M}{\partial X_1} = 0.6x \qquad \frac{\partial F}{\partial X_1} = -0.8$$
$$0 < x < 8 \qquad L = 8'$$

*From D to A,*

$$M = 0.6xX_1 - 10(x - 8)$$
$$\frac{\partial M}{\partial X_1} = 0.6x$$
$$8 < x < 16$$
$$F = -0.8X_1$$
$$\frac{\partial F}{\partial X_1} = -0.8$$
$$L = 8'$$

**Primary Structure**

*Evaluating Eq. (1),*

$$\int_0^8 (0.6xX_1)(0.6x)\frac{dx}{I_2} + \int_8^{16}(0.6xX_1 - 10x + 80)(0.6x)\frac{dx}{I_2} + \frac{X_1(20)(1)}{A_1}$$
$$+ \frac{(-0.8X_1)(16)(-0.8)}{A_2} = 0$$

*from which*

$$X_1\left(491.52 + 20\frac{I_2}{A_1} + 10.24\frac{I_2}{A_2}\right) = 2{,}560,$$

$$\frac{I_2}{A_1} = \frac{25}{12}, \qquad \frac{I_2}{A_2} = \frac{5}{24}$$

$$\therefore 535.32X_1 = 2{,}560$$
$$\therefore X_1 = +4.80^k$$

*whereas, if the axial stress term were neglected,*

$$X_1 = +5.20^k$$
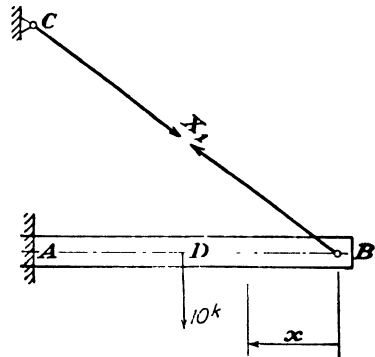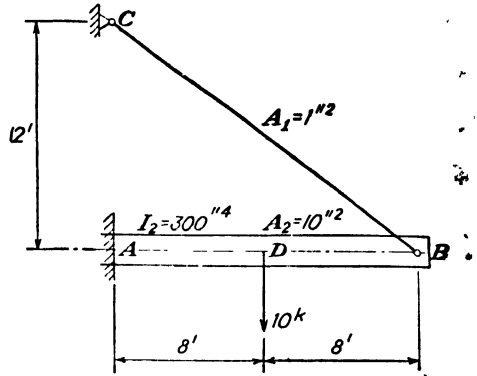
**14·9 Development of the Three-moment Equation.** The *three-moment equation* was first presented in 1857 by the French engineer Clapeyron. This equation is a relationship that exists between the moment at three points in a continuous member. It is particularly helpful in solving for the moments at the supports of indeterminate beams.

Designate three points on a continuous member as $L$, $C$, and $R$ as shown in Fig. 14·6. Suppose that the moment of inertia is constant between point $L$ and $C$ and equal to $I_L$ and likewise constant between $C$
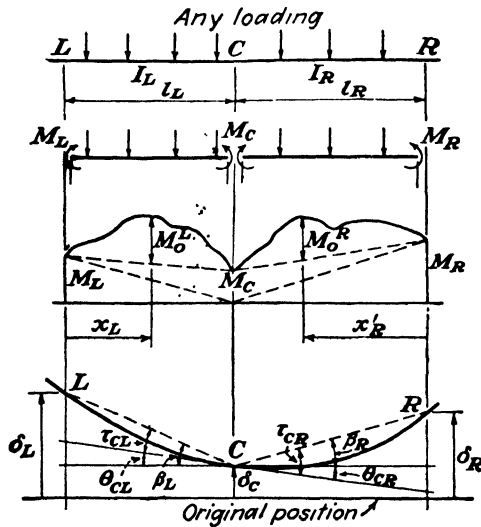


FIG 14·6

and $R$ and equal to $I_R$.* The member is assumed to be straight initially, and the deflections from the original position are assumed to be $\delta_L$, $\delta_C$, and $\delta_R$ at points $L$, $C$, and $R$, respectively—all to be considered positive when upward as shown.

Let the moments at these three points be $M_L$, $M_C$, and $M_R$. Bending moments are to be considered plus when causing tension on the lower fibers of a member. The bending-moment diagram for the portion $LC$ or $CR$ may be considered to be that resulting from superposition of three separate effects: the contribution of each of the end moments acting separately, which is given by the ordinates of the triangles indicated by the dashed lines; and the contribution of the applied load acting by itself with the end moments removed, which is given by the ordinates $M_o^L$ in the portion $LC$ and by $M_o^R$ in the portion $CR$.

* Theoretically it is possible, though cumbersome, to include the effect of variable $I$ in this development.

From the sketch of the elastic curve,

$$\theta_{CL} = \beta_L - \tau_{CL} \quad \text{and} \quad \theta_{CR} = \tau_{CR} - \beta_R$$

However, since the elastic curve is continuous through point $C$,

$$\theta_{CL} = \theta_{CR}$$

Hence,

$$\beta_L - \tau_{CL} = \tau_{CR} - \beta_R \qquad (a)$$

Since these are all small angles, it is permissible to consider that

$$\left.\begin{array}{l} \beta_L = \dfrac{\delta_L - \delta_C}{l_L} \\[2mm] \beta_R = \dfrac{\delta_R - \delta_C}{l_R} \end{array}\right\} \qquad (b)$$

If the bending-moment diagram were converted into an $M/EI$ diagram, $\tau_{CL}$ and $\tau_{CR}$ could be evaluated easily by means of the second moment-area theorem,

$$\left.\begin{array}{l} \tau_{CL} = \dfrac{1}{EI_L l_L}\left(\dfrac{M_L l_L^2}{6} + \dfrac{M_C l_L^2}{3} + \displaystyle\int_0^{l_L} M_o^L x_L \, dx_L\right) \\[4mm] \tau_{CR} = \dfrac{1}{EI_R l_R}\left(\dfrac{M_R l_R^2}{6} + \dfrac{M_C l_R^2}{3} + \displaystyle\int_0^{l_R} M_o^R x_R' \, dx_R'\right) \end{array}\right\} \qquad (c)$$

Let

$$\left.\begin{array}{l} (\mathfrak{M}_o)_L = \displaystyle\int_0^{l_L} M_o^L x_L \, dx_L \\[4mm] (\mathfrak{M}_o)_R = \displaystyle\int_0^{l_R} M_o^R x_R' \, dx_R' \end{array}\right\} \qquad (14\cdot4)$$

Substituting from Eq. (14·4) in Eq. (c) and then from Eqs. (b) and (c) in Eq. (a), we obtain the so-called "*three-moment equation*,"

$$M_L \frac{l_L}{I_L} + 2M_C\left(\frac{l_L}{I_L} + \frac{l_R}{I_R}\right) + M_R \frac{l_R}{I_R} = -\frac{\mathcal{L}_o}{I_L} - \frac{\mathfrak{R}_o}{I_R}$$

$$+ 6E\left[\frac{\delta_L}{l_L} - \delta_C\left(\frac{1}{l_L} + \frac{1}{l_R}\right) + \frac{\delta_R}{l_R}\right] \qquad (14\cdot5)$$

where the load terms are

$$\left.\begin{array}{l} \mathcal{L}_o = + \dfrac{6(\mathfrak{M}_o)_L}{l_L} \\[4mm] \mathfrak{R}_o = + \dfrac{6(\mathfrak{M}_o)_R}{l_R} \end{array}\right\} \qquad (14\cdot6)$$

In using these equations, note particularly that

(1) $M_L$, $M_C$, and $M_R$ are plus when causing tension on the lower fibers.

(2) $\delta_L$, $\delta_C$, and $\delta_R$ are plus when upward from the original position.

(3) $\mathcal{L}_o$ and $\mathcal{R}_o$ are load terms dependent on the applied load in the spans $LC$ and $CR$, respectively.

(4) The $M_o$ diagram for a member is the bending-moment diagram drawn for the member if it is assumed to be a simple end-supported beam. $(\mathfrak{M}_o)_L$ represents the static moment of the area under this diagram, taken about an axis through the left end, while $(\mathfrak{M}_o)_R$ represents the static moment about an axis through the right end. The sign of both these static moments depends simply on the sign of the ordinates of the $M_o$ diagram.



$(a)$-CONCENTRATED LOAD



$(b)$-FULL UNIFORMLY DISTRIBUTED LOAD
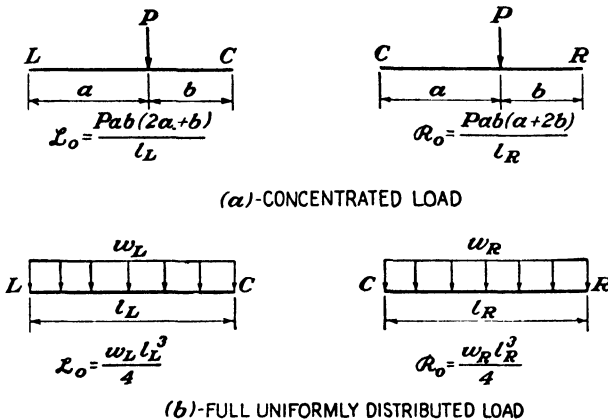
Fig. 14·7

In the special case where $I_L = I_R = I$, Eq. (14·5) simplifies to

$$M_L l_L + 2M_C(l_L + l_R) + M_R l_R = -\mathcal{L}_o - \mathcal{R}_o$$
$$+ 6EI \left[ \frac{\delta_L}{l_L} - \delta_C \left( \frac{1}{l_L} + \frac{1}{l_R} \right) + \frac{\delta_R}{l_R} \right] \quad (14\cdot 7)$$

The load terms in the cases of full uniform load and concentrated load are shown in Fig. 14·7.

**14·10 Application of Three-moment Equation.** The three-moment equation is applicable to any three points on a beam as long as there are no discontinuities, such as hinges in the beam within this portion. In applying the equation to a continuous beam, if we select three successive support points as being $L$, $C$, and $R$, the deflection terms on the right-hand side of the equation will be equal either to zero or to the known movements of the support points. We thus obtain an equation involving the moments at the support points as the only unknowns.

In this manner, we may write an independent equation for any three successive support points along a continuous beam. We shall obtain $n$ independent equations involving $n$ unknown support moments, which may then be obtained from the simultaneous solution of these equations.

There is a slight ambiguity in handling a fixed end on a continuous beam, but the technique of solving such problems is explained in the illustrative examples that follow.

The analysis of continuous beams by this method is straightforward. Be careful, however, to follow the sign convention noted in Art. 14·9. Likewise, be careful to use consistent units, particularly when there are movements of the supports.

**Example 14·16**  *Compute the reactions and draw the bending-moment diagram for this beam:*

Apply Eq. (14·5), taking a, b, and c as L, C, and R.

$$M_a = 0, \qquad M_b = ?, \qquad M_c = 0,$$
$$\delta_a = \delta_b = \delta_c = 0$$

$$\mathcal{L}_0 = \frac{(30)(6)(9)(21)}{15} = 2{,}268,$$

$$\mathcal{R}_0 = \frac{(4)(18)^3}{4} = 5{,}832$$

$$0 + 2M_b\left(\frac{15}{I} + \frac{18}{1.5I}\right) + 0$$
$$= \frac{-2{,}268}{I} - \frac{5{,}832}{1.5I}$$

$$54M_b = -6{,}156, \qquad M_b = -114^{k'}$$
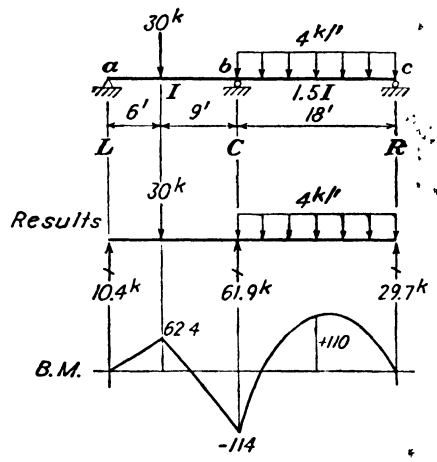


**Example 14·17**  *Compute the moments at the support points of this beam.   E and I are constant.*
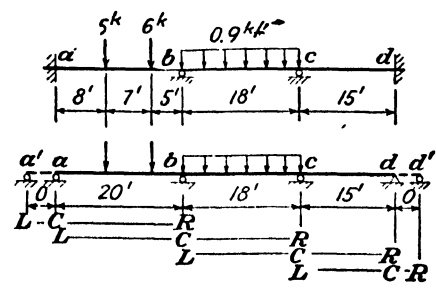


In this case, there are four unknown moments $M_a$, $M_b$, $M_c$, and $M_d$. Four equations are therefore required. A fixed end may be handled by replacing it by an additional span of essentially zero length as shown. The required equations may then be obtained by applying Eq. (14·7) four separate times, considering L, C, R in turn as indicated.

$$\delta_{a'} = \delta_a = \delta_b = \delta_c = \delta_d = \delta_{d'} = 0$$

Consider a', a, and b as L, C, and R.

$$\mathcal{L}_0 = 0, \qquad \mathcal{R}_0 = \frac{(5)(8)(12)(32)}{20} + \frac{(6)(15)(5)(25)}{20} = 1{,}330.5, \qquad M'_a = 0$$

$$\therefore 40M_b + 20M_c = -1{,}330.5 \tag{1}$$

*Consider a, b, and c as L, C, and R.*

$$\mathcal{L}_0 = \frac{(5)(8)(12)(28)}{20} + \frac{(6)(15)(5)(35)}{20} = 1,459.5, \qquad \mathcal{R}_0 = \frac{(6)(9)(18)}{4} = 1,312.2$$

$$\therefore 20M_a + 76M_b + 18M_c = -2,771.7 \tag{2}$$

*Consider b, c, and d as L, C, and R.*

$$\mathcal{L}_0 = 1,312.2, \qquad \mathcal{R}_0 = 0 \qquad \therefore 18M_b + 66M_c + 15M_d = -1,312.2 \tag{3}$$

*Consider c, d, and d' as L, C, and R.*

$$\mathcal{L}_0 = \mathcal{R}_0 = 0, \qquad M'_d = 0 \qquad \therefore 15M_c + 30M_d = 0 \tag{4}$$

*Solving Eqs. (1), (2), (3), and (4) simultaneously leads to the following values for the unknown moments:*

$$M_a = -19.17^{k'} \qquad M_b = -28.20^{k'} \qquad M_c = -13.74^{k'} \qquad M_d = +6.87^{k'}$$

---

**Example 14·18**  *Compute the reactions and draw the bending-moment diagram for this beam, due to the following support settlement:*

> *Support a rotates 0 005 radian clockwise.*
> *Support b settles 0.0208 ft down..*
> *E = 30 × 10³ kips per sq in.*

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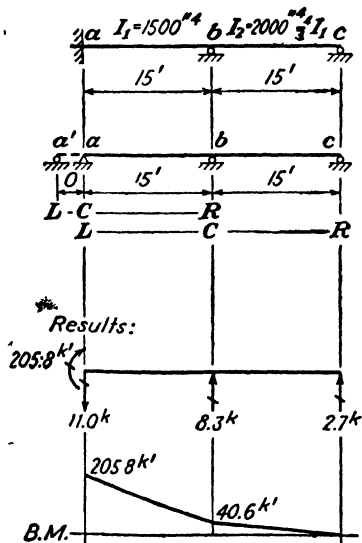


*Consider a', a, and b as L, C, and R.*

$$M'_a = 0, \qquad \mathcal{L}_0 = \mathcal{R}_0 = 0, \qquad \delta_a = 0,$$
$$\delta_b = -0.0208'$$

*As a result of rotation of support at a, $\delta_a'/l_L$ approaches +0.005. Applying Eq. (14·5),*

$$2M_a \left(\frac{15}{I_1}\right) + M_b \left(\frac{15}{I_1}\right) = 6E(0.005)$$
$$+ 6E \left(-\frac{0.0208}{15}\right)$$
$$\therefore 30M_a + 15M_b = 0.02166EI_1, \tag{1}$$

*Consider a, b, and c as L, C, and R.*

$$M_c = 0, \qquad \delta_a = 0, \qquad \delta_b = -0.0208,$$
$$\delta_c = 0$$

*Applying Eq. (14·5),*

$$M_a \left(\frac{15}{I_1}\right) + 2M_b \left(\frac{15}{I_1} + \frac{15}{\frac{4}{3}I_1}\right)$$
$$= -6E(-0.0208) \left(\frac{1}{15} + \frac{1}{15}\right)$$
$$\therefore 15M_a + 52.5M_b = +0.01664EI_1 \tag{2}$$

*Solving Eqs. (1) and (2) simultaneously and substituting $EI_1 = 31.25 \times 10^4$ kip-ft²,*

$$M_a = +0.000658EI_1 = +205.8^{k'}$$
$$M_b = +0.0001291EI_1 = +40.6^{k'}$$

**14·11   Development of the Slope-deflection Equation.**   The
*slope-deflection method* was presented by Prof. G. A. Maney in 1915 as a
general method to be used in the analysis of rigid-joint structures.   This
method extended the use of equations, originally proposed by Manderla
and Mohr for computing secondary stresses in trusses.   It is useful in its
own right; and even more important, it provides an excellent means of
introducing the method of moment distribution.

The following fundamental equations are derived by means of the
moment-area theorems.   Thus, these equations consider distortion
caused by bending moment but neglect that due to shear and axial stress.
Since the effect of axial stress and shear distortion on the stress analysis
of most indeterminate beams and frames is very small, the error that
results from using these equations as a basis for the slope-deflection
method of analysis is also very small (for corroboration, see the results
of Example 14·14).   The fundamental slope-deflection equation is an
expression for the moment on the end of a member in terms of four quan-
tities, *viz.*, the rotation of the tangent at each end of the elastic curve of
the member, the rotation of the chord joining the ends of the elastic curve,
and the external loads applied to the member.   It is convenient in the
application of this equation to use the following sign convention:

1.  Moments acting on the ends of a member are positive when *clock-
wise.*

2.  Let $\theta$ be the rotation of the tangent to the elastic curve at the end
of a member referred to the original direction of the member.   The
angle $\theta$ is positive when the tangent to the elastic curve has rotated
*clockwise* from its original direction.

3.  Let $\psi$ be the rotation of the chord joining the ends of the elastic
curve referred to the original direction of the member.   The angle $\psi$
is positive when the chord of the elastic curve has rotated *clockwise* from
its original direction.

In designating the end moments, two subscripts will be used; these
subscripts together designate the member under consideration, and the
first one designates the end of the member to which the moment is applied.
For example, $M_{AB}$ designates the moment acting on the $A$ end of member
$AB$; $M_{BA}$, the moment on the $B$ end of that member.   The $\theta$ angles will
be designated by one subscript indicating the end of the member.   The
$\psi$ angles will be designated by two subscripts indicating the chord and,
likewise, the member.

Using the above notation and convention, consider a member $AB$
that has a constant $E$ and $I*$ throughout its length and that is initially

* It is theoretically possible, of course, to set up the slope-deflection equation con-
sidering the effect of variable $I$.

straight. Suppose that this member is acted upon by the positive end moments $M_{AB}$ and $M_{BA}$ and any condition of an applied load, as shown in Fig. 14·8. Let $AB$ be the elastic curve of this beam and $A'B'$ represent



its original unstrained position. $\theta_A$, $\theta_B$, and $\psi_{AB}$ are positive as shown.
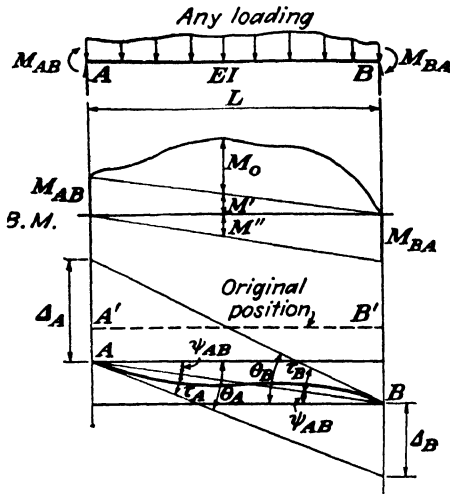
The bending-moment diagram for this member may be considered to be the superposition of three separate effects: the contribution of each of the end moments acting separately, which is given by the ordinates of the triangular portions, $M'$ and $M''$; and the contribution of the applied load acting by itself with the end moments removed, which is given by the ordinates $M_o$. In other words, the $M_o$ ordinates are the ordinates of the simple beam

FIG. 14·8

bending-moment diagram. The total bending moment at any point will be the algebraic sum of $M_o$, $M'$, and $M''$, but for this present derivation it is easier to consider these three contributions separately.

If the bending-moment diagram is converted into an $M/EI$ diagram, $\Delta_A$ and $\Delta_B$ may be evaluated by the second moment-area theorem. Then,

$$\Delta_A = -\frac{L^2}{6EI} M_{AB} + \frac{L^2}{3EI} M_{BA} - \frac{(\mathfrak{M}_o)_A}{EI} \qquad (a)$$

$$\Delta_B = \frac{L^2}{3EI} M_{AB} - \frac{L^2}{6EI} M_{BA} + \frac{(\mathfrak{M}_o)_B}{EI} \qquad (b)$$

in which $(\mathfrak{M}_o)_A$ is the static moment about a vertical axis through $A$ of the area under the $M_o$ portion of the bending-moment diagram and $(\mathfrak{M}_o)_B$ is a corresponding static moment about an axis through $B$.

Realizing that the angles and distortion shown in Fig. 14·8 are actually so small that an angle, its sine, and its tangent may all be considered equal, we see from the figure that

$$\left. \begin{array}{l} \dfrac{\Delta_A}{L} = \tau_B = \theta_B - \psi_{AB} \\[3mm] \dfrac{\Delta_B}{L} = \tau_A = \theta_A - \psi_{AB} \end{array} \right\} \qquad (c)$$

Solving Eqs. (a) and (b) simultaneously for $M_{AB}$ and $M_{BA}$ and substituting in the resulting expressions for $\Delta_A/L$ and $\Delta_B/L$ from Eqs. (c), we obtain

$$\left. \begin{array}{l} M_{AB} = \dfrac{2EI}{L}\,(2\theta_A + \theta_B - 3\psi_{AB}) + \dfrac{2}{L^2}\,[(\mathfrak{M}_o)_A - 2(\mathfrak{M}_o)_B] \\[3mm] M_{BA} = \dfrac{2EI}{L}\,(2\theta_B + \theta_A - 3\psi_{AB}) + \dfrac{2}{L^2}\,[2(\mathfrak{M}_o)_A - (\mathfrak{M}_o)_B] \end{array} \right\} \qquad (d)$$

Up to this point, the condition of loading has not been defined, and Eqs. (d) are valid for any condition of transverse loading. The last term in brackets in each of these equations is a function of the type of loading, and it is important that its physical significance be recognized. Suppose that $\theta_A$, $\theta_B$, and $\psi_{AB}$ are all equal to zero. Then the last terms of Eqs. (d) are, respectively, equal to the moment at the $A$ end and the moment at the $B$ end of the member. If, however, $\theta_A$, $\theta_B$, and $\psi_{AB}$ are all equal to zero, it means physically that both ends of the member are completely fixed against rotation or translation and, therefore, that this member is what we call a fixed-end beam. These last terms of Eqs. (d) are therefore equal to the so-called "fixed end moments." Calling fixed end moments FEM,

$$\left. \begin{array}{l} \mathrm{FEM}_{AB} = \dfrac{2}{L^2}\,[(\mathfrak{M}_o)_A - 2(\mathfrak{M}_o)_B] \\[3mm] \mathrm{FEM}_{BA} = \dfrac{2}{L^2}\,[2(\mathfrak{M}_o)_A - (\mathfrak{M}_o)_B] \end{array} \right\} \qquad (14\cdot8)$$

Substituting in Eq. (d) from Eqs. (14·8), we obtain

$$\left. \begin{array}{l} M_{AB} = \dfrac{2EI}{L}\,(2\theta_A + \theta_B - 3\psi_{AB}) + \mathrm{FEM}_{AB} \\[3mm] M_{BA} = \dfrac{2EI}{L}\,(2\theta_B + \theta_A - 3\psi_{AB}) + \mathrm{FEM}_{BA} \end{array} \right\} \qquad (14\cdot9)$$

Closer inspection of Eqs. (14·9) reveals that these two equations can be summarized by one general equation by calling the near end of a member $N$ and the far end $F$. Also, if we let

$$K_{NF} = \textit{stiffness factor} \text{ for member } NF = \frac{I_{NF}}{L_{NF}} \qquad (14\cdot10)$$

then the fundamental slope-deflection equation may be written as follows·

$$M_{NF} = 2EK_{NF}(2\theta_N + \theta_F - 3\psi_{NF}) + \mathrm{FEM}_{NF} \qquad (14\cdot11)$$

Of course, the FEM may easily be determined for any given loading. If, in addition, the rotation of the tangent at each end and the rotation

of the chord joining the ends of a member are known, the end moments in the member may easily be computed from Eq. (14·11). In Art. 14·12, the use of this equation in the solution of indeterminate beams and frames is discussed.

The FEM for any given loading may be evaluated by means of Eq. (14·8) in the following manner:

*Concentrated load* (see Fig. 14·9):

$$(\mathfrak{M}_o)_A = \frac{Pab}{L}\left[\frac{a}{2}\left(\frac{2a}{3}\right) + \frac{b}{2}\left(a + \frac{b}{3}\right)\right] = \frac{Pab}{6}(2a+b)$$

$$(\mathfrak{M}_o)_B = \frac{Pab}{L}\left[\frac{b}{2}\left(\frac{2b}{3}\right) + \frac{a}{2}\left(b + \frac{a}{3}\right)\right] = \frac{Pab}{6}(2b+a)$$

$$\left.\begin{array}{l} \text{FEM}_{AB} = \dfrac{2}{L^2}\left[\dfrac{Pab}{6}(2a+b) - 2\dfrac{Pab}{6}(2b+a)\right] = -\dfrac{Pab^2}{L^2} \\[3mm] \text{FEM}_{BA} = \dfrac{2}{L^2}\left[2\dfrac{Pab}{6}(2a+b) - \dfrac{Pab}{6}(2b+a)\right] = +\dfrac{Pa^2b}{L^2} \end{array}\right\}$$

$$(14·12)$$
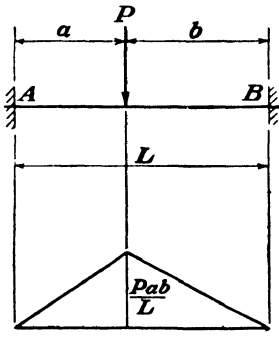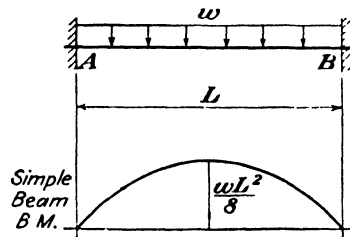


Fig. 14·9



Fig. 14·10

*Full uniform load* (see Fig. 14·10):

$$(\mathfrak{M}_o)_A = (\mathfrak{M}_o)_B = \frac{wL^4}{24}$$

$$\left.\begin{array}{l} \text{FEM}_{AB} = -\dfrac{wL^2}{12} \\[3mm] \text{FEM}_{BA} = +\dfrac{wL^2}{12} \end{array}\right\}$$

$$(14·13)$$

It will be noted that the proper signs of the FEM work out automatically from these calculations. In most cases, we know the direction of the end moments by inspection, and in this way we are usually able to verify the signs of the FEM.

Note again the sign convention to be used in applying the slope-deflection method. Note also that the above equations have been derived for a member that is initially straight and of constant $E$ and $I$.

**14·12 Application of Slope-deflection Method to Beams and Frames.** Consider first the application of the slope-deflection method to continuous beam problems, such as the beam shown in Fig. 14·11, the supports of which are assumed to be unyielding. Think of this beam as being composed of two members, $AB$ and $BC$, rigidly connected together at joint $B$. We could write expressions for the end moments at each end of each member, using Eq. (14·11).

These four end moments $M_{AB}$, $M_{BA}$, $M_{BC}$, and $M_{CB}$ would then be expressed in terms of the $\theta$ and $\psi$ angles and the FEM, which could be computed from Eqs. (14·12) and (14·13).
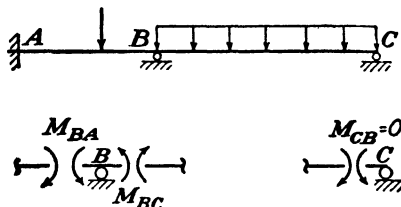
Since the supports are unyielding, we know in this case that $\theta_A$, $\psi_{AB}$, and $\psi_{BC}$ are all equal to zero. Further,



FIG. 14·11

since members $BA$ and $BC$ are rigidly connected together at joint $B$, the tangent to the elastic curve at the $B$ end of member $AB$ must rotate with respect to its original direction algebraically the same amount $\theta_B$ as the tangent at the $B$ end of member $BC$. Only the values of $\theta_B$ and $\theta_C$ are unknown, therefore, and involved in the expressions for the four end moments. If we could in some way find the values of $\theta_B$ and $\theta_C$, we could then compute all the end moments; and, having them, we could compute by statics any other moment, shear, or reaction we desired. In other words, the stress analysis of this beam would be reduced to a problem in statics if we knew the values of $\theta_B$ and $\theta_C$.

In this case, we are able to solve for these two unknowns by virtue of the fact that there are two convenient equations of statics which these end moments must satisfy. These equations are obtained by isolating joints $B$ and $C$ as shown in Fig. 14·11 and writing the equations $\Sigma M = 0$ for each of these joints. Thus,

$$\text{From } \Sigma M_B = 0, \quad M_{BA} + M_{BC} = 0$$
$$\text{From } \Sigma M_C = 0, \quad M_{CB} = 0$$

By substituting in these two equations the expressions for the end moments obtained by applying Eq. (14·11), we shall obtain two equations involving the two unknowns $\theta_B$ and $\theta_C$. After solving these equations simultaneously for these unknowns, we shall then be able to compute the end moments and complete the stress analysis of the beam. The actual numerical solution of such a problem is illustrated by

Example 14·19. In Example 14·20, these ideas have been extended and applied to a case where the supports move.

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**Example 14·19** *Compute the reactions and draw the shear and bending-moment diagrams for this beam. Unyielding supports.*

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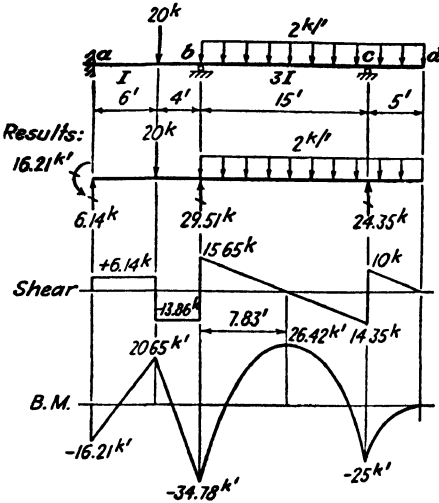


*Analyzing the $\theta$ and $\psi$ angles,*

$$\theta_a = 0, \qquad \psi_{ab} = \psi_{bc} = 0$$
$$\theta_b = ? \qquad \theta_c = ?$$

$$K_{ab} = \frac{I}{10} = K, \qquad K_{bc} = \frac{3I}{15} = 2K$$

$$FEM_{ab} = -\frac{(20)(6)(4)^2}{(10)^2} = -19.2^{k\prime}$$

$$FEM_{ba} = +\frac{(20)(6)^2(4)}{(10)^2} = +28.8^{k\prime}$$

$$FEM_{bc} = -\frac{(2)(15)^2}{12} = -37.5^{k\prime}$$

$$FEM_{cb} = +37.5^{k\prime}$$

*Using Eq. (14·11), write expressions for the end moments*

$$M_{ab} = 2EK\theta_b - 19.2$$
$$M_{ba} = 4EK\theta_b + 28.8$$
$$M_{bc} = 8EK\theta_b + 4EK\theta_c - 37.5$$
$$M_{cb} = 8EK\theta_c + 4EK\theta_b + 37.5$$

*Isolate joints b and c; write the joint equations; and substitute for $M_{ba}$, $M_{bc}$, etc.*

$$\Sigma M_b = 0, \qquad M_{ba} + M_{bc} = 0 \qquad \therefore 12EK\theta_b + 4EK\theta_c - 8.7 = 0 \quad (1)$$
$$\Sigma M_c = 0, \qquad M_{cb} - 25 = 0 \qquad \therefore 4EK\theta_b + 8EK\theta_c + 12.5 = 0 \quad (2)$$
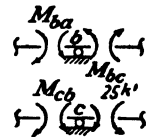
*Solving Eqs. (1) and (2), we find*

$$EK\theta_b = \underline{+1.495}, \qquad EK\theta_c = \underline{-2.31}$$

*Hence, substituting back in end-moment expressions,*

$$M_{ab} = +2.99 - 19.2 = \underline{-16.21^{k\prime}}$$
$$M_{ba} = +5.98 + 28.8 = \underline{+34.78^{k\prime}}$$
$$M_{bc} = +11.96 - 9.24 + 37.5 = \underline{-34.78^{k\prime}}$$
$$M_{cb} = -18.48 + 5.98 + 37.5 = \underline{+25.0^{k\prime}}$$

*The remaining results may be computed by statics.*

**Discussion:**

The cantilever portion *cd* adds no complications since the bending moment in this portion is statically determinate. The cantilever does, however, affect the joint equation at

joint c.   When isolating joints, assume unknown moments to be positive (i.e., to act clockwise on the end of the member and therefore counterclockwise on the end of the joint stub).   Any known moment, however, should be shown acting with its known value in its known direction.

A convenient way of handling the K factor is to select one K factor as a standard and express all others in terms of this.

Units will always be consistent if all values are substituted in kip and foot units.

An alternate way of handling the cantilever effect is to reason that the moment of c causes tension in the top fibers of the beam and therefore acts clockwise on the c end of member bc.   Hence, $M_{cb} = +25$ kip-ft, which is the same result as stated by Eq. (2) above.   This method fails, of course, if there is more than one member with an unknown end moment connected to this joint.   Note that the final results satisfy Eqs. (1) and (2).   This check does not verify the work prior to setting up these equations, of course.

---

**Example 14·20**   Compute the end moments for the beam in Example 14·19 caused only by the following support movements (no load acting):

> Support a, vertically 0.01 ft down, rotates 0.001 radian clockwise
> Support b, vertically 0.04 ft down
> Support c, vertically 0.0175 ft down
> Assume $E = 30 \times 10^3$ kips per sq in., $I = 1,000$ in.$^4$

---

In this case,

$$\theta_a = +0.001, \qquad \psi_{ab} = \frac{0.04 - 0.01}{10} = +0.003, \qquad \psi_{bc} = -\frac{(0.04 - 0.0175)}{15}$$
$$= -0.0015$$

but

$$\theta_b = ? \qquad and \qquad \theta_c = ?$$

There are no loads; therefore all FEM's are zero.   Using Eq. (14·11) to write expressions for end moments,

$$M_{ab} = 2EK(0.002 + \theta_b - 0.009) \quad = 2EK\theta_b - 0.014EK$$
$$M_{ba} = 2EK(2\theta_b + 0.001 - 0.009) = 4EK\theta_b - 0.016EK$$
$$M_{bc} = 2E(2K)(2\theta_b + \theta_c + 0.0045) = 8EK\theta_b + 4EK\theta_c + 0.018EK$$
$$M_{cb} = 2E(2K)(2\theta_c + \theta_b + 0.0045) = 4EK\theta_b + 8EK\theta_c + 0.018EK$$

From the joint equations,

$$\Sigma M_b = 0, \qquad M_{ba} + M_{bc} = 0, \qquad 12EK\theta_b + 4EK\theta_c + 0.002EK = 0 \qquad (1)$$
$$\Sigma M_c = 0, \qquad M_{cb} = 0, \qquad 4EK\theta_b + 8EK\theta_c + 0.018EK = 0 \qquad (2)$$

Solving Eqs. (1) and (2) simultaneously,

$$\theta_b = +0.0007 \qquad \theta_c = -0.0026$$

Therefore substituting back

$$M_{ab} = -0.0126EK = \underline{-262.5^{k'}}$$
$$M_{ba} = -0.0132EK = \underline{-275.0^{k'}}$$
$$M_{bc} = +0.0132EK = \underline{+275.0^{k'}}$$
$$M_{cb} = \qquad 0 \qquad = \underline{\quad 0 \quad}$$

*since*

$$EK = 30 \times 10^3 \times 144 \times \frac{1,000}{144^2 \times 10} = 20,833 \; kip\text{-}ft$$

**Discussion:**

   *In this type of problem, be particularly careful to insert the correct signs for the known $\theta$ and $\psi$ angles.   Also be careful to keep units consistent.*

Consider the rigid frames shown in Figs. 14·12a to *d*.   Suppose that we neglect the change in length of the members due to axial stress, as is usually permissible in rigid frames, and consider only the effect of bending distortion.   With this assumption, it is easy to show in each of these four frames that the $\psi$ angle is zero for every member (with the exception
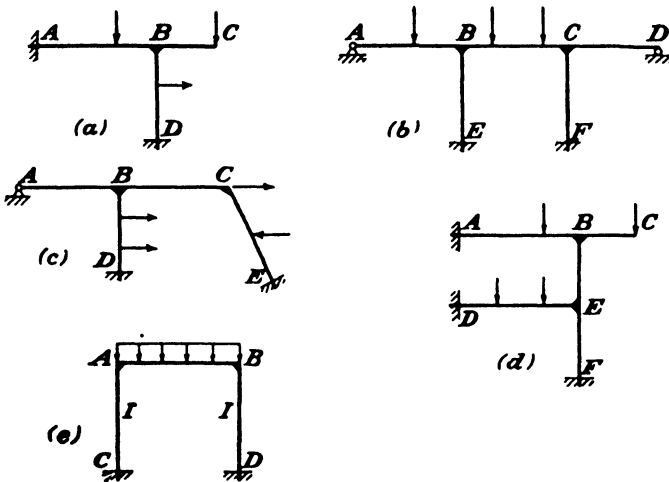


Fig. 14·12

of the statically determinate cantilever portions, of course).   In Fig. 14·12b, for example, neglecting the axial change in length of members $AB$ and $BE$, it is obvious that joint $B$ cannot move unless the supports move. If joint $B$ does not move, we can reason in the same manner that joint $C$ cannot move.   The $\psi$ angles must therefore be zero for all five members $AB$, $BC$, $CD$, $BE$, and $CF$.

   At any particular joint, the $\theta$ angle will be the same for the ends of all members that are rigidly connected together at that joint.   In the case of this frame, therefore, there will be only four unknown $\theta$ angles, $\theta_A$, $\theta_B$, $\theta_C$, and $\theta_D$.   Hence, using Eq. (14·11), we shall obtain expressions for the end moments involving these four $\theta$ angles as the only unknowns.   Since we may write a joint equation $\Sigma M = 0$ at each joint where there is an

unknown $\theta$ we are thereby able to obtain four equations containing the four $\theta$ angles as unknowns, in the same manner as in Example 14·19. After solving for the values of these $\theta$ angles, we may substitute back to obtain the values of the end moments, thus reducing the remainder of the stress analysis to a problem in statics.

In effect, we have shown that the slope-deflection solution of any frame in which there are no unknown $\psi$ angles is essentially the same as that of a continuous beam. The solution of the frame in Fig. 14·12e is likewise in this category, provided that both the make-up and the loading are symmetrical. In this special case, the deflections will also be symmetrical and therefore there can be no horizontal deflection of the column tops. As a result, the $\psi$ angles of all three members are zero.

In the most general type of rigid frame, both unknown $\theta$ and unknown $\psi$ angles are involved even when we consider only the effect of bending distortion. In other words, there are both joint rotations and chord rotations, or "sidesway." Several examples of such frames are shown in Fig. 14·15. Certain new ideas must be introduced to handle such problems. For this purpose, consider the frame shown in Fig. 14·13.
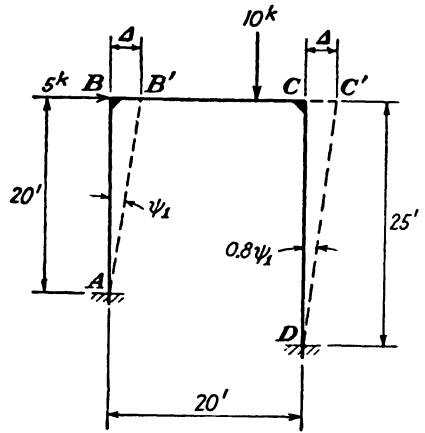


Fig. 14·13

If the supports are unyielding, there are only two unknown $\theta$ angles, $\theta_B$ and $\theta_C$. In this case, however, there is nothing that would prevent joint $B$ from moving. Since we are neglecting distortion due to axial stress and since the chord rotations of the members are small, joint $B$ moves essentially perpendicular to member $AB$, or, in this case, horizontally. Suppose that this movement is $\Delta$. In the same manner, we should reason that joint $C$ at the top of column $CD$ must likewise move horizontally. Since the axial change in length of $BC$ is also neglected, the horizontal movement of $C$ must also be $\Delta$.

The deflected position of the chords is shown by the dashed lines in Fig. 14·13. Note that these dashed lines indicate, not the elastic curve of the frame, but simply the chords of the elastic curve. From the sketch it is apparent that

$$\psi_{BC} = 0$$

$$\psi_{AB} = \frac{\Delta}{20} = \psi_1$$

and therefore

$$\psi_{CD} = \frac{\Delta}{25} = \frac{4}{5}\psi_1$$

In this case, therefore, the $\psi$ angles of the members of the frame may all be expressed in terms of one independent unknown, which we can call $\psi_1$.

Applying Eq. (14·11) in this case leads to expressions for the end
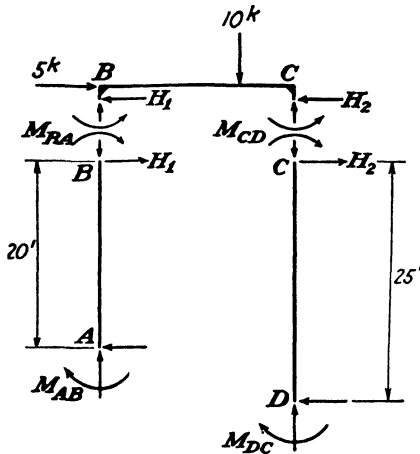


FIG. 14·11

moments that involve $\theta_B$, $\theta_C$, and $\psi_1$ as unknowns. Two of the three equations required to solve for these unknowns may be obtained from the familiar joint equations at joints $B$ and $C$. The third independent equation of statics that the end moments must satisfy must be obtained in some other manner. It may be derived as follows: Isolate each of the columns by cutting them out just below the girder and just above the foundation as shown in Fig. 14·14. Likewise, isolate the girder by cutting it out just below the top of the columns. Then, taking moments about the base of each isolated column,

$$M_{AB} + M_{BA} + 20H_1 = 0 \qquad (a)$$
$$M_{DC} + M_{CD} + 25H_2 = 0 \qquad (b)$$

Likewise from $\Sigma F_x = 0$ on the girder,

$$H_1 + H_2 = 5 \qquad (c)$$

Substituting for $H_1$ and $H_2$ in Eq. (c) from Eqs. (a) and (b), we obtain

$$M_{AB} + M_{BA} + 0.8M_{DC} + 0.8M_{CD} + 100 = 0 \qquad (d)$$

which is the required third independent equation of statics. From the two joint equations and this so-called "shear equation," we are able to arrive at three equations from which we can solve for $\theta_B$, $\theta_C$, and $\psi_1$ and then to complete the solution as in previous problems.

Examples 14·21 and 14·22 illustrate the slope-deflection solution of certain typical frames where sidesway is involved.

**Example 14·21**  *Compute the end moments and draw the bending-moment diagram for the frame:*

*Analyzing the $\theta$ and $\psi$ angles,*

$$\theta_A = \theta_D = \psi_{BC} = 0$$
$$\theta_B = ? \qquad \theta_C = ? \qquad \psi_{AB} = \psi_{CD} = \psi_1$$
$$K_{AB} = K_{CD} = \frac{I_1}{15} = K, \quad K_{BC} = \frac{4I_1}{20} = 3K$$
$$FEM_{BC} = -\frac{(100)(8)(12)^2}{(20)^2} = -288^{k'}$$
$$FEM_{CB} = +\frac{(100)(8)^2(12)}{(20)^2} = +192^{k'}$$

*Using Eq. (14·11),*

$$M_{AB} = 2EK\theta_B - 6EK\psi_1$$
$$M_{BA} = 4EK\theta_B - 6EK\psi_1$$
$$M_{BC} = 12EK\theta_B + 6EK\theta_C - 288$$
$$M_{CB} = 12EK\theta_C + 6EK\theta_B + 192$$
$$M_{CD} = 4EK\theta_C - 6EK\psi_1$$
$$M_{DC} = 2EK\theta_C - 6EK\psi_1$$

**Joint B:**

$$\Sigma M_B = 0, \qquad M_{BA} + M_{BC} = 0$$
$$\therefore 16EK\theta_B + 6EK\theta_C - 6EK\psi_1 - 288 = 0 \tag{1}$$

**Joint C:**

$$\Sigma M_C = 0, \qquad M_{CB} + M_{CD} - 500 = 0$$
$$\therefore 6EK\theta_B + 16EK\theta_C$$
$$- 6EK\psi_1 - 308 = 0 \tag{2}$$
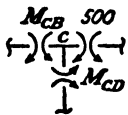
**Shear equation:**

Col. AB,  $M_{AB} + M_{BA} + 15H_1 = 0$
Col. DC,  $M_{DC} + M_{CD} + 15H_2 = 0$
Girder,  $H_1 + H_2 = 20$
$\therefore M_{AB} + M_{BA} + M_{CD} + M_{DC} + 300 = 0$  (3)

*Solving Eqs. (1), (2), and (3),*

$$EK\theta_B = +18.63 \qquad EK\theta_C = +20 60 \qquad EK\psi_1 = +22.34$$

*Then,*

$$M_{AB} = -96.9^{k'} \qquad M_{BA} = -59.5^{k'} \qquad M_{BC} = +59.2^{k'}$$
$$M_{CB} = +551.0^{k'} \qquad M_{CD} = -51.6^{k'} \qquad M_{DC} = -92.8^{k'}$$
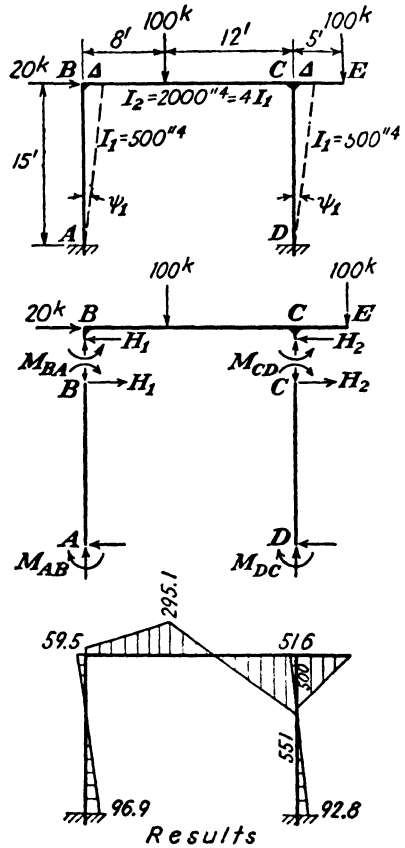
**Discussion:**

Note that the dashed lines in the first sketch indicate the deflected position of the chords of the elastic curve of the columns, not the elastic curve itself.

When various portions of a frame are isolated as shown in the second sketch, all unknown end moments should be assumed to be positive, i.e., to act clockwise on the ends of the members. Shears and axial stresses may be assumed to act in either direction, of course; but it having been assumed that a force such as $H_1$ acts in a certain direction on one isolated portion, it must be assumed that the force acts consistently on any subsequently isolated portions.

When plotting bending-moment diagrams for such frames, plot the ordinates on the side of the member that is in compression under the moment at that section of the frame.

**Example 14·22**   Compute the end moments and draw the bending-moment diagram for this frame:
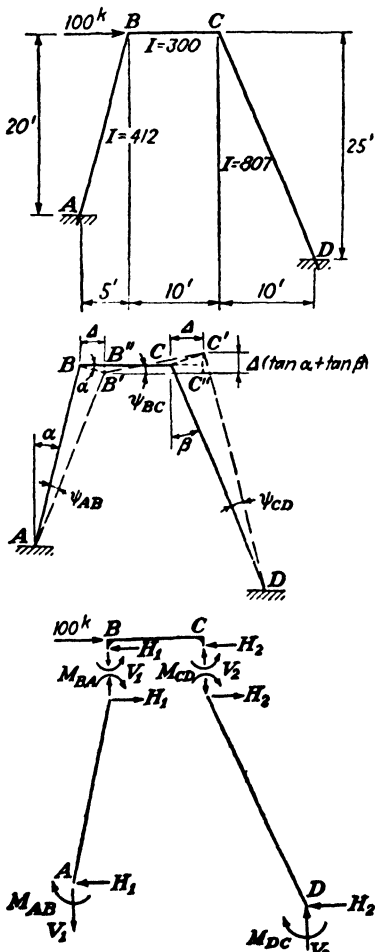


To analyze $\psi$-angle relationships, imagine that the members are temporarily disconnected at the joints and then reassembled one by one. First connect member $AB$ at the base. If the chord of this member did rotate an amount $\psi_{AB}$, the $B$ end would move essentially perpendicular to $AB$ along the path $BB'$. Translate member $BC$ parallel to itself so that the $B$ ends of $AB$ and $BC$ can be connected together at point $B'$. If the chord of $BC$ were now rotated, the $C$ end would move along a path $C''C'$ perpendicular to its original direction. Likewise, if member $CD$ were connected to the support at $D$ and the chord were rotated, the $C$ end would have to move along a path $CC'$ perpendicular to $CD$. The $C$ ends of members $BC$ and $CD$ could be joined at point $C'$ where these paths intersect. Then,

$$\psi_{AB} = \frac{BB'}{L_{AB}} = \frac{\Delta}{(\cos\alpha)L_{AB}} = \frac{\Delta}{20} = \frac{10}{13}\psi_1$$

$$\psi_{BC} = -\frac{C''C'}{10} = -\frac{\Delta}{10}(\tan\alpha + \tan\beta)$$

$$= -\frac{13}{200}\Delta = -\psi_1$$

$$\psi_{CD} = \frac{CC'}{L_{CD}} = \frac{\Delta}{(\cos\beta)L_{CD}} = \frac{\Delta}{25} = \frac{8}{13}\psi_1$$

*Therefore all $\psi$ angles can be expressed in terms of one unknown, $\psi_1$.  Since $\theta_A = \theta_D = 0$, the independent unknown are $\theta_B$, $\theta_C$, and $\psi_1$.*

$$K_{AB} = \frac{412}{20.6} = 20 = K, \qquad K_{BC} = \frac{300}{10} = 30 = 1.5K, \qquad K_{CD} = \frac{807}{26.9} = 1.5K$$

*All FEM's are zero since no loads are applied between joints.  Applying Eq. (14·11),*

$$M_{AB} = 2EK\theta_B - {}^{60}\!\!/_{13}EK\psi_1$$
$$M_{BA} = 4EK\theta_B - {}^{60}\!\!/_{13}EK\psi_1$$
$$M_{BC} = 6EK\theta_B + 3EK\theta_C + 9EK\psi_1$$
$$M_{CB} = 3EK\theta_B + 6EK\theta_C + 9EK\psi_1$$
$$M_{CD} = 6EK\theta_C - {}^{72}\!\!/_{13}EK\psi_1$$
$$M_{DC} = 3EK\theta_C - {}^{72}\!\!/_{13}EK\psi_1$$

*Joint B:*

$$\Sigma M_B = 0, \qquad M_{BA} + M_{BC} = 0 \qquad \therefore\ 10EK\theta_B + 3EK\theta_C + 4.385EK\psi_1 = 0 \quad (1)$$

*Joint C:*

$$\Sigma M_C = 0, \qquad M_{CB} + M_{CD} = 0 \qquad \therefore\ 3EK\theta_B + 12EK\theta_C + 3.462EK\psi_1 = 0 \quad (2)$$

*Shear equation: Col. AB, $\Sigma M_A = 0$,* $\qquad M_{AB} + M_{BA} + 20H_1 - 5V_1 = 0$ $\qquad$ (a)

$\qquad\qquad$ Col. CD, $\Sigma M_D = 0$, $\qquad M_{DC} + M_{CD} + 25H_2 - 10V_2 = 0$ $\qquad$ (b)

$\qquad\qquad$ Girder, $\Sigma M_C = 0$, $\qquad M_{BA} + M_{CD} + 10V_1 = 0$ $\qquad$ (c)

$\qquad\qquad\qquad \Sigma F_y = 0$, $\qquad V_1 = V_2$ $\qquad$ (d)

$\qquad\qquad\qquad \Sigma F_x = 0$. $\qquad H_1 + H_2 = 100$ $\qquad$ (e)

*Substitute in (a) and (b) for $V_1$, $V_2$, and $H_2$ from (c), (d), and (e); then eliminate $H_1$ from the modified Eqs. (a) and (b); and obtain*

$$M_{AB} + 2.3M_{BA} + 2.1M_{CD} + 0.8M_{DC} + 2,000 = 0$$

*or*

$$11.2EK\theta_B + 15EK\theta_C - 31.292EK\psi_1 = -2,000 \quad (3)$$

*Now solve Eqs. (1), (2), and (3) in the following tabular form:*

| Eq. | Operation | $EK\theta_B$ + | $EK\theta_C$ + | $EK\psi_1$ | $= const. \times 10^{-2}$ | Check |
|---|---|---|---|---|---|---|
| 1 | | +10 | + 3 | + 4 385 | 0 | +17.385 |
| 2 | | + 3 | +12 | + 3.462 | 0 | +18.462 |
| 3 | | +11.2 | +15 | −31.292 | −20 | −25.092 |
| 3' | 3 × 0.893 | +10 | +13.393 | −27.939 | − 17 857 | −22.404 |
| 3'' | 3 × 0.268 | + 3 | + 4.018 | − 8.382 | − 5.357 | − 6.721 |
| 4 | 1 − 3' | | − 10.393 | +32.324 | +17.857 | +39.789 |
| 5 | 2 − 3'' | | + 7.982 | +11.844 | + 5.357 | +25.183 |
| 4' | 4 × 0.768 | | − 7.982 | +24.825 | +13.715 | +30.559 |
| 6 | 5 + 4' | | | +36.669 | +19.072 | +55.742 |
| | | | | + 1 0 | + 0.5201 | |

$$7.982 EK\theta_C = 5.357 - 6.159 = -0.802, \qquad EK\theta_C = -0.1005$$
$$3EK\theta_B = -5.357 + 4.359 + 0.404 = -0.594, \qquad EK\theta_B = -0.1980$$

*The above results are for constants that are 0.01 of the actual constants. The actual results are 100 times the above.*

$$EK\theta_B = -19.80, \qquad EK\theta_C = -10.05, \qquad EK\psi_1 = +52.01$$

*The end moments are therefore*
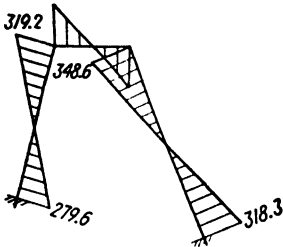


$$M_{AB} = -39.6 - 240.0 = -279.6^{k'}$$
$$M_{BA} = -79.2 - 240.0 = -319.2^{k'}$$
$$M_{BC} = -118.8 - 30.2 + 468.1 = +319.1^{k'}$$
$$M_{CB} = -59.4 - 60.4 + 468.1 = +348.7^{k'}$$
$$M_{CD} = -60.4 - 288.1 = -348.5^{k'}$$
$$M_{DC} = -30.2 - 288.1 = -318.3^{k'}$$

**Discussion:**

   *When analyzing the relations between the $\psi$ angles, be sure to record the proper signs for them, thus indicating whether the chord rotates clockwise or counterclockwise.*

   *In this problem, the solution of the simultaneous equations has been carried through in detail to illustrate a tabular procedure that is very convenient when there are three or more simultaneous equations to be solved. The important features of this procedure are as follows:*

   *1. When eliminating an unknown, select the equation having the highest coefficient in that column. Operate on this equation so as to obtain several reduced versions of it, each version adjusted so as to match the coefficient of the unknown, which is being eliminated, in one of the original group of equations. Proceeding in this manner tends to minimize errors rather than to increase them.*

   *2. Keep track of the operations performed on each equation, to facilitate checking.*

   *3. In the check column, record the algebraic sum of all the coefficients and constant terms in each equation. Operate on this figure the same as on all other terms in an equation. After any operation, the new sum of the coefficients and constant terms should be equal to the new figure in the check column. Note that this is a necessary check but is not sufficient to catch occasional compensating errors.*

   *4. The order of magnitude of the constant should be adjusted so as to be of the same order as the coefficients of the unknowns. This is done to make the check column most effective. After the solution of the equations is completed, the values of the unknowns may then be adjusted to give the answers corresponding to the actual constants.*

---

   By this time, the student should have started comparing the advantages and disadvantages of the different methods of stress analysis. He should have begun to form some idea as to when to use one method and when to use another. For example, he should study the structures in Figs. 14·12 and 14·15 and decide in each case whether the approach

using superposition equations or Castigliano's theorem is or is not superior to the slope-deflection method.

The amount of computation is more or less proportional to the square of the number of simultaneous equations involved in the solution. Generally speaking, the superior method is that involving the fewest unknowns. Thus, comparing the superposition-equation approach with the slope-deflection method consists largely in comparing the number of redundant stress components with the number of unknown $\theta$ and $\psi$ angles.



(a)          (b)          (c)

(d)

(e)

(f)

(g)          (h)
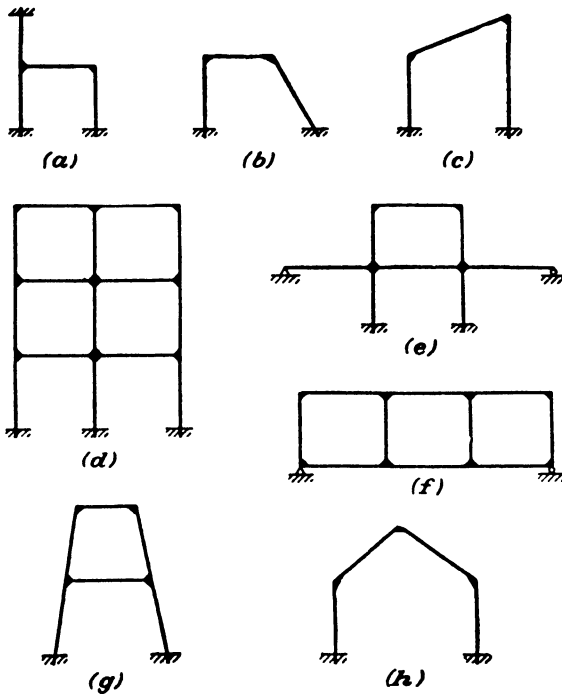
Fig. 14·15

14·13  Fundamentals of the Moment-distribution Method. The moment-distribution method is an ingenious and convenient method of handling the stress analysis of rigid-joint structures.[1]  All the methods discussed previously involve the solution of simultaneous equations, which constitutes a major part of the computational work when there are more than three or four unknowns.  The method of moment dis-

[1] The moment-distribution method was presented by Prof. Hardy Cross in an article in *Trans. ASCE*, vol. 96, Paper 1793, 1932, and in several prior publications and is without a doubt the most important contribution to structural analysis in recent years.

tribution usually does not involve simultaneous equations and is often much shorter than any of the methods already discussed. It has the further advantage of consisting of a series of cycles, each converging on the precise final result; therefore the series may be terminated whenever one reaches the degree of precision required by the problem at hand.

If we refer to Eq. (14·11), the fundamental slope-deflection equation, we observe that the moment acting on the end of a member is the algebraic sum of four separate effects, *viz.*:

1. The moment due to the applied loads on the member if it is a fixed-end beam, *i.e.*, the FEM
2. The moment caused by the rotation of the tangent to the elastic curve at the near end
3. The moment caused by the rotation of the tangent to the elastic curve at the far end
4. The moment caused by the rotation of the chord of the elastic curve joining the two ends of the member

Since the end moment is the superposition of these four effects, it is suggested that it might be possible to allow them to take place separately and thus arrive at their sum total.
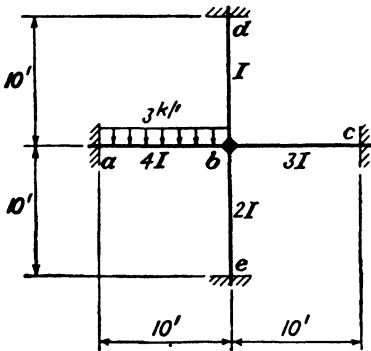


FIG. 14·16

To simplify the present discussion, we shall confine our attention to structures which are composed of prismatic members, *i.e.*, members which have a constant $I$ throughout their length; and also to structures in which there is no joint translation and in which, therefore, the $\psi$ angles are all equal to zero.

Consider such a structure as that shown in Fig. 14·16. If the supports are unyielding, there is no joint rotation at $a$, $d$, $c$, or $e$, but there will be some joint rotation at $b$ when the load is applied. However, suppose that we first consider the unloaded structure and imagine that we temporarily apply an external clamp which locks joint $b$ against rotation. Then, if the load is applied, fixed end moments will be developed in member $ab$ and these may be computed from Eqs. (14·13).

The moment $FEM_{ba}$ causes a counterclockwise moment on the locked joint $b$. If the clamp is released, this moment will cause the joint to rotate counterclockwise. When this joint rotates, certain moments are developed throughout the length of all members meeting at this joint. The joint will continue to rotate until sufficient end moments are devel-

oped in the $b$ ends of these members to balance the effect of $FEM_{ba}$. Of course, simultaneously certain end moments have been developed in the far ends of these members. When equilibrium is established at joint $b$, the structure in this case will have attained its final distorted position and the total end moment at the ends of the various members will be the algebraic sum of the fixed end moment and the moment caused by the rotation of joint $b$.

The procedure outlined above is essentially the moment-distribution method. It is convenient to adopt a certain terminology to simplify the description of the above procedure. We are already familiar with the term used to describe the end moments developed when the loads are applied to the structure with all joints locked against rotation— such moments are *fixed end moments*. When a joint is unlocked, it will rotate if the algebraic sum of the fixed end moments acting on the joint does not add up to zero; this resultant moment acting on the joint is therefore called the *unbalanced moment*. When the unlocked joint rotates under this unbalanced moment, end moments are developed in the ends of the members meeting at the joint. These finally restore equilibrium at the joint and are called *distributed moments*. As the joint rotated, however, and bent these members,



Fig. 14·17

end moments were likewise developed at the far ends of each; these are called *carry-over moments*.

Before evaluating these various moments numerically, it is desirable to adopt a convenient sign convention. Three different conventions are in use in the literature, but the authors prefer to use the same convention as that previously suggested for the slope-deflection method, *viz.*, end moments are plus when they act clockwise on the *ends* of the members.

Expressions for the *fixed end moments* have already been developed
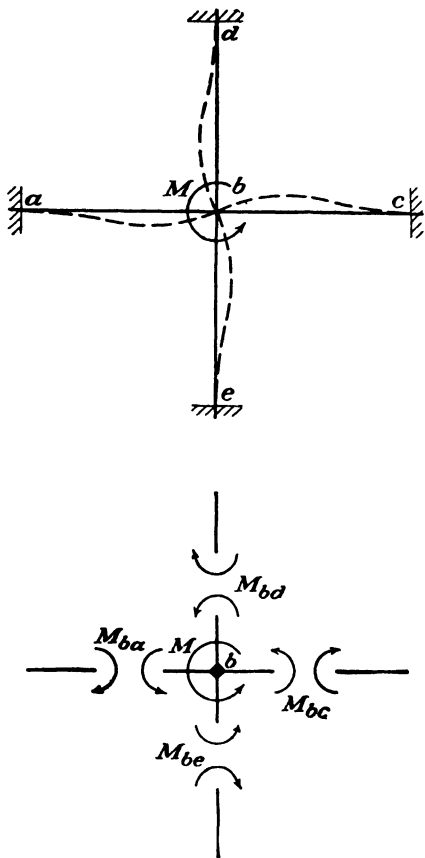
in Art. 14·11.   The *unbalanced moment* acting on a joint $n$ is simply the algebraic sum of the end moments at the $n$ end of all members that are rigidly connected to this joint.

Some further analysis is necessary to explain how to compute the *distributed moments*.   Consider the structure in Fig. 14·16 under the condition where joint $b$ is unlocked and allowed to rotate under the unbalanced moment $M$.   The structure distorts as shown in Fig. 14·17 and develops distributed moments $M_{ba}$, $M_{bd}$, etc., which restore the equilibrium of joint $b$.   These distributed moments, being unknown, are assumed to be positive and therefore to act clockwise on the ends of the members and counterclockwise on the joint.   The unbalanced moment $M$ is assumed to be the resultant of positive fixed end moments and therefore also acts counterclockwise on the joint.   Since $\Sigma M_b = 0$,

$$M_{ba} + M_{bc} + M_{bd} + M_{be} + M = 0 \tag{a}$$

However, the distributed moments may be evaluated by means of Eq. (14·11), it being noted in this case that $\theta_a = \theta_d = \theta_c = \theta_e = 0$ and that all $\psi$ angles are zero.

$$\left.\begin{aligned}
M_{ba} &= 4EK_{ab}\theta_b \\
M_{bd} &= 4EK_{bd}\theta_b \\
M_{bc} &= 4EK_{bc}\theta_b \\
M_{be} &= 4EK_{be}\theta_b
\end{aligned}\right\} \tag{b}$$

By substituting in Eq. (a) from Eq. (b), we can solve for $\theta_b$, substitute this value back in Eq. (b), and obtain expressions for $M_{ba}$, etc.   For example,

$$M_{ba} = \frac{-K_{ba}}{K_{ba} + K_{bd} + K_{bc} + K_{be}} M, \text{ etc.} \tag{c}$$

or, in general, the distributed moment in any bar $bm$ is given by

$$M_{bm} = -\frac{K_{bm}}{\sum\limits_{b} K} M \tag{d}$$

where the summation is meant to include all members meeting at joint $b$.   Let

$DF_{bm} = $ *distribution factor* for the $b$ end of member $bm = \dfrac{K_{bm}}{\sum\limits_{b} K}$    (14·14)

Then,

$$M_{bm} = -DF_{bm}M \tag{14·15}$$

Equation 14·15 may be interpreted as follows:

*The distributed moment developed at the b end of member bm as joint b is unlocked and allowed to rotate under an unbalanced moment M is equal to*

*the distribution factor $DF_{bm}$ times the unbalanced moment M with the sign reversed.*

An expression for *carry-over moment* must also be developed. Consider any member, one end $b$ of which has rotated $\theta_b$, developing a distributed moment $M_{bm}$ as shown in Fig. 14·18. It should be noted that, as joint $b$ was unlocked and allowed to rotate, the joint at $m$ remained locked and therefore $\theta_m$ is equal to zero. Since $\psi_{bm}$ is also zero, applying Eq. (14·11) leads to the following:

$$M_{bm} = 4EK_{bm}\theta_b \quad \text{and} \quad M_{mb} = 2EK_{bm}\theta_b$$

Hence,

$$M_{mb} = \tfrac{1}{2}M_{bm} \tag{14·16}$$

or, in words,

*The carry-over moment is equal to one-half of its corresponding distributed moment and has the same sign.*

We have now developed all the fundamental ideas and relations required to solve the simpler moment-distribution problems. Note, however, that the above discussion is restricted to structures composed of prismatic



FIG. 14·18

members. These ideas are extended to members having variable $I$ in Art. 14·15.
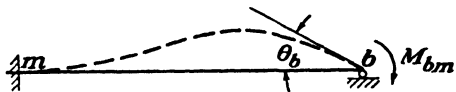
**14·14 Application of Moment Distribution to Beams and Frames.** For the first illustration of the application of moment distribution, consider the continuous beam shown in Fig. 14·19, the supports of which are unyielding. In this case, the $\psi$ angles of both members are zero, and support $a$ is permanently locked against rotation. Suppose that joints $b$ and $c$ are temporarily locked and the loads are applied, thereby developing the following fixed end moments:

$$\begin{aligned}
&FEM_{ab} = -9.6 \text{ kip-ft;} \quad && FEM_{bc} = -18.75 \text{ kip-ft} \\
&FEM_{ba} = +14.4 \text{ kip-ft;} \quad && FEM_{cb} = +18.75 \text{ kip-ft}
\end{aligned}$$

In preparation for unlocking joints $b$ and $c$ and distributing the unbalanced moments, the stiffness factors $K$ are computed, and from them the distribution factors at these joints are evaluated.

$$\begin{aligned}
&\text{At } b: \Sigma K = 0.3I, \quad && DF_{ba} = \tfrac{1}{3}, \quad && DF_{bc} = \tfrac{2}{3} \\
&\text{At } c: \Sigma K = 0.2I, \quad && DF_{cb} = 1
\end{aligned}$$

These distribution factors are then recorded in the appropriate box on the working diagram in Fig. 14·19. All the computations for the end moments are recorded on this diagram, the numbers referring to a particu-

lar end moment being recorded in a column normal to the member and running away from the member on the side first encountered in proceeding clockwise about the joint.   Such an arrangement of the computations is, of course, not imperative but will be found highly desirable for frames.

After recording the fixed end moments, joints $b$ and $c$ are successively unlocked and allowed to rotate gradually into their equilibrium positions. Upon unlocking joint $c$, it rotates under an unbalanced moment of $+18.75$ until a distributed moment of $-18.75$ is developed to restore equilibrium.
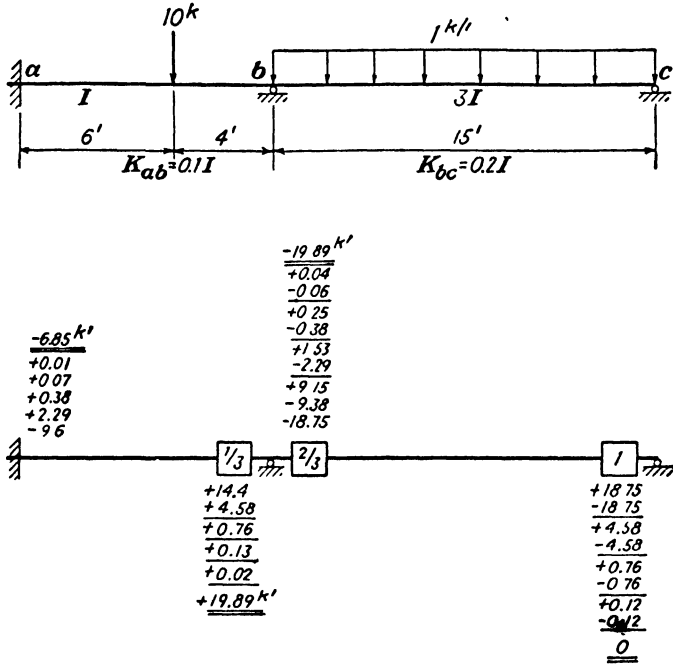


Fig. 14·19

Joint $c$ is relocked in this new position and a line is drawn under the $-18.75$ to indicate that the joint is now in equilibrium.   As joint $c$ rotated, a carry-over moment of one-half of the distributed moment was developed at the $b$ end of member $bc$, or in this case $-9.38$.

If joint $b$ is now unlocked, it will rotate under an unbalanced moment equal to the algebraic sum of the two fixed end moments at this joint and the above carry-over moment, or $-13.73$.   Using the appropriate distribution factors, the distributed moments that restore equilibrium are computed and are found to be $+4.58$ and $+9.15$, they are recorded and underlined, indicating that the joint is now in equilibrium.   Relocking this joint in this new position and recording the carry-over moments

developed at $a$ and $c$ by the rotation of joint $b$, leads to $+2.29$ and $+4.58$, respectively.

Returning now to joint $c$ and unlocking it a second time, it will rotate under an unbalanced moment of $+4.58$, developing a balancing distributed moment of $-4.58$. Again half of this distributed moment is carried over to the $b$ end of this member. Joint $c$ is locked in its new position. Now unlocking joint $b$ a second time, it will rotate under an unbalance of $-2.29$, developing distributed moments of $+0.76$ and $+1.53$, which in turn induce carry-over moments at $a$ and $c$ of $+0.38$ and $+0.76$.

We may proceed in this manner, unlocking first joint $c$ and then joint $b$, until the effects are so small that we are willing to neglect them. This problem has been carried through more cycles than required for practical purposes simply to illustrate the procedure. After the moment-distribution procedure is completed, the final end moments are obtained by adding algebraically all the figures in the various columns.

The convergence in the above solution has been rather slow owing to the fact that joint $c$ is a hinged end and continually throws back sizable carry-over moments to joint $b$. Whenever there is such a hinged end at the extremity of a structure, the convergence may be improved by modifying the above procedure as indicated in Fig. 14·20.

We start out as before, locking all joints against rotation, applying the loads, and developing the fixed end moments. Again as a first step, we unlock joint $c$, and let it rotate and develop the distributed moment of $-18.75$, which then carries over $-9.38$ to joint $b$. At this point, we leave joint $c$ unlocked so that it can rotate freely and hence can develop no end moment. Under these conditions, when we now proceed to joint $b$ and unlock it, it will rotate under the unbalanced moment of $-13.73$ with the joint $c$ unlocked instead of locked. Physically, this means that member $bc$ is not as stiff as it was previously and hence does not take so much of the unbalanced moment. We shall now evaluate just how much its stiffness has been reduced.

Refer to Figs. 14·16 and 14·17 and the development of an expression for distributed moments. Suppose that the support at $c$ were a roller or hinge support instead of a fixed support. Then, there would be a $\theta$ angle at $c$ as well as $b$ under the action of the unbalanced moment $M$, but $M_{cb}$ would be equal to zero. Upon applying Eq. (14·11), the previous analysis would be modified as follows:

$$M_{cb} = 4EK_{bc}\theta_c + 2EK_{bc}\theta_b = 0$$

from which

$$\theta_c = -\frac{\theta_b}{2}$$

**and** then

$$M_{bc} = 4EK_{bc}\theta_b + 2EK_{bc}\theta_c = 3EK_{bc}\theta_b = 4E(\tfrac{3}{4}K_{bc})\theta_b = 4EK_{bc}^R\theta_b$$

where,

$$K_{bc}^R = \text{reduced stiffness factor} = \tfrac{3}{4}K_{bc} \qquad (14\cdot17)$$

Hence, our previous expressions for distribution factors may be used for this new case provided that a reduced stiffness factor $K^R$ instead of the usual stiffness factor $K$ is used for a member where the far end is hinged.

We shall therefore revise the distribution factors at $b$, using the reduced stiffness factor for member $bc$.

$$\text{At } b\colon \quad \Sigma K = 0.25I, \qquad DF_{ba} = 0.4, \qquad DF_{bc} = 0.6$$

As a result, distributed moments of $+5.19$ and $+8.24$ are developed to balance the unbalanced moment of $-13.73$. There is no carry-over to
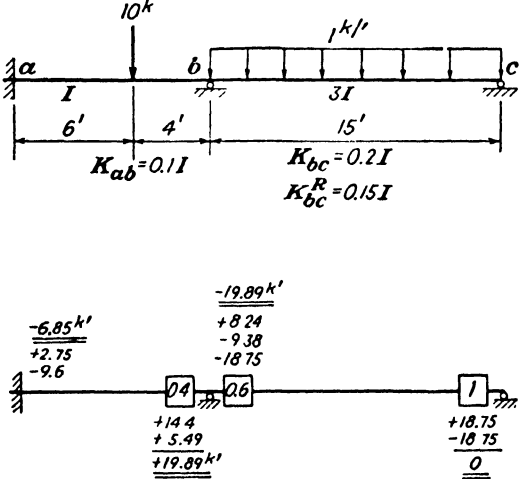


Fig. 14·20

$c$, of course, since this joint was left unlocked. There is the usual carry-over,· though, from $b$ to $a$. In this particular case, our solution is now complete since joint $c$ has not been unbalanced by the unlocking at $b$. All joints are therefore in equilibrium and will not be disturbed if the temporary locking device is permanently removed from joint $b$. Adding up the various columns will, in this case, now give us the exact end moments for this beam.

The revised procedure will not always give exact results as it did in this problem, but in any case it will give more rapid convergence than the original procedure. Example 14·23 is designed to illustrate the

application of the above ideas to a somewhat more elaborate structure that does not involve any unknown chord rotations. In Example 14·24, these ideas are extended to cover a case where the $\psi$ angles are not zero but their values are known.

**Example 14·23**  *Compute the end moments and draw the bending-moment diagram for this frame.   Unyielding supports.*



**Distribution factors:**

| At B, $\Sigma K = 2.8$ | At C, $\Sigma K = 3.0$ | At D, $\Sigma K = 3.788$ |
|---|---|---|
| $DF_{BA} = 0.268$ | $DF_{CB} = 0.417$ | $DF_{DC} = 0.330$ |
| $DF_{BC} = 0.446$ | $DF_{CD} = 0.417$ | $DF_{DJ} = 0.212$ |
| $DF_{BG} = 0.286$ | $DF_{CH} = 0.167$ | $DF_{DE} = 0.248$ |
| $\overline{1.000}$ | $\overline{1.000}$ | $DF_{DI} = 0.212$ |
| | | $\overline{1.002}$ |

**Fixed end moments:**

$$FEM_{AB} = -\frac{(30)(6)(4)^2}{(10)^2} = -28.8^{k\prime} \qquad FEM_{BA} = +\frac{(30)(6^2)(4)}{(10)^2} = +43.2^{k\prime}$$

$$FEM_{BC} = -\frac{(3)(16)^2}{12} = -64^{k\prime} \qquad FEM_{CB} = +64^{k\prime}$$

$$FEM_{CD} = -\frac{(20)(12)}{8} = -30^{k\prime} \qquad FEM_{DC} = +30^{k\prime}$$

**Cantilever moment,**

$$M_{EF} = -(10)(6) = -60^{k\prime}$$

*Moment distribution:*
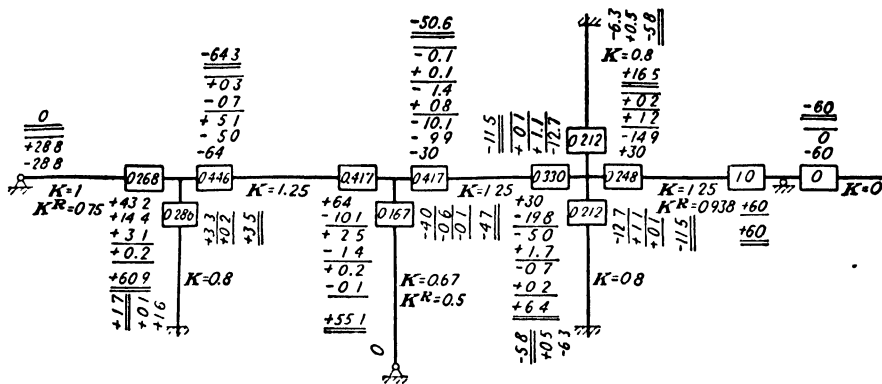


*Final results:*
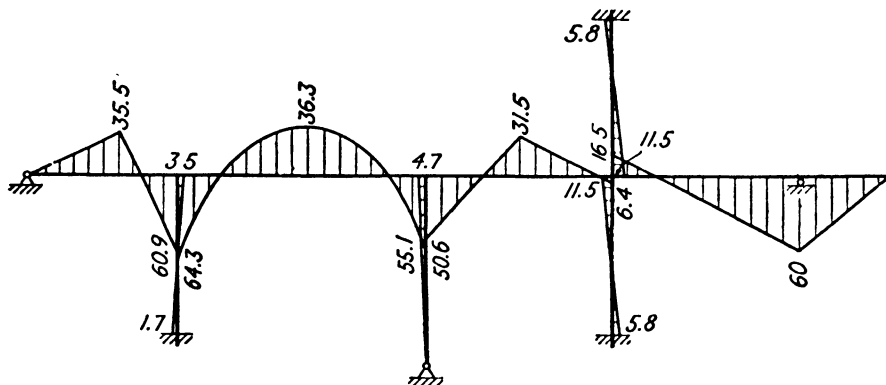


**Discussion:**

In this structure, when the loads are applied after all the joints have been locked against rotation, a moment will be developed at the E end of the cantilever, in addition to the usual fixed end moments at other points in the frame.   The moment of 60 kip-ft contributes to the unbalanced moment at joints E, just as a fixed end moment would.

Note that the cantilever arm has no restraining effect on the rotation of joint E; that is, in effect, its stiffness factor is equal to zero.   Any unbalanced moment is therefore carried entirely by the other members that meet at this joint.

Following the modified procedure, all hinged extremities must be unlocked first.   This includes not only joints A and H but also E.

The remaining joints are unlocked in turn and gradually rotated into their equilibrium positions.   If in doing this we start with the joint which has the largest unbalanced moment, the convergence of the solution will be somewhat more rapid, although the final results are independent of the order in which the joints are unlocked.

**Example 14·24** *Solve Example 14·20, using moment distribution.*

*Suppose that we temporarily lock all joints against rotation and that we then introduce the specified support movements, so that*

$$\theta_a = +0.001 \ radian$$
$$\psi_{ab} = +0.003 \ radian$$
$$\psi_{bc} = -0.0015 \ radian$$

*In effect, we have bent the members and developed certain initial end moments that can be evaluated by using Eq. (14·11), or*

$$M_{ab} = 2E(0.1I)(0.002 - 0.009)$$
$$= -0.0014EI = -292^{k'}$$
$$M_{ba} = 2E(0.1I)(0.001 - 0.009)$$
$$= -0.0016EI = -334^{k'}$$
$$M_{bc} = 2E(0.2I)(+0.0045)$$
$$= +0.0018EI = +376^{k'}$$
$$M_{cb} = +0.0018EI = +376^{k'}$$

*since*

$$EI = 30 \times 10^3 \times \frac{1,000}{144}$$
$$= 208,333 \ kip\text{-}ft^2$$

*The remainder of the solution is handled in exactly the same manner as if these initial end moments were fixed end moments. Thus, the distribution, etc., is done exactly as it was in Fig. 14·20.*
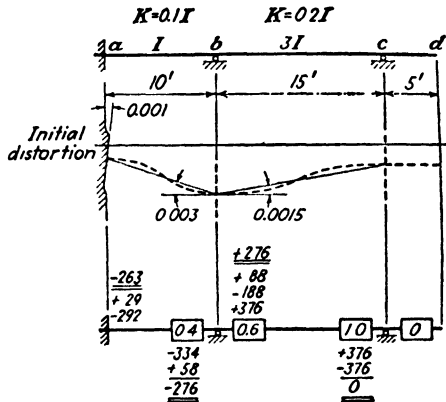


Once we understand the fundamental philosophy of moment distribution and the details of its application to the cases discussed above, it is quite easy to extend these ideas to more complicated cases involving sidesway, *i.e.*, unknown $\psi$ angles. Consider, for example, the frame shown in Fig. 14·21. In this case, there is nothing to prevent a horizontal deflection of the column tops. In addition to rotation of joints $B$ and $C$, we therefore have certain unknown chord rotations developed in the columns.

In order to handle this problem by moment distribution, we may break it down into two separate parts. First, suppose that we introduce a horizontal holding force $R$ which prevents any horizontal movement of joint $B$. With the structure restrained in this manner, we can apply all the given loads that are applied to the members between joints and

determine the resulting end moments by moment distribution just as we would for any structure that involved no sidesway. Having the end moments, we can then back-figure the holding force $R$ by statics. This part of the solution is called Case $A$ in Fig. 14·21.

In the second, or Case $B$, part of the solution, we imagine that we lock the joints against rotation, push on the frame at joint $B$, and introduce some arbitrary horizontal displacement $\Delta$. We can then analyze the $\psi$ angles thus introduced into the columns. Using Eq. (14·11), we can compute the corresponding initial end moments developed in the members. These end moments will be expressed in terms of $E\Delta$; but since $\Delta$ is arbitrary, we can let $E\Delta$ equal unity or any convenient amount and



Case A
Hold joint B:
Prevent sidesway

Case B
Push joint B:
Introduce sidesway

Fig. 14·21
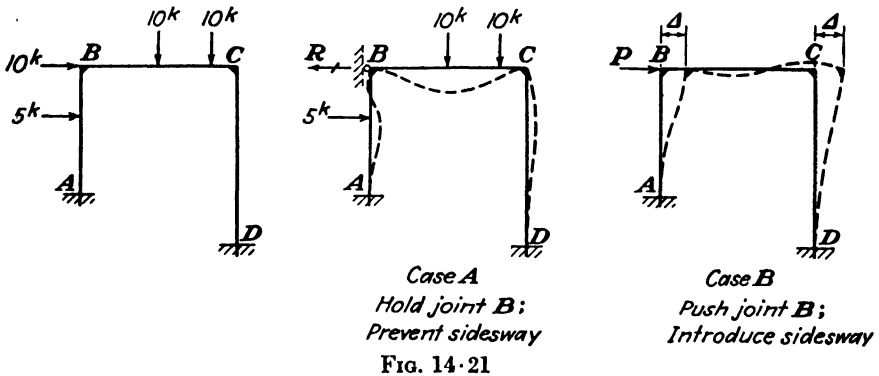
obtain numerical values for the initial end moments. If we then unlock, distribute, and carry over through several cycles at joints $B$ and $C$, we shall arrive at a set of end moments for the frame. Again by statics we can back-figure the force $P$ that has pushed the frame into this position and developed these end moments.

We are now in a position to superimpose the Case $A$ and Case $B$ parts of the solution so as to obtain the answers for the specified loads. To simplify the discussion of this superposition, let us assume that $R$ was computed to be a force of 2 kips acting to the left and $P$ a force of 6 kips acting to the right. We must take the Case $A$ moments as they are, since some of the external loads for this case are the same as the specified ones; but we can combine these results with any multiple of the Case $B$ results that we desire. Using the assumed values for $R$ and $P$, if we take twice the Case $B$ loads and superimpose them on the Case $A$ loads, we shall obviously obtain the specified load system. The final answers for the end moments are therefore obtained by adding algebraically twice the Case $B$ moments to the Case $A$ moments.

**Example 14·25**   *Compute the end moments in this frame:*

At *B*, $K_{AB} = 15$     $DF_{BA} = 0.429$
      $K_{BD} = 20$     $DF_{BD} = 0.571$
      $\Sigma K = 35$
At *D*, $K_{DB} = 20$     $DF_{DB} = 0.426$
      $K_{DE}^{R} = 15$     $DF_{DE} = 0.319$
      $K_{DC}^{R} = 12$     $DF_{DC} = 0.255$
      $\Sigma K = 47$
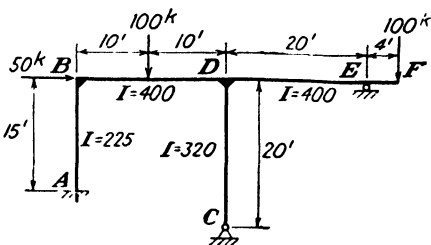


**Case A:** *Prevent sidesway.   Hold frame at joint B.*

$$FEM_{BD} = -250$$
$$FEM_{DB} = +250$$
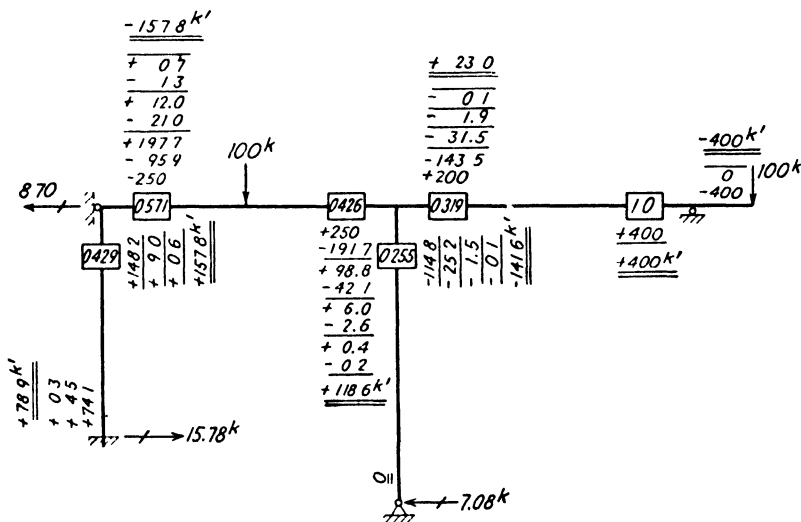


*By isolating columns, horizontal reactions at A and C may be computed.   Then, applying*
$\Sigma H = 0$ *to the entire structure, we find the holding force at B to be $8.70^k$ acting toward the left.*

**Case B:** *Introduce sidesway by pushing at point B, having first temporarily locked all joints (including C) against rotation.   Compute initial end moments from Eq. (14·11).*
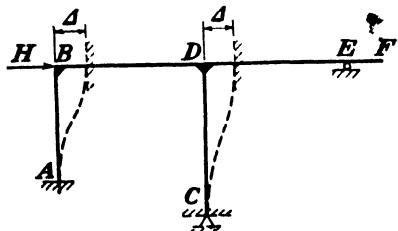


$$M_{AB} = M_{BA} = -6E(15)\left(\frac{\Delta}{15}\right) = -6E\Delta$$

$$M_{CD} = M_{DC} = -6E(16)\left(\frac{\Delta}{20}\right) = -4.8E\Delta$$

*Let* $E\Delta = 100;$ *then*
$$M_{AB} = M_{BA} = -600$$
$$M_{CD} = M_{DC} = -480$$

$+349.3$

$\begin{array}{r} - 0.5 \\ + 0.9 \\ - 8.3 \\ + 14.6 \\ \hline +342.6 \end{array}$

$+23.3$

$\begin{array}{r} + 0.1 \\ + 1.3 \\ \hline +21.9 \end{array}$

$66.0^k$

$\boxed{0.571}$    $\boxed{0.426}$    $\boxed{0.319}$    $\boxed{1.0}$    $\underset{\overline{777}}{0}$

$\boxed{2.429}$ $\underset{}{-600 \atop +257.4}$ $\begin{array}{c} 6.3 \\ 0.4 \end{array}$ $-349.3$

$\boxed{0.255}$

$+171.3$
$+29.3$
$-4.2$
$+1.8$
$-0.3$
$+0.1$

$-480 \atop +240$ $+17.5 \atop +1.1$ $+0.1 \atop -221.3$

$\underset{\overline{777}}{0}$

$-474.6^{k'}$ $\begin{array}{c} 0.2 \\ -3.1 \\ +128.7 \\ -600 \end{array}$

$54.93^k$

$+198.0$

$\underset{\overline{777}}{0} \, \begin{array}{|c} +480 \\ -480 \end{array} \, 11.07^k$

*As in case A, the pushing force H may be computed by statics*
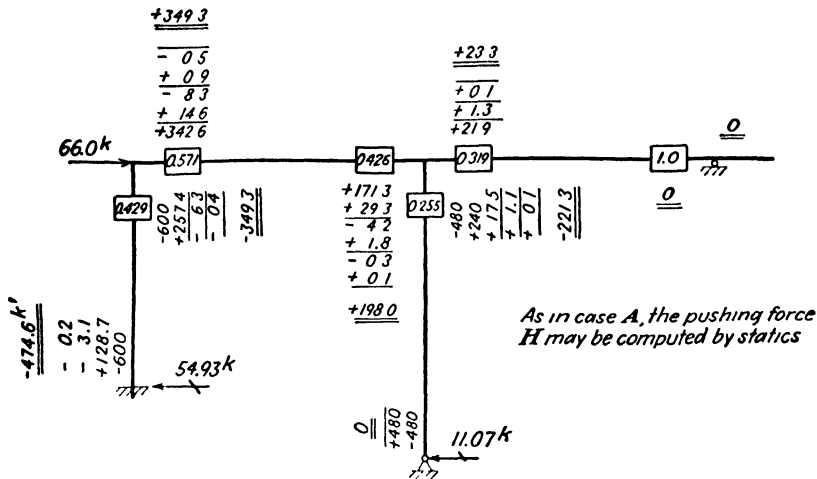
***Combination of Case A and Case B:*** *Let k = factor by which we must multiply the case B solution. Then,*

$$66.0k - 8.70 = 50, \qquad k = 0.89$$

$$\therefore final\ results = case\ A + (k)\ (case\ B)$$

$$M_{AB} = +78.9 - 421.5 = -342.6^{k'} \qquad M_{DC} = -141.6 - 196.7 = -338.3^{k'}$$

$$M_{BA} = +157.8 - 310.5 = -152.7^{k'} \qquad M_{DE} = +23.0 + 20.7 = +43.7^{k'}$$

$$M_{BD} = -157.8 + 310.5 = +152.7^{k'} \qquad M_{ED} = +400 + 0 = +400^{k'}$$

$$M_{DB} = +118.6 + 176.0 = +294.6^{k'}$$

**Discussion:**

When computing the initial end moments developed by a chord rotation, note that all joints, including hinged supports, are assumed to be locked against rotation. Note further that the stiffness factor $K$, not the reduced stiffness factor $K^R$, is used in these computations.

The reduced stiffness factor is used only to compute distribution factors and for no other purpose.

The end moments produced by any given horizontal load acting by itself at joint $B$ can be computed by straight proportion from the Case B part of the solution.

---

The ideas illustrated by Example 14·25 may be applied without modification to any frame involving only one independent $\psi$ angle. Space limitations prevent extending these ideas to frames involving more than one independent $\psi$ angle, but the extension is not difficult. In a frame having $n$ independent $\psi$ angles, or, as they are often called, $n$ degrees of freedom, we may break the problem down into a Case $A$ solution plus $n$ other cases in each of which only one independent $\psi$ angle is allowed to occur at a time. Then $n$ simultaneous equations must be set up and solved to find the $n$ factors that are to be used in

superimposing these $n$ cases on the Case $A$ solution. Each of these equations is similar to the equation written in Example 14·25 for this purpose.

An alternate procedure that does not require simultaneous equations can be used for cases involving sidesway. In this method, we introduce joint displacements at the same time as we apply the loads that develop fixed end moments. After each cycle of distribution, we check the equilibrium equations and back-figure the joint loads. If the joint loads do not agree with the given ones, we introduce some additional joint displacements, carry through another cycle of distribution, check the forces again, etc.

**14·15 Moment-distribution Method Applied to Nonprismatic Members.** The fundamental philosophy of the moment-distribution procedure developed in Art 14 13 is applicable to any beam or frame whether composed of

members of constant or varying $E$ and $I$. The expressions for fixed end moment, stiffness factor, and carry-over moment are derived specifically, however, for members of constant $E$ and $I$ and are not applicable to nonprismatic members. We shall now develop new expressions for these quantities for *the case of originally straight members of varying E and I*.

1. *Carry-over factor* $C$: The carry-over moment is the moment induced at a fixed end of a member when the opposite end is rotated by an end moment. It is convenient to express the carry-over moment at $B$ as being equal to the applied moment at $A$ multiplied by the



Fig 14·22

carry-over factor $C_{AB}$, the order of the subscripts indicating the direction in which the effect is carried over, i.e., from $A$ to $B$ in this case, or

$$M_{BA} = C_{AB}M_{AB} \qquad (4·18)$$

From this equation, the carry-over factor could be defined as the end moment induced at the fixed end of a member when the opposite end is rotated by an end moment of unity. For a prismatic member, the carry-over factor is ½. Note further that $C_{AB}$ is equal to $C_{BA}$ only when a member is symmetrical about its mid-point

An outline of the procedure for computing the carry-over factor in any given case follows: Consider any member $AB$ as shown in Fig. 14·22, and apply a moment of unity at $A$, thus inducing a moment $C_{AB}$ at $B$. The resulting deflection of point $A$ on the elastic curve from the tangent at $B$ is equal to zero. Upon applying the second moment-area theorem, the static moment of the composite $M/EI$ diagram taken about an axis through point $A$ must therefore be equal to zero. $C_{AB}$ may easily be evaluated from this equation.
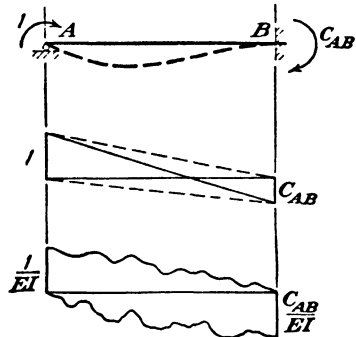
2. *True stiffness factor* $K'$: Let us refer to Art. 14·13 where the evaluation of distributed moments is discussed (page 418) and consider that Figs. 14·16 and 14·17 now represent structures composed of nonprismatic members. In considering the effect of the rotation of joint $B$ under the unbalanced moment $M$, we can no longer evaluate the distributed moments by Eq. (14·11). We can write, however,

$$\left. \begin{array}{l} M_{BA} = K'_{BA}\theta_B \\ M_{BC} = K'_{BC}\theta_C, \text{ etc.} \end{array} \right\} \qquad (a)$$

where $K'_{BA}$, $K'_{BC}$, etc., are called the *true stiffness factors* for the $B$ end of member $BA$, for the $B$ end of member $BC$, etc. From these equations, the true stiffness factor $K'_{BA}$ may be defined as the end moment required to rotate the tangent at the $B$ end of member $BA$ through a unit angle when the far end $A$ is fixed.

For a prismatic member, the true stiffness factor is $4EK$. Note that $K$, previously called simply the "stiffness factor" of a member, gives the relative value of the true stiffness factors for prismatic members. Perhaps $K$ should have been called the "relative stiffness factor," but this term is used so much that it is desirable to keep it as short as possible. Note further that, in general, a member has a different true stiffness factor at each end; only when the member is symmetrical will the true stiffness factor be the same at each end.



Fig. 14·23

The true stiffness factor $K'_{AB}$ for the $A$ end of member $AB$ shown in Fig. 14·23 may be computed as follows: By definition, an end moment of $K'_{AB}$ applied at $A$ will rotate the tangent at $A$ through a unit angle when the far end $B$ is fixed. Upon applying the first moment-area theorem, the net area under the composite $M/EI$ diagram must therefore be equal to unity. If $C_{AB}$ has previously been computed as described above, $K'_{AB}$ may now be determined from this equation.



Fig. 14·24

3. *True reduced stiffness factor* $K'^R$: The true reduced stiffness factor $K'^R_{AB}$ may be defined as the end moment required to rotate the tangent at the $A$ end through a unit angle when the far end is *hinged*. $K'^R_{AB}$ may easily be computed by proceeding as follows, provided that $K'_{AB}$, $C_{AB}$, and $C_{BA}$ have already been computed: Consider a bar $AB$, and suppose temporarily that we lock the $B$ end against rotation. A moment of $K'_{AB}$ applied at the $A$ end would produce a rotation of $A$ of unity and induce a carry-over moment at the far end of $C_{AB}K'_{AB}$. If we now lock the $A$ end in this position and unlock the $B$ end, the latter end
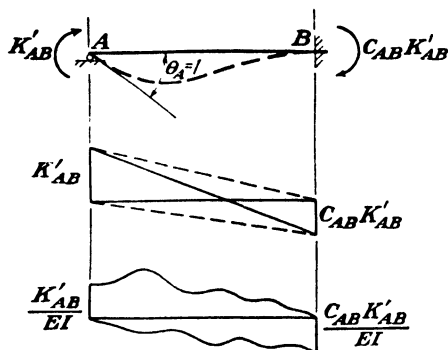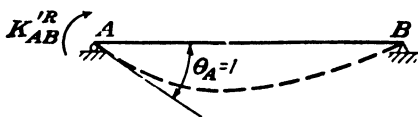
will rotate under the unbalanced moment $C_{AB}K'_{AB}$ and develop a distributed moment of $-C_{AB}K'_{AB}$. As a result, a carry-over moment of $-C_{BA}C_{AB}K'_{AB}$ is induced at $A$. The member will now be in the distorted condition shown in Fig. 14·24, and the total end moment at $A$ will be equal to the true reduced stiffness factor $K'^R_{AB}$. Hence,

$$K'^R_{AB} = K'_{AB}(1 - C_{BA}C_{AB}) \qquad (14·19)$$

For a prismatic member where both $C_{AB}$ and $C_{BA}$ are equal to one-half, we see from Eq. (14·19) that $K'^R_{AB}$ is equal to three-fourths of $K'_{AB}$, as it should be.

4. *Check relationship involving stiffness and carry-over factors:* A useful relationship between stiffness and carry-over factors may be derived and used for checking purposes. Consider the member $AB$ shown in



Fig. 14·25

Fig. 14·25, where it is acted upon by two separate loading systems (I) and (II). Applying Betti's law, we find that,

$$C_{AB}K'_{AB} = C_{BA}K'_{BA} \qquad (14·20)$$

This expression proves very helpful when used as a check on the values of the carry-over and stiffness factors.

5. *Distribution factors and distributed moments:* If we complete the evaluation of distributed moments using the expressions in Eqs. (a) of this article and proceeding as in Art. 14·13, we obtain an expression for the distribution factor which is the same as Eq. (14·14) except that true stiffness factors $K'$ are substituted for the stiffness factors $K$.

6. *Sidesway factor:* It is likewise easy to develop expressions for the sidesway factor $J$ that may be used to compute the initial end moments developed by a chord rotation. We can write the following expressions for such end moments:



Fig. 14·26

$$M_{AB} = J_{AB}\psi_{AB}$$
$$M_{BA} = J_{BA}\psi_{AB} \qquad (14·21)$$

The sidesway factor $J_{AB}$ may be defined as the end moment developed at the $A$ end of member $AB$ by a chord rotation of unity, both ends of the member being locked against joint rotation. For a prismatic member, the sidesway factor is $-6EK$. The sidesway factor will be the same at each end only when the member is symmetrical.

The sidesway factor for member $AB$ in Fig. 14·26a may be expressed in terms of stiffness and carry-over factors by reasoning as follows: Suppose that the fixity is removed from the $A$ and $B$ ends temporarily and that the member is dis-

placed as shown in Fig. 14·26b. Now lock the $A$ end and rotate the $B$ end until $\theta_B$ is restored to zero, thus developing end moments of $-K'_{BA}$ at $B$ and $-C_{BA}K'_{BA}$ at $A$. Lock the $B$ end in this position and rotate the $A$ end until $\theta_A$ is likewise restored to zero, thus developing end moments of $-K'_{AB}$ at $A$ and $-C_{AB}K'_{AB}$ at $B$. The member will now be in the same condition of distortion as shown in Fig. 14·26a, and the total end moments will be equal to the sidesway factor, or
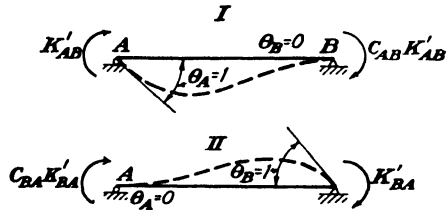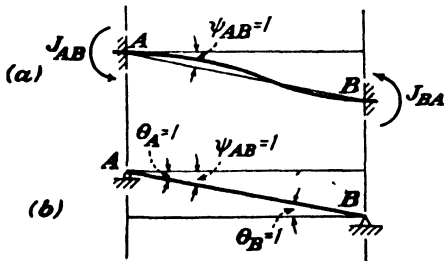
$$\left.\begin{array}{l} J_{AB} = -(K'_{AB} + C_{BA}K'_{BA}) \\ J_{BA} = -(K'_{BA} + C_{AB}K'_{AB}) \end{array}\right\} \quad (14\cdot22)$$

7. *Fixed end moments:* The fixed end moments in a nonprismatic member may be computed as follows: Consider member $AB$ acted upon by any loading. Assume temporarily that the load is applied to this member acting as an end-supported beam. For this condition, compute the resulting rotations of the tangents $\theta_{Ao}$ and $\theta_{Bo}$, using the moment-area theorems. Then if we imagine that we lock the $B$ end and rotate the $A$ end back to a zero slope, and then lock the $A$ end in this position and rotate the $B$ end back the resulting end moments will be found to be
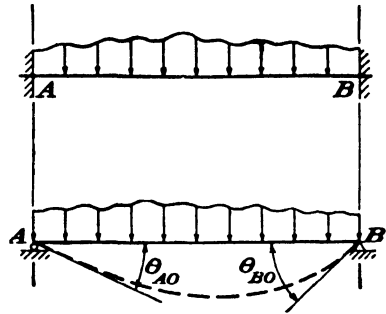


Fig. 14·27

$$\left.\begin{array}{l} \text{FEM}_{AB} = -K'_{AB}\theta_{Ao} + C_{BA}K'_{BA}\theta_{Bo} \\ \text{FEM}_{BA} = +K'_{BA}\theta_{Bo} - C_{AB}K'_{AB}\theta_{Ao} \end{array}\right\} \quad (14\cdot23)$$

8. *General remarks:* The above expressions are strictly applicable only to members the axes of which are originally straight. They may be used with satisfactory accuracy for members with slightly curved axes provided that the structure is arranged so that the axial thrust developed in such members is relatively small. Consideration of more accurate expressions for curved members is beyond the scope of this book.

**14·16 Stress Analysis of Statically Indeterminate Space Frameworks.** Statically indeterminate space frameworks may be analyzed by means of superposition equations or Castigliano's theorem. Fundamentally, the stress analysis of such structures by these methods is accomplished in exactly the same manner as for an indeterminate truss. The detail of the analysis is more tedious, of course, for the stress analysis of the statically determinate primary structure is more complicated when it is a three-dimensional framework instead of a planar truss. All the ideas discussed above for indeterminate trusses are applicable here, but a detailed extension of these ideas to indeterminate space frameworks is not within the scope of this book.

**14·17 Deflection of Statically Indeterminate Structures.** Once the stress analysis of an indeterminate structure has been completed, then the strains and the resulting deflections of the structure may be computed without difficulty by the same procedures as those discussed and applied to statically determinate structures in Chap. 13.

For example, suppose that we have computed the bar stresses and the corresponding changes in length of the members of the truss shown in Fig. 14·28a, due to a given loading. Suppose that we then wish to compute the resulting vertical deflection of joint $f$ by the method of virtual work. We could proceed in the usual manner and apply a unit vertical load at joint $f$ of the actual structure shown in Fig. 14·28a. This load would cause $F_Q$ stresses in most of the bars of the indeterminate truss. After these stresses had been computed, we could apply Eq. (13·5) and compute the required deflection of joint $f$. But this procedure, though straightforward, is laborious since most of the bars of the truss are involved in the computations.

Deflection computations for indeterminate structures may be simplified considerably if we simply recognize that the redundants of such structures have been computed so that the stresses, strains, and deflections of the primary and actual structures are identical. Deflections may be computed, therefore, by using the statically determinate primary structure rather than the actual indeterminate structure. It is much easier to use the primary structure, of course. For example, in the above structure, suppose that we use the primary structure shown in Fig. 14·28b and proceed to compute the vertical deflection of joint $f$ by the method of virtual work. In this case, the unit load at joint $f$ would develop $F_Q$ stresses in only four bars, namely, $ef$, $fg$, $eF$, and $Fg$. As a result, when Eq. (13·5) is applied, only these four bars contribute to the sum of $F_Q \, \Delta L$, the computations being thus tremendously reduced in this particular case.
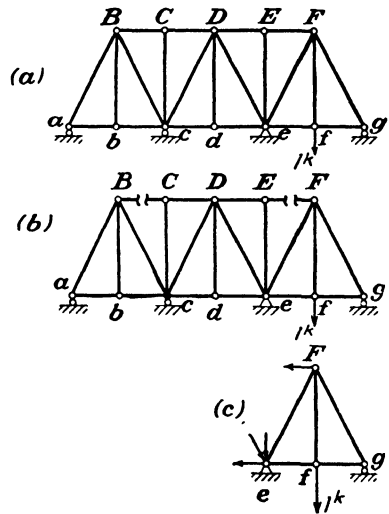


FIG. 14·28

To some, a descriptive explanation such as is given above is not adequate, and they prefer a mathematical demonstration. Although it will not be done here, it can be proved mathematically that the results are identical whether the unit $Q$ load is considered to be applied to the actual indeterminate structure or to the statically determinate primary structure.

Physically, it seems obvious that the deflections computed for the right span of the actual structure must be identical with the deflections of this portion isolated as a simple span and considered to be acted upon by any given external loads and also by external forces corresponding to the bar stresses in the members cut by the isolating section as shown in Fig. 14·28c. These ideas are even more obvious from a physical viewpoint in the case of the deflections of a continuous beam. In such a structure, it seems perfectly natural to isolate each span and compute its deflections by considering it as a simple end-supported beam acted upon by any loads which act in that span and also by end moments which are equivalent to the bending moments at the corresponding support points of the actual continuous beam.

14·18   **Secondary Stresses in Trusses.**   In Chap. 4, it will be recalled that the elementary stress analysis of a truss is based upon the following assumptions:

1. The members are connected together at their ends by frictionless pin joints.

2. All external loads and reactions (including the weight of the members) are applied to the truss at the joints.

3. The centroidal axes of all members are straight, coincide with the lines connecting the joint centers, and lie in a plane that also contains the lines of action of all the loads and reactions.

A stress analysis based on these assumptions leads to the determination of so-called "primary stresses."

Stresses caused by conditions not considered in the primary stress analysis are called *secondary stresses.*   Of these the most important are caused by the fact that the joints are rigid and hence the members are not free to change their relative directions when the truss is distorted.   Several classical methods[1] are available for making approximate analyses for secondary stresses.   This problem can likewise be solved very efficiently by means of moment distribution.   The latter method is the only one that will be discussed here.

A truss with rigid joints is actually a rigid frame.   Theoretically, we could analyze such a frame by the use of superposition equations or Castigliano's theorem, considering distortion due to both bending and axial stress.   Such structures would be so highly statically indeterminate, however, that it would not be practical to carry through an exact stress analysis.   We have already noted that the deflection of the joints is primarily a function of the axial stresses in the members since the bending of the members has only a second-order effect on their axial change in length.   Further, it can be shown that the axial stresses in the members of a truss with rigid joints are essentially the same as those in an ideal pin-jointed truss, *i.e.,* that the presence of shears and moments in the members has a small effect on the axial stresses.[2]   This suggests that the axial stresses and the resulting joint deflections can be computed assuming the truss to be pin-jointed.   If the joint deflections are known, all the $\psi$ angles of the members can be computed and the remainder of the problem solved either by slope deflection or moment distribution, in the same manner as for any frame in which only the joint rotations or $\theta$ angles are unknown.

The solution of a secondary stress problem using moment distribution may be outlined as follows:

1. Compute the bar stresses, *i.e.,* primary stresses, assuming the truss to be pin-jointed.

2. Compute the joint deflections and the corresponding chord rotations of the members.   This can be done conveniently by means of the Williot method or the angle-change procedure used in Example 14 26.

[1] A very comprehensive paper on these methods was presented by Cecil Vivian von Abo, *Trans. ASCE,* vol. 89, 1926.

[2] See Parcel, J. I., and G. A. Maney, "Statically Indeterminate Stresses," 1st ed., Chap. VII, John Wiley & Sons, Inc., New York, 1926.

3. Compute the initial end moments developed if all the joints are locked against rotation and then the above joint deflections and chord rotations introduced.    Likewise, compute any unbalanced moments that may act on the joints as a result of eccentricities of the bar stresses.

4. Distribute and carry over through as many cycles as is necessary.    The final end moments thus obtained will be the first approximation of the secondary end moments, from which the secondary stresses can be computed.    From these end moments, the shears in the members can be computed as well and also, by statics, new values of the bar stresses in the members.

5. If these new values of the bar stresses are markedly different from those computed in step 1, repeat steps 2, 3, and 4, and thus arrive at a second approximation for the secondary stresses.

This procedure is illustrated by Example 14·26.

---

**Example 14·26**    *Compute the secondary stresses in the members of this truss.*

---

| Bar | L | A | I | c | K | Make-up |
|---|---|---|---|---|---|---|
| | '' | ''² | ''⁴ | '' | ''³ | |
| ab bc | 300 | 18 0 | 174.9 | 6 375 | 0 583 | 4 ⌐ s6 × 3½ × ½ |
| Bb | 336 | 15 88 | 153.4 | 6 375 | 0 456 | 4 ⌐ s6 × 3½ × 7⁄16 |
| Bc | 450 4 | 13 68 | 131.4 | 6.375 | 0 292 | 4 ⌐ s6 × 3½ × 3⁄8 |
| Cc | 336 | 11.44 | 78 9 | 5.375 | 0 235 | 4 ⌐ s5 × 3 × 3⁄8 |
| aB | 450 4 | 27 68 | 960 9 | 5.739 / 9 699 | 2 134 | 2[s 15″ — 33 9# / 1Pl 18 × 7⁄16 |
| BC | 300 | 26.55 | 922 7 | 5.921 / 9.453 | 3.076 | 2[s 15″ — 33 9# / 1Pl 18 × 3⁄8 |



*Bar stresses ⎮ Stress intensities*
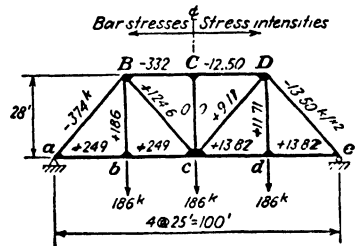
NOTES: (1) *Members ab, bc, Bb, Bc, and Cc assembled so:*

(2) *Working lines of members aB and BC lie on ℄ of channels.*

**Step 1:** *Primary bar stresses and stress intensities shown on line diagram.*
**Step 2:** *Compute ψ angles using angle changes computed by Eq. (13·11b).*

| Angle | $f_2 - f_1$ | cot $\beta_1$ | $f_2 - f_2$ | cot $\beta_2$ | 1st term | 2d term | $E \Delta\gamma$ |
|---|---|---|---|---|---|---|---|
| B-a-b | +11 71+13 50 = +25 21 | 1.12 | | 0 | +28 25 | 0 | +28 25 |
| b-B-a | +13 82+13 50 = +27 32 | 0 893 | | 0 | +24 40 | 0 | +24 40 |
| a-b-B | -13 50-13 82 = -27.32 | 0 893 | -13.50-11 71 = -25 21 | 1 12 | -24 40 | -28 25 | -52 65 |
| c-B-b | +13 82- 9 11 = +4 71 | 0 893 | | 0 | + 4 21 | 0 | + 4 21 |
| B-b-c | + 9.11-11 71 = -2 60 | 1.12 | + 9.11-13 82 = -4 71 | 0 893 | - 2 91 | - 4 21 | - 7 12 |
| b-c-B | | 0 | +11.71- 9 11 = +2 60 | 1 12 | 0 | + 2 91 | + 2 91 |
| C-B-c | | 0 | 0 -9 11 = - 9 11 | 1 12 | 0 | - 10 20 | - 10 20 |
| c-C-B | + 9 11 - 0 = +9.11 | 1 12 | + 9 11+12 50 = +21 61 | 0 893 | +10 20 | + 19 30 | +29 50 |
| B-c-C | | 0 | -12 50 - 9 11 = - 21 61 | 0 893 | 0 | - 19 30 | - 19 30 |

Then, $E\psi_{Cc} = 0$   $E\psi_{aB} = +47.91$   In this case, $\psi_{Cc}$ is known to equal zero by
   $E\Delta_{BcC} = -19.30$   $E\Delta_{aBb} = +24.40$   symmetry.   All these $\psi$ angles are there-
   $E\psi_{Bc} = +19.30$   $E\psi_{Bb} = +23.51$   fore oriented correctly.   However, any
   $E\Delta_{Bcb} = + 2.91$   $E\Delta_{bBc} = + 4.21$   $\psi$ angle may be assumed to be zero and
   $E\psi_{bc} = +16.39$   $E\psi_{Bc} = +19.30$   all other $\psi$ angles computed to correspond.
   $E\Delta_{Bbc} = - 7.12$   $E\Delta_{cBC} = -10.20$   This is obviously permissible since it
   $E\psi_{Bb} = +23.51$   $E\psi_{BC} = +29.50$   means in effect that the truss has been
   $E\Delta_{abB} = -52.65$   $E\Delta_{BCc} = +29.50$   rotated as a rigid body which does not
   $E\psi_{ab} = +76.16$   $E\psi_{Cc} = 0$   alter the condition of stress.
   $E\Delta_{Bab} = +28.25$
   $E\psi_{aB} = +47.91$

**Step 3:** Compute initial end moments and eccentric moments.

### Joint a:



| Bar | K | $E\psi$ | $-6EK\psi$ |
|---|---|---|---|
| ab | 0 583 | +76 16 | -266 0$^{k''}$ |
| bc | 0 583 | +16.39 | - 57 3$^{k''}$ |
| aB | 2 134 | +47.91 | -613.5$^{k''}$ |
| BC | 3 076 | +29 50 | -544 0$^{k''}$ |
| Bb | 0.456 | +23.51 | - 64.3$^{k''}$ |
| Cc | 0.235 | 0 | 0 |
| Bc | 0.292 | +19.30 | - 33.8$^{k''}$ |

$$M_e = +(374)(2.20) = +823^{k''}$$

### Joint B:

$$M_e = -(374)(2.20)' + (332)(1.953)$$
$$= -174^{k''}$$
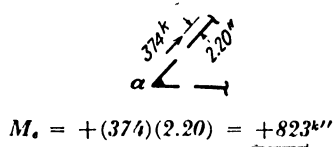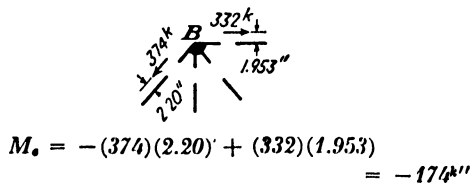
**Step 4:** Distribute, carry over, and obtain secondary end moments.   These end moments
are underlined.   With the section moduli known, it is simple to compute the
secondary stress intensities from these.

$+224.2^{k''}$   $-159.8^{k''}$

$+ 0.3$
$+ 1.5$
$+ 274$
$+739$
$-544$

$+ 0.2$
$+ 0.8$
$+ 13.7$
$+369.5$
$-544$

$B$   $0.516$   $C$

$(M_e = +823)$

$a$   $0.215$   $0.359$   $b$   $0.359$   $c$

$-266$      $-266$      $-573$      $-573$
$- 42.9$    $- 21.5$    $+127.0$    $+63.5$
$+ 63.5$    $+127.0$    $+ 2.1$     $+ 1.0$
$- 13.7$    $- 6.9$     $+71.8$     $+7.2$
$- 2.0$     $- 1.0$
$+ 1.0$     $+ 2.1$
$- 0.3$     $- 0.2$

$-260.4$    $-166.5$
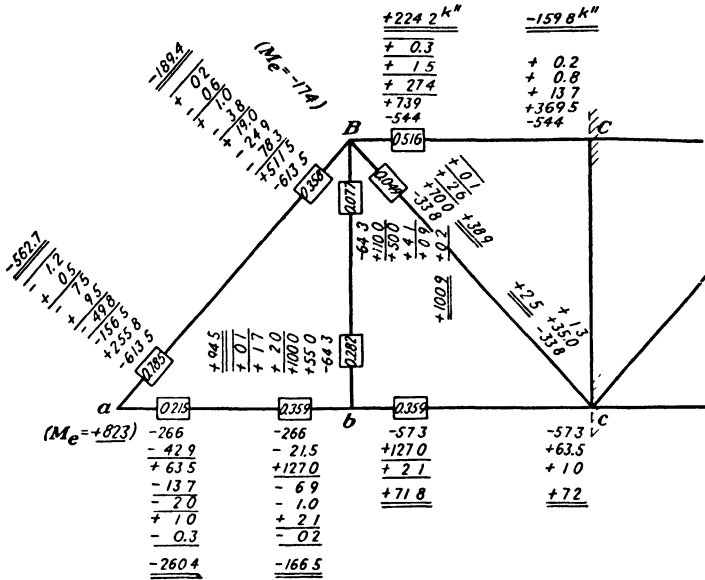
Note in this case that because of symmetry only half the truss needs to be considered; further, it is known, therefore, that joints $C$ and $c$ do not rotate.   These joints never have to be unlocked.

**14·19 Supplementary Remarks—Symmetrical and Antisymmetrical Loads, Elastic Center, and Column Analogy.**   In presenting an elementary discussion of any subject, the writer must draw a rather arbitrary line between the fundamental and advanced aspects of the field, since there are always marginal topics that may or may not be included.   It is desirable, however, to mention some of these topics briefly so as to suggest transitional reading material for those who intend to study further.

Here and there in previous discussions we have mentioned and used symmetry to a limited extent.   Whenever we are dealing with a symmetrical structure, we should always be on the alert to utilize symmetry as effectively as possible.   For example, consider the closed-ring type of structure shown in Fig. 14·29.   Strictly speaking, this structure is indeterminate to the third degree.   If both the frame and the loading are symmetrical about both the $x$ and the $y$ axis, however, then we can reason from symmetry that the shear is zero and the axial stress is 6 kips compression at the mid-point cross section of the girder.   Only the moment remains unknown.   Because of symmetry, therefore, only one statically indeterminate quantity remains, instead of three.

The gain from symmetry in the above situation suggests that we can gain considerable simplification even when we have an *unsymmetrical load* acting on a *symmetrical structure*.   Consider such a situation as that shown in Fig. 14·30.   Suppose that this frame (called a "Vierendeel truss") is symmetrical about the

vertical axis only.  Consider the solution of the frame under the given unsymmetrical load as shown in Fig. 14·30a.  If the slope-deflection method is used, there is a total of 11 unknowns—8θ and 3ψ angles.  Suppose that the load is broken down into two separate systems, one a symmetrical system shown in Fig. 14·30b and one an antisymmetrical system shown in Fig. 14·30c.  Obviously, the sum of these two systems is equal to the given load, and therefore the sum of the results for the two separate systems is equal to the results for the given load.
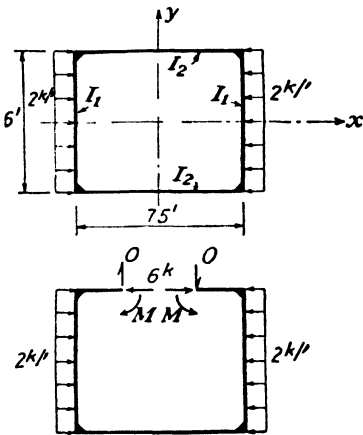


FIG. 14·29
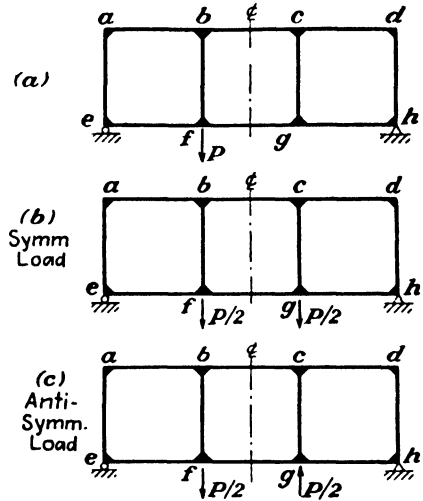


FIG. 14·30

Let us now compare the computational work of analyzing the structure for the given unsymmetrical load with the sum of the computational works for the symmetrical and antisymmetrical loads.  For the case of the symmetrical loads, there are only five independent unknowns—four θ angles and one ψ angle—since, by symmetry,

$$\theta_a = -\theta_d, \qquad \theta_b = -\theta_c, \qquad \theta_e = -\theta_h, \qquad \theta_f = -\theta_g, \qquad \psi \text{ cols} = \psi_{fg} = 0,$$
$$\psi_{ef} = -\psi_{gh}$$

For the case of the antisymmetrical loads, there are six independent unknowns—four θ and two ψ angles—since in this case, by antisymmetry,

$$\theta_a = \theta_d, \qquad \theta_b = \theta_c, \qquad \theta_e = \theta_h, \qquad \theta_f = \theta_g, \qquad \psi_{ef} = \psi_{gh}$$

Since the computational work is roughly proportional to the square of the number of unknowns, using symmetrical and antisymmetrical loads almost cuts the computational work in half.  This idea is very useful when we are dealing with symmetrical structures and deserves further study.[1]

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[1] NEWELL, J. S., Symmetric and Anti-symmetric Loadings, *Civ. Eng.*, April, 1939, pp. 249–251.  ANDRÉE, W. L., Das B = U Verfahren, R. Oldenbourg, Munich and Berlin, 1919.

The so-called "elastic center" is another useful idea that we have not discussed.[1]   The elastic-center technique is applicable to the closed-ring type of structures; a frame such as that shown in Fig. 14·31 would be classified as this type if we consider the ground as being the closing side of the ring.   We may analyze this structure by means of superposition equations, selecting the redundants as shown in Fig. 14·31b, which is the same selection as that used in Example 14·14.   Proceeding with this solution involves three simultaneous equations of the following form:

$$\Delta_{ao} + X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} = 0$$
$$\Delta_{bo} + X_a\delta_{ab} + X_b\delta_{bb} + X_c\delta_{bc} = 0$$
$$\Delta_{co} + X_a\delta_{ac} + X_b\delta_{bc} + X_c\delta_{cc} = 0$$

If we can select the three redundants in such a manner that $\delta_{ab}$, $\delta_{ac}$, and $\delta_{bc}$ are all equal to zero, then each of these equations will contain only one redundant and
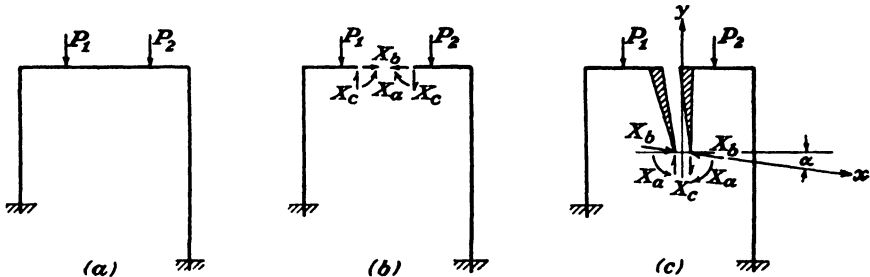


FIG. 14·31

we will not have to solve them simultaneously.   We can do this if we apply the redundants at some point o as shown in Fig. 14·31c, where the two coordinates of point o and the inclination $\alpha$ of the $x$ axis are selected so as to make $\delta_{ab}$, $\delta_{ac}$, and $\delta_{bc}$ all be equal to zero simultaneously.   The redundants acting at point o are assumed to be connected by two rigid (nondeformable) arms to the two sides of the cut in the girder.   Note that the two sets of redundants in Figs. 14·31b and c are statically equivalent but do not have the same values, of course.

The detailed calculations involved in using this elastic-center procedure are found to be similar to computing static moments and products and moments of inertia of areas.   Prof. Hardy Cross recognized this and suggested organizing the computations as in computing the stresses in a column subjected to combined bending and direct stress.   He called this suggested procedure "column analogy."[2]

## 14·20   Problems for Solution.

**Problem 14·1**   Find all the reactions of the structure shown in Fig. 14·32, using superposition equations as the basis for the solution.   $L/A = 1$ for all members.

[1] FIFE, W. M., and J. B. WILBUR, "Theory of Statically Indeterminate Structures," pp. 114–120, McGraw-Hill Book Company, Inc., New York, 1937.

[2] CROSS, H., and N. D. MORGAN, "Continuous Frames of Reinforced Concrete," John Wiley & Sons, Inc., New York, 1932.
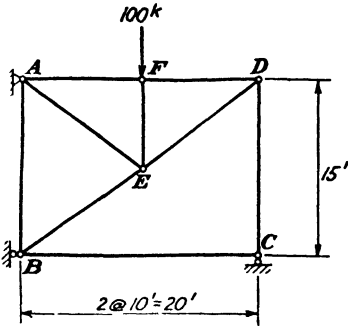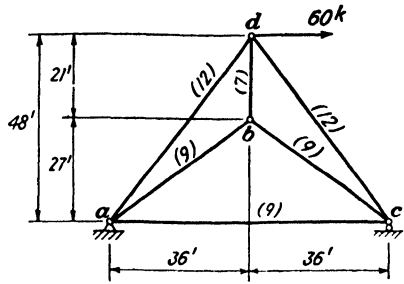
FIG. 14·32          FIG. 14·33

**Problem 14·2**   Compute the stresses in the members of the truss shown in Fig. 14·33, using the superposition-equation method.   Cross-sectional areas of members, in square inches, are shown in parentheses.

**Problem 14·3**   Using superposition equations, compute the stress in the tie rod *ad* of the structure of Fig. 14·34.   Cross-sectional areas of the members, in square inches, are shown in parentheses.
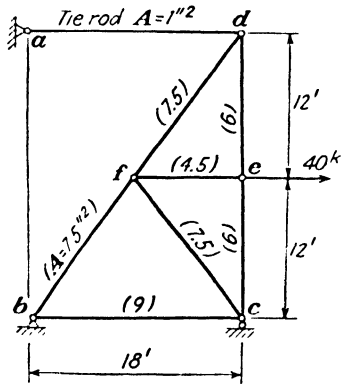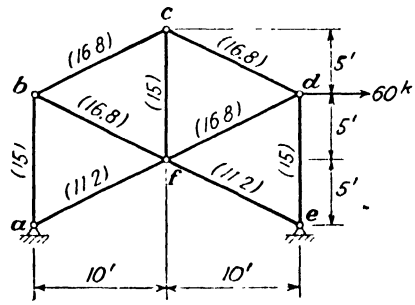


FIG. 14·34          FIG. 14·35

**Problem 14·4**   Using superposition equations, compute the stress in all members of the arch of Fig. 14·35.   Cross-sectional areas in square inches are shown in parentheses; $E = 30 \times 10^3$ kips per sq in.; $\epsilon = 1/150,000$ per °F.

**Problem 14·5**   Find the horizontal component of the right reaction of the arch in Prob. 14·4, due to each of the following conditions: (*a*) Increase of temperature of 50°F in bars *ab*, *bc*, *cd*, *de*.   No change in remaining bars.   (*b*) If bars *bc* and *cd* are ¼ in. too short and bar *cf* is ⅛ in. too long owing to errors in fabrication and it was necessary to force them into place.   (*c*) If the supports settle as follows:

Left support:   Vertical      = 0.48 in. down
                Horizontal = 0.24 in. to left
Right support: Vertical      = 0.24 in. down
                Horizontal = 0.36 in. to right

**Problem 14·6**   Using the superposition equations as a basis for the solution, draw the shear and bending-moment diagram for the beam shown in Fig. 14·36,
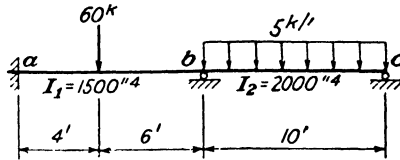


FIG. 14·36

(a) due to load shown; (b) due only to the effect of a vertical settlement of support b of 0.24 in. and a clockwise rotation of support a of 0.005 radian;

$$E = 30 \times 10^3 \text{ kips per sq in.}$$

**Problem 14·7**   Draw the bending moment diagram for the frame shown in Fig. 14 37.   Use Castigliano's theorem as a basis for the solution.   Consider only the effect of bending distortion.
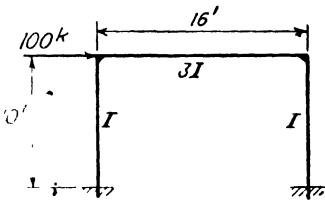


FIG. 14 37
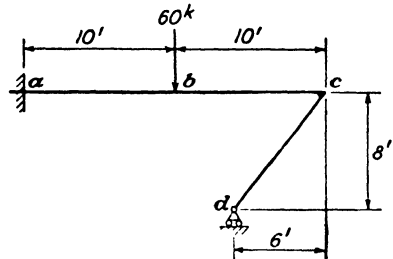


FIG. 14·38

**Problem 14·8**   Compute the reactions of the structure shown in Fig. 14·38. Neglect distortion due to direct stress.   E and I are constant.

**Problem 14·9**   Compute the reactions on the structure shown in Fig. 14·39. Neglect distortion due to direct stress.   E and I are constant.
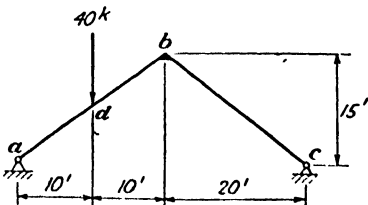


FIG. 14·39
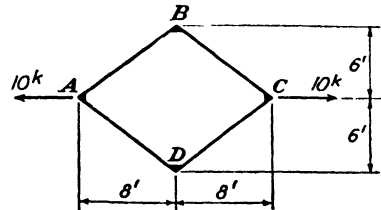


FIG. 14·40

**Problem 14·10**   Draw the shear and bending-moment diagrams for member AB of the frame shown in Fig. 14·40.   Neglect distortion due to axial stress.

**Problem 14·11**   Referring to Fig. 14 41, draw the bending-moment curve for the beam AB, using Castigliano's theorem.
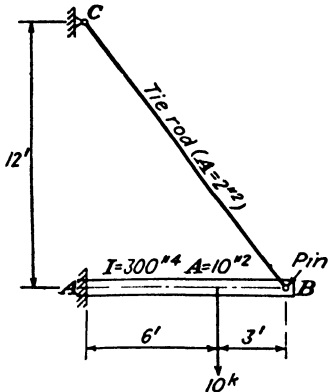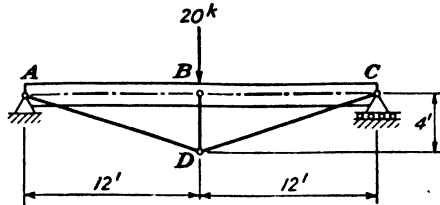
FIG. 14·41                                    FIG. 14·42

**Problem 14·12**  Referring to Fig. 14·42, compute the axial stresses in the members of the king post truss, due to the load shown.  Also, draw the bending-moment curve for member $AC$.

> Member $AC$: $A = 12$ sq in.      and      $I = 432$ in.[4]
> Member $AD$: $A = 3$ sq in.
> Member $DC$: $A = 3$ sq in.
> Member $BD$: $A = 2$ sq in.
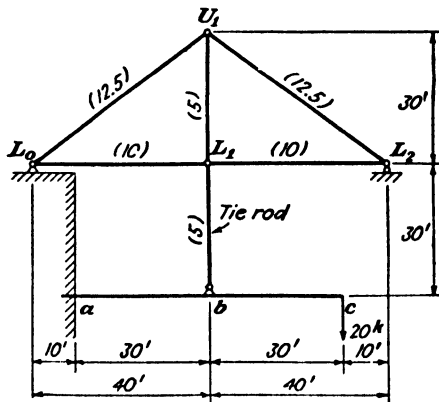


FIG. 14·43

**Problem 14·13**  In the structure shown in Fig. 14·43, the cross-sectional areas of members, in square inches, are shown in parentheses; $E = 30 \times 10^3$ kips per sq in.; $I$ of beam $= 4,000$ in.[4]  Compute the stress in the tie rod, which is connected to the beam and the truss by hinged ends.

**Problem 14·14**  Compute the reactions of the structure shown in Fig. 14·44.
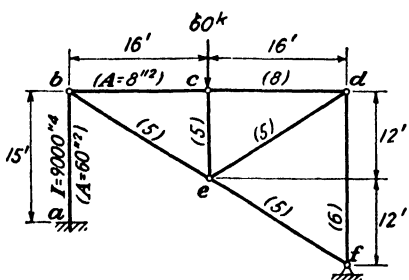
FIG. 14·44
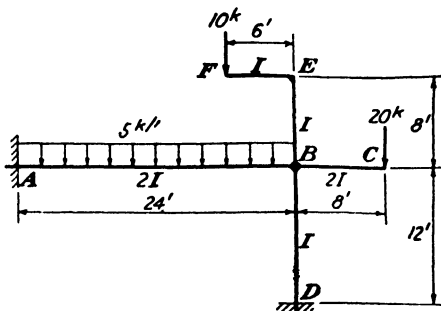


FIG. 14·45

**Problem 14·15**   Refer to Fig. 14·45.   Compute the end moments in all the members of this frame, using the slope-deflection method.   Draw the moment curves for members $AB$ and $BD$.

**Problem 14·16**   Refer to Fig. 14·46.   Compute the end moments in all members of this frame, using the slope-deflection method.   Draw the shear and moment curves for member $ab$.
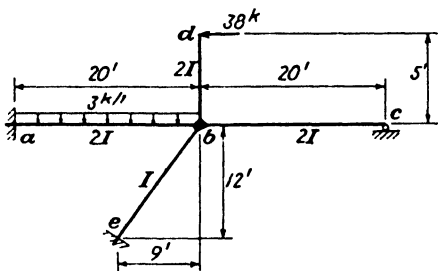


FIG. 14·46
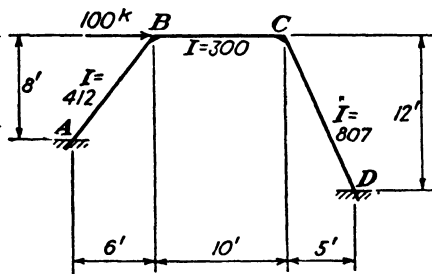


FIG. 14·47

**Problem 14·17**   Solve both parts of Prob. 14·6, using the slope-deflection method.

**Problem 14·18**   Using the slope-deflection method, find all the end moments and support reactions of the frame shown in Fig. 14·47.

**Problem 14·19**   Using the slope-deflection method, find all the end moments and support reactions of the frame shown in Fig. 14·48.
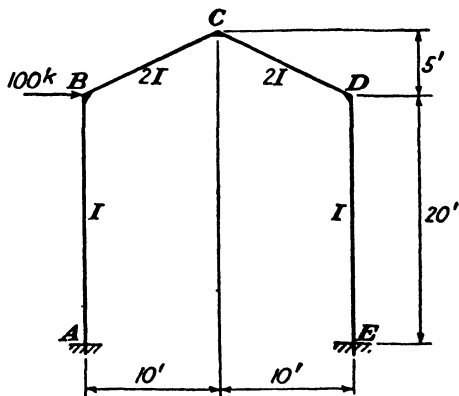


FIG. 14·48

**Problem 14·20**   Refer to Fig. 14·49.   Using moment distribution, compute the end moments in the members of this frame, and draw the shear and moment curves for member $AB$.
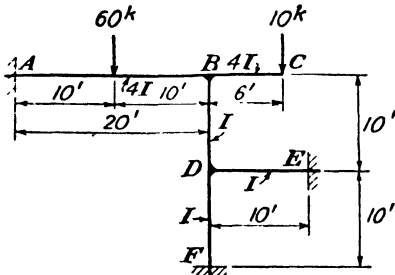


Fig. 14·49
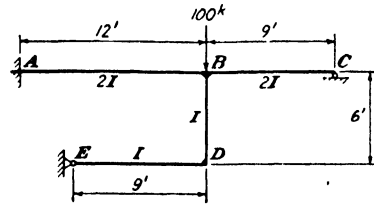


Fig. 11·50

**Problem 14·21**   Solve both parts of Prob. 11 6, using moment distribution.

**Problem 14·22**   Refer to Fig. 14 50.   Find the end moments of this frame, using the moment-distribution method.
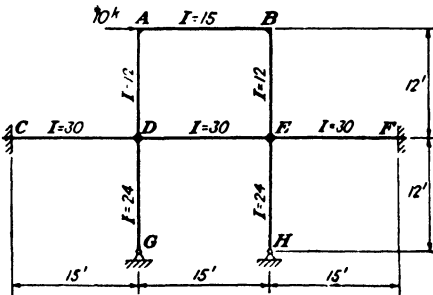


Fig. 14·51
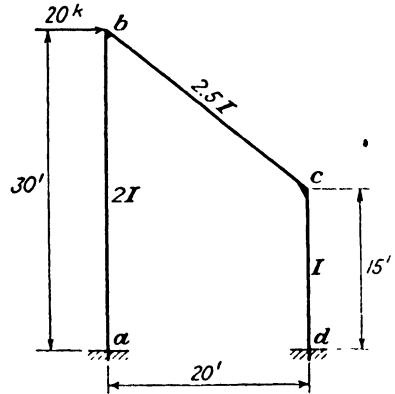


Fig. 11·52

**Problem 14·23**   Compute the end moments in the members of the frame shown in Fig. 14·51, using the moment-distribution method.
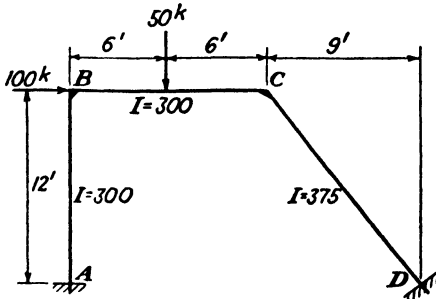
**Problem 14·24**   Refer to Fig. 14·52.   Find the end moments in the members of this frame, using the moment-distribution method.



Fig. 14·53

**Problem 14·25**   Refer to Fig. 14·53.   Find the end moments in this frame, using the moment-distribution method.

**Problem 14·26**   For the frame shown in Fig. 14·54, draw the bending-moment curves, using the moment-distribution method.
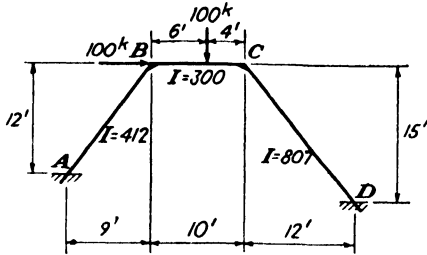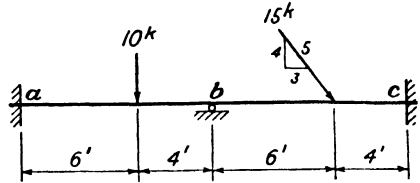


Fig. 14·54                                    Fig. 14·55

**Problem 14·27**   Consider the beam shown in Fig. 14·55, in which $I = 500$ in.$^4$, $A = 30$ sq in., and $E = 30 \times 10^3$ kips per sq in.   Compare the solution of this problem by the theorem of three moments, slope deflection, and superposition equations.   HINTS: To what degree is the beam indeterminate?   Can you find the horizontal reactions by three moments or slope deflection?   Which method do you consider to be superior?   How many simultaneous equations are involved in each solution?

**Problem 14·28**   Determine the number of independent $\theta$ and $\psi$ angles in each of the structures shown in Figs. 14·12 and 14·15.

# CHAPTER 15

## INFLUENCE LINES FOR
## STATICALLY INDETERMINATE STRUCTURES

**15·1 Introduction.** In Chap. 14, various methods of analyzing indeterminate structures are discussed; in all cases, however, the structures are subjected to some particular condition of loading. Often it is necessary to analyze an indeterminate structure for the effect of a movable or a moving load. In such cases, it is convenient to prepare influence lines or influence tables for the various stress components, since from these we can determine both how to load the structure to produce a maximum effect and likewise the magnitude of this maximum effect.

In Chap. 6, the preparation and use of influence lines for statically determinate structures are discussed. In such cases, we found that after a little practice we could draw an entire influence line by figuring a few key ordinates and connecting them with straight lines. Influence lines for indeterminate structures cannot be drawn so easily, however, since in general they are curved lines or, at best, a series of chords of a curved line.

Fortunately, influence lines of the latter type occur quite frequently. When an indeterminate structure is loaded at the panel points by floor beams, which in turn support stringers, which act as simple beams between the floor beams, it is easy to show that the various influence lines for the indeterminate structure are straight lines between panel points. If, however, the stringers are continuous over several floor beams, the influence lines are curved lines between panel points, though in most practical cases the departure of these curves from a straight line between panel points is rather slight. If the moving load does not act through a stringer–floor-beam system and can be applied to the structure at any point along its length, the influence lines are likewise curved lines.

The first step in preparing influence lines for the various stresses in an indeterminate structure is to determine the influence lines for the redundants. Once this has been done, the influence lines for any other reaction, bar stress, shear, or moment, can be computed by statics.

**15·2 Influence Line by Successive Positions of Unit Load.** In Chap. 6, it is pointed out that the ordinates of an influence line for some particular quantity can always be obtained by placing a unit load successively at each possible load point on the structure and computing

the value of that quantity for each of these positions. This same procedure may be followed to obtain the influence lines for the redundants of an indeterminate structure. Doing so means solving a number of problems in just the same way as in Chap. 14.

Offhand this might seem a long and tedious process. It will be found, however, that the computations may be organized very efficiently, as is illustrated in Example 15·1. Moreover, once the influence lines for the redundants have been obtained, the influence lines for all other stress components can be obtained very easily, simply by superimposing quantities already computed. In Example 15·1, once the influence-line ordinates have been computed for the redundants $X_1$ and $X_2$, the ordinates for all other bar stresses may be computed in tabular form, by means of the relation,

$$F = F_o + X_1F_1 + X_2F_2$$

Note that this is very easy, for $F_o$ stresses have already been computed for all bars for every position of the unit load. While it takes longer to compute simply the influence lines for the redundants of an indeterminate truss by this method, influence lines for all the bar stresses and reactions may be obtained essentially as fast by this method as by any other.

**Example 15·1** *Assuming A to be constant for all bars, prepare influence lines for the stresses in the redundant bars of this truss:*
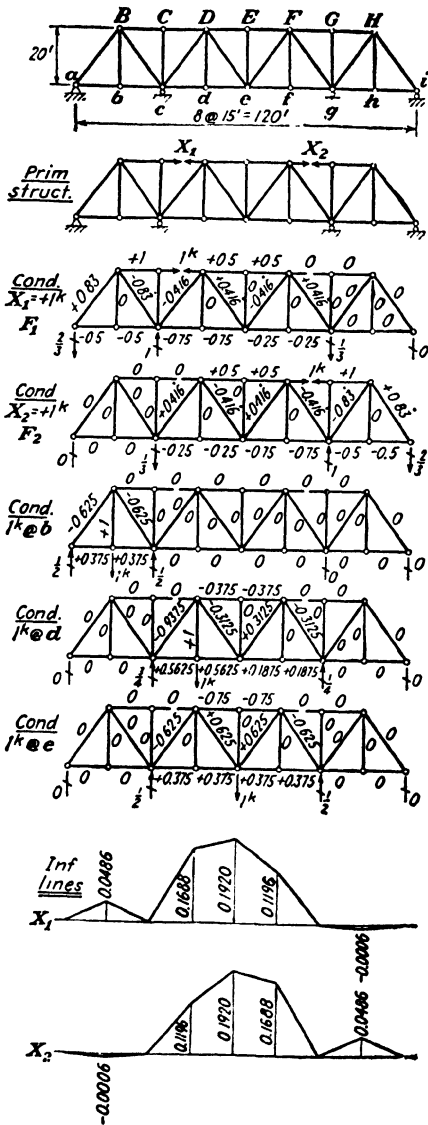


Select stresses in bars $CD$ and $FG$ as redundants. Compute these redundants for a unit load at any joint $n$, using superposition equations.

$$\Delta_1^{\longrightarrow} = \Delta_{1n} + X_1\delta_{11} + X_2\delta_{12} = 0 \quad (1)$$
$$\Delta_2^{\longrightarrow} = \Delta_{2n} + X_1\delta_{21} + X_2\delta_{22} = 0 \quad (2)$$

By symmetry, $\delta_{11} = \delta_{22}$
By Maxwell's law, $\delta_{12} = \delta_{21}$
Equations (1) and (2) therefore become

$$\Delta_{1n} + X_1\delta_{11} + X_2\delta_{12} = 0 \quad (1a)$$
$$\Delta_{2n} + X_1\delta_{12} + X_2\delta_{11} = 0 \quad (2a)$$

Using the method of virtual work to evaluate these deflection terms, the constants of the structure are

$$(1^k)(\delta_{11}) = \frac{1}{EA} \sum F_1{}^2 L$$
$$(1^k)(\delta_{12}) = \frac{1}{EA} \sum F_1 F_2 L$$

while the load terms are

$$(1^k)(\Delta_{1n}) = \frac{1}{EA} \sum F_1 F_n L$$
$$(1^k)(\Delta_{2n}) = \frac{1}{EA} \sum F_2 F_n L$$

Since $EA$ is constant for all bars and the right-hand sides of Eqs. (1a) and (2a) are zero, for convenience we can let $EA = 1$ for the rest of the solution. By symmetry, $X_1$ due to a unit load at $b$ is equal to $X_2$ due to a unit load at $h$, etc. As a result, if we compute both $X_1$ and $X_2$ for a unit load placed successively at points $b$, $d$, and $e$, we shall have enough information to plot the complete influence lines for both $X_1$ and $X_2$.

*Constants of structure:*

| Bar | L | $F_1$ | $F_2$ | $F_1F_2L$ | $F_1^2L$ |
|---|---|---|---|---|---|
| ac | 30 | −0.5 | 0 | 0 | + 7.5 |
| ce | 30 | −0.75 | −0 25 | +5.625 | + 16 875 |
| eg | 30 | −0.25 | −0 75 | +5.625 | + 1.875 |
| BD | 30 | +1.0 | 0 | 0 | + 30.0 |
| DF | 30 | +0.5 | +0 5 | +7 5 | + 7.5 |
| aB | 25 | +0.83 | 0 | 0 | + 17 35 |
| Bc | 25 | −0.83 | 0 | 0 | + 17 35 |
| cD | 25 | −0 416 | +0.416 | −4.345 | + 4.345 |
| De | 25 | +0 416 | −0.416 | −4 345 | + 4.345 |
| eF | 25 | −0 416 | +0.416 | −4 345 | + 4 345 |
| Fg | 25 | +0 416 | −0 416 | −4 345 | + 4 345 |
| Σ | . | . . . . | . . . | +1 37 | +115 83 |

$$\therefore \delta_{11} = +115.83, \qquad \delta_{12} = +1.37$$

*Load terms:*

| | | Unit load in left span | | | | | Unit load in center span | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | At b | | | | | At d | | | At e | |
| Bar | L | $F_1$ | $F_n$ | $F_1F_nL$ | Bar | L | $F_1$ | $F_2$ | $F_n$ | $F_1F_nL$ | $F_2F_nL$ | $F_n$ | $F_1F_nL$ |
| ac | 30 | −0 5 | +0 375 | − 5 625 | ce | 30 | −0 75 | −0 25 | +0 563 | −12 66 | − 4 22 | +0 375 | − 8 44 |
| aB | 25 | +0 83 | −0 625 | −13 0 | eg | 30 | −0 25 | −0 75 | +0 188 | − 1 41 | − 4 22 | +0 375 | − 2 81 |
| Bc | 25 | −0 83 | −0 625 | +13 0 | DF | 30 | +0.5 | +0 5 | −0 375 | − 5 63 | − 5 63 | −0 75 | −11 25 |
| Σ | | | | − 5 625 | cD | 25 | −0.416 | +0 416 | −0 938 | + 9 78 | − 9 78 | −0 625 | + 6 50 |
| | | | | | De | 25 | +0 416 | −0 416 | −0 313 | − 3 26 | + 3 26 | +0 625 | + 6 50 |
| | | | | | eF | 25 | −0 416 | +0 416 | +0 313 | − 3 26 | + 3 26 | +0 625 | − 6 50 |
| | | | | | Fg | 25 | +0 416 | −0 416 | −0 313 | − 3 26 | + 3 26 | −0 625 | − 6 50 |
| | | | | | Σ | | | | . . | −19 70 | −14 07 | . . . . . . | −22 50 |

NOTE:
When load is at b, $\Delta_{2b} = 0$.
When load is at e, $\Delta_{1e} = \Delta_{2e}$.

*Solution of equations:*

| Eq No. | Operation | $X_1$ | $X_2$ | b | d | e | Check |
|--------|-----------|-------|-------|---|---|---|-------|
| (1) | ........... | +115.83 | + 1.37 | +5.625 | +19.70 | +22.50 | +165.025 |
| (2) | ........... | + 1.37 | +115.83 | +0 | +14.07 | +22.50 | +153.77 |
| (1') | (1) × 0.01182 | + 1.37 | + 0.016 | +0 066 | + 0.233 | + 0.266 | + 1.951 |
| (3) | (2) − (1') | | +115.814 | −0 066 | +13.837 | +22.234 | +151.819 |
| | | | + 1.0 | −0.00057 | +0.1196 | + 0.1920 | + 1.3100 |
| | | − 1.37 | | +0.00078 | −0.1638 | − 0 2610 | |
| | | +115.83 | | +5.6258 | +19.5362 | +22 2390 | |
| | | + 1.0 | | +0.0486 | + 0.1688 | + 0.1920 | |

## 15·3 Müller-Breslau's Principle for Obtaining Influence Lines.

Muller-Breslau's principle provides a very convenient method of computing[1] influence lines and likewise is the basis for certain indirect methods of model analysis. This principle may be stated as follows:

*The ordinates of the influence line for any stress element (such as axial stress, shear, moment, or reaction) of any structure are proportional to those of the deflection curve which is obtained by removing the restraint corresponding to that element from the structure and introducing in its place a corresponding deformation into the primary structure which remains.*

This principle is applicable to any type of structure, whether beam, truss, or frame or whether statically determinate or indeterminate. In the case of indeterminate structures, this principle is limited to structures the material of which is elastic and follows Hooke's law. This limitation is not particularly important, however, since the vast majority of practical cases fall into this category.

The validity of this principle may be demonstrated in the following manner: For this purpose, consider the two-span continuous beam shown in Fig. 15·1a. Suppose that the influence line for the vertical reaction at $a$ is required. The influence line may be plotted after the reaction has been evaluated for a unit vertical load applied successively at various points $n$ along the structure, each

[1] Likewise, by applying Müller-Breslau's principle, it is often possible to sketch the shape of an influence line roughly and thus determine, accurately enough for design purposes, how to load the structure so as to produce a maximum effect.

evaluation being carried out as follows: Temporarily remove the roller support at *a* from the actual structure, leaving the primary structure shown in Fig. 15·1*b*. Suppose that this primary structure is acted upon by the unit load at a point *n* and a vertical upward force $R_a$ at point *a*. If this force has the same value as the vertical reaction at point *a* on the actual structure, then the stresses—and hence the distortion—of the primary structure will be exactly the same as those of the actual structure. The elastic curve of the primary structure under such conditions will therefore be as indicated in Fig. 15·1*b*, the vertical deflection at point *a* being zero.

Suppose that we now consider the primary structure to be acted upon by simply a vertical force *F* at point *a*. In this case, the primary structure will deflect as shown in Fig. 15·1*c*. Thus we have considered the primary structure under the action of two separate and distinct force systems, (1) the forces in sketch *b* and (2) those in sketch *c*. Applying Betti's law to this situation, we may write,

$$(R_a)(\Delta_{aa}) + (1)(\Delta_{na}) = (F)(0)$$

and therefore

$$R_a = -\frac{\Delta_{na}}{\Delta_{aa}}(1) \qquad (15\cdot1)$$

where the same notation is used to designate these deflections as that in the superposition-equation method.



Fig. 15·1

From this equation, it is apparent that the reaction $R_a$ when the unit vertical load is at point *n* is proportional to the deflection $\Delta_{na}$ at that point. The shape of the influence line for $R_a$ is therefore the same as the shape of the elastic curve of the structure when it is acted upon by a force *F* at point *a*. The magnitude of the influence-line ordinate at any point *n* may be obtained by dividing the deflection at that point on this elastic curve by the deflection at point *a*. In this manner, we have demonstrated that influence lines may be obtained in the manner outlined by Müller-Breslau's principle.

In a similar manner, the validity of this principle may be demonstrated for any stress element of any structure. In the general case, Eq. (15·1) may be written as follows for any stress element $X_a$,

$$X_a = -\frac{\Delta_{na}}{\Delta_{aa}}(1) \qquad (15\cdot2)$$

It is important to note the sign convention of this equation, *viz.*: $X_a$ is plus when in the same sense as the introduced deflection $\Delta_{aa}$; and $\Delta_{na}$ is
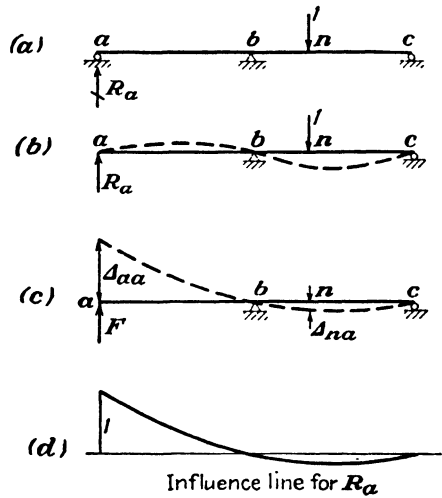
plus when in the same sense as the applied unit load, the influence of which is given by the ordinates of the influence line. Note further that $X_a$ may represent either a force or a couple. If $X_a$ is a force, the corresponding $\Delta_{aa}$ is a linear deflection; but if $X_a$ is a couple, the corresponding $\Delta_{aa}$ is an angular rotation.

It is also important to note that the magnitude of any influence-line ordinate is independent of the magnitude of the force $F$ which must be applied to introduce the deflection $\Delta_{aa}$ into the primary structure. For computation of the influence-line ordinates, $F$ is usually taken as unity and $\Delta_{aa}$ and $\Delta_{na}$ computed to correspond.

Examples 15·2 and 15·3 illustrate the application of this procedure to continuous beams. When applying this method, note that, if the actual structure is indeterminate to more than the first degree, the primary structure which remains after the removal of a redundant is still a statically indeterminate structure. This does not cause any real difficulty, however. It simply means that before the deflections $\Delta_{aa}$ and $\Delta_{na}$ can be computed, the statically indeterminate primary structure must be analyzed by one of the methods of Chap. 14.

**Example 15·2**  *Prepare an influence line for the vertical reaction at b of this beam:*
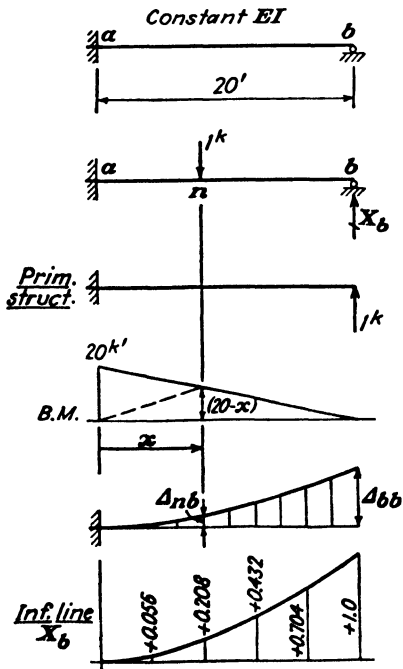


*Applying Müller-Breslau's principle, we may select the primary structure shown and write that*

$$X_b^{\uparrow} = -\frac{\Delta_{nb}^{\downarrow}}{\Delta_{bb}^{\uparrow}} \quad (1^k)$$

*where the plus direction of the various terms is shown.*

*If, for convenience, we decide to call $\Delta_{nb}$ plus when up, we must reverse the sign of the right-hand side of this equation and write*

$$X_b^{\uparrow} = \frac{\Delta_{nb}^{\uparrow}}{\Delta_{bb}^{\uparrow}} \quad (1^k)$$

*By the second moment-area theorem,*

$$EI\Delta_{nb} = (20)\left(\frac{x}{2}\right)\left(\frac{2x}{3}\right)$$
$$+ (20-x)\left(\frac{x}{2}\right)\left(\frac{x}{3}\right) = \frac{x^2}{6}(60-x)$$
$$EI\Delta_{bb} = \frac{(20)^2}{6}(60-20) = \frac{8,000}{3}$$
$$\therefore X_b^{\uparrow} = \frac{x^2(60-x)}{16,000}$$

**Example 15·3**  *Prepare an influence line for the moment at support b of this beam:*
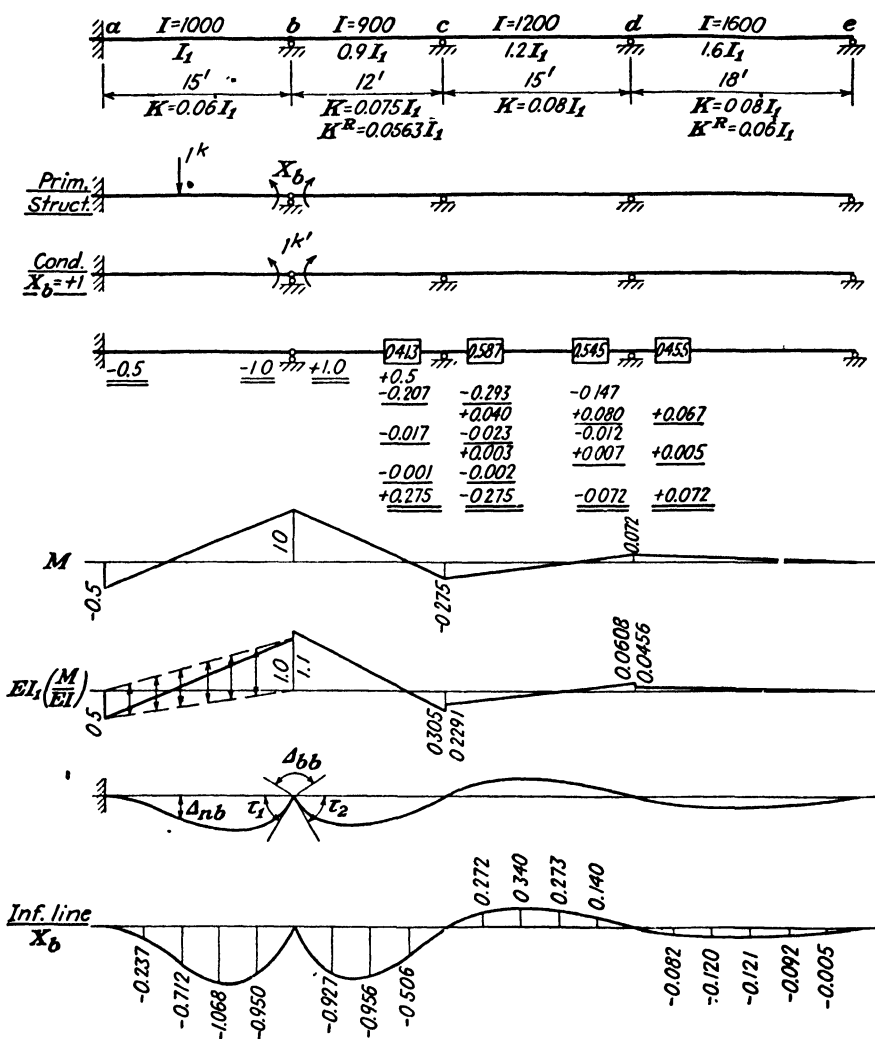


Using Müller-Breslau's principle,

$$X_b^{\phantom{b}} = -\frac{\Delta_{nb}^{\downarrow}}{\Delta_{bb}^{\phantom{bb}}}\; (1^k)$$

or

$$X_b^{\phantom{b}} = +\frac{\Delta_{nb}^{\uparrow}}{\Delta_{bb}^{\phantom{bb}}}\; (1^k)$$

*if we reverse the sign convention for $\Delta_{nb}$.*

*Computing $\Delta_{nb}$ and $\Delta_{bb}$ by the moment-area or elastic-load procedure,*

$$EI_1\tau_1 = (0.5)(7.5) = 3.75$$
$$EI_1\tau_2 = (1.1)(6)(\tfrac{2}{3}) - (0.305)(6)(\tfrac{1}{3}) = 3.834 \Big\} \quad \therefore EI_1\Delta_{bb} = 3.75 + 3.83 = \underline{7.58}$$

**Span ab:**

$$EI_1\Delta_{nb}^{\uparrow} = \left(\frac{x}{15}\right)\left(\frac{x^2}{6}\right) - \left(\frac{1}{2}\right)\left(\frac{x}{2}\right)\left(\frac{2x}{3}\right) - \left(\frac{1}{2} - \frac{x}{30}\right)\left(\frac{x^2}{6}\right) = \frac{x^3}{60} - \frac{x^2}{4}$$

$$\therefore X_b = \frac{1}{7.58}\left(\frac{x^3}{60} - \frac{x^2}{4}\right)$$

**Span bc:**

$$EI_1\Delta_{nb}^{\uparrow} = -3.834x + (1.1)\left(\frac{x^2}{3}\right) + (1.1 - 0.0928x)\left(\frac{x^2}{6}\right) - (0.02542)\left(\frac{x^3}{6}\right)$$

$$= -3.834x + 0.556x^2 - 0.0197x^3$$

$$\therefore X_b = \frac{1}{7.58}\left(-3.834x + 0.556x^2 - 0.0197x^3\right)$$

*etc. for spans cd and de. From these equations, the ordinates every 3 ft are found to be as noted.*

---

**15·4   Influence Lines Obtained by Superposition of the Effects of Fixed End Moments.** This method is very useful for determining influence lines for the end moments of indeterminate beams or frames. It involves using moment distribution to determine the separate effects of each of the fixed end moments of the various loaded members. These separate effects may then be superimposed to give the total end moments. When properly organized as illustrated in Example 15·4, it is a very effective method of obtaining influence lines for the end moments.

Essentially this method consists of the following steps:

1. Apply a fixed end moment of unity at one end of a member. Using the moment-distribution procedure, compute the resulting end moments in all members. Repeat this process for each end of every member that can have a fixed end moment developed by the applied loads.

2. Compute the fixed end moments developed by a unit load placed in turn at each of the various load points.

3. Combine the data from steps 1 and 2 to find the end moments throughout caused by a unit load at each of the various load points of the structure.

Of course, when the influence lines have been computed for the end moments, other influence lines may be computed by statics.

When computing step 1 of the above procedure, note that actually only one moment-distribution solution is required for each joint of the frame. This may be done for a fixed end moment of unity in any member at a joint. Once this is done, the effect of a fixed end moment of unity in any of the other members meeting at that joint may be determined by inspection. This is evident in the solution of Example 15·4.

**Example 15·4**  *By superimposing the effects of fixed moments, prepare an influence table for the end moments of the frame used in Example 14·25.   Give ordinates every 5 ft along girder BDEF.*

In this case, there can be a sidesway of the columns.   To consider this we need to combine a Case A and a Case B type of solution.

Use the Case B end moments from Example 14·25, dividing the data there by 60.
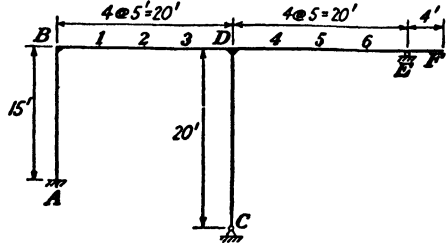


**Step 1:** *Find the end moments caused by a FEM = +1 in ends of girder members.*

$F.E.M._{DB}=+10$

```
           -0.118                    -0.340
           ‾‾‾‾‾                      ‾‾‾‾‾
           -0.021*                    -0.001*
           -0.097                     -0 339
           ‾‾‾‾‾                      ‾‾‾‾‾
           +0 007                     -0 001
           -0 013                     -0 019
           +0.122                     -0 319
  0.0039   -0.213                     ‾‾‾‾‾
   →      |2.571|        |0.426|      |2.319|      ////
        |2.429|  |+0.091|         |-0.255| |-0.272|
              |+0.026| +1.000     |-0.016| |+0.013*|
              |+0.097| -0.426     |-0.001| |-0.259|
              |+0.021*|+0.061  |0.255|
              |+0.118| -0.026
                       +0.004
                       -0 002
   |+0.077|            +0 611
   |+0.028*| |+0.003|  -0 012*
   |+0.049|  |+0.046|  +0.599
   ////  0.0097
            ////
                       //// 0.0136
```
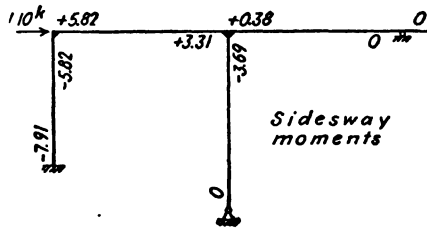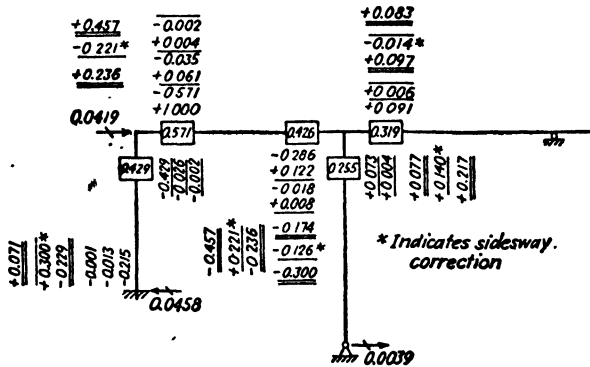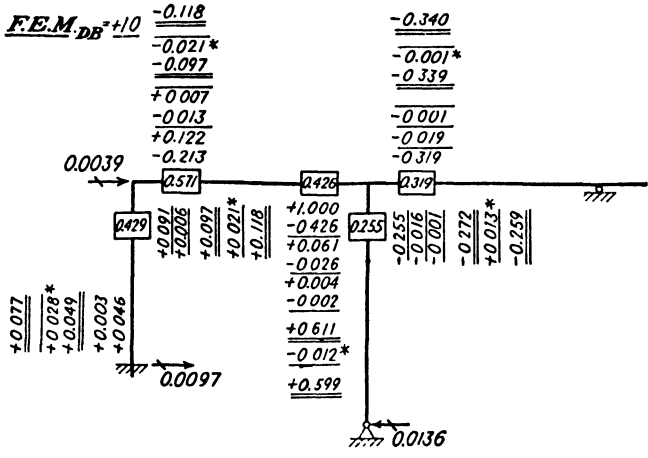
These are the only actual moment-distribution solutions that need to be carried through.
The end moments caused by $FEM_{DE} = +1.000$ can be written by inspection simply by
shifting the $+1.000$ figure from the $DB$ to the $DE$ column of figures. Likewise, the end
moments due to either $FEM_{ED} = +1.0$ or $FEM_{EF} = +1.0$ will be equal to one-half those
due to $FEM_{DE} = +1$, with the exception of the ends $ED$ and $EF$.

**Summary of end moments caused by $FEM = +1$ at the various points:**

| $FEM = +1$ at point | $M_{AB}$ | $M_{BD}$ | $M_{DB}$ | $M_{DE}$ | $M_{DC}$ | $M_{ED}$ |
|---|---|---|---|---|---|---|
| BD | +0.071 | +0 236 | −0 300 | +0 083 | +0 217 | 0 |
| DB | +0.077 | −0.118 | +0.599 | −0 340 | −0 259 | 0 |
| DE | +0.077 | −0.118 | −0.401 | +0 660 | −0 259 | 0 |
| ED | −0 038 | +0.059 | +0 201 | −0 330 | +0 129 | 0 |
| EF | −0 038 | +0 059 | +0 201 | −0 330 | +0.129 | −1 0 |

**Step 2:** *FEM due to unit load at various load points:*

| Load at | FEM developed at end | | | | |
|---|---|---|---|---|---|
| | BD | DB | DE | ED | EF |
| 1 | −2.8125 | +0.9375 | | | |
| 2 | −2.50 | +2.50 | | | |
| 3 | −0.9375 | +2.8125 | | | |
| 4 | ........ | ........ | −2.8125 | +0.9375 | |
| 5 | ........ | ..... .. | −2.50 | +2.50 | |
| 6 | ........ | ........ | −0.9375 | +2.8125 | |
| F | ........ | ........ | ........ | ........ | −4.00 |

*Step 3: Influence table for end moments:*

| Load at | Factor × [end moment corresponding to FEM = +1 at point (−)] | $M_{AB}$ | $M_{BD}$ | $M_{DB}$ | $M_{DE}$ | $M_{DC}$ | $M_{ED}$ |
|---|---|---|---|---|---|---|---|
| 1 | −2.8125 × BD | −0 199 | −0 664 | +0.844 | −0.233 | −0 610 | |
|   | +0.9375 × DB | +0 072 | −0 111 | +0 561 | −0.318 | −0 243 | |
|   |  | −0.127 | −0.775 | +1.405 | −0.551 | −0 853 | 0 |
| 2 | −2.50 × BD | −0 172 | −0.590 | +0 750 | −0.207 | −0.543 | |
|   | +2 50 × DB | +0 192 | −0.295 | +1 498 | −0.850 | −0 648 | |
|   |  | +0 020 | −0 885 | +2 248 | −1.057 | −0 191 | 0 |
| 3 | −0 9375 × BD | −0 067 | −0 221 | +0 281 | −0 078 | −0.203 | |
|   | +2 8125 × DB | +0 216 | −0.332 | +1 682 | −0 956 | −0 729 | |
|   |  | +0 149 | −0.553 | +1 963 | −1.034 | −0 932 | 0 |
| 4 | −2 8125 × DE | −0 216 | +0 332 | +1 128 | −1 852 | +0.729 | |
|   | +0 9375 × ED | −0 036 | +0.054 | +0 188 | −0 308 | +0.120 | |
|   |  | −0 252 | +0 386 | +1 316 | −2 160 | +0 849 | 0 |
| 5 | −2 50 × DE | −0 192 | +0.295 | +1.002 | −1 648 | +0.648 | |
|   | +2 50 × ED | −0 095 | +0.145 | +0.500 | −0 820 | +0.320 | |
|   |  | −0 287 | +0.440 | +1.502 | −2 468 | +0 968 | 0 |
| 6 | −0 9375 × DE | −0 072 | +0 111 | +0 376 | −0 619 | +0 243 | |
|   | +2 8125 × ED | −0 107 | +0 163 | +0.565 | −0 922 | +0 360 | |
|   |  | −0 179 | +0.274 | +0.941 | −1 541 | +0 603 | 0 |
| F | −4 0 × EF | +0 152 | −0 232 | −0 804 | +1 321 | −0.521 | +4 0 |

## 15·5   Problems for Solution

**Problem 15·1**   *a.* Assuming $L/A$ to be constant for all bars, prepare an influence line for the stress in bar *CD* of the truss shown in Fig. 15 2.
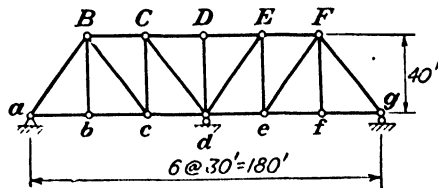


Fig. 15·2

*b.* Using the data from part *a*, likewise prepare influence lines for the stress in bars *Bc* and *bc*.

**Problem 15·2**   Prepare influence lines for the stresses in bars *bC* and *Cd* of the truss shown in Example 14·5.

**Problem 15·3**   Using Müller-Breslau's principle, prepare influence lines for (*a*) the moment at point *c* of the beam shown in Example 15 3; (*b*) the vertical reaction at point *b* of this same beam.

**Problem 15·4**   Using the method described in Art. 15 4, prepare influence lines for the end moments of the beam shown in Example 15·3.

# CHAPTER 16

## OTHER STRUCTURES

**16·1  Introduction.**  While the illustrations used in previous chapters are drawn largely from civil engineering structures, these methods of analysis are applicable to many other types of structures, including those which are of importance in other branches of engineering. In this chapter, the foregoing statement is amplified by considering the analysis of a few typical structures beyond the scope of the practice of most civil engineers.  In this brief treatment, no attempt is made to discuss in detail the methods of stress analysis for these other structures, since lack of space precludes such a procedure and, moreover, standard textbooks are available that fulfill that purpose.  The object is, rather, to show the student in civil engineering that, with a proper background in the analysis of civil engineering structures, he has at his command principles and methods applicable to the analysis of structures in general.

**16·2  Airplane Design.**[1]  While economy is of primary importance in the design of any engineering structure, the factors affecting the over-all economy of a structure vary greatly.  Thus, in most civil engineering structures, the weight of the structure is not of importance, except insofar as it influences the cost of the structure, and it often proves economical to design a heavy structure where a lighter one might have been functionally satisfactory.  In an airplane, the situation is, of course, quite different.  In the over-all economics, it becomes necessary to hold the weight of the structure to a practical minimum, in order that the "pay load" of the airplane can be large.  This not only justifies but makes necessary the use of lighter and more expensive materials; it leads to the use of structural forms and connections that are often more difficult to analyze and fabricate; and it requires structural design based on a smaller margin of safety against failure than is customary for civil engineering structures.

For airplane design, it is customary to base design on *limit loads*, which are the actual maximum loads to which the structure can be subjected under various loading conditions.  A factor of safety of perhaps 1.5 is then specified; but instead of using this factor of safety to determine permissible fiber stresses, as is the case in most other fields of

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[1] Much of the following material dealing with the stress analysis of airplanes is based on "Airplane Structures," by A. S. Niles and J. S. Newell, John Wiley & Sons, Inc., New York, 1943.

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structural engineering, the limit load is multiplied by the factor of safety, leading to the so-called *design load*. A member is then designed by computing the *ultimate load*, which is the load which would cause the member to fail, and so proportioning the member that the ultimate load is equal to or greater than the design load. This procedure is usually carried out in terms of fiber stresses, by designating the quantity computed by the expression

$$\frac{\text{Ultimate fiber stress}}{\text{Fiber stress due to design load}} - 1$$

as the margin of safety and requiring that the margin of safety be equal to or greater than zero.

In computing ultimate fiber stresses, failure may be due to elastic buckling, tension, fatigue, or other causes. It is usually assumed that methods of analysis which are strictly applicable only when the stresses remain within the elastic limit can be used even though that limit is exceeded. To prevent permanent set, it is however further specified that the stresses due to the limit load shall not exceed the yield point. Since the limit loads have not been increased by the factor of safety, this latter specification will not influence the design unless the yield point is less than two-thirds of the ultimate fiber stress due to other causes.

**16·3  Airplane Loadings.** The principal types of loads to which an airplane is subjected may be classified as follows:

*a. Weight of airplane and contents:* These loads, depending upon mass and gravity, are essentially constant in magnitude but vary in direction relative to the airplane, depending upon its line of flight and orientation.

*b. Inertia forces of airplane and contents:* For stationary structures, the laws of statics state that, along any coordinate axis, $\Sigma F = 0$, while, about any coordinate axis, $\Sigma M = 0$. Since airplanes are subject to movements and hence to accelerations, the foregoing relations must be revised to the following: $\Sigma F - (W/g)a = 0$ and $\Sigma M - (I/g)\alpha = 0$, in which $(W/g)a$ may be called an inertia force and acts in a direction opposite to that of the linear acceleration and $(I/g)\alpha$ may be called an inertia moment and acts in a direction opposite to that of the circular acceleration. If inertia forces and moments are considered among those acting on an airplane or a portion of an airplane, the structure may, for purposes of analysis, be considered in static equilibrium. It should be noted that, while rotation at a constant speed about the center of gravity of a body causes no resultant inertia forces, each element of the body undergoes an acceleration toward the center of gravity; moreover, rotation at a constant speed about a point other than the center of gravity of a body results in a rotating resultant radial inertia force that acts toward the center of gravity.

*c. Aerodynamic forces:* These loads are due to air pressure. They are, in reality, inertia forces developed within the mass of air that undergoes acceleration

as it is displaced by the airplane passing through it.  Aerodynamic forces are divided into three components: (1) *lift*, the components perpendicular to the line of flight and parallel to the plane of symmetry of the airplane; (2) *drag*, the components parallel to the line of flight; (3) *transverse forces*, the components perpendicular to the plane of symmetry of the airplane.

*d. Propeller thrust:* This is the force used to pull the airplane through air.  In some airplanes, a similar thrust, or propulsive force, is developed by a jet engine.

*e. Engine torque forces:* This is a necessary by-product of supplying propeller thrust.

*f. Ground reactions:* These are forces resulting from contact of the landing gear with the ground.  The values of these forces both while the airplane is landing and when it is at rest must be considered.

**16·4  Airplane Loading Conditions.**  Each part of an airplane must, of course, be designed to withstand every reasonable combination of loads that can occur.  Such combinations are usually classified under one of the following loading conditions:

*a. Main flying conditions:* These are the conditions resulting during flight from a sudden change in the effective angle of attack of the aerodynamic forces acting on the wings or tail, due either to the encountering of a gust or to the intentional act of the pilot.

*b. Main landing conditions:* These are the range of flight attitudes at which satisfactory landings can be made, including intermediate as well as extreme conditions.

*c. Minor loading conditions:* These are special conditions of flight or landing that must be considered to ensure satisfactory design of parts not adequately covered by the main flying and landing conditions.

**16·5  Stress Analysis for Airplanes.**  With the magnitude of the loads acting on an airplane or portion of an airplane known, the procedures involved in computing reactions, shears, bending moments, bar stresses, etc., are essentially the same as those hitherto used for civil engineering structures.  This statement holds for the determination of the position of movable loads that leads to maximum stresses as well as for the computation of the stresses themselves.  We shall now consider a series of illustrative problems, based on airplane structures, that will demonstrate the foregoing.

**16·6  Computation of Reactions.**  Consider the reactions on the wing $AB$ shown in Fig. 16·1 and supported by the fuselage at $B$ and by the strut $CD$.  Since the direction of the reaction at $B$ is not known, both the vertical and horizontal components of the reaction at that point constitute independent unknowns.  Although the reaction at $C$ has both vertical and horizontal components, the stress in strut $CD$ must lie along the axis of $CD$, so that the direction of $R_c$ is known.  Hence there are in all only three independent unknown reactions for the wing $AB$, and

these may be determined by the usual application of the equations of statics.

In airplane construction, it is often impractical to construct points of support in a manner such that reactions can be determined by statics only. Thus in the horizontal drag truss of a wing, as shown in Fig. 16·2, the direction of neither reaction is known, so that there are four independent reactions at the supports of the drag truss on the fuselage.
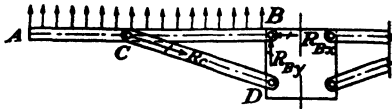


FIG. 16·1

For this condition, one may resort to an elastic analysis or, for preliminary design, proceed on the basis of one or more assumptions. Thus one may assume that $R_{Az} = R_{Bz}$; or, instead, since the diagonals $a$ and $b$ may be wires, so that $a$ would tend to be in compression and hence carry no load, one may assume that the entire reaction in the $Z$ direction occurs at $B$, where $b$ meets the fuselage. Only one of these assumptions is necessary for the determination of the reactions; but it is well to note that, in problems of this kind, successive analyses, based on successive reasonable assumptions, often serve as an important basis for design.



FIG. 16·2

Each member of the structure can then be proportioned to withstand the largest stress computed for that member.
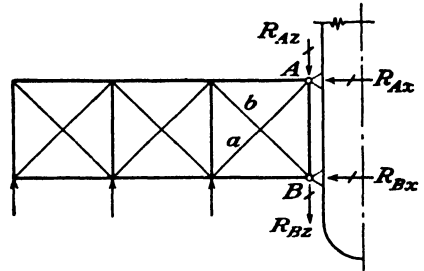
**16·7  Truss Analysis.** The framing of an airplane often involves planar trusses that may be analyzed by the usual analytical or graphical
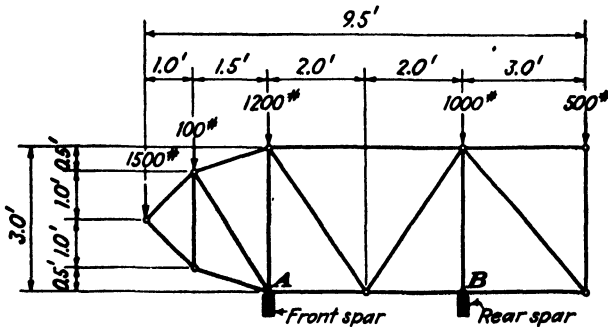


FIG. 16·3

methods. The side truss of a nacelle, as shown in Fig. 16·3, is an example of such a truss. Because the points of support of this truss at both $A$

and $B$ are capable of developing horizontal as well as vertical reactions, this truss is statically indeterminate to the first degree with respect to its outer forces. The vertical reactions can be determined by statics; but, to determine the horizontal reactions on the basis of statics only, an assumption must be made. For this case, the truss is acted upon by vertical loads only, and it thus is reasonable to assume that the horizontal reaction at one point of support equals zero; it then follows by statics that the horizontal reaction at the other point of support also equals zero.
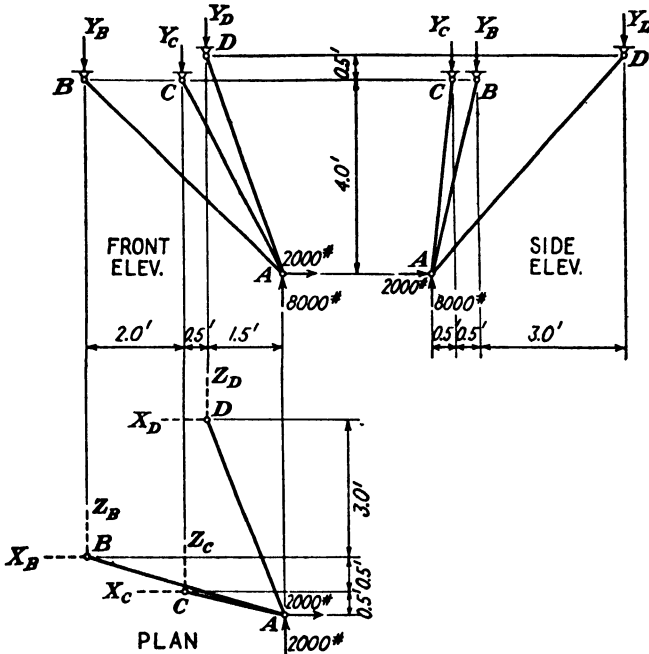


Fig. 16·4

Of course, one can carry out an elastic analysis to determine the horizontal reactions. This, however, either involves an assumption as to the relative horizontal movement of points $A$ and $B$ or necessitates the inclusion of the supporting structure in the elastic analysis.

With the reactions assumed or computed, the bar stresses can be readily computed, by either analytical or graphical methods.

**16·8 Landing-gear Tripod.** A typical landing-gear tripod, as shown in Fig. 16·4, is a statically determinate three-dimensional structure and may readily be analyzed. This particular landing gear has 3 bars plus 9 reactions on the fuselage, or a total of 12 unknowns; it has 4 joints with 3 equations of statics each, or a total of 12 independent equations of statics.

One may, for example, commence the analysis by solving joint $A$ by the method of joints. This will require the solution of three simultaneous equations involving the stresses in the three legs of the tripod. With these leg stresses known, the reactions are easily computed by the method of joints.

**16·9   Externally Braced Wing.**   The structure of a typical externally braced wing is shown in Fig. 16·5. The primary load-carrying structure on each side of the fuselage consists of a three-dimensional framework supported by the fuselage. This framework is composed of
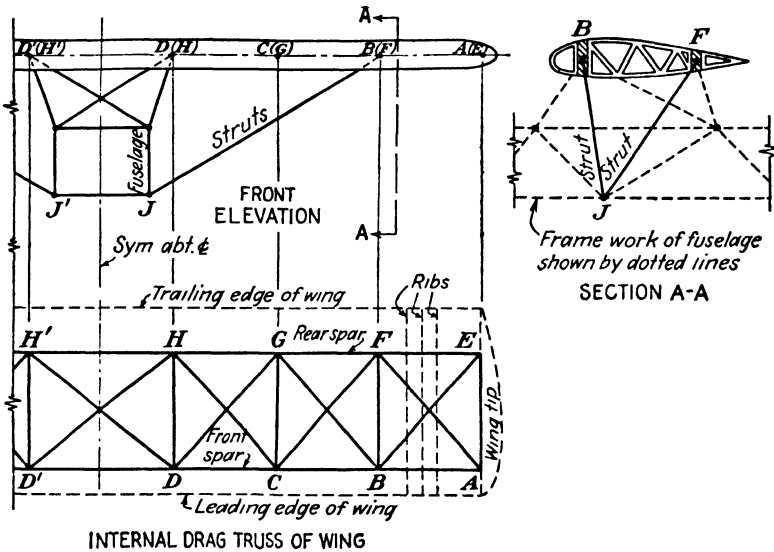


INTERNAL DRAG TRUSS OF WING
FIG. 16·5

the internal drag truss of the wing and of the two supporting struts $BJ$ and $FJ$.

The lift and drag forces are applied to the skin of the wings and are carried by the skin to transverse wing ribs. The ribs may be considered as statically determinate beams supported on the wing spars, so that the rib reactions furnish the spar loadings. While the spars carry axial stress as members of the three-dimensional framework, they also act as beams that carry in bending the loads applied by the ribs. As beams, they develop reactions at the points where they are supported by the struts and by the fuselage; the reactions at the struts are the panel loads for which the three-dimensional framework on one side of the fuselage is to be analyzed.

For this illustration, the spars are each supported at four points (for example, $B$, $D$, $D'$, and $B'$), and the spar reactions at these points

may be determined by an elastic analysis,[1] using, for example, the three-moment equation, provided that one assumes that the points of support of the spar do not yield.

If the loads applied by the spars to the struts at points $B$ and $F$ are known, the stresses in the bars of the three-dimensional framework on that side of the fuselage can be determined by conventional methods. The diagonals of the drag truss may be of wire and hence not capable of carrying compression, which will simplify the analysis. For the wing spars, total fiber stresses are obtained by superimposing the fiber stresses due to the axial stresses that they receive as members of the three-dimensional structure and those due to bending as continuous beams.

The foregoing examples are typical of the fabric-covered airplane having a framework of wood, plastic, or metal members that act as beams or parts of trusses. The civil engineer finds little difficulty in analyzing such structures once he has become familiar with the strength properties of the material of which they are made and is accustomed to the difference in the scale of things as represented by the dimensions of the members.

Many elements in the structures of all-metal stressed-skin airplanes are amenable to analysis by methods with which the civil engineer is familiar, but others require the application of basic principles of mechanics in ways not ordinarily used in the design of bridges or buildings. The very thin sections used in aircraft require more consideration in respect to shear and compression than is normally given standard structural sections. Some aircraft members are designed so that they will not buckle locally, whereas others are permitted to buckle under their working loads so long as their ultimate strengths exceed their design loads. Such buckling, whether attributed to shear or to compression, entails a redistribution of stress throughout the member in which it appears and requires the use of methods and assumptions in the analysis of the member which are new to structural engineering. To master the details of these methods requires some time for study and practice, but it entails no insurmountable difficulty for a well-trained civil engineer.

**16·10  Ship Structures—General.** Like an airplane structure, a ship structure differs from typical civil engineering structures in two important respects. (1) Instead of being supported by a relatively unyielding foundation, the structure of a ship receives its support from fluid pressures. (2) Because a ship undergoes motion, its structure is subjected to inertia forces.

The important loads for which a ship must be designed include the

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[1] The usual method for determining these reactions is by applying a form of the three-moment equation that provides for the effects of axial and transverse loads on the spar and for deflection at the points of support. An engineer who is familiar with the ordinary three-moment equation can readily master the more general form that is applicable to such "beam columns."

weight of the ship itself, the weight of the cargo (which must be treated as a movable load), the hydrostatic pressures of the supporting water (which must also be treated as a movable load since its distribution depends on wave formations), the dynamic pressures resulting from wave action, the impact loads resulting from contact with piers, ice, etc., and inertia forces.

The analysis of ship structures is usually divided into two parts: (1) the analysis of the strength of local parts such as deck beams and pillars; (2) the analysis of the strength of the hull of the ship, which is considered as a girder carrying in bending and shear the loads to which it is subjected.

**16·11   Local Parts.**   The analysis of stresses in local parts includes the determination of stresses in structural members supporting heavy weights such as masts, and engines.   It also includes investigating the more or less localized stresses that occur during docking, grounding, etc., where comparatively small areas of the surface of the hull are subject to large intensities of loading.

For such conditions, the analysis of the stresses in beams, columns, deck plates, bulkheads, etc., can be carried out, once the loadings have been determined, by the same general methods as those used for civil engineering structures.   However, in naval architecture, there is a much greater tendency to resort to rules such as those in "Rules for Building and Classing Steel Vessels,"[1] than to carry out detailed stress analyses.   Such rules specify the size of members that should be employed under various standard conditions in such detail that the use of structural analysis is largely limited to conditions of an unusual nature.

**16·12   Shape of Supporting Wave.**   It is usually assumed that the hull of a ship, acting as a girder, receives its greatest stresses either



SAGGING CONDITION                    HOGGING CONDITION
(a)                                         (b)

FIG. 16·6

when the crests of the waves are at the end of the ship, as shown in Fig. 16·6a, which is known as the sagging condition, or with the crest of a wave at the center of the ship, as shown in Fig. 16·6b, which is known as the hogging condition.

In either case, the distance between wave crests is taken as equal to the length of the ship, and the depth of the wave is usually assumed to equal one-twentieth of that distance.   The wave itself is assumed to have the shape of a trochoid, which for the case under consideration has

[1] Published by the American Bureau of Shipping.

the following relative ordinates for each fifth point from the hollow to the crest: 0.000; 0.066; 0.260; 0.552; 0.856; 1.000.

In considering the sagging condition, it is usually assumed that the ship is fully loaded; for the hogging condition, while the ship is assumed to be carrying its cargo, it is usually assumed to carry neither fuel nor water.

**16·13  Analysis of Longitudinal Strength.**  In the analysis of the hull, the position of the ship with respect to the surface of the water is determined by trial.  The procedure is similar whether the sagging or hogging condition is considered; for purposes of illustration, we shall consider the sagging condition shown in Fig. 16·7a, in which the distances $d_1$ and $d_2$ must be tentatively assumed.

A weight diagram is then constructed, as shown in Fig. 16·7b. This curve shows the distribution from bow to stern of the total weight of the ship including cargo, fuel, and other contents.

On the basis of the assumed values of $d_1$ and $d_2$, the buoyancy curve of Fig. 16·7c is then drawn, based, of course, on the weight of displaced water.
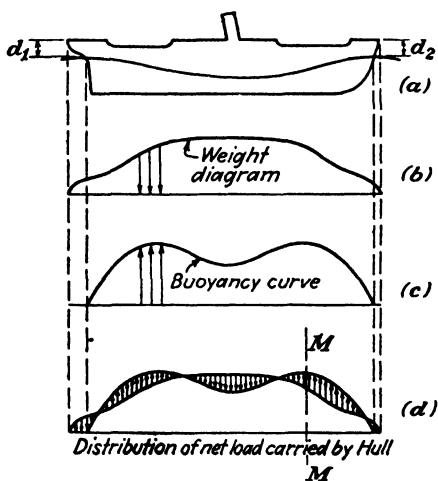


FIG. 16·7

The weight diagram and buoyancy curve are then superimposed as shown in Fig. 16·7d, leading to differential ordinates that define the net load that is carried by the hull if $d_1$ and $d_2$ have been properly assumed. For this analysis, the ship is assumed to be stationary, so that the net loading curve must be in static equilibrium.  Under this circumstance, two conditions must be satisfied.  (1) The sum of the vertical forces acting on the hull must equal zero, so that the positive areas under the net loading curve must equal the negative areas under the same curve. (2) The sum of the moments, about any point lying in the vertical plane of symmetry of the hull, of all the vertical forces acting on the hull must equal zero, so that the moments of the positive areas under the net loading curve, taken about any vertical line such as $MM$, must equal the moments of the negative areas under the same curve, about the same vertical line. If these two conditions are not satisfied, one must, by successive analyses, adjust $d_1$ and $d_2$ by trial until equilibrium is obtained.

With equilibrium obtained, the net loading curve on the hull is

defined.    Moments and shears at any section along the hull can then be computed by statics.

To determine fiber stresses in the hull, a transverse section through the hull is treated as a built-up section, for which the section modulus and other properties may be computed by the usual methods.

**16·14    Chemical Engineering Structures—General.**    Many of the structures of importance to the chemical engineer may be analyzed by the methods already presented in connection with civil engineering structures.    The problem of corrosion, which is, of course, to be reckoned with in all structures, is of particular importance in many structures that come into contact with corrosive chemicals.    Pipes, tanks, and stacks of chemical plants may, for example, be exposed to corrosive material in either the liquid or the gaseous state.    To allow for this, one may either design on the basis of relatively low fiber stresses; or, as an alternate, additional thickness of material may be provided over and above that required for stress-carrying purposes.

Such structures may also be exposed to much greater variations in temperature than one is likely to encounter in civil engineering structures. An arch in a process chamber may, for example, be subjected to temperatures as high as 1200°F.    Such conditions not only indicate the unusual importance of analyses for stresses due to changes in temperature but call for the use of materials, such as firebrick, that are resistant to high temperatures.

We shall now consider the analysis of two typical structures of importance to the chemical engineer.

**16·15    Temperature Stresses in Pipes.**    An analysis frequently encountered in the design of chemical plants is that of determining the



$E = 30,000,000$ lb. per sq. in.
$A = 3.00$ sq. in.
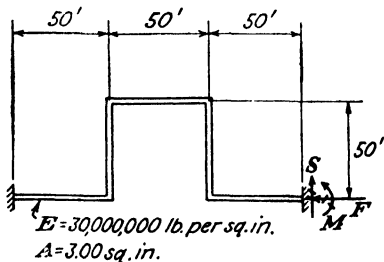
FIG. 16·8

stresses in a pipe caused by changes in temperature.    Consider the layout shown in Fig. 16·8, for which we shall assume a temperature change of $\Delta t°$F. Since each end of the pipe is completely fixed, the structure is statically indeterminate to the third degree. The three reaction components at one end of the pipe may be chosen as the redundants, and their values, for the temperature change under consideration, may be determined by any of the standard methods available for investigating stresses in statically indeterminate structures, due to temperature changes.

For long flexible pipes, laid out in a manner similar to that shown in Fig. 16·8, it is sometimes assumed that the moment $M$ and the shear $S$

at the end of the pipe are each equal to zero, since comparative analyses have shown that the maximum moments in the pipe will not be greatly affected by these assumptions. This leads to the single redundant $F$, which can then be evaluated without recourse to simultaneous equations. If, with this single redundant, the deflections entering into the indeterminate analysis are computed by the moment-area method, one arrives at the Walker-Crocker method of analyzing the problem of temperature stresses in pipes.

16·16  **Analysis of Stacks.**  Consider the stack shown in Fig. 16·9. Such stacks are sometimes designed to carry lateral loads by cantilever
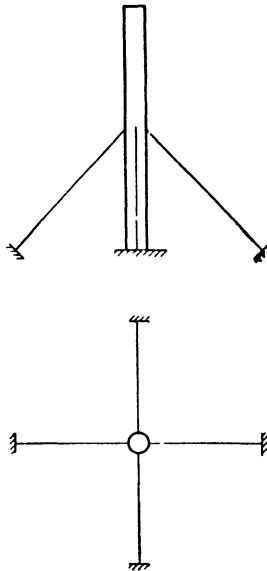


FIG. 16·9

action only; the guys are then added to prevent excessive lateral deflections of the stack and supposedly to give a certain indeterminate increase to the strength of the stack to resist lateral loads. Such a procedure is open to criticism, however, since the tension in the guys develops compression in the stack below the elevation where the guys are attached. The procedures outlined in Art. 11·9 for guyed structures should be followed.

# CHAPTER 17

## MODEL ANALYSIS OF STRUCTURES

**17·1   Application of Model Analysis.**   In the field of structural engineering, the use of models has steadily increased in the last 25 years. Today, model analysis of structures not only is extremely important as a tool for research and development but also forms an important supplement to the mathematical methods used in the actual design of structures. Perhaps the most widely publicized use of models in this latter respect has been in connection with the design of most of the important and well-known suspension bridges erected during the last 20 years.   Further evidence of the importance of model analysis in the field of structural design is furnished by the well-equipped laboratories that have been established by several governmental agencies.   There are also many academic institutions that possess fine model-analysis laboratories established primarily for educational and research purposes.

Model analysis of structural problems encountered in either research or actual design may be used for one or more of three reasons: (1) because mathematical analysis of the problem concerned is virtually impossible; (2) because the analysis, though possible, is so complex and tedious that the model analysis offers an advantageous short cut; (3) because the importance of the problem is such that verification of the mathematical solution by model test is warranted.   The stress distribution in an irregularly shaped member may be investigated by use of a model for the first reason; a model test may serve as the basis for the analysis of a complex building frame for the second reason; a model study of the proposed design of a suspension bridge may come under the third classification.

The objective of tests of a structural model may generally be placed in one of the following four categories: (1) stress analysis of the model; (2) determination of stress distribution; (3) determination of critical or buckling loads; (4) analysis of the characteristics of the normal modes of vibration.   As used in this chapter, *stress analysis* means the determination of the total axial stress, the total shear stress, and the resisting moment acting on any cross section of the model, whereas *stress distribution* is the term used to designate the manner in which the stress intensities vary across any cross section of a member.

**17·2   Standard Methods of Model Stress Analysis.**   Certain methods are commonly used for the stress analysis of a model, among

them being the *brass-wire model method*, the *Beggs method*, the *Eney deformeter*, the *Gottschalk continostat*, the *moment indicator*, and the *moment deformeter*. All these methods are discussed in Chapter 18. The *photoelastic method* is also used to a limited extent in the stress analysis of structural models, but its principal application is in the solution of stress-distribution problems.

**17·3  Design of Models.**  Whenever a reduced-scale model is used to study an actual structure, it is necessary, of course, for the model to be designed so that full-scale behavior of the prototype may be deduced from the observations of the behavior of the model. For this to be accomplished, the dimensions of the model and the characteristics of the material used in its construction must bear certain definite relations to the dimensions and material of the prototype. The principles governing the relationship between a model and its prototype are called the principles of similitude. Certain of these principles govern the design of the model, and others establish the means of extrapolating the results of the model tests to predict the performance of the prototype.

The determination of the principles of similitude is discussed briefly in Chap. 18.[1]

The choice of the proper material for the contruction of models is of great importance. Not only must the material be such that its structural action is suitable to its use, but one should also bear in mind the ease with which it can be fabricated for a small model. For many models, the materials of the prototype may be used. Steel is often used and reinforced concrete may be used if the model is sufficiently large.

It is often desirable to use a material having a lower modulus of elasticity than the material of the prototype so that distortions which are large enough to be measured accurately may be obtained without the application of forces which are too great. The use of duralumin or brass in place of steel is sometimes convenient for this reason. Brass has the additional advantage that it may be soldered easily, thus facilitating the construction of the model.

Celluloid is one of the most widely used materials in the construction of the models used in conjunction with the more common model methods of stress analysis; its properties are discussed in more detail in the next article.

The selection of the scale of a model depends on many factors, some of the more important of which are the properties of the materials avail-

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[1] See also CONRAD, R. D., Structural Models. Part I: Theory, *U.S. Navy Dept., Bur. Construction and Repair, C and R Bull.* 13, 1938; BEGGS, G. E., R. E. DAVIS, and H. E. DAVIS, "Tests on Structural Models of Proposed San Francisco-Oakland Suspension Bridge," University of California Press, Berkeley, 1933.

able for its construction, the capacity of the equipment to be used in loading the model, the dimensions of the instruments to be used in testing the model, the limitations of machinery to be used in fabricating the model, and the funds and time available for the experimental program. As the scale of a model is reduced, it becomes increasingly difficult to maintain geometric similarity, and the duplication of all the details of the prototype is physically impossible. Some details of the design are obviously unimportant and may be omitted from the model. In other cases, the details of the structural connections have a great influence on the result, and a large enough scale must be used so that the structural action of the model is adequate.

**17·4  Properties of Model Materials.**  The properties of model materials, such as steel, brass, duralumin, wood, and concrete, are well known and need not be reviewed here. Celluloid is widely used for structural models and has, in common with certain other plastic materials, certain properties that are not well known and require further discussion.

Celluloid (or cellulose nitrate) has some very desirable properties as a model material, but it also has some that are very undesirable. It is very readily machined, has a low modulus of elasticity, is homogeneous, and may be readily welded by using acetone. On the other hand, its elastic properties change decidedly with age, temperature, and humidity. More serious still, celluloid creeps under a constant load; *i.e.*, if a load is applied, while some 85 per cent of the deformation occurs within a few seconds, the remaining 15 per cent takes place more slowly. It is necessary to wait an appreciable period, in the neighborhood of 15 min, before motion is essentially complete. Even then, small movements will still occur.

This creep phenomenon may be more easily understood by referring to Fig. 17·1. Suppose that a weight $W$ is hung on the celluloid member shown. Almost instantaneously, the member will undergo about 85 per cent of its total elongation, but the remaining 15 per cent will take place gradually, as shown in Fig. 17·1b. If the member is loaded instead with a weight of $2W$, the elongation will vary with time, as is also shown in Fig. 17·1b. The elongation that takes place up to certain times after loading, such as $t = 1, 2, 5, 10$, etc., may be read off from the curves of Fig. 17·1b for various loads $W$, $2W$, $3W$, etc. If these elongations are plotted against the loads as shown in Fig. 17·1c, it is found that all the elongations measured at 1 min after loading lie along a straight line for all practical purposes. The same is true of the elongations measured at times $t = 2, 5, 10$, etc. All these straight lines likewise pass through the origin. This latter plot discloses a very important characteristic of the creep of celluloid. At any particular instant after loading, the instan-

taneous elongation (or strain) is directly proportional to the load (or stress intensity); *i.e.*, at any instant the material is following Hooke's law and has an instantaneous modulus of elasticity $E_t$. The effect of creep is to lower this instantaneous value of the modulus with time.

It is extremely important that the creep of celluloid has this characteristic; otherwise, the usefulness of the material for structural models
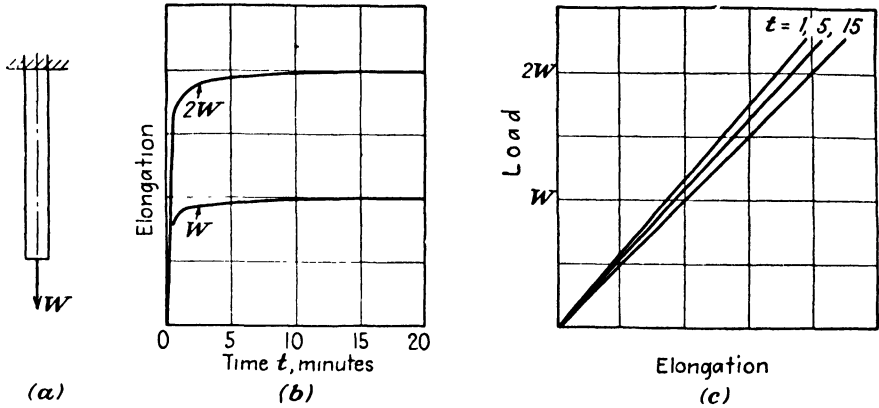


Fig. 17·1

would be impaired.     In view of the creep of celluloid, if a constant load $W$ is applied to the end of a cantilever beam, the elastic curve progressively assumes different positions as the time after loading increases, as shown in Fig. 17·2a.    Of course, after about 15 min, the rate of creep has become so small that the beam may be considered to have come to rest.    Suppose, however, that a fixed deflection $\Delta_b$ is introduced at the end of the beam, as shown in Fig. 17·2b. The force $P_t$ applied to the beam by the pin maintaining the fixed deflection at $b$ decreases with time owing to the creep and the resultant lowering of the instantaneous modulus $E_t$. Since, at a given instant, $E_t$ does not vary with the stress intensity and is therefore



Fig. 17·2

constant for the entire beam, the deflection at point $b$ may be expressed as $\Delta_b = K_b(P_t/E_t)$, where $K_b$ is a constant that depends only on the dimensions of the beam.    Hence, $P_t/E_t = \Delta_b/K_b = C$, a constant that does not vary with time.    The deflection at any point $n$ may be expressed as $\Delta_n = K_n(P_t/E_t) = K_nC$, also constant and independent of
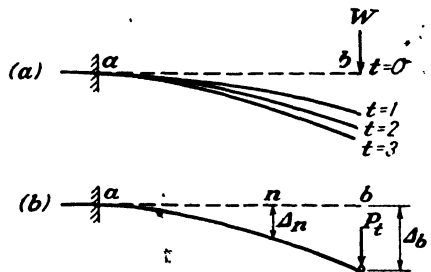
time, since $K_n$ is a constant depending only on the dimensions of the beam and the location of point $n$. It is apparent therefore, that not only does the end of the cantilever remain fixed in position but the entire elastic curve of the beam also does so. It may therefore be concluded that the deflected position of a celluloid model will not change with time if a fixed deformation is applied to the model instead of a constant load.

**17·5   Use of Spring Balance to Overcome Creep.**   It is shown in the previous article that the deflected shape of a celluloid model does not change with time if a fixed deflection is introduced at one point on the model.   Owing to creep, however, the effective modulus of celluloid does change with time, and there-



fore the external forces and internal stresses of the model likewise change with time. Thus, it is difficult to interpret the strains and deflections, even though they do not change with time and therefore may be measured without difficulty. In other words, we cannot conclude simply from the measured strains that a certain load will produce certain stresses in the model—all we know is that some unknown load produces certain strains and deflections.

$\Delta_s$ = Relative deflection of points $b$ and $c$

FIG. 17·3

Through the use of a so-called celluloid "spring balance," we may circumvent this difficulty, however, and interpret the measurements of the strains and deflections.   Such balances may be designed in a number of different forms and shapes, depending on the problem at hand. Suppose, for example, that we wish to analyze the celluloid model of the rigid frame shown in Fig. 17·3 when it is acted upon by a horizontal force $H$ at the top of the right column.   To do this, the model may be connected to a balance, as shown in Fig. 17·3a.   Then, by pulling point $d$ to the right, a horizontal deflection $\Delta$ may be applied to the combined system of model and balance.

On the assumption that the steel straps $ab$ and $cd$ may be considered as being infinitely rigid in comparison with the celluloid model and spring balance, part of this total deflection $\Delta$ is introduced into the model and the remainder into the spring balance.   Thus

$$\Delta = \Delta_m + \Delta_s \qquad\qquad (a)$$
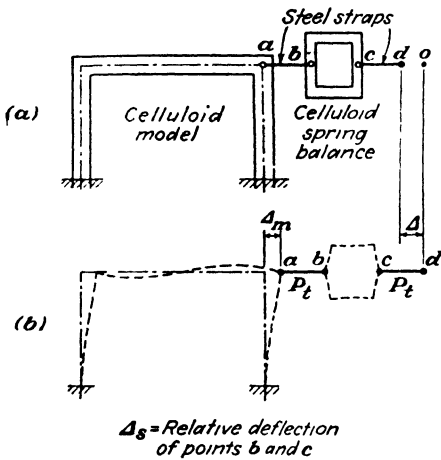
If both the balance and the model have been made out of the same piece of celluloid, it is legitimate to assume that the instantaneous modulus $E_t$ is the same for both. Recognizing that the instantaneous value of the tension in both straps is the same and equal to $P_t$ and, therefore, that the distorting force on both model and balance is also the same and equal to $P_t$, we may express the deflections $\Delta_m$ and $\Delta_s$ as follows:

$$\Delta_m = K_m \frac{P_t}{E_t} \qquad (b)$$

$$\Delta_s = K_s \frac{P_t}{E_t} \qquad (c)$$

where $K_m$ and $K_s$ are constants, the value of which depends only on the geometry and dimensions of the model and balance, respectively.

Substituting in Eq. ($a$) from Eqs. ($b$) and ($c$), we find that

$$\frac{P_t}{E_t} = \frac{\Delta}{K_m + K_s} \qquad (d)$$

and therefore conclude that the ratio of $P_t$ to $E_t$ remains constant and does not change with time. In other words, while both $P_t$ and $E_t$ change with time, they always bear a constant relationship to each other. Substituting now from Eq. ($d$) back into Eqs. ($b$) and ($c$), we find that

$$\Delta_m = \frac{K_m \Delta}{K_m + K_s} \qquad (e)$$

$$\Delta_s = \frac{K_s \Delta}{K_m + K_s} \qquad (f)$$

Since the right-hand sides of both these equations contain only constants that do not change with time, both $\Delta_m$ and $\Delta_s$ remain constant and do not change with time. In other words, by using a celluloid spring balance in this manner and introducing a fixed deflection $\Delta$ into the combined system of model and balance, we produce a distortion of both balance and model that remains constant and does not change with time.

Suppose that the model has been distorted in this manner and that the resulting strains in it have been measured. Suppose that we then wish to interpret these strains so as to obtain the stresses in the model due to a horizontal force $H$. While the stresses in the model vary with time, we may express the instantaneous value of the stress intensity, $\sigma_t$, in terms of $E_t$ and its corresponding strain $e$.
Thus,

$$\sigma_t = E_t e \qquad (g)$$

From Eq. ($c$),

$$E_t = \frac{K_s}{\Delta_s} P_t \qquad (h)$$

Therefore,

$$\sigma = \frac{K_s}{\Delta_s} eP_t \qquad (i)$$

Assume that $K_s$, the constant of the spring balance, has been either computed or determined previously from a calibration test. The desired stress intensity $\sigma_t$ has been expressed, therefore, in terms of the known constant $K_s$, the measured quantities $e$ and $\Delta_s$, and the unknown value of $P_t$. By assigning various values to $P_t$, the corresponding stress intensities may be obtained. If $P_t = H$, the corresponding stress intensity $\sigma_H$ is found to be

$$\sigma_H = \frac{K_s e}{\Delta_s} H \qquad (j)$$

In this manner, the strain measurements may be interpreted to give the stresses in terms of a horizontal force $H$.

The use of the spring balance is discussed further in subsequent articles. The determination of the constant of the spring balance by calibration test is also discussed in detail later. The purpose of the present discussion is simply to introduce the idea of the balance and illustrate how it may be used to overcome the difficulties associated with the creep properties of celluloid.

**17·6  Planning a Model Analysis.** It is difficult, if not impossible, to give a complete list of instructions that will cover the planning of the model analysis of any and all problems that might be encountered. There are, however, certain factors that are more or less common to most such problems and that should therefore be discussed.

The first factor that should be considered is the purpose of the proposed model study. If the model is being used to check the design of an actual structure, then the principles of similitude must be established for the given problem and observed as closely as is possible and practical in designing and constructing the model. Contrasted to this type of model study is the case where the model is being used to develop or study the mathematical theory for a certain type of problem. In such cases, the model may be considered as actually a small-sized structure for which the mathematical results are computed and compared with the experimental results. In other words, model and prototype are synonymous, and similitude is not a factor. When models are used in this manner, a number of different models should be selected so that the relative dimensions of the various elements are varied sufficiently to cover the entire range that might be encountered in large-sized structures of this type.

The next step in planning a model study is to select the most appropriate method of model analysis for the particular problem at hand. Quite

often the choice of the method is limited by such factors as availability of equipment and suitable model material or experience of the laboratory personnel.    Assuming that there are no such limitations, the method of analysis may be selected simply on the basis of the advantages and disadvantages of the various methods applicable to the problem at hand.

Once the method of analysis is selected, the limiting dimensions of the model may then be established.    To illustrate how to do this, suppose that it has been decided to use the Beggs method for the stress analysis of a given problem.    First, the width of the model at any point where the deformeter gage is to be attached should not exceed $\frac{3}{4}$ in., which is the normal capacity of the clamping device on the gage.    On the other hand, the minimum width at any point in the model should preferably be not less than $\frac{1}{4}$ in.    This is controlled by the working tolerances that should be permitted in constructing celluloid models.    Such models can be finished quite easily to satisfy a tolerance of ±0.002 in.    If the width of a member is less than $1\frac{1}{4}$ in., the tolerance in the width will cause a variation in the moment of inertia of more than ±2 per cent, which is about the maximum that should be permitted.    The length scale and the thickness of the members should be selected so that the model is neither too stiff nor too flexible.    If the model is too stiff, the springs of the deformeter gage will not be strong enough to distort the model.    If the model is too flexible, some of the compression members may buckle.

Similar factors are involved in establishing the limiting dimensions of any model.    Careful consideration must be given to the limitations imposed by the proposed methods of measuring strains and deflections or by the proposed technique of loading or distorting the model.    Of course, the more experience one has, the easier it is to make such decisions.

In planning a model study, the fact is often overlooked that some of the most informative model studies can be conducted with relatively simple models.    Elaborate, complex, and expensive models may be impressive, but they do not necessarily produce the most valuable results. If it still seems necessary and desirable to make such a model, precede such a study with enough studies of simplified models to be sure that the proposed model and method of analysis will yield satisfactory results.

**17·7   Interpretation of Model Results.**    The first step in interpreting model results comes when the experimental data are actually being measured and recorded.    One should continually be studying the data as they are accumulated, being sure that the data from duplicate tests are consistent, checking to see that the measurements from point to point vary in a reasonable and orderly manner, taking precautions to see that constant test conditions and techniques are being maintained, etc.    In this way, if questionable or unexpected data are being obtained,

the measurements may be checked and verified before the test setup is altered. It is often helpful to make rough plots of data as they are being obtained in order to "spot" quickly any inconsistent measurements.

After the test data have been analyzed and the test results computed, the information should be plotted or tabulated in such a manner that its reasonableness may be estimated. The best way of judging the results, of course, is to apply any physical checks available. For example, in any static problem, the test results must satisfy the laws of static equilibrium: $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$. Of course, static checks may be applied to the entire model or any portion of it. Such checks are extremely useful. There is perhaps no better way of judging the reliability of model results for a static problem.

It is often difficult to load a model exactly as specified. For example, suppose that the model shown in Fig. 17·4 is analyzed for a vertical load



To counterweight
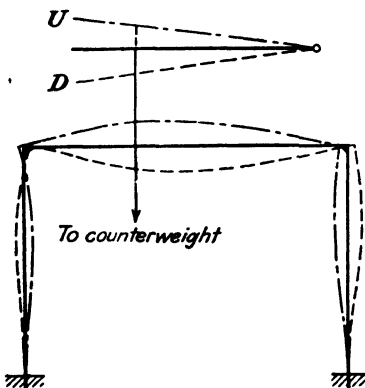
**FIG. 17·4**

by means of the moment indicator. To do this, the distorting device is arranged as indicated. Just how successful we are in applying the desired vertical load is indicated by the model results. Suppose that the joints are in equilibrium under the experimental end moments in the members but that the shears in the columns which are computed by statics from these end moments are not equal and opposite. This indicates that the load which was actually applied to the model had a horizontal as well as a vertical component.

To correct these results for this error in loading we now analyze the model for a horizontal load. Then, using this information we can adjust the values of the first test so as to remove the effect of the undesired horizontal component of the load. In this manner, therefore, the laws of statics may often be used to assist us in correcting for the effect of errors in loading.

Whenever the results obtained from a model are to be extrapolated in order to predict the behavior of its prototype, the principles of similitude must be fulfilled in designing, constructing, and testing the model. The requirements of these principles can seldom be met exactly, and in almost all practical cases it will be found that the behavior of a model differs to some extent from that of its prototype. Scale effect is defined as the degree to which a prediction made from a model test will not be fulfilled in the full-scale structure. This effect may be caused by unavoidable inaccuracies in test conditions. As the scale of a model is reduced,

it becomes impossible to reproduce all the details of the prototype and such elimination also may contribute to the effect.  The excellence of the workmanship in building the model must increase as the scale is reduced in order to minimize such results.

It is evident that scale effect may arise from a variety of sources, some of which are beyond control.  It should always be suspected.  If the purpose of the test is to provide an accurate indication of the behavior in full scale, similar models of different scales may be built and tested and the laws of behavior determined empirically from the results of the series of model tests.

# CHAPTER 18

## MODEL METHODS OF STRESS ANALYSIS

**18·1 General.** One of the most frequent uses of structural models is to obtain the stress analysis of a model of a statically indeterminate structure. Numerous methods[1] and techniques have been developed for this purpose. In using some of these methods, the required results may be obtained by loading the model in the same manner as the prototype. The elastic deformation of the model is then similar to that of the prototype, and strain measurements then lead to the required results. Such a method is called a direct method of model stress analysis. Suspension-bridge model studies are usually carried out in this manner; the moment indicator is also used in a direct manner.

Contrasted to direct methods of model analysis are those methods in which the model is loaded in a manner bearing no direct relation to the actual loading on the prototype. Such methods are called indirect methods. Usually they involve first finding influence lines for the model. These results may be extrapolated to the prototype, and the stresses in the prototype due to the given condition of loading may then be computed from the extrapolated influence lines. The methods of using the Beggs deformeter and the moment deformeter are examples of indirect methods of model analysis.

**18·2 Theory of Certain Indirect Methods.** Certain of the indirect model methods of stress analysis are merely different experimental techniques of applying Müller-Breslau's principle, which is stated in Art. 15·3. The spline method, the brass-wire model method, the Beggs method, and the Gottschalk Continostat all are examples of such methods.

In the general case for any stress element $X_a$, this principle may be stated mathematically by the following equation:

$$X_a = - \frac{\Delta_{na}(1)}{\Delta_{aa}} \tag{18·1}$$

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[1] McCULLOUGH, C. B., and E. S. THAYER, "Elastic Arch Bridges," Chap. VII, John Wiley & Sons, Inc., New York, 1931; WILBUR, J. B., Structural Analysis Laboratory Research, *Mass. Inst. Tech., Dept. Civil Sanitary Eng.,* Ser. 65, 1938, Ser. 68, 1939, Ser. 73, 1940, Ser. 80, 1941; NORRIS, C. H., Model Analysis of Structures, *Exptl. Stress Analysis,* vol. 1, No. 2, July, 1944; WILBUR, J. B., and C. H. NORRIS, Model Analysis of Structures, "Handbook of Experimental Stress Analysis," Chap. 15, John Wiley & Sons, Inc., in preparation.

It is important to note the sign convention of this equation, namely: $X_a$ is plus when in the same sense as the introduced deflection $\Delta_{aa}$, and $\Delta_{na}$ is plus when in the same sense as the applied unit load, the influence of which is given by the ordinates of the influence line. Note further that $X_a$ may represent either a force or a couple. If $X_a$ is a force, the corresponding $\Delta_{aa}$ is a linear deflection; but if $X_a$ is a couple, the corresponding $\Delta_{aa}$ is an angular rotation.

According to Müller-Breslau's principle, the influence line for any reaction element $X_a$ can be obtained experimentally by applying the following procedure: Make a model of the structure, and temporarily remove the support restraint that supplies this reaction element to the model. Now introduce a displacement $\Delta_{aa}$ in the direction of the removed restraint. This will distort the model into the shape of the influence line for the reaction element $X_a$. To obtain the absolute magnitude of any ordinate, measure the deflection $\Delta_{na}$ which is produced at that point, and divide it by the introduced displacement $\Delta_{aa}$. Since any suitable constant displacement $\Delta_{aa}$ may be introduced and the force necessary to produce this displacement is not involved in the calculations, influence lines may be determined in this manner from celluloid models, if this is desirable, without encountering any difficulty due to creep.

**18·3   Instruments for Measurement of Deflections.** Practically all the model methods of stress analysis involve measuring linear deflections of the model. Such deflections are commonly measured by one of the following methods: steel scale or cross-section paper; dial gage; micrometer barrel; or filar micrometer microscope.

If the deflections of a model are rather large, they do not have to be measured by precise methods. In such cases, a steel scale, graduated in hundredths of an inch, or ordinary cross-section paper may be mounted adjacent to a model and the deflection measured in this manner with the aid of a magnifying glass. Cross-section paper is quite useful for such purposes, for on inspection it will be found that the lines are not solid but actually are composed of a series of dots spaced about a fiftieth of an inch apart.

When the deflections are small, it is necessary to use one of the more precise methods of measurements. Dial gages graduated in thousandths of an inch may be used. Such gages have the disadvantage, however, that the spring attached to the plunger applies sufficient force to alter the deflections of a flexible model by an appreciable amount. In such cases where a dial gage cannot be used for this reason, it will be necessary to use either a micrometer or a microscope.

The principal disadvantage involved in using a micrometer is that it is difficult to establish the exact point of contact between the model

and the micrometer. This difficulty may be overcome by using some type of contact indicator. The simplest type of indicator is a battery-and-lamp system in which the model and micrometer are part of the circuit and the micrometer acts as the switch which closes the circuit by making contact with the model. A much improved contact indicator using a 6E5 "magic-eye" radio tube has been suggested.[1] The wiring diagram for this device is shown in Fig. 18·1. In this arrangement, the contacts on the model and micrometer are connected to the grid circuit of the tube, and their coming together causes the magic-eye cathode-ray target to become completely illuminated. This device is very sensitive and will establish the point of contact of polished steel contacts within three or four millionths of an inch.
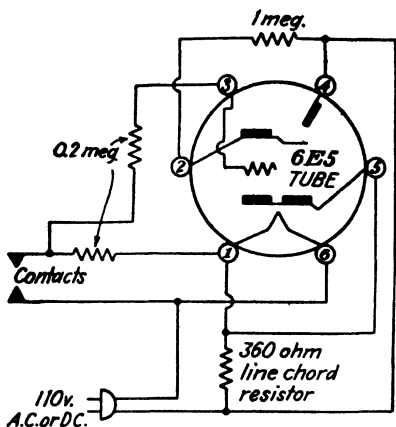


Fɪɢ. 18·1

All things considered, the filar micrometer microscope is perhaps the most useful instrument for measuring model deflections. The type of microscope supplied with the Beggs deformeter apparatus is very convenient. The apparent field of view of this microscope is shown in Fig. 18·2. There are two orthogonal cross hairs, which may be moved across the field by turning the micrometer head. There is likewise an index, which moves with the cross hairs along a fixed scale that makes an angle of 45° with each of the cross hairs. One complete turn of the micrometer head causes the index to move one full division along the fixed scale. By bringing the cross hairs tangent



APPARENT FIELD OF VIEW

Fɪɢ. 18·2

to two successive positions of the target, it is possible to obtain the movement of the target from the difference of the micrometer readings for the two settings. This arrangement of the cross hairs and the fixed scale makes it possible to read both the horizontal and the vertical movements of the target with one orientation of the microscope. The
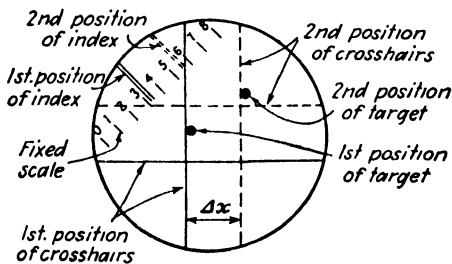
[1] Mɪʟʟs, B., A Sensitive Contact Indicator, *Rev. Sci. Instr.*, vol. 12, No. 2, p. 105, February, 1941.

value of one of the micrometer divisions in inches may easily be calibrated by observing with the microscope a known movement of a target placed on the plunger of a dial gage.

After a little practice, almost anyone can learn to use a micrometer microscope successfully. There are certain important rules that should be observed, however, *viz.:*

1. Choose clear, well-defined, and readily identified targets.
2. Adjust the eyepiece carefully to eliminate parallax.
3. Focus the microscope on the target as sharply as possible.
4. Orient the cross hairs accurately.
5. Do not unnecessarily touch the microscope or the table on which it is mounted.

If parallax is present, the image of the target will not lie in the plane of the cross hairs and this will lead to difficulty in reproducing readings. Poor focusing not only makes it difficult to bring the cross hairs tangent to the target but also in effect changes the magnification of the microscope and therefore changes its calibration factor.

Poor orientation leads to errors of the type shown in Fig. 18·3. Suppose that the target *a* moves to *a'*. To obtain the vertical component *d* of this movement, the cross hair *XX* should be oriented in a horizontal direction. The difference
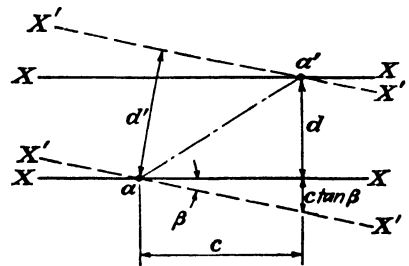


Fig. 18·3

between the two solid-line positions of cross hair *XX* will then give the correct distance *d*. Suppose, however, that the cross hair is oriented poorly and is set at an angle $\beta$ to the horizontal, as indicated by the dashed-line positions $X'X'$. The difference between the two positions of $X'X'$ will then indicate that the supposedly vertical movement is the distance $d'$. From the sketch,

$$d' = \cos \beta(d + c \tan \beta) = d \cos \beta + c \sin \beta \qquad (a)$$

This equation may now be used to study the effect of poor orientation:

If $c = 0$      and $\beta = 1°$, $d' = 0.9999d$      ∴ 0.0% error
If $c = 0.5d$ and $\beta = 1°$, $d' = 1.0086d$      ∴ 0.9% error
If $c = 5d$    and $\beta = 1°$, $d' = 1.0871d$      ∴ 8.7% error

This comparison shows that, if the resultant movement of a point is essentially in the same direction as the component which is being measured, then an error in orientation does not have much effect on the measurement. If, however, the resultant movement of a point is such that the component being measured is small in comparison with the component normal to this, then a small error in orientation makes an appreciable error in the measurement.

## 18·4  Certain Indirect Methods.

Perhaps the simplest method utilizing Müller-Breslau's principle is the spline method of obtain-

ing influence lines for the reactions of continuous beams. The procedure consists simply in selecting a long flexible spline of steel, brass, or wood and laying it down on a board on which a piece of cross-section paper has been mounted. The spline may be held in place between two nails, driven into the board on each side of the spline at the support points. A vertical displacement may then be introduced at the reaction the influence line for which is desired. The elastic curve of the spline in such a case may be marked on the paper and the influence-line ordinates obtained by reading the deflection ordinates on the cross-section paper and dividing each of them by the introduced displacement.

Very good accuracy may be obtained by using a $\frac{1}{8}$-in.-square steel spline and introducing deflections of about one-sixth of the span length either way from its mean position. Introducing equal displacements both ways from the undeflected position of a model and measuring the movement of a point between these two deflected positions constitute a technique used in many methods of model analysis. Such a technique not only has the advantage of producing larger deflections without the possibility of overstressing the model but also in some cases minimizes errors in the measurements due to changing the geometry of the model.
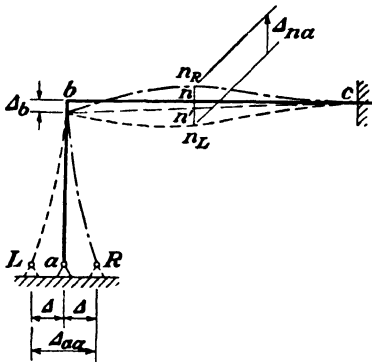


FIG. 18·4

It happens that this technique does not affect the geometrical error encountered in the case of the reaction influence lines of a continuous beam. This technique is effective, however, in the case shown in Fig. 18·4. In this case, the influence line for the horizontal reaction at point $a$ may be obtained by introducing a horizontal displacement either to the right or to the left at this point. In either case, deflecting the column into a curve causes a small drop of the top of the column $\delta_b$, which is greatly exaggerated in the sketch. As a result, a point $n$ would drop to the position of $n'$ owing simply to the rotation of chord $bc$ of the elastic curve of the girder. Thus, if the deflection of point $n$ were measured from $n$ to $n_L$, it would be too large by the amount $nn'$; or if it were measured from $n$ to $n_R$ when the introduced displacement was to the right at point $a$, the deflection of $n$ would be too small by the same amount $nn'$. Note, however, that if point $a$ were displaced from $L$ to $R$ and the resulting displacement of $n$ were measured from $n_L$ to $n_R$, the error $nn'$ due to the change in geometry would have been eliminated from this measurement. The displacement from $n_L$ to $n_R$ would give, therefore, the correct

value for the deflection $\Delta_{na}$ corresponding to an introduced displacement of $a$ of $\Delta_{aa}$. Introducing equal displacements both ways from the mean position therefore eliminates certain errors resulting from changing the geometry of the structure. This technique is not a "cure-all," however, and does not eliminate all errors due to changing the geometry of a structure.

The so-called brass-wire model method[1] is simply a more or less generalized version of the spline method, in which the models are fabricated out of brass wire. A rather large variety of two- and three-dimensional models can be built up in this manner. Even members with varying moments of inertia may be simulated by soldering end to end a number of small pieces of different-sized brass wires. Brass wires may normally be obtained in a wide range of gage sizes; and, of course, brass may easily be soldered to facilitate fabrication. Simple templates may be improvised to introduce the displacements, and the resulting deflections may be measured by using a magnifying glass in combination with cross-section paper or a steel scale, by a micrometer barrel, or by some other simple device.

The Gottschalk continostat[2] and the Eney deformeter[3] are two other well-known indirect methods. They are described in the references given below.

All the methods referred to above utilize rather simple models and simple means of introducing displacements and measuring the resulting deflections To obtain suitable accuracy, it is necessary to introduce somewhat large distortions so that the lack of precision in the measuring devices does not introduce too large an error in the results. As a result in certain cases, errors due to changes in geometry may become significant and may be impossible to minimize. In such cases, it may be necessary to use a method utilizing more refined instruments and techniques.

**18·5   The Beggs Method.**   The Beggs method[4] is the most general and usually the most satisfactory experimental method of those based on Müller-Breslau's principle. It was developed by the late Prof. George

[1] BULL, ANDERS, Brass Wire Models Used to Solve Indeterminate Structures, *Eng. News-Record,* vol. 99, No. 23, Dec. 8, 1927.

[2] GOTTSCHALK OTTO, Mechanical Calculation of Elastic Systems, *J. Franklin Inst.,* vol. 202, No 1, pp. 61–88, July, 1926.

[3] ENEY, W. J , New Deformeter Apparatus, *Eng. News-Record,* Feb. 16, 1939, p 221; Model Analysis of Continuous Girders, *Civil Eng.,* vol. 11, No. 9, p. 521, September, 1941.

[4] BEGGS, G E., An Accurate Mechanical Solution of Statically Indeterminate Structures by 1 se of Paper Models and Special Gages, *Proc. ACI,* vol. 18, pp. 58–82, 1922; Discussion of "Design of a Multiple-arch System," *Trans. ASCE,* vol. 88, pp. 1208–1230, 1925; The Use of Models in the Solution of Indeterminate Structures, *J. Franklin Ins* vol. 203, No. 3, pp. 375–386, March, 1927.

E. Beggs of Princeton University.   The equipment consists of a set of deformeter gages and plugs for introducing the deformations and a micrometer microscope for measuring the resulting deflections.   In order to obtain influence lines for reactions, the deformeter gage is used to replace the support of the model at that point, one half of the gage being attached to the model and the other half to the mounting surface. To obtain influence lines for internal moments, shears, and axial stresses, it is necessary to cut the model and connect one half of the deformeter gage to each side of the cut.   By inserting different types of plugs between the two halves of the gage, an axial, shear, or angular deformation may
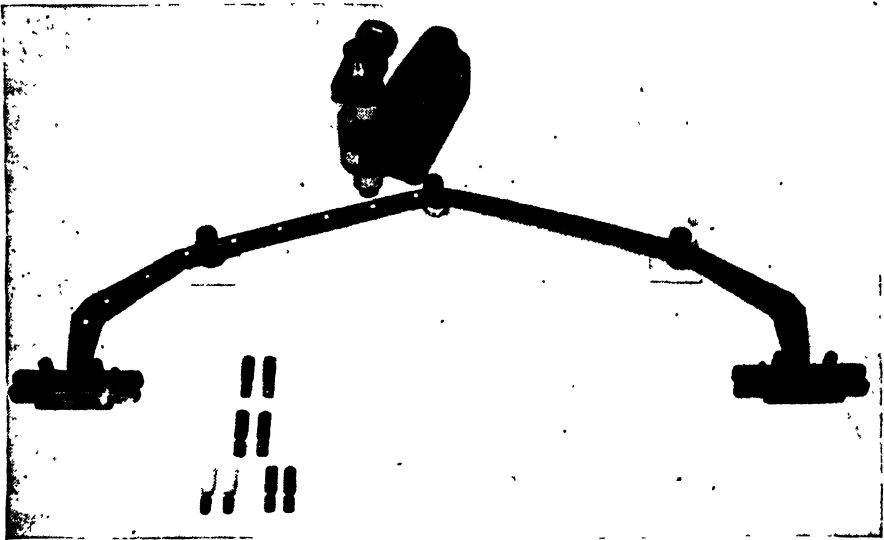


FIG. 18 5

be introduced into the model, thus distorting it into the shape of the corresponding influence line.   A typical application of the Beggs method is shown in Fig. 18·5.

This method has the advantage that a more reliable and accurate means is used for introducing the deformations and a more precise instrument is used for measuring the deflections that are produced. This enables one to introduce smaller distortions and thus reduce errors due to changing the geometry of the structure.   The models used with the Beggs method are ordinarily made from celluloid or a high-grade cardboard.   Since the deformeter gages introduce constant distortions rather than apply constant loads to the structure, no trouble is encountered from the creep characteristics of celluloid.   The method can be used without modification for any planar structure, regardless of

whether the members are straight or curved or of constant or varying moment of inertia.

The usual precautions must be observed in using the microscope in this method. Care must also be taken in mounting the deformeter gages to be sure that the long axis of the gage is normal to the axis of the member. If the gage is not so mounted, the influence data obtained for the shear and axial stress will be in error. In Fig. 18·6, if the gage is mounted in the direction indicated by the dashed line, the test data will give the values of the thrust $T_t$ and shear $S_t$ perpendicular and parallel to this direction. To obtain the true thrust and shear $T$ and $S$, it is necessary to measure the angle $\phi$ and convert the test data as indicated below:

$$T = T_t \cos \phi + S_t \sin \phi$$
$$S = S_t \cos \phi - T_t \sin \phi$$

It should be apparent that such an error in the orientation of the gage does not affect the data obtained for the moment at this point.

One should also be careful to attach the model to the gage so that the axis of the member lies at the center of the gage. If this is not done, the influence-line ordinates for the moment at this point will be in error; for the angular deformation introduced by the gage not only rotates the cross section of the model but also introduces an axial displacement of its centroid.
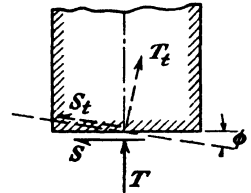


FIG. 18·6

The Beggs plugs may be calibrated in the following manner: Attach a strip to one half of the gage, and fasten the other half to the mounting surface. Insert the various plugs into the gage, and measure the resulting displacement of a target on the strip with the microscope. In this manner, the deformations introduced by the various plugs may be calibrated in micro-units and thence converted to inch units, if desired, using the calibration constant of the microscope.

**18·6   The Moment Deformeter.**   The moment deformeter[1] is an instrument which deforms a model so that it takes the shape of the influence line for the bending moment at the section located at the center of the instrument. This deformation is accomplished without cutting the model as is necessary in using the Beggs method.

The action of this instrument depends upon a relationship that exists between bending moment and deflection. Consider a segment $ab$ of a member, this segment being initially straight and having a constant $EI$. Suppose that a load $P$ is applied at point $o$ as shown in Fig. 18 7. The effect of the load $P$ is to distort

[1] NORRIS, *loc. cit.*; WILBUR and NORRIS, *loc. cit.*

the structure and produce bending moments throughout.   The bending moments thus produced at the ends of this segment are $M_a$ and $M_b$, which are assumed to act as shown.   The bending-moment diagram for this portion may then easily be drawn.   By means of the moment-area theorems, $\Delta_c$, the deflection of point $c$ on the elastic curve from the chord $ab$, may be computed and the resulting expression simplified to

$$\Delta_c = \frac{M_c L^2}{8EI} - \frac{Pd^3}{12EI} \qquad (18 \cdot 2)$$

Thus the moment at $c$ due to the load $P$ acting at any point $o$ in the segment $ab$ is found to be
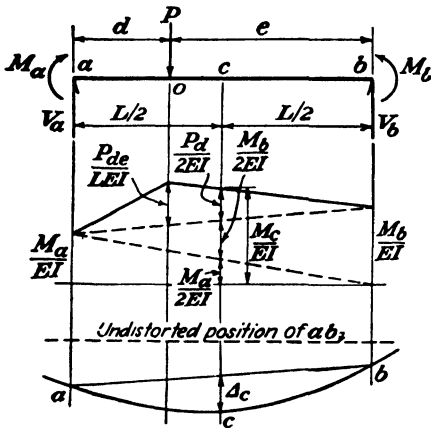


FIG. 18·7

$$M_c = \frac{8EI}{L^2}\Delta_c + \frac{2}{3}\frac{Pd^3}{L^2} \qquad (18 \cdot 3a)$$

where $d$ is the distance from point $o$ to either end $a$ or $b$, whichever is closer. If the load $P$ is applied at a point outside the segment $ab$, the second term of Eq. $(18\cdot 3a)$ vanishes and

$$M_c = \frac{8EI}{L^2}\Delta_c \qquad (18 \cdot 3b)$$

Thus it is apparent that the bending moment at $c$ could easily be computed if the deflection $\Delta_c$ could easily be measured on a model.

There is a convenient way of obtaining $\Delta_c$ that may be explained by the following considerations: First consider the structure to be acted upon by the force $P$ applied at point $o$.   This causes the structure to be distorted; the deflected shape of the segment $ab$ is shown in Fig. 18·8a.   Next consider the structure to be acted upon by the special force system shown in Fig. 18·8b.   The deflected shape of the segment $ab$ under such conditions is also shown in this sketch. Applying Betti's law to this situation, we know that the virtual work done by the external force system in sketch $a$ during the distortion produced by the external force system shown in sketch $b$ is equal to the virtual work done by the system in sketch $b$ during the distortion produced by the system in sketch $a$; therefore,

$$(P)(\Delta_o) = -\left(\frac{P}{2}\right)(\Delta_{ao}) + (P)(\Delta_{co}) - \left(\frac{P}{2}\right)(\Delta_{bo})$$

or

$$\Delta_o = \Delta_{co} - \frac{\Delta_{ao} + \Delta_{bo}}{2} \qquad (a)$$

From the geometry of Fig. 18·8a, it is apparent that

$$\Delta_c = \Delta_o - \frac{\Delta_{ao} + \Delta_{bo}}{2} \qquad (b)$$

and therefore, from Eqs. (a) and (b), it may be concluded that

$$\Delta_o = \Delta_c \tag{18·4}$$

Thus, if a system of loads such as that shown in Fig. 18·8b is applied to a segment ab, the model distorts so that it takes the shape of the influence line for $\Delta_c$.

In the development leading to Eqs. (18·3a) and (18·3b), $M_c$ denotes the bending moment at c caused by a load P acting on the model. If $m_c$ denotes the bending moment at c due to a unit value of P, then

$$m_c = \frac{M_c}{P} \tag{18·5}$$

and, from Eq. (18·3a), the following relation is apparent:

$$m_c = \frac{8EI}{L^2}\frac{\Delta_c}{P} + \frac{2}{3}\frac{d^3}{L^2} \tag{18·6}$$

If the values of $\Delta_c$ are obtained by using the loading condition shown in Fig.
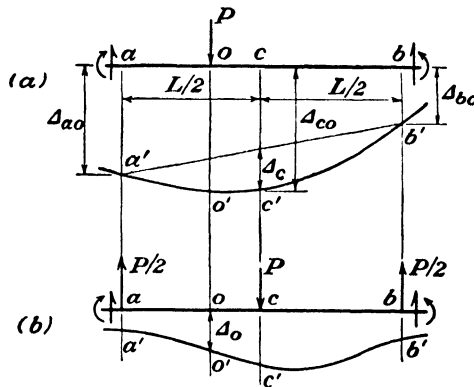


Fig. 18·8

18·8b, then, by Eq. (18·4),

$$m_c = \frac{8EI}{L^2}\frac{\Delta_o}{P} + \frac{2}{3}\frac{d^3}{L^2} \tag{18·7a}$$

As before, when $\Delta_o$ is measured at a point o outside the segment ab, the second term in Eq. (18·7a) vanishes and

$$m_c = \frac{8EI}{L^2}\frac{\Delta_o}{P} \tag{18·7b}$$

Thus, for the portion of the model lying outside the segment ab, if the loading of Fig. 18·8b is applied, the deformed model takes the shape not only of the influence line for $\Delta_c$ but also of the influence line for $m_c$. To obtain the actual ordinates for $m_c$ at points o outside of the segment ab, values of $\Delta_o$ must be multiplied by $8EI/L^2P$, where E, I, L, and P refer to the segment ab. To obtain the actual

ordinates for $m_c$ at points $o$ within the segment $ab$, values of $\Delta_o$ are substituted into Eq. (18·7a).

To determine the values of the influence-line ordinates, one must know, in addition to $\Delta_o$, the values of $E$, $I$, $L$, $d$, and $P$. With the exception of $P$ and $E$, all these are easily determined. When a celluloid model is used, it is necessary to apply a fixed distortion corresponding to the type of loading shown in Fig. 18·8b instead of constant loads $P$ and $P/2$. A moment deformeter that embodies the principles shown diagrammatically in Fig. 18·9 may be shown to produce a distorted shape of both the model and the balance beam that does not change with time and thus effectively eliminates creep. If a plug that has a diameter larger by an amount $\Delta$ than the undistorted distance between the model and balance
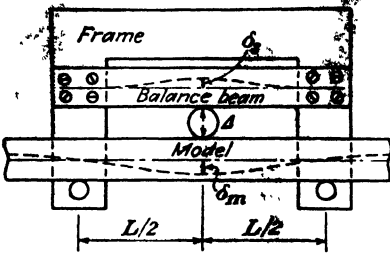


beam is forced into place between the two pieces, a distorted shape of both balance and model will be produced that will not change with time. This assumes that both model and balance have been made out of the same piece of celluloid and therefore have the same creep characteristics. It also assumes that the frame of the instrument is made out of metal and is heavy enough to be assumed rigid as compared with the flexible celluloid members. The pressure $P_l$ applied by this plug to the model and balance will, of course, change with time, proportionally to the change in effective modulus $E_l$, but the elastic curve of both members will not change.

FIG. 18·9

If, in addition to measuring the deflections $\Delta_o$ at the various points $o$ on the model, the deflection $\delta_s$ of the balance is also measured, then, upon recalling the discussion in Art. 17·5, the following relationships are apparent:

$$\delta_s = K_s \frac{P_l}{E_l} \qquad \text{or} \qquad \frac{E_l}{P_l} = \frac{K_s}{\delta_s} \tag{18·8}$$

where $K_s$ is a geometrical constant of the balance beam, which depends on its dimensions and the manner in which it is attached to the instrument frame. Thus, if $K_s$ is computed and $\delta_s$ is measured, the ratio of $E_l$ to $P_l$ will be known not only for the balance beam but also for the model, provided that both balance beam and model are made from the same sheet of celluloid.

Upon substituting from Eq. (18·8), Eqs. (18·7a) and (18·7b) may be written in the following form for use on a celluloid model where $\delta_s$ has been measured on the balance beam of the instrument. The influence-line ordinates for the bending moment at $c$ are obtained from Eq. (18·9a) for points $o$ within the segment $ab$,

$$m_c = \frac{8IK_s}{L^2} \frac{\Delta_o}{\delta_s} + \frac{2}{3} \frac{d^2}{L^2} \tag{18·9a}$$

and from Eq. (18·9b) for points o outside the segment ab,

$$m_c = \frac{8IK_s}{L^2}\frac{\Delta_o}{\delta_s} \tag{18·9b}$$

The value of $K_s$ may be computed, but the accuracy of the result **may not be good** because of the uncertain fixity conditions at the points of attachment of the balance beam.   As a result, it is more satisfactory **to** obtain $K_s$ by a calibration test run on a statically determinate cantilever. beam, where the bending moments are known by statics.
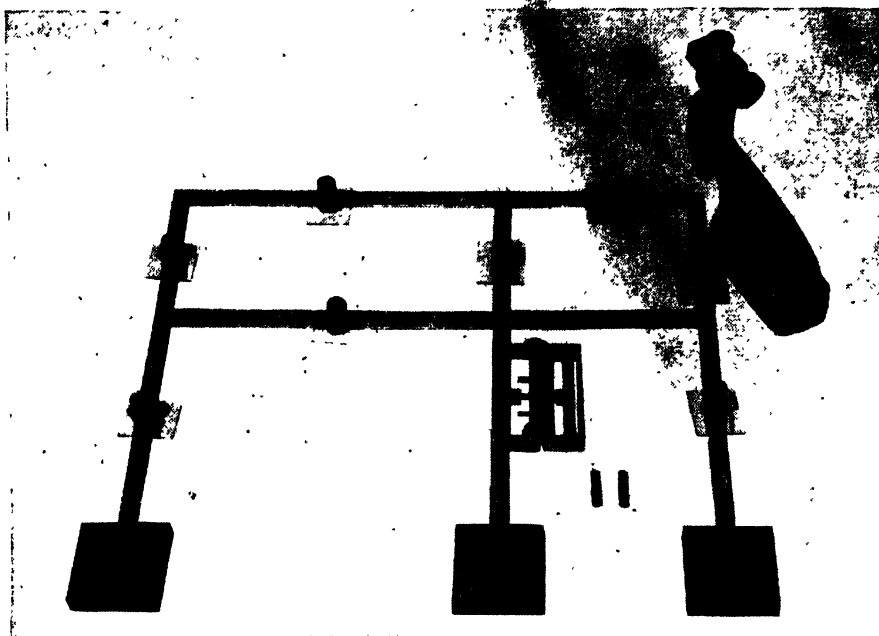


Fig. 18·10

The latest model of the moment deformeter is shown mounted on a celluloid model in Fig. 18·10.   It will be noted that the principle of this instrument is essentially the same as that shown in simplified form in Fig. 18·9.   The deflections $\delta_s$ and $\Delta_o$ are measured with the same type of micrometer microscope as that used in the Beggs method.

This instrument provides an effective means of obtaining influence lines for bending moments at internal sections of a model without the necessity of cutting the model.   The method does have the disadvantage that, in order to use the above simple interpretation of the measurements, it is necessary to apply the instrument to a segment of the model which is initially straight and has a constant $I$.

**18·7   The Moment Indicator.**   The moment indicator[1] is a convenient instrument that furnishes a direct method of obtaining bending moments in a model.   The theory of this instrument is based upon the Manderla-Winkler equations, which are used primarily in the analysis of secondary stresses in trusses.   These equations are applicable to a member or a portion of a member which is initially straight and has constant $E$ and $I$ and to which no external loads are applied between the ends of the portion under consideration.

The Manderla-Winkler equations are simply expressions for the moments acting on the ends of a member in terms of the slopes of the elastic curve of these two ends of the member.   Let $M_{AB}$ be the moment acting on the $A$ end of member $AB$, and $M_{BA}$ be the moment on the $B$ end, the end moments being positive when clockwise on the end of the member.   Also, let $\tau_A$ and $\tau_B$ be the slopes of the tangents to the elastic curve at points $A$ and $B$, respectively.   Such slopes are measured with reference to the chord $AB$ of the elastic curve and are positive when the tangent rotates clockwise with reference to the chord.   The Manderla-Winkler equations may be derived by means of the moment-area theorems and may be stated as follows:

$$M_{AB} = \frac{2EI}{L}\,(2\tau_A + \tau_B)$$
$$M_{BA} = \frac{2EI}{L}\,(\tau_A + 2\tau_B)$$

$$(18 \cdot 10)$$

Keeping these relationships in mind, we may now proceed to the development of the theory of the moment indicator.

Referring to Fig. 18·11, suppose that two arms are attached at points $A$ and $B$ of a member.   As the model is loaded and distorted by loads applied outside of this segment, the two arms will rotate through the same angles as the tangents to the elastic curve at their point of attachment   Upon recognizing that the rotations are actually through small angles, the following relative movements of the targets $a$ and $a'$ and $b$ and $b'$ may easily be computed.   Let $\Delta_a$ and $\Delta_b$ represent the relative movements of the targets, being positive when the targets move apart; then

$$\Delta_a = \frac{2L}{3}\,\tau_A + \frac{L}{3}\,\tau_B = \frac{L}{3}\,(2\tau_A + \tau_B)$$
$$\Delta_b = \frac{L}{3}\,\tau_A + \frac{2L}{3}\,\tau_B = \frac{L}{3}\,(\tau_A + 2\tau_B)$$

$$(18 \cdot 11)$$

[1] RUGE, A. C., and E. O. SCHMIDT, Mechanical Structural Analysis by the Moment Indicator, *Proc. ASCE*, October, 1938.

Upon substituting from Eq. (18·11) in Eq. (18·10), the following expres-
sions are obtained for the end moments at points $A$ and $B$:

$$M_{AB} = \frac{6EI}{L^2}\, \Delta_a \Bigg\}$$
$$M_{BA} = \frac{6EI}{L^2}\, \Delta_b \Bigg\} \qquad\qquad (18\cdot12)$$

in which a positive $\Delta$ (or a relative movement apart) indicates a positive
end moment (or clockwise on the end of the segment).   Hence if $\Delta_a$ and
$\Delta_b$ are measured with a microscope and if $E$ is known, the moments can
be determined at the points of attachment of the moment indicator.

Here again, if celluloid is used as the model material, the creep prob-
lem must be overcome.   Of course, constant loads cannot be applied to
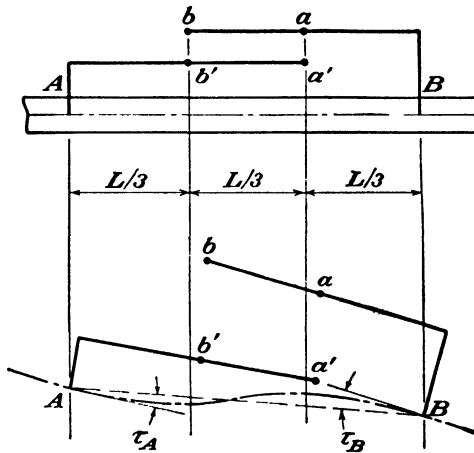


Fig. 18·11

a celluloid model, for its deflection will change with time.   If, however, a
fixed deformation is applied to the model at the point and in the direction
of the specified loading, the deflected position of the model will not change
with time.   The movements of the targets of the moment indicator may
then be measured without difficulty with a microscope.   However, it
would still be impossible to compute the end moments from Eq. (18·12),
for $E$ is unknown and changing with time.   Further, the corresponding
load on the model is also unknown and is changing with time in direct
proportion to the change in $E$.   In such a case, however, it is possible to
obtain the relative value of the moments, for they are proportional to
the product of $I$ and the moment-indicator deflections.

Quite often these relative values of the end moments may be con-
verted into absolute values in terms of the load, through the use of

conditions of static equilibrium. Consider as an example the simple frame shown in Fig. 18·12. Suppose that the moment indicator has been mounted in turn on each column and thus relative values of the moments at two points on each column have been obtained for some fixed horizontal displacement imposed at the top of column 1-2. In this case, the relative value of the end moments in each column may be extrapolated from the relative values measured at the points of attachment of the indicator. Let these relative values at the ends of the columns be $m_{12}$, $m_{21}$, etc.; then the absolute values may be expressed as

$$M_{12} = Cm_{12} \qquad M_{34} = Cm_{34}$$
$$M_{21} = Cm_{21} \qquad M_{43} = Cm_{43}$$

If each column is isolated as a free body, this enables us to compute the shear in each column in terms of the unknown constant $C$. The girder



Fig. 18·12

may likewise be isolated as a free body by cutting it free from the columns just below the column tops. Since the shears on the stub ends of the columns have been expressed in terms of $C$, the statics equation $\Sigma F_x = 0$ applied to the isolated girder now enables us to find $C$ in terms of $P$. Having $C$ in terms of $P$, we may obtain without difficulty the absolute values of the end moments in terms of $P$.
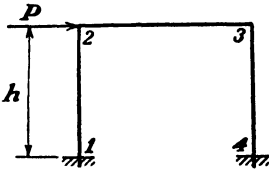
The above procedure of converting relative values into absolute values of moments can often be done very easily. There are cases, however, when either there are no suitable equations of statics available or, if such equations are available, to solve them is too laborious. In such cases, the use of a celluloid spring balance in the loading system leads to a direct solution for the absolute values of the moments in the model.

The use of a spring balance to overcome creep is discussed in Art. 17·5. In Fig. 17·3, the balance is shown connected to the same type of model as shown in Fig. 18·12. Suppose that a moment indicator were mounted on the model in Fig. 17·3 and that readings were taken on the targets of both the indicator and the spring balance. Referring to Eq. (h) in Art. 17·5, we recall that, having measured $\Delta_s$, $E$ may then be expressed as

$$E = \frac{K_s}{\Delta_s} P \qquad\qquad (18 \cdot 13)$$

Substituting from Eq. (18·13) in Eq. (18·12),

$$M_{AB} = \frac{6IK_s}{L^2} \frac{\Delta_a}{\Delta_s} P \qquad\qquad (18 \cdot 14)$$

By using the celluloid spring balance in this manner, therefore, it is easy to obtain the absolute values of the moments in terms of the load $P$.

This presumes, of course, that the constant $K_s$ of the balance is known. The value of this constant can be computed, but it can be obtained more accurately from a calibration test on a statically determinate canti- lever beam such as that shown in Fig. 18·13. Upon attaching the indicator as shown, the deflections $\Delta_a$ and $\Delta_s$ may be measured with the microscope. Knowing that the moment $M_{AB}$ is equal to $P$ times $d$, we can then easily back-figure $K_s$ by applying Eq. (18·14).
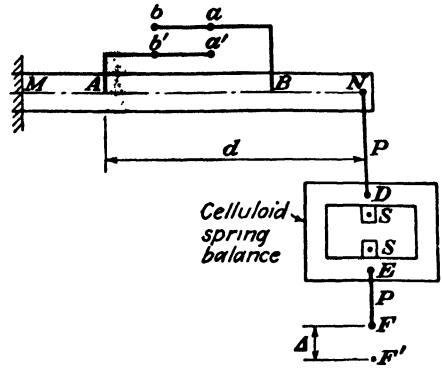
The moment indicator provides a very satisfactory model solution for many problems. A typical



FIG. 18·13

setup of the moment indicator in combination with a celluloid spring balance is shown in Fig. 18·14. It should be recalled that the theory of the moment indicator is based on the Manderla-Winkler equations. In order to use the above simple interpretation of the measurement, it is
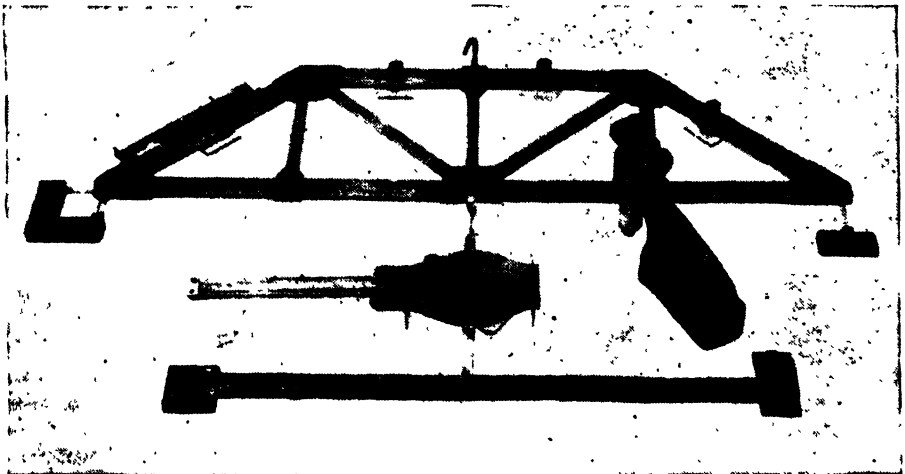


FIG. 18·14

necessary, therefore, to apply the indicator to a portion of a member that is initially straight and has a constant $I$. It is possible to interpret the indicator readings in cases where $I$ varies within the portion, but the computations would be very cumbersome.

'18·8  **Principles of Similitude.** The principles of similitude governing the relationships between a model and its prototype may be determined by either of two approaches. The conditions of similarity may be expressed in mathematical form by means of established laws of structural mechanics, and the principles of similitude rigorously deduced from them; or the principles may be deduced by using the methods of dimensional analysis. The first method is generally employed for structural models since the mathematical laws that structures follow are usually well known. However, in cases where the mathematical laws are not known but the factors affecting the phenomena are known, the principles of similitude may be determined by the ·dimensional-analysis approach.
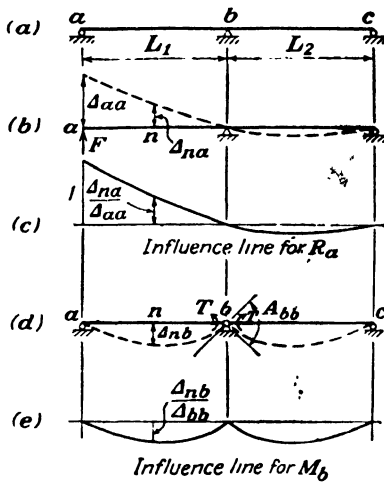
FIG. 18·15

18·9  **Derivation of Principle of Similitude Using Established Laws of Structural Mechanics.** To illustrate the derivation of the principle of similitude by this approach, consider the problem of extrapolating influence-line data from a model to its prototype. For this purpose, consider a beam which is statically indeterminate to the first degree, such as that shown in Fig. 18·15.

Suppose that we wished to obtain the influence lines for $R_a$, the vertical reaction at $a$, and for $M_b$, the moment at $b$. In either case, we may apply Müller-Breslau's principle and thereby obtain the desired influence lines as indicated in Fig. 18·15. Of course, this procedure may be applied either to the prototype or to its model. Upon applying this procedure to the prototype, the expressions for the ordinates of these two influence lines are

$$R_a^P = -\frac{\Delta_{na}^P}{\Delta_{aa}^P}\,(1) \qquad\qquad (18·15)$$

and

$$M_b^P = -\frac{\Delta_{nb}^P}{\Delta_{bb}^P}\,(1) \qquad\qquad (18·16)$$

where the index $P$ indicates that these quantities refer to the prototype.

On the other hand, upon applying Müller-Breslau's procedure to the model, the ordinates for the two influence lines for the model are

$$R_a^M = -\frac{\Delta_{na}^M}{\Delta_{aa}^M}\,(1) \qquad\qquad (18\cdot17)$$

and

$$M_b^M = -\frac{\Delta_{nb}^M}{\Delta_{bb}^M}\,(1) \qquad\qquad (18\cdot18)$$

where the index $M$ refers to the model. Of course, if the principles of similarity between model and prototype are known, the ordinates of these influence lines may be measured experimentally on a small-scale model and then these results extrapolated to the prototype. In other words, to do this we need to know the relation between $R_a^M$ and $R_a^P$ and between $M_b^M$ and $M_b^P$.

From Eqs. (18·15) to (18·18), it is apparent that these relationships between the reactions or moments on the model and those on the prototype depend upon the relationship between deflections. The relationship between deflections of the model and the prototype may be investigated by considering the computations of such deflections by the method of virtual work. To compute beam deflections, the law of virtual work may be expressed as

$$\sum Q\delta = \int M_P M_Q \frac{dx}{EI} \qquad\qquad (18\cdot19)$$

If a force $F^P$ acts at point $a$ of the primary structure of the prototype, as shown in Fig. 18·15$b$, the vertical deflection of point $n$ may be computed from

$$(1)\,(\Delta_{na}^P) = \int M_P^P M_Q^P \frac{dx^P}{E^P I^P} \qquad\qquad (18\cdot20)$$

In the same manner, Eq. (18·19) may be applied to compute a similar deflection on the model, due to a force $F^M$ at $a$, or

$$(1)\,(\Delta_{na}^M) = \int M_P^M M_Q^M \frac{dx^M}{E^M I^M} \qquad\qquad (18\cdot21)$$

Suppose that the model has been constructed so that the following relations exist between the model and prototype:

$$L^M = kL^P, \qquad I^M = \alpha I^P, \qquad E^M = \beta E^P, \qquad F^M = \gamma F^P \quad (18\cdot22)$$

In view of these relations, it is apparent that the following additional relations are true for the statically determinate primary structure shown in Fig. 18·15$b$:

$$M_P^M = k\gamma M_P^P, \qquad M_Q^M = kM_Q^P \qquad\qquad (18\cdot23)$$

Substituting in Eq. (18·21) from Eqs. (18·22) and (18·23) and comparing with Eq. (18·20),

$$(1)\ (\Delta_{na}^M) = \frac{k^3\gamma}{\alpha\beta} \int M_F^P M_Q^P \frac{dx^P}{E^P I^P} = \frac{k^3\gamma}{\alpha\beta} (1)(\Delta_{na}^P)$$

Therefore,

$$\Delta_{na}^P = \frac{\alpha\beta}{k^3\gamma} \Delta_{na}^M \tag{18·24}$$

These relationships are true for any of the vertical deflections in Fig. 18·15b. Substituting therefore in Eq. (18·15) from Eq. (18·24) and comparing with Eq. (18·17),

$$R_a^P = R_a^M \tag{18·25}$$

and therefore the corresponding influence-line ordinates for $R_a$ are exactly the same for both model and prototype.

In a similar manner, the vertical deflection $\Delta_{nb}$ and the relative angular rotation $\Delta_{bb}$ of the primary structure shown in Fig. 18·15d may be investigated for model and prototype. In this case, of course, the deflections are caused by couples $T^P$ on the prototype and couples $T^M$ on the model. Proceeding as before with the exception that we shall let

$$T^M = \gamma T^P$$

we shall find that

$$\Delta_{nb}^P = \frac{\alpha\beta}{k^2\gamma} \Delta_{nb}^M \tag{18·26}$$

$$\Delta_{bb}^P = \frac{\alpha\beta}{k\gamma} \Delta_{bb}^M \tag{18·27}$$

Substituting in Eq. (18·16) from Eqs. (18·26) and (18·27) and comparing with Eq. (18·18),

$$M_b^P = \frac{1}{k} M_b^M \tag{18·28}$$

and therefore the influence-line ordinates for $M_b$ on the model should be multiplied by $1/k$ to obtain the corresponding ordinates on the prototype.

Of course, this discussion is limited to a beam that is indeterminate to the first degree, but it could now be extended successively to include beams or frames which were indeterminate to the second, third, or any degree. Such considerations would lead to the following general conclusion for any indeterminate beam or frame, the stress analysis of which can be carried out satisfactorily by considering only the effect of bending distortion:

*A model should be dimensioned so that the axial lengths of its members are*
*k times those of the prototype; the moments of inertia of its cross sections are*
*α times those of the prototype; and the modulus of elasticity of the model is β*
*times that of the prototype.    If this is done, then the ordinates of the influ-*
*ence line for any reactive force, shear, or axial stress on the prototype are*
*equal to the corresponding ordinates on the model; but the ordinates of the*
*influence line for any moment on the prototype are equal to 1/k times the*
*corresponding ordinates on the model.*

In this manner, the principles of similitude may be developed for
models of trusses and other types of structures, provided that the solution
of the problem can be formulated mathematically.

# CHAPTER 19

## OTHER USES OF STRUCTURAL MODELS

**19·1 General.** Model analysis has been extremely helpful in solving stress-distribution problems, *i.e.*, in determining the manner in which the stress intensities vary across the cross sections of a member. Several experimental methods are available for solving such problems; among these, the two most important at the present time are the photoelastic method and the approach based on first measuring surface strains.

The photoelastic method has been used extensively in recent years, particularly by the mechanical engineer, to study the complex stress distributions that he often encounters in the design of machine elements. The civil engineer to a less, but still important, extent has used this method in analyzing similar problems for structural elements.

The art of measuring surface strains has developed tremendously since 1938. Wartime research produced astounding techniques for measuring such strains under both static and dynamic loading conditions. Numerous types of mechanical and electrical strain gages have been developed. Today that most generally used is the so-called *SR-4* wire-resistance strain gage.[1]

Model analysis is used most commonly to study either stress analysis or stress-distribution problems. In addition, however, models are also frequently used to determine buckling loads or vibrational characteristics.

**19·2 Photoelastic Method—General.** Relatively simple stress-distribution problems can be handled successfully by mathematical methods. There are innumerable practical problems, however, where mathematical methods are inadequate and recourse to experimental procedures is necessary. The photoelastic method[2] is one of the most useful of these experimental procedures and is widely used to solve for the stress distribution in such locations as those around holes and notches,

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[1] Produced and distributed by the Baldwin-Southwark Corporation.

[2] The following are excellent references on the photoelastic method: COKER, E. G., and L. N. G. FILON, "A Treatise on Photoelasticity," Cambridge University Press, London, 1931 (this is the classic reference book on photoelasticity); FROCHT, M. M., "Photoelasticity," John Wiley & Sons, Inc., New York, 1941 (this book is an excellent textbook on photoelasticity and contains a number of valuable illustrative examples and photographs); MINDLIN, R. D., Review of the Photoelastic Method of Stress Analysis, *J. Applied Phys.*, vol. 10, Nos. 4 and 5, April, May, 1939 (these two articles give an excellent summary of the photoelastic method and contain a very good bibliography on the subject).

in riveted and welded joints, or in other irregularly shaped structural elements.

The photoelastic method is based upon the fact that, when certain transparent materials are stressed, their optical properties are changed. It is possible to measure these changes and relate them to the state of stress in the material. Thus, by making a model from a suitable transparent material and loading it in the same manner as the prototype, certain optical effects may be measured and interpreted to give the stress distribution in the model.

In a plane (two dimensional) stress problem, the complete condition of stress at any point may be defined in terms of the magnitudes and directions of the two principal stress intensities. Since the two principal stresses[1] are perpendicular to each other, the magnitudes of the two principal stress intensities and the direction of one of them are sufficient to define the condition of stress. Three independent quantities must be determined, therefore, at each and every point of a body subjected to a condition of plane stress, in order to define the stress distribution in the body completely.

When the photoelastic method is used for such problems, it is usually most convenient to determine three slightly different independent quantities at the various points, viz., the algebraic difference between the principal stress intensities, their algebraic sum, and the direction of one of them. From these data, it is easy to compute the normal and shear stress intensities on any plane through a point.

The photoelastic method is likewise applicable to three-dimensional stress systems. In such cases, however, the experimental techniques and the interpretation of the results are quite complex. This discussion will therefore be limited to the two-dimensional application of the method.

**19·3 Review of Certain Principles of Optics.** Before the fundamental theory of two-dimensional photoelasticity can be developed it is necessary to review briefly certain ideas of physical optics. The photoelastic phenomena may be explained satisfactorily by means of the ether-wave theory of light. In this theory, light is assumed to result from the transverse vibration, in a plane containing the ray, of all the particles lying along the ray. It is further assumed that at any instant all these particles lie along a sine curve, the *amplitude* of which is 2a and the wave length of which is λ. Thus, a particle m is assumed to oscillate between the positions A and B as shown in Fig. 19·1 and some other particle n between positions C and D. Suppose that particles m and n lie as shown on the solid sine curve at time $t = 0$. Suppose further that at some other time $t$, the particles have moved to the position $m_1$ and $n_1$ and now lie on the dashed-line curve. It is clear that the effect of the transverse movement of the particles

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[1] In this chapter, for the sake of brevity, the term *stress* instead of *stress intensity* is used where the meaning is obvious.

has been to cause the sine curve connecting them to move to the right along the ray a distance $\Delta z$.

The rate of the apparent movement of the sine curve along the ray is called the *velocity of light* and is constant for a given medium through which the ray is passing. For one complete wave of the sine curve to pass a given point, it is necessary for the various particles to undergo one complete cycle of transverse vibration. The frequency of the waves passing a given point is therefore the same as the frequency of the transverse vibration of the individual particles. The *frequency* of the vibration of the particles, $f$, determines the *color of light* and is constant for any given color regardless of the medium through which the ray may be passing. White light is composed of waves of all frequencies.
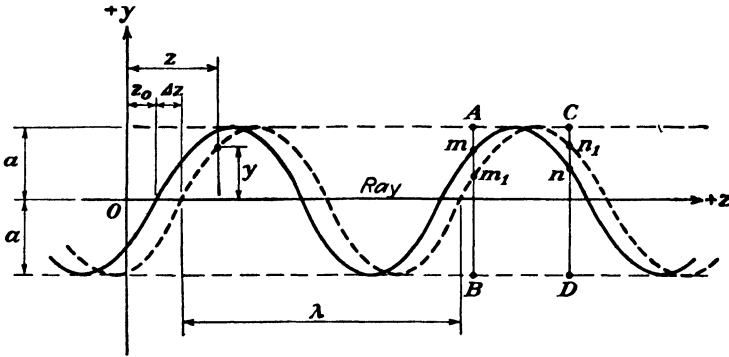


FIG. 19·1

The velocity of light is equal to the product of the length of one complete wave and the frequency with which the waves pass a given point, or

$$v = \lambda f \qquad (19 \cdot 1)$$

Since the frequency for any given color of light remains constant, the wave length of that color varies directly with the velocity, which in turn depends upon the medium through which the light is passing.

The *intensity of light* is proportional to the amplitude of the transverse vibration of the particles. If there is no transverse vibration, *i.e.*, if the amplitude is zero, the intensity of light is zero—that is, there is no light.

In subsequent discussions, it will be necessary to use an expression for the instantaneous transverse position of a particle vibrating at a section located a distance $z$ from the origin. In Fig. 19·1, the transverse displacement $y$ of this particle at any time $t$ may be evaluated as follows:

$$y = a \sin \frac{2\pi}{\lambda} [ z - (z_o + \Delta z)] \qquad (a)$$

However, $\Delta z$ is the distance the sine curve has moved since $t = 0$ and therefore is equal to

$$\Delta z = vt = \lambda f t \qquad (b)$$

Substituting in Eq. (a) from Eq. (b),

$$y = a \sin \left[ \frac{2\pi}{\lambda} (z - z_o) - 2\pi f t \right] \qquad (19\cdot2)$$

From this equation, it is apparent that the transverse vibration of such particles is a simple harmonic motion.

*Polarized light* is another term that should be defined. A ray of ordinary light consists of waves vibrating in all the planes that can be passed through the ray. If, however, the vibrations are controlled in some manner, the light is said to be *polarized*. If the vibrations are constrained to one plane only, the light is said to be *plane-polarized*. Such light need not be monochromatic and may be produced by passing ordinary light through a Nicol prism, a Polaroid sheet, or some other device.

If, on the other hand, the light is the resultant of two waves of equal amplitude and wave length but vibrating in perpendicular planes and one-quarter of a wave length out of phase, it is said to be *circularly polarized*. Such light may be produced by passing a ray of plane-polarized monochromatic light through a *quarter-wave plate*. The quarter-wave plate is a sheet of mica or cellophane or some other permanently doubly refractive material, the axes of which are set at 45° to the plane of polarization of the incident ray of plane-polarized light. The thickness of the quarter-wave plate is such that one of the two component waves is slowed up a quarter of a wave length with respect to the other. The two component waves emerge on the far side of the quarter-wave plate one-quarter of a wave length out of phase, equal in amplitude, and vibrating in perpendicular planes and therefore combine to produce circularly polarized light.

*Double refraction* is a property possessed permanently by certain crystalline substances such as mica. Such materials may be considered to break an incident ray of plane-polarized light down into two rectangular components. These two components travel through the material at different velocities and therefore may be out of phase when they emerge on the far side. Other materials such as bakelite and celluloid become temporarily doubly refractive when stressed, breaking an incident ray down into two components that travel through the material in the planes of the two principal stresses. This property is the basis of the photoelastic method of stress determination.

Two light waves having the same intensity and wave length and vibrating in the same plane may, of course, be superimposed. The phenomenon associated with such superposition is called *interference*. If these two waves are in phase, they will reinforce each other and produce a combined wave of greater intensity. If, however, the two waves are one-half a wave length out of phase, they will cancel each other and total darkness will result. As the phase difference is varied from zero to $\lambda/2$ to $\lambda$, etc., the intensity of the combined effect varies through various stages of interference from brilliance to darkness, to brilliance, etc. For white light, the phase difference that eliminates a color of one wave length will not completely eliminate colors of other wave lengths. The combined light will therefore have a color that depends upon the color eliminated.

 ·· 19·4  Photoelastic Method—Fundamental Theory.  The prin-
cipal apparatus used in applying photoelasticity to two-dimensional
problems is called a *polariscope*.  The simplest type of polariscope is
shown diagrammatically in Fig. 19·2.  The essential parts of this polari-
scope are the light source and two devices for producing plane-polarized
light, one called the *polarizer* and the other the *analyzer*.

The light coming from the light source is vibrating in various planes,
but after passing through the polarizer it is converted into plane-polar-
ized light vibrating, in this case, in a vertical direction.  This plane-
polarized light then passes through the loaded bakelite model, which, as
a result of its stressed condition, has temporarily become doubly refrac-
tive.  The plane-polarized light is therefore broken down into two
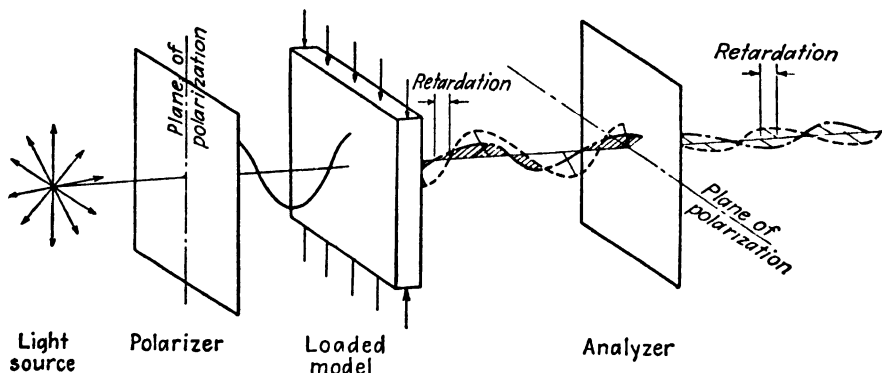


Fɪɢ. 19·2

rectangular components vibrating in the planes of the principal stresses
and traveling through the model at different velocities.  Since one of
these components has been retarded with respect to the other, they may
be partly or completely out of phase when they emerge on the far side
of the model.  The amount by which they are out of phase may be
measured by putting them into interference with one another.  This can
be done by passing them through the analyzer, which permits only the
horizontal components of the incident waves to pass through, thus putting
them into interference with one another in a plane normal to the plane
of polarization of the polarizer.  If these horizontal components are in
phase with each other, there will be maximum brilliance; but if they are a
half wave length out of phase, there will be complete interference, or
darkness.  The resulting pattern that one sees coming through the
analyzer may be related quantitatively to the state of stress in the model.

Consider the differential particle of the model shown in Fig. 19·3a.  Suppose
that this particle has been isolated so that its sides are planes of principal stress.

It is so small that the principal stress intensities $\sigma_x$ and $\sigma_y$ may be assumed to be uniformly distributed over its sides. Suppose that a ray of *plane-polarized monochromatic light* of amplitude 2a and vibrating in a plane *OP* strikes the particle. Because of the doubly refractive properties of the stressed bakelite in the planes of principal stress, this ray will be broken down into two rectangular components, one of which goes through the particle vibrating in the plane *YOZ*, and the other in the plane *XOZ*. The amplitudes of these two components will be $2a \cos \alpha$ and $2a \sin \alpha$, respectively.

Consider now the component vibrating in the plane *YOZ* and shown in Fig. 19·3b. Suppose first that this component is traveling through air; in this case, the instantaneous position of the particles will lie on the solid-line sine curve. However, when this component passes through the stressed bakelite, it is retarded
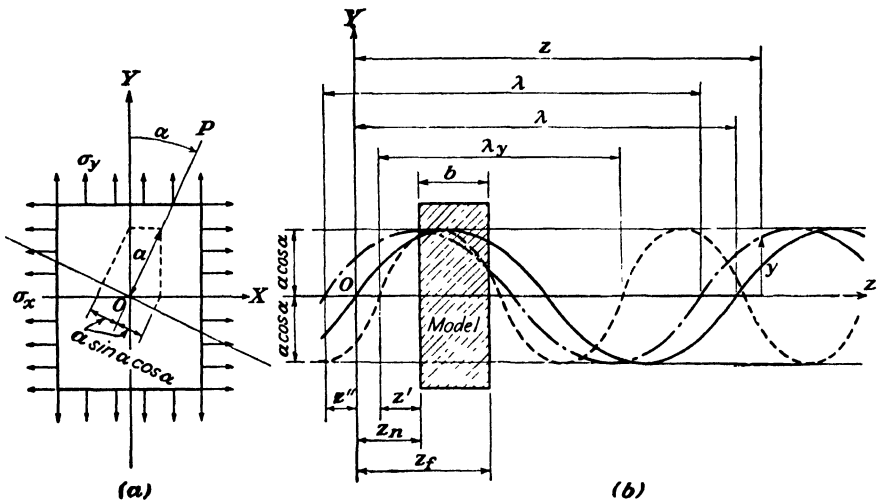


Fig. 19·3

and hence its wave length is shortened from $\lambda$ to $\lambda_y$. If all the medium were bakelite, the instantaneous position of the particles would lie on the dashed-line sine curve. In the actual case, though, the instantaneous positions of the particles are located on the solid-line sine curve up to the model, on the dashed-line sine curve through the model, and on the dot-dash sine curve to the right of the model, where the wave returns to air as a medium and resumes a wave length of $\lambda$.

In order to evaluate the instantaneous displacement $y$ of a particle located a distance $z$ from the origin and on the dot-dash sine curve, it is first necessary to compute the distances $z'$ and $z''$. First,

$$\frac{z'}{\lambda_y} = \frac{z_n}{\lambda} \qquad \text{or} \qquad z' = \frac{\lambda_y}{\lambda} z_n \tag{a}$$

Also,

$$\frac{z'' + z_f}{\lambda} = \frac{z' + b}{\lambda_y} \qquad \text{or} \qquad z'' = \frac{\lambda}{\lambda_y}(z' + b) - z_f \tag{b}$$

Substituting in Eq. (b) from Eq. (a) and rewriting, using Eq. (19·1),

$$z'' = \frac{b}{\lambda_y}\lambda - b = \frac{b}{v_y}f\lambda - b = f\lambda t_y - b \qquad (c)$$

where $v_y$ is the velocity of the light through the bakelite in the YOZ plane and $t_y$ is the time it takes it to pass through the model. However, using Eq. (19·2),

$$y = a \cos \alpha \sin \left[ \frac{2\pi}{\lambda}(z + z'') - 2\pi ft \right] \qquad (d)$$

Substituting from Eq. (c) and simplifying,

$$y = a \cos \alpha \sin \left[ \frac{2\pi}{\lambda}(z - b) - 2\pi f(t - t_y) \right] \qquad (19 \cdot 3)$$

In a similar manner, the instantaneous displacement $x$ of the wave vibrating in the XOZ plane may be evaluated and found to be

$$x = a \sin \alpha \sin \left[ \frac{2\pi}{\lambda}(z - b) - 2\pi f(t - t_x) \right] \qquad (19 \cdot 4)$$

When these two waves are passed through the analyzer and put into interference with one another, the resultant displacement $s$ obtained from the superposition of these two components is

$$s = y \sin \alpha - x \cos \alpha \qquad (19 \ 5)$$

Substituting in Eq. (19·5) from Eqs. (19·3) and (19 4) and simplifying,

$$s = a \sin 2\alpha \sin [\pi f(t_y - t_x)] \cos \left[ \frac{2\pi}{\lambda}(z - b) - 2\pi f\left(t - \frac{t_y + t_x}{2}\right) \right] \qquad (19 \cdot 6)$$

It is apparent from Eq. (19·6) that the resulting displacement is a harmonic vibration, the amplitude of which is

$$\text{Amplitude} = 2a \sin 2\alpha \sin [\pi f(t_y - t_x)] \qquad (19 \cdot 7)$$

This expression may be put in slightly different form, since

$$t_y - t_x = \frac{b}{v_y} - \frac{b}{v_x} = b\frac{v_x - v_y}{v_x v_y} \approx \frac{b}{v_b^2}(v_x - v_y) \qquad (e)$$

The latter simplification is permissible since $v_x$ and $v_y$ are essentially the same as $v_b$ (the velocity of light through unstressed bakelite) and therefore the denominator may be replaced by $v_b^2$. Experiments show that $v_x - v_y$ is proportional to the difference in the principal stresses, and hence Eq. (e) may be written thus:

$$t_y - t_x = \frac{kb}{v_b^2}(\sigma_y - \sigma_x) \qquad (f)$$

Now, by using Eq. (f), Eq. (19·7) may be rewritten as follows:

$$\text{Amplitude} = 2a \sin 2\alpha \sin \left[ \frac{\pi f k b}{v_b^2} (\sigma_y - \sigma_x) \right] \qquad (19·8)$$

This is the fundamental equation of two-dimensional photoelasticity, since from it we can determine the amplitude and hence the intensity of the light that comes through the analyzer.

Inspection of Eq. (19·8) reveals that the intensity may be zero for either or both of two reasons. First, the amplitude will be zero if $\sin 2\alpha$ equals zero. This will be so when $\alpha$ is a multiple of $\pi/2$, in other words, when the plane of polarization of the polarizer coincides with the direction of either of the principal stresses. Thus, regardless of the color of light or the magnitudes of the principal stresses, every point on the model where the direction of one of the principal stresses coincides with the plane of polarization of the polarizer will be black. There will be a number of points at which the principal stresses have the same direction. They will all be joined together by a black line, called an *isoclinic*.

Second, the amplitude is also dependent on the difference of the principal stresses, which affects the value of $\sin [(\pi f k b/v_b^2)(\sigma_y - \sigma_x)]$. When the model is unloaded and unstressed, the value of this term, and hence the amplitude, is zero; *i.e.*, an unstressed model appears black all over when viewed through the analyzer. If we focus our attention on a particular point in the model and gradually apply the load, we notice this point getting lighter and lighter, attaining maximum brilliance when the value of $(\pi f k b/v_b^2)(\sigma_y - \sigma_x)$ equals $\pi/2$ so that the sine of this angle equals unity. Further increase in load, however, makes the point get darker and darker until it is black again when the stresses are such that $(\pi f k b/v_b^2)(\sigma_y - \sigma_x)$ equals $\pi$. Further increase of load causes the point to undergo additional cycles from black through brilliance and back to black again. All this discussion presumes that the model is being viewed with monochromatic (or one-color) light, as is assumed at the beginning of this article.

As we view this one particular point on the model, there are other points going through the same number of cycles. In fact, when this particular point is black, the other points are also black; all together, they form a black line, called an *isochromatic*. At all points along such an isochromatic, the difference in principal stresses must therefore be the same. Of course, at this particular instant there are other series of points that have undergone a different number of cycles, and each of these series forms another black isochromatic line. Thus, the model as a whole at a given instant appears to have several black isochromatic

lines on a white background, as shown in Fig. 19·4. The various different isochromatics are designated as being the first-order isochromatic, second-order isochromatic, etc., depending on the number of cycles of black through brilliance to black that have occurred along that particular line.

It is easy to evaluate the difference in principal stress along the $n$th-order isochromatic, because we know that along this isochromatic every point has undergone $n$ cycles, or

$$\frac{\pi f k b}{v_b^2} (\sigma_y - \sigma_x) = n\pi$$

Hence,

$$\sigma_y - \sigma_x = \frac{n}{b} \frac{v_b^2}{fk} \tag{19·9}$$

Knowing $n$ and the thickness of the model $b$ and certain constants of the light and model material, we may evaluate $\sigma_y - \sigma_x$ from this equation.

From the theory developed in this article, we now see how the photo-elastic phenomena can be related quantitatively to the condition of stress in the model. How to use this information in solving a stress distribution problem is discussed in the next article. Before doing so, however, several additional factors should be noted.

We have found that, when we observe a bakelite model through the analyzer, using monochromatic light, we see two different types of lines, isoclinics and isochromatics, both of which are black lines on a light background. The question immediately arises as to how we are able to tell one from the other. If we used white light instead of monochromatic, the isoclinic would still be black, for it does not depend on the frequency of the light. The isochromatic lines, however, would assume the colors of the spectrum instead of being black, for only when the difference in the principal stresses is zero can there be complete interference for all frequencies simultaneously. Since such a situation usually exists at isolated points, there is seldom a black isochromatic—usually they are colored lines. By using white light, therefore, it is usually easy to identify the isoclinics.

For convenience, isochromatics are usually photographed, and for this purpose it is most satisfactory to use monochromatic light so that the isochromatics are black lines on a light background. It is desirable, however, to eliminate the isoclinic from the photograph. This can be accomplished by using circularly polarized monochromatic light. In such a case, the theory may be developed as is done above for plane-polarized light, and the amplitude of the resulting vibration seen through

the analyzer will be found to be

$$\text{Amplitude} = 2a \sin\left[\frac{\pi f k b}{v_b^2}(\sigma_y - \sigma_z)\right] \qquad (19\cdot10)$$

This expression is the same as Eq. (19·8) except that the sin $2\alpha$ term has disappeared and there is no directional effect. In practice, circularly polarized light is produced by introducing one quarter-wave plate on each side of the model. Actually, however, the same effect could be produced by rotating the polarizer and analyzer, these being always kept crossed. If we did this, the isoclinic line would move around the model as the plane of polarization was rotated. We can visualize, however, that, if we rotated the polarizer and analyzer fast enough, the isoclinic would disappear, in much the same way as the spokes of a wheel do if the latter is whirled fast enough. This is one way of visualizing the effect of circularly polarized light.

**19·5   Stress Distribution Determined by Photoelastic Method.** In the previous article, it is shown that the optical pattern which one views through the analyzer can be interpreted to give the direction of the principal stresses and the differences between them. This gives us two independent quantities concerning the condition of stress at any point in the model. In this article, we discuss how to obtain a third independent quantity and how to use this information to obtain the complete stress distribution in a two-dimensional system.

The polariscopes actually used in experimental work may be arranged in several different ways. The essential parts, however, are those shown in Fig. 19·2, with certain refinements added so that plane-polarized white light, plane-polarized monochromatic light, or circularly polarized monochromatic light may be obtained easily when desired. A mercury-vapor lamp with a suitable filter is widely used for monochromatic light, and the mercury-vapor lamp without the filter or an incandescent-filament lamp is used for white light. The polarizer and analyzer are fitted with quarter-wave plates arranged so that it is easy to change from plane- to circularly polarized light. A suitable lens system should be provided so that the model is in a field of parallel light. The loading machine should be arranged so that the entire loading device and model can be moved and aligned without removing or varying the load. The camera used with the polariscope should be a plate-type camera fitted with a screen for viewing the image.

With a polariscope such as this, it is a simple matter to set up a model and photograph the isochromatics (or isochromatic fringes) by using circularly polarized monochromatic light. A photograph of the isochromatics of a simple end-supported beam loaded by a concentrated

load at mid-span is shown in Fig. 19·4. The fringe orders on such a photograph may be identified by watching the model as the load is applied and counting the number of cycles at certain points. Knowing the fringe orders $n$, we may use Eq. (19·9) to compute the difference in the principal stresses at various points in the model.

For such computations, it is convenient to change the form of Eq. (19·9) and denote the two principal stresses by $p$ and $q$ instead of $\sigma_y$

Load=165#

Thickness = 0.350 in.
Optical constant = 87.0 lbs. per in.

1.845"

4.50"

82.5#
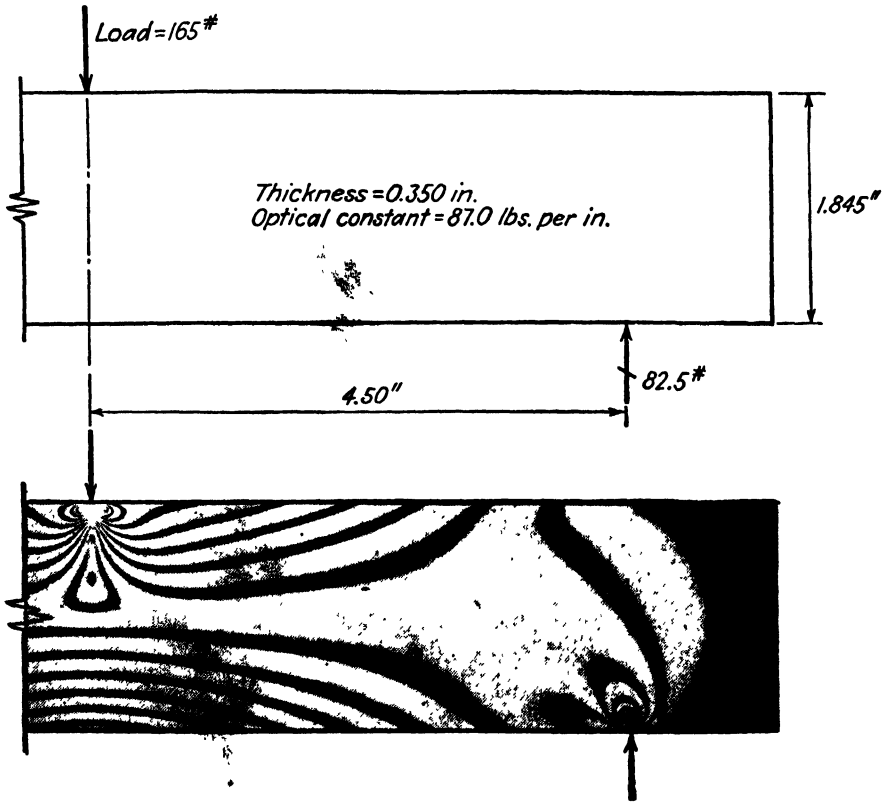
FIG. 19·4

and $\sigma_z$, or

$$p - q = \frac{n}{b}\, OC \qquad (19·11)$$

where $OC =$ optical constant of model material $= v_b^2/fk$. The optical constant for a given material may be determined from an auxiliary test on a tension specimen where $p - q$ is known.

While, in general, the isochromatics give us only the difference between the principal stresses, such information is sufficient to compute

the edge stresses of a model.   At a free unloaded edge, there is no shear or normal stress on the boundary surface, and therefore the boundary is a principal plane on which the principal stress is zero.   At such points, the other principal stress is parallel to the boundary and is therefore numerically equal to the edge value of $p - q$ obtained from the iso-chromatics.   In many problems, the maximum stresses occur at the edges of the model so that the investigation does not need to be carried beyond photographing the isochromatics.

Isoclinics are obtained by using plane-polarized white light and may be sketched on a piece of tracing paper mounted on the screen of the camera.   The plane of polarization of the polarizer is set successively at various positions between the vertical and horizontal, the polarizer and analyzer being kept crossed at all times.   For each of these positions, the corresponding isoclinic on the image is sketched on the tracing paper.   We thus obtain a sketch from which we can determine the direction of the principal stresses at any point in the model, since we know that at the points lying along any particular isoclinic, such as the 20° isoclinic, one of the principal stresses is in a direction which is 20° to the vertical and the other principal stress is perpendicular to that.
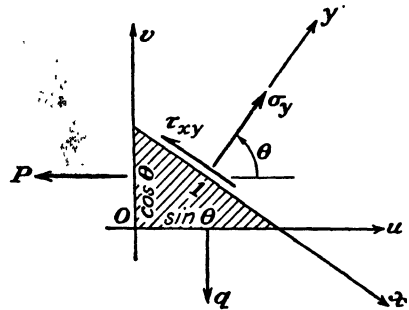


FIG. 19·5

If, in addition to knowing $p - q$ and the direction of the principal stresses at any point in a model, we know $p + q$, that is, the sum of the principal stresses, it will then be easy to compute the normal and shear stress intensity on any plane through any point in the model.   By considering the equilibrium of the small particle shown in Fig. 19·5, suitable equations may be derived for the normal stress $\sigma_y$ and the shear stress $\tau_{xy}$ on a plane the normal of which makes an angle $\theta$ with the direction of the principal stress $p$ at point $O$.   The perpendicular sides of this element are principal planes of stress, and it is assumed that the length of the hypotenuse and the thickness of the particle are both unity. From $\Sigma F_y = 0$,

$$(\sigma_y)(1)(1) - (p)(\cos\theta)(1)(\cos\theta) - (q)(\sin\theta)(1)(\sin\theta) = 0$$

which simplifies to

$$\sigma_y = \frac{p+q}{2} + \frac{p-q}{2}\cos 2\theta \qquad (19\cdot12)$$

Likewise, from $\Sigma F_x = 0$, we find that

$$\tau_{xy} = -\frac{p-q}{2}\sin 2\theta \qquad\qquad (19\cdot13)$$

From these equations, it is now apparent that both $\sigma_y$ and $\tau_{xy}$ can be computed if $p + q$ is also known.

Several methods are available for determining $p + q$, one of the best of which is measuring the change in thickness of the model. The change in thickness of a plate depends upon the principal stresses in the plane of the plate in exactly the same way as the contraction of the cross section of a tension specimen depends upon the axial stress in the specimen.

$$\Delta b = -\frac{\mu b}{E}(p+q) \qquad\qquad (19\cdot14)$$

Several lateral extensometers have been developed specifically for measuring $\Delta b$. The de Forest–Anderson extensometer[1] is one of the best of these. With such instruments, $p + q$ may be measured at certain specific points, and, by means of this information, curves may be drawn to assist in interpolating the values of $p + q$ at other points.

The values of $p + q$ may also be determined by several other experimental procedures or by an iterative numerical method suggested by Liebmann for solving problems involving Laplace's equation. Instead of determining $p + q$ by one of the methods that have been suggested, it would be possible to compute the separate values of $p$ and $q$ simply through the information that can be obtained from the isochromatics and the isoclinics. To do this, however, involves some rather tedious graphical and analytical computations not required by the other methods.

The use of the photoelastic method in stress distribution problems may be summarized as follows:

1. *To determine edge stress intensities.* Only isochromatics are required.

2. *To determine shear stress intensities on any plane at every point.* Isochromatics and isoclinics are required. Note, however, that maximum shear stress at a point can be determined simply from the isochromatics since

$$\tau_{max} = -\frac{p-q}{2}$$

3. *To determine both normal and shear stress intensities on any plane at every point.* In addition to isochromatics and isoclinics, the value of

[1] DE FOREST, A. V., and A. R. ANDERSON, A New Lateral Extensometer, *Proc. the Tenth Semi-annual Eastern Photoelasticity Conf.*, Dec., 1939.

$p + q$ or some third independent quantity must be determined for every
point.

**19·6 Strain Measurement.** It is impossible to discuss the art
of strain measurement in the limited space available here.[1] Many
different strain gages have been developed, but the principles on which
they are based are limited in number. The various instruments may
be classified in one of the following groups: mechanical gages; optical
gages; electric resistance gages; electric inductance gages; or electric
capacity gages. At the present time, the most widely used strain gage
is the SR-4 resistance wire gage.

One should note, however, that the minimum size of a structural
model may be controlled by the method selected for the strain measure-
ments. Of course, if the strain varies linearly along a gage line, the
average strain over any gage length will be exactly equal to the strain at
the mid-point of the gage. If, however, the strain varies in a distinctly
nonlinear manner, the average strain may be considerably different
from the strain at the mid-point of the gage. It is necessary, therefore,
to design the model large enough or to select a strain gage with a gage
length small enough so that the strain variation between the gage points
is essentially linear. If proper consideration is not given to this matter,
the results of the model study may not be sufficiently accurate.

**19·7 Stress Distribution by Surface-strain Measurement.**
When a thin flat plate is loaded by forces acting in the plane of the plate,
a condition of plane stress is developed in the plate, *i.e.*, the stresses
developed in the plate may be assumed to act parallel to the plane of
the plate and the stress intensities to remain constant through the thick-
ness of the plate. Such a condition of stress is the one most commonly
encountered in structural members. In such cases, the complete state
of stress throughout the interior of the plate is defined as soon as the
stress condition is determined on the surface.

In the case of a thick plate or in the most general case of a stressed
body, a three-dimensional stress system is involved, and the stresses
vary through the thickness of the plate. Determining the surface
stresses in such problems, therefore, does not enable one to define the
condition of stress in the interior. Fortunately, however, in such cases
the maximum stresses usually occur at the surface, and hence determining
these surface stresses gives a practical solution to many problems.

The surface stresses at a point on the unloaded surface of any body
form a two-dimensional stress system. As stated in Art. 19·2, the com-
plete condition of stress at such a point may be defined in terms of three

[1] The reader is referred to the "Handbook of Experimental Stress Analysis,"
which will soon be published by the Society of Experimental Stress Analysis.

unknowns, the magnitude of the two principal stress intensities and the direction of one of them.   Of course, stresses cannot be measured directly, but we can measure surface strains.   If three independent linear strains are measured, it then is possible to compute the principal strains and their directions.   By using the physical relations between stress and strain, these principal strains can be converted into the principal stresses. The directions of the principal stresses and the principal strains are the same.   Once the magnitude and directions of the principal stresses are known, the normal and shear stress intensities on any plane through a point can be computed.
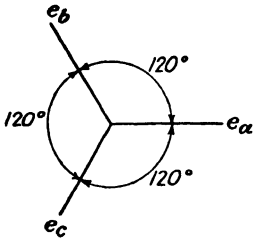
The gage lines of the three linear strains that must be measured may be chosen in such a way as to simplify the computation of the principal strains and stresses.[1]   One such arrangement, known as the *equiangular strain rosette*,[2] is shown in Fig. 19·6.   The three linear strains $e_a$, $e_b$, and $e_c$ are measured along gage lines that make angles of 120° with each other.   Then it can be shown that the major (or algebraically larger) principal strain $e_1$ and the minor (or algebraically smaller) principal strain $e_2$ may be computed from the following formula:

FIG. 19·6

$$e_1 = A_1 + B_1 \Big\}$$
$$e_2 = A_1 - B_1 \Big\} \qquad (19·15)$$

where

$$A_1 = \tfrac{1}{3}(e_a + e_b + e_c)$$
$$B_1 = \frac{\sqrt{2}}{3}\sqrt{(e_a - e_b)^2 + (e_b - e_c)^2 + (e_c - e_a)^2} \Bigg\} \qquad (19·16)$$

All strains, when they are elongations, should be considered plus.   The direction of the major principal strain $e_1$ is given by an angle $\alpha_1$, which is measured counterclockwise from the $e_a$ gage line, where

$$\tan 2\alpha_1 = \frac{\sqrt{3}\,(e_c - e_b)}{2e_a - e_b - e_c} \qquad (19·17)$$

The quadrant in which the angle $2\alpha_1$ lies can be identified by noting the signs of both numerator and denominator when the numerical values of the strains are substituted in Eq. (19·17).   The direction of the minor principal strain $e_2$ is located by an angle $\alpha_1 + 90°$ measured counterclockwise from the $e_a$ gage line.

[1] MURRAY, W. M., An Adjunct to the Strain Rosette, *Exptl. Stress Analysis*, vol. 1, No. 1, pp. 128–133.

[2] MINDLIN, R. D., The Equiangular Strain Rosette, *Civil Eng.*, vol. 8, No. 8, pp. 546–547, August, 1948.

Another suitable arrangement is known as the *rectangular strain rosette* and is shown in Fig. 19·7. In this case the principal strains may be computed from the following formulas:

$$\left. \begin{aligned} e_1 &= A_2 + B_2 \\ e_2 &= A_2 - B_2 \end{aligned} \right\}$$

(19·18)

where

$$\left. \begin{aligned} A_2 &= \frac{e_a + e_c}{2} \\ B_2 &= \tfrac{1}{2}\sqrt{(2e_b - e_a - e_c)^2 + (e_a - e_c)^2} \end{aligned} \right\}$$

(19·19)

The direction of the major principal strain $e_1$ is given by the following angle $\alpha_1$, which is measured counterclockwise from the $e_a$ gage line,

$$\tan 2\alpha_1 = \frac{2e_b - e_a - e_c}{e_a - e_c}$$

(19·20)

whereas the direction of the minor principal strain $e_2$ makes an angle of $\alpha_1 + 90°$ with the $e_a$ gage line.

The principal strains having been computed from the above equations, the major and minor principal stress intensities, $\sigma_1$ and $\sigma_2$, respectively, may be computed from the following equations:

$$\left. \begin{aligned} \sigma_1 &= \frac{E}{1 - \mu^2}(e_1 + \mu e_2) \\ \sigma_2 &= \frac{E}{1 - \mu^2}(\mu e_1 + e_2) \end{aligned} \right\}$$

(19·21)
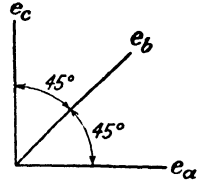


Fig. 19·7

where $E$ is the modulus of elasticity of the material and $\mu$ is Poisson's ratio. For either type of rosette, the principal stresses may be computed directly without first computing the principal strains by using the following equations, which have been obtained by substituting either Eq. (19·15) or (19·18) in Eq. (19·21):

$$\left. \begin{aligned} \sigma_1 &= E\left(\frac{A}{1 - \mu} + \frac{B}{1 + \mu}\right) \\ \sigma_2 &= E\left(\frac{A}{1 - \mu} - \frac{B}{1 + \mu}\right) \end{aligned} \right\}$$

(19·22)

The directions of $\sigma_1$ and $e_1$ and of $\sigma_2$ and $e_2$ are the same.

In certain cases, it is convenient to establish the directions of the principal stresses and strains by some auxiliary means such as symmetry or through the use of brittle lacquer or Stress Coat. If this is done, then the two principal strains may be measured in these directions, the strain readings required at every point being thus reduced from three to two.

**19·8 Determination of Buckling Loads.** Structural models may also be used advantageously to determine critical or buckling loads of columns, beams, and other structural elements. Such information may be obtained most readily by running tests under loads that are somewhat less than the buckling load and then interpreting the test data by Southwell's method[1] or some adaptation of it. In such tests, the load is applied in a series of increments, and certain deflections are measured after each increment of loading. By plotting this information in a certain manner, the plotted points lie on a straight line the slope of which may be interpreted to give the critical buckling load. The chief advantage of this procedure is that necessary data may be obtained from measurements taken at loads smaller than the buckling load, and as a result the model is not destroyed in conducting the test. The model may be retested, therefore, after any desired modifications in its design have been made.

In Southwell's original paper, the theoretical proof of his method was given only for the case of a simple strut. He demonstrated that, if the ratio of lateral deflection to the corresponding axial load was plotted as an ordinate against the deflection itself as an abscissa, the plotted points would lie along a straight line the inverse slope of which was equal to the critical load of the strut.

Subsequently Lundquist[2] suggested a modification of Southwell's procedure. Donnell[3] and other investigators have suggested and applied variations of Southwell's method for more complex types of buckling. These studies indicate that this procedure or some variation of it may be applied to all cases in which buckling does not introduce appreciable second-order stresses.

[1] SOUTHWELL, R. V., On the Analysis of Experimental Observations in Problems of Elastic Stability, *Proc. Roy. Soc. (London)*, vol. 135A, p. 601, 1932.

[2] LUNDQUIST, E. E., Generalized Analysis of Experimental Observations in Problems of Elastic Stability, *NACA Tech. Note* 658.

[3] DONNELL, L. H., "On the Application of Southwell's Method for the Analysis of Buckling Tests," Contributions to the Mechanics of Solids, Dedicated to S. Timoshenko, p. 27, The Macmillan Company, New York, 1938.

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