ENGINEERING MECHANICS

Engineering Mechanics

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Second Edition

NEW YORK
JOHN WILEY & SONS, INC.
LONDON: CHAPMAN & HALL, LIMITED

SECOND EDITION
Fourth Printing, March, 1947

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE TO SECOND EDITION

The general arrangement of subject matter in the second edition is substantially the same as that employed in the first edition. Changes made in the textual material pertain largely to details in the manner of presentation. Chief among these is the greater emphasis that has been placed on methods of solution in which reference is made to relationships set forth verbally, as principles, rather than symbolically, as formulas.

A large percentage of the problems of the first edition have been replaced by new ones, and the total number, both of illustrative examples and of practice problems, has been increased. Answers to approximately one-half the problems have been furnished. The cuts are entirely new, and the book has been completely reset. A few of the more advanced topics not widely used in undergraduate classes have been omitted.

As in the first edition, the author has endeavored to provide a full explanation of each essential point, with the purpose of relieving the instructor from the necessity of consuming a large portion of each class period in a general discussion of the fundamental principles of the subject.

FRANK L. BROWN

LAWRENCE, KANSAS
January 7, 1942

PREFACE TO FIRST EDITION

This book is designed primarily as a textbook on mechanics for students of engineering. For this reason, and because of a desire to provide a book that could be very nearly, or perhaps fully, covered in the usual undergraduate course, the author has omitted the treatment of many of the applications usually included in such textbooks. A few of the more instructive applications have been retained, together with all the fundamental principles that are of importance to the engineer in the ordinary course of events.

The book also represents an effort to lift from the shoulders of the instructor a portion of the burden of explanation and amplification.

A large number of illustrative problems and practice problems have been included. Many of the problems are of a practical nature; others are purely imaginative. The author believes that the latter type is often more instructive. An effort has been made to state each problem fully, so as to leave no doubt as to the conditions. The problems are all appended to the articles with which they are most closely associated. Answers are given in nearly all cases.

Statics is treated first, in Part I. The topics of statics whose study requires a knowledge of the calculus are placed at the end of Part I. In Part II, Kinematics and Kinetics alternate, as the various types of motion are discussed.

The formulas of kinetics have been developed from a statement of laws of motion that does not include the term, mass. Naturally, the quantity $\frac{W}{g}$ appears in the kinetic formulas in lieu of M. In Art. 120 the author has attempted to explain mass, and has given his reasons for not utilizing it in the remainder of the book.

Every instructor learns that beginners in the subject are wont to make certain typical errors. The author has made an effort to warn the student with regard to some of the more common types of errors.

Coherence, and rigor in analysis, have been the aim throughout the book.

F. L. B.

LAWRENCE, KANSAS
August, 1930

CONTENTS

PART I. STATICS

CHAPTER I. GENERAL PRINCIPLES

		PAGE
1.	Mechanics in Engineering	1
2.	Force	1
3.	Force by Contact	2
4.	External and Internal Forces	2
5.	The Occurrence of Forces in Pairs	3
6.	Characteristics of a Force	4
7.	Units of Force; Weight	4
8.	Effects of Force; Rigid and Non-Rigid Bodies	5
9.	Equivalence of Force Systems	5
10.	Classification of Force Systems	6
11.	Resultants and Components	6
12.	The Principle of Transmissibility	7
13.	Accuracy of Calculations	7
14.	Smooth Surfaces	8
15.	Vector Quantities	9
16.	The Measurement of Angles of Inclination	9

CHAPTER II. RESULTANTS OF COPLANAR FORCE SYSTEMS

17.	Resultant of Two Concurrent Forces; The Parallelogram Law	11
18.	Resultant of Two Concurrent Forces at Right Angles to Each Other	14
19.	Resolution of a Force into Two Concurrent Components at Right Angles to Each Other	16
20.	The Use of the Algebraic Sum in Mechanics	18
21.	Conventions for Signs	19
22.	Interpretation of Signs of Results	19
23.	Resultant of the Collinear System	19
24.	Resultant of the Coplanar Concurrent System; Graphic Solution	20
25.	Resultant of the Coplanar Concurrent System; Algebraic Solution	23
26.	Moment of a Force about a Line; Special Case	27
27.	The Principle of Moments for Two Concurrent Forces; Varignon's Theorem	30
28.	Couples; The Moment of a Couple	32
29.	Equivalence of Couples in the Same Plane	36
30.	Equivalence of Couples in Parallel Planes	36
31.	Summary of Permissible Operations with a Couple	37
32.	Resultant of the Coplanar Parallel System; Graphic Solution	37
33.	Resultant of the Coplanar Parallel System; Algebraic Solution	42

PAGE

34. Resultant of the General Coplanar System; Graphic Solution 47
35. Resultant of the General Coplanar System; Algebraic Method 50
36. Resultant of a System of Couples; Special Method 56
37. Resultant of a Coplanar Couple and Single Force; Special Method 58
38. Resolution of a Force into a Force and Couple 59

CHAPTER III. RESULTANTS OF NON-COPLANAR FORCE SYSTEMS

39. Resultant of Three Non-Coplanar Concurrent Forces at Right Angles to One Another ... 61
40. Resolution of a Force into Three Concurrent Components at Right Angles to One Another ... 62
41. Resultant of the Non-Coplanar Concurrent System 64
42. Moment of a Force about a Line; General Case 66
43. Representation of a Couple by Means of a Single Vector 69
44. Resultant of Three Couples in Planes at Right Angles to One Another .. 71
45. Resolution of a Couple into Three Component Couples in Perpendicular Planes ... 75
46. Resultant of a System of Non-Coplanar Couples 78
47. Resultant of the Non-Coplanar Parallel System 81
48. Resultant of the General Non-Coplanar System 85
49. The Principle of Moments for Two Concurrent Forces; General Case... 90
50. General Nature of the Principles of Composition 91

CHAPTER IV. EQUILIBRIUM OF COPLANAR FORCE SYSTEMS

51. Meaning and Physical Significance of Equilibrium 93
52. The General Graphic Conditions for Equilibrium 93
53. The General Algebraic Conditions for Equilibrium 94
54. General Features of Graphic Solutions 94
55. General Features of Algebraic Solutions 94
56. The Free-Body Diagram 95
57. Limitations in the Matter of Equilibrium Equations 95
58. Selection of Axes 96
59. Statically Indeterminate Cases 96
60. Equilibrium of the Collinear System 97
61. Two-Force Members 97
62. Equilibrium of the Coplanar Concurrent System; Graphic Solution 98
63. Equilibrium of the Coplanar Concurrent System; Algebraic Solution ... 101
64. Equilibrium of the Coplanar Parallel System; Graphic Solution 106
65. Equilibrium of the Coplanar Parallel System; Algebraic Solution 108
66. Equilibrium of the General Coplanar System; Graphic Solution 111
67. Equilibrium of the General Coplanar System; Algebraic Solution 114

CHAPTER V. PLANE TRUSSES

68. Axial Stress 126
69. Trusses in General 127
70. Primary Stresses; Algebraic Solution 128
71. Primary Stresses; Graphic Solution 139

CHAPTER VI. EQUILIBRIUM OF NON-COPLANAR FORCE SYSTEMS

PAGE

72. Equilibrium of the Non-Coplanar Concurrent System 144
73. Equilibrium of the Non-Coplanar Parallel System 149
74. Equilibrium of the General Non-Coplanar System 153

CHAPTER VII. FRICTION

75. Friction in General 160
76. Static Friction; Kinetic Friction 161
77. Limiting Friction; Impending Motion 161
78. Relation between Limiting Friction and Normal Pressure 162
79. Angle of Static Friction; Angle of Repose 168
80. Classification of Problems Involving Static Friction 173
81. Kinetic Friction 176
82. The Inclined Plane 177
83. The Screw 180
84. The Screw Jack 183
85. Brake Bands 185
86. Journal Bearings; Friction Circle 189

CHAPTER VIII. SUSPENDED CABLES

87. Flexible and Inextensible Cables 194
88. Loads Carried by Suspended Cables 194
89. The Parabolic Cable 195
90. The Catenary Cable 200

CHAPTER IX. CENTER OF GRAVITY

91. Center of Gravity of a Body 206
92. Center of Gravity of a Line, Area or Volume 207
93. Coordinates of the Center of Gravity; General Formulas 208
94. Axes and Planes of Symmetry 209
95. First Moments; Gravity Axes 210
96. Derivation of Special Formulas for Lines 210
97. Derivation of Special Formulas for Areas 212
98. Derivation of Special Formulas for Volumes 214
99. First Moments by Finite Summation 216
100. Concerning Subsequent Problems 217
101. Applications of the Special Formulas for Lines 217
102. Applications of the Special Formulas for Areas 218
103. Applications of the Special Formulas for Volumes and Bodies 222
104. Theorems of Pappus and Guldinus 225
105. Graphic Methods 228
106. Experimental Methods 228

PART II. KINEMATICS AND KINETICS

107. General 231

CHAPTER X. RECTILINEAR MOTION OF A POINT

PAGE
108. Linear Velocity 232
109. Linear Acceleration 235
110. Uniform Motion 239
111. Motion with Constant Acceleration 240
112. The Crank-and-Connecting-Rod Mechanism 245
113. Simple Harmonic Motion 247

CHAPTER XI. CURVILINEAR MOTION OF A POINT

114. Linear Velocity 251
115. X- and Y-Components of Velocity 255
116. Linear Acceleration 259
117. Tangential and Normal Components of Acceleration 261
118. X- and Y-Components of Acceleration 266

CHAPTER XII. KINETICS OF A PARTICLE

119. The Laws of Motion 271
120. Inertia 272
121. The Fundamental Kinetic Formula for the Particle 272
122. Mass 274
123. The Effective Force for the Particle 275

CHAPTER XIII. KINETICS OF A BODY IN GENERAL

124. Fundamental Difference between the Formulas of Statics and Kinetics .. 277
125. Effective Forces and External Forces 277
126. The Motion of the Center of Gravity of a Finite Body 278
127. The Resultant of the External Forces 285

CHAPTER XIV. KINEMATICS OF TRANSLATION

128. Kinematics of a Translating Body 286

CHAPTER XV. KINETICS OF TRANSLATION

129. The Resultant of the External Forces 288
130. Methods of Solving Problems 289

CHAPTER XVI. MOMENT OF INERTIA

131. Moment of Inertia of a Line, Area, or Volume; General Formulas 298
132. Moment of Inertia of a Body; General Formulas 299
133. Radius of Gyration 300
134. The Parallel-Axis Theorem 301
135. Derivation of Special Formulas for Lines 303
136. Polar Moment of Inertia 304
137. Derivation of Special Formulas for Areas 305
138. Moments of Inertia of Slender Rods and Thin Plates; Approximate Methods 309

PAGE
139. Derivation of Special Formulas for Homogeneous Bodies 310
140. Moment of Inertia by Finite Summation 314
141. Concerning Subsequent Problems 314
142. Applications of the Special Formulas for Lines 314
143. Applications of the Special Formulas for Areas 316
144. Applications of the Special Formulas for Bodies 319
145. Moments of Inertia with Respect to Planes and Points 321
146. Moments of Inertia of Irregular Forms 321

CHAPTER XVII. KINEMATICS OF ROTATION

147. Definition of Rotation 323
148. Angular Velocity 323
149. Angular Acceleration 325
150. Relations between the Motion of a Rotating Body and the Motions of Its Particles 329
151. Rotation with Constant Angular Velocity 332
152. Rotation with Constant Angular Acceleration 333

CHAPTER XVIII. KINETICS OF ROTATION

153. Special Nature of the Discussion 336
154. The Resultant of the External Forces 336
155. Methods of Solving Problems 339
156. Superelevation of Curves 349

CHAPTER XIX. KINEMATICS OF PLANE MOTION

157. Definition of Plane Motion 351
158. Angular Velocity 351
159. Angular Acceleration 352
160. Rolling Bodies 353
161. Relation between the Motions of Two Particles of the Body 355
162. The Instantaneous Axis 364

CHAPTER XX. KINETICS OF PLANE MOTION

163. Special Nature of the Discussion 371
164. The Resultant of the External Forces 371
165. Methods of Solving Problems 374
166. Rolling Resistance 383

CHAPTER XXI. RELATIVE MOTION

167. The Path of a Point 389
168. Relative Velocity 390
169. Relation between Relative and Absolute Velocities 391
170. Relative Acceleration 401
171. Relation between Relative and Absolute Accelerations 402
172. Motion of a Point Relative to a Second Point 415

CHAPTER XXII. WORK

PAGE
173. Work Done by a Single Concentrated Force 419
174. Work Done by a Force with a Constant Working Component 424
175. Total Work 425
176. Work Done by a Constant Force 427
177. Work Done by Distributed Forces 430
178. Work Done by Gravity 430
179. Work in Terms of Torque 433
180. Work Done by Frictional Forces 434
181. Work Done by Two Collinear, Equal, and Opposite Forces 437

CHAPTER XXIII. POWER

182. Power in General 441
183. Measurement of Power 445

CHAPTER XXIV. ENERGY

184. Energy in General 447
185. Kinetic Energy 448
186. Kinetic Energy of a Particle 449
187. Kinetic Energy of a Translating Body 450
188. Kinetic Energy of a Rotating Body 451
189. Kinetic Energy of a Body Having Any Plane Motion 453
190. Potential Energy 455
191. Efficiency 457

CHAPTER XXV. WORK AND ENERGY

192. The Work-Energy Method 459

CHAPTER XXVI. LINEAR IMPULSE AND LINEAR MOMENTUM

193. Elementary Linear Impulse 465
194. Finite Linear Impulse; General Case 465
195. Finite Linear Impulse; Special Cases 469
196. Linear Momentum of a Particle 472
197. Linear Momentum of a Body 472
198. Relation between External Forces and Linear Momentum 476
199. Conservation of Linear Momentum 477
200. Relation between Linear Impulse and Linear Momentum 479
201. Pressure Caused by the Diversion of a Jet or Stream 482

CHAPTER XXVII. ANGULAR IMPULSE AND ANGULAR MOMENTUM

202. Elementary Angular Impulse 486
203. Finite Angular Impulse; General Case 486
204. Finite Angular Impulse; Special Case 488

PAGE

205. Angular Momentum of a Particle 489
206. Angular Momentum of a Rotating Body about the Axis of Rotation 489
207. Relation between External Forces and Angular Momentum of a Rotating Body 491
208. Conservation of Angular Momentum of a Rotating Body 492
209. Relation between Angular Impulse and Angular Momentum for a Rotating Body 494

INDEX 499

PART I. STATICS

CHAPTER I

GENERAL PRINCIPLES

1. Mechanics in Engineering. Mechanics deals primarily with force and motion. Engineers are vitally interested in these phenomena. Safety, utility, and economy in the design of structures and machines are attained only through careful study of the forces involved and of the motions to be expected.

The more specialized subjects, such as strength of materials, hydraulics, structural design, and machine design, serve to link up the principles of mechanics with the physical properties of the substances utilized in engineering works, developing out of the combination an array of methods and formulas constituting a large and important part of modern engineering science.

2. Force. Under certain conditions bodies at rest can be set in motion. Also, bodies in a state of motion of one description can be caused to assume a motion of a different description, or can be brought to rest. All such phenomena will be referred to in this book as *changes of motion.*

A change of motion of a given body can always be traced to the influence of one or more other bodies, and is often accompanied by a change of motion of the latter. The obvious conclusion is that changes of motion are caused by an *interaction,* or *mutual action,* between bodies. For the sake of clearness this interaction between two bodies is usually thought of as consisting of two actions. Each of these actions is called a *force.* Thus, if body A pushes against body B, it is stated that A exerts a force on B. Experience teaches that B inevitably and simultaneously reacts by pushing against A, and so it is also stated that B exerts a force on A. Similar statements are made when the interaction is in the nature of a pull.

The foregoing statements should not be interpreted as meaning that force always results in change of motion. Two or more forces often act on a given body in such a manner as to balance, or neutralize, one another, in which case their combined effect on the body, so far as its state of motion as a whole is concerned, is nil. Change of motion is, however, the typical effect of force, and will always occur when the forces acting on a given body are in an unbalanced condition.

PROBLEM

1. A rifle bullet embeds itself in a block of wood lying on a horizontal surface. The block moves forward. What is the cause of the change of motion of the block? Of the change of motion of the bullet?

3. Force by Contact. With certain exceptions, bodies cannot exert forces on one another unless they are in contact. Electric and magnetic attractions and repulsions, and gravitational forces, are the principal exceptions. Problems involving electric and magnetic forces will not be dealt with in this book. In ordinary engineering problems the only case of gravitational attraction great enough to require consideration is that which exists between the earth and a body on or near the earth's surface. Therefore, with the exception of the earth's pull, it will be assumed that no force of importance exists between bodies which are not in contact.

Bodies in contact do not always exert forces on one another, but the existence of such forces must be assumed unless the conditions of the problem clearly show it to be impossible.

PROBLEMS

2. A wheel rests on a horizontal floor and is in contact with a low step. A horizontal force is applied at the center of the wheel and is gradually increased until it causes the wheel to roll over the edge of the step. Make a sketch of the wheel and step, and by means of vectors show the forces acting on the wheel at the instant when motion impends. In what manner could the sketch be altered to represent conditions at an earlier instant? Values need not be calculated.

3. A cubical block rests on a horizontal surface. A horizontal force is applied to the block at one of the upper edges. If this force is gradually increased, and if the friction on the bottom of the block prevents sliding, the block will eventually tip. Sketch the block, and show the forces acting on it at the instant when tipping impends. Values need not be calculated.

4. The water in a reservoir exerts a pressure on the dam. The dam exerts a simultaneous pressure on the body of water. Forces such as that exerted by the water are sometimes called active forces, and forces resembling that exerted by the dam are called passive forces. Explain these terms.

4. External and Internal Forces. An *external* force is a force exerted on a body by another, or different, body. An *internal* force is a force exerted on one part of a body by another part of the same body. In mechanics *a body is any definite portion of matter.*

The majority of the principles of mechanics, so far as they refer to forces, refer to external forces only. For this reason, when information is desired regarding a force, it is necessary to select a body on which the desired force acts externally. This often renders it necessary to establish bodies having imaginary boundaries, and it accounts for the broad definition of a body stated above.

Whether the boundaries of the body under consideration at any time are real or imaginary, it is highly important that they be definite, and that they be clearly fixed in the mind of the student before he attempts any calculations with reference to the body.

PROBLEMS

5. In the operation of an automobile many forces are brought into action. Considering the entire car and its passengers as "the body," state whether each of the following forces is external or internal: the pressure of the gas on one of the pistons; the pressure of the air against the moving car; the weight of the car; the pressure of the operator's foot against the brake pedal; the force exerted on one of the tires, by the roadway; the pull of the tow rope when the car is being "hauled in"; the pressure of the differential pinion against the ring gear; the pull of the drag link on the steering knuckle.

6. Under what circumstances would the gravitational force exerted on the earth by the moon be considered an external force? When would it be considered internal?

7. Under what circumstances would the gravitational forces exerted by the sun and the planets on one another be classified as internal forces?

5. The Occurrence of Forces in Pairs. In Art. 2 it was explained that forces always occur in pairs; that if body A exerts a force on body B, body B simultaneously exerts a force on A. To this statement may be added the important fact that the two forces thus produced are equal in magnitude and have the same line of action, but act in opposite directions along that line. All the foregoing facts are comprehended in Newton's third law of motion (Art. 119).

Two forces that are equal, collinear, and opposite are said to be *balanced*. When a definite body is under consideration the internal forces constitute a balanced system, since they are forces between parts of the body and therefore include nothing but complete pairs of balanced forces as described above.

The external forces acting on a given body do not necessarily constitute a balanced system. Each external force is a member of one of the balanced pairs referred to in Newton's third law, but the other member of the pair does not act on the given body; it is a force exerted by the given body on some other body. Therefore, the system of external forces acting on a given body may be, and often is, unbalanced.

PROBLEMS

8. A bat strikes a pitched ball. The motion of the ball is abruptly changed, and yet the two forces acting between the bat and the ball are balanced forces. Explain.

9. The forces within the cylinders of a locomotive when steam is admitted are balanced forces, since they are internal with respect to the locomotive, and yet if these forces become sufficiently large the train starts. Explain.

10. Explain, in the language of mechanics, why " a man cannot lift himself over a fence by his boot straps."

11. Make a sketch of a body resting on the surface of the earth. Represent by vectors the two forces exerted on the body by the earth. Call these forces " actions," and number them 1 and 2. Now show the reaction which, according to Newton's third law, will accompany each of the two actions. Give each reaction the same number as the action with which it belongs. Which pair ceases to act if the body is thrown into the air?

6. Characteristics of a Force. A force is completely identified only when certain distinguishing features, called characteristics, have been ascertained. The characteristics of a force are *magnitude, line of action, and sense.*

The magnitude of a force is its size, or amount. The line of action is the line along which the force exerts its effect. The sense of a force is the direction in which it acts along the line of action. The *point of application* of a force is the particular point on the line of action at which the force may be considered to be applied to the body.

It follows from the foregoing definitions that force is a vector quantity.

7. Units of Force; Weight. In the English system the fundamental unit of force used by engineers is called the *pound.* The pound is the force with which the earth pulls on a certain piece of platinum, when the latter is placed at sea level, at 45° north latitude. This original platinum standard is in the possession of the British government.

The gravitational pull exerted by the earth on a body is called the weight of that body. Since gravitational force depends on the distance between the two bodies, the weight of a body varies slightly when the elevation is changed. Furthermore, since the earth is not quite spherical, a change in latitude has a slight effect on the weight of a body.

Nevertheless, if a body is weighed on a platform scale, or on any other type of scale operating through a system of levers, the reading obtained is the same regardless of the locality in which the weighing is done. The reason for the apparent contradiction is that the earth-pull on the standard weights used in connection with the scale and the earth-pull on the body being weighed vary in the same ratio when the position relative to the earth is changed.

Since the weights used with a scale of the lever type are derived from the standard platinum unit, this form of apparatus does not give the true local weight of a body. The reading obtained is merely the weight that the body would have at the standard locality. This is called the *standard weight* of the body.

A spring balance, if it were sufficiently sensitive and had been carefully calibrated with standard weights at the standard locality, would give the true *local weight* of a body.

Because of the methods used by engineers in obtaining the weights of bodies it is usually the standard weight, and not the true local weight, that is actually introduced into the calculations. In any event, the possible variations in the weight of a body caused by changes in elevation or in latitude are so small that they may be entirely disregarded in the average engineering problem.

Units. As was stated, the fundamental unit of force in the English system is the pound. The ton, the ounce, and other multiples and submultiples of the pound are sometimes used. The poundal, a unit of force based on acceleration and used principally by physicists, is of little interest in engineering practice.

8. Effects of Force; Rigid and Non-Rigid Bodies. Forces always cause more or less change in the dimensions of bodies. This occurs whether or not there is any change in the motion of the body as a whole. The effect of forces in deforming a body is called the *deformation effect.* The effect of forces in changing the motion of a body is called the *motion effect.*

A perfectly *rigid body* would be one that could not be deformed. No such body really exists, but some bodies are relatively so rigid that in many problems their deformations may be disregarded.

Engineering mechanics usually concerns itself with motion effect only. This kind of effect will be meant whenever the word " effect " is used without qualification. The study of deformation effect belongs properly to the subject of strength of materials.

Even when the information desired by the engineer necessitates the consideration of the deformations suffered by a body, a knowledge of the mechanics of rigid bodies is prerequisite. Some of the principles and formulas of mechanics apply with equal rigor to rigid and to non-rigid bodies.

PROBLEM

12. Give your opinion as to whether motion effect or deformation effect is more in evidence in each of the following: the force exerted by a pneumatic riveter on a heated rivet; the force exerted by a bat on a pitched ball; the force exerted by a wall on a snowball thrown against it; the pull of the earth on a boulder falling from an overhanging cliff.

9. Equivalence of Force Systems. Any selection of forces dealt with or referred to as a group is called a *force system.* The system need not include all the forces involved in any particular problem, or even in a single step of the problem.

Two or more systems which would have the same motion effect on a given rigid body are called *equivalent systems.* The possible number of equivalent systems is unlimited. Equivalent systems would not necessarily have the same motion effect on a non-rigid body, nor would they necessarily have the same deformation effect on such a body.

10. Classification of Force Systems. In this book force systems will be classified as follows: (1) the collinear system, (2) the coplanar concurrent system, (3) the coplanar parallel system, (4) the general coplanar system, (5) the non-coplanar concurrent system, (6) the non-coplanar parallel system, (7) the general non-coplanar system.

A *coplanar system* is one in which the lines of action of all the forces lie in the same plane. A *concurrent system* is one in which all the lines of action intersect at a common point. A *collinear system* is one in which all the forces have the same line of action. A *parallel system* is one in which all the lines of action are parallel. In collinear systems and in parallel systems the senses of the forces do not necessarily agree.

The general principles of mechanics apply to all force systems. There are various special principles, however, which pertain only to certain systems. It is for this reason, and also for the sake of greater ease of study, that the foregoing classification has been adopted.

11. Resultants and Components. The *resultant* of a given force system is the simplest equivalent system. It follows that all equivalent systems have the same resultant.

As defined in Art. 9, equivalent systems are systems that would have the same motion effect on a given rigid body. This statement should be interpreted to mean that if any or all of the forces acting on a rigid body were replaced by their resultant, or by any other equivalent system, the subsequent state of motion of the body would not be altered by such replacement.

A *balanced force system* is one which would have no motion effect on a rigid body. All balanced systems are equivalent, their common lack of effect being thought of as a special case of motion effect. The resultant of a balanced system is nil; in other words, a *balanced system has no resultant.*

The process of finding a simpler equivalent system is called *composition.* The process of finding a more complex equivalent system, or any equivalent system having a larger number of forces, is called *resolution.*

The forces of a system that has a resultant are called *components* of that resultant. Composition begins with a system of forces and usually is continued until the resultant is found; resolution usually begins with a resultant and consists in the finding of some equivalent system of forces called the components.

The reversed resultant of a system of forces is called the *equilibrant* of that system.

PROBLEMS

13. An unbalanced force system acts on a body, producing a certain motion effect. What motion effect would have resulted if the entire system had been removed and the resultant had been applied? What would have occurred if the resultant had been applied without the removal of the original forces?

14. In Prob. 13, what would have occurred if the original forces had been removed and their equilibrant had been applied to the body? What would have occurred if the equilibrant had been applied without the removal of the original forces?

15. What name could be given to any one of the forces of a balanced system, in view of the relationship of that force to the remaining forces of the system?

12. The Principle of Transmissibility. *Two forces whose magnitudes are equal, and which have the same line of action and sense, are equivalent, regardless of where their actual points of application are situated.*

The foregoing principle shows that a vector representing a force acting on a rigid body can be shifted along the line of action, in either direction, without change of effect, provided that the magnitude and sense of the force are not changed and that the line of action is not disturbed. The shifting of force vectors in this manner is often of convenience in mechanics.

There is no satisfactory mathematical proof of the principle of transmissibility, but its validity has been amply demonstrated by observation and experiment.

13. Accuracy of Calculations. Many of the calculations in engineering problems involve quantities that are the result of physical measurements. The accuracy of such measurements is always limited. The degree of accuracy attainable in the result of any calculation is limited by the least accurate of the data used in the calculation.

For example, let it be supposed that it is desired to ascertain the tensile strength of a certain round bar of structural steel. An actual test in the laboratory would constitute the most accurate method of obtaining the desired information. If this is impossible, the strength can be calculated from the following formula: $P = \pi D^2 S/4$; in which P is the total tensile strength of the bar in pounds, D is the diameter of the bar in inches, and S is the ultimate tensile strength of the material in pounds per square inch.

The diameter can be measured, but the value of S must be assumed. An examination of the laboratory reports of several students, on actual tests of structural steel, showed values of S ranging from 60,800 to 63,500 lb per sq in. This constitutes a variation of 2.7 in the second significant figure, or 27 in the third significant figure. Comparable variations might be expected in any series of routine laboratory tests of structural steel. For this reason, average values of S for materials of this nature are seldom stated to more than two significant figures.

For example, a certain book states that the ultimate tensile strength of structural steel is 63,000 lb per sq in. A cipher is used for the third figure because of its convenience, and because it is as likely to be correct for any given specimen of that material as is any other number.

The diameter of a bar is usually measured at several points. Accurate measurement would probably show a percentage variation in diameter considerably less than the percentage variation in values of S obtained from a series of tests. Furthermore, the instrument used for the measurement of the diameters probably would be somewhat more precise than the testing apparatus. These facts show that S is undoubtedly the factor which limits the accuracy obtainable in the calculated result. It would be unnecessary and undesirable, therefore, to use more than two or, at the most, three significant figures in measuring the diameter and in calculating its average value.

Let it be supposed that the value finally adopted as the average diameter of this particular bar is 0.998 in. It will be consistent, then, to use for π the value 3.14. In each step of the calculation all significant figures beyond the third may be dropped. The calculation could be made as follows: $P = 3.14 \times (0.998)^2 \times 63{,}000/4 = 49{,}500$ lb.

It is impossible to state a general rule governing the number of significant figures that should be retained in actual engineering calculations. The decision in each case is one that requires good judgment and an understanding of all the factors involved. It may be stated, however, that calculations carried out to three significant figures are sufficiently accurate in the majority of the problems of engineering in which mechanics is used. This is about the average accuracy attainable with the ordinary 10-in. slide rule.

In general, intermediate results and final answers in the problems of this book will be calculated or given to the nearest third significant figure. Because of the many possible methods of arranging and making calculations, two persons working independently on the same problem, and attempting to use the same degree of accuracy, often find a slight disagreement in the last significant figure retained. Therefore, so far as the importance of the final result is concerned, the student need feel no anxiety if his answers and those given differ slightly in the third significant figure.

14. Smooth Surfaces. Friction is present whenever two bodies slide, or tend to slide, each on the other. In many engineering problems the frictional forces are relatively small and can be disregarded without serious effect on the results. In others, however, the frictional forces are relatively large and must be included in the solution if accurate results are desired.

In this book friction will be disregarded in a somewhat greater percentage of cases than would be justified in good engineering practice. Many of the problems have for their primary purpose the illustration of methods and principles to which the inclusion of frictional forces is not essential.

The intentions of the author regarding the omission of frictional forces will be revealed in various ways. In some cases it is stated that one, or both, of the bodies in contact is "smooth." This means that perfect smoothness is to be assumed, and that friction does not exist between the two bodies. The significant conclusion from such an assumption is that *the pressure between the bodies is at right angles to the common tangent plane at the point of contact.*

Rollers placed between two bodies are usually considered to produce the effect of smooth surfaces. In this case the inaccuracy involved in the assumption depends on the amount of rolling resistance present. A stationary body supported by wheels or casters may be treated as though it were resting on a smooth surface, if the rolling resistance and axle friction are relatively small.

Pulleys are sometimes referred to as being smooth. This means that friction on the axle of the pulley is to be disregarded. The tension in a light, flexible rope or cable on a smooth pulley may be assumed to be the same at all points of contact, if equilibrium exists.

Rolling resistance also will be disregarded in the majority of the problems in this book.

15. Vector Quantities. Many of the quantities used in mechanics are vector quantities. This means that in addition to magnitude they have inclination and sense and, in some cases, line of action. Force, velocity, acceleration, impulse, and momentum are examples of vector quantities occurring in mechanics.

The methods of dealing with vectors are essentially the same, regardless of the particular use to which the vector is being put. In this book, the methods for the handling of vector quantities will be developed in the next few chapters, but these methods will be developed in connection with a particular vector quantity (force) rather than with vectors in general. The student should understand, however, that the methods of composition, resolution, calculation of moments, etc., by means of which forces are manipulated, also apply in large measure to velocities, accelerations, and all other vector quantities.

16. The Measurement of Angles of Inclination. When a vector quantity is shown on a drawing, the inclination of its line of action should be indicated in the simplest manner possible. For example, in Fig. 1 the inclination of the force P is clearly shown by the angle of 30°

between the line of action and the x-axis. The sense of the force is revealed by the arrowhead.

It is desirable, however, to establish a system of indicating the inclination and sense of a vector quantity without the use of a drawing. In such cases the angle of inclination will be measured from the positive end of the reference axis to that end of the line of action toward which the vector points. Thus, the angle of inclination of the force in Fig. 1, with the x-axis, would be measured from the positive end of that axis to the end, A, of the line of action, and would be 210°.

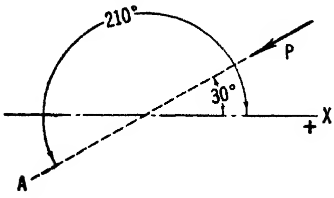

FIG. 1

If all the vectors in a given problem are coplanar it will be understood that the angles of inclination are measured " counterclockwise," starting from the axis of reference. In such cases angles of inclination may have values from 0° up to, but not including, 360°.

If the vectors are non-coplanar the term " counterclockwise " will not be used in this connection. Instead, angles of inclination will be measured by the shorter of the two possible routes. Thus, if the force in Fig. 1 were one of the forces of a non-coplanar system the angle of inclination with the x-axis would be considered to be 150°, instead of 210°. In this case it would also be necessary to give the angle with the y-axis (120°) and with the z-axis (90°) in order to establish definitely the inclination and sense of the force.

CHAPTER II

RESULTANTS OF COPLANAR FORCE SYSTEMS

17. Resultant of Two Concurrent Forces; The Parallelogram Law. *If vectors representing any two concurrent forces are drawn to scale at the point of concurrence, in such a manner that their senses agree with respect to that point, and if lines are added to form a parallelogram, that diagonal of the parallelogram which touches the point of concurrence will represent the resultant of the two forces. The sense of the resultant will agree with the senses of the two component forces, with respect to the point of concurrence.*

The foregoing statement is called the *parallelogram law.* It is a fundamental principle upon which much of the science of mechanics is based. No satisfactory mathematical proof can be offered, but the validity of the law can easily be verified by observation and experiment. This law applies to all vector quantities. The resultant of vector quantities is often called the *vector sum* of those quantities.

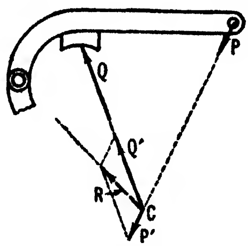

Fig. 2

In Fig. 2 let P and Q represent two of the forces acting on one of the arms of a double-block brake. The point of concurrence of P and Q is at C. Vectors P' and Q', counterparts of P and Q, are plotted at C. Their senses are in agreement, since both are directed away from the point of concurrence. Two lines are added, to form a parallelogram, and the diagonal through C is drawn. This diagonal, R, represents the resultant of P and Q. The sense of R is also away from the point of concurrence.

Great care must always be exercised to ensure the necessary agreement in the senses of the two vectors and their resultant, with respect to the point of concurrence. It is also highly important to remember that the line of action of the resultant passes through the point of concurrence of the two forces.

In the application of the parallelogram law either a graphic or an algebraic solution may be utilized. In the graphic solution the vectors are plotted accurately, to scale, and the desired results are taken from the drawing, by measurement. In order to avoid complexity in the original drawing the parallelogram is sometimes constructed in a separate figure. The resultant is finally shown on the original drawing, in its correct position.

In the algebraic solution a freehand sketch is sufficient, and the desired results are calculated by mathematical analysis.

The parallelogram law may be used, naturally, to resolve a given force into two concurrent components. It should be carefully observed that the point of concurrence for the two components must lie on the line of action of the given force, although it may be placed anywhere on that line. This point is often called the *point of resolution.*

The Triangle Law. In the application of the parallelogram law it is not necessary to draw the entire parallelogram. Instead, the triangle constituting one of the halves of the parallelogram may be drawn, and the other half may be omitted. The relations thus existing, when only the triangle is utilized, are sometimes referred to as the *triangle law.*

Illustrative Problems

16. Find the resultant of the following concurrent forces: 40 lb, $\theta_x = 160°$; 85 lb, $\theta_x = 225°$.

Graphic Solution. Plot the two forces to scale, as in Fig. 3, and complete the parallelogram. Make certain that the plotting is done in such a manner that the senses of the two vectors agree with respect to the point of concurrence, as specified in the statement of the parallelogram law. Draw the diagonal, from the point of concurrence, and indicate its sense in agreement with the senses of the two component forces. This vector, R, represents the resultant.

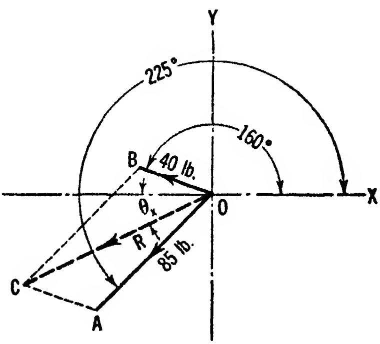

Fig. 3

Obtain the magnitude and angle of inclination of R from the figure, by measurement.

Algebraic Solution. Make a sketch similar to Fig. 3. A neat, freehand sketch is sufficient.

From the figure: $\angle AOB = 225° - 160° = 65°$. From the geometry of the parallelogram: $\angle OAC = \frac{1}{2}(360° - 2 \times 65°) = 115°$. Also, $AC = OB = 40$ lb. In the triangle AOC, the side AC, the side AO, and the included angle OAC are now known. From trigonometry, by the law of cosines,

$$R = \sqrt{(40)^2 + (85)^2 - 2 \times 40 \times 85 \times \cos 115°}$$

$$= \sqrt{1600 + 7225 - 6800(-0.423)} = 108 \text{ lb}$$

From trigonometry, by the law of sines,

$$\frac{\sin AOC}{\sin OAC} = \frac{AC}{OC} \qquad \sin AOC = \frac{0.906 \times 40}{108} = 0.336 \qquad \angle AOC = 19^\circ\,40'$$

From the figure,

$$\theta_x = (225^\circ - 180^\circ) - \angle AOC = 45^\circ - 19^\circ\,40' = 25^\circ\,20'$$

The resultant of the two given forces is, then, as follows: $R = 108$ lb, $\theta_x = 25^\circ\,20'$, its line of action passing through the point of concurrence, O, as shown in the figure.

If it were desired to indicate the inclination of R without reference to the figure the angle θ_x would be given the value $205^\circ\,20'$, in accordance with the convention adopted for such cases in Art. 16.

17. Given a force of 4.25 tons, $\theta_x = 70^\circ$; resolve into two concurrent components, at O, making angles of 120° and 345° with the x-axis.

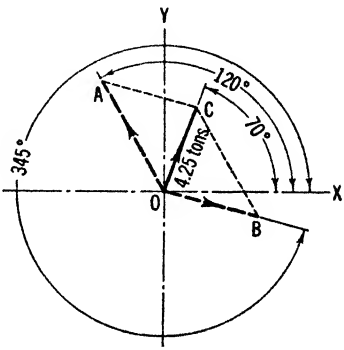

FIG. 4

Graphic Solution. Plot the given force to scale, as in Fig. 4. Construct a parallelogram having the given force for its diagonal, by drawing lines through each extremity of the vector, at angles with the x-axis equal to those specified for the two components. The sides OA and OB represent the two required components. The components are directed away from O, since the given force is so directed. Obtain their magnitudes from the drawing, by scaling.

Algebraic Solution. Make a sketch similar to Fig. 4. From the figure: $\angle AOC = 120^\circ - 70^\circ = 50^\circ$; $\angle OCA = \angle BOC = 70^\circ + (360^\circ - 345^\circ) = 85^\circ$; $\angle CAO = 180^\circ - (50^\circ + 85^\circ) = 45^\circ$. In the triangle OCA one side and all the angles are now known. From trigonometry, by the law of sines,

$$\frac{AC}{OC} = \frac{\sin AOC}{\sin CAO} \qquad AC = \frac{4.25 \times 0.766}{0.707} = 4.60 \text{ tons}$$

The required component, BO, is equal to AC. Also,

$$\frac{AO}{OC} = \frac{\sin OCA}{\sin CAO} \qquad AO = \frac{4.25 \times 0.996}{0.707} = 5.99 \text{ tons}$$

PROBLEMS

18. Find the resultant of the following forces: 75 lb, acting from the origin toward the point $(-3, +4)$; 68 lb, acting from the origin toward the point $(-8, -15)$. *Ans.* $R = 77$ lb; $\theta_x = 180^\circ$.

19. Find the resultant of the following forces, concurrent at the origin: 400 lb, $\theta_x = 27^\circ$; 220 lb, $\theta_x = 168^\circ$.

20. A force of 6.3 tons acts toward the right, parallel to the x-axis, through the point $(-6, +10)$; a force of 2.5 tons acts downward through the points $(-6, +10)$ and $(-13.5, 0)$. Calculate the resultant of the two forces, and locate the point at which its line of action intersects the x-axis. *Ans.* 5.2 tons; $\theta_x = 337° \ 20'$; 18 units to the right of O.

21. Find the equilibrant of the following concurrent forces: 1.2 tons, $\theta_x = 170°$; 2.8 tons, $\theta_x = 195°$.

22. Given a force of 75 lb, $\theta_x = 315°$; resolve into concurrent components, inclined at angles of 20° and 245° to the x-axis. *Ans.* 99.7 lb; 96.1 lb.

23. A force of 400 lb acts through O, at 215° with the x-axis. It is resolved, at O, into two concurrent components. One component is 160 lb, $\theta_x = 160°$. Find the other component.

24. Find the resultant of the following coplanar concurrent forces: 80 lb, $\theta_x = 160°$; 80 lb, $\theta_x = 280°$; 60 lb, $\theta_x = 310°$. First find the resultant of two of the forces; then compound this resultant with the third force.

18. Resultant of Two Concurrent Forces at Right Angles to Each Other. This constitutes a highly important special case in the application of the parallelogram law. The two concurrent forces to be compounded are at right angles to each other, and the parallelogram is a rectangle. *The majority of problems in the algebraic composition of force systems are solved wholly, or in part, by the use of this special case.*

Graphic Solution. In the graphic solution an accurate drawing is made, to scale, of the parallelogram of forces, and the desired results are obtained from the drawing, by measurement. If preferred, the triangle law (Art. 17) may be used. The parallelogram of forces is a rectangle, and the triangle of forces is a right triangle, in this special case.

Algebraic Solution. In the algebraic solution a neat, freehand sketch showing the vectors in their correct relationship is usually sufficient. The desired quantities are then calculated by mathematical methods.

Illustrative Problem

25. Find the resultant of the forces P and Q, acting on the bell crank in Fig. 5. Their magnitudes are 42 lb and 70 lb, respectively.

Graphic Solution. Produce the lines of action of P and Q to their point of concurrence, C. Plot the forces to scale, at C, in the manner indicated. Complete the rectangle, and draw the diagonal, R, through C. Obtain the magnitude and inclination of R, by measurement.

Algebraic Solution 1. From Fig. 5,

$$\theta = \text{arc tan}\,\frac{Q}{P} = \text{arc tan}\,\frac{70}{42} = \text{arc tan } 1.67 = 59° \ 05'$$

$$R = \frac{P}{\cos\theta} = \frac{42}{0.514} = 81.7 \text{ lb}$$

The line of action of R passes through the point of concurrence, C, as shown in the figure.

Algebraic Solution 2. From Fig. 5,

$$R = \sqrt{P^2 + Q^2} = \sqrt{(42)^2 + (70)^2}$$
$$= \sqrt{6664} = 81.6 \text{ lb}$$

$$\theta = \text{arc tan } \frac{Q}{P} = \text{arc tan } \frac{70}{42}$$
$$= \text{arc tan } 1.67 = 59° \ 05'$$

PROBLEMS

26. Find the resultant of the following concurrent forces: 35 lb, $\theta_x = 90°$; 84 lb, $\theta_x = 180°$. *Ans.* 91 lb; $\theta_x = 157° \ 20'$.

27. Find the resultant of the following concurrent forces: 4.20 tons, $\theta_x = 192°$; 3.15 tons, $\theta_x = 282°$.

28. A body weighing 60 lb is suspended from a single wire attached to an overhead support. A horizontal force of 25 lb, toward the left, is applied to the body, thus deflecting the wire from its vertical position. Find the resultant of this force and the weight of the body. What relation would exist between this resultant and the pull of the wire on the body? *Ans.* 65 lb, $\theta_x = 247° \ 25'$.

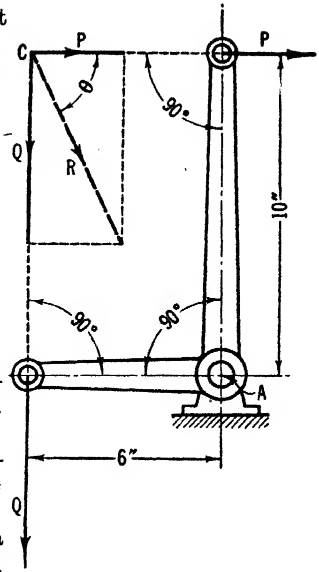

FIG. 5

29. Figure 6 represents a cross section of a masonry dam of the gravity type. P represents the total water pressure on a section of the dam having a width of 1 ft at right angles to the plane of the figure. W represents the weight of this section. $P = 18{,}000$ lb, and $W = 34{,}800$ lb. Find the resultant of the two forces, and locate the point at which its line of action intersects the base, AB, of the dam. Ordinarily this point should fall within the middle third of the length AB. Is such the case in this problem?

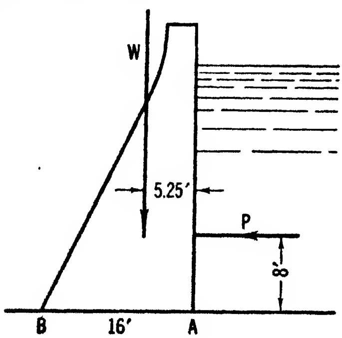

FIG. 6

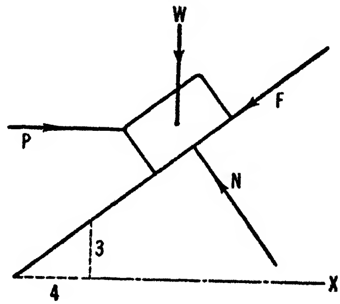

FIG. 7

30. Figure 7 represents a body resting on an inclined plane. The forces acting on the body are as follows: $W = 310$ lb, $P = 420$ lb, $F = 150$ lb, $N = 500$ lb. Find the resultant of W and P. *Ans.* 522 lb, $\theta_x = 323° \ 35'$.

31. Find the resultant of F and N, in Prob. 30, Fig. 7. Compare this resultant with that of W and P in Prob. 30. What is the significance of this comparison?

32. Prove that the following coplanar concurrent system is in equilibrium: 35 lb, $\theta_x = 20° 10'$; 21 lb, $\theta_x = 147°$; 28 lb, $\theta_x = 237°$.

33. Find the resultant of the following coplanar concurrent system: 45 lb, $\theta_x = 0°$; 60 lb, $\theta_x = 90°$; 180 lb, acting from the point of concurrence (0, 0) through the point (−4, +3). *Ans.* 195 lb, $\theta_x = 120° 30'$.

34. A certain force system consists of the following: 24 tons, $\theta_x = 0°$; 10 tons, $\theta_x = 90°$. Another system consists of: 16.5 tons, $\theta_x = 0°$; 12.5 tons, acting from (0, 0) through (+3, +4). All the forces are coplanar, and are concurrent at (0, 0). Prove that the two systems are equivalent.

19. Resolution of a Force into Two Concurrent Components at Right Angles to Each Other. This is another important special case of the application of the parallelogram law. The process is the reverse of that discussed in Art. 18.

A rectangle is constructed, in such a manner that the given force forms one of its diagonals. The position of the rectangle is governed by the requirements of the two components. Careful attention should be paid to the fact that the given force and its two components must be concurrent.

Component of a Force along a Line. When a force has been resolved into two rectangular components, one of which is parallel to a given line, the latter is referred to as the *component of the force "along" the given line.* This terminology is abbreviated, however, whenever possible. For example, the component of a force along a vertical line is referred to as the *vertical component,* along the x-axis as the x-*component,* etc.

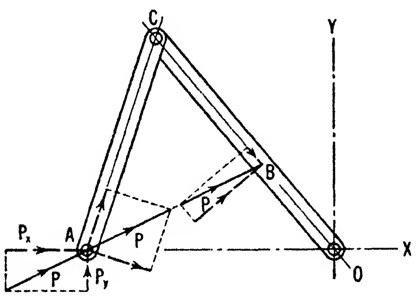

Fig. 8

Figure 8 represents a force, P, which has been resolved into rectangular components in three different ways. At A it has been resolved into its x- and y-components. It has also been resolved at A into components along, and at right angles to, the line AC. At B it has been resolved into components along, and at right angles to, the line CO.

The component of a force along any line is equal to the product of the magnitude of the force and the cosine of the angle that the force makes with the line.

The correctness of the foregoing statement is obvious from an examination of Fig. 8.

Illustrative Problem

35. In Fig. 9, $P = 1200$ lb and $Q = 2290$ lb. Resolve P, at the point B, into components along, and at right angles to, the line AB. Resolve Q into its x- and y-components, at A.

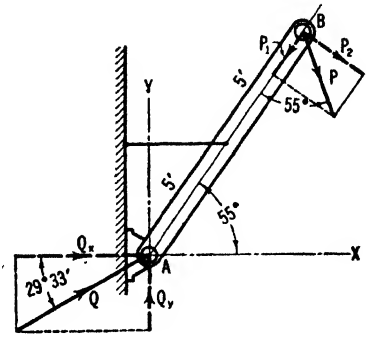

Fig. 9

Graphic Solution. Plot the given forces to scale and complete the rectangles in conformity with the requirements of the problem. Obtain the magnitudes of the desired components by scaling.

Algebraic Solution. Make a sketch similar to Fig. 9. From the figure,

$$P_1 = P \cos 55° = 1200 \times 0.574 = 689 \text{ lb}$$

$$P_2 = P \cos 35° = 1200 \times 0.819 = 983 \text{ lb}$$

$$Q_x = Q \cos 29° 33' = 2290 \times 0.870 = 1990 \text{ lb}$$

$$Q_y = Q \cos 60° 27' = 2290 \times 0.493 = 1130 \text{ lb}$$

In the calculation of P_2, sin 55° instead of cos 35° could have been used, and sin 29° 33′ instead of cos 60° 27′ in the calculation of Q_y.

It should be noticed that in each of the resolutions performed above the two components have been shown correctly on the figure in conformity with the rule that they must be concurrent with the given force at the point of resolution.

PROBLEMS

36. Resolve the following force into its x- and y-components, at the origin: 119 lb, $\theta_x = 151° 57'$. *Ans.* -105 lb; $+55.9$ lb.

37. A force of 8 tons acts from the origin through the point $(-3, -4)$. Resolve it into its x- and y-components, at the origin.

38. Given: $P = 283$ lb, $\theta_x = 125°$; calculate the component of P along a line at 72° with the x-axis. *Ans.* 170 lb.

39. A force of 650 lb acts from the point $(-24, -17)$ through the point $(-12, -12)$. Resolve it into its x- and y-components at the point $(-12, -12)$; at the point where it intersects the y-axis; at the point where it intersects the x-axis.

40. Figure 10 represents a body resting on an inclined plane. The forces acting on the body are as follows: $W = 520$ lb, $P = 260$ lb, $F = 40$ lb, $N = 580$ lb. Resolve W into rectangular components, one of which is parallel to the incline. *Ans.* 200 lb; 480 lb.

41. The force P in Prob. 40, Fig. 10, is horizontal. Resolve it into components parallel to, and at right angles to, the incline.

42. The body in Prob. 40, Fig. 10, is supposed to be in equilibrium. If so, the components of the forces acting on the body, along any chosen line, will balance. Ascertain whether this is true of components along the incline.

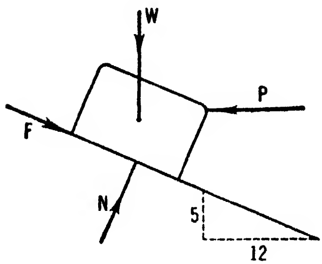

FIG. 10

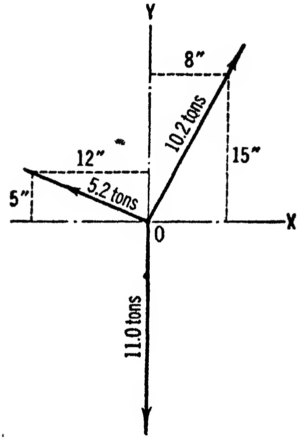

FIG. 11

43. Calculate the algebraic sum of the horizontal components of the forces acting on the body in Prob. 40, Fig. 10. Also calculate the algebraic sum of the vertical components. What is the significance of the results?

44. Given: 80 lb, $\theta_x = 155°$; 50 lb, $\theta_x = 195°$. Resolve each force into its x- and y-components, at the origin. Calculate the algebraic sum of the x-components. This will be the x-component of the resultant. Obtain the y-component in a similar manner. Find the resultant. As a check, find the resultant by applying the parallelogram law directly to the two original forces. *Ans.* 123 lb, $\theta_x = 170°\ 10'$.

45. Prove that the system of forces shown in Fig. 11 is in equilibrium.

46. Prove, by means of the parallelogram law, that a force can be resolved into two parallel components.

20. The Use of the Algebraic Sum in Mechanics. Algebraic summations are utilized extensively in the solution of problems in mechanics. For example, it is frequently desirable to write the algebraic sum of the components of the forces of a system along some established line, or axis. This is done by giving each component an algebraic sign in accordance with a previously adopted convention. All components having one sense are given plus signs, and those of the opposite sense are given minus signs. In any given problem this algebraic sum is known to equal a certain quantity, rendering it possible to form an equation containing one or more of the unknown quantities. Algebraic summations are utilized in a similar manner in connection with velocities, accelerations and other vector quantities, and in connection with moments.

The student should avoid using the algebraic sum in merely mechanical fashion. He should be aware of the significance of the process. For example, he should notice that *the algebraic sum of the components of a system of forces along an axis represents the amount by which the total force in one direction exceeds the total force in the opposite direction* or, in other words, that *it represents the amount of unbalanced force along the given axis.*

In this book the Greek letter Σ will be used to represent the phrase, "the algebraic sum of."

21. Conventions for Signs. In this book, vectors or components directed toward the right along the x-axis will be considered positive. Along the y-axis, upward will be taken as positive. Vectors along the z-axis will be considered positive if directed toward the reader. Moments, angular velocities, etc., will be considered positive if counterclockwise.

22. Interpretation of Signs of Results. If the *plus* sign is obtained when an unknown quantity is solved for, it may be concluded that the quantity was given the *correct* sign in the original equations. If the *minus* sign is obtained the conclusion is that the quantity was given the *wrong* sign. In the problems of mechanics the signs of the unknown quantities are often unknown, as well as the magnitudes. In order that complete equations may be formed it is necessary to assume the senses of the unknowns. The signs obtained with the results will then show whether the assumptions were correct. A negative sign means that the sense of the accompanying quantity was assumed incorrectly. The numerical value, however, is usually correct, provided that no other errors were made.

It is important, of course, to indicate all results finally in such a manner that no confusion can arise concerning them. Further details in the matter of algebraic signs will be brought out in the illustrative problems.

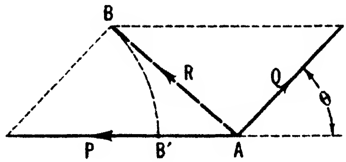

FIG. 12

23. Resultant of the Collinear System. The composition of two collinear forces may be thought of as a limiting case in the application of the parallelogram law. Let P and Q, in Fig. 12, represent any two concurrent forces, with their resultant, R. Let the angle θ be decreased, the magnitudes of P and Q remaining constant. The point B will approach the position B'. R will approach AB', and its sense will agree with that of the larger force, P.

Thus it is seen that in the limiting position, when the two forces are collinear, the resultant is equal to their algebraic sum. With this result as a basis it can easily be shown that *the resultant of any collinear system is equal to the algebraic sum of all the forces, and that its line of action coincides with that of the system.*

If the algebraic sum of the forces is zero, the system has no resultant and is, therefore, in equilibrium.

24. Resultant of the Coplanar Concurrent System; Graphic Solution. A *space diagram* is a scale drawing in which the lines of action of the forces of a system are shown in their correct space relationship.

A *force diagram,* or *force polygon,* is a scale drawing in which each force is correctly represented in magnitude, inclination, and sense, but in which no attempt is made to preserve the true linear spacing of the lines of action. Consequently, a force diagram reveals the magnitude, inclination, and sense of the resultant, but ordinarily does not give the correct position of the line of action.

Let Fig. 13 represent the space diagram for any coplanar concurrent system of forces. Each force has been designated by two lower-case letters.

Solution by Means of Parallelograms. Draw the space diagram for the system, Fig. 14. Compound *ab* and *bc* by means of the parallelogram law. Their resultant is R_1. Compound R_1 with the third force of the system, *cd*. Their resultant is R_2, which is also the resultant of *ab*, *bc*, and *cd*. Compound R_2 with the remaining force, *de*. Their resultant is R, which is also the resultant of the original system.

Notice that *the line of action of the resultant of a coplanar concurrent system necessarily passes through the point of concurrence.*

Solution by Means of the Force Polygon. An examination of Fig. 14 reveals the fact that several unnecessary lines were drawn. The drawing of the four lines AB, BC, CD, and DE, and of the resultant AE, would have been sufficient. The polygon $ABCDE$, formed by these vectors, is the force polygon for the system. It is shown alone, in Fig. 15.

In the actual solution of a problem the space diagram is drawn, and the force polygon is then constructed near, but usually not on, the former. The inclinations of the vectors in the force polygon are obtained by " taking parallels " from the space diagram.

The vectors in the force polygon are designated by upper-case letters corresponding to the lower-case letters used on the space diagram. These upper-case letters are placed at the extremities of the vectors, and are arranged so that their alphabetical sequence agrees with the sense of the force. For example, the sense of *CD* in Fig. 15 is from *C* toward *D*.

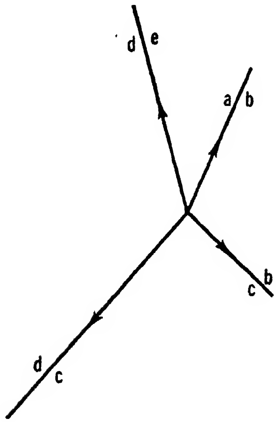

FIG. 13

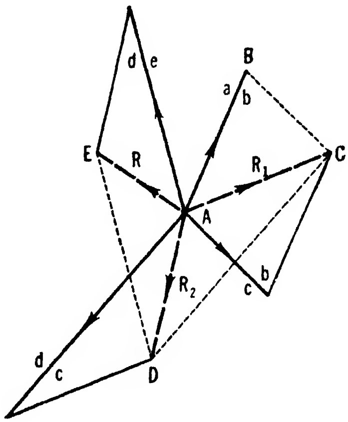

FIG. 14

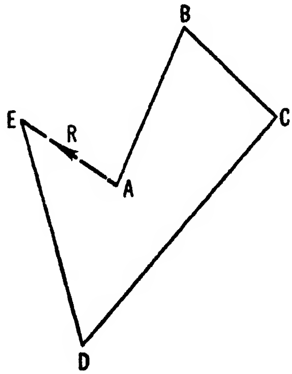

FIG. 15

The order in which the forces are lettered in the space diagram is immaterial, but the vectors in the force polygon must be confluent. In other words, all the vectors must lead toward the terminus of the polygon. It should be observed, also, that the resultant acts from the initial point of the polygon toward the terminal point. After the magnitude, inclination, and sense of the resultant have been ascertained from the force polygon, the resultant can be shown on the space diagram in its correct position, passing through the point of concurrence.

Solution by means of the force polygon is to be preferred when the system contains more than two forces.

Figure 14 shows that the resultant of a coplanar concurrent system, if a resultant exists, is a single force. It follows that, if the force polygon closes, the system has no resultant and is in equilibrium.

Illustrative Problem

47. Find the resultant of the five coplanar concurrent forces shown in Fig. 16.

Solution. Draw the space diagram carefully to some convenient scale, as in Fig. 16. Plot the vectors in a separate force polygon, as in Fig. 17, using a convenient scale for the magnitudes and obtaining the inclinations by

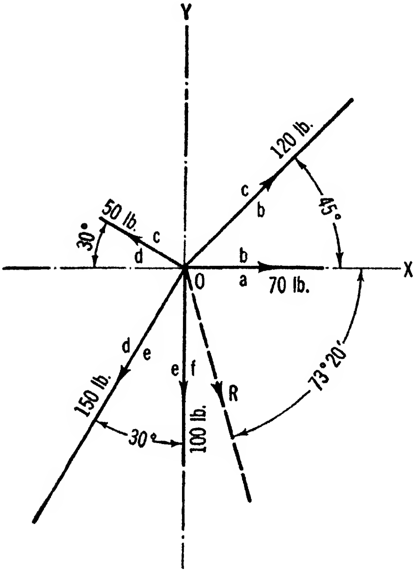

Fig. 16

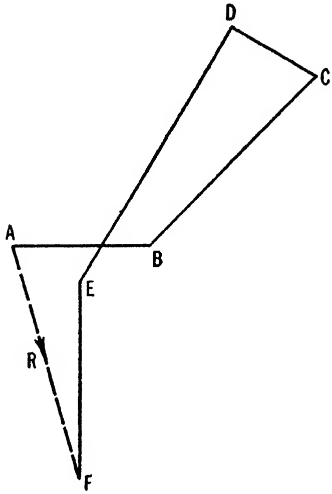

Fig. 17

taking parallels from the space diagram. In Fig. 16 the forces have been lettered in a counterclockwise sequence, starting with the 70-lb force. The vectors in the force polygon should be plotted in alphabetical sequence, as designated on the space diagram.

The order in which the forces are originally lettered on the space diagram is of importance only when it affects accuracy in drawing, or makes it possible to keep the force polygon within the compass of the paper.

The resultant is represented in magnitude, inclination, and sense by the vector AF, in Fig. 17, the sense being from A toward F. It has been shown finally in its true position on the space diagram, acting through the point of concurrence.

The results, obtained by measurement from Fig. 17, are as follows: $R =$ 125 lb, $\theta_x = 286° \ 40'$.

PROBLEMS

48. The following system is concurrent at the origin, and the forces act outward through the points indicated: 3.40 tons, $(-15, +8)$; 5.20 tons, $(-5, -12)$; 6.25 tons, $(+4, -3)$. Find the resultant.

49. Solve Prob. 47, handling the forces in a different order from that followed in the author's solution.

50. What relation would a vector drawn from the terminal point to and toward the initial point of the force polygon bear to the system?

51. The following coplanar system is concurrent at the origin: 160 lb, $\theta_x = 50°$; 120 lb, $\theta_x = 315°$; 80 lb, $\theta_x = 255°$; 135 lb, $\theta_x = 180°$. Find the resultant. Handle the forces in the order in which they are given above. Make a check solution, handling them in a different order.

52. The following system is concurrent at the origin, and the forces act outward through the points indicated: 850 lb, (+8, +15); 160 lb, (−10, 0); 300 lb, (−8, −6); 570 lb, (0, −12). Prove, by plotting the force polygon, that the system is in equilibrium.

25. Resultant of the Coplanar Concurrent System; Algebraic Solution. *The Principle of Components. The algebraic sum of the components of the forces of a coplanar concurrent system, along any axis in the plane of the forces, is equal to the component of the resultant along that axis.*

Proof. Let Fig. 13, Art. 24, represent any coplanar concurrent force system. In Fig. 18, *ABCDE* is the force polygon for the system, and *R* represents the resultant, in magnitude, inclination, and sense. Draw any axis, such as x–x, in the plane of the forces. The vectors at the top and bottom of the figure, drawn parallel to x–x, represent the x-components of the forces and of their resultant.

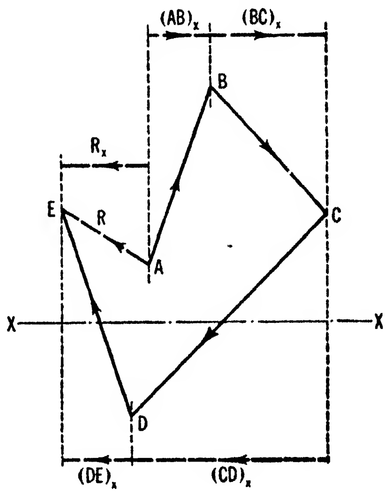

FIG. 18

From the figure it is obvious that R_x is numerically equal to the difference between the sum of the positive x-components of the forces and the sum of the negative x-components, and that its sense agrees with that of the larger sum. In other words, R_x is equal to the algebraic sum of the x-components of all the forces of the system. It is also obvious that a similar result would be obtained with any other axis in the plane of the system. Therefore, the principle of components, as stated above, is valid.

It was proved in Art. 24 that the line of action of the resultant of a coplanar concurrent system passes through the point of concurrence.

Since the resultant of a coplanar concurrent system is a single force, it follows from the principle of components that, if the component sums of the forces are zero along two non-parallel axes in the plane of the system, there is no resultant and the system is in equilibrium.

Application. In the algebraic solution of a problem two rectangular axes are established in some convenient manner, in the plane of the system. The component of the resultant along each of these axes is found, from the principle of components, by calculating the algebraic sum of the components of the forces along that axis.

The resultant is then found from its two components by means of the special case of the parallelogram law discussed in Art. 18. The magnitude and angle of inclination of the resultant are calculated, and finally the resultant is shown on the sketch in its correct position, passing through the point of concurrence.

Neat, freehand sketches are usually satisfactory in connection with algebraic solutions.

Illustrative Problems

53. Find the resultant of the system shown in Fig. 19.

Solution. The student should learn to give each component its correct algebraic sign by means of a careful inspection of the inclination and sense of the force itself, as shown in the sketch. By the principle of components:

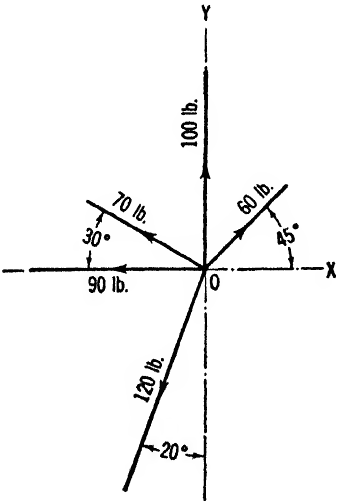

Fig. 19

$$\begin{aligned}R_x &= +60\cos 45° - 70\cos 30° \\ &\quad - 90 - 120\cos 70° \\ &= +42.4 - 60.6 - 90 - 41.0 \\ &= -149.2 \text{ lb} \\ R_y &= +60\cos 45° + 100 + 70\cos 60° \\ &\quad - 120\cos 20° \\ &= +42.4 + 100 + 35.0 - 113 \\ &= +64.4 \text{ lb}\end{aligned}$$

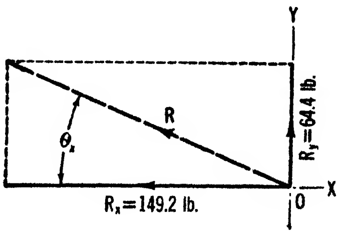

Fig. 20

R_x and R_y were tacitly assumed to be positive in the foregoing calculations. The minus sign was finally obtained for R_x and the plus sign for R_y. This shows that R_x was assumed incorrectly and that R_y was assumed correctly. Therefore, R_x is toward the left, and R_y is upward.

R_x and R_y, and the resultant, R, are shown in their correct relationship in Fig. 20. The angle θ_x will now be calculated. From Fig. 20,

$$\tan \theta_x = \frac{R_y}{R_x} = \frac{64.4}{149.2} = 0.432 \qquad \theta_x = 23^\circ\ 20'$$

Finally, from Fig. 20,

$$\sin \theta_x = \frac{R_y}{R} \qquad R = \frac{R_y}{\sin \theta_x} = \frac{64.4}{0.396} = 163 \text{ lb}$$

The line of action of the resultant passes through the point of concurrence, as was shown in Art. 24.

54. Find the resultant of the system shown in Fig. 21.

Solution. In Fig. 21 the slope of the line of action of each force is indicated, instead of the angle of inclination. The components of the forces can be calculated without the use of trigonometric tables, if preferred.

From Fig. 21, by the principle of components,

$$\begin{aligned} R_x &= -91 \times (\tfrac{5}{13}) - 204 \times (\tfrac{15}{17}) + 150 \times (\tfrac{3}{5}) \\ &= -35.0 - 180 + 90.0 \\ &= -125 \text{ lb} \\ R_y &= +132 + 91 \times (\tfrac{12}{13}) - 204 \times (\tfrac{8}{17}) - 150 \times (\tfrac{4}{5}) \\ &= +132 + 84.0 - 96.0 - 120 \\ &= 0 \end{aligned}$$

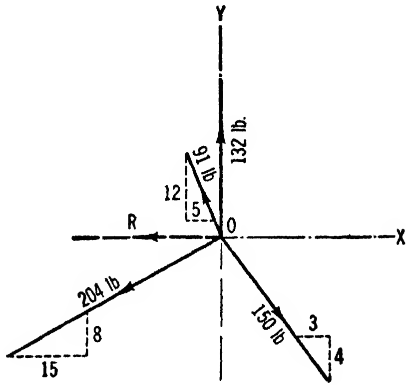

FIG. 21

In this problem, therefore, since the resultant has no y-component, it is identical with its x-component. The resultant is, then, a force of 125 lb, toward the left, coinciding with the x-axis.

PROBLEMS

55. Solve Prob. 48 by the algebraic method. *Ans.* 6.95 tons; $\theta_x = 270^\circ$.

56. Solve Prob. 51 by the algebraic method.

57. Prove algebraically that the system of forces in Prob. 52 is in equilibrium.

58. Figure 22 represents a board subjected to two concurrent force systems. Make an exact comparison of the effects that these systems would have on the state of motion of the board, if applied to it at different times.

59. Solve Prob. 47 by the algebraic method. *Ans.* 125 lb, $\theta_x = 286^\circ\ 55'$.

60. Figure 23 represents a body, B, weighing 860 lb, suspended by means of a system of wires, all of which lie in the same vertical plane. The four forces acting on the connection, at A, are in equilibrium; therefore the resultant of the pulls exerted by the three inclined wires balances the weight of the suspended body, B. Using this fact as a basis, calculate the unknown pulls, P and Q.

61. Figure 24 represents a body weighing 322 lb, being drawn along a horizontal plane by a constant force, P, acting as shown. The vector Q represents the force exerted on the body by the supporting plane. From the kinetics of a translating body it is known that the resultant, R, of the three forces acts as shown, and that it is equal to $(W/g)a$. Assuming that $R = 25$ lb, calculate the magnitudes of P and Q.

Ans. 96.1 lb; 280 lb.

62. In Fig. 25, find the resultant of the 2.4-ton, 5.1-ton, and

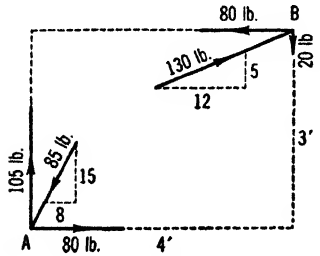

FIG. 22

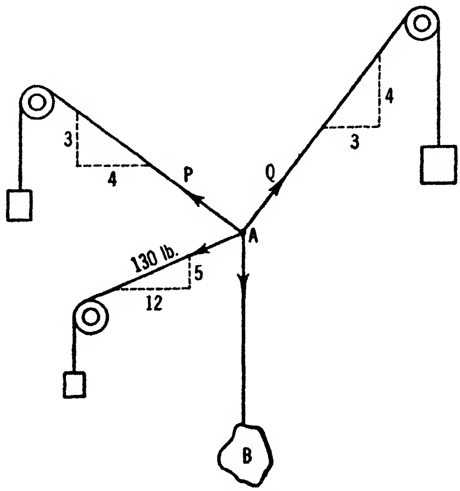

FIG. 23

8.0-ton forces. Also find the resultant of the 3.9-ton, 6.7-ton, and 4.5-ton forces. From the two resultants thus found, find the resultant of the entire system.

63. Figure 26 represents a board subjected to three coplanar forces concurrent at O. It is desired to replace these three forces by two forces whose lines of action will

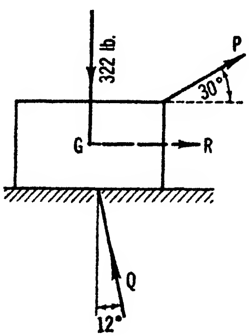

FIG. 24

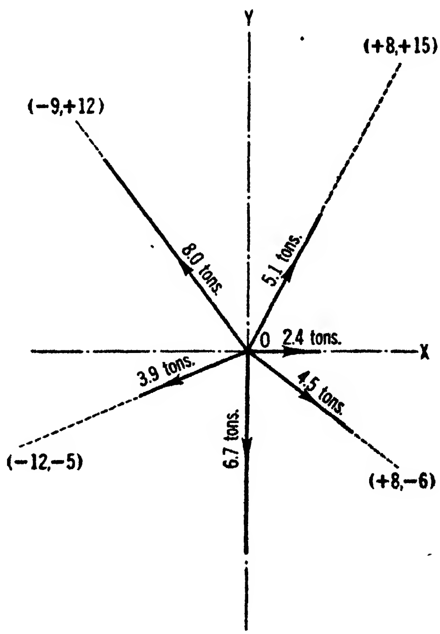

FIG. 25

coincide with OA and OB, without altering the state of motion of the body. Find the magnitudes and senses of the two forces needed. *Ans.* 25.5 lb, A to O; 25.5 lb, O to B.

64. The system of forces in Fig. 25 has no resultant. With this fact in mind, ascertain by inspection what the resultant of the system would be if the 8-ton force were doubled. What would the resultant be if the 8-ton force were reversed in sense? If it were omitted altogether?

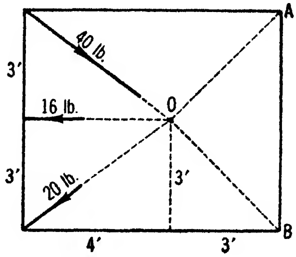

FIG. 26

26. Moment of a Force about a Line; Special Case.

The line about which a moment is taken is called the *axis of moments.* The discussion in the present article will be confined to the case in which the axis of moments is at right angles to some plane containing the line of action of the force. In this case the moment of a force about the axis is equal to the product of the magnitude of the force and the perpendicular distance between the axis and the line of action of the force. This distance is called the *moment-arm* or, in some cases, the *lever-arm.*

The term " torque " is used more or less interchangeably with " moment," although the former is more likely to be used in connection with the moments of the couples exerted between the rotating parts of machines.

Moment of a Force about a Point. In the employment of moments in connection with coplanar forces the axis of moments is usually placed at right angles to the plane of the forces. The point at which the axis intersects the plane of the forces is called the *center of moments.* In such cases the necessary moment-arms can be measured from the center of moments, and the axis of moments need not be shown on the sketch. Under such conditions a moment is usually referred to as a " moment about a point," the center of moments. It is important, however, to remember that moments are really associated with axes.

Sign of a Moment. A moment is usually given an algebraic sign. In this book counterclockwise moments will be considered positive, and clockwise moments negative.

In calculating a moment it is advisable to write the product of the force and the arm without regard to signs, and then to give the product its correct sign solely from the consideration of whether the tendency is to cause rotation about the axis of moments in a clockwise or counterclockwise direction. It is not advisable to attempt to ascertain the sign of a moment by giving separate signs to the force and to the arm.

In Fig. 27, let it be understood that axes OX and OY are in the plane of the paper, and that OZ is at right angles to that plane. The force,

P, is in the plane of the paper. The moment-arm of P with respect to the axis OZ is the line OA. Let the length of the arm be represented by a. The moment of P about OZ, and also about the point O, is: $M_z = M_0 = -P \times a$.

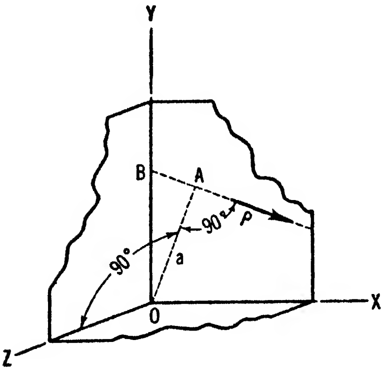

Fig. 27

Physical Significance of a Moment. The moment of a force is a measure of the tendency of the force to cause rotation about the axis of moments. For example, consider a wheel mounted on a shaft. If several forces are applied successively, and if there are no forces resisting rotation, each force will give the wheel a definite angular acceleration. These angular accelerations will be proportional to the moments of the forces about the axis of the shaft.

It is learned in kinetics that the net rotational tendency produced by a number of forces applied simultaneously to a given body is proportional, in general, to the algebraic sum of the respective moments about the given axis.

Units. The unit of moment is compound, and has no name peculiar to itself. It is stated by naming the unit of distance and the unit of force employed. The foot-pound, foot-ton, inch-pound, inch-ton, and kilogram-centimeter are some of the units commonly used.

Illustrative Problem

65. In Fig. 28, calculate the moment of the 3200-lb force about the point A; about C; about B.

Solution. The moment-arm of the given force with respect to A as a center of moments is the distance $AD = 8.94$ ft. The moment is clockwise, or negative.

$$M_A = -3200 \times 8.94 = -28{,}600 \text{ ft-lb}$$

The arm with respect to C is the distance $CD = 8.94$ ft. The moment is counterclockwise, or positive.

$$M_C = +3200 \times 8.94 = +28{,}600 \text{ ft-lb}$$

The arm with respect to B is the distance BG, which must be calculated. From the figure,

$$\beta = \text{arc tan} \frac{8}{24 - 8} = \text{arc tan } 0.5 = 26° \, 35'$$

In the right triangle ADF, $\overline{AF} = \dfrac{\overline{AD}}{\cos \beta} = \dfrac{8.94}{0.894} = 10.0$ ft. From the figure, $\overline{FB} = \overline{AB} - \overline{AF} = 24 - 10 = 14$ ft. Also, $\angle FBG = \beta = 26° 35'$. In the

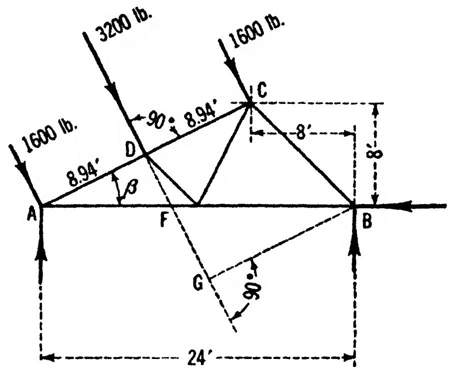

FIG. 28

right triangle BFG, $\overline{BG} = \overline{FB} \cos FBG = \overline{FB} \cos \beta = 14 \times 0.894 = 12.5$ ft. The moment of the force about B is counterclockwise, and is as follows:

$$M_B = +3200 \times 12.5 = +40{,}000 \text{ ft-lb}$$

PROBLEMS

66. In Fig. 27, the angle of inclination of the force P, with the x-axis, is $340° 20'$, and the magnitude is 260 lb. The distance OB is 18 in. Calculate the moment of P about the z-axis. *Ans.* -4410 in-lb.

67. A pulley 3 ft in diameter is mounted on a shaft. A belt encircles three-fifths of the circumference of the pulley. The tension in the belt on one side is 360 lb, and on the other side is 12 lb. Calculate the combined torque affecting the rotation of the pulley.

68. In Prob. 29, calculate the algebraic sum of the moments of P and W, about A. Find the resultant of P and W, and calculate its moment about A. Compare the two results. *Ans.* $+327{,}000$ ft-lb; $+327{,}000$ ft-lb.

69. In Fig. 28, calculate the moment of the upper 1600-lb force about A; about B; about C. *Ans.* $-28{,}600$ ft-lb; $+5700$ ft-lb; 0.

70. In Fig. 28, calculate the moment of the lower 1600-lb force about A; about B; about C. *Ans.* 0. $+34{,}300$ ft-lb; $+28{,}600$ ft-lb.

71. In Fig. 28, find the resultant of the three parallel forces, by inspection. Calculate the moment of this resultant about B. In Probs. 65, 69, and 70, the moments of these three forces about B were calculated separately. Calculate the moment-sum, and compare with the result of the present problem.

72. In Fig. 22, calculate the moment of each of the forces acting at A, about the point B. Calculate the combined turning effect of the three forces, about B.

73. In Fig. 22, calculate the moment of each of the forces acting at B, about the point A. Calculate the combined turning effect of the three forces, about A. *Ans.* $+240$ ft-lb; -160 ft-lb; -80 ft-lb; 0.

74. Let it be imagined that the board shown in Fig. 26 is attached to a shaft which passes through A and is at right angles to the plane of the figure. The three forces shown in the figure would tend to rotate the board. It is desired to apply a force along the line OB to counteract this rotative effect. Ascertain the necessary magnitude and sense of such a force. Could a force applied along OA accomplish the same purpose?

27. The Principle of Moments for Two Concurrent Forces; Varignon's Theorem. *The algebraic sum of the moments of two concurrent forces, about any point in the plane of the forces, is equal to the moment of the resultant about that point.*

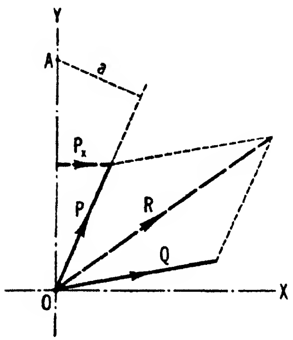

FIG. 29

Proof. In Fig. 29, let P and Q represent any two concurrent forces, and let R represent their resultant. Let A be used as the center of moments, representing any point in the plane of the forces.

Draw x- and y-axes, placing the origin at the point of concurrence of the forces, with the y-axis passing through A.

The distance a is the moment-arm of P with respect to A, and P_x is the x-component of P. Let the moments of P, Q, and R, about A, be represented by M_P, M_Q, and M_R. By definition,

$$M_P = P \times a \qquad [1]$$

From the figure, by similar triangles, $\dfrac{P}{P_x} = \dfrac{\overline{OA}}{a}$; therefore, $P = \dfrac{P_x(\overline{OA})}{a}$.

Substituting this value of P in Eq. 1,

$$M_P = P_x \times \overline{OA} \qquad [2]$$

Since P is any force passing through O, it follows that

$$M_Q = Q_x \times \overline{OA} \qquad [3]$$

and that

$$M_R = R_x \times \overline{OA} \qquad [4]$$

From [2] and [3],

$$M_P + M_Q = P_x \times \overline{OA} + Q_x \times \overline{OA} = (P_x + Q_x) \times \overline{OA} \qquad [5]$$

By the principle of components, Art. 25, $P_x + Q_x = R_x$. Substituting in [5],

$$M_P + M_Q = R_x \times \overline{OA} \qquad [6]$$

From [4], $R_x \times \overline{OA} = M_R$; therefore,

$$M_P + M_Q = M_R \qquad [7]$$

which verifies the principle of moments as stated above.

The principle of moments for two concurrent forces is called *Varignon's theorem.*

As will be seen, this principle not only can be used directly to good advantage in the solution of many problems, but also can be amplified to include all force systems, and can be used in the development of much of the theory of mechanics.

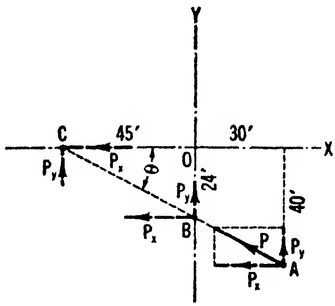

Fig. 30

Illustrative Problems

75. The force P in Fig. 30 has a magnitude of 13.6 tons. By means of the principle of moments, calculate the moment of P about O, by resolving the force into its x- and y-components at A; by resolving at B; by resolving at C.

Solution. From the figure, $\overline{AC} = \sqrt{(75)^2 + (40)^2} = 85$ ft; $\cos\theta = \frac{75}{85} = \frac{15}{17}$; $\sin\theta = \frac{40}{85} = \frac{8}{17}$; $P_x = P\cos\theta = 13.6 \times \frac{15}{17} = 12.0$ tons; $P_y = P\sin\theta = 13.6 \times \frac{8}{17} = 6.4$ tons.

Resolving at A,

$$M_0 = -P_x \times 40 + P_y \times 30 = -12 \times 40 + 6.4 \times 30 = -288 \text{ ft-tons}$$

Resolving at B,

$$M_0 = -P_x \times 24 + P_y \times 0 = -12 \times 24 = -288 \text{ ft-tons}$$

Resolving at C,

$$M_0 = P_x \times 0 - P_y \times 45 = -6.4 \times 45 = -288 \text{ ft-tons}$$

Thus it is seen that the principle of moments provides a convenient alternative method of calculating the moment of a force about a point. In many problems the force is resolved into components for purposes other than that of moment calculation. The components are, therefore, already available, and frequently their moment-arms can be taken directly from the figure. In many cases the calculations are further simplified by so choosing the point of resolution that the moment of one of the components is equal to zero, as was done at B and C in the foregoing example.

76. A force of 480 lb makes an angle of 330° with the x-axis. It is known that the moment of the force about the origin is equal to -1380 ft-lb, but the location of its line of action is unknown. Locate the line of action, by finding the point at which it intersects the x-axis.

Solution. The sense of the force is downward and toward the right. The moment about O is negative, or clockwise. It follows from these facts that the point at which the line of action intersects the x-axis will lie to the right of O.

Let R, in Fig. 31, represent the given force. The desired distance is x_R. Resolve R into its x- and y-components, at A.

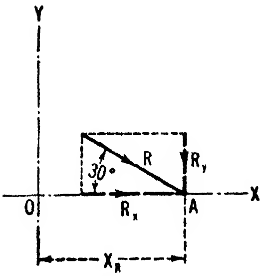

FIG. 31

By the principle of moments, the moment of R about O is equal to the moment-sum of R_x and R_y about that point. The moment of R_x is zero; therefore the value of R_x is not needed. $R_y = R \sin 30° = 480 \times 0.5 = 240$ lb. By the principle of moments,

$$-1380 = R_x \times 0 - R_y x_R \qquad -1380 = -240\, x_R$$
$$x_R = 5.75 \text{ ft}$$

PROBLEMS

77. In Prob. 25, Fig. 5, calculate the moment of the resultant R, about A, by the use of the principle of moments in connection with the components P and Q. What does the result reveal regarding the line of action of R?

78. In Prob. 35, Fig. 9, calculate the moment of P about A, using the principle of moments. Check by calculating the moment directly, without using the principle of moments. *Ans.* -9830 ft-lb.

79. In Fig. 28, calculate the moment of the 3200-lb force, about B, resolving it into components at D and utilizing the principle of moments.

80. In Prob. 29, Fig. 6, locate the point at which the line of action of the resultant intersects the base of the dam, without using the resultant itself in any manner.

81. In Prob. 76, locate the point at which R intersects the y-axis, using the moment of R about O, as given, and the principle of moments.

82. In Fig. 32, calculate the moment of the resultant of the two forces, about O, without calculating the resultant itself. *Ans.* $+144$ ft-lb.

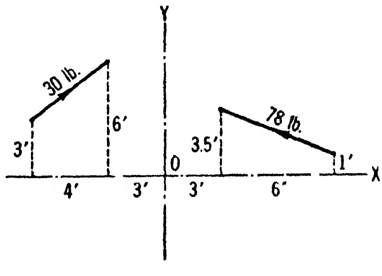

FIG. 32

83. In Fig. 32, locate the point at which the resultant of the two forces intersects the y-axis, without finding the line of action of the resultant itself.

84. In Fig. 32, prove that the line of action of the resultant of the two forces passes through the point $(+5', -2')$, without finding the resultant itself or the inclination of its line of action.

85. Prove the principle of moments for the case in which the center of moments, A, lies in the space between the lines of action of the two components, instead of in the position assigned to it in Fig. 29.

28. Couples; The Moment of a Couple. *Two forces that are equal in magnitude, parallel, but opposite in sense, constitute a couple.*

The algebraic sum of the moments of the two forces of a couple about any point in the plane of the forces is referred to, for the sake of brevity, as the *moment of the couple.*

The perpendicular distance between the lines of action of the two forces is called the *arm of the couple.*

Special Method of Calculating the Moment of a Couple. The moment of a couple is the same about all points in the plane of the couple, and is equal to the magnitude of either of the forces multiplied by the arm of the couple.

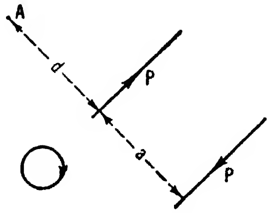

FIG. 33

Proof. Let the two forces in Fig. 33 represent any couple. Let the magnitude of each force be represented by P, and the arm of the couple by a. Let A represent any point in the plane of the couple. Let C represent the moment of the couple. By definition,

$$C = P \times d - P(a + d) = P \times d - P \times a - P \times d = -P \times a$$

which proves the principle stated above.

The foregoing special method of calculating the moment of a couple lessens the labor necessary in finding the moment-sum of a system of forces consisting wholly, or in part, of couples.

Sign of the Moment of a Couple. In calculating the moment of a couple by the special method, the sign of the moment is ascertained by inspection. If the rotational tendency of the couple is counterclockwise the sign of the moment is positive; if clockwise, the moment is negative.

An inspection of the couple in Fig. 33 shows that the rotational tendency is clockwise; the moment, therefore, is negative. This agrees with the sign obtained in the foregoing proof, in which the moments of the two forces were calculated separately.

It is learned in kinetics that the natural tendency of a couple is to cause a body to rotate about an axis through its center of gravity, at right angles to the plane of the couple.

ILLUSTRATIVE PROBLEM

86. Calculate the moment-sum of the forces in Fig. 34, about any point in the plane of the system.

Solution. The system is composed entirely of couples; therefore its moment-sum is the same for all points in its plane, and no particular center of moments need be specified if the problem is solved by the special method.

Let C_1, C_2, and C_3 represent the moments of the three couples, numbered in the order of the magnitudes of the forces. Let a_1, a_2, and a_3 represent the arms of the couples.

From the figure, $a_1 = 5 + 3 + 3 = 11$ ft; $a_2 = 2 + 4 = 6$ ft; $a_3 = \overline{BC} = 3 \cos \beta = 3 \times \frac{4}{5} = 2.4$ ft. By inspection, C_1 is clockwise, or negative; C_2 and C_3 are counterclockwise, or positive.

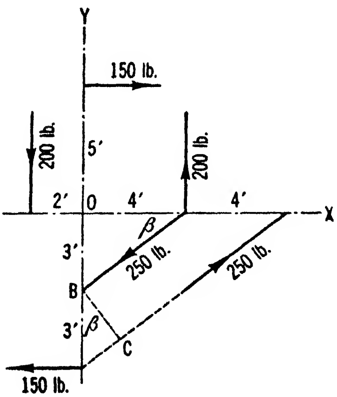

FIG. 34

$$C_1 = -150 \times a_1 = -150 \times 11 = -1650 \text{ ft-lb}$$

$$C_2 = +200 \times a_2 = +200 \times 6 = +1200 \text{ ft-lb}$$

$$C_3 = +250 \times a_3 = +250 \times 2.4 = +600 \text{ ft-lb}$$

$$\Sigma M = \Sigma C = -1650 + 1200 + 600 = +150 \text{ ft-lb}$$

Even in a case in which the system is not composed entirely of couples, the moment-sum of such forces as are members of couples can be conveniently calculated by the foregoing method.

PROBLEMS

87. Calculate the moment of each of the couples shown in Fig. 70, Art. 35. *Ans.* -1300 ft-lb; $+1500$ ft-lb; -1440 ft-lb.

88. In Fig. 34, calculate separately the moment of each of the six forces, about O, and then calculate the algebraic sum. Compare with the solution given in Prob. 86. Select a different center of moments and solve again.

89. In Fig. 6 two forces, $P = 18{,}000$ lb and $W = 34{,}800$ lb, are shown, acting on a dam. These forces are balanced by a frictional component, F, acting toward the right along AB, and a vertical, upward pressure, N. Since the dam is in equilibrium, $F = P$ and $N = W$. Thus, the entire system consists of two couples. The moments of the two couples also must balance; otherwise the dam would overturn. From these obvious facts calculate the distance from A to the line of action of N. *Ans.* 9.39 ft.

90. Explain why a system of forces composed entirely of couples has no tendency to cause a body, as a whole, to move definitely in any direction.

91. It was proved that the moment of the couple in Fig. 33, about the point A, is $-P \times a$. Show that the same result is obtained, both in sign and in amount, when the center of moments is placed in the space between the lines of action of the two forces. Also, place the center of moments in the space to the right of the couple, and prove.

92. Can a single force be equivalent to a couple? Explain. Can a coplanar concurrent force system have the same effect on the motion of a body as a couple, or system of couples? Explain.

93. In the simplification of some force systems it is found that only a couple remains. What is the relation of such a couple to the original system?

29. Equivalence of Couples in the Same Plane. *Any two couples in the same plane are equivalent if their moments are equal.*

It is understood that "equal" moments signifies agreement in sign as well as in magnitude.

Proof. Let the couples PP and QQ in Fig. 35 represent any two coplanar couples having equal moments. Therefore, $-Pa = -Qb$, or $Pa = Qb$.

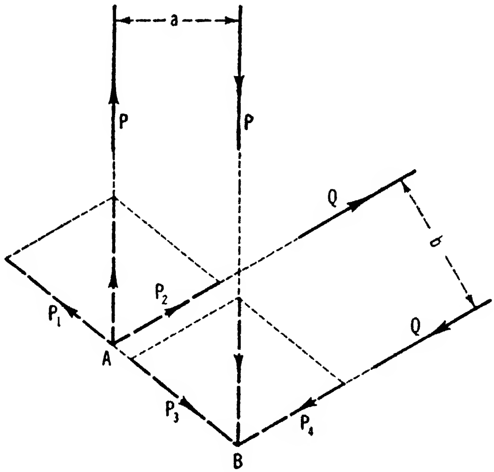

FIG. 35

Prolong the line of action of one force of each couple to the point of intersection, A. Prolong the other two lines of action to their intersection, B. At A, resolve P into a component P_1 along the line AB, and a component P_2 collinear with Q, as shown in the figure. At B, resolve the other force P into a component P_3 along the line AB, and a component P_4 collinear with the remaining force Q.

The two parallelograms have equal diagonals, and the corresponding sides are parallel; therefore they are equal in all respects. It follows that $P_1 = P_3$, and that $P_2 = P_4$. P_1 and P_3 are therefore collinear, equal, and opposite; their combined effect is nil, and they may be disregarded. P_2 and P_4 are parallel, equal, and opposite, but are not collinear. They constitute a couple, which is obviously equivalent to the original couple PP, and whose moment is $-P_2b$.

Applying the principle of moments, Art. 27, to the forces P_1 and P_2, and to their resultant, P, using B as the center of moments: $-P_1 \times 0 - P_2 \times b = -P \times a$; therefore, $P_2b = Pa$. By the original assumption, $Pa = Qb$. Therefore, $P_2b = Qb$, and $P_2 = Q$.

The foregoing results, together with the principle of transmissibility, Art. 12, prove that couples $P_2 P_4$ and QQ are equivalent. And since couples PP and QQ are both equivalent to couple $P_2 P_4$ they are equivalent to each other.

In the special case in which all the forces of the two given couples are parallel, the foregoing principle shows that each of the two couples is equivalent to a third couple whose forces are inclined to those of the given couples. Since each given couple is equivalent to this third couple, the two given couples are equivalent to each other.

PROBLEMS

94. Each force of a certain couple has a magnitude of 0.32 ton, and the arm of the couple is 5 ft. The moment is positive. It is desired to replace this couple with a new couple having an arm of 8 in., without changing the effect. Describe the new couple.

95. A certain shaft is subjected to a twisting moment, or torque, of 900 in-lb. A handwheel 15 in. in diameter is keyed to the shaft. A couple is applied to the rim of the handwheel, with its forces tangent to the wheel. Calculate the necessary magnitude of each force, to produce an effect equivalent to the given torque. *Ans.* 60 lb.

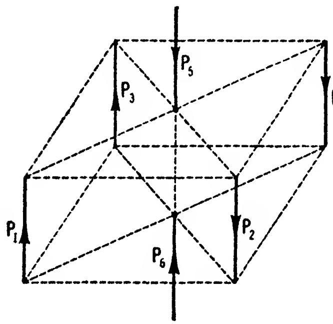

FIG. 36

30. Equivalence of Couples in Parallel Planes.

Any two couples in parallel planes are equivalent if their moments are equal.

It should be understood that the moments of two couples are not equal unless the direction of rotational effect is the same.

Proof. Figure 36 represents six forces, equal in magnitude and parallel, but not agreeing in sense, and not coplanar. P_1 and P_2 constitute a couple. P_3 and P_4 constitute a couple which is the counterpart of couple $P_1 P_2$, but which is in a parallel plane. P_5 and P_6 are collinear, equal, and opposite, and lie midway between P_1 and P_4, and also midway between P_2 and P_3.

The forces P_1, P_2, P_5, and P_6 may be thought of as a system consisting of two couples; namely, $P_1 P_5$ and $P_2 P_6$. The forces P_3, P_4, P_5, and P_6 may be thought of as a system consisting of two couples; namely, $P_4 P_6$ and $P_3 P_5$.

The couple $P_1 P_5$ is equivalent to the couple $P_4 P_6$, since the two couples are coplanar and have equal moments (Art. 29). The couple $P_2 P_6$ is equivalent to the couple $P_3 P_5$, for similar reasons. Therefore, the system P_1, P_2, P_5, P_6 is equivalent to the system P_3, P_4, P_5, P_6. The two forces, P_5 and P_6, appearing in each of the foregoing systems, are balanced, and contribute nothing to the effect of either system. These forces may be ignored, from which it follows that

couple $P_1\ P_2$ is equivalent to couple $P_3\ P_4$

This proves that two like couples in parallel planes are equivalent. By Art. 29, either of these couples is equivalent to any other couple in its own plane, having an equal moment.

PROBLEMS

96. A couple, each force of which is 200 lb, and whose arm is 18 in., is applied to one end of a shaft. Find the arm of a couple having 120-lb forces, applied in a parallel plane at the other end of the shaft, and having the same effect as that of the first couple. *Ans.* 30 in.

97. A certain shaft has a 15-in. handwheel at one end, and a 20-in. handwheel at the other end. Two 40-lb. forces are applied tangentially to the rim of the smaller wheel, forming a couple. Find the magnitudes of the forces of a couple applied in a similar manner to the larger wheel, and having the same effect as that of the first couple.

98. Under what circumstances would it be possible for the resultant of a coplanar force system to lie in a plane different from that containing the system? Explain.

31. Summary of Permissible Operations with a Couple. The principles of Arts. 29 and 30 show that any of the following changes can be made in a given couple without changing its effect, provided that the moment of the couple is not altered, either in magnitude or in sign:

(*a*) *The couple can be rotated through any angle in its own plane.*

(*b*) *It can be shifted to any other position in its own plane.*

(*c*) *It can be shifted to any parallel plane.*

(*d*) *The forces can be changed to any desired value, if the arm of the couple is changed in inverse proportion.*

(*e*) *The arm can be changed to any desired value, if the forces are changed in inverse proportion.*

It has been shown, and will be further shown in subsequent articles, that couples have various properties peculiar to themselves. The conception of the couple is not indispensable in mechanics, but the knowledge of these properties makes it possible to simplify many of the discussions and problems in which couples are involved. As a result of such facts the term "couple" is of common occurrence in engineering.

32. Resultant of the Coplanar Parallel System; Graphic Solution. Let Fig. 37 represent any coplanar parallel force system. It is the space diagram, and has been drawn to scale. The four forces have been designated as *ab*, *bc*, *cd*, and *de*.

Let the force *ab* be resolved into two concurrent components at any convenient point on its line of action, such as point 1. Figure 39 is the force diagram for the resolution, and for the sake of clearness the

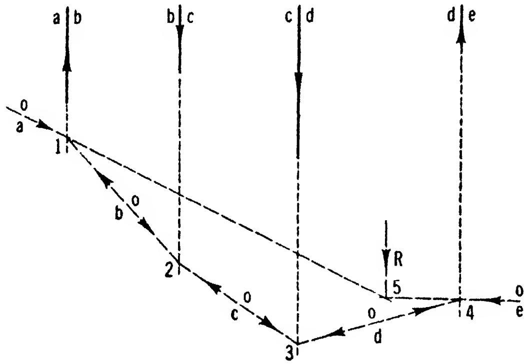

Fig. 37

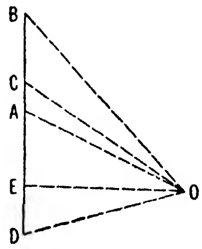

Fig. 38

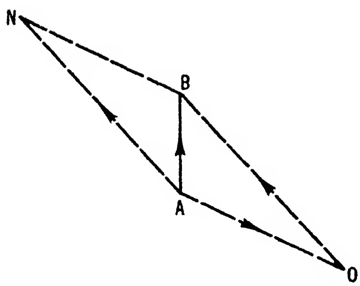

Fig. 39

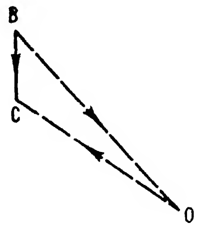

Fig. 40

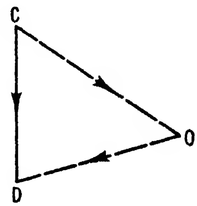

Fig. 41

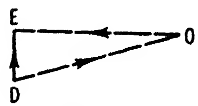

Fig. 42

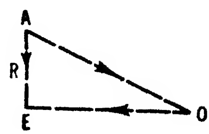

Fig. 43

entire parallelogram has been drawn, although the lines *AN* and *BN* are not necessary, and are omitted in the actual solution of a problem.

In Fig. 39 the force has been designated by upper-case letters placed at the extremities of the vector. *A* has been placed at the lower end and *B* at the upper end. This is done so that the alphabetical sequence of the letters indicates the sense of the force, which in the present case is upward. The letter *O* has been placed at the outer vertex of the force triangle. The lettering for the two components should be read "AO" and "OB," A and B remaining in alphabetical order. The sequence in which the letters are read thus indicates the senses of the components also.

AO and OB are now drawn on the space diagram as *ao* and *ob*, acting through point 1, the chosen point of resolution. It is not necessary to show the magnitudes of the components to scale on the space diagram. The magnitude of any force or component can be obtained from the force diagram. In the actual solution of a problem even the arrowheads showing senses are necessary only on the original forces in the space diagram, and are used more liberally in the present discussion to promote clearness of explanation.

Next, the second force, *bc*, is also resolved into two concurrent components, as shown in Fig. 40. In the resolution of *ab* the point O was chosen arbitrarily, but in the resolution of *bc* the point O is so placed that the component BO will be equal and parallel to the component OB of the first force, *ab*. It should be noticed, however, that BO and OB are opposite in sense.

The point 2, in Fig. 37, is the point of resolution for *bc*, and instead of being chosen arbitrarily it must be placed where the component *ob*, of the first force, intersects the line of action of *bc*. The purpose of this plan is to render the components *ob* and *bo* collinear on the space diagram, and since they are also equal in magnitude and opposite in sense they will balance, and will be of no effect in determining the final resultant.

The general scheme of analysis now can be understood. The remaining forces of the system are resolved in order, in accordance with the plan, one component of each force balancing one component of the preceding force. When the resolution is completed the entire original system can be considered to have been replaced by an equivalent system of components, all of which may be disregarded except one unbalanced component, *ao*, of the first force, and one unbalanced component, *oe*, of the last force. These two components are all that remain of the original system, and, since no step has been taken that would change the effect, they are equivalent to the original system.

The Case in Which the Resultant is a Single Force. In this case the two unbalanced components, *ao* and *oe*, will be concurrent, as in Fig. 37. Point 5 is their point of concurrence. Figure 43 shows *ao* and *oe* and their resultant, AE, which is also the resultant of the original system. AE is finally shown on the space diagram, Fig. 37, as R, and acts through point 5, the point of concurrence of its two components.

An examination of Figs. 39–43 shows that if a common scale is used the five force diagrams can be combined in such a manner that all points designated by the same letter will coincide, and an unnecessary duplication of lines will be avoided. A single diagram, Fig. 38, is drawn, and serves the purpose of the five separate diagrams.

In Fig. 38, $ABCDE$ is the force polygon. The general principles of its construction are precisely the same as in the coplanar concurrent system in Fig. 15, Art. 24. For a parallel system the force polygon obviously is a straight line.

The point O in Fig. 38 is called the pole, and the components OA, OB, OC, OD, and OE are called *rays*. The polygon 1, 2, 3, 4, 5, in Fig. 37, is called a *funicular polygon*, or an *equilibrium polygon*. The lines 1, 2; 2, 3; 3, 4; etc., in the funicular polygon are called *strings*.

The Case in Which the Resultant is a Couple. In this case the force polygon closes. If such had been the situation in the foregoing discussion the final point E in Fig. 38 would have coincided with the starting point A. The construction would have been carried out in the same manner, but the rays AO and OE would have been coincident, and the corresponding components ao and oe, in Fig. 37, would have been parallel instead of concurrent. They would thus have constituted a couple.

Since all couples in the same plane or in parallel planes are equivalent if their moments are equal, it is necessary, in describing such a resultant, merely to state that it is a couple in the plane of the system or in any plane parallel thereto, and to give its moment, accompanied by the correct sign.

The Case in Which There is no Resultant. If the force polygon closes, and if the components ao and oe are found to be collinear, then all the components balance, and the system has no resultant.

Illustrative Problems

99. Find the resultant of the coplanar parallel system shown in Fig. 44.

Solution. Draw the space diagram to some convenient scale, as in Fig. 44. Plot the force diagram, $ABCDEF$, as in Fig. 45. The diagram begins at A and ends at F; therefore AF represents the resultant, in magnitude, inclination, and sense. The magnitude is found, by scaling, to be 100 lb. Obviously the resultant is parallel to the forces of the system. The sense is from A toward F, or upward.

Choose a pole, O, and draw the rays, AO, BO, etc. Take parallels from the rays and construct the funicular polygon, ao, bo, co, etc. The components ao and of are the only ones that are not neutralized. They intersect at G; therefore the resultant passes through that point.

Finally, show the resultant in its correct position, acting through G. In the present case the distance from the line of action of R to the line of action of *ab* is found, by scaling, to be 3.6 ft. The exact value, calculated by the algebraic method, is 3.5 ft.

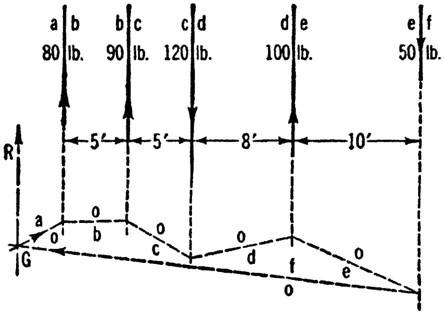

FIG. 44

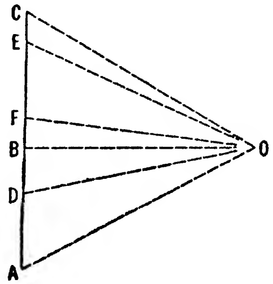

FIG. 45

100. Find the resultant of the coplanar parallel system shown in Fig. 46.

Solution. Draw the space diagram to a suitable scale, as in Fig. 46. Plot the force diagram, *ABCDEF*, as in Fig. 47. In the present problem the force diagram closes, the final point *F* coinciding with the starting point *A*. This means that the resultant, if one exists, is a couple.

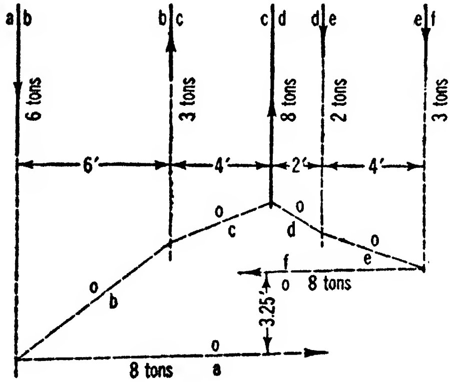

FIG. 46

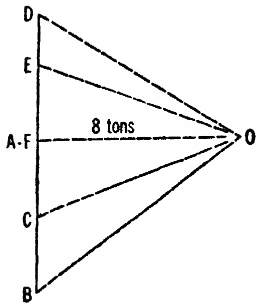

FIG. 47

Choose a pole, draw the rays, and construct the funicular polygon, following the same procedure as in the case in which the resultant is a single force. The unbalanced components, *ao* and *of*, Fig. 46, are found to be parallel, of course, since they are drawn parallel to the same ray, *AO* (or *OF*). In the present problem *ao* and *of* do not coincide, on the space diagram; therefore they constitute a couple, which is the resultant of the system. The arm of the couple is found, by scaling, to be 3.25 ft.

The magnitude of each of the forces of the resultant couple is represented by the ray AO, in Fig. 47, and is found, by scaling, to be 8.0 tons. Therefore, the moment of the resultant couple is equal to $8 \times 3.25 = 26.0$ ft-tons. The senses of AO and OF, in Fig. 47, and the relative positions of their lines of action, ao and of, in Fig. 46, show that the moment of the resultant couple is counterclockwise, or positive.

In some problems of the type in which the force polygon closes, the components ao and of would actually coincide on the space diagram. They would then be collinear, as well as equal and opposite, and would balance. In such a case the system has no resultant and is, therefore, in equilibrium.

PROBLEMS

101. Find the resultant of the system shown in Fig. 48, by the graphic method.

102. In Fig. 48, change the magnitude of the 30-lb force to 150 lb, and then find the resultant of the system. Make a second solution, placing the pole in a different position relative to the force polygon.

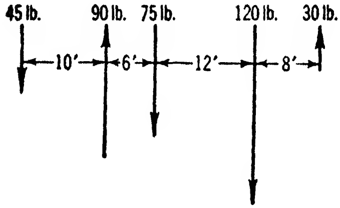

FIG. 48

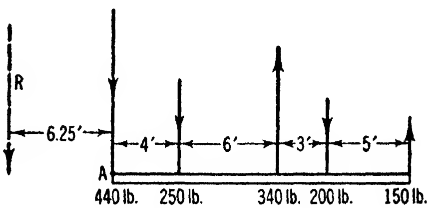

FIG. 49

103. In Fig. 48, insert an additional parallel force of 120 lb, acting upward, at a point 5.5 ft to the right of the 75-lb force. Find the resultant of the six forces.

104. Find the resultant of the system shown in Fig. 49, by the graphic method.

105. A certain system of coplanar, vertical forces consists of the following (from left to right): -20 lb; -50 lb; $+90$ lb; $+40$ lb; -60 lb. The distances to their lines of action, measured from the 20-lb force, are: 6 ft; 10 ft; 15 ft; 20 ft. Prove that the system is in equilibrium, by constructing the force polygon and the funicular polygon.

33. Resultant of the Coplanar Parallel System; Algebraic Solution.

The Principle of Components. The algebraic sum of the forces of a coplanar parallel system is equal to the resultant of the system if the resultant is a single force, and is equal to zero if the resultant is a couple.

Proof. Figure 37, Art. 32, represents any coplanar parallel system whose resultant is a single force. In Fig. 38 the forces have been plotted consecutively and lettered in accordance with their senses. It was shown in Art. 32 that AE, Fig. 38, represents the resultant, in magnitude and sense. A study of the relationship of the four vectors, AB, BC, CD, and DE, and their resultant, AE, readily leads to the conclusion that the resultant is equal to the algebraic sum of the forces.

In the case in which the resultant is a couple, the force polygon closes, as was explained in Art. 32. It is clear that the force polygon will close only when the sum of the positive forces is numerically equal to the sum of the negative forces or, in other words, only when the algebraic sum of all the forces is equal to zero.

It could be shown that the principle of components applies in its more general form, as in Art. 25, but the special form stated above is simpler and is adequate for all purposes.

The Principle of Moments. The algebraic sum of the moments of the forces of a coplanar parallel system, about any point in the plane of the forces, is equal to the moment of the resultant about that point.

Proof. The principle will be proved first for the case in which the resultant is a single force. Let any point in Fig. 37, Art. 32, be chosen as a center of moments. Applying the special form of the principle of moments, as proved in Art. 27, it can be seen that

the moment of *ab* = the moment-sum of its components *ao* and *ob*
" " " *bc* = " " " " " " *bo* and *oc*
" " " *cd* = " " " " " " *co* and *od*
" " " *de* = " " " " " " *do* and *oe*

Equating the sum of the left-hand members of these equations to the sum of the right-hand members, it follows that

the moment-sum of the original forces =
the moment-sum of all the components

But the components *ob* and *bo* are collinear, equal, and opposite, and their moment-sum is equal to zero. The same is true of *oc* and *co*, and of *od* and *do*. Therefore,

the moment-sum of all the components = the moment-sum of *ao* and *oe*

And so

the moment-sum of the original forces = the moment-sum of *ao* and *oe*

Again by Art. 27,

the moment-sum of *ao* and *oe* = the moment of their resultant, *ae*

And finally,

the moment-sum of the original forces = the moment of their resultant

When the resultant is a couple the phrase "moment of the resultant about that point" is interpreted as meaning the moment-sum of the two forces of the resultant couple or, simply, the moment of the couple as defined in Art. 28. Formal proof would closely resemble the proof given above for the case in which the resultant is a single force.

Application. The first step in the actual solution of a problem is the calculation of the algebraic sum of the forces of the system. If this sum is not equal to zero the resultant is a single force, parallel to the forces of the system. The magnitude of the resultant is equal to the magnitude of this sum, and the sense of the resultant is shown by the sign obtained.

The principle of moments is then used to locate the line of action of the resultant. Any convenient center of moments is selected, and the moment-sum of the forces is calculated. The moment of the resultant about the chosen center is equal to this moment-sum. The magnitude and sense of the resultant are already known. An equation can then be formed in which the moment-arm of the resultant will be the only unknown quantity. The moment-arm serves to locate the line of action of the resultant, since it is the distance to that line of action from a known point, the center of moments.

Care must be exercised to avoid placing the resultant on the wrong side of the center of moments. The sense of the resultant is known, also the sign of its moment. The matter can be decided on the basis of these facts alone.

If the algebraic sum of the forces of the system is found to be equal to zero, the resultant, if one exists, is a couple. A convenient center of moments is then chosen, and the moment-sum of the forces is calculated. This sum is equal to the moment of the resultant couple. It is usually considered a sufficient solution to state the moment, mentioning the fact that the resultant can be any couple having such a moment, situated in the plane of the system or in any plane parallel thereto.

If both the algebraic sum of the forces and their moment-sum are found to be zero the system will have no resultant.

Illustrative Problems

106. Find the resultant of the five coplanar parallel forces in Fig. 49.

Solution. Calculate the algebraic sum of the five forces.

$$R = -440 - 250 + 340 - 200 + 150 = -400 \text{ lb}$$

The resultant is now known to be a single force of 400 lb, parallel to the forces of the system, and acting downward. The location of its line of action is as yet unknown.

Calculate the moment-sum of the forces, using the point A as the center of moments.

$$\begin{aligned} \Sigma M_A &= -250 \times 4 + 340 \times 10 - 200 \times 13 + 150 \times 18 \\ &= -1000 + 3400 - 2600 + 2700 = +2500 \text{ ft-lb} \end{aligned}$$

By the principle of moments, the foregoing value is also the moment of the resultant about A. The magnitude of the resultant is 400 lb. Its moment about A is $+2500$ ft-lb. Therefore, the distance of its line of action from A is equal to $2500/400 = 6.25$ ft.

There remains only the decision as to whether the resultant lies to the right or to the left of A. The resultant acts downward and has a positive, or counterclockwise, moment about A. These two circumstances, considered together, clearly show that the resultant lies to the left of A.

Finally, show the resultant on the sketch, in its correct position, as in Fig. 49.

107. Alter the system of forces in Fig. 49 by reversing the 200-lb force in sense. Find the resultant.

Solution. Calculate the algebraic sum of the five forces.

$$R = -440 - 250 + 340 + 200 + 150 = 0$$

The foregoing result does not necessarily mean that the system has no resultant, but it does mean that the resultant, if one exists, is a couple.

Calculate the moment-sum of the forces, using any convenient point, such as A, as the center of moments.

$$\Sigma M_A = -250 \times 4 + 340 \times 10 + 200 \times 13 + 150 \times 18$$
$$= -1000 + 3400 + 2600 + 2700 = +7700 \text{ ft-lb}$$

Since the moment-sum is not equal to zero, the resultant is a couple. By the principle of moments, the moment-sum obtained above is also the moment of the resultant couple.

Summarizing, the resultant of the given system is any couple in the plane of the system, or in any plane parallel thereto, having a counterclockwise moment of 7700 ft-lb.

PROBLEMS

108. Find the resultant of the six forces shown in Fig. 50. *Ans.* -30 lb; 4 ft to the left of the 50-lb force.

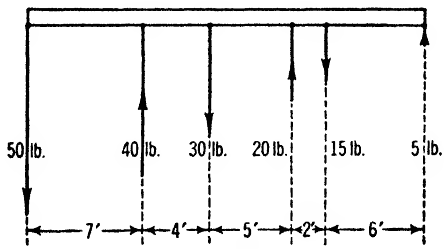

FIG. 50

109. In Fig. 50, increase the magnitude of the 40-lb force to 70 lb, and find the resultant of the altered system.

110. In Fig. 50, increase the magnitude of the 5-lb force to 34 lb, leaving the other forces as they are shown, and find the resultant of the system. *Ans.* -1 lb; 816 ft to the left of the 50-lb force.

111. In engineering problems it is often desired to calculate the moment-sum of the forces of a system about an axis, or point. In some cases it is easier to find the resultant of the forces and calculate its moment than it is to calculate the moments of the forces individually. Calculate the moment-sum of the five dead loads in Fig. 169, Art. 67, about the point A, by this method. Check the result by calculating the moments of the forces separately.

112. In Fig. 46, transfer the line of action of the force, *de*, to a position 9 ft to the right of the force, *ef*. Let the other forces remain as they are. Find the resultant.

113. Solve Prob. 99, Art. 32, by the algebraic method. *Ans.* $+100$ lb; 3.5 ft to the left of the 80-lb force.

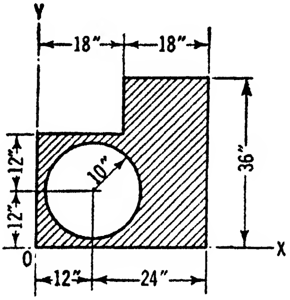

Fig. 51

114. Solve Prob. 100, Art. 32, by the algebraic method.

115. Solve Prob. 101, Art. 32, by the algebraic method. *Ans.* -120 lb; 21.5 ft to the right of the 45-lb force.

116. Calculations for locating the center of gravity are similar to those for locating the line of action of the resultant of a system of parallel forces. Locate the center of gravity of the shaded area in Fig. 51, by the following process:

Calculate the area of a full square, 36×36 in. Represent this area by a vector, downward in sense, parallel to the y-axis and passing through the center of the square. In a similar manner establish upward vectors to represent the areas of the missing parts, namely, the rectangle and the circle. Locate the resultant of the three parallel vectors. Now rotate the vectors until they are parallel to the x-axis, and again locate the resultant. The center of gravity of the shaded area will be at the intersection of the two resultants. Give its coordinates. *Ans.* $(+23.0, +17.1)$.

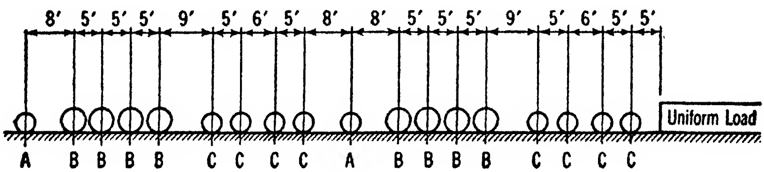

Fig. 52

117. Figure 52 represents Cooper's conventional system of loads, often used in the design of railway bridges. It represents two consolidation locomotives, followed by a uniform train loading. In the present problem the loads marked A are 30,000 lb each; loads B are 60,000 lb each; loads C are 39,000 lb each. The train load is 6000 lb per linear foot of track. Find the resultant of the loads caused by the two locomotives and the first 41 ft of the train. *Ans.* -549 tons; 68.9 ft to the right of the foremost wheel.

118. The resultant of two parallel forces which have the same sense lies between them, and is so situated that the distances from its line of action to the lines of action of the two forces are inversely proportional to the magnitudes of the forces. Make a general algebraic proof of this statement.

119. The resultant of two parallel forces which are opposite in sense lies in such a position that the larger force is between the resultant and the smaller force, and that the distances from the line of action of the resultant to the lines of action of the two forces are inversely proportional to the magnitudes of the forces. Make a general algebraic proof of this statement.

34. Resultant of the General Coplanar System; Graphic Solution. Any coplanar system in which not all the forces are concurrent, or in which not all are parallel, may be classified as a general coplanar system.

The process of finding the resultant of a general coplanar system by the graphic method is essentially the same as for the coplanar parallel system (Art. 32), and the full details of the solution will not be repeated here. The chief difference is in the general appearance of the various diagrams. For example, the force polygon for the general coplanar system is not a straight line, but more nearly resembles that of the coplanar concurrent system as shown in Figs. 15 and 17.

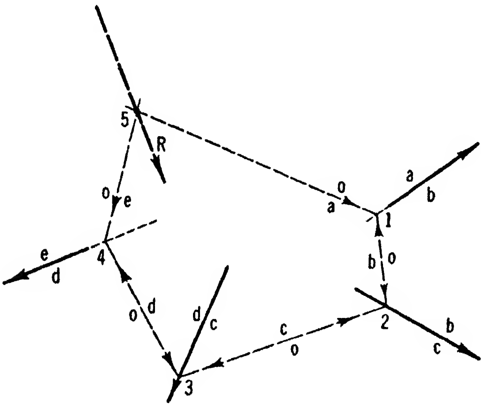

FIG. 53

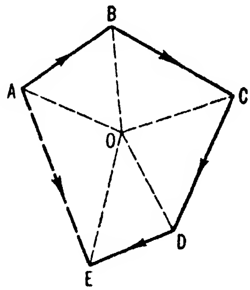

FIG. 54

The Case in Which the Resultant is a Single Force. Figures 53 and 54 represent such a case. The magnitude, inclination, and sense of the resultant are ascertained by means of the force polygon, *ABCDE*, in Fig. 54. The resultant is represented by *AE*. A pole, *O*, is selected, rays are drawn, and the funicular polygon 1, 2, 3, 4, 5 is constructed. The line of action of the resultant passes through the point 5, at the intersection of the two unbalanced components *ao* and *oe*.

The Case in Which the Resultant is a Couple. In this case the force polygon is constructed in the usual manner, but is found to close. The funicular polygon is then constructed. The two unbalanced components are parallel. The perpendicular distance between their lines of action is the arm of the resultant couple, and that ray in the force diagram which runs to the closing point of the force polygon represents the magnitude of each of the forces of the couple. The moment of the resultant couple is calculated from these values.

The Case in Which There is No Resultant. If both the force polygon and the funicular polygon close, the system has no resultant, and is in equilibrium.

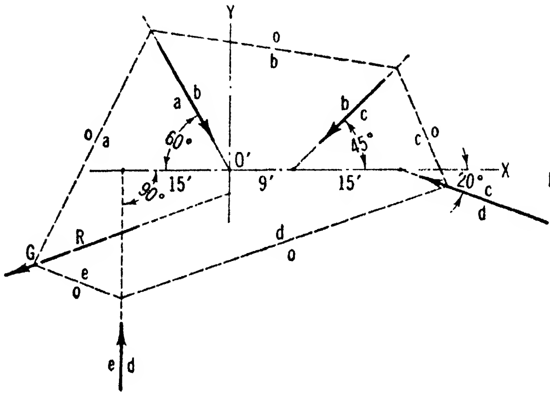

FIG. 55

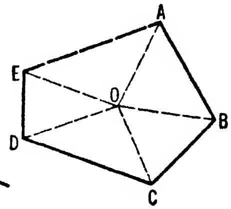

FIG. 56

ILLUSTRATIVE PROBLEMS

120. In Fig. 55, ab = 7.5 tons, bc = 6.0 tons, cd = 9.0 tons, and de = 4.5 tons. Find the resultant of the system, by the graphic method.

Solution. Draw the space diagram to some convenient scale, as in Fig. 55. Plot the force polygon, $ABCDE$, as in Fig. 56, to a suitable scale. AE represents the resultant, in magnitude, inclination, and sense. The magnitude is found, by scaling, to be 9.52 tons. The angle of inclination with the x-axis is found, by the tangent method, to be 19° 40′. The sense is from A toward E.

Choose a pole, O, and draw the rays, ao, bo, co, etc. The components ao and oe are the only ones that are not neutralized. They intersect at G; therefore the resultant passes through that point.

Finally, show the resultant in its correct position on the space diagram, acting through G. In the present case the line of action of the resultant intersects the y-axis at a point found, by scaling, to be 3.54 ft below the origin, O'.

121. In Fig. 57, ab = 30 lb, bc = 75 lb, cd = 85 lb, and de = 130 lb. Find the resultant of the system, by the graphic method.

Solution. Draw the space diagram to some convenient scale, as in Fig. 57. Plot the force polygon, $ABCDE$, to a suitable scale, as in Fig. 58. In the present case it is found that the force polygon closes, point E coinciding with point A. This shows that the resultant, if one exists, is a couple.

Choose a pole, O, draw the rays, AO, BO, etc., and by taking parallels therefrom construct the funicular polygon, just as in the case in which the resultant is a single force. In the present case, however, the two unbalanced components, ao and oe, are parallel, since both are drawn parallel to the ray

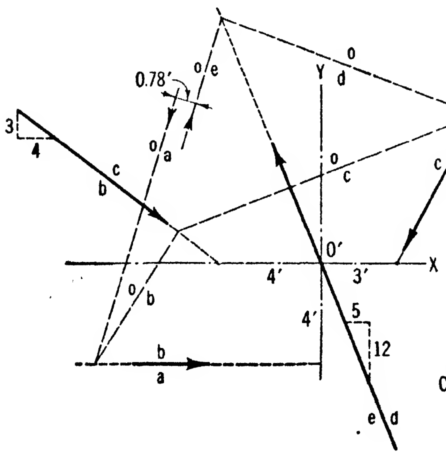

Fig. 57

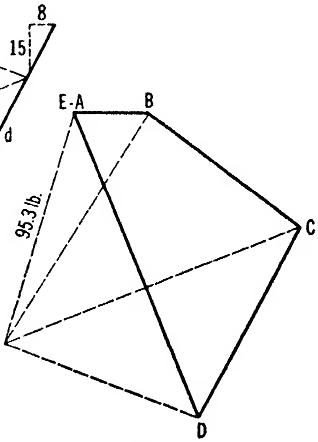

Fig. 58

OA. On the space diagram it is found that ao and oe do not coincide; therefore they constitute a resultant couple. The perpendicular distance between them is the arm of the resultant couple, and is found, by scaling, to be 0.78 ft. Each force of the couple is equal to AO, in Fig. 58, and is equal to 95.3 lb. The moment of the resultant couple is equal to $95.3 \times 0.78 = 74.3$ ft-lb. The senses of ao and oe, and their position in Fig. 57, show that the couple is counterclockwise. If the lines of action of ao and oe had coincided the system would have had no resultant.

The results are summarized by stating that the resultant of the system is any couple, in the plane of the system or in any plane parallel thereto, having a counterclockwise moment of 74.3 ft-lb.

PROBLEMS

122. Find the resultant of the five coplanar forces shown in Fig. 59, by the graphic method.

123. Find the resultant of the following coplanar force system: 150 lb $(-5', 0)$, $(-5', +5')$; 500 lb $(+3', 0)$, $(0, -4')$; 650 lb $(+12', 0)$, $(0, +5')$; 900 lb $(0, +2')$, $(+5', +2')$. The values in parentheses after each force are coordinates of points on its line of action. The sense is from the first point toward the second.

124. Find the resultant of the following coplanar force system: 1 ton $(0, +4')$, $(-3', 0)$; 1.1 tons $(-5', 0)$, $(-5', +5')$; 1.3 tons $(0, +5')$, $(+5', +5')$; 1.3 tons

$(-5', 0)$, $(0, -12')$; 1.5 tons $(+4', 0)$, $(0, +3')$. The values in parentheses after each force are coordinates of points on its line of action. The sense is from the first point toward the second.

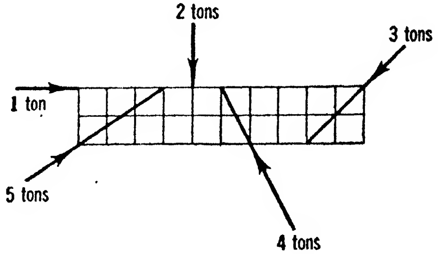

FIG. 59

125. Find the resultant of the system shown in Fig. 60, Art. 35, by the graphic method.

126. Find the resultant of the system shown in Fig. 64, Art. 35, by the graphic method.

35. Resultant of the General Coplanar System; Algebraic Method. *The Principle of Components. The algebraic sum of the components of the forces of the general coplanar system, along any axis in the plane of the forces, is equal to the component of the resultant along that axis.*

Proof. The proof of the principle of components in Art. 25, applied therein to the coplanar concurrent system, may be considered sufficient for the present case. The force diagram for the general coplanar system is constructed in the same manner as that for the coplanar concurrent system, and the reasoning based on the one is applicable to the other.

In the case in which the resultant is a couple the force polygon closes, and the component-sum of the forces along any axis is equal to zero. In such a case it is understood that the phrase " component of the resultant along that axis " means the component-sum of the two forces of the resultant couple, and since this sum is equal to zero the principle is valid.

The Principle of Moments. The algebraic sum of the moments of the forces of the general coplanar system, about any point in the plane of the forces, is equal to the moment of the resultant about that point.

Proof. This principle was proved in Art. 33, with reference to Fig. 37, for the case of the coplanar parallel system. That proof can be applied without essential change to the general coplanar system as represented in Fig. 53, and will not be repeated here.

Application. In the case in which the resultant is a single force, its magnitude, inclination, and sense are ascertained by means of the principle of components, exactly as in the case of the coplanar concurrent system.

It is known that the resultant of a concurrent system passes through the point of concurrence, but the position of the resultant of the general coplanar system must be found by calculation. This is done by means of the principle of moments. A convenient point is chosen as a center of moments, and the moment-sum of the forces is calculated. This gives the moment of the resultant about the chosen center. The resultant is then placed in such a position as to give it the required moment about that point.

The rather common error of placing the resultant on the wrong side of the center of moments can be most surely avoided by following the suggestions in Art. 33 for the avoidance of a similar error.

If the component-sums of the forces along two non-parallel axes are found to be zero, the resultant, if one exists, is a couple. The principle of moments is then used to find the moment of the resultant couple. If the moment-sum is also zero the system has no resultant and is in equilibrium.

Illustrative Problems

127. Find the resultant of the coplanar system shown in Fig. 60, by the algebraic method.

Solution. By the principle of components,

$$R_x = +100 \cos 45° - 160 \cos 34° - 220 \cos 67° 30'$$

$$= +70.7 - 133 - 84.3 = -146.6 \text{ lb}$$

$$R_y = -100 \cos 45° + 160 \cos 56° - 220 \cos 22° 30' + 300$$

$$= -70.7 + 89.4 - 203 + 300 = +115.7 \text{ lb}$$

Figure 61 shows R_x, R_y, and R in their correct relationship under the parallelogram law. From Fig. 61,

$$\theta_x = \text{arc tan } \frac{R_y}{R_x} = \text{arc tan } \frac{115.7}{146.6} = \text{arc tan } 0.789 = 38° 15'$$

$$R = \frac{R_x}{\cos \theta_x} = \frac{146.6}{0.785} = 187 \text{ lb}$$

The magnitude, inclination, and sense of the resultant are now known, but the position of its line of action relative to the forces of the system must be found.

Calculate the moment-sum of the forces about any convenient point, such as O, in Fig. 60. In so doing, it is convenient to resolve each force into its x- and y-components at the point where its line of action intersects the x-axis.

The moment of each x-component about O is then equal to zero, leaving only the moments of the y-components to be calculated. The magnitudes of the components were calculated above.

$$\Sigma M_0 = +70.7 \times 14 + 89.4 \times 12 - 300 \times 9$$
$$= +990 + 1070 - 2700 = -640 \text{ ft-lb}$$

By the principle of moments, the foregoing value is also the moment of the resultant about O. Now calculate a, the arm of the resultant with respect to O, disregarding signs.

$$a = \frac{\Sigma M_0}{R} = \frac{640}{187} = 3.42 \text{ ft}$$

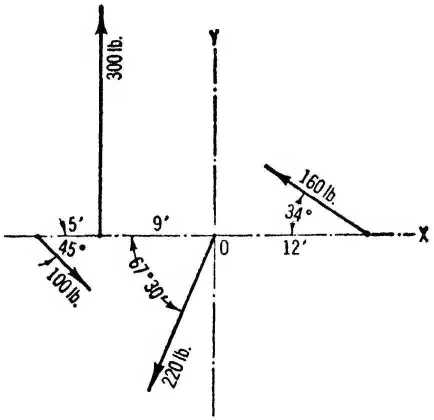

Fig. 60

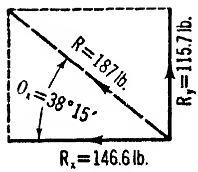

Fig. 61

It is now known that the perpendicular distance from O to the line of action of R is 3.42 ft. The foregoing calculation, however, does not show whether R intersects the x-axis to the right or to the left of O. The sense of R is upward and toward the left (Fig. 61). The moment of R about O is negative, or clockwise. These two facts, considered together, clearly show that R intersects the x-axis to the left of O.

Figure 62 shows R in its correct position, as revealed by the foregoing calculations.

Alternative Method of Locating R. In certain cases it is more satisfactory to show the position of the line of action of R by locating the point at which it intersects one of the coordinate axes. Figure 63 shows R_x, R_y, and R, the point of resolution having been placed at the intersection of the line of action of R with the x-axis. This renders the moment of R_x about O equal to zero. Since the moment of R is equal to the moment-sum of R_x and R_y, in the present case it is equal to the moment of R_y alone. Therefore,

$$\Sigma M_0 = R_y x_R \qquad x_R = \frac{\Sigma M_0}{R_y} = \frac{640}{115.7} = 5.53 \text{ ft}$$

The fact that R intersects the x-axis at a point to the left of O is ascertained by a process of reasoning similar to that employed in the first method. The point at which R intersects the y-axis could be located, if preferred, by dividing ΣM_0 by R_x, instead of by R_y.

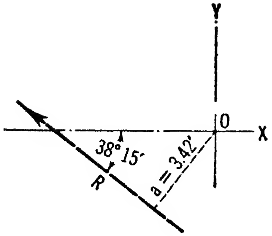

FIG. 62

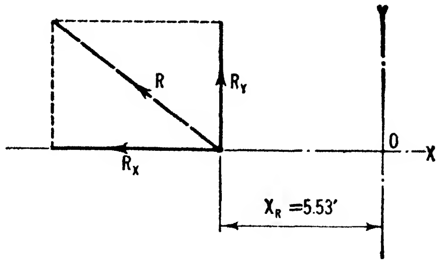

FIG. 63

128. Find the resultant of the coplanar system shown in Fig. 64.

Solution. By the principle of components,

$$R_x = -3 \times \frac{4}{5} + 5.2 \times \frac{6}{6.5} + 6.8 \times \frac{15}{17} - 8.4$$
$$= -2.4 + 4.8 + 6.0 - 8.4 = 0$$
$$R_y = -3 \times \frac{3}{5} - 5.2 \times \frac{2.5}{6.5} - 6.8 \times \frac{8}{17} + 7.0$$
$$= -1.8 - 2.0 - 3.2 + 7.0 = 0$$

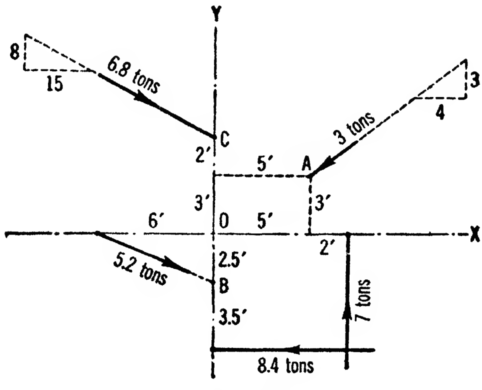

FIG. 64

It is thus learned that $R_x = 0$ and $R_y = 0$. This means that the resultant, if one exists, is a couple.

Now calculate the moment-sum of the forces about any convenient point, such as O. For this purpose resolve the three inclined forces into their x- and y-components, at the points A, B, and C.

$$\Sigma M_0 = (+2.4 \times 3 - 1.8 \times 5) + 4.8 \times 2.5 - 6.0 \times 5 + 7.0 \times 7 - 8.4 \times 6$$
$$= +7.2 - 9.0 + 12.0 - 30.0 + 49.0 - 50.4$$
$$= -21.2 \text{ ft-tons}$$

It is now known that the resultant is a couple. By the principle of moments, the moment of the resultant couple is equal to -21.2 ft-tons. If ΣM_0 had been found to be zero, there would have been no resultant.

The results may be summarized by stating that the resultant of the system is any couple, in the plane of the system or in any plane parallel thereto, having a clockwise moment of 21.2 ft-tons.

PROBLEMS

129. Figure 65 represents a cross section of a common type of reinforced-concrete retaining wall. Calculations relative to the strength and stability of retaining walls are usually made with reference to a unit section, or slice, of the wall, having a dimension of 1 ft in the direction of the length of the wall. In Fig. 65, W_1 represents the weight of such a unit section, and W_2 represents the weight of the prism of earth, *CDEF*, resting directly on the section. The force of 17,300 lb is the pressure of the remainder of the earth filling against the plane *BC*. $W_1 = 5000$ lb, and $W_2 = 10{,}600$ lb. Find the resultant of these three forces, and locate the point where its line of action intersects the base, *AB*, of the wall. *Ans.* 28,500 lb; $\theta_x = 238° 20'$; 6.62 ft to the left of *B*.

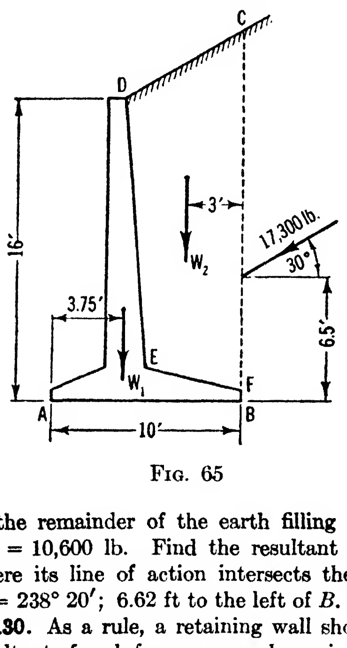

FIG. 65

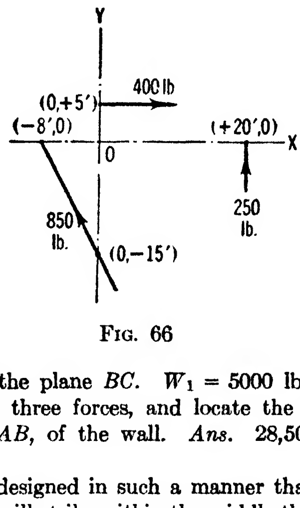

FIG. 66

130. As a rule, a retaining wall should be designed in such a manner that the resultant of such forces as are shown in Fig. 65 will strike within the middle third of the base, *AB*. Is such the case in Prob. 129? What would be likely to occur if the resultant should strike to the left of *A*?

131. Find the resultant of the system of forces shown in Fig. 66. *Ans.* 1000 lb; $\theta_x = 90°$; $x_R = -3$ ft.

132. Solve Prob. 121, Art. 34, by the algebraic method.

133. Find the resultant of the system of forces shown in Fig. 67.

134. The system of forces shown in Fig. 67 has no resultant, and is in equilibrium. Prove this by finding the resultant of the 20-lb, 50-lb, and 80-lb forces, and comparing it with the 130-lb force.

135. Solve Prob. 120, Art. 34, by the algebraic method. *Ans.* 9.49 tons; $\theta_x = 199° 25'$; $y_R = -3.55$ ft.

136. Solve Prob. 123, Art. 34, by the algebraic method.

137. Solve Prob. 124, Art. 34, by the algebraic method.

138. The car shown in Fig. 68 is being pulled along a horizontal track by a 40-lb force, applied as shown. The total weight of the car is 400 lb, and the center of gravity is at G. There is a total resistance of 20 lb, along the track. The other

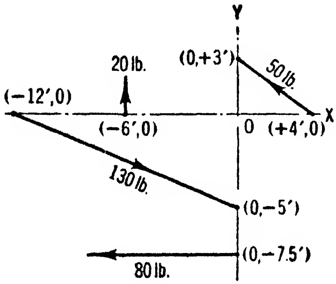

FIG. 67

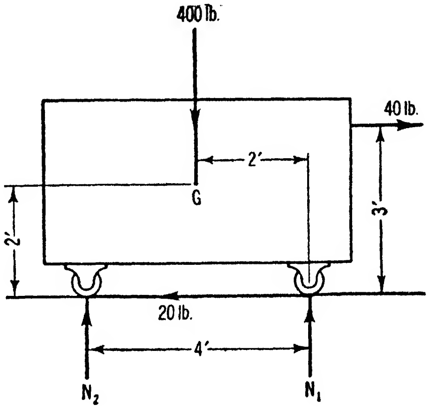

FIG. 68

external forces acting on the car are N_1 and N_2. The motion of the car is rectilinear translation. In kinetics it is learned that under such circumstances the resultant of all the external forces is a single force, parallel to the acceleration and passing through the center of gravity of the body. In the present case this means that the resultant is horizontal and toward the right, and passes through G. Calculate N_1 and N_2. *Ans.* +220 lb; +180 lb.

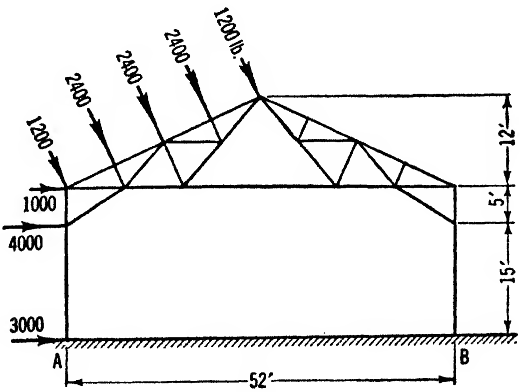

FIG. 69

139. Solve Prob. 122, Art. 34, by the algebraic method.

140. Figure 69 represents a transverse bent of a steel mill building. Eight coplanar wind loads are shown. The inclined loads are spaced uniformly. Find the resultant of all the loads, and locate the point at which it intersects the line AB.

The problem can be simplified by first finding the resultant of the five inclined loads by inspection. *Ans.* 14,900 lb; $\theta_x = 324°$; 34.2 ft to the right of A.

141. In Prob. 140, the reactions exerted on the transverse bent by its supports at A and B, calculated by a method often used, are: at A, 6010 lb to the left, 2990 lb upward; at B, 6010 lb to the left, 5730 lb upward. Find the resultant of these reactions, and compare it with the resultant of the wind loads in Prob. 140. What is the significance of the comparison?

142. Find the resultant of the system of forces shown in Fig. 70. It may be seen that the system consists entirely of couples. In Art. 36 a special method for the solution of such cases is presented. For the purpose of comparison, however, this problem should be solved by the method of the present article. Compare with Prob. 143.

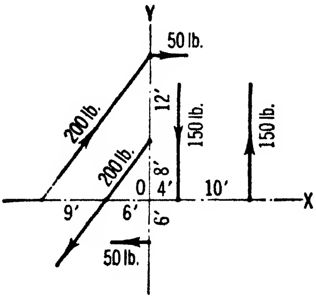

FIG. 70

36. Resultant of a System of Coplanar Couples; Special Method. *The resultant of a system of coplanar couples is a couple, situated in the plane of the system or in any parallel plane; and the moment of the resultant couple is equal to the algebraic sum of the moments of the couples comprising the system.*

Proof. It is possible to compound a system of couples by the more general methods followed in Arts. 33 and 35. Such a procedure would naturally be followed if the student did not notice that he was dealing with a system of couples. The use of the special principle stated above results, however, in an appreciable saving of time. Even when the system does not consist entirely of couples the principle sometimes can be applied to advantage in connection with the couples themselves.

The validity of the principle follows, almost without further comment, from the principles established in Arts. 33 and 35, and formal proof will not be given.

If the algebraic sum of the moments of the couples is equal to zero, the system has no resultant and is in equilibrium.

ILLUSTRATIVE PROBLEM

143. Find the resultant of the system of couples in Prob. 142, Fig. 70, by the method of the present article.

Solution. Let the three couples be numbered 1, 2, and 3, in the order of the magnitudes of their forces. Let a_1, a_2, and a_3 represent the arms of the couples, and let C_1, C_2, and C_3 represent the moments.

$$a_1 = 6 + 8 + 12 = 26 \text{ ft} \qquad C_1 = -50 \times 26 = -1300 \text{ ft-lb}$$

$$a_2 = 10 \text{ ft} \qquad C_2 = +150 \times 10 = +1500 \text{ ft-lb}$$

$$a_3 = 9 \times \tfrac{4}{5} = 7.2 \text{ ft} \qquad C_3 = -200 \times 7.2 = -1440 \text{ ft-lb}$$

$$\Sigma C = -1300 + 1500 - 1440 = -1240 \text{ ft-lb}$$

Summarizing, the resultant of the system is any couple, in the plane of the system or in any plane parallel thereto, having a clockwise moment of 1240 ft-lb.

PROBLEMS

144. Find the resultant of the system of couples shown in Fig. 71. *Ans.* +320 in-lb.

145. It is desired to find a specific resultant couple for the system shown in Fig. 71. It is required that one of its forces coincide with the y-axis, that it be upward, and that its magnitude be 80 lb. Find the other force.

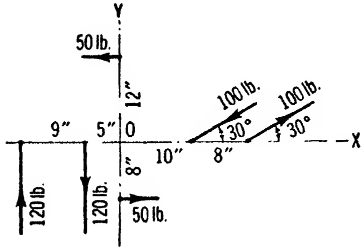

FIG. 71

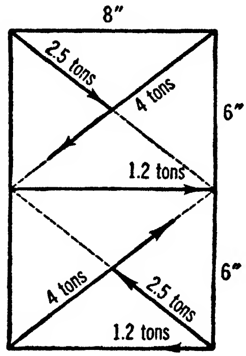

FIG. 72

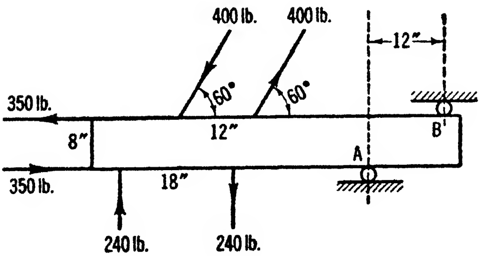

FIG. 73

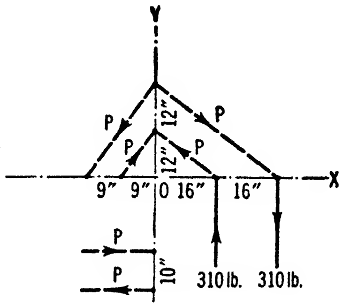

FIG. 74

146. Find the resultant of the system of couples shown in Fig. 72.

147. Figure 73 represents a beam supported by rollers at A and B (see Art. 14). Three coplanar couples are applied to the beam as shown. Will the beam move? Calculate the forces exerted on the beam by its supports. Solve by first finding the resultant of the three given couples. Disregard the weight of the beam.

148. Figure 74 shows a couple having 310-lb forces and a 16-in. arm. It is desired to replace the given couple by an equivalent system of six forces having senses and lines of action as shown, and having equal magnitudes. Calculate the common magnitude of the six forces. *Ans.* 400 lb.

37. Resultant of a Coplanar Couple and Single Force; Special Method. *The resultant of a coplanar system consisting of a couple and a single force is a single force equal to, parallel to, and agreeing in sense with, the original single force. The line of action of the resultant is in the plane of the system and is so situated that the moment of the resultant about any point on the line of action of the original single force is equal to the moment of the couple.*

Proof. A system consisting of a couple and a single force in the same plane could be classified as a general coplanar system, or possibly a coplanar parallel system. The principle of components and the principle of moments are valid in either case. The theorem stated at the beginning of the article can be verified with the aid of these two principles. Formal proof will be left as an exercise for the student.

The use of the foregoing theorem results in a saving of time in the solution of certain problems.

Illustrative Problems

149. Find the resultant of the couple and single force shown in Fig. 75.

Solution. By the principle stated at the beginning of the article, the resultant is a single force equal and parallel to the original single force, and agreeing with it in sense. Therefore, in the present case, $R = 6.4$ tons, and $\theta_x = 315°$.

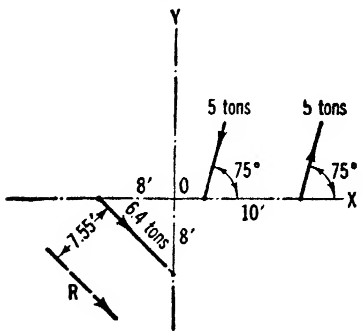

Fig. 75

Calculate the moment of the given couple, as follows:

$$a = 10 \times \sin 75° = 10 \times 0.966 = 9.66 \text{ ft}$$

$$C = +5 \times 9.66 = +48.3 \text{ ft-tons}$$

The line of action of R is so situated that the moment of R about any point on the line of action of the original 6.4-ton force is equal to +48.3 ft-tons. Therefore, the distance between the two lines of action is $48.3/6.4 = 7.55$ ft. Also, since the moment is counterclockwise, R lies below the original force. R has been shown in Fig. 75, in its correct position.

150. Reverse the couple in Prob. 149, Fig. 75, leaving the single force unchanged. Find the resultant of the altered system.

Solution. The only difference between the results in the present problem and those of Prob. 149 is that the resultant will lie above, instead of below, the original single force, since it must now have a negative moment about any point in the line of action of that force.

PROBLEMS

151. Find the resultant of the system shown in Fig. 76. *Ans.* 600 lb; $\theta_x = 216° 50'$; 7.2 in. from the original force, and below.

152. Solve Prob. 151 by the following method: Replace the given couple with an equivalent couple having 600-lb forces. Place the new couple in the plane of the forces, in such a position that one of its forces will neutralize the original single force.

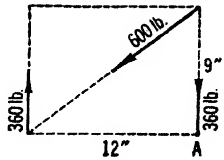

FIG. 76

153. Find the resultant of the system in Fig. 70, Art. 35, in the following manner: Compound the upper 50-lb force and the two 200-lb forces, by the method of the present article; then compound the other three forces by the same method. Find the moment of the resultant couple.

154. In Fig. 71, Art. 36, insert an additional single force of 40 lb, coinciding with the x-axis and acting toward the left. Find the resultant of the entire system, first compounding the three couples by the method of Art. 36, and then compounding this resultant couple with the 40-lb single force by the method of the present article. *Ans.* 40 lb; $\theta_x = 180°$; 8 in. above the x-axis.

155. The three couples in Fig. 72, Art. 36, are in equilibrium. Prove this statement by compounding the upper 1.2-ton force with the two 4-ton forces, and the lower 1.2-ton force with the two 2.5-ton forces. If the system is in equiibrium the two single-force resultants thus obtained will balance.

38. Resolution of a Force into a Force and Couple.

A given force can be resolved into a component force and couple. The component force must be equal and parallel to the original force, and must agree with it in sense, but the point of application can be chosen anywhere in space. The couple is any couple in the plane of the original force and the component force, having a moment equal to the moment of the original force about any point in the line of action of the component force.

Proof. The relations involved in the foregoing principle are essentially the same as in the case discussed in Art. 37, and their validity may be considered as having been established.

ILLUSTRATIVE PROBLEM

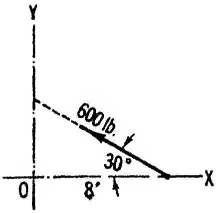

FIG. 77

156. Resolve the 600-lb force shown in Fig. 77 into a single force acting through O, and a couple.

Solution. Calculate the moment of the given force about the chosen point, O.

$$M_0 = +600 \sin 30° \times 8 = +300 \times 8$$
$$= +2400 \text{ ft-lb}$$

The results are as follows: a single force of 600 lb, acting through O, $\theta_x = 150°$; and any couple in the plane of the figure, or in any parallel plane, having a counterclockwise moment of 2400 ft-lb.

PROBLEMS

157. As was stated in Art. 9, equivalent systems of forces would not necessarily have the same deformation effect on a body. In some cases, however, they would have the same effect. Figure 78 represents a short column supporting a vertical load, W, applied eccentrically at a distance, e, from the axis of the column. It is learned in strength of materials that, if W is resolved into a single force applied at O, on the axis, and a couple applied near the end of the column, as shown, the deformation effect and the resulting stresses in the column are not altered. Let $W = 1200$ lb, and let $e = 4.5$ in. Let the arm, a, of the couple be 2.7 in. Calculate the component force at O, and the forces of the couple.

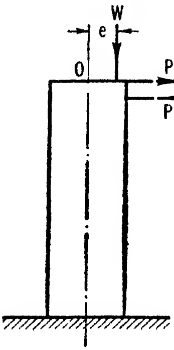

FIG. 78

158. A force of 8400 lb is applied to a crank keyed to the end of a shaft. The distance from the center of the shaft to the center of the crankpin is 16.5 in. The force is tangent to the circle in which the center of the crankpin moves. The stresses in the shaft would not be altered if the force were resolved into a single force at the center of the shaft, and a couple, both lying in a plane at right angles to the shaft and containing the original force. Find the component force, and couple. *Ans.* 8400 lb; 138,600 in-lb.

159. Resolve each of the forces in Fig. 32, Art. 27, into a force acting through O, and a couple.

160. Find the resultant of the coplanar system shown in Fig. 66, Art. 35, by the following method: Resolve each of the given forces into a force through O, and a couple. Find the resultant of the concurrent system thus established at O, by the method of Art. 25. Find the resultant of the three couples by the method of Art. 36. Compound the resultant single force and the resultant couple by the method of Art. 37.

The foregoing method is sometimes suggested as the standard method of compounding coplanar force systems. It differs only slightly from the method of Art. 35. *Ans.* 1000 lb; $\theta_x = 90°$; $x_R = -3$ ft.

161. Find the resultant of the system shown in Fig. 67, Art. 35, by the method described in Prob. 160.

CHAPTER III

RESULTANTS OF NON-COPLANAR FORCE SYSTEMS

39. Resultant of Three Non-Coplanar Concurrent Forces at Right Angles to One Another. *If vectors representing three non-coplanar, concurrent and mutually perpendicular forces are constructed to scale at the point of concurrence, in such a manner that their senses agree with respect to that point, and if lines are added to form a parallelepiped, that diagonal of the parallelepiped which touches the point of concurrence will represent the resultant of the three forces. The sense of the resultant will agree with the senses of the three component forces with respect to the point of concurrence.*

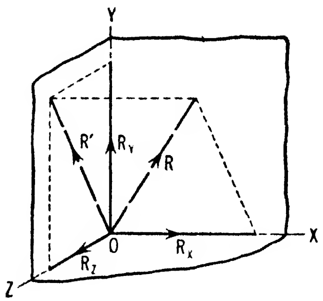

FIG. 79

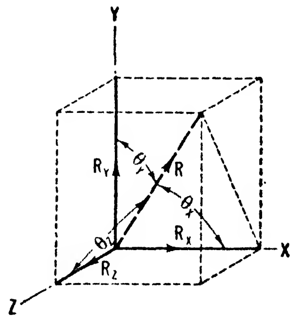

FIG. 80

Proof. In Fig. 79, let R_x, R_y, and R_z represent any three concurrent forces at right angles to one another. R_x and R_y are in the plane of the figure, but R_z is at right angles to that plane.

By the parallelogram law, Art. 17, R' is the resultant of R_y and R_z. Again by the parallelogram law, R is the resultant of R' and R_x. Therefore, R is also the resultant of R_x, R_y, and R_z.

A careful study of Fig. 79 shows that by the addition of six lines a parallelepiped can be constructed in which all the relations will be as stated in the foregoing theorem. The completed construction is shown in Fig. 80.

The theorem could be generalized to include any three non-coplanar concurrent forces, whether they were mutually perpendicular or not.

However, the theorem is generally used in the special form in which it was stated above.

Formulas. Let θ_x, θ_y, and θ_z represent the angles between the given forces and their resultant. From Fig. 80,

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \tag{8}$$

$$\cos\theta_x = \frac{R_x}{R} \tag{9}$$

$$\cos\theta_y = \frac{R_y}{R} \tag{10}$$

$$\cos\theta_z = \frac{R_z}{R} \tag{11}$$

Signs of Components in Non-Coplanar Problems. Three coordinate axes will be used frequently in subsequent problems and discussions. In general the x- and y-axes will be placed in the plane of the paper; consequently the z-axis will be at right angles to that plane. The right-hand end of the x-axis, and the upper end of the y-axis, will be considered positive. The positive end of the z-axis will be the end toward the reader. Angles of inclination will be stated in accordance with the conventions established in Art. 16.

PROBLEMS

162. The lines of action of the following forces coincide with the x-, y-, and z-axes, respectively. Their senses are indicated by signs. +30 lb, +60 lb, +60 lb. Find the resultant. *Ans.* 90 lb; $\theta_x = 70° 35'$; $\theta_y = 48° 10'$; $\theta_z = 48° 10'$.

163. Given: $R_x = +120$ lb, $R_y = -90$ lb, $R_z = +360$ lb; find the resultant.

164. Given: $R_x = -2.4$ tons, $R_y = +7.5$ tons, $R_z = -3.2$ tons; find the resultant. *Ans.* 8.50 tons; $\theta_x = 106° 25'$; $\theta_y = 28° 05'$; $\theta_z = 112° 05'$.

165. The following forces coincide with the x-axis: +280 lb, −85 lb, −195 lb; the following coincides with the y-axis: −300 lb; the following coincides with the z-axis: −160 lb. Find the resultant. *Ans.* 340 lb; $\theta_x = 90°$; $\theta_y = 151° 55'$; $\theta_z = 118\ 05'$.

166. Given: two x-components as follows: +45 lb, −85 lb; two y-components as follows: +25 lb, −100 lb; two z-components as follows: −70 lb, −134 lb. Find the resultant of the system.

40. Resolution of a Force into Three Concurrent Components at Right Angles to One Another. In this case R is given, and R_x, R_y, and R_z are to be found. The relations between R and its components are the same as those established in Art. 39.

In Art. 19 the following statement appears: "The component of a force along any line is equal to the product of the magnitude of the force and the cosine of the angle that the force makes with the given line."

The case under discussion in Art. 19 was that in which the force is to be resolved into two rectangular components. Formulas 9, 10, and 11, Art. 39, show that the same principle applies when the force is to be resolved into three rectangular components.

Close attention should be paid to the fact that the point of concurrence of the three components must lie on the line of action of the given force. This point is also called the point of resolution.

Formulas. Formulas 9, 10, and 11, Art. 39, may be used for the present case. Formula 8 is valid, also, and could be used as a convenient check.

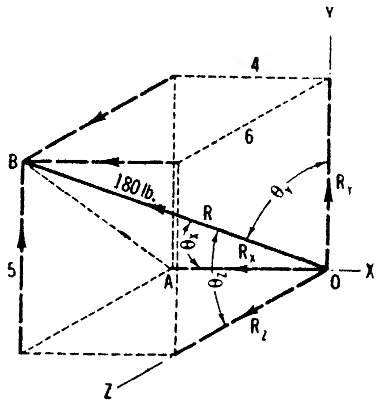

Fig. 81

Illustrative Problem

167. A force of 180 lb acts through the origin, and through the point $(-4, +5, +6)$, its sense being toward the latter point. Resolve the force, at the origin, into its x-, y-, and z-components. Also resolve at the second point.

Solution. Figure 81 shows the given force in its specified position. Complete the parallelepiped, as shown in the figure.

Triangle AOB is a right triangle; therefore, $\cos\theta_x = \dfrac{AO}{OB}\cdot$ From the figure, $AO = 4$. By geometry, $OB = \sqrt{(4)^2 + (5)^2 + (6)^2} = 8.77$. Therefore, $\cos\theta_x = \dfrac{4}{8.77} = 0.456$. Similarly, $\cos\theta_y = \dfrac{5}{8.77} = 0.570$, and $\cos\theta_z = \dfrac{6}{8.77} = 0.684$. And so,

$$R_x = R\cos\theta_x \qquad R_x = 180 \times 0.456 = 82.1 \text{ lb}$$
$$R_y = R\cos\theta_y \qquad R_y = 180 \times 0.570 = 103 \text{ lb}$$
$$R_z = R\cos\theta_z \qquad R_z = 180 \times 0.684 = 123 \text{ lb}$$

To complete the solution it is necessary to show the three components on the figure, acting through the chosen point of resolution. This has been done in Fig. 81, for the two points of resolution specified in the problem.

A distinctive feature of the foregoing solution is the fact that $\cos\theta_x$, $\cos\theta_y$, and $\cos\theta_z$ were calculated directly from the figure, and that the angles themselves were not calculated. This is the logical method in problems in which the position of a force is shown by means of coordinates, or linear dimensions, rather than by means of angles, and in which the values of the angles are not required for any other purpose.

PROBLEMS

168. A force of 136 lb acts from the origin through the point (+9, +12, +8). Resolve the force, at the origin, into its x-, y-, and z-components. Also resolve at the second point. *Ans.* +72 lb; +96 lb; +64 lb.

169. A force of 13 tons acts from the origin through the point (−7, −24, +60). Resolve the force, at the origin, into its x-, y-, and z-components. Also resolve at the second point.

170. A force of 70 lb has the following angles of inclination: $\theta_x = 54° 20'$ and $\theta_y = 132° 40'$. It is also known that the z-component is negative. Calculate the x-, y-, and z-components, and θ_z. *Ans.* +40.8 lb; −47.5 lb; −31.3 lb; 116° 35′.

171. A force of 300 lb has the following angles of inclination: $\theta_x = 123° 16'$; $\theta_y = 105° 42'$; $\theta_z = 37° 42'$. Resolve the force into its x-, y-, and z-components, at the origin. Check by recomposition of the three components (Art. 39).

172. Prove that the 130-lb force in Fig. 82 would have the same effect on a body as the other four forces combined.

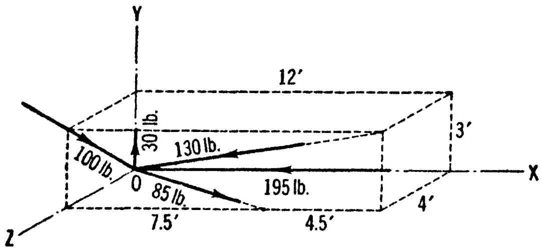

FIG. 82

41. Resultant of the Non-Coplanar Concurrent System. *The Principle of Components. The algebraic sum of the components of the forces of a non-coplanar concurrent system, along any axis, is equal to the component of the resultant along that axis.*

Proof and Application. In the solution of a problem a convenient set of three rectangular axes is selected, having its origin at the point of concurrence of the forces. Each force is then resolved, at the origin, into its x-, y-, and z-components. This transforms the original system into three collinear systems, each of which coincides with one of the axes.

The algebraic sum of the x-components is then calculated. This gives their resultant, a single force along the x-axis. The y- and z-components are treated similarly. The system now has been reduced to a single force coinciding with each of the axes. The three forces are then compounded by the method of Art. 39. Their resultant is obviously the resultant of the original system.

The resultant, if one exists, is necessarily a single force. If each of the three component-sums is found to be equal to zero there is no resultant, and the system is in equilibrium.

ILLUSTRATIVE PROBLEM

173. Find the resultant of the four non-coplanar concurrent forces shown in Fig. 83.

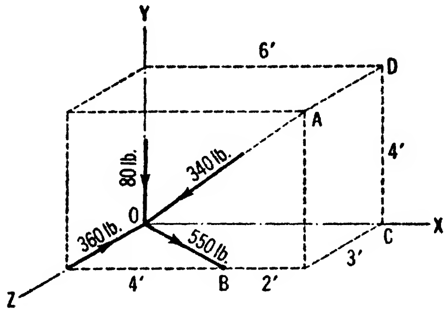

FIG. 83

Solution. Resolve each force into its x-, y-, and z-components, at O. Calculate the cosines of the various angles of inclination, from the figure, by the method used in Prob. 167, Art. 40.

For the 340-lb force: $\overline{OA} = \sqrt{(6)^2 + (4)^2 + (3)^2} = \sqrt{61} = 7.81$; $\cos\theta_x = \dfrac{6}{7.81}$; $\cos\theta_y = \dfrac{4}{7.81}$; $\cos\theta_z = \dfrac{3}{7.81}$. For the 550-lb force: $\overline{OB} = \sqrt{(4)^2 + (3)^2} = \sqrt{25} = 5$; $\cos\theta_x = \frac{4}{5}$; $\cos\theta_z = \frac{3}{5}$.

By the principle of components,

$$R_x = -340 \times \frac{6}{7.81} + 550 \times \frac{4}{5} = -261 + 440 = +179 \text{ lb}$$

$$R_y = -340 \times \frac{4}{7.81} - 80 = -174 - 80 = -254 \text{ lb}$$

$$R_z = -340 \times \frac{3}{7.81} + 550 \times \frac{3}{5} - 360 = -131 + 330 - 360 = -161 \text{ lb}$$

By Art. 39,

$$R = \sqrt{(179)^2 + (254)^2 + (161)^2} = 350 \text{ lb}$$

$$\theta_x = \text{arc cos}\,\frac{R_x}{R} = \text{arc cos}\,\frac{179}{350} = \text{arc cos}\,0.511 = 59°\,15'$$

$$\theta_y = \text{arc cos}\,\frac{R_y}{R} = \text{arc cos}\,\frac{254}{350} = \text{arc cos}\,0.726 = 43°\,25'$$

$$\theta_z = \text{arc cos}\,\frac{R_z}{R} = \text{arc cos}\,\frac{161}{350} = \text{arc cos}\,0.460 = 62°\,35'$$

Figure 84 shows the resultant and its three components in their correct positions. The angles calculated above are the acute angles between the resultant and the axes, as indicated in the figure.

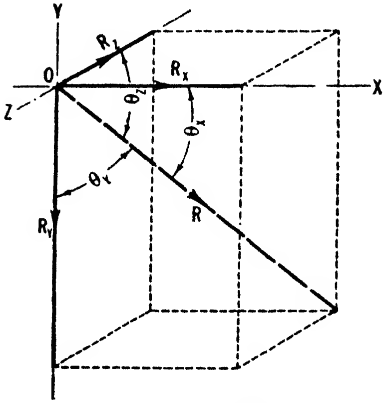

FIG. 84

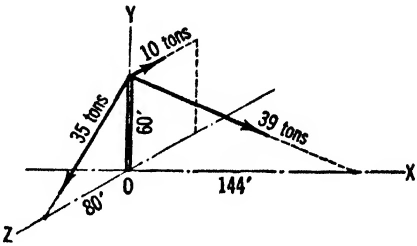

FIG. 85

PROBLEMS

174. The following forces are concurrent at the origin, and act toward and through the points indicated: 45 lb (0, −10, 0); 65 lb (0, −12, −5); 90 lb (+4, +4, +2); 175 lb (0, 0, +10). Find the resultant. *Ans.* 195 lb; $\theta_x = 72° 05'$, $\theta_y = 103° 20'$, $\theta_z = 22° 40'$.

175. The following forces are concurrent at the origin, and act toward and through the points indicated: 780 lb (+4, −12, −3); 850 lb (−8, 0, +15); 160 lb (+10, 0, 0); 570 lb (0, 0, −10). Find the resultant.

176. Find the resultant of the following concurrent system: 1 ton, $\theta_x = 180°$, $\theta_y = 90°$, $\theta_z = 90°$; 2 tons, $\theta_x = 90°$, $\theta_y = 180°$, $\theta_z = 90°$; 5 tons, $\theta_x = 144°$, $\theta_y = 70° 22'$, $\theta_z = 61° 11'$; 6 tons, $\theta_x = 90°$, $\theta_y = 60°$, $\theta_z = 150°$.

177. The following statement, referring to the non-coplanar concurrent system, appears above: " The resultant, if one exists, is necessarily a single force." Prove that this statement is correct.

178. The following forces are concurrent at the origin, and act toward and through the points indicated: 80 lb (0, 0, +10); 88 lb (0, 0, −10); 89 lb (0, +10, 0); 120 lb (−8, −4, −8); 175 lb (0, −7, +24). Find the resultant. *Ans.* 113 lb; $\theta_x = 135°$, $\theta_y = 90°$, $\theta_z = 45°$.

179. Figure 85 represents a vertical tower, 60 ft high, resting on level ground and subjected to pulls from three cables, as shown. It is desired to brace the tower by means of a guy 90 ft long, to be attached to the top of the tower and anchored at the ground. The guy is to be drawn up to a tension such that the resultant pull of the entire system will be vertical. Calculate the necessary tension in the guy, and find the x- and z-coordinates of its point of anchorage.

42. Moment of a Force about a Line; General Case. By "general case " is meant that in which the force does not lie in a plane at right angles to the line.

The moment of a force about a line is the moment of that component which lies in a plane at right angles to the line, provided that the force has been so resolved that the other component is parallel to the line. The line about which the moment is taken is called the axis of moments.

It follows from the foregoing definition that the moment of a force about a line parallel to the force is equal to zero. It is also obvious that the moment of a force about a line that intersects the line of action of the force is equal to zero.

For example, let P, in Fig. 86, represent the force, and let AB represent any line that is not at right angles to a plane containing P. P has been resolved, at C, into a component, P_1, parallel to AB, and a component, P', in a plane perpendicular to AB. The line BD is the moment-arm of P' with respect to AB.

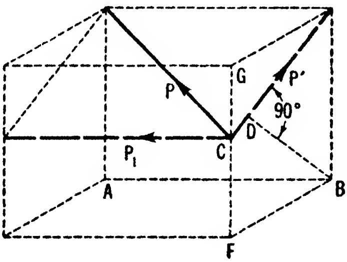

FIG. 86

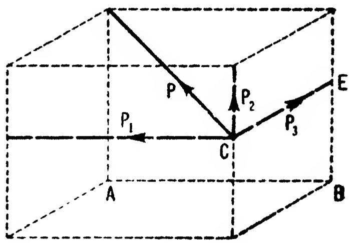

FIG. 87

By the definition, the moment of P about the line AB is the moment of P' about that line and is equal, therefore, to $-P' \times \overline{BD}$. The negative sign is used because the moment appears clockwise from the point of observation.

In Fig. 87 the same force, P, has been shown, but has been resolved at C into three rectangular components, instead of two. P_1 is the same as in Fig. 86, and P_2 and P_3 are in a plane perpendicular to AB. P_2 and P_3 are rectangular components of P' (Fig. 86).

By the principle of moments, Art. 27, the moment of P' is equal to the moment-sum of P_2 and P_3. Therefore, the moment of P about AB is equal to $-P_2 \times \overline{CE} - P_3 \times \overline{EB}$.

If preferred, then, the given force may be resolved into three rectangular components, instead of two. One of the components must be parallel to the axis of moments, and the moment of the force is then found by calculating the moment-sum of the other two components.

The Sign of a Moment. In this book, whenever a moment is calculated with respect to one of the coordinate axes, or with respect to a line parallel thereto, the point of observation will be considered to be so situated that the line of sight is parallel to the axis of moments and the observer faces in the negative direction. As in previous discussions, a moment appearing counterclockwise will be given the positive sign.

The physical significance of a moment, and the units, were discussed in Art. 26 and apply also to the present case.

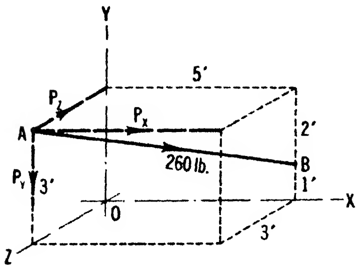

FIG. 88

ILLUSTRATIVE PROBLEMS

180. Calculate the moment of the 260-lb force, in Fig. 88, about each of the coordinate axes.

Solution. Resolve the force into its x-, y-, and z-components, at the point A. $\overline{AB} = \sqrt{(5)^2 + (2)^2 + (3)^2} = \sqrt{38} = 6.16$.

$$P_x = P \cos \theta_x \qquad P_x = 260 \times \frac{5}{6.16} = 211 \text{ lb}$$

$$P_y = P \cos \theta_y \qquad P_y = 260 \times \frac{2}{6.16} = 84.4 \text{ lb}$$

$$P_z = P \cos \theta_z \qquad P_z = 260 \times \frac{3}{6.16} = 127 \text{ lb}$$

Show the three components on the sketch, passing through the point of resolution, A, as in Fig. 88. The moment of the 260-lb force about the x-axis is equal to the moment-sum of P_y and P_z about that axis. Only one component, P_x, has a moment about the y-axis. P_x is also the only component having a moment about the z-axis. Therefore,

$$M_x = +P_y \times 3 - P_z \times 3 = +84.4 \times 3 - 127 \times 3 = -128 \text{ ft-lb}$$

$$M_y = +P_x \times 3 = +211 \times 3 = +633 \text{ ft-lb}$$

$$M_z = -P_x \times 3 = -211 \times 3 = -633 \text{ ft-lb}$$

The signs given to the moments calculated above are in accordance with the conventions adopted. For example, the moment of P_x about the y-axis was given the positive sign since, to an observer looking in the negative direction along that axis, the moment appears counterclockwise.

181. Solve Prob. 180, resolving the given force at B (Fig. 88), instead of at A.

Solution. The magnitudes and senses of the x-, y-, and z-components are the same as in Prob. 180.

$$M_x = -P_z \times 1 = -127 \times 1 = -127 \text{ ft-lb}$$

$$M_y = +P_z \times 5 = +127 \times 5 = +635 \text{ ft-lb}$$

$$M_z = -P_x \times 1 - P_y \times 5 = -211 \times 1 - 84.4 \times 5 = -633 \text{ ft-lb}$$

PROBLEMS

182. A force of 1200 lb passes through the points (+8′, 0, 0) and (0, +8′, +4′), its sense being toward the latter point. Calculate the moment of the force about each of the coordinate axes. Check the results, using a new point of resolution. *Ans.* 0; −3200 ft-lb; +6400 ft-lb.

183. Solve Prob. 180, resolving the given force into its components at a point midway between A and B, instead of at A. Compare the solution with that of Prob. 180, both as to results obtained and as to ease of solution.

184. In Fig. 86, assume the following dimensions: AB = 10 ft, FC = 4 ft, CG = 4 ft, BF = 8 ft. Assume P = 500 lb. Calculate the moment of P about the line AB, first by resolving as in Fig. 86, and then by resolving as in Fig. 87. *Ans.* −2390 ft-lb.

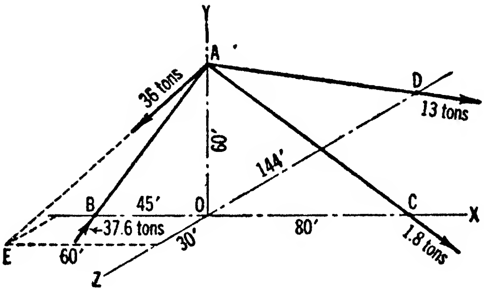

FIG. 89

185. Figure 89 represents a framework consisting of three members, AB, AC, and AD, and subjected to a load of 36 tons, applied as shown. Calculate the moment of this load about each of the coordinate axes.

186. Calculate the combined turning effect of the four forces in Fig. 83, Prob. 173, about the line CD. *Ans.* −966 ft-lb.

187. The framework shown in Fig. 89 is in equilibrium; therefore the system of external forces to which it is subjected is a balanced system. This system consists of four forces, as shown. If the magnitudes assigned to them are correct their moment-sum, about any axis, will equal zero. Ascertain whether this is the case, as regards the three coordinate axes.

188. Calculate the turning effect of the couple shown in Fig. 90, about each of the coordinate axes. *Ans.* −27,000 ft-lb; −27,000 ft-lb; 0.

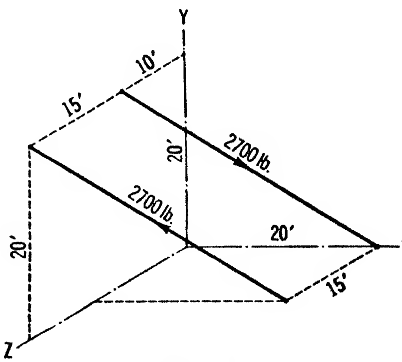

FIG. 90

43. Representation of a Couple by Means of a Single Vector. Heretofore, in this book, when it was desired to depict a couple, vectors were drawn representing the forces themselves. It is possible, however, and sometimes more convenient, to represent an entire couple by means of a single vector. This is done by letting the length of the vector represent, to a suitable scale, the moment of the couple, and by placing the vector at right angles to the plane of the couple. The direction of the rotational tendency of the couple is shown by the sense of the vector. This is accomplished by pointing the vector in the direction in which a right-handed screw would advance if the couple were turning the screw.

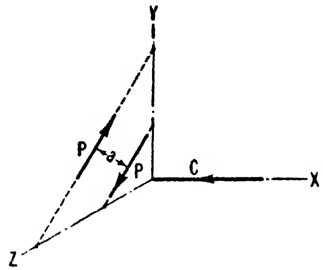

FIG. 91

In Fig. 91, let the forces P, P represent any couple, and let a represent its arm. The moment of the couple is equal to $-P \times a$. Any vector, such as C, whose length to some chosen scale is equal to $P \times a$, which is at right angles to the plane containing PP, and whose sense is toward the left, would correctly represent the couple.

A given vector does not represent one specific couple, to the exclusion of all others; it represents any one of an unlimited number of equivalent couples.

It should be noticed that when a given couple is in one of the coordinate planes, or in a plane parallel thereto, its vector will be parallel to one of the coordinate axes and can be given a sign in accordance with the convention adopted for components of forces, in Art. 39. It should also be noticed that the sign of the moment of the couple and the sign of the vector will agree. Thus, the couple in Fig. 91 has a negative moment, and its vector has the negative sense.

PROBLEMS

189. One of the couples in Fig. 92 has 215-lb forces. Make a sketch showing the coordinate axes, and place thereon a single vector, at O, representing this couple. Find the magnitude of the vector. Draw a second vector, through A, representing the same couple. *Ans.* -1935 ft-lb.

190. The vector C_1, in Fig. 92, is a couple vector. Make a sketch showing the three coordinate axes. Draw a couple, represented by the vector C_1, having 320-lb forces, one of which is positive and coincides with the x-axis.

191. One of the couples shown in Fig. 92 has 12-ton forces. Make a sketch showing the three coordinate axes, and place thereon a single vector, at O, representing this couple. Find the magnitude of the vector. Draw a second vector, through A, representing the same couple. *Ans.* $+57.6$ ft-tons.

192. Draw a single vector through O, in Fig. 92, representing the couple which has 850-lb forces. Calculate the angle of inclination of this vector with each of the coordinate axes. Find the magnitude of the vector. Resolve the vector into its x-, y-, and z-components. *Ans.* $C_x = +1520$ ft-lb; $C_y = 0$; $C_z = -3040$ ft-lb.

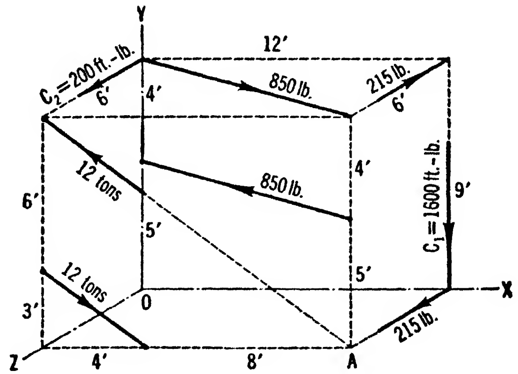

FIG. 92

193. The vector C_2, in Fig. 92, is a couple vector. Make a sketch showing the three coordinate axes. Draw a couple, represented by the vector C_2, one of whose forces coincides with the y-axis and is negative. Give the couple an arm of 5 ft.

194. Calculate the moment-sum of the two 850-lb forces, in Fig. 92, about each of the three coordinate axes. Compare these moment-sums with the components of the couple vector, as found in Prob. 192.

44. Resultant of Three Couples in Planes at Right Angles to One Another.

The resultant of three couples in planes at right angles to one another is a couple, and its vector is the resultant of the vectors of the three given couples.

Proof. In Fig. 93, let PP, QQ, and SS represent any three couples in perpendicular planes. For convenience the couples have been shown in the three coordinate planes. Let C_x, C_y, and C_z, shown for convenience at O, represent the vectors of couples PP, QQ, and SS, respectively.

Replace PP by an equivalent couple, $P'P'$, one of whose forces coincides with the z-axis, as shown. AB is the arm of $P'P'$. Replace QQ by an equivalent couple, $Q'Q'$, whose forces are equal to the forces of couple $P'P'$. Place couple $Q'Q'$ in such a position that one of its forces also coincides with the z-axis, but is opposite in sense to the force P' already placed there. AC is the arm of $Q'Q'$. Since the two forces P' and Q' occupying the z-axis are equal and opposite, they balance, and may be disregarded henceforth. This leaves two equal forces, P' and Q', forming a couple in a diagonal plane, having the arm BC. This couple is the resultant of the original couples PP and QQ. Let C' represent its vector.

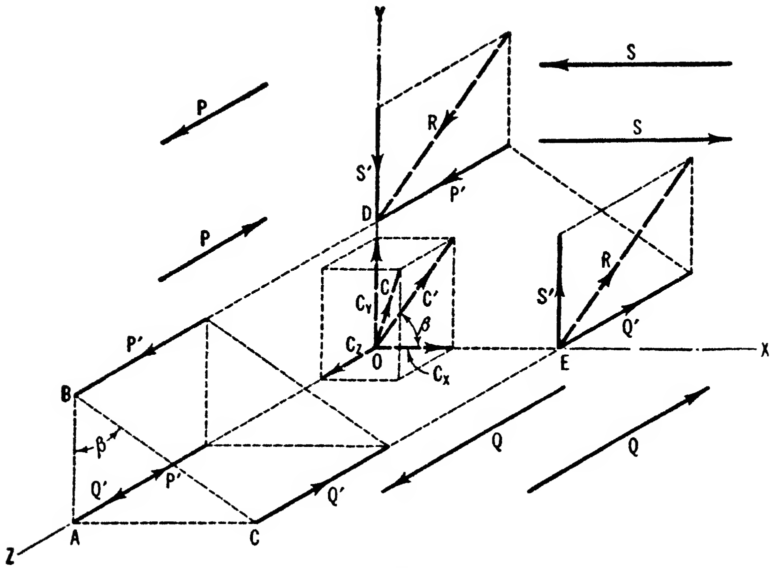

FIG. 93

It will now be proved that the vector C' is the resultant of the vectors C_x and C_y. By definition,

$$C' = P' \times \overline{BC} \qquad [12]$$

From the figure,

$$\overline{BC} = \sqrt{\overline{AB}^2 + \overline{AC}^2} \qquad [13]$$

Substituting in [12],

$$C' = P'\sqrt{\overline{AB}^2 + \overline{AC}^2} = \sqrt{(P' \times \overline{AB})^2 + (P' \times \overline{AC})^2} \qquad [14]$$

But, $P' \times \overline{AB} = C_x$ and $P' \times \overline{AC} = C_y$. Therefore,

$$C' = \sqrt{C_x^2 + C_y^2} \qquad [15]$$

From the figure,

$$\cos\beta = \frac{\overline{AB}}{\overline{BC}} = \frac{P' \times \overline{AB}}{P' \times \overline{BC}} = \frac{C_x}{C'} \qquad [16]$$

Equations 15 and 16 prove that the relationship of the couple vectors C_x, C_y, and C' is that of the sides and diagonal of a rectangle, showing that the parallelogram law applies, and that C' is the resultant of C_x and C_y.

The foregoing proof is sufficient to show that the resultant of any two couples in perpendicular planes is a couple, and that its vector is the resultant of their vectors. The couple $P'Q'$ and the remaining original couple, SS, are in perpendicular planes; it follows that their resultant is a couple and that its vector is the resultant of vectors C' and C_z. Since couple $P'Q'$ is the resultant of the original couples PP and QQ, and since vector C' is the resultant of vectors C_x and C_y, the general principle stated at the beginning of the article may be considered proved.

Although not needed in the foregoing discussion, a specific resultant couple, RR, has been shown in Fig. 93. It was constructed by replacing original couple SS by an equivalent couple $S'S'$ having vertical forces passing through the points D and E, and by compounding the forces of couple $P'Q'$ with the forces of $S'S'$, at those points.

The vector, C, of the final resultant couple has also been shown in the figure. The relationship of vectors C_x, C_y, C_z, and C is the same as that of R_x, R_y, R_z, and R in Fig. 80, Art. 39, and the algebraic formulas for the present case differ from those of Art. 39 in notation, only.

Formulas.

$$C = \sqrt{C_x^2 + C_y^2 + C_z^2} \qquad [17]$$

$$\cos \theta_x = \frac{C_x}{C} \qquad [18]$$

$$\cos \theta_y = \frac{C_y}{C} \qquad [19]$$

$$\cos \theta_z = \frac{C_z}{C} \qquad [20]$$

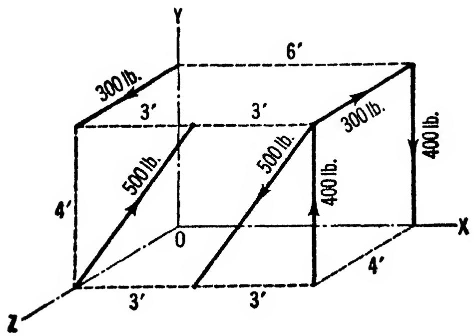

Fig. 94

Illustrative Problem

195. Find the resultant of the three couples shown in Fig. 94.

Solution. Let C_x, C_y, and C_z, in Fig. 95, represent the vectors of the three given couples, and let C represent the vector of the resultant couple.

From Fig. 94,

$$C_x = -400 \times 4 = -1600 \text{ ft-lb}$$

$$C_y = +300 \times 6 = +1800 \text{ ft-lb}$$

$$C_z = -500 \times (3 \times \tfrac{4}{5}) = -1200 \text{ ft-lb}$$

$$C = \sqrt{C_x^2 + C_y^2 + C_z^2} \qquad C = \sqrt{(1600)^2 + (1800)^2 + (1200)^2}$$

$$= 2690 \text{ ft-lb}$$

$$\cos\theta_x = \frac{C_x}{C} \qquad \cos\theta_x = \frac{1600}{2690} = 0.595 \qquad \theta_x = 53° \, 30'$$

$$\cos\theta_y = \frac{C_y}{C} \qquad \cos\theta_y = \frac{1800}{2690} = 0.669 \qquad \theta_y = 48° \, 00'$$

$$\cos\theta_z = \frac{C_z}{C} \qquad \cos\theta_z = \frac{1200}{2690} = 0.446 \qquad \theta_z = 63° \, 30'$$

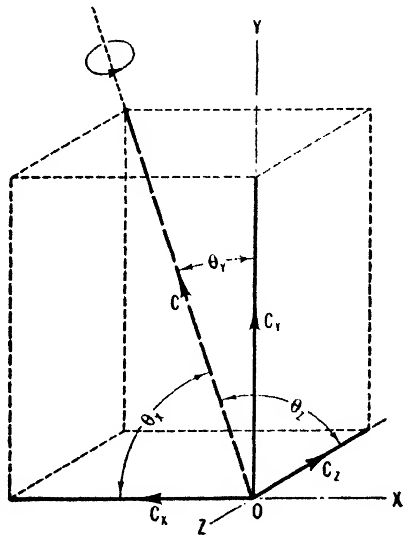

FIG. 95

Figure 95 shows the position of C, the vector of the resultant couple. The resultant of the three given couples is, then, any couple whose moment is 2690 ft-lb, lying in any plane at right angles to the vector C, and whose rotational tendency agrees with that indicated by the curved arrow shown in Fig. 95.

The solution given above is sufficient for the problem as stated, since it is not required that a specific resultant couple be found.

PROBLEMS

196. Three couples whose moments are: +27 in-tons, −36 in-tons, and −108 in-tons, lie in the *yz*-, *xz*-, and *xy*-planes, respectively. Find the vector of the resultant couple. *Ans.* 117 in-tons; $\theta_x = 76° 40'$, $\theta_y = 107° 55'$, $\theta_z = 157° 20'$.

197. Three couples, whose moments are: −800 ft-lb, −1920 ft-lb, and +3900 ft-lb, lie in planes at right angles to the *x*-, *y*-, and *z*-axes, respectively. ·Find the vector of the resultant couple.

198. In Fig. 93, assume that couple *PP* has 40-lb forces and an 8-in. arm, that couple *QQ* has 50-lb forces and a 6-in. arm, and that couple *SS* has 30-lb forces and a 10-in. arm. Find the vector of the resultant couple. *Ans.* 531 in-lb; $\theta_x = 52° 55'$, $\theta_y = 55° 35'$, $\theta_z = 55° 35'$.

199. Find the resultant of the three couples shown in Fig. 96.

200. Find the resultant of the system shown in Fig. 97. *Ans.* 180 ft-lb; $\theta_x = 109° 25'$, $\theta_y = 131° 50'$, $\theta_z = 48° 10'$.

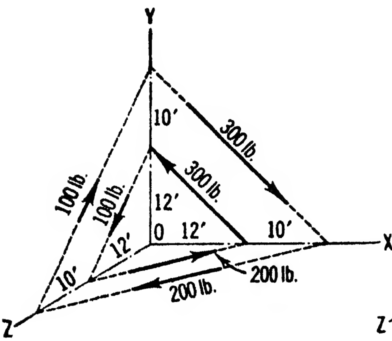

FIG. 96

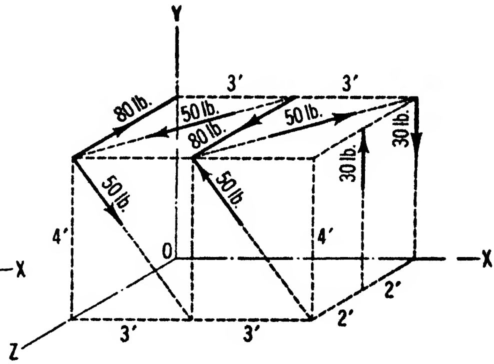

FIG. 97

45. Resolution of a Couple into Three Component Couples in Perpendicular Planes. *The vector of the given couple may be resolved into three component vectors at right angles to the planes in which it is desired to place the three component couples. The component vectors thus obtained are the vectors of the desired component couples.*

Proof. The procedure suggested above is merely the reverse of that described in Art. 44, and the proof in that article is sufficient for both cases.

Ordinarily the *x*-, *y*-, and *z*-axes are used in resolving the vector of the given couple, and the component couples are placed in the coordinate planes. A component couple in the *yz*-plane would be referred to as the "*x*-component couple," since its vector would be parallel to the *x*-axis.

First Method. If the problem is one in which the magnitude and inclination of the vector of the given couple are stated, or can be ascertained readily, the vector can be resolved into components by the method described for a force, in Art. 40.

Second Method. In the majority of cases the couple itself, and not its vector, is given, and it is usually difficult to make the computations that would be necessary in establishing the vector and in resolving it into its components. In such cases the resolution can be performed by means of the following principle:

The component, along any axis, of the vector of a given couple is equal to the moment-sum of the forces of the couple about that axis, and the sign of the component agrees with the sign of the moment-sum of the forces.

Proof. In Art. 44, Fig. 93, PP, QQ, and SS were used to represent any three couples in perpendicular planes, and RR was shown to be the resultant couple.

Let RR now be thought of as the given couple. Its vector is C, Fig. 93. Couple PP is the x-component couple, and C_x is its vector. It will now be proved that C_x is equal to the moment-sum of the two forces of the given couple, RR, about the x-axis.

Figure 93 shows one of the forces of the couple RR resolved into components S' and P' at the point D, and the other force resolved into the components S' and Q' at the point E.

By the principle of moments, the moment-sum of the two forces RR about the x-axis is equal to the moment-sum of their four components, S', P', S', and Q', about that axis. By inspection, the moments of three of these components, S', S', and Q', are equal to zero. Therefore, the moment-sum of R, R is equal to the moment of P' alone. The moment of P' is equal to $P' \times \overline{AB}$, which, in turn, is equal to the moment of couple $P'P'$ and to the moment of the x-component couple, PP. C_x is the vector of couple PP.

Furthermore, the figure shows that the sign of the component vector C_x agrees with the sign of the moment-sum of the forces of the given couple, RR, about the x-axis.

Proofs involving the other two component vectors, C_y and C_z, would be similar to that given above for C_x.

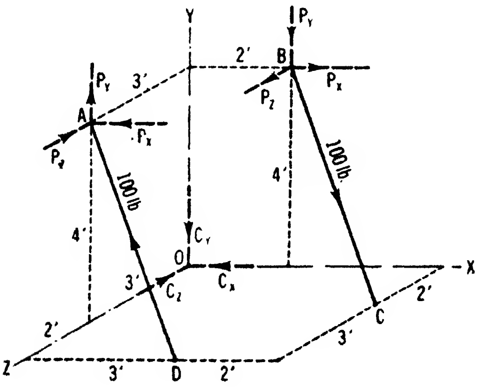

Fig. 98

Illustrative Problem

201. Resolve the couple shown in Fig. 98 into its x-, y-, and z-component couples.

Solution. The second method of resolution described in the present article is preferable in this problem. Resolve the two forces of the given couple into their x-, y-, and z-components at the points A and B. From the figure, $\overline{BC} = \overline{AD} = \sqrt{(3)^2 + (4)^2 + (2)^2} = \sqrt{29} = 5.39$ ft. For both forces, $\cos\theta_x = \dfrac{3}{5.39}$, $\cos\theta_y = \dfrac{4}{5.39}$, and $\cos\theta_z = \dfrac{2}{5.39}$.

$$P_x = P\cos\theta_x \qquad P_x = 100 \times \frac{3}{5.39} = 55.7 \text{ lb}$$

$$P_y = P\cos\theta_y \qquad P_y = 100 \times \frac{4}{5.39} = 74.2 \text{ lb}$$

$$P_z = P\cos\theta_z \qquad P_z = 100 \times \frac{2}{5.39} = 37.1 \text{ lb}$$

Now find the x-, y-, and z-components of the vector of the given couple, by calculating the moment-sum of the forces about each coordinate axis.

$$C_x = -P_y \times 3 - P_z \times 4 + P_z \times 4 = -74.2 \times 3 = -223 \text{ ft-lb}$$

$$C_y = -P_x \times 3 - P_z \times 2 = -55.7 \times 3 - 37.1 \times 2 = -241 \text{ ft-lb}$$

$$C_z = +P_x \times 4 - P_x \times 4 - P_y \times 2 = -74.2 \times 2 = -148 \text{ ft-lb}$$

The vectors C_x, C_y, and C_z, representing the three component couples, have been shown in the figure, at O.

Summarizing, the original couple is equivalent to any system of three couples of the following nature: any couple in any plane perpendicular to the x-axis, having a clockwise moment of 223 ft-lb; any couple in any plane perpendicular to the y-axis, having a clockwise moment of 241 ft-lb; and any couple in any plane perpendicular to the z-axis, having a clockwise moment of 148 ft-lb.

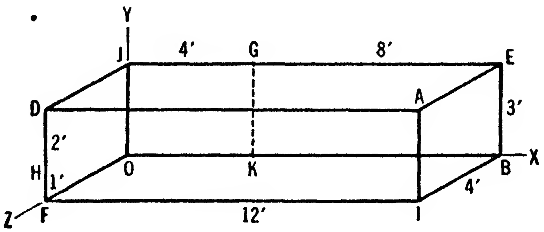

Fig. 99

PROBLEMS

202. A force of 200 lb acts from O through D, in Fig. 99. A second force of 200 lb acts from A through B. Resolve the couple thus formed into its x-, y-, and z-component couples. *Ans.* 0; +1920 ft-lb; −1440 ft-lb.

203. A force of 10 tons acts from E through D, in Fig. 99. A second force of 10 tons acts from F through B. Resolve the couple thus formed into its x-, y-, and z-component couples.

204. A force of 750 lb acts from G through H, in Fig. 99. An equal, opposite, and parallel force acts through A. Resolve the couple thus formed into its x-, y-, and z-component couples. *Ans.* -1000 ft-lb; $+6000$ ft-lb; $+2000$ ft-lb.

205. Resolve the couple shown in Fig. 90, Art. 42, into its x-, y-, and z-component couples.

206. A force of 39 tons acts from O through A, in Fig. 99. An equal, opposite, and parallel force acts through B. Resolve the couple thus formed into its x-, y-, and z-component couples. *Ans.* 0; $+144$ ft-tons; -108 ft-tons.

207. A force of 6 tons acts from the point $(0, 0, +4')$ through the point $(+8', 0, +8')$. An equal, opposite, and parallel force acts through the point $(+8', +5', +8')$. Resolve the couple thus formed into its x-, y-, and z-component couples.

208. A force of 650 lb acts from A through O, in Fig. 99. An equal, opposite, and parallel force acts through G. It is desired to replace this couple by an equivalent system of three couples, as follows: one whose forces lie on the lines AI and EB, one whose forces lie on the lines AD and EJ, and one whose forces lie on the lines JO and GK. Find the magnitude and sense of each of the six forces.

46. Resultant of a System of Non-Coplanar Couples. In the solution of this problem a set of three coordinate axes is selected, and each couple of the given system is resolved into its x-, y-, and z-component couples, by one of the methods described in Art. 45.

The algebraic sum of the moments of all the x-component couples is then calculated. This gives the moment of their resultant, which is a couple in a plane at right angles to the x-axis. The y-component couples, and finally the z-component couples, are treated in a similar manner.

The three resultant couples thus obtained are the components of the final resultant couple, which is then calculated by the method of Art. 44.

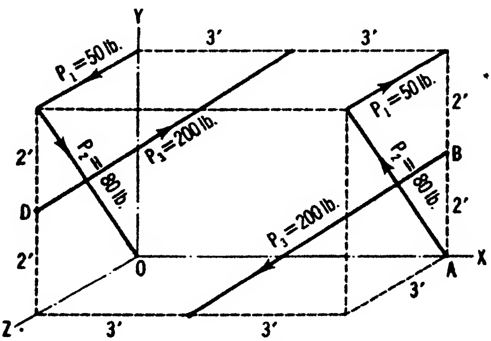

FIG. 100

ILLUSTRATIVE PROBLEM

209. Find the resultant of the three non-coplanar couples shown in Fig. 100.

Solution. Let C_1, C_2, and C_3 represent the vectors of the given couples, P_1P_1, P_2P_2, and P_3P_3, respectively. The vectors need not be shown in the figure. Calculate the x-, y-, and z-components of the individual forces of the couples. By inspection, $P_{1x} = 0$, $P_{1y} = 0$, and $P_{1z} = P_1 = 50$ lb. Also, $P_{2x} = 0$.

$$P_{2y} = P_2 \cos \theta_{2y} \qquad P_{2y} = 80 \times \frac{4}{5} = 64 \text{ lb}$$

$$P_{2z} = P_2 \cos \theta_{2z} \qquad P_{2z} = 80 \times \frac{3}{5} = 48 \text{ lb}$$

$$P_{3x} = P_3 \cos \theta_{3x} \qquad P_{3x} = 200 \times \frac{3}{4.69} = 128 \text{ lb}$$

$$P_{3y} = P_3 \cos \theta_{3y} \qquad P_{3y} = 200 \times \frac{2}{4.69} = 85.3 \text{ lb}$$

$$P_{3z} = P_3 \cos \theta_{3z} \qquad P_{3z} = 200 \times \frac{3}{4.69} = 128 \text{ lb}$$

The force acting at O need not be resolved, since its moment about each coordinate axis is zero. Resolve the other force P_2 at A, and resolve the forces P_3, P_3 at B and D. Find the x-, y-, and z-components of the vector of each couple, by the second method, Art. 45.

$$C_{1x} = \Sigma M_{1x} \qquad C_{1x} = +P_1 \times 4 - P_1 \times 4 = 0$$

$$C_{1y} = \Sigma M_{1y} \qquad C_{1y} = +P_1 \times 6 = +50 \times 6 = +300 \text{ ft-lb}$$

$$C_{1z} = \Sigma M_{1z} \qquad C_{1z} = 0$$

$$C_{2x} = \Sigma M_{2x} \qquad C_{2x} = 0$$

$$C_{2y} = \Sigma M_{2y} \qquad C_{2y} = -P_{2z} \times 6 = -48 \times 6 = -288 \text{ ft-lb}$$

$$C_{2z} = \Sigma M_{2z} \qquad C_{2z} = +P_{2y} \times 6 = +64 \times 6 = +384 \text{ ft-lb}$$

$$C_{3x} = \Sigma M_{3x} \qquad C_{3x} = +P_{3z} \times 2 - P_{3z} \times 2 - P_{3y} \times 3 = -85.3 \times 3 = -256 \text{ ft-lb}$$

$$C_{3y} = \Sigma M_{3y} \qquad C_{3y} = -P_{3z} \times 6 + P_{3x} \times 3 = -128 \times 6 + 128 \times 3 = -384 \text{ ft-lb}$$

$$C_{3z} = \Sigma M_{3z} \qquad C_{3z} = +P_{3x} \times 2 - P_{3y} \times 6 - P_{3x} \times 2 = -85.3 \times 6 = -512 \text{ ft-lb}$$

The components of the resultant couple are, then,

$$\Sigma C_x = 0 + 0 - 256 = -256 \text{ ft-lb}$$

$$\Sigma C_y = +300 - 288 - 384 = -372 \text{ ft-lb}$$

$$\Sigma C_z = 0 + 384 - 512 = -128 \text{ ft-lb}$$

and the resultant couple is

$$C = \sqrt{(\Sigma C_x)^2 + (\Sigma C_y)^2 + (\Sigma C_z)^2} \qquad C = \sqrt{(256)^2 + (372)^2 + (128)^2}$$
$$= 469 \text{ ft-lb}$$

$$\cos\theta_x = \frac{\Sigma C_x}{C} \qquad \cos\theta_x = \frac{256}{469} = 0.546 \qquad \theta_x = 56°\ 55'$$

$$\cos\theta_y = \frac{\Sigma C_y}{C} \qquad \cos\theta_y = \frac{372}{469} = 0.793 \qquad \theta_y = 37°\ 30'$$

$$\cos\theta_z = \frac{\Sigma C_z}{C} \qquad \cos\theta_z = \frac{128}{469} = 0.273 \qquad \theta_z = 74°\ 10'$$

Figure 101 shows the component vectors, C_x, C_y, and C_z, and the resultant vector, C. The resultant of the original system is any couple having a moment of 469 ft-lb, lying in any plane at right angles to the vector C, and whose rotational tendency agrees with that indicated by the curved arrow shown in the figure.

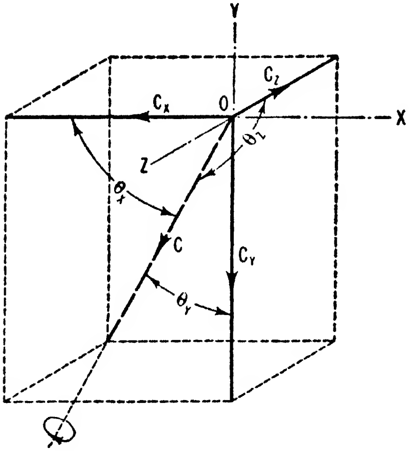

Fig. 101

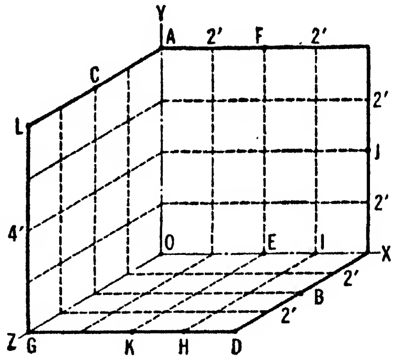

Fig. 102

PROBLEMS

210. Insert the following forces in Fig. 102: 180 lb, from A through B; 180 lb, from D through C; 360 lb, from A through O; 360 lb, from E through F; 140 lb, from O through G; 140 lb, from H through I. Find the resultant. *Ans.* 780 ft-lb; $\theta_x = 107°\ 55'$, $\theta_y = 76°\ 40'$, $\theta_z = 22°\ 40'$.

211. Insert the following forces in Fig. 102: 6 tons, from G through J, and an equal, opposite, and parallel force through O; 5 tons, from A through I, and an e.o.p. force through O; 3 tons, from C through K, and an e.o.p. force through G. Find the resultant.

212. Insert the following forces in Fig. 102: 1500 lb, from K through A, and an equal, opposite, and parallel force through G; 500 lb, from O through E, and an e.o.p. force through D; 1000 lb, from G through L, and an e.o.p. force through K. Find the resultant.

213. Insert the following forces in Fig. 102: 24 tons, from D through F, and an equal, opposite, and parallel force through C; 20 tons, from I through G, and an e.o.p. force through O; 8 tons, from L through A, and an e.o.p. force through O. Find the resultant. *Ans.* +32 ft-tons; in any plane perpendicular to the z-axis.

214. Insert the following forces in Fig. 99, Art. 45: 91 lb, from E through F, and an equal, opposite, and parallel force through K; 50 lb, from J through K, and an e.o.p. force through D; 60 lb, from A through I, and an e.o.p. force through B; 80 lb, from A through D, and an e.o.p. force through F. Find the resultant.

215. Find the resultant of the following system: 80 lb, upward, coinciding with the y-axis, and an equal, opposite, and parallel force through the point $(0, 0, +3')$; 50 lb, from $(0, +4', 0)$ through $(+3', 0, 0)$, and an e.o.p. force through $(+7', 0, 0)$; 65 lb, from $(+4', +6', 0)$ through $(0, +3', +12')$, and an e.o.p. force through $(-5', +3', 0)$. *Ans.* 689 ft-lb; $\theta_x = 52° 25'$, $\theta_y = 141° 40'$, $\theta_z = 82° 55'$.

47. Resultant of the Non-Coplanar Parallel System. *Principle of Components. The algebraic sum of the forces of a non-coplanar parallel system is equal to the resultant of the system, if the resultant is a single force; and is equal to zero, if the resultant is a couple.*

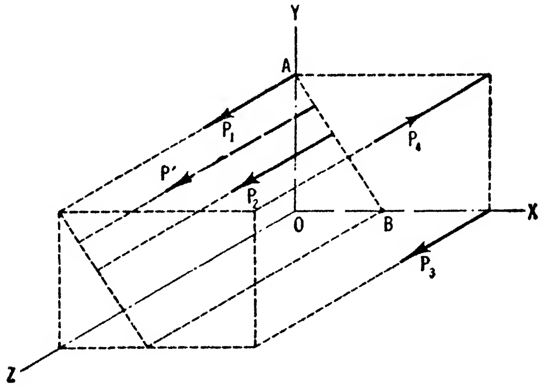

FIG. 103

Proof. In Fig. 103, let P_1, P_2, P_3, and P_4 represent any non-coplanar parallel system. The z-axis has been placed parallel to the forces. Let P' represent the resultant of P_1 and P_2. P' is coplanar with, and parallel to, P_1 and P_2, and is equal to their algebraic sum.

P' could now be compounded with P_3, giving a second resultant, P'', which would be not only the resultant of P' and P_3, but also the resultant of the three original forces, P_1, P_2, and P_3, and would be equal to their algebraic sum.

If it were found that P'' and the remaining original force, P_4, were equal, and opposite in sense, they would constitute a couple. The couple would be the resultant of the original system. Then the algebraic sum of P'' and P_4 would be zero; therefore, the algebraic sum of the forces of the original system would be zero.

If P'' and P_4 were not equal and opposite they could be further compounded into a single force, which would be the resultant of the original system. It would be equal to the algebraic sum of the forces of the system and would be parallel to them.

Principle of Moments. The algebraic sum of the moments of the forces of a non-coplanar parallel system, about any axis at right angles to the forces of the system, is equal to the moment of the resultant about that axis.

Proof. In Fig. 103, the point B is on the x-axis, and is also in the plane containing P_1 and P_2. By Art. 33, the moment-sum of P_1 and P_2, about B, is equal to the moment of their resultant, P', about that point. If this relation is written in the form of an equation, and is multiplied through by the sine of angle OBA, it will be proved that the moment-sum of P_1 and P_2 about the x-axis is equal to the moment of P' about that axis. With this result as a basis the principle of moments can be established for the entire system, by a process of reasoning similar to that used above in connection with the principle of components.

In the case in which the resultant is a couple, the phrase "moment of the resultant" is interpreted as meaning the moment-sum of the forces of the resultant couple about the chosen axis.

The principle of moments could be generalized to include any axis, but axes perpendicular to the forces are more convenient and can be employed in the great majority of cases.

Application. First, the algebraic sum of the forces is calculated. If this sum is not zero, the resultant is a single force, and its magnitude and sense are shown by the result of the summation. Two coordinates are necessary in showing the position of the line of action of the resultant. These are calculated by means of the principle of moments.

If the summation of the forces is zero, the resultant, if one exists, is a couple and lies in a plane parallel to the forces of the system. The components of the vector of the resultant couple are found by calculating the moment-sums of the forces of the system about two axes at right angles to the forces. The vector of the resultant couple can then be found.

If the moment-sums about two non-parallel axes at right angles to the forces are equal to zero the system has no resultant and is in equilibrium.

Illustrative Problems

216. Find the resultant of the four non-coplanar parallel forces shown in Fig. 104. The forces are parallel to the z-axis and pierce the xy-plane at the points shown.

Solution. First, calculate the algebraic sum of the forces.

$$R = +100 - 700 + 500 - 300 = -400 \text{ lb}$$

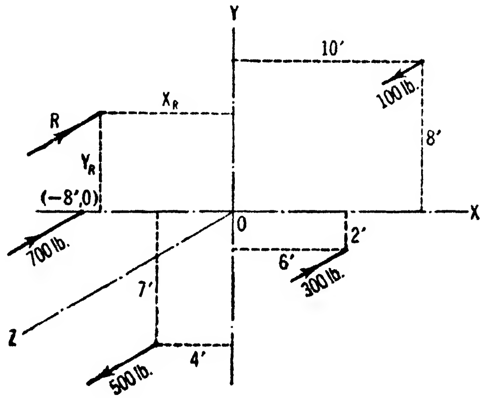

Fig. 104

It is now known that the resultant is a single force, of 400 lb, parallel to the forces of the system, and that it acts in the negative direction (away from the reader). Calculate the moment-sum of the forces, about the y-axis.

$$\Sigma M_y = -100 \times 10 - 700 \times 8 + 500 \times 4 + 300 \times 6 = -2800 \text{ ft-lb}$$

It is now known, by the principle of moments, that the resultant lies in such a position that it has a negative (clockwise) moment of 2800 ft-lb about the y-axis. Now calculate the moment-arm of the resultant, without regard to sign.

$$\Sigma M_y = R \times (x_R) \qquad x_R = \frac{\Sigma M_y}{R} = \frac{2800}{400} = 7.00 \text{ ft}$$

This shows that the line of action of R is 7 ft from the y-axis.

Ascertain whether R lies to the right or to the left of the y-axis by the following process: R acts away from the reader (Fig. 104), and has a clockwise moment about the y-axis. To fulfill these requirements it must lie to the left of that axis.

Now calculate the moment-sum of the forces about the x-axis.

$$\Sigma M_x = +100 \times 8 - 500 \times 7 + 300 \times 2 = -2100 \text{ ft-lb}$$

$$\Sigma M_x = R \times (y_R) \qquad y_R = \frac{\Sigma M_x}{R} = \frac{2100}{400} = 5.25 \text{ ft}$$

Since the resultant acts away from the reader, and has a clockwise moment about the x-axis, its line of action must lie above that axis.

Summarizing, the resultant of the given system is a single force of 400 lb, parallel to the z-axis, negative in sense, and piercing the xy-plane at the point $(-7', +5.25', 0)$.

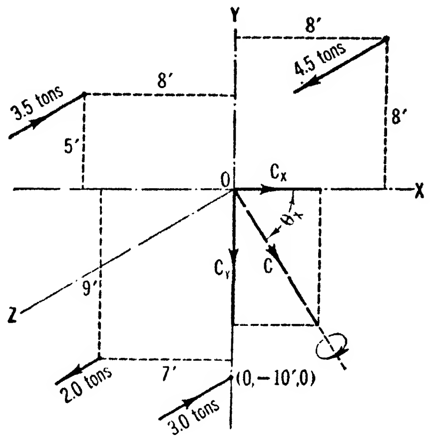

FIG. 105

217. Find the resultant of the four non-coplanar parallel forces shown in Fig. 105. The forces are parallel to the z-axis.

Solution. Calculate the algebraic sum of the forces.

$$R = +4.5 - 3.5 + 2.0 - 3.0 = 0$$

This shows that the resultant, if one exists, is a couple.

Calculate the moment-sum of the forces about the x-axis, and also about the y-axis.

$$C_x = +4.5 \times 8 - 3.5 \times 5 - 2.0 \times 9 + 3.0 \times 10 = +30.5 \text{ ft-tons}$$

$$C_y = -4.5 \times 8 - 3.5 \times 8 + 2.0 \times 7 = -50.0 \text{ ft-tons}$$

C_x and C_y, the vectors of the x- and y-component couples, have been shown at O, in Fig. 105. Compound C_x and C_y, as follows:

$$C = \sqrt{C_x^2 + C_y^2} \qquad C = \sqrt{(30.5)^2 + (50)^2} = 58.6 \text{ ft-tons}$$

$$\cos\theta_x = \frac{C_x}{C} \qquad \cos\theta_x = \frac{30.5}{58.6} = 0.520 \qquad \theta_x = 58^\circ\, 40'$$

Figure 105 shows C_x, C_y, and C in their correct relationship. Summarizing, the resultant of the system is any couple having a moment of 58.6 ft-tons, lying in any plane at right angles to the vector C, and whose rotational tendency agrees with that indicated by the curved arrow, Fig. 105.

PROBLEMS

218. The following forces are parallel to the z-axis and pierce the xy-plane at the points indicated: $+200$ lb $(+10', +8')$; -300 lb $(+5', -8')$; $+400$ lb $(-10', -3')$; -500 lb $(-7', +8')$. Find the resultant. *Ans.* -200 lb; $(0, +6')$.

219. The following forces are parallel to the z-axis and pierce the xy-plane at the points indicated: -2 tons $(+12', +10')$; $+4$ tons $(+6', -8')$; $+6$ tons $(-8', -2')$; -8 tons $(-6', +4')$. Find the resultant.

220. The following forces are parallel to the z-axis and pierce the xy-plane at the points indicated: $+650$ lb $(+6', +6')$; $+800$ lb $(+3', -5')$; -1000 lb $(+6', -8')$; -950 lb $(-4', +2')$. Find the resultant. *Ans.* -500 lb; $(-8.2', -12')$.

221. The following forces are parallel to the z-axis and pierce the xy-plane at the points indicated: -10 tons $(+6', +15')$; $+12$ tons $(-5', +3')$; -6 tons $(-6', -8')$; $+16$ tons $(+12', 0)$. Find the resultant.

222. The following forces are parallel to the z-axis and pierce the xy-plane at the points indicated: -2000 lb $(+16', +4')$; $+4000$ lb $(0, -10')$; $+6000$ lb $(-12', 0)$; -8000 lb $(-13', -6')$. Find the resultant.

223. The following forces are parallel to the z-axis and pierce the xy-plane at the points indicated: -100 lb $(-7', -7')$; $+300$ lb $(-5', +7')$; $+500$ lb $(+6', -4')$; -700 lb $(+10', +4')$. Find the resultant. *Ans.* A couple; $C = 5200$ ft-lb; $\theta_x = 112° \, 40'$; $\theta_z = 90°$.

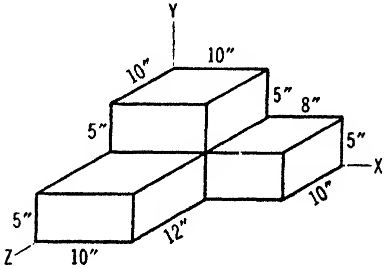

FIG. 106

224. The process of locating the center of gravity of a body is similar to that of locating the resultant of a system of parallel forces. Figure 106 represents a homogeneous body. Divide the body into rectangular prisms and calculate the volume of each. Represent each volume by a vector parallel to the z-axis, positive in sense, passing through the center of gravity of the prism. The resultant of these vectors will pass through the center of gravity of the entire body. Find the x- and y-coordinates of the center of gravity by locating the resultant. Then turn the vectors until they are parallel to the x-axis, and find the z-coordinate by a similar process. *Ans.* $(+6.8'', +3.75'', +8.3'')$.

48. Resultant of the General Non-Coplanar System. This is the general force system, free from all geometric restrictions. The method of compounding this system, as it will be developed, is an adaptation of methods previously used in this book.

Theory. In Fig. 107, let P_1 represent one of the forces of any general non-coplanar system. Let any convenient set of rectangular coordinate axes be introduced.

Insert at O two forces, P_1' and P_1'', opposite to each other in sense, parallel to P_1 and equal to it in magnitude. Since P_1' and P_1'' are in themselves balanced, their insertion does not change the effect of the system.

Careful attention should now be paid to two significant facts: namely, the fact that P_1 and P_1'' are equal, opposite, and parallel, thus constituting a couple; and the fact that in addition to the couple there is a single force, P_1', which acts through O, but which otherwise is an exact duplicate of the original force, P_1.

Now let it be imagined that the remaining forces of the original system (not shown in the figure) are treated precisely as was P_1, the point O being utilized in each case. The original system is thereby transformed into two elementary systems, namely, a system of non-coplanar

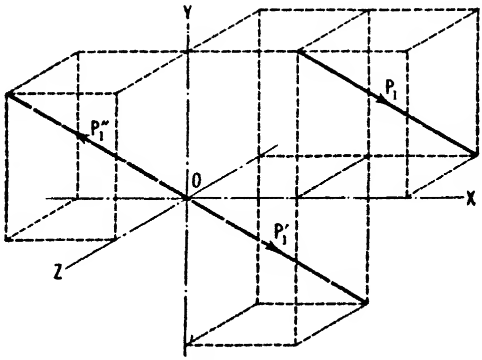

FIG. 107

couples, and a non-coplanar concurrent system of forces acting through the point O. The resultants of these two systems can then be found by the methods developed in Arts. 41 and 46.

The resultant of a concurrent system is a single force, and the resultant of a system of couples is a couple. Therefore, the process described above will eventuate in a resultant single force through O, and a couple in space. Ordinarily, further composition would not be possible, and a full description of the single-force resultant at O, and of the vector of the resultant couple, would be considered as completing the solution.

Application. The actual steps in the solution of a problem follow the foregoing theory, except that the various forces which are introduced at O in the explanation of the theory need not be shown or dealt with in an actual solution.

For example, in compounding the concurrent system at O, the x-, y-, and z-components of its forces are needed. Each force is the duplicate of one of the original forces. Therefore, it is merely necessary to calculate the components of the original forces, and the solution can proceed without the necessity of showing the concurrent system on the figure.

Also, in compounding the system of couples, the components of the vector of each couple are needed. In Art. 45 it is shown that the component of a couple-vector, along any axis, is equal to the moment-sum of the forces of the couple about that axis. Each of the couples has a force passing through O, such as P_1'', Fig. 107. The moment of this force is zero, about each of the coordinate axes. Therefore, the moment-sum of the two forces of the couple is equal to the moment of the one belonging to the original system. It follows that the components of the couple-vectors can be found by calculating the moments of the original forces, about the coordinate axes. The forces acting at O need not be shown.

Further Simplification. If the following equation is satisfied, the resultant single force at O is parallel to the plane of the resultant couple:

$$R_x \times C_x + R_y \times C_y + R_z \times C_z = 0 \qquad [21]$$

in which R_x, R_y, and R_z are the components of the resultant single force and C_x, C_y, and C_z are the components of the vector of the resultant couple.

In such a case, the resultant couple can be shifted into a parallel plane containing the resultant single force, and the two can then be compounded into a final resultant single force by the method of Art. 37.

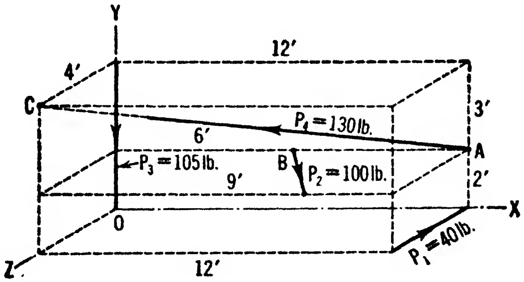

FIG. 108

ILLUSTRATIVE PROBLEM

225. Find the resultant of the four non-coplanar forces shown in Fig. 108.

Solution. Resolve each of the forces into its x-, y-, and z-components. From the figure, $\overline{AC} = \sqrt{(12)^2 + (3)^2 + (4)^2} = \sqrt{169} = 13$ ft.

$P_{1x} = 0$	$P_{1y} = 0$	$P_{1z} = -40$ lb
$P_{2x} = +100 \times \frac{3}{5} = +60$ lb	$P_{2y} = 0$	$P_{2z} = +100 \times \frac{4}{5} = +80$ lb
$P_{3x} = 0$	$P_{3y} = -105$ lb	$P_{3z} = 0$
$P_{4x} = -130 \times \frac{12}{13} = -120$ lb	$P_{4y} = +130 \times \frac{3}{13} = +30$ lb	$P_{4z} = +130 \times \frac{4}{13} = +40$ lb

Resultant of the Concurrent Forces at O. The point O will be chosen as the point of concurrence for the system of single forces.

$$R_x = 0 + 60 + 0 - 120 = -60 \text{ lb}$$

$$R_y = 0 + 0 - 105 + 30 = -75 \text{ lb}$$

$$R_z = -40 + 80 + 0 + 40 = +80 \text{ lb}$$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \qquad R = \sqrt{(60)^2 + (75)^2 + (80)^2} = 125 \text{ lb}$$

$$\cos\theta_x = \frac{R_x}{R} \qquad \cos\theta_x = \frac{60}{125} = 0.480 \qquad \theta_x = 61^\circ\, 20'$$

$$\cos\theta_y = \frac{R_y}{R} \qquad \cos\theta_y = \frac{75}{125} = 0.600 \qquad \theta_y = 53^\circ\, 10'$$

$$\cos\theta_z = \frac{R_z}{R} \qquad \cos\theta_z = \frac{80}{125} = 0.640 \qquad \theta_z = 50^\circ\, 10'$$

Figure 109 shows R_x, R_y, R_z, and R in their correct positions with reference to the coordinate axes.

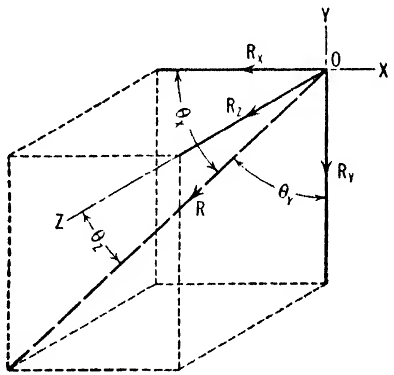

FIG. 109

Resultant of the Non-Coplanar Couples. Calculate the components of the vector of each couple by calculating the moments of the forces of the couples, about the coordinate axes. Resolve P_2 at B, and P_4 at A, for this purpose.

$$C_{1x} = 0 \qquad C_{1y} = +40 \times 12 = +480 \text{ ft-lb} \qquad C_{1z} = 0$$

$$C_{2x} = +P_{2z} \times 2 = +80 \times 2 = +160 \text{ ft-lb}$$

$$C_{2y} = -P_{2z} \times 6 = -80 \times 6 = -480 \text{ ft-lb}$$

$$C_{2z} = -P_{2x} \times 2 = -60 \times 2 = -120 \text{ ft-lb}$$

$$C_{3x} = 0 \qquad C_{3y} = 0 \qquad C_{3z} = 0$$

$$C_{4x} = +P_{4z} \times 2 = +40 \times 2 = +80 \text{ ft-lb}$$
$$C_{4y} = -P_{4z} \times 12 = -40 \times 12 = -480 \text{ ft-lb}$$
$$C_{4z} = +P_{4x} \times 2 + P_{4y} \times 12 = +120 \times 2 + 30 \times 12 = +600 \text{ ft-lb}$$

Now calculate C, the vector of the resultant couple.

$$C_x = 0 + 160 + 0 + 80 = +240 \text{ ft-lb}$$
$$C_y = +480 - 480 + 0 - 480 = -480 \text{ ft-lb}$$
$$C_z = 0 - 120 + 0 + 600 = +480 \text{ ft-lb}$$

$$C = \sqrt{C_x^2 + C_y^2 + C_z^2} \qquad C = \sqrt{(240)^2 + (480)^2 + (480)^2} = 720 \text{ ft-lb}$$

$$\cos\theta_x = \frac{C_x}{C} \qquad \cos\theta_x = \frac{240}{720} = 0.333 \qquad \theta_x = 70° \, 35'$$

$$\cos\theta_y = \frac{C_y}{C} \qquad \cos\theta_y = \frac{480}{720} = 0.667 \qquad \theta_y = 48° \, 10'$$

$$\cos\theta_z = \frac{C_z}{C} \qquad \cos\theta_z = \frac{480}{720} = 0.667 \qquad \theta_z = 48° \, 10'$$

Figure 110 shows C_x, C_y, C_z, and C in their correct relative positions, the vectors having been placed, for convenience, at O. Now substitute in Eq. 21, to ascertain whether the resultant force is parallel to the plane of the resultant couple.

$$(-60) \times (+240) + (-75) \times (-480) + (+80) \times (+480) = 60{,}000$$

Equation 21 is not satisfied; therefore, the system cannot be simplified further.

Summarizing, the resultant of the given system consists of a single force of 125 lb, acting through O in the manner indicated in Fig. 109; and any couple having a moment of 720 ft-lb, lying in any plane at right angles to the vector C shown in Fig. 110, and having the rotational sense indicated by the curved arrow.

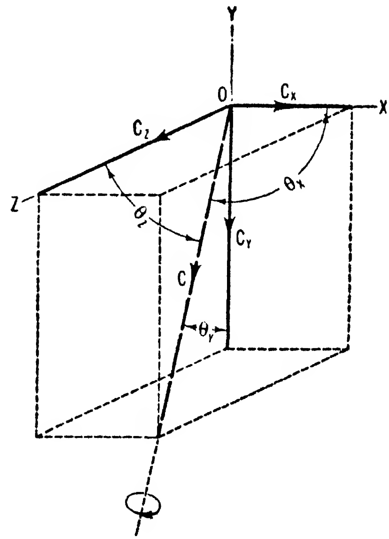

Fig. 110

PROBLEMS

226. Find the resultant of the following system: 600 lb, acting from the point (+10′, +10′, 0) through the point (+10′, 0, 0); 750 lb, from (0, +5′, 0) through (0, +2′, +4′); 900 lb, from (+8′, +2′, +4′) through (0, +10′, 0); 1500 lb, from

(0, 0, 0) through (0, 0, +10′). *Ans.* 1950 lb, $\theta_x = 107°\,55'$, $\theta_y = 103°\,20'$, $\theta_z = 22°\,40'$; no couple.

227. Find the resultant of the system shown in Fig. 111. *Ans.* 50 tons, $\theta_x = 126°\,50'$, $\theta_y = 61°\,20'$, $\theta_z = 129°\,50'$; 170 ft-tons, $\theta_x = 58°\,05'$, $\theta_y = 134°\,55'$, $\theta_z = 118°\,05'$.

228. Find the resultant of the following system: 300 lb, acting from the point (0, +12′, +15′) through the point (0, +12′, 0); 400 lb, from (+16′, 0, 0) through (0, +12′, 0); 320 lb, from (0, −18′, +15′) through (+10′, −18′, +15′); 480 lb, from (+16′, 0, 0) through (0, 0, 0); 500 lb, from (+16′, +12′, 0) through (0, 0, +15′).

229. Find the resultant of the following system: 5 tons, acting from the point (0, +4′, 0) through the point (0, 0, −3′); 6 tons, from (0, −5.5′, 0) through (0, −5.5′, +10′); 12 tons, from (0, 0, 0) through (+12′, 0, 0); 13 tons, from (+12′, 0, 0) through (0, +4′, −3′).

230. Find the resultant of the following system: 275 lb, acting from the point (0, 0, 0) through the point (+12′, 0, 0); 300 lb, from (0, +12′, 0) through (0, 0, 0); 500 lb, from (0, +3′, +8′) through (+6′, +3′, 0); 575 lb, from (+12′, +12′, 0) through (0, +12′, 0); 850 lb, from (0, +12′, +8′) through (+12′, +3′, 0). *Ans.* 1250 lb, $\theta_x = 61°\,20'$, $\theta_y = 126°\,50'$, $\theta_z = 129°\,50'$; 7680 ft-lb, $\theta_x = 108°\,15'$, $\theta_y = 20°\,15'$, $\theta_z = 99°\,00'$.

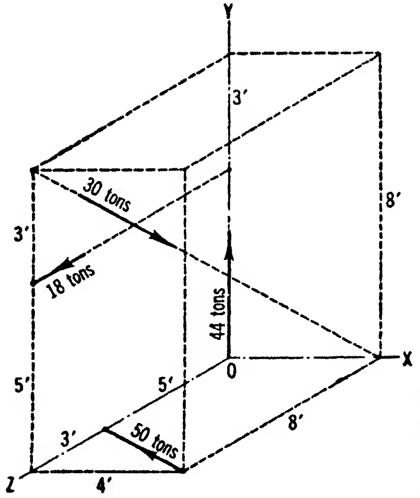

FIG. 111

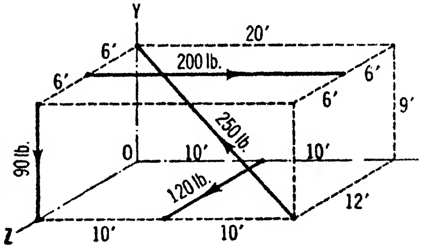

FIG. 112

231. Find the resultant of the system shown in Fig. 112.

232. Find the resultant of the following system: 120 lb, acting from the point (+15′, −9′, 0) through the point (+15′, −9′, −10′); 200 lb, from (0, −12′, 0) through (+15′, −12′, 0); 150 lb, from (0, 0, +9′) through (0, +12′, 0); 250 lb, from (0, +12′, +9′) through (+20′, 0, 0). *Ans.* a single force of 500 lb, acting from point (0, 0, +9′) through point (+12′, 0, 0).

49. The Principle of Moments for Two Concurrent Forces; General Case. *The algebraic sum of the moments of two concurrent forces, about any axis, is equal to the moment of the resultant about that axis.*

Proof. A special case of the foregoing principle, limited to a center of moments lying in the plane of the two forces or to an axis of moments at right angles to that plane, was discussed in Art. 27. It now becomes desirable to extend the principle to include any axis of moments.

In Fig. 113, let P, Q, and R represent any two concurrent forces and their resultant. The point of concurrence is at O. Let the line BC represent any axis of moments. Let the xz-plane be so placed as to contain the line BC and the point of concurrence, O. Let the line OA be drawn, at right angles to BC.

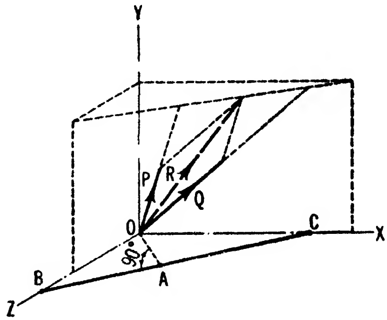

FIG. 113

If P, Q, and R are resolved, at O, into their x-, y-, and z-components, all the x- and z-components will intersect BC and their moments about that line will be zero. Consequently the moments of P, Q, and R about BC will be equal to the moments of their y-components alone. Since the line OA is at right angles both to the y-axis and to BC, its length will be the moment-arm of the y-component in each case. Therefore, taking moments about BC,

$$M_R = R_y \times (OA) \qquad M_P = P_y \times (OA) \qquad M_Q = Q_y \times (OA) \quad [22]$$

The foregoing equations may be written as follows:

$$R_y = \frac{M_R}{(OA)} \qquad P_y = \frac{M_P}{(OA)} \qquad Q_y = \frac{M_Q}{(OA)} \quad [23]$$

By the principle of components, Art. 41,

$$R_y = P_y + Q_y \quad [24]$$

Substituting in Eq. 24 the values of R_y, P_y, and Q_y given by **Eq. 23,**

$$M_R = M_P + M_Q \quad [25]$$

This proves the principle of moments in the general form, as set forth at the beginning of the article.

50. General Nature of the Principles of Composition. *The Principle of Components. The algebraic sum of the components of any system, along any axis, is equal to the component of the resultant along that axis.*

The Principle of Moments. The algebraic sum of the moments of the forces of any system, about any axis, is equal to the moment of the resultant about that axis.

Proof. It was shown in Art. 48 that any force system can be replaced, without change of effect, by a system of concurrent forces and a system of couples. An examination of Fig. 107, which shows the method used, leads to the conclusion that the component-sum of all the forces of the two latter systems, along any axis, is equal to the component-sum of the forces of the original system along that axis. A similar conclusion can be reached with regard to the moment-sums.

Consequently, if the principle of components and the principle of moments are valid for a non-coplanar concurrent system and for a system of non-coplanar couples, with reference to any axes, they are valid for any system.

The principle of components was proved for a non-coplanar concurrent system in Art. 41. It is also valid for a system of non-coplanar couples because of the fact that the component-sum of the forces of a couple or of any number of couples, along any axis, is equal to zero.

The principle of moments was proved for two concurrent forces, with reference to any axis, in Art. 49. That proof can readily be extended to include any concurrent system by compounding the forces of the system in pairs. That the principle of moments applies to a system of non-coplanar couples should be clear from a study of the methods used in resolving and in compounding such couples in Arts. 45 and 46.

Therefore, these two important principles are valid for any force system, and the choice of axes is unlimited. The ability to make unrestricted use of these principles is of value in subsequent discussions.

CHAPTER IV

EQUILIBRIUM OF COPLANAR FORCE SYSTEMS

51. Meaning and Physical Significance of Equilibrium. *The situation that exists when a system of forces has no resultant is described by stating that the system is in equilibrium.* Such a system is also referred to as a *balanced system.* When the entire system of external forces acting on a body is in equilibrium, *the body also is referred to as being in equilibrium.*

In the study of kinetics it is learned that, when the external forces acting on a given body are in equilibrium, one of the following situations exists:

(*a*) The center of gravity of the body moves in a straight line at constant speed, and the angular velocity of the body is constant.

(*b*) The center of gravity of the body moves in a straight line at constant speed, and the angular velocity of the body is zero.

(*c*) The center of gravity of the body is at rest, and the angular velocity of the body is constant.

(*d*) The entire body is at rest.

Conditions (*a*), (*b*), and (*c*) are usually classified under kinetics. Condition (*d*) is the status for the entire subject of statics.

The foregoing conditions are also correct when stated conversely, as follows: Whenever the center of gravity of a body is at rest, or moves in a straight line at constant speed, and the angular velocity of the body is constant, or zero, the external forces acting on the body are in equilibrium. In all other motions the external forces are unbalanced.

52. The General Graphic Conditions for Equilibrium. If a coplanar force system is in equilibrium the following conditions exist:

(*a*) *The force polygon closes,* and
(*b*) *The funicular polygon closes.*

These two statements can be recognized as the principle of components and the principle of moments, for the special case in which no resultant exists, expressed in the language of graphics.

The two principles also are valid when stated conversely, if combined as follows: *If both the force polygon and the funicular polygon close, the system is in equilibrium.*

53. The General Algebraic Conditions for Equilibrium. A force system in equilibrium has no resultant; therefore the principle of components and the principle of moments may be stated as follows:

The Principle of Components. When a system of forces is in equilibrium, the algebraic sum of the components of the forces, along any axis, is equal to zero.

The Principle of Moments. When a system of forces is in equilibrium, the algebraic sum of the moments of the forces, about any axis, is equal to zero.

The two principles are also valid when stated conversely, if combined as follows: *If the algebraic sum of the components of the forces, along any axis, is equal to zero, and if the algebraic sum of the moments of the forces, about any axis, is equal to zero, the system is in equilibrium.*

54. General Features of Graphic Solutions. The solution of a problem in equilibrium, by the graphic method, consists in plotting to scale all the forces that are completely known, and then in finding the unknown quantities by closing the polygon. If necessary, the funicular polygon also is plotted and made to close, in order to supply additional information needed for the proper closing of the force polygon.

The term " equilibrium polygon " is appropriate for a closed funicular polygon.

55. General Features of Algebraic Solutions. The fact that the body is in equilibrium is usually known in advance. Some of the external forces will be completely known; the others will be partially or wholly unknown. In problems of a practical nature a force is seldom wholly unknown; usually the position of at least one point on its line of action is known.

Magnitudes of forces, angles of inclination, and coordinates of points on lines of action constitute the unknown quantities requiring for their determination the use of the principles of statics. The sense of an unknown force is determined by the sign obtained when the magnitude is found, and does not of itself constitute an additional unknown quantity.

The algebraic solution of a problem in the equilibrium of forces consists in the formation and solution of equations based on the two fundamental conditions stated in Art. 53. Sometimes, additional relations obtained from the laws of friction, elasticity of materials, or from other empirical sources are necessary to a complete solution.

The principle of components and the principle of moments are always valid, but at times certain special methods are of advantage. The more important of these special methods will be pointed out in subsequent discussions.

56. The Free-Body Diagram. In the solution of a problem each body which is to be subjected to analysis should be represented in a separate sketch. In order to preserve the simplicity of the sketch all other bodies are omitted; only the forces exerted by them on the given body are shown. Such a sketch is called a *free-body diagram.*

All the known characteristics of the external forces acting on the given body must be shown correctly on the free-body diagram. All the unknown characteristics are then assumed, and are shown definitely on the sketch in conformity with the assumptions, even though considerable doubt may exist as to their correctness. This procedure provides a complete and definite figure on which to base the solution. As will be seen, the results can be made to reveal the correctness or incorrectness of the various assumptions.

Failure to draw a complete and definite sketch before the analysis is attempted is a prolific source of error in the solution of the problems of mechanics.

57. Limitations in the Matter of Equilibrium Equations. There is a definite limit to the number of independent equations that can be written for each type of force system, whether these equations are to be used in finding the resultant of the system, or in finding the unknown quantities in a problem involving a balanced system. Any equations beyond this standard number would be dependent; in other words, they would be mere repetitions of relations involved in the equations already written.

The maximum numbers of independent equations that the principles of statics can furnish in problems involving the various force systems are as follows: the collinear system (1); the coplanar concurrent system (2); the coplanar parallel system (2); the general coplanar system (3); the non-coplanar concurrent system (3); the non-coplanar parallel system (3); the general non-coplanar system (6).

With a proper selection of axes, any or all of these equations may be of the moment-sum type. In some cases, however, they may not all be of the component-sum type. The following list gives the maximum numbers of equations of the component-sum type that can be of use in the calculation of the unknown quantities: the collinear system (1); the coplanar concurrent system (2); the coplanar parallel system (1); the general coplanar system (2); the non-coplanar concurrent system (3); the non-coplanar parallel system (1); the general non-coplanar system (3).

58. Selection of Axes. In general, axes of resolution for component-sum equations and axes of moments for moment-sum equations should be selected in such a manner as to yield the simplest possible solution. With each force system, however, there are certain combinations of axes that will not yield independent equations. It is possible to formulate rules by which such contingencies can be avoided, but such rules are difficult to memorize.

It may be stated that when the component-sum of the forces along a chosen axis has been placed equal to zero it is simply an expression of the fact that the resultant, if the system possessed one, could have no component along that particular axis. Similarly, a moment-sum equated to zero expresses the fact that the resultant has no moment about the chosen axis. After one equation has been written, each succeeding equation must express some essentially new fact that further restricts the possibility of the existence of a resultant, and must not simply express a relation that would follow from the equations already written.

Fortunately, in the methods of solution that present themselves most naturally to the mind, the danger of so choosing the axes that the resulting equations will not be independent is comparatively small.

The ability to find the simplest solution is best cultivated by solving each problem by several methods and comparing them, until the judgment is developed to the point where one can feel confident of his ability to choose an efficient solution by means of a rapid inspection of the general features of the problem.

59. Statically Indeterminate Cases. Any quantity that cannot be found without the use of principles beyond the scope of the subject of statics is called a statically indeterminate quantity. The laws of friction are usually included in the subject matter of statics, but laws relating to elasticity are generally reserved for consideration in strength of materials. Many quantities that are statically indeterminate can be found with the assistance of the laws of elasticity. The principles of statics are valid in any case in which the forces are in equilibrium, but in a statically indeterminate case these principles alone are insufficient for a complete solution. In any event it should be remembered that it is futile to write equations of equilibrium in excess of the standard number, except for the purpose of checking.

A few problems involving statically indeterminate cases will be given in this book, but partial answers will be included in the statements of the problems, making it possible to complete the solution by the principles of statics.

Complete treatments of the important cases in engineering practice involving statically indeterminate quantities are found in many books on higher structures.

60. Equilibrium of the Collinear System. The algebraic method of solving a problem involving a collinear force system in equilibrium consists in forming an equation by placing the algebraic sum of the forces equal to zero, and then solving for the unknown quantity.

Such an equation is in fact an application of the principle of components, in which the resolution of the forces is rendered unnecessary by placing the axis parallel to the forces of the system. The use of the principle of moments would not be advantageous in this case. The graphic method would have little to recommend it in connection with the collinear system.

Special Condition. Two forces, to be in equilibrium, must be collinear, equal, and opposite in sense.

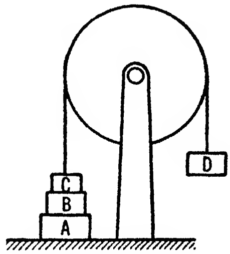

FIG. 114

PROBLEM

233. Figure 114 represents three blocks arranged in a pile, in such a manner that the centers are on the same vertical line. The weights of blocks A, B, and C are 100 lb, 80 lb, and 60 lb, respectively. A cord, attached to C, passes over a frictionless pulley and supports a block D weighing 40 lb. Draw a free-body diagram of each of the four blocks, showing the external forces acting on each, and calculate the unknown magnitudes.

61. Two-Force Members. The parts of a structure or of a machine are often referred to as *members*. A member that is found to be subjected to only two external forces, when considered as a free body, is called a *two-force member*. If the member is in equilibrium the two external forces are necessarily collinear, equal, and opposite.

In actual fact a member is usually subjected to more than two external forces. Even in such cases the methods of solution used by engineers often render it permissible to disregard all but two of the forces.

Figure 115 represents a simple pin-connected framework in equilibrium, consisting of two members, AB and CD. The framework supports a vertical load, P, and is itself supported at the points A and D. Let it be assumed that the forces caused by the weights of the members are negligible in comparison with those caused by the load P.

Since the weights of the members are to be disregarded, AB is a three-force member, subjected to external forces at A, B, and C; and CD is a two-force member, subjected to external forces at C and D only.

Figure 116 is a free-body diagram of member CD. The two external forces acting on CD are in equilibrium; therefore they are collinear, equal, and opposite. If the connecting pins are on the center line of the member the two external forces will coincide with that line. This leads to the important conclusion that the force exerted on the framework by the support D acts along the line CD. The inclination of this force thus becomes a known quantity. It should be carefully observed, however, that the force exerted on the framework at A cannot be assumed to act along AB, since AB is not a two-force member.

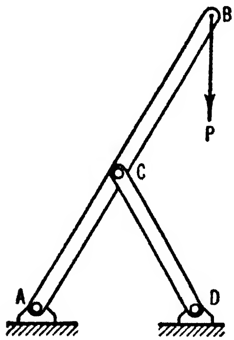

FIG. 115

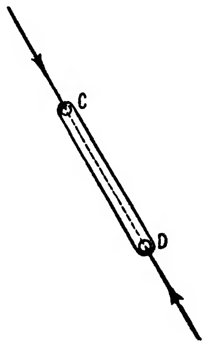

FIG. 116

The recognition of the fact that certain members are "two-force," or that they may be so considered, always tends to simplify the solution of a problem, and often renders determinate a problem which otherwise would be indeterminate. It is necessary that the student be alert for the existence of two-force members.

PROBLEM

234. Answer the following questions, proving your answer in each case: In Fig. 129, Prob. 250, is it permissible to assume that the force acting on the crane at A is collinear with AB? In Fig. 132, Prob. 253, what assumption is permissible regarding the force acting on the crane at C? Would such an assumption be permissible if the weight of the member CD were to be regarded? Is it permissible to assume the inclination of the force acting at B? In Fig. 159, the force supporting the truss at B is represented as being collinear with BC. Is this permissible? In Fig. 160, what assumption may be made regarding the inclination of the force at B? In Fig. 169, Prob. 297, what assumption may be made regarding the supporting force at K? In Fig. 176, what assumption may be made regarding the force at B?

62. Equilibrium of the Coplanar Concurrent System; Graphic Solution. In the solution of a problem all the forces that are completely known are plotted, to scale, in a force polygon and then, with the assistance of whatever information may be available concerning the remaining forces, the polygon is made to close.

Since a single force is the only kind of resultant possible to a coplanar concurrent system, and since the closing of the force polygon eliminates that possibility, nothing more can be accomplished through the principles of statics, and the use of the equilibrium polygon is unnecessary.

Special Condition. If the total number of forces in a system is three, and the system is in equilibrium, the forces are either coplanar parallel or coplanar concurrent.

Proof. If the three forces were not coplanar, an axis of moments could be inserted in such a manner as to intersect two of them, but not the third. The moment-sum of the three forces about such an axis would not be zero; consequently the forces would not be in equilibrium.

If the three forces were coplanar, but were neither concurrent nor parallel, their lines of action would intersect in at least two points, and their moment-sum about any such point of intersection would not be zero. This shows that equilibrium is impossible unless the three forces are either parallel or concurrent.

The foregoing special condition for three forces in equilibrium can be used to advantage in certain types of problems, especially in connection with the graphic method of solution.

Illustrative Problem

235. Five wires are arranged in a vertical plane as shown in Fig. 117. Wires *ab* and *bc* pass over smooth pulleys and carry weights as indicated. Wire *cd* carries its load directly. Wires *de* and *ef* are anchored. Each pulley is 15 in. in diameter. Find the tensions in wires *de* and *ef*.

Solution. Draw a space diagram, similar to Fig. 117, to a convenient scale. Plot the known forces, $ab = 400$ lb, $bc = 1000$ lb, and $cd = 300$ lb, to scale, in a force polygon, such as $ABCD$ in Fig. 118.

The force polygon must close when the two unknown forces, DE and EF, are inserted, since the system is in equilibrium. Therefore, draw a line through D parallel to *de*, and a line through A parallel to *ef*. The missing point, E, is necessarily at the intersection of these two lines.

By scaling, $DE = 575$ lb, and $EF = 1130$ lb. Their senses show that both wires are under tension. Flexible wires, cables, cords, etc., cannot sustain compression. If the results had indicated compression in one or both of the wires the conclusion would have been that the system could not remain in the position shown in Fig. 117.

PROBLEMS

236. A certain coplanar concurrent system in equilibrium is as follows: 3.4 tons, $\theta_x = 42° 50'$; 2.7 tons, $\theta_x = 164° 20'$; 1.1 tons, $\theta_x = 321° 00'$; 2.6 tons, $\theta_x = ?$? tons, $\theta_x = 192° 00'$. Find the two unknown quantities.

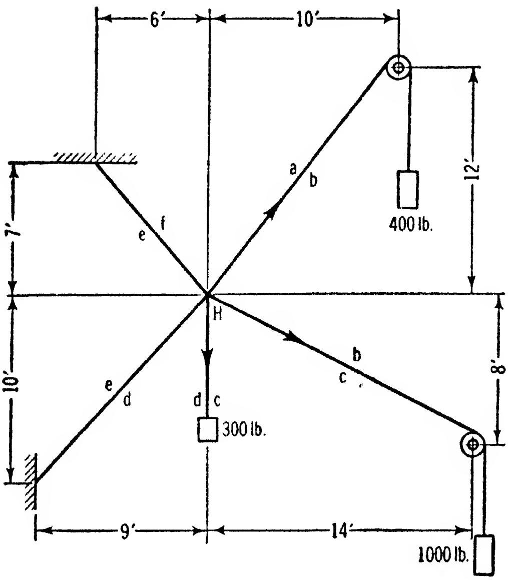

FIG. 117

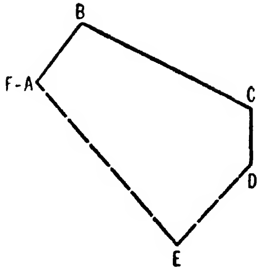

FIG. 118

237. A certain coplanar concurrent system in equilibrium is as follows: 3.4 tons, $\theta_x = 42° 50'$; 2.7 tons, $\theta_x = 164° 20'$; 1.1 tons, $\theta_x = 321° 00'$; 2.8 tons, $\theta_x = ?$; 3.0 tons, $\theta_x = ?$. Find the unknown angles of inclination.

238. In Fig. 119, the beam AC is supported by a smooth pin at A, and by the tie, BD. Find the forces acting on the beam at A and B. Solve as a concurrent

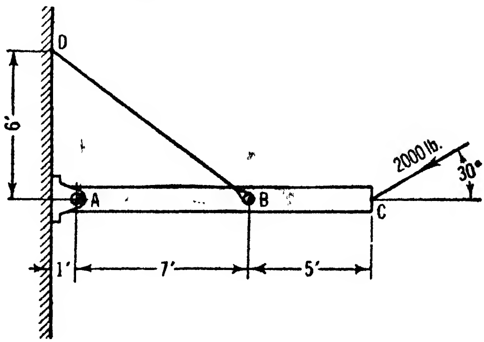

FIG. 119

system, by the special condition for three forces in equilibrium. Disregard the weights of the members.

239. A certain coplanar concurrent system in equilibrium consists of five forces, each of which acts from the origin through the point designated: 100 lb (−2, −8); 400 lb (−4, +6); 500 lb (+7, +3); P_1 (+4, −4); P_2 (−8, −3). Find P_1 and P_2.

240. Solve Prob. 252, Art. 63, by the graphic method. Utilize the hint given in that problem.

63. Equilibrium of the Coplanar Concurrent System; Algebraic Solution. In the algebraic solution of a problem equations are written, based on the principle of components or the principle of moments, or both. The choice of principles and the choice of axes are governed by the circumstances of each problem.

Special Condition. The special condition for three forces in equilibrium was stated and proved in Art. 62. The use of this condition is sometimes of advantage in an algebraic solution, when the number of forces in the system is three, or when the number can be reduced quickly to three by composition.

The force polygon for three forces in equilibrium obviously is a triangle. A convenient solution often can be performed by making a freehand sketch of the force triangle, and then by solving the triangle by the methods of geometry or trigonometry.

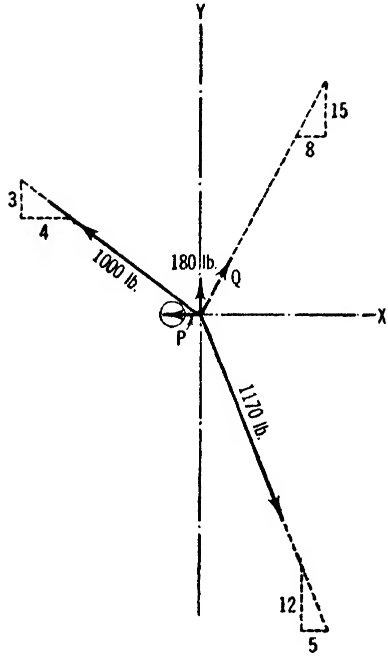

FIG. 120

ILLUSTRATIVE PROBLEMS

241. Figure 120 represents a coplanar concurrent system of five forces in equilibrium. Find the two unknown magnitudes, P and Q.

Solution. A quick inspection of the system does not reveal with certainty the senses of the unknown forces. Assume them to be as shown in the figure. The use of the principle of components, Art. 53, is preferable in this problem. From the figure,

$$\Sigma F_x = 0 \qquad +Q \times \tfrac{8}{17} - 1000 \times \tfrac{4}{5} - P + 1170 \times \tfrac{5}{13} = 0$$

$$\Sigma F_y = 0 \qquad +Q \times \tfrac{15}{17} + 180 + 1000 \times \tfrac{3}{5} - 1170 \times \tfrac{12}{13} = 0$$

The solution of these equations gives $P = -190$ lb, and $Q = +340$ lb.

The minus sign accompanying P does not mean that P is actually negative, or toward the left. It means that the sense of P was assumed incorrectly, and consequently that P acts toward the right. This interpretation of the signs of results was discussed in Art. 22.

In Fig. 120 a small circle has been drawn around the arrowhead of the vector representing P. This is to be interpreted as meaning that the sense of the force was found to be opposite to that assumed and indicated by the vector. Such a convention will be used in any similar cases that may occur in this book.

The fact that the positive sign was obtained with Q shows that Q was correctly assumed.

242. Figure 121 represents a bar, AB, pin-connected to a wall at B, and supported by rollers resting on a horizontal surface at A. A load of 500 lb is applied at C as shown. Find the reactions exerted on the bar by the supporting pins at A and B. Disregard the weight of the bar.

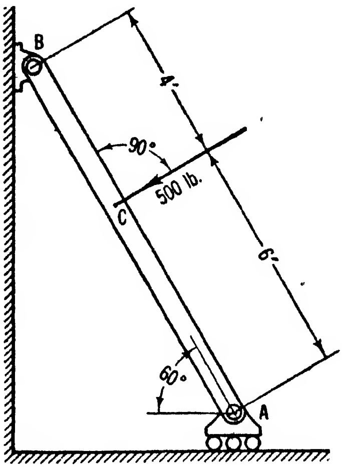

FIG. 121

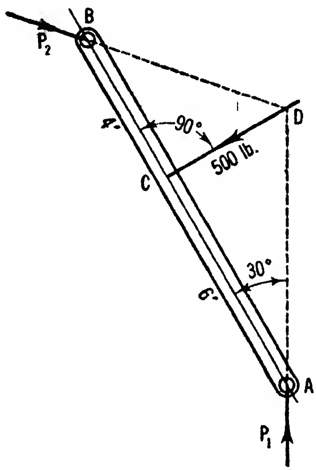

FIG. 122

Solution. Since the weight of the bar is to be disregarded, the total number of external forces is three, and the special condition applies. The forces must be either coplanar parallel or coplanar concurrent. Because of the rollers, the force at A is vertical (Art. 14). Since the 500-lb load is not vertical, the system cannot be parallel; therefore it must be concurrent.

Figure 122 is a free-body diagram of the bar. P_1 and the 500-lb load intersect at D; consequently D is the point of concurrence for the system. It is now known that P_2 passes through B and D, and that its inclination can be ascertained from the geometry of the figure. Assume the senses of P_1 and P_2 to be as shown in Fig. 122. From Fig. 122, $\overline{CD} = 6 \tan 30° = 6 \times 0.577 = 3.46$ ft; also, $\angle BDC = \text{arc tan } 4/3.46 = \text{arc tan } 1.16 = 49° 15'$.

Two simple equations now can be formed by the use of the principle of moments, with the centers of moments at A and B.

$$\Sigma M_A = 0 \qquad -(P_2 \cos 49° 15') \times 10 + 500 \times 6 = 0$$

$$\Sigma M_B = 0 \qquad +(P_1 \sin 30°) \times 10 - 500 \times 4 = 0$$

The solution of these equations gives: $P_1 = +400$ lb, and $P_2 = +459$ lb. The positive sign was obtained in each case, showing that the sense of each force was assumed correctly and is as shown in the figure.

243. Solve Prob. 242 by the method in which the force triangle is sketched, and solved by trigonometry.

Solution. The method specified is the one suggested in the present article as an alternative method in problems involving three concurrent forces in equilibrium.

Figure 123 represents the force triangle. P_1 is known to be vertical, because of the rollers. The angle between the 500-lb load and P_1 is 60°. The angle β is equal to the angle BDC in Fig. 122. From Prob. 242, $\angle BDC = \beta = 49° 15'$. From Fig. 123, by geometry, $\alpha = 180° - (60° + \beta) = 70° 45'$. By the law of sines, from trigonometry,

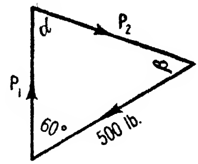

FIG. 123

$$\frac{P_1}{\sin 49° 15'} = \frac{500}{\sin 70° 45'} \qquad P_1 = 401 \text{ lb}$$

$$\frac{P_2}{\sin 60°} = \frac{500}{\sin 70° 45'} \qquad P_2 = 459 \text{ lb}$$

These results are in essential agreement with those obtained in Prob. 242.

PROBLEMS

244. Figure 124 represents a body weighing 3000 lb, suspended by means of wires from the points A and B. Calculate the tension in each wire. *Ans.* $AC = 2400$ lb; $BC = 1800$ lb.

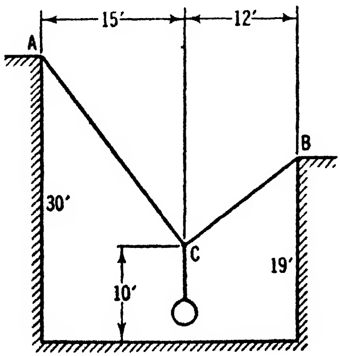

FIG. 124

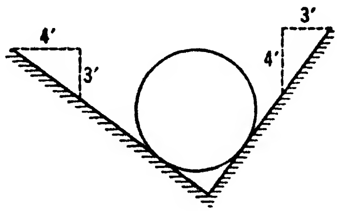

FIG. 125

245. Figure 125 represents a homogeneous right circular cylinder weighing 300 lb, resting in the notch between two inclined planes. Calculate the force exerted on the cylinder by each plane. Assume the planes to be smooth.

246. Figure 126 represents a portion of a plane roof truss. Calculate the unknown stresses, S_1 and S_2. Disregard the weight of the body.

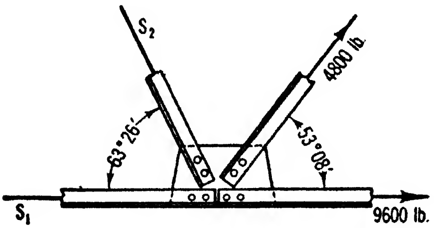

FIG. 126

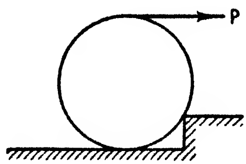

FIG. 127

247. Figure 127 represents a homogeneous right circular cylinder 24 in. in diameter, weighing 200 lb, resting on a horizontal floor, and in contact with a step 6 in. high. A horizontal force, P, is applied to the top of the cylinder, and is gradually increased. Calculate the magnitude of P at the instant when the cylinder is on the verge of rising from the floor. *Ans.* 116 lb.

248. In Prob. 247, apply the force P at the center of the cylinder, instead of at the top, and solve the problem.

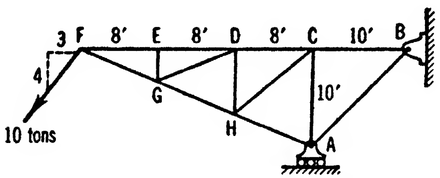

FIG. 128

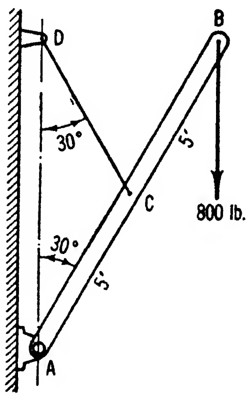

FIG. 129

249. Figure 128 represents a truss supported at A and B, and subjected to a load of 10 tons, applied as shown. The truss rests on rollers at A. Find the forces exerted on the truss by its supports.

250. Figure 129 represents a wall crane carrying a vertical load of 800 lb. Find the tension in the cable CD, and the force exerted on the crane by its support at A. *Ans.* 924 lb; 462 lb, $\theta_x = 0$.

251. Figure 130 represents a brake lever supported at A and subjected to a horizontal force of 50 lb at the handle. Find the unknown force Q, and the supporting force at A.

252. Figure 131 represents a roof truss subjected to three wind loads acting at right angles to the roof. Find the forces exerted on the truss by its supports. First find the resultant of the three wind loads by inspection; then solve by the use of the special condition. *Ans.* $A = 3190$ lb, $\theta_x = 90°$; $B = 5020$ lb, $\theta_x = 149° \ 15'$.

253. Find the forces exerted on the crane in Fig. 132, by its supports at B and C.

254. Figure 133 represents a cable, $ABCD$, supporting two loads, W_1 and W_2. Ascertain the ratio, W_1/W_2, necessary in maintaining the cable in the position shown. If $W_1 = 2000$ lb, calculate the tension in each portion of the cable.

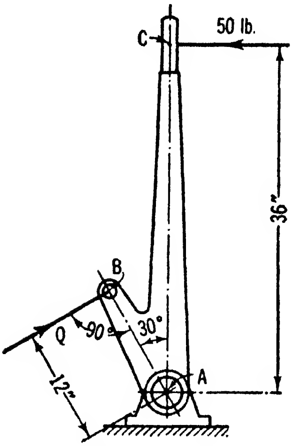

FIG. 130

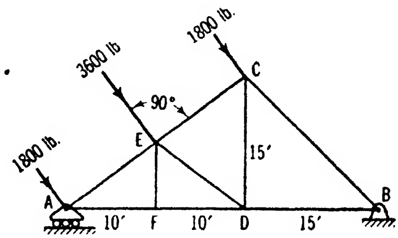

FIG. 131

255. Let Fig. 127 represent a homogeneous right circular cylinder whose weight is W and whose radius is r. The cylinder rests on a horizontal floor and is in contact

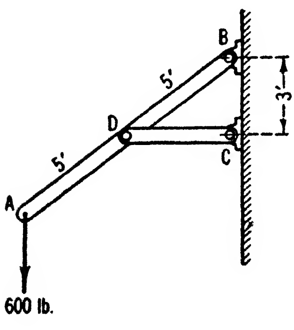

FIG. 132

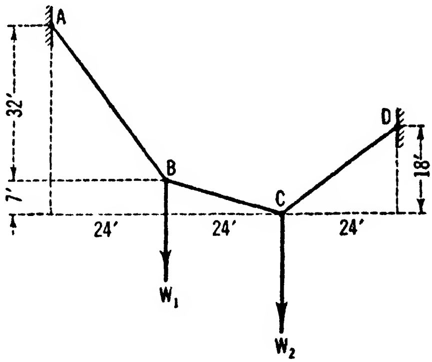

FIG. 133

with a step whose height is h. A horizontal force, P, is applied to the top of the cylinder and is gradually increased. Derive a formula for the magnitude of P, in terms of W, r, and h, at the instant when the cylinder is on the verge of moving. Solve Prob. 247 by the use of this formula. *Ans.* $P = W\sqrt{1/(2\frac{r}{h} - 1)}$; 116 lb.

256. One of the methods of erecting towers for transmission lines is indicated in Fig. 134. The tower is assembled in a horizontal position and raised into place by means of a shear legs, OB, straddling the tower. The guy, BC, is drawn in by means of a block and tackle, and the shear and tower rotate about an axis through O at right angles to the plane of the paper. In the present case the distance AB remains constant. The coordinates of the center of gravity of the tower are $(+22', +2.75')$. The tower weighs 1500 lb. The shear legs does not touch the tower. Find

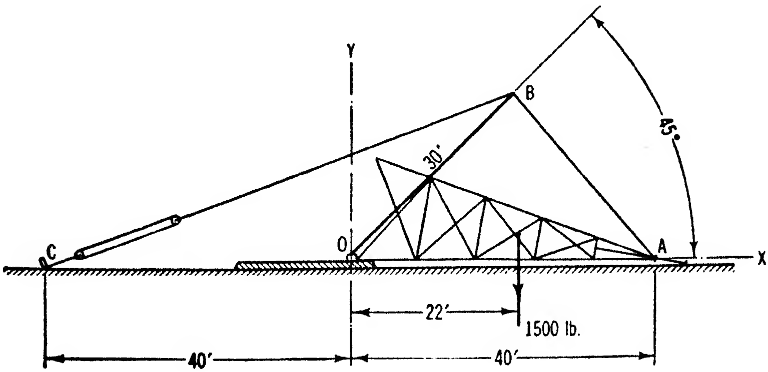

FIG. 134

the tension in AB, and the reaction exerted on the tower at O, at the instant when the top of the tower just clears the ground. *Ans.* 1100 lb; 995 lb, $\theta_x = 42° 45'$.

257. Using the data and results of Prob. 256, find the tension in the guy BC, and the reaction exerted on the shears at O. Disregard the weight of the shears.

64. Equilibrium of the Coplanar Parallel System; Graphic Solution.

The graphic method of finding the resultant of the coplanar parallel system was described in Art. 32. The force polygon and the funicular polygon are used for this purpose. If the system has no resultant, it is clear that both these polygons will close. Consequently, in a problem in which the system is known to be in equilibrium the two polygons are plotted and are made to close. The desired information regarding the unknown quantities can then be obtained from the polygons.

As has been stated, the funicular polygon for a balanced system is often called the equilibrium polygon.

In some problems the force polygon can be closed before the equilibrium polygon is begun, but in others the closing of the force polygon must await certain information obtained through the closing of the equilibrium polygon.

Special Condition. In Art. 62 it was shown that three forces, to be in equilibrium, must be either coplanar parallel or coplanar concurrent. If, in a problem involving three forces in equilibrium, two of the forces are known to be parallel, it follows from the special condition that all three are parallel.

ILLUSTRATIVE PROBLEM

258. Figure 135 represents a horizontal beam supporting three concentrated loads. The weight of the beam, which is 60 lb per lin ft, is also to be considered. The beam rests on supports at J and K. Find the reactions exerted on the beam by its supports.

Solution. When a rigid beam simply rests on level supports and sustains vertical loads only, the reactions at the supports are assumed to be vertical. The weight of a beam is usually distributed uniformly, and in the calculation of external forces the weight may be treated as a concentrated load applied at the middle of the length.

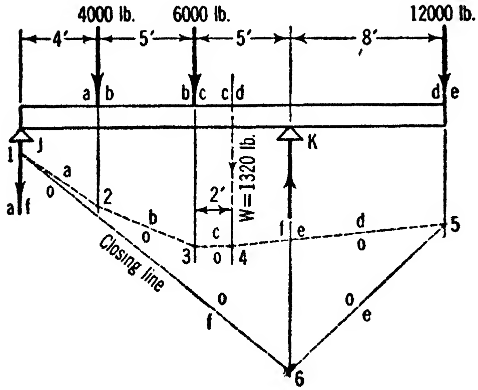

FIG. 135

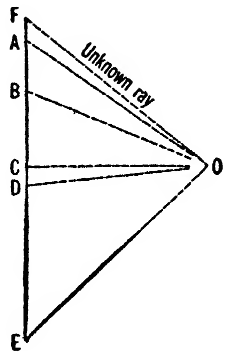

FIG. 136

Letter the forces consecutively in a clockwise direction around the beam, leaving the unknown forces to the last. Plot the known forces in a force polygon, Fig. 136. This yields the polygon $ABCDE$. The point F, which will determine the values of the reactions, is as yet unknown.

Start the equilibrium polygon at the point 1, Fig. 135, which may be any convenient point on the line of action of the unknown reaction, af. Draw the string ao parallel to the ray AO, intersecting the load ab at point 2. Proceed in like fashion until points 3, 4, 5, and 6 are located. One more string remains to be drawn, and since the polygon must close, this last string must be drawn from point 6 to the starting point 1. This string is called the closing line. The missing ray in the force polygon will be parallel to this closing line. Draw it so, through the pole O. This locates the missing point F in the force polygon. The vector EF represents, in magnitude and sense, the reaction ef, and the vector FA represents the reaction fa. The sequence of the letters in the force polygon indicates the sense of the force. Therefore, reaction EF is upward, and reaction FA is downward. The latter result shows that the beam must be anchored to the left-hand support. By scaling, $EF = 24{,}900$ lb, and $FA = 1550$ lb.

PROBLEMS

259. Solve Prob. 264, Art. 65, by the graphic method.

260. Solve Prob. 265, Art. 65, by the graphic method.

261. A rigid beam, 12 ft long, rests on two level supports placed at the extreme ends of the beam. The beam carries three concentrated loads, as follows: 500 lb, 240 lb, and 360 lb, placed at distances of 2 ft, 5 ft, and 9 ft, respectively, from the left end of the beam. Find the reactions exerted by the supports.

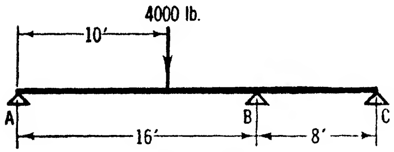

Fig. 137

262. Given a balanced system of seven vertical forces whose magnitudes, from left to right, are as follows: +10 lb; P_1; −20 lb; −40 lb; P_2; +60 lb; −80 lb. The distances to the lines of action, measured from the 10-lb force, are respectively: 2 ft; 4 ft; 8 ft; 9 ft; 11 ft; 13 ft. Find P_1 and P_2.

263. Figure 137 represents a beam with three supports. Beams having more than two supports are of common occurrence in structures; they are called continuous beams. The reactions cannot be calculated by the principles of statics alone, since the forces constitute a coplanar parallel system having more than two unknown quantities (Art. 57). In the present case the reaction at the extreme left-hand support was calculated by means of a formula from strength of materials, and is 992 lb, upward. Find the reactions at the other two supports.

65. Equilibrium of the Coplanar Parallel System; Algebraic Solution.

The simplest solution of a problem is usually achieved by writing one equilibrium equation based on the principle of components and one based on the principle of moments, although moment-sum equations can be used exclusively, if desired. The axis for a component-sum equation is naturally taken parallel to the forces of the system. The equation then expresses the fact that the algebraic sum of the forces themselves is equal to zero.

Special Condition. The special condition for the equilibrium of a coplanar parallel system was discussed in Art. 64.

Illustrative Problem

264. Figure 138 represents an overhanging beam, supported at two points. The beam weighs 50 lb per lin ft, and sustains two concentrated loads. Calculate the reactions exerted on the beam by its supports, taking into account the weight of the beam.

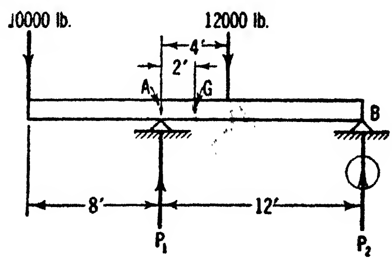

Fig. 138

Solution. The total weight of the beam is 1000 lb, and for the purposes of the problem may be treated as a single, concentrated force, applied at the center of gravity, G, 2 ft to the right of A. Assume both reactions, P_1 and P_2, to be vertical and upward.

By the principle of moments, with center of moments at B,

$$\Sigma M_B = 0 \qquad -P_1 \times 12 + 10{,}000 \times 20 + 12{,}000 \times 8 + 1000 \times 10 = 0$$
$$P_1 = 25{,}500 \text{ lb}$$

By the principle of components, using a vertical axis,

$$\Sigma F = 0 \qquad +25{,}500 - 10{,}000 - 12{,}000 - 1000 + P_2 = 0$$
$$P_2 = -2500 \text{ lb}$$

The minus sign accompanying P_2 means that the sense of this force was assumed incorrectly. Therefore, P_2 acts downward, and the beam must be anchored to the support in such a manner as to make this possible. As in Prob. 241, draw a small circle around the arrowhead, to show that the assumed sense was found to be incorrect.

PROBLEMS

265. In Fig. 139, calculate the reactions exerted on the beam by its supports. Disregard the weight of the beam. *Ans.* +2510 lb; +1690 lb.

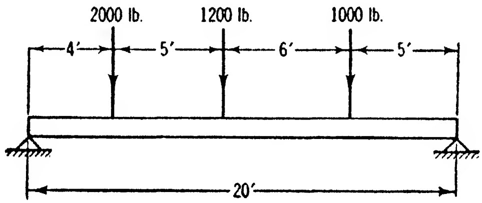

FIG. 139

266. Solve Prob. 262, Art. 64, by the algebraic method.

267. The beam in Fig. 140 weighs 100 lb per lin ft and sustains two concentrated loads as shown. Calculate the supporting forces. Consider the weight of the beam. *Ans.* −16,300 lb; +39,500 lb.

268. Figure 141 represents a system of pulleys in which a weight, W, is lifted by means of a downward pull, P. Assume that the bearings of the pulleys are frictionless, and that equilibrium exists. It follows that the tension in any given piece of the rope is the same at all points (Art. 14). Calculate W in terms of P.

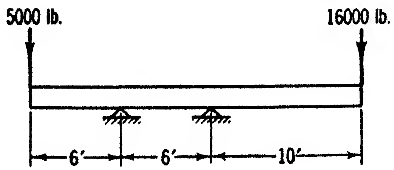

FIG. 140

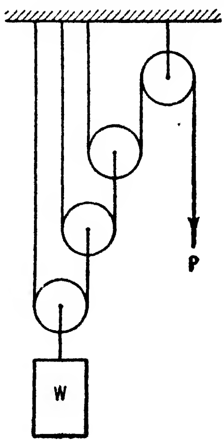

FIG. 141

269. Solve Prob. 258, Art. 64, by the algebraic method. *Ans.* −1570 lb; +24,900 lb.

270. The beam in Fig. 142 weighs 25 lb per lin ft. There is an additional uniform load of 300 lb per lin ft over a portion of the length, as shown, and a single, concentrated load. Calculate the reactions exerted by the supports.

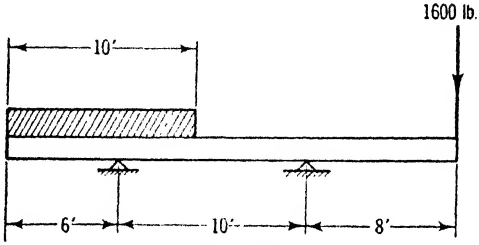

Fig. 142

271. In the system of pulleys shown in Fig. 143, calculate P in terms of W. Make assumptions similar to those in Prob. 268. *Ans.* $W = 4P$.

272. Figure 144 represents a continuous beam having four supports and overhanging the left-hand support. There is a uniform load of 400 lb per lin ft over the entire beam. By strength of materials the reactions at A and D were found to be 8700 lb and 3300 lb, respectively, both upward. Calculate the other two reactions. *Ans.* +4500 lb; +7500 lb.

273. In the system of beams and pulleys shown in Fig. 145 a load of 1 ton is lifted by means of a downward pull, P. Each pulley is 1 ft in diameter. Calculate P, and the reaction exerted on the beam at A. Make assumptions similar to those in Prob. 268.

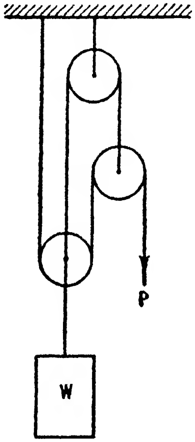

Fig. 143

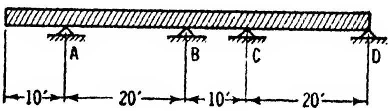

Fig. 144

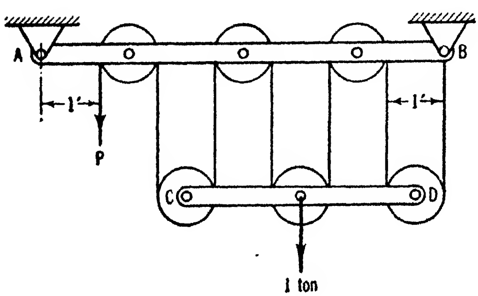

Fig. 145

274. Figure 146 shows the principle of the weighing system of a certain materials-testing machine. P represents the force applied to the specimen. W represents the weight of the poise. When the poise is at O it just balances the dead weight, with no load on the specimen. Derive a formula for the ratio P/W, in terms of the lengths of the various lever arms. *Ans.* $P/W = 2ABCD/ab(Cd + cD)$.

275. A certain bridge span, supported at the ends, is 180 ft long. The locomotives and train described in Prob. 117, Fig. 52, have stopped on the span with the foremost wheel 10 ft from the forward end of the span. Calculate the reactions at

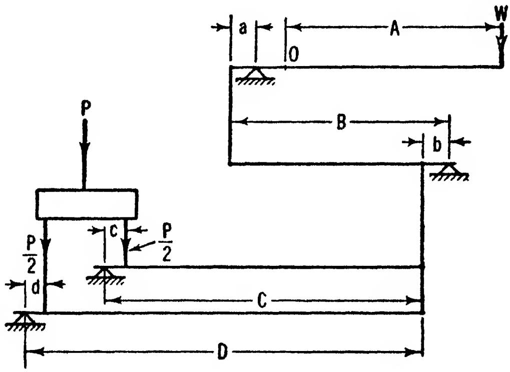

FIG. 146

the two supports. The loads given in Prob. 117 represent axle loads; therefore, the reactions obtained are the total reactions for the span, and not for any individual truss or girder. *Ans.* +312 tons; +297 tons.

66. Equilibrium of the General Coplanar System; Graphic Solution. In the graphic solution of a problem involving the general coplanar system in equilibrium the unknown quantities are found by constructing a force polygon and an equilibrium polygon, and by causing both to close. In exceptional cases the force polygon can be closed before the equilibrium polygon is begun, but usually the force polygon cannot be closed until certain additional information is obtained through the construction of the equilibrium polygon.

Special Conditions. If the total number of forces in the system is three, the system must be either concurrent or parallel (Art. 62), and frequently can be handled more easily under one of these classifications.

If the total number of forces is four, it may be possible to save a certain amount of time by utilizing the fact that the resultant of any portion of the system must balance the resultant of the remaining portion. This condition is true, of course, for any system in equilibrium.

ILLUSTRATIVE PROBLEMS

276. Figure 147 represents a Pratt roof truss, symmetrical in design. The five loads shown are wind loads and are at right angles to the roof surface. The loads are spaced at equal distances. The truss rests on rollers at B, and is fastened at A by means of a smooth pin. Find the reactions exerted on the truss by its supports.

Solution. Plot the known forces in a force polygon (Fig. 148), to a convenient scale. This gives that portion of the polygon represented by *ABCDEF*. It is desirable to select the forces in the order in which they are encountered in passing around the diagram of the truss. The order used in Figs. 147 and 148 is that which results from passing around the truss in a clockwise direction. Next in order comes the reaction at *B*, designated as *fg*. Since it is known that this reaction is vertical, it will be represented in the force polygon by a vertical vector drawn through *F*. The magnitude of this vector is, as yet, unknown; consequently, the position of the point *G* remains to be found.

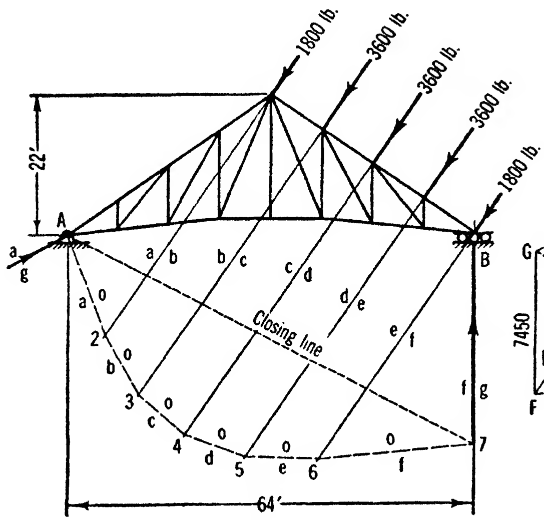

FIG. 147

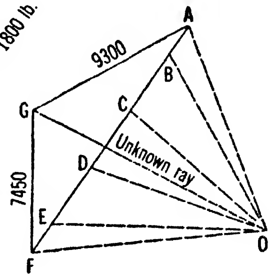

FIG. 148

Select a pole, *O*, and draw the known rays. Nothing is known about the remaining reaction *ga*, except that it must pass through the center of the pin, at *A*. Since *A* is the only known point on the line of action of *ga*, it must be chosen as the point of resolution for that force. In other words, the equilibrium polygon must be started at *A*. Draw the string *ao* parallel to the ray *AO*, the string *bo* parallel to the ray *BO*, etc. This process is continued until the point 7, on the line of action of *fg*, is located. One more string remains to be drawn, and, since the equilibrium polygon must close, it must be drawn from 7 through *A*. This is the closing line of the equilibrium polygon. The missing ray, *GO*, in the force polygon, must be parallel to this closing line. Draw a line through *O* parallel to the closing line, and locate the intersection of this ray with the vertical line previously drawn through *F*. This intersection is the missing point *G*. Since the force polygon also must close, the vector *GA* represents the reaction *ga*, in magnitude, inclination, and sense.

The magnitudes of the two reactions, as found by scaling their lengths in Fig. 148, are as follows: $ga = 9300$ lb, $fg = 7450$ lb. From the beginning it has been obvious that fg is upward, and that ga is upward and to the right. The sequence of the letters in the force polygon agrees with this observation.

277. Figure 149 represents a transverse bent in a steel-frame mill building. The six loads shown are wind loads, each one acting at 90° to the surface to which it is applied. The bent is supported on pin-bearings at M and N, but

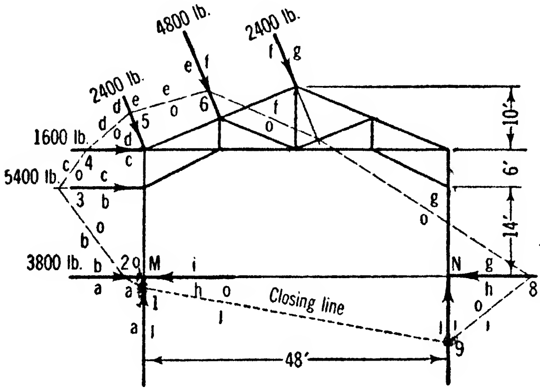

Fig. 149

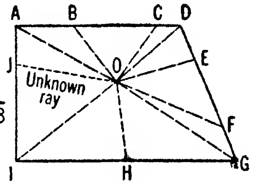

Fig. 150

no rollers are used. The reactions are statically indeterminate. The horizontal components of the reactions are frequently assumed to be equal. This brings the problem within the scope of statics. Find the horizontal and the vertical components of the reactions.

Solution. Letter the forces in the clockwise direction around the structure, starting with the first known force, the 3800-lb load. Leave the vertical components of the reactions to be lettered last. Plot the portion, $ABCDEFG$, of the force polygon, as in Fig. 150. Draw a horizontal line through G, and a vertical line through A. This locates I. Since GH and HI are to be assumed equal, the point H will be at the center of the line GI. The point J, which divides the two vertical components of the reactions, remains to be located.

Plot the equilibrium polygon, starting at point 1, on the line of action of ja, and ending at point 9, on the line of action of ij. Since gh and hi are collinear, the ray OH, in Fig. 150, is not used.

Draw the closing line jo, of the equilibrium polygon, through points 9 and 1. The missing ray, JO, must be parallel to this closing line. Draw it so, thus locating the point J, and completing the solution.

The magnitudes of the components of the reactions, as obtained by scaling, are as follows: $gh = hi = 7250$ lb; $ij = 6370$ lb; $ja = 2500$ lb.

PROBLEMS

278. Figure 151 represents a beam supported by a pin-connection at B and by a roller at A. Three concentrated loads are applied as shown. Disregard the weight of the beam, and find the reactions exerted on the beam by its supports.

279. Figure 152 represents an unsymmetrical roof truss. Three wind loads are shown, acting at right angles to the roof surface. The truss is supported by a pin-bearing at M, and by rollers at N. Find the reactions.

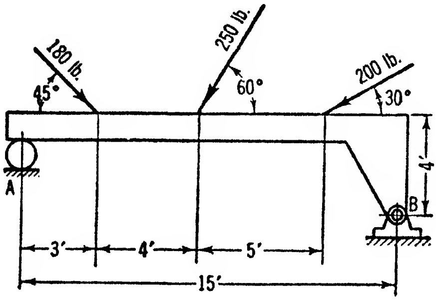

FIG. 151

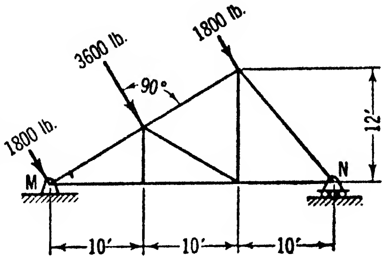

FIG. 152

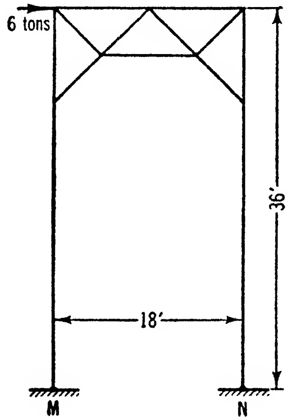

FIG. 153

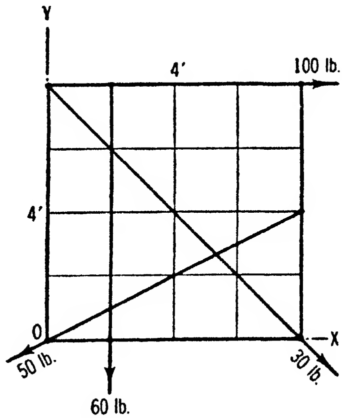

FIG. 154

280. Solve Prob. 279, with the rollers transferred to the left-hand support.

281. Figure 153 represents a bridge portal, supported by pin-bearings at M and N, and subjected to a single concentrated load, applied horizontally. There are no rollers, and the reactions are statically indeterminate. Assume the horizontal components of the reactions to be equal, as was done in Prob. 277. Find the horizontal and vertical components of the reactions.

282. Figure 154 represents a coplanar force system in equilibrium. Four of the forces are completely known, and are as shown. One force is completely unknown. Find the unknown force.

67. Equilibrium of the General Coplanar System; Algebraic Solution. The algebraic solution of a problem involving the general coplanar force system in equilibrium is usually accomplished by the formation of equations based on the principle of components, or the principle of moments, or both.

In the selection of the equations it should be observed that a point on the line of action of one or more of the unknown forces is likely to be advantageous for use as a center of moments. Also, a component-sum taken along an axis at right angles to one or more of the unknown forces may yield a comparatively simple equation.

Special Conditions. Special conditions for the general coplanar system were discussed in Art. 66.

Illustrative Problems

In some of the problems of the present article the number of external forces acting on the given body is three, and in any case the number could be reduced to three by composition. Therefore, the special condition for three forces in equilibrium, Art. 62, could be utilized in any of these problems. However, the problems of the present article are intended to provide practice in the general methods of solution, and the special condition should not be used.

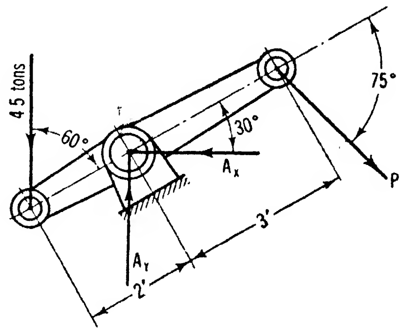

Fig. 155

283. Figure 155 represents a beam supported by a frictionless pin-bearing at A and subjected to forces as indicated. Find the unknown forces.

Solution. Let the reaction at A be resolved into its horizontal and vertical components. By the principle of moments,

$$\Sigma M_A = 0 \qquad -(P \cos 15°) \times 3 + (4.5 \cos 30°) \times 2 = 0 \qquad P = 2.69 \text{ tons}$$

By the principle of components, using the result obtained above,

$$\Sigma F_x = 0 \qquad -A_x + 2.69 \times \cos 45° = 0 \qquad A_x = 1.90 \text{ tons}$$

$$\Sigma F_y = 0 \qquad +A_y - 4.5 - 2.69 \times \cos 45° = 0 \qquad A_y = 6.40 \text{ tons}$$

The positive sign was obtained in each case, showing that the senses of the unknown forces were assumed correctly.

284. Figure 156 represents a Howe roof truss subjected to four wind loads at right angles to the roof surface. The truss is supported by a pin-bearing at B, and by rollers resting on a horizontal surface at A. Find the reactions at the supports.

Solution. This is a problem of the type in which time can be saved by compounding some, or all, of the unknown forces. The four loads are parallel, and are arranged symmetrically with respect to the point D. Their resultant is, by inspection, a force of 10,200 lb, parallel to the loads and acting through D. The resultant now may be used as a substitute for the four loads in any calculation involving forces external to the truss. D is at the middle of the line BC.

Because of the rollers the reaction at A may be assumed to be vertical.

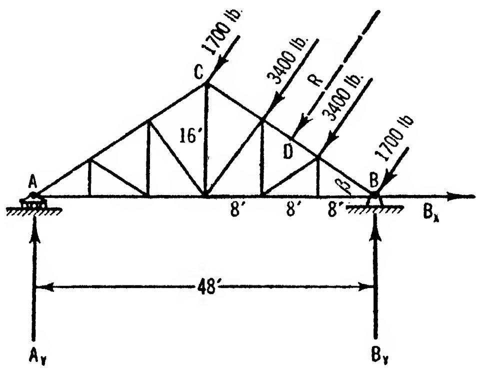

FIG. 156

Resolve the reaction at B into components, as shown. From the figure: β = arc tan $\frac{16}{24}$ = arc tan 0.667 = 33° 40′. $\overline{BD} = \dfrac{12}{\cos\beta} = \dfrac{12}{0.832} = 14.4$ ft. By the principle of moments,

$$\Sigma M_B = 0 \qquad -A_y \times 48 + 10{,}200 \times 14.4 = 0 \qquad A_y = 3060 \text{ lb}$$

By the principle of components,

$$\Sigma F_x = 0 \qquad +B_x - 10{,}200 \sin\beta = 0 \qquad B_x = 5650 \text{ lb}$$

$$\Sigma F_y = 0 \qquad +B_y + A_y - 10{,}200 \cos\beta = 0 \qquad B_y = 5430 \text{ lb}$$

The positive signs accompanying the results show that the unknown forces were assumed correctly.

285. Figure 157 represents a three-hinged arch of a type sometimes used to support roofs. The three hinges, or pins, are at A, B, and C. They are usually assumed to be smooth (Art. 14). The arch is subjected to four wind loads. Find the reactions at the supports B and C.

Solution. Resolve the reaction at B into components B_x and B_y. The reaction at C passes also through A. This fact can be proved by means of Fig. 158, which represents the right half of the arch as a free body. Only two external forces act on this half; namely, the forces exerted by the pins at A and C. Two forces, to be in equilibrium, must be collinear. Therefore, force C must pass through C and A.

From the figure: $\angle EDA$ = arc tan $\frac{6}{12}$ = 26° 35′. $\angle DBF$ = arc tan $\frac{18}{6}$ = 71° 35′. $\angle ACF$ = arc tan $\frac{18}{24}$ = 36° 50′. By the principle of moments, using B as the center of moments and resolving the loads into their x- and y-components,

$$\Sigma M_B = 0 \qquad +(C \sin 36° \, 50') \times 48 - 12{,}000 \times 19 - (6800 \sin 26° \, 35') \times 18$$
$$-(6800 \cos 26° \, 35') \times 6 - (6800 \sin 26° \, 35') \times 24$$
$$-(6800 \cos 26° \, 35') \times 18 = 0$$
$$C = 17{,}400 \text{ lb}$$

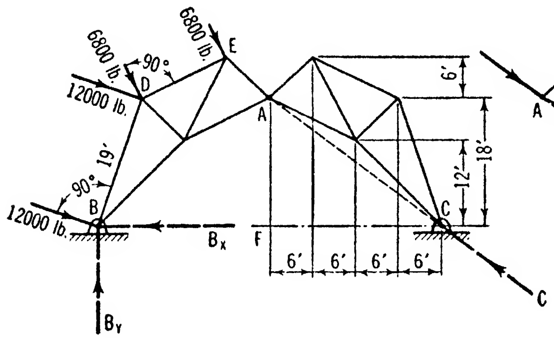

FIG. 157

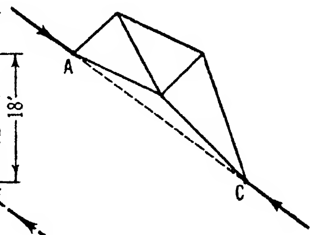

FIG. 158

By the principle of components, using the result obtained above,

$$\Sigma F_x = 0 \qquad -B_x + 24{,}000 \sin 71° \, 35' + 13{,}600 \sin 26° \, 35'$$
$$-17{,}400 \cos 36° \, 50' = 0$$
$$B_x = 14{,}900 \text{ lb}$$
$$\Sigma F_y = 0 \qquad +B_y - 24{,}000 \cos 71° \, 35' - 13{,}600 \cos 26° \, 35'$$
$$+17{,}400 \sin 36° \, 50' = 0$$
$$B_y = 9300 \text{ lb}$$

The results have the positive sign, showing that the sense was assumed correctly in each case.

286. Figure 159 represents a Fink roof truss of the cantilever type, subjected to five wind loads at right angles to the roof surface, and spaced equally. The truss is supported by pin-connections at A and B. Find the reactions.

Solution. The weights of the members are not given and are, therefore, to be disregarded. Consequently, BC is a two-force member (Art. 61), and the reaction at B may be assumed to be collinear with the member. The resultant, R, of the wind loads is 18,400 lb and acts through F. From the figure: $\overline{AD} = \sqrt{(15)^2 + (36)^2} = 39$ ft; $\overline{DF} = \frac{1}{2}\overline{AD} = 19.5$ ft. From the similar triangles, CDF and ADE: $\overline{CD}/\overline{DF} = \overline{AD}/\overline{DE}$, from which $\overline{CD} = 21.1$ ft. Also, from the figure, $\overline{EC} = 36 - 21.1 = 14.9$ ft; $\angle BCE = \text{arc tan } \dfrac{10}{14.9} = \text{arc tan } 0.671 = 33° 50'$.

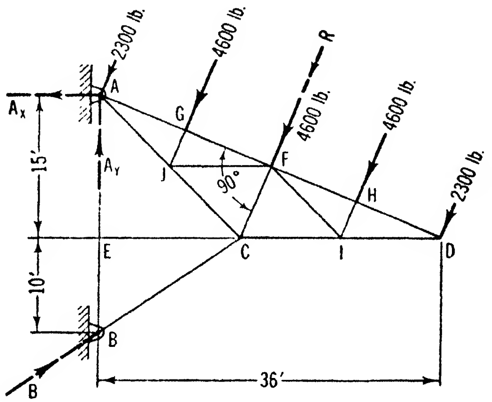

FIG. 159

By the principle of moments, using A as the center and replacing the loads by R,

$\Sigma M_A = 0 \qquad +(B \cos 33° 50') \times 25 - 18{,}400 \times 19.5 = 0 \qquad B = 17{,}300 \text{ lb}$

By the principle of components, using the result obtained above,

$\Sigma F_x = 0 \qquad -A_x + 17{,}300 \cos 33° 50' - 18{,}400 \times \frac{15}{38} = 0 \quad A_x = 7300 \text{ lb}$

By the principle of moments, with C as the center,

$\Sigma M_C = 0 \qquad -A_y \times 14.9 + 7300 \times 15 = 0 \qquad A_y = 7350 \text{ lb}$

The positive signs of the results show that the senses of the unknown forces were assumed correctly.

287. The pin-connected framework shown in Fig. 160 sustains two concentrated loads as indicated. Find all the forces acting on each member.

Solution. Figure 161 shows the member AC as a free body. Let A_x and A_y represent the components of the force exerted on AC by the support at A, and let C_x and C_y represent the components of the force exerted on AC by the member BC through the medium of the connecting pin. The correct sense of A_x is uncertain; let it be assumed that the component acts toward the left.

It should be observed that neither AC nor BC is a two-force member; therefore the resultant forces at its extremities may not be assumed to be collinear with the member. By the principle of moments, and the principle of components,

$$\Sigma M_C = 0 \qquad +A_x \times 8 + A_y \times 6 - 400 \times 5 = 0$$
$$\Sigma F_x = 0 \qquad -A_x - C_x + 400 \times 0.8 = 0$$
$$\Sigma F_y = 0 \qquad +A_y - C_y - 400 \times 0.6 = 0$$

The three equations thus obtained contain four unknown quantities, and, as matters stand, a solution is impossible. The two fundamental principles used are valid for all axes; therefore it would be possible to write additional equations by selecting new axes. Such additional equations would be entirely correct, but they would be of no avail in the present difficulty, since only three independent equations can be written, from the principles of statics, for the general coplanar system (Art. 57).

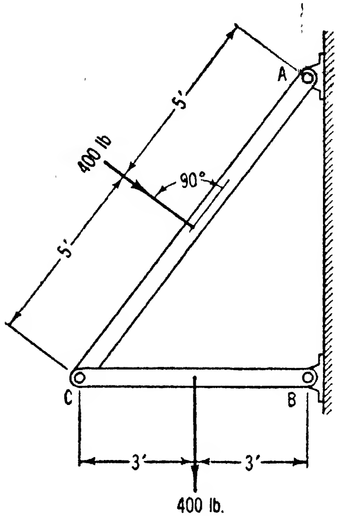

FIG. 160

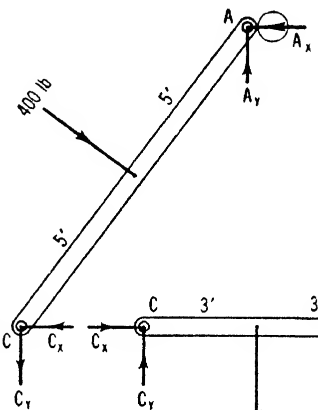

FIG. 161

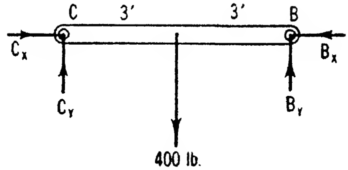

FIG. 162

Therefore, let the three equations obtained above be held in abeyance, and let member BC now be considered. Figure 162 represents member BC as a free body. C_x and C_y are the components of the force exerted on BC by member AC, and B_x and B_y are the components of the force exerted on BC by the support at B.

Careful attention should be paid to the fact that components C_x and C_y in Fig. 162 are not identical with components C_x and C_y in Fig. 161. For example, the two designated as C_x are collinear, and are equal in magnitude, but they are not the same force, and are directly opposite in sense. They constitute an example of the fact that forces always occur in pairs, the two forces of each pair being collinear, equal, and opposite (Art. 5). Ordinarily when the sense of a force is unknown it may be assumed arbitrarily, and the correctness or incorrectness of the assumption will be revealed by the sign obtained when the magnitude is solved for. However, it is necessary to avoid assumptions that obviously violate the fundamental laws of mechanics. For this reason, C_x and C_y in Fig. 162 must be taken as opposite in sense to C_x and C_y in Fig. 161.

Now let three independent equations be written for member BC, as follows:

$$\Sigma M_C = 0 \qquad +B_y \times 6 - 400 \times 3 = 0$$

$$\Sigma F_x = 0 \qquad -B_x + C_x = 0$$

$$\Sigma F_y = 0 \qquad +B_y + C_y - 400 = 0$$

A total of six independent equations, containing six unknown quantities, are now available. The solution of these equations gives: $A_x = -80$ lb; $A_y = +440$ lb; $B_x = +400$ lb; $B_y = +200$ lb; $C_x = +400$ lb; $C_y = +200$ lb.

The negative sign accompanying A_x shows that the sense of that component was assumed incorrectly. All the other components were assumed correctly.

PROBLEMS

288. The beam shown in Fig. 163 is supported by a pin-connection at B and by a roller at A. Find the reactions exerted on the beam by its supports. *Ans.* $A_y = +1700$ lb; $A_x = 0$; $B_x = +360$ lb; $B_y = -820$ lb.

289. Solve Prob. 278, Art. 66, by the algebraic method.

290. Figure 164 represents a portion of a lever-and-toggle mechanism for exerting heavy pressures. Find all the forces acting on the lever, AD. *Ans.* $A_x = -91.5$ lb; $A_y = -108$ lb; $C = 183$ lb, $\theta_x = 60°$.

291. Solve Prob. 238, Art. 62, by the method of the present article. Represent the force at A by its horizontal and vertical components.

292. Figure 165 represents a uniform, rectangular plate weighing 400 lb. The plate is suspended, in the position indicated, by a pin-connection at B, and by cables attached at A and D. The cables pass over smooth pulleys and are weighted as shown. Calculate the weight, C, and the reaction exerted on the plate by the support B. *Ans.* $C = 200$ lb; $B_x = +100$ lb; $B_y = +200$ lb.

293. Figure 166 represents an unsymmetrical roof truss subjected to three wind loads. Panel lengths AD and CD are equal. The truss is supported by a fixed pin-bearing at A and by rollers at B. Calculate the reactions exerted by the supports.

294. Figure 167 represents, in side elevation, an apparatus that was used at the University of Kansas to ascertain the pressure of sand against a vertical surface. F and N are the frictional and normal components of the resultant pressure exerted by the sand. The forces P_1, P_2, and P_3 were measured by means of platform scales. The weight of the apparatus was eliminated from the computations by the use of initial readings. In one of the tests the following data were obtained: $P_1 = 121.4$ lb; $P_2 = 118.5$ lb; $P_3 = 356.2$ lb. Calculate F, N, and h. *Ans.* 239.9 lb; 356.2 lb; 23.04 in.

295. Solve Prob. 253, Art. 63, resolving the reaction at B into its horizontal and vertical components, and using the general methods of the present article.

296. Figure 168 represents a Thénard shutter dam. The resultant pressure of the water on the dam is 4000 lb, as shown. Assuming smooth hinges at B and C, and disregarding the weight of the structure, find the reactions at C and D. *Ans.* $C_x = -1690$ lb; $C_y = -1140$ lb; $D = 3080$ lb, $\theta_x = 135°$.

297. Figure 169 represents a cantilever truss sustaining five vertical loads. Find the supporting forces at A and K

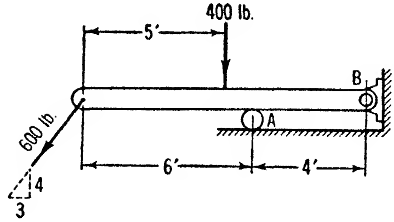

Fig. 163

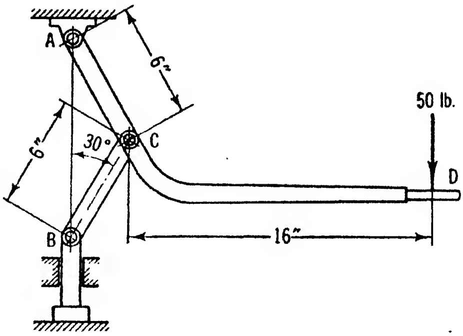

Fig. 164

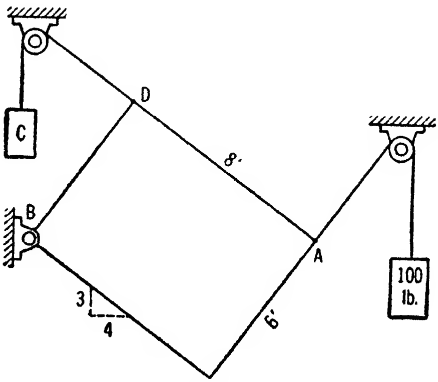

Fig. 165

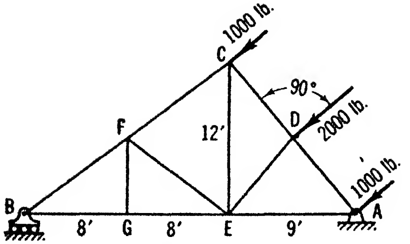

Fig. 166

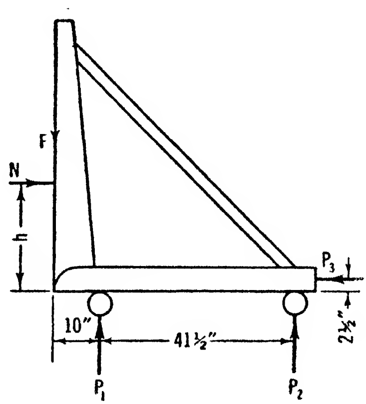

Fig. 167

298. Figure 170 represents a ship's davit for handling lifeboats. The ship has a list of 15° from the vertical. Find the reactions at A and B, assuming that the latter is at right angles to AB. Disregard the weight of the davit. *Ans.* A_x = 3670 lb, right, $\perp$ to AB; A_y = 2900 lb, up, along AB; B_x = 4450 lb, left, $\perp$ to AB; B_y = 0.

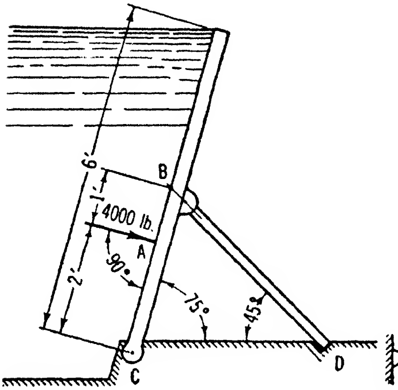

Fig. 168

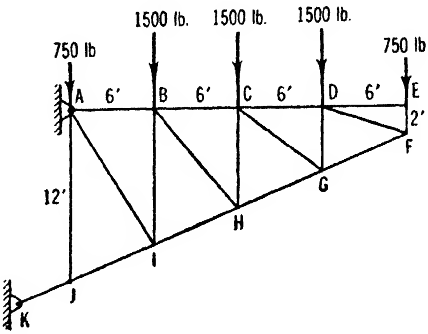

Fig. 169

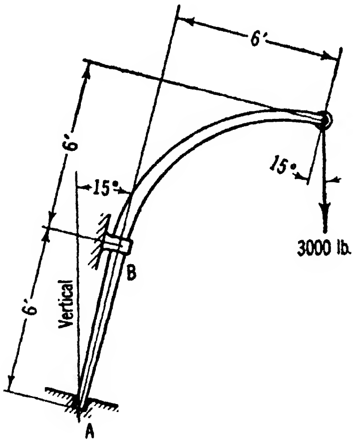

Fig. 170

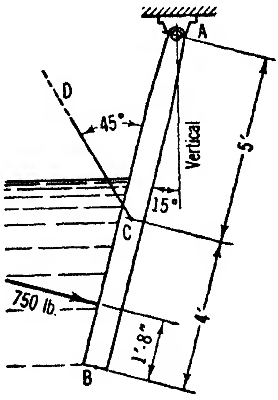

Fig. 171

299. Solve Prob. 298 for the case where the list of the ship is 15° to the left of the vertical. Assume that the 3000-lb load remains vertical.

300. Figure 171 represents a curtain dam. The dam is hinged at A, and can be raised and lowered by means of the chain CD. In the position shown the resultant water pressure is 750 lb, at right angles to the face of the dam. Find the force supporting the dam at A, and the tension in the chain. Disregard the weight of the dam. *Ans.* A_x = +53.4 lb; A_y = −1150 lb; C = 1560 lb.

301. The truss shown in Fig. 172 is supported by a pin-connection at E and by rollers at A. Find the reactions at the supports.

302. Figure 173 represents a spandrel-braced, three-hinged arch of a type often used in bridges. The hinges are at A, B, and C, and are assumed to be smooth pins.

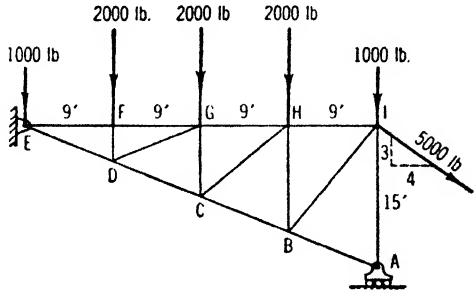

FIG. 172

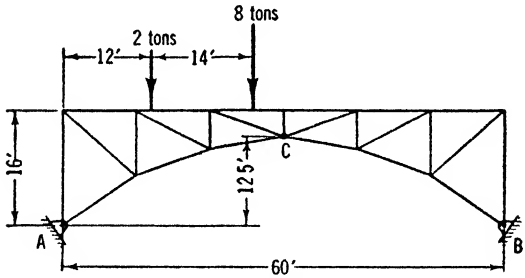

FIG. 173

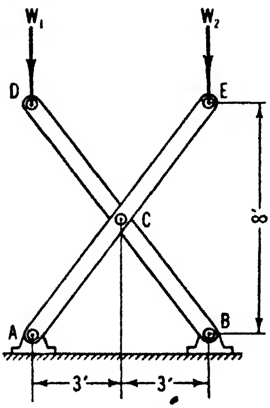

FIG. 174

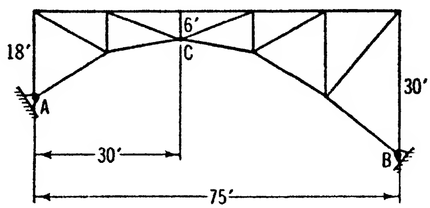

FIG. 175

The two loads shown are caused by a heavy truck. Find the reactions at the supports, caused by the two loads. *Ans.* $A_x = +9.28$ tons; $A_y = +6.13$ tons; $B = 10.1$ tons, from B through C.

303. Figure 174 represents a pin-connected framework consisting of two members, AE and BD, supporting vertical loads as shown. Let $W_1 = W_2 = 500$ lb. Owing to the symmetry of the structure and its loading, it is permissible to assume that the force exerted by each member on the other at C is horizontal. Making this assumption, find all the forces acting on each member. Now solve the problem again, without making the foregoing assumption.

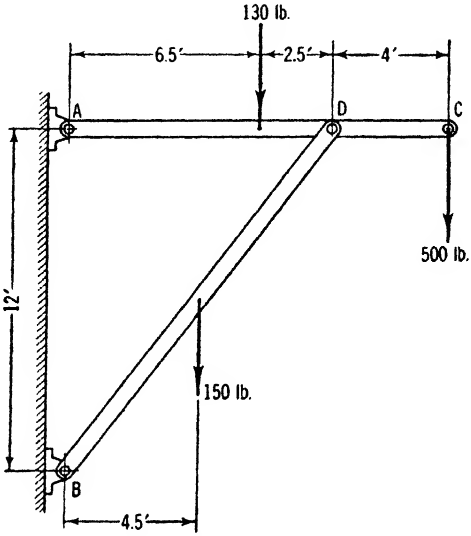

FIG. 176

304. In Fig. 174 let $W_1 = 500$ lb and $W_2 = 300$ lb. Find the forces acting on each member. Is the assumption made in Prob. 303 permissible in the present case? *Ans.* On AE: $A_x = +600$ lb; $A_y = +500$ lb; $C_x = -600$ lb; $C_y = -200$ lb. On BD: $B_x = -600$ lb; $B_y = +300$ lb; $C_x = +600$ lb; $C_y = +200$ lb.

305. In Fig. 174 let $W_1 = 500$ lb. Change W_2 so that it is horizontal and toward the right, and is equal to 300 lb. Find the forces acting on each member.

306. Figure 175 represents an unsymmetrical three-hinged arch, with hinges at A, B, and C. Assume that the right-hand portion of the arch is supporting a uniformly distributed live load of 500 lb per horizontal linear foot, and that the left-hand portion is not loaded. Assume that the hinges are frictionless pins. Find the reactions at the supports, A and B. *Ans.* $A = 13{,}000$ lb, from A through C; $B_x = -12{,}100$ lb; $B_y = +17{,}700$ lb.

307. In Prob. 306, assume that the live load of 500 lb per lin ft covers the entire span, and solve the problem.

308. Find all the forces acting on each member of the crane shown in Fig. 176.

309. Figure 177 represents a pair of tongs used in connection with a crane, for gripping and lifting heavy objects. The body being lifted weighs 300 lb. Find all the forces acting on the member AC. *Ans.* $A_x = -263$ lb; $A_y = -150$ lb; $B_x = +363$ lb; $B_y = 0$; $C = 180$ lb, upward, along DC.

310. Figure 178 represents a pillar crane, sustaining a load of 4 tons. Find all the forces acting on the mast AD, and on the boom CE. Assume the force at B to be horizontal. Disregard the weight of the crane.

311. Figure 179 represents a body which is being drawn along a horizontal plane by a constant force of 650 lb, applied as indicated. The center of gravity is at G, and the weight, W, is 1500 lb. The remaining forces are F_1, N_1, F_2, and N_2, as shown. The body has a horizontal acceleration, toward the right, of 3.22 ft per sec per sec. Since the body is being accelerated the system of external forces is not in

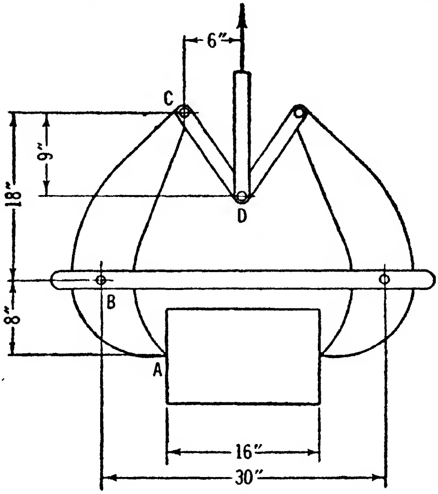

FIG. 177

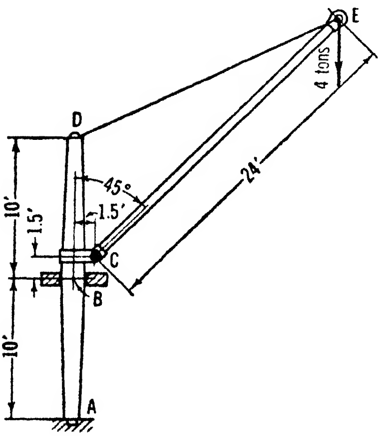

FIG. 178

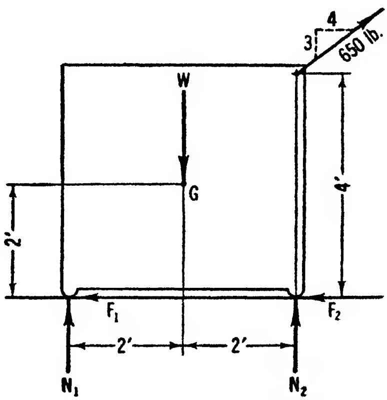

FIG. 179

equilibrium and the problem is not a static one. However, in kinetics it is learned that in a motion of this particular type the equilibrant of the external forces is a vector whose magnitude is $(W/g)a$, parallel to the acceleration but opposite to it in sense, and passing through G. If the equilibrant is inserted in the figure the entire system of vectors will balance.

Assuming these statements to be correct, and also assuming that $F_1 = \frac{1}{3}N_1$ and $F_2 = \frac{1}{3}N_2$, calculate F_1, N_1, F_2, and N_2. *Ans.* -102 lb; $+305$ lb; -268 lb; $+805$ lb.

CHAPTER V

PLANE TRUSSES

68. Axial Stress. The upper portion of Fig. 180 depicts a "two-force member" (Art. 61). It is a straight bar, in equilibrium, subjected to two external forces, P_1 and P_2, applied at points lying on the axis of the member, at its extremities. Sometimes a member is subjected to a concurrent system, instead of a single force, at one or both ends.

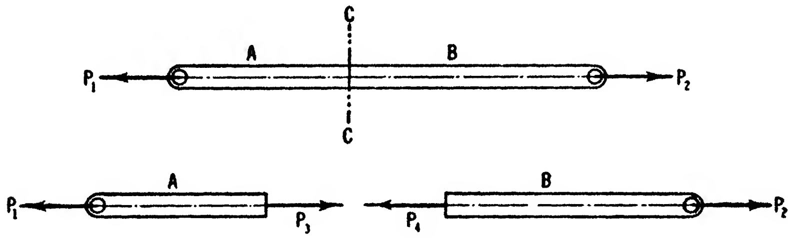

FIG. 180

In that case P_1 and P_2 may be taken to represent the resultants of such concurrent systems. Let the weight of the bar be disregarded.

For equilibrium, P_1 and P_2 must be collinear, equal, and opposite; therefore, $P_1 = P_2$. Since the points of application of P_1 and P_2 are on the axis of the bar, their lines of action coincide with that axis. Under such circumstances the member is said to be in a condition of *axial stress.*

Let CC represent an imaginary plane at right angles to the axis of the bar and situated anywhere between the ends. For the purposes of the discussion this plane divides the bar into two parts, A and B. Part A exerts a force on Part B, and vice versa. In the lower portion of Fig. 180 each of these parts is shown as a free body. Let P_3 represent the force exerted by Part B on Part A, and let P_4 represent the force exerted by A on B. Obviously, $P_1 = P_2 = P_3 = P_4$.

Either of the forces P_3 or P_4 is referred to as the total stress in the member. For the sake of brevity the word "total" will be omitted in this discussion. In engineering practice the term "stress," when used without qualification, is usually taken to mean force per unit area, rather than total stress.

The particular situation represented in Fig. 180 is called *tension*, and the stress is called *tensile stress*. Each force constitutes a pull. If all the forces in the figure were reversed in sense the member would be in *compressive stress*.

Strict attention should be paid to the fact that the assumption of axial stress is justifiable only if the member is straight, and is subjected to only two forces, the lines of action of which coincide with the axis of the member; or is subjected to forces which, by composition, can be reduced to two resultants whose lines of action coincide with the axis of the member.

In this book the study of stresses will be confined to those which are axial.

PROBLEMS

312. Is the bar in Fig. 116 in the condition of axial stress? Is the stress tensile or compressive?

313. Which member of the framework in Fig. 132 is in the condition of axial stress, if the weights of the members are disregarded? Is the stress tensile or compressive? What would be the situation if the weights of the members were included?

314. In Fig. 176 the 130-lb and 150-lb forces represent the weights of the members. Is either member under axial stress? What would be the situation if the weights of the members were disregarded?

315. In Fig. 177, which member or members are under axial stress, if the weights of the members are disregarded? Answer a similar question for Fig. 178.

69. Trusses in General. A truss usually consists of straight pieces, called members, connected in such a manner as to divide the area within the perimeter of the truss into triangular spaces. If the longitudinal axes of all the members lie in the same plane the truss is called a *plane truss*. In pin-connected trusses the members are connected at each joint by means of a cylindrical pin. In riveted trusses the members are connected by means of gusset plates, to which they are firmly riveted. Welded connections also have been used to a considerable extent during recent years. Those joints which are situated on the perimeter of a truss are called *panel points*. The loads and reactions sustained by a truss are usually applied at the panel points.

Loads. Trusses are designed for all the permanent loads that they will be called upon to carry, and for such combinations of intermittent or varying loads as may reasonably be expected to occur. The permanent loading is called the *dead load*. The weight of the truss itself is a part of the dead load. The weight is usually estimated for the truss as a whole, and distributed among the other permanent loads at the various panel points. Other loads that may require consideration include the effects of snow, vehicles, tractors, human beings, and any other factors whose action on the truss can be foreseen and estimated. Each member is designed for the stress resulting from that combination of loading which would most greatly endanger the member.

Stresses. After the loads have been estimated the *primary stresses* in the members are calculated. The primary stress is the total stress in a member, calculated under the assumption that each member of the truss is acted on by only two forces, or resultants, applied at the extremities of its longitudinal axis. In other words, primary stresses are calculated by assuming a condition of axial stress in each member. It is necessary, therefore, in the calculation of primary stresses, to disregard the weight of the individual member, although, as has been intimated, the effect of such weight on the truss as a whole is provided for by including it with the dead load.

Axial stress cannot exist throughout a member that is subjected to bending. Loads applied between joints, weights of individual members not in a vertical position, failure of the joints to permit members to adjust themselves to the deformation of the truss, and the tendency of compressive forces to cause bending in members even when applied axially constitute the principal reasons why bending usually does occur. Consequently, the ideal conditions assumed in the calculation of primary stresses are never completely realized in practice. The ideal conditions are more nearly attained in pin-connected trusses, as a rule, than in riveted or welded trusses.

After the primary stresses have been calculated, any serious bending in individual members caused by their weights or by other transverse loads is provided for by means of calculations based on the principles of strength of materials. Sometimes the bending is partially eliminated by introducing certain eccentricities into the connection details at the joints. The primary stresses play a part in all such calculations. In relatively slender members under compression it is assumed that bending is inevitable, and various column formulas are used, in which the primary stress is taken as the total load on the column.

Stresses arising from the failure of the joints to permit the ends of members to adjust themselves to the deformation of the truss are called *secondary stresses.* Their calculation is laborious, and is seldom made except on large jobs. In the examples that follow only primary stresses will be considered.

70. Primary Stresses; Algebraic Solution. The detailed method of calculating primary stresses in plane trusses by algebraic analysis will be shown by means of illustrative problems.

Illustrative Problems

316. Figure 181 represents a Howe roof truss. The lines represent the axes of the various members. The loads are a combination of dead loads and snow loads. The lines ADG and AJG are called the upper chord and the lower chord, respectively. The panel lengths AB, BC, CD, etc., in the

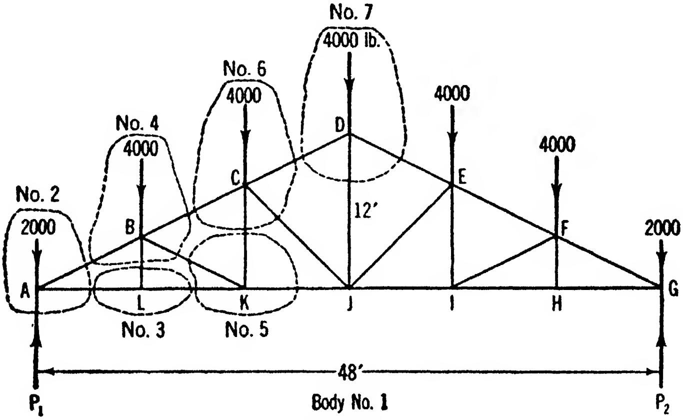

Fig. 181

upper chord, are equal. The members BL, CK, DJ, etc., are vertical. It follows that the panel lengths in the lower chord are equal. The entire truss and its loads are symmetrical with respect to a vertical axis through D. Assume a condition of axial stress throughout the truss, and calculate the primary stresses in the members.

Solution. Certain data will be needed in the solution, as follows: $\overline{BL} = \frac{1}{3} \times 12 = 4$ ft; $\overline{CK} = \frac{2}{3} \times 12 = 8$ ft. $\angle BAL = \angle BKL = \text{arc tan } \frac{4}{8} = 26° \, 35'$. $\angle ABL = \angle KBL = \angle BCK = \angle CDJ = 90° - 26° \, 35' = 63° \, 25'$. $\angle CBK = 180° - 2 \times \angle ABL = 180° - 126° \, 50' = 53° \, 10'$. $\angle CJK = \text{arc tan } \frac{8}{8} = 45° \, 00'$. $\angle KCJ = 90° - \angle CJK = 90° - 45° = 45° \, 00'$. $\angle DCJ = 180° - \angle BCK - \angle KCJ = 180° - 63° \, 25' - 45° \, 00' = 71° \, 35'$.

The complete solution will necessitate the consideration of several free bodies, and it will be advantageous to number these bodies in the order in which they are studied.

Body 1

Consider the entire truss, Fig. 181, as Body 1. The external forces acting on the truss are the seven loads, and the reactions, P_1 and P_2, of the supports. The reactions may be assumed to be vertical. Since the truss and its external forces are symmetrical,

$$P_1 = P_2 = \tfrac{1}{2} \times 24{,}000 = 12{,}000 \text{ lb}$$

Since at present the body under consideration is the entire truss, the stresses in the members are internal forces. Internal forces cannot be calculated, as such; they must first be made external. This is accomplished by selecting for consideration various segments or portions of the truss in such a manner that the desired stresses will be external, instead of internal, forces. The unknown stresses are then calculated by the principles of equilibrium.

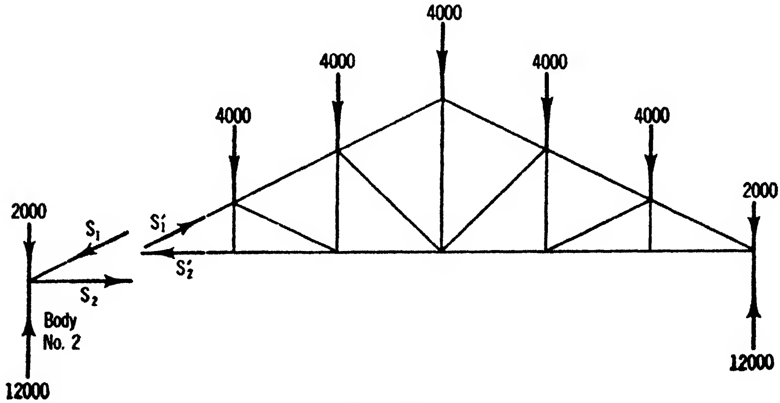

FIG. 182

Body 2

As Body 2 consider that portion of the truss enclosed within the line marked "No. 2," in Fig. 181. Figure 182 shows this portion and the remainder of the truss, separated slightly in order that the stresses may be clearly shown. S_1 and S_2 are the forces exerted on Body 2 by the larger portion of the truss. The reaction P_1 and one 2000-lb load also act on the body. While Body 1 was being considered, S_1 and S_2 were internal; now they are external, since they are exerted on Body 2 by a different body (Art. 4).

The forces S_1' and S_2' are exerted on the larger portion of the truss, by Body 2. S_1 and S_1' are equal in magnitude, opposite in sense, and collinear, as are S_2 and S_2'. Either of the forces S_1 or S_1' is the stress in member AB, while either S_2 or S_2' is the stress in AL. If the senses of S_1 and S_2 have been assumed correctly, in Fig. 182, member AB is under compression and member AL is under tension (Art. 68).

For greater clearness Body 2 has been shown to a larger scale, in Fig. 183. From Fig. 183, by the principle of components,

$$\Sigma F_y = 0 \qquad -S_1 \sin 26° 35' - 2000 + 12{,}000 = 0$$

$$S_1 = 22{,}300 \text{ lb, in } AB$$

By the principle of moments, with the center at B,

$$\Sigma M_B = 0 \qquad +S_2 \times 4 + 2000 \times 8 - 12{,}000 \times 8 = 0$$

$$S_2 = 20{,}000 \text{ lb, in } AL$$

The positive sign was obtained in each case, showing that the senses of S_1 and S_2 were assumed correctly. Therefore, AB is under compressive stress, and AL is under tensile stress.

The student sometimes experiences difficulty in thinking of a portion of a body as a body in itself. He must clearly understand that Body 2, for example, is a definite body and is subject to all the laws of equilibrium, even though a portion of its boundary surface is purely imaginary.

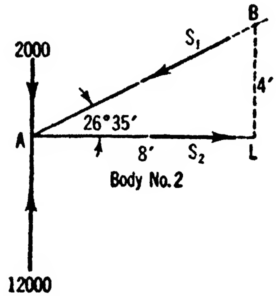

FIG. 183

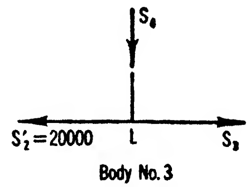

FIG. 184

Body 3

Care must be exercised to avoid the selection of a body involving too many unknown forces. In a coplanar concurrent system the number of unknowns must not exceed two. Figure 184 represents the part chosen as Body 3. It is now known that $S_2' = S_2 = 20{,}000$ lb. Since S_2' is now a known force, its sense may not be assumed arbitrarily; it must be correct. Extreme care should be exercised to avoid any error in this regard. The student should remind himself that the stress in AL was found to be tensile, and that S_2' must be represented as a pull, in accordance with that result.

A brief study of Fig. 184 shows that $S_3 = S_2' = 20{,}000$ lb, and that $S_4 = 0$. Member LK, then, is under a tensile stress of 20,000 lb, and BL is under zero stress.

In the calculation of primary stresses some of the members of a truss are occasionally found to be under zero stress. Frequently such members are stressed when the truss sustains a loading different from that under consideration at the moment. Also, such members often perform useful secondary functions even when the primary stress analysis does not represent them as sustaining stress under any type of loading.

Body 4

Figure 185 shows the part selected as Body 4. The stress in AB was found to be a compressive stress of 22,300 lb; therefore, $S_1' = 22{,}300$ lb and acts as a push. One of the 4000-lb loads acts on Body 4; also the unknown stresses S_5 and S_6. Let S_5 and S_6 be assumed to be compressive. By the principle of components,

$$\Sigma F_x = 0 \qquad -S_5 - S_6 \cos 53° \, 10' - 4000 \cos 63° \, 25' + 22{,}300 = 0$$

$$\Sigma F_y = 0 \qquad +S_6 \sin 53° \, 10' - 4000 \sin 63° \, 25' = 0$$

from which,

S_5 = 17,800 lb, compressive stress in BC

S_6 = 4480 lb, compressive stress in BK

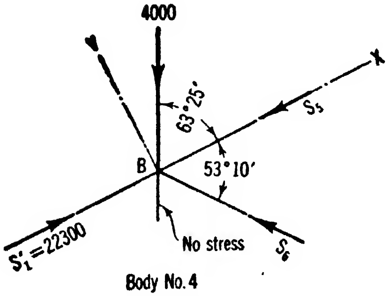

Fig. 185

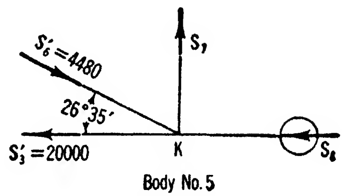

Fig. 186

Body 5

Figure 186 shows the body selected. S_7 has been assumed to be tensile, and S_8 compressive. By the principle of components, using horizontal and vertical axes,

$$\Sigma F_x = 0 \qquad -S_8 + 4480 \cos 26° \, 35' - 20{,}000 = 0$$

$$S_8 = -16{,}000 \text{ lb}$$

$$\Sigma F_y = 0 \qquad +S_7 - 4480 \sin 26° \, 35' = 0$$

$$S_7 = 2010 \text{ lb}$$

The negative sign accompanying S_8 shows that the sense of that stress was assumed incorrectly. S_7 was assumed correctly. Therefore, the stress in JK is 16,000 lb, tensile; and in CK is 2010 lb, also tensile.

Body 6

Figure 187 shows the body. One of the 4000-lb loads acts on this body. Placing the y-axis at right angles to the roof,

$$\Sigma F_y = 0 \qquad +S_{10} \sin 71° \, 35' - (4000 + 2010) \sin 63° \, 25' = 0$$

S_{10} = +5650 lb, compressive stress, in CJ

Placing the x-axis in a horizontal position, for greater convenience,

$$\Sigma F_x = 0 \qquad -S_9 \cos 26^\circ 35' + 17{,}800 \cos 26^\circ 35' - 5650 \cos 45^\circ = 0$$

$$S_9 = +13{,}300 \text{ lb, compressive stress, in } CD$$

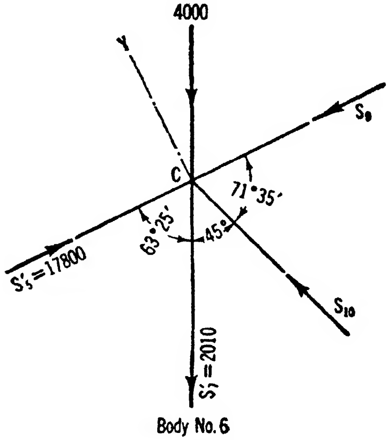

Fig. 187

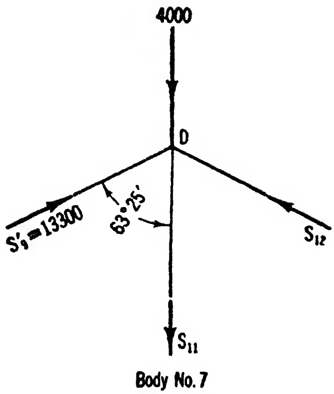

Fig. 188

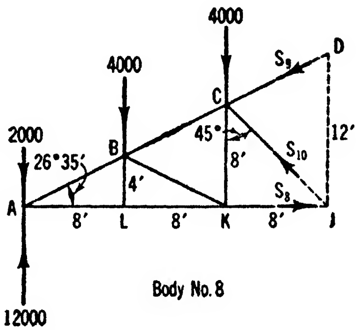

Fig. 189

Body 7

Figure 188 shows the body. From symmetry, $S_{12} = S_9' = 13{,}300$ lb, compressive stress.

$$\Sigma F_y = 0 \qquad -S_{11} - 4000 + 2 \times 13{,}300 \times \cos 63^\circ 25' = 0$$

$$S_{11} = +7920 \text{ lb, tensile stress, in } DJ$$

Because of the symmetry of the truss and of its loading it is not necessary to calculate the stresses in the remaining members. The stress in any member may be assumed to be equal to that in the corresponding member on the opposite side of the truss. Nevertheless it will be instructive to consider now a larger portion of the truss, as shown in Fig. 189.

Body 8

From Fig. 189, by the principle of moments,

$$\Sigma M_C = 0 \qquad +S_8 \times 8 - 12{,}000 \times 16 + 2000 \times 16 + 4000 \times 8 = 0$$

$$\Sigma M_J = 0 \qquad +(S_9 \cos 26^\circ\, 35') \times 12 - 12{,}000 \times 24 + 2000 \times 24$$
$$+\ 4000 \times 16 + 4000 \times 8 = 0$$

$$\Sigma M_A = 0 \qquad +(S_{10} \sin 45^\circ) \times 24 - 4000 \times 16 - 4000 \times 8 = 0$$

The solution of these equations gives: S_8 = 16,000 lb, tensile stress, in JK; S_9 = 13,400 lb, compressive stress, in CD; and S_{10} = 5660 lb, compressive stress, in CJ.

These results check satisfactorily the stresses previously calculated in JK, CD, and CJ. If the student will review the calculations of the stresses in the other members of the truss he will discover that this check also verifies all previous results except that for the stress in DJ. Let the student devise a simple check for this case. The foregoing check obviously does not verify the various geometrical calculations made in the beginning.

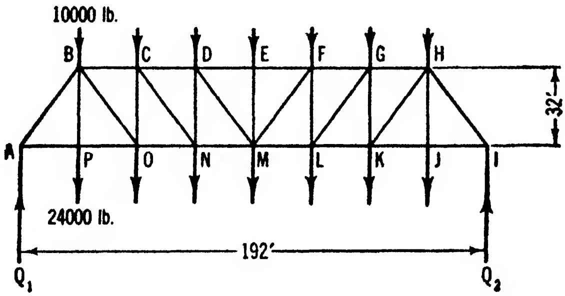

Fig. 190

317. Figure 190 represents a Pratt bridge truss. The panel lengths are equal. The truss sustains a vertical load of 10,000 lb at each joint in the upper chord, and 24,000 lb at each joint in the lower chord. Find the reactions at the supports by inspection, and then find the stress in member DE by means of a single equilibrium equation. Find the stresses in DM and MN in a similar manner, without using any stresses previously calculated.

Solution. The problem can be solved by considering as a free body that portion of the truss shown in Fig. 191. $\angle MDN$ = arc tan $\frac{24}{32}$ = 36° 50′. cos $\angle MDN$ = 0.8. By symmetry, $Q_1 = Q_2 = \frac{1}{2}(7 \times 10{,}000 + 7 \times 24{,}000)$ = 119,000 lb.

The external forces acting on the body in Fig. 191 are as follows: three of the 10,000-lb loads, three of the 24,000-lb loads, $Q_1 = 119{,}000$ lb, and the three unknown stresses, S_1, S_2, and S_3. Assume S_1 to be compressive, and S_2 and S_3 to be tensile.

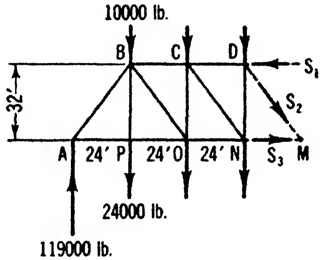

FIG. 191

The problem requires that S_1 be found by means of one equilibrium equation. An equation must be selected, therefore, which will not contain S_2 or S_3. This can be accomplished by means of the principle of moments, with the center of moments at M, since S_2 and S_3 pass through that point. Replacing the six loads by their resultant, which is equal to 102,000 lb,

$$\Sigma M_M = 0 \qquad +S_1 \times 32 + 102{,}000 \times 48 - 119{,}000 \times 96 = 0$$

$$S_1 = +204{,}000 \text{ lb, compressive stress, in } DE$$

S_2 cannot be found independently, and by means of a single equation, if moments are used, because S_1 and S_3 are parallel. However, if the principle of components is used, with reference to a vertical axis, S_1 and S_3 will not appear in the equation.

$$\Sigma F_y = 0 \qquad -S_2 \cos 36° \, 50' - 102{,}000 + 119{,}000 = 0$$

$$S_2 = +21{,}250 \text{ lb, tensile stress, in } DM$$

S_3 can be found, under the conditions laid down in the problem, by taking moments about D.

$$\Sigma M_D = 0 \qquad +S_3 \times 32 + 102{,}000 \times 24 - 119{,}000 \times 72 = 0$$

$$S_3 = +191{,}000 \text{ lb, tensile stress, in } MN$$

Let the student check the foregoing results by considering the remainder of the truss as a free body.

PROBLEMS

318. Find the amount and kind of stress in each member of the small Warren truss shown in Fig. 192.

319. Figure 193 represents a small fan truss. The panel lengths in the upper chord are equal. Calculate the reactions at the supports. Find the amount and kind of stress in member HI, by means of one equilibrium equation. *Ans.* 4500 lb (T).

320. Find the amount and kind of stress in each member of the unsymmetrical roof truss in Prob. 293, Fig. 166. Members AD and DC are equal in length. *Ans.* $BF = 2000$ lb (C); $BG = 1600$ lb (T); $CF = 2000$ lb (C); $FE = 0$; $FG = 0$; $CD = 1340$ lb (C); $CE = 1670$ lb (T); $DE = 2090$ lb (C); $AD = 750$ lb (C); $AE = 2850$ lb (T); $EG = 1600$ lb (T).

321. In the Howe roof truss shown in Fig. 194 the panel lengths in the upper chord are equal, and members BH, CG, and DF are vertical. Find the amount and kind of stress in each member.

322. Figure 195 represents a small Pratt roof truss. This type of truss is more often seen in bridges, but is sometimes used for roofs when the span is not too long. The panel lengths in the upper chord are equal, and members BG and DF are vertical. Find the amount and kind of stress in each member. *Ans.* $AB = 4870$ lb (C);

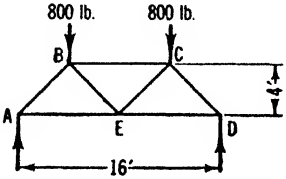

Fig. 192

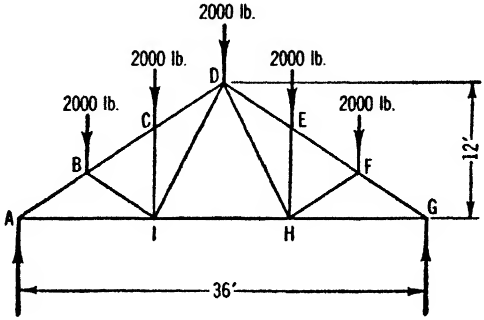

Fig. 193

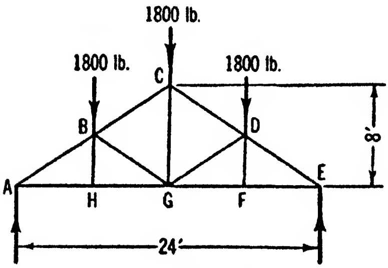

Fig. 194

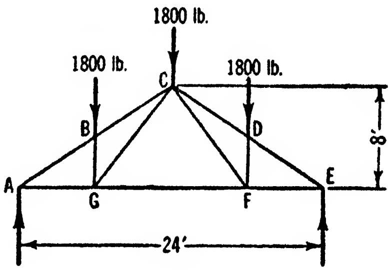

Fig. 195

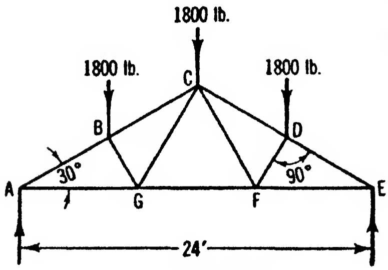

Fig. 196

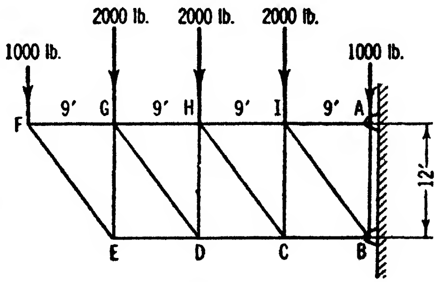

Fig. 197

$AG = 4050$ lb (T); $BC = 4870$ lb (C); $BG = 1800$ lb (C); $CG = 2250$ lb (T); $FG = 2700$ lb (T).

323. Figure 196 represents a small Fink roof truss. The panel lengths in the upper chord are equal, and the member BG is at right angles to the roof slope. Find the amount and kind of stress in each member.

324. Find the amount and kind of stress in members HI, HC, and DC, of the cantilever truss in Fig. 197. Use only one equation in each case, and make each result independent of the others. *Ans.* $HI = 6750$ lb (T); $HC = 6250$ lb (C); $DC = 3000$ lb (C).

325. Find the amount and kind of stress in each member of the truss in Prob. 249, Fig. 128.

326. Find the amount and kind of stress in members GH, CH, and BC of the truss in Prob. 301, Fig. 172. Solve by dividing the truss into two parts and considering one part as a free body. Then check the results by considering the other part of the truss. *Ans.* $GH = 800$ lb (C); $CH = 3120$ lb (T); $BC = 2600$ lb (T).

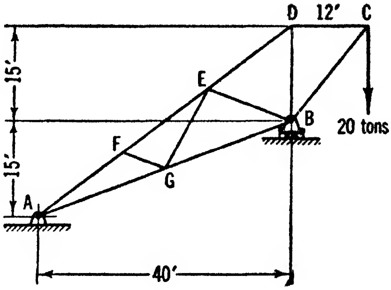

Fig. 198

327. Find the amount and kind of stress in each member of the truss in Fig. 198. The truss is supported by rollers at B. Panel length $\overline{AF} = \overline{FE} = \overline{ED}$.

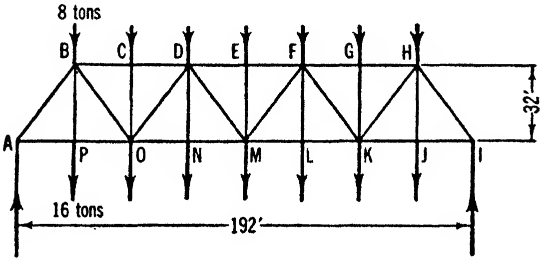

Fig. 199

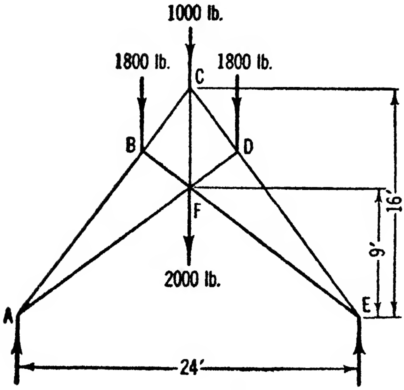

Fig. 200

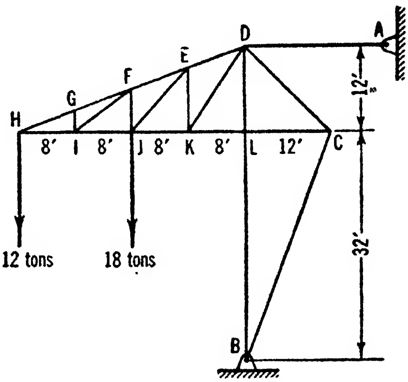

Fig. 201

328. Find the amount and kind of stress in members BC, BH, and HI, of the truss in Prob. 297, Fig. 169. *Ans.* $BC = 2570$ lb (T); $BH = 2600$ lb (T); $HI = 4620$ lb (C).

329. Find the amount and kind of stress in each member of the truss in Prob. 286, Fig. 159. Members GJ, FC, and HI are at right angles to the upper chord of the truss.

330. Figure 199 represents a Warren bridge truss. The panel lengths are equal. There is a vertical load of 8 tons at each joint in the upper chord, and of 16 tons at each joint in the lower chord. Find the amount and kind of stress in each member. *Ans.* $AB = 105$ tons (C); $AP = 63$ tons (T); $BP = 16$ tons (T); $OP = 63$

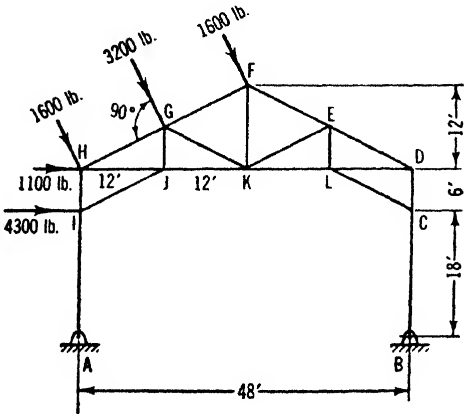

Fig. 202

tons (T); $BO = 75$ tons (T); $BC = 108$ tons (C); $CO = 8$ tons (C); $CD = 108$ tons (C); $DO = 45$ tons (C); $NO = 135$ tons (T); $MN = 135$ tons (T); $DM = 15$ tons (T); $DE = 144$ tons (C); $EM = 8$ tons (C).

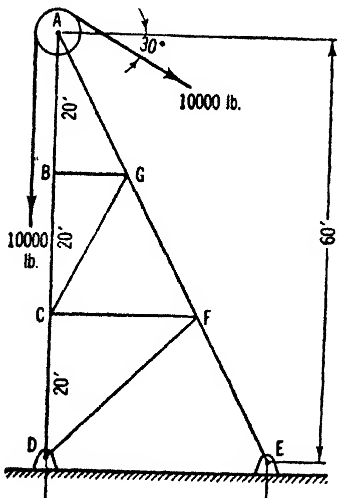

Fig. 203

331. Figure 200 represents a type of framework known as the scissors truss. Find the amount and kind of stress in each member.

332. Find the amount and kind of stress in each member of the structure shown in Fig. 201.

333. Figure 202 represents a transverse bent of a mill building. Calculate the reactions exerted by the supports at A and B, assuming that their horizontal components are equal. Then calculate the stress in the member JK, using only one equilibrium equation. *Ans.* $A_x = -4130$ lb; $A_y = +342$ lb; $B_x = -4130$ lb; $B_y = +5380$ lb; 8050 lb (T).

334. In Fig. 202, the column AH is not a two-force member and is not under axial stress. Calculate all the forces acting on the column. Make use of the answers to Prob. 333.

335. In Fig. 202, find the amount and kind of stress in member EK, by means of one equilibrium equation. Make use of the answers to Prob. 333. *Ans.* 9250 lb (T).

336. Find the amount and kind of stress in members AB and AG of the mine headframe truss shown in Fig. 203, caused by the two tensile forces in the cable.

337. In the mine headframe truss of Prob. 336, Fig. 203, assume horizontal wind loads at B and C of 3200 lb, each, and one of 1600 lb at A, all acting toward the right. Find the amount and kind of stress in each member, caused by the wind loads only. *Ans.* AG = 3400 lb (C); AB = 3000 lb (T); BG = 3200 lb (C); BC = 3000 lb (T); CG = 3400 lb (T); FG = 6800 lb (C); CF = 4800 lb (C); CD = 6000 lb (T); EF = 10,200 lb (C); DF = 4380 lb (T).

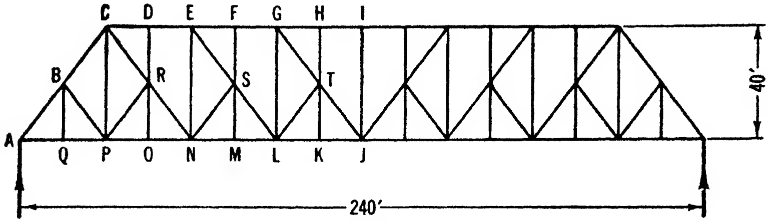

FIG. 204

338. Figure 204 represents a Baltimore bridge truss. The truss sustains a vertical load of 1600 lb at each joint in the upper chord, and a load of 3200 lb at each joint in the lower chord. There is no load at B or at the corresponding point at the other end of the truss. Find the amount and kind of stress in AB, AQ, QP, BQ, BP, BC, CP, CR, CD, DE, and PR.

339. In Prob. 338, Fig. 204, find the amount and kind of stress in members HI, TJ, and KJ. *Ans.* HI = 57,000 lb (C); TJ = 3000 lb (T); KJ = 55,200 lb (T).

71. Primary Stresses; Graphic Solution.

The details of the graphic method for calculating primary stresses in plane trusses will be shown by means of an illustrative problem.

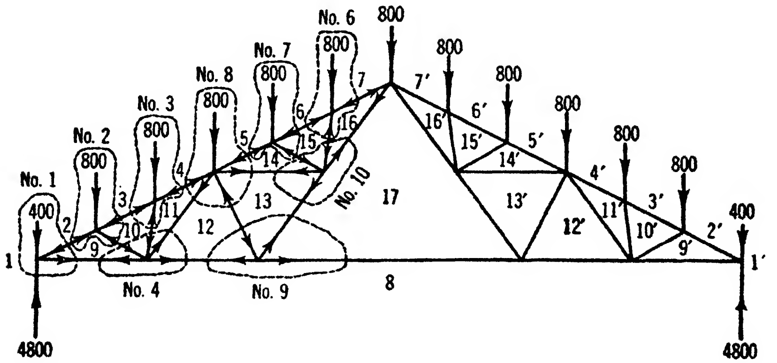

FIG. 205

ILLUSTRATIVE PROBLEM

340. Find the stresses in the truss shown in Fig. 205, by the graphic method.

Solution. Figure 205 will serve as the space diagram.. It is customary to number or letter the spaces between the members, and around the exterior of the truss.

Take Body 1 as indicated by the dotted line, marked "No. 1," in Fig. 205. The forces acting on Body 1 are the 4800-lb reaction, the 400-lb load, and two unknown stresses. Each force is designated by the numbers appearing in the spaces adjacent to it. These numbers will be read, and the forces will be plotted, in the sequence in which they occur as the eye is made to pass around Body 1 in a clockwise direction.

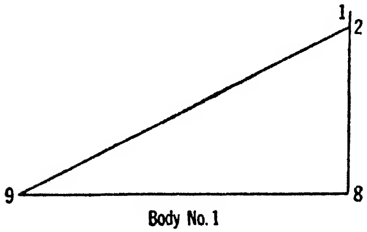

FIG. 206

Figure 206 is the force polygon, or stress diagram, for Body 1. In accordance with the plan of procedure adopted, and in order that the two unknown forces may come last in order, it is necessary to observe the following sequence in plotting the forces: (8, 1), (1, 2), (2, 9), and (9, 8). The reaction (8, 1) is, therefore, plotted first. The numbers 8 and 1 are placed at the extremities of the vector, with the 1 at the upper end. It will be understood that the sequence 8, 1 on the space diagram shall mean that the sense of the vector in the stress diagram is from 8 toward 1. This explains why the 1 must be placed at the upper end of the vector, since the sense of the reaction is upward.

The load (1, 2) is then laid off downward from 1, and the point 2 is thereby located. The unknown stress, (2, 9), is next in order. A line is drawn through 2, with the correct slope, but with a length which is, as yet, indefinite. One force, (9, 8), remains. Since the forces are in equilibrium the diagram must close at the point 8. The force (9, 8) is horizontal, and so a horizontal line is drawn through 8 until it intersects the inclined line previously drawn through 2. The intersection of these lines is the point 9. The forces (2, 9) and (9, 8) can now be scaled. Remembering the sequence adopted for the numbers, 8, 1, 2, 9, 8, and following that sequence along the stress diagram, it is seen that (2, 9) acts downward and toward the left, or toward the joint, and that (9, 8) acts toward the right, or away from the joint. Therefore, (2, 9) is a push, and (9, 8) is a pull; in other words, (2, 9) is compressive stress and (9, 8) is tensile stress. Arrowheads may be placed on the space diagram in the manner shown, as a convenient record of the kind of stress in each member.

It should be noticed that the sequence for the numbers is obtained from the space diagram, and that the observance of that sequence in reading the stress diagram does not necessarily cause the eye to move in a clockwise direction around the diagram.

Figure 207 is the stress diagram for Body 2, constructed in accordance with the same general scheme. It should be noticed that the force (9, 2) of the present case is not identical with the force (2, 9) of Body 1. These two forces are necessarily equal, but are opposite in sense. The force (9, 2) must be plotted first in the present case, and it is correct to place point 2 at the upper end of the vector. It is seen that the points 2 and 9 have the same relative positions as in Fig. 206, but that the order in which they are read is reversed.

The stress diagram for Body 3 is shown in Fig. 208, and the stress diagram for Body 4 is shown in Fig. 209.

At this juncture it is discovered that a free body taken at any one of the remaining joints of the truss would involve three unknown quantities. This

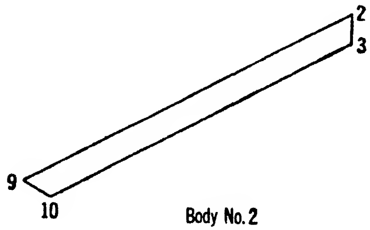

Fig. 207

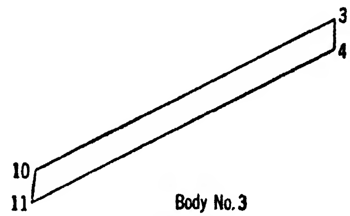

Fig. 208

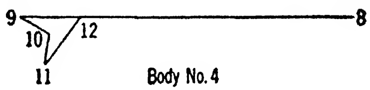

Fig. 209

obstacle can be surmounted as follows: in Fig. 210 is shown a truss having the same general form as the truss in Fig. 205, but with twelve of the interior members missing. The five 800-lb loads have been replaced by their resultant, a 4000-lb force acting at D. At first it might seem that such changes

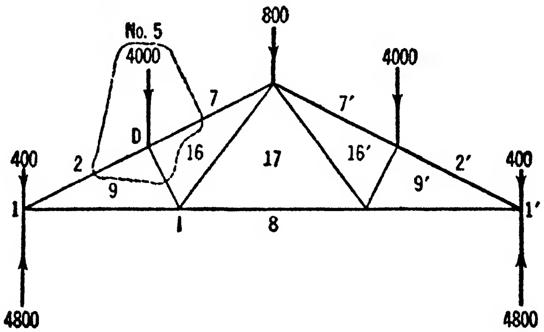

Fig. 210

would so seriously alter the conditions as to be of no value, but upon examination it becomes apparent that the stresses (2, 9), (7, 16), (16, 17), (17, 8), and (9, 8) in the new truss (Fig. 210) have the same values as those in the corresponding members of the old truss (Fig. 205). For example, the stresses (2, 9) and (7, 16) could be found by writing exactly the same algebraic equations as those by which these stresses would be calculated in the old truss (Fig. 205). Body 5 now may be taken from Fig. 210, as shown. Figure 211 is the stress diagram for Body 5. The stress (16, 9) in the new truss is not equal to any of the stresses in the old truss, and no use will be made of it. The true stress, (7, 16), has been found, however, and this makes it possible to construct the stress diagrams for Bodies 6, 7, 8, and 9, in their

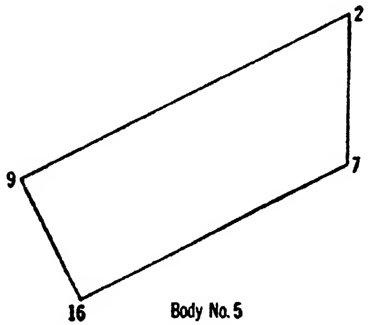

Fig. 211

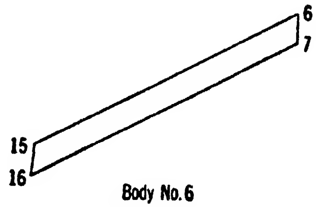

Fig. 212

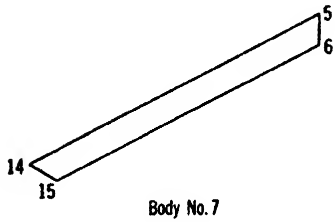

Fig. 213

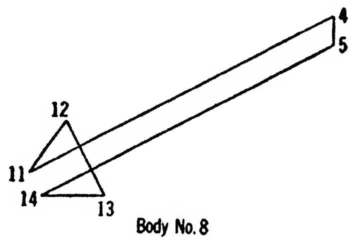

Fig. 214

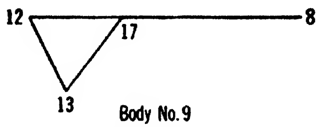

Fig. 215

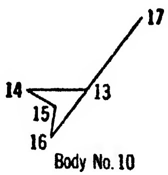

Fig. 216

order. The diagram for Body 10 may be constructed as a check. These stress diagrams are shown in Figs. 212, 213, 214, 215, and 216, respectively. In Fig. 216, the points 16, 13, and 17 should fall on a straight line.

It will be noticed that in the construction of the various stress diagrams in the foregoing solution several lines have been duplicated. For example, the line 2, 9 in Fig. 206 could have been made to serve for the first line in Fig. 207. Such a scheme, if followed throughout the solution, results in a compact stress diagram (Fig. 217) in which no unnecessary lines have been drawn. This replaces the ten separate stress diagrams constructed for illustrative purposes in the foregoing solution.

In the construction of the compact stress diagram as shown in Fig. 217, the entire truss was first taken as a free body, and the loads and reactions were plotted in accordance with the clockwise sequence around the truss. This sequence was used throughout the construction. If preferred, the counter-clockwise sequence may be adopted. It is important, however, to follow one plan consistently throughout the solution.

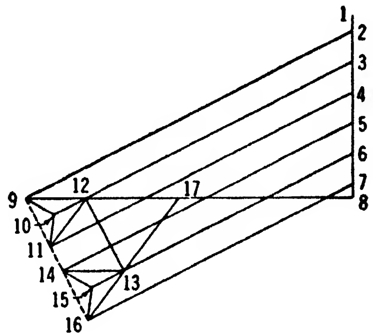

FIG. 217

The graphic method is widely used by engineers for the calculation of stresses in trusses, and may result in a considerable saving of time, especially for complicated trusses. The compact stress diagram is the one actually employed, of course, and should be used in the solution of the problems of the present article.

PROBLEMS

341. Solve Prob. 318, Fig. 192, by the graphic method.

342. Solve Prob. 293, Fig. 166, by the graphic method.

343. Solve Prob. 321, Fig. 194, by the graphic method.

344. Solve Prob. 322, Fig. 195, by the graphic method.

345. Solve Prob. 323, Fig. 196, by the graphic method.

346. Solve Prob. 331, Fig. 200, by the graphic method.

347. Solve Prob. 316, Fig. 181, by the graphic method.

348. In the truss shown in Fig. 193, panel lengths $\overline{AB}$, $\overline{BC}$, $\overline{DC}$, etc., are equal, and panel lengths $\overline{AI}$, $\overline{IH}$, and $\overline{HG}$ are equal. Find the amount and kind of stress in each member, by the graphic method.

349. Find the amount and kind of stress in each member of the truss in Prob. 317, Fig. 190, by the graphic method.

350. Solve Prob. 330, Fig. 199, by the graphic method.

351. Solve Prob. 332, Fig. 201, by the graphic method.

CHAPTER VI

EQUILIBRIUM OF NON-COPLANAR FORCE SYSTEMS

72. Equilibrium of the Non-Coplanar Concurrent System. Methods for the solution of problems involving non-coplanar force systems in equilibrium do not differ fundamentally from those used in connection with coplanar systems. When using the principle of moments with non-coplanar forces it is of especial importance to remember that moments are taken about axes, and not about points. A judicious choice of axes for the resolution of forces and for the calculation of moments is, as always, necessary to an efficient solution.

When a force is to be resolved into three rectangular components, the method used in Prob. 167, Art. 40, for calculating the cosines of the angles of inclination, is often superior.

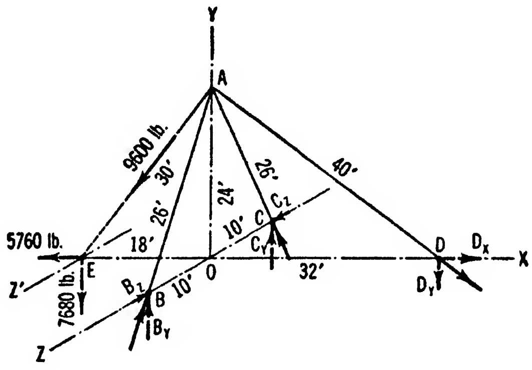

Fig. 218

Illustrative Problems

352. Figure 218 represents a framework consisting of two main members, *AB* and *AC*, braced by a cable, or guy, *AD*, and sustaining an inclined load of 9600 lb acting along the line *AE*. Disregarding the weights of the members, find the amount and kind of stress in *AB*, *AC*, and *AD*.

Solution. *AB*, *AC*, and *AD* are two-force members; therefore the forces exerted on them at the points of support, *B*, *C*, and *D*, are collinear with their respective members (Art. 61). The stress in each member is axial (Art. 68), and is equal to the force at the support. Let the letters *B*, *C*, and

D represent also the forces at the corresponding supports. Resolve the four external forces into their components, as shown, assuming that AB and AC are under compression and that AD is under tension.

The x-component of the 9600-lb load = 9600 cos $\angle AEO$ = 9600 × 18/30 = 5760 lb. The y-component = 9600 sin $\angle AEO$ = 9600 × 24/30 = 7680 lb. By the principle of components,

$$\Sigma F_x = 0 \qquad +D_x - 5760 = 0 \qquad D_x = 5760 \text{ lb}$$

By the principle of moments, using as the axis of moments a line through E parallel to the z-axis, and calling it z',

$$\Sigma M_{z'} = 0 \qquad +B_y \times 18 + C_y \times 18 - D_y \times 50 = 0$$

Since the structure and its loading are symmetrical with respect to the xy-plane, it may be assumed that $B_y = C_y$. From the figure,

$$B_y = B \times \tfrac{24}{26} \qquad C_y = C \times \tfrac{24}{26} \qquad D_x = D \times \tfrac{32}{40} \qquad D_y = D \times \tfrac{24}{40}$$

From these relationships, and from the values previously calculated,

$B = C = 6500$ lb, compressive stress, in AB and in AC

$D = 7200$ lb, tensile stress, in AD

It is suggested that the student check the foregoing results by calculating the algebraic sum of the y-components.

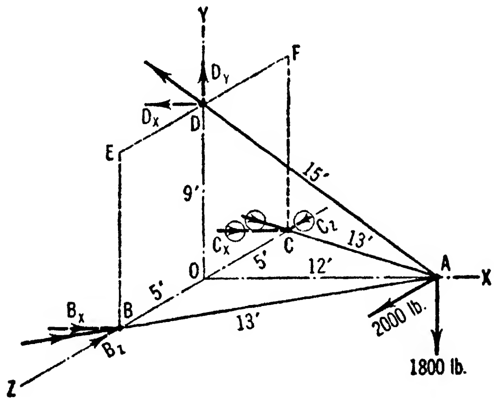

Fig. 219

353. Figure 219 represents a bracket consisting of three members, AB, AC, and AD. The points of support, B, C, and D, lie in the yz-plane. The bracket sustains a vertical load of 1800 lb and a load of 2000 lb parallel to the z-axis, as shown. Find the amount and kind of stress in each member.

Solution. AB, AC, and AD are two-force members (Art. 61). Therefore, the forces B, C, and D, exerted on the bracket by its supports, are collinear with the axes of the members. The stress in each member is equal to the force at the support.

The bracket itself is symmetrical with respect to the xy-plane, but the loading is not symmetrical; therefore it may not be assumed that B and C are equal. Let each of the supporting forces be resolved into components as shown in the figure. Let D be assumed tensile, and B and C compressive. By the principle of components,

$$\Sigma F_y = 0 \qquad +D_y - 1800 = 0 \qquad D_y = 1800 \text{ lb}$$

By the principle of moments, using FC as the axis of moments,

$$\Sigma M_{FC} = 0 \qquad +B_x \times 10 - D_x \times 5 - 2000 \times 12 = 0$$

Similarly, using EB as the axis of moments,

$$\Sigma M_{EB} = 0 \qquad -C_x \times 10 + D_x \times 5 - 2000 \times 12 = 0$$

Also, from the figure,

$$B_x = B \times \tfrac{12}{13} \qquad C_x = C \times \tfrac{12}{13} \qquad D_x = D \times \tfrac{12}{15} \qquad D_y = D \times \tfrac{9}{15}$$

The solution of the foregoing equations gives: $B = +3900$ lb; $C = -1300$ lb; and $D = +3000$ lb. The minus sign accompanying C shows that the sense was assumed incorrectly. Therefore, B is compressive, C is tensile, and D is tensile.

Let the student check the results by ascertaining whether the x-components balance.

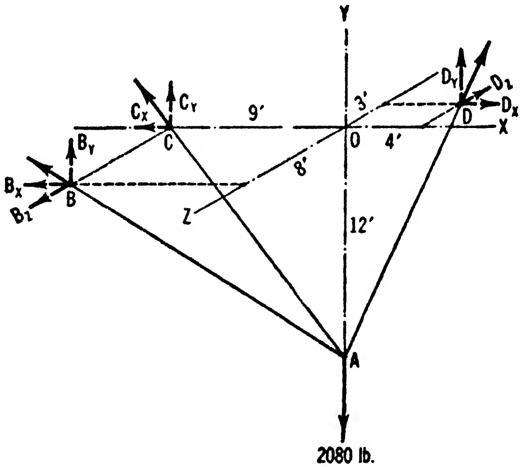

Fig. 220

354. In Fig. 220 three cables, AB, AC, and AD, are shown, joined at A, and supporting a vertical load of 2080 lb. The upper ends of the cables are anchored at the points B, C, and D, in the same horizontal plane. Calculate the stress in each cable.

Solution. Length of cable $\overline{AB} = \sqrt{(9)^2 + (12)^2 + (8)^2} = 17$ ft; $\overline{AC} = \sqrt{(9)^2 + (12)^2} = 15$ ft; $\overline{AD} = \sqrt{(4)^2 + (12)^2 + (3)^2} = 13$ ft. Let the stresses in the three cables be represented by the letters B, C, and D. Resolve each stress, at the point of support, into its components parallel to the coordinate axes.

By the principle of moments, using the line BC as the axis of moments,

$$\Sigma M_{BC} = 0 \qquad +D_y \times 13 - 2080 \times 9 = 0 \qquad D_y = 1440 \text{ lb}$$

By the principle of components,

$$\Sigma F_z = 0 \qquad +B_z - D_z = 0$$

$$\Sigma F_x = 0 \qquad -B_x - C_x + D_x = 0$$

From the figure,

$$B_x = B \times \tfrac{9}{17} \qquad B_z = B \times \tfrac{8}{17} \qquad C_x = C \times \tfrac{9}{15}$$

$$D_x = D \times \tfrac{4}{13} \qquad D_y = D \times \tfrac{12}{13} \qquad D_z = D \times \tfrac{3}{13}$$

The solution of the foregoing equations gives: $B = +765$ lb; $C = +125$ lb; $D = +1560$ lb. The positive sign was obtained in each case, showing that the senses of the unknown forces were assumed correctly and that the cables are under tension. A flexible cable is not capable of sustaining any considerable amount of compression.

Let the student check the results by means of an equilibrium equation different from any used in the solution.

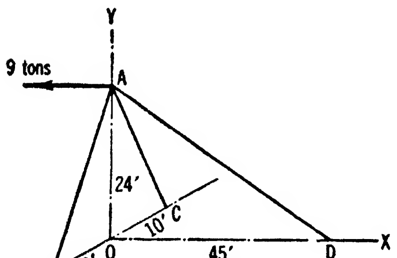

FIG. 221

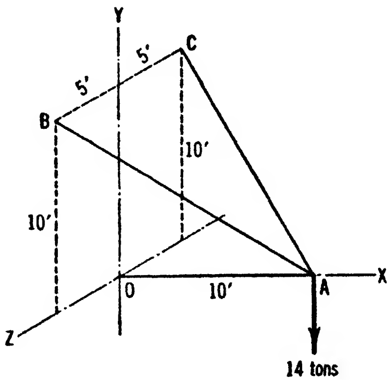

FIG. 222

PROBLEMS

355. Figure 221 represents an A-frame, consisting of the members AB and AC, and braced by a guy, AD. The frame sustains a load of 9 tons, parallel to the x-axis. Find the stress in each member, and in the guy. *Ans.* $B = C = 2.60$ tons (C); $D = 10.2$ tons (T).

356. Change the 9-ton load in Prob. 355 so that its line of action is inclined and intersects the x-axis at a point 18 ft to the left of O. Solve the problem, leaving all other conditions unchanged.

357. Figure 222 represents a bracket consisting of three members, AB, AC, and AO. It sustains a single, vertical load of 14 tons. The points of support, B, C, and O, are in the yz-plane. Find the stress in each member. *Ans.* $BA = CA =$ 10.5 tons (T); OA = 14.0 tons (C).

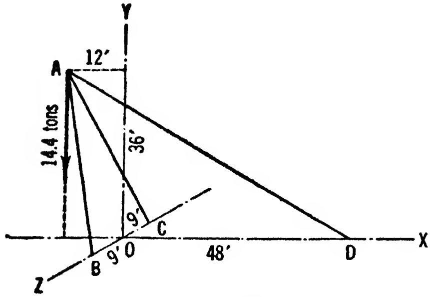

FIG. 223

358. In Prob. 357 replace the 14-ton load by a load of 10 tons parallel to the z-axis, positive in sense, and solve the problem. Leave all the other data unchanged.

359. Solve Prob. 357 after making the following change: replace the member AO by a member extending from A to a point on the y-axis 7.5 ft below the origin. Call the new point D. *Ans.* $B = C =$ 6 tons (T); $D =$ 10 tons (C).

360. Figure 223 represents a type of derrick known as a shear legs, or **shears.** AB and AC are the legs, and are braced by the guy, AD. The derrick sustains a vertical load of 14.4 tons. Calculate the stress in each leg, and in the guy.

361. A uniform, circular plate, 10 ft in diameter and weighing 1800 lb, is to be suspended in a horizontal position by means of three wires, each 13 ft long. The upper ends of the wires are joined at a point, and the lower ends are attached to the plate at points on its periphery 120° apart. Calculate the stresses in the wires. *Ans.* 650 lb (T), in each.

362. Let the conditions in Prob. 361 be altered so that the points at which the wires are attached to the plate are separated by angles of 60°, 150°, and 150°, measured at the center of the plate. Let all other data remain unchanged, and solve the problem.

363. A uniform, rectangular platform, 16 by 18 ft, weighing 24,000 lb, is to be suspended in a horizontal position by means of three cables. The lower ends of two of the cables are attached to the platform at the corners terminating one of the 16-ft sides. The third cable is attached at the middle of the other 16-ft side. The upper ends of the cables are joined at a point 12 ft vertically above the center of the platform. Calculate the stress in each cable. *Ans.* 8500 lb (T); 8500 lb (T); 15,000 lb (T).

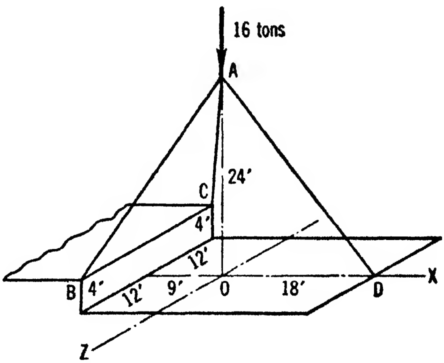

FIG. 224

364. In Fig. 224, AB, AC, and AD are the members of a tripod which supports a vertical load of 16 tons. Calculate the stress in each leg.

365. Figure 225 represents a tripod carrying a vertical load of 980 lb. The stress in leg AB is known to be 630 lb, compression, and in AC it is 540 lb, compression. Find the stress in AD, and the distances x_D and z_D. *Ans.* D = 650 lb (C); $x_D = -24$ ft; $z_D = +6$ ft.

366. Figure 226 represents the familiar stiff-leg derrick. Members AB and AC are capable of sustaining either tension or compression. In the present problem the angle OAD is 90°. Calculate the stress in AD by solving the coplanar concurrent system acting at the point D; then calculate the stresses in the legs AB and AC, by solving the non-coplanar concurrent system acting at A.

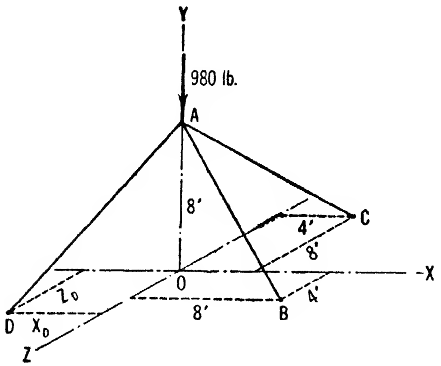

Fig. 225

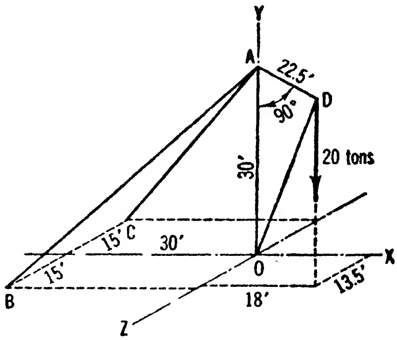

Fig. 226

367. In Prob. 366, Fig. 226, the boom, OD, can be rotated about the vertical axis, OA. In what position of the boom would the legs AB and AC sustain equal tensile stresses? In what position would the stress in AB be equal to zero?

368. Solve Prob. 366 with the boom OD in the yz-plane. Let all the other data of the problem remain unchanged. *Ans.* $AD = 15$ tons (T); $AB = 22.5$ tons (C); $AC = 22.5$ tons (T).

73. Equilibrium of the Non-Coplanar Parallel System.

In the solution of a problem involving the non-coplanar parallel system in equilibrium, the principle of components is able to furnish only one independent equation. This necessitates the use of the principle of moments in obtaining the additional equations needed in the solution. A problem can be solved by the use of moment equations alone, if so desired. Usually, however, the simplest solution is attained by means of one component equation with reference to an axis parallel to the forces, together with moment equations referred to axes at right angles to the forces.

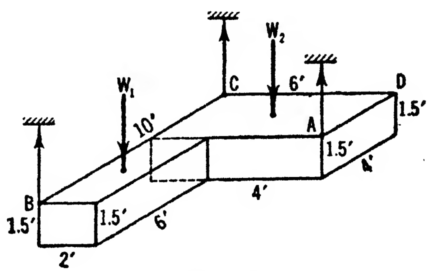

Fig. 227

Illustrative Problems

369. Figure 227 represents an L-shaped block of homogeneous material weighing 150 lb per cu ft. The block is suspended, with its larger faces horizontal, by means of three vertical wires, attached at the points A, B, and C. Calculate the stress in each wire.

Solution. In this problem the only load is the weight of the block. For convenience in dealing with the weight the block may be considered to be divided into two rectangular prisms, as indicated in the figure. Let the weights of the two portions be represented by W_1 and W_2. Since the material is homogeneous, W_1 and W_2 may be considered to be applied at the centers of the prisms.

From the figure: $W_1 = 6 \times 2 \times 1.5 \times 150 = 2700$ lb, and $W_2 = 6 \times 4 \times 1.5 \times 150 = 5400$ lb. The principle of components yields one convenient equation, as follows:

$$\Sigma F = 0 \qquad +A + B + C - 2700 - 5400 = 0$$

By the principle of moments, using the lines BC and CD as axes,

$$\Sigma M_{BC} = 0 \qquad +A \times 6 - 2700 \times 1 - 5400 \times 3 = 0$$

$$\Sigma M_{CD} = 0 \qquad -A \times 4 - B \times 10 + 2700 \times 7 + 5400 \times 2 = 0$$

The solution of these equations gives: $A = +3150$ lb, tension; $B = +1710$ lb, tension; $C = +3240$ lb, tension.

If two parallel axes had been used as axes of moments the equations would not have been independent, and a complete solution for the three unknowns would have been impossible.

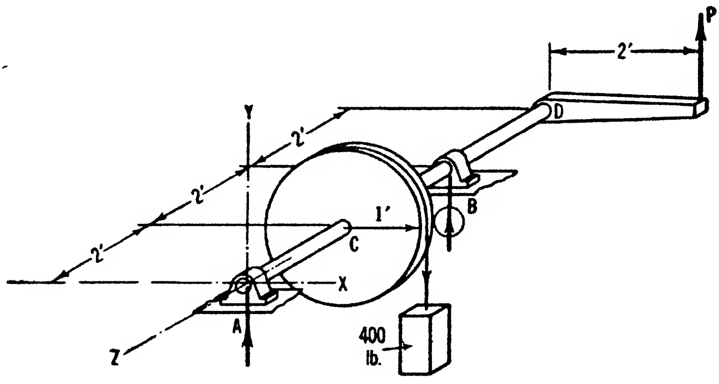

FIG. 228

370. Figure 228 represents a horizontal shaft supported in bearings at A and B. A wheel 2 ft in diameter is keyed to the shaft at C. A wire, depending from the wheel in the manner indicated, sustains a body weighing 400 lb. A vertical force, P, applied at the end of a crank, prevents the shaft from turning. Disregarding friction, calculate the forces exerted on the shaft by its bearings. Disregard also the weight of the mechanism.

Solution. Since friction is to be disregarded, and since the loads are vertical, it may be assumed that the reactions at the two bearings are vertical or, in other words, that they have no x- or z-components. Assume that both reactions are upward.

The following equilibrium equations provide a convenient solution:

$$\Sigma M_z = 0 \qquad +P \times 2 - 400 \times 1 = 0$$

$$\Sigma M_x = 0 \qquad +P \times 6 + B \times 4 - 400 \times 2 = 0$$

$$\Sigma F_y = 0 \qquad +A + B + P - 400 = 0$$

The solution of these equations gives: $A = +300$ lb; $B = -100$ lb; $P = +200$ lb. The minus sign accompanying B shows that the sense was assumed incorrectly, and that the force acts downward.

PROBLEMS

371. A steel plate of constant thickness, weighing 360 lb, has the shape of a right triangle whose legs are 3 ft and 6 ft. The plate is suspended with its triangular faces horizontal, by means of three vertical wires, attached at the corners. Calculate the stress in each wire. *Ans.* 120 lb (T), in each.

372. Solve Prob. 370, Fig. 228, with the crank, D, in a position 180° from that shown in the figure.

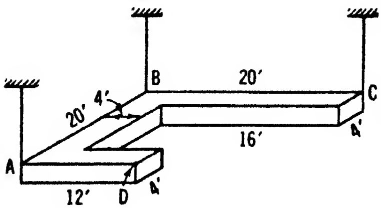

Fig. 229

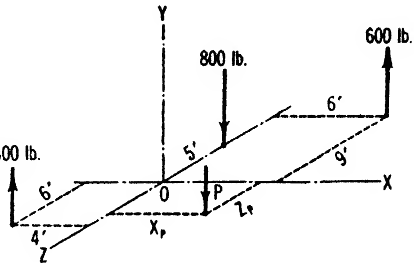

Fig. 230

373. Figure 229 represents a homogeneous slab weighing 150 lb per cu ft, suspended with its larger faces horizontal, by means of three vertical wires attached at A, B, and C. The slab is 2 ft thick. Calculate the stress in each wire. *Ans.* A = 22,560 lb (T); B = 12,480 (T); C = 17,760 lb (T).

374. In Prob. 373, Fig. 229, calculate the maximum vertical load that could be applied to the slab at the point D, without disturbing its equilibrium.

375. Figure 230 represents a system of four parallel, non-coplanar forces in equilibrium. Calculate the magnitude of the unknown force P, and the coordinates x_P and z_P. *Ans.* $P = -200$ lb; $x_P = +10$ ft; $z_P = +5$ ft.

376. A uniform, homogeneous, circular plate, 5 ft in diameter and weighing 1600 lb, is suspended in a horizontal position by means of three vertical wires, attached at points on the periphery of the plate. The distance between two of the wires is 4 ft. The third wire is equidistant from the others. Calculate the stress in each wire.

377. A uniform, homogeneous, square plate is to be suspended, with its square faces horizontal, by means of three vertical wires, in such a manner that each wire will sustain one-third of the weight. The wires are to be attached only at the edges of the plate. Ascertain two methods by which the desired result can be accomplished.

378. A table having a uniform circular top is to be supported by three vertical legs equidistant from the center of the table. The table top weighs 90 lb, and the angular spacing of the legs is such that they sustain 25 lb, 25 lb, and 40 lb. Calculate the angular spacing of the legs about the center of the table.

379. Figure 231 represents an end elevation and the foundation plan of a pier supported on twenty piles. The weight, W, of the pier is 300 tons. There is also a vertical, eccentric load, P, of 240 tons. The eccentricity, e, is 3 ft. The load P has no eccentricity in a direction at right angles to the plane of the figure. The piles are 4 ft apart, in both directions. Let the supporting reaction of each pile in row A be represented by A, in row B by B, etc. The problem is to find the reaction exerted on the pier by each pile. Certain assumptions must be made. Let it be assumed that all the piles in any one row exert equal reactions. Let it be further assumed that $B = \frac{1}{2}(A + C)$, and that $C = \frac{1}{2}(B + D)$. *Ans.* $A = 16.2$ tons; $B = 23.4$ tons; $C = 30.6$ tons; $D = 37.8$ tons.

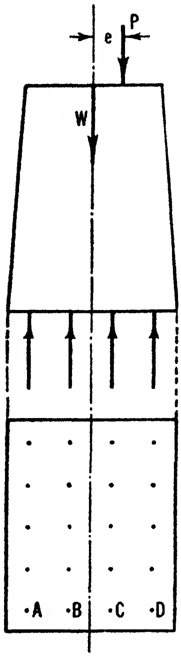

FIG. 231

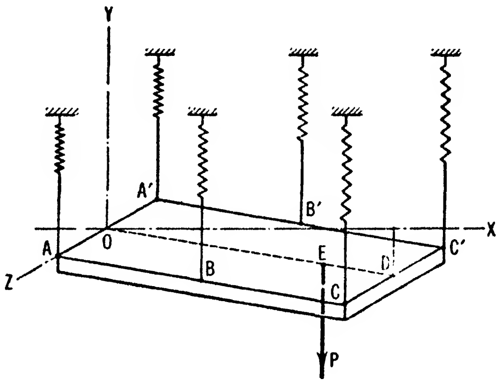

FIG. 232

380. Figure 232 represents a uniform, rectangular board supported by six vertical wires attached to helical springs. The board sustains an eccentric load, P, in addition to its own weight, W. The six springs have the same elastic properties, and when not loaded are of equal length. The wires also are equal in length. When the board alone is hung from the wires the springs are elongated equally, but when the eccentric load, P, is applied the board assumes an inclined position, as shown.

The distance $\overline{OE} = \frac{3}{4}\overline{OD}$; also, $\overline{AB} = \overline{A'B'} = \overline{BC} = \overline{B'C'}$. From symmetry, it may be assumed that the tensions in the springs A and A' are equal. A similar assumption may be made for springs B and B', and for C and C'. The tension in each spring is proportional to its elongation.

Assuming that the board does not deform, and that the springs and wires are vertical at all times, prove that the tension in $A = \dfrac{W}{6} + \dfrac{P}{24}$; in $B = \dfrac{W}{6} + \dfrac{P}{6}$; in $C = \dfrac{W}{6} + \dfrac{7P}{24}$.

74. Equilibrium of the General Non-Coplanar System. Because of the great variety of conditions that may exist in connection with systems of this class, it is difficult to make helpful suggestions as to the choice of equilibrium equations. The simplest method of solution for one problem may not be at all desirable in another. Each problem requires careful, individual consideration.

As always, the illustrative problems should be studied carefully.

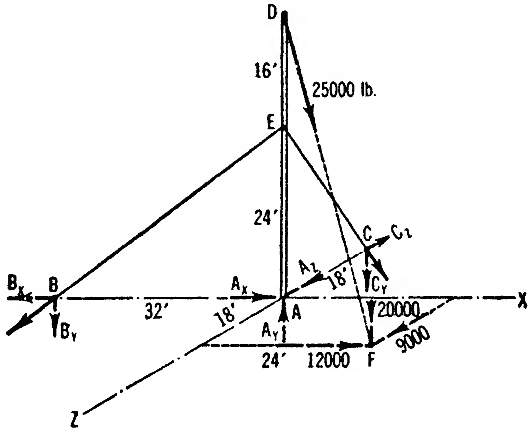

FIG. 233

ILLUSTRATIVE PROBLEMS

381. Figure 233 represents a vertical pole, AD, 40 ft in height, resting in a socket at A and receiving additional support from the braces, BE and CE. An inclined load of 25,000 lb is applied to the pole at D, its line of action passing through the point F. Find the reactions exerted on the structure at the points of support, A, B, and C. Disregard all weights.

Solution. Certain distances can be used in the solution, as follows:

$$\overline{BE} = \sqrt{(32)^2 + (24)^2} = 40 \text{ ft} \qquad \overline{CE} = \sqrt{(18)^2 + (24)^2} = 30 \text{ ft}$$

$$\overline{DF} = \sqrt{(24)^2 + (40)^2 + (18)^2} = 50 \text{ ft}$$

Since BE and CE are two-force members (Art. 61), the forces exerted on them at their supports are collinear with the members. Let these forces be represented by B and C, and their components by B_x, B_y, C_y, and C_z, as shown in the figure. Since the pole is not a two-force member, it is not permissible to assume that the reaction at A is collinear with the pole. Let A_x, A_y, and A_z represent the components of this reaction.

Resolve the 25,000-lb load into its components, at the point F, by the method used in Prob. 167, Art. 40, as follows: x-component $= 25{,}000 \times 24/50 = 12{,}000$ lb; y-component $= 25{,}000 \times 40/50 = 20{,}000$ lb; z-component $= 25{,}000 \times 18/50 = 9000$ lb.

From the principles of equilibrium,

$$\Sigma F_x = 0 \qquad +A_x - B_x + 12{,}000 = 0$$

$$\Sigma F_y = 0 \qquad +A_y - B_y - C_y - 20{,}000 = 0$$

$$\Sigma F_z = 0 \qquad +A_z - C_z + 9000 = 0$$

$$\Sigma M_x = 0 \qquad -C_y \times 18 + 20{,}000 \times 18 = 0$$

$$\Sigma M_z = 0 \qquad +B_y \times 32 - 20{,}000 \times 24 = 0$$

and from the figure,

$$B_x = B \times \tfrac{32}{40} \qquad B_y = B \times \tfrac{24}{40} \qquad C_y = C \times \tfrac{24}{30} \qquad C_z = C \times \tfrac{18}{30}$$

From the foregoing equations the following results are obtained: $A_x = +8000$ lb; $A_y = +55{,}000$ lb; $A_z = +6000$ lb; $B = +25{,}000$ lb; $C = +25{,}000$ lb. The positive sign was obtained in each case, showing that the senses of the unknown forces were assumed correctly. It is seen that members BE and CE are in tension. The resultant force at A could be found, if desired, but it will be considered that the foregoing results are sufficient.

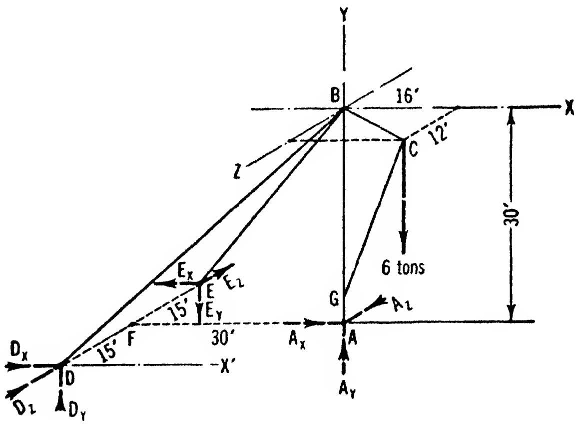

FIG. 234

382. Figure 234 represents a stiff-leg derrick. The mast, AB, is vertical, and rests in a bearing at A. The mast is held in position by the braces, or legs, BD and BE, which are capable of sustaining either tension or compression. The points of support, A, D, and E, are in the same horizontal plane, and DE is at right angles to AF. The boom, GC, and the mast can be rotated about the vertical axis, AB. The boom also can be rotated in a vertical plane. In this problem the end of the boom, C, is in the xz-plane, and its coordinates are as shown in the figure. A load of 6 tons is supported by the derrick at C. Find the forces exerted on the derrick by its supports at A, D, and E. Disregard the weights of the members.

Solution. In the figure the x-axis is parallel to AF, and the z-axis is parallel to DE. Let the three reactions be represented by the letters A, D, and E, and let their components be distinguished by the appropriate subscripts. Assume that BE is in tension and BD in compression. The components of D and E as shown in the figure are consistent with these assumptions. For the sake of simplicity only the components have been shown in the figure.

Since weights are to be disregarded, BD and BE are two-force members (Art. 61). Consequently, reactions D and E are collinear with their respective members. The mast, AB, is not a two-force member; therefore the inclination of reaction A is unknown.

From the figure: distance $\overline{BD} = \overline{BE} = \sqrt{(30)^2 + (30)^2 + (15)^2} = 45$ ft. By the principle of moments, with the fact in mind that reactions D and E are collinear with BD and BE and pass through B,

$$\Sigma M_x = 0 \qquad -A_z \times 30 + 6 \times 12 = 0 \qquad A_z = +2.4 \text{ tons}$$

$$\Sigma M_z = 0 \qquad +A_x \times 30 - 6 \times 16 = 0 \qquad A_x = +3.2 \text{ tons}$$

$$\Sigma M_{DE} = 0 \qquad +A_y \times 30 - 6(30 + 16) = 0 \qquad A_y = +9.2 \text{ tons}$$

$$\Sigma M_{x'} = 0 \qquad -E_y \times 30 + A_y \times 15 - 6(15 - 12) = 0 \qquad E_y = +4.0 \text{ tons}$$

By the principle of components,

$$\Sigma F_y = 0 \qquad +D_y + A_y - E_y - 6 = 0 \qquad D_y = +0.8 \text{ ton}$$

From the figure, using the method shown in Prob. 167, Art. 40,

$$D_y = D \times \tfrac{30}{45} \qquad D = \tfrac{45}{30} D_y = +1.2 \text{ tons}$$

$$E_y = E \times \tfrac{30}{45} \qquad E = \tfrac{45}{30} E_y = +6.0 \text{ tons}$$

All the foregoing results are accompanied by positive signs, showing that the senses of the unknown forces were assumed correctly. It follows that BD is in compression and that BE is in tension. The resultant force at A could be calculated, but the foregoing results will be considered sufficient.

383. Figure 235 represents a gate weighing 2000 lb, supported by a vertical shaft, AB. The shaft is supported by bearings at A and B, the latter being a collar bearing capable of providing horizontal support only. The bearings have not been shown in the figure. A force of 270 lb is applied to the gate, its line of action parallel to the z-axis. A wire, DE, anchored at E, prevents the gate from rotating. Calculate the stress in the wire, and the components of the reactions at A and B. Disregard friction.

Solution. This problem is of a general nature, requiring for its solution six equilibrium equations. Assume the various components of the unknown forces to act as shown. The force at D is collinear with the wire DE; therefore its components must be so assumed as to be consistent with that fact.

From the figure: distance $\overline{DE} = \sqrt{(12)^2 + (20)^2 + (9)^2} = 25$ ft. By the principle of moments,

$$\Sigma M_y = 0 \qquad -D_z \times 12 + 270 \times 12 = 0 \qquad D_z = 270 \text{ lb}$$

By the method of resolution described in Prob. 167, Art. 40,

$$D_z = D \times \tfrac{9}{25} \qquad D = \tfrac{25}{9} D_z = \tfrac{25}{9} \times 270 = 750 \text{ lb}$$

$$D_x = D \times \tfrac{12}{25} \qquad D_x = 750 \times \tfrac{12}{25} = 360 \text{ lb}$$

$$D_y = D \times \tfrac{20}{25} \qquad D_y = 750 \times \tfrac{20}{25} = 600 \text{ lb}$$

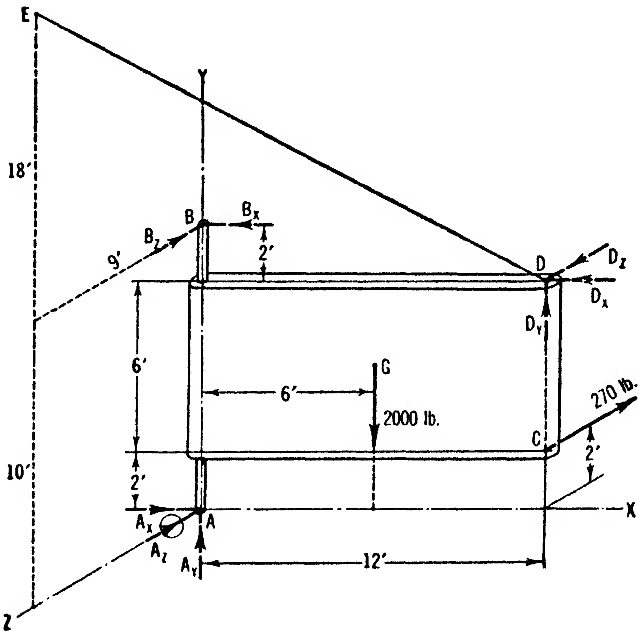

Fig. 235

The solution can now be completed as follows:

$$\Sigma F_y = 0 \qquad +A_y + D_y - 2000 = 0$$

$$A_y = 2000 - D_y = 2000 - 600 = 1400 \text{ lb}$$

$$\Sigma M_x = 0 \qquad -B_z \times 10 + D_z \times 8 - 270 \times 2 = 0$$

$$B_z = \tfrac{1}{10}(D_z \times 8 - 270 \times 2) = \tfrac{1}{10}(270 \times 8 - 270 \times 2) = 162 \text{ lb}$$

$$\Sigma F_z = 0 \qquad -A_z - B_z + D_z - 270 = 0$$

$$A_z = -B_z + D_z - 270 = -162 + 270 - 270 = -162 \text{ lb}$$

$$\Sigma M_z = 0 \qquad +B_x \times 10 + D_x \times 8 + D_y \times 12 - 2000 \times 6 = 0$$

$$B_x = \tfrac{1}{10}(-D_x \times 8 - D_y \times 12 + 2000 \times 6)$$

$$B_x = \tfrac{1}{10}(-360 \times 8 - 600 \times 12 + 2000 \times 6) = 192 \text{ lb}$$

$$\Sigma F_x = 0 \qquad +A_x - B_x - D_x = 0$$

$$A_x = B_x + D_x = 192 + 360 = 552 \text{ lb}$$

The minus sign accompanying A_z shows that the sense of that component was assumed incorrectly. The circle enclosing the arrowhead has been inserted to indicate that the true sense of A_z is opposite to that assumed. The senses of all the other components were assumed correctly.

PROBLEMS

384. Figure 236 represents a pole, AE, 50 ft long, resting in a socket at A, and further supported by the cables BD and CD. The pole sustains a vertical load of 3600 lb at its upper end. Calculate the stress in each cable, and the components of the force acting on the pole at A. Disregard the weight of the pole. *Ans.* $BD = CD = 4330$ lb (T); $A_x = +8000$ lb; $A_y = +3600$ lb; $A_z = 0$.

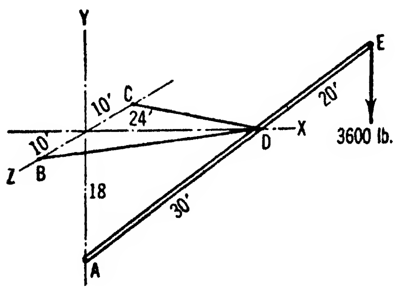

Fig. 236

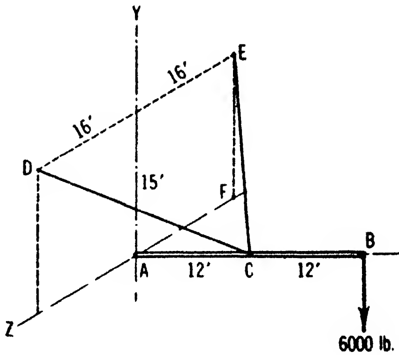

Fig. 237

385. Solve Prob. 384, Fig. 236, after inserting an additional load of 1000 lb, parallel to the z-axis, positive in sense, and applied to the pole at E. Let all the other data of Prob. 384 remain unchanged.

386. Figure 237 represents a pole, AB, 24 ft long, supported in a socket at A, and receiving further support from the cables CD and CE. The pole sustains a vertical load of 6000 lb at its outer end. Calculate the stress in each cable, and the components of the force acting on the pole at A. Disregard the weight of the pole. *Ans.* $CD = CE = 10{,}000$ lb (T); $A_x = +9600$ lb; $A_y = -6000$ lb; $A_z = 0$.

387. In Prob. 386, Fig. 237, replace the cable CE by one extending from C to F, and solve the problem. Let all the other data remain unchanged.

388. In Prob. 382, Fig. 234, move the boom until the point C is on the x-axis, 20 ft to the right of B. Let all the other data remain unchanged, and solve the problem. *Ans.* $BD = BE = 3$ tons (T); $A_x = +4$ tons; $A_y = +10$ tons; $A_z = 0$.

389. Figure 238 represents a pole, AD, supported in a socket at D, and supported also by the braces AB and AC. The pole sustains two loads, applied at its center point. The 2400-lb load is vertical, and the 1200-lb load is parallel to the z-axis.

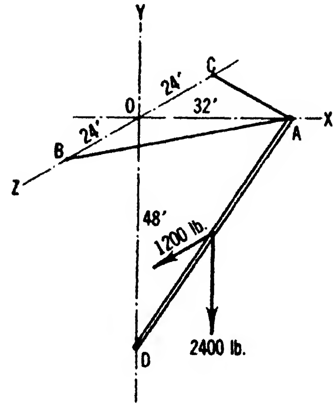

Fig. 238

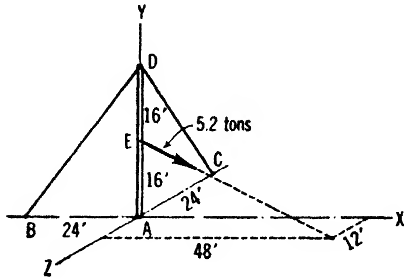

Fig. 239

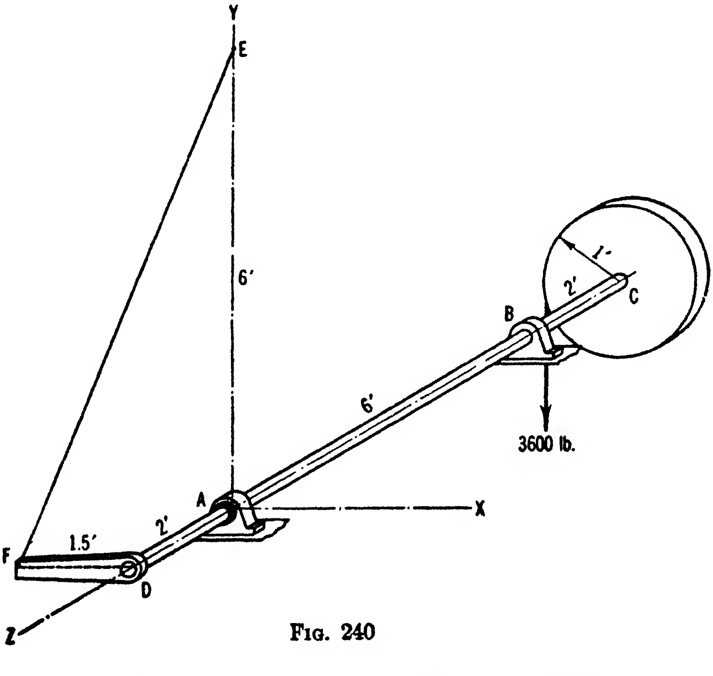

Fig. 240

Find the stress in each brace, and the components of the force acting on the pole at D.

390. If the 270-lb load in Prob. 383, Fig. 235, is increased sufficiently in magnitude, the component B_x will reverse in sense and will act toward the right. Find all the unknown forces acting on the gate at the instant of reversal. *Ans.* $A_x = +429$ lb; $A_y = +1290$ lb; $A_z = +193$ lb; $B_x = 0$; $B_y = 0$; $B_z = -193$ lb; $DE = 893$ lb (T); load $= -321$ lb.

391. In Prob. 383, Fig. 235, replace the wire DE with one extending from D to a point 20 ft vertically below E. Let all the other data of the problem remain unchanged, and solve.

392. Figure 239 represents a vertical pole, AD, resting in a socket at A and receiving additional support from the guys BD and CD. The pole sustains an inclined load of 5.2 tons, applied as shown. Calculate the stress in each guy, and the force exerted on the pole at A. Disregard the weight of the pole. *Ans.* $BD = 4$ tons (T); $CD = 1$ ton (T); $A_x = -2.4$ tons; $A_y = +5.6$ tons; $A_z = -0.6$ ton.

393. Figure 240 represents a horizontal shaft supported in bearings at A and B. At one end is a drum, C, 2 ft in diameter, on which is wrapped a cable supporting a load of 3600 lb. At the other end, D, is a crank 1.5 ft long, in a horizontal position. Rotation of the shaft is prevented by a cable, FE, anchored at the point E, 6 ft vertically above A. The shaft is provided with collars at A, which render that bearing capable of providing support along the axis of the shaft. The bearing at B is not so provided. Calculate the tension in FE, and the components of the forces exerted on the shaft by its bearings. Disregard friction and the weight of the shaft and accessories. *Ans.* $A_x = -600$ lb; $A_y = -4400$ lb; $A_z = +800$ lb; $B_x = 0$; $B_y = +5600$ lb; $B_z = 0$; $EF = 2600$ lb (T).

394. In Prob. 393, Fig. 240, move the drum to a position halfway between A and B. Let all the other data remain unchanged, and solve the problem.

395. In Prob. 392, Fig. 239, move the point C to the position whose coordinates are $(+24', 0, -30')$. Let all the other data remain unchanged, and solve the problem.

CHAPTER VII

FRICTION

75. Friction in General. When one body exerts a pressure on another body, the line of action of the force is frequently oblique to the common tangent plane at the point of contact. In such a case the pressure has one component lying in the common tangent plane, and one at right angles thereto. The existence of this tangential component is possible even when there is no connection or adhesion between the two surfaces, and under such circumstances it is known as *frictional force,* or simply *friction.* The normal component is usually referred to as the *normal pressure.*

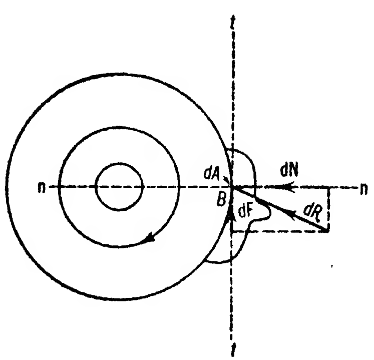

FIG. 241

Figure 241 represents a brakeshoe pressing against a car wheel. The point B is any point in the surface of contact. The line tt shows the position of the common tangent plane, and the line nn is the common normal. The element of area in the surface of contact, at B, is represented by dA. The resultant pressure of the shoe on the area dA is represented by dR. The wheel is represented as rotating, or tending to rotate, in a clockwise direction, in which case dR will be inclined to tt and nn in the manner shown. The friction and the normal pressure on the area dA are represented by dF and dN, respectively.

The existence of normal pressure does not necessitate the existence of friction. Friction does not exist unless there are conditions causing, or tending to cause, sliding between the two bodies. Friction opposes sliding, or the tendency to slide. In the case of the brake-shoe on the wheel, for example, there will be no friction unless the wheel is turning under the shoe, or unless there is a tendency for it so to turn.

In many problems, one or both of the bodies are assumed to be smooth. Under such an assumption friction does not exist, and the resultant pressure at any point coincides with the normal. Surfaces that are not to be assumed smooth are designated as *rough* surfaces.

This expression is not to be interpreted as necessarily meaning a high degree of roughness, but simply as indicating that friction is to be taken into consideration in the solution of the problem.

PROBLEMS

396. Assume that the wheel shown in Fig. 241 tends to rotate in the counterclockwise direction. Make separate sketches of the wheel and the shoe. Show in each sketch the friction, normal pressure, and resultant pressure acting at the surface of contact.

397. A body is placed on an inclined plane, and a horizontal force, tending to move the body up the plane, is applied by means of a rope. Make a sketch showing all the forces acting on the body. Assume that the body is about to move up the plane. Numerical values need not be used.

76. Static Friction; Kinetic Friction. Friction between two bodies that have no relative motion at the surface of contact is called *static friction*. Both bodies may be at rest, one may be in motion, or both may be in motion, but the friction at any point at which sliding is not in progress is classified as static friction.

Friction between two bodies that have relative motion at the surface of contact is called *kinetic friction*.

PROBLEM

398. A box is placed on the horizontal floor of a car. An acceleration is given to the car, thus causing frictional force to come into existence between the box and the car floor. If the acceleration is insufficient to cause the box to change its position relative to the car, is the friction static, or kinetic? What is the nature of the friction if the box moves along the floor of the car?

An automobile driver applies the brakes, but not with sufficient force to lock the wheels or to cause slipping between the tires and the roadway. What kind of friction exists between the brake-shoes and the drums? Between the tires and the roadway? Answer similar questions for the case in which the driver locks the wheels while the car is in motion.

77. Limiting Friction; Impending Motion. Experiments have shown that the amount of static friction which can exist between two bodies has a definite maximum limit, and that this limit is fixed by two factors: namely, the normal pressure and the degree of roughness at the surface of contact. This maximum possible value of the static friction for any given case is called the *limiting friction*.

A good illustration of limiting friction is found in the efforts of a locomotive driver to start a train that is too heavy for the locomotive. The normal pressure between the rails and the driving wheels is determined by the weight on those wheels, and cannot be altered by any effort on the part of the driver. By dropping sand on the rails he can increase the degree of roughness, but only up to a certain point. When he opens the throttle, steam enters the cylinders and presses against the pistons, and static friction immediately develops between the rails and the driving wheels. The frictional force exerted by the rails on the driving wheels is forward, and tends to start the train. As the steam pressure rises in the cylinders the static friction increases, but if the train refuses to start the friction soon reaches the limiting value corresponding to the fixed normal pressure. From this point any further increase in the steam pressure causes the wheels to slip on the rails. Sliding the wheels is of no avail, for friction usually decreases somewhat as it passes from the static to the kinetic form.

The situation that exists when static friction has its limiting value is referred to as *impending motion.* Impending motion is, more specifically, impending *relative* motion, or impending slipping. When motion impends, friction is in the static form, but is on the verge of changing to the kinetic form.

78. Relation between Limiting Friction and Normal Pressure. It was indicated in Art. 77 that the value of the limiting friction depends only on the normal pressure and the roughness conditions at the surface of contact. Experimental research has established the following formula for use in problems in which the frictional force has its limiting value:

$$dF = \mu\, dN \qquad [26]$$

In the foregoing formula dF and dN represent the friction and the normal pressure, respectively, on any element of area at which relative motion is impending, and μ represents a coefficient whose value depends on the roughness conditions. This coefficient, μ, is called the *coefficient of static friction.* Its value depends on the materials of which the contact surfaces are composed, and on any special treatment that may have been given to those surfaces. If the contact surfaces are dry, or nearly so, the value of μ is considered to be independent of the manner of distribution of the normal pressure. It is also considered to be independent of the intensity of the normal pressure at any particular point. For well-lubricated surfaces, however, μ is subject to considerable variation, and greater care is necessary in the selection of its value.

Equation 26 is of a general nature and may be used in any case of impending motion. Frequently, however, the conditions are such that the following special form may be used:

$$F = \mu N \qquad [27]$$

in which F represents the total friction on the entire area of contact, and N represents the total normal pressure on that area.

The limiting friction for the locomotive discussed in Art. 77 could be calculated by means of Eq. 27. The value of μ commonly used for locomotive wheels on steel rails is about 0.25. The use of sand would increase this to about 0.3. The total weight carried by the driving wheels, including the weight of the wheels themselves, would be used for N. The resulting value of F would represent the maximum static friction that could exist between the rails and the driving wheels of the locomotive. It would also represent the maximum pull that the locomotive could possibly exert in starting a train on level track.

The choice between Eqs. 26 and 27 in each case depends on various matters having to do with the distribution of the forces over the contact area and, in some cases, on the nature of the experiments by which the values of μ were originally obtained.

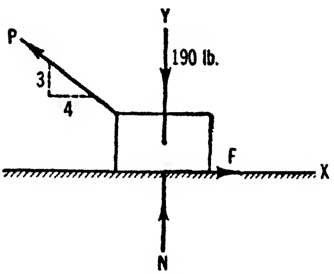

Fig. 242

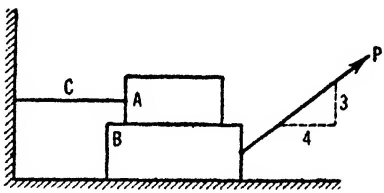

Fig. 243

Illustrative Problems

399. Figure 242 represents a body weighing 190 lb, resting on a horizontal surface. The coefficient of static friction is 0.25. Calculate the magnitude of the force P, if motion toward the left impends. Also calculate the frictional force and the normal pressure at the instant of impending motion.

Solution. The student should carefully observe that the normal pressure, N, is not equal to the weight of the body, in this problem.

The principle of components may be used. Equation 27 applies, also, since sliding is impending.

$$\Sigma F_x = 0 \qquad -P \times \tfrac{4}{5} + F = 0$$

$$\Sigma F_y = 0 \qquad +P \times \tfrac{3}{5} + N - 190 = 0$$

$$F = \mu N \quad F = 0.25 \times N$$

The solution of these equations gives: $P = 50$ lb; $F = 40$ lb; $N = 160$ lb.

400. Figure 243 represents a body A resting on a body B, which rests on a horizontal plane. A horizontal wire, C, prevents A from moving. An inclined force, P, is applied to B, and is gradually increased. A weighs 150 lb and B weighs 300 lb. The coefficient of static friction is $\frac{1}{3}$, for all surfaces. Calculate the value of P, and the tension in the wire, for the instant when motion of B impends.

Solution. Figure 244 represents A as a free body. C represents the tension in the wire, and F_1 and N_1 represent, respectively, the frictional force and the normal pressure exerted by B on A. The student should consider carefully whether the sense assumed for F_1 is in accordance with the conditions. Since sliding of B is impending, Eq. 27 may be used. For body A,

$$\Sigma F_x = 0 \qquad +F_1 - C = 0$$

$$\Sigma F_y = 0 \qquad +N_1 - 150 = 0$$

$$F = \mu N \quad F_1 = \tfrac{1}{3}N_1$$

The solution of these equations gives: $N_1 = 150$ lb; $F_1 = 50$ lb; $C = 50$ lb.

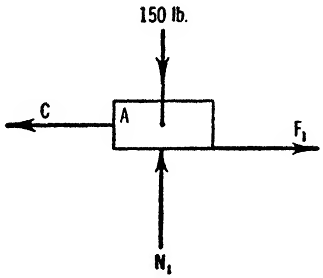

Fig. 244

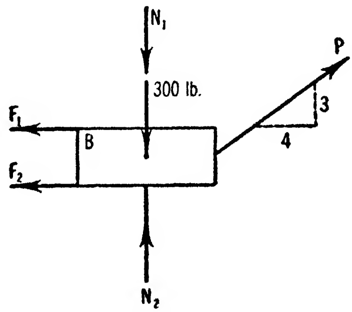

Fig. 245

Figure 245 represents B as a free body. F_1 and N_1 represent the frictional force and normal pressure exerted on B by A. The student should carefully observe that these forces are opposite in sense, although equal in magnitude, to the F_1 and N_1 exerted on A, by B, shown in Fig. 244. F_2 and N_2 represent the frictional force and normal pressure exerted on B by the supporting plane. Here, as always, the student must consider whether the sense of the frictional force has been chosen in accordance with the conditions of the problem. Utilizing results previously obtained,

$$\Sigma F_x = 0 \qquad +\tfrac{4}{5}P - 50 - F_2 = 0$$

$$\Sigma F_y = 0 \qquad +\tfrac{3}{5}P - 150 - 300 + N_2 = 0$$

$$F = \mu N \quad F_2 = \tfrac{1}{3}N_2$$

The solution of these equations gives: $P = 200$ lb; $F_2 = 110$ lb; $N_2 = 330$ lb.

It is suggested that the student check the answers by considering the two bodies, A and B, together as a single free body.

401. Figure 246 represents a homogeneous, prismatic bar, AB, resting on a horizontal plane at A, and leaning against the edge of a wall at D. The coefficient of static friction for both surfaces of contact is 0.2. The bar weighs 500 lb. A horizontal force, P, is gradually applied at C. Calculate the magnitude of P at the instant when motion of the bar impends. Find the other forces acting on the bar at this instant.

Solution. Show all the forces acting on the bar. Be sure that the senses of the frictional and normal components are assumed in a manner consistent with the conditions of the problem. The lower end of the bar is on the verge of moving toward the right. Therefore, the frictional force F_A acts toward the left, since it will oppose the tendency of the bar to move. Likewise, F_D acts downward, as shown in the figure. The senses of N_A and N_D are obvious.

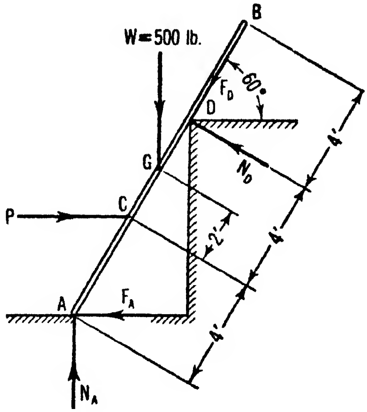

FIG. 246

$$\Sigma M_C = 0 \qquad +N_D \times 4 - N_A \times (4 \cos 60°) - F_A \times (4 \cos 30°) - 500 \times (2 \cos 60°) = 0$$

$$\Sigma F_y = 0 \qquad +N_D \cos 60° - F_D \cos 30° + N_A - 500 = 0$$

Motion impends both at A and at D. Therefore, Eq. 27 may be utilized,

$$F = \mu N \qquad F_A = 0.2N_A \qquad F_D = 0.2N_D$$

Substitute these expressions for F_A and F_D in the two equations written above, and simplify.

$$+4N_D - 2.69N_A = 500$$

$$+0.327N_D + N_A = 500$$

The solution of these equations gives: $N_A = 376$ lb; $N_D = 378$ lb; $F_A = 75.2$ lb; $F_D = 75.6$ lb.

The value of P can now be obtained as follows:

$$\Sigma F_x = 0 \qquad +P - 75.2 - 75.6 \cos 60° - 378 \cos 30° = 0$$

$$P = 440 \text{ lb}$$

PROBLEMS

402. A body whose weight is W rests on an inclined plane. The inclination of the plane is gradually increased until the body begins to slide. Let θ represent the angle of inclination of the plane at the instant just before motion starts. Prove that $\mu = \tan \theta$.

403. Reverse the force P in Prob. 399, Fig. 242, and solve, leaving the other data unchanged.

404. In Fig. 247, body A weighs 100 lb, and B weighs 200 lb. A is prevented from moving by the horizontal cord, C, attached to a wall. The coefficient of static friction for all surfaces of contact is 0.2. Calculate the value of P, assuming that B is on the verge of moving toward the left. Calculate the tension in the cord. *Ans.* 80 lb; 20 lb.

405. Solve Prob. 404, with the cord C at an angle of 30° with the horizontal, and sloping upward toward the right.

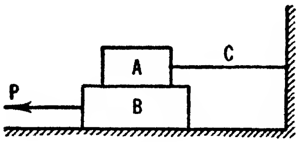

Fig. 247

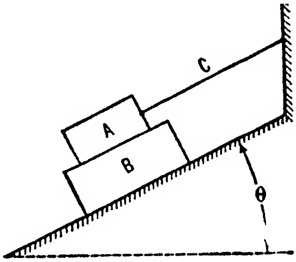

Fig. 248

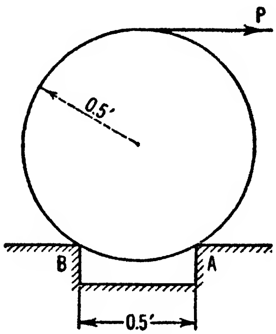

Fig. 249

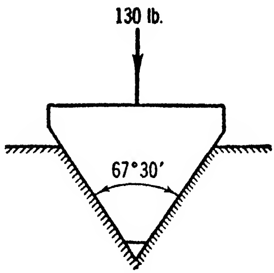

Fig. 250

406. In Fig. 248, A weighs 50 lb and B weighs 100 lb. The coefficient of static friction for all surfaces is 0.25. A is prevented from moving by the cord, C, parallel to the incline. Assume that motion of B down the incline is impending. Calculate θ, and the tension in the cord. *Ans.* 26° 35′; 33.5 lb.

407. Figure 249 represents a homogeneous cylinder weighing 100 lb. It is to be rolled, without slipping, over the edge, A. Calculate the minimum possible value of the force P, and of the coefficient of static friction.

408. If the coefficient of friction in Prob. 407 is only 0.15, what kind of motion will occur as the force P is increased? Calculate the frictional force and the normal pressure at A, and also at B, at the instant of impending motion. *Ans.* F_A = 12.9 lb; N_A = 85.9 lb; F_B = 3.64 lb; N_B = 24.2 lb; P = 16.5 lb.

409. Figure 250 is an end view of a wedge arranged to slide in a V-shaped groove, in a direction at right angles to the plane of the figure. What horizontal force, parallel to the groove, would be necessary to cause motion to impend, if the coefficient of static friction is 0.1? Assume that the frictional forces on either side of the wedge are all at right angles to the plane of the figure. *Ans.* 23.4 lb.

410. In Fig. 251 the coefficient of static friction for the brake-shoe on the wheel is 0.3. Calculate the minimum value of the force, P, necessary to prevent the wheel from turning, if the load, W, is 250 lb. Disregard friction in the bearings. In what manner does the curvature of the brake arm assist the action of this brake?

411. Theoretically, it would be possible so to locate the pin, A, in Fig. 251, Prob. 410, that the slightest touch at the handle would lock the brake. Find this position of the pin.

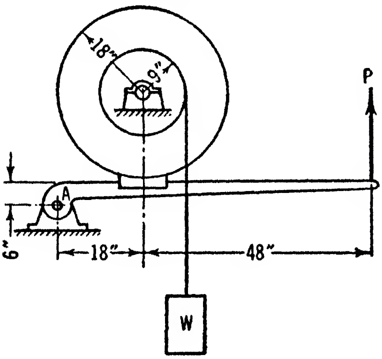

FIG. 251

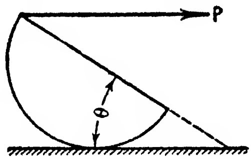

FIG. 252

412. In Fig. 252 is shown a homogeneous half-cylinder, to which is applied a horizontal force, P, as indicated. If the coefficient of static friction is 0.15, at what angle, θ, will the cylinder stand when slipping impends? If the body weighs 50 lb, what will be the value of P at the instant of impending motion? *Ans.* 33° 05′; 7.5 lb.

413. A straight, prismatic and homogeneous bar rests on a horizontal floor at its lower end, and against a vertical wall at its upper end. Derive a formula for the minimum angle with the horizontal at which such a bar can stand without slipping, the coefficient of static friction being the same for all surfaces. *Ans.* $\theta = \arctan (1 - \mu^2)/2\mu$.

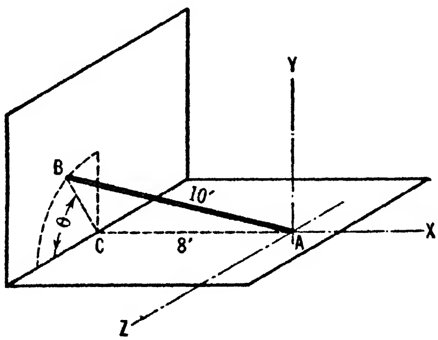

FIG. 253

414. In Prob. 406, let the cord C be attached at a point higher on the wall, so that it makes an angle of 30° with the incline. Calculate the value of θ, and the tension in the cord.

415. Figure 253 represents a uniform bar, AB, 10 ft long and weighing 200 lb. The lower end, A, rests on a horizontal surface, and is prevented from shifting by a frictionless ball-and-socket joint. The upper end, B, rests against a vertical wall. The coefficient of static friction for the bar against the wall is 0.35. Calculate the minimum possible value of the angle θ, if the bar is not to slide on the wall. Calculate the components of the reaction exerted on the bar at A. *Ans.* 65°; $A_x = -121$ lb; $A_y = +182$ lb; $A_z = +38.3$ lb.

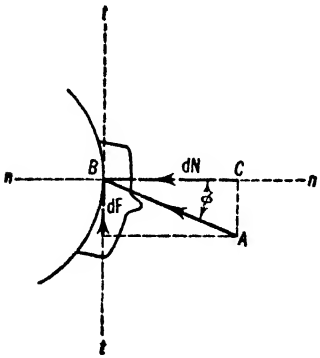

FIG. 254

79. Angle of Static Friction; Angle of Repose. The angle between the normal and the line of action of the resultant pressure on any element of area at which motion is impending is called the *angle of static friction.*

Figure 254 represents a brake-shoe pressing against a car wheel. Let it be assumed that clockwise rotation of the wheel is impending. The friction, then, on an element of area at the point B has its limiting value, dF, and the angle ABC is the angle of static friction. Let ϕ represent the angle of static friction. From the figure,

$$\tan \phi = \frac{dF}{dN} \tag{28}$$

In many cases, also,

$$\tan \phi = \frac{F}{N} \tag{29}$$

Substituting in Eq. 28 the value of dF/dN from Eq. 26, of Art. 78, it follows that

$$\tan \phi = \mu \tag{30}$$

Values of ϕ are sometimes found in tables, but can at any time be computed from values of μ, by means of Eq. 30.

The introduction of the concept, " angle of friction," simply facilitates the procedure in cases of impending motion in which it may seem more desirable to deal with the resultant pressures and their angles of inclination to the normal than with the frictional and normal components. The use of the angle of friction is especially convenient in graphic solutions, and in algebraic solutions in which the total number of forces acting on the given body does not exceed three.

If the resultant pressure at any point makes an angle with the normal less than the angle of static friction, motion is not impending at that point.

The Angle of Repose. Figure 255 represents a body resting on a plane which is inclined at such an angle that motion of the body down the plane is impending. The angle of inclination, β, of the plane to the horizontal under the foregoing conditions is called the *angle of repose.* For equilibrium, R must be collinear with W. The angle ABC between R and the normal is, therefore, equal to β. But since motion is impending the angle ABC is also equal to ϕ. Therefore, β is equal to ϕ. The angle of repose for solids is equal to the angle of static friction.

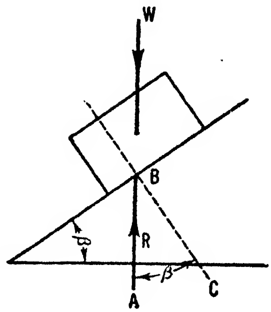

FIG. 255

For granular materials, such as sand, the angle of repose is the greatest angle with the horizontal at which the material will stand in a pile. It has been found that in such cases there is a slight difference between the angle of repose and the angle of static friction.

Tables of average values of the coefficient and the angle of static friction are to be found in the various engineering handbooks. The best value for use in any given problem is that value which was originally obtained under conditions most nearly like those of the problem.

The facts concerning friction are the result of the researches of many experimenters, notable among whom were Morin and Coulomb. The laws and formulas cannot be considered to be exact, but if their use is accompanied by good judgment in the selection of coefficients they can usually be made to yield satisfactory results. Where greater accuracy is desired special experiments should be performed.

ILLUSTRATIVE PROBLEMS

416. Figure 256 represents a body weighing 1000 lb, resting on a horizontal plane. The angle of static friction is 14°. The force P, applied as shown, is gradually increased until motion impends. Calculate the magnitude of P at this instant. Also calculate the magnitude of the reaction, R, exerted on the body by the supporting plane.

Solution. Before P is applied R will be vertical, and collinear with the weight of the body. The magnitude of R will then be equal to the weight of the body. After P is applied R will assume an inclination, as shown, and this inclination will increase as P increases. The point of application of R will move toward the left. R will eventually reach its maximum possible inclination to the normal. At this point the angle of inclination of R to the normal will reach a value equal to the angle of static friction, ϕ. If P is further increased, motion will begin.

$$\Sigma F_x = 0 \qquad +R \sin 14^\circ - P \cos 30^\circ = 0$$

$$\Sigma F_y = 0 \qquad +R \cos 14^\circ - P \sin 30^\circ - 1000 = 0$$

The solution of the foregoing equations gives

$$P = 337 \text{ lb} \qquad R = 1210 \text{ lb}$$

The coordinate axes in the foregoing solution could have been selected in such a manner that each equation would have contained only one unknown quantity. However, such a procedure would have necessitated additional geometrical calculations, and it is doubtful that much time would have been saved.

Another method favored by some consists in making a sketch of the force polygon for the three external forces acting on the body, and in solving the polygon by the methods of trigonometry. Since the three forces P, W, and

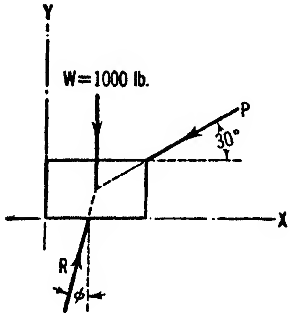

Fig. 256

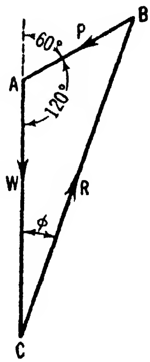

Fig. 257

R are in equilibrium, they will form a closed polygon, as shown in Fig. 257. One side of this triangle is known, and the three angles are easily calculated from the geometry of Fig. 256. The triangle is then solved by means of the law of sines. If the triangle is plotted accurately to scale the method constitutes a graphic solution of the problem. The results are scaled from the diagram.

417. Figure 258 represents a wedge, A, resting between two blocks, B and C. These blocks rest on horizontal surfaces, D and E. A horizontal force of 2000 lb is applied to each block, as shown. The angle of static friction is 15° for all surfaces. The wedge is symmetrical with respect to a vertical plane. Calculate the magnitude of the vertical force, P, assuming that downward motion of the wedge is impending. Disregard the weights of the wedge and the blocks.

Solution. Figure 259 is a free-body diagram of block B. The external forces acting on B are the given 2000-lb force, the force R, exerted by the supporting surface D, and the force Q, exerted by the wedge. Since sliding is impending at both surfaces of contact, R and Q will be inclined to their normals at angles of 15°. Great care must be exercised to incline them on the proper side of the normal. It is known from the conditions of the problem that the block B is on the point of moving toward the left. Therefore, R must be inclined as in Fig. 259, in order that its frictional component may resist the tendency of B to move. Since the wedge A is about to move downward, the frictional force that it exerts on B must be downward, and as a consequence Q must be inclined as shown.

Figure 260 is the force polygon for the three external forces acting on B.

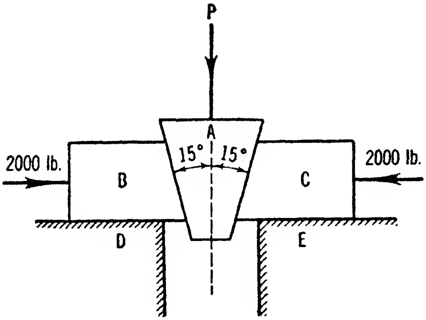

FIG. 258

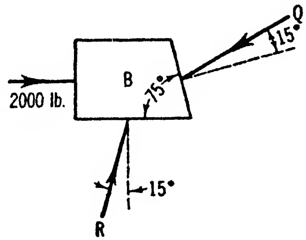

FIG. 259

The three angles of the triangle are easily calculated from the geometry of Fig. 259, and are as shown in Fig. 260. From Fig. 260, by the law of sines,

$$\frac{Q}{\sin 105°} = \frac{2000}{\sin 45°} \qquad Q = \frac{2000 \times 0.966}{0.707} = 2730 \text{ lb}$$

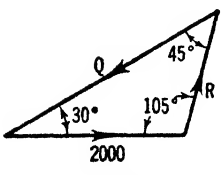

FIG. 260

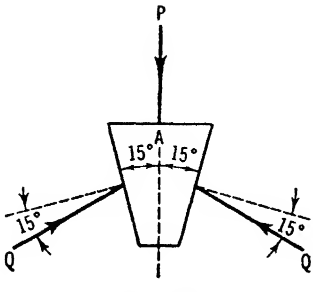

FIG. 261

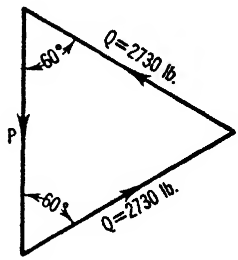

FIG. 262

Figure 261 is a free-body diagram of the wedge A. From symmetry, the forces QQ, exerted on the wedge by the blocks B and C, may be assumed equal. The force Q acting on the left-hand face of the wedge is collinear with, and equal to, the force Q that the wedge exerts on the block B. The magnitude of this force was found to be 2730 lb, in the foregoing calculation. The two forces are, however, opposite in sense. Figure 262 is the force polygon for the three forces acting on the wedge. The triangle is equilateral; therefore

$$P = 2730 \text{ lb}$$

The equilibrium equations could have been written, and solved in the usual manner, in the foregoing problem. The method used is a very convenient one, however, and is a favorite with many persons.

PROBLEMS

418. A body weighing 260 lb rests on a horizontal plane. The angle of static friction for the surfaces of contact is 15°. A force is applied to the body, its line of action having an inclination of 45° to the horizontal. The sense of the force is upward. Calculate the magnitude of this force at the instant of impending motion. *Ans.* 77.7 lb.

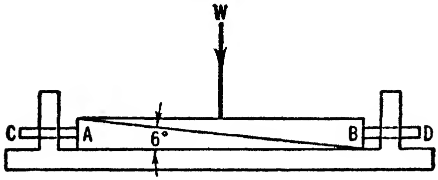

FIG. 263

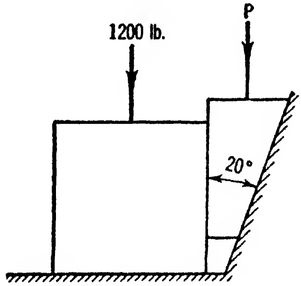

FIG. 264

419. Calculate the minimum force that could cause motion to impend in Prob. 418, and the angle at which it would be necessary to apply it. *Ans.* 67.2 lb; upward, at 15° to the horizontal.

420. Figure 263 represents an adjustable baseplate of a type often placed under the bearing at one end of a shaft. The bearing can be raised by advancing either, or both, of the capscrews, C and D. Assume a load, W, of 500 lb, and an angle of static friction of 11° at all points. Calculate the pressure that screw C must exert against wedge A, in order to cause A to start toward the right. Assume that this force has no vertical component.

421. Calculate the magnitude of the force, P, in Fig. 264, necessary to start motion. Assume an angle of static friction of 10° at all surfaces of contact. *Ans.* 165 lb.

422. Calculate the force P, in Fig. 265, that would cause motion of the wedge-shaped block to impend. Block A is attached to a wall by means of a horizontal cord, C. Calculate the tension in the cord. The angle of static friction is 8° at all contact surfaces.

423. Figure 266 represents a cotter key, C, used to connect a bar, B, to a plate, A. The bar passes loosely through a hole in the plate, and the key is inserted through a slot in that portion of the bar projecting beyond the plate. The angle between the upper face of the key and the horizontal is 12°. The angle of static friction for all contact surfaces is 14°. Calculate the force, P, necessary to start the key inward. Assume that the bar comes into contact with the plate at the left-hand side of the hole, only. *Ans.* 1260 lb.

424. In Prob. 423, Fig. 266, calculate the force that would be required to start the cotter key outward. Assume that the bar would come into contact with the plate at the right-hand side of the hole, only.

425. The device shown in Fig. 267 consists of a stationary, vertical bar, or shaft, passing through a hole in a horizontal arm, or hanger. The hole in the hanger is slightly larger than the shaft. If the arm is held in a strictly horizontal position it can be slid up or down at will. If it is permitted to tilt slightly downward it will

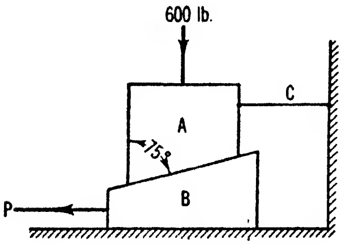

FIG. 265

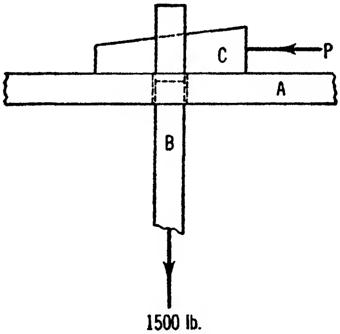

FIG. 266

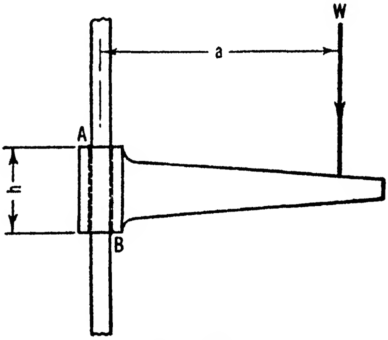

FIG. 267

come into contact with the shaft at the points A and B, thus giving rise to vertical frictional forces tending to prevent the hanger from sliding downward. Prove that a load W, of any magnitude, can be hung on the arm without causing the arm to slide downward, provided that the distance, a, is made equal to, or greater than, $h \cot \phi/2$, in which ϕ represents the angle of static friction.

80. Classification of Problems Involving Static Friction. Problems involving static friction may be classified in two groups. In problems of the one group sliding is impending, and this fact is known in advance, either from a direct statement or by inference from the conditions of the problem. In problems of this class Eqs. 26, 27, 28, 29, and 30 are valid.

In problems of the other class it is not known in advance that sliding is impending. Therefore, it cannot be assumed that the foregoing formulas apply. The important point to be held in mind is that these formulas involve the limiting friction, which is the maximum friction possible with a given normal pressure, and that the frictional force in a problem may or may not have this maximum value, depending on the conditions. In many problems the frictional forces can be found only by means of the general principles of mechanics, and the formulas referred to above may be used solely as a means of ascertaining whether the amount of friction demanded by the conditions of the problem really could exist.

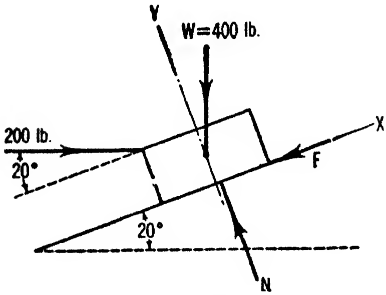

FIG. 268

ILLUSTRATIVE PROBLEMS

426. Figure 268 represents a body weighing 400 lb, resting on a 20° inclined plane. The coefficient of static friction is 0.25. A horizontal force of 200 lb is applied to the body. Calculate the friction and the normal pressure exerted on the body by the supporting plane.

Solution. Select axes as shown.

$$\Sigma F_y = 0 \qquad +N - 400 \cos 20° - 200 \sin 20° = 0 \qquad N = 444 \text{ lb}$$

There is nothing to indicate that motion of the body, either up or down the incline, is impending. Therefore, the formula $F = \mu N$ may not be used to calculate the value of the frictional force actually exerted on the body. This force can be calculated by the following process:

$$\Sigma F_x = 0 \qquad -F - 400 \cos 70° + 200 \cos 20° = 0 \qquad F = 51.2 \text{ lb}$$

This is the amount of friction that must actually exist if, under the conditions, the body is to remain in equilibrium. The formula $F = \mu N$ may be used only to test the conditions. If the value of 444 lb obtained above for N, and the value of 0.25 given for the coefficient of friction, be substituted in this formula, F will be equal to 111 lb. This is simply the maximum amount of friction that could possibly exist between the two given surfaces, with the given normal pressure. Since the amount of friction required for equilibrium is only 51.2 lb, motion will not occur, and the results obtained are the correct ones.

427. Figure 269 represents a homogeneous, rectangular prism weighing 480 lb, resting on a horizontal plane. An inclined force, P, is applied as indicated. The coefficient of static friction is $\frac{1}{3}$. If the force P is gradually increased, will the body slide, or tip? Calculate the value of P at the instant of impending motion.

Solution. Assume first that the body will tip. In such an event, at the instant when tipping impends the frictional force and the normal pressure will be concentrated at the forward edge of the bottom of the prism, as shown in the figure. Since it is not known whether sliding also impends, the formula $F = \mu N$ may not be used. From the conditions of equilibrium,

$$\Sigma M_A = 0 \qquad -(P \times \tfrac{4}{5}) \times 2 + 480 \times 0.5 = 0$$

from which $P = 150$ lb. This is the value of P which would be necessary to cause tipping to impend, assuming that sliding would not occur first.

Now assume that the body will slide without tipping. In this case, at the instant of impending sliding, the formula $F = \mu N$ does apply, but the point of application of N is unknown.

$$\Sigma F_x = 0 \qquad +\tfrac{4}{5} P - F = 0$$

$$\Sigma F_y = 0 \qquad +\tfrac{3}{5} P + N - 480 = 0$$

$$F = \mu N \qquad F = \tfrac{1}{3} N$$

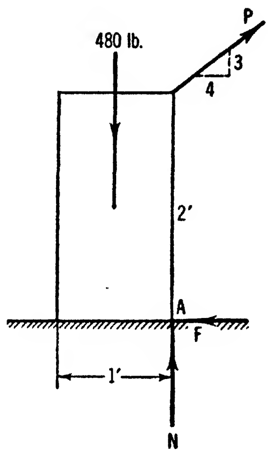

FIG. 269

The solution of these equations gives: $P = 160$ lb. It is now known that the body will tip, and that this will occur at the instant when $P = 150$ lb, since this is the smaller of the two values.

PROBLEMS

428. A body weighing 1200 lb rests on a horizontal plane. The coefficient of static friction for the surfaces of contact is 0.2. A force of 200 lb is applied to the body, at an angle of 22° 30′ with the horizontal, acting upward. Calculate the frictional force and the normal pressure exerted on the body by the supporting plane. Is motion impending? *Ans.* $F = 185$ lb; $N = 1120$ lb.

429. A homogeneous cube weighing 1000 lb is placed on a 30° inclined plane. The coefficient of static friction is 0.3. A force, parallel to the incline and directed up the slope, is applied to the upper edge of the block. This force is gradually increased until motion begins. Will the block slide, or tip? Calculate the magnitude of the applied force, and of the frictional force, for the instant of impending motion.

430. Change the coefficient of friction in Prob. 429 to 0.15, and solve the problem. *Ans.* 630 lb; 130 lb.

431. Assume that slipping and tipping of the block in Prob. 429 impend at the same instant. What is the value of the coefficient of friction under these conditions?

432. Figure 249, Art. 78, represents a cylinder weighing 80 lb, resting in a rectangular groove. The coefficient of static friction is 0.4. If the force, P, is gradually increased, will the cylinder finally slip and rotate in the groove, or will it roll out of the groove? Calculate the magnitude of the frictional force at A, at the instant when motion is impending. *Ans.* 21.4 lb.

433. Figure 270 represents one of the rear wheels of an automobile that is being held at rest on a 10 per cent grade by means of the brakes. The wheel is 28 in. in diameter, and the total load, W, carried by the wheel is 1000 lb. The forces, PP, represent the couple exerted on the wheel by the brake. Calculate the moment of this couple, and the magnitude of the frictional force, F, at the roadway.

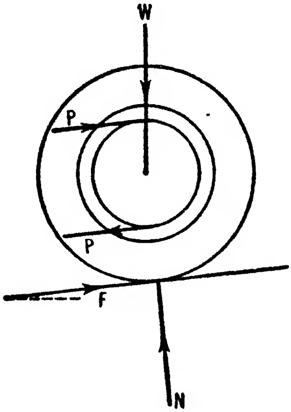

FIG. 270

434. If the coefficient of static friction for pneumatic tires on pavement is assumed to be equal to 0.6, on what maximum grade could the automobile of Prob. 433 stand, provided that the brakes were capable of holding the car? Calculate the moment of the couple that the brake would exert on the wheel under these conditions. *Ans.* 60% grade; 600 ft-lb.

81. Kinetic Friction. The *coefficient of kinetic friction* is the ratio of the kinetic friction to the normal pressure.

Let dF represent the kinetic friction on an elementary portion, dA, of the surface of contact between the two bodies. Let dN represent the normal pressure on the area dA. Let μ represent the coefficient of kinetic friction. Experiments have indicated that the relation between the foregoing quantities is as follows:

$$dF = \mu\, dN \tag{31}$$

Equation 31 may be used in any case of kinetic friction. In some cases the following special form may be used:

$$F = \mu N \tag{32}$$

in which F represents the kinetic frictional force on the entire area of contact, and N represents the total normal pressure on that area.

The angle of kinetic friction is the angle between the resultant pressure and the normal at any point at which relative motion is in progress. Let ϕ represent the angle of kinetic friction.

$$\tan \phi = \mu \tag{33}$$

The proof of Eq. 33 is similar to that of Eq. 30, in Art. 79. It is seen that the formulas for kinetic friction are identical, in form, with those for the case of limiting static friction. *The formulas for kinetic friction may be used, however, in any case in which kinetic friction exists, while the formulas involving limiting static friction apply only to the particular situation in which relative motion impends.*

The coefficient of kinetic friction is subject to greater variations than the coefficient of static friction. Its value is affected by many factors such as the relative velocity of the two surfaces, the duration of the rubbing, the intensity of the normal pressure, and the temperature. It is also affected, of course, by the roughness of the surfaces and the degree of lubrication. If accurate results are needed, these variations must be considered. If only roughly approximate results are required, an average value may be used for the coefficient. This average value is then treated as a constant in the calculations. Extensive discussions of the coefficient of kinetic friction are found in many of the handbooks and textbooks of mechanical engineering.

PROBLEMS

435. In Prob. 399, Fig. 242, let the force P have a magnitude of 100 lb, and let the coefficient of kinetic friction be 0.2. Acceleration will occur, and the horizontal components of the forces acting on the body will be unbalanced. The vertical components will be balanced. Calculate the frictional force. *Ans.* 26 lb.

436. A body weighing 180 lb is placed on a horizontal plane. It is found that a horizontal force of 18 lb, applied to the body, causes the body to move at a constant speed in a straight line. It is known from the principles of kinetics that under these conditions the external forces acting on the body are in equilibrium. Calculate the coefficient of kinetic friction.

437. A body weighing 1200 lb is placed on a 20° inclined plane. The coefficient of kinetic friction is 0.15. A force is applied to the body, its line of action parallel to the incline, and its sense being up the slope. If the body moves at constant speed, what is the magnitude of the applied force? *Ans.* 580 lb.

82. The Inclined Plane. The *inclined plane* is one of the oldest and simplest mechanical devices. It is used primarily to raise or lower heavy bodies by means of comparatively small applied forces. Friction plays an important part in determining the effectiveness of the device. In the raising of a body by means of an inclined plane, friction is detrimental, but in the lowering of a body friction may be useful in helping to control the descent. Sometimes friction alone is depended upon to hold the body at any desired point. Inclined tramways and chutes of various kinds are familiar examples of the application of the principle of the inclined plane to commercial use. Wedges and screws are adaptations of the inclined plane frequently found in machinery.

Algebraic formulas will be derived for the force required to start a body up an inclined plane. Since kinetic friction is, in general, less than static friction, the force that will start the body is usually sufficient to maintain motion.

General Formula for Force to Start Body up Plane. Figure 271 represents a body of weight, W, resting on a plane inclined to the horizontal at any angle, β. P is a force having a magnitude just sufficient to cause motion to impend, up the plane. P makes any angle, θ, with the plane, as shown. R represents the force exerted on the body by the plane, and since motion up the plane is impending R has the position shown, making an angle with the normal equal to the angle of static friction, ϕ.

Figure 272 is the force polygon for the three external forces acting on the body. In Fig. 271 it can be seen that the angle between R and the vertical is equal to $\phi + \beta$. This is also the angle ACB in Fig. 272.

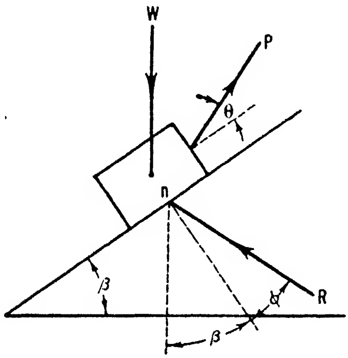

FIG. 271

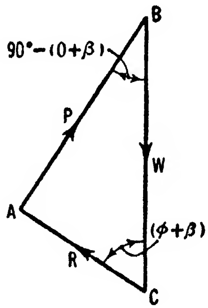

FIG. 272

Figure 271 also shows that the angle between P and the vertical is equal to $90° - (\theta + \beta)$. This is the angle ABC in Fig. 272. The remaining angle, BAC, of the force triangle is, then, equal to $180° - (\phi + \beta) - [90° - (\theta + \beta)] = 90° - (\phi - \theta)$. By the law of sines,

$$\frac{P}{\sin(\phi + \beta)} = \frac{W}{\sin[90° - (\phi - \theta)]}$$

$$P = W\frac{\sin(\beta + \phi)}{\cos(\phi - \theta)} \tag{34}$$

An examination of Eq. 34 shows, as would be supposed, that for any fixed values of β and ϕ the value of P can be greatly changed by varying the value of θ. For example, if θ is given the negative value, $-(90° - \phi)$, $\cos(\phi - \theta)$ becomes $\cos(\phi + 90° - \phi) = \cos 90° = 0$. The corresponding value of P is infinity. For such a value of θ, then, the body would not start at all, and for neighboring values of θ, P would be very large. On the other hand, if θ is given the positive value $(90° - \beta)$, $\cos(\phi - \theta)$ becomes $\cos(\phi - 90° + \beta) = \sin(\beta + \phi)$, and Eq. 34 will then give $P = W$. In such a case no advantage would be gained from the use of the inclined plane. Somewhere between these two extremes the value of P is a minimum.

Formula for Minimum Force to Start Body up Plane. An examination of Eq. 34 shows that, for any fixed values of β and ϕ, P will be a minimum when cos $(\phi - \theta)$ is a maximum. The maximum possible value of the cosine is unity, in which case $(\phi - \theta)$ is equal to zero. The formulas for the minimum value of P are, then, as follows:

$$\theta' = \phi \tag{35}$$

$$P' = W \sin (\phi + \beta) \tag{36}$$

In kinetics it is learned that if all points of a body are moving in straight lines at constant speed the forces acting on the body are in equilibrium. Therefore, the formulas of the present article apply to the case in which the body moves uniformly up the plane, provided that the angle of static friction is replaced by the angle of kinetic friction.

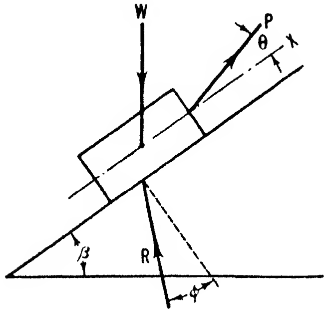

FIG. 273

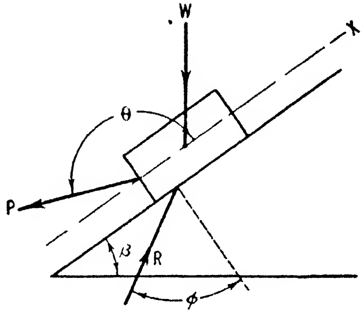

FIG. 274

PROBLEMS

438. Derive Eq. 34 by writing the equations of equilibrium, in lieu of the method used in the text. Refer to Fig. 271.

439. Figure 273 represents a body whose weight is W, resting on an inclined plane that makes an angle of β with the horizontal. The case represented is one in which the angle of static friction, ϕ, is smaller than β. This means that the body will slide down the incline of its own accord, unless prevented from doing so. Let the force P, applied as shown, be just great enough to prevent the body from sliding downward. Derive a formula for P, in terms of W, ϕ, β, and θ. *Ans.* $P = W \sin (\beta - \phi)/\cos (\phi + \theta)$.

440. In Prob. 439, for any fixed set of values of W, β, and ϕ, there is a value of θ for which P will be a minimum. Ascertain this value of θ, and obtain a formula for the minimum value of P. Use the results of Prob. 439. *Ans.* $\theta' = -\phi$; $P' = W \sin (\beta - \phi)$.

441. A body weighing 260 lb rests on a 30° inclined plane. The coefficient of static friction is 0.3. A force, P, is applied as shown in Fig. 273, in magnitude just sufficient to prevent the body from sliding down the incline. The angle θ is 22° 30′. Calculate the value of P. Also calculate the minimum value of P to accomplish the same purpose. *Ans.* 77.2 lb; 59.8 lb.

442. Figure 274 shows a body whose weight is W, resting on an inclined plane that makes an angle β with the horizontal. The case represented is one in which the angle of static friction, ϕ, is equal to, or greater than, β. Let the force P, applied as shown, be of a magnitude just sufficient to cause motion to impend, down the incline. Derive a formula for P, in terms of W, ϕ, β, and θ. *Ans.* $P = W \sin (\beta - \phi)/\cos (\phi + \theta)$.

443. In Prob. 442, for any fixed set of values of W, β, and ϕ, there is a value of θ for which P will be a minimum. Ascertain this value of θ, and obtain a formula for the minimum value of P. Use the answer obtained in Prob. 442. *Ans.* $\theta = 180° - \phi$; $P = -W \sin (\beta - \phi)$.

444. A body weighing 260 lb rests on a 10° inclined plane. The coefficient of static friction is 0.5. A force, P, applied as in Fig. 274, has a magnitude just sufficient to cause motion to impend, down the incline. The angle θ is 173°. Calculate the value of P. Calculate the minimum value of P that would accomplish the same purpose. *Ans.* 78.7 lb; 74.1 lb.

445. A body weighing 1000 lb rests on a horizontal surface. The coefficient of static friction is 0.3. Calculate the minimum force that would cause motion to impend. Solve as a special case of the inclined plane. *Ans.* 287 lb.

83. The Screw. The screw is used most frequently as a fastening device, but in many forms of machinery it is designed to play an active part in the mechanism, transmitting force or power from one part of the machine to another. The screw is an adaptation of the inclined plane; it is, in effect, an inclined plane wrapped around a cylinder.

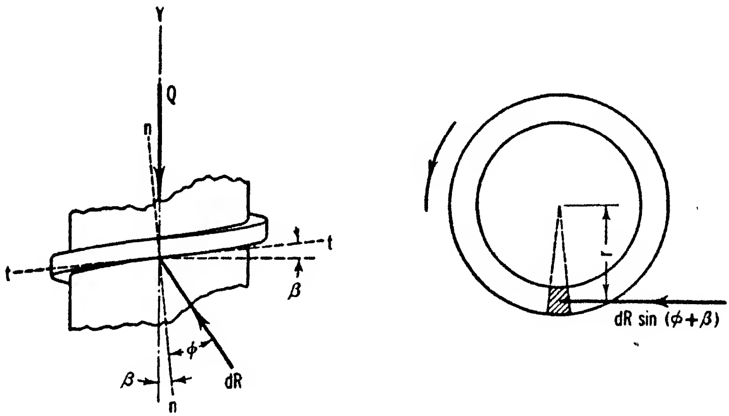

FIG. 275 FIG. 276

There is one general formula that can be used as a basis for special formulas applying to particular machines. This formula gives the moment, or torque, about the longitudinal axis of the screw, of the forces acting on the threaded portion, at the instant of impending motion. This formula will be derived for the square-threaded screw. This is the type of screw generally used where large forces are to be transmitted.

Figure 275 represents a portion of a square-threaded screw. Q represents the resultant of all the longitudinal forces acting on the screw, other than those which act on the threads. The line tt is tangent to the thread. The angle β, which is the angle of inclination of the thread, is called the pitch angle. Let dR represent the resultant pressure on an elementary portion of the surface of the thread. The exact position of the point of application of dR depends on the distribution of pressure on the thread, but it is usually assumed to be at a distance from the axis of the screw equal to the mean radius of the thread. Let the mean radius of the thread be represented by r.

Figure 276 is an end view, and shows the element of area, the mean radius, and the horizontal component of dR.

Formula for the Case in Which Q Opposes the Motion. In this case the impending rotation of the portion of the screw shown in Fig. 275 would be counterclockwise, if viewed from above. Obviously, dR would be inclined in the manner shown in that figure. Let M_T represent the moment-sum of all the elementary forces, dR, with respect to the axis of the screw. From Fig. 275 it can be seen that the horizontal component of dR is equal to $dR \sin(\phi + \beta)$. The moment of dR about the axis of the screw is equal to this component multiplied by r.

$$M_T = \Sigma[r\,dR \sin(\phi + \beta)] = r \sin(\phi + \beta)\Sigma\,dR \qquad [37]$$

It is understood that the summation indicated in Eq. 37 extends to all parts of the screw thread that make contact with the thread of the nut. Continuing, from Fig. 275,

$$\Sigma F_y = 0 \qquad \Sigma[dR \cos(\phi + \beta)] - Q = 0 \qquad \Sigma\,dR = \frac{Q}{\cos(\phi + \beta)} \qquad [38]$$

Substituting in Eq. 37 the value of $\Sigma\,dR$ obtained in Eq. 38, the following formula for M_T is obtained:

$$M_T = Qr \tan(\phi + \beta) \qquad [39]$$

Formula for the Case in Which Q Assists the Motion. In this case the position of dR would be on the opposite side of the normal from that shown in Fig. 275. The resulting formula for M_T is as follows:

$$M_T = Qr \tan(\phi - \beta) \qquad [40]$$

The derivation of Eq. 40 is similar to that of Eq. 39. These equations give that portion of the torque required to start the screw which can be attributed to the resistance of the threads.

The formulas may also be used in the case of a screw that is actually turning at constant speed, provided that the angle of kinetic friction is used for ϕ.

Illustrative Problem

446. Figure 277 represents a screw, A, passing through a nut, B, and operated by means of a handwheel, C. When the screw is advanced it pushes the block, D, along the horizontal surface, E. The block weighs 5000 lb. The screw has square threads, and there are 4 threads per inch. The diameter at the root of the threads is 0.781 in., and at the top of the threads it is 1 in. The coefficient of static friction for the threads is 0.1, and for the block on the supporting plane it is 0.25. Calculate the torque that must be applied to the handwheel, in order to cause motion to impend toward the left. Disregard the friction between the end of the screw and the block.

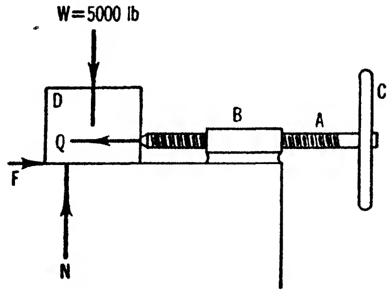

Fig. 277

Solution. The four forces shown in Fig. 277 are the external forces acting on the block.

$$\Sigma F_y = 0 \qquad N - 5000 = 0 \qquad N = 5000 \text{ lb}$$

And, since motion is impending,

$$F = \mu N \qquad F = 0.25 \times 5000 = 1250 \text{ lb}$$

$$\Sigma F_x = 0 \qquad -Q + 1250 = 0 \qquad Q = 1250 \text{ lb}$$

The longitudinal thrust, Q, exerted by the block on the screw is equal to the force Q found above, although opposite in sense.

The mean radius, r, of the threads is equal to $(0.781 + 1)/4 = 0.445$ in. The angle of static friction, ϕ, is equal to arc tan $0.1 = 5° 45'$. The pitch angle of the threads is calculated by dividing the rise of one thread by the mean circumference of the thread. This quotient gives the tangent of the pitch angle. Therefore, $\beta = \text{arc tan } 0.25/(2\pi 0.445) = \text{arc tan } 0.0894 = 5° 05'$. By substitution in Eq. 39,

$$M_T = Qr \tan(\phi + \beta)$$

$$M_T = 1250 \times 0.445 \tan(5° 45' + 5° 05') = 1250 \times 0.445 \times 0.191 = 106 \text{ in-lb}$$

PROBLEMS

447. A certain square-threaded screw has $1\frac{3}{4}$ threads per inch. The diameter at the root of the thread is 2.5 in., and the outside diameter is 3 in. The axial load on the screw is 20,000 lb, and the coefficient of friction is 0.1. Calculate the moment required to cause the screw to advance in a direction opposite to that in which the load is applied. Calculate the moment required to retract the screw. Assume that no friction must be overcome except that of the nut. *Ans.* 383 ft-lb; 77.4 ft-lb.

448. A certain machine for testing materials is equipped with two square-threaded screws. Each screw has 2 threads per inch, and the mean radius of the threads is

1.27 in. Assume the coefficient of friction to be 0.15. The two screws are threaded through the pulling head of the machine. They are prevented from moving longitudinally, and as they rotate they cause the pulling head to rise or fall. Calculate the total torque that must be exerted on the two screws, in applying a load of 100,000 lb to a test specimen. Assume that the only friction to be overcome is that on the threads. Calculate the torque required to remove the load.

449. Figure 278 represents a hand-power press, equipped with a square-threaded screw having 2 threads per inch. The bar by means of which the screw is turned has a length, b, of 18 in. The mean radius of the screw thread is 1.14 in. The coefficient of friction is 0.15. Calculate the axial force that the screw can be made to exert, by means of the application of a tangential force of 50 lb at the end of the bar. Calculate the force that would be necessary at the end of the bar to remove this load. Disregard the friction at the end of the screw. *Ans.* 3550 lb.

450. If the pitch angle of a screw is sufficiently large, and the angle of static friction is sufficiently small, the axial force Q will cause the screw to run backward, unless prevented from so doing. Assume such conditions, and derive a formula for the minimum torque, M_T, necessary to prevent the screw from turning under the influence of the force Q. *Ans.* $M_T = Qr \tan (\beta - \phi)$.

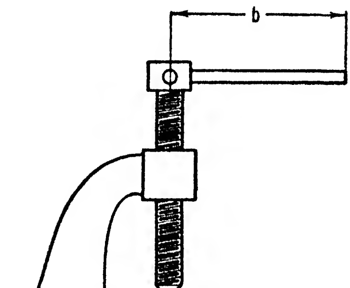

FIG. 278

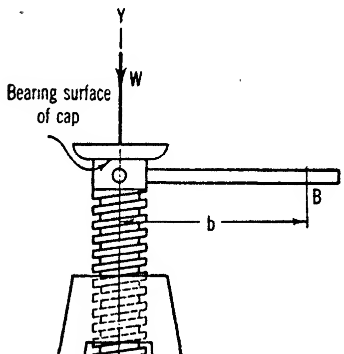

FIG. 279

84. The Screw Jack. The ordinary screw jack furnishes an illustration of the application of the screw to a useful mechanical device. In most cases the body against which the head of the jack is placed does not turn when the jack is operated. This means that friction at the head of the jack must be overcome, in addition to the resistance at the threads. The best form of jack is provided with a bearing surface at the upper end of the screw, and with a cap that transmits the load to this bearing surface. The relative motion at the head of the screw is thus confined to an adequate bearing surface.

Figure 279 represents a jack of the type described above. W represents the load on the jack, and if the weight of the cap is disregarded the normal pressure on the bearing surface will be equal to W. The limiting friction between the cap and the screw will be equal to μW. The bearing surface is usually in the form of a hollow circle, and the friction may be assumed to be concentrated at a distance from the axis of the screw equal to the mean radius of the bearing surface. Let r_c represent this mean radius. Let P represent the force required to start the screw, assumed to be applied in a horizontal plane, and at right angles to the lever at the point B. Let b represent the moment-arm of P with respect to the axis of the screw.

Formula for Force Required to Start the Screw Upward. From Fig. 279

$$\Sigma M_y = 0 \qquad Pb - \mu W r_c - M_T = 0 \tag{41}$$

in which M_T has the same meaning as in Art. 83. Equation 39, Art. 83, gives the correct value of M_T for the present case. The weight of the screw itself may be disregarded. Substituting in Eq. 41 the value of M_T given by Eq. 39, replacing Q by W, and solving for P, the following formula is obtained:

$$P = \frac{W}{b}\,[\mu r_c + r \tan\,(\phi + \beta)] \tag{42}$$

Formula for Force Required to Start the Screw Downward. It would be possible, of course, to give the screw threads a pitch angle so great that the screw would, of its own accord, run back when the load was removed. In such a case, if it were desired to hold the load at any given point, it would be necessary to retain a certain amount of force on the lever. However, the pitch angle ordinarily used in jacks is so small as to make the jack self-locking. In this case the force P must be reversed when the load is to be lowered. The formula for the required force is as follows:

$$P = \frac{W}{b}\,[\mu r_c + r \tan\,(\phi - \beta)] \tag{43}$$

The derivation of Eq. 43 is similar to that of Eq. 42. In the present case, however, the value of M_T given by Eq. 40 is used.

PROBLEMS

451. A certain screw jack has 4 threads per inch. The mean radius of the threads is 0.445 in. The inner and outer diameters of the bearing surface under the cap are 0.75 and 1.5 in., respectively. A lever 15 in. long is used to turn the screw. Assuming a coefficient of friction of 0.15 at all surfaces, calculate the necessary tangential force at the end of the lever, to raise a load of 1000 lb with the jack. Calculate the force necessary to lower this load. *Ans.* 12.8 lb; 7.4 lb.

452. A certain screw jack has 3 threads per inch. The mean radius of the threads is 0.648 in. The inner and outer diameters of the bearing surface under the cap are 1 in. and 2 in., respectively. A lever 24 in. long is used to turn the screw. Assuming a coefficient of friction of 0.15 at all surfaces, calculate the load that can be raised by the jack, when a tangential force of 40 lb is applied at the end of the lever. Calculate the force necessary to lower this load.

85. Brake Bands. Band brakes are used extensively in automobiles, hoisting engines, and in many other forms of machinery. The ability to calculate the tensions in the band is an important matter in the design of such brakes.

Band brakes are usually applied by hand, or foot, through the agency of a lever, or system of levers. One situation of great importance in the use of a brake is that which exists when, with a given force at the lever, relative motion impends between the band and the drum. An analysis based on such conditions shows the maximum holding torque of which the brake is capable, with a given force at the lever.

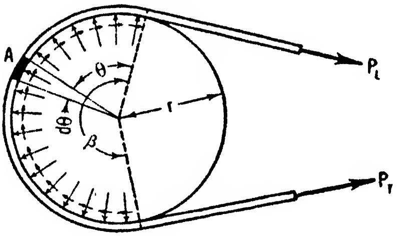

Fig. 280

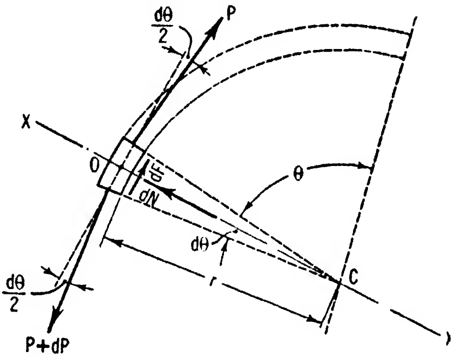

Fig. 281

The formula that will be derived gives the relation between the tensions in the tight and loose portions of the band, in terms of the coefficient of static friction, and the total angle of contact between the band and the drum.

Figure 280 represents the drum and a portion of the band. It will be assumed that clockwise rotation of the drum under the band is impending. Let P_T and P_L represent the tensions in the tight and loose sides, respectively. The terms " tight " and " loose " are used relatively, the former referring to that side of the band sustaining the greater tension. Figure 280 shows that P_T is pulling against both P_L and the frictional force, which accounts for the fact that it is greater than P_L. Let the total angle of contact be represented by β.

Figure 281 represents, to a larger scale, the elementary sector of the band, shown at point A in Fig. 280. The forces acting on the elementary portion are as shown, the tension at one end being represented by P, and at the other end by $P + dP$. From the figure,

$$\Sigma F_x = 0 \qquad (P + dP) \sin \frac{d\theta}{2} + P \sin \frac{d\theta}{2} - dN = 0 \qquad [44]$$

The sine of an infinitesimal angle may be replaced by the angle itself, if it is understood that the angle is expressed in radians. Equation 44 then becomes

$$P\, d\theta + \frac{dP\, d\theta}{2} - dN = 0 \qquad [45]$$

The term $\frac{dP\, d\theta}{2}$, in Eq. 45, is an infinitesimal of the second order, and may be dropped. Equation 45 then becomes

$$P\, d\theta - dN = 0 \qquad [46]$$

It is assumed that relative motion is impending between the brake band and the drum; therefore,

$$dF = \mu\, dN \qquad [47]$$

Eliminating dN between Eqs. 46 and 47, it follows that

$$P\, d\theta - \frac{dF}{\mu} = 0 \qquad [48]$$

The moment-arm of P, and also of $(P + dP)$, with respect to the center of the drum, may be taken equal to the radius of the drum, without serious error.

$$\Sigma M_C = 0 \qquad -Pr + (P + dP)r - r\, dF = 0 \qquad dF = dP \qquad [49]$$

Substitute the foregoing value of dF in Eq. 48.

$$P\, d\theta = \frac{dP}{\mu} \qquad \int_{P_L}^{P_T} \frac{dP}{P} = \int_0^\beta \mu\, d\theta \qquad \log_e P_T - \log_e P_L = \mu\beta \qquad [50]$$

$$\log_e \frac{P_T}{P_L} = \mu\beta \qquad [51]$$

$$\log_{10} P_T - \log_{10} P_L = 0.434\, \mu\beta \qquad [52]$$

Equation 52 can be solved with the assistance of a table of common logarithms. It should be noted that β must be expressed in radians, and that if the band completely encircles the drum one or more times the value 2π must be added to β for each full turn. The formulas may be used for ropes, belts, and flexible cables of any sort, on cylindrical surfaces, provided that the thickness is small compared with the radius of the drum. The formulas also apply when the drum is in motion and the band is stationary, if the coefficient of kinetic friction is used.

The foregoing formulas are also used to a certain extent in the case of a belt, where both belt and pulley are in motion, but if the motion is rapid they cannot give accurate results, because of the fact that equilibrium along the radius of the drum does not exist, and Eq. 44 is, therefore, incorrect. Under such conditions satisfactory results can be obtained only by the use of more general formulas that take into account the increase in the tension caused by the rapid motion in the circular path.

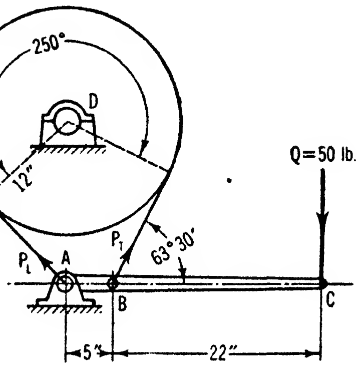

Fig. 282

Illustrative Problem

453. Figure 282 represents a drum, 24 in. in diameter, partially encircled by a brake band. The band is tightened by means of the lever, AC, and the vertical force, Q. Rotation of the drum is impending in a counterclockwise direction. The coefficient of static friction for the band on the drum is 0.25. Calculate the tensions in the straight portions of the band, and the torque that they produce with respect to the axis of the drum. Disregard the friction in the bearings of the drum and lever.

Solution. Three of the forces acting on the lever are shown in Fig. 282. The bearing reaction at A need not be shown. Considering the lever as a free body, it is seen that

$$\Sigma M_A = 0 \qquad +(P_T \sin 63^\circ\, 30')5 - 50 \times 27 = 0 \qquad P_T = \frac{50 \times 27}{0.895 \times 5} = 302 \text{ lb}$$

Equation 52 may now be used to obtain the value of P_L. $\text{Log}_{10} P_T = \log_{10} (302) = 2.48$. The value of β in radians is equal to $250 \times 2\,\pi/360 = 4.36$ radians. From Eq. 52, by algebra,

$$P_L = \text{antilog } (\log P_T - 0.434\,\mu\beta)$$

$$P_L = \text{antilog } (2.48 - 0.434 \times 0.25 \times 4.36) = \text{antilog } 2.01 = 102 \text{ lb}$$

Disregarding the thickness of the brake band, the moment-sum of P_T and P_L with respect to the shaft of the drum is as follows:

$$M_D = -12\,P_T + 12\,P_L = -12 \times 302 + 12 \times 102 = -2400 \text{ in-lb} = -200 \text{ ft-lb}$$

This torque is a measure of the effectiveness of the brake, in preventing counterclockwise rotation of the drum, when a force of 50 lb is used at the handle.

PROBLEMS

454. A workman lowers a heavy casting into a pit by means of a rope wrapped around an 8-in. round pole placed across the top of the pit. The coefficient of static friction for the rope on the pole is 0.35. The rope makes one complete turn around the pole. Calculate the greatest weight that the man can sustain by exerting a force of 50 lb at his end of the rope. *Ans.* 450 lb.

455. Calculate the greatest weight that the man in Prob. 454 can sustain, if he takes two complete turns of the rope around the pole.

456. A belt, running at slow speed, passes halfway around a 6-ft pulley. The maximum permissible tension in the belt is 200 lb. The coefficient of static friction for the belt on the pulley is 0.3. Calculate the maximum torque that the belt can transmit to the pulley under these conditions. *Ans.* 366 ft-lb.

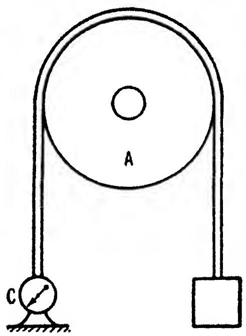

FIG. 283

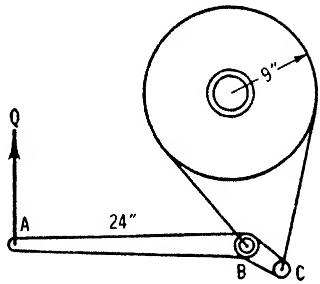

FIG. 284

457. Figure 283 represents a dynamometer of a simple type. A rope passes over a pulley, A. One end of the rope is attached to a spring balance, C, and from the other end is suspended a weight. The radius of the pulley is 4 in., and the suspended weight is 10 lb. The pulley is on the point of turning in a counterclockwise direction. The coefficient of static friction for the rope on the pulley is 0.25. Calculate the reading of the spring balance.

458. Assume that the pulley in Fig. 283, Prob. 457, is on the point of turning in a clockwise direction. The balance reads 17.5 lb. Calculate the suspended weight, and the torque exerted on the rope by the pulley. *Ans.* 7.98 lb; 3.17 ft-lb.

459. Figure 284 represents a band brake of a simple type. The brake lever is pivoted at B. One end of the brake band is fastened to the pin at B, and the other end is attached to the lever at C. The moment-arm of the tension at C, with respect to B, is 2 in. The arm AB is 24 in. The force Q is 25 lb, and the coefficient of friction for the brake band is 0.2. The band encircles 65 per cent of the circumference of the pulley. The diameter of the pulley is 18 in. Calculate the maximum torque to which the pulley can be subjected, in a counterclockwise direction, without causing it to rotate.

460. Reverse the direction of the impending motion of the pulley in Prob. 459, and solve. *Ans.* 284 ft-lb.

461. Reverse the direction of the impending motion of the drum in Prob. 453, and solve. *Ans.* P (left side) = 897 lb; P (right side) = 302 lb; M_D = 595 ft-lb.

86. Journal Bearings; Friction Circle. The ordinary form of bearing in which a shaft or an axle rotates is called a *journal bearing.* The *journal* is that portion of the shaft or axle which lies within the bearing. When a shaft is placed in a new, well-fitted journal bearing, contact will occur over a considerable portion, if not all, of the surface of the bearing. The friction depends on the tightness of the fit, and the calculation of its amount is practically impossible. After the machine has been in operation for a time, however, the bearing will become worn in, and the contact area will take the approximate form of a very narrow rectangle, the longer sides of which will be parallel to the axis of the shaft. Since the width of this rectangle is small, the contact may be assumed to be along a single line.

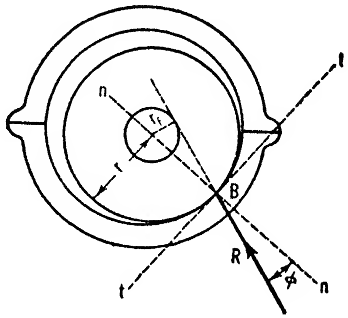

FIG. 285

Figure 285 is an exaggerated representation of a journal in a bearing that has been worn in. Let it be assumed that clockwise rotation of the journal in the bearing is impending. Let R represent the resultant pressure of the bearing on the journal. R may be assumed to be concentrated at B, the center point in the line of contact. Let r represent the radius of the journal. Since clockwise rotation of the journal is impending, R will be inclined to the normal in the manner shown by the figure. Obviously, R will be tangent to a certain circle whose center is on the axis of the journal. This circle is called the *friction circle.* Let the radius of the friction circle be represented by r_F. From the figure,

$$r_F = r \sin \phi \qquad [53]$$

If ϕ is small, $\sin \phi$ may be replaced by $\tan \phi$, and

$$r_F = r \tan \phi \quad \text{or} \quad r_F = r\mu \qquad [54]$$

If the impending rotation of the journal were counterclockwise the position of R would be on the opposite side of the normal from that represented in Fig. 285. However, R would still be tangent to the friction circle.

The friction circle is especially useful in graphic solutions, but also in algebraic solutions it may facilitate the construction of the sketch.

The friction circle may be applicable if the journal is rotating in its bearing, provided that the angle, and coefficient, of kinetic friction are used in its construction.

Illustrative Problem

462. Figure 286 represents a block, A, sliding in vertical guides, and a second block, B, sliding in horizontal guides. A rod, AB, connects the two blocks, by means of cylindrical pins at A and B. Each pin is 3 in. in diameter. The coefficient of static friction for all rubbing surfaces is 0.2. Block A carries a load, W, of 10,000 lb. The angle β is 45°. Calculate the magnitude of the force P at the instant when upward motion of block A impends. Disregard the weights of the parts of the mechanism.

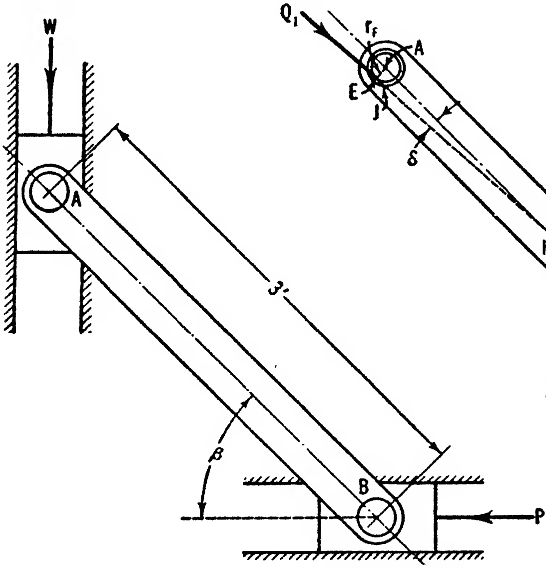

Fig. 286

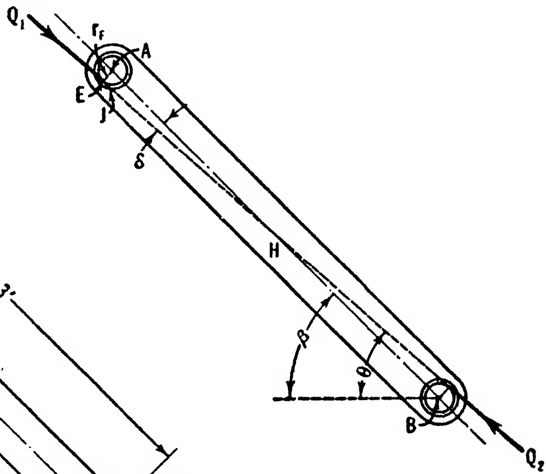

Fig. 287

Solution. Figure 287 is a free-body diagram of the rod AB. The inner circles at A and B represent the friction circles for the bearings. Q_1 and Q_2 are the forces exerted on the rod by the pins. They are tangent to the friction circles. Furthermore, they are collinear, since the rod is in equilibrium and they are the only external forces acting on it. It is possible to draw four different lines tangent to the two friction circles, and great care must be exercised in choosing the correct one for the common line of action of Q_1 and Q_2. It can be seen that the impending rotation of pin A, relative to the rod, is in a counterclockwise direction. Q_1 represents the pressure of the pin against the inner surface of the hole in the rod. Its point of application is at J. Its frictional component must act toward the right. Therefore, it is evident that Q_1 must be tangent to the friction circle at E, rather than at a point on the opposite side of that circle. A similar process of reasoning establishes the fact that Q_2 touches the upper side of its friction circle, as shown in Fig. 287. By Eq. 54,

$$r_F = r\mu \qquad r_F = 1.5 \times 0.2 = 0.3 \text{ in.}$$

From the right triangle AEH,

$$\delta = \arcsin \frac{r_F}{AH} = \arcsin \frac{0.3}{18} = \arcsin 0.01667 = 0° \; 57'$$

The angle θ, which is the angle of inclination of the line of action of Q_1 and Q_2 to the horizontal, can now be calculated.

$$\theta = \beta - \delta = 45° - 0° \; 57' = 44° \; 03'$$

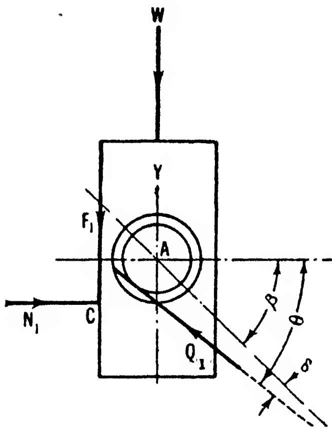

Fig. 288

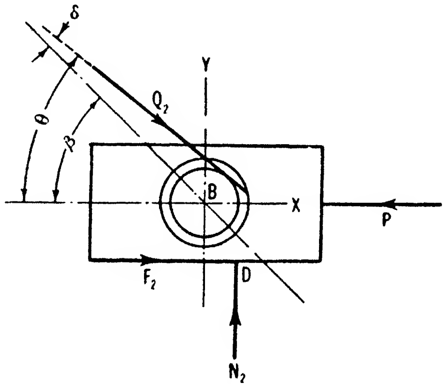

Fig. 289

Figure 288 is a free-body diagram for the block A. Four external forces act on the block, as shown. The force Q_1 is collinear with, equal to, and opposite to the force Q_1 shown in Fig. 287. From Fig. 288,

$$\Sigma F_x = 0 \qquad +N_1 - Q_1 \cos \theta = 0 \qquad +N_1 - 0.7187\, Q_1 = 0$$

$$\Sigma F_y = 0 \qquad -F_1 + Q_1 \sin \theta - W = 0 \qquad -F_1 + 0.6953\, Q_1 = 10{,}000$$

$$F = \mu N \qquad F_1 = \mu N_1 \qquad F_1 = 0.2\, N_1$$

The solution of the foregoing equations gives $Q_1 = 18{,}130$ lb. From Fig. 287,

$$Q_2 = Q_1 \qquad Q_2 = 18{,}130 \text{ lb}$$

Figure 289 is a free-body diagram for the block B. Four external forces act on the block, as shown. The force Q_2 is collinear with, equal to, and opposite to the force Q_2 shown in Fig. 287. From Fig. 289,

$$\Sigma F_x = 0 \qquad +F_2 + Q_2 \cos\theta - P = 0 \qquad +F_2 + 18{,}130 \times 0.7187 - P = 0$$

$$\Sigma F_y = 0 \qquad +N_2 - Q_2 \sin\theta = 0 \qquad +N_2 - 18{,}130 \times 0.6953 = 0$$

$$F = \mu N \qquad F_2 = \mu N_2 \qquad F_2 = 0.2\, N_2$$

The solution of the foregoing equations gives $P = 15{,}560$ lb.

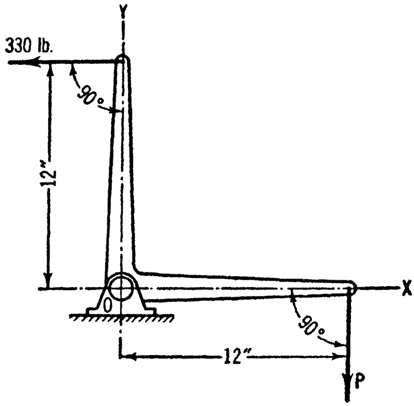

Fig. 290

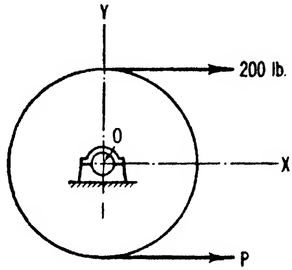

Fig. 291

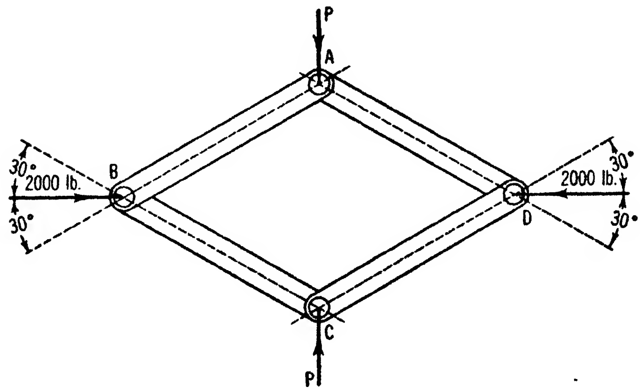

Fig. 292

PROBLEMS

463. Figure 290 represents a bell crank mounted on a shaft at O. Rotation of the crank is impending in a counterclockwise direction. The coefficient of friction for the journal is 0.2, and the diameter of the journal is 2 in. Find the reaction exerted on the journal by the bearing. Calculate the magnitude of the force P. Disregard the weight of the crank. *Ans.* 461.4 lb, $\theta_x = 44°\,20'$; $P = 322.4$ lb.

464. The pulley shown in Fig. 291 is 3 ft in diameter, and weighs 160 lb. It is subjected to two tangential forces, as shown, and is on the point of rotating in a clockwise direction. The journal is 4 in. in diameter, and the coefficient of friction is 0.15. Calculate the force P. Find the reaction exerted on the journal by the bearing. Take into account the weight of the pulley.

465. Each link in the toggle mechanism shown in Fig. 292 is 18 in. long, measured from center to center of pins. Each pin is 2 in. in diameter, and the coefficient of friction is 0.2. The impending motion is such that pins A and C are on the point of approaching each other. Calculate the magnitude of the force P. *Ans.* 1214 lb.

466. In Prob. 465, calculate the magnitude of the force P that would be just sufficient to prevent the pins B and D from approaching each other.

467. Reverse all the forces in Prob. 465. Assume that the impending motion is such that pins B and D are about to approach each other. Calculate the magnitude of P. *Ans.* 1214 lb.

468. In Prob. 462, Fig. 286, calculate the minimum magnitude of P necessary to prevent block A from descending.

469. In Prob. 462, Fig. 286, change the angle β to 30°. Let the other data of the problem remain unchanged, and solve. *Ans.* 31,260 lb.

CHAPTER VIII

SUSPENDED CABLES

87. Flexible and Inextensible Cables. Ropes, wires, and cables will all be referred to as "cables." In the fundamental theory suspended cables are usually assumed to be perfectly flexible and perfectly inextensible. These assumptions lead to a group of reasonably simple formulas sufficiently accurate for many cases.

A perfectly *flexible* cable would be one that offered no resistance to bending. No real cable could be perfectly flexible, but most of the cables used by engineers offer comparatively little resistance to bending. In a perfectly flexible cable the resultant stress on any cross section would necessarily be a tensile stress, and its line of action would be tangent to the curve of the center line of the cable.

A perfectly *inextensible* cable would be one whose length, measured along its center line, could not be changed. Sometimes in practice the results obtained under this assumption are modified to take into account the lengthening and shortening of the cable caused by temperature variations, and stress, but in the discussions in this book the cable will be assumed to be flexible and inextensible.

88. Loads Carried by Suspended Cables. Suspended cables must necessarily carry their own weight, and are frequently designed to support additional loads. The suspension bridge furnishes an important example of cables that carry applied loads much greater than their own weights. Steel messenger cables are sometimes used to support trolley wires. Electric transmission wires and other outdoor cables are subjected to the pressure of the wind, and in the winters of northern latitudes are called upon to sustain occasional ice loads of considerable magnitude. In this book the discussion will be limited to cables in vertical planes, carrying vertical loads.

The curve assumed by the center line of a suspended cable depends on the manner in which the load is distributed. The load is usually distributed uniformly, either along the center line of the cable, or along the horizontal. When the load is distributed uniformly along the center line of the cable, the cable hangs in a curve called the *catenary*. When the load is distributed uniformly along the horizontal, the resulting curve is a *parabola*.

89. The Parabolic Cable. As was indicated in Art. 88, the load is assumed to be distributed uniformly along a horizontal line. Practically all cables used in engineering are of constant cross section. Therefore, the weight of the cable itself is distributed uniformly along its own center line, rather than along the horizontal, and tends to cause the cable to assume the catenary form. For this reason it is impossible for any ordinary cable to conform exactly to the conditions assumed above, but there are many cases in which there is a horizontally uniform load so much greater than the weight of the cable itself that the effect of the latter is almost obscured. Furthermore, if the cable is drawn up fairly taut there is only a slight difference between the results ob-

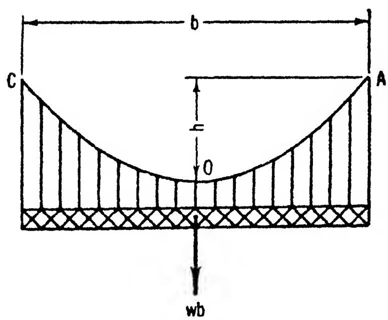

FIG. 293

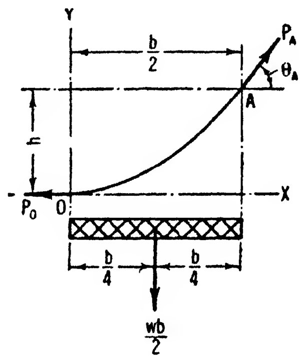

FIG. 294

tained from the catenary and the parabolic formulas. Under such conditions the parabolic formulas are generally used, because of their greater simplicity, even when the entire load is distributed uniformly along the center line of the cable.

The discussion will be confined to the special case in which the two points of support are at the same elevation, causing the cable to hang symmetrically about the vertical center line. Figure 293 represents such a cable. The distance b is called the span; h is called the sag; w represents the load per unit of horizontal distance. The total load on the cable is wb. The weight of the cable will be assumed to be distributed in the same manner as the applied load. It will be assumed that the magnitude of wb includes the weight of the cable.

Figure 294 represents the right half of the cable as a free body. It is in equilibrium under the two tensile forces, P_0 and P_A, and the total load, $wb/2$. Because of the manner in which the load is distributed, the resultant load, $wb/2$, will bisect the half span. From the symmetry of the entire cable it is obvious that O is the lowest point, and that P_0 is horizontal.

Tension at the Lowest Point. From Fig. 294,

$$\Sigma M_A = 0 \qquad -P_0 h + \frac{wb}{2}\frac{b}{4} = 0 \qquad P_0 = \frac{wb^2}{8\,h} \qquad [55]$$

Equation of the Curve. Figure 295 represents a portion of the cable, extending from O to any other point, B. The origin of coordinates has been placed at O. From Fig. 295,

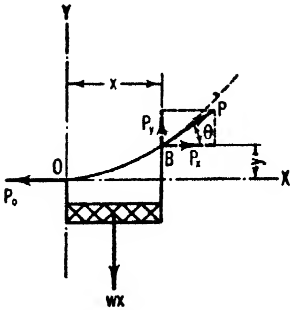

Fig. 295

$$\Sigma M_B = 0 \qquad -P_0 y + wx\frac{x}{2} = 0 \qquad [56]$$

Eliminating P_0 between Eqs. 55 and 56, it follows that

$$-\frac{wb^2}{8\,h}y + \frac{wx^2}{2} = 0 \qquad x^2 = \frac{b^2}{4\,h}y \qquad [57]$$

Equation 57 is the equation of a parabola whose axis coincides with the y-axis of the figure.

Tension at Any Point. From Fig. 295,

$$\Sigma F_x = 0 \qquad -P_0 + P_x = 0 \qquad P_x = P_0 \qquad [58]$$

Substituting in Eq. 58 the value of P_0 given by Eq. 55, it follows that

$$P_x = \frac{wb^2}{8\,h} \qquad [59]$$

Again from Fig. 295,

$$\Sigma F_y = 0 \qquad -wx + P_y = 0 \qquad P_y = wx \qquad [60]$$

Also

$$P = \sqrt{P_x^2 + P_y^2} \qquad [61]$$

Substituting in Eq. 61 the values of P_x and P_y given by Eqs. 59 and 60, the following formula is obtained:

$$P = \sqrt{\frac{w^2b^4}{64\,h^2} + w^2x^2} = w\sqrt{\frac{b^4}{64\,h^2} + x^2} \qquad [62]$$

By a similar process

$$\tan\theta = \frac{P_y}{P_x} \qquad \tan\theta = \frac{8\,h}{b^2}x \qquad [63]$$

Maximum Tension in the Cable. An examination of Eq. 62 shows that for any given cable P is a maximum at the point where x is a maximum. The maximum value of x is $b/2$. The force P_A, in Fig. 294, is, therefore, the maximum tension in the cable.

Substituting the value $x = b/2$ in Eqs. 62 and 63, the following formulas for the maximum tension are obtained:

$$P_A = \frac{wb}{2}\sqrt{\frac{b^2}{16\,h^2} + 1} \qquad [64]$$

$$\tan\theta_A = \frac{4\,h}{b} \qquad [65]$$

Length of the Cable; Exact Formula. Let l represent the total length of the cable, measured along its center line. From the calculus,

$$ds = \sqrt{dx^2 + dy^2} \qquad [66]$$

From Eq. 57, by differentiation,

$$2\,x\,dx = \frac{b^2}{4\,h}\,dy \qquad dy = \frac{8\,h}{b^2}\,x\,dx \qquad [67]$$

Substituting in Eq. 66 the value of dy given by Eq. 67, it follows that

$$ds = \sqrt{dx^2 + \frac{64\,h^2}{b^4}\,x^2\,dx^2} = \sqrt{1 + \frac{64\,h^2}{b^4}\,x^2}\,dx \qquad [68]$$

$$s = \int_0^x \sqrt{1 + \frac{64\,h^2}{b^4}\,x^2}\,dx \qquad [69]$$

And, by integration,

$$s = \frac{x}{2}\sqrt{1 + \frac{64\,h^2}{b^4}\,x^2} + \frac{b^2}{16\,h}\log_e\left(\frac{8\,h}{b^2}\,x + \sqrt{1 + \frac{64\,h^2}{b^4}\,x^2}\right) \qquad [70]$$

Equation 70 gives the length of a portion of the cable extending from the lowest point to any other point. A formula for the total length can be obtained by substituting in Eq. 70 the value $x = b/2$, and multiplying by 2. Thus,

$$l = \frac{b}{2}\left[\sqrt{1 + \frac{16\,h^2}{b^2}} + \frac{b}{4\,h}\log_e\left(\frac{4\,h}{b} + \sqrt{1 + \frac{16\,h^2}{b^2}}\right)\right] \qquad [71]$$

Length of the Cable; Approximate Formula. Equation 71 is somewhat cumbersome, and a more convenient, approximate formula, satisfactory for most practical cases, can be obtained in the following manner:

Rewrite Eq. 68, as follows:

$$ds = \left(1 + \frac{64\,h^2}{b^4}\,x^2\right)^{\frac{1}{2}} dx \qquad [72]$$

Expand the right-hand side of Eq. 72, by means of the binomial theorem, until four terms of the series have been written.

$$ds = \left(1 + \frac{32\,h^2}{b^4}x^2 - \frac{512\,h^4}{b^8}x^4 + \frac{16{,}384\,h^6}{b^{12}}x_6 \cdots\right)dx \qquad [73]$$

Integrating Eq. 73 between the limits 0 and x, substituting the value $x = b/2$, and multiplying by 2, the following approximate formula for the total length of the cable is obtained:

$$l = b\left[1 + \frac{8}{3}\left(\frac{h}{b}\right)^2 - \frac{32}{5}\left(\frac{h}{b}\right)^4 + \frac{256}{7}\left(\frac{h}{b}\right)^6 \cdots\right] \qquad [74]$$

In this series the exponent of h/b increases by 2 in each succeeding term. This ratio, h/b, is called the sag ratio. In most of the cases occurring in engineering practice the value of the sag ratio is considerably less than 1. For this reason the fourth term of Eq. 74 is usually so small as to be negligible, and in many cases the third term can also be disregarded, without serious error.

Illustrative Problems

470. A certain cable is suspended between two supports at the same elevation and 500 ft apart. The load is 500 lb per horizontal foot, and includes the weight of the cable. The sag of the cable is 30 ft. Calculate the total length of the cable, and the maximum tension.

Solution. The sag ratio $= h/b = 30/500 = 0.06$. Using the approximate formula, Eq. 74, for the length of the cable,

$$l = b\left[1 + \frac{8}{3}\left(\frac{h}{b}\right)^2 - \frac{32}{5}\left(\frac{h}{b}\right)^4 + \frac{256}{7}\left(\frac{h}{b}\right)^6\right]$$

$$l = 500\left[1 + \frac{8}{3}(0.06)^2 - \frac{32}{5}(0.06)^4 + \frac{256}{7}(0.06)^6\right]$$

It is obvious that the last two terms in the foregoing equation are negligible. Discarding these,

$$l = 500\ (1 + 0.0096) = 504.8 \text{ ft}$$

Equations 64 and 65 give the maximum tension in the cable.

$$P_A = \frac{wb}{2}\sqrt{\frac{b^2}{16\,h^2} + 1} = \frac{500 \times 500}{2}\sqrt{\frac{(500)^2}{16\,(30)^2} + 1} = 536{,}000 \text{ lb}$$

$$\theta = \text{arc tan}\,\frac{4\,h}{b} = \text{arc tan}\,\frac{4 \times 30}{500} = \text{arc tan } 0.24 = 13°\ 30'$$

471. A wire 310 ft in length is to be suspended from two supports at the same elevation and 300 ft apart. Calculate the sag.

Solution. Equation 74 contains all the necessary quantities for the solution of this problem. For the purpose of ascertaining the number of terms that may be discarded from Eq. 74, the sag may be roughly estimated to be in the neighborhood of 30 ft. This would correspond to a sag ratio of $30/300 = 0.1$, making it obvious that the last two terms of the formula are negligible. Discarding these,

$$l = b\left[1 + \frac{8}{3}\left(\frac{h}{b}\right)^2\right] \qquad h = b\sqrt{\frac{3}{8}\left(\frac{l}{b} - 1\right)}$$

$$h = 300\sqrt{\frac{3}{8}\left(\frac{310}{300} - 1\right)} = 33.5 \text{ ft}$$

PROBLEMS

472. A cable is to be hung between two points at the same elevation and 800 ft apart. The cable is to be drawn up until the sag is 30 ft. The total load to be carried, including the weight of the cable, is 90 lb per horizontal linear foot. Calculate the maximum and minimum tensions in the cable. Calculate the length of the cable, using the approximate formula. *Ans.* 243,000 lb; 240,000 lb; 803 ft.

473. A wire weighing 0.06 lb per lin ft is to be strung between poles 120 ft apart, the points of support being at the same elevation. The maximum allowable tension in the wire is 200 lb. If the weight of the wire is the only load to be carried, what is the minimum sag that may be used?

474. It is desired to hang a wire weighing 0.06 lb per lin ft in such a manner that the sag will be 1 ft. The maximum allowable tension is 200 lb. Calculate the maximum distance at which the poles may be spaced. Calculate the angle of inclination of the tangent to the curve of the wire at the point of support. *Ans.* 163 ft; 1° 24′.

475. A certain cable weighs 10 lb per lin ft, and the maximum tension to which it can be safely subjected is 50 tons. It is suspended between two supports at the same elevation and 500 ft apart. It is to be drawn up until the sag is 20 ft. Calculate the load per horizontal foot that the cable can safely support, in excess of its own weight.

476. A cable 1000 ft in length is to be suspended from two supports at the same elevation, in such a manner as to give it a sag of 40 ft. Calculate the necessary distance between supports. *Ans.* 995.7 ft.

477. A cable 2005 ft in length is to be strung between two towers standing on level ground. The towers are 80 ft high, and the points of support are 2000 ft apart. Calculate the clearance between the ground and the lowest point of the cable.

478. A wire weighing 0.2 lb per lin ft is to be strung between two points of support at the same elevation and 600 ft apart. The wire is to be drawn up until the maximum tension reaches 350 lb. The new wire is required to have a vertical clearance of 6 ft above an old line that intersects the plane of the new line at a point 200 ft from the center of the new span, and 30 ft above the ground. Calculate the necessary height of the points of support of the new span, assuming level ground. *Ans.* 50.5 ft.

479. A certain span in a pole line has a width of 1000 ft and a sag of 50 ft. The adjacent span is 400 ft, and the sag is 20 ft. The supports are all at the same elevation, and the wire weighs 0.64 lb per lin ft. Find the magnitude and inclination of the resultant pull of the wire on the intermediate tower.

480. A cable weighing 3.1 lb per lin ft is strung between two poles resting on level ground, each pole being 40 ft high. The span is 300 ft, and the cable is drawn up to a maximum allowable tension of 3 tons. Each pole is braced by means of a guy wire placed at an angle of 45° with the horizontal, and attached to the pole at a point 30 ft above the ground. Assume that the guys alone prevent the poles from overturning under the pulls from the cable, and calculate the tension that each guy sustains under these conditions. *Ans.* 5.64 tons.

90. The Catenary Cable. The load in this case is assumed to be distributed uniformly along the center line of the cable. Any cable of constant cross section, sustaining no load except its own weight, con-

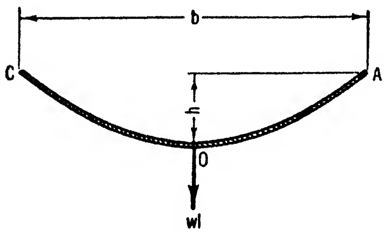

FIG. 296

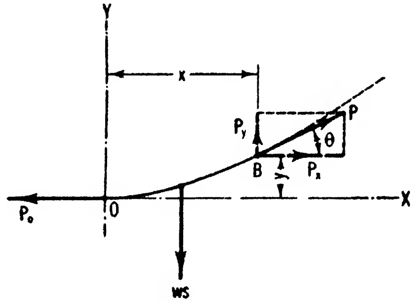

FIG. 297

forms to the foregoing assumption. Sometimes certain additional loads, such as coatings of ice, are assumed to be distributed in this manner, also.

Figure 296 represents such a cable. Let w represent the load per unit distance along the center line of the cable. Let l represent the total length of the cable, between supports. The total load on the entire cable is, then, equal to wl.

Equation of the Curve. Figure 297 represents a portion of the cable, extending from the lowest point, O, to any other point, B. Let s represent the length of this portion. The body is in equilibrium under the action of the two tensile forces, P_0 and P, and the load, ws.

In the theory of the parabolic cable, Art. 89, the position of the line of action of the load could be expressed at once in terms of x. In the present case this is impossible, and the procedure will necessarily be somewhat different. From Fig. 297,

$$\Sigma F_x = 0 \qquad -P_0 + P_x = 0 \qquad P_x = P_0 \tag{75}$$

$$\Sigma F_y = 0 \qquad -ws + P_y = 0 \qquad P_y = ws \tag{76}$$

$$\tan\theta = \frac{P_y}{P_x} = \frac{ws}{P_0} \tag{77}$$

Since P is tangent to the curve at B,

$$\tan\theta = \frac{dy}{dx} \qquad \frac{dy}{dx} = \frac{ws}{P_0} \tag{78}$$

From the calculus

$$ds^2 = dx^2 + dy^2 \tag{79}$$

Dividing Eq. 79 by dy^2,

$$\left(\frac{ds}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1 \tag{80}$$

Inverting Eq. 78 and substituting in Eq. 80,

$$\left(\frac{ds}{dy}\right)^2 = \left(\frac{P_0}{ws}\right)^2 + 1 \tag{81}$$

$$dy = w\frac{s\,ds}{\sqrt{P_0^2 + w^2s^2}} \tag{82}$$

$$y = w\int_0^s \frac{s\,ds}{\sqrt{P_0^2 + w^2s^2}} \tag{83}$$

$$y = \frac{\sqrt{P_0^2 + w^2s^2}}{w} - \frac{P_0}{w} \tag{84}$$

Dividing Eq. 79 by dx^2,

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \tag{85}$$

Substituting in Eq. 85, from Eq. 78,

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{ws}{P_0}\right)^2 \tag{86}$$

$$dx = P_0\frac{ds}{\sqrt{P_0^2 + w^2s^2}} \tag{87}$$

$$x = P_0\int_0^s \frac{ds}{\sqrt{P_0^2 + w^2s^2}} \tag{88}$$

$$x = \frac{P_0}{w}\left[\log_e\left(ws + \sqrt{P_0^2 + w^2s^2}\right) - \log_e P_0\right] \tag{89}$$

$$\frac{wx}{P_0} = \log_e \frac{ws + \sqrt{P_0^2 + w^2s^2}}{P_0} \tag{90}$$

$$e^{wx/P_0} = \frac{ws + \sqrt{P_0^2 + w^2s^2}}{P_0} \tag{91}$$

$$s = \frac{P_0}{2\,w}\left(e^{wx/P_0} - e^{-wx/P_0}\right) \tag{92}$$

Eliminating s between Eqs. 84 and 92, it is found that

$$y = \frac{P_0}{w}\left[\tfrac{1}{2}\left(e^{wx/P_0} + e^{-wx/P_0}\right) - 1\right] \tag{93}$$

Equation 93 is the equation of the catenary, with the origin of coordinates at the center point of the curve.

Tension at the Lowest Point. The quantities $b/2$ and h are the coordinates of the right-hand point of support, A. Therefore, these quantities will satisfy Eq. 93. Substituting $x = b/2$ and $y = h$ in

Eq. 93, the following equation results:

$$h = \frac{P_0}{w}\left[\tfrac{1}{2}\left(e^{wb/2P_0} + e^{-wb/2P_0}\right) - 1\right] \tag{94}$$

Tension at Any Point. From Fig. 297,

$$P = \sqrt{P_x^2 + P_y^2} \tag{95}$$

Substituting in Eq. 95 the values of P_x and P_y given by Eqs. 75 and 76,

$$P = \sqrt{P_0^2 + w^2s^2} \tag{96}$$

Substituting in Eq. 96 the value of s given by Eq. 92,

$$P = P_0\sqrt{1 + \tfrac{1}{4}\left(e^{wx/P_0} - e^{-wx/P_0}\right)^2} \tag{97}$$

Substituting in Eq. 77 the value of s given by Eq. 92,

$$\tan\theta = \tfrac{1}{2}\left(e^{wx/P_0} - e^{-wx/P_0}\right) \tag{98}$$

Maximum Tension in the Cable. An examination of Eq. 97 shows that for any given cable P is a maximum when x is a maximum. The maximum value of x is $b/2$, showing that the tension is greatest at the points of support. Let this maximum tension be represented by P_A. Substituting $x = b/2$ in Eqs. 97 and 98, the following formulas are obtained:

$$P_A = P_0\sqrt{1 + \tfrac{1}{4}\left(e^{wb/2P_0} - e^{-wb/2P_0}\right)^2} \tag{99}$$

$$\tan\theta_A = \tfrac{1}{2}\left(e^{wb/2P_0} - e^{-wb/2P_0}\right) \tag{100}$$

Length of the Cable. A formula for the length of the cable can be obtained by substituting $x = b/2$ in Eq. 92, and multiplying by 2. Let the length of the cable, between the two supports, be represented by l.

$$l = \frac{P_0}{w}\left(e^{wb/2P_0} - e^{-wb/2P_0}\right) \tag{101}$$

It is evident that some of the quantities involved in the formulas for the catenary cable cannot be solved for by the usual methods of algebra. In such cases the trial method can be used, various values being substituted for the unknown quantity until one is found that closely satisfies the equation. The use of tables of logarithms facilitates the calculations. Various curves and diagrams have also been devised to simplify the solution of problems. These are to be found in many of the books and papers that deal more extensively with the application of mechanics to suspended cables.

The Formulas Expressed in Terms of Hyperbolic Functions. Calculations involving the use of Eqs. 92, 93, 94, 97, 98, 99, 100, and 101 can be expedited by means of tables of the hyperbolic functions. The expression $\frac{1}{2}(e^x - e^{-x})$ is the hyperbolic sine of x, and is written $\sinh x$. Also, the expression $\frac{1}{2}(e^x + e^{-x})$ is the hyperbolic cosine of x, and is written $\cosh x$. Therefore, such an expression as $\frac{1}{2}(e^{wx/P_0} + e^{-wx/P_0})$ may be written $\cosh wx/P_0$. By virtue of these facts, the principal formulas for the catenary may be written as follows:

Equation of the Curve.

$$y = \frac{P_0}{w}\left(\cosh \frac{wx}{P_0} - 1\right) \qquad [102]$$

Tension at the Lowest Point.

$$h = \frac{P_0}{w}\left(\cosh \frac{wb}{2\,P_0} - 1\right) \qquad [103]$$

Tension at Any Point.

$$P = P_0 \cosh \frac{wx}{P_0} \qquad [104]$$

$$\tan \theta = \sinh \frac{wx}{P_0} \qquad [105]$$

Maximum Tension in the Cable.

$$P_A = P_0 \cosh \frac{wb}{2\,P_0} \qquad [106]$$

$$\tan \theta_A = \sinh \frac{wb}{2\,P_0} \qquad [107]$$

Length of the Cable.

$$l = \frac{2\,P_0}{w} \sinh \frac{wb}{2\,P_0} \qquad [108]$$

The relation, $\cosh^2 x - \sinh^2 x = 1$, was also used in obtaining some of the foregoing formulas.

Illustrative Problem

481. A wire weighing 2 lb per lin ft is suspended between two points at the same elevation, 600 ft apart. The sag is 100 ft. Find the maximum tension. Calculate the length of the wire.

Solution. First, it is necessary to calculate P_0, the tension at the lowest point. This can be done by means of Eq. 94. The value 2.718 may be used for e.

$$h = \frac{P_0}{w}\left[\tfrac{1}{2}\left(e^{wb/2P_0} + e^{-wb/2P_0}\right) - 1\right]$$

$$100 = \frac{P_0}{2}\left[\tfrac{1}{2}\left(2.718^{(2\times 600)/2P_0} + 2.718^{-(2\times 600)/2P_0}\right) - 1\right]$$

P_0 cannot be solved for in the foregoing equation by the usual methods. Its value can be found by trial, however, without great difficulty. That value which will satisfy the equation is, of course, the correct one. Some idea of the correct value can be formed by the use of Eq. 55, which gives the magnitude of P_0 for the parabolic cable. Thus, $P_0 = wb^2/8\,h = 2 \times (600)^2/(8 \times 100) = 900$ lb. This value of P_0, substituted in the equation above, gives $100 = 103.7$, showing that 900 lb is only roughly correct.

After a few trials it will be found that the value $P_0 = 931$ lb gives $100 = 100.08$, which is sufficiently accurate. The maximum tension can now be found from Eq. 99.

$$P_A = P_0\sqrt{1 + \tfrac{1}{4}\left(e^{wb/2P_0} - e^{-wb/2P_0}\right)^2}$$

$$P_A = 931\sqrt{1 + \tfrac{1}{4}\left(2.718^{(2\times 600)/(2\times 931)} - 2.718^{-(2\times 600)/(2\times 931)}\right)^2}$$

$$P_A = 1130 \text{ lb}$$

Equation 100 gives the angle of inclination of P_A.

$$\tan\theta_A = \tfrac{1}{2}\left(e^{wb/2P_0} - e^{-wb/2P_0}\right)$$
$$= \tfrac{1}{2}\left(2.718^{(2\times 600)/(2\times 931)} - 2.718^{-(2\times 600)/(2\times 931)}\right)$$
$$\tan\theta_A = 0.690 \qquad \theta_A = 34^\circ\,35'$$

Equation 101 gives the length of the wire.

$$l = \frac{P_0}{w}\left(e^{wb/2P_0} - e^{-wb/2P_0}\right)$$
$$= \frac{931}{2}\left(2.718^{(2\times 600)/(2\times 931)} - 2.718^{-(2\times 600)/(2\times 931)}\right)$$
$$l = 642 \text{ ft.}$$

PROBLEMS

482. A cable weighing 8.4 lb per lin ft is suspended between two points at the same elevation, 1000 ft apart. The sag is 200 ft. Calculate the tension at the lowest point. Calculate the magnitude of the maximum tension. *Ans.* 5510 lb; 7190 lb.

483. Calculate the length of the cable described in Prob. 482, using the value of P_0 given in the answers to that problem.

484. Solve Prob. 482 by means of the parabolic formulas of Art. 89. Compare the results thus obtained with those obtained in Prob. 482. *Ans.* 5250 lb; 6720 lb.

485. A cable weighing 2.1 lb per lin ft is to be suspended between two supports at the same elevation, 600 ft apart. The maximum allowable tension in the cable is 1000 lb. Calculate the minimum sag to which the cable may be drawn up. Calculate the length of the cable.

486. A wire 680 ft long, weighing 1.002 lb per lin ft, is suspended between two supports at the same elevation, 500 ft apart. Calculate the sag, and the magnitude of the maximum tension. *Ans.* 205 ft; 385 lb.

CHAPTER IX

CENTER OF GRAVITY

91. Center of Gravity of a Body. *The center of gravity of a body is that point through which the resultant weight passes, regardless of the position in which the body is placed.*

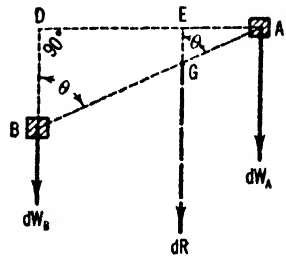

FIG. 298

That a body really possesses such a point will be shown by means of a simple proof.

Proof. In Fig. 298 let A and B represent any two particles of a given body. The remainder of the body is not shown. Let dW_A and dW_B represent the weights of the two particles, and let dR represent the resultant of the two weights.

By the principle of moments, Art. 50, using point A as the center of moments,

$$dW_B \times \overline{AD} = dR \times \overline{AE} \qquad [109]$$

From Fig. 298,

$$\overline{AD} = \overline{AB} \sin \theta \quad \text{and} \quad \overline{AE} = \overline{AG} \sin \theta \qquad [110]$$

Substituting in Eq. 109 the values of $\overline{AD}$ and $\overline{AE}$ given by Eq. 110,

$$dW_B \times \overline{AB} \sin \theta = dR \times \overline{AG} \sin \theta \qquad [111]$$

which can be written as follows:

$$\overline{AG} = \overline{AB} \frac{dW_B}{dR} \qquad [112]$$

The point G is the point of intersection of dR with the line AB, drawn between the two particles. Equation 112 shows that the position of G depends only upon the distance between the two particles and upon the ratio of their weights, and that it does not depend upon the value of the angle θ. It follows that, regardless of the position in which the body may be placed, dR will always pass through G, provided that the dimensions of the body itself are not altered.

Since the resultant, dR, always passes through G, regardless of the position of the body, its behavior when the position of the body is changed is the same as that of the weight of any single particle. Consequently, a process of reasoning similar to that above can be followed in connection with dR and the weight of a third particle of the body. Thus it can be proved that any three of the particles possess a center of gravity of their own, conforming to the definition. An extension of the reasoning proves the existence of a center of gravity for the entire body.

Strictly, the weights of the particles of a body do not constitute a parallel system, but from the viewpoint of the engineer their departure from parallelism is negligible.

In a problem in mechanics it is usually permissible to treat the total weight of a body as a single, concentrated force acting through the center of gravity, thereby simplifying many calculations. This is one of the many reasons why the conception of the center of gravity is of great importance to the engineer.

92. Center of Gravity of a Line, Area, or Volume. *The center of gravity of a line, area, or volume is a point whose position relative to the elementary portions of such line, area, or volume is essentially the same as the position of the center of gravity of a body relative to the particles of the body.*

A line, for example, may be considered to be made up of elementary lengths. These elementary lengths may be thought of as corresponding to the particles of a body. The lengths of the elements may be represented by a system of parallel vectors, similar to the vectors representing the weights of the particles of a body, the magnitude of each vector being made proportional to the length of the element that it represents. Also, as was true of the body, it is to be understood that each vector always passes through its corresponding element, regardless of the inclination of the vectors relative to the line.

Under the foregoing conception, and from the proof in Art. 91, it is seen that there exists a point through which the resultant of the system always passes, regardless of the inclination of the system of vectors relative to the line. This point is called *the center of gravity of the line.*

In the area, or the volume, the existence of the center of gravity can be demonstrated by a process of reasoning similar to that used above for the line.

In connection with lines, areas, and volumes, some writers prefer the term " centroid " to " center of gravity," limiting the use of the latter term exclusively to bodies. From the academic viewpoint this distinction is desirable, but in engineering practice the term center of gravity is widely used for all cases, and it will be so used in this book.

The term " mass-center " appears in most of the books on physics or mechanics. The center of gravity and the mass-center have the same position in a body, and the latter term will not be used in this book.

93. Coordinates of the Center of Gravity; General Formulas. General formulas for the rectangular coordinates of the center of gravity of a body, line, area, and volume will now be derived.

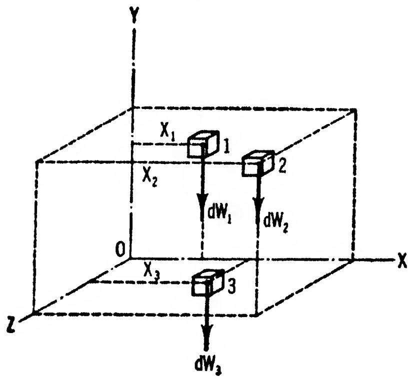

FIG. 299

Bodies. Let Fig. 299 represent any three of the particles of a body, and let dW_1, dW_2, and dW_3 represent their weights. Let the resultant of the weights of all the particles or, in other words, the total weight of the body, be represented by W (not shown in the figure).

Since, by Art. 91, the resultant weight always passes through the center of gravity of the body, it follows that the x-coordinate of the line of action of W is equal to the x-coordinate of the center of gravity. Let this coordinate be represented by $\bar{x}$.

By the principle of moments, Art. 50, using the z-axis as the axis of moments,

$$W\bar{x} = (dW_1)x_1 + (dW_2)x_2 + (dW_3)x_3 \cdots \qquad [113]$$

with the understanding that the summation of moments in the right-hand member of Eq. 113 is extended to include all the particles of the body. This summation may be represented, more briefly, by $\int x\, dW$. Equation 113 may now be written as follows:

$$\bar{x} = \frac{\int x\, dW}{W} \qquad [114]$$

If the body, together with the coordinate axes, is now rotated through an angle of 90° about the z-axis, and a similar analysis is made, the following formula will result:

$$\bar{y} = \frac{\int y\, dW}{W} \qquad [115]$$

If, in Fig. 299, the x-axis is taken as the axis of moments, it will be found that

$$\bar{z} = \frac{\int z\, dW}{W} \qquad [116]$$

Lines, Areas, and Volumes. By a process of reasoning similar to that employed in the case of the body, together with the conception of the center of gravity of a line, area, or volume presented in Art. 92, the following formulas can be derived:

For a Line.

$$\bar{x} = \frac{\int x\,dL}{L} \qquad \bar{y} = \frac{\int y\,dL}{L} \qquad \bar{z} = \frac{\int z\,dL}{L} \qquad [117]$$

in which L represents the total length of the line.

For an Area.

$$\bar{x} = \frac{\int x\,dA}{A} \qquad \bar{y} = \frac{\int y\,dA}{A} \qquad \bar{z} = \frac{\int z\,dA}{A} \qquad [118]$$

in which A represents the total area.

For a Volume.

$$\bar{x} = \frac{\int x\,dV}{V} \qquad \bar{y} = \frac{\int y\,dV}{V} \qquad \bar{z} = \frac{\int z\,dV}{V} \qquad [119]$$

in which V represents the total volume.

In the foregoing discussion it is tacitly assumed that there will be no uncertainty as to the coordinate of a given element. There is no uncertainty, of course, if the element is so designed that all its dimensions are infinitesimals. However, some persons prefer to simplify the integration by so choosing the element that some of its dimensions are finite. This may be done if certain precautions are observed. For example, if it is desired to calculate $\bar{x}$ for a given volume, the element dV may be so selected that its coordinate, x, has a single value for all points in the element, even though some of the dimensions are finite. In addition, an element of any form may be used if the position of its center of gravity is known. In this case x should be taken as the coordinate of the center of gravity of the element.

The quantities $\bar{x}$, $\bar{y}$, and $\bar{z}$, in the foregoing formulas, are read "gravity x," "gravity y," and "gravity z," or "bar x," "bar y," and "bar z."

94. Axes and Planes of Symmetry. *If a line, area, volume, or homogeneous body has one or more axes, or planes, of symmetry, the center of gravity will lie in all such axes or planes.*

The foregoing principle is useful in reducing the amount of labor in many problems. The proof is simple, and will be omitted.

95. First Moments; Gravity Axes. In the development of many of the formulas used by engineers, and in the solution of many engineering problems, integrals of the form $\int x\,dV$, $\int y\,dA$, $\int x\,dL$, $\int z\,dW$, etc., are frequently encountered. Such integrals are called *first moments, static moments*, or simply *moments*. For example, $\int y\,dA$ is the first moment of the area, A, with respect to either the x-axis or the z-axis. Since y also represents the distance from the element to the xz-plane, the expression $\int y\,dA$ is sometimes referred to as the first moment of the area A with respect to the xz-plane. It can be seen that these moments bear a certain resemblance to the moments of forces about axes.

Equation 118, Art. 93, shows that $\int y\,dA = A\bar{y}$. Therefore, the expression $A\bar{y}$ also gives the first moment of A with respect to the x-axis, the z-axis, or the xz-plane. Expressions of this type are usually more convenient for calculating first moments than the expressions containing integrals, by virtue of the fact that simple special formulas not involving the use of the calculus are generally available for the computation of such quantities as $\bar{x}$, $\bar{y}$, $\bar{z}$, L, A, V, and W. Methods for deriving these special formulas will be shown in Arts. 96, 97, and 98.

Frequently the ability to find the position of the center of gravity is important in itself, entirely apart from the matter of first moments. This is an additional reason for the importance of the special formulas mentioned above.

It can easily be shown that such a quantity as $\bar{x}$ is equal to the arithmetic mean of the x-coordinates to all the points in a given figure.

Any straight line passing through the center of gravity is called a *gravity axis*. It is obvious that *the first moment of any line, area, volume, or weight with respect to a gravity axis is equal to zero.*

96. Derivation of Special Formulas for Lines. The position of the center of gravity of a line, as such, is of limited importance in engineering problems. However, formulas giving the position of the center of gravity of a line may be used to find the position of the center of gravity of a slender rod, wire, or other object, if the object is homogeneous and of constant cross section. The line used is the center line of the rod or wire. Such results are approximate, but usually sufficiently accurate, the degree of approximation depending on the slenderness of the object.

Illustrative Problem

487. Derive a formula showing the position of the center of gravity of a circular arc.

Solution. Figure 300 shows a circular arc, whose central angle is 2β. Let the length of the arc be represented by L, and the radius by r. Let the origin of coordinates be placed at the center of the arc, the x-axis coinciding with the axis of symmetry. By Eq. 117,

$$\bar{x} = \frac{\int x\, dL}{L}$$

From Fig. 300,

$$x = r \cos \theta; \qquad dL = r\, d\theta \quad \text{and} \quad L = 2\, r\beta$$

Substituting these values in the foregoing formula,

$$\bar{x} = \frac{\int_{-\beta}^{+\beta} r \cos \theta\, r\, d\theta}{2\, r\beta}$$

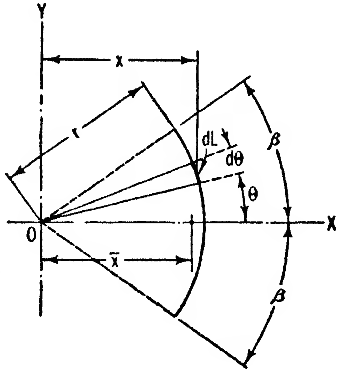

Fig. 300

The first moments of the upper and lower halves of the arc, with respect to the y-axis, are equal. For greater convenience, then, the integral may be rewritten, and solved, as follows:

$$\bar{x} = \frac{2\, r^2 \int_0^{+\beta} \cos \theta\, d\theta}{2\, r\beta} = \frac{2\, r^2 \Big[\sin \theta\Big]_0^{\beta}}{2\, r\beta} = \frac{r \sin \beta}{\beta}$$

From symmetry (Art. 94) it is clear that $\bar{y} = 0$.

PROBLEMS

488. Derive a formula showing the position of the center of gravity of a circular arc whose central angle is 90°. Place the x- and y-axes as in Fig. 300. Solve by integration, from Eq. 117. Check by substituting in the formula derived in Prob. 487. *Ans.* $\bar{x} = 2\sqrt{2}\, r/\pi$.

489. Derive a formula showing the position of the center of gravity of a circular arc whose central angle is 180°. Place the x- and y-axes as in Fig. 300. Solve by integration, from Eq. 117. Check by substituting in the formula derived in Prob. 487. *Ans.* $\bar{x} = 2r/\pi$.

490. Place a 90° circular arc in the first quadrant formed by the x- and y-axes, and derive formulas for $\bar{x}$ and $\bar{y}$. Solve by integration, from Eq. 117. Check by using the answer obtained for Prob. 488. Also, compare the result with that of Prob. 489, and explain why the two results are the same. *Ans.* $\bar{x} = \bar{y} = 2r/\pi$.

491. Solve Prob. 487, using rectangular instead of polar coordinates.

97. Derivation of Special Formulas for Areas. The position of the center of gravity of an area is a matter of great importance in the solution of many engineering problems, particularly in those involving strength of materials and hydraulics.

Illustrative Problems

492. Derive a formula showing the position of the center of gravity of any triangular area.

Solution. Figure 301 shows a triangular area of a general type. The most useful formula for practical purposes is one that gives the perpendicular distance from the center of gravity to any side of the triangle. Let b represent the length of one of the sides, and let h represent the altitude of the triangle, measured from that side as a base.

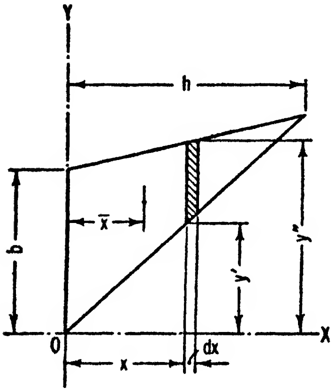

Fig. 301

$$\bar{x} = \frac{\int x\,dA}{A}$$

From Fig. 301,

$$dA = (y'' - y')\,dx \qquad \text{and} \qquad A = \frac{bh}{2}$$

From similar triangles,

$$\frac{y'' - y'}{h - x} = \frac{b}{h} \qquad y'' - y' = \frac{b}{h}(h - x)$$

Therefore,

$$\bar{x} = \frac{\int_0^h x\frac{b}{h}(h-x)\,dx}{bh/2} = \frac{2}{h^2}\int_0^h (hx - x^2)\,dx = \frac{2}{h^2}\left[\frac{hx^2}{2} - \frac{x^3}{3}\right]_0^h = \frac{h}{3}$$

Thus, it is seen that the distance from any side of a triangle to the center of gravity depends only on the altitude measured from that side, and is independent of the shape of the triangle.

The center of gravity of a triangle lies on any median. This follows from the fact that the moment of each element of area, Fig. 301, with respect to the median drawn to the side b, is equal to zero.

493. Derive a formula showing the position of the center of gravity of any circular sector.

Solution. Figure 302 shows a circular sector having any central angle, 2β. Let the radius of the sector be represented by r.

$$\bar{x} = \frac{\int x\,dA}{A}$$

From Fig. 302,

$$x = \rho \cos\theta; \quad dA = \rho\, d\rho\, d\theta \quad \text{and} \quad A = r^2\beta$$

$$\bar{x} = \frac{\int_0^r \int_{-\beta}^{+\beta} \rho\cos\theta\,\rho\,d\rho\,d\theta}{r^2\beta} = \frac{2\int_0^r \int_0^{+\beta} \rho^2\,d\rho\cos\theta\,d\theta}{r^2\beta} = \frac{2\sin\beta \int_0^r \rho^2\,d\rho}{r^2\beta}$$

$$= \frac{2\sin\beta\,\frac{r^3}{3}}{r^2\beta} = \frac{2}{3}\,\frac{r\sin\beta}{\beta}$$

From the symmetry of the figure (Art. 94), $\bar{y} = 0$.

494. Derive a formula showing the position of the center of gravity of any circular sector, by means of polar coordinates as in Prob. 493, but by the use of a single integration only.

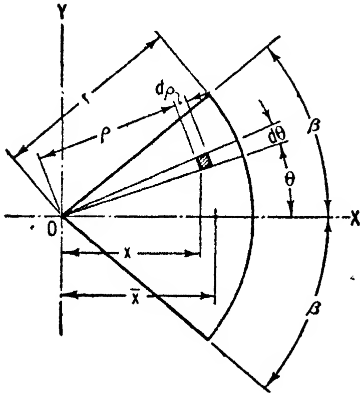

FIG. 302

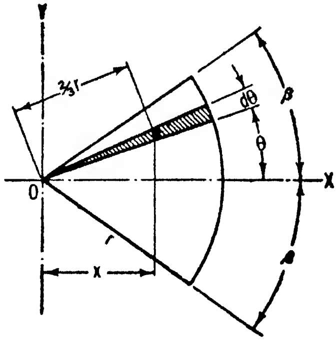

FIG. 303

Solution. Figure 303 shows a circular sector having any central angle, 2β Select an element of triangular form, as shown. This makes it possible to solve by means of a single integration. However, since the dimension of the element measured parallel to the x-axis is finite, it is necessary to know the position of its center of gravity, as was stated in Art. 93. Referring to Prob. 492, it follows that the distance from O to the center of gravity of the element is $\frac{2}{3}r$.

$$\bar{x} = \frac{\int x\,dA}{A} \qquad x = \tfrac{2}{3}r\cos\theta \qquad dA = \tfrac{1}{2}(r\,d\theta)\,r \qquad A = r^2\beta$$

$$\bar{x} = \frac{\frac{r^3}{3}\int_{-\beta}^{+\beta} \cos\theta\,d\theta}{r^2\beta} = \frac{\frac{2r^3}{3}\int_0^{+\beta} \cos\theta\,d\theta}{r^2\beta} = \frac{2r}{3\beta}\Big[\sin\theta\Big]_0^{+\beta} = \frac{2}{3}\,\frac{r\sin\beta}{\beta}$$

From the symmetry of the figure (Art. 94), $\bar{y} = 0$.

PROBLEMS

495. Derive a special formula showing the position of the center of gravity of a circular sector whose central angle is 90°, placing the axes as in Fig. 302. Solve by integration, from Eq. 118, and then check by substitution in the formula obtained in Prob. 493. *Ans.* $\bar{x} = (4\sqrt{2})r/3\pi$.

496. Derive a special formula showing the position of the center of gravity of a semicircular area. Solve by integration, from Eq. 118, and then check by substitution in the formula obtained in Prob. 493. *Ans.* $\bar{x} = 4r/3\pi$.

497. Derive a formula giving the distance from the center of gravity of a 90° circular sector to one of the bounding radii. Solve by using the answer obtained in Prob. 495, and check by an examination of Prob. 496. Explain this agreement in results. *Ans.* $\bar{x} = 4r/3\pi$.

498. The equation of a semi-cubical parabola is $ay^2 = x^3$. Derive formulas for locating the center of gravity of the area bounded by this curve and by the line whose equation is $x = b$. *Ans.* $\bar{x} = \frac{5}{7}b$; $\bar{y} = 0$.

499. Derive formulas showing the position of the center of gravity of the area enclosed within the parabola whose equation is $y^2 = ax$, the x-axis, and the line whose equation is $x = b$. *Ans.* $\bar{x} = \frac{3}{5}b$; $\bar{y} = \frac{3}{8}\sqrt{ab}$.

500. Derive formulas showing the position of the center of gravity of that quarter of an elliptical area lying in the first quadrant. The equation of the ellipse is $x^2b^2 + y^2a^2 = a^2b^2$. *Ans.* $\bar{x} = 4a/3\pi$; $\bar{y} = 4b/3\pi$.

501. The equation of a cubical parabola is $a^2y = x^3$. Derive formulas for locating the center of gravity of the area bounded by the upper half of this curve, by the x-axis, and by the line whose equation is $x = b$. *Ans.* $\bar{x} = \frac{4}{5}b$; $\bar{y} = 2b^3/7a^2$.

502. Derive formulas for locating the center of gravity of the area under the first arch of the sine curve, $y = \sin x$. *Ans.* $\bar{x} = \pi/2$; $\bar{y} = \pi/8$.

503. Derive formulas for locating the center of gravity of the shaded area in Fig. 304. The curved boundary is the parabola whose equation is $y^2 = ax$. *Ans.* $\bar{x} = \frac{3}{10}b$; $\bar{y} = \frac{3}{4}\sqrt{ab}$.

504. Derive a formula for locating the center of gravity of the surface area of a right circular cone, exclusive of the base. *Ans.* $\bar{y} = \dfrac{h}{3}$.

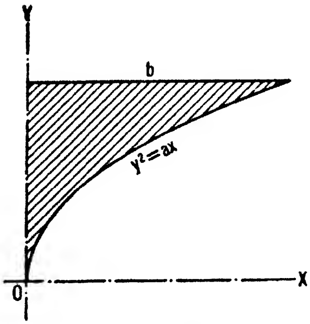

FIG. 304

505. Derive formulas for locating the center of gravity of the curved surface of a hemisphere. *Ans.* $\bar{y} = \dfrac{r}{2}$.

506. Solve Prob. 493, using rectangular instead of polar coordinates.

98. Derivation of Special Formulas for Volumes. In engineering, the importance of formulas locating the centers of gravity of volumes rests chiefly on the fact that such formulas can be used in locating the centers of gravity of homogeneous bodies having the same geometrical forms as the given volumes.

ILLUSTRATIVE PROBLEMS

507. Derive a formula showing the position of the center of gravity of any cone or pyramid.

Solution. Let Fig. 305 represent a side elevation of any cone or pyramid. Use a lamina, parallel to the base, as the element of volume. Let a represent the area of either of the plane faces of the lamina. Let A represent the area of the base of the cone or pyramid. By Eq. 119,

$$\bar{x} = \frac{\int x\,dV}{V}$$

From solid geometry it is learned that parallel sections of a cone or pyramid are to each other as the squares of their distances from the vertex. Therefore,

$$\frac{a}{A} = \frac{(h-x)^2}{h^2} \qquad a = \frac{A}{h^2}(h-x)^2 \qquad dV = a\,dx = \frac{A}{h^2}(h-x)^2\,dx$$

$$\bar{x} = \frac{\int_0^h x\frac{A}{h^2}(h-x)^2\,dx}{Ah/3} = \frac{3}{h^3}\int_0^h (h^2x - 2hx^2 + x^3)\,dx$$

$$= \frac{3}{h^3}\left[\frac{h^2x^2}{2} - \frac{2hx^3}{3} + \frac{x^4}{4}\right]_0^h = \frac{h}{4}$$

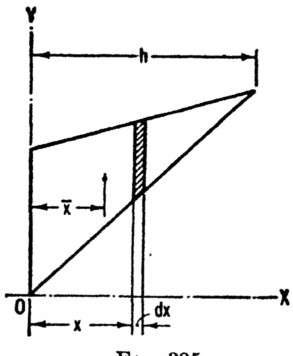

FIG. 305

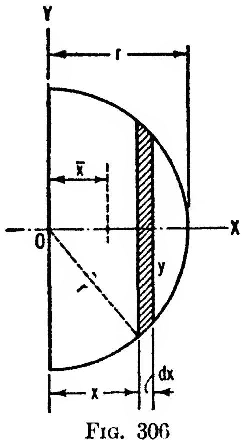

FIG. 306

The center of gravity of a cone or pyramid also lies on a line drawn through the vertex, and through the center of gravity of the base. This is clear from the fact that such a line passes through the center of gravity of each laminar element, and that the first moment of each lamina with respect to the line is equal to zero (Art. 95).

508. Derive a formula showing the position of the center of gravity of a hemisphere.

Solution. Let Fig. 306 represent a side view of a hemisphere. Use a lamina, parallel to the base, as the element of volume. Let r represent the radius of the hemisphere. Let a represent the area of either of the plane faces of the lamina.

$$\bar{x} = \frac{\int x\,dV}{V}$$

From the figure,

$$dV = a\,dx = \pi(r^2 - x^2)\,dx$$

$$\bar{x} = \frac{\int_0^r x\pi(r^2 - x^2)\,dx}{2\pi r^3/3} = \frac{\pi\left[\int_0^r r^2x\,dx - \int_0^r x^3\,dx\right]}{2\pi r^3/3}$$

$$= \frac{\left[r^2x^2/2\right]_0^r - \left[x^4/4\right]_0^r}{2r^3/3} = \frac{3}{8}r$$

From symmetry (Art. 94), it is obvious that $\bar{y} = 0$.

PROBLEMS

509. The equation of the semi-cubical parabola is $ay^2 = x^3$. Let the area enclosed between such a curve and the line whose equation is $x = b$ be revolved about the x-axis, through an angle of 180°. Derive formulas for locating the center of gravity of the volume thus generated. *Ans.* $\bar{x} = \frac{4}{5}b$; $\bar{y} = 0$.

510. Let the area bounded by the parabola $y^2 = ax$, the x-axis, and the straight line whose equation is $x = b$ be revolved about the x-axis, through an angle of 360°. Derive formulas for locating the center of gravity of the volume thus generated. *Ans.* $\bar{x} = \frac{2}{3}b$; $\bar{y} = 0$.

511. Let the shaded area in Fig. 304 be revolved about the x-axis, through an angle of 360°. Derive formulas for locating the center of gravity of the volume thus generated. *Ans.* $\bar{x} = \frac{1}{3}b$; $\bar{y} = 0$.

512. Let a quarter of an elliptical area be revolved about the y-axis, through an angle of 360°. The equation of the ellipse is $x^2b^2 + y^2a^2 = a^2b^2$. Derive formulas for locating the center of gravity of the volume thus generated. *Ans.* $\bar{x} = 0$; $\bar{y} = \frac{3}{8}b$.

513. Let the shaded area in Fig. 304 be revolved about the y-axis, through an angle of 360°. Derive formulas for locating the center of gravity of the volume thus generated. *Ans.* $\bar{x} = 0$; $\bar{y} = \frac{5}{6}\sqrt{ab}$.

99. First Moments by Finite Summation. *The first moment of an entire body, line, area, or volume is equal to the algebraic sum of the first moments of its parts.* This fact is of great usefulness in locating centers of gravity.

For example, a line frequently can be divided into finite parts in such a manner that the center of gravity of each part can be located either by inspection or by means of a special formula. The first moment of each part can then be calculated and the results added algebraically to give the first moment of the entire line. The first moment of the line also can be expressed as $L\bar{x}$, $L\bar{y}$, or $L\bar{z}$ (Art. 95). The two expressions for the first moment then can be equated and the equation solved for $\bar{x}$, $\bar{y}$, or $\bar{z}$.

In some cases it can be seen that the arbitrary addition of one or more portions to the figure will produce a new figure whose first moment can be calculated more easily than that of the original figure. Then the first moments of the added portions are calculated separately and subtracted from the first moment of the altered figure. This process gives the first moment of the original figure. The proofs of the foregoing statements are very simple and will be omitted.

The methods suggested above usually render it possible to avoid entirely the use of integration.

100. Concerning Subsequent Problems. The remaining problems in this chapter are designed primarily to illustrate the use of the various special formulas in numerical problems and, in some cases, in the derivation of other useful special formulas. It will be understood, unless otherwise specified, that in the solution of these problems any formulas or results obtained in previous discussions or problems may be utilized.

101. Applications of the Special Formulas for Lines. As has been indicated, the engineer is seldom concerned with the position of the center of gravity of a line, as such. However, he sometimes utilizes formulas pertaining to lines in finding the approximate positions of the centers of gravity of slender rods, wires, and similar objects.

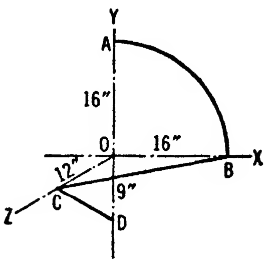

FIG. 307

ILLUSTRATIVE PROBLEM

514. Figure 307 represents the center line of a uniform, slender rod bent into the form $ABCD$, as shown. Calculate approximate values of $\bar{x}$, $\bar{y}$, and $\bar{z}$ for the rod.

Solution. For the purposes of the solution the figure divides naturally into three finite parts, AB, BC, and CD. Length of $\overline{AB} = 2\pi \times 16/4 = 25.1$ in.; $\overline{BC} = \sqrt{(12)^2 + (16)^2} = 20$ in.; $\overline{CD} = \sqrt{(9)^2 + (12)^2} = 15$ in. Total length $= 60.1$ in.

The coordinates of the centers of gravity of the three parts are as follows:

For AB, using the answer to Prob. 490, Art. 96,

$$\bar{x} = \bar{y} = \frac{2r}{\pi} = \frac{2 \times 16}{\pi} = +10.2 \text{ in.} \qquad \bar{z} = 0$$

For BC, by inspection,

$$\bar{x} = +8 \text{ in.} \qquad \bar{y} = 0 \qquad \bar{z} = +6 \text{ in.}$$

For CD, by inspection,

$$\bar{x} = 0 \qquad \bar{y} = -4.5 \text{ in.} \qquad \bar{z} = +6 \text{ in.}$$

Using the method suggested in Art. 99,

$$60.1\,\bar{x} = 25.1 \times 10.2 + 20 \times 8 + 15 \times 0 \qquad \bar{x} = +6.92 \text{ in.}$$

$$60.1\,\bar{y} = 25.1 \times 10.2 + 20 \times 0 - 15 \times 4.5 \qquad \bar{y} = +3.14 \text{ in.}$$

$$60.1\,\bar{z} = 25.1 \times 0 + 20 \times 6 + 15 \times 6 \qquad \bar{z} = +3.49 \text{ in.}$$

PROBLEMS

515. Locate the center of gravity of the line $AOBC$, shown in Fig. 308. *Ans.* ($-1.04''$, $+2.46''$, $+6.92''$).

516. A uniform, slender rod is bent into the form $OABO$, Fig. 309. Locate the center of gravity.

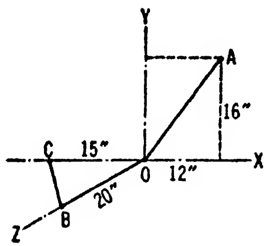

FIG. 308

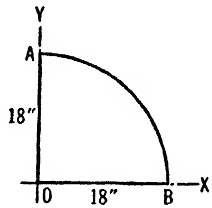

FIG. 309

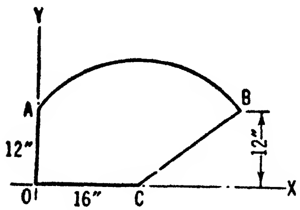

FIG. 310

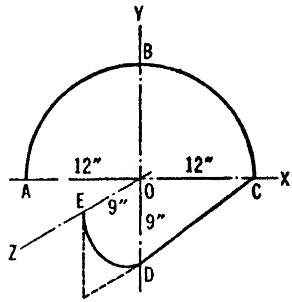

FIG. 311

517. A uniform, slender rod is bent into the form $OABCO$, Fig. 310. The portion AB is a circular arc whose center is at C. Locate the center of gravity. *Ans.* ($+14.1''$, $+9.79''$).

518. A uniform, slender rod is bent into the form $ABCDE$, Fig. 311. The portion ABC is semicircular, and the portion DE is a 90° circular arc. Locate the center of gravity.

102. Applications of the Special Formulas for Areas. The special formulas obtained in the problems of Art. 97 will now be utilized in the solution of various numerical problems. The method of utilizing the special formulas in the derivation of additional special formulas, thus avoiding integration, will also be illustrated.

Illustrative Problems

519. Locate the center of gravity of the shaded area, *OABCDE*, shown in Fig. 312. The portion *OAB*, of the boundary, is a semicircular arc.

Solution. The first moment of the shaded area can be calculated most easily by calculating the moment of the rectangle, *OCFE*, and by subtracting from this the moments of the semicircular area, *OAB*, and the triangular area, *DFE* (Art. 99).

For the rectangle *OCFE*, by inspection,

$$\bar{x} = +12 \text{ in.} \qquad \bar{y} = +8 \text{ in.}$$

For the semicircle *OAB*, utilizing Prob. 496,

$$\bar{x} = +8 \text{ in.} \qquad \bar{y} = \frac{4r}{3\pi} = \frac{4 \times 8}{3\pi} = +3.39 \text{ in.}$$

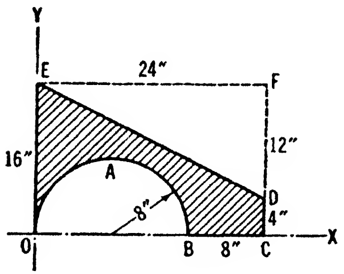

Fig. 312

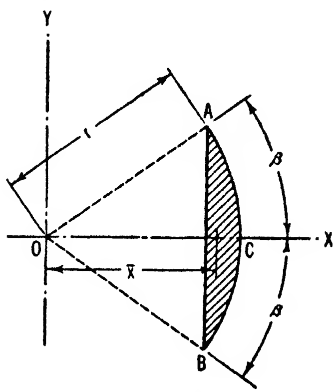

Fig. 313

For the triangle *DFE*, utilizing Prob. 492,

$$\bar{x} = 24 - \frac{h}{3} = 24 - \frac{24}{3} = +16 \text{ in.} \qquad \bar{y} = 16 - \tfrac{12}{3} = +12 \text{ in.}$$

For the shaded area,

$$\bar{x} = \frac{24 \times 16\,(+12) - \frac{1}{2} \times 24 \times 12\,(+16) - \frac{1}{2}\pi \times 8^2\,(+8)}{24 \times 16 - \frac{1}{2} \times 24 \times 12 - \frac{1}{2}\pi \times 8^2} = +10.8 \text{ in.}$$

$$\bar{y} = \frac{24 \times 16\,(+8) - \frac{1}{2} \times 24 \times 12\,(+12) - \frac{1}{2}\pi \times 8^2\,(+3.39)}{24 \times 16 - \frac{1}{2} \times 24 \times 12 - \frac{1}{2}\pi \times 8^2} = +7.19 \text{ in.}$$

520. Derive a formula showing the position of the center of gravity of a segment of a circular area.

Solution. The shaded area in Fig. 313 represents the segment. The problem can be solved most easily by finite summation, advantage being taken of special formulas obtained in Art. 97. The moment of the sector $OACB$ is calculated first, and from this is subtracted the moment of the triangle OAB.

For the sector, utilizing Prob. 493,

$$A = r^2\beta \qquad \bar{x} = \frac{2\,r \sin\beta}{3\,\beta}$$

For the triangle, utilizing Prob. 492,

$$A = r^2 \sin\beta \cos\beta \qquad \bar{x} = \tfrac{2}{3}r\cos\beta$$

For the segment,

$$\bar{x} = \frac{r^2\beta\,\dfrac{2r\sin\beta}{3\,\beta} - (r^2\sin\beta\cos\beta)\,\tfrac{2}{3}r\cos\beta}{r^2\beta - r^2\sin\beta\cos\beta} = \frac{2r\sin\beta\,(1-\cos^2\beta)}{3\,(\beta - \sin\beta\cos\beta)}$$

$$\bar{x} = \frac{4r\sin^3\beta}{3\,(2\beta - \sin 2\beta)}$$

From symmetry (Art. 94),

$$\bar{y} = 0.$$

PROBLEMS

521. Locate the center of gravity of the standard 8 by 4 by 1 in. angle section shown in Fig. 314. *Ans.* (+1.05″, +3.05″).

522. Locate the center of gravity of the standard tee section shown in Fig. 315.

523. Locate the center of gravity of the small channel section shown in Fig. 316. *Ans.* (+0.182″, +1.00″).

524. Locate the center of gravity of the shaded area in Fig. 304, Prob. 503, without using the calculus. Utilize the answer given in Prob. 499. Assume that the shaded area $= \frac{1}{3}b\sqrt{ab}$.

525. Locate the center of gravity of the shaded area shown in Fig. 317. *Ans.* (+6.40″, +4.42″).

526. A certain water tank is cylindrical in form, having a conical cover and a hemispherical bottom. The diameter is 12 ft, and the height of the cylindrical portion is 16 ft. The height of the conical cover is 3 ft. Locate the center of gravity of the empty tank, assuming that it will coincide with the center of gravity of the outer surface. Assume that the edge of the cover does not project beyond the cylindrical surface. *Ans.* $\bar{y} = +12.6$ ft.

527. Figure 318 represents a bin having the form of a trapezoidal prism. The bin has no lid. Locate the center of gravity of the empty bin, assuming that it coincides with the center of gravity of the outer surface. *Ans.* (+2.63′, +4.42′, +6.00′).

528. Derive a formula giving the distance from the center of gravity of a trapezoidal area to the longer base. Represent the length of the longer base by B, the shorter by b, and the distance between the two bases by h. Solve without using the calculus. *Ans.* $\dfrac{h(B+2b)}{3(B+b)}$.

529. Locate the center of gravity of the area enclosed within the line $OABCO$, in Fig. 310, Art. 101.

530. Locate the center of gravity of the area shown in Fig. 319. *Ans.* (+24″, +10.0″).

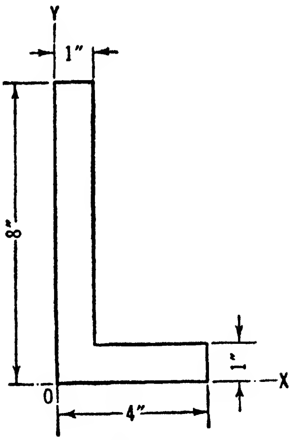

FIG. 314

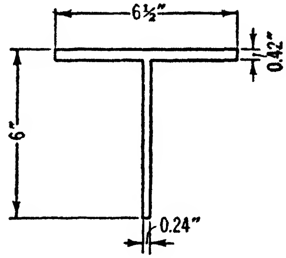

FIG. 315

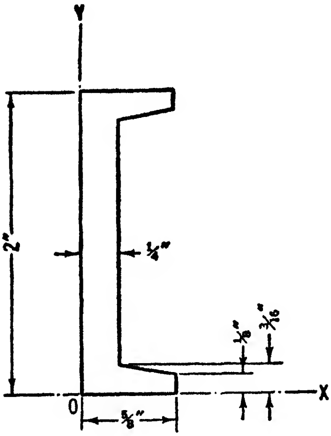

FIG. 316

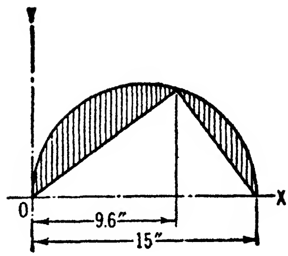

FIG. 317

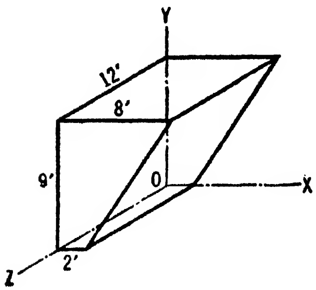

FIG. 318

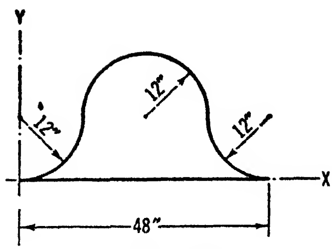

FIG. 319

103. Applications of the Special Formulas for Volumes and Bodies. Formulas locating the centers of gravity of volumes are of interest to the engineer chiefly because of the fact that they enable him to locate the centers of gravity of homogeneous bodies having the same forms as the given volumes.

It may happen that a body, when considered in its entirety, is not homogeneous, but can be divided into finite parts, each of which is homogeneous in itself. In such bodies the center of gravity can be located by the method of finite summation, if the position of the center of gravity of each part is known. The differences in the densities of the various parts are taken into account in the calculation of their weights.

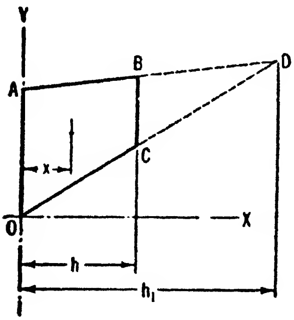

FIG. 320

ILLUSTRATIVE PROBLEMS

531. Derive a formula showing the position of the center of gravity of a frustum of any pyramid or cone.

Solution. Let $OABC$, in Fig. 320, represent a side view of the frustum of any pyramid or cone. Let A represent the area of the larger base of the frustum, and let a represent the area of the smaller base. Let h represent the altitude of the frustum, and let h_1 represent the altitude of the completed cone or pyramid.

The first moment of the frustum is calculated most easily by subtracting the moment of the part BCD from the moment of the completed figure, ADO.

For the frustum, utilizing the results of Prob. 507,

$$\bar{x} = \frac{\dfrac{Ah_1}{3}\dfrac{h_1}{4} - \dfrac{a}{3}(h_1 - h)\left[h + \dfrac{h_1 - h}{4}\right]}{\dfrac{Ah_1}{3} - \dfrac{a(h_1 - h)}{3}}$$

From solid geometry (Prob. 507),

$$\frac{a}{A} = \frac{(h_1 - h)^2}{h_1^2} \qquad h_1 = h\frac{A + \sqrt{Aa}}{A - a}$$

Substitute the foregoing value of h_1 in the equation for $\bar{x}$.

$$\bar{x} = \frac{h}{4}\,\frac{\dfrac{(A + \sqrt{Aa})^2}{A - a} - 2a\dfrac{A + \sqrt{Aa}}{A - a} + 3a}{A + \sqrt{Aa} + a}$$

The foregoing equation can be reduced to

$$\bar{x} = \frac{h}{4}\,\frac{A + 2\sqrt{Aa} + 3a}{A + \sqrt{Aa} + a}$$

The center of gravity of the frustum also lies on a straight line drawn through the centers of gravity of the two bases.

532. Figure 321 represents a homogeneous wooden block, weighing 40 lb per cu ft. A hole 4 in. in diameter is drilled entirely through the block in a direction parallel to the x-axis, the hole being centered in the face $ABCD$. A 4-in. steel pin, weighing 490 lb per cu ft, is driven into the hole, completely filling it. Locate the center of gravity of the body.

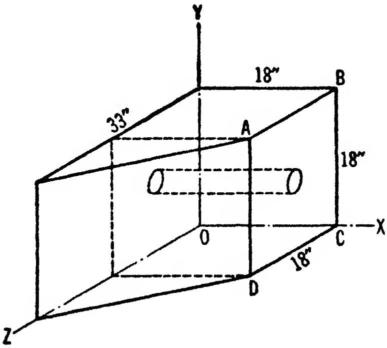

Fig. 321

Solution. Let the block be divided as indicated in the figure, into a cube and a triangular prism. The moment of the body may be found as follows: calculate the moment of the block as it was before the hole was bored; subtract from this the moment of the wooden cylinder which is to be removed; add the moment of the steel pin which is inserted.

$$\text{Weight of solid wood cube} = \frac{18 \times 18 \times 18}{1728}\,40 = 135 \text{ lb}$$

$$\text{Weight of triangular prism} = \frac{\frac{1}{2} \times 18 \times 15 \times 18}{1728}\,40 = 56.3 \text{ lb}$$

$$\text{Weight of wood cylinder} = \frac{\pi \times 4 \times 18}{1728}\,40 = 5.24 \text{ lb}$$

$$\text{Weight of steel cylinder} = \frac{\pi \times 4 \times 18}{1728}\,490 = 64.1 \text{ lb}$$

$$\text{Actual weight of body} = 135 + 56.3 - 5.24 + 64.1 = 250.2 \text{ lb}$$

$$\bar{x} = \frac{135 \times 9 + 56.3 \times 6 - 5.24 \times 9 + 64.1 \times 9}{250.2} = +8.32 \text{ in.}$$

By inspection, $\bar{y} = +9.00$ in.

$$\bar{z} = \frac{135 \times 9 + 56.3 \times 23 - 5.24 \times 9 + 64.1 \times 9}{250.2} = +12.2 \text{ in.}$$

PROBLEMS

533. Locate the center of gravity of the volume shown in Fig. 322. *Ans.* (+3.6″, +1.80″, +5.40″).

534. The frustum of a certain right circular cone is 20 in. high. The diameters of the ends are 8 in. and 16 in. Locate the center of gravity. Solve without reference to Prob. 531; then check the result by means of the formula derived in that problem.

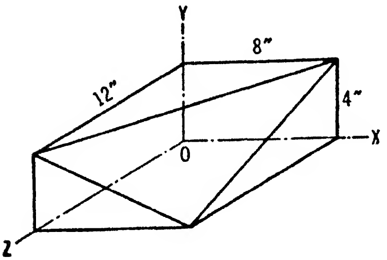

Fig. 322

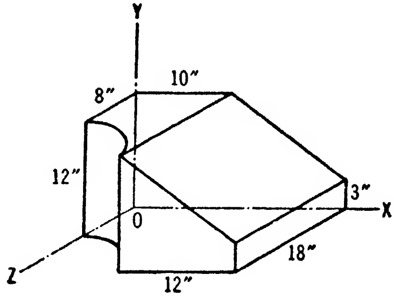

Fig. 323

535. The bin described in Prob. 527, Art. 102, weighs 1000 lb. It is filled with wheat weighing 50 lb per cu ft. Locate the center of gravity of the full bin, utilizing the answers to Prob. 527. *Ans.* (+2.79′, +5.37′, +6.00′).

536. The water tank described in Prob. 526 weighs 8 tons, empty. Locate the center of gravity of the full tank, the water surface being at the top of the cylindrical portion. Water weighs 62.4 lb per cu ft. Utilize the answer to Prob. 526.

537. A certain homogeneous block has the form of a rectangular prism, 12 by 12 by 24 in. In one end is a recess having the form of a right pyramid with a square base, 12 by 12 in. The altitude of the pyramid is 12 in., and its base coincides with the end of the prism. Locate the center of gravity. *Ans.* 10.2 in. from the heavy end.

538. A certain homogeneous right circular cylinder is 20 in. in diameter and 40 in. long. In one end is a recess having the form of a right circular cone, whose base is 20 in. in diameter and whose height is 24 in. The base of the recess coincides with the end of the cylinder. It is desired to fill the recess with homogeneous material such that the center of gravity of the entire body will be 18 in. from the recessed end. Calculate the ratio of the density of the filling material to that of the cylinder.

539. Locate the center of gravity of the homogeneous body shown in Fig. 323. The curved recess shown at one edge has the form of a quarter of a circular cylinder having a radius of 10 in. *Ans.* (+10.8″, +4.96″, +7.43″).

540. A certain solid, right circular cone is 8 in. high, and its base is 4 in. in diameter. The tip, or upper portion, of the cone is made of cast iron, weighing 450 lb per cu ft. The lower part is of aluminum, weighing 165 lb per cu ft. The boundary between the two materials is a plane, 4 in. from the base of the cone, and parallel thereto. Locate the center of gravity of the body.

541. Figure 324 shows side and end elevations of a wooden beam, 12 by 12 by 48 in., to which two steel plates, each 12 by 12 by $\frac{1}{2}$ in., have been fastened. A hole 6 in. in diameter has been bored through the beam and the plates. The wood and the steel weigh 40 lb per cu ft and 490 lb per cu ft, respectively. Locate the center of gravity of the body. *Ans.* 2.2 ft from the left end.

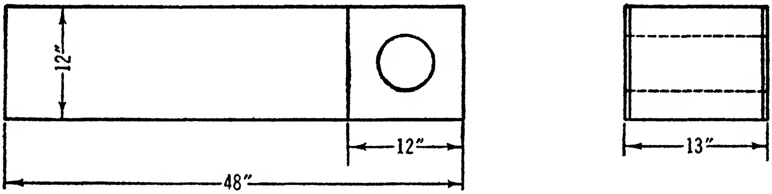

FIG. 324

542. Figure 325 represents a pier having the form of a truncated right pyramid, 20 ft square at the base and 10 ft square at the top. The height is 24 ft. Locate the center of gravity. Solve by dividing the figure into simpler portions and applying the method of finite summation of moments; then check the result by using the answers to Prob. 531.

543. A certain homogeneous body has the form of a hemisphere and a right circular cone, the base of the cone coinciding with the base of the hemisphere. The center of gravity of the entire body coincides with the center of the common base of the hemisphere and cone. Derive a formula showing the relationship between the altitude of the cone and the common diameter of the bases. *Ans.* $h = D\dfrac{\sqrt{3}}{2}$.

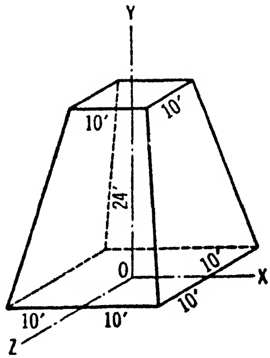

FIG. 325

104. Theorems of Pappus and Guldinus.

The foregoing title is given to two simple theorems that are useful in the solution of a certain type of problem, and in the derivation of certain formulas.

For Areas. The area of a surface generated by revolving a plane curve about any axis in its plane is equal to the product of the length of the generating curve and the distance described by the center of gravity of that curve, provided that the generating curve lies entirely on one side of the axis of revolution.

Proof. Let BC, in Fig. 326, represent any plane curve, which is to be rotated, through any desired angle, about the y-axis. Let dL represent any element of length of the curve, and let L represent the total length. Let θ, represent the angle through which the curve BC is to be rotated. Let A represent the area of the surface generated. From the calculus,

$$A = \int (x\theta)\, dL = \theta \int x\, dL = \theta \bar{x} L \qquad [120]$$

The foregoing theorem, or Eq. 120, can be used to calculate the area of a surface of revolution when the position of the center of gravity of the generating curve is known; or it can be used to locate the center of gravity of the generating curve when the area of the surface of revolution is known.

If the generating curve does not lie entirely on one side of the axis of revolution, it can be divided into two parts at that axis, and the theorem can then be applied to the two portions separately.

For Volumes. The volume generated by revolving a plane area about any axis in its plane is equal to the product of the generating area and the distance described by the center of gravity of that area, provided that the generating area lies entirely on one side of the axis of revolution.

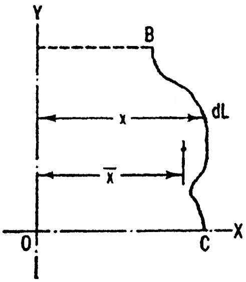

Fig. 326

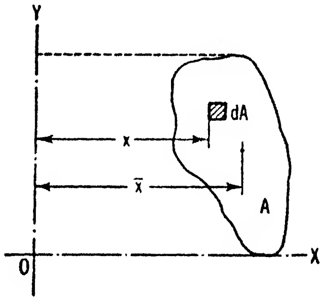

Fig. 327

Proof. Let A, in Fig. 327, represent any plane area, which is to be rotated, through any desired angle, about the y-axis. Let dA represent any element of the area. Let θ represent the angle through which the area is to be rotated. Let V represent the volume of the solid of revolution generated. From the calculus,

$$V = \int (x\theta)\, dA = \theta \int x\, dA = \theta \bar{x} A \qquad [121]$$

The foregoing theorem, or Eq. 121, can be used to calculate the volume of a solid of revolution when the position of the center of gravity of the generating area is known; or it can be used to locate the center of gravity of the generating area when the volume of the solid of revolution is known.

If the generating area does not lie entirely on one side of the axis of revolution, it can be divided into two parts at that axis, and the theorem can then be applied to the two portions separately.

If either Eq. 120 or Eq. 121 is used, the angle θ must be expressed in radians.

Illustrative Problems

544. A casting is made in the form of the solid of revolution generated by revolving the semicircular area shown in Fig. 328 about the y-axis, through an angle of 360°. Calculate the volume of the casting.

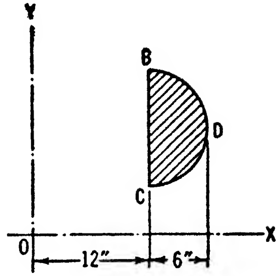

Fig 328

Solution. It will be assumed that the formula for the area of a semicircle is known. $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi 6^2 = 56.55$ sq in. The next step is the calculation of $\bar{x}$ for the generating area. Utilizing the answer given to Prob. 496, Art. 97,

$$\bar{x} = 12 + \frac{4r}{3\pi} = 12 + \frac{4 \times 6}{3\pi} = 12 + 2.55 = 14.55 \text{ in.}$$

The distance described by the center of gravity of the generating area is as follows:

$$2\pi\bar{x} = 2\pi \times 14.55 = 91.43 \text{ in.}$$

And so

$$V = 56.55 \times 91.43 = 5170 \text{ cu in.}$$

545. Calculate the entire surface area of the casting described in Prob. 544.

Solution. The generating line is the boundary of the semicircle in Fig. 328. Utilizing the answer given to Prob. 489, Art. 96, the x-coordinate of the center of gravity of the semicircular arc is found to be $12 + 2r/\pi = 12 + 2 \times 6/\pi = 15.82$ in. The length of the arc is $6\pi = 18.85$ in.

Area generated by the semicircular arc,

$$A = 18.85\ (2\pi\ 15.82) = 1870 \text{ sq in.}$$

Area generated by the straight line, BC,

$$A = 12\ (2\pi\ 12) = 905 \text{ sq in.}$$

Total surface area of casting,

$$A = 1870 + 905 = 2775 \text{ sq in.}$$

PROBLEMS

546. A circular area may be generated by revolving a straight line in the proper manner. Derive the formula for the area of a circle, basing the calculation on the foregoing fact. *Ans.* $A = \pi r^2$.

547. Derive the formula for the area of the curved surface of a right circular cone, in terms of the slant height, s, and the radius of the base, r. *Ans.* $A = \pi rs$.

548. Derive the formula for the area of the surface of a sphere, assuming that the necessary formulas pertaining to the semicircular arc are known. *Ans.* $A = 4\pi r^2$.

549. Derive a formula for the area of the surface generated by revolving a 90° circular arc about a tangent drawn at one of the extremities of the arc, through an angle of 360°. *Ans.* $A = \pi r^2(\pi - 2)$.

550. Derive a formula for the volume of a sphere, by the methods of the present article. *Ans.* $V = \frac{4}{3}\pi r^3$.

551. Assuming that the volume of a right circular cone is given by the formula, $V = \frac{1}{3}\pi r^2 h$, use the methods of the present article to locate the center of gravity of a right triangle.

552. A 45° right triangle, each leg of which is 3 in. long, is rotated about an axis in its own plane, through an angle of 360°. The axis is parallel to one of the legs, and 3 in. distant therefrom, and does not intersect the triangle. Calculate the volume by the methods of the present article. Check the result by ordinary geometrical methods. *Ans.* 113 cu in.

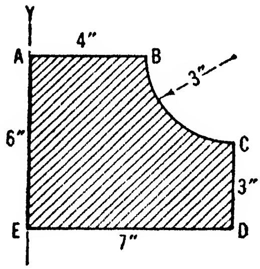

Fig. 329

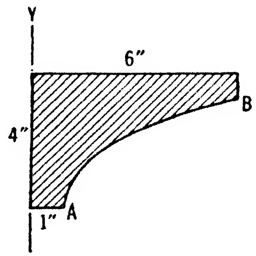

Fig. 330

553. Calculate the area of the surface generated by revolving the line *ABCDE*, in Fig. 329, about the y-axis, through an angle of 360°.

554. Calculate the volume of the solid generated by revolving the shaded area in Fig. 329, about the y-axis, through an angle of 360°. *Ans.* 669 cu in.

555. The area bounded by the parabola $y^2 = ax$, the x-axis, and the straight line $x = b$ is revolved about the x-axis, through an angle of 360°. Calculate the volume of the solid thus generated. *Ans.* $V = \frac{1}{2}\pi ab^2$.

556. Calculate the volume generated by revolving the shaded area shown in Fig. 330, about the y-axis, through an angle of 360°. The curved line AB is a parabola with its origin at A, and having the equation $y^2 = 1.8\,x$. *Ans.* 201 cu in.

557. Locate the center of gravity of the standard angle section in Fig. 314, Art. 102, by the methods of the present article.

558. Locate the center of gravity of the tee section in Fig. 315, Art. 102, by the methods of the present article.

105. Graphic Methods. Sometimes in practice an irregular figure is encountered the relations between whose dimensions cannot be expressed by means of mathematical equations, or can be expressed only by equations of an extremely complex form. To some of these figures approximate algebraic or graphic methods are applicable. Such methods, though of some importance in practice, are of minor interest from the standpoint of fundamental mechanics, and they will not be included in this book.

106. Experimental Methods. Sometimes if the object under consideration is irregular, or extremely complex, experimental methods are resorted to for the purpose of locating the center of gravity. The principles of statics are usually needed in the calculations accompanying these methods.

PROBLEMS

559. In many of the calculations necessary in automotive engineering the position of the center of gravity of a car is of importance. A completely assembled automobile is so complicated as to render any exact mathematical determination of the coordinates of the center of gravity extremely difficult. The determination can be made, with sufficient accuracy, by a combination of experimental and mathematical methods. The following method is recommended by the New Departure Manufacturing Company:

The car is first placed in a level position, with the two wheels on one side resting on a scale platform. Let s_1 represent the reading of the scales in this case. Let W represent the total weight of the car, and let b represent the distance between the centers of contact of the front, or rear, wheels. The distance b is commonly called the tread. The standard tread is 56 in., but there are slight variations from this value. It will be assumed that the tread is the same, whether measured at the front or rear wheels. Obviously, the formula for the transverse position of the center of gravity is as follows: $\bar{x} = s_1b/W$. That side of the car which rests on the ground is then elevated to some convenient height, c_1, as shown in Fig. 331. Let s_2 represent the new reading of the scales. The formula for $\bar{y}$, giving the vertical position of the center of gravity when the car is level is as follows:

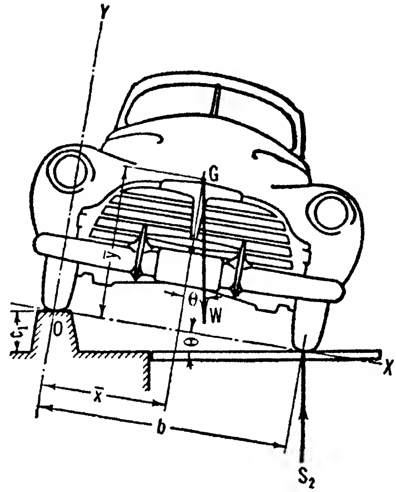

FIG. 331

$$\bar{y} = \frac{b(s_2 - s_1)\sqrt{b^2 - c_1^2}}{c_1W}$$

Derive this formula.

560. The longitudinal position of the center of gravity of an automobile is also of importance. This can be ascertained by resting the two rear wheels of the car on a scale platform, with all four tire contacts at the same elevation. Placing the origin of coordinates at the center of contact of one of the front wheels, with the z-axis parallel to the wheelbase, representing the scale reading by s_3, and the length of the wheelbase by l, derive the formula for $\bar{z}$. Show how the vertical position of the center of gravity obtained by the method of Prob. 559 could now be checked by elevating the front end of the car. Derive the formula for $\bar{y}$ in this case, representing the elevation of the front wheels by c_2, and the new scale reading by s_4.

561. For a given amount of elevation of the wheels, which method for finding $\bar{y}$ would be more accurate; that suggested in Prob. 560, or that suggested in Prob. 559?

562. A class of University of Kansas students performed the experiment described in Prob. 559, using four cars of different manufacture. In one case the following data were obtained: $s_1 = 1475$ lb; $W = 2925$ lb; $b = 54\frac{3}{4}$ in.; $c_1 = 1.006$ ft;

s_2 = 1810 lb. Calculate $\bar{x}$ and $\bar{y}$ for this car. *Ans.* $\bar{x}$ = 27.6 in.; $\bar{y}$ = 27.7 in.

563. In the design of a ship a knowledge of the vertical position of the center of gravity is a matter of extreme importance. It is possible to obtain this information by calculation alone, but the process is long and laborious. After a ship has been launched the position of the center of gravity can be ascertained very accurately and fairly easily by experiment, supplemented by a simple calculation. The information thus obtained can be used in the design of other ships of a similar type, as well as in the operation of the particular ship on which the experiment is performed.

Figure 332 represents a cross section of a ship in the water. A known weight, W', which can be shifted about on the deck, at will, is used in performing the experiment. This "inclining" weight must be of a magnitude such that when it is shifted to the edge of the deck it will cause the ship to assume an appreciable inclination, or heel.

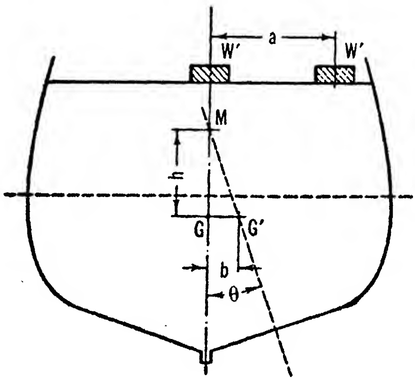

Fig. 332

Let W represent the total weight, or displacement, of the ship, including W'. Let G represent the center of gravity of the total weight when the inclining weight is at the center, and the ship is trim. Let a represent the distance through which the inclining weight is moved transversely to its second position. This transverse movement of W' causes the center of gravity to move to G', through a distance, b. Let the line MG' represent a vertical line drawn through G' when the ship has assumed its inclined position. The point M is called a metacenter. It is at the intersection of the center line of the ship and a vertical line drawn through the center of buoyancy of the ship in the inclined position. The position of the metacenter can be accurately calculated from the known dimensions of the hull and the position of the water line. Therefore, if the distance h can be obtained from the experiment, the vertical position of the center of gravity will then be known. Let θ represent the angle of inclination. The value of $\tan\theta$ is readily found by means of a plumb line, during the experiment. The formula used in obtaining h from the data of the experiment is as follows: $h = \dfrac{W'a}{W\tan\theta}$. Prove that this formula is correct.

PART II. KINEMATICS AND KINETICS

107. General. *Kinetics* is the study of motion. Changes in the motions of bodies are caused by forces. The immediate result of an unbalanced force is acceleration, and the relation between the force and the acceleration is influenced by a third quantity, the weight of the body. Acceleration results in changes in velocity.

There are important relations among the purely geometric quantities, distance, velocity, and acceleration, that can well be studied and fixed in the memory, as a preliminary to the study of the complete kinetic relations. This study of the purely geometric features of motion is called *kinematics.*

In this book the term " point " will be used as in mathematics. The term " particle " will be used to designate an elementary body, or portion of a body, all of whose dimensions are infinitesimals.

It will be noticed that the *kinematics of a point* and the *kinetics of a particle* are discussed before bodies of finite size are considered. There are various reasons why such a method of study is desirable. Certain terms in kinetics are likely to lack clearness unless used solely in connection with points or particles. The term " linear velocity " is an example. At any one instant the linear velocities of various points throughout a moving body may differ widely. Hence the term linear velocity cannot with clearness be applied to the motion of the body as a whole; it should be applied only to individual points in or on the body, unless the case happens to be the special one in which the velocities of all points are alike. Possible confusion in the use of such terms is more readily avoided if the motion of points is given separate attention.

The fundamental laws of motion will be stated and discussed with reference to the single particle. The transition from the laws for the particle to the practical formulas for the finite body will then be made by means of summations and other algebraic processes. It is desirable, therefore, to master the kinetics of the particle thoroughly before attempting to analyze the case of the finite body, with its more complex motion.

CHAPTER X

RECTILINEAR MOTION OF A POINT

108. Linear Velocity. *The linear velocity of a moving point at any instant is the time rate at which the point is traversing distance at that instant.* In many of the problems of mechanics the direction of motion is of importance, and for this reason velocity is treated as a vector quantity. The magnitude of the vector shows the rate of motion, and the inclination and sense of the vector show the direction of motion.

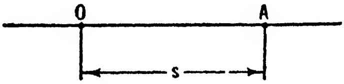

FIG. 333

Let Fig. 333 represent the path of a point in rectilinear motion. Let A represent the moving point. Let O represent a stationary point, placed at any convenient position on the path. Let s represent the distance between A and O, considered positive when the moving point is on one side of O, and negative when it is on the opposite side. Let t represent the time, and let v represent the magnitude of the velocity.

Obviously, the rate at which A is traversing distance is measured by the rate at which s is changing at the given instant. The velocity at any instant, then, is equal to the rate at which s is changing with respect to t. The principles of the calculus teach that the differentiation of s with respect to t gives the rate of change of s with respect to t. The general formula for the magnitude of velocity in rectilinear motion is, therefore,

$$v = \frac{ds}{dt} \qquad [122]$$

The inclination of the velocity is equal to the inclination of the path. The sense of the velocity is shown by the sign obtained for v when Eq. 122 is used in a problem. For example, a minus sign indicates that s and t are changing in the opposite manner. Since t is bound to increase, the minus sign is interpreted as meaning that s is decreasing, algebraically. In other words, a negative velocity means that the point is moving in the negative direction, or toward the negative end of the path.

For the sake of brevity the adjective " linear " is frequently dropped when linear velocities and accelerations are under consideration. The term should be used, however, in any case in which there is danger of confusion between linear and angular quantities. When the terms velocity and acceleration are used without qualification it is understood that the linear types are meant.

Graphic Methods. Equation 122 is commonly used to find the velocity when the algebraic relation between s and t is known. In some cases, simultaneous values of s and t are obtained from instrumental observations in the field or laboratory. Frequently it is difficult to obtain a satisfactory algebraic relation between the quantities. In such cases the points represented by the values can be plotted on coordinate paper, and an average curve drawn. A tangent to the curve can be drawn at any desired point. The slope of this tangent, with proper attention to the scales used in the plotting, will give the velocity at the chosen point. If desired, a velocity-time curve can then be plotted.

The foregoing process is sometimes referred to as graphic, or geometric, differentiation. The reverse process, graphic integration, can be used to obtain the (s, t) curve when the (v, t) curve is given. This process consists in measuring areas under the curve, with proper attention to scales.

Units. The unit of velocity is a combination of the units employed for distance and for time. For example, if s is in feet and t is in seconds, v will be in feet per second. The unit may be abbreviated in various ways.

Illustrative Problems

564. A point moves in a straight line in such a manner that $s = t^{3/2} + 2t - 60$, in which s is expressed in feet and t is expressed in seconds. Calculate the values of s and v for the instant when $t = 9$ sec.

Solution. The value of s for the given instant is obtained by direct substitution in the equation of the motion,

$$s = t^{3/2} + 2t - 60 = \sqrt{(9)^3} + 2 \times 9 - 60 = -15 \text{ ft}$$

The negative sign shows that at the given instant the point is on the negative side of the origin.

$$v = \frac{ds}{dt} = \frac{d(t^{3/2} + 2t - 60)}{dt} = \frac{3}{2}t^{1/2} + 2$$

At the instant when $t = 9$ sec,

$$v = \frac{3}{2}t^{1/2} + 2 = \frac{3}{2}\sqrt{9} + 2 = +6.5 \text{ ft/sec}$$

The positive sign shows that at the given instant the point is moving toward the positive end of the path.

565. A point moves in a straight line in accordance with the law: $v = 3t^2 - 50$, in which v is expressed in feet per second and t is expressed in seconds. It is also known that s has the value $+10$ ft at the instant when t is zero. Calculate the values of v and s for the instant when $t = 3$ sec.

Solution. The value of v for the given instant is obtained by direct substitution in the equation of the motion,

$$v = 3t^2 - 50 = 3 \times (3)^2 - 50 = -23 \text{ ft/sec}$$

The minus sign shows that at the given instant the point is moving toward the negative end of the path.

$$v = \frac{ds}{dt} \qquad ds = v\,dt \qquad ds = (3t^2 - 50)\,dt$$

Integrating,

$$s = t^3 - 50t + C$$

in which C is the constant of integration. The value of this constant must be found before a numerical result can be obtained for s. The problem states that $s = +10$ ft at the instant when $t = 0$. Since the equation for s obtained above is valid for any values of s and t that may occur, the simultaneous values, $s = +10$ and $t = 0$, will satisfy the equation. Substituting,

$$10 = 0 - 0 + C \qquad C = +10$$

The completed equation for s is as follows:

$$s = t^3 - 50t + 10$$

At the instant when $t = 3$ sec,

$$s = (3)^3 - 50 \times 3 + 10 = -113 \text{ ft}$$

The minus sign shows that at the given instant the moving point is on the negative side of the origin.

PROBLEMS

566. A point moves in a straight line in accordance with the law $s = 4t^2 + 8$, in which s is expressed in feet, and t is expressed in seconds. Calculate the velocity of the point for the instant when $t = 20$ sec. *Ans.* $+160$ ft per sec.

567. A point moves in a straight line in accordance with the law $v = \frac{4}{3}t^{1/3} - 2t$, in which v is in feet per minute and t is in minutes. It is also known that $s = +10$ ft at the instant when $t = 0$. Calculate s and v for the instant when $t = 8$ min.

568. A point moves in a straight line in accordance with the law $s = 10 - \sqrt{t^3}$, in which s is in inches and t is in minutes. Calculate the velocity of the point at the instant when $t = 15$ sec. *Ans.* -0.75 in. per min.

569. A point moves in a straight line in accordance with the law $v = 2t - 8$, in which v is in miles per hour and t is in hours. It is also known that $t = 0$ at the instant when $s = 0$. Calculate the values of v and s at the instant when $t = 30$ min. *Ans.* -7 mi per hr; -3.75 mi.

570. A point moves in a straight line in conformity with the law $s = 2t^{2.4} - 10$, in which s is in feet and t is in seconds. Calculate s and v for the instant when $t =$ 15 sec.

571. A point moves in a straight line in accordance with the law $v = 3t^2 - 4t + 6$, in which v is in feet per second and t is in seconds. It is also known that $s = +10$ ft at the instant when $t = 0$. Calculate the value of s for the instant when $t =$ 10 sec. *Ans.* +870 ft.

572. A certain point in rectilinear motion follows the law $v = t^{-1/2} - 6t^{1/2}$, in which v is in feet per second and t is in seconds. It is also known that $s = +8$ ft at the instant when $t = 0$. Calculate v and s for the instant when $t =$ 4 sec.

573. A point moves in a straight line in accordance with the law $s = 10 \cos (2t)$, in which s is in inches and t is in seconds. The angle $(2t)$ is in radians. Calculate the velocity of the point at the instant when $t =$ 4 sec. *Ans.* −19.8 in. per sec.

574. A point moves in a straight line in accordance with the law $s = t^{7/2} - 10t^{3/2} + 4$, in which s is in feet and t is in minutes. Calculate s and v for the instant when $t =$ 4 min.

575. A point moves in a straight line in accordance with the law $v = 2 \cos (3t)$, in which v is in feet per second and t is in seconds. The angle $(3t)$ is in radians. It is also known that $s = 0$ at the instant when $t = 0$. Calculate v and s for the instant when $t =$ 10 sec. *Ans.* +0.312 ft per sec; −0.659 ft.

109. Linear Acceleration. *The linear acceleration of a point at any instant is the time rate at which the linear velocity of the point is changing at that instant.* Linear acceleration is a vector quantity.

Let a represent the linear acceleration of any point moving in a straight line. By the definition, a is equal to the rate at which v is changing with respect to t. The general formula for the magnitude of the acceleration in rectilinear motion is, then, as follows:

$$a = \frac{dv}{dt} \tag{123}$$

From an inspection of Eqs. 123 and 122 it can also be seen that

$$a = \frac{d^2s}{dt^2} \tag{124}$$

In rectilinear motion the inclination of the acceleration is equal to the inclination of the path. In other words, the acceleration is "along" the path. The sense of the acceleration is revealed by the sign obtained for a when either of the foregoing formulas is used in a problem. For example, a minus sign indicates that v is decreasing, algebraically. This does not necessarily mean that the rapidity of motion is decreasing. If the velocity is positive at a given instant, a negative acceleration at that instant means that the point is "slowing down"; but if the velocity is negative, a negative acceleration means that the point is "speeding up," although in either case the velocity is decreasing, algebraically. In short, when the velocity and the acceleration agree in sign, the rapidity of motion is increasing; when they disagree, the rapidity of motion is decreasing.

Thus, the term "acceleration" is correct whether the rapidity of motion is increasing or decreasing. Frequently, however, the term "deceleration" is used in lieu of "acceleration" where the rapidity of motion is decreasing. The term "retardation" is also applied, to a limited extent, under these conditions. In this book the term "acceleration" will be used in its general sense, although the term "deceleration" may be found occasionally in problems in which that particular condition obtains.

The impression is sometimes received that if at a certain instant the velocity becomes zero the acceleration necessarily becomes zero. If the point remains at rest for any finite interval of time the acceleration must be zero during that interval, but if the point remains at rest for only an instant, or, more properly, if it does not remain at rest at all, the acceleration can have any value except zero at the instant of zero velocity. For example, if a ball is thrown vertically into the air its velocity becomes zero for an instant at the upper end of its path, but the acceleration has the same value at that instant as at any other instant during the motion, if air resistance and the slight variation in the earth-pull are disregarded. The value of the velocity simply passes through zero at the given point, its rate of variation at that instant being the same as at any other instant. In some motions the acceleration actually has its maximum value at the instant of zero velocity.

In certain problems the following formula is more convenient than either of those given at the beginning of the article:

$$v\,dv = a\,ds \tag{125}$$

This formula is obtained by eliminating dt between Eq. 122 and Eq. 123.

Graphic Methods. When necessary, values of acceleration can be found by graphic differentiation from a (v, t) curve. If desired, the (a, t) curve can then be plotted. By means of graphic integration, a (v, t) curve can be constructed from an (a, t) curve.

Units. The unit of acceleration is a combination of the units used for velocity and for time. Usually the same unit is used for the time in expressing the rate of change of the velocity as is used in expressing the velocity itself. For example, the value of the acceleration of a freely falling body is usually given as 32.2 ft per sec per sec. This may also be written 32.2 ft/sec^2.

Illustrative Problems

576. A point moves in a straight line in obedience to the law $s = t^{3/2} - 10t - 6$, in which s is in feet and t is in minutes. Calculate the velocity and acceleration of the point for the instant when $t = 4$ min.

Solution. From the problem,

$$s = t^{3/2} - 10t - 6$$

$$v = \frac{ds}{dt} = \frac{d(t^{3/2} - 10t - 6)}{dt} = 1.5t^{1/2} - 10$$

At the instant when $t = 4$ min,

$$v = 1.5\sqrt{4} - 10 = -7 \text{ ft/min}$$

The minus sign shows that at the given instant the point is moving toward the negative end of the path.

$$a = \frac{dv}{dt} = \frac{d(1.5t^{1/2} - 10)}{dt} = 0.75t^{-1/2}$$

At the instant when $t = 4$ min,

$$a = \frac{0.75}{\sqrt{4}} = +0.375 \text{ ft/min}^2$$

Since a is positive and v is negative at the given instant, the speed of the point is decreasing, although v is increasing algebraically. In other words, a, although positive, is a " deceleration " at the given instant.

577. A point moves in a straight line in accordance with the law $a = 12t - 3.75t^{1/2}$, in which a is in feet per second per second and t is in seconds. It is also known that $v = 0$ and $s = +12$ ft at the instant when $t = 0$. Calculate a, v, and s for the instant when $t = 9$ sec.

Solution. From the problem,

$$a = 12t - 3.75t^{1/2}$$

At the instant when $t = 9$ sec,

$$a = 12 \times 9 - 3.75\sqrt{9} = +96.8 \text{ ft/sec}^2$$

$$a = \frac{dv}{dt} \qquad dv = a\,dt \qquad dv = (12t - 3.75t^{1/2})\,dt$$

Integrating,

$$v = 6t^2 - 2.5t^{3/2} + C_1$$

in which C_1 represents the constant of integration. From the problem, $v = 0$ when $t = 0$. This simultaneous pair of values must satisfy the equation for v obtained above. Substituting,

$$0 = 0 - 0 + C_1 \qquad C_1 = 0$$

Therefore,

$$v = 6t^2 - 2.5t^{3/2}$$

At the instant when $t = 9$ sec,

$$v = 6 \times 81 - 2.5\sqrt{(9)^3} = +419 \text{ ft/sec}$$

In this problem the signs of the velocity and of the acceleration agree at the given instant; therefore, the speed of the point is increasing.

$$v = \frac{ds}{dt} \qquad ds = v\,dt \qquad ds = (6t^2 - 2.5t^{3/2})\,dt$$

Integrating,

$$s = 2t^3 - t^{5/2} + C_2$$

From the problem, $s = +12$ ft at the instant when $t = 0$. Substituting these in the foregoing equation,

$$12 = 0 - 0 + C_2 \qquad C_2 = +12$$

The completed equation is

$$s = 2t^3 - t^{5/2} + 12$$

At the instant when $t = 9$ sec,

$$s = 2 \times (9)^3 - \sqrt{(9)^5} + 12 = +1230 \text{ ft}$$

Summarizing: at the given instant the point is 1230 ft from the origin and on the positive side thereof, moving in the positive direction at a speed of 419 ft per sec, and with a positive acceleration of 96.8 ft per sec per sec.

PROBLEMS

578. At a certain instant the linear acceleration of a moving point is 70 mi per hr per min. Express this acceleration in feet per second per second. *Ans.* 1.71 ft/sec^2.

579. A certain table contains a large number of accelerations expressed in miles per hour per second. It is desired to calculate a second table, giving the same accelerations, but expressed in feet per minute per minute. Calculate the coefficient by which the values in the first table should be multiplied.

580. A certain table contains a large number of accelerations expressed in feet per second per second. It is desired to calculate a second table, giving the same accelerations, but expressed in inches per minute per second. Calculate the coefficient by which the values in the first table should be multiplied. *Ans.* 720.

581. A point moves in a straight line in accordance with the law $s = 4 - 2t - 3t^2$, in which s is in feet and t is in seconds. Calculate v and a for the instant when $t =$ 5 sec. Is the point "speeding up" or "slowing down" at the given instant? *Ans.* -32 ft/sec; -6 ft/sec^2; speeding up.

582. A point moves in a straight line in obedience to the law $a = -3\sqrt{t}$, in which a is in feet per second per second and t is in seconds. It is known, also, that $v = -4$ ft per sec and $s = 0$ at the instant when $t = 0$. Calculate v and s for the instant when $t = 4$ sec.

583. A point moves in a straight line in accordance with the law $v = \frac{4}{3}\sqrt[3]{t} - 2t$, in which v is in feet per second and t is in seconds. It is also known that $s = +10$ ft at the instant when $t = 0$. Calculate s, v, and a for the instant when $t = 8$ sec. Is the point "speeding up" or "slowing down" at that instant? *Ans.* -38.0 ft; -13.3 ft/sec; -1.89 ft/sec^2.

584. A point moves in a straight line in accordance with the law $a = 3.75\sqrt{t} - 3/\sqrt{t}$, in which a is expressed in feet per second per second, and t is in seconds. It is also known that $v = 0$ and $s = +2$ ft at the instant when $t = 0$. Calculate s, v, and a for the instant when $t = 4$ sec.

585. A certain rectilinear motion is governed by the following law: $v = t^{1.6} - 10$, in which v is in feet per minute and t is in minutes. It is also known that $s = 0$ at the instant when $t = 0$. Calculate s, v, and a for the instant when $t = 5$ min. Is the point "speeding up" or "slowing down" at the given instant? *Ans.* -24.7 ft; $+3.13$ ft/min; $+4.20$ ft/min^2.

586. A point moves in a straight line in accordance with the law $a = -4\,s$, in which a is in feet per second per second and s is in feet. It is also known that $v = +2$ ft per sec and $s = 0$ at the instant when $t = 0$. Derive the formula $s = \sin(2t + n\pi)$, in which n is any positive integer. Start by using Eq. 125.

587. A certain rectilinear motion is governed by the following law: $a = 0.75/\sqrt[3]{s}$, in which a is in feet per second per second and s is in feet. It is also known that s, v, and t have the value zero at the same instant. Derive the equation $a = \frac{3}{4}t^{-\frac{1}{2}}$.

110. Uniform Motion. *Uniform rectilinear motion* is motion in a straight line at constant velocity. It should be remembered that velocity is a vector quantity. The term "constant," when applied to a vector quantity, means that there is no change in magnitude, inclination, or sense. Thus, it is seen that *rectilinear motion is the only type of motion in which the linear velocity of a point can remain constant.* In curvilinear motion the magnitude of the velocity may remain constant, but the inclination necessarily changes.

It follows, from the definition, that *in a uniform rectilinear motion the acceleration is zero.* Equation 122, Art. 108, is valid for any case.

$$v = \frac{ds}{dt} \qquad ds = v\,dt$$

Integrating, and remembering that in the present case v is constant,

$$s = vt + C \tag{126}$$

The value of the constant of integration, C, depends upon the position of the moving point in its path at the instant when $t = 0$. In a specific problem it is usually possible and convenient to measure t from the instant when the moving point is at the origin. Adopting this procedure, it follows that $s = 0$ when $t = 0$. Substituting these simultaneous values in Eq. 126,

$$0 = 0 + C \qquad C = 0$$

Therefore,

$$s = vt \tag{127}$$

PROBLEMS

588. Express a velocity of 60 mi per hr in feet per second. From the foregoing result, express velocities of 15, 20, 45, and 50 mi per hr in feet per second, by proportion.

589. In nautical parlance the knot is the unit of velocity for expressing the speed of a ship. One knot is equivalent to one nautical mile per hour. The length of the nautical mile is 6080.27 ft. A certain vessel has a speed of 30 knots. Express this speed in ordinary statute miles per hour. *Ans.* 34.5 mi/hr.

590. A certain automobile maintains an average speed of 40 ft per sec between two points 265 mi apart. How many hours will the journey consume?

591. The average speed of the pistons in the cylinders of an engine is called the "piston speed." It is usually expressed in feet per minute. Calculate the piston speed of an automobile engine having a $4\frac{1}{2}$-in. stroke, when the crankshaft is rotating at the rate of 3400 rpm. *Ans.* 2550 ft/min.

592. A certain army in marching order is 15 mi long. It travels at constant speed. A messenger starts at the rear end of the line, rides to the head, and returns to the rear. The army meanwhile moves forward 15 mi. The messenger rides at constant speed. Calculate the actual distance traveled by the messenger. Disregard the time lost in turning at the head of the line. *Ans.* 36.2 mi.

593. Car A, traveling at a constant speed of 70 mi per hr, overtakes and passes car B, which is traveling at 60 mi per hr. Car C approaches, traveling at 70 mi per hr in the opposite direction. Car A turns out at the instant when the clear space between it and B is 100 ft. At the instant when A returns to its own lane, ahead of B, the clear space between A and B is again 100 ft. At the same instant the front end of car C is just even with the rear end of A. Each car has a total length of 18 ft. Calculate the clear distance between cars A and C at the beginning of the maneuver. Disregard the slight additional distance traveled by car A owing to its deviation from a straight path.

111. Motion with Constant Acceleration. In this type of rectilinear motion the velocity varies, but the acceleration is constant. It follows that equal increments of velocity occur during equal increments of time. This motion is also frequently called *uniformly accelerated motion.*

From Eq. 123, Art. 109,

$$a = \frac{dv}{dt} \qquad dv = a\,dt$$

Integrating, and remembering that a is constant,

$$v = at + C_1 \tag{128}$$

Let v_0 represent the velocity of the point at the instant when $t = 0$. This is called the initial velocity. Substituting the simultaneous values, $v = v_0$ and $t = 0$, in Eq. 128,

$$v_0 = 0 + C_1 \qquad C_1 = v_0$$

Therefore,

$$\boldsymbol{v = v_0 + at} \tag{129}$$

From Eq. 122, Art. 108,

$$v = \frac{ds}{dt} \qquad ds = v\,dt$$

Substituting in this equation the value of v from Eq. 129, and integrating,

$$ds = (v_0 + at)\,dt \qquad s = v_0t + \tfrac{1}{2}at^2 + C_2 \tag{130}$$

The value of the constant of integration, C_2, depends upon the position of the moving point in its path at the instant when $t = 0$. Let t be measured from the instant when the moving point is at the origin. Consequently, $s = 0$ when $t = 0$. Substituting in Eq. 130,

$$0 = 0 + 0 + C_2 \qquad C_2 = 0$$

Therefore,

$$s = v_0t + \tfrac{1}{2}at^2 \tag{131}$$

From Eq. 125, Art. 109,

$$v\,dv = a\,ds$$

Integrating, and remembering that a is constant,

$$\frac{v^2}{2} = as + C_3 \tag{132}$$

The initial conditions adopted in deriving Eqs. 129 and 131 were: $v = v_0$ when $t = 0$, and $s = 0$ when $t = 0$. Therefore, $v = v_0$ when $s = 0$. Substituting in Eq. 132,

$$C_3 = \frac{v_0^2}{2}$$

Substituting this value of C_3 in Eq. 132, and rearranging,

$$v = \sqrt{v_0^2 + 2as} \tag{133}$$

Another useful formula can be obtained by eliminating a between Eqs, 129 and 133; it is as follows:

$$s = \frac{v_0 + v}{2}\,t \tag{134}$$

Illustrative Problems

594. A point moves in a horizontal straight line. The initial velocity is 100 ft per sec, toward the right. Two minutes later the velocity is 200 ft per sec, toward the left. The acceleration is constant. Calculate the acceleration, and find the position of the point at the end of the interval.

Solution. By Eq. 129,

$$v = v_0 + at$$

From the problem, with due regard for signs and units, $v = -200$, $v_0 = +100$, and $t = 120$. Substituting,

$$-200 = +100 + a \times 120 \qquad a = -2.5\text{ ft/sec}^2$$

From the statement of the problem it is clear that the acceleration is toward the left. However, the minus sign obtained above verifies this conclusion. By Eq. 131,

$$s = v_0 t + \tfrac{1}{2}at^2$$

Substituting,

$$s = 100 \times 120 + \tfrac{1}{2}(-2.5)(120)^2 = -6000 \text{ ft}$$

Therefore, at the final instant the moving point is 6000 ft to the left of the origin. It is suggested that the student check the solution by the use of Eq. 133 or 134.

595. A ball is thrown vertically upward from the top of a tower, with an initial velocity of 80 ft per sec. The velocity of the ball is 120 ft per sec at the instant when it reaches the ground. Calculate the height of the tower.

Solution. Equation 133 provides the most direct solution.

$$v = \sqrt{v_0^2 + 2as}$$

In the present case $v = -120$, $v_0 = +80$, and $a = -32.2$. Substituting and squaring,

$$(-120)^2 = (+80)^2 + 2(-32.2)s \qquad s = -124 \text{ ft}$$

The use of Eq. 133 necessitates placing the origin of coordinates at the point of initial velocity, v_0, which in the present case is at the top of the tower. s is the coordinate of the moving point. The value actually obtained above is the coordinate of the bottom of the tower, measured from the origin at the top. This accounts for the negative sign.

596. A point, A, starts from rest and moves in a horizontal straight line, with a constant acceleration of 2 ft per sec per sec, toward the right. A second point, B, moves along the same line, in the same direction, with a constant velocity of 20 ft per sec. B passes A's starting point 4 sec after A departs. How much time will elapse, after A starts, until the two points coincide? How far from the starting point will the conjunction occur?

Solution. Let t_A and t_B represent the elapsed times of the two points. Let s_A and s_B represent the coordinates of the points, measured from an origin at A's starting point.

Point A moves with constant acceleration; therefore, Eq. 131 may be used.

$$s = v_0 t + \tfrac{1}{2}at^2$$

Substituting,

$$s_A = 0 \times t_A + \tfrac{1}{2} \times 2 \times t_A^2 \qquad s_A = t_A^2$$

B moves with constant velocity; therefore,

$$s = vt \qquad s_B = 20t_B$$

When the points coincide, $s_A = s_B$; therefore,

$$t_A^2 = 20t_B$$

And, from the conditions of the problem,

$$t_A = t_B + 4$$

The solution of the two preceding equations yields two values of t_A, as follows: 5.53 sec and 14.5 sec. This outcome means that the two points coincide twice. This was to be expected; B must overtake A if they are to coincide at all, but A will eventually overtake B since B has no acceleration.

Substituting the values of t_A thus found, in the equation for s_A obtained above,

$$s_A = t_A^2 \qquad s_A = (5.53)^2 \quad \text{or} \quad (14.5)^2 = +30.6 \text{ ft} \quad \text{or} \quad +210 \text{ ft}$$

It is suggested that the student check the solution by calculating s_B.

PROBLEMS

597. The building laws of the City of New York, July 17, 1917, contained a provision to the effect that, with certain exceptions, elevators should have a governor or speed regulator of such a nature as to bring the car to an easy and gradual stop in a distance not greater than 8 ft, from a speed of 700 ft per min. Calculate the deceleration, assumed constant, represented by the foregoing figures. *Ans.* 8.51 ft/sec^2.

598. The curve in Fig. 334 shows the distances in which it was claimed, some years ago, that a certain automobile could be brought to rest, on level ground, by the use of the brakes. Assuming constant deceleration, calculate the time consumed in coming to rest from speeds of 20, 40, and 60 mi per hr.

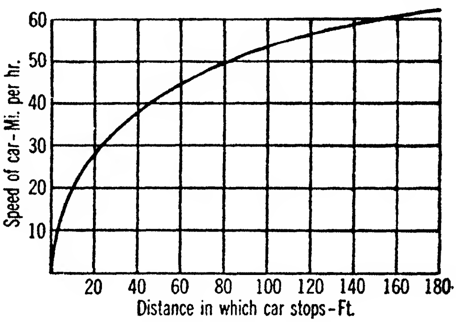

FIG. 334

599. The Uniform Vehicle Code (1938) recommended by the National Conference on Street and Highway Safety specifies that vehicles equipped with brakes on all wheels shall be able to stop in a distance of 30 ft, from a speed of 20 mi per hr, or that they shall be able to decelerate at the rate of 14 ft per sec per sec. Assuming constant deceleration, ascertain whether the two requirements are reasonably consistent.

600. The driver of an automobile traveling at a speed of 80 mi per hr sees danger ahead, applies the brakes, and stops the car. If he is of average quickness the car will travel 88 ft before deceleration begins. It will travel an additional 352 ft in coming to rest. Calculate the total time required for stopping, from the instant when the danger is first discovered. *Ans.* 6.75 sec.

601. A train, moving at a speed of 30 mi per hr, slows down to 20 mi per hr, and then accelerates and resumes its former speed. The deceleration is accomplished at the rate of 1.8 mi per hr per sec, and the rate of acceleration in resuming speed is 0.125 mi per hr per sec. Calculate the amount of time lost because of this change of speed.

602. The average speed that a train makes, based on the total time consumed in running and in stopping at stations, is sometimes called the "schedule speed." A certain electric train has an average acceleration of 2.2 ft per sec per sec, and its average braking retardation is 3 ft per sec per sec. Its maximum constant running speed is 45 mi per hr. Calculate the best schedule speed that the train can maintain if the stations are 1 mi apart and the stops are of 30-sec duration. *Ans.* 26.5 mi/hr.

603. A ball is dropped from a captive balloon at a point 1000 ft above the earth, without initial velocity. At the same instant a ball is thrown vertically upward from the ground. What initial velocity must be given to the second ball, to cause it to meet the first ball at a point 300 ft above the ground?

604. A shell is fired vertically upward with an initial velocity of 2000 ft per sec. It is timed to burst in 5 sec. Two seconds after the firing of the first shell a second shell is fired with the same initial velocity. This shell is timed to burst in 4 sec. An observer stationed in a balloon near the line of fire hears both bursts at the same instant. How far is the balloon above the earth? Assume the velocity of sound to be 1100 ft per sec. *Ans.* 8120 ft.

605. Two points, A and B, move along the same horizontal line. They pass the origin at the same instant, A having a velocity of 10 ft per sec, toward the right, and B having a velocity of 20 ft per sec, toward the left. A has a constant acceleration of 4 ft per sec per sec, toward the left, and B has a constant acceleration of 2 ft per sec per sec, toward the right. When and where will the two points be together again, and what velocities will they have?

606. Observations indicate that a parachutist of average weight whose "chute" failed to open would attain a velocity of approximately 140 ft per sec at the end of the first 6 sec of his fall, and that at the end of the next 7 sec his velocity would be 175 ft per sec, remaining constant thereafter. If he "bails out" at an altitude of 3000 ft, calculate the total time required to reach the earth. Assume constant acceleration in each of the acceleration periods. *Ans.* 21.5 sec.

607. Solve Prob. 606, assuming that the parachutist opens his chute at a distance of 1000 ft above the earth, and that his velocity thereby decreases to 18 ft per sec in a distance of 200 ft and remains constant thereafter. Let the other conditions of the problem remain unchanged.

608. A man standing at the top of a cliff wishes to ascertain its approximate height. He drops a rock over the edge, and hears the sound of the rock striking the ground at the base of the cliff 10 sec after the rock leaves his hand. Calculate the height of the cliff. Assume the velocity of sound to be 1100 ft per sec. *Ans.* 1270 ft.

609. A certain automobile has a top speed of 60 mi per hr. It can attain that speed, from rest, at an average acceleration of 3 ft per sec per sec, and can be stopped in a distance of 250 ft. Assuming constant acceleration during the starting period and also during the stopping period, calculate the minimum time in which the car could be driven between two points $\frac{1}{2}$ mi apart, starting from rest at one point and coming to rest again at the other.

610. One car can accelerate from rest to its top speed of 50 mi per hr, in 25 sec. A second car requires 45 sec to reach its top speed of 70 mi per hr. The two cars start from rest at the same point, the slower car leaving the starting point 40 sec ahead of the faster. In what length of time after the departure of the second car will it overtake the other? How far from the starting point will this occur? *Ans.* 147.5 sec; 12,830 ft.

611. The first car mentioned in Prob. 610 starts from rest at station A, and makes all possible speed toward station B, 10 mi away. At the same instant the other car starts at B and makes all possible speed toward A. How much time elapses before the two cars meet? At what distance from A do they meet?

112. The Crank-and-Connecting-Rod Mechanism. Figure 335 is a diagrammatic representation of the crank-and-connecting-rod mechanism, as used in ordinary steam and gas engines. The point A is on the axis of the crankshaft, and the point B is on the axis of the crankpin. B moves in a circular path, called the crank circle, about A as a center. The point D is on the axis of the crosshead pin, or wristpin. All points on the crosshead, piston rod, or piston have the same motion as D. The distance OE is called the stroke, and is equal to the diameter of the crank circle. The point O is called the outer dead-center position, and the point E is called the inner dead-center position.

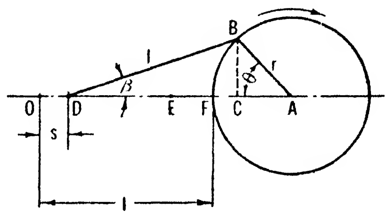

FIG. 335

Formulas will be derived for the motion of D, assuming the crankshaft to be rotating in a clockwise direction at constant speed. The origin will be placed at the outer dead-center position, O. The angle BAC is called the crank angle. Let the crank angle be represented by θ, expressed in radians.

Exact Formulas. Let n represent the number of revolutions of the crankshaft per unit time. Let t have the value zero at the instant when D is at the origin and B is at F. In a unit of time the crank angle will have the value $2\pi n$ radians. This quantity is the angular velocity of the crankshaft. Let it be represented by ω. The relation between θ and t may be expressed, then, by the equation $\theta = \omega t$. From Fig. 335,

$$OF + FA = OD + DC + CA \qquad [135]$$

$$l + r = s + l \cos \beta + r \cos \theta \qquad [136]$$

$$\cos \beta = \sqrt{1 - \sin^2 \beta} = \sqrt{1 - \left(\frac{BC}{l}\right)^2} = \sqrt{1 - \left(\frac{r \sin \theta}{l}\right)^2} \qquad [137]$$

Substituting in Eq. 136 the value of $\cos \beta$ given by Eq. 137, replacing θ by its value ωt, and rearranging, the following relation between s and t is obtained:

$$s = l + r - \sqrt{l^2 - r^2 \sin^2 \omega t} - r \cos \omega t \qquad [138]$$

Equation 122, Art. 108, can now be utilized.

$$v = \frac{ds}{dt} \qquad v = r\omega \left(\sin \omega t + \frac{r \sin 2\,\omega t}{2\,(l^2 - r^2 \sin^2 \omega t)^{1/2}} \right) \qquad [139]$$

By Eq. 123, Art. 109,

$$a = \frac{dv}{dt} \qquad a = r\omega^2\left(\cos\omega t + \frac{rl^2\cos 2\,\omega t + r^3\sin^4\omega t}{(l^2 - r^2\sin^2\omega t)^{3/2}}\right) \qquad [140]$$

Equations 138, 139, and 140 completely describe the motion. They show that the motion is of the type having variable acceleration.

Approximate Formulas. Equations 138, 139, and 140 are exact, but somewhat cumbersome. Simpler, approximate formulas can be obtained in the following manner:

Expanding the radical in Eq. 138 by the binomial theorem, to four terms,

$$(l^2 - r^2\sin^2\omega t)^{1/2} = \left(l - \frac{r^2\sin^2\omega t}{2\,l} - \frac{r^4\sin^4\omega t}{8\,l^3} - \frac{r^6\sin^6\omega t}{16\,l^5}\cdots\right) \qquad [141]$$

In practice the ratio r/l is not likely to exceed 1/2.5. The maximum possible value of $\sin\omega t$ is unity. Substituting these extreme values in Eq. 141,

$$(l^2 - r^2\sin^2\omega t)^{1/2} = \left(l - \frac{r}{5} - \frac{r}{125} - \frac{r}{1562.5}\cdots\right) \qquad [142]$$

An examination of Eq. 142 reveals the fact that the terms of the series rapidly diminish in magnitude, and that no serious error can result if all terms except the first two are discarded. Equation 138 can now be written as follows:

$$s = r + \frac{r^2\sin^2\omega t}{2\,l} - r\cos\omega t \qquad [143]$$

from which, by the use of Eqs. 122 and 123,

$$v = r\omega\left(\frac{r}{2\,l}\sin 2\,\omega t + \sin\omega t\right) \qquad [144]$$

$$a = r\omega^2\left(\frac{r}{l}\cos 2\,\omega t + \cos\omega t\right) \qquad [145]$$

In many of the applications of kinematics, including the one with which the present article is concerned, graphic methods have been devised, to eliminate much of the labor of solution. The algebraic solutions are useful, however, where only a few values are desired. Algebraic solutions are of special importance to the student, in that they serve to fix in the mind the fundamental relations between the quantities, and show the connection between principles and practice.

PROBLEMS

612. In a certain automobile engine the length of the stroke is 4 in., and the length of the connecting rod is 8 in. Calculate the velocity and the acceleration of the piston when the crankshaft is rotating at a constant speed of 3000 rpm, for the instant when the crank angle, θ, is equal to 90°. Use the exact formulas. *Ans.* $v = +52.4$ ft/sec; $a = -4250$ ft/sec^2.

613. Solve Prob. 612 by means of the approximate formulas.

614. Solve Prob. 612 for the instant when the crank angle is equal to 180°, using the exact formulas. *Ans.* $v = 0$; $a = -12{,}300$ ft/sec^2.

615. Solve Prob. 612 for the instant when the crank angle is equal to 180°, using the approximate formulas.

616. Calculate the value of the crank angle at which the piston in Prob. 612 has its maximum velocity. Use the approximate formulas. Calculate the maximum velocity. *Ans.* $\theta = 77°$; $v = +53.9$ ft/sec.

617. Calculate the value of the crank angle at which the piston in Prob. 612 has its maximum acceleration. Calculate the maximum acceleration. Use the approximate formulas.

113. Simple Harmonic Motion. Rectilinear motion in which the acceleration is always directed toward a fixed point in the path, and in which the magnitude of the acceleration is proportional to the distance between the moving point and the fixed point, is called *simple harmonic motion*.

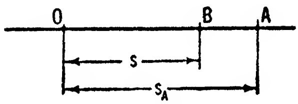

FIG. 336

Let Fig. 336 represent the path of a point having simple harmonic motion. Let B represent the position of the moving point at any instant. O, the fixed point, will be used as the origin from which to measure s. The fundamental facts of the motion, as stated in the definition above, can be expressed in algebraic form, as follows:

$$a = -ks \qquad [146]$$

in which k is a constant in any particular problem. It can be seen that a will be negative for positive values of s, and positive for negative values of s, thus satisfying the requirement that the acceleration always be directed toward O.

Let it be imagined that the motion is started by projecting B through O with the initial velocity v_0. A formula for v in terms of s can be obtained by substituting the value $a = -ks$ in the general formula $v\,dv = a\,ds$, and integrating.

$$\int_{v_0}^{v} v\,dv = \int_0^s -ks\,ds \qquad v = \sqrt{v_0^2 - ks^2} \qquad [147]$$

A formula for s in terms of t can be obtained by substituting, in the general formula $v = ds/dt$, the value of v obtained in Eq. 147, and integrating. It will also be specified that t is to have its zero value at the instant when s is zero and the point is passing through O in the positive direction.

$$\int_0^t dt = \int_0^s \frac{ds}{\sqrt{v_0^2 - ks^2}} \qquad s = \frac{v_0}{\sqrt{k}} \sin \sqrt{k}\, t \tag{148}$$

in which $\sqrt{k}\, t$ represents an angle, expressed in radians.

A formula for v in terms of t can be obtained by eliminating s between Eqs. 147 and 148, or by differentiating Eq. 148 with respect to t. The resulting formula is as follows:

$$v = v_0 \cos \sqrt{k}\, t \tag{149}$$

The following formula for a in terms of t can be obtained in a similar manner:

$$a = -v_0 \sqrt{k} \sin \sqrt{k}\, t \tag{150}$$

An examination of Eq. 147 reveals the fact that, when s has the value $\pm \dfrac{v_0}{\sqrt{k}}$, the value of v is zero. The substitution of the foregoing value of s in Eq. 146 gives $a = \mp v_0\sqrt{k}$. These results show that the point comes to rest at the instant when $s = + \dfrac{v_0}{\sqrt{k}}$, but that the acceleration at this instant is $a = -v_0 \sqrt{k}$. Therefore, the point does not remain at rest, but immediately starts back toward O. It passes through O in the negative direction, with the velocity v_0, and comes to rest again at the instant when $s = - \dfrac{v_0}{\sqrt{k}}$. The point immediately starts back toward O, and repasses O in the positive direction with the velocity v_0.

The foregoing analysis shows that simple harmonic motion consists of a series of vibrations through a common center, O, with a symmetrical arrangement of velocities and accelerations on either side of that center. The distance from the center point, O, to the extreme limit of motion on either side of the center, is called the amplitude of the vibration. Let the amplitude be represented by s_A. The formula for s_A is as follows:

$$s_A = \frac{v_0}{\sqrt{k}} \tag{151}$$

The time required to execute one complete cycle, or vibration, is called the period of vibration. It is the time that elapses between two successive passages through O in the same direction. Let the period of vibration be represented by T. Let Eq. 148 be rearranged, as follows:

$$t = \frac{1}{\sqrt{k}} \text{arc sin} \frac{\sqrt{k}\, s}{v_0} \qquad [152]$$

Substituting $s = 0$ in Eq. 152,

$$t = \frac{1}{\sqrt{k}} \text{arc sin } 0 = \frac{n\pi}{\sqrt{k}} \qquad [153]$$

in which n represents any integer. When n is zero, t is zero. When n is unity, the point has returned to the origin after a half-vibration. When n is 2, the full vibration has been completed. The formula for the period is, therefore,

$$T = \frac{2\,\pi}{\sqrt{k}} \qquad [154]$$

In physics many cases arise in which motions are found to be very nearly simple harmonic. Vibrations of tuning forks, of the strings of musical instruments, of springs, and of air that is transmitting sound waves are examples. Pendulums of various types frequently have simple harmonic rotational motions. In the engineering field are found many examples of bodies that move more or less closely in accordance with the laws of simple harmonic motion. It can easily be shown that the motion of the piston of an engine, as discussed in Art. 112, is approximately simple harmonic and approaches that motion more closely in those cases in which the ratio of the length of the connecting rod to the length of the crank is large. If this ratio were infinite the two motions would agree exactly.

Figure 337 represents a mechanism in which the connecting rod is dispensed with altogether. This device is called the Scotch crosshead. When the crankshaft is rotating at constant speed the crosshead executes an exact simple harmonic motion. This mechanism has been used to a limited extent in pumping machinery, where compactness is of paramount importance.

PROBLEMS

618. Figure 338 represents a block resting on a smooth, horizontal plane, between two helical springs. The block is pulled toward the left a distance of 1 ft, and is then released, from rest. It then executes a simple harmonic motion through O as the central point. The stiffness of the springs, and the weight of the block, are such that the motion follows the law, $a = -96s$, in which a is expressed in feet per second per second, and s is expressed in feet. Calculate the maximum velocity of the block, and the period of vibration. Calculate the maximum acceleration. *Ans.* $v_0 = 9.8$ ft/sec; $T = 0.641$ sec; $a = 96$ ft/sec^2.

619. In Prob. 618, let t be counted from the instant when the block passes O, moving in the positive direction. Calculate the values of s, v, and a, for the instant when $t = 60$ sec.

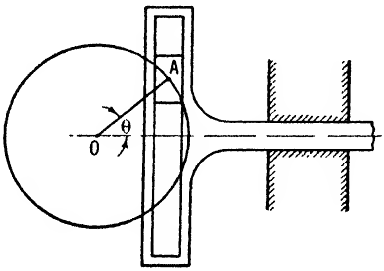

FIG. 337

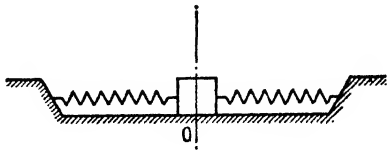

FIG. 338

620. Assume that the motion of the Scotch crosshead, in Fig. 337, is simple harmonic motion, when the crankshaft is rotating at constant speed. If the radius, OA, of the crank circle is 6 in., and the crankshaft has a constant angular velocity of 180 rpm, calculate the maximum velocity and acceleration of the crosshead. *Ans.* $v_0 = 9.43$ ft/sec; $a = 178$ ft/sec^2.

621. Calculate the velocity and acceleration of the Scotch crosshead described in Prob. 620, at the instant when its position is 3 in. from the positive end of its stroke.

622. Calculate the velocity and acceleration of the Scotch crosshead described in Prob. 620, at the instant when the crank angle, θ (Fig. 337), is equal to 150°. *Ans.* $v = -4.71$ ft/sec; $a = +154$ ft/sec^2.

CHAPTER XI

CURVILINEAR MOTION OF A POINT

114. Linear Velocity. The definition of velocity given in Art. 108 for rectilinear motion will serve as a formal definition of velocity in *curvilinear motion.* The magnitude of the velocity at any instant is the time rate at which the moving point is traversing distance at that instant, and it can be calculated by means of a method similar to that developed for rectilinear motion.

However, for a clear understanding of curvilinear motion it is necessary to be fully aware of the vectorial nature of velocity. *A change of inclination is just as surely a change of velocity as is a change of magnitude.* A person accustomed only to the popular usage of the terms " velocity " and " acceleration " finds it somewhat difficult to adjust his thinking to the technical usage. Acceleration is one of the important fundamental quantities in mechanics. *The general conception of acceleration,* involving, as it does, change of velocity, *contemplates velocity as a quantity that changes whenever its magnitude, or its inclination, or both, change.*

Constant velocity cannot exist, then, in curvilinear motion. The only feature of the velocity that may remain constant in such motion is the magnitude. The word " speed " is sometimes used as a shorter term for the magnitude of velocity. This term is not supposed to arouse in the mind any thought of the inclination of the velocity. Thus, in a curvilinear motion there may be constant speed, but never constant velocity.

The inclination of the velocity at any instant in curvilinear motion is equal to the inclination of the tangent to the path. It is a matter of common observation that a particle which is moving in a curved path has a tendency to " fly off on a tangent." A more scientific statement of this situation is that a particle will not follow a curved path at all unless it is subjected to unbalanced forces, and if at any time the forces become balanced the particle will assume a rectilinear motion. Observation indicates that the rectilinear path will be tangent to the curved path at the point at which the curvilinear motion ceases.

Let Fig. 339 represent the path of a point in curvilinear motion. Let A represent the moving point. Let O' represent a stationary point at any convenient position on the path. Let s represent the distance, measured along the curve, between A and O', considered positive when the moving point is on one side of O', and negative when it is on the opposite side. Since the moving point traverses distance along a curved path, and since s is measured along the path, the magnitude of the velocity at any instant is equal to the rate at which s is changing with respect to t at that instant. The general formula for the speed of a point in curvilinear motion is, then, the same as in rectilinear motion.

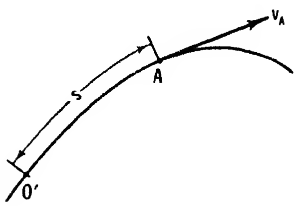

FIG. 339

$$v = \frac{ds}{dt} \qquad [155]$$

It was stated above that the inclination of the velocity agrees with the inclination of the tangent to the path at the position under consideration. This inclination can be found by the usual methods of the calculus. The interpretation of the algebraic signs obtained for v in the use of Eq. 155 is similar to that for the case of rectilinear motion, as discussed in Art. 108.

ILLUSTRATIVE PROBLEMS

623. Figure 340 represents a point, P, moving in a circle 200 ft in diameter. The distance along the path from the fixed point, A, to the moving particle at any instant is represented by s. The motion is such that $s = 0.25\,t^3$, in which s is expressed in feet and t in seconds. Find the velocity of the moving particle at the instant when $t = 10$ sec.

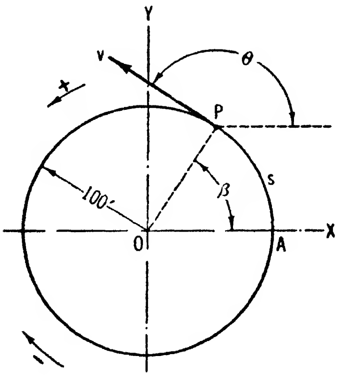

FIG. 340

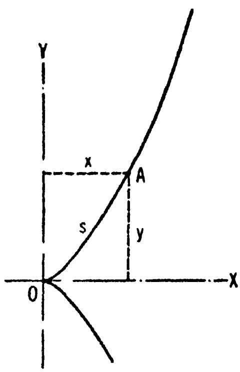

FIG. 341

Solution. By utilizing Eq. 155, and the law of the motion as stated in the problem, the magnitude of the velocity can be readily calculated.

$$s = 0.25\,t^3 \qquad v = \frac{ds}{dt} \qquad v = 0.75\,t^2$$

At the instant when $t = 10$ sec,

$$v = 0.75\ (10)^2 = 75 \text{ ft/sec}$$

The angle of inclination of the velocity can be found as follows:

At the instant when $t = 10$ sec, $s = 0.25\ t^3 = 0.25\ (10)^3 = 250$ ft

$$\beta = \frac{s}{r} = \frac{250}{100} = 2.5 \text{ radians} = 143° \ 13'$$

from which

$$\theta = 233° \ 13'$$

624. Figure 341 represents a semi-cubical parabola, whose equation is $2\ y^2 = x^3$, in which x and y are both expressed in feet. A point, A, moves along the upper portion of the curve, following the law $s = 2\ t^2$, in which s is expressed in feet and t in seconds. Find the velocity of the point for the instant at which $t = 2$ sec.

Solution. The speed, or magnitude of the velocity, is readily calculated by means of Eq. 155.

$$s = 2\ t^2$$

$$v = \frac{ds}{dt} \qquad v = 4\ t$$

At the instant when $t = 2$ sec,

$$v = 4 \times 2 = 8 \text{ ft/sec}$$

The inclination of the velocity is the same as that of the tangent to the curve, and may be found as follows:

The equation of the curved path is

$$2\ y^2 = x^3 \quad \text{or} \quad y = \frac{1}{\sqrt{2}}\ x^{3/2}$$

By differentiation,

$$4\ y\ dy = 3\ x^2\ dx$$

from which

$$\frac{dy}{dx} = \frac{3\ x^2}{4\ y}$$

From the equation of the curve,

$$y = \sqrt{\frac{x^3}{2}}$$

Differentiating,

$$\frac{dy}{dx} = \frac{3}{4}\sqrt{2}\sqrt{x}$$

The foregoing equation will give the value of the slope for any value of x. However, the value of x corresponding to the value $t = 2$ sec must first be found. This can be done as follows:

From the calculus,

$$ds^2 = dx^2 + dy^2$$

Substituting in this expression the value $dy = \frac{3}{4}\sqrt{2}\sqrt{x}\,dx$, as given by the preceding equation, and solving for ds,

$$ds = \sqrt{\tfrac{9}{8}x + 1}\,dx$$

Integrating,

$$s = \tfrac{16}{27}\left(\tfrac{9}{8}x + 1\right)^{3/2} + C$$

From the figure, when $x = 0$, $s = 0$. These values will satisfy the foregoing equation. Therefore,

$$0 = \tfrac{16}{27} + C \qquad C = -\tfrac{16}{27}$$

The completed (s, x) equation is, then,

$$s = \tfrac{16}{27}\left(\tfrac{9}{8}x + 1\right)^{3/2} - \tfrac{16}{27}$$

From the problem, $s = 2t^2$. For $t = 2$ sec, $s = 2 \times 4 = 8$ ft. The corresponding value of x can now be found by substituting this value of s in the (s, x) equation obtained above.

$$8 = \tfrac{16}{27}\left(\tfrac{9}{8}x + 1\right)^{3/2} - \tfrac{16}{27}$$

Transposing, rationalizing, and solving for x,

$$x = +4.4 \text{ ft}$$

Substituting this value of x in the equation for dy/dx obtained above, it is found that

$$\frac{dy}{dx} = \tfrac{3}{4}\sqrt{2}\sqrt{4.4} = 2.22$$

From which the angle of inclination of the velocity is found to be

$$\theta = \text{arc tan } 2.22 = 65° \, 45'$$

Thus, it has been learned that, at the instant when $t = 2$ sec, the point is moving at a speed of 8 ft per sec, and at an angle of 65° 45′ with the x-axis.

PROBLEMS

625. The point P in Fig. 340 moves along its circular path in accordance with the law $s = 4\sqrt{t^3}$, in which s is in feet and t is in seconds. Find the velocity of P at the instant when $t = 16$ sec. *Ans.* 24 ft/sec; $\theta_x = 236° \, 40'$.

626. The point P in Fig. 340 moves in accordance with the law $s = 2t^2 - 20t$, in which s is in feet and t is in seconds. Find the velocity of P at the instant when $t = 2$ sec.

627. The point P in Fig. 340 moves in accordance with the law $v = 3\sqrt{t} + 20$, in which v is in feet per second and t is in seconds. It is also known that $s = 0$ at the instant when $t = 0$. Find the velocity of P at the instant when $t = 4$ sec. *Ans.* 26 ft/sec; $\theta_x = 145° \, 00'$.

628. The point P in Fig. 340 moves in accordance with the law $\beta = \sqrt{t}$, in which β is expressed in radians and t is expressed in seconds. Find the velocity of the point for the instant when $t = 4$ sec.

629. The point P in Fig. 340 moves in accordance with the law $v = 1/\sqrt{\beta}$, in which v is in feet per second and β is in radians. It is also known that $s = 0$ at the instant when $t = 0$. Find the velocity of the point at the instant when $t = 125$ sec. *Ans.* 0.811 ft/sec; $\theta_x = 177° \, 05'$.

630. A point moves along a curve whose equation is $y = x^{3/2}$, in which x and y are expressed in feet. The point moves along the upper portion of the curve in such a manner that $s = 4t$, in which s is in feet and t is in seconds. Find the velocity of the point at the instant when $t = 1$ sec.

115. *X*- and *Y*-Components of Velocity. The methods of Art. 114 are convenient for the calculation of velocity in cases in which the relation between s and t is known, or can be readily ascertained, and in which the slope of the path can be found without especial difficulty. Frequently, however, the velocity can be calculated more readily through the medium of its components parallel to a set of rectangular coordinate axes.

Velocity is a vector quantity, and it can be resolved into components, or can be found from its components, by methods identical with those used for forces and other vector quantities.

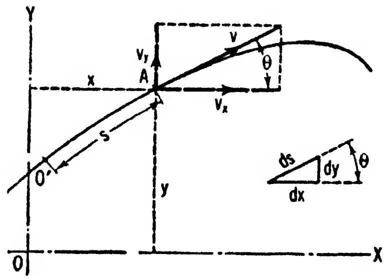

Fig. 342

Let A, in Fig. 342, represent a point moving along the curved path shown. Let O' represent any convenient fixed point in the path. Let v represent the velocity of A, and θ the angle of inclination between v and the x-axis. Let the coordinates of the moving point, A, be represented by x and y, as shown. Obviously, x and y vary in some manner with respect to time. Let s represent the distance, measured along the path, from O' to A. From the figure,

$$v_x = v \cos \theta \qquad [156]$$

From the calculus, $\cos \theta = dx/ds$. Substituting this value of $\cos \theta$ in Eq. 156,

$$v_x = v\frac{dx}{ds} = \frac{ds}{dt}\frac{dx}{ds} = \frac{dx}{dt} \qquad [157]$$

Equation 157 shows that the x-component of the velocity at any instant is equal to the time rate at which the x-coordinate of the moving point is changing.

In a similar manner it can be shown that

$$v_y = \frac{dy}{dt} \qquad [158]$$

Illustrative Problems

631. The curve shown in Fig. 343 is a cubical parabola, whose equation is $16\,y = x^3$. In this equation x and y are both expressed in feet. A point, A, moves along this curve in such a manner that $x = 2\,t^2$, in which x is expressed in feet, and t is expressed in seconds. Find the velocity of the point for the instant when $t = 1$ sec.

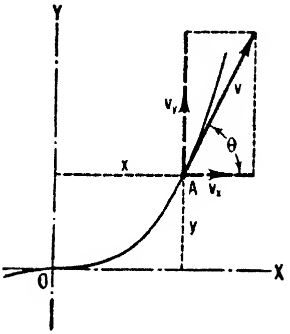

Fig. 343

Solution. The x-component of the velocity can be found directly from the law of the motion, as stated in the problem, with the assistance of Eq. 157.

$$x = 2\,t^2 \qquad v_x = \frac{dx}{dt} \qquad v_x = 4\,t$$

At the instant when $t = 1$ sec,

$$v_x = 4 \times 1 = 4 \text{ ft/sec}$$

In the calculation of v_y, it is first necessary to ascertain the relation between y and t. This can be done by substituting the value $x = 2\,t^2$ in the equation of the curve, as given in the problem.

$$16\,y = x^3 \qquad x = 2t^2 \qquad 16\,y = 8t^6 \qquad y = \tfrac{1}{2}t^6$$

From the foregoing equation, and from Eq. 158,

$$y = \tfrac{1}{2}t^6 \qquad v_y = \frac{dy}{dt} \qquad v_y = 3t^5$$

At the instant when $t = 1$ sec,

$$v_y = 3\,(1)^5 = 3 \text{ ft/sec}$$

$$v = \sqrt{v_x + v_y^2} = \sqrt{(4)^2 + (3)^2} = \sqrt{25} = 5 \text{ ft/sec}$$

$$\theta = \text{arc tan}\,\frac{v_y}{v_x} = \text{arc tan}\,\frac{3}{4} = 36°\ 50'$$

632. A point moves along a curve whose equation is $16y^2 = x^3$, in which x and y are expressed in feet. The motion is such that $v_x = +4$ ft per sec, and is constant. It is also known that $t = 0$ at the instant when the moving point is at the origin. Calculate the velocity at the instant when $t = 9$ sec.

Solution. From Eq. 157, and from the problem,

$$v_x = \frac{dx}{dt} \qquad v_x = +4 \qquad \frac{dx}{dt} = 4 \qquad dx = 4dt$$

Integrating,

$$x = 4t + C$$

From the problem, $x = 0$ when $t = 0$. Substituting,

$$0 = 0 + C \qquad C = 0$$

Therefore,

$$x = 4t$$

From the problem,

$$16y^2 = x^3$$

Eliminating x between the two preceding equations,

$$16y^2 = 64t^3 \qquad y = 2t^{3/2}$$

From the y, t relationship thus obtained, and from Eq. 158,

$$v_y = \frac{dy}{dt} \qquad v_y = \frac{d(2t^{3/2})}{dt} = 3t^{1/2}$$

At the instant when $t = 9$ sec,

$$v_y = 3(9)^{1/2} = +9 \text{ ft/sec}$$

From the problem,

$$v_x = +4 \text{ ft/sec}$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(4)^2 + (9)^2} = 9.85 \text{ ft/sec}$$

$$\theta_x = \text{arc tan}\,\frac{v_y}{v_x} = \text{arc tan}\,\frac{9}{4} = 66° \, 00'$$

PROBLEMS

633. A point travels in a curvilinear path whose equation is $xy = 48$, in which x and y are in feet. The motion follows the law $y = 24/t^2$, in which y is in feet and t is in minutes. Find the velocity of the point at the instant when $t = 2$ min. *Ans.* 10 ft/min; $\theta_x = 323° \, 10'$.

634. A point travels in a path whose equation is $4y = x^3$, in which x and y are in feet. The motion is governed by the law $v_y = 3\sqrt{t}$, in which v_y is in feet per second and t is in seconds. It is also known that $t = 0$ at the instant when the moving point is at the origin. Find the velocity of the point at the instant when $t = \frac{1}{4}$ sec.

635. A point travels in a plane curve in such a manner that $v_x = 3\sqrt{t}$ and $v_y = 3/t^{1/4}$. It is also known that $t = 0$ at the instant when the point is at the origin. Derive the equation of the path. *Ans.* $y^2 = 8x$.

636. Figure 344 represents the upper portion of a parabola whose equation is $y^2 = 2x$. The point B moves along the x-axis in accordance with the law $v_B = 8t^3$. It is also known that $t = 0$ at the instant when B leaves the origin. The point C moves along the y-axis, and A moves along the curve. These motions are so coordinated that the figure $OBAC$ is always a rectangle. Prove that $v_C = 4t$.

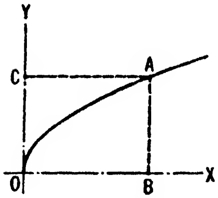

Fig. 344

637. Figure 344 represents the upper portion of a parabola whose equation is $y^2 = 2x$, in which x and y are in feet. The point C moves along the y-axis in accordance with the law $y = 2t^{1.6}$, in which y is in feet and t is in seconds. B moves along the x-axis, and A moves along the curve, in such a manner that the figure $OBAC$ is always a rectangle. Find the velocity of the point A at the instant when $t = \frac{1}{4}$ sec. *Ans.* 1.42 ft/sec; $\theta_x = 77° 45'$.

638. A straight piece of pipe, 25 ft long, is fastened rigidly on a flat car, in a vertical position. Both ends of the pipe are open. The car moves toward the right along a straight, horizontal track, with a constant acceleration, also toward the right, of 4 ft per sec per sec. At the instant when the speed of the car is 30 ft per sec, a ball is released in the upper end of the pipe, with no initial velocity in the vertical direction. Calculate the velocity of the ball at the instant when it emerges from the lower end of the pipe. Disregard friction.

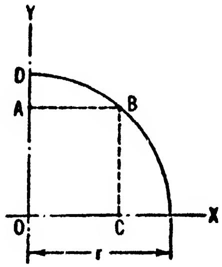

Fig. 345

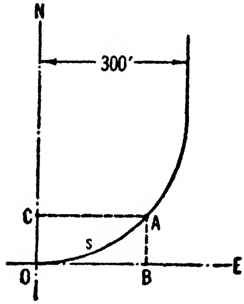

Fig. 346

639. The point C, in Fig. 345, starts from rest at O and moves toward the right along the x-axis, in accordance with the law $x = t^4$. At the same instant a second point, A, starts at D and moves toward O, along the y-axis, and a third point, B, starts at D, and moves along the circular path. The three points move in such a manner that the figure $ABCO$ is always a rectangle. Derive a formula for the magnitude of the velocity of B, in terms of r and t. *Ans.* $v = \dfrac{4rt^3}{\sqrt{r^2 - t^8}}$.

640. Three automobiles, A, B, and C, start from rest at the point O, in Fig. 346. Car A follows the curved road, in accordance with the law $s = \frac{1}{4}t^2$. The curve is a circular arc, having a radius of 300 ft. Cars B and C start simultaneously with A, car B going due east, and car C going due north. Cars B and C move in such a manner that the figure $ABOC$ is always a rectangle. Derive general formulas for the velocities of A, B, and C, in terms of t. If all distances are expressed in feet, and t is expressed in seconds, calculate the velocity of each car for the instant when $t = 40$

sec. *Ans.* $v_A = 20$ ft/sec, N 13° 35′ E; $v_B = 4.7$ ft/sec, east; $v_C = 19.4$ ft/sec, north.

641. A point moves in a plane in such a manner that its x-coordinate follows the law $x = c \cos kt$, and its y-coordinate follows the law $y = c \sin kt$. Derive a formula for the velocity of the point at any instant, and find the equation of the curve in which the point moves. The quantities c and k are constants.

642. Figure 347 represents a familiar mechanism, known as an isosceles linkage. The link OB rotates about O. The point A, at the end of the link AC, can move only in the straight line AOX. The lengths AB and OB are equal, and each will be represented by r. The length BC will be represented by l. In the present case the

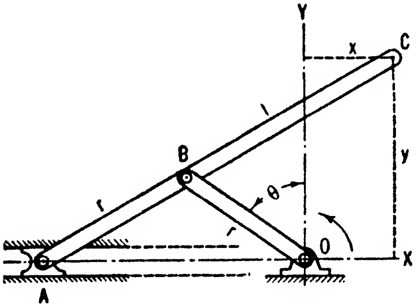

FIG. 347

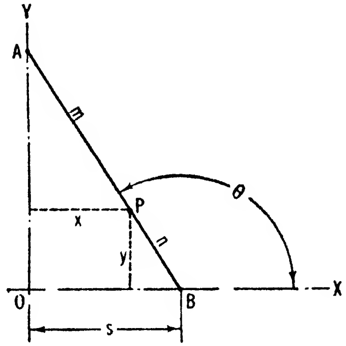

FIG. 348

link OB is rotating in a counterclockwise direction in such a manner that $\theta = 4t$, in which θ is expressed in radians and t is expressed in seconds. Derive general formulas for x and y, in terms of l, r, and t. From these, derive formulas for the x- and y-components of the velocity of the point C. Calculate the velocity of C for the instant when $t = 3$ sec. Assume $r = 12$ in. and $l = 20$ in. Derive the equation of the path in which C moves, in terms of r and l. *Ans.* $v_C = 73.8$ in./sec, $\theta_x = 68° 30'$.

643. Figure 348 represents a rigid bar, AB, whose upper end moves along the y-axis, and whose lower end moves along the x-axis. P is a point marked on the bar at a distance, m, from the upper end. The bar moves in such a manner as to cause the angle θ to vary in accordance with the law $\theta = (\pi/2) + 0.5t$, in which θ is expressed in radians and t is expressed in seconds. Derive formulas for the x- and y-components of the velocity of P, in terms of m, n, and t. Find the equation of the curve in which P moves. Calculate the velocity of P for the instant when $t = 8$ sec, assuming that $m = 8$ ft and $n = 4$ ft.

116. Linear Acceleration. *The linear acceleration of a point in curvilinear motion, at any instant, is the time rate at which the linear velocity of the point is changing at that instant.* In its wording, this definition is the same as that stated for acceleration in rectilinear motion (Art. 109). It must be remembered, however, that velocity is a vector quantity, and is capable of being changed in inclination, as well as in magnitude and sense. In rectilinear motion change in the inclination of velocity does not occur, but *in curvilinear motion change in inclination is in progress at all times,* whether magnitude is changing or not. It is necessary to gain a clear conception of what is meant by " rate of change of velocity " in the general case in which both magnitude and inclination are changing.

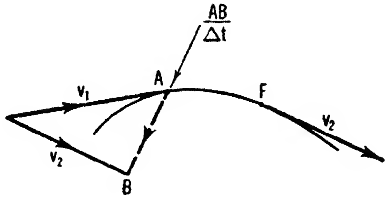

FIG. 349

In Fig. 349, let the curved line represent the path of a moving point. Let v_1 represent the velocity at A, and let v_2 represent the velocity at any later instant in the motion. Let Δt represent the interval of time between these two instants.

Change of Velocity. Let a triangle be formed, consisting of the vectors v_1 and v_2, and a third vector, AB, directed toward B, as shown in the figure.

This third vector, AB, is called the change of velocity of the moving point, for the given interval. That AB actually does represent the change that has occurred in the velocity can be demonstrated as follows:

From the figure, v_2 is the vector sum, or resultant, of v_1 and AB. In other words, v_2 is the result of adding AB, vectorially, to v_1. Thus, the velocity v_1 can be " changed " into the velocity v_2 by the vectorial addition of AB.

Average Acceleration for the Interval. The vector designated in the figure as $AB/\Delta t$ is called the average acceleration for the given interval, Δt. It has the same inclination and sense as AB, but its magnitude is equal to the magnitude of AB divided by the time interval, Δt. If one such vector were added, vectorially, to v_1 per unit time, the resultant velocity at the end of the interval would be v_2. $AB/\Delta t$ represents, therefore, the average vectorial change per unit time, for the given interval.

The " average acceleration for the interval " must not be confused with the " acceleration at A." By acceleration at A is meant the rate at which the velocity is changing at the particular instant when the moving point passes through A. This may have a value entirely different from that of the average acceleration for the interval.

Acceleration at A. Now let Δt represent a shorter time interval than in the first case, but having the same point of beginning, A. There will now be a new average acceleration, represented by a new vector, $AB/\Delta t$, whose magnitude and inclination will be different, in general, from those in the first case.

Imagine this process to be repeated with successively shorter intervals, all beginning at A. A whole series of vectors representing average accelerations is thus established. The magnitude of this vector variable approaches a definite limit at A as Δt approaches zero. The inclination of the vector also approaches a definite limit. *The vector whose magnitude is the limit of this series of magnitudes, and whose inclination is the limit of the series of inclinations, is called the acceleration of the moving point, at A.*

There is no reason for concluding that the acceleration at A will be tangent to the path. A study of the possibilities will show that it may make any angle, other than zero, with the tangent, but that *it will always be directed inward, or toward the concave side of the curve.*

Methods for the actual calculation of acceleration in curvilinear motion will be developed in subsequent articles.

117. Tangential and Normal Components of Acceleration. The case in which the acceleration of a point is resolved into components tangent and normal to the path can be made to yield special formulas of great utility.

In Fig. 350, let the curved line represent the path of a moving point. Let v_1 represent the velocity at A, and let v_2 represent the velocity at any later instant in the motion. Let Δt represent the interval of time between these two instants.

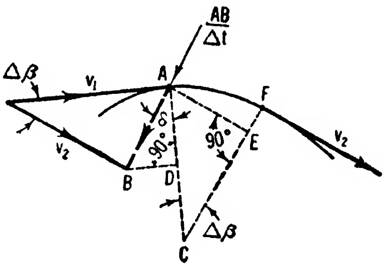

FIG. 350

As was explained in Art. 116, the vector AB represents the change of velocity for the interval, and the vector $AB/\Delta t$ represents the average acceleration. It was explained, also, that, as the interval Δt, always beginning at A, is made to approach zero, the vector $AB/\Delta t$ approaches as its limit a certain definite vector called the acceleration at A.

The lines AC and FC, in Fig. 350, are drawn normal to the path. Let the angle between them be represented by $\Delta\beta$. Obviously, $\Delta\beta$ is also the angle between v_1 and v_2. Let the lines BD and AE be drawn, parallel to v_1 and v_2, respectively. Let the angle DAB be represented by δ.

Tangential Component. From the figure, the tangential component of the average acceleration is $\dfrac{AB}{\Delta t}\sin\delta = \dfrac{AB\sin\delta}{\Delta t} = \dfrac{BD}{\Delta t}$. The limit approached by the magnitude, $BD/\Delta t$, of the tangential component of the average acceleration, as Δt approaches zero, is the magnitude of the tangential component of the acceleration at A. Let this be represented by a_T. In algebraic language,

$$a_T = \lim_{\Delta t \to 0} \frac{BD}{\Delta t} \tag{159}$$

From the figure,

$$BD = v_1 - v_2 \cos \Delta\beta \tag{160}$$

Substituting in Eq. 159, and using the theory of limits,

$$a_T = \lim_{\Delta t \to 0} \left(\frac{v_1 - v_2 \cos \Delta\beta}{\Delta t} \right) = \lim_{\Delta t \to 0} \left(\frac{v_1}{\Delta t} - \frac{v_2 \cos \Delta\beta}{\Delta t} \right)$$

$$= \lim_{\Delta t \to 0} \left(\frac{v_1}{\Delta t} \right) - \lim_{\Delta t \to 0} \left(\frac{v_2}{\Delta t} \right) \lim_{\Delta t \to 0} (\cos \Delta\beta) \tag{161}$$

From the figure, as Δt approaches zero, $\Delta\beta$ also approaches zero as a limit. Therefore, $\lim\limits_{\Delta t \to 0} (\cos \Delta\beta) = 1$. Continuing, from Eq. 161,

$$a_T = \lim_{\Delta t \to 0} \left(\frac{v_1}{\Delta t} \right) - \lim_{\Delta t \to 0} \left(\frac{v_2}{\Delta t} \right) = \lim_{\Delta t \to 0} \left(\frac{v_1 - v_2}{\Delta t} \right) \tag{162}$$

In Eq. 162, $\lim\limits_{\Delta t \to 0} \left(\frac{v_1 - v_2}{\Delta t} \right)$ may be expressed as the derivative, dv/dt. Also, from Eq. 155, Art. 114, $v = ds/dt$. Therefore,

$$a_T = \frac{dv}{dt} = \frac{d^2s}{dt^2} \tag{163}$$

The tangential component of the acceleration is often referred to, more briefly, as the *tangential acceleration.*

Normal Component. From Fig. 350, the normal component of the average acceleration for the interval is $\frac{AB}{\Delta t} \cos \delta = \frac{AB \cos \delta}{\Delta t} = \frac{AD}{\Delta t}$. The limit approached by the magnitude, $AD/\Delta t$, of this normal component, as Δt approaches zero, is the magnitude of the normal component of the acceleration at A. Let this be represented by a_N. In algebraic language,

$$a_N = \lim_{\Delta t \to 0} \frac{AD}{\Delta t} \tag{164}$$

From the figure,

$$AD = v_2 \sin \Delta\beta \tag{165}$$

Substituting in Eq. 164, and referring to the figure,

$$a_N = \lim_{\Delta t \to 0} \left(\frac{v_2 \sin \Delta\beta}{\Delta t} \right) = \lim_{\Delta t \to 0} \left(\frac{v_2 \times AE/AC}{\Delta t} \right)$$

$$= \lim_{\Delta t \to 0} \left(\frac{v_2 \times AE}{AC \times \Delta t} \right) \qquad [166]$$

By the theory of limits,

$$a_N = \lim_{\Delta t \to 0} \left(\frac{v_2}{AC} \times \frac{AE}{\Delta t} \right) = \frac{\lim\limits_{\Delta t \to 0} (v_2)}{\lim\limits_{\Delta t \to 0} (AC)} \lim_{\Delta t \to 0} \left(\frac{AE}{\Delta t} \right) \qquad [167]$$

From the figure, as Δt approaches zero, AE approaches equality with the arc AF. Therefore, $\lim\limits_{\Delta t \to 0} \left(\frac{AE}{\Delta t} \right) = \lim\limits_{\Delta t \to 0} \left(\frac{AF}{\Delta t} \right) = \frac{ds}{dt} = v_1$. Also, $\lim\limits_{\Delta t \to 0} (v_2) = v_1$. From the figure, $\lim\limits_{\Delta t \to 0} (AC) =$ the radius of curvature of the path at A. Let this be represented by r.

Equation 167 now can be written $a_N = \frac{v_1^2}{r}$, or, since v_1 may be considered to be the velocity at any instant,

$$a_N = \frac{v^2}{r} \qquad [168]$$

The normal component of the acceleration is often referred to more briefly as the *normal acceleration.* It is also called the *centripetal acceleration.*

Constant Speed. The normal acceleration of a point in curvilinear motion cannot be zero. Sometimes, however, the tangential acceleration is zero, and as a result the point travels at constant speed. The formula $s = vt$, Art. 110, may be used in this case.

Constant Tangential Acceleration. Sometimes the tangential acceleration is constant in magnitude. In such an event all the formulas in Art. 111, derived therein for the case of constant acceleration in rectilinear motion, may be used, provided that a is replaced by a_T.

Effects of Tangential and Normal Acceleration. Equation 163 shows that the tangential component of the acceleration is equal to the rate at which the magnitude of the velocity (speed) is changing. Equation 168 shows that the normal component depends upon the speed at the particular instant, but not on the rate at which the speed is changing. However, the normal component depends upon the rate at which the inclination of the velocity is changing, since it is proportional to the curvature of the path, and since velocity is always tangent to the path.

It may be stated, then, that the *tangential acceleration results only in changes in the speed*, and that the *normal acceleration results only in changes in the inclination of the velocity.* Thus, a normal component of acceleration is essential to a curvilinear motion, and if it becomes zero, and remains so for any finite interval of time, the path changes to a straight line, and remains straight throughout the interval. Curvilinear motion can continue, however, without tangential acceleration. In such a case the point simply moves along the curved path at constant speed.

It is difficult, at first, to accept the idea that a point which is moving at constant speed in a curved path is being accelerated. The source of the difficulty is in the popular conception of the term "acceleration." The technical meaning of the term, as discussed in Art. 116, is the meaning assumed in the statement of the fundamental laws of motion, and, like any other definition, simply must be accepted.

Illustrative Problems

644. Figure 351 represents a point, P, moving in a circular path whose radius is 10 ft. The point moves in accordance with the law $s = t^{5/2} - 10t + 50$, in which s is in feet and t is in seconds. s is measured as shown in the figure. Calculate the acceleration of the point at the instant when $t = 4$ sec.

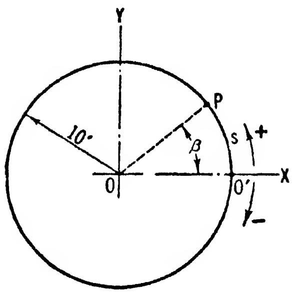

Fig. 351

Solution. From Eq. 155, Art. 114, and from the problem,

$$v = \frac{ds}{dt} \qquad s = t^{5/2} - 10t + 50$$

$$v = \frac{d(t^{5/2} - 10t + 50)}{dt} = 2.5t^{3/2} - 10$$

At the instant when $t = 4$ sec,

$$v = 2.5(4)^{3/2} - 10 = +10 \text{ ft/sec}$$

From Eq. 163, and from the foregoing,

$$a_T = \frac{dv}{dt} \qquad v = 2.5t^{3/2} - 10 \qquad a_T = \frac{d(2.5t^{3/2} - 10)}{dt} = 3.75t^{1/2}$$

At the instant when $t = 4$ sec,

$$a_T = 3.75(4)^{1/2} = +7.5 \text{ ft/sec}^2$$

From Eq. 168, using the value of v calculated above,

$$a_N = \frac{v^2}{r} = \frac{(10)^2}{10} = 10 \text{ ft/sec}^2$$

The position of the moving point in its path at the given instant can be found by calculating β. From the figure, $\beta = s/r$. At the instant when $t = 4$ sec,

$$s = t^{5/2} - 10t + 50 = (4)^{5/2} - 10 \times 4 + 50 = +42 \text{ ft}$$

$$\beta = \frac{s}{r} = \frac{42}{10} = 4.2 \text{ radians} = 240° \, 40'$$

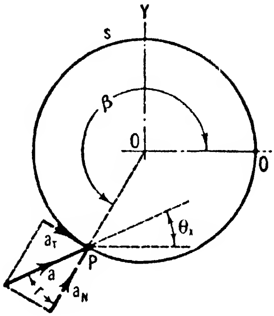

FIG. 352

Therefore, the position of the point at the given instant is as shown in Fig. 352. The resultant acceleration is as follows:

$$a = \sqrt{a_T^2 + a_N^2} = \sqrt{(7.5)^2 + (10)^2} = 12.5 \text{ ft/sec}^2$$

$$\gamma = \text{arc tan} \frac{a_T}{a_N} = \text{arc tan} \frac{7.5}{10} = 36° \, 50'$$

$$\theta_x = \beta - 180° - \gamma = 240° \, 40' - 180° - 36° \, 50' = 23° \, 50'$$

645. A point, P, moves in the circular path shown in Fig. 351, in such a manner that $a_T = -2$ ft per sec per sec, and is constant. The negative sign means that the sense of a_T is clockwise with reference to the circular path. It is also known that $s = 0$ and $v = +36$ ft per sec at the instant when $t = 0$. Find the resultant acceleration of P at the instant when $t = 15$ sec.

Solution. Although a_T is constant, a_N necessarily varies; therefore a is variable. The problem belongs to the special case discussed in the present article, in which the tangential acceleration is constant in magnitude. The formulas of Art. 111 will be used, a being replaced by a_T. From Eq. 129, and from the problem,

$$v = v_0 + a_T t \qquad v = +36 - 2t$$

At the instant when $t = 15$ sec,

$$v = 36 - 2 \times 15 = +6 \text{ ft/sec}$$

From Eq. 168,

$$a_N = \frac{v^2}{r} \qquad a_N = \frac{(6)^2}{10} = 3.6 \text{ ft/sec}^2$$

$$a = \sqrt{a_T^2 + a_N^2} = \sqrt{(2)^2 + (3.6)^2} = 4.12 \text{ ft/sec}^2$$

$$\gamma = \text{arc tan} \frac{a_T}{a_N} = \text{arc tan} \frac{2}{3.6} = 29° \, 05'$$

From Eq. 134, Art. 111,

$$s = \frac{v_0 + v}{2} t \qquad s = \frac{36 + 6}{2} \times 15 = +315 \text{ ft}$$

$$\beta = \frac{s}{r} = \frac{315}{10} = 31.5 \text{ radians} = 1804° \, 57' = 4° \, 57'$$

The angle of inclination of a with the x-axis is calculated as follows:

$$\theta_x = 180° + \beta + \gamma = 180° + 4° \, 57' + 29° \, 05' = 214° \, 02'$$

PROBLEMS

646. A train travels on a curve having a radius of 1000 ft, at a constant speed of 40 mi per hr. Calculate the acceleration of the train in miles per hour per second, and in feet per second per second. *Ans.* 2.35; 3.45.

647. The normal acceleration of a point on the rim of a 6-ft flywheel has a constant magnitude of 20,000 ft per sec per sec. Calculate the speed of the flywheel in revolutions per minute.

648. The crankshaft of an automobile engine is rotating at the constant speed of 2500 rpm. The crank is 2 in. long, on centers. Calculate the centripetal acceleration of a point at the center of the crankpin. *Ans.* 11,400 ft/sec^2.

649. The speed of a point moving in a 12-ft circle decreases uniformly from 200 ft per sec to 100 ft per sec, in an interval of 2.5 sec. Calculate the tangential and normal accelerations of the point at the instant ending the interval.

650. A point starts from rest and gains speed at a uniform rate with respect to time, attaining a speed of 50 ft per sec at the instant when it has traveled 20 ft. The path is a circle whose diameter is 100 ft. Calculate the tangential and normal accelerations of the point at the instant when it has traveled 30 ft. *Ans.* 62.5 ft/sec^2; 75.0 ft/sec^2.

651. The point P, in Fig. 351, moves in such a manner that $a_T = +1.5$ ft per sec per sec. It is also known that $s = 0$ and $v = +3$ ft per sec at the instant when $t = 0$. Calculate the resultant acceleration of the point at the instant when $t = 2$ sec.

652. The point P, in Fig. 351, moves in accordance with the law $s = \frac{1}{4}t^4 + 2t + 20$, in which s is in feet and t is in seconds. Calculate the resultant acceleration of the point at the instant when $t = 2$ sec. *Ans.* 15.6 ft/sec^2; $\theta_x = 290° \, 15'$.

653. The point P, in Fig. 351, moves in accordance with the law $v = 0.6t^2$, in which v is in feet per second and t is in seconds. It is also known that $s = 0$ at the instant when $t = 0$. Calculate the resultant acceleration of the point at the instant when $t = 5$ sec.

654. The centrifugal pull of a rotating body on a shaft is equal to the weight of the body, multiplied by the normal acceleration of its center of gravity, and divided by the acceleration due to gravity. A certain body, weighing 60 lb, is attached to a shaft in such a manner that its center of gravity is 18 in. from the center of the shaft. It is desired to balance the centrifugal pull of the body by means of a counterweight placed exactly opposite, with its center of gravity 8 in. from the center of the shaft. Calculate the necessary weight of the second body. *Ans.* 135 lb.

655. A point moves in the parabolic path whose equation is $y^2 = 4x$, in which x and y are in feet. The law governing the motion is $x = t^2$, in which x is in feet and t is in seconds. Calculate the x- and y-components of the acceleration, at the instant when $t = 2$ sec. By means of the values thus obtained, and from the slope of the curve, calculate the tangential and normal components of the acceleration for the given instant. *Ans.* 1.79 ft/sec^2; 0.894 ft/sec^2.

118. *X*- and *Y*-Components of Acceleration. In some cases acceleration in curvilinear motion can be dealt with most conveniently through the medium of rectangular components not parallel to the tangent or to the normal.

In Fig. 353, let the curved line represent the path of a moving point. Let v_1 represent the velocity at A, and let v_2 represent the velocity at any later instant in the motion. Let Δt represent the time interval between these two instants.

As was explained in Art. 116, the vector AB represents the change of velocity for the interval, and the vector $AB/\Delta t$ represents the average acceleration. It was also explained that as the interval Δt, always considered as beginning at A, is made to approach zero, the vector $AB/\Delta t$ approaches as its limit a certain vector, which is called the acceleration at A.

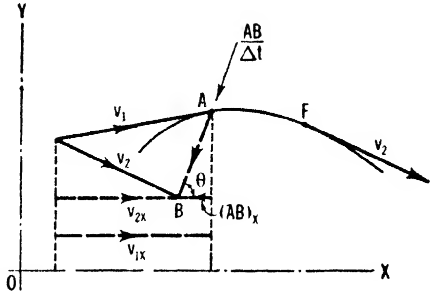

Fig. 353

X-Component. The x-component, at A, of the average acceleration $\frac{AB}{\Delta t}$ is: $\frac{AB}{\Delta t}\cos\theta = \frac{AB\cos\theta}{\Delta t} = \frac{(AB)_x}{\Delta t}$. The limit approached by this component, $\frac{(AB)_x}{\Delta t}$, as Δt approaches zero, is the x-component of the acceleration of the moving point, at A. Let this be represented by a_x. In algebraic language,

$$a_x = \lim_{\Delta t \to 0} \frac{(AB)_x}{\Delta t} \tag{169}$$

From the figure,

$$(AB)_x = v_{2x} - v_{1x} \tag{170}$$

Substituting,

$$a_x = \lim_{\Delta t \to 0}\left(\frac{v_{2x} - v_{1x}}{\Delta t}\right) \tag{171}$$

In Eq. 171, $\lim_{\Delta t \to 0}\left(\frac{v_{2x} - v_{1x}}{\Delta t}\right)$ can be expressed as the derivative, $\frac{dv_x}{dt}$.

Also, from Eq. 157, Art. 115, $v_x = \frac{dx}{dt}$. Therefore,

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \tag{172}$$

Y-Component. By a process of reasoning similar to that employed in deriving Eq. 172, it can be shown that

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \tag{173}$$

Thus, it is seen that the component of the acceleration of a point, along any axis, is equal to the time rate at which the component of the velocity, along that axis, is changing.

ILLUSTRATIVE PROBLEMS

656. A point travels in a parabolic path whose equation is $y^2 = 16x$, in which x and y are in feet. The motion is governed by the law $x = \frac{1}{4}t^4$, in which x is in feet and t is in seconds. Find the acceleration of the point for the instant when $t = 1$ sec.

Solution. From Eq. 157, Art. 115, and from the problem,

$$v_x = \frac{dx}{dt} \qquad x = \tfrac{1}{4}t^4 \qquad v_x = \frac{d(\frac{1}{4}t^4)}{dt} = t^3$$

From Eq. 172, and from the foregoing result,

$$a_x = \frac{dv_x}{dt} \qquad v_x = t^3 \qquad a_x = \frac{d(t^3)}{dt} = 3t^2$$

When $t = 1$ sec,

$$a_x = 3 \times (1)^2 = +3 \text{ ft/sec}^2$$

From the problem,

$$y^2 = 16x \qquad x = \tfrac{1}{4}t^4$$

Eliminating x,

$$y^2 = 16 \times (\tfrac{1}{4}t^4) \qquad y = 2t^2$$

From Eq. 158, Art. 115, and from the foregoing result,

$$v_y = \frac{dy}{dt} \qquad y = 2t^2 \qquad v_y = \frac{d(2t^2)}{dt} = 4t$$

From Eq. 173, and from the foregoing result,

$$a_y = \frac{dv_y}{dt} \qquad v_y = 4t \qquad a_y = \frac{d(4t)}{dt} = +4 \text{ ft/sec}^2$$

Thus, it is seen that a_y is constant. Therefore, for the instant when $t = 1$ sec,

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(3)^2 + (4)^2} = 5 \text{ ft/sec}^2$$

$$\theta_x = \text{arc tan}\,\frac{a_y}{a_x} = \text{arc tan}\,\frac{4}{3} = 53^\circ\ 10'$$

657. A point moves along a plane curve in such a manner that $a_x = 4$ and $a_y = 15t^4$, in which a_x and a_y are in feet per second per second and t is in seconds. It is also known that $v = 0$ and $t = 0$ at the instant when the moving point is at the origin. Derive the equation of the path.

Solution. From Eq. 172, and from the problem,

$$a_x = \frac{dv_x}{dt} \qquad a_x = 4 \qquad \frac{dv_x}{dt} = 4 \qquad dv_x = 4dt$$

Integrating,

$$v_x = 4t + C_1$$

From the initial conditions stated in the problem it follows that, when $t = 0$, $v_x = 0$. Substituting in the foregoing equation,

$$0 = 0 + C_1 \qquad C_1 = 0$$

Therefore,

$$v_x = 4t$$

From Eq. 157, Art. 115, and from the foregoing result,

$$v_x = \frac{dx}{dt} \qquad v_x = 4t \qquad \frac{dx}{dt} = 4t \qquad dx = 4t\,dt$$

Integrating,

$$x = 2t^2 + C_2$$

From the initial conditions stated in the problem, $x = 0$ when $t = 0$. Substituting in the foregoing equation,

$$0 = 0 + C_2 \qquad C_2 = 0$$

Therefore,

$$x = 2t^2$$

Reverting to the statement of the problem, and from Eq. 173,

$$a_y = \frac{dv_y}{dt} \qquad a_y = 15t^4 \qquad dv_y = 15t^4$$

If this equation is then integrated, and a series of steps are taken similar to those in the first half of the solution, it will be found that

$$y = \tfrac{1}{2}t^6$$

The equation of the path of the moving point now can be found by eliminating t between the foregoing equation for y and the equation for x obtained earlier in the solution.

$$x = 2t^2 \qquad y = \tfrac{1}{2}t^6$$

Eliminating t,

$$16y = x^3$$

which is the equation of a cubical parabola.

PROBLEMS

658. Calculate the acceleration of the point in Prob. 631, Art. 115, at the given instant. *Ans.* 15.5 ft/sec^2; $\theta_x = 75° \, 05'$.

659. Calculate the acceleration of the point in Prob. 632, Art. 115, at the given instant.

660. A point moves along a plane curve in such a manner that $a_x = 0$, and $a_y = +2$ ft per sec per sec, throughout the motion. It is also known that $t = 0$, $v_y = 0$, and $v_x = +2$ ft per sec, at the instant when the moving point is at the origin.

Find the equation of the path. Calculate the coordinates of the moving point at the instant when $t = 3$ sec. *Ans.* $x^2 = 4y$; $(+6', +9')$.

661. A point moves along a plane curve in such a manner that $x = t^2$ and $y = 2t$, in which x and y are in feet and t is in seconds. Calculate the velocity and acceleration of the point for the instant when $t = 1.5$ sec. Find the equation of the path.

662. A point moves along a plane curve in accordance with the following: $v_x = 4t^3$ and $v_y = 4t^3 - 12t$, in which v_x and v_y are expressed in feet per minute and t is in minutes. It is also known that $x = 0$ and $y = +9$ ft at the instant when $t = 0$. Calculate the acceleration of the point at the instant when $t = 2$ min. Derive the equation of the path. *Ans.* 60 ft/min^2; $\theta_x = 36° 50'$; $x^{1/2} \pm y^{1/2} = 3$.

663. A belt conveyor discharges its material at a velocity of 400 ft per min, at an upward angle of 20° with the horizontal. Air resistance may be disregarded, which means that the motion of the material after it leaves the conveyor will be such that $a_x = 0$ and $a_y = -32.2$ ft per sec per sec. If the ground level is 20 ft below the point of discharge, calculate the horizontal distance from that point to the point at which the stream of material strikes the ground. This problem was suggested by the Link-Belt Company.

664. In Prob. 641, derive a formula for the acceleration of the moving point at any instant. Also derive the equation of the path. *Ans.* $a = ck^2$; $\theta_x = kt$; $x^2 + y^2 = c^2$.

665. Calculate the acceleration of the point C in Prob. 642, Fig. 347, for the given instant.

666. Calculate the acceleration of the point P, in Prob. 643, Fig. 348, for the given instant. *Ans.* 1.65 ft/sec^2; $\theta_x = 23° 25'$.

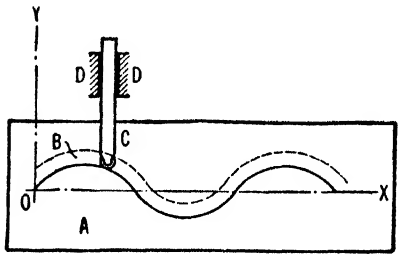

FIG. 354

667. In Fig. 354, A represents a plate cam in which a groove, B, has been cut. The groove engages a follower, C, which is constrained to move in a vertical straight line by the stationary guides, DD. In the present case the groove has been cut to conform to the sine curve whose equation is $y = 0.5 \sin 2x$, in which x and y are in inches, and the angle $2x$ is in radians. The cam moves toward the left with a constant velocity of 10 in. per sec. Prove that the follower executes a simple harmonic motion. Calculate the maximum velocity and maximum acceleration of the follower.

668. A shell is fired from a gun at an angle θ with the horizontal. The initial velocity is v_0. Derive the equation of the path, referred to an origin of coordinates at the muzzle of the gun. Disregard atmospheric resistance. Assume that $a_x = 0$ and that $a_y = -g$. *Ans.* $y = -\dfrac{g}{2v_0^2 \cos^2 \theta} x^2 + (\tan \theta)x$.

CHAPTER XII

KINETICS OF A PARTICLE

119. The Laws of Motion. Sir Isaac Newton was the first to state the fundamental laws of kinetics in a clear and comprehensive manner. He published them in 1687, and since that time they have been referred to as Newton's laws of motion. The wording of the laws in this book is different from that of Newton, but it has the same fundamental significance, leads to the same results, and is more in accord with modern methods of presenting the subject.

The First Law. A particle that is subjected to the action of balanced forces has no acceleration.

The Second Law. A particle that is subjected to the action of unbalanced forces has an acceleration whose inclination and sense at any instant are the same as the inclination and sense of the resultant force, and whose magnitude is directly proportional to the magnitude of the resultant force.

The Third Law. For every force there is an equal, opposite, and simultaneous force.

Discussion of the Laws. It must be remembered that the term "acceleration," in the first and second laws, has the general meaning ascribed to it in Art. 116.

The first law shows that a particle which, for any interval, is subjected solely to balanced forces, will either be at rest or will move at constant speed in a straight line, throughout the interval. These are the only situations possible when there is no acceleration. The first law is really the statement of a special case under the second law.

The second law shows that unbalanced force causes acceleration of the particle, and gives the fundamental relation between the force and the resulting acceleration. Since all the dimensions of a particle are infinitesimal, the system of forces acting on any single particle is necessarily a concurrent system, and the resultant is a single force. It might be added that the acceleration comes into existence simultaneously with the condition of unbalanced force, beginning at the instant when the forces become unbalanced, and ceasing at the instant when they become balanced.

The third law is the basis for the statements in Art. 5 to the effect that, if a body A exerts a force on body B, B will "react" on A with an equal, opposite, and simultaneous force.

Because of the elementary nature of the particle the term "particle" may be used interchangeably with the term "point" in any statement of purely kinematic facts. In any statement of kinetic facts, however, the particle takes on an additional significance because of its inertia, and its weight.

120. Inertia. To give a particle an acceleration it is necessary to apply to the particle a single force, or its equivalent, an unbalanced system of forces. Matter may be considered, therefore, to possess an inherent property by virtue of which it "resists" acceleration. This property of matter is called *inertia.* To accelerate a particle it is necessary to "overcome" inertia, by the application of unbalanced forces. Inertia is not force, but it causes the particle to exert "reacting" forces on other particles. Reacting forces having their origin in inertia alone disappear when the forces acting on the given particle become balanced and the acceleration ceases.

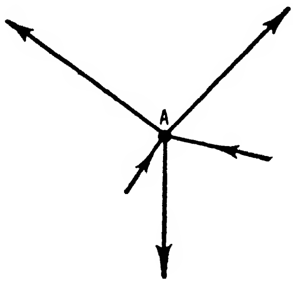

FIG. 355

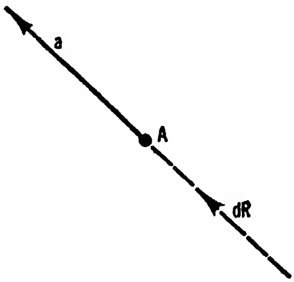

FIG. 356

FIG. 357

121. The Fundamental Kinetic Formula for the Particle. The second law of motion states that the magnitude of the acceleration of a particle is proportional to the magnitude of the resultant force.

Let A, in Fig. 355, represent any particle. Since a particle is simply an elementary portion of a body, and is partially or wholly surrounded by many other particles, it would seem that each particle must be subjected to a great number of elementary forces, caused in various ways. However, let the few forces shown in Fig. 355 represent the entire system to which A is subjected.

Let A, in Fig. 356, represent the same particle as in Fig. 355. Let dR represent the resultant of the entire system of forces acting on the particle. According to the second law of motion, as stated in Art. 119, this resultant force will produce an acceleration whose inclination and sense are the same as the inclination and sense of the resultant force. Let a represent this acceleration.

In Fig. 357, let A represent the same particle as in Figs. 355 and 356, but now considered to be falling freely in a vacuum. Let dW represent the weight of the particle. Obviously, dW is the only force acting on the particle, and in this case itself constitutes the resultant force. Let g represent the acceleration produced by dW. This acceleration is usually referred to as " the acceleration due to gravity."

The second law of motion covers all cases, regardless of the number of individual forces that may be acting on the particle. Therefore, it includes the case represented by Fig. 357. The law includes a statement to the effect that the magnitude of the acceleration is directly proportional to the magnitude of the resultant force. It follows, then, that for the cases represented by Figs. 356 and 357,

$$\frac{a}{g} = \frac{dR}{dW} \qquad [174]$$

This is usually written,

$$dR = \frac{dW}{g} a \qquad [175]$$

Equation 175 is the fundamental formula of kinetics. Practical formulas for the kinetics of finite bodies moving in various ways will be obtained from it by direct mathematical analysis.

It can easily be shown that $dR_x = (dW/g)a_x$, in which dR_x represents the x-component of the resultant, and a_x represents the x-component of the acceleration. In statics it was learned that $R_x = \Sigma F_x$, in which ΣF_x represents the component-sum of the forces of the system, along the x-axis. Therefore, the following formulas, usually more convenient of application than Eq. 175, may be written:

$$\Sigma F_x = \frac{dW}{g} a_x \qquad [176]$$

$$\Sigma F_y = \frac{dW}{g} a_y \qquad [177]$$

$$\Sigma F_z = \frac{dW}{g} a_z \qquad [178]$$

In Art. 7 it was explained that the weight of a body really varies slightly as the position of the body, relative to the earth, is changed. The acceleration, g, caused by the weight, varies in the same ratio.

Strict correctness in the use of any of the formulas obtained above would require the use of values for dW and g that correspond as regards locality. Observance of this precaution would necessitate consideration of the methods by which the values were obtained. However, the accuracy of the average engineering problem is limited by other factors, to such an extent that variations in weight, and in the acceleration caused by weight, attributable to changes in locality, can usually be disregarded. Ordinarily, g may be given the value 32.2 ft per sec per sec. The foregoing value for g will be used in this book.

Units. Ordinarily, in kinetic problems, any convenient unit may be used for expressing the magnitudes of the forces acting on a particle, or body, but the weight of the particle or body must be expressed in the same unit as is used for the other forces. A similar statement may be made regarding the units to be used for a and g.

122. Mass. Statements of the second law of motion usually include a clause to the effect that the acceleration is inversely proportional to the " mass " of the particle. However, no reference was made to mass in the statement of the laws of motion in Art. 119, and in Art. 121 the fundamental formula of kinetics was derived without the inclusion of the term.

Various definitions of mass have been given in textbooks, but they differ considerably. The use of the term " mass " is not vitally necessary to the development of the formulas of kinetics, nor is it indispensable in the solution of the problems of that subject. The latter statement is especially applicable in problems of the type with which the engineer is usually concerned. The term mass will not be utilized in this book. However, because of the frequent occurrence of the term in technical writings, it is desirable to gain some idea of its meaning.

Inertia Concept. The mass of a finite body is always understood to be the sum of the masses of the particles of which the body is composed. The mass of a particle is, in one sense at least, a measure of the degree of inertia possessed by that particle. Since the accelerations produced by a given resultant force, applied successively to different particles, are inversely proportional to the masses of the particles, it follows that mass is a measure of the resistance of a particle to acceleration. Resistance to acceleration is the tangible result of the property of inertia.

The foregoing is sometimes referred to as the " inertia concept " of mass, and it conveys the meaning of mass to the mind of the average individual as well, perhaps, as does any other concept.

Units. When the term mass is used, Eq. 175, Art. 121, appears in the form $dR = dMa$, or some equivalent thereof, in which dM represents the mass of the particle. The engineering student who has completed a course in physics, and has begun the study of engineering mechanics, usually finds that his difficulty in understanding the meaning of mass is to be accompanied by another possible source of confusion. This new difficulty has its origin in the fact that the engineer ordinarily uses units for mass and for force differing from those used by the physicist. For example, the physicist, if he happens to be working under the English system of units, takes the value of dM in the formula as numerically equal to the standard weight of the particle, in pounds. In so doing he uses a unit of mass that is also called the pound. This procedure he accompanies by the use of the poundal as the unit of force. The poundal is equal to the standard pound force divided by the standard value of g. The engineer uses the pound for the unit of force, but expresses mass in terms of a unit equal to the pound mass multiplied by the standard value of g. This unit of mass is called the slug, gee-pound, engineer's unit of mass, etc.

In this book, then, the fundamental formula is dealt with as a relation between force and acceleration only. The quantity dW/g performs the same service as that performed by dM in other books, and in the formulas for finite bodies W/g appears in lieu of M.

123. The Effective Force for the Particle. In Arts. 119 and 121, laws were stated and formulas derived for the kinetics of the individual particle of matter. A particle, however, is only an elementary portion of a body, and engineers are interested in principles and formulas that can be applied directly to the motions of finite bodies. In subsequent articles it will be shown how the transition can be made from the kinetics of the particle to the kinetics of the finite body.

Any individual particle of a body is subjected to countless minute forces, many of which are exerted on the given particle by other particles of the same body, in contact with the given particle, or in the immediate neighborhood. Some of the forces acting on the given particle, however, are exerted by bodies other than that which contains the particle. In other words, some of the forces acting on a given particle are internal, and some are external, to the body to which the particle belongs.

The resultant of all the forces that act on a given particle is called the *effective force* for that particle. In Fig. 356, Art. 121, the resultant dR is the effective force for particle A. By Eq. 175, the magnitude of the effective force is $(dW/g)a$.

In Fig. 358, let A represent a finite body moving in any manner. Let B represent another body in contact with A, and exerting a force on A at the surface of contact. Two of the particles of which A is composed are represented in the figure. No. 1 represents a particle in the interior of A, and No. 2 represents a particle lying in that part of the surface of A

which is in contact with B. A few forces are shown acting on particle 1, as representative of the entire system of forces acting on that particle. One of the forces shown, however, is a specific force, the weight of the particle, and is represented by dW. So far as particle 1 is concerned all these forces are external, but when body A as a whole is under consideration all are internal except dW.

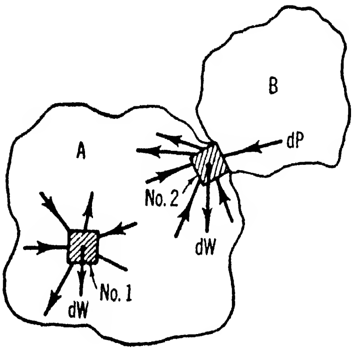

FIG. 358

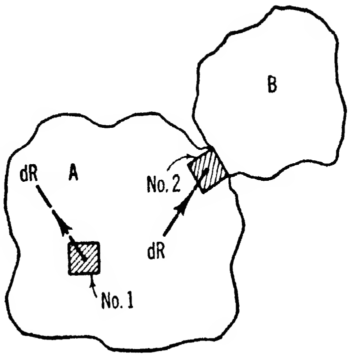

FIG. 359

The effective force for particle 1 is the resultant of all the forces acting on that particle, whether they are internal or external to body A. The system of forces acting on a given particle is necessarily a concurrent system, and the resultant is a single force. The effective force for a particle is, then, as the name implies, a single force. In Fig. 359 the effective force for each of the two particles is shown, and its magnitude is represented by dR.

Particle 2 is in contact with body B, and one of the forces that act on this particle is a part of the total force that body B exerts on body A at the surface of contact. This force is represented by dP, in Fig. 358. In the case of particle 2, then, both dW and dP are external to body A.

In the discussions to follow there will be imagined a single effective force for each particle of a given body, equivalent to all the forces actually exerted on that particle.

CHAPTER XIII

KINETICS OF A BODY IN GENERAL

124. Fundamental Difference between the Formulas of Statics and Kinetics. Statics is really a special case under kinetics. In this book, as in many others, statics is presented first. This method of presentation has many advantages. It also has certain disadvantages. Chief among the disadvantages is the danger that the student will become so habituated to the formulas and principles of statics that he will find himself attempting to use them indiscriminately in kinetic problems.

For example, the student who has completed a course in statics has learned that when a body is in equilibrium the external forces acting on the body are related as follows: $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_z = 0$, $\Sigma M_x = 0$, $\Sigma M_y = 0$, and $\Sigma M_z = 0$. In kinetics, however, the body is not in equilibrium; a different situation exists, and, in general, the quantities ΣF_x, ΣF_y, ΣF_z, ΣM_x, ΣM_y, and ΣM_z are not equal to zero. What these quantities are equal to in the various cases will be revealed as the study of kinetics progresses. It is true that some of them are equal to zero in certain cases, even in kinetics. However, the danger mentioned above will be entirely avoided if the student holds firmly in mind the fact that he is now dealing with the more general phases of mechanics, and that each case will inevitably require careful consideration and the acquirement of much information that will be entirely new to him.

To the certain avoidance of confusion it is necessary to classify carefully each problem as it is attacked, and to be sure that no principles or formulas are used except those that are known to be valid for the case at hand.

125. Effective Forces and External Forces. *The resultant of the effective forces for all the particles of a given body is identical with the resultant of the external forces acting on that body.*

The foregoing statement is, in effect, what is known as d'Alembert's principle, and, as will be seen, its application is an important step in the transition from the kinetics of the particle to the kinetics of the finite body.

Proof. Each effective force is, in itself, the resultant of all the forces actually exerted on one particle of the given body. The resultant of the effective forces for all the particles is nothing more or less than the resultant of all the forces acting on all the particles, or, in other words, it is the resultant of all the external and internal forces acting on, or within, the given body. The internal forces occur in balanced pairs and would simply cancel out of any summation into which they might be introduced. A composition of all the internal and external forces would amount to the same thing as a composition of the external forces alone, and the same resultant would be obtained whether the internal forces were introduced or simply ignored. Thus, the effective forces for all the particles of the body have the same resultant as the external forces acting on the body.

The effective force for a single particle is a single force, but the resultant of the effective forces, or of the external forces, is not necessarily a single force. It may be a single force, or a couple, or a single force and couple.

126. The Motion of the Center of Gravity of a Finite Body. The relation between the forces acting on a single particle and the motion of that particle follows directly from the laws of motion and is of a simple nature, as was shown in Art. 121. The relation between the external forces acting on a finite body and the motion of the body or of its particles cannot, in general, be expressed in so simple a manner. These relations will be fully discussed, in later articles, for bodies in translation, rotation, and plane motion.

Certain general relations of a simple nature do exist, however, between the external forces acting on a finite body and the motion of a particular point of that body: namely, the center of gravity. These relations are valid for any kind of motion, and they apply to non-rigid bodies as well as to rigid bodies. They are sufficient for the complete solution of some problems and are useful in many others. The relations will now be established, and they will be utilized frequently in the remainder of the book.

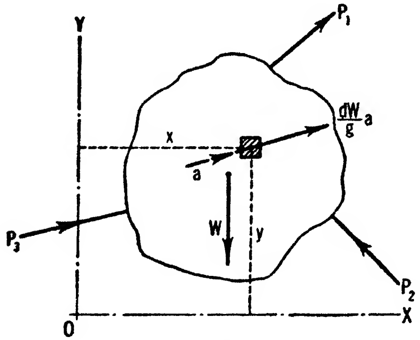

FIG. 360

Let Fig. 360 represent any body, rigid or non-rigid, moving in any manner. Let OX and OY represent any convenient pair of stationary rectangular axes. Let any single particle of the body be shown, and let a represent the linear acceleration of the particle at any given instant. Let P_1, P_2, P_3, $\cdots$ represent the external forces acting on the body at the given instant. Let ΣF_x represent the component-sum of the external forces, along the x-axis, at the given instant. Let $(dW/g)a$ represent the effective force for the particle at the given instant, agreeing with the acceleration in the manner shown.

In Art. 125 it was shown that the resultant of the external forces acting on a body at any instant is identical with the resultant of the effective forces for all the particles at that instant. It follows that the component-sum of the external forces along any axis is equal to the component-sum of the effective forces along that axis. The x-component of the effective force for the given particle, Fig. 360, is readily seen to be $(dW/g)a_x$, in which a_x is the x-component of the acceleration of the particle. The following equation may, then, be written:

$$\Sigma F_x = \frac{dW_1}{g} a_{1x} + \frac{dW_2}{g} a_{2x} + \frac{dW_3}{g} a_{3x} \cdots \text{ etc.} \tag{179}$$

The summation indicated by the right-hand member of Eq. 179 is to be thought of as including all the particles of the body, the numerical subscripts referring to the different particles. Replacing a_{1x} by dv_{1x}/dt, a_{2x} by dv_{2x}/dt, etc., in Eq. 179, and integrating with respect to t, there is obtained

$$\int (\Sigma F_x)\, dt = \frac{dW_1}{g} v_{1x} + \frac{dW_2}{g} v_{2x} + \frac{dW_3}{g} v_{3x} \cdots \text{ etc.} \tag{180}$$

Replacing v_{1x} by dx_1/dt, v_{2x} by dx_2/dt, etc., in Eq. 180, and again integrating with respect to t,

$$g\int\int (\Sigma F_x)\, dt\, dt = dW_1x_1 + dW_2x_2 + dW_3x_3 \cdots \text{ etc.} \tag{181}$$

The right-hand member of Eq. 181 is equivalent to $\int x\, dW$ for the entire body. This may be written $\bar{x}W$ in which $\bar{x}$ represents the x-coordinate of the center of gravity of the body at the given instant. Equation 181 may then be written as follows:

$$\int\int (\Sigma F_x)\, dt\, dt = \frac{W}{g}\bar{x} \tag{182}$$

Differentiating twice with respect to t, there is obtained

$$\Sigma F_x = \frac{W}{g}\frac{d^2\bar{x}}{dt^2} \tag{183}$$

The quantity $d^2\bar{x}/dt^2$, in Eq. 183, is the x-component of the acceleration of the center of gravity of the body at the given instant. Let this be represented by $\bar{a}_x$. Equation 183 may now be written as follows:

$$\Sigma F_x = \frac{W}{g}\bar{a}_x \tag{184}$$

For the sake of simplicity the constants of integration were not introduced in the foregoing integrations, inasmuch as they would have disappeared in the subsequent differentiations. By means of analyses similar to that by which Eq. 184 was obtained, it can be shown that

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \tag{185}$$

$$\Sigma F_z = \frac{W}{g}\bar{a}_z \tag{186}$$

Units. When the English system of units is used, the value 32.2 ft per sec per sec, or thereabouts, is usually assigned to g. This renders it necessary to express a in feet per second per second, also, in the formulas of the present chapter and in all the other kinetic formulas in which g and a appear together. Any desired unit may be used for W, provided that the same unit is used in expressing the other forces involved in the formula.

A comparison of the results obtained above for the finite body, with those obtained in Art. 121 for the single particle, shows that the motion of the center of gravity of a body is related to the external forces in the same manner as the motion of a single particle is related to the forces acting on the particle. As was stated, the foregoing formulas apply in all cases of motion. In some problems, of which those in the present article are examples, they are in themselves sufficient for the solution.

Illustrative Problems

669. Figure 361 represents a car having an inclined floor, running on a horizontal track. A body weighing 161 lb rests on the incline. The coefficient of static friction for the contact between the body and the car is 0.1. The car is given a constant horizontal acceleration of 20 ft per sec per sec, toward the left. Calculate the frictional force and normal pressure exerted on the body. Also, calculate the maximum acceleration that the car could have, toward the left, without causing the body to move up the incline.

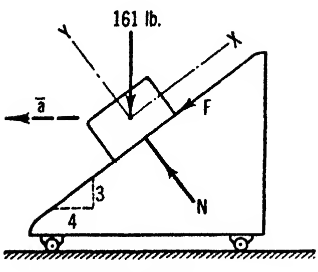

Fig. 361

Solution. In the first part of the problem there is no certainty as to whether the body tends to slide up the incline, or down. Let it be assumed that it tends to slide upward, in which case F would act downward, as shown in the figure. Furthermore, there is no assurance that sliding is impending; therefore the coefficient of static friction cannot be used (Art. 80).

Let the x- and y-axes be placed as shown. Using Eqs. 184 and 185, with careful attention to algebraic signs,

$$\Sigma F_x = \frac{W}{g}\bar{a}_x \qquad -161 \times \tfrac{3}{5} - F = \frac{161}{32.2}(-20 \times \tfrac{4}{5})$$

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad -161 \times \tfrac{4}{5} + N = \frac{161}{32.2}(+20 \times \tfrac{3}{5})$$

The solution of these equations gives

$$F = -16.6 \text{ lb} \qquad N = +189 \text{ lb}$$

The negative sign accompanying the value of F shows that the sense of the frictional force was assumed incorrectly. Therefore, F acts up the incline, and the body tends to slide downward. If the acceleration were greater this situation would be reversed.

In the second part of the problem it is desired to find the maximum acceleration that could be imposed without causing the body to slide up the incline. It should be assumed, therefore, that upward sliding is impending. In such a case F is necessarily directed down the incline. Also, Eq. 27, Art. 78, may be used.

$$\Sigma F_x = \frac{W}{g}\bar{a}_x \qquad -161 \times \tfrac{3}{5} - F = \frac{161}{32.2}(-\bar{a} \times \tfrac{4}{5})$$

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad -161 \times \tfrac{4}{5} + N = \frac{161}{32.2}(+\bar{a} \times \tfrac{3}{5})$$

$$F = \mu N \qquad F = 0.1N$$

The solution of these equations gives: $\bar{a} = 29.6$ ft/sec².

670. Figure 362 represents a ball weighing 64.4 lb, swinging in a vertical plane, as a simple pendulum, at the end of a wire. At the instant depicted by the figure the velocity of the center of gravity of the ball is 12 ft per sec. Calculate the tangential and normal components of the acceleration of the center of gravity, and the tension in the wire, at the given instant.

Solution. In the figure the tangential and normal accelerations of the center of gravity are represented by $\bar{a}_x$ and $\bar{a}_y$, respectively. From Eq. 168, Art. 117,

$$a_N = \frac{v^2}{r} \qquad \bar{a}_N = \bar{a}_y = \frac{(12)^2}{4} = 36 \text{ ft/sec}^2$$

From Eqs. 184 and 185,

$$\Sigma F_x = \frac{W}{g}\bar{a}_x \qquad -64.4 \sin 30° = \frac{64.4}{32.2}(-\bar{a}_T)$$

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad -64.4 \cos 30° + P = \frac{64.4}{32.2}(+36)$$

The solution of these equations gives

$$\bar{a}_T = 16.1 \text{ ft/sec}^2 \qquad P = 128 \text{ lb}$$

671. Figure 363 represents two bodies, weighing 16.1 lb and 48.3 lb, suspended from a slender wire passing over two small pulleys. Calculate the velocity of the bodies at the instant when they have moved 20 ft, starting from rest. Calculate the tension in the wire. Disregard friction, and the inertia of the pulleys and wire.

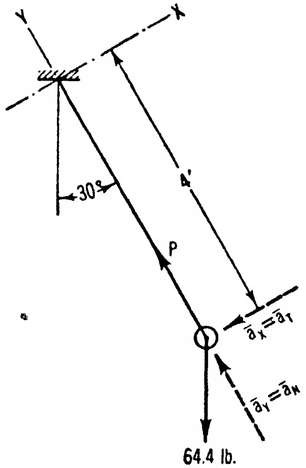

Fig. 362

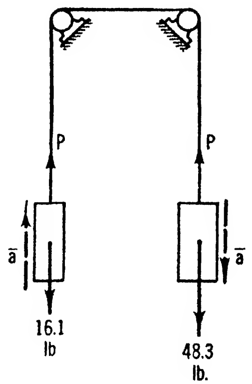

Fig. 363

Solution. Since friction and the inertia of the pulleys and wire are to be disregarded, the tension in the cable will be equal at the points of connection with the two bodies. Let P represent the tension. The accelerations of the bodies are equal and opposite, as shown.

Using Eq. 185 in connection with each body,

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad +P - 16.1 = \frac{16.1}{32.2}(+\bar{a})$$

$$+P - 48.3 = \frac{48.3}{32.2}(-\bar{a})$$

The solution of these equations gives

$$\bar{a} = 16.1 \text{ ft/sec}^2 \qquad P = 24.2 \text{ lb}$$

672. Two elevators, each weighing 2 tons, start at the same instant, from the same elevation. The cable supporting one of the elevators is under a constant tension of 1.5 tons, and the cable supporting the other is under a constant tension of 2.5 tons. What is the position of the second elevator when the first has descended 40 ft? Solve by treating the two elevators as one body, and by considering the motion of their common center of gravity.

Solution. Figure 364 represents the two elevators, in their starting position. G represents the common center of gravity. Since the formulas for the motion of the center of gravity are valid for non-rigid bodies, they may be applied in the present case to the two elevators, considered as one body.

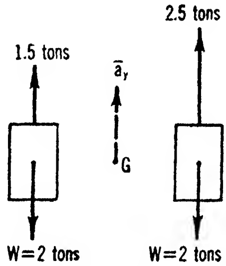

FIG. 364

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad +1.5 + 2.5 - 2 - 2 = \frac{4}{32.2}(+\bar{a}_y)$$

$$\bar{a}_y = 0$$

Thus, it is seen that the common center of gravity of the two elevators remains stationary. Therefore, since the weights of the two are equal, the second elevator will be at a point 40 ft above the starting point, at the instant when the first elevator is 40 ft below the starting point.

PROBLEMS

673. A certain elevator, rising at a speed of 720 ft per min, is brought to rest in a distance of 12 ft, with constant deceleration. A body weighing 30 lb is suspended from a spring balance attached to the ceiling of the elevator. What will be the reading of the spring balance while the elevator is coming to rest? What would be the reading of ordinary platform scales under similar circumstances? *Ans.* 24.4 lb; 30 lb.

674. In Prob. 673, assume that the elevator is descending, and solve. Let all the other data remain unchanged.

675. A block weighing 160 lb is drawn along a horizontal plane by a constant force of 100 lb. The 100-lb force is inclined at a slope of 3 (horizontal) to 4 (vertical). The coefficient of kinetic friction is 0.25. Calculate the time that will elapse while the velocity of the body changes from 20 to 40 ft per sec. If the 100-lb force is suddenly reversed in sense at the instant when the velocity is 40 ft per sec, what will be the velocity at the end of the next 20 sec? *Ans.* 2.48 sec; 0.

676. A body weighing 300 lb slides down an incline whose slope is 4 (horizontal) to 3 (vertical). A constant force, P, horizontal and directed toward the incline, is applied to the body. The coefficient of kinetic friction is $\frac{1}{5}$. The body has an acceleration of 3.22 ft per sec per sec, down the incline. Calculate P.

677. In Prob. 598, find the constant net retarding force, expressed in terms of the weight of the car, for each of the three cases mentioned in the problem.

678. An aviator performs an "outside loop" in an airplane. The vertical circle described by the plane is 2000 ft in diameter. The speed at the lowest point of the circle, when the man is directly under the plane, is 280 mi per hr. The man's weight is 160 lb. Calculate the vertical component of the total pull of the life belt on the man's body, assuming that the belt provides the entire pull necessary to hold the man in his seat. *Ans.* 1000 lb.

679. Figure 365 represents a body weighing 50 lb, resting on the floor of a car. A wire is attached to the body and to the car in the manner indicated. Calculate the maximum acceleration, toward the left, that could be given to the car without causing the body to rise from the car floor. Calculate the tension in the wire when the foregoing acceleration is given to the car.

680. Figure 365 represents a body weighing 50 lb, resting on the floor of a car. A wire is attached to the body and to the car in the manner indicated. The car is given an acceleration of 10 ft per sec per sec, toward the left. Calculate all the forces acting on the 50-lb body. Disregard friction. *Ans.* $P = 25.9$ lb; $N = 29.3$ lb.

681. A car travels on an incline whose slope is 4 (horizontal) to 3 (vertical). The car is so designed that when it is on the incline its floor is horizontal. A body weighing 100 lb rests on the car floor. Calculate the frictional force and normal pressure exerted on the 100-lb body when the car has an acceleration of 8 ft per sec per sec, up the incline. Assume that the body does not slip.

682. The floor of a certain elevator slopes at an angle of 30° to the horizontal. A body weighing 500 lb rests on the floor. A vertical, upward acceleration of 5 ft per sec per sec is given to the elevator. Calculate the frictional force and normal pressure exerted on the 500-lb body, assuming that no slipping occurs. *Ans.* 289 lb; 500 lb.

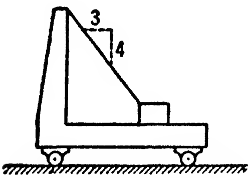

FIG. 365

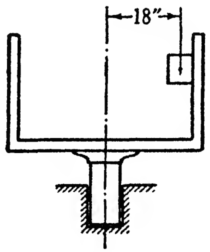

FIG. 366

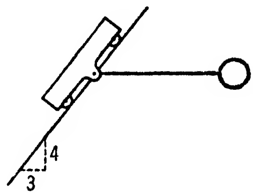

FIG. 367

683. Figure 366 represents a cylindrical drum rotating at constant speed on a vertical shaft. A body weighing 48.3 lb rests against the interior wall of the drum, prevented from falling by friction alone. The coefficient of static friction is 0.2. Calculate the minimum speed, in revolutions per minute, at which the drum could rotate without permitting the body to slip downward. Calculate the forces acting on the body at this minimum speed.

684. Figure 367 represents a trolley running on an inclined track. A body weighing 96.6 lb is suspended from a wire attached to the trolley. The trolley is given a constant acceleration, down the incline, and the wire assumes a horizontal position. Calculate the acceleration, and the tension in the wire. *Ans.* 40.3 ft/sec^2; 72.5 lb.

685. A ball weighing 20 lb is attached to the lower end of a wire. The upper end of the wire is attached to a vertical shaft. The distance from the center of the ball to the upper end of the wire is 6 ft. The system rotates as a conical pendulum, about the vertical axis of the shaft, the center of the ball moving in a horizontal circle and the wire describing a conical surface. The radius of the circle is 3.6 ft. Calculate the linear velocity of the center of the ball, and the tension in the wire.

686. A ball weighing 20 lb, suspended from a ceiling by means of a fine wire, swings freely in a vertical plane, as a simple pendulum. The distance from the point of suspension to the center of the ball is 5 ft. At the extremity of the swing the center of the ball is 1 ft, measured vertically, above its lowest position. Calculate the tension in the wire, and the acceleration of the center of the ball. at the extremity of the swing. Also, solve with the center of the ball at the extremity of the swing at the same elevation as the point of attachment. *Ans.* 16 lb; 19.3 ft/sec^2, θ_x = 323° 10′; 0; 32.2 ft/sec^2, θ_x = 270°.

687. Figure 368 represents three bodies, attached to fine wires passing over two pulleys as shown. The bodies A, B, and C weigh 120, 180, and 200 lb, respectively. Calculate the acceleration of the bodies and the tension in each wire. Disregard friction and the inertia of the pulleys and wire.

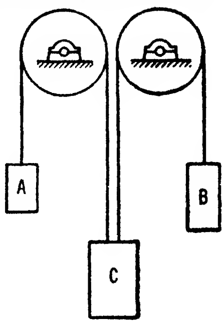

FIG. 368

688. In Fig. 364, change the tension in the left-hand cable to 1.8 tons, and in the right-hand cable to 1.4 tons. Find the position of the common center of gravity of the two elevators at the end of 4 sec, starting from rest. Solve first by writing equations for the two elevators considered as different bodies, and then finding the final position of the common center of gravity by geometry. Then solve by considering the two elevators as a single, non-rigid body, dealing solely with the motion of the common center of gravity. *Ans.* −51.5 ft.

127. The Resultant of the External Forces. Since, by Art. 50, the component-sum of the forces of any system, along any axis, is equal to the component of the resultant along that axis, Eqs. 184, 185, and 186 may be thought of as expressing relations between the resultant of the external forces acting on a body, and the acceleration of the center of gravity of the body.

The resultant of the external forces may be a single force, a couple, or a non-coplanar single force and couple. If the resultant is a single force, its component along any axis is equal to W/g multiplied by the component of the acceleration of the center of gravity, along that axis. If the resultant is a couple, its component along any axis is zero, and the center of gravity has no acceleration. In such a case the center of gravity is either at rest or is moving with constant speed in a straight line. If the resultant is a non-coplanar single force and couple, the resultant couple has no effect on the value of ΣF_x, ΣF_y, or ΣF_z, and the motion of the center of gravity depends solely on the resultant single force, the relation being the same as when the entire resultant is a single force.

If the external forces are in equilibrium, the center of gravity may be at rest, or may be moving at constant speed in a straight line, as in the case in which the resultant is a couple.

CHAPTER XIV

KINEMATICS OF TRANSLATION

128. Kinematics of a Translating Body. *Translation is that motion of a body in which the straight line passing through any two particles of the body, at any instant, is parallel to the line passing through the same two particles, at any other instant.* The term " translation " is applicable only to bodies that are assumed rigid.

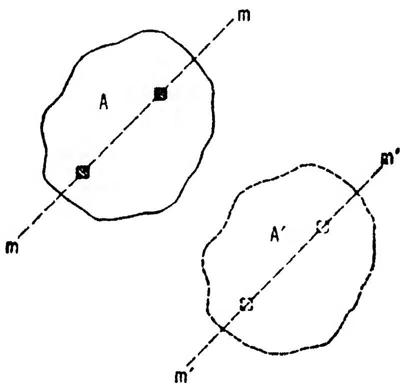

FIG. 369

In Fig. 369, let A represent a translating body, at any given instant during the motion. Two of the particles of the body have been shown, and the straight line mm has been drawn through them. Let A' represent the position of the body at any other instant. The same two particles have been shown in the new position, with the line $m'm'$ passing through them. By the definition of translation, the lines mm and $m'm'$ are parallel. Since the body is assumed to be rigid, the two particles are at a constant distance from each other throughout the motion.

The foregoing definition of translation does not preclude the possibility that the particles will move in curved paths. Such a motion is called *curvilinear translation.* Motion in which the particles move only in straight lines is called *rectilinear translation.*

The motion of a piston in an automobile engine is rectilinear translation, as long as the car itself is stationary, and becomes curvilinear translation if the car moves straight ahead on a plane surface.

Motions of the Particles. The following are the important kinematic facts peculiar to the motions of the particles of a translating body: (1) The paths described by all the particles of the body are alike in every respect. (2) The velocities of all the particles at any given instant have the same magnitude, inclination, and sense. (3) The accelerations of all the particles at any given instant have the same magnitude, inclination, and sense.

The foregoing principles would be accepted by many persons, as obvious from the definition of the motion. The mathematical proof is of a very simple nature and will be omitted.

PROBLEM

689. In each of the following cases state whether or not the motion is translation; and, if so, state whether it is rectilinear or curvilinear: the side rod of a locomotive when the locomotive is running on a track having neither horizontal nor vertical curvature; the connecting rod of the same locomotive; the wheels of the same locomotive; the crosshead of the same locomotive; the side rod of the locomotive if the track has horizontal curvature, if the track has vertical curvature; the car of a Ferris wheel if the car is not swinging to and fro.

CHAPTER XV

KINETICS OF TRANSLATION

129. The Resultant of the External Forces. *The resultant of the system of external forces acting on a translating body at any instant is a single force. Its magnitude is equal to* $(W/g)a$*; its inclination and sense are the same as the inclination and sense of the acceleration; and its line of action passes through the center of gravity of the body.*

Proof. The weights of the particles of a body constitute, practically, a non-coplanar parallel system of forces agreeing in sense. Their resultant is a single force, W, equal to their sum, parallel to them and agreeing with them in sense. The resultant always passes through the center of gravity, regardless of the position or orientation of the body (Art. 91).

The resultant of all the forces acting on a single particle of a body is called the effective force for that particle (Art. 123). In Art. 121 it is shown that $dR = (dW/g)a$, in which dR represents the effective force, dW the weight of the particle, and a the acceleration of the particle. Also, the inclination and sense of dR are the same as the inclination and sense of a (Art. 119).

The accelerations of all the particles of a translating body at any given instant have the same magnitude, inclination, and sense (Art. 128). Therefore, the effective forces for the particles, like the weights, constitute a non-coplanar parallel system of forces agreeing in sense. Let R represent the resultant of the effective forces. $R = \int dR = \int \frac{dW}{g} a.$ At any given instant in the motion, a and g are constants throughout the body. Therefore, $R = \frac{a}{g}\int dW = \frac{W}{g} a$, as was stated in the beginning.

Since the magnitude of each effective force is equal to $(dW/g)a$, and, since a and g are constants throughout the body at a given instant, the effective force is equal to the weight of the particle multiplied by a constant. It follows that the effective forces are proportional to the weights of the particles.

Thus, it has been shown that the system of effective forces has all the characteristics of the system formed by the weights of the particles. Therefore, the resultant of the effective forces also will pass through the center of gravity of the body at all times.

Finally, since the resultant of the external forces is identical with the resultant of the effective forces (Art. 125), the validity of the principles stated at the beginning is established.

It should be noticed that, although the effective forces for the particles constitute a non-coplanar parallel system, the external forces acting on the body do not necessarily constitute such a system.

130. Methods of Solving Problems. Three methods for the solution of problems in the kinetics of translation will be described.

The Resultant Method. A convenient method is suggested by the principles stated and proved in Art. 129. A sketch is made of the moving body, and all the external forces acting on the body are shown. The resultant of the external forces is also shown, in conformity with its description in Art. 129. Certain unknown quantities will exist, of course, associated with the forces, or with the resultant, or both.

Equations are then formed on the basis of the known relationships between a system of forces and its resultant, as discussed in statics (Arts. 11–50). These equations are solved for the various unknown quantities.

The Equilibrant Method. This method differs but little from the resultant method, although it is preferred by some persons. A sketch is made of the body and the external forces acting thereon, as in the resultant method, but the equilibrant of the external forces is inserted instead of the resultant. The equilibrant is the resultant, reversed in sense (Art. 11).

Any system of forces, together with its equilibrant, would constitute a balanced system of vectors. Therefore, the equations for the solution can be formed on the basis of the principles of equilibrium (Arts. 51–74).

If the student uses this method he should not permit himself to receive any impression that the body actually is in equilibrium. When a body has acceleration it is definitely not in equilibrium; neither are the external forces in that state. The equilibrant, like the resultant, is not an actual force but merely a device used for the purposes of convenient solution.

The equilibrant method is also called the inertia-force method, or the reversed-effective-force method.

Solution by Formulas. Many problems in translation are so simple that the use of moment-sum equations is not necessary. If desired, such problems can be solved completely by direct substitution in Eqs. 184, 185, and 186, Art. 126. Illustrative Probs. 669 and 671, of Art. 126, and the majority of the practice problems in that article, are of this type.

Illustrative Problems

690. Figure 370 represents a homogeneous body weighing 750 lb, having the form of a triangular prism, and mounted on a car running on an incline. The prism is supported by a pin-bearing at A, and rests on a roller at B. The car has an acceleration of 3.22 ft per sec per sec, up the incline. Calculate the forces exerted on the prism by its supports.

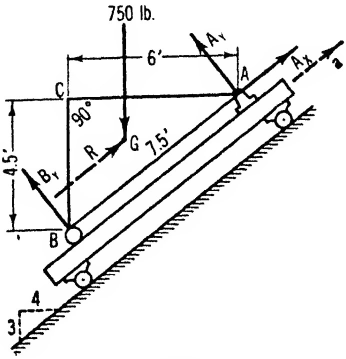

Fig. 370

Solution. The problem will be solved by the "resultant method." Because of the roller, the force at B will have no component parallel to the incline (Art. 14).

Let R represent the resultant of the external forces acting on the prism. From Art. 129, $R = (W/g)a = (750/32.2) \times 3.22 = 75$ lb. The acceleration is parallel to the incline, and upward. Also, R acts through the center of gravity, G, of the body.

By the principle of moments, Art. 50, the moment-sum of the external forces is equal to the moment of the resultant, about any point or axis. Using A as the center of moments,

$$+750(\tfrac{2}{3} \times 6) - B_y \times 7.5 = -75(\tfrac{1}{3} \times 3.6) \qquad B_y = 412 \text{ lb}$$

By the principle of components (Art. 50), the component-sum of the external forces is equal to the component of the resultant, along any axis. Using an axis parallel to the incline,

$$-750 \times \tfrac{3}{5} + A_x = +75 \qquad A_x = 525 \text{ lb}$$

By the principle of components, using an axis at right angles to the incline,

$$-750 \times \tfrac{4}{5} + 412 + A_y = 0 \qquad A_y = 188 \text{ lb}$$

It is suggested that the student check the results by using the principle of moments, with the center of moments at G.

691. Solve Prob. 690, Fig. 370, by the equilibrant method.

Solution. The equilibrant of a system of forces is the reversed resultant. Figure 370 shows the resultant, R, of the external forces acting on the prism. Its magnitude was found, in Prob. 690, to be 75 lb. The equilibrant is, then, equal to 75 lb, and is the same as R in all respects, except that its sense is down the incline.

The entire vectorial system, consisting of the external forces and their equilibrant, is a balanced system. The principles of equilibrium may be applied to the complete system, although the external forces acting on the body are not in equilibrium among themselves.

By the principle of moments, Art. 53, the moment-sum of the system about any axis or point is equal to zero. Using point A as the center of moments, and remembering that the equilibrant is directed down the incline,

$$+750(\tfrac{2}{3} \times 6) - B_y \times 7.5 + 75(\tfrac{1}{3} \times 3.6) = 0 \qquad B_y = 412 \text{ lb}$$

By the principle of components, Art. 53, the component-sum of the system along any axis is equal to zero. Using an axis parallel to the incline,

$$-750 \times \tfrac{3}{5} + A_x - 75 = 0 \qquad A_x = 525 \text{ lb}$$

By the principle of components, using an axis at right angles to the incline,

$$-750 \times \tfrac{4}{5} + 412 + A_y = 0 \qquad A_y = 188 \text{ lb}$$

A comparison of the solution of the present problem with that of Prob. 690 shows that there is no essential difference between the two methods. The mere shifting of one term in some of the equations written under one method renders the entire set of equations identical with those used in the other method.

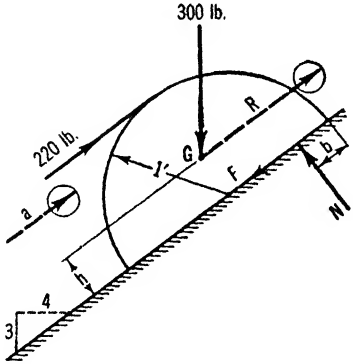

Fig. 371

692. Figure 371 represents a homogeneous, hemispherical body weighing 300 lb, sliding up an inclined plane. A constant, applied force of 220 lb acts on the body as shown. The coefficient of kinetic friction is 0.3. The radius of the hemisphere is 1 ft. Calculate F, N, b, and the acceleration of the body.

Solution. The resultant method of solution will be used. In this problem it is intended that it shall not at first be known whether the acceleration of the body is up, or down, the incline. It is intended, however, that it shall be known that the body is moving up the incline. The frictional force, therefore, acts downward. N represents the resultant of the distributed normal force acting on the bottom of the body. It is desired to locate the line of action of N, by means of the distance b.

From Prob. 508, Art. 98: $h = \frac{3}{8}r = \frac{3}{8} \times 1 = \frac{3}{8}$ ft. Let it be assumed that the acceleration is up the incline. Let R represent the resultant of the external forces. $R = (W/g)a = (300/32.2)a = 9.32a$. If the foregoing assumption is correct, R will be as shown in the figure.

From the principle of components, Art. 50, using an axis at right angles to the incline,

$$-300 \times \tfrac{4}{5} + N = 0 \qquad N = 240 \text{ lb}$$

From Eq. 32, Art. 81,

$$F = \mu N \qquad F = 0.3 \times 240 = 72 \text{ lb}$$

Again by the principle of components, using an axis parallel to the incline,

$$-300 \times \tfrac{3}{5} - 72 + 220 = 9.32a \qquad a = -3.43 \text{ ft/sec}^2$$

The negative sign shows that the sense of a was assumed incorrectly; therefore, a and R are directed down the incline. Since it is known that the body is moving up the incline, the fact that a is downward does not mean that F should be reversed. Therefore, the acceleration is the only quantity that was incorrectly assumed.

By the principle of moments, Art. 50, using G as the center of moments,

$$-220 \times (\tfrac{5}{8} \times 1) - 72 \times (\tfrac{3}{8} \times 1) + 240(1 - b) = 0$$

$$b = 0.315 \text{ ft}$$

It is suggested that the student check the results by using the principle of moments, with a center of moments different from that used above.

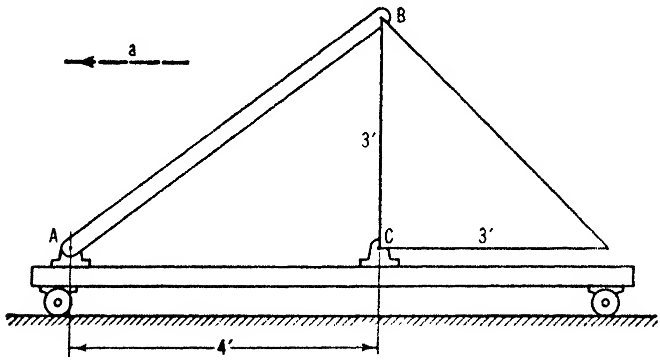

Fig. 372

693. Figure 372 represents a homogeneous body weighing 400 lb, having the form of a triangular prism, mounted on a car. The prism is supported by a pin-bearing at C, and by the brace, AB. The brace is homogeneous, of uniform cross section, and weighs 160 lb. The car has an acceleration of 6.44 ft per sec per sec, toward the left. Calculate all the forces acting on the prism, and on the brace.

Solution. The resultant method of solution will be used. First, let the brace, AB, be considered. Figure 373 is a free-body sketch of AB. The senses of the components of the unknown forces are difficult to predict. Let them be assumed as shown. R_1 represents the resultant of the external forces. By Art. 129, $R_1 = (W/g)a = (160/32.2) \times 6.44 = 32$ lb, and acts through the center of gravity, G, as shown in the figure.

By the principle of moments, Art. 50, using A as the center of moments,

$$-160 \times 2 - B_x \times 3 + B_y \times 4 = +32 \times 1.5$$

By the principle of components, Art. 50, using horizontal and vertical axes,

$$-A_x + B_x = -32$$

and

$$-160 + A_y + B_y = 0$$

Three independent equations have thus been obtained. They contain, however, four unknown quantities. No additional independent equations can be written for the brace, AB.

Let the prism now be considered, as shown in Fig. 374. By Art. 5, it is known that the components acting on the prism at B are equal and opposite to those acting on the brace at that point. It is necessary that these components

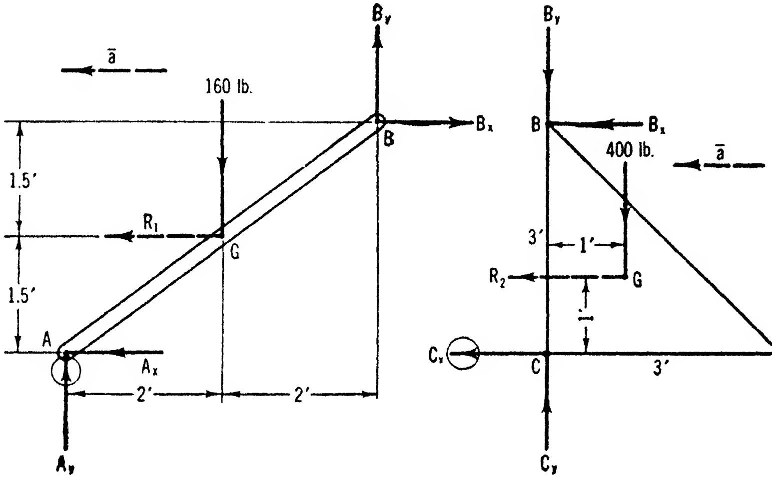

Fig. 373 Fig. 374

be assumed strictly in accordance with this law, as has been done in the figure. Let R_2 represent the resultant of the external forces. $R_2 = (W/g)a = (400/32.2) \times 6.44 = 80$ lb.

By the principle of moments, using C as the center of moments,

$$-400 \times 1 + B_x \times 3 = +80 \times 1$$

By the principle of components, using horizontal and vertical axes,

$$-B_x - C_x = -80$$

and

$$-400 - B_y + C_y = 0$$

There are now six independent equations and six unknown quantities. The solution of these equations gives: $A_x = +192$ lb; $A_y = -52$ lb; $B_x = +160$ lb; $B_y = +212$ lb; $C_x = -80$ lb; $C_y = +612$ lb.

The negative signs accompanying A_y and C_x show that the senses of these components were assumed incorrectly. Therefore, A_y acts downward and C_x acts toward the right. Circles have been drawn around the arrowheads to indicate this fact. All the other components were assumed correctly.

It is suggested that the student check the results by using the principle of moments, with the center of moments at G, or at B.

PROBLEMS

694. A box weighing 1000 lb rests on the floor of an elevator. A second box, weighing 500 lb, rests on the larger one. The elevator has a vertical, upward acceleration of 6.44 ft per sec per sec. Calculate the force acting on the bottom of each box. Reverse the acceleration and solve again.

695. The following problem was given in a registration examination in highway engineering: a car weighing 5000 lb is started from rest and pulled 1000 ft up an incline making an angle of 30° with the horizontal, in 50 sec. The force causing the motion is constant, and parallel to the incline. The resistance to traction due to friction and rolling resistance is 200 lb. Find the magnitude of the constant force causing the motion. *Ans.* 2820 lb.

696. A homogeneous cube, 2 by 2 by 2 ft, weighing 800 lb, is placed on a horizontal plane. A constant, horizontal force of 320 lb is applied at the top of the cube. The coefficient of kinetic friction is 0.3. Calculate the acceleration, the frictional force, and the normal pressure. Locate the point of application of the resultant normal pressure. *Ans.* 3.22 ft/sec^2; 240 lb; 800 lb; 0.3 ft from front edge.

697. Let the 320-lb force in Prob. 696 be increased gradually. What maximum value can be attained without causing the cube to tip?

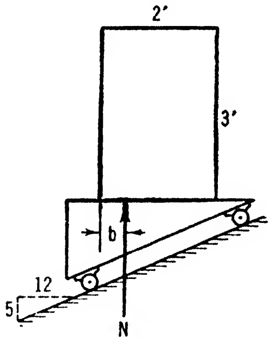

FIG. 375

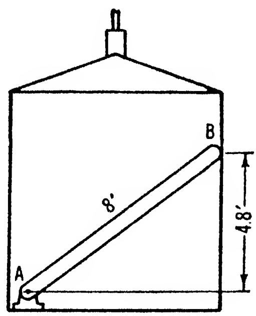

FIG. 376

698. Figure 375 represents a car running on an incline. The floor of the car is horizontal. A homogeneous block weighing 644 lb is placed on the car, and the car is given an acceleration of 4.83 ft per sec per sec, up the incline. Calculate the frictional force, the normal pressure, and the distance b. Assume that the block does not slip. *Ans.* 89.2 lb; 681 lb; 0.804 ft.

699. Calculate the maximum acceleration, down the incline, that could be given to the car in Prob. 690, Fig. 370, without causing the prism to break contact with the roller at B.

700. Remove the 220-lb force in Prob. 692, Fig. 371. Then solve the problem, with the understanding that the body is still in motion up the incline. Also solve the problem with the body in motion down the incline. *Ans.* 1st part: 72 lb; 240 lb; 0.888 ft; 27 ft/sec^2. 2nd part: 72 lb; 240 lb; 1.11 ft; 11.6 ft/sec^2.

701. Reverse the acceleration in Prob. 693, Fig. 372, and solve the problem.

702. Figure 376 represents a uniform, homogeneous bar, AB, weighing 644 lb, in an elevator having a vertical, upward acceleration of 6 ft per sec per sec. The bar is supported by a pin-bearing at A, and rests against the wall of the elevator at B.

Calculate all the unknown forces acting on the bar. Disregard friction. *Ans.* $A_x = +509$ lb; $A_y = +764$ lb; $B_x = -509$ lb.

703. When four-wheel brakes for automobiles were first introduced, stories were circulated to the effect that it was possible to cause a car to turn a front somersault by the too sudden application of such brakes. A car on the market at that time had a wheelbase of 104 in., and a center of gravity 62 in. to the rear of the front hubs. Assume that the coefficient of static friction for the tires on the roadway is unity. Calculate the minimum height of the center of gravity, above the roadway, that would make the front somersault possible, through the action of the brakes alone. Calculate the deceleration of the car. Assume that the entire car has a motion of rectilinear translation, and that the maximum braking effect is being attained, at the front wheels.

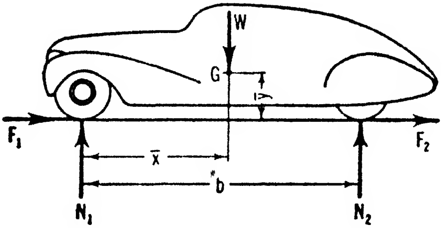

FIG. 377

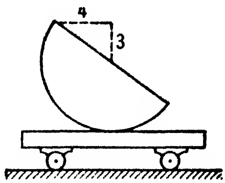

FIG. 378

704. Figure 377 represents an automobile equipped with four-wheel brakes, being brought to rest by the action of the brakes. The center of gravity is at G. Assume $W = 3500$ lb, $b = 112$ in., $\bar{x} = 68$ in., $\bar{y} = 28$ in., and that the coefficient of static friction for the tires on the roadway is 0.6. Assume that the arrangement of the brakes is such that $F_1 = F_2$. Assume that slipping of the tires on the roadway impends at the rear wheels, only. Calculate the minimum distance in which, under ideal conditions, the car can be brought to rest from a speed of 60 mi per hr. Calculate F_1, N_1, F_2, and N_2. *Ans.* 215 ft; 981 lb; 1870 lb; 981 lb; 1630 lb.

705. Solve Prob. 704 for a car equipped with rear-wheel brakes only, all other data remaining the same as in that problem. Assume $F_1 = 0$. Explain, in the terms of mechanics, why four-wheel brakes are more effective than two-wheel brakes. In what manner does the deceleration of the car in this problem influence the effectiveness of the brakes?

706. Figure 378 represents a homogeneous half-cylinder weighing 100 lb, and having a radius of 6 in., resting on a car. The car is given a constant acceleration toward the left, and the cylinder assumes the position shown. Calculate the acceleration, and the frictional force, assuming that the cylinder does not slip. *Ans.* 12.4 ft/sec^2; 38.6 lb.

707. Figure 379 represents a body being drawn up an inclined plane by a force, P, applied as shown. The body weighs 1000 lb, and its center of gravity is at G. The body is supported by wheels at B, and at A by a leg which slides on the incline. The coefficient of kinetic friction at A is 0.2. The wheels will be assumed frictionless. The acceleration is 3.22 ft per sec per sec, up the incline. Calculate all the forces acting on the body. *Ans.* $F_A = 25.8$ lb; $N_A = 129$ lb; $N_B = 671$ lb; $P = 726$ lb.

708. In Prob. 707, Fig. 379, calculate the maximum value that could be given to the force P without causing the body to tip. Calculate the maximum acceleration that the body could have under the same conditions.

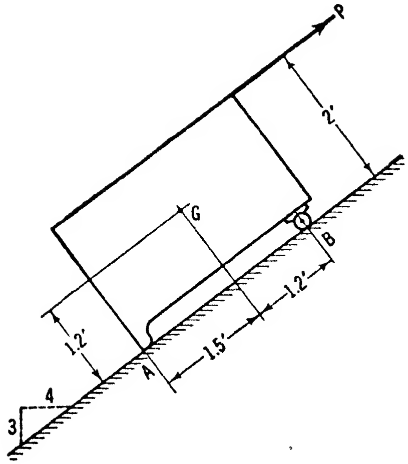

Fig. 379

709. Omit the force P in Prob. 707, Fig. 379, and calculate the velocity of the body after it has moved 20 ft, starting from rest. Calculate the forces acting on the body at A and B. *Ans.* 25.9 ft/sec; $F_A = 78$ lb; $N_A = 390$ lb; $N_B = 410$ lb.

710. In Prob. 706, Fig. 378, let the acceleration of the car be represented by a, the radius of the half-cylinder by r, and the weight by W. Let the angle of inclination, to the horizontal, of the rectangular face of the half-cylinder be represented by θ. Prove that

$$a = \frac{g \sin \theta}{0.75\pi - \cos \theta} \cdot$$

711. Figure 380 represents an inclined plane mounted on wheels. A cylinder rests on the incline, against a small block. The block is attached to the incline. Derive a formula for the maximum acceleration, toward the left, that could be given to the car without causing the cylinder to roll up the incline. *Ans.* $a = g \tan \theta$.

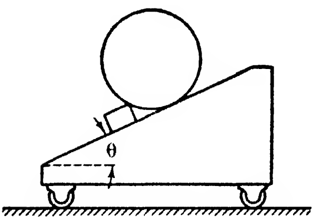

Fig. 380

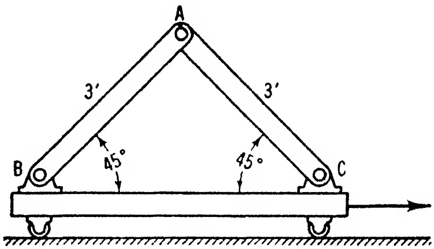

Fig. 381

712. Figure 380 represents an inclined plane mounted on wheels. A cylinder rests on the car, against a small block, as shown. The block is attached to the incline. The height of the block, measured at right angles to the incline, is equal to one-half the radius of the cylinder. Derive a formula for the maximum acceleration, toward the right, that could be given to the car without causing the cylinder to roll over the block. *Ans.* $a = g \cot (\theta + 30°)$.

713. Two uniform, homogeneous bars, AB and AC, in Fig. 381, are connected at A by a smooth pin, and at B and C are attached to the floor of the car by smooth pins, as shown. Each bar weighs 50 lb. The car is drawn along a straight, horizontal track, with an acceleration of 4 ft per sec per sec, toward the right. Find the horizontal and vertical components of the forces exerted on the bars by the pins.

714. Figure 382 represents two uniform, homogeneous bars, AB and CD, weighing 322 lb and 161 lb, respectively. The bars are connected by a pin at C, and are supported on the floor of a car by pin-bearings at A and D. The bearing shoes are fastened rigidly to the car floor. The car has an acceleration of 10 ft per sec per sec, toward the left. Calculate the horizontal and vertical components of all the forces acting on each bar. *Ans.* On AB: $A_x = +221$ lb; $A_y = +143$ lb; $C_x = -321$ lb; $C_y = +179$ lb. On CD: $C_x = +321$ lb; $C_y = -179$ lb; $D_x = -371$ lb; $D_y = +340$ lb.

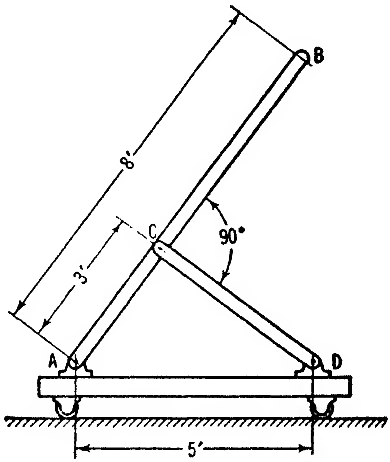

FIG. 382

715. Reverse the acceleration in Prob. 714, Fig. 382, and solve the problem.

716. A rectangular block, instead of a cylinder, rests on the car of Prob. 711, Fig. 380. The coefficient of static friction is μ. Derive a formula for the maximum acceleration, toward the left, that could be given to the car without causing the block to slide up the incline. *Ans.* $a = g\,\dfrac{\mu + \tan\theta}{1 - \mu\tan\theta}$.

CHAPTER XVI

MOMENT OF INERTIA

131. Moment of Inertia of a Line, Area, or Volume; General Formulas. The *moment of inertia of a line* with respect to a given axis is the quantity expressed by the following formula:

$$I = \int q^2\, dL \qquad [187]$$

in which dL represents an elementary portion of the length of the line, and q represents the distance between dL and the given axis, measured at right angles to that axis. The element dL must be formed in such a manner that q has a single value for all the points in any one element.

In many cases the element can be formed in such a way that multiple integration is unnecessary. If this is done, some of the dimensions of the element will be finite, but it must be done so as to satisfy the specification given above. The impression is sometimes received that the element may be formed in some manner other than that specified, and that q may be taken as the distance from the given axis to the center of gravity of the element. In general, such a procedure gives incorrect results.

The *moment of inertia of an area* with respect to a given axis is the quantity expressed by the following formula:

$$I = \int q^2\, dA \qquad [188]$$

The *moment of inertia of a volume* with respect to a given axis is the quantity expressed by the following formula:

$$I = \int q^2\, dV \qquad [189]$$

The specification limiting the formation of the element, given above for the line, applies also for an area or a volume.

Units. Moment of inertia is a scalar quantity, and is always positive. The unit has no special name. If the inch is used for the unit of distance, the moment of inertia of a line is expressed in inches^3, the moment of inertia of an area in inches^4, and the moment of inertia of a volume in inches^5.

Physical Significance. A moment of inertia of a volume, area, or line has no definite physical significance of its own. The term is simply a convenient name for the result of a certain calculation that has been found to occur in many engineering problems. Moments of inertia of areas are of special importance in engineering because of their frequent occurrence in strength of materials. Moments of inertia of lines are of importance chiefly because of the fact that simple approximate formulas for thin rods and wires can be obtained from them. Moments of inertia of volumes as such are of little importance in engineering.

132. Moment of Inertia of a Body; General Formulas. The *moment of inertia of a body* with respect to a given axis, as calculated in engineering practice, is the quantity expressed by the following formula:

$$I = \int q^2 \frac{dW}{g} \qquad [190]$$

in which dW represents the weight of an elementary portion of the body. The distance, q, has the same meaning as in Art. 131, and the specification in that article regarding the formation of the element applies also in the present case.

Let w represent the weight per unit volume of the body. Equation 190 can be written, for homogeneous bodies, as follows:

$$I = \int q^2 \frac{w\,dV}{g} = \frac{w}{g}\int q^2\,dV \qquad [191]$$

Equation 191 shows that the moment of inertia of a homogeneous body is equal to the moment of inertia of the volume of the body, multiplied by w/g.

Units. There is no generally accepted name for the unit moment of inertia. Engineers use the value 32.2 feet per second per second, or thereabouts, for g, and express q in feet. The weight of the body may be expressed in any unit of force, but the pound is usually employed. When feet, pounds, and seconds are used, the moment of inertia is sometimes spoken of as being expressed in " engineer's units." The foregoing term will be used in this book.

Physical Significance. The term " moment of inertia " may be thought of simply as a name for the result of a certain calculation of frequent occurrence in engineering problems. When used in connection with bodies, the term is of special importance in problems involving rotation. The moment of inertia of a body is, in a sense, a measure of the resistance of the body to angular acceleration about the axis to which the value refers.

Some writers use the term " second moment " instead of " moment of inertia " when referring to lines, areas, and volumes, thus limiting the use of the latter term to the case of bodies. Because of the similarity in the methods by which the quantity is obtained, however, engineers usually prefer the use of the one term, moment of inertia, for all cases.

133. Radius of Gyration. The *radius of gyration of a line* with respect to a given axis is the quantity expressed by the following formula:

$$k = \sqrt{\frac{I}{L}} \tag{192}$$

in which I represents the moment of inertia of the line with respect to the given axis, and L represents the total length of the line.

The formulas for the *radii of gyration of areas and volumes* are as follows:

$$k = \sqrt{\frac{I}{A}} \tag{193}$$

$$k = \sqrt{\frac{I}{V}} \tag{194}$$

The formula for the *radius of gyration of a body* is as follows:

$$k = \sqrt{I\frac{g}{W}} \tag{195}$$

In many problems the radius of gyration is known, or calculated, in advance. In such cases the foregoing formulas can be used more conveniently in the following forms:

$$I = Lk^2 \qquad I = Ak^2 \qquad I = Vk^2 \qquad I = \frac{W}{g}k^2 \tag{196}$$

Through his desire to simplify the conception of the radius of gyration, the beginner often attempts to convince himself that it is the distance from the given axis to the center of gravity, or to some other tangible point on the figure. Such attempts are futile. The radius of gyration is a definite distance in any concrete problem, but not to any point whose position can readily be found in advance. The radius of gyration should be thought of simply as a certain linear quantity, related to the moment of inertia in the manner indicated by the formulas of the present article, and often appearing in various engineering formulas in which the moment of inertia otherwise would appear.

Units. In motion problems the radius of gyration must ordinarily be expressed in feet. In the problems of strength of materials it is usually expressed in inches.

PROBLEMS

717. The moment of inertia of a 6 by 12 in. rectangular area, with respect to one of its 6-in. sides, is 3456 in.^{4} Calculate the radius of gyration. Compare the result with the distance from the axis to the center of gravity of the rectangle. *Ans.* 6.93 in.

718. The moment of inertia of a triangular area with respect to its base is given by the formula $I = \frac{1}{12} bh^3$, in which b is the length of the base, and h is the altitude measured from that base Derive the corresponding formula for the radius of gyration.

719. A certain flywheel weighs 30 tons. Its radius of gyration, with respect to the axis of its shaft, is 8.5 ft. Calculate the moment of inertia. *Ans.* 135,000 engineer's units.

720. The moment of inertia of a solid, homogeneous sphere with respect to a diameter is given by the formula, $I = \frac{2}{5}(W/g)r^2$. Derive the corresponding formula for the radius of gyration. *Ans.* $\sqrt{0.4}\, r$.

721. The rim of a certain flywheel weighs 62,500 lb. The outside diameter of the rim is 23 ft, and the radial thickness is 18 in. An approximate value of the moment of inertia in such cases is sometimes found by assuming that the radius of gyration of the rim is equal to the arithmetic mean of the outer and inner radii. Calculate the moment of inertia of this flywheel, making the foregoing assumption.

134. The Parallel-Axis Theorem. *The moment of inertia of a line, area, or volume with respect to any axis is equal to the moment of inertia with respect to the parallel gravity axis, plus the product of the length, area, or volume and the square of the distance between the two axes.*

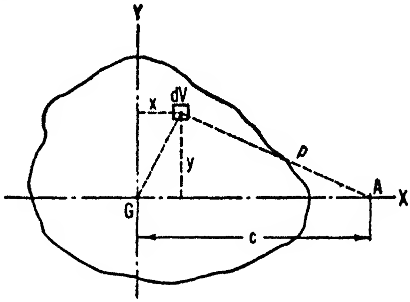

Fig. 383

The foregoing principle also applies to the moment of inertia of a body, if the ratio W/g is used instead of the length, area, or volume. It is called the *parallel-axis theorem* for moments of inertia.

Proof. Let Fig. 383 represent any volume. Let G represent the center of gravity, and A any other point. Let the x-axis be placed, for convenience, so as to pass through G and A. Let $\bar{I}$ represent the moment of inertia of the volume with respect to an axis through G at right angles to the plane of the paper. Let I represent the moment of inertia with respect to an axis through A, also at right angles to the plane of the paper. Let c represent the distance between these two axes. From the figure,

$$I = \int p^2\, dV \qquad [197]$$

$$p^2 = y^2 + (c - x)^2 \qquad [198]$$

Substituting in Eq. 197 the value of p^2 given by Eq. 198,

$$I = \int [y^2 + (c - x)^2]\, dV = \int (y^2 + c^2 - 2\,cx + x^2)\, dV \qquad [199]$$

Equation 199 may be written

$$I = \int y^2\, dV + c^2 \int dV - 2\,c \int x\, dV + \int x^2\, dV \qquad [200]$$

The first and last terms in Eq. 200 may be combined, as follows:

$$\int y^2\, dV + \int x^2\, dV = \int (x^2 + y^2)\, dV = \int q^2\, dV = \bar{I}$$

Furthermore,

$$c^2 \int dV = c^2 V \quad \text{and} \quad 2\,c \int x\, dV = 2\,c\bar{x}V = 0, \quad \text{since } \bar{x} = 0$$

Therefore,

$$I = \bar{I} + Vc^2 \qquad [201]$$

Equation 201 is the algebraic statement of the parallel-axis theorem for volumes. The proof for the case of a line, area, or body is similar to the foregoing proof for volumes. Formulas for these cases are as follows:

$$I = \bar{I} + Lc^2 \qquad [202]$$

$$I = \bar{I} + Ac^2 \qquad [203]$$

$$I = \bar{I} + \frac{W}{g} c^2 \qquad [204]$$

In the use of the parallel-axis theorem it must be remembered that one of the two parallel axes is a gravity axis.

PROBLEMS

722. The moment of inertia of a certain plane area with respect to an axis passing through the center of gravity of the area is equal to 432 in.^{4} The area itself is equal to 50 sq in. Calculate the moment of inertia of the area with respect to an axis parallel to the given gravity axis, and at a distance of 8.6 in. therefrom. *Ans.* 4130 in.^{4}

723. The moment of inertia of a certain body with respect to a given axis outside the body is equal to 539 engineer's units. The weight of the body is 322 lb. Calculate the radius of gyration of the body with respect to the gravity axis parallel to the given axis, if the distance between the two axes is 7 ft.

724. The formula for the moment of inertia of a triangular area with respect to a gravity axis parallel to the base, b, in terms of the altitude, h, is: $I = \frac{1}{36}bh^3$. Derive a formula for the moment of inertia with respect to the base, by means of the parallel-axis theorem. *Ans.* $\frac{1}{12}bh^3$.

725. The formula for the moment of inertia of a rectangular area with respect to the side b is $I = \frac{1}{3}bh^3$, in which h is the length of the other side. Derive a formula for the moment of inertia with respect to a gravity axis parallel to b. *Ans.* $I = \frac{1}{12}bh^3$.

726. The moment of inertia of a certain area with respect to an axis 5 in. from the center of gravity is 4000 in.[4] The area is 120 sq in. Calculate the moment of inertia with respect to a parallel axis 10 in. from the center of gravity.

135. Derivation of Special Formulas for Lines. The engineer seldom finds it necessary to use the general formulas for moment of inertia given in Arts. 131 and 132. Many of the handbooks and textbooks of engineering contain special formulas for the moments of inertia of standard geometrical forms, in terms of convenient dimensions. Such formulas usually make it possible to obtain the desired value without the use of the calculus. A few examples of the methods used in deriving these special formulas will be given in the present article, and in articles to follow.

The moment of inertia of a line is of importance to the engineer chiefly because of the fact that it often provides him with an approximate method of calculating the moment of inertia of a slender bar, by a shorter process than that necessitated by the exact method.

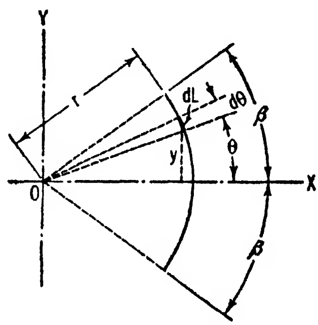

FIG. 384

ILLUSTRATIVE PROBLEM

727. Derive a formula for the moment of inertia of any circular arc, with respect to the axis of symmetry. Also derive a formula for the radius of gyration of the arc with respect to the given axis.

Solution. Let Fig. 384 represent any circular arc, whose radius is r, and whose central angle is 2β. Let the x-axis be taken as the axis of symmetry of the arc. Let dL represent any elementary portion of the length, and let L represent the total length. By Eq. 187,

$$I_x = \int y^2\, dL$$

From Fig. 384,

$$y = r \sin\theta \quad \text{and} \quad dL = r\, d\theta$$

Substituting, and integrating,

$$I_x = \int_{-\beta}^{+\beta} r^2 \sin^2\theta\, r\, d\theta = 2\, r^3 \int_0^{+\beta} \sin^2\theta\, d\theta = 2\, r^3 \left[\frac{\theta}{2} - \frac{\sin 2\,\theta}{4}\right]_0^{\beta}$$

$$= r^3 \left(\beta - \frac{\sin 2\,\beta}{2}\right)$$

which is a convenient formula for the desired moment of inertia. In the use of this formula the angle β must be expressed in radians.

A formula for the radius of gyration can now be obtained from the foregoing formula, by substitution in Eq. 192.

$$k_x = \sqrt{\frac{I_x}{L}} = \sqrt{\frac{r^3\left(\beta - \dfrac{\sin 2\beta}{2}\right)}{2r\beta}} = \frac{r}{2}\sqrt{2 - \frac{\sin 2\beta}{\beta}}$$

PROBLEMS

728. Derive a formula for the moment of inertia of a straight line, with respect to a gravity axis making any angle, θ, with the line. Also, derive a formula for the radius of gyration. *Ans.* $I = \frac{1}{12}L^3 \sin^2 \theta;\ k = \dfrac{L \sin \theta}{\sqrt{12}}$.

729. Derive formulas for the moment of inertia and radius of gyration of a straight line, with respect to a gravity axis at right angles to the line. *Ans.* $I = \frac{1}{12}L^3$; $k = \dfrac{L}{\sqrt{12}}$.

730. Derive formulas for the moment of inertia and radius of gyration of a straight line, with respect to an axis through one end of the line, making any angle, θ, with the line. *Ans.* $I = \frac{1}{3}L^3 \sin^2 \theta;\ k = \dfrac{L \sin \theta}{\sqrt{3}}$.

731. Derive formulas for the moment of inertia and radius of gyration of a straight line, with respect to an axis through one end of the line, making an angle of 90° with the line. *Ans.* $I = \frac{1}{3}L^3;\ k = \dfrac{L}{\sqrt{3}}$.

732. Derive formulas for the moment of inertia and radius of gyration of the circular arc in Prob. 727, Fig. 384, with respect to the y-axis. *Ans.* $I = r^3\left(\beta + \dfrac{\sin 2\beta}{2}\right)$; $k = \dfrac{r}{2}\sqrt{2 + \dfrac{\sin 2\beta}{\beta}}$.

733. Derive formulas for the moment of inertia and radius of gyration of any circular arc, whose radius is r, and whose central angle is 2β, with respect to an axis through the geometric center at right angles to the plane of the arc. *Ans.* $I = 2r^3\beta$; $k = r$.

734. Solve Prob. 727 by the use of rectangular, instead of polar, coordinates.

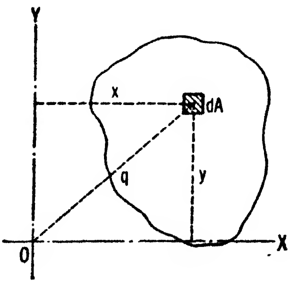

FIG. 385

136. Polar Moment of Inertia.

The moment of inertia of a plane area, with respect to an axis at right angles to the plane of the area, is called the *polar moment of inertia* of the area. The following special principle regarding polar moments of inertia is of considerable utility in many cases:

The polar moment of inertia of a plane area is equal to the sum of the moments of inertia with respect to any two rectangular axes in the plane of the area concurrent with the given polar axis.

Proof. Let Fig. 385 represent any plane area. Let OX and OY represent any two rectangular axes in the plane of the area. Let a polar axis, OZ, be imagined, passing through O, at right angles to the plane of the paper. From the figure,

$$I_z = \int q^2\,dA = \int (x^2 + y^2)\,dA = \int x^2\,dA + \int y^2\,dA \qquad [205]$$

In Eq. 205,

$$\int x^2\,dA = I_y, \quad \text{and} \quad \int y^2\,dA = I_x$$

Therefore,

$$I_z = I_y + I_x \qquad [206]$$

A formula similar to Eq. 206 can easily be derived for plane lines.

Parallel-Axis Theorem. The parallel-axis theorem, as stated and proved in Art. 134, is valid for polar moments of inertia, as is evident from the general nature of the proof given in that article.

PROBLEMS

735. Assuming that the formulas for moment of inertia obtained in Probs. 727 and 732 are correct, derive a formula for the polar moment of inertia of any circular arc with respect to an axis through the geometric center of the arc. Use the principle developed in the present article. Compare the result with that of Prob. 733.

736. The moment of inertia of a circular area with respect to a diameter is equal to $\frac{1}{4}\pi r^4$. Derive a formula for the moment of inertia with respect to a polar axis passing through the center of the circle. *Ans.* $\frac{1}{2}\pi r^4$.

737. The moment of inertia of a square with respect to a gravity axis parallel to two of the sides is equal to $\frac{1}{12}\,b^4$, in which b is the length of each side. Prove that the polar moment of inertia with respect to an axis through one corner is $\frac{2}{3}\,b^4$. Solve in two ways, using the principles of the present article in both cases.

738. The moment of inertia of a circular area with respect to a diameter is $\frac{1}{4}\pi r^4$. Calculate the moment of inertia of a 6-in. circular area with respect to an axis at right angles to the plane of the circle and 8 in. from the center.

137. Derivation of Special Formulas for Areas.

137. Derivation of Special Formulas for Areas. As has been stated, moments of inertia of areas are of great importance in strength of materials, and in the many branches of study based thereon. Furthermore, the moment of inertia of an area may often be used in an approximate method of calculating the moment of inertia of a homogeneous, thin plate.

ILLUSTRATIVE PROBLEMS

739. Derive a formula for the moment of inertia of a rectangular area, with respect to a gravity axis parallel to any side. Also derive a formula for the radius of gyration with respect to the given axis.

Solution. Let Fig. 386 represent any rectangular area. Let b represent the width, and h the height. Let an element of area, dA, be selected as shown. When an element is formed in this manner, a single integration is sufficient, and yet the element conforms to the limitation imposed upon it in Art. 131. By Eq. 188,

$$I_x = \int y^2 \, dA$$

From the figure,

$$dA = b \, dy$$

Therefore,

$$I_x = \int_{-h/2}^{+h/2} y^2 b \, dy = 2\,b \int_0^{+h/2} y^2 \, dy = 2\,b \left[\frac{y^3}{3}\right]_0^{+h/2} = \frac{1}{12}\,bh^3$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{bh^3}{12\,bh}} = \frac{h}{\sqrt{12}}$$

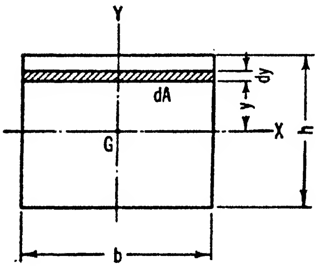

FIG. 386

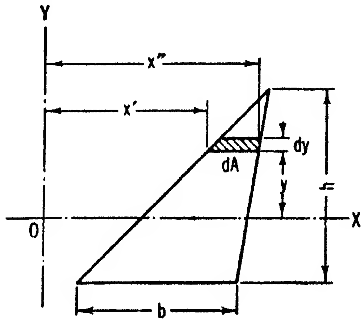

FIG. 387

Particular attention should be given to the fact that the foregoing formulas give the values for a gravity axis parallel to the side b. In a given problem b may be used to designate either side, as may be desired.

740. Derive formulas for the moment of inertia and radius of gyration of a triangular area, with respect to a gravity axis parallel to any side.

Solution. Let Fig. 387 represent any triangular area. Let OX be the axis for which the values are desired. Let the side, or base, parallel to OX be represented by b. Let the altitude, measured to the side b as a base, be represented by h. Let the element be formed as shown, with a view to avoiding a multiple integration. By Eq. 188,

$$I_x = \int y^2 \, dA$$

From the figure,

$$dA = (x'' - x') \, dy$$

From similar triangles,

$$\frac{x'' - x'}{\frac{2}{3}h - y} = \frac{b}{h} \qquad x'' - x' = \frac{b}{h}\left(\tfrac{2}{3}h - y\right)$$

Substituting in the first equation, and integrating,

$$I_x = \int_{-\frac{1}{3}h}^{+\frac{2}{3}h} y^2 \frac{b}{h} \left(\tfrac{2}{3} h - y\right) dy = \frac{b}{h} \int_{-\frac{1}{3}h}^{+\frac{2}{3}h} \left(\tfrac{2}{3} hy^2 - y^3\right) dy$$

$$= \frac{b}{h}\left[\frac{2\,hy^3}{9} - \frac{y^4}{4}\right]_{-\frac{1}{3}h}^{+\frac{2}{3}h} = \frac{1}{36}\,bh^3$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{bh^3}{36}\,\frac{2}{bh}} = \frac{h}{\sqrt{18}}$$

It should be remembered that the gravity axis to which the formulas are referred is parallel to the side b.

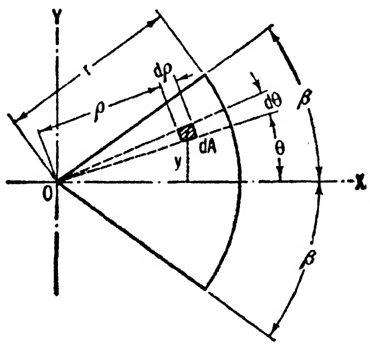

FIG. 388

741. Derive formulas for the moment of inertia and radius of gyration of any circular sector, with respect to its axis of symmetry.

Solution. Let Fig. 388 represent any circular sector. Let the radius be represented by r, and the central angle by 2β. OX is the axis of symmetry. Polar coordinates will be used. Let dA represent any element of area. It would be possible so to select the element as to avoid multiple integration, but the integrations are simple and little would be gained thereby.

$$I_x = \int y^2\, dA$$

From the figure,

$$y = \rho \sin\theta \quad \text{and} \quad dA = \rho\, d\rho\, d\theta$$

$$I_x = \int_0^r \int_{-\beta}^{+\beta} \rho^2 \sin^2\theta\, \rho\, d\rho\, d\theta = 2\int_0^r \int_0^{+\beta} \rho^3\, d\rho \sin^2\theta\, d\theta$$

$$= 2\left[\frac{\theta}{2} - \frac{\sin 2\,\theta}{4}\right]_0^{\beta}\left[\frac{\rho^4}{4}\right]_0^r = \frac{r^4}{4}\left(\beta - \frac{\sin 2\,\beta}{2}\right)$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{\frac{r^4}{4}\left(\beta - \frac{\sin 2\,\beta}{2}\right)}{r^2\beta}} = \frac{r}{2}\sqrt{1 - \frac{\sin 2\,\beta}{2\,\beta}}$$

PROBLEMS

742. Derive formulas for the moment of inertia and radius of gyration of a rectangular area, with respect to any side, b. Use the general formulas, and check the result by the use of the parallel-axis theorem in connection with the answers to Prob. 739. *Ans.* $I = \frac{1}{3}bh^3$; $k = \frac{h}{\sqrt{3}}$.

743. Derive formulas for the moment of inertia and radius of gyration of a rectangular area, with respect to a polar axis through the center of gravity. Use the general formulas, and then check the result by means of Eq. 206, in connection with the answers to Prob. 739. *Ans.* $I = \frac{1}{12}bh(b^2 + h^2)$; $k = \sqrt{\frac{b^2 + h^2}{12}}$.

744. Derive a formula for the moment of inertia of a square area with respect to a diagonal. Let b represent the length of each side. Solve by using the general formulas; then check by using the answer to Prob. 739. *Ans.* $\frac{1}{12}b^4$.

745. Derive formulas for the moment of inertia and radius of gyration of a circular area, with respect to a diameter. Use the general formulas, and check the results by substitution in the formulas obtained in Prob. 741. *Ans.* $I = \frac{1}{4}\pi r^4$; $k = \frac{r}{2}$.

746. Derive formulas for the polar moment of inertia and radius of gyration of a circular area, with respect to a gravity axis. Solve by the general method; then check the results by means of Eq. 206, in connection with the answers to Prob. 745. *Ans.* $I = \frac{1}{2}\pi r^4$; $k = \frac{r}{\sqrt{2}}$.

747. Derive formulas for the moment of inertia and radius of gyration of a triangular area, with respect to the side b. Solve by the use of the general formulas; then check the results by means of the parallel-axis theorem, in connection with the answers to Prob. 740. *Ans.* $I = \frac{1}{12}bh^3$; $k = \frac{h}{\sqrt{6}}$.

748. Derive formulas for the moment of inertia and radius of gyration of the area enclosed within the parabola whose equation is $y^2 = ax$, and the straight line whose equation is $x = b$, with respect to the x-axis. *Ans.* $I = \frac{4}{15}a^{3/2}b^{5/2}$; $k = \sqrt{\frac{ab}{5}}$.

749. Derive formulas for the moment of inertia and radius of gyration of the parabolic area described in Prob. 748, with respect to the y-axis. *Ans.* $I = \frac{4}{7}a^{1/2}b^{7/2}$; $k = b\sqrt{\frac{3}{7}}$.

750. The general equation of the semi-cubical parabola is $ay^2 = x^3$, in which a is a constant. Derive a formula for the moment of inertia of the total area included between this curve and the straight line whose equation is $x = b$, with respect to the x-axis. *Ans.* $I = \frac{4}{33}\frac{b^{5.5}}{a^{1.5}}$.

751. Derive a formula for the moment of inertia of the area in Prob. 750, with respect to the y-axis. *Ans.* $\frac{4}{9}\frac{b^{4.5}}{a^{0.5}}$.

752. Derive formulas for the moment of inertia and radius of gyration of the area enclosed within the ellipse whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with respect to the x-axis. *Ans.* $I = \frac{1}{4}\pi ab^3$; $k = \frac{b}{2}$.

753. Derive formulas for the polar moment of inertia and radius of gyration of the elliptical area described in Prob. 752, with respect to a gravity axis. Solve by the use of the general formulas; then check the results by means of Eq. 206, in connection with the answers to Prob. 752. *Ans.* $I = \frac{1}{4}\pi ab(a^2 + b^2)$; $k = \frac{1}{2}\sqrt{a^2 + b^2}$.

138. Moments of Inertia of Slender Rods and Thin Plates; Approximate Methods. *The radius of gyration of any relatively slender, homogeneous rod or bar of constant cross section, with respect to any axis, is approximately equal to the radius of gyration of the center line of the body with respect to the given axis.*

The radius of gyration of any relatively thin, homogeneous plate or shell of constant thickness, with respect to any axis, is approximately equal to the radius of gyration of the area of either face of the plate with respect to the given axis.

The foregoing principles are almost self-evident, but a simple mathematical proof can be devised, if desired. The principles may be used to save time in problems in which extreme accuracy is unnecessary.

ILLUSTRATIVE PROBLEM

754. Derive an approximate formula for the moment of inertia of a straight, slender, and homogeneous bar, of constant cross section, with respect to gravity axis at any angle, θ, with the long axis of the bar.

Solution. In the answers to Prob. 728, the following formula is given for the radius of gyration of a straight line with respect to a gravity axis making any angle with the line:

$$k = \frac{L \sin \theta}{\sqrt{12}}$$

This value is approximately correct for the bar described in the present problem. Let the total weight of the bar be represented by W.

$$I = \frac{W}{g} k^2 = \frac{W}{g} \frac{L^2 \sin^2 \theta}{12}$$

PROBLEMS

755. Derive an approximate formula for the moment of inertia of a straight, slender, and homogeneous bar, of constant cross section, with respect to a gravity axis at right angles to the length of the bar. *Ans.* $I = \frac{1}{12}(W/g)L^2$.

756. Derive an approximate formula for the moment of inertia of a straight, slender, and homogeneous bar, of constant cross section, with respect to an axis through one end, at any angle, θ, with the length of the bar. Use the answers given in Prob. 730. *Ans.* $I = \frac{1}{3}(W/g)L^2 \sin^2 \theta$.

757. Derive an approximate formula for the moment of inertia of a straight, slender, and homogeneous bar, of constant cross section, with respect to an axis through one end, at right angles to the length of the bar. *Ans.* $I = \frac{1}{3}(W/g)L^2$.

758. Derive an approximate formula for the moment of inertia of a thin, circular, and homogeneous plate, of constant thickness, with respect to a diametric, gravity axis. Use the answers given in Prob. 745. *Ans.* $I = \frac{1}{4}(W/g)r^2$.

759. Given a thin, homogeneous, and rectangular plate, of constant thickness, whose length is h and whose width is b; derive an approximate formula for the moment of inertia with respect to a gravity axis parallel to b. Use the answers to Prob. 739. *Ans.* $I = \frac{1}{12}(W/g)h^2$.

139. Derivation of Special Formulas for Homogeneous Bodies. As the title indicates, the discussion in the present article will be limited to the case of the homogeneous body. The bodies with which the engineer deals, if not entirely homogeneous, can usually be divided into finite parts each of which is homogeneous in itself, in which event the moment of inertia of each homogeneous portion can be calculated separately, and these results can then be added to give the moment of inertia of the entire body.

Formulas for moments of inertia of volumes, as such, do not often appear in engineering books, and will not be derived or given in this book. Any formula giving the moment of inertia of a homogeneous body could, if desired, be transformed into a formula for the moment of inertia of the volume of that body, by dividing by the quantity w/g, in which w represents the weight per unit volume.

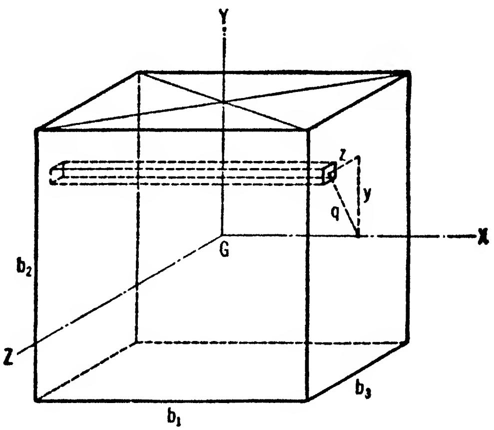

Fig. 389

ILLUSTRATIVE PROBLEMS

760. Derive formulas for the moment of inertia and radius of gyration of a rectangular parallelepiped, with respect to a gravity axis parallel to any edge.

Solution. Let Fig. 389 represent any homogeneous rectangular parallelepiped. Let b_1, b_2, and b_3 represent the lengths of the edges. Let the x-axis be the axis for which the values are desired. A filament, parallel to the x-axis, is convenient for use as the element, dV.

$$I_x = \frac{w}{g}\int q^2\,dV$$

From the figure,

$$q^2 = y^2 + z^2, \quad \text{and} \quad dV = b_1\, dy\, dz$$

Substituting these values in the first equation, and integrating,

$$I_x = \frac{w}{g}\int_{-b_3/2}^{+b_3/2}\int_{-b_2/2}^{+b_2/2} (y^2 + z^2) b_1\, dy\, dz = 4\, b_1 \frac{w}{g}\int_{0}^{+b_3/2}\int_{0}^{+b_2/2} (y^2 + z^2)\, dy\, dz$$

$$= 4\, b_1 \frac{w}{g}\int_0^{b_3/2}\left[\frac{y^3}{3} + z^2 y\right]_0^{b_2/2} dz = 4\, b_1 \frac{w}{g}\int_0^{b_3/2}\left(\frac{b_2^3}{24} + \frac{b_2 z^2}{2}\right) dz$$

$$= 4\, b_1 \frac{w}{g}\left[\frac{b_2^3 z}{24} + \frac{b_2 z^3}{6}\right]_0^{b_3/2} = \frac{b_1 b_2 b_3 w}{12\, g}\,(b_2^2 + b_3^2)$$

In the last expression, $b_1 b_2 b_3 w = W$, the total weight of the body, and so

$$I_x = \frac{1}{12}\frac{W}{g}\,(b_2^2 + b_3^2)$$

$$k_x = \sqrt{\frac{I_x g}{W}} = \sqrt{\frac{b_2^2 + b_3^2}{12}}$$

It should be noticed that in the foregoing formulas b_2 and b_3 are the lengths of the two edges which are at right angles to the x-axis.

761. Derive formulas for the moment of inertia and radius of gyration of a right circular cylinder, with respect to a gravity axis parallel to the base.

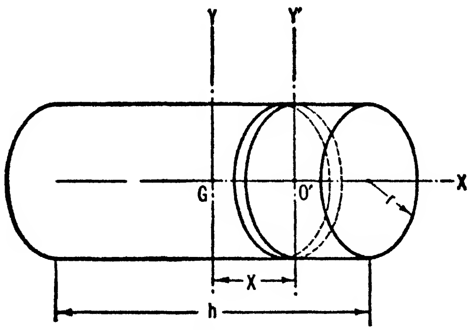

FIG. 390

Solution. Let Fig. 390 represent any right circular cylinder. Let r represent the radius of the cylinder, and let h represent the length. A thin slice, or lamina, will be used as the element, as shown in the figure. This element does not satisfy the requirement that q have a single value for all points in the element. Therefore, the general formulas may not be used. However, the principles stated in Art. 138 lead to a simple solution, as follows:

The axis $O'Y'$, in Fig. 390, is a diameter of the lamina. The radius of gyration of the lamina with respect to $O'Y'$ is equal to the radius of gyration of either of its circular faces with respect to that axis. The answers to Prob. 745 show that the radius of gyration of a circular area with respect to a diameter is equal to $\frac{1}{2}r$. Let the moment of inertia of the lamina with respect to $O'Y'$ be represented by dI'. Then,

$$dI' = \frac{dW}{g}\left(\frac{r}{2}\right)^2 = \frac{\pi r^2\,dx\,w}{g}\left(\frac{r^2}{4}\right) = \frac{\pi r^4 w\,dx}{4\,g}$$

Let the moment of inertia of the lamina with respect to the desired axis, GY, be represented by dI_y. By the parallel-axis theorem, Art. 134,

$$dI_y = dI' + \frac{dW}{g}x^2$$

$$dI_y = \frac{\pi r^4 w\,dx}{4\,g} + \frac{\pi r^2 w\,dx}{g}x^2$$

Integrating,

$$I_y = 2\int_0^{h/2}\left(\frac{\pi r^4 w\,dx}{4\,g} + \frac{\pi r^2 w\,dx}{g}x^2\right) = \frac{2\,\pi r^2 w}{g}\left[\frac{r^2 x}{4} + \frac{x^3}{3}\right]_0^{h/2}$$

$$I_y = \frac{1}{4}\frac{W}{g}\left(r^2 + \frac{h^2}{3}\right)$$

$$k_y = \frac{1}{2}\sqrt{r^2 + \frac{h^2}{3}}$$

The principles of Art. 138 give approximate results for thin plates of finite thickness. For plates of infinitesimal thickness they are exact. Therefore, the foregoing results are exact.

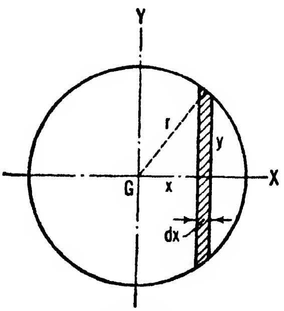

Fig. 391

762. Derive formulas for the moment of inertia and radius of gyration of a sphere, with respect to a diameter.

Solution. Let Fig. 391 represent any sphere. Let GX represent the desired axis. Let a lamina be used for the element, placed at right angles to GX. From the principles of Art. 138, utilizing the answer to Prob. 746,

$$I_x = 2\int_0^r \frac{\pi y^2\,dx\,w}{g}\left(\frac{y}{\sqrt{2}}\right)^2$$

From the figure,

$$y^2 = r^2 - x^2$$

$$I_x = \frac{\pi w}{g}\int_0^r (r^2 - x^2)^2\,dx = \frac{\pi w r^4}{g}\Big[x\Big]_0^r - \frac{2\,\pi w r^2}{g}\left[\frac{x^3}{3}\right]_0^r + \frac{\pi w}{g}\left[\frac{x^5}{5}\right]_0^r$$

$$I_x = \frac{2}{5}\frac{W}{g}r^2$$

$$k_x = \sqrt{\frac{I_x g}{W}} = r\sqrt{\frac{2}{5}}$$

PROBLEMS

763. Solve Prob. 760, using a lamina for the element, and following a method similar to that used in Prob. 761.

764. Derive formulas for the moment of inertia and radius of gyration of a right circular cylinder, with respect to its geometric axis. *Ans.* $I = \frac{1}{2}(W/g)r^2$; $k = \dfrac{r}{\sqrt{2}}$.

765. Derive formulas for the moment of inertia and radius of gyration of a right circular cone, with respect to its geometric axis. *Ans.* $I = \frac{3}{10}(W/g)r^2$; $k = r\sqrt{0.3}$.

766. The equation of the semi-cubical parabola is $ay^2 = x^3$, in which a is a constant. Let the total plane area enclosed between this curve and the straight line $x = b$ be rotated around the x-axis, through an angle of 180°. Let the solid of revolution thus generated be regarded as a homogeneous body. Derive formulas for the moment of inertia and radius of gyration of the body with respect to the x-axis. *Ans.* $I = \dfrac{2}{7}\dfrac{W}{g}\dfrac{b^3}{a}$; $k = \sqrt{\dfrac{2}{7}\dfrac{b^3}{a}}$.

767. Derive a formula for the moment of inertia of a right circular cone, with respect to an axis through the apex, parallel to the base. *Ans.* $\dfrac{3}{20}\dfrac{W}{g}(4h^2 + r^2)$.

768. The area bounded by the parabola $y^2 = ax$, the x-axis, and the straight line $x = b$ is revolved about the x-axis, through an angle of 360°. Derive a formula for the moment of inertia of a homogeneous body having such a form, with respect to the x-axis. *Ans.* $I = \frac{1}{3}(W/g)ab$.

769. An elliptical area is revolved about the x-axis, generating a complete solid of revolution. The equation of the ellipse is $x^2b^2 + y^2a^2 = a^2b^2$. Derive a formula for the moment of inertia of a homogeneous body having this form, with respect to the x-axis. *Ans.* $I = \frac{2}{5}(W/g)b^2$.

770. Derive formulas for the moment of inertia and radius of gyration of a right circular cylinder with respect to a diameter of the base. *Ans.* $I = \frac{1}{4}(W/g)(r^2 + \frac{4}{3}h^2)$; $k = \frac{1}{2}\sqrt{r^2 + \frac{4}{3}h^2}$.

771. Derive formulas for the moment of inertia and radius of gyration of a right circular cone, with respect to a diameter of the base. *Ans.* $I = \dfrac{1}{10}\dfrac{W}{g}(h^2 + \frac{3}{2}r^2)$; $k = \sqrt{\dfrac{h^2 + 1.5\,r^2}{10}}$.

772. Derive formulas for the moment of inertia and radius of gyration of a right circular cone, with respect to a gravity axis parallel to the base. Use the parallel-axis theorem, in connection with the results of Prob. 771.

Ans. $I = \dfrac{3}{20}\dfrac{W}{g}\left(\dfrac{h^2}{4} + r^2\right)$; $k = \sqrt{\dfrac{3(\frac{1}{4}h^2 + r^2)}{20}}$.

140. Moment of Inertia by Finite Summation. *The figure representing a line, area, volume, or body may be divided into finite parts, and the moment of inertia of each part may then be calculated. The moment of inertia of the whole is equal to the sum of the moments of inertia of the parts.*

In some problems it can be seen that the arbitrary addition of one or more portions to the figure will produce a new figure whose moment of inertia can be calculated more easily than the moment of inertia of the original figure. The moments of inertia of all such added portions are calculated, and are then subtracted from the moment of inertia of the altered figure. The result is the moment of inertia of the original figure.

The proofs of the foregoing statements are simple, and will not be given.

141. Concerning Subsequent Problems. The remaining problems in the present chapter are designed primarily to illustrate the use of the various special formulas in the solution of numerical problems and, in some cases, in the derivation of other useful special formulas. The method of finite summation will be used extensively. Unless otherwise specified, it will be understood that any formulas or results obtained in previous problems may be utilized.

142. Applications of the Special Formulas for Lines. As has been indicated, the engineer is not often concerned with the moment of inertia of a line, as such, but is interested in the fact that the radius of gyration of a slender, uniform, and homogeneous rod is approximately equal to the radius of gyration of the center line of the rod (Art. 138).

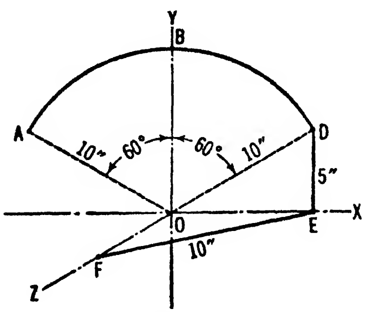

Fig. 392

Illustrative Problem

773. Figure 392 represents a slender, homogeneous rod, of constant cross section, which has been bent into the form $ABDEF$. The rod weighs 1.5 lb per lin ft. The portion ABD is in the form of a circular arc, and lies in the xy-plane. DE is parallel to the y-axis, and EF lies in the xz-plane. Calculate an approximate value of the moment of inertia of the rod, with respect to the y-axis.

Solution.

$$\text{Weight of } ABD = \left(2\,\pi\,\tfrac{10}{12} \times \tfrac{120}{360}\right)1.5 = 2.62 \text{ lb}$$

$$\text{Weight of } DE = \left(\tfrac{5}{12}\right)1.5 = 0.625 \text{ lb}$$

$$\text{Weight of } EF = \left(\tfrac{10}{12}\right)1.5 = 1.25 \text{ lb}$$

Moment of Inertia of ABD. In Prob. 727, the radius of gyration of a circular arc, with respect to its axis of symmetry, was found to be

$$k = \frac{r}{2}\sqrt{2 - \frac{\sin 2\beta}{\beta}}$$

By Art. 138, the foregoing formula is approximately correct for the radius of gyration of ABD, with respect to the y-axis. Therefore,

$$k_y^2 = \left(\frac{10}{2 \times 12}\right)^2 \left(2 - \frac{\sin 120^\circ}{\pi/3}\right) = 0.204 \text{ ft}^2$$

The moment of inertia of ABD is, then, approximately

$$I_y = \frac{W}{g}k_y^2 = \frac{2.62}{32.2}0.204 = 0.0166 \text{ engineer's unit}$$

Moment of Inertia of DE. The moment of inertia of the portion DE, with respect to a gravity axis parallel to the y-axis, is approximately equal to zero. The distance, OE, is equal to $10 \cos 30^\circ/12 = 0.722$ ft. Therefore, by the parallel-axis theorem, Art. 134,

$$I_y = 0 + \frac{W}{g}(OE)^2 = \frac{0.625}{32.2}(0.722)^2 = 0.0101 \text{ engineer's unit}$$

Moment of Inertia of EF. First, the moment of inertia of the portion EF must be calculated with respect to a gravity axis parallel to the y-axis. The answer to Prob. 755 is the desired formula.

$$\bar{I} = \frac{1}{12}\frac{W}{g}L^2 = \frac{1.25\,(\frac{10}{12})^2}{12 \times 32.2} = 0.00225 \text{ engineer's unit}$$

The distance, c, from the center of gravity of EF, to the y-axis, is equal to 5 in. Therefore, by the parallel-axis theorem,

$$I_y = 0.00225 + \frac{1.25}{32.2}\left(\frac{5}{12}\right)^2 = 0.00899 \text{ engineer's unit}$$

Total Moment of Inertia of the Rod. The moment of inertia of the entire rod can now be obtained.

$$I_y = 0.0166 + 0.0101 + 0.00899 = 0.0357 \text{ engineer's unit}$$

PROBLEMS

774. A certain slender, uniform bar is 48 in. long and weighs 35 lb. Calculate the moment of inertia with respect to an axis through one end, at right angles to the length, by the approximate method. *Ans.* 5.80 engineer's units.

775. A certain piece of fir lumber is 12 by 12 in. in cross section, and 12 ft long. The material weighs 30 lb per cu ft. Calculate the moment of inertia of the body with respect to a gravity axis at right angles to two of the lateral faces. Solve by the exact method, using the results of Prob. 760. Solve also by the approximate method for slender rods (Art. 138). Calculate the percentage error of the approximate result.

776. Calculate the moment of inertia of a uniform, round steel bar, 2 in. in diameter and 3 ft long, with respect to an axis through one end at an angle of 60° with the axis of the bar. The material weighs 490 lb per cu ft. Use the approximate method of Art. 138. *Ans.* 2.24 engineer's units.

777. Derive a formula for the moment of inertia of the arc of a full circle, with respect to a tangent. Solve without using the calculus. *Ans.* $3\pi r^3$.

778. The sides of a certain right triangle are 6, 8, and 10 in. long. Calculate the moment of inertia of the boundary line of the triangle, with respect to the 10-in. side.

779. Calculate the moment of inertia of the boundary line of the triangle in Prob. 778, with respect to an axis at right angles to the plane of the triangle and passing through the intersection of the 6- and 8-in. sides.

780. Calculate the moment of inertia of the line $AOBC$, in Fig. 308, Art. 101, with respect to the x-axis. *Ans.* 7710 in.³

781. Calculate the moment of inertia of the line $OABO$, in Fig. 309, Art. 101, with respect to the x-axis.

782. Calculate the moment of inertia of the line $ABCDE$, in Fig. 311, Art. 101, with respect to the z-axis. *Ans.* 7130 in.³

783. Calculate the moment of inertia of the line $ABCD$, in Fig. 307, Art. 101, with respect to the z-axis.

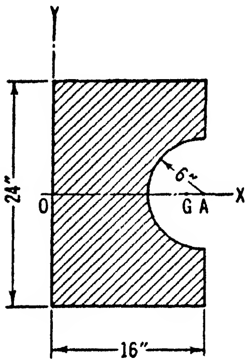

Fig. 393

143. Applications of the Special Formulas for Areas. The problems to follow include a few examples of areas whose moments of inertia are often of importance in structural design.

Illustrative Problem

784. Calculate the moment of inertia of the area shown in Fig. 393, with respect to the y-axis.

Solution. The problem can be solved most readily by calculating the moment of inertia of the full rectangle, 16 by 24 in., and by subtracting therefrom the moment of inertia of the semicircular area.

Moment of Inertia of the Rectangle. The answers to Prob. 742, Art. 137, contain a formula directly applicable to the present case.

$$I = \tfrac{1}{3}\,bh^3 = \frac{24\,(16)^3}{3} = 32{,}800 \text{ in.}^4$$

Moment of Inertia of the Semicircular Area. Let A represent the geometric center of the semicircle, and let G represent its center of gravity. By Prob. 496, Art. 97,

$$AG = \frac{4r}{3\pi} = \frac{4 \times 6}{3\pi} = 2.55 \text{ in.}$$

$$OG = 16 - AG = 16 - 2.55 = 13.5 \text{ in.}$$

The moment of inertia of the semicircle with respect to a vertical axis through

A can be calculated from the formula given in the answers to Prob. 745, Art. 137,

$$I' = \tfrac{1}{2}(\tfrac{1}{4}\pi r^4) = \frac{\pi(6)^4}{8} = 509 \text{ in.}^4$$

By the parallel-axis theorem, Art. 134, the moment of inertia of the semicircle with respect to a vertical axis through G is as follows:

$$I = \bar{I} + Ac^2 \qquad 509 = I_G + \frac{\pi(6)^2}{2} \times (2.55)^2 \qquad I_G = 141 \text{ in.}^4$$

Again by the parallel-axis theorem, I_y for the semicircle now can be calculated, as follows:

$$I_y = 141 + \frac{\pi(6)^2}{2} \times (13.5)^2 = 10{,}400 \text{ in.}^4$$

Moment of Inertia of the Shaded Area. The moment of inertia of the shaded area now can be obtained by subtraction.

$$I_y = 32{,}800 - 10{,}400 = 22{,}400 \text{ in.}^4$$

PROBLEMS

785. Calculate the moment of inertia of the shaded area in Fig. 393, with respect to the x-axis. *Ans.* 17,900 in.^{4}

786. Calculate the moment of inertia of the shaded area in Fig. 317, Art. 102, with respect to the x-axis.

787. Calculate the moment of inertia of the standard angle section in Fig. 394, with respect to the x-axis. *Ans.* 62.4 in.^{4}

788. Calculate the moment of inertia of the standard angle section in Fig. 394, with respect to the y-axis.

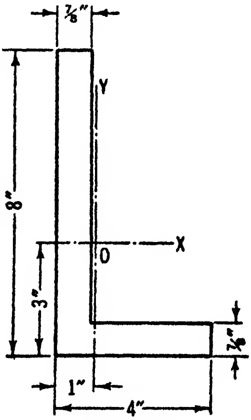

FIG. 394

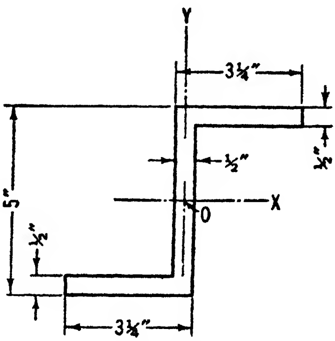

FIG. 395

789. Calculate the moment of inertia and radius of gyration of the standard zee section, in Fig. 395, with respect to the x-axis. *Ans.* 19.2 in.^{4}

790. Calculate the moment of inertia of the standard zee section in Fig. 395, with respect to the y-axis. Also calculate the moment of inertia with respect to an axis through O at right angles to the plane of the area, using the results of this problem and of Prob. 789.

791. Calculate the moment of inertia of the shaded area in Fig. 312, Art. 102, with respect to the x-axis.

Ans. 9270 in.^{4}

792. Derive a formula for the moment of inertia of the shaded area in Fig. 313, Art. 102, with respect to the x-axis.

793. Derive a formula for the moment of inertia of any trapezoidal area, with respect to the longer base. Represent the longer and shorter bases by B and b, respectively, and the height by h. *Ans.*

$$I = \frac{h^3(B + 3b)}{12}.$$

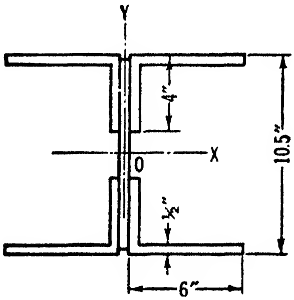

Fig. 396

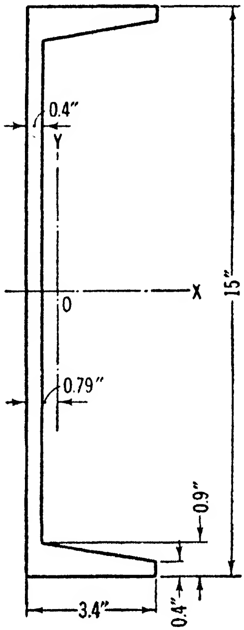

Fig. 397

794. Derive a formula for the moment of inertia of any trapezoidal area with respect to a gravity axis parallel to the two bases. Use the notation of Prob. 793, and also the result of that problem. The answer to Prob. 528 also may be used.

$$Ans. \quad I = \frac{h^3}{36}\frac{(B^2 + 4Bb + b^2)}{(B + b)}.$$

795. A piece is cut from a flat steel plate, in the form of an ellipse. The major and minor axes of the piece are 24 in. and 18 in., respectively. The thickness is 1 in. throughout. The material weighs 490 lb per cu ft. Calculate the moment of inertia of the plate with respect to the major axis, using the approximate method for thin plates (Art. 138) in connection with the results of Prob. 752, Art. 137.

796. Figure 396 represents the cross section of a type of column frequently used, consisting of four angles riveted to a rectangular web plate. In the present case the web plate is 10 by $\frac{1}{2}$ in. in cross section. The four angles are alike, each having the dimensions shown. Calculate the moment of inertia and radius of gyration of the column section, with respect to the x-axis. *Ans.* 412 in.^{4}; 4.14 in.

797. Figure 397 represents the cross section of a standard channel. Calculate the moment of inertia and radius of gyration of the section with respect to the gravity axis, OX.

798. Calculate the moment of inertia of the area in Fig. 319, Art. 102, with respect to the x-axis.

144. Applications of the Special Formulas for Bodies. Bodies composed of more than one kind of material can be dealt with by finite summation, if it is possible to divide the body into finite parts, each of which is homogeneous in itself.

ILLUSTRATIVE PROBLEMS

799. A certain piece of oak lumber is 4 by 4 by 24 in. The material weighs 45 lb per cu ft. Calculate the moment of inertia of the piece with respect to an axis lying in one of the 4 by 4 in. faces, passing through the center of the face and parallel to two of its sides. Solve by the exact method. Solve also by the approximate method of Art. 138. Calculate the percentage error of the approximate result.

Solution. The weight of the body, $W = 4 \times 4 \times 24 \times 45/1728 = 10$ lb. First calculate the moment of inertia of the body with respect to a gravity axis at right angles to its length. This can be done by means of the formula in Prob. 760, Art. 139. In the present case: $b_2 = 4$ in. $= \frac{1}{3}$ ft, $b_3 = 24$ in. $=$ 2 ft. Substituting,

$$I_x = \frac{1}{12}\frac{W}{g}(b_2^2 + b_3^2)$$

$$\bar{I} = \frac{10}{12 \times 32.2}\left[\left(\frac{1}{3}\right)^2 + (2)^2\right] = 0.1064 \text{ engineer's unit}$$

Then, by the parallel-axis theorem (Art. 134),

$$I = \bar{I} + \frac{W}{g}c^2 \qquad I = 0.106 + \frac{10}{32.2}\left(\frac{12}{12}\right)^2 = 0.4170 \text{ engineer's unit}$$

Approximate Method. In Prob. 731, Art. 135, the radius of gyration of a straight line with respect to an axis at right angles to the line, through one end, is found to be $L/\sqrt{3}$. By Art. 138, this value should be approximately correct for the timber in the present problem. Therefore, $k = 2/\sqrt{3}$ ft.

$$I = \frac{W}{g}k^2 = \frac{10}{32.2}\left(\frac{2}{\sqrt{3}}\right)^2 = 0.4141 \text{ engineer's unit}$$

The percentage error in the approximate result is equal to $(0.4170 - 0.4141) \times 100/0.4170 = 0.695$ per cent. The given body would hardly be called slender, in the ordinary sense of the term, and yet the approximate method gives a fairly accurate result.

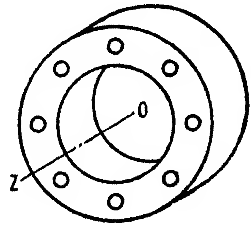

FIG. 398

800. Figure 398 represents a cylindrical ring, having inner and outer diameters equal to 12 in. and 20 in., respectively, and having a length of 6 in. The ring is made of wood, weighing 45 lb per cu ft. Eight holes are drilled entirely through the ring, in a direction parallel to the geometric axis, OZ. Each hole is 2 in. in diameter, and its own axis is at a distance of 8 in. from OZ. A solid, steel pin is fitted into each hole, completely filling it. Calculate the moment of inertia of the entire body with respect to OZ.

Solution.

$$\text{Wt. of 20-in. solid wood cylinder} = \frac{\pi(10)^2 6}{1728} 45 = 49.1 \text{ lb}$$

$$\text{Wt. of 12-in. solid wood cylinder} = \frac{\pi(6)^2 6}{1728} 45 = 17.7 \text{ lb}$$

$$\text{Wt. of 2-in. solid wood cylinder} = \frac{\pi(1)^2 6}{1728} 45 = 0.491 \text{ lb}$$

$$\text{Wt. of 2-in. solid steel cylinder} = \frac{\pi(1)^2 6}{1728} 490 = 5.35 \text{ lb}$$

In the answers to Prob. 764 is found the formula for the moment of inertia of a right circular cylinder, with respect to its geometric axis.

$$I = \frac{1}{2}\frac{W}{g}r^2$$

For the 20-in. solid wood cylinder,

$$I_z = \frac{49.1(10)^2}{2 \times 32.2 \times 144} = 0.529 \text{ engineer's unit}$$

For the 12-in. solid wood cylinder,

$$I_z = \frac{17.7(6)^2}{2 \times 32.2 \times 144} = 0.0687 \text{ engineer's unit}$$

For the 2-in. wood cylinders, using the parallel-axis theorem, Art. 134,

$$I_z = 8\left[\frac{0.491(1)^2}{2 \times 32.2 \times 144} + \frac{0.491}{32.2}\left(\frac{8}{12}\right)^2\right] = 0.0546 \text{ engineer's unit}$$

For the 2-in. steel cylinders,

$$I_z = 8\left[\frac{5.35(1)^2}{2 \times 32.2 \times 144} + \frac{5.35}{32.2}\left(\frac{8}{12}\right)^2\right] = 0.595 \text{ engineer's unit}$$

The moment of inertia of the entire body can now be calculated, as follows:

$$I_z = 0.529 - 0.0687 - 0.0546 + 0.595 = 1.00 \text{ engineer's unit}$$

PROBLEMS

801. The rim of a certain cast-iron flywheel has an outside diameter of 22 ft, and a radial thickness of 30 in. The width across the face is 24 in. Calculate the moment of inertia of the rim, with respect to the axis of rotation of the wheel, assuming that the radius of gyration is equal to the arithmetic mean of the inner and outer radii. *Ans.* 407,000 engineer's units.

802. Solve Prob. 801 by the exact method. Compare the result with the answer to that problem.

803. Calculate the moment of inertia of a solid, steel sphere, 6 in. in diameter, with respect to an axis 18 in. from the center of the sphere. The material weighs 490 lb per cu ft. Solve by the exact method. Also solve approximately by assuming that the radius of gyration is equal to the distance between the given axis and the center of the sphere. *Ans.* 2.27 engineer's units; 2.24 engineer's units.

804. A certain cylindrical disk of cast iron is 18 in. in diameter and 4 in. thick. The material weighs 450 lb per cu ft. Calculate the moment of inertia with respect to an axis at right angles to the plane faces and 3 in. from the center of the disk.

805. A certain round, steel bar is 2 in. in diameter and 16 in. long. The material weighs 490 lb per cu ft. Calculate the moment of inertia with respect to an axis through one end, making an angle of 60° with the length of the bar. Solve by the approximate method of Art. 138. *Ans.* 0.197 engineer's unit.

806. A certain right circular cone is 12 in. high and 4 in. in diameter at the base. A cylindrical hole 2 in. in diameter and 4 in. deep is drilled, axially, in the base. The material is aluminum weighing 165 lb per cu ft. Calculate the moment of inertia with respect to an axis through the apex, parallel to the base.

807. Given a square, flat plate of cast iron, 18 by 18 by 1 in., weighing 450 lb per cu ft. Calculate the moment of inertia with respect to a gravity axis parallel to four of the 18-in. edges. Solve by the exact method. Check by the approximate method of Art. 138. Calculate the percentage error of the approximate result. *Ans.* 0.4930 engineer's unit; 0.4915 engineer's unit; 0.304 %.

808. Calculate the moment of inertia of the cone in Prob. 806, with respect to its geometric axis, if the hole is filled with lead weighing 700 lb per cu ft.

809. A certain block of wood measures 12 by 12 by 15 in. A hole, 6 in. in diameter, is bored through the center of the block, in a direction at right angles to two of the 12 by 15 in. faces. The hole is filled by a plug of steel weighing 490 lb per cu ft. The wood weighs 30 lb per cu ft. Calculate the moment of inertia of the body with respect to a central axis parallel to the 15-in. edges. *Ans.* 0.472 engineer's unit.

810. Derive a formula for the moment of inertia of a homogeneous rectangular parallelepiped, the lengths of whose edges are b_1, b_2, and b_3, with respect to one of the b_1 edges. *Ans.* $I = \frac{1}{3}\frac{W}{g}(b_2^2 + b_3^2)$.

811. A solid, wooden disk, 36 in. in diameter and 4 in. thick, is fitted with a steel rim whose outer diameter is 38 in., and whose sides are flush with the surface of the disk. Calculate the moment of inertia of the body with respect to its geometric axis. Use 40 and 490 lb per cu ft for the unit weights of the two materials.

145. Moments of Inertia with Respect to Planes and Points. In the preceding articles the term "moment of inertia" was usually accompanied by the phrase "with respect to," followed by the designation of an axis. The term moment of inertia is occasionally encountered in situations in which a plane, or a point, is designated, instead of an axis. Moments of inertia with respect to planes and points are of limited importance in engineering, however, and will not be discussed in this book.

146. Moments of Inertia of Irregular Forms. In the case of an irregular figure that cannot be subjected to exact mathematical analysis, or a figure which is so complex that a mathematical analysis would be extremely laborious, recourse may be had to various approximate methods. Graphical constructions are usually of assistance in such solutions. Experimental methods are also used in many cases. The description of these special methods is beyond the scope of this book, but can be found in some of the more extended treatments of the subject.

CHAPTER XVII

KINEMATICS OF ROTATION

147. Definition of Rotation. *Rotation* is that motion of a body in which the particles move in circular paths whose centers lie on a fixed straight line. The term rotation is applicable only to bodies that are assumed rigid; therefore, the planes of the circles in which the particles move are at right angles to the line on which the centers lie. This fixed line is called the *axis of rotation.*

148. Angular Velocity. The terms " linear velocity " and " linear acceleration " refer properly to single points or particles. With the exception of the translating body, the use of these terms with reference to bodies should be avoided, unless it is distinctly understood that some definite point, or particle, of the body is in mind. The terms " angular velocity " and " angular acceleration," however, refer primarily to finite bodies; in some books they are also used in connection with points, but they will not be so used in this book.

The linear velocity of a moving point at any instant is the time rate at which the point is traversing distance at that instant. In this definition the word " distance " is used in the ordinary sense, implying linear measurement. The *angular velocity of a rotating body may be defined as the time rate at which the body is traversing angular distance at the given instant.* By angular distance is meant the angle through which the body turns in any given time or, more specifically, the angle described by any straight line lying in a plane at right angles to the axis of rotation and always passing through the same particles of the body. Since the body is assumed rigid, all such lines describe equal angles during any given interval of time.

Let Fig. 399 represent a body rotating about an axis through O at right angles to the plane of the paper. Let A represent any given particle of the body. Let OA represent a line which, throughout the motion, passes through the moving particle A, lies in a plane at right angles to the axis of rotation, and intersects that axis at O. Let OX represent any convenient stationary axis through O, at right angles to the axis of rotation. Let θ represent the angle between the stationary axis OX and the moving line OA, at any instant. The direction of measurement of θ will be considered to be from OX to OA. If this measurement is made in a counterclockwise direction, θ will be considered positive; if the measurement is made in a clockwise direction, θ will be considered negative.

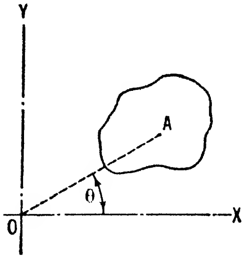

FIG. 399

The angle θ is the angular distance from OX to OA. The time rate at which the body is traversing angular distance at any instant is equal to the rate at which θ is changing with respect to t at that instant. Let ω represent the angular velocity of the body. The general formula for the angular velocity of a rotating body at any instant is, then,

$$\omega = \frac{d\theta}{dt} \tag{207}$$

A plus sign accompanying an angular velocity obtained from Eq. 207 indicates that the body is rotating in the counterclockwise direction. A minus sign indicates clockwise rotation.

Units. The units most commonly used for angular velocity are radians per second and revolutions per minute. The unit of an angular velocity obtained from Eq. 207 will depend on the units in which θ and t are expressed.

ILLUSTRATIVE PROBLEMS

812. A certain body rotates in accordance with the law $\theta = t^3 + 10$, in which θ is expressed in radians and t is expressed in seconds. Calculate the angular velocity of the body at the instant when $t = 5$ sec.

Solution.

$$\theta = t^3 + 10$$

$$\omega = \frac{d\theta}{dt} \qquad \omega = \frac{d(t^3 + 10)}{dt} = 3t^2$$

The foregoing equation can be used to calculate the value of ω for any value of t. At the instant when $t = 5$ sec,

$$\omega = 3(5)^2 = 75 \text{ rad/sec}$$

813. A certain body rotates in accordance with the law $\omega = 6 - 4t$, in which ω is expressed in radians per second, and t is expressed in seconds. It is also known that $\theta = 2$ radians at the instant when $t = 0$. Calculate the value of θ for the instant when $t = 10$ sec.

Solution.

$$\omega = 6 - 4t \qquad \omega = \frac{d\theta}{dt} \qquad d\theta = \omega\, dt$$

Substituting,

$$d\theta = (6 - 4\,t)\,dt$$

Integrating,

$$\theta = 6\,t - 2\,t^2 + C$$

in which C represents the constant of integration. The problem states that $\theta = 2$ radians at the instant when $t = 0$. These values must satisfy the foregoing equation. Substituting,

$$2 = 0 - 0 + C \qquad C = 2$$

The completed (θ, t) equation is, then,

$$\theta = 6\,t - 2\,t^2 + 2$$

For the instant when $t = 10$ sec,

$$\theta = 6(10) - 2(10)^2 + 2 = -138 \text{ rad}$$

PROBLEMS

814. The crankshaft of a certain automobile rotates at a speed of 3000 rpm. Express this angular velocity in radians per second.

815. Express an angular velocity of 480 rad per sec, in revolutions per minute.

816. The flywheel of a certain engine is 12 ft in diameter. The wheel rotates at a constant angular velocity of 20 rad per sec. Calculate the linear velocity of a point on the rim of the wheel. *Ans.* 120 ft/sec.

817. A body rotates in accordance with the law $\theta = 4t^2 - 10$, in which θ is in revolutions and t is in minutes. Calculate the angular velocity of the body, in radians per second, for the instant when $t = 4$ min.

818. A certain pulley rotates in accordance with the law $\theta = 6t$, in which θ is in radians and t is in seconds. Calculate the angular velocity of the pulley, in revolutions per minute. *Ans.* 57.3 rpm.

819. A body rotates in accordance with the law $\omega = 2t^{3/2}$, in which ω is in radians per second and t is in seconds. It is also known that $\theta = +4$ rad at the instant when $t = 0$. Calculate θ and ω for the instant when $t = 9$ sec.

820. A certain flywheel, 4 ft in diameter, rotates in accordance with the law $\theta = 100t^2 + 50t - 4$, in which θ is in revolutions and t is in minutes. Calculate the linear velocity of a point on the rim of the wheel at the instant when $t = 2$ min.

821. A body rotates in accordance with the law $\omega = t^{-0.6}$, in which ω is in radians per second and t is in seconds. It is also known that $\theta = 0$ at the instant when $t = 0$. Calculate ω and θ for the instant when $t = 10$ sec. *Ans.* 0.251 rad/sec; 6.28 rad.

149. Angular Acceleration. *The angular acceleration of a rotating body at any instant is the time rate at which the angular velocity of the body is changing at that instant.*

Let α represent the angular acceleration at any instant. The angular acceleration, then, is equal to the rate at which ω is changing with respect to t, and is given by the following general formula:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \qquad [208]$$

If the angular velocity and the angular acceleration agree in sign the rapidity of the rotary motion is increasing; if they disagree, the rapidity is decreasing.

In certain problems the following formula is more convenient than is Eq. 208. It is obtained by eliminating dt between Eqs. 207 and 208.

$$\omega\, d\omega = \alpha\, d\theta \qquad [209]$$

Units. The units most commonly used for angular acceleration are radians per second per second and revolutions per minute per minute. The unit of an angular acceleration obtained from Eq. 208 will depend on the units in which ω and t, or θ and t, are expressed.

Illustrative Problems

822. A certain body rotates in accordance with the law $\theta = t^{5/2} - t^2 + 10$. In this equation, θ is expressed in revolutions, and t is expressed in minutes. Calculate the angular velocity and angular acceleration of the body, for the instant when $t = 4$ min.

Solution.

$$\theta = t^{5/2} - t^2 + 10$$

$$\omega = \frac{d\theta}{dt} \qquad \omega = \frac{d(t^{5/2} - t^2 + 10)}{dt} = 2.5\, t^{3/2} - 2\,t$$

At the instant when $t = 4$ min,

$$\omega = 2.5(4)^{3/2} - 8 = 20 - 8 = 12 \text{ rev/min}$$

$$\alpha = \frac{d\omega}{dt} \qquad \alpha = \frac{d(2.5\, t^{3/2} - 2\,t)}{dt} = 3.75\, t^{1/2} - 2$$

At the instant when $t = 4$ min,

$$\alpha = 3.75(4)^{1/2} - 2 = 7.5 - 2 = 5.5 \text{ rev/min}^2$$

At the given instant ω and α are both positive, showing that the body is rotating in the counterclockwise direction, at increasing speed.

823. A body rotates in accordance with the law $\alpha = 0.84t^{0.4}$, in which α is in radians per second per second and t is in seconds. It is also known that $\theta = +4$ rad and $\omega = -6$ rad per sec at the instant when $t = 0$. Calculate θ, ω, and α for the instant when $t = 10$ sec.

Solution. From the problem, $\alpha = 0.84t^{0.4}$. At the instant when $t = 10$ sec.

$$\alpha = 0.84(10)^{0.4} = +2.11 \text{ rad/sec}^2$$

$$\alpha = \frac{d\omega}{dt} \qquad d\omega = \alpha\, dt \qquad d\omega = (0.84t^{0.4})\, dt$$

Integrating,

$$\omega = 0.6t^{1.4} + C_1$$

in which C_1 represents the constant of integration. From the problem, $\omega = -6$ when $t = 0$. This pair of simultaneous values must satisfy the equation for ω obtained above. Substituting, to find the value of C_1,

$$-6 = 0 + C_1 \qquad C_1 = -6$$

The general equation for ω in this problem is, therefore,

$$\omega = 0.6t^{1.4} - 6$$

At the instant when $t = 10$ sec,

$$\omega = 0.6(10)^{1.4} - 6 = +9.07 \text{ rad/sec}$$

$$\omega = \frac{d\theta}{dt} \qquad d\theta = \omega\, dt \qquad d\theta = (0.6t^{1.4} - 6)\, dt$$

Integrating,

$$\theta = 0.25t^{2.4} - 6t + C_2$$

From the problem, $\theta = +4$ when $t = 0$. Substituting these values in the foregoing equation,

$$+4 = 0 - 0 + C_2 \qquad C_2 = +4$$

The completed equation for θ is, therefore,

$$\theta = 0.25t^{2.4} - 6t + 4$$

At the instant when $t = 10$ sec,

$$\theta = 0.25(10)^{2.4} - 6(10) + 4 = +6.80 \text{ rad}$$

824. A body rotates in accordance with the law, $\alpha^3 = 6\theta$, in which α is expressed in radians per second, and θ is expressed in radians. It is also known that $\omega = 0$ and $t = 0$ at the instant when $\theta = 0$. Derive the equation expressing the relation between α and t.

Solution. This is a case in which Eq. 209 is directly useful. Solving for α in the equation of the motion,

$$\alpha = \sqrt[3]{6\,\theta} = (6)^{1/3}\,\theta^{1/3}$$

Substituting this value of α in Eq. 209,

$$\omega\, d\omega = \alpha\, d\theta \qquad \omega\, d\omega = (6)^{1/3}\,\theta^{1/3}\, d\theta$$

Integrating,

$$\frac{\omega^2}{2} = (6)^{1/3}\,\tfrac{3}{4}\,\theta^{4/3} + C_1$$

The problem states that $\omega = 0$ at the instant when $\theta = 0$. These values must satisfy the foregoing equation. Substituting,

$$0 = 0 + C_1 \qquad C_1 = 0$$

Therefore, the (ω, θ) relation for the given motion is

$$\omega^2 = 1.5(6)^{1/3}\,\theta^{4/3} \quad \text{or} \quad \omega = (1.5)^{1/2}(6)^{1/6}\,\theta^{2/3}$$

Substituting the foregoing expression for ω in Eq. 207, and transposing,

$$dt = (1.5)^{-1/2}(6)^{-1/6}\,\theta^{-2/3}\,d\theta$$

Integrating,

$$t = (1.5)^{-1/2}(6)^{-1/6}\,3\,\theta^{+1/3} + C_2$$

By the use of the initial conditions stated in the problem, the value of C_2 is found to be zero. Therefore, the completed (θ, t) relation is as follows:

$$t = (6)^{1/3}\,\theta^{1/3} \quad \text{or} \quad \theta = \tfrac{1}{6}t^3$$

from which, by means of two successive differentiations, in accordance with Eqs. 207 and 208, it is found that,

$$\omega = \tfrac{1}{2}t^2 \quad \text{and} \quad \alpha = t$$

the latter being the desired equation.

PROBLEMS

825. A flywheel has an angular acceleration of 1.25 rad per sec per sec. Express this acceleration in revolutions per minute per minute. *Ans.* 716 rev/min^2.

826. A flywheel has an angular acceleration of 400 rev per min per min. Express this acceleration in radians per second per second.

827. A rotating body has an angular acceleration of 10 rev per min per sec. Express this acceleration in rad per sec per sec. *Ans.* 1.05 rad/sec^2.

828. A motor attains a speed of 1750 rpm, during an interval of 5 sec, starting from rest. Assuming that the speed increases at a constant rate with respect to time, calculate the angular acceleration, in radians per second per second.

829. A 24-in. pulley starts from rest and accelerates uniformly with respect to time. At the end of 20 sec a point on the periphery of the pulley has a linear velocity of 1 mi per min. Calculate the angular acceleration of the pulley, in revolutions per minute per minute. *Ans.* 2520 rev/min^2.

830. A body rotates in accordance with the law $\theta = 0.3t^3 + 4t - 2$, in which θ is in radians and t is in seconds. Calculate ω and α for the instant when $t = \frac{1}{4}$ min.

831. A body rotates in accordance with the law $\omega = 2t + 4$, in which ω is in radians per second and t is in seconds. It is also known that $\theta = 0$ when $t = 0$. Calculate ω, θ, and α for the instant when $t = 3$ sec. *Ans.* +10 rad/sec; +21 rad; +2 rad/sec^2.

832. A body rotates in accordance with the law $\alpha = 4.5t^{1/4}$, in which α is in revolutions per minute per minute and t is in minutes. It is also known that $\omega = 0$ and $\theta = 0$ at the instant when $t = 0$. Calculate the value of θ for the instant when $t = 16$ min.

833. A body rotates in accordance with the law $4\theta = \alpha^2$. It is also known that $\omega = 0$ and $\theta = 0$ at the instant when $t = 0$. Derive the equation showing the relation between α and t. *Ans.* $\alpha = \frac{1}{3}t^2$.

834. A body rotates in accordance with the law $\theta = t^3$, in which θ is in revolutions and t is in minutes. Calculate the average angular acceleration of the body during the interval from $t = 4$ min to $t = 12$ min.

835. A body rotates in accordance with the law $\alpha = -5\theta$, in which α is in radians per second per second and θ is in radians. It is known, also, that $\theta = \pi/4$ rad at the instant when $\omega = 0$. Calculate the angular velocity for the instant when $\theta = \pi/8$ rad. *Ans.* 1.52 rad/sec.

836. A certain pendulum rotates in accordance with the law $\alpha = -2 \sin \theta$, in which α is in radians per second per second and θ is in radians. It is known, also, that $\theta = \pi/6$ rad at the instant when $\omega = 0$. Calculate the value of ω at the instant when $\theta = 0$.

150. Relations between the Motion of a Rotating Body and the Motions of Its Particles. *Distances. The linear distance through which a particle of a rotating body moves in a given interval of time is equal to the product of the radius of the path of the particle and the angular distance, in radians, traversed by the body in the given interval.*

The truth of the foregoing statement is obvious from principles of plane geometry.

Velocities. The linear velocity of a particle of a rotating body at any instant is equal to the product of the radius of the path of the particle and the angular velocity of the body at that instant.

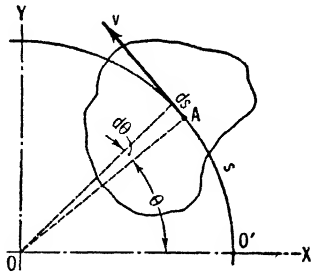

FIG. 400

Proof. Let Fig. 400 represent a body rotating about an axis through O at right angles to the plane of the paper. Let A represent the given particle, and let r represent the radius of the circular path in which the particle moves. Let s represent the distance from A to any fixed point, O', on the path, measured along the path. Let θ represent the angle between the fixed axis, OX, and the moving line, OA, at any instant. The angle θ will be understood to be expressed in radians, in all the equations of the present article. Let v represent the linear velocity of the particle A, at the given instant. From the figure,

$$ds = r\,d\theta \tag{210}$$

Differentiating in Eq. 210, with respect to t,

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \tag{211}$$

Equation 211 is an algebraic statement of the relations stated above.

Obviously, v is at right angles to the radius of rotation of the particle, and its sense must be consistent with the direction of rotation of the body.

Accelerations. The tangential component of the linear acceleration of a particle of a rotating body is equal to the product of the radius of the path of the particle and the angular acceleration of the body.

Proof. From Eq. 163, Art. 117, $a_T = dv/dt$. Substituting in this equation the value of v given by Eq. 211,

$$a_T = \frac{d\,(r\omega)}{dt} = r\frac{d\omega}{dt} = r\alpha \qquad [212]$$

which proves the principle stated above. The tangential component, a_T, is also at right angles to the radius of rotation of the particle, and its sense must be such as to be consistent with the sign of α.

The normal component of the linear acceleration of a particle of a rotating body is equal to the product of the radius of the path of the particle and the square of the angular velocity of the body.

Proof. Equation 168, Art. 117, gives the value of a_N for any case of curvilinear motion. In the present case the radius of curvature of the path is the radius of the circle in which A moves. Substituting in Eq. 168, the value of v given by Eq. 211,

$$a_N = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2 \qquad [213]$$

The normal acceleration, a_N, coincides with the radius of rotation, r, and its sense is toward the center of the circular path. It is sometimes called the *centripetal acceleration.*

Units. It must be kept firmly in mind that the equations of the present article are in forms that necessitate the use of the radian for the unit of angular measure.

Illustrative Problems

837. The speed of a certain 12-ft flywheel increases uniformly at the rate of 600 rev per min per min. Calculate v, a_T, and a_N, for a point on the rim of the wheel at the instant when the speed is 90 rpm.

Solution. As was stated, substitution in Eqs. 211, 212, or 213 necessitates the use of the radian as the unit of angular measure. Any unit of time, and of length, may be used, but the second and the foot are most common, and are required by many kinetic formulas. The foregoing units will be used in the present problem. Expressing ω and α in radians and seconds,

$$\omega = 90 \text{ rpm} = \frac{90 \times 2\pi}{60} = 9.42 \text{ rad/sec}$$

$$\alpha = 600 \text{ rev/min}^2 = \frac{600 \times 2\pi}{3600} = 1.05 \text{ rad/sec}^2$$

From Eq. 211,

$$v = r\omega \qquad v = 6 \times 9.42 = 56.5 \text{ ft/sec}$$

From Eq. 212,

$$a_T = r\alpha \qquad a_T = 6 \times 1.05 = 6.30 \text{ ft/sec}^2$$

From Eq. 213,

$$a_N = r\omega^2 \qquad a_N = 6(9.42)^2 = 532 \text{ ft/sec}^2$$

838. A body rotates in accordance with the law $\theta = t^3 - 2t + 4$, in which θ is expressed in revolutions, and t is expressed in minutes. Calculate v, a_T, and a_N for a particle of the body at a distance of 18 in. from the axis of rotation, at the instant when $t = 60$ sec. Express the results in terms of feet and seconds.

Solution.

$$\theta = t^3 - 2t + 4$$

$$\omega = \frac{d\theta}{dt} \qquad \omega = \frac{d(t^3 - 2t + 4)}{dt} = 3t^2 - 2$$

$$\alpha = \frac{d\omega}{dt} \qquad \alpha = \frac{d(3t^2 - 2)}{dt} = 6t$$

At the instant when $t = 60 \text{ sec} = 1 \text{ min}$

$$\omega = 3(1)^2 - 2 = 1 \text{ rev/min}$$

$$\alpha = 6(1) = 6 \text{ rev/min}^2$$

Changing to radians and seconds,

$$\omega = \frac{1 \times 2\pi}{60} = 0.105 \text{ rad/sec}$$

$$\alpha = \frac{6 \times 2\pi}{(60)^2} = 0.0105 \text{ rad/sec.}^2$$

Substituting in Eqs. 211, 212, and 213,

$$v = r\omega = 1.5(0.105) = 0.158 \text{ ft/sec}$$

$$a_T = r\alpha = 1.5(0.0105) = 0.0158 \text{ ft/sec}^2$$

$$a_N = r\omega^2 = 1.5(0.105)^2 = 0.0165 \text{ ft/sec}^2$$

PROBLEMS

839. The crankshaft of a certain engine is rotating at a constant speed of 2400 rpm. The stroke is 6 in. Calculate the linear velocity of the center of the crankpin. *Ans.* 62.8 ft/sec.

840. A 36-in. pulley is rotating at a constant speed of 1200 rpm. Calculate the linear velocity and acceleration of a point on the rim of the pulley.

841. Calculate the angular acceleration necessary to cause a 12-ft flywheel to attain a rim speed of 1 mi per min, during an interval of 40 sec, starting from rest. *Ans.* 0.367 rad/sec^2.

842. A belt accelerates a 3-ft pulley uniformly, from a speed of 1200 rpm to a speed of 2400 rpm, during an interval of 45 sec. Calculate the linear velocity and tangential acceleration of the belt at the instant ending the interval.

843. A 4-ft pulley is retarded uniformly from a speed of 120 rpm to a speed of 30 rpm, during an interval of 3 sec. Calculate the resultant acceleration of a point on the rim, at the instant when 2 sec have elapsed. Also calculate the angle between the resultant acceleration and the normal component. *Ans.* 79.2 ft/sec^2; 4° 35′.

844. At a given instant a point on the rim of a 2-ft pulley has a resultant linear acceleration of 360 ft per sec per sec. The angle between the acceleration and the radius of the pulley is 30°. Calculate the angular velocity and angular acceleration of the pulley at the given instant.

845. A body rotates in accordance with the law $\theta = 2t^{3/2}$, in which θ is in revolutions and t is in minutes. Calculate v, a_T, and a_N for a point 2 ft from the axis of rotation, at the instant when $t = 4$ min.

846. A particle on the rim of an 18-in flywheel moves in accordance with the law $s = t^{7/2}$, in which s is in feet and t is in seconds. Calculate v, a_T, and a_N for the instant when $t = 16$ sec. *Ans.* 3580 ft/sec; 560 ft/sec^2; 17,100,000 ft/sec^2.

847. The tangential acceleration of the center of the crankpin of a certain engine has a constant magnitude of 2 ft per sec per sec. The stroke of the engine is 16 in. Calculate the velocity of the point, its normal acceleration, and the angular velocity of the crankshaft at the instant when $\frac{1}{2}$ min has elapsed, starting from rest.

848. A certain body rotates in accordance with the law $\theta = 2t^3$. It is desired to prepare a table giving values of the magnitude of the resultant acceleration of a point at a distance, r, from the axis of rotation, at several instants during the motion. Derive a formula for the magnitude of a, in terms of r and t.

Ans. $a = 12rt\sqrt{1 + 9t^6}$.

151. Rotation with Constant Angular Velocity. Rotation in which the angular velocity is constant is often called *uniform rotation.* It follows, from the title, that the angular acceleration is zero.

The special formula for uniform rotation is as follows:

$$\theta = \omega t \tag{214}$$

The derivation of Eq. 214 is similar to that of Eq. 127, Art. 110. The two formulas are alike, except for differences in notation.

In the use of Eq. 214, the stationary axis OX, Fig 399, from which θ is measured, must be placed in the position occupied by the moving line OA at the instant when t is zero.

PROBLEMS

849. A certain pulley describes 1,000,000 rev in 24 hr, at constant speed. Calculate the angular velocity of the pulley, in radians per second.

850. Calculate the angular velocity of the earth on its axis, in radians per second. *Ans.* 0.0000727 rad/sec.

851. A point on the rim of a 10-ft flywheel travels 3,000,000 ft in 24 hr. Calculate the angular velocity of the wheel in radians per second, assuming uniform rotation.

852. A certain flywheel rotates at a constant speed of 4 rad per sec. How many revolutions will the wheel describe in 8 hr? Calculate the linear velocity, in miles per hour, of a point on the wheel at a distance of 18 in. from the axis of the shaft. *Ans.* 4.09 mi/hr.

853. The 1927 model of a certain automobile had a gear ratio, in "high," of 46/11. This means that the crankshaft made 46 revolutions about its own axis while the rear wheels made 11 revolutions about their axis, when the car was running in a straight line in high gear. The tires were 29 in. in diameter. Calculate the number of revolutions made by the engine while the car traveled 1 mi. Calculate the angular velocity of the crankshaft, in revolutions per minute, and in radians per second, when the car had a speed of 60 mi per hr.

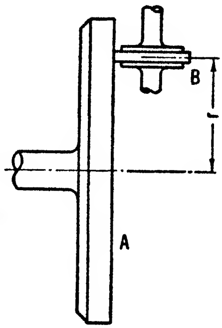

FIG. 401

854. In Fig. 401, A and B represent a pair of friction wheels, used in transmitting small amounts of power. The small wheel is 4 in. in diameter, and has a constant speed of 240 rpm. The distance, r, is 6 in. Calculate the angular velocity of the larger wheel. Assume that no slipping occurs. *Ans.* 80 rpm.

855. A certain electric motor has a speed of 1750 rpm. The pulley on the motor shaft is 4 in. in diameter. The motor drives a lineshaft, by means of a belt running on an 18-in. pulley on the lineshaft. A machine is to be driven from the lineshaft by means of a second belt. The machine must run at a speed of 250 rpm and its pulley is 14 in. in diameter. Calculate the necessary diameter of the second line-shaft pulley.

856. A power punch is to be operated at the rate of 20 strokes per min. It makes one stroke during each revolution of its mainshaft. The mainshaft has a spur gear having 120 teeth. This gear is driven by a 24-tooth pinion on a countershaft. The pulley on the countershaft is 24 in. in diameter, and is belt-connected to a lineshaft. The lineshaft has a speed of 300 rpm. Calculate the necessary diameter of the lineshaft pulley. *Ans.* 8 in.

857. A certain body rotates in accordance with the law $\theta = 4t + 2$, in which θ is in radians and t is in seconds. Calculate the angular velocity of the body, in revolutions per minute, for the instant when $t = 2$ sec, and also for $t = 4$ sec.

858. A body rotates in accordance with the law $\theta = 4 - 2t$, in which θ is in revolutions and t is in minutes. Calculate the angular velocity of the body, in radians per second. *Ans.* -0.209 rad/sec.

152. Rotation with Constant Angular Acceleration. In this type of rotation the angular velocity varies, but the angular acceleration is constant. The angular velocity varies uniformly with respect to time. The special formulas for this type of rotation are as follows:

$$\omega = \omega_0 + \alpha t \qquad [215]$$

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 \qquad [216]$$

$$\omega = \sqrt{\omega_0^2 + 2\alpha\theta} \qquad [217]$$

$$\theta = \frac{\omega_0 + \omega}{2}\, t \qquad [218]$$

The derivations of the foregoing formulas are similar to those of the formulas in Art. 111. It will be noticed that the two sets of formulas are exactly alike except for differences in notation.

When Eqs. 215, 216, 217, and 218 are used in problems it is necessary to place the stationary axis OX, in Fig. 399, in the position occupied by the moving line OA at the instant when t is zero.

The quantity ω_0 is the particular value of ω at the instant when t is zero, and is sometimes called the initial angular velocity.

Rotation in which the angular acceleration is constant is often called *uniformly accelerated rotation.*

Illustrative Problems

859. A certain flywheel has a constant angular acceleration of 20 rad per sec per sec. The initial angular velocity is 1800 rpm. Calculate the total number of revolutions described in the next 12 sec.

Solution. Expressing the given angular acceleration in revolutions per minute per minute,

$$\alpha = 20 \text{ rad/sec}^2 = \frac{20 \times 3600}{2\pi} = 11{,}500 \text{ rev/min}^2$$

By Eq. 216,

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 \qquad \theta = 1800 \times (\tfrac{12}{60}) + \tfrac{1}{2} \times 11{,}500\,(\tfrac{12}{60})^2 = 590 \text{ rev}$$

860. The angular velocity of a certain wheel increases uniformly from 500 rpm to 3000 rpm in $\frac{1}{2}$ min. Calculate the angular acceleration of the wheel, in radians per second per second. Calculate the total number of revolutions described by the wheel during the interval. Calculate the angular velocity of the wheel, in revolutions per minute, at the instant when 500 rev have been described, counting from the beginning of the interval.

Solution. Substituting in Eq. 215,

$$\omega = \omega_0 + \alpha t \qquad 3000 = 500 + \alpha(0.5)$$

$$\alpha = 5000 \text{ rev/min}^2 = 8.73 \text{ rad/sec}^2$$

Substituting in Eq. 218,

$$\theta = \frac{\omega_0 + \omega}{2}t \qquad \theta = \frac{500 + 3000}{2} \times \frac{1}{2} = 875 \text{ rev}$$

Substituting in Eq. 217,

$$\omega = \sqrt{\omega_0^2 + 2\alpha\theta} \qquad \sqrt{(500)^2 + 2 \times 5000 \times 500} = 2290 \text{ rpm}$$

PROBLEMS

861. A certain wheel rotates with a constant angular acceleration of 3.25 rad per sec per sec. Calculate the angular velocity, in revolutions per minute, that the wheel will attain in 45 sec, starting from rest. *Ans.* 1390 rpm.

862. What angular deceleration, in revolutions per minute per minute, would be necessary to reduce the speed of a rotating body from 3600 to 1200 rpm, during an interval of 30 sec?

863. A flywheel is accelerated uniformly at the rate of 3 rev per min per sec. How much time will elapse while the speed is changing from 50 to 200 rad per sec? How many revolutions will be described by the wheel during the interval? *Ans.* 478 sec; 9510 rev.

864. A brakewheel is rotating at a speed of 3200 rpm. When the brake is applied the wheel is brought to rest, uniformly with respect to time, while it describes 212 rev. Calculate the angular deceleration, in radians per second per second, produced by the brake, and the time consumed in bringing the wheel to rest.

865. What angular acceleration, in radians per second per second, is necessary in order to increase the speed of a wheel uniformly, with respect to time, from 2500 to 4500 rpm, while the wheel describes 3000 rev? How much time is required? *Ans.* 4.08 rad/sec^2; 51.4 sec.

866. An elevator is raised and lowered by means of a cable which is wound on a hoisting drum 6 ft in diameter. It is desired to accelerate the elevator from rest to a speed of 650 ft per min, in a distance of 15 ft. Calculate the constant angular acceleration that the drum should have. How many revolutions would be described by the drum during the acceleration period? How much time would be required?

867. A certain rotating body describes 20,000 rev in 8 min, with a constant angular acceleration of 0.5 rad per sec per sec. Calculate the initial and final angular velocities. *Ans.* 142 rad/sec; 382 rad/sec.

868. A wheel has a constant angular acceleration of 4 rev per min per sec. The initial angular velocity is 1000 rpm. Calculate the angular velocity at the instant when the wheel has described 750 rev. Calculate the elapsed time.

869. A body rotates in accordance with the law $\theta = t^2 + 4t - 12$, in which θ is in radians and t is in seconds. Calculate the angular acceleration. *Ans.* 2 rad/sec^2.

870. A certain flywheel rotates in accordance with the kinetic equation, $\Sigma M_z = 22.5\alpha$, in which ΣM_z represents the torque of the external forces about the axis of rotation, in foot-pounds, and α represents the angular acceleration of the body, in radians per second per second. Calculate the torque necessary to accelerate the flywheel from 1200 to 3600 rpm, while the wheel describes 200 rev. *Ans.* 1130 ft-lb.

CHAPTER XVIII

KINETICS OF ROTATION

153. Special Nature of the Discussion. The study of the kinetics of rotation in this book will be confined to the case in which the rotating body is homogeneous and has a plane of symmetry at right angles to the axis of rotation.

The majority of the problems in rotation encountered in engineering practice conform to these special conditions. In some problems, when the body as a whole lacks such a plane of symmetry, it is possible to divide it into two or more parts, each of which does have such a plane. Then the problem can be solved by treating each symmetrical part individually by means of the methods to be developed in the present chapter. The results thus obtained, properly combined, are valid for the entire body.

154. The Resultant of the External Forces. *The resultant of the external forces acting on a homogeneous, rotating body having a plane of symmetry at right angles to the axis of rotation is a force lying in the plane of symmetry, whose magnitude is equal to $(W/g)\bar{a}$, whose inclination and sense are the same as the inclination and sense of $\bar{a}$, and whose moment about the axis of rotation is equal to $I_z\alpha$.*

In the foregoing, $\bar{a}$ is the linear acceleration of the center of gravity, α is the angular acceleration of the body, and I_z is the moment of inertia with respect to the axis of rotation.

In the special case in which the center of gravity is on the axis of rotation the resultant reduces to a couple whose moment is equal to $I_z\alpha$, lying in any plane at right angles to the axis of rotation. The sense of the couple agrees with that of α.

Proof. In Fig. 402, A represents any particle of the body, at any given instant. The body itself is not shown. The xy-plane represents the plane of symmetry of the body. The center of gravity, represented by G, is necessarily in the plane of symmetry. The z-axis is the axis of rotation, and ρ represents the radius of rotation of A. β represents certain equal angles, as indicated in the figure. A' represents that particle of the body which is symmetrical with A on the opposite side of the xy-plane. A and A' are equal in weight.

The vector $(dW/g)a$ represents the effective force for the particle

A (Art. 123). The vectors $(dW/g)a_T$ and $(dW/g)a_N$ are the tangential and normal components of the effective force. Obviously the effective force for particle A' is in all respects equal to the effective force for particle A.

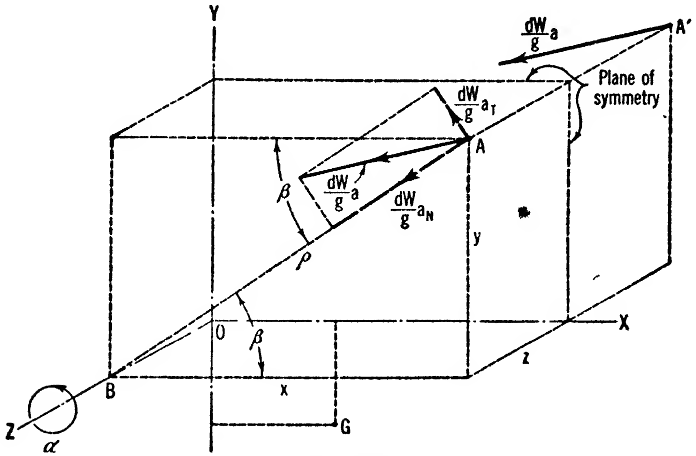

Fig. 402

By the principle of components, Art. 50, $R_x = \Sigma F_x$. By Eq. 184, $\Sigma F_x = (W/g)\bar{a}_x$. Therefore,

$$R_x = \frac{W}{g}\bar{a}_x \qquad [219]$$

In a similar manner it can be shown that

$$R_y = \frac{W}{g}\bar{a}_y \qquad [220]$$

Since G moves in a plane at right angles to the axis of rotation, $\bar{a}_z = 0$. From this fact, and by the process of reasoning used above, $R_z = \Sigma F_z$, and $\Sigma F_z = (W/g)\bar{a}_z = 0$. Therefore,

$$R_z = 0 \qquad [221]$$

By compounding R_x, R_y, and R_z, as given by Eqs. 219, 220, and 221, it can be shown that

$$R = \frac{W}{g}\bar{a} \qquad [222]$$

The foregoing results also show that R has the same inclination and sense as has $\bar{a}$.

The fact that the resultant lies in the plane of symmetry can be shown from Fig. 402. As was stated, the effective forces for particles A and A' are equal, parallel, and alike in sense. Their resultant lies in the plane of symmetry. Since the body is homogeneous and symmetrical, all the effective forces will be paired in this manner. It follows that the resultant of all the effective forces will lie in the plane of symmetry. By Art. 125, the resultant of the external forces is identical with the resultant of the effective forces. Therefore, the resultant of the external forces also lies in the plane of symmetry.

It now remains to be proved that the moment of R about the axis of rotation is equal to $I_z\alpha$. Since, by Art. 125, the resultant of the external forces is identical with the resultant of the effective forces, it follows that the moment of R about the axis of rotation is equal to the moment-sum of the effective forces about that axis. Let this moment-sum be represented by ΣM_z. From the figure,

$$\Sigma M_z = \int \left(\frac{dW}{g} a_T \right) \rho \qquad [223]$$

By Eq. 212, Art. 150, $a_T = \rho\alpha$. Substituting in Eq. 223,

$$\Sigma M_z = \int \left(\frac{dW}{g} \rho^2 \right) \alpha \qquad [224]$$

The angular acceleration, α, is a constant for the entire body at any one instant. Therefore, Eq. 224 may be written as follows:

$$\Sigma M_z = \alpha \int \frac{dW}{g} \rho^2 \qquad [225]$$

In Eq. 225, the expression $\int \frac{dW}{g} \rho^2$ represents the moment of inertia of the body with respect to the axis of rotation (Art. 132). Representing this by I_z,

$$\Sigma M_z = I_z\alpha \qquad [226]$$

By the principle of moments, Art. 50, the moment of R is equal to the moment-sum of the external forces. Therefore, ΣM_z also represents the moment-sum of the external forces about the axis of rotation.

Figure 403 represents the resultant of the external forces as a single force, in accordance with the conception set forth in the beginning. The moment-arm of R with respect to the axis of rotation must be such that the moment of R will be equal to $I_z\alpha$.

Alternative Form of the Resultant. Any force can be resolved into an equivalent force and couple (Art. 38). Thus, the resultant as shown in Fig. 403 can be resolved into an equal force in the plane of symmetry, intersecting the axis of rotation, and a couple in that plane whose moment is equal to $I_z\alpha$ and whose sense agrees with that of α. This conception of the resultant is shown in Fig. 404. It is preferred by some persons.

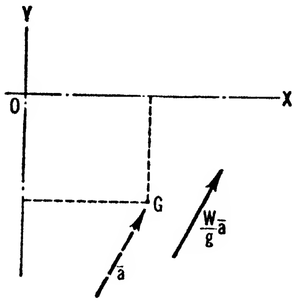

FIG. 403

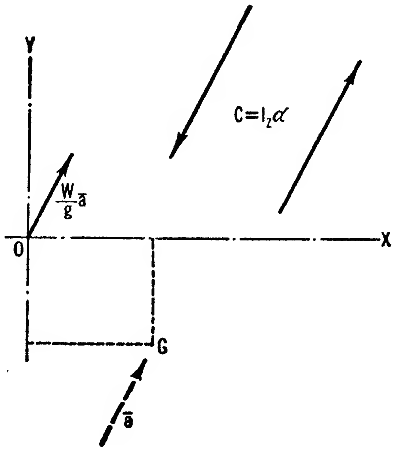

FIG. 404

If desired, the x-axis may be placed so that it will pass through the center of gravity of the body. If this is done, $\bar{a}_x$ will be equal to $\bar{r}\omega^2$, and $\bar{a}_y$ will be equal to $\bar{r}\alpha$, in which $\bar{r}$ represents the radius of rotation of the center of gravity.

155. Methods of Solving Problems. Three slightly different methods for the solution of problems will be described.

The Resultant Method. In this method the external forces, together with their resultant, are shown on the sketch. The resultant may be represented in accordance with either of the conceptions discussed in Art. 154. The equations for the solution are then formed on the basis of the principle of components, Art. 50, or the principle of moments, or both.

The Equilibrant Method. In this method the equilibrant, or reversed resultant, of the external forces is used instead of the resultant. The principles of equilibrium, Arts. 51–74, are then applied to the entire system, external forces and equilibrant, to provide the necessary equations. If the student uses this method he should be careful to avoid any impression that the body actually is in equilibrium. Equilibrium of the body exists only under the special conditions that the center of gravity lies on the axis of rotation and the body has no angular acceleration.

Solution by Formulas. In many problems direct substitution in Eq. 226, Art. 154, is sufficient for a complete solution. Equations 184, 185, and 186, of Art. 126, may be used if needed. If additional equations are necessary the following principle may be used:

The moment-sum of the external forces about any axis lying in the plane of symmetry of the body is equal to zero.

The foregoing principle follows from the fact that the resultant of the external forces lies entirely in the plane of symmetry (Art. 154), and consequently has no moment about any axis in that plane.

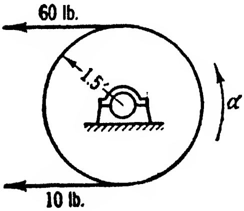

FIG. 405

ILLUSTRATIVE PROBLEMS

871. Figure 405 represents a pulley 3 ft in diameter, weighing 200 lb, mounted in bearings and subjected to two constant belt pulls, as shown. The radius of gyration of the pulley with respect to the axis of rotation is 1.2 ft. How many revolutions will the pulley describe while its speed changes from 1000 to 2000 rpm? Disregard friction.

Solution. The moment of inertia of the pulley with respect to the axis of rotation is $I_z = \frac{W}{g}k^2 = \frac{200}{32.2}(1.2)^2 = 8.94$ engineer's units. Equation 226, Art. 154, is sufficient for the calculation of α.

$$\Sigma M_z = I_z\alpha \qquad +60 \times 1.5 - 10 \times 1.5 = 8.94\alpha \qquad \alpha = 8.39 \text{ rad/sec}^2$$

Expressing α in revolutions per minute per minute,

$$\alpha = \frac{8.39 \times 3600}{2\pi} = 4810 \text{ rev/min}^2$$

By Eq. 217, Art. 152,

$$\omega = \sqrt{\omega_0^2 + 2\alpha\theta} \qquad \theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{(2000)^2 - (1000)^2}{2 \times 4810} = 312 \text{ rev}$$

The resultant method, or the equilibrant method, could have been used, of course. These methods offer little or no advantage, however, in such a simple problem.

872. Figure 406 represents a drum, A, keyed to a horizontal shaft. A body, B, is suspended from a light cable which is wrapped around the drum. The drum and its shaft weigh 966 lb, and have a radius of gyration of 1.6 ft, with respect to the axis of rotation. The suspended body, B, weighs 644 lb. Calculate the angular acceleration of the drum, the linear acceleration of B, and the tension in the cable. Disregard friction, and the weight and stiffness of the cable.

Solution. The drum has a motion of rotation, and body B has a motion of rectilinear translation. Thus far in the book no method has been devised in the use of which it would be possible to treat this entire mechanism as a single body. Therefore, it is necessary to deal with A and B individually.

Figure 407 represents the drum alone. The external forces acting on the drum are its weight, W_A, the reaction at the shaft bearings, Q, and the pull exerted by the cable, P. A very common error in problems of this type consists in assuming that P, which is the tension in the cable, is equal to the weight of the body B, suspended from the cable. This would be correct if B were in equilibrium, but there is no reason for expecting a condition of equilibrium in the present case.

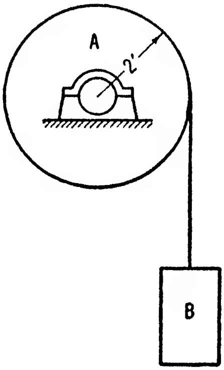

FIG. 406

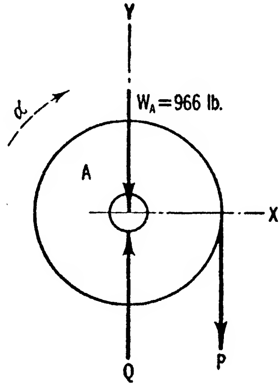

FIG. 407

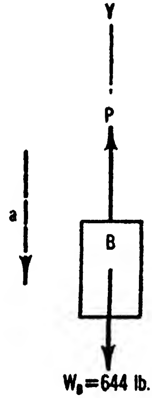

FIG. 408

Since friction is to be disregarded, the force Q will be normal to the periphery of the shaft.

Figure 408 represents B as a free body. The external forces acting on B are its own weight, W_B, and the upward pull, P, of the cable.

The angular acceleration, α, of the drum has been assumed clockwise. The linear acceleration, a, of body B has been assumed downward. Great care must be exercised to make assumptions that are consistent with the possibilities of a given mechanism. In the present case it would be plainly inconsistent to assume α clockwise and a upward. If desired, however, α could be assumed counterclockwise and a upward. The results obtained under such an assumption would be numerically correct, but would be accompanied by minus signs, showing the entire assumption regarding directions to be incorrect.

For the drum, applying Eq. 226, Art. 154, with careful attention to signs,

$$\Sigma M_s = I_s\alpha \qquad -2P = \left[\frac{966}{32.2}(1.6)^2\right](-\alpha) \qquad P = 38.4\alpha$$

For body B, applying Eq. 185, Art. 126, with careful attention to signs,

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad P - 644 = \frac{644}{32.2}(-a) \qquad P = 644 - 20a$$

By Eq. 212, Art. 150,

$$a_T = r\alpha \qquad a = 2\alpha$$

The solution of the three equations obtained above gives,

$$\alpha = 8.21 \text{ rad/sec}^2 \qquad a = 16.4 \text{ ft/sec}^2 \qquad P = 315 \text{ lb}$$

It should be noticed that the tension in the cable is decidedly different from the weight of body B.

873. Figure 409 represents a horizontal shaft, AB, turning in bearings at A and B. The bearings themselves are not shown. An arm, OG, is attached rigidly to the shaft, and carries at its outer end a solid cylinder, weighing 322 lb. The cylinder is 1 ft in diameter, and its geometric axis is parallel to

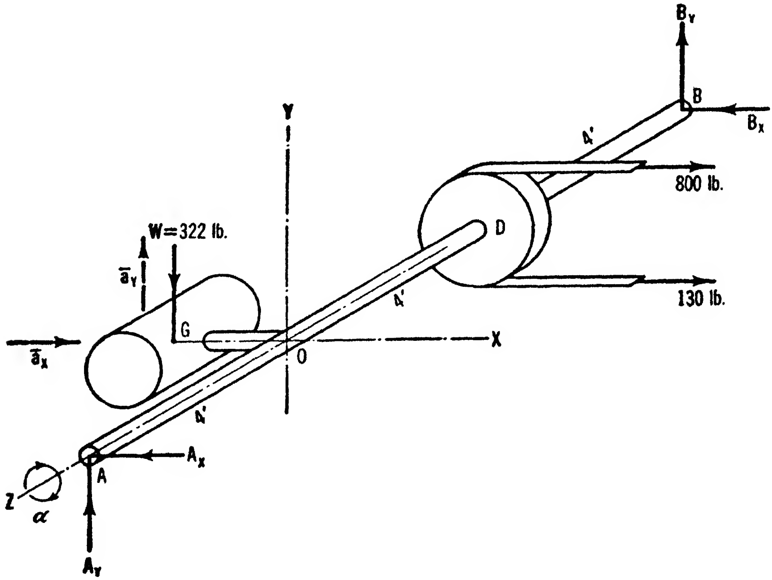

Fig. 409

the axis of the shaft, the distance between the two axes being 1.5 ft. At the instant under consideration the axis of the cylinder is in the horizontal plane containing the axis of the shaft. At D an 18-in. pulley is keyed to the shaft, and is subjected to horizontal belt pulls, as shown. Distance $AO = OD = DB = 4$ ft. The angular velocity of the system, at the given instant, is 30 rpm. Calculate the angular acceleration of the system, and the components of the reactions exerted on the shaft by its bearings, for the given instant. Disregard all weights except that of the cylinder, and disregard all friction.

Solution. It will be assumed that α is clockwise, as indicated by the curved arrow. $\bar{a}_x$ is the normal acceleration of G and is directed toward O. $\bar{a}_y$ is the tangential acceleration of G, and must be assumed upward in order to be consistent with the assumed sense of α. Assume the x- and y-components of the bearing reactions in the manner shown. Assume that these reactions have no z-components.

For the cylinder, using the parallel-axis theorem, Art. 134, and the answers to Prob. 764,

$$I_z = \bar{I} + \frac{W}{g}c^2$$

$$I_z = \frac{1}{2}\frac{W}{g}r^2 + \frac{W}{g}\overline{OG}^2 = \frac{322 \times (0.5)^2}{2 \times 32.2} + \frac{322 \times (1.5)^2}{32.2} = 23.8 \text{ engineer's units}$$

Substituting in Eqs. 184 and 185, Art. 126,

$$\Sigma F_x = \frac{W}{g}\bar{a}_x \qquad -A_x - B_x + 800 + 130 = \frac{322}{32.2}(+\bar{a}_x)$$

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad +A_y + B_y - 322 = \frac{322}{32.2}(+\bar{a}_y)$$

In the present article it is shown that the moment-sum of the external forces about any axis in the plane of symmetry is equal to zero. Using the x- and y-axes as axes of moments,

$$\Sigma M_x = 0 \qquad -4A_y + 8B_y = 0$$

$$\Sigma M_y = 0 \qquad -4A_x + 8B_x - 800 \times 4 - 130 \times 4 = 0$$

From Eq. 226, Art. 154, and from Eqs. 212 and 213, Art. 150,

$$\Sigma M_z = I_z\alpha \qquad +322 \times 1.5 - 800 \times 0.75 + 130 \times 0.75 = 23.8(-\alpha)$$

$$a_T = r\alpha \qquad \bar{a}_y = 1.5\alpha$$

$$a_N = r\omega^2 \qquad \bar{a}_x = 1.5\left(\frac{30 \times 2\pi}{60}\right)^2$$

The solution of the foregoing equations gives

$$\alpha = 0.819 \text{ rad/sec}^2$$

$$A_x = 211 \text{ lb} \qquad A_y = 223 \text{ lb} \qquad B_x = 571 \text{ lb} \qquad B_y = 111 \text{ lb}$$

Positive signs were obtained with all results, showing that the senses were assumed correctly in all cases.

874. Solve Prob. 873 by the resultant method.

Solution. Figure 410 is a simplified representation of the system. In Art. 154 it is shown that the resultant of the external forces may be indicated as a single force whose magnitude is $(W/g)\bar{a}$, lying in the plane of symmetry and intersecting the axis of rotation, and a couple, also lying in the plane of symmetry, whose moment is $I_z\alpha$. In Fig. 410 the single force has been resolved, for greater convenience, into its components, $(W/g)\bar{a}_x$ and $(W/g)\bar{a}_y$. The sense of the couple must be assumed in conformity with the assumed (clockwise) sense of α. From Prob. 873, $I_z = 23.8$ engineer's units. By the principle of components, Art. 50,

$$\Sigma F_x = R_x \qquad -A_x - B_x + 800 + 130 = +\frac{322}{32.2}\bar{a}_x$$

$$\Sigma F_y = R_y \qquad +A_y + B_y - 322 = +\frac{322}{32.2}\bar{a}_y$$

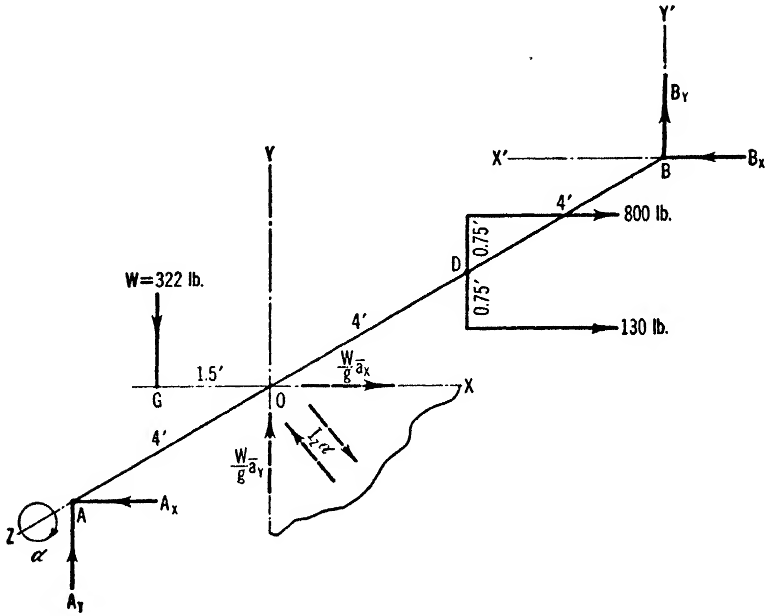

Fig. 410

By the principle of moments, Art. 50, using the axes x' and y', at B, as the axes of moments,

$$\Sigma M_{x'} = M(R)_{x'} \qquad -12A_y + 322 \times 8 = -\left(\frac{322}{32.2}\bar{a}_y\right)8$$

$$\Sigma M_{y'} = M(R)_{y'} \qquad -12A_x + 800 \times 4 + 130 \times 4 = +\left(\frac{322}{32.2}\bar{a}_x\right)8$$

$$\Sigma M_z = M(R)_z \qquad +322 \times 1.5 - 800 \times 0.75 + 130 \times 0.75 = 23.8(-\alpha)$$

By Eqs. 212 and 213, Art. 150,

$$a_T = r\alpha \qquad \bar{a}_y = 1.5\alpha$$

$$a_N = r\omega^2 \qquad \bar{a}_x = 1.5\left(\frac{30 \times 2\pi}{60}\right)^2$$

The solution of the seven equations thus formed gives the same results as in Prob. 873. It should be noticed that the resultant method is of advantage because of its general nature, offering complete freedom in the choice of axes. The equations are more easily solved than those of Prob. 873, because of the fact that the unknown quantities appear less frequently.

875. Solve Prob. 873 by the equilibrant method.

Solution. In this method the equilibrant, instead of the resultant, of the external forces is shown in the figure. The figure would be like Fig. 410, except that the four vectors representing the resultant in that figure would be reversed. The equations would be formed by equating the algebraic sum of the components of the forces and their equilibrant, along any desired axis, to zero; and by equating the algebraic sum of the moments of the forces and their equilibrant, about any desired axis, to zero.

If the axes were selected as in Prob. 874, the resulting equations would be identical with those of that problem, except for the position of the equality sign and for the consequent differences in algebraic signs. Obviously they would yield the same results. The details of the solution will not be given.

PROBLEMS

876. A certain flywheel weighs 60,000 lb, and its radius of gyration about the axis of rotation is 8 ft. Calculate the torque necessary to increase the angular velocity of the wheel from 100 to 200 rpm, in 6 min. Calculate the total number of revolutions described during the interval. *Ans.* 3460 ft-lb; 900 rev.

877. A solid, homogeneous cylinder 1 ft in diameter, weighing 120 lb, is keyed to a short, horizontal shaft. The shaft is 2 in. in diameter, and the total friction between it and its bearings is 2 lb. The body is caused to rotate at a speed of 1200 rpm, and is then permitted to coast to rest. How much time will be consumed in coming to rest, and how many revolutions will be described? Disregard the inertia of the projecting portions of the shaft. Treat the friction as a single force tangent to the shaft.

878. The following problem, furnished by the Westinghouse Electric and Manufacturing Company, is of the type that the engineers of that company are called upon to solve in connection with the application of electrical equipment: a direct motor-driven centrifugal extractor, such as is used in the textile and laundry industries, has a basket 48 in. in diameter and 24 in. deep. The empty basket, together with the attached shaft and rotor, weighs 1050 lb. The capacity of the basket is 900 lb of wet goods. The full-load speed is 700 rpm. Assuming the radius of gyration to be 0.7 of the radius of the basket, what torque is it necessary for the motor to develop in order to bring the machine to full speed in 1.5 min, assuming that 15 per cent of the torque is utilized in overcoming friction? *Ans.* 114 ft-lb.

879. A wheel 4 ft in diameter, weighing 644 lb, is mounted on a shaft. The radius of gyration is 1.8 ft. A brake is applied with a constant normal pressure of 25 lb, and brings the wheel to rest from an initial speed of 2000 rpm, while describing 20,000 rev. Calculate the coefficient of kinetic friction for the brake-shoe.

880. A wheel weighing 322 lb, 2 ft in diameter, is mounted on a horizontal shaft in the manner indicated in Fig. 411. The radius of gyration of the wheel with respect to its geometric axis is 0.7 ft. In the position shown in the figure the angular velocity

of the wheel is 10 rad per sec in a counterclockwise sense. Calculate the horizontal and vertical components of the reaction exerted on the wheel by its bearings.
Ans. $P_x = -599$ lb; $P_y = -493$ lb.

881. Solve Prob. 880, with the center of gravity, G, on a horizontal line through the center of the shaft, and to the right. Let all the other conditions remain as in Prob. 880.

882. In Fig. 406, Prob. 872, it is desired to give the body B an upward velocity of 30 ft per sec, in a distance of 20 ft, starting from rest. This is to be done by applying a constant torque directly to the shaft of the drum. Calculate the necessary torque. *Ans.* 3040 ft-lb.

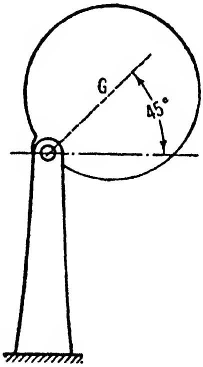

Fig. 411

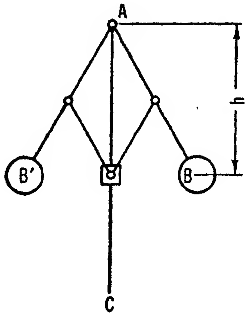

Fig. 412

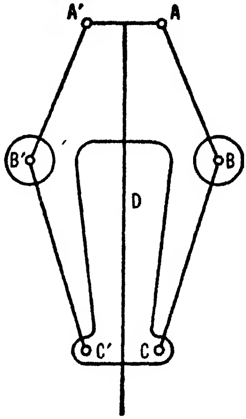

Fig. 413

883. The radius of gyration of the drum and brake-wheel of Fig. 251, Art. 78, is 12 in., and their combined weight is 241.5 lb. The coefficient of kinetic friction for the brake-shoe on the wheel is 0.22, and the suspended weight, W, is 250 lb. Calculate the force, P, which it would be necessary to use on the brake in order to permit W to attain a velocity of 8 ft per sec in 4 sec, starting from rest. Calculate the tension in the cable, and the total horizontal and vertical components of the reactions exerted on the shaft by its bearings. Disregard axle friction.

884. An elevator weighing 2 tons is raised by means of a cable which is wrapped around a hoisting drum. The drum is 4 ft in diameter, and weighs 1200 lb. The radius of gyration of the drum with respect to its axis of rotation is 22.5 in. Calculate the constant torque that must be supplied to the drum by the motor, in order to give the elevator an upward acceleration of 8 ft per sec per sec. Calculate the tension in the cable. Disregard friction, and the weight of the cable. *Ans.* 10,500 ft-lb; 4990 lb.

885. Figure 412 represents a simple Watt governor, which is the simplest of all flyball governors. It is now seldom used on engines, without certain modifications. Disregarding the weights of all parts of the mechanism except the balls, B and B', prove that $h = g/\omega^2$, in which ω represents the angular velocity of the governor. If the distance AB is 15 in., and the weight of each ball is 14 lb, calculate the tension in the arm AB when the governor is rotating uniformly at a speed of 60 rpm.

886. Figure 413 represents a Porter governor. The inclusion of the central weight, D, constitutes the essential difference between this governor and the simple Watt governor. In the figure the distance $AA' = CC' = 3$ in., $AB = 6$ in., and $BC = 8$ in. The weight of each ball is 5 lb, and the central weight, D, is 25 lb. Disregarding all other weights, calculate that constant angular velocity of the governor at which

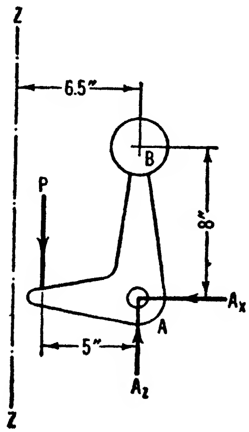

FIG. 414

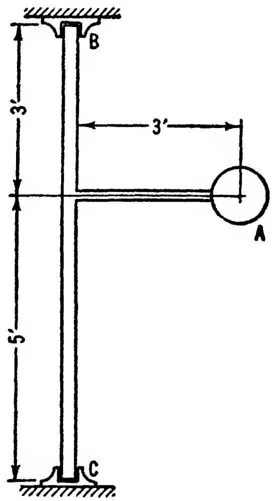

FIG. 415

the arm BC will make an angle of 30° with the vertical. Calculate the tensions in AB and BC, at this speed. *Ans.* 171 rpm; 23.5 lb; 14.4 lb.

887. In Fig. 414 is shown a portion of a spring-controlled governor. The ball, B, is attached to a small bellcrank lever, which is pivoted at A. A spring exerts a vertical, downward force, P, on the bellcrank, as shown. The ball and lever rotate about the vertical axis, ZZ. The ball weighs 12 lb, and the weight of the bellcrank will be disregarded. Assume that the ball takes the position shown in the figure when the governor is rotating at a constant speed of 240 rpm. Calculate the spring pressure, P, and the components, A_x and A_z, of the reaction exerted on the bellcrank by the pin at A.

888. In Fig. 415, A represents a homogeneous sphere, 1 ft in diameter, weighing 250 lb. The sphere is attached to the vertical shaft, BC, by means of a horizontal arm, as shown. The entire system is rotating about the axis of the shaft at a constant speed of 60 rpm. Calculate the reactions exerted on the shaft by its bearings, B and C, when the system is in the position shown by the figure. Assume that the reaction at B is horizontal. Disregard friction, and the weight of the shaft and the connecting arm. *Ans.* $B_x = -669$ lb; $C_x = -251$ lb; $C_z = +250$ lb.

889. In Fig. 416, B represents a square, homogeneous bar, 4 by 4 in. in cross section, weighing 48.3 lb. The bar is held in a yoke, by means of a pin at A, and the bar and its yoke rotate about a horizontal shaft through O at right angles to the plane of the figure. The bar presses against the stop C, or D, depending on circumstances. In the position shown in the figure the angular velocity of the system is 4 rad per sec, and the angular acceleration is 6 rad per sec per sec, both counterclockwise. Calculate the reaction exerted on the bar by the pin, A. Also ascertain which stop is acting, and calculate its reaction on the bar, assuming the force to be vertical. *Ans.* $A_x = -24$ lb; $A_y = +29.9$ lb; $C = +27.4$ lb.

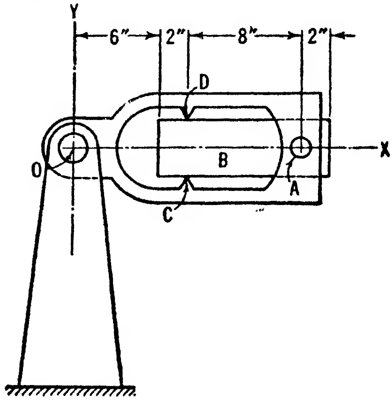

FIG. 416

890. Solve Prob. 889 with the bar and yoke in a position 180° from that shown in Fig. 416. Let all the other data remain unchanged.

891. Solve Prob. 889 with the bar and yoke in a position 60°, clockwise, from that shown in Fig. 416. Let all the other data remain unchanged.

892. Figure 417 represents a horizontal shaft, *AD*, rotating in bearings at *A* and *B*. At *C* a solid, cylindrical disk, 24 in. in diameter, weighing 322 lb, is keyed eccentrically to the shaft, the center of the disk being 4 in. from the center of the shaft. At the instant under consideration the center of the disk is vertically above the shaft. At *D* a light pulley, 18 in. in diameter, is keyed to the shaft, and is subjected to belt pulls as shown. At the given instant the system is rotating at a speed of 360 rpm. Calculate the components of the reactions exerted on the shaft by its bearings. Disregard friction, and the weight of the pulley and shaft. *Ans.* $A_x = +17.4$ lb; $A_y = -1890$ lb; $B_x = -56.9$ lb; $B_y = -2520$ lb.

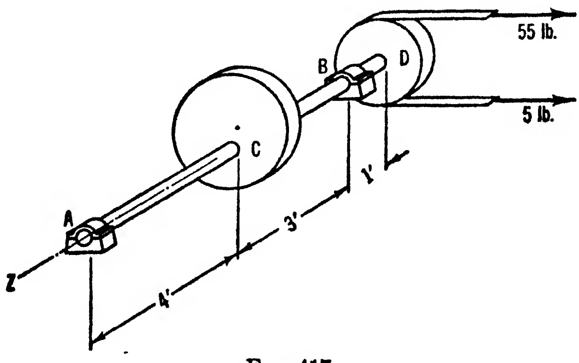

FIG. 417

893. Solve Prob. 892 with the center of disk *C* in a position 90°, counterclockwise, from that shown in Fig. 417. Let all the other data of the problem remain unchanged.

894. Solve Prob. 873, Fig. 409, with the cylinder in such a position that its center of gravity, G, is directly below the axis of the shaft. Let all the other data of the problem remain unchanged. *Ans.* $A_x = -522$ lb; $A_y = +313$ lb; $B_x = -726$ lb; $B_y = +157$ lb.

895. Solve Prob. 873, with the line OG in a position 45°, clockwise, from that shown in Fig. 409. Let all the other data of the problem remain unchanged.

156. Superelevation of Curves. Those portions of railway lines, highways, race tracks, etc., which have horizontal curvature are usually built with the outer rail or boundary at a higher elevation than the inner, causing vehicles to lean inward, toward the center of curvature. This arrangement is called *superelevation.* It may be omitted where the curvature is very slight.

If superelevation were omitted from a highway curve, friction exerted on the wheels by the roadway would be the sole dependence in maintaining the normal, or centripetal, acceleration necessary to curvilinear motion (Art. 117). At the higher speeds the danger of skidding would be extreme. In the railway the forces exerted by the outer rail on the flanges of the wheels largely supplant friction in maintaining normal acceleration, but danger of overturning would still exist, and excessive stresses in rolling stock and track might result in accident. Also, the wheels would tend to climb the outer rails and cause derailment.

Excessive superelevation would create somewhat similar dangers, although in lesser degree. In the highway, excessive superelevation would create additional dangers in icy weather, associated with the lower speeds prevailing under such conditions.

The superelevation chosen for a given curve will be theoretically correct only for one speed. The speed assumed in the calculation is usually higher than the average speed expected on such a curve, but lower than the extreme speeds used by a small minority of drivers.

A formula commonly used for highways, for the case in which transverse friction is to be avoided entirely at the chosen speed, is as follows:

$$\tan\theta = \frac{v^2}{15r} \qquad [227]$$

in which θ is the transverse angle of inclination between the roadway and the horizontal, v is the velocity of the vehicle in miles per hour, and r is the radius of the curve in feet. Equation 227 also applies to railways, for the case in which flange pressures are to be avoided at the chosen speed.

In some cases in the design of highways a certain limited amount of transverse friction against the wheels is considered safe, and the following approximate formula is sometimes used:

$$\tan\theta = \frac{v^2}{15r} - f \qquad [228]$$

in which θ, v, and r have the same meanings as in Eq. 227, and in which f represents a quantity called the *safe side friction factor*. The quantity f is the ratio of the transverse frictional force to the normal pressure, but should be considerably less than the coefficient of static friction, the difference representing the margin of safety. The value 0.16 is sometimes used for f, for moderate speeds. For the highest speeds the value of f is reduced.

PROBLEMS

896. Calculate the correct angle of superelevation for a highway curve whose radius is 2000 ft, for a speed of 50 mi per hr. Assume that transverse friction is to be avoided at the designated speed. *Ans.* 4° 45′.

897. Calculate the correct angle of superelevation for the curve in Prob. 896, using a safe side friction factor of 0.16.

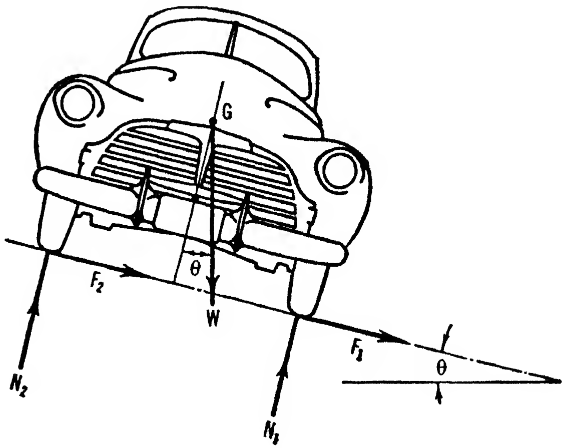

FIG. 418

898. Figure 418 represents an automobile traveling on a superelevated curve. Equation 227 gives the value of tan θ for the case in which no transverse friction is to be permitted at the chosen speed. Therefore, F_1 and F_2 should be taken equal to zero. Derive Eq. 227, using Eqs. 184 and 185, Art. 126.

899. In Eq. 228, f is called the safe side friction factor. Derive Eq. 228, referring to Fig. 418, and assuming that $F_1 = fN_1$ and that $F_2 = fN_2$. Disregard the vertical components of F_1 and F_2.

CHAPTER XIX

KINEMATICS OF PLANE MOTION

157. Definition of Plane Motion. *Plane motion* is that motion of a body in which all the particles move in parallel planes. That particular plane in which the center of gravity of the body moves is called the *plane of the motion.* Rotation is a special case of plane motion. Translation may, or may not, be plane motion.

The term "plane motion" is applicable only to rigid bodies. It follows that *the motions of all the particles lying on any given straight line at right angles to the plane of the motion will be alike.* At any given instant *the velocities of the particles lying on such a line are equal in magnitude and inclination, and agree in sense. This is also true of their accelerations.*

Let a locomotive be imagined to be running on a stretch of track that has neither horizontal nor vertical curvature. Under such conditions the motion of the side rod of the locomotive is an example of a plane motion which is also a translation. The motion of the connecting rod is not translation, but is a plane motion of a more general nature. The motion of any wheel of the locomotive is an example of plane motion, but not of translation. If the track has vertical curvature only, the side rod, connecting rod, and wheels still have plane motion, but the motion of the side rod is no longer translation.

158. Angular Velocity. *The angular velocity of a body in plane motion may be defined as the time rate at which the body is describing angular distance at the instant under consideration.*

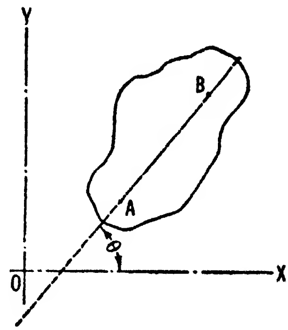

FIG. 419

Let Fig. 419 represent a body having plane motion, in which the plane of the motion is parallel to the plane of the paper. Let A and B represent any two of the particles of the body that move in the same plane. Let OX represent any convenient stationary axis. Let θ represent the angle between the line AB and the axis OX at any instant. As the body moves the line AB turns and the angle θ varies, except in the special case in which the motion is one of translation. The time rate at which the body describes angular distance is the rate at which θ changes with respect to t. The general formula for the angular velocity, at any instant, of a body in plane motion is, therefore, as follows:

$$\omega = \frac{d\theta}{dt} \tag{229}$$

The formula is the same as that for the angular velocity of a rotating body, and the statements in Art. 148 regarding signs and units apply also in the present case.

Obviously, the angular velocity of a translating body is equal to zero.

159. Angular Acceleration. *The angular acceleration of a body in plane motion, at any instant, is the time rate at which the angular velocity of the body is changing at that instant.*

The general formula for the angular acceleration of a body in plane motion is, therefore, as follows:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{230}$$

Naturally, the formula is the same as that for the angular acceleration of a rotating body, and the statements in Art. 149 regarding signs and units apply also in the present case.

The following formula, obtained by eliminating dt between Eqs. 229 and 230, may be used, if desired:

$$\omega\, d\omega = \alpha\, d\theta \tag{231}$$

Because of the fact that in a motion of rotation there is a certain specific and important axis, called the axis of rotation, students sometimes receive the impression that the angular velocity of a body will not have a definite value unless it is referred to some such axis at right angles to the plane of the motion. Such an impression is erroneous. *At any specific instant, in a specific problem, the angular velocity of the body has one, and only one, value,* and, consequently, that value does not depend on any selection of axes that may be made during the solution of the problem. For example, let it be stated that the angular velocity of a certain body in plane motion is 2 rad per sec at a given instant. It is not necessary to mention an axis in order to clarify the understanding of the situation; the statement is definite and complete in itself, from the definition of angular velocity. *The foregoing statements are also true for angular acceleration.*

Illustrative Problem

900. Figure 420 represents a straight bar, AB, of which one end slides on a horizontal floor and the other on a vertical wall. The plane of the motion is a vertical plane at right angles to the wall. Derive a formula showing the relation between the angular velocity of the bar and the linear velocity of its lower end, at any instant.

Solution. Let x represent the distance from the lower end, A, of the bar, to the fixed point, O, at any instant. From the figure,

$$\theta = \text{arc cos}\left(-\frac{x}{l}\right)$$

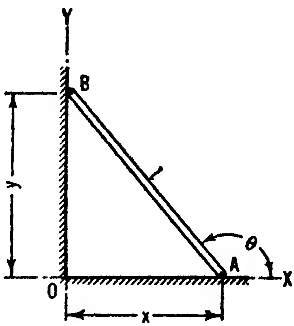

Fig. 420

By Eq. 229,

$$\omega = \frac{d\theta}{dt} = \frac{d\,[\text{arc cos }(-x/l)]}{dt} = \frac{1}{\sqrt{l^2 - x^2}}\frac{dx}{dt}$$

But, $dx/dt = v_A$. Therefore,

$$\omega = \frac{v_A}{\sqrt{l^2 - x^2}} = \frac{v_A}{y} = \frac{v_A}{l \sin \theta}$$

PROBLEMS

901. Calculate the angular velocity and angular acceleration of the bar AC in Prob. 642, Art. 115. *Ans.* -4 rad/sec; 0.

902. Calculate the angular velocity and angular acceleration of the bar AB in Prob. 643, Art. 115.

903. The bar in Fig. 420 moves in such a manner that $\theta = \pi - t^{2.5}$, in which radians and seconds are the units used. Calculate the angular velocity and angular acceleration of the bar, for the instant when $t = 4$ sec. *Ans.* -20 rad/sec; -7.5 rad/sec^2.

904. The bar in Fig. 420 moves in such a manner that $x = l \cos (t^2)$, in which the units used are feet, seconds, and radians. The length of the bar is 2 ft. Calculate the angular velocity and angular acceleration of the bar for the instant when $t = 2$ sec.

905. Derive a formula for the angular velocity of the bar in Prob. 900, in terms of l, θ, and the linear velocity of the upper end, B. *Ans.* $\omega = \dfrac{v_B}{l \cos \theta}$.

906. Derive a formula for the angular acceleration of the bar in Prob. 900, in terms of l, θ, v_A, and a_A. *Ans.* $\alpha = \dfrac{a_A - v_A\omega \cot \theta}{l \sin \theta}$.

160. Rolling Bodies. Motion in which a wheel, cylinder, sphere, or similar object rolls on a track or other surface is of wide importance in engineering. If no slipping occurs at the surface of contact the motion is called *perfect rolling.* In this book perfect rolling will always be assumed unless there is some statement or indication to the contrary.

The discussion in the present article will be limited to perfect rolling of wheels or other bodies having circular peripheries. Certain useful formulas will be derived, showing the relationship between the angular motion of the body and the linear motion of its geometric center.

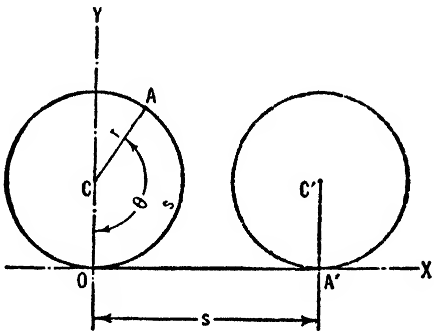

Fig. 421

Let Fig. 421 represent a wheel or other body having a circular periphery, rolling on a plane surface, without slipping, its center point moving in a straight line parallel to the supporting surface. Obviously this is plane motion.

Let C' represent the position of the center point at a certain initial instant. Let A' represent the position of the lowest particle on the periphery of the body at that instant. Let C and A represent the positions of the same particles at any later instant. Let OX and OY represent a pair of stationary axes. Let s represent the distance between the moving point C and the fixed initial point C'. The arc, OA, is also equal to s, since no slipping occurs. Let v_C and a_C represent the linear velocity and linear acceleration of C at any instant.

From geometry, $s = r\theta$. By Eq. 122, Art. 108,

$$v = \frac{ds}{dt} \qquad v_C = \frac{ds}{dt} = \frac{d(r\theta)}{dt} = r\frac{d\theta}{dt} \tag{232}$$

But, from Eq. 229, Art. 158, $\omega = d\theta/dt$. Substituting in Eq. 232,

$$v_C = r\omega \tag{233}$$

By Eq. 123, Art. 109,

$$a = \frac{dv}{dt} \qquad a_C = \frac{dv_C}{dt} = \frac{d(r\omega)}{dt} = r\frac{d\omega}{dt} \tag{234}$$

From Eq. 230, Art. 159, $\alpha = d\omega/dt$. Substituting in Eq. 234,

$$a_C = r\alpha \tag{235}$$

Units. Because of the fact that the formula $s = r\theta$ is valid only when the radian is the unit of angular measure, it follows that the radian must be used in applying Eqs. 233 and 235. Any desired units may be employed for r and t, and their choice will determine the units in which v_C and a_C will be expressed.

PROBLEMS

907. A wheel, 6 ft in diameter, rolls on a plane surface, without slipping. The linear acceleration of the center point is 1.25 mi per hr per sec. Calculate the angular acceleration of the wheel, in radians per second per second. Calculate the angular velocity, in radians per second, at the end of 10 sec, starting from rest. *Ans.* 0.610 rad/sec^2; 6.10 rad/sec.

908. A sphere, 24 in. in diameter, starts from rest and rolls on a plane surface, without slipping. It has a constant angular acceleration of 0.18 rad per sec per sec. Calculate the linear velocity of the center point at the instant when $\frac{1}{2}$ min has elapsed.

909. A wheel 3 ft in diameter rolls on a plane surface, without slipping. The velocity of the center point increases from 10 to 30 mi per hr, in a distance of 100 ft. Calculate the angular acceleration, in revolutions per minute per minute. *Ans.* 3290 rev/min^2.

910. A wheel 4 ft in diameter rolls on a plane surface, in accordance with the law $\theta = t^{5/4}$, in which θ is in revolutions and t is in minutes. Calculate the linear acceleration of the center point, for the instant when $t = 2\frac{1}{4}$ min.

911. Explain, on the basis of the similarity of the motions, why Eqs. 233 and 235 are the same in form as Eqs. 211 and 212, of Art. 150.

161. Relation between the Motions of Two Particles of the Body. Certain useful formulas will be derived, showing the relation between the linear velocities, and between the linear accelerations, of any two particles of a body in plane motion.

Velocities. Let Fig. 422 represent a body in plane motion, the plane of the motion being parallel to the plane of the paper. Let particles 1 and 2 in the figure represent any two particles of the body, lying in the plane of the motion or in any parallel plane. Let OX and OY represent any convenient pair of stationary rectangular axes in the plane of the motion. Let v_1, v_2, a_1, and a_2 represent the linear velocities and accelerations of the two chosen particles at the instant under consideration. Let θ represent the angle between the fixed axis, OX, and the moving line, (1, 2), at the given instant. Let q represent the distance between the two particles. The body is assumed to be rigid; therefore, the distance q remains constant.

From the figure,

$$x_2 - x_1 = q \cos \theta \qquad [236]$$

Differentiating in Eq. 236, with respect to t,

$$\frac{dx_2}{dt} - \frac{dx_1}{dt} = -q \sin \theta \frac{d\theta}{dt} \qquad [237]$$

By Eq. 157, Art. 115, $dx_2/dt = v_{2x}$ and $dx_1/dt = v_{1x}$. By Eq. 229, Art. 158, $d\theta/dt = \omega$. Substituting in Eq. 237,

$$v_{2x} - v_{1x} = -q\omega \sin \theta \qquad [238]$$

By a similar process of analysis it can be shown that

$$v_{2y} - v_{1y} = q\omega \cos \theta \qquad [239]$$

Accelerations. Differentiating in Eq. 238, with respect to t,

$$\frac{dv_{2x}}{dt} - \frac{dv_{1x}}{dt} = -q\left(\omega \cos \theta \frac{d\theta}{dt} + \sin \theta \frac{d\omega}{dt}\right) \qquad [240]$$

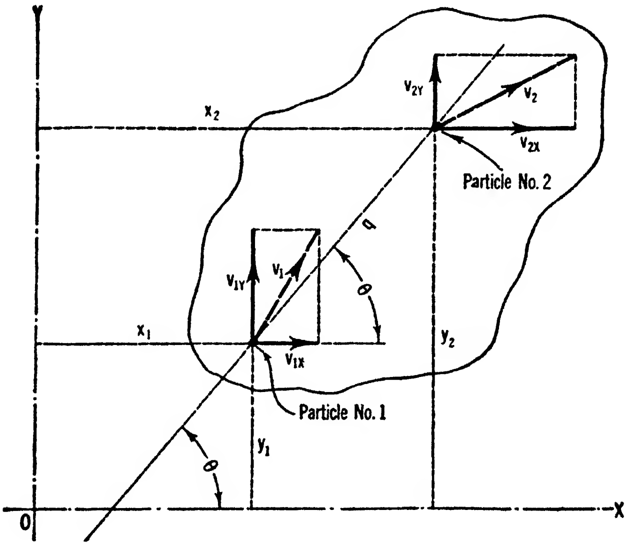

FIG. 422

By Eq. 172, Art. 118, $dv_{2x}/dt = a_{2x}$ and $dv_{1x}/dt = a_{1x}$. By Eq. 229, Art. 158, $d\theta/dt = \omega$. By Eq. 230, Art. 159, $d\omega/dt = \alpha$. Substituting in Eq. 240,

$$a_{2x} - a_{1x} = -q\omega^2 \cos \theta - q\alpha \sin \theta \qquad [241]$$

By a similar process of analysis it can be shown that

$$a_{2y} - a_{1y} = -q\omega^2 \sin \theta + q\alpha \cos \theta \qquad [242]$$

Equations 238, 239, 241, and 242 can be expressed more concisely, as follows; From Fig. 422, $q \sin \theta = y_2 - y_1$, and $q \cos \theta = x_2 - x_1$.

Substituting in Eqs. 238, 239, 241, and 242,

$$v_{2x} - v_{1x} = -(y_2 - y_1)\omega \qquad [243]$$

$$v_{2y} - v_{1y} = (x_2 - x_1)\omega \qquad [244]$$

$$a_{2x} - a_{1x} = -(x_2 - x_1)\omega^2 - (y_2 - y_1)\alpha \qquad [245]$$

$$a_{2y} - a_{1y} = -(y_2 - y_1)\omega^2 + (x_2 - x_1)\alpha \qquad [246]$$

In the majority of problems it is convenient to place the origin of coordinates at the point occupied by particle 1. In this case let x and y represent the coordinates of particle 2. The four equations then assume the following simple forms:

$$v_{2x} - v_{1x} = -y\omega \qquad [247]$$

$$v_{2y} - v_{1y} = x\omega \qquad [248]$$

$$a_{2x} - a_{1x} = -x\omega^2 - y\alpha \qquad [249]$$

$$a_{2y} - a_{1y} = -y\omega^2 + x\alpha \qquad [250]$$

When any of the formulas of the present article are used in the solution of problems it is immaterial which of the two particles is called 1, provided that strict attention is paid to the algebraic signs of all quantities. The velocity and acceleration components, and the angular velocity and angular acceleration of the body, are given signs in accordance with the conventions previously adopted in this book. The coordinates x and y, of particle 2 in Eqs. 247–250, are given signs in accordance with the customary practice in analytical geometry. Similar precautions must be observed in connection with the various coordinates in Eqs. 243–246.

Special Case. Equations 247–250 can be further simplified if the x-axis is placed in a position such that it passes through the two particles. If this is done, $y = 0$, and the four formulas appear as follows:

$$v_{2x} - v_{1x} = 0 \qquad [251]$$

$$v_{2y} - v_{1y} = x\omega \qquad [252]$$

$$a_{2x} - a_{1x} = -x\omega^2 \qquad [253]$$

$$a_{2y} - a_{1y} = x\alpha \qquad [254]$$

In the application of these formulas the origin of coordinates must be made to coincide with particle 1, as in the preceding case.

Equations 251–254 show that either of the two particles may be considered to have a motion of rotation about the other particle as a center, superimposed on a motion identical with that of the other. In other words, general plane motion may be regarded as rotation and translation occurring simultaneously with the same body.

Units. In applying any of the formulas of the present article the radian must be chosen as the unit of angular measure. Any convenient units may be used for time, and for linear distance, provided that consistency is observed.

ILLUSTRATIVE PROBLEMS

912. Figure 423 represents a straight, rigid bar, AB, 5 ft long. The upper end moves on a wall, in a vertical straight line, and the lower end moves in a horizontal straight line at right angles to the wall. The lower end, A, has a constant velocity of 8 ft per sec, toward the left. Calculate the angular velocity and angular acceleration of the bar, and the linear velocity and acceleration of B, at the instant represented by the figure.

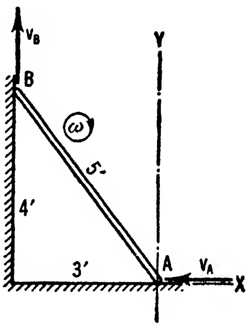

FIG. 423

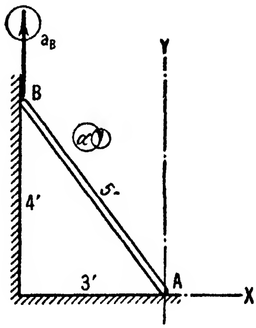

FIG. 424

Solution. Let A be chosen as particle 1, and B as particle 2. Assume the senses of ω and v_B in the manner indicated. Solution will be made by means of Eqs. 247–250. The use of these particular formulas renders it mandatory to place the origin of coordinates at particle 1. Therefore, $x = -3$ ft, $y = +4$ ft. Also, $v_A = -8$ ft/sec. Substituting in Eqs. 247 and 248, and paying strict attention to algebraic signs,

$$v_{2x} - v_{1x} = -y\omega \qquad 0 - (-8) = -(+4)(-\omega)$$

$$v_{2y} - v_{1y} = x\omega \qquad +v_B - 0 = (-3)(-\omega)$$

The solution of these equations gives

$$\omega = +2 \text{ rad/sec} \qquad v_B = +6 \text{ ft/sec}$$

The positive signs accompanying the results show that the senses of ω and v_B were assumed correctly. Therefore ω is clockwise and v_B is upward.

Let α and a_B be assumed as shown in Fig. 424. From the problem, since

v_A is constant, $a_A = 0$. Substituting in Eqs. 249 and 250,

$$a_{2x} - a_{1x} = -x\omega^2 - y\alpha \qquad 0 - 0 = -(-3)(-2)^2 - (+4)(-\alpha)$$

$$a_{2y} - a_{1y} = -y\omega^2 + x\alpha \qquad +a_B - 0 = -(+4)(-2)^2 + (-3)(-\alpha)$$

from which

$$\alpha = -3 \text{ rad/sec}^2 \qquad a_B = -25 \text{ ft/sec}^2$$

In this case negative signs are obtained, showing that α and a_B were assumed incorrectly. Therefore α is counterclockwise and a_B is downward. Draw small circles enclosing the arrowheads, to show that the correct senses are opposite to those assumed.

It is suggested that the student check the solution by using Eqs. 251–254. If this is done, the x-axis must be made to pass through A and B.

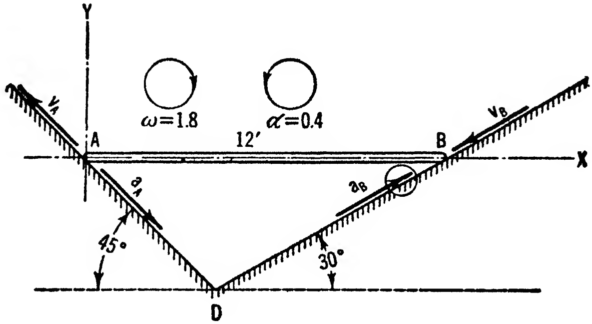

FIG. 425

913. Figure 425 represents a straight bar, AB, the ends of which slide on inclined planes, as shown. The bar moves in a vertical plane. At the instant under consideration the bar is horizontal, has a clockwise angular velocity of 1.8 rad per sec, and a counterclockwise angular acceleration of 0.4 rad per sec per sec. Calculate the linear velocity and linear acceleration of each end of the bar at the given instant.

Solution. A moves along the straight line AD, and B moves along BD. Therefore, v_A and a_A are parallel to AD, and v_B and a_B are parallel to BD. Assume the senses of all quantities as shown in the figure. In this problem it will be convenient to use Eqs. 251–254, in which the x-axis is placed so as to pass through the two chosen particles, with the origin at particle 1. Let A be chosen as particle 1 and B as particle 2. The arrangement of axes will be as shown in the figure. Therefore, $x = +12$ ft. Also, $\omega = -1.8$ rad/sec, and $\alpha = +0.4$ rad/sec^2. Substituting in Eqs. 251 and 252, paying close attention to the signs of all quantities,

$$v_{2x} - v_{1x} = 0 \qquad -v_{Bx} - (-v_{Ax}) = 0$$

$$v_{2y} - v_{1y} = x\omega \qquad -v_{By} - (+v_{Ay}) = (+12)(-1.8)$$

From the figure, $v_{Ax} = v_A \cos 45°$, $v_{Ay} = v_A \sin 45°$, $v_{Bx} = v_B \cos 30°$, $v_{By} = v_B \sin 30°$. Substituting these values in the preceding equations,

$$-0.866v_B + 0.707v_A = 0$$

$$-0.5v_B - 0.707v_A = -21.6$$

The solution of these equations gives

$$v_A = +19.4 \text{ ft/sec} \qquad v_B = +15.8 \text{ ft/sec}$$

Substituting now in Eqs. 253 and 254, with careful attention to signs,

$$a_{2x} - a_{1x} = -x\omega^2 \qquad +a_{Bx} - a_{Ax} = -(+12)(-1.8)^2$$

$$a_{2y} - a_{1y} = x\alpha \qquad +a_{By} - (-a_{Ay}) = (+12)(+0.4)$$

From the figure, $a_{Ax} = a_A \cos 45°$, $a_{Ay} = a_A \sin 45°$, $a_{Bx} = a_B \cos 30°$, and $a_{By} = a_B \sin 30°$. Substituting these values in the two preceding equations,

$$0.866a_B - 0.707a_A = -38.9$$

$$0.5a_B + 0.707a_A = +4.8$$

The solution of these equations gives

$$a_A = +24.5 \text{ ft/sec}^2 \qquad a_B = -24.9 \text{ ft/sec}^2$$

The negative sign accompanying the value of a_B shows that the sense of that acceleration was incorrectly assumed. Therefore, a_B is directed toward D. A small circle has been drawn around the arrowhead to indicate these facts.

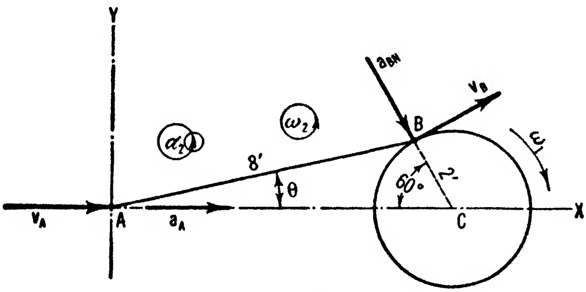

Fig. 426

914. Figure 426 represents a specific example of the familiar crank-and-connecting-rod mechanism. The crank rotates in a clockwise direction at a constant angular velocity of 180 rpm. At the instant under consideration the crank angle, ACB, is 60°. Other dimensions are as shown. Calculate the angular velocity and angular acceleration of the connecting rod, and the linear velocity and linear acceleration of the crosshead pin, A, at the given instant.

Solution. The connecting rod executes a plane motion of a general type. The crank, BC, is in simple rotation. The point B is common to the crank and the connecting rod. The angle θ can be calculated from the triangle ABC, by the law of sines, as follows:

$$\frac{\sin\theta}{2} = \frac{\sin 60°}{8} \qquad \theta = 12° \, 30'$$

The velocity of B can be calculated from the rotational motion of the crank. By Eq. 211, Art. 150,

$$v = r\omega \qquad v_B = 2\left(\frac{180 \times 2\pi}{60}\right) = 37.7 \text{ ft/sec}$$

$$v_{Bx} = v_B \cos 30° = 37.7 \times 0.866 = 32.6 \text{ ft/sec}$$

$$v_{By} = v_B \cos 60° = 37.7 \times 0.5 = 18.9 \text{ ft/sec}$$

The problem will be solved by means of Eqs. 247–250, using A and B as the two particles. A will be chosen as particle 1. This necessitates placing the origin at that point. Arrange the axes as shown in the figure. Let ω_2 and α_2 represent the angular velocity and angular acceleration of the connecting rod, and let their senses be assumed as shown. In the formulas, x and y are the coordinates of particle 2, in this case, B. From the figure, $x = +8 \cos\theta = +8 \cos 12° \, 30' = +7.81$ ft, and $y = +2 \sin 60° = +1.73$ ft. By Eqs. 247 and 248,

$$v_{2x} - v_{1x} = -y\omega \qquad +32.6 - v_A = -(+1.73)(+\omega_2)$$

$$v_{2y} - v_{1y} = x\omega \qquad +18.9 - 0 = +7.81(+\omega_2)$$

The solution of these equations gives

$$\omega_2 = +2.42 \text{ rad/sec} \qquad v_A = +36.8 \text{ ft/sec}$$

Since the angular velocity of the crank is constant, B has normal acceleration only. By Eq. 213, Art. 150,

$$a_N = r\omega^2 \qquad a_{BN} = 2\left(\frac{180 \times 2\pi}{60}\right)^2 = 711 \text{ ft/sec}^2$$

$$a_{Bx} = 711 \cos 60° = 356 \text{ ft/sec}^2$$

$$a_{By} = 711 \cos 30° = 616 \text{ ft/sec}^2$$

By Eqs. 249 and 250, using the value of ω_2 previously obtained,

$$a_{2x} - a_{1x} = -x\omega^2 - y\alpha \qquad +356 - a_A = -(+7.81)(+2.42)^2 - (+1.73)(+\alpha_2)$$

$$a_{2y} - a_{1y} = -y\omega^2 + x\alpha \qquad -616 - 0 = -(+1.73)(+2.42)^2 + (+7.81)(+\alpha_2)$$

The solution of these equations gives

$$\alpha_2 = -77.6 \text{ rad/sec}^2 \qquad a_A = +267 \text{ ft/sec}^2$$

The negative sign obtained with α_2 shows that the sense of the angular acceleration was incorrectly assumed; therefore, it is clockwise.

PROBLEMS

915. The wheel shown in Fig. 427 rolls on a horizontal plane, without slipping. The diameter is 4 ft. At the instant represented by the figure the velocity of the center is 4 ft per sec, toward the right. The acceleration of the center is 6 ft per sec per sec, toward the left. Calculate the velocity and acceleration of B_1. *Ans.* 8 ft/sec, $\theta_x = 0°$; 14.4 ft/sec², $\theta_x = 213° 40'$.

916. Calculate the velocity and acceleration of B_2, of the wheel in Prob. 915, Fig. 427.

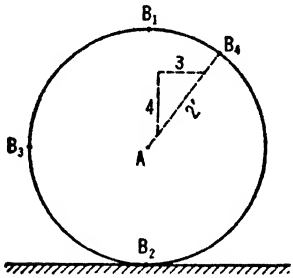

Fig. 427

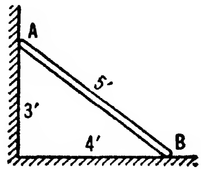

Fig. 428

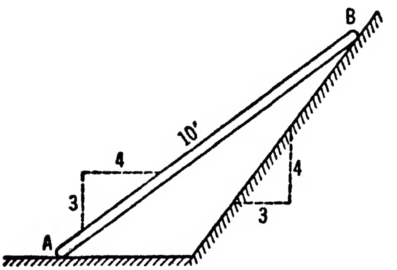

Fig. 429

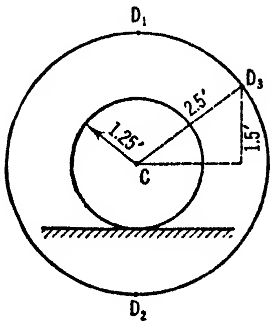

Fig. 430

917. Calculate the velocity and acceleration of particle B_3, of the wheel in Prob. 915, Fig. 427. *Ans.* 5.66 ft/sec, $\theta_x = 45°$; 6.32 ft/sec², $\theta_x = 288° 25'$.

918. Calculate the velocity and acceleration of particle B_4, of the wheel in Prob. 915, Fig. 427.

919. Figure 428 represents a rigid bar, AB, 5 ft long. The upper end, A, moves in a vertical straight line on a wall, and the lower end, B, moves in a horizontal straight line at right angles to the wall. A has a constant velocity of 12 ft per sec, upward. Calculate the velocity and acceleration of B, and the angular velocity and angular acceleration of the bar, for the position shown in the figure. *Ans.* -9 ft/sec; -56.3 ft/sec²; -3 rad/sec; -6.75 rad/sec².

920. In Fig. 428, let the point B have a velocity of 6 ft per sec, toward the right, and an acceleration of 4 ft per sec per sec, toward the left. Calculate the velocity and acceleration of A for the position shown by the figure.

921. Figure 429 represents a rigid bar AB, 10 ft long, in plane motion. The plane of the motion is parallel to the plane of the figure. The upper end has a velocity of 10 ft per sec, down the incline, and a deceleration of 5 ft per sec per sec, in the position shown. Calculate the velocity and acceleration of the lower end, A.

922. The bar in Fig. 429 has a constant angular velocity of 2 rad per sec, clockwise. Calculate the velocity and acceleration of each end of the bar, for the position shown by the figure. *Ans.* v_A = 24 ft per sec, left; v_B = 20 ft/sec, down; a_A = 14 ft/sec^2, right; a_B = 30 ft/sec^2, down.

923. The body shown in Fig. 430 consists of a large disk 5 ft in diameter, to which a pair of wheels 2.5 ft in diameter are rigidly attached. The wheels roll, without slipping, on a horizontal track, which is elevated sufficiently to permit the disk to move without touching the ground. At the instant represented by the figure the velocity of the center point, C, is 5 ft per sec, toward the left. At the same instant C is gaining speed at the rate of 2.5 ft per sec per sec. Calculate the velocity and acceleration of D_1 at the given instant.

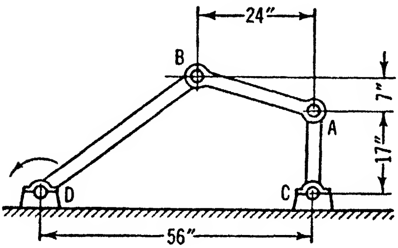

Fig. 431

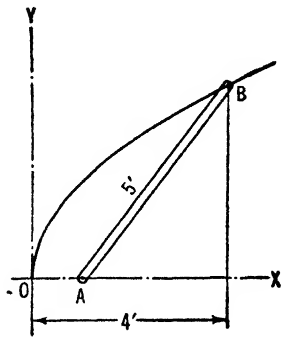

Fig. 432

924. Calculate the velocity and acceleration of the point D_2, in Prob. 923, Fig. 430. *Ans.* 5 ft/sec, $\theta_x = 0°$; 40.1 ft/sec^2, $\theta_x = 86° 25'$.

925. Calculate the velocity and acceleration of the point D_3, in Prob. 923, Fig. 430.

926. Calculate the velocity and acceleration of a point on the connecting rod in Prob. 914, Fig. 426, 5 ft from A. Make use of any of the results obtained in the solution of that problem. *Ans.* 36.2 ft/sec, $\theta_x = 19° 00'$; 502 ft/sec^2, $\theta_x = 309° 50'$.

927. Calculate the velocity and acceleration of the crosshead pin, A, in Prob. 914, Fig. 426, for the instant when the crank angle, ACB, is equal to 216° 52′. Let all the other conditions of the problem remain unchanged.

928. Calculate the velocity and acceleration of a point at the middle of the bar AB in Prob. 913, Fig. 425. Use any of the results obtained in that problem which may be needed. *Ans.* 14.0 ft/sec, $\theta_x = 168° 00'$; 15.0 ft/sec^2, $\theta_x = 261° 55'$.

929. Mechanisms similar in principle to that shown in Fig. 431 are frequently utilized in machines. The links, AC and BD, can rotate about the fixed points, C and D, respectively. In the present case BD is rotating in a counterclockwise direction with an angular velocity of 1.25 rad per sec. Calculate the angular velocity of AC, for the position shown by the figure. *Ans.* +2.46 rad/sec.

930. In Prob. 929, Fig. 431, the angular acceleration of link BD is zero at the given instant. Calculate the angular acceleration of link AC at that instant.

931. In Fig. 432, the upper end of the bar AB moves along a parabola whose equation is $y^2 = 4x$, in which x and y are expressed in feet. The lower end moves along the x-axis. In the position shown by the figure the velocity of the lower end is 10 ft per sec, toward the right. Calculate the velocity of the upper end. *Ans.* 6.71 ft/sec, $\theta_x = 26° 35'$.

932. In Prob. 931, the acceleration of the lower end of the bar at the given instant is zero. Calculate the acceleration of the upper end.

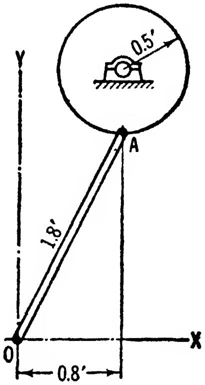

FIG. 433

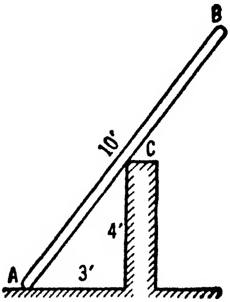

FIG. 434

933. Figure 433 represents a variation of the crosshead-and-connecting-rod mechanism. The crankpin, A, travels in the circle at a constant angular velocity of 20 rad per sec, clockwise. The crosshead pin, O, travels along the y-axis. Calculate the velocity and acceleration of the crosshead pin for the given position. *Ans.* -4.97 ft/sec; $+278$ ft/sec^2.

934. Solve Prob. 933 with the crankpin in a position one-half revolution from that shown in Fig. 433. Let all the other conditions of the problem remain unchanged.

935. Figure 434 represents a straight, rigid bar 10 ft in length. The lower end, A, slides in a horizontal, straight line parallel to the plane of the paper. The bar leans against a wall, at C, and remains in contact therewith at all times. In the position shown the velocity of A is 2.5 ft per sec, toward the left. Calculate the velocity of B for the given position. *Ans.* 2.5 ft/sec, $\theta_x = 286° 15'$.

162. The Instantaneous Axis. At each instant during plane motion there are certain particles of the body, or of the body extended, that have zero velocity. These particles fall on a straight line at right angles to the plane of the motion. This line is called the *instantaneous axis*. *The relation existing, at the given instant, between the linear velocity of any particle of the body, the angular velocity of the body, and the distance from the instantaneous axis to the particle, is precisely the same as in a rotating body (Art. 150), the instantaneous axis corresponding to the axis of rotation.* The point at which the instantaneous axis intersects the plane of the motion is called the *instantaneous center*.

The sole exception to the foregoing statements occurs when the conditions are those of translation, in which case there is no instantaneous axis.

Proof. Let Fig. 435 represent a body in plane motion. Let A represent any particle lying in the plane of the motion, and let v_A represent the velocity of A at any given instant. Place the origin of coordinates at A, with the x-axis lying in the plane of the motion, at right angles to v_A. Let B represent any other particle lying on the x-axis.

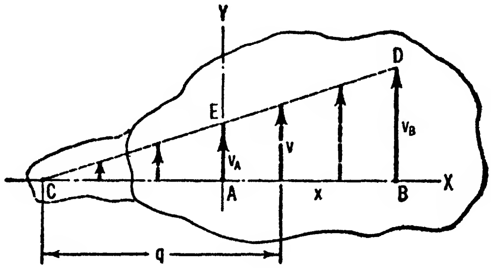

Fig. 435

By Eq. 251, Art. 161, $v_{2x} - v_{1x} = 0$. Therefore, $v_{Bx} - v_{Ax} = 0$. But, by the construction, $v_{Ax} = 0$. This proves that the velocities of all the particles lying on the x-axis are at right angles to that axis.

By Eq. 252, Art. 161, $v_{2y} - v_{1y} = x\omega$. Therefore, $v_B - v_A = x\omega$. Since ω is constant for the entire body at any single instant, it follows that the difference between the velocities of any two particles lying on the x-axis is proportional to the distance between the particles. Therefore, the velocities of all the particles lying on the x-axis are proportional to the ordinates to the straight line DE. Obviously, the velocity of the particle at C, where DE intersects the x-axis, is zero.

Now let the origin of coordinates be shifted to C, without disturbing the position of the x-axis. Let the velocity of any other particle on the x-axis be represented by v, and its abscissa by q. By Eq. 252, $v_{2y} - v_{1y} = x\omega$. But $v_{1y} = v_C = 0$. Therefore, $v = q\omega$.

Thus, it has been proved that if a line is drawn through any particle at right angles to the velocity of the latter, one particle on that line will have zero velocity at the given instant, and the relation among the velocities of all the particles lying on any such line is precisely the same as in a rotating body (Art. 150), the point of zero velocity corresponding to the center of rotation.

Obviously, if a second line is drawn, similar to that studied above, but passing through a different set of particles, similar results will be obtained. If the body is not translating, the two lines will intersect. Since the point of intersection lies on both lines its velocity must be at right angles to both. This is impossible; therefore, the velocity of the point of intersection is zero. It follows that there is only one particle in the plane of the motion having zero velocity. This point is the instantaneous center.

The entire discussion would apply to the particles in any plane parallel to the plane of the motion, leading to the conclusion that for the entire body at any given instant there is a line of particles, at right angles to the plane of the motion, whose velocity is zero. This line is the instantaneous axis.

As the motion continues, the instantaneous axis shifts with respect to the body and also with respect to the earth. The instantaneous axis is sometimes called the instantaneous axis of zero velocity, in order to emphasize the fact that it is not an axis of zero acceleration. It is possible to locate an axis of zero acceleration, but space for that purpose will not be used in this book.

Methods for locating the instantaneous axis in a specific problem follow naturally from the various relationships revealed in the foregoing proof, the detailed procedure in any given case depending upon the manner in which the data are presented.

Illustrative Problems

936. Figure 436 represents a wheel rolling on a plane surface, without slipping. Prove that the instantaneous axis is at the point of contact between the wheel and the supporting plane.

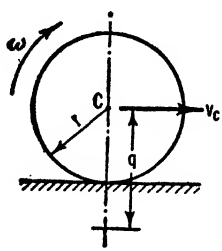

Fig. 436

Solution. Let v_C represent the velocity of the center point, and let ω represent the angular velocity of the wheel, at any instant. Let r represent the radius of the wheel.

Obviously, v_C is horizontal. From the discussion in the text it is learned that the instantaneous center lies on a line drawn through any particle in the plane of the motion, at right angles to the velocity of that particle. Therefore, for the wheel the instantaneous center lies on a vertical line through C.

Also, the relation between the position of the instantaneous center, the angular velocity of the body, and the linear velocities of the particles is the same as in simple rotation. Therefore, the instantaneous center lies below C, and $v_C = q\omega$, in which q represents the distance from C to the instantaneous center.

From Art. 160, $v_C = r\omega$. Equating the two expressions for v_C: $q\omega = r\omega$. Therefore, $q = r$. This proves that the instantaneous center is at the point of contact. The instantaneous axis is at right angles to the plane of the motion, through the instantaneous center.

937. Calculate the angular velocity of the connecting rod, and the linear velocity of the crosshead pin, A, in Prob. 914, by the instantaneous axis method.

Solution. The particles A and B, in Fig. 437, are two particles of the connecting rod. The inclinations of their velocities are known. v_A is along the line AC, and v_B is at right angles to BC. It was shown in the present article that a line drawn at right angles to the velocity of any particle intersects the

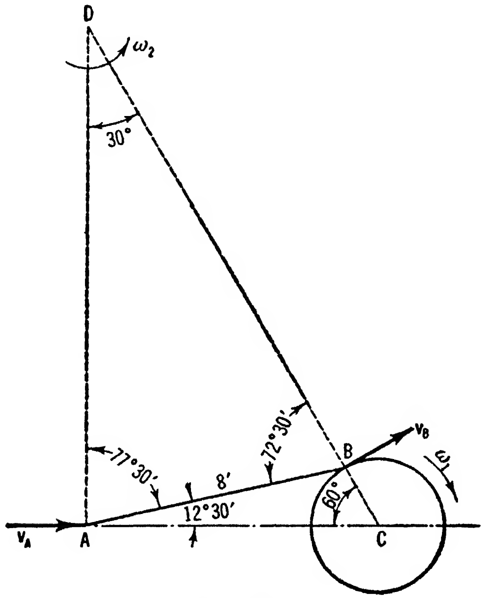

Fig. 437

instantaneous axis. Therefore, when the inclinations of the velocities of two particles of the body are known, the obvious method of locating the instantaneous axis is to draw lines through the two particles at right angles to their velocities. The instantaneous axis must lie at the intersection of these two lines.

In the present case, the line BD is drawn at right angles to v_B, and the line AD is drawn at right angles to v_A. The two lines intersect at D. The instantaneous axis passes through D, at right angles to the plane of the paper.

From Prob. 914, $\angle BAC = 12° 30'$, and $v_B = 37.7$ ft per sec. The three angles of triangle ABD are easily calculated, and are as shown in the figure. By the law of sines,

$$\frac{\overline{BD}}{8} = \frac{\sin 77° 30'}{\sin 30°} \qquad \overline{BD} = 15.6 \text{ ft}$$

$$\frac{\overline{AD}}{8} = \frac{\sin 72° 30'}{\sin 30°} \qquad \overline{AD} = 15.3 \text{ ft}$$

From the present article,

$$v = q\omega \qquad \omega_2 = \frac{v_B}{BD} = \frac{37.7}{15.6} = 2.42 \text{ rad/sec}$$

$$v_A = \overline{AD} \times \omega_2 = 15.3 \times 2.42 = 37.0 \text{ ft/sec}$$

938. Figure 438 represents a car on which is mounted a wheel 5 ft in diameter. The wheel is carried on a horizontal shaft which is at right angles to the

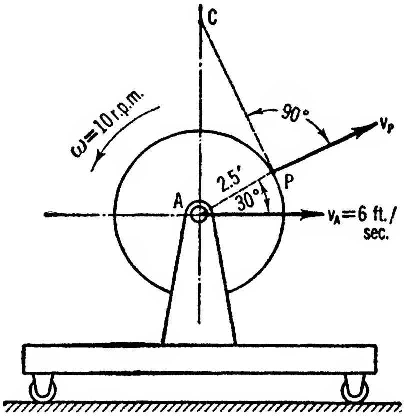

FIG. 438

plane of the figure. At the instant shown the car has a velocity of 6 ft per sec, toward the right, and the wheel has an angular velocity of 10 rpm, counterclockwise. Find the linear velocity of the particle, P, on the rim of the wheel, at the given instant.

Solution. Expressing the angular velocity of the wheel in radians per second,

$$\omega = \frac{10 \times 2\pi}{60} = \frac{\pi}{3} = 1.05 \text{ rad/sec}$$

The linear velocity of particle A, of the wheel, is 6 ft per sec, and is horizontal and toward the right. The instantaneous axis for the wheel intersects the plane of motion somewhere on a vertical line through A. The senses of v_A and ω show that the instantaneous axis will lie above A. Let C represent its position. The distance AC now can be calculated.

$$\overline{AC} = \frac{v_A}{\omega} \qquad \overline{AC} = \frac{6}{1.05} = 5.71 \text{ ft}$$

In the triangle ACP the sides AP and AC, and the included angle, PAC, are known. By the law of cosines,

$$\overline{CP} = \sqrt{(5.71)^2 + (2.5)^2 - 2 \times 5.71 \times 2.5 \cos 60^\circ} = 4.96 \text{ ft}$$

$$v_P = \overline{CP}\omega = 4.96 \times 1.05 = 5.21 \text{ ft/sec}$$

The inclination of v_P to the horizontal is equal to the angle ACP, since v_P is at right angles to CP. By the law of sines,

$$\frac{\sin (ACP)}{2.5} = \frac{\sin 60^\circ}{4.96} \qquad \angle ACP = 25^\circ\ 55'$$

PROBLEMS

939. The wheel in Fig. 427, Art. 161, is 4 ft in diameter, and rolls without slipping. The velocity of the center point is 4 ft per sec, toward the right. Calculate the velocity of B_1, and of B_3, by the instantaneous-axis method. *Ans.* 8 ft/sec, $\theta_x = 0^\circ$; 5.66 ft/sec, $\theta_x = 45^\circ$.

940. Calculate the velocity of the point B_4, in Fig. 427, Prob. 939, by the instantaneous-axis method.

941. Calculate the velocity of the point B, in Fig. 428, Prob. 919, by the instantaneous-axis method. Also calculate the velocity of a point at the middle of the bar. *Ans.* 9 ft/sec, $\theta_x = 180^\circ$; 7.5 ft/sec, $\theta_x = 126^\circ\ 50'$.

942. Calculate the velocity of point A, in Fig. 429, Prob. 921, by the instantaneous-axis method. Also calculate the velocity of a point on the bar 7.2 ft from A.

943. Solve Prob. 922, Fig. 429, for velocities only, by the instantaneous-axis method.

944. Calculate the velocities of points D_1, D_2, and D_3, in Fig. 430, Prob. 923, by the instantaneous-axis method.

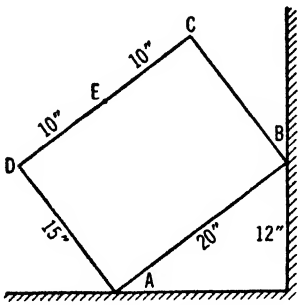

FIG. 439

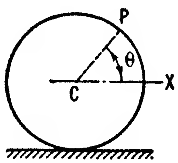

FIG. 440

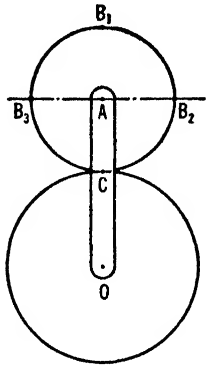

FIG. 441

945. Figure 439 represents a rectangular block in plane motion. Point B slides on a wall, in a vertical line, and point A slides in a straight line at right angles to the wall. In the position shown the velocity of B is 100 in. per min, downward. Calculate the velocities of points A, C, D, and E, for the given position. *Ans.* v_A = 75 in./min, $\theta_x = 180^\circ$; $v_C = 86.9$ in./min, $\theta_x = 329^\circ\ 45'$; $v_D = 56.3$ in./min, $\theta_x = 90^\circ$; $v_E = 38$ in./min, $\theta_x = 9^\circ\ 30'$.

946. Solve Prob. 929, Fig. 431, by the instantaneous-axis method.

947. Solve Prob. 931, Fig. 432, by the instantaneous-axis method.

948. Figure 440 represents a wheel rolling on a plane surface, without slipping, and P represents any point on the rim. θ is the angle of inclination of the radius, CP, measured as indicated. Prove that $v_P = v_C\sqrt{2 + 2 \sin \theta}$, at any given instant.

949. Solve Prob. 933, Fig. 433, for velocity only, by the instantaneous-axis method.

950. Figure 441 represents an epicyclic gear train. The larger gear is stationary, and the arm OA rotates about O, causing the smaller gear to roll on the circumference of the larger one. The gears are 12 and 16 in. in diameter. The arm OA rotates in a clockwise direction at a constant speed of 60 rpm. Find the linear velocities of the points B_1, B_2, and B_3. *Ans.* $v_1 = 14.7$ ft/sec, $\theta_x = 0°$; $v_2 = 10.4$ ft/sec, $\theta_x = 315°$; $v_3 = 10.4$ ft/sec, $\theta_x = 45°$.

951. Solve Prob. 935, Fig. 434, for velocity only, by the instantaneous-axis method.

952. Calculate the velocity of the middle point of the connecting rod in Prob. 937, Fig. 437. *Ans.* 36.3 ft/sec, $\theta_x = 15° 05'$.

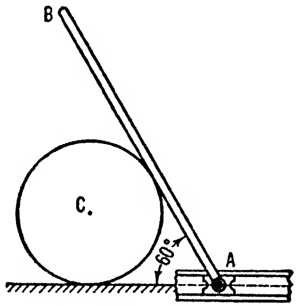

Fig. 442

953. Figure 442 represents a cylinder, 2 ft in diameter, rolling on a horizontal plane, without slipping. AB represents a straight bar, whose lower end moves in a horizontal straight line. The bar rests against the cylinder at all times. No slipping occurs between the bar and the cylinder or between the cylinder and the supporting surface. In the position shown the velocity of A is 6 ft per sec, toward the right. Calculate the angular velocity of the cylinder. *Ans.* -2 rad/sec.

CHAPTER XX

KINETICS OF PLANE MOTION

163. Special Nature of the Discussion. The study of the kinetics of plane motion will be restricted to the case in which the body is homogeneous, and is symmetrical with respect to the plane of the motion.

The majority of practical problems in plane motion conform to this special case. In some problems, when the body as a whole is not symmetrical with respect to the plane of the motion, it is possible to divide it into two or more parts, each of which, considered individually, is symmetrical with respect to its own plane of motion. The problem can then be solved by the methods of the present chapter, by treating each part separately and combining the results.

164. The Resultant of the External Forces. *In plane motion the resultant of the external forces acting on a homogeneous body which is symmetrical with respect to the plane of the motion is a force lying in that plane, whose magnitude is equal to* $(W/g)\bar{a}$, *whose inclination and sense are the same as the inclination and sense of* $\bar{a}$, *and whose moment about the center of gravity is equal to* $\bar{I}\alpha$.

In the foregoing, $\bar{a}$ is the linear acceleration of the center of gravity, α is the angular acceleration of the body, and $\bar{I}$ is the moment of inertia of the body with respect to a gravity axis at right angles to the plane of the motion.

In the special case in which $\bar{a} = 0$, the resultant reduces to a couple, lying in the plane of the motion, whose moment is equal to $\bar{I}\alpha$, and whose sense agrees with that of α.

Proof. Let A, in Fig. 443, represent any particle of the body. The body itself is not shown. Let G represent the center of gravity. Let the xy-plane be so placed as to contain G, and let the origin of coordinates be placed at G. The xy-plane is both the plane of the motion and the plane of symmetry of the body. The z-axis passes through G, at right angles to the plane of the motion. Let B represent a particle on the z-axis, moving in the same plane with particle A. Let q represent the distance AB, and let θ represent the angle indicated in the figure.

Let $\bar{a}_x$ and $\bar{a}_y$ represent the components of the acceleration of G. Obviously, $\bar{a}_x$ and $\bar{a}_y$ also will represent the components of the acceleration of B. The z-component of the acceleration of any particle of the body is equal to zero.

Let a_x and a_y represent the components of the acceleration of A. Let $(dW/g)a_x$ and $(dW/g)a_y$ represent the components of the effective force for particle A (Art. 123).

The fact that the resultant of the external forces acting on the body at a given instant is equal to $(W/g)\bar{a}$, that its inclination and sense agree with those of $\bar{a}$, and that it lies in the plane of the motion, can be demonstrated by a process of reasoning essentially the same as that

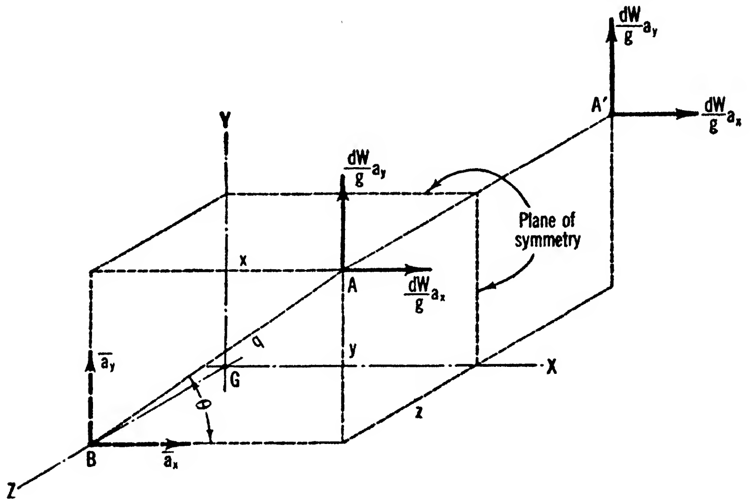

Fig. 443

followed in the case of the rotating body (Art. 154). That portion of the proof will be considered to have been accomplished, and will not be repeated. It remains to be proved, however, that the moment of the resultant about a gravity axis at right angles to the plane of the motion is equal to $\bar{I}\alpha$.

Let Eqs. 249 and 250, of Art. 161, be adapted to the present case. Let B be selected as particle 1 and A as particle 2.

$$a_{2x} - a_{1x} = -x\omega^2 - y\alpha \qquad a_x = \bar{a}_x - x\omega^2 - y\alpha \qquad [255]$$

$$a_{2y} - a_{1y} = -y\omega^2 + x\alpha \qquad a_y = \bar{a}_y - y\omega^2 + x\alpha \qquad [256]$$

Let $\Sigma\bar{M}$ represent the moment-sum of the effective forces for all the particles of the body, about the z-axis. By the principle of moments, Art. 50, $\Sigma\bar{M}$ is equal to the moment of the resultant of the effective forces about the z-axis. Furthermore, by Art. 125, the resultant of the effective forces is identical with the resultant of the external forces. Consequently, $\Sigma\bar{M}$ also represents the moment of the resultant of the external forces, about the z-axis. From Fig. 443,

$$\Sigma\overline{M} = \int\left(-\frac{dW}{g}a_x y + \frac{dW}{g}a_y x\right) \quad [257]$$

Substituting in Eq. 257 the values of a_x and a_y given by Eqs. 255 and 256,

$$\Sigma\overline{M} = -\frac{\bar{a}_x}{g}\int y\,dW + \omega^2\int xy\frac{dW}{g} + \alpha\int y^2\frac{dW}{g} + \frac{\bar{a}_y}{g}\int x\,dW$$
$$-\omega^2\int xy\frac{dW}{g} + \alpha\int x^2\frac{dW}{g} \quad [258]$$

Since the coordinate axes pass through the center of gravity of the body the integrals, $\int y\,dW$ and $\int x\,dW$, in Eq. 258, are equal to zero (Art. 95). The expressions $\alpha\int x^2\frac{dW}{g}$ and $\alpha\int y^2\frac{dW}{g}$ may be combined into the single expression, $\alpha\int(x^2+y^2)\frac{dW}{g}$. From the figure, $x^2 + y^2 = q^2$. Equation 258 can now be reduced as follows:

$$\Sigma\overline{M} = \alpha\int(x^2+y^2)\frac{dW}{g} = \alpha\int q^2\frac{dW}{g} = I_z\alpha$$

The integral $\int q^2\frac{dW}{g}$ is the moment of inertia of the body with respect to the z-axis. Denoting this by $\bar{I}$, the following formula is obtained:

$$\Sigma\overline{M} = \bar{I}\alpha \quad [259]$$

thus completing the proof.

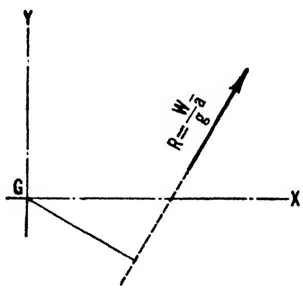

FIG. 444

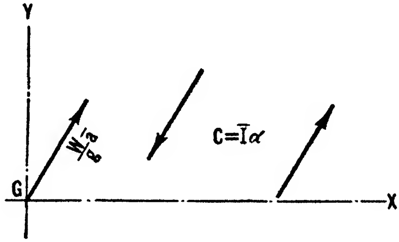

FIG. 445

Figure 444 depicts the resultant of the external forces in conformity with the foregoing discussion. The moment-arm of R with respect to the center of gravity, G, must be such that the moment of R will be equal to $\bar{I}\alpha$.

Alternative Form of the Resultant. Any force can be resolved into an equivalent force and couple (Art. 38). Thus, the resultant of the external forces as shown in Fig. 444 can be resolved into an equal force passing through the center of gravity, and a couple in the plane of the motion whose moment is equal to $\bar{I}\alpha$, and whose sense agrees with that of α. This conception of the resultant is depicted in Fig. 445. It is preferred by some persons.

165. Methods of Solving Problems. The usual methods of solving problems in the kinetics of plane motion correspond closely to those described for the case of rotation, in Art. 155.

The Resultant Method. In this method the external forces, together with their resultant, are shown on the sketch. The resultant may be represented in accordance with either of the conceptions discussed in Art. 164. The equations for the solution are then formed on the basis of the principle of components (Art. 50), or of the principle of moments, or of both.

The Equilibrant Method. In this method the equilibrant, or reversed resultant, of the external forces is used instead of the resultant. The principles of equilibrium, Arts. 51–74, are then applied to the entire system, external forces and equilibrant, to provide the necessary equations. If the student uses this method he should be careful to avoid any impression that the body actually is in equilibrium. Equilibrium of a body in plane motion exists only in the special case in which the acceleration of the center of gravity is zero and the angular acceleration of the body is zero.

Solution by Formulas. Equations 184, 185, and 186, Art. 126, may be used in any problem. Equation 259, Art. 164, is frequently useful. If additional equations are needed, the following principle may be used:

The moment-sum of the external forces about any axis lying in the plane of the motion is equal to zero.

The foregoing principle follows from the fact that the resultant of the external forces lies entirely in the plane of the motion (Art. 164), and consequently has no moment about any axis in that plane.

Illustrative Problems

954. Figure 446 represents a solid, homogeneous cylinder 3 ft in diameter, weighing 644 lb. The cylinder rolls on a horizontal plane, without slipping, under a constant, horizontal pull of 30 lb, applied as shown. Calculate $\bar{a}$, α, F, and N.

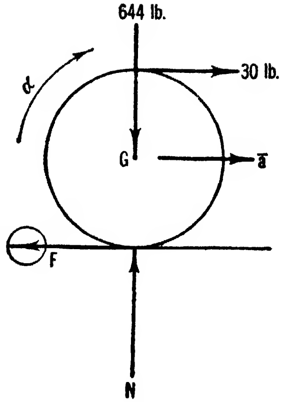

Fig. 446

Solution. It is obvious that $\bar{a}$ will be toward the right, and that α will be clockwise. In problems of this kind it is particularly important to make assumptions that are consistent with each other and with the conditions of the problem. No harm would be done in the present case if $\bar{a}$ were inadvertently assumed toward the left, provided that α were assumed counterclockwise, which would be consistent with the fact that no slipping occurs.

N is obviously upward, but the correct sense of F is less apparent. Let F be assumed toward the left.

The resultant method and the equilibrant method are of doubtful advantage in the simpler problems. Solution will be made in the present case by direct substitution in various formulas. Using the answer to Prob. 764, Art. 139,

$$I = \frac{1}{2}\frac{W}{g}r^2 \qquad \bar{I} = \frac{1}{2} \times \frac{644}{32.2} \times (1.5)^2 = 22.5 \text{ engineer's units}$$

By Eqs. 184 and 185, Art. 126,

$$\Sigma F_x = \frac{W}{g}\bar{a}_x \qquad +30 - F = \frac{644}{32.2}(+\bar{a})$$

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad +N - 644 = 0$$

By Eq. 259, Art. 164,

$$\Sigma \bar{M} = \bar{I}\alpha \qquad -30 \times 1.5 - F \times 1.5 = 22.5(-\alpha)$$

By Eq. 235, Art. 160,

$$a_C = r\alpha \qquad \bar{a} = 1.5\alpha$$

The solution of the foregoing equations gives: $\bar{a} = +2$ ft per sec per sec; $\alpha = +1.33$ rad per sec per sec; $F = -10$ lb; $N = +644$ lb. The negative sign accompanying the value of F shows that the sense was incorrectly assumed; therefore, F acts toward the right. The student may be somewhat puzzled by the fact that α was given the minus sign in the moment equation above, but was given the plus sign in the last equation. The moment equation and the two preceding equations involve algebraic sums, ΣF_x, ΣF_y, and $\Sigma \bar{M}$, and the algebraic signs of all quantities must be carefully regarded. The last formula, $a_C = r\alpha$, is designed merely to give the relationship of the magnitudes of a_C, r, and α, and has nothing to do with algebraic signs. The student should always distinguish carefully between these two types of equations.

955. Figure 447 represents a drum, A, 3 ft in diameter. The drum is mounted on a shaft, and a wheel, B, 1 ft in diameter, is keyed to each end of the shaft. The drum, shaft, and wheels are fastened rigidly together. Their total weight is 966 lb, and their radius of gyration with respect to the axis of the shaft is 1 ft. The wheels roll on an elevated, inclined track, as shown. The coefficient of static friction for the wheels on the rails is 0.5. A light cable is wrapped around the drum, and is subjected to a constant, horizontal force of 100 lb, as shown. Calculate the angular acceleration of the body, the linear acceleration of its center point, and the total friction and normal pressure exerted on the two wheels by the rails.

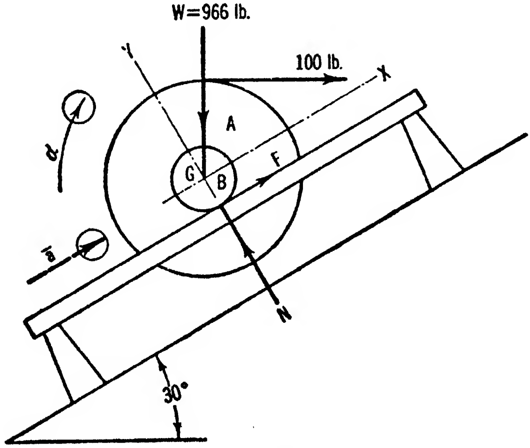

FIG. 447

Solution. The problem will be solved by the use of formulas. A casual examination of the figure does not reveal whether the system will be accelerated up, or down, the incline. Let it be assumed that $\bar{a}$ is up the incline. It is extremely important that α be assumed in such a manner as to be consistent with $\bar{a}$ and with the condition that slipping does not occur. Therefore, α will be assumed clockwise. Let F and N be assumed as shown. $\bar{I} = (W/g)k^2 = (966/32.2)(1)^2 = 30$ engineer's units. By Eqs. 184 and 185, Art. 126,

$$\Sigma F_x = \frac{W}{g}\bar{a}_x \qquad +F + 100 \cos 30^\circ - 966 \sin 30^\circ = \frac{966}{32.2}(+\bar{a})$$

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad +N - 100 \sin 30^\circ - 966 \cos 30^\circ = \frac{966}{32.2}(0)$$

By Eq. 259, Art. 164,

$$\Sigma \bar{M} = \bar{I}\alpha \qquad +F \times 0.5 - 100 \times 1.5 = 30(-\alpha)$$

By Eq. 235, Art. 160,

$$a_C = r\alpha \qquad \bar{a} = 0.5\alpha$$

The solution of the foregoing equations gives: $\alpha = -1.29$ rad per sec per sec; $\bar{a} = -0.643$ ft per sec per sec; $F = +377$ lb; $N = +887$ lb.

The signs obtained show that $\bar{a}$ and α were assumed incorrectly as regards sense; therefore, the acceleration of the system is down the incline. F and N were assumed correctly.

The coefficient of static friction could not be used in the solution, since it was not known whether sliding was impending between the wheels and the track. The coefficient now may be used, however, to ascertain whether the conditions of the problem are possible. The actual frictional force called for by the conditions is 377 lb. The maximum possible amount of friction, with a normal pressure of 887 lb, is equal to $\mu N = 0.5 \times 887 = 444$ lb. The actual friction needed is well within this limiting value; therefore, the system can roll without slipping, as assumed.

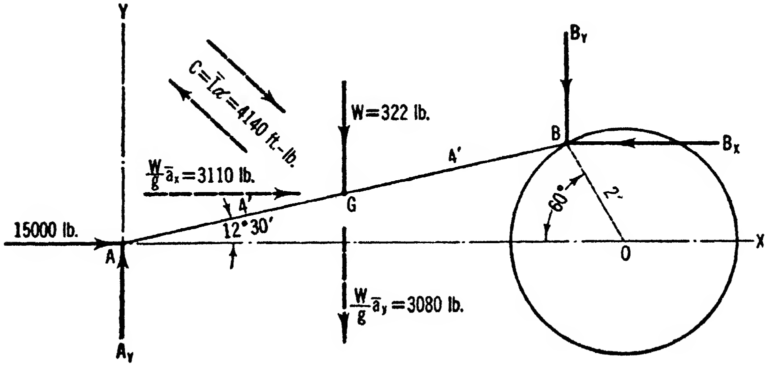

FIG. 448

956. The connecting rod described in Prob. 914, Art. 161, weighs 322 lb. The horizontal component of the force exerted on the rod by the crosshead pin, at A (Fig. 448), is 15,000 lb. Find the vertical component of this force, and the horizontal and vertical components of the force exerted on the rod by the crankpin, at B. Assume that all conditions are the same as in Prob. 914. Disregard friction. Assume the rod to be a slender, uniform bar.

Solution. The problem will be solved by the resultant method. Let the forces acting on the rod be assumed as shown in Fig. 448. Let $\bar{a}_x$ be assumed toward the right, and let $\bar{a}_y$ be assumed downward.

First it is necessary to calculate $\bar{a}_x$ and $\bar{a}_y$ for the rod. In the solution of Prob. 914 it was found that: $a_A = 267$ ft per sec per sec, toward the right; $\alpha = 77.6$ rad per sec per sec, clockwise; and $\omega = 2.42$ rad per sec, counterclockwise. Choosing A as particle 1 and G as particle 2, for use in Eqs. 249 and 250, Art. 161, the coordinates of G are as follows: $x = +4 \cos 12° 30' = +3.91$ ft; and $y = +4 \sin 12° 30' = +0.866$ ft. Substituting,

$$a_{2x} - a_{1x} = -x\omega^2 - y\alpha \qquad +\bar{a}_x - (+267) = -(+3.91)(+2.42)^2 - (+0.866)(-77.6)$$

$$a_{2y} - a_{1y} = -y\omega^2 + x\alpha \qquad -\bar{a}_y - 0 = -(+0.866)(+2.42)^2 + (+3.91)(-77.6)$$

The solution of these equations gives: $\bar{a}_x = +311$ ft per sec per sec; $\bar{a}_y = +308$ ft per sec per sec. The positive signs show that $\bar{a}_x$ and $\bar{a}_y$ were correctly assumed; therefore, $\bar{a}_x$ is toward the right and $\bar{a}_y$ is downward.

The moment of inertia of the connecting rod may be calculated by the approximate formula obtained in Prob. 755, Art. 138.

$$\bar{I} = \frac{1}{12}\frac{W}{g}L^2 \qquad \bar{I} = \frac{322(8)^2}{12 \times 32.2} = 53.3 \text{ engineer's units}$$

The resultant of the external forces acting on the connecting rod consists of the following:

A component, $\frac{W}{g}\bar{a}_x = \frac{322}{32.2} \times 311 = +3110$ lb, through G, toward the right

A component, $\frac{W}{g}\bar{a}_y = \frac{322}{32.2}(-308) = -3080$ lb, through G, downward

A couple, $C = \bar{I}\alpha = 53.3\ (-77.6) = -4140$ ft-lb, in the plane of the motion

The resultant has been completely shown in the figure.

The figure now contains a system of forces, partially known, and their resultant, completely known. The necessary equations now can be obtained by means of the principle of components and the principle of moments. Any convenient axes may be used, a fact which lends a certain advantage to the resultant method in this problem.

By the principle of moments, the moment-sum of the forces is equal to the moment of the resultant. Choosing B as the center of moments, and remembering that the moment of the couple must be included,

$$-A_y\ (8 \cos 12^\circ\ 30') + 15{,}000\ (8 \sin 12^\circ\ 30') + 322\ (4 \cos 12^\circ\ 30')$$
$$= +3110\ (4 \sin 12^\circ\ 30') + 3080\ (4 \cos 12^\circ\ 30') - 4140$$

from which,

$$A_y = +2130 \text{ lb}$$

By the principle of components, the component-sum of the forces is equal to the component of the resultant. Using the x- and y-axes, and remembering that the component-sum for the couple along any axis is zero,

$$\Sigma F_x = R_x \qquad -B_x + 15{,}000 = 3110 \qquad B_x = +11{,}900 \text{ lb}$$
$$\Sigma F_y = R_y \qquad -B_y + 2130 - 322 = -3080 \qquad B_y = +4890 \text{ lb}$$

The fact that positive signs were obtained with A_y, B_x, and B_y shows that these components were correctly assumed, and that they act as shown in the figure.

957. Solve Prob. 956 by the equilibrant method.

Solution by the equilibrant method would be like the solution by the resultant method, used in Prob. 956, except that the equilibrant of the external forces would be used, instead of the resultant.

In the present case the equilibrant would consist of two components, $(W/g)\bar{a}_x$ and $(W/g)\bar{a}_y$, acting at G as in Fig. 448, but reversed in sense, and a couple having the moment $\bar{I}\alpha$, like the couple in Fig. 448, but reversed in sense.

The principles of equilibrium would then be applied to the entire system, forces and equilibrant, to obtain the necessary kinetic equations. If the same axes as in Prob. 956 were used, the only difference between the resulting kinetic equations and those obtained in that problem would be the position of the equality sign.

Because of the similarity of the two methods, actual solution by the equilibrant method will be omitted.

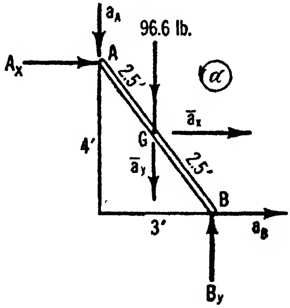

FIG. 449

958. Figure 449 represents a uniform bar 5 ft long, weighing 96.6 lb, whose upper end rests against a vertical plane and whose lower end rests on a horizontal plane. The bar is held momentarily in the position shown and then is suddenly released. Calculate the forces acting on the bar at the instant following its release. The bar may be assumed to be slender. Disregard friction.

Solution. The problem will be solved by direct substitution in the necessary formulas. Since friction is to be disregarded, the force acting on the bar at A will be horizontal, and that at B will be vertical. Assume $\bar{a}_x$, $\bar{a}_y$, and α as shown.

Since the bar is slender, $\bar{I}$ may be calculated by the approximate formula from Prob. 755, Art. 138.

$$\bar{I} = \frac{1}{12}\frac{W}{g}L^2 \qquad \bar{I} = \frac{96.6(5)^2}{12 \times 32.2} = 6.25 \text{ engineer's units}$$

Four equations can be obtained by the use of Eqs. 249 and 250, Art. 161. First, let A be selected as particle 1, and B as particle 2. Since the origin must be placed at particle 1, $x = +3$ ft, and $y = -4$ ft. The angular velocity of the body, ω, obviously is equal to zero.

$$a_{2x} - a_{1x} = -x\omega^2 - y\alpha \qquad +a_B - 0 = -(+3)(0) - (-4)(+\alpha)$$

$$a_{2y} - a_{1y} = -y\omega^2 + x\alpha \qquad 0 - (-a_A) = -(-4)(0) + (+3)(+\alpha)$$

Simplifying,

$$a_B = 4\alpha$$

$$a_A = 3\alpha$$

Now let A again be selected as particle 1, but let G be selected as particle 2.

In this case, $x = +1.5$ ft, and $y = -2$ ft.

$$a_{2x} - a_{1x} = -x\omega^2 - y\alpha \qquad +\bar{a}_x - 0 = -(+1.5)(0) - (-2)(+\alpha)$$

$$a_{2y} - a_{1y} = -y\omega^2 + x\alpha \qquad -\bar{a}_y - (-a_A) = -(-2)(0) + (+1.5)(+\alpha)$$

Simplifying,

$$\bar{a}_x = 2\alpha$$

$$\bar{a}_y - a_A = -1.5\alpha$$

From Eqs. 184 and 185, Art. 126,

$$\Sigma F_x = \frac{W}{g}\bar{a}_x \qquad +A_x = \frac{96.6}{32.2}\bar{a}_x$$

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad -96.6 + B_y = \frac{96.6}{32.2}(-\bar{a}_y)$$

From Eq. 259, Art. 164,

$$\Sigma \bar{M} = \bar{I}\alpha \qquad -A_x \times 2 + B_y \times 1.5 = 6.25\,(+\alpha)$$

From the seven equations obtained above the values of A_x and B_y are found to be $A_x = +34.8$ lb and $B_y = +70.5$ lb. The positive signs show that A_x and B_y act as assumed.

PROBLEMS

959. A solid, homogeneous cylinder is given an initial velocity and is then permitted to roll freely up an inclined plane. The angle of inclination of the plane is represented by θ. No slipping occurs. Prove that the deceleration of the center point of the cylinder is equal to 21.5 sin θ. Prove that the frictional force is equal to $\frac{1}{3}W$ sin θ.

960. A solid, homogeneous cylinder of any diameter rolls freely down an inclined plane. The coefficient of static friction for the cylinder on the plane is 0.3. Calculate the angle of inclination of the steepest plane down which the cylinder could roll, without slipping. *Ans.* 42°.

961. A solid, homogeneous cylinder 3 ft in diameter, weighing 966 lb, rolls down an inclined plane, without slipping. The slope of the incline is 4 (horizontal) to 3 (vertical). The only forces acting on the cylinder are its own weight, and the reaction exerted by the plane. Calculate $\bar{a}$, α, F, and N. Calculate the minimum possible value of the coefficient of static friction consistent with the assumption that no slipping occurs.

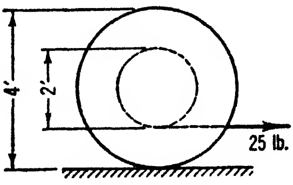

FIG. 450

962. Figure 450 represents a 2-ft drum, rigidly attached to a pair of 4-ft wheels. The entire assembly rolls on a horizontal plane, without slipping. A light, flexible wire has been wound on the drum, its free end being subjected to a constant pull of 25 lb, as shown. The assembly weighs 322 lb, and its radius of gyration with respect to its geometric axis is 1.5 ft. Calculate the linear acceleration of the center point, and the frictional force. Calculate the minimum coefficient of static friction consistent with "no slipping." *Ans.* +0.8 ft/sec^2; −17 lb; 0.0528.

963. In Prob. 962, Fig. 450, change the position of the free end of the wire so that it comes vertically upward off the drum, on the left-hand side. Assume that this direction is maintained throughout the motion. Solve the problem, letting all the other data remain unchanged. Calculate the distance through which the center point moves in 18 sec, starting from rest.

964. Change the wire in Prob. 962, Fig. 450, in such a manner that the 25-lb pull will act tangentially at the top of the drum, and toward the left. Solve the problem, letting the other data remain unchanged. *Ans.* -2.4 ft/sec^2; $+1$ lb; 0.00311.

965. Change the wire in Prob. 962, Fig. 450, in such a manner that the 25-lb pull will be directed upward and toward the right, at an angle of 60° with the horizontal. The point at which the wire leaves the drum is to the right of, and below, the center. Solve the problem, letting the other data remain unchanged.

966. A wheel 3 ft in diameter, weighing 64.4 lb, rolls on a horizontal plane, without slipping. The radius of gyration is 1 ft. A constant horizontal force of 10 lb is applied at the center of the wheel. Calculate the distance described by the center point while its velocity changes from 10 to 30 ft per sec. *Ans.* 116 ft.

967. The wheel in Prob. 966 arrives at the foot of a 30° inclined plane at the instant when the velocity of its center is 30 ft per sec. The 10-lb force continues to act horizontally at the center. How far up the incline will the wheel roll before coming to rest?

968. Reverse the sense of the 100-lb force in Prob. 955, Fig. 447, and solve the problem. *Ans.* $+11.6$ rad/sec^2; 5.80 ft/sec^2, downward; 396 lb, upward; 787 lb.

969. A drum fitted with a pair of wheels, similar to that shown in Fig. 450, rests on a horizontal plane. A light, flexible wire has been wound on the drum. The free end of the wire comes off the drum at the top, horizontally, and is subjected to a constant pull. Let P represent the pull on the wire, and let r_1 and r_2 represent the radii of the drum and of the wheels, respectively. Let k represent the radius of gyration of the assembly with respect to its geometric axis. Prove that the drum would roll perfectly, even though the supporting plane were frictionless, if $r_1 r_2 = k^2$.

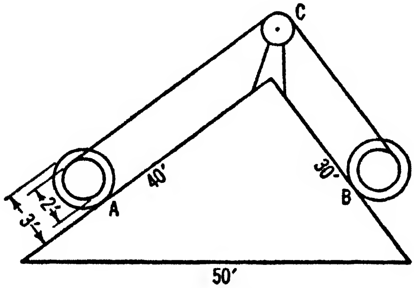

FIG. 451

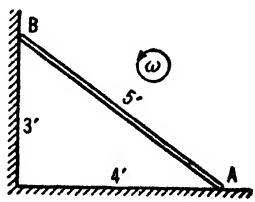

FIG. 452

970. Figure 451 represents a drum, A, 2 ft in diameter, to which a pair of 3-ft wheels are rigidly attached. The assembly weighs 322 lb, and its radius of gyration is 1 ft. At B is a second drum, the counterpart of the first one. A light, flexible wire unwinds from drum A, passes over a small pulley C, and winds onto drum B. The pulley is to be considered weightless and frictionless. The drums roll without slipping. Calculate the acceleration of the center of each drum, and the frictional force exerted on each, by the plane. Calculate the tension in the wire.

971. Figure 452 represents a slender, uniform bar, AB, weighing 32.2 lb. The upper end, B, moves in a vertical line on a vertical plane, and the lower end, A, moves along a horizontal plane in a straight line at right angles to the vertical plane. At the instant represented by the figure the angular velocity of the bar is 2 rad per sec, counterclockwise. Calculate the forces acting on the bar at A and B. Disregard friction. *Ans.* $A_y = +10.7$ lb; $B_x = +3.59$ lb.

972. Change the angular velocity of the bar in Prob. 971, Fig. 452, to zero. Solve the problem, letting all the other data remain unchanged.

973. Figure 453 represents a plank moving down an incline on two rollers. The plank weighs 64.4 lb. Each roller weighs 32.2 lb, and is 1 ft in diameter. Assume that no slipping occurs. Calculate the acceleration of the center point of each roller. Calculate the frictional forces acting on the rollers. *Ans.* $a = 8.78$ ft/sec^2; $F_A = F_C = 5.85$ lb, up the incline; $F_B = F_D = 1.46$ lb, up the incline.

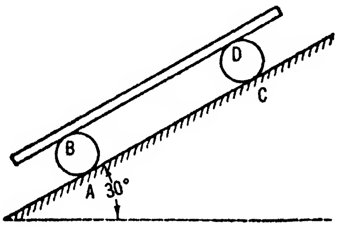

FIG. 453

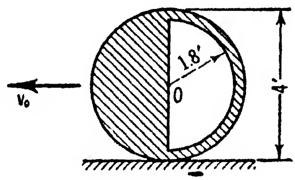

FIG. 454

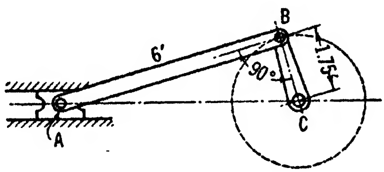

FIG. 455

974. The plank and rollers described in Prob. 973 are placed on a horizontal plane. A constant pull of 5.5 lb is applied horizontally to the plank, at the right end. Assume that no slipping occurs, and calculate the acceleration of the center point of each roller. Calculate the frictional forces acting on the rollers.

975. Figure 454 represents a 4-ft cylinder, with a hole through it in the form of a half-cylinder, as shown. The body weighs 1610 lb. It rolls toward the left, without slipping, and in the position shown the velocity of the point O is 4 ft per sec. The only forces acting on the body are its own weight, and the reaction of the plane. Calculate the angular acceleration, and the frictional force and normal pressure. *Ans.* $+3.34$ rad/sec^2; -230 lb; $+1520$ lb.

976. Turn the body in Prob. 975 through a counterclockwise angle of 90°, and solve the problem. Let all the other data remain unchanged.

977. Figure 455 represents a crank, BC, and a connecting rod, AB. The crank rotates in a clockwise direction with a constant angular velocity of 20 rad per sec. Nothing is attached to the rod at A except a crosshead whose weight shall be disregarded. Friction also may be disregarded. The connecting rod weighs 322 lb, and may be treated as a slender, uniform bar. The angle ABC is 90°. Calculate the forces acting on the connecting rod. *Ans.* $A_y = -1070$ lb; $B_x = +1070$ lb; $B_y = -1970$ lb.

978. Solve Prob. 977, Fig. 455, for the position in which the crank angle, ACB, is 90°. Let the other data remain unchanged.

979. Solve Prob. 977, Fig. 455, for the position in which the crank angle, ACB, is 180°. Let the other data remain unchanged. *Ans.* $A_y = +161$ lb; $B_x = -5980$ lb; $B_y = +161$ lb.

166. Rolling Resistance. When a cylinder, wheel, sphere, or other round body rolls on any surface, it deforms that surface. Furthermore, the rolling body itself is deformed. These deformations may be small or they may be large, they may be temporary or they may be permanent, but they always occur. Experiments indicate that among the various resistances that can impede rolling motion there is one that can be attributed solely to the deformations just mentioned. This is called *rolling resistance.*

A comparatively small amount of information is available regarding the amount and exact nature of rolling resistance. In many cases the experiments have been performed in such a manner that the results include the effects of various other resistances, and the amount of rolling resistance has not been obtained as a separate item. This is especially true of experiments made with railway trains, motor cars, and other vehicles. However, a certain amount of experimenting has been done on rolling bodies in such a manner as to make it possible to estimate roughly the amount of pure rolling resistance to be expected under similar conditions.

The Case of the Rigid Body. As an approach to the discussion of the manner in which deformations are capable of causing rolling resistance, let a case be considered in which the bodies are assumed to be perfectly rigid, and in which, therefore, rolling resistance could not exist. Let Fig. 456 represent such a case. In the figure, W represents the total load carried by the rolling body, and N represents the reaction exerted on that body, by the supporting surface.

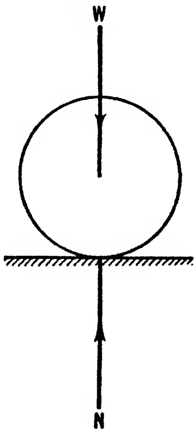

FIG. 456

Since the two bodies are to be assumed rigid they can be in contact at a single point only, or along a line. For this reason N is necessarily at right angles to the direction of motion, and can act neither as a resistance nor as a propulsive force. The existence of axle or brake friction would tend to produce a frictional force at the point of contact between the body and the supporting surface, opposing the motion, but it is obvious that in the hypothetical case now under consideration the supporting surface is unable to offer resistance to the motion by virtue of any action inherent in itself.

Two cases more typical of actual conditions will now be considered. The bodies will not be assumed rigid, and rolling resistance will occur, as a result of the deformations of the bodies. First consider a rolling body which serves as a wheel, carrying its load by means of an axle. It is desired to study the effect of rolling resistance alone, and for that reason axle friction and other resistances will be disregarded. It is generally considered that the magnitude of the rolling resistance is not affected by the addition or removal of other resistances.

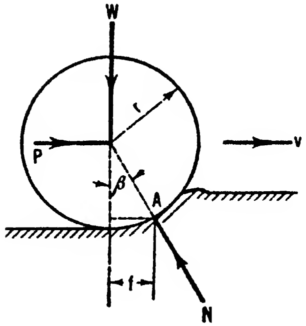

FIG. 457

Body Serving as a Wheel. Let Fig. 457 represent the rolling body that serves as a wheel. In the figure the supporting surface is depicted as suffering a deformation of considerable magnitude. The figure is also typical of the case in which the roadway is inelastic. Frequently the roadway is elastic, and returns to its original condition as soon as the rolling body has passed over it. The general nature of rolling resistance is the same in all cases, but an elastic roadway tends to offer a resistance of lesser magnitude.

Sometimes the magnitude of the rolling resistance is more seriously affected by the deformation of the rolling body than by the deformation of the supporting surface. This situation is especially noticeable when a wheel with a pneumatic tire rolls on a hard road or pavement.

In any case there is a finite surface of contact between the two bodies. It is obvious that the force N will have its point of application in an advanced position, such as A. Thus, N will assume a rearward inclination, and will have a component opposing the motion of the wheel. Let W represent the total load carried by the wheel, and let P represent the force that propels the wheel. This propulsive force will be assumed to be just sufficient to overcome the rolling resistance, and since this is the only resistance being considered, the wheel will move at constant speed. The distance f is called the *coefficient of rolling resistance.* Values for f have been obtained for rolling bodies and surfaces of various materials. Since f is a distance, and not a true coefficient, the unit in which it is expressed must be known before any practical use can be made of it. It is usually given in inches.

Since the center of gravity of the wheel moves at constant speed in a direction parallel to the supporting surface, the following equations can be formulated:

$$P - N \sin \beta = 0 \qquad [260]$$

$$N \cos \beta - W = 0 \qquad [261]$$

Eliminating N between Eqs. 260 and 261, and solving for P,

$$P = W \tan \beta = W \frac{f}{\sqrt{r^2 - f^2}} \qquad [262]$$

If the coefficient of rolling resistance is comparatively large, Eq. 262 should be used. In many cases, however, f is small, compared with r. If so, $\tan \beta$ may be replaced by $\sin \beta$, and the formula becomes

$$P = W \frac{f}{r} \qquad [263]$$

Body Serving as a Roller. Equations 262 and 263 were derived for the case in which the body serves as a wheel, and thus has rolling contact with only one surface. Now the case will be discussed in which the body serves as a roller between two parallel surfaces and, naturally, has rolling contact with both surfaces. This situation exists in roller bearings, ball bearings, and in rollers used for moving heavy objects. It is called *double rolling.*

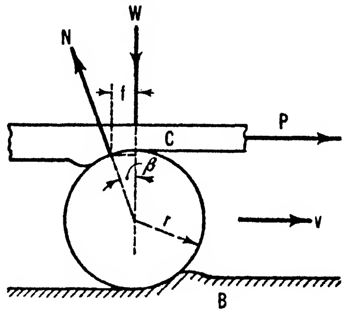

Fig. 458

Figure 458 represents the case. The surface B will be assumed stationary. The superimposed load, W, is applied through the medium of body C. It is desired to obtain a formula for the force, applied to the body C, that will maintain uniform motion against rolling resistance alone. Let P represent this force. The analysis will be limited to the case in which the coefficient of rolling resistance, f, is equal for the two surfaces of contact.

Considering the motion of body C only, the following equations can be written:

$$P = N \sin \beta \qquad [264]$$

$$N \cos \beta - W = 0 \qquad [265]$$

From which,

$$P = W \tan \beta = W \frac{f}{\sqrt{r^2 - f^2}} \qquad [266]$$

or, approximately,

$$P = W \frac{f}{r} \qquad [267]$$

It will be observed that Eqs. 266 and 267 are identical, in form, with Eqs. 262 and 263. In the case of the roller there are two surfaces at which rolling contact occurs, while in the case of the wheel there is only one. It would be expected, therefore, that the roller would experience a resistance to rolling twice as great as that experienced by the wheel. This would be true if the propulsive force, P, were applied in the same manner in the two cases. In the roller, however, P is applied in such a manner as to render it twice as effective in maintaining motion. This accounts for the fact that the same propulsive force will suffice in both cases, if equal loads are carried.

The following table gives a few values of the coefficient of rolling resistance, as offered by Goodman:

	f (in inches)
Iron or steel wheels on steel rails	0.007 to 0.015
" " " wood	0.06 to 0.10
" " " macadam	0.05 to 0.20
" " " soft ground	3.00 to 5.00
Pneumatic tires on good roads or asphalt	0.020 to 0.022
" " in heavy mud	0.04 to 0.06
Solid rubber tires on good roads or asphalt	0.04
" " " in heavy mud	0.09 to 0.11

For the sake of uniformity and simplicity, rolling resistance has been, and will be, disregarded in all the problems in this book, with the exception of those appended to the present article. In some problems involving the motion of vehicles, however, the total vehicle or train resistances are given, and the values ordinarily used for these include the effects of rolling resistance. In practice, the decision as to whether rolling resistance should be taken into account calls for good judgment and an appreciation of the relative importance of the various factors influencing the motion.

PROBLEMS

980. A solid cylinder, 2 ft in diameter and weighing 644 lb, rolls on a horizontal surface. The coefficient of rolling resistance is 0.06 in. The initial velocity of the center of the cylinder is 4 ft per sec. How far will the cylinder move while coming to rest under the influence of rolling resistance? *Ans.* 74.5 ft.

981. The cylinder described in Prob. 980 starts from rest on a 10° inclined plane, and rolls down the plane a distance of 20 ft. It then reaches a horizontal plane, and continues to roll until brought to rest by rolling resistance. The coefficient of rolling resistance is 0.06 in., throughout the motion. How far does the cylinder roll on the horizontal surface? Assume that the rolling resistance on the incline is equal to that on the level surface.

982. A weight of 500 lb is placed on rollers on a horizontal surface. The rollers are 3 in. in diameter, and are in contact with similar materials at top and bottom. A horizontal force of 6 lb, applied to the weight, is just sufficient to cause motion. Calculate the coefficient of rolling resistance. *Ans.* 0.018 in.

983. A machine weighing 2 tons is to be moved on a set of rollers 2 in. in diameter. The coefficient of rolling resistance is 0.02 in. Calculate the horizontal force that must be applied to the machine to cause uniform motion *Ans.* 80 lb.

984. A set of rollers, each of which is 2 in. in diameter, is placed on a horizontal surface. A board is laid across the rollers. A second set of 2-in. rollers is placed on the board, and a weight of 1000 lb is placed on these rollers. The weight is blocked in such a manner that it is prevented from moving. What force will be required to start the board from beneath the weight, if the coefficient of rolling resistance is 0.02 in. at all points?

CHAPTER XXI

RELATIVE MOTION

167. The Path of a Point. The term "path" has been frequently used in previous discussions in this book, in connection with the motions of points, but no attempt was made to formulate a definition of it. It was considered that the reader's conception of what is meant by the "path of a point" would be adequate for the more elementary studies in kinematics and kinetics. It now becomes desirable, as a basis for the study of relative motion, to acquire a definite understanding of the meaning of the term. The term "path" is strictly a relative one, the nature of the path described by a moving point depending entirely upon the body on which, or within which, the path is considered to be traced.

The path of a point relative to a given body may be defined as a line joining the successive particles of the given body touched by the point. However, a point may be considered to have a path relative to a given body, even though the motion of the point is entirely outside of the material confines, or boundaries, of the body. In such a case the definition can still be considered applicable if the path is conceived of as a line joining those particles which would be touched by the moving point if the body were sufficiently large. In other words, the relative path, under such conditions, lies in the body "extended." Such an imaginary "extension" of a body is to be considered rigid, having whatever motion would belong to it as an integral portion of the body, the motion of the body being in no way altered by the extension. The definition of relative path contemplates that the body to which the path is referred is a rigid body.

The path of a point relative to the earth is usually called the absolute path, the earth being considered stationary. It can be seen that a point which is stationary has no absolute path, but does have a path relative to any moving body.

As an example of relative paths let the familiar steam-engine indicator, as illustrated in Fig. 459, be considered. A small cylinder, A, is connected to the cylinder of the engine, by suitable piping. The piston, B, is forced upward by the pressure of the steam from the engine cylinder, against the resistance of a calibrated spring. The motion of the piston, B, is communicated to the pencil, D. The mechanism that transmits the motion is so designed as to give the pencil a rectilinear motion parallel to that of the small piston, but multiplied several times in order to furnish a diagram of convenient size. The pencil touches the drum, *E*, which is covered with a card. This drum rotates about its own geometric axis, receiving its motion from the crosshead of the engine in such a manner that its angular motion is proportional to the linear motion of the crosshead, but greatly reduced in magnitude. The motion of the pencil-point furnishes an excellent illustration of the meaning of the terms discussed above, as will be shown.

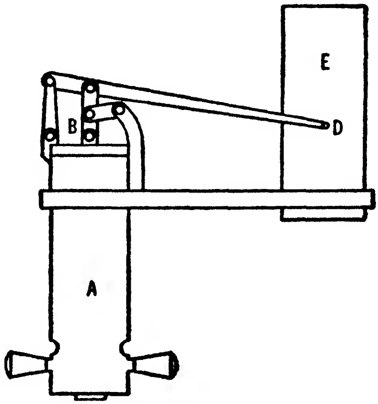

Fig. 459

First, let it be imagined that the indicator drum is disconnected from the crosshead of the engine, and that steam is admitted into the indicator cylinder. If the pencil-point is brought into contact with the drum, a straight vertical line will be drawn on the card. Since the card is stationary it is essentially a part of the earth, and the line drawn upon it by the pencil is the absolute path of the pencil-point.

Now let it be imagined that the drum is connected to the crosshead, but that the steam is shut off from the indicator cylinder, so that the pencil does not move and, therefore, has no absolute path. The pencil will, however, mark out a horizontal circular arc on the card, showing the successive particles of the card that are touched by the pencil-point. This arc is, therefore, the path of the pencil-point relative to the drum. In actual practice the line drawn in this manner is called the atmospheric line.

Fig. 460

As a final illustration let it be imagined that the steam is admitted into the indicator cylinder, with the drum connected to the crosshead as in the preceding case. The absolute path of the pencil-point is a vertical straight line, as in the first illustration, but the path relative to the drum is the one that is actually marked on the paper, and is quite of a different nature. When the card is removed from the drum and flattened out, the diagram usually has an appearance similar to that shown in Fig. 460. The indicator diagram is much used by engineers in ascertaining the power developed in the cylinders of steam engines, gasoline engines, air compressors, etc., and in obtaining other information regarding the performance of such machines.

The illustrations given above are cases in which the paths are actually made visible by means of a pencil line. Usually, however, relative and absolute paths are purely imaginary.

PROBLEMS

985. A certain locomotive runs on a track that has neither horizontal nor vertical curvature. What is the general nature of the absolute path of a point on the rim of one of the drivers? What is the nature of the path of this point relative to the frame of the locomotive? Answer the same questions for a point on one of the cross-heads.

986. A certain automobile runs straight forward on a level road. What is the general nature of the absolute path of a point on the rim of the flywheel? What is the nature of the path of this point relative to the body of the car? Answer the same questions for a point on one of the pistons.

168. Relative Velocity. The linear velocity of a moving point was defined, in Art. 105, as the time rate at which the point traverses distance, and was shown to be given by the expression ds/dt. It will be remembered that in the foregoing expression s represents the linear distance to the moving point from some point fixed on the path, and is measured along the path. The discussion in Art. 167 shows, however, that in each specific case of motion there exists a definite path for each body to which the motion of the point may be referred, or related, and that these relative paths may be quite different in form and in dimensions. Thus it can be seen that the relation between s and t will depend upon which of these paths is to be used for the measurement of s.

It becomes evident, therefore, that velocity is a relative quantity, and that a given point at a given instant has any number of relative velocities, one for each relative path along which the point is considered to be moving. Consequently, *if s is measured along the path of the point relative to a given body the formula $v = ds/dt$ gives the velocity of the point relative to that particular body.* The vector intended to represent the relative velocity should be drawn tangent to the relative path. When velocity is mentioned without any statement regarding the body to which it is referred, it is understood that *absolute velocity* is meant; that is, velocity relative to the earth.

It can be seen that the formulas $v_x = dx/dt$ and $v_y = dy/dt$, Art. 115, will give components of absolute velocity if x and y are measured from stationary axes; but if the coordinate axes are considered to be fixed to some moving body, thus partaking of the motion of the body, these formulas will give the components of the velocity of the point relative to the body.

To illustrate the foregoing discussion, let the steam-engine indicator of Art. 167 again be considered. If the distances s, x, and y are measured from points and axes fixed to the earth, the various velocity formulas will determine the absolute velocity of the pencil-point. If, however, s is measured along the indicator diagram, from a point fixed on that diagram, and if the coordinate axes are drawn in a fixed position on the indicator card, the results obtained will pertain to the velocity of the pencil-point relative to the drum.

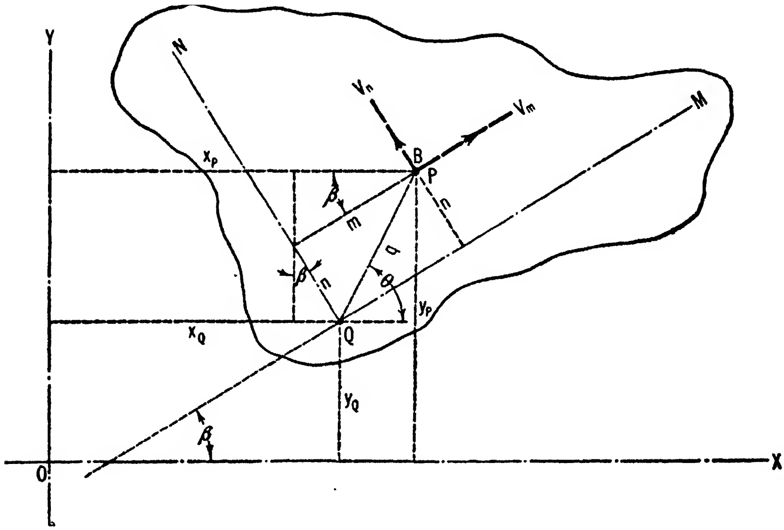

FIG. 461

169. Relation between Relative and Absolute Velocities. *The absolute velocity of a moving point at any instant is the vector sum, or resultant, of the following:*

I. *The velocity of the moving point relative to any moving body at the given instant,* and

II. *The absolute velocity of that particle belonging to the moving body, or extension thereof, which coincides with the moving point at the given instant.*

Proof. The foregoing relation will be proved only for the case in which the point moves in a plane, and the moving body has plane motion, the plane in which the point moves being parallel to the plane of motion of the body. The majority of examples in practice conform essentially to this special case.

The moving body is represented in Fig. 461, the plane of its motion being parallel to the plane of the paper. Let P represent the moving point. Let OX and OY represent a pair of stationary coordinate axes. Let QM and QN represent a pair of coordinate axes fixed in the moving body and, naturally, partaking of its motion. Let x_P and y_P represent the coordinates of P with respect to the stationary axes OX and OY. Let m and n represent the coordinates of P with respect to the moving axes QM and QN. Since P has a motion relative to the moving body, as well as a motion relative to the earth, it can be seen that x_P, y_P, m, and n are all variables, the variation of x_P and y_P being governed by the absolute motion of P, and the variation of m and n being governed by the motion of P relative to the moving body. Let x_Q and y_Q represent the coordinates of the point Q with respect to the stationary axes OX and OY. The point Q is the origin of the moving axes QM and QN, and is fixed in the moving body. Let q represent the distance QP, as shown. Let β and θ represent the various angles indicated in the figure. Let ω represent the angular velocity of the moving body at any instant.

It will be noticed that the letter B also has been placed at the point P. B is intended to represent that particle, belonging to the moving body, with which P coincides at the instant under consideration. It must be kept in mind that P moves with respect to both sets of coordinate axes, while Q and B are particles belonging to the moving body itself and move with respect to OX and OY only. The body is assumed to be rigid.

Absolute and relative velocities will be distinguished by the use of v (lower-case) to represent absolute velocity, and V (upper-case) to represent relative velocity. When it is desired to represent the velocity of P the various symbols, v, V, V_x, v_n, etc., will be used without additional subscripts. In the cases of Q and B additional subscripts will be used; for example, v_{Qx}, v_{By}, etc.

From Fig. 461,

$$x_P = x_Q - n \sin \beta + m \cos \beta \qquad [268]$$

Since $v_x = dx_P/dt$, an expression for v_x can now be obtained by differentiating in Eq. 268 with respect to t.

$$v_x = \frac{dx_Q}{dt} - n \cos \beta \left(\frac{d\beta}{dt}\right) - \frac{dn}{dt} \sin \beta - m \sin \beta \left(\frac{d\beta}{dt}\right) + \frac{dm}{dt} \cos \beta \qquad [269]$$

In Eq. 269, $dx_Q/dt = v_{Qx}$; $d\beta/dt = \omega$; $dn/dt = V_n$, the n-component of the relative velocity; and $dm/dt = V_m$, the m-component of the relative velocity. Equation 269 may now be written as follows:

$$v_x = v_{Qx} - \omega\,(n \cos \beta + m \sin \beta) + (V_m \cos \beta - V_n \sin \beta) \qquad [270]$$

From the figure, $(n \cos \beta + m \sin \beta) = y_P - y_Q$. The figure also shows that $(V_m \cos \beta - V_n \sin \beta) = V_x$, the x-component of the relative velocity. Substituting these values in Eq. 270, and replacing the symbol y_P by its equivalent, y_B,

$$v_x = V_x + v_{Qx} - (y_B - y_Q)\omega \tag{271}$$

Particles Q and B are two particles of a body in plane motion. Equation 243, Art. 161, states that $v_{2x} - v_{1x} = -(y_2 - y_1)\omega$. Selecting Q as particle 1 and B as particle 2, Eq. 243 becomes, in the notation of Fig. 461,

$$v_{Bx} - v_{Qx} = -(y_B - y_Q)\omega \tag{272}$$

Eliminating the expression $-(y_B - y_Q)\omega$ between Eqs. 271 and 272, the following simple relationship is obtained:

$$v_x = V_x + v_{Bx} \tag{273}$$

Referring again to Fig. 461, it can be seen that

$$y_P = y_Q + n \cos \beta + m \sin \beta \tag{274}$$

From Eq. 274, by an analysis similar to that used in deriving Eq. 273, the following formula can be obtained:

$$v_y = V_y + v_{By} \tag{275}$$

Equations 273 and 275 constitute an algebraic statement of the principle set forth at the beginning of the article and, therefore, the derivation of these equations is a proof of the principle. Equations 273 and 275 may be used directly in the solution of problems, if desired. However, the principle itself is easily remembered and can be applied in any particular case in conjunction with whatever method of vector composition is most convenient.

Illustrative Problems

987. Figure 462 represents a car, on which is mounted a wheel 5 ft in diameter. The wheel is carried on a horizontal shaft at right angles to the plane of the figure. At the instant shown the car has a velocity of 6 ft per sec, toward the right, and the wheel has an angular velocity of 10 rpm, counterclockwise. Find the absolute velocity of the particle, P, on the rim of the wheel, at the given instant.

Solution. Let P be taken as the moving point referred to in the principle stated at the beginning of the article. Let the car be taken as the moving body.

The path of P relative to the car is the circle representing the periphery of the wheel. The velocity of P relative to the car is tangent to this path. The relative velocity can be calculated from the radius, and angular velocity, of the wheel.

$$V_P = \frac{10(2\pi \times 2.5)}{60} = 2.62 \text{ ft/sec}$$

Since ω is counterclockwise, the relative velocity is directed upward and toward the left, as shown in the figure.

Let B represent that particle belonging to the moving body, with which P coincides at the given instant. In other words, B is a particle belonging to an imaginary extension of the body of the car. Since the car is translating, the velocities of all its particles are equal. Therefore, $v_B = 6$ ft per sec, horizontal and toward the right, as shown.

According to the principle, v_P is the resultant of V_P and v_B. Therefore, v_P is represented by the diagonal of the parallelogram, as indicated in the figure. By the law of cosines, from trigonometry,

$$v_P = \sqrt{V_P^2 + v_B^2 - 2 \times V_P \times v_B \cos 60^\circ}$$

$$= \sqrt{(2.62)^2 + (6)^2 - 2 \times 2.62 \times 6 \times 0.5} = 5.21 \text{ ft/sec}$$

The angle of inclination of v_P can be calculated by the law of sines.

$$\frac{\sin \theta}{2.62} = \frac{\sin 60^\circ}{5.21} \qquad \theta = 25^\circ\, 45'$$

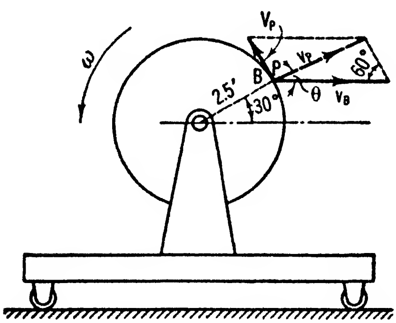

FIG. 462

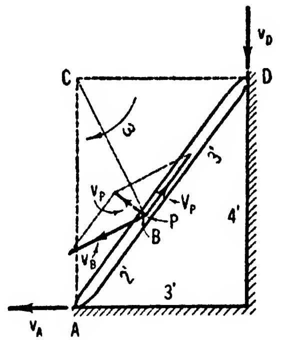

FIG. 463

988. Figure 463 represents a bar, AD, in plane motion. The plane of the motion is parallel to the plane of the paper. A moves horizontally and D moves vertically. A point, P, moves along the center line of the bar, toward D. At the instant represented by the figure, P is 2 ft from the lower end of the bar, and its velocity relative to the bar is 5 ft per sec. At the same instant the velocity of D is 6 ft per sec, downward. Calculate the absolute velocity of P.

Solution. Let P be chosen as the moving point, and let the bar be chosen as the moving body. Let B represent that particle belonging to the bar, with which P coincides at the given instant. The absolute velocity of P is the resultant of the velocity of P relative to the bar and the absolute velocity of B. The absolute velocity of B can be calculated conveniently by the instantaneous-axis method, Art. 162.

It is known that the velocities of A and D are horizontal and vertical, respectively. The instantaneous center lies on a line drawn in the plane of the motion, passing through any particle, at right angles to the velocity of the particle. Therefore the instantaneous center in Fig. 463 is at C. Also,

$$v_D = \overline{CD}\omega \qquad \omega = \tfrac{8}{4} = 2 \text{ rad/sec}$$

Obviously, ω is clockwise. In calculating v_B, the distance BC and the angle ACB will be needed. From the figure,

$$\angle BAC = \text{arc tan } \tfrac{3}{4} = 36° \, 50'$$

In the triangle ABC, by the law of cosines,

$$\overline{BC} = \sqrt{\overline{AB}^2 + \overline{AC}^2 - 2\overline{AB} \times \overline{AC} \cos (BAC)}$$

$$= \sqrt{(2)^2 + (4)^2 - 2 \times 2 \times 4 \cos 36° \, 50'} = 2.68 \text{ ft}$$

By the law of sines,

$$\frac{\sin (ACB)}{2} = \frac{\sin 36° \, 50'}{2.68} \qquad \angle ACB = 26° \, 35'$$

Therefore,

$$v_B = \overline{BC}\omega = 2.68 \times 2 = 5.36 \text{ ft/sec}$$

Since v_B is at right angles to BC, its angle of inclination to the horizontal is equal to $\angle ACB = 26° \, 35'$. v_P, being the resultant of V_P and v_B, is represented by the diagonal of the parallelogram, as shown in the figure.

Let the magnitude and angle of inclination of v_P be calculated by the principle of components,

$$v_{Px} = V_{Px} - v_{Bx} = 5 \times \tfrac{3}{5} - 5.36 \cos 26° \, 35' = -1.79$$

$$v_{Py} = V_{Py} - v_{By} = 5 \times \tfrac{4}{5} - 5.36 \sin 26° \, 35' = +1.60$$

$$v_P = \sqrt{(1.79)^2 + (1.60)^2} = 2.40 \text{ ft/sec}$$

$$\theta_x = \text{arc tan } \frac{1.60}{1.79} = 41° \, 50'$$

989. Figure 464 represents a cylinder 4 ft in diameter, rolling toward the left, without slipping, on a horizontal plane. In the position shown the linear velocity of the center of the cylinder is 5 ft per sec. A circular arc, DCB, having its center at E, is inscribed on the end of the cylinder. The radius of the arc is 2 ft. A point, P, traverses this arc, moving toward B. The point reaches B at the instant when the cylinder is in the position shown by the figure, and its velocity relative to the cylinder at this instant is 6 ft per sec. Find the absolute velocity of P at the given instant.

Solution. Let P be chosen as the moving point and the cylinder as the moving body. Let V_P represent the velocity of P relative to the cylinder. From the problem, the arc DCB is the relative path. V_P is tangent to the relative path, at B. Let β represent the angle of inclination of V_P at the given instant. From the figure,

$$\beta = 90° - \angle CEB = 90° - 60° = 30°$$

Let B represent that particle belonging to the cylinder, with which the moving point P coincides at the given instant. It is now necessary to find the absolute velocity of B. Let Eqs. 247 and 248, Art. 161, be utilized, and let C be

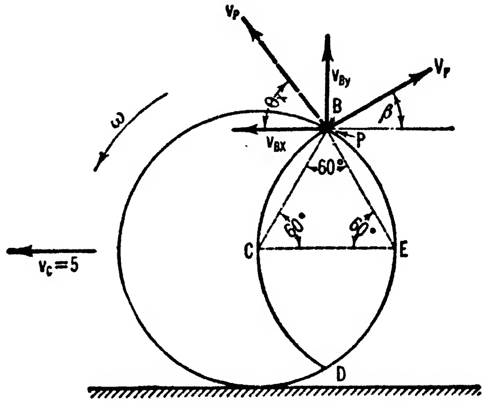

FIG. 464

chosen as particle 1 and B as particle 2. Then, since the origin is to be at C, $x = +1$ ft, and $y = +2 \sin 60° = +1.73$ ft. By Eq. 233, Art. 160, $v_C = r\omega$. Therefore, $\omega = v_C/r = \frac{5}{2} = 2.5$ rad per sec, counterclockwise. Assume v_{Bx} and v_{By} as shown in the figure.

$$v_{2x} - v_{1x} = -y\omega \qquad -v_{Bx} - (-5) = -(+1.73)(+2.5)$$

$$v_{Bx} = +9.33 \text{ ft/sec}$$

$$v_{2y} - v_{1y} = +x\omega \qquad +v_{By} - 0 = +(+1)(+2.5)$$

$$v_{By} = +2.50 \text{ ft/sec}$$

The components of v_B, as found above, can be used directly in the final calculation, and v_B itself need not be calculated.

By the principle stated at the beginning of the article, v_P is the resultant of V_P and v_B. Using the principle of components,

$$v_{Px} = V_{Px} - v_{Bx} = +6 \cos 30° - 9.33 = -4.13 \text{ ft/sec}$$

$$v_{Py} = V_{Py} + v_{By} = +6 \sin 30° + 2.50 = +5.50 \text{ ft/sec}$$

$$v_P = \sqrt{(4.13)^2 + (5.50)^2} = 6.88 \text{ ft/sec}$$

$$\theta_x = \text{arc tan} \frac{5.50}{4.13} = 53° \, 05'$$

990. Figure 465 represents a pair of cams, A and D. They rotate about the shafts E and F. The line CC' shows the position of the common tangent plane to the two cams at their point of contact. The angular velocity of cam A, at the instant shown by the figure, is 2.4 rad per sec, clockwise. Calculate the angular velocity of cam D at the given instant, assuming that contact is being maintained between the two cams.

Solution. Let P represent that particle, belonging to cam A, which lies at the point of contact. Let P be chosen as the moving point, and cam D as the moving body. The absolute velocity of P can be found as follows:

$$v_P = \overline{EP}\omega_A = \sqrt{(3)^2 + (1.5)^2} \times (2.4) = 8.04 \text{ in./sec}$$

v_P is at right angles to EP; therefore,

$$\theta_x = \text{arc tan } \frac{3}{1.5} = \text{arc tan } 2.0 = 63° \ 25'$$

v_P is directed upward and toward the left, as shown in the figure.

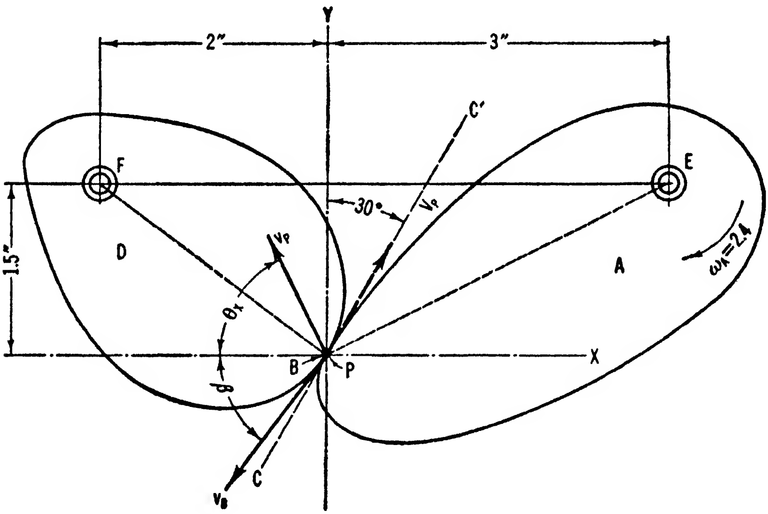

Fig. 465

The path of P relative to cam D is a curve lying somewhere between the two cams. So far as the present problem is concerned the important fact is that P touches cam D only once, at B. Therefore, the common tangent, CC', also is the tangent to the relative path, and the relative velocity, V_P, coincides with CC'. Obviously, the sense of V_P will be as shown.

Let B represent that particle, belonging to cam D, with which P coincides at the instant under consideration. The absolute velocity of B can be found as follows:

$$v_B = \overline{FB} \times \omega_D = \sqrt{(2)^2 + (1.5)^2} \times \omega_D = 2.5\omega_D$$

v_B is at right angles to FB; therefore,

$$\beta = \text{arc tan } \frac{2}{1.5} = \text{arc tan } 1.33 = 53° \ 05'$$

By the principle stated at the beginning of the article, v_P is the resultant of V_P and v_B. Therefore, by the principle of components, using an axis at right angles to CC',

$$v_P \sin [30° + (90° - \theta_x)] = v_B \sin (60° - \beta)$$

$$8.04 \sin 56° 35' = 2.5\omega_D \sin 6° 55'$$

$$\omega_D = \frac{8.04 \times 0.835}{2.5 \times 0.120} = 22.4 \text{ rad/sec, clockwise}$$

PROBLEMS

991. Solve Prob. 987, Fig. 462, for the case in which the point P is in a position 180° from that shown in the figure. Let all the other data of the problem remain unchanged. *Ans.* 7.65 ft/sec, $\theta_x = 342° 45'$.

992. A certain automobile has tires 30 in. in diameter, and a high-gear ratio of 4.77. The stroke is 4 in. Find the absolute velocity of a point at the center of one of the crankpins, when the crank is horizontal, the crankpin is moving downward, and the car has a speed of 70 mi per hr on a straight, horizontal roadway.

993. A certain motor boat has a speed of 40 mi per hr, in still water. It crosses a river 1 mi wide, steered in such a manner as to keep it headed at right angles to the stream. When it reaches the opposite bank it is 100 ft downstream from the starting point. Calculate the velocity of the stream, assuming that the speed of the boat relative to the water is the same as in still water. *Ans.* 0.758 mi/hr.

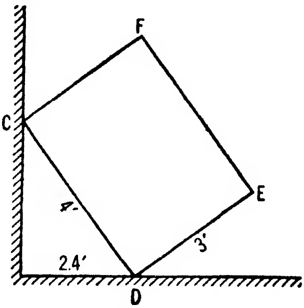

Fig. 466

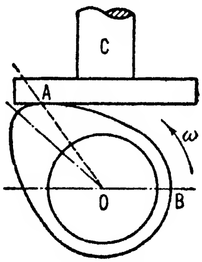

Fig. 467

994. Figure 466 represents a rectangular, flat plate, 3 by 4 ft, in plane motion, the plane of the motion being vertical and parallel to the plane of the paper. C moves vertically up a wall, and D moves horizontally along a floor. The velocity of C is 4.8 ft per sec, upward. A point, P, moves along the edge CF, toward F, at a constant speed of 4 ft per sec, relative to the plate. At the instant represented by the figure, P is 1.92 ft from C. Find the absolute velocity of P at the given instant.

995. Let the point P, in Prob. 994, Fig. 466, move along the edge EF, toward F, reaching F when the plate is in the position shown in the figure. Find the absolute velocity of P at the given instant. Let all the other data remain as in Prob. 994. *Ans.* 3.42 ft/sec, $\theta_x = 69° 30'$.

996. A point moves along the edge CD of the plate in Fig. 466, Prob. 994. At the instant depicted by the figure the absolute velocity of the point is zero. How far is the point from C? Calculate the velocity of the point relative to the plate.

997. Figure 467 represents a cam similar to those used to operate the valves in an automobile engine. The tappet, C, can move in a vertical direction only. If the speed of the camshaft is 1500 rpm, calculate the absolute velocity of the tappet at the instant when the distance OA is 1.4 in., and the angle AOB is 126°. Also calculate the velocity of a point on the cam at A, relative to the tappet. *Ans.* 10.8 ft/sec, downward; 14.8 ft/sec, horizontal and toward the left.

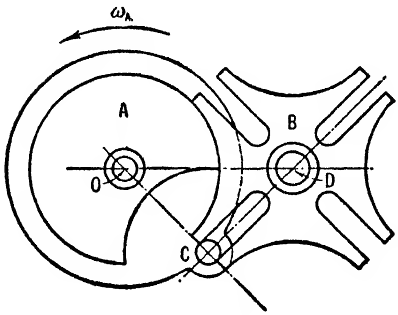

FIG. 468

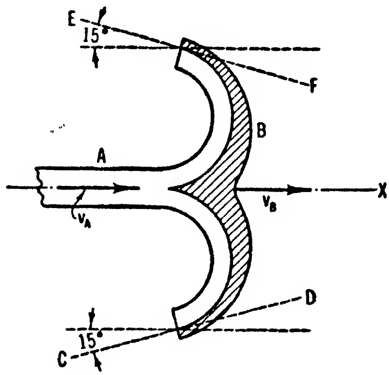

FIG. 469

998. Figure 468 represents a type of intermittent gearing usually known as the Geneva wheel. The driving wheel, A, rotates at constant speed, and carries the small roller, C. The driven wheel, B, has four radial slots, 90° apart. In the figure the roller is shown just as it enters one of the slots, at which instant angle COD = angle ODC = 45°. The roller leaves the slot after each wheel has made $\frac{1}{4}$ rev. Wheel B then remains stationary while A completes its revolution. If $OC = 4.24$ in., $OD = 6$ in., and $\omega_A = 60$ rpm, calculate the angular velocity of wheel B for the instants when angle COD has the following values: 45°; 22° 30′; 0°. Calculate also the velocity of the center of roller C, relative to wheel B, for those positions

999. On a certain clock, at a certain instant, the velocity of the point at the tip of the hour hand, relative to the minute hand, is 1.1π in. per min, and is directed downward and toward the right, at an angle of 70° with the horizontal. What time is it? How long is the hour hand? *Ans.* 8:20; 3 ft.

1000. In Fig. 469, B represents a cross section of one of the buckets of a hydraulic turbine of the type known as impulse wheels. A represents the jet of water striking the bucket. The stream divides and, in the present case, is turned through an angle of 165°, relative to the bucket. The lines CD and EF are drawn tangent to the curve of the bucket at the point where the water leaves. If the velocity, v_A, of the jet is 200 ft per sec, and the velocity, v_B, of the bucket is 100 ft per sec, ascertain the amount and inclination of the absolute velocity of the water at the instant when it leaves the bucket. Assume that the velocity of the water relative to the bucket remains constant in magnitude.

1001. Figure 470 represents a circular cam, 10 in. in diameter, mounted on a horizontal shaft. The follower, A, moves in a vertical direction only. The cam has a counterclockwise angular velocity of 120 rpm. Calculate the absolute velocity of the follower for the position shown. *Ans.* 3.34 ft/sec, upward.

1002. Figure 471 represents a quick-return mechanism of a type often used to drive certain machines. The wheel, A, carries a block, B, mounted on a pin. As the wheel rotates, B slides up and down in the link CD, causing CD to oscillate about the fixed center, C. D is connected to the machine by means of the link DE. While B

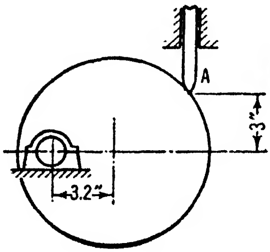

Fig. 470

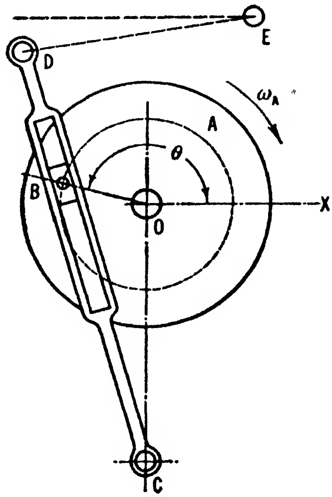

Fig. 471

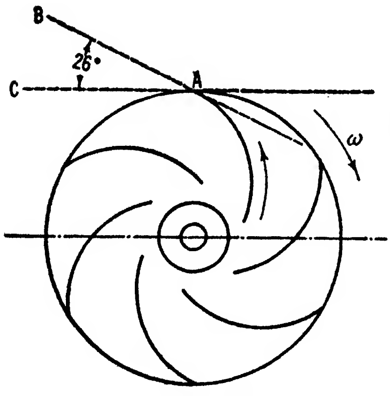

Fig. 472

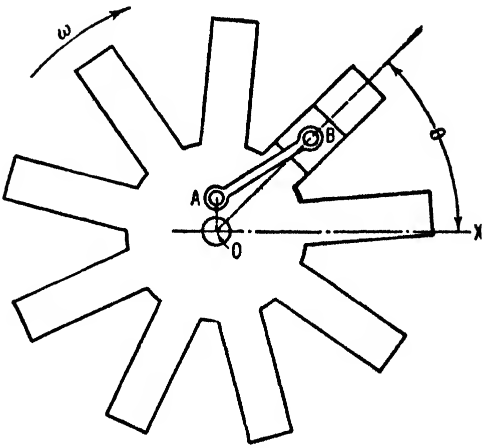

Fig. 473

traverses the upper portion of its arc a slow stroke is executed. The return stroke is accomplished more quickly, thus saving considerable time. $OB = 6$ in., $OC = 18$ in., and $DC = 30$ in. If the wheel A rotates at a constant speed of 10 rpm, calculate the velocity of B relative to CD, and the absolute linear velocity of D, for the instant when $\theta = 150°$.

1003. Figure 472 represents the impeller of a centrifugal pump. Water issues from the impeller at A with a velocity, relative to the impeller, of 7 ft per sec, along the line AB, which is drawn tangent to the vane. If the impeller is 4 ft in diameter, and is rotating in a clockwise direction at a speed of 400 rpm, find the absolute velocity of the water at A as it leaves the impeller. *Ans.* 77.5 ft/sec, $\theta_x = 2° 16'$.

1004. Figure 473 illustrates the principle of a rotary engine formerly used in aviation, known as the Gnome engine. The cylinders rotate about the crankshaft, O. All the connecting rods turn on the fixed crankpin, A. Because of the eccentric position of the crankpin, each piston executes a reciprocating motion within its

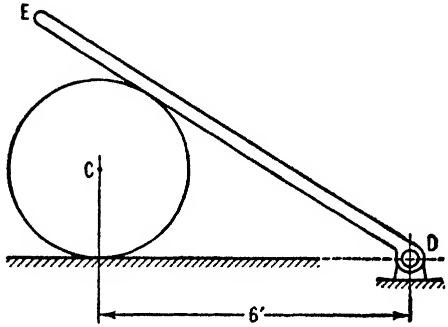

FIG. 474

cylinder as the cylinder revolves about O. Only one piston and connecting rod are shown in the figure. If $OA = 3.5$ in. and $AB = 11.5$ in., and the speed of the engine is 3000 rpm, clockwise, calculate the absolute velocity of the point B, and its velocity relative to the cylinder, at the instant when $\theta = 45°$.

1005. Figure 474 represents a cylinder, 3.5 ft in diameter, rolling on a horizontal plane, without slipping. A bar, DE, rotates on a shaft at D, and rests on the cylinder. At the instant represented by the figure the velocity of the center of the cylinder is 5 ft per sec, toward the left. Calculate the angular velocity of the bar at the given instant. *Ans.* +0.448 rad/sec.

170. Relative Acceleration. Linear acceleration, as well as linear velocity, is a relative term, depending on the body to which the path of the moving point is referred. *The formula $a_T = d^2s/dt^2$ gives the tangential component of the absolute acceleration of a point if s is measured along the absolute path, but it gives the tangential component of the relative acceleration if s is measured along the relative path.* The alternative formula, $a_T = dv/dt$, also gives the absolute or relative tangential acceleration, depending on whether v represents the absolute or relative velocity. *The formula $a_N = v^2/r$ gives the normal component of the absolute acceleration if v represents the absolute velocity, and it gives the normal component of the relative acceleration if v represents the relative velocity.* In the first case the value used for r would be the radius of curvature of the absolute path, and in the second it would be the radius of curvature of the relative path.

Likewise, *the formulas for the axial components of linear acceleration will give components of the absolute acceleration if they refer to a stationary* *pair of axes, and will give the relative acceleration if they refer to a pair of axes fixed in the moving body.*

Obviously, a vector intended to represent the tangential component of an absolute acceleration would be drawn tangent to the absolute path, and a vector intended to represent a normal absolute acceleration would be drawn normal to that path. Similarly, vectors representing relative accelerations would be drawn in their proper positions with regard to the relative path.

171. Relation between Relative and Absolute Accelerations. *The absolute acceleration of a moving point at any instant is the vector sum, or resultant, of the following:*

I. *The acceleration of the moving point relative to any moving body at the given instant,* and

II. *The absolute acceleration of that particle belonging to the moving body, or extension thereof, which coincides with the moving point at the given instant,* and

III. *A third acceleration, called the "acceleration of Coriolis";* also called the complementary acceleration. The acceleration of Coriolis is made up as follows:

(*a*) Magnitude. Its magnitude is equal to twice the product of the velocity of the moving point relative to the moving body, and the angular velocity of the body, at the given instant.

(*b*) Inclination. It is at right angles to the relative velocity.

(*c*) Sense. Let the reader imagine that he stands on the plane of motion of the body and faces in the direction of the relative velocity. If the angular velocity of the body is clockwise, the sense of Coriolis' acceleration is to the reader's right; if the angular velocity of the body is counterclockwise, the sense of Coriolis' acceleration is to his left.

Proof. The foregoing relations will be proved only for the case in which the point moves in a plane, and the moving body has plane motion, the plane in which the point moves being parallel to the plane of motion of the body. The majority of problems in practice conform essentially to this special case.

Figure 461, and the various symbols used in Art. 169, will also be employed in the present proof. In addition to these, the following symbols will be used: a_Q and a_B will represent the absolute accelerations of particles Q and B, respectively; a (lower-case) will represent the absolute acceleration of the moving point, P; A (upper-case) will represent the acceleration of P relative to the moving body; and α will represent the angular acceleration of the moving body at the given instant.

For convenience, Eq. 270, of Art. 169, will be repeated here, slightly changed in form.

$$v_x = v_{Qx} - \omega n \cos\beta - \omega m \sin\beta + V_m \cos\beta - V_n \sin\beta \qquad [276]$$

Since $a_x = dv_x/dt$, an expression for a_x can now be obtained by differentiating in Eq. 276, with respect to t. The following equation results:

$$a_x = \frac{dv_{Qx}}{dt} + \omega n \sin\beta\frac{d\beta}{dt} - \omega\cos\beta\frac{dn}{dt} - n\cos\beta\frac{d\omega}{dt}$$
$$- \omega m\cos\beta\frac{d\beta}{dt} - \omega\sin\beta\frac{dm}{dt} - m\sin\beta\frac{d\omega}{dt} - V_m\sin\beta\frac{d\beta}{dt}$$
$$+ \cos\beta\frac{dV_m}{dt} - V_n\cos\beta\frac{d\beta}{dt} - \sin\beta\frac{dV_n}{dt} \qquad [277]$$

In Eq. 277,

$$\frac{dv_{Qx}}{dt} = a_{Qx}, \frac{d\beta}{dt} = \omega, \frac{dn}{dt} = V_n, \frac{d\omega}{dt} = \alpha, \frac{dm}{dt} = V_m, \frac{dV_m}{dt} = A_m \text{ and } \frac{dV_n}{dt} = A_n$$

Substituting these values in Eq. 277, and rearranging,

$$a_x = a_{Qx} - \omega^2(m\cos\beta - n\sin\beta) - \alpha(m\sin\beta + n\cos\beta)$$
$$+ (A_m\cos\beta - A_n\sin\beta) - 2\,\omega\,(V_m\sin\beta + V_n\cos\beta) \qquad [278]$$

From Fig. 461, Art. 169, $(m\cos\beta - n\sin\beta) = x_P - x_Q$, and $(m\sin\beta + n\cos\beta) = y_P - y_Q$. Also, from the figure, $(A_m\cos\beta - A_n\sin\beta) = A_x$, and $(V_m\sin\beta + V_n\cos\beta) = V_y$. Substituting these values in Eq. 278, and replacing the symbols x_P and y_P by their equivalents, x_B and y_B,

$$a_x = A_x + a_{Qx} - 2V_y\omega - (x_B - x_Q)\omega^2 - (y_B - y_Q)\alpha \qquad [279]$$

Particles Q and B are two particles of a body in plane motion. Equation 245, Art. 161, states that $a_{2x} - a_{1x} = -(x_2 - x_1)\omega^2 - (y_2 - y_1)\alpha$. Selecting Q as particle 1 and B as particle 2, and using the notation of Fig. 461, Eq. 245 becomes,

$$a_{Bx} - a_{Qx} = -(x_B - x_Q)\omega^2 - (y_B - y_Q)\alpha \qquad [280]$$

Eliminating the expression $-(x_B - x_Q)\omega^2 - (y_B - y_Q)\alpha$, between Eqs. 279 and 280,

$$a_x = A_x + a_{Bx} - 2V_y\omega \qquad [281]$$

An analysis similar to the foregoing, and based on Eq. 246, Art. 161, leads to the following formula for a_y:

$$a_y = A_y + a_{By} + 2V_x\omega \qquad [282]$$

It will now be shown that Eqs. 281 and 282 constitute a proof of the principles stated at the beginning of the article. The absolute acceleration, a, of the moving point, P, is the resultant of a_x and a_y. Therefore, from Eqs. 281 and 282, a is the resultant of the six vector quantities, A_x, a_{Bx}, $(-2V_y\omega)$, A_y, a_{By}, and $(+2V_x\omega)$. The resultant of A_x and A_y alone is A, the relative acceleration. The resultant of a_{Bx} and a_{By} is a_B, the absolute acceleration of particle B.

It only remains to be shown that the resultant of the two vectors, $(-2V_y\omega)$ and $(+2V_x\omega)$, is the acceleration of Coriolis, as described at the beginning of the article. The magnitude of the resultant of these two vectors is equal to $\sqrt{(2\,V_x\omega)^2 + (-\,2\,V_y\omega)^2} = 2\,\omega\sqrt{V_x^2 + V_y^2} = 2\,V\omega$. This last expression is the magnitude ascribed to the acceleration of Coriolis, at the beginning of the article. The angle that this resultant, $2\,V\omega$, makes with OX is arc tan $\dfrac{2V_x\omega}{-2V_y\omega}$ = arc tan $-\dfrac{V_x}{V_y}$. The angle that the relative velocity, V, makes with OX is arc tan $\dfrac{V_y}{V_x}$. A comparison of the two expressions shows that the acceleration of Coriolis is at right angles to the relative velocity.

The sense of Coriolis' acceleration depends on the algebraic signs that its components, $(-2V_y\omega)$ and $(+2V_x\omega)$, have in any particular problem. These signs depend, in turn, on the signs of V_x, V_y, and ω. A study of all possible combinations of these signs readily shows that the sense of the vector, $2V\omega$, representing the acceleration of Coriolis, will always be as stated in the beginning; that is, as seen by a person facing in the direction of V, the sense of $2V\omega$ will be toward the right if ω is clockwise, and toward the left if ω is counterclockwise.

Methods of Solving Problems. The study made in this article furnishes a choice of two procedures in the solution of specific problems involving relative and absolute accelerations. The general principles stated at the beginning of the article can be learned, and a sketch made of the various vectors involved in the problem at hand. The unknown quantities can then be found by whatever method seems best suited to the situation. A graphic solution could be used, if desired.

In the alternative method a convenient pair of x- and y-axes can be selected, and the problem solved by the direct use of Eqs. 281 and 282. If the quantities A_x, A_y, a_{Bx}, a_{Bx}, V_x, V_y, and ω are given their proper algebraic signs, Eqs. 281 and 282 will apply in any problem.

Illustrative Problems

1006. Figure 475 represents a car on which is mounted a wheel 5 ft in diameter. The wheel has a constant angular velocity of 10 rpm, counterclockwise. The car has a linear acceleration of 2 ft per sec per sec, toward the right. Find the absolute acceleration of a particle, P, on the rim of the wheel, for the position shown in the figure.

Solution. Let P be selected as the moving point, and the body of the car as the moving body. By the principle stated at the beginning of the article, the absolute acceleration of P is the resultant of various components, as follows:

I. *The acceleration of P relative to the body of the car.* The path of P relative to the car is the circle representing the periphery of the wheel. Since the angular velocity of the wheel is constant, and since the angular velocity of the

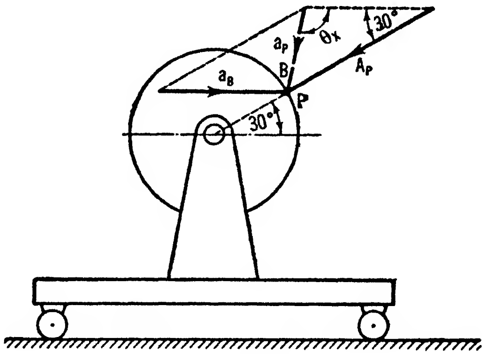

FIG. 475

body of the car is zero, the angular velocity of the wheel relative to the car also is constant and is equal to 10 rpm. Therefore, the linear acceleration of P relative to the body of the car has a normal component, only.

$$A_P = r\omega^2 = 2.5\left(\frac{10 \times 2\pi}{60}\right)^2 = 2.74 \text{ ft/sec}^2$$

A_P is situated as shown in the figure, along the radius through P and toward the center of the wheel.

II. *The absolute acceleration of that point belonging to the moving body, with which the moving point coincides at the given instant.* Let B represent that particle belonging to the body of the car (extended) with which P coincides. The body has a motion of translation; therefore, the accelerations of all its points are equal. The absolute acceleration of B is, then,

$$a_B = 2 \text{ ft/sec}^2$$

horizontal, and toward the right.

III. *The acceleration of Coriolis.* The magnitude of this component is equal to $2V\omega$, in which ω represents the angular velocity of the moving body. Since the car body is translating, its angular velocity is zero; therefore, the acceleration of Coriolis in this problem is zero.

It follows, then, that a_P is the resultant of A_P and a_B. The parallelogram of accelerations is as shown in the figure. By the law of cosines,

$$\begin{aligned} a_P &= \sqrt{A_P^2 + a_B^2 - 2A_Pa_B\cos 30°} \\ &= \sqrt{(2.74)^2 + (2)^2 - 2 \times 2.74 \times 2 \times 0.866} \\ &= 1.42 \text{ ft/sec}^2 \end{aligned}$$

By the law of sines,

$$\frac{\sin \theta_x}{A_P} = \frac{\sin 30°}{a_P} \qquad \theta_x = \arc\sin \frac{2.74 \times 0.5}{1.42} = 105° \, 10'$$

1007. Figure 476 represents a wheel 8 ft in diameter, mounted on a shaft. A straight line, AB, is inscribed on the wheel at a distance of 2 ft from the center. A point, P, traverses this line, moving from A toward B. At the instant when P reaches B its velocity relative to the wheel is 2 ft per sec and is decreasing at the rate of 3 ft per sec per sec. At the same instant the angular velocity of the wheel is 1 rad per sec, counterclockwise, and is decreasing at the rate of 2 rad per sec per sec. Find the absolute acceleration of P at the given instant.

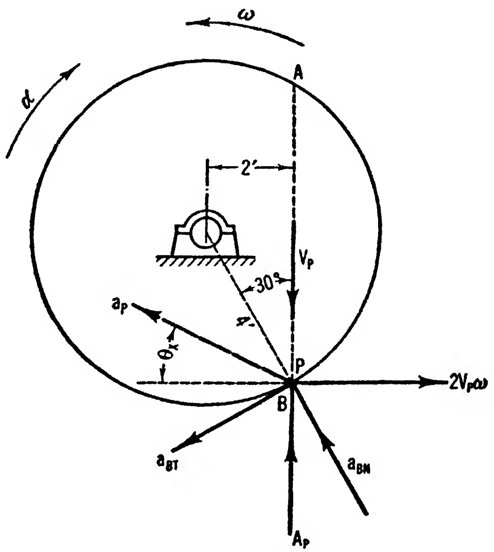

Fig. 476

Solution. Naturally, P will be chosen as the moving point and the wheel as the moving body. The absolute acceleration of P is the resultant of the following:

I. *The acceleration of P relative to the wheel.* Obviously the line AB is the relative path. From the problem, $A_P = 3$ ft/sec². Since AB is a straight line, A_P is parallel to the path, and since V_P is decreasing in magnitude, A_P is upward, as shown in the figure.

II. *The absolute acceleration of that particle belonging to the moving body which coincides with the moving point at the given instant.* Let B represent this particle. Its acceleration has a tangential, and a normal, component. By Eqs. 212 and 213, Art. 150,

$$a_T = r\alpha \qquad a_{BT} = 4 \times 2 = 8 \text{ ft/sec}^2$$

$$a_N = r\omega^2 \qquad a_{BN} = 4(1)^2 = 4 \text{ ft/sec}^2$$

Since the angular acceleration of the wheel is clockwise, the sense of a_{BT} is as shown in the figure. a_{BN} is radial and toward the center of the wheel.

III. *The acceleration of Coriolis.* The magnitude of the acceleration of Coriolis $= 2V_P\omega = 2 \times 2 \times 1 = 4$ ft per sec per sec. It is at right angles to V_P; consequently it is horizontal in the figure. To ascertain its correct sense, the observer imagines himself standing on the plane of motion of the wheel, facing in the direction of V_P. In the present problem he faces downward, in the figure. Since the angular velocity of the wheel is counterclockwise, the acceleration of Coriolis will be directed toward the observer's left. This will be as shown.

All the components of a_P are now known. By the principle of components, assuming a_P to be inclined upward and toward the left,

$$-a_{Px} = -a_{BT}\cos 30° - a_{BN}\sin 30° + 2V_P\omega$$

$$= -8 \times 0.866 - 4 \times 0.5 + 4$$

$$+a_{Py} = -a_{BT}\sin 30° + a_{BN}\cos 30° + A_P = -8 \times 0.5 + 4 \times 0.866 + 3$$

From which, $a_{Px} = +4.93$ ft per sec per sec and $a_{Py} = +2.46$ ft per sec per sec. The positive signs show that the senses of a_{Px} and a_{Py} were correctly assumed, and that the sense of a_P is upward and toward the left.

$$a_P = \sqrt{(4.93)^2 + (2.46)^2} = 5.51 \text{ ft/sec}^2$$

$$\theta_x = \text{arc tan}\,\frac{2.46}{4.93} = 26°\ 30'$$

1008. Figure 477 represents a bar, AB, 10 ft long, in plane motion. The plane of the motion coincides with the plane of the paper. The upper end moves in a vertical line, and the lower end in a horizontal line. In the given position the angular velocity of the bar is 2.5 rad per sec, counterclockwise, and the angular acceleration is 2 rad per sec per sec, clockwise. A point, P, moves along the bar, toward B. At the instant when P reaches B its velocity relative to the bar is 5 ft per sec, and is increasing at the rate of 6 ft per sec per sec. Find the absolute acceleration of P at the given instant.

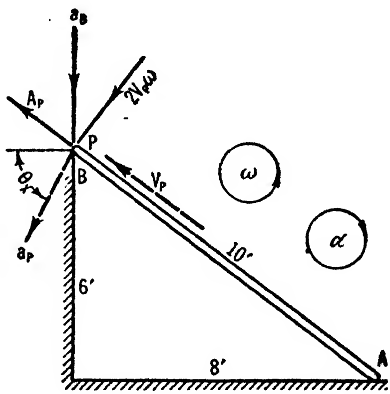

FIG. 477

Solution. Naturally, P will be chosen as the moving point and the bar as the moving body. The various components of the absolute acceleration of P are calculated as follows:

I. *The acceleration of P relative to the bar.* By the problem, this acceleration, A_P, is 6 ft per sec per sec. The relative path of P is a straight line along the axis of the bar. Therefore, A_P is along the bar and, since the relative velocity is increasing, is directed upward and toward the left, as shown.

II. *The absolute acceleration of the point B, belonging to the bar and coinciding with P.* a_B can be calculated from Eq. 250, Art. 161. Let A be chosen as particle 1 and B as particle 2. The origin must be placed at A; therefore, $x = -8$ ft and $y = +6$ ft. Let a_B be assumed downward. By Eq. 250,

$$a_{2_y} - a_{1_y} = -y\omega^2 + x\alpha \qquad -a_B - 0 = -(+6)(+2.5)^2 + (-8)(-2)$$

$$a_B = +21.5 \text{ ft/sec}^2$$

The positive sign shows that a_B was assumed correctly and is, therefore, downward.

III. *The acceleration of Coriolis.* The magnitude of this component is equal to $2V_P\omega = 2 \times 5 \times 2.5 = 25$ ft/sec². It is at right angles to V_P and to the bar. To ascertain the sense of the component the observer imagines himself standing on the plane of motion of the bar, facing in the direction of V_P. The angular velocity of the bar is counterclockwise; therefore, the acceleration of Coriolis is directed toward the observer's left, as shown in the figure.

All the components of a_P are now known. By the principle of components, using horizontal and vertical axes, and assuming a_P to be inclined downward and toward the left,

$$-a_{P_x} = -A_P \times \tfrac{4}{5} - (2V_P\omega) \times \tfrac{3}{5} = -6 \times \tfrac{4}{5} - 25 \times \tfrac{3}{5}$$

$$-a_{P_y} = +A_P \times \tfrac{3}{5} - a_B - (2V_P\omega) \times \tfrac{4}{5} = +6 \times \tfrac{3}{5} - 21.5 - 25 \times \tfrac{4}{5}$$

Solving, $a_{P_x} = +19.8$ ft/sec² and $a_{P_y} = +37.9$ ft/sec². The positive signs show that the sense of a_P was assumed correctly.

$$a_P = \sqrt{(19.8)^2 + (37.9)^2} = 42.8 \text{ ft/sec}^2$$

$$\theta_x = \text{arc tan} \frac{37.9}{19.8} = 62° \, 20'$$

1009. Figure 478 represents a wheel, C, mounted on a horizontal shaft at E. A small pin, P, is fastened to the wheel, at a distance of 2.5 ft from the axis of the shaft. A straight bar, D, mounted on a horizontal shaft at F, rests on the pin P. In the position shown the wheel has a counterclockwise angular velocity of 2 rad per sec, and a clockwise angular acceleration of 3 rad per sec per sec. Calculate the angular velocity and angular acceleration of the bar D, for the given position, assuming that the bar is in contact with the pin at all times.

Solution. Let the center of the pin, P, be chosen as the moving point, and the bar D as the moving body. Certain distances and angles will be needed in the solution. In the triangle BEF, by the law of cosines,

$$\overline{BF} = \sqrt{(2.5)^2 + (6)^2 - 2 \times 2.5 \times 6 \cos 45°} = 4.59 \text{ ft}$$

In the triangle BEF, by the law of sines,

$$\frac{\sin (EBF)}{\sin 45°} = \frac{6}{4.59} \qquad \angle EBF = 112° \, 30'$$

$$\angle EPH = 112° \, 30' - 90° = 22° \, 30'$$

Absolute Velocity of P. In this problem the absolute velocity of the moving point can be calculated immediately, since P is a particle of a rotating body.

$$v = r\omega \qquad v_P = \overline{BE} \times \omega_C = 2.5 \times 2 = 5 \text{ ft/sec}$$

v_P is at right angles to BE, and its sense is as shown. v_P is the resultant of the following:

I. *Velocity of P relative to the bar.* Since the pin is small, it may be assumed that its center point, P, is in contact with the lower surface of the bar at all times. Therefore, the path of P relative to the bar is a straight line along the lower surface of the bar. V_P is along this line, directed as shown in the figure. Its magnitude must remain unknown until the final solution is performed.

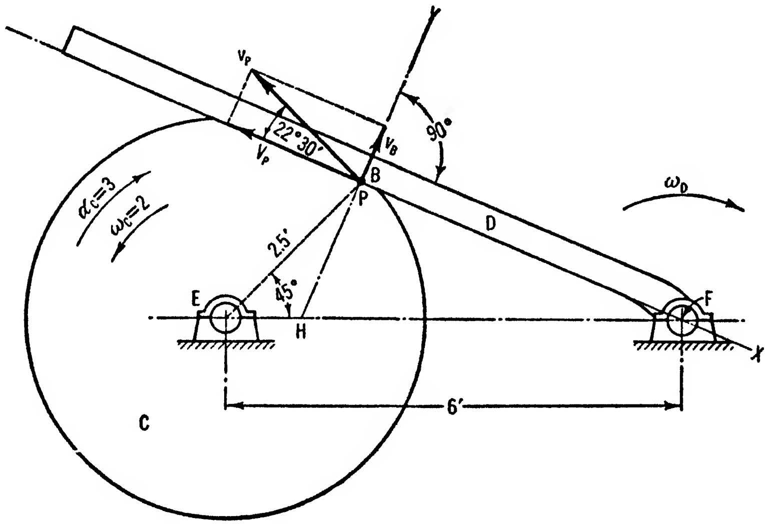

FIG. 478

II. *Absolute velocity of B.* Let B represent that particle belonging to the bar, with which P is in contact. The bar rotates about an axis at F. Therefore,

$$v = r\omega \qquad v_B = \overline{BF} \times \omega_D = 4.59\omega_D$$

v_B is at right angles to BF, and is directed as shown.

By the principle of Art. 169, v_P is the resultant of V_P and v_B. Therefore,

$$V_P = v_P \cos 22° \, 30' = 5 \times 0.924 = 4.62 \text{ ft/sec}$$

$$v_B = v_P \sin 22° \, 30' = 5 \times 0.383 = 1.91 \text{ ft/sec}$$

The angular velocity of the bar now can be calculated, since the radius of rotation, and linear velocity, of one of its particles are known.

$$v = r\omega \qquad \omega = \frac{v}{r} \qquad \omega_D = \frac{v_B}{\overline{BF}} = \frac{1.91}{4.59} = 0.416 \text{ rad/sec}$$

The solution for accelerations will be shown in Fig. 479.

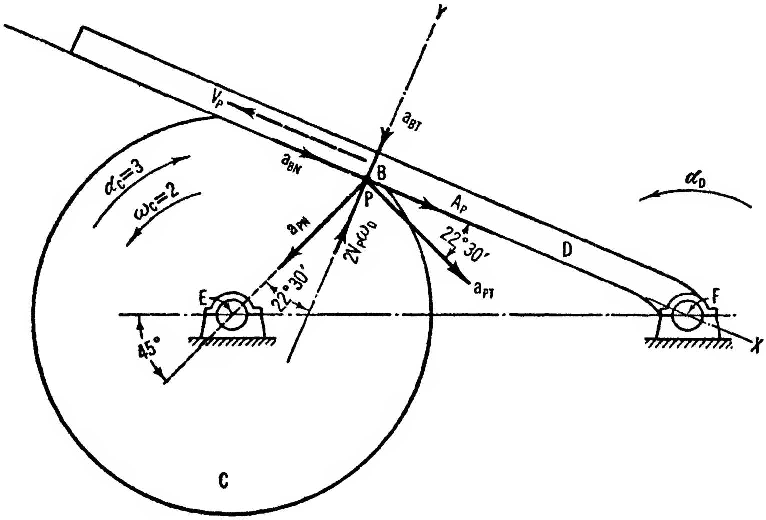

FIG. 479

Absolute acceleration of P. The absolute acceleration of the moving point also can be calculated immediately. Since the absolute path of P is a circle and the wheel has an angular acceleration, a_P will have both a tangential and a normal component. a_P itself need not be calculated.

$$a_T = r\alpha \qquad a_{PT} = \overline{BE} \times \alpha_C = 2.5 \times 3 = 7.5 \text{ ft/sec}^2$$

$$a_N = r\omega^2 \qquad a_{PN} = \overline{BE}(\omega_C)^2 = 2.5 \times (2)^2 = 10 \text{ ft/sec}^2$$

a_P is also the resultant of various components, as follows:

I. *Acceleration of P relative to the bar.* Since the relative path is a straight line along the lower surface of the bar, A_P will be directed along this line, and its sense probably will be as assumed in the figure. Its magnitude is unknown.

II. *Absolute acceleration of B.* Since B moves in a circle whose center is at F, its acceleration has a normal component, and probably a tangential component. Let α_D represent the angular acceleration of the bar.

$$a_T = r\alpha \qquad a_{BT} = \overline{BF} \times \alpha_D = 4.59\alpha_D$$

$$a_N = r\omega^2 \qquad a_{BN} = \overline{BF}(\omega_D)^2 = 4.59\ (0.416)^2 = 0.794 \text{ ft/sec}^2$$

These components will be directed as shown in Fig. 479.

III. *Acceleration of Coriolis.* The magnitude of the acceleration of Coriolis is as follows:

$$2V\omega = 2V_P\omega_D = 2 \times 4.62 \times 0.416 = 3.84 \text{ ft/sec}^2$$

This acceleration is at right angles to V_P. To ascertain its correct sense the observer imagines himself standing on the plane of motion of the bar, facing in the direction of V_P. Since the angular velocity of the bar is clockwise, the acceleration of Coriolis is directed toward the observer's right, as shown in the figure.

a_P now has been expressed in two ways: as the resultant of a_{PT} and a_{PN}, and as the resultant of A_P, a_{BT}, a_{BN}, and $2V_P\omega_D$. Since these two groups of vectors have the same resultant they are equivalent, and their component-sums along any axis are equal. Using the y-axis,

$$-a_{PN} \cos 22° \, 30' - a_{PT} \sin 22° \, 30' = -a_{BT} + 2V_P\omega_D$$

$$-10 \times 0.924 - 7.5 \times 0.383 = -4.59\alpha_D + 3.84$$

From which,

$$\alpha_D = 3.84 \text{ rad/sec}^2$$

The positive sign shows that the sense of α was assumed correctly, and that the angular acceleration of the bar is counterclockwise.

1010. Find the absolute acceleration of the point P in Prob. 989, Art. 169. Assume that the center of the cylinder has a linear acceleration of 1.6 ft per sec per sec, toward the left, and that the velocity of P relative to the cylinder is decreasing at the rate of 4 ft per sec per sec. All other conditions are to be the same as in Prob. 989.

Solution. Figure 480 shows the cylinder. As in Prob. 989, P will be chosen as the moving point, and the cylinder as the moving body. The absolute acceleration of P is the resultant of the following:

I. *Acceleration of P relative to the cylinder.* The path of P relative to the cylinder is the circular arc, DCB. The statement of the problem shows that the tangential component of the relative acceleration, $A_{PT} = 4$ ft/sec². Since V_P is decreasing, A_{PT} is opposite to V_P in sense, as shown in the figure. By Eq. 213, Art. 150,

$$a_N = \frac{v^2}{r} \qquad A_{PN} = \frac{V_P^2}{r} = \frac{(6)^2}{2} = 18 \text{ ft/sec}^2$$

This component is normal to the relative path, and is directed toward E, the center of curvature.

II. *Absolute acceleration of B.* Let B represent that particle belonging to the cylinder, with which P coincides at the given instant. From Prob. 989, the angular velocity of the cylinder $\omega = 2.5$ rad/sec, counterclockwise. By Eq. 235, Art. 160,

$$a_C = r\alpha \qquad \alpha = \frac{a_C}{r} = \frac{1.6}{2} = 0.8 \text{ rad/sec}^2$$

Components of the absolute acceleration of B now can be calculated by means of Eqs. 249 and 250, Art. 161. Let C be chosen as particle 1 and B as particle 2. The origin is placed at C; hence, $x = +1$ ft and $y = +2 \times 0.866 = +1.73$ ft. Assuming a_{Bx} and a_{By} as shown in the figure,

$$a_{2x} - a_{1x} = -x\omega^2 - y\alpha$$

$$-a_{Bx} - (-1.6) = -(+1)(+2.5)^2 - (+1.73)(+0.8)$$

$$a_{Bx} = +9.23 \text{ ft/sec}^2$$

$$a_{2y} - a_{1y} = -y\omega^2 + x\alpha$$

$$-a_{By} - 0 = -(+1.73)(+2.5)^2 + (+1)(+0.8)$$

$$a_{By} = +10.0 \text{ ft/sec}^2$$

The positive signs show that the senses of a_{Bx} and a_{By} were assumed correctly.

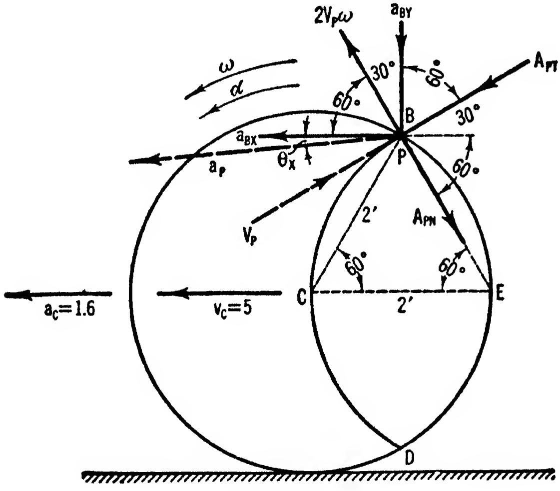

FIG. 480

III. *Acceleration of Coriolis.* The magnitude of this component is calculated as follows:

$$2V\omega = 2V_P\omega = 2 \times 6 \times 2.5 = 30 \text{ ft/sec}^2$$

The acceleration of Coriolis is at right angles to V_P. To ascertain its correct sense the observer imagines himself standing on the plane of motion of the wheel, facing in the direction of V_P. The angular velocity of the wheel is counterclockwise; therefore, the acceleration of Coriolis is directed toward the observer's left, as shown in the figure.

All the components of the absolute acceleration of P are now known. By the principle of components, using horizontal and vertical axes, and assuming that a_P is inclined downward and toward the left,

$$-a_{Px} = -A_{PT}\cos 30° + A_{PN}\cos 60° - a_{Bx} - (2V_P\omega)\cos 60°$$
$$= -4\cos 30° + 18\cos 60° - 9.23 - 30\cos 60°$$
$$a_{Px} = +18.7 \text{ ft/sec}^2$$
$$-a_{Py} = -A_{PT}\sin 30° - A_{PN}\sin 60° - a_{By} + (2V_P\omega)\sin 60°$$
$$= -4\sin 30° - 18\sin 60° - 10 + 30\sin 60°$$
$$a_{Py} = +1.61 \text{ ft/sec}^2$$

The positive signs show that a_P was assumed correctly in sense, and that it is directed downward and toward the left. Compounding,

$$a_P = \sqrt{(18.7)^2 + (1.61)^2} = 18.8 \text{ ft/sec}^2$$
$$\theta_x = \text{arc tan}\,\frac{1.61}{18.7} = 4°\,55'$$

PROBLEMS

1011. Calculate the absolute acceleration of the point in Prob. 992. Assume that the car is moving at constant speed. *Ans.* 25,600 ft/sec^2, horizontal and at right angles to the crankshaft.

1012. An automobile, moving along a straight, level road, at a speed of 60 mi per hr, is brought to rest in a distance of 250 ft, with constant deceleration. The tires are 28 in. in diameter. Find the velocity and acceleration of the highest point on one of the tires, relative to the body of the car, at the instant when the speed of the car is 5 mi per hr. Find the absolute velocity and acceleration of the point at the given instant.

1013. Solve Prob. 1012 for the lowest point on one of the tires. *Ans.* $V = 7.33$ ft/sec, horizontal and toward the rear; $A = 48.6$ ft/sec^2, upward and forward, $\theta_x = 71°\,30'$; $v = 0$; $a = 46.1$ ft/sec^2, vertical and upward.

1014. Figure 481 represents a car, C, having an inclined floor. A block, B, is placed at the upper end of the incline. Both the car and the block start from rest, the car being pulled toward the right with a constant acceleration of 2 ft per sec per sec in that direction. The block descends the incline with a constant acceleration, relative to the car, of 5 ft per sec per sec. Find the absolute velocity, and acceleration, of the block at the instant when it leaves the car.

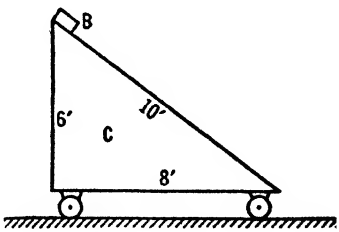

Fig. 481

1015. A certain box car runs on a straight, horizontal track. At a given instant the velocity of the car is 4 ft per sec, toward the right, and the acceleration is 3 ft per sec per sec, also toward the right. A circle 6 ft in diameter is drawn on the side of the car. A point, P, traverses the circle in a clockwise direction, at a constant speed of 3 ft per sec, relative to the car. Calculate the absolute velocity and acceleration of P, assuming that it is in the "3 o'clock" position on the circle at the given instant. Solve also for the "9 o'clock" position. *Ans.* $v = 5$ ft/sec, $\theta_x = 323° 10'$; $a = 0$; $v = 5$ ft/sec, $\theta_x = 36° 50'$; $a = 6$ ft/sec^2, $\theta_x = 0$.

1016. Solve Prob. 1015, assuming P to be in the "1 o'clock" position. Solve also for the "7 o'clock" position.

1017. Calculate the linear acceleration of the follower in Prob. 1001, Fig. 470, assuming that the angular velocity of the cam is constant. *Ans.* 56.3 ft/sec^2, $\theta_x = 270°$.

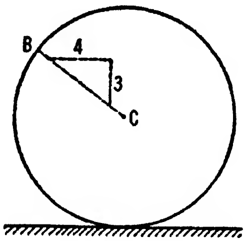

FIG. 482

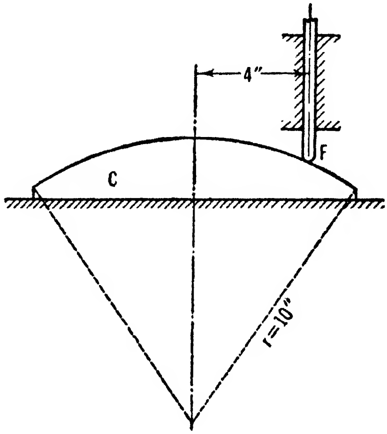

FIG. 483

1018. Figure 482 represents a cylinder 4 ft in diameter, rolling on a horizontal surface, without slipping. At the given instant the velocity of the center point is 4 ft per sec, toward the left, and its acceleration is 5 ft per sec per sec, toward the right. A point P moves along the radius BC, toward B, at a constant velocity of 3 ft per sec, relative to the cylinder. Calculate the absolute velocity and acceleration of P at the instant when it reaches B, assuming that the position of the cylinder at that instant is as shown.

1019. In Fig. 483, C represents a sliding cam, whose profile is a circular arc. At the instant represented by the figure the velocity of the cam is 3 ft per sec, toward the right, and its acceleration is 2 ft per sec per sec, also toward the right. The follower, F, can move only in the vertical direction. Calculate the absolute acceleration of the follower at the given instant.

1020. Turn the wheel in Prob. 1009 until the radius BE is at right angles to the bar D, and solve the problem. Let all the other data of the problem remain unchanged. *Ans.* $\omega = 0$; $\alpha = +1.83$ rad/sec^2.

1021. Calculate the absolute acceleration of the point D in Prob. 1002, Fig. 471.

1022. A wheel 4 ft in diameter rotates on a horizontal shaft. A ball is dropped from a point 9 ft vertically above the center of the wheel. At the instant when the ball is about to touch the wheel, the angular velocity of the wheel is 5 rad per sec, counterclockwise, and the angular acceleration is 4 rad per sec per sec, clockwise. Calculate the acceleration of the ball, relative to the wheel, at the given instant. *Ans.* 234 ft/sec^2, $\theta_x = 200° 25'$.

1023. Find the absolute acceleration of the point B, in the rotary engine of Prob. 1004, Fig. 473. Assume that the engine is rotating at constant speed.

1024. Figure 484 represents a flat, rectangular plate, 3 by 4 ft, moving in a vertical plane. Corner C moves vertically and D moves horizontally. C moves downward at a constant velocity of 2.4 ft per sec. A point, P, moves along the edge CF, toward F, at a constant velocity of 2.5 ft per sec, relative to the plate. Calculate the absolute velocity and acceleration of P for the instant when it reaches F, assuming that the plate at that instant is in the position shown by the figure. *Ans.* 1.51 ft/sec, $\theta_x = 82° 25'$; 3.16 ft/sec^2, $\theta_x = 198° 15'$.

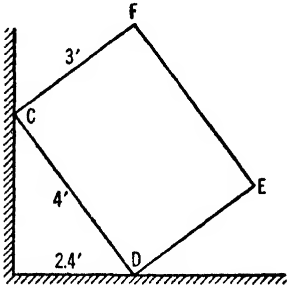

FIG. 484

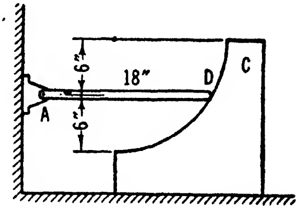

FIG. 485

1025. A point, P, moves along the edge FE of the plate in Prob. 1024, Fig. 484. It moves toward E at a constant velocity of 2.5 ft per sec, relative to the plate. Calculate the absolute velocity and acceleration of P for the instant when it reaches E, assuming that the plate at that instant is in the position shown. Let all the other data of Prob. 1024 remain unchanged.

1026. Figure 485 represents a sliding cam, C, whose profile is a circular arc having a radius of 12 in. The follower, AD, is a straight bar 18 in. long, mounted on a shaft at A and resting against the cam at D. In the position shown the cam has a velocity of 2 ft per sec toward the right, and an acceleration of 3 ft per sec per sec toward the left. Calculate the angular acceleration of the follower for the given instant. *Ans.* $+15.6$ rad/sec^2.

1027. Calculate the angular acceleration of the bar DE in Prob. 1005, Fig. 474. Assume that the cylinder is rolling at constant velocity.

172. Motion of a Point Relative to a Second Point. In the preceding articles such phrases as "the path of a point relative to a body," "the velocity of a point relative to a body," and "the acceleration of a point relative to a body" were explained, and various relations and formulas, useful in engineering practice, were obtained.

Expressions are sometimes encountered that refer the motion of a point to a second point, instead of referring the motion to a body. Thus, a point A may be said to have a path, a velocity and an acceleration relative to a second point B. It will not be necessary to acquire any new conception of relative motion in order to understand what is meant by such expressions; the understanding of relative motion already reached in the preceding articles can be made to suffice.

Let two points, A and B, be imagined. Regardless of the type of motion actually described by the body to which B belongs, let a body be imagined, each particle of which moves in exactly the same manner as B. The motion of this imaginary body is, consequently, a motion of translation. *The path, velocity, and acceleration of the point A relative to the point B are identical with the path, velocity, and acceleration of A relative to this imaginary translating body.* The conception of the motion of a point relative to a second point is thus made a special case of the motion of a point relative to a body. Furthermore, the general principles involving relative and absolute motion, proved in the preceding articles, now will be shown to apply in a simplified form to the case under discussion in the present article, as follows:

I. *The absolute velocity of a moving point at any instant is the vector sum, or resultant, of the velocity of the moving point relative to a second moving point, and the absolute velocity of the second point.*

II. *The absolute acceleration of a moving point at any instant is the vector sum, or resultant, of the acceleration of the moving point relative to a second moving point, and the absolute acceleration of the second point.*

Proof. The fact that the absolute velocities of the particles of a translating body are exactly alike at any given instant, that the absolute accelerations of the particles are also alike, and that the acceleration of Coriolis is equal to zero in the case of the motion of a point relative to a translating body, constitutes the verification of the foregoing principles.

Illustrative Problem

1028. Figure 486 represents two wheels, each of which is 4 ft in diameter, mounted on horizontal shafts in the manner indicated. The wheels have angular velocities and angular accelerations as shown, at the instant depicted by the figure. Find the linear velocity and linear acceleration of point C, relative to point D, at the given instant. Points C and D are fixed on the rims of the wheels.

Solution. The absolute velocities of C and D can be calculated at once, from the given conditions.

$$v_C = r\omega_C = 2 \times 2 = 4 \text{ ft/sec}$$

$$v_D = r\omega_D = 2 \times 1.5 = 3 \text{ ft/sec}$$

v_C is at right angles to the radius CE, and v_D is at right angles to DF. Vectors representing v_C and v_D are shown in Fig. 487.

By the principle stated in the article, v_C is the resultant of v_D and the velocity of C relative to D. The relative velocity is represented by V_C, in Fig. 487, and the three vectors must form a triangle, as shown in that figure. From the figure, by the law of cosines,

$$V_C = \sqrt{(4)^2 + (3)^2 - 2 \times 4 \times 3 \cos 75°} = 4.33 \text{ ft/sec}$$

By the law of sines,

$$\frac{\sin (LHO)}{\sin 75°} = \frac{4}{4.33} \qquad \angle LHO = 63° \, 10'$$

$$\theta_x = 63° \, 10' - 45° = 18° \, 10'$$

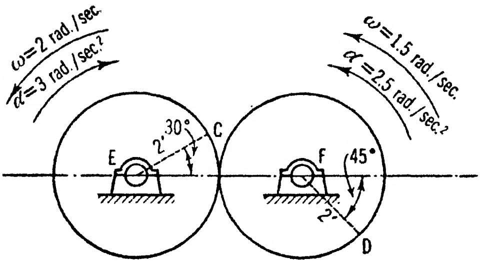

FIG. 486

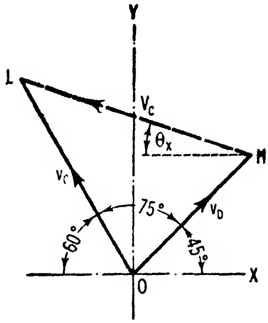

FIG. 487

The tangential and normal components of the absolute accelerations of C and D are as follows:

$$a_{CT} = r\alpha_C = 2 \times 3 = 6 \text{ ft/sec}^2$$
$$a_{CN} = r\omega_C^2 = 2(2)^2 = 8 \text{ ft/sec}^2$$
$$a_{DT} = r\alpha_D = 2 \times 2.5 = 5 \text{ ft/sec}^2$$
$$a_{DN} = r\omega_D^2 = 2(1.5)^2 = 4.5 \text{ ft/sec}^2$$

These components are positioned as shown in Fig. 488. Let A_{Cx} and A_{Cy} represent the x- and y-components of the acceleration of C relative to D. Let these two components be assumed positive, as shown. By the principle stated in the article, the resultant of a_{CT} and a_{CN} will be the resultant of the four remaining vectors. Therefore, resolving along OX and OY,

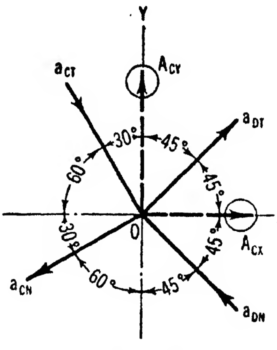

FIG. 488

$$+a_{CT} \sin 30° - a_{CN} \cos 30° = +a_{DT} \cos 45° - a_{DN} \cos 45° + A_{Cx}$$

$$-a_{CT} \cos 30° - a_{CN} \sin 30° = +a_{DT} \sin 45° + a_{DN} \sin 45° + A_{Cy}$$

Substituting in these equations the values obtained above,

$$+6 \sin 30° - 8 \cos 30° = +5 \cos 45° - 4.5 \cos 45° + A_{Cx}$$

$$-6 \cos 30° - 8 \sin 30° = +5 \sin 45° + 4.5 \sin 45° + A_{Cy}$$

$$A_{Cx} = -4.28 \text{ ft/sec}^2$$

$$A_{Cy} = -15.9 \text{ ft/sec}^2$$

$$A_C = \sqrt{(4.28)^2 + (15.9)^2} = 16.5 \text{ ft/sec}^2$$

$$\theta_x = \text{arc tan}\,\frac{15.9}{4.28} = \text{arc tan } 3.71 = 74°\ 55'$$

Negative signs were obtained for both components of A_C, showing that the senses of these components were incorrectly assumed in the figure. Therefore, A_C is directed downward, and toward the left, at an angle of 74° 55′ with OX.

PROBLEMS

1029. At a certain instant a moving point, C, is traveling along the y-axis in the negative direction with a velocity of 10 ft per sec. At the same instant a second point, D, has a velocity of 6 ft per sec, at an angle of 30° with the x-axis. Find the velocity of D relative to C at the given instant. *Ans.* 14 ft/sec, $\theta_x = 68°\ 15'$.

1030. At a certain instant a point, C, has a velocity of 30 mi per hr, due east. At the same instant the velocity of C relative to a second moving point, D, is 60 mi per hr, due south. Find the absolute velocity of D.

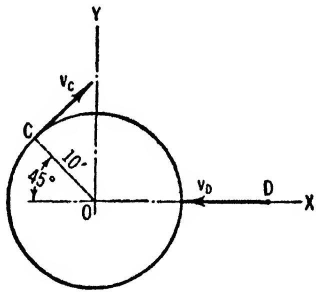

FIG. 489

1031. A point, C, travels due north, and a second point, D, travels due west. At a certain instant the velocity of C relative to D is 100 ft per sec, in a direction N. 30° 00′ E. Calculate the absolute velocity of each point. *Ans.* $v_C = 86.6$ ft/sec; $v_D = 50$ ft/sec.

1032. The point, C, in Fig. 489, moves in the circular path, as indicated, with a constant speed of 8 ft per sec. A second point, D, moves in the negative direction along the x-axis. At the instant represented by the figure the speed of D is decreasing at the rate of 4 ft per sec per sec. Calculate the acceleration of C relative to D.

1033. Solve Prob. 1032, if the speed of point C is decreasing, at the given instant, at the rate of 6 ft per sec per sec, all other data remaining the same as in that problem. *Ans.* 9.52 ft/sec², $\theta_x = 247°\ 00'$.

CHAPTER XXII

WORK

173. Work Done by a Single Concentrated Force. If a force that acts on a moving particle has a component tangent to the path of the particle during any interval of the motion, the force is said to do *work* on the particle during that interval. *The component tangent to the path is called the working component.* The sense of the working component may agree with the direction of the motion of the particle, in which case the work done by the force is considered to be positive; or the sense of the working component may disagree with the direction of the motion of the particle, in which case the work is negative.

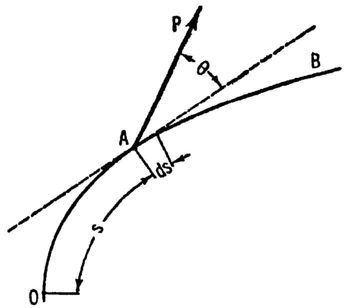

FIG. 490

Let Fig. 490 represent a particle, A, moving in the curved path, OAB. Let s represent the distance, measured along the path, from the fixed point O to the moving particle, at any instant. Let P represent a force acting on the particle, and let θ represent the angle between the line of action of P and the tangent to the path, drawn through the moving particle at the instant under consideration. The amount of work done by P during any interval of the motion is calculated by integrating the product of the working component and the differential of the distance traversed by the particle. The working component is equal to $P \cos \theta$. Let U represent the work done by P during any interval. The value of U can be expressed by a formula, as follows:

$$U = \int P \cos \theta \, ds \qquad [283]$$

Equation 283 is the general formula for the work done by any single, concentrated force, whether the magnitude of the force and the value of θ are variable or constant.

Units. The unit of work is usually named from the units used to express the distance and the force. In the English system the foot-pound is the most common unit, although the inch-ton and the foot-ton are sometimes encountered. In problems involving power a unit of work is frequently used which is a combination of units of power and and time, namely, the horsepower-hour. The kilowatt-hour is another power-time unit of work widely used in electrical engineering practice. The fundamental unit of work in the metric system is the centimeter-dyne, also called the erg. Another unit, called the joule, is equal to 10,000,000 ergs. The following equivalents are often useful:

$$\text{One horsepower-hour} = 1{,}980{,}000 \text{ foot-pounds}$$

$$\text{One kilowatt-hour} = 3{,}600{,}000 \text{ joules}$$

Illustrative Problems

1034. Figure 491 represents a helical spring, having a normal length of 12 in. One end of the spring is attached to a wall, as shown. The modulus of the spring is 10 lb per in. This means that a force of 10 lb will hold the spring at an elongation, or shortening, of 1 in. It is known, also, that the force required to maintain any longitudinal deformation of the spring is proportional to that deformation, provided that the elastic limit of the material is not exceeded. Calculate the work done on this spring in elongating it gradually to a length of 16 in., assuming that the elastic limit is not exceeded.

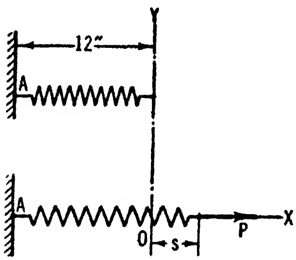

Fig. 491

Solution. The upper portion of the figure represents the spring in its normal, or starting, position, when the force P is equal to zero. The lower view shows the spring under any elongation. Let s represent the elongation, in inches.

Since $P = 10$ lb at the instant when $s = 1$ in., and since the magnitude of P is proportional to s, it follows that $P = 10s$. The angle θ in the present case is equal to zero, since P acts along the path of the particle to which it is applied. By Eq. 283,

$$U = \int P \cos\theta \, ds \qquad U = \int (10\, s)\,(+1)\, ds$$

The limits for s are $s = 0$ and $s = 16 - 12 = 4$. Therefore,

$$U = \int_0^4 10\, s\, ds = \Big[5\, s^2\Big]_0^4 = +80 \text{ in-lb}$$

The result was given the positive sign because of the fact that the working component of the force is in the same direction as the motion of the particle to which the force is applied.

1035. Prove that the work done on a spring by an axial force, when one end of the spring is held stationary and the inclination of the axis is constant, is equal to the arithmetic mean of the initial and final values of the force multiplied by the deformation of the spring.

Solution. Let m represent the modulus of the spring. Let P_1 and P_2 represent the initial and final values of the axial force. Let s represent the deformation of the spring. By Eq. 283,

$$U = \int P \cos\theta \, ds \qquad U = \int_{s_1}^{s_2} (ms)\, ds = m\left[\frac{s^2}{2}\right]_{s_1}^{s_2}$$

$$= m\,\frac{s_2^2 - s_1^2}{2} = m\,\frac{(s_2 - s_1)(s_2 + s_1)}{2} = \frac{ms_2 + ms_1}{2}(s_2 - s_1)$$

In the foregoing equation, $ms_2 = P_2$ and $ms_1 = P_1$. Therefore,

$$U = \frac{P_2 + P_1}{2}(s_2 - s_1)$$

The expression $\frac{P_2 + P_1}{2}$ represents the arithmetic mean of the initial and final values of P, and $(s_2 - s_1)$ represents the elongation or shortening of the spring.

1036. Solve Prob. 1034 by the method of Prob. 1035.

Solution. The initial value of P in this case is equal to zero. The final value $= 10 \times 4 = 40$ lb. Therefore,

$$U = \frac{0 + 40}{2}\,4 = +80 \text{ in-lb}$$

1037. Figure 492 represents a helical spring whose normal length is 12 in. One end is fastened at A. The other end moves from O to B, in a vertical, straight line. P is the force that stretches the spring, and acts along the axis of the spring. Calculate the work done on the spring by P. The modulus of the spring is 10 lb per in.

Solution. By Eq. 283,

$$U = \int P \cos\theta \, ds$$

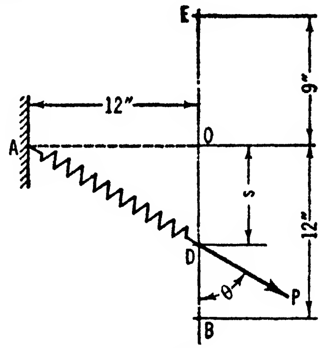

FIG. 492

From the figure, the elongation of the spring at any instant is as follows:

$$AD - AO = \sqrt{s^2 + (12)^2} - 12$$

And since the modulus of the spring is 10 lb per in.,

$$P = 10\left[\sqrt{s^2 + (12)^2} - 12\right]$$

Also, from the figure,

$$\cos\theta = \frac{OD}{AD} = \frac{s}{\sqrt{s^2 + (12)^2}}$$

Substituting these values of P and $\cos\theta$ in the original formula,

$$U = \int_0^{12} 10\,[\sqrt{s^2 + (12)^2} - 12]\,\frac{s}{\sqrt{s^2 + (12)^2}}\,ds$$

$$= 10\int_0^{12} s\,ds - 120\int_0^{12} \frac{s\,ds}{\sqrt{s^2 + (12)^2}}$$

$$= 10\left[\frac{s^2}{2}\right]_0^{12} - 120\left[\sqrt{s^2 + (12)^2}\right]_0^{12} = +124 \text{ in-lb}$$

This problem can be solved more easily by a special method developed in Art. 181, and is introduced here merely as a convenient illustration of the method of applying the general formula for work.

1038. Figure 493 represents a body weighing 500 lb, being drawn along a horizontal plane by a cable. The cable runs over a small pulley at B. The coefficient of kinetic friction is 0.2. Calculate the work done on the body by P, the pull of the cable, while the body moves from a point 100 ft from A to a point 50 ft from A. The velocity of the body is constant.

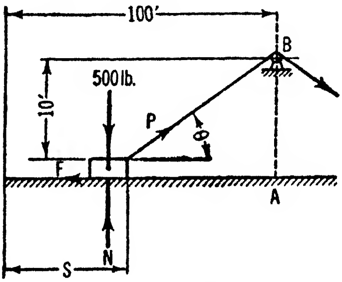

Fig. 493

Solution. Equations 184 and 185, Art. 126, and Eq. 32, Art. 81, can be used to ascertain the relation between P and θ.

$$\Sigma F_x = \frac{W}{g}\bar{a}_x \qquad P\cos\theta - F = 0$$

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad P\sin\theta + N - 500 = 0$$

$$F = \mu N \qquad F = 0.2N$$

Eliminating F and N from the foregoing equations,

$$P = \frac{100}{0.2\sin\theta + \cos\theta}$$

Substituting in Eq. 283,

$$U = \int P\cos\theta\,ds \qquad U = \int \frac{100\cos\theta}{0.2\sin\theta + \cos\theta}\,ds = \int \frac{100\,ds}{0.2\tan\theta + 1}$$

From the figure, $\tan\theta = \dfrac{10}{100 - s}$. Substituting in the foregoing, and simplifying,

$$U = 100\int_0^{50} \frac{(100 - s)\,ds}{102 - s} = 100\int_0^{50} \frac{100\,ds}{102 - s} - 100\int_0^{50} \frac{s\,ds}{102 - s}$$

$$= 10{,}000\left[-\log_e(102 - s)\right]_0^{50} - 100\left[102 - s - 102\log_e(102 - s)\right]_0^{50}$$

which gives

$$U = +4870 \text{ ft-lb}$$

PROBLEMS

1039. A body weighing 90 lb is dragged along a horizontal plane by a constant force of 25 lb, which acts upward at a constant inclination of 4 (horizontal) to 3 (vertical). The coefficient of kinetic friction is 0.2. Calculate the work done by each of the external forces acting on the body, while the body moves 30 ft. *Ans.* $U_W = 0$; $U_N = 0$; $U_{25} = +600$ ft-lb; $U_F = -450$ ft-lb.

1040. A block slides along a straight path on a horizontal plane. A horizontal force acts on the block in the direction of the motion. The force remains horizontal, but its magnitude varies in accordance with the law $P = \sqrt{s}$, in which P is in pounds and s is in feet. Calculate the work done on the body by the force in the interval during which s varies from +16 to +81 ft.

1041. In Prob. 1034, calculate the additional work that would be necessary to stretch the spring an additional 2.5 in. Solve by integration, using Eq. 283, and check by the shorter method, as developed in Prob. 1035. *Ans.* +131 in-lb.

1042. The working component of a certain force varies in accordance with the law $P \cos \theta = s^{3/2}$, in which P is in tons and s is in inches. Calculate the work done by the force in the interval during which the value of s varies from 4 to 16 in.

1043. An airplane executes a circular loop in a vertical plane. Prove that the work done by the weight of the plane, during the interval in which the plane moves from the bottom to the top of the loop, is equal to $-WD$, in which W represents the weight of the plane and D represents the diameter of the loop. With due regard to signs, how much work would be done by gravity while the plane made a complete loop?

1044. In Prob. 1037, Fig. 492, calculate the work that would be done by P, if the end of the spring were moved from E to O.

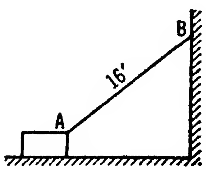

Fig. 494

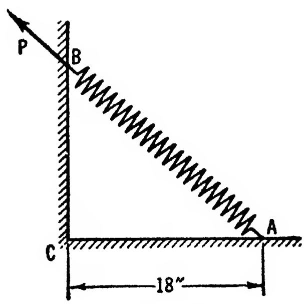

Fig. 495

1045. Figure 494 represents a body being drawn along a horizontal plane by a rope, AB. The rope is 16 ft long. Its lower end is attached to the body at A, and its upper end, B, moves vertically up a wall. A tension having a constant magnitude of 100 lb is maintained in the rope. Calculate the work done on the body by this force while the point A moves from a position 15 ft from the wall to a position 5 ft from the wall. *Ans.* +625 ft-lb.

1046. Figure 495 represents a helical spring having a modulus of 10 lb per in. The lower end is fixed to the floor at A. The upper end, B, is caused to move vertically up a wall, by a force P whose line of action coincides with the axis of the spring. The normal length of the spring is 15 in. Calculate the work done on the spring, by P, while B moves from a position 6 in. above C to a position 18 in. above C.

1047. The formula $e = Pl/aE$ is a very important one in strength of materials. It gives the elongation, or shortening, of any prismatic bar of elastic material, when subjected to a steady axial load, P, not large enough to stress the material beyond the elastic limit. Calculate the work that would be done on a 3 by 3 by $\frac{1}{2}$ in. angle, 12 ft long, by an axial load, gradually applied from zero up to a final magnitude of 40,000 lb. The modulus of elasticity, E, may be assumed to be 30,000,000 lb per sq in., and the cross-sectional area, a, of the angle is 2.75 sq in. Use pounds and inches throughout. *Ans.* 1400 in-lb.

1048. The formula $f = Pl^3/48EI$ gives the deflection at the center of a simply supported beam, caused by a steady load applied at that point. In the formula, P is the load in pounds, l is the length of the beam in inches, E is the modulus of elasticity of the material in pounds per square inch, and I is the moment of inertia of the cross-sectional area of the beam about a horizontal axis in the plane of the section and passing through its center of gravity. Assume a Douglas fir beam, 7.5 by 11.5 in. in section, and 115 in. long, the modulus of elasticity being 1,600,000 lb per sq in. Calculate the work done in gradually applying a load at the center of the beam, reaching a final value of 2000 lb.

• **1049.** A small body moves in a circular path whose radius is r. A force, P, having a constant magnitude but a variable inclination acts on the body. The inclination of P varies in such a manner that the line of action always passes through a fixed point on the circular path. Prove that the work done by P while the body moves halfway around the circle, starting from the fixed point, is equal to $\pm 2Pr$, the sign depending on the sense of P.

174. Work Done by a Force with a Constant Working Component. If the working component of the force is constant in magnitude and in sense during a given interval, Eq. 283, of Art. 173, may be simplified, as follows:

$$U = \int P \cos\theta \, ds = P \cos\theta \int_{s_1}^{s_2} ds = P \cos\theta \, (s_2 - s_1) \qquad [284]$$

The foregoing formula shows that *the work done by a force whose working component is constant may be calculated by multiplying the magnitude of the working component by the distance traversed by the particle on which the force acts.*

The most common case of a force with a constant working component is that in which the particle moves in a straight line, and in which the magnitude of the force and its inclination to the path are both constant. It is possible, however, for P and θ to remain constant while the particle moves in a curved path. It is also possible for the magnitude of the working component, $P \cos\theta$, to remain constant, with P and θ variable, and with the path of the particle either straight or curved.

PROBLEMS

1050. An elevator weighing 4 tons is being raised by means of a cable in which there is a constant tension of 4.2 tons. Calculate the work done on the elevator by the tension in the cable, and by gravity, in a distance of 60 ft. *Ans.* +252 ft-tons; −240 ft-tons.

1051. A locomotive exerts a constant drawbar pull of 60,000 lb, and thereby accelerates a train at the rate of 1.25 mi per hr per sec. Calculate the quantity of work done on the train by the locomotive, during the interval while the speed changes from 30 to 70 mi per hr.

1052. A belt passes halfway around a 36-in. pulley. The tension in the belt at the point where it leaves the pulley has a constant value of 75 lb. At the point where the belt goes onto the pulley the tension is 15 lb. Calculate the work done on the pulley by each of these pulls in one minute, while the pulley rotates at a constant speed of 1000 rpm.

1053. A body weighing 322 lb slides down an incline whose slope is 4 (horizontal) to 3 (vertical). The angle of kinetic friction is 14°. Calculate the work done on the body by gravity, and by the reaction of the plane, during the first 10 sec, starting from rest. *Ans.* +125,000 ft-lb; −41,500 ft-lb.

1054. The average pressure in the cylinders of a certain six-cylinder automobile engine, during the working stroke, is 100 lb per sq in. The cylinders are $3\frac{1}{4}$ in. in diameter, and the stroke is $4\frac{1}{8}$ in. Each piston has one working stroke during two revolutions of the crankshaft. Calculate the work done in all the cylinders, per second, by the gas, if the crankshaft rotates at a constant speed of 3000 rpm.

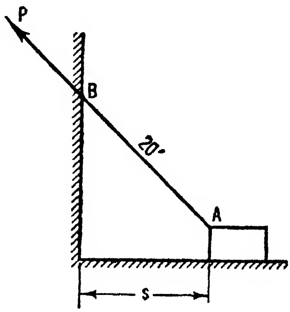

FIG. 496

1055. The body shown in Fig. 496 is drawn along a horizontal plane by the cord, AB. One end of the cord is attached to the body at A, and the other end moves up a vertical wall. The tension, P, in the cord, varies in accordance with the law $P = 200/s$, in which P is in pounds and s is in feet. The cord is 20 ft long. Calculate the work done by P while the body moves 10 ft. *Ans.* +100 ft-lb.

175. Total Work. If a given force does both positive and negative work during an interval, the total work done by the force during the interval is the algebraic sum of the various quantities of positive and negative work. *The algebraic sum of the quantities of work done by several forces during a given interval is called the total work of these forces for the interval.* Thus it is seen that even though the forces have different inclinations in space, the total work done by them is calculated by adding the various amounts algebraically, and not vectorially. Work is treated, therefore, as a scalar, and not a vector, quantity, and so it does not have such attributes as inclination and sense.

Equation 283, of Art. 173, gives the total work done by the force, P, during any interval, if proper attention is paid to the limits for the integration, and to the algebraic signs of the various functions.

Illustrative Problem

1056. Figure 497 represents a body weighing 50 lb, resting on a horizontal plane. A helical spring, having a modulus of 4 lb per in., is attached to the body at A, at an angle of 15° with the horizontal. A force P is gradually applied to the spring, its magnitude being slowly increased until the body begins to move. At this instant P becomes constant. Assume that the coefficient of static friction and the coefficient of kinetic friction are equal, and that their value is 0.2. Calculate the total work done by all the external forces, from the instant when P is first applied until the body has moved 6 in.

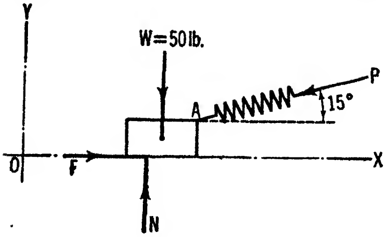

FIG. 497

Solution. For the instant when motion of the body is impending,

$$\Sigma F_x = 0 \qquad -P\cos 15° + F = 0$$

$$\Sigma F_y = 0 \qquad -P\sin 15° + N - 50 = 0$$

$$F = \mu N \qquad F = 0.2\,N$$

The solution of these equations gives

$$P = 10.9 \text{ lb} \qquad F = 10.6 \text{ lb} \qquad N = 52.8 \text{ lb}$$

Work Done by P before the Body Moves. The force P compresses the spring, and does a certain amount of work before the body begins to move. The initial value of P is zero. The final value is the value calculated above, 10.9 lb. The amount by which the spring is shortened is 10.9/4 = 2.73 in. By the method of Prob. 1035, Art. 173,

$$U_P = \frac{0 + 10.9}{2}\,2.73 = +14.9 \text{ in-lb}$$

Work Done by P after the Body Begins to Move. P remains constant after the body begins to move.

$$U_P = (P\cos\theta)\,s = (10.9\cos 15°)6 = +63.2 \text{ in-lb}$$

Work Done by F.

$$U_F = -(10.6 \times 6) = -63.6 \text{ in-lb}$$

Total Work. Obviously, the forces W and N do no work. The total work is as follows:

$$U = +14.9 + 63.2 - 63.6 = +14.5 \text{ in-lb}$$

PROBLEMS

1057. A locomotive exerts a constant drawbar pull of 52,000 lb in hauling a 900-ton train up a 2 per cent grade. The train resistance is equal to 12 lb per ton of weight. Calculate the total work done by all the external forces acting on the train in a distance of $\frac{1}{2}$ mile. *Ans.* +1.3 ton-miles.

1058. A body weighing 300 lb slides down an incline whose slope is 3 (horizontal) to 4 (vertical). A constant force of 200 lb is applied horizontally to the body, in such a manner as to oppose the motion. The coefficient of kinetic friction is 0.2.

Calculate the total work done by the external forces during a displacement of 100 ft.

1059. A certain automobile weighing 3200 lb is driven up a 6 per cent grade at a constant speed of 20 mi per hr. The constant propulsive force necessary is equal to 250 lb. Calculate the work done by each of the external forces acting on the automobile during a displacement of 100 ft. Calculate the total work. *Ans.* Work of tractive effort = +25,000 ft-lb; work of gravity = −19,200 ft-lb; work of frictional resistances = −5800 ft-lb; total work = 0.

1060. A cylinder 3 ft in diameter, weighing 350 lb, is rolled up a 30° inclined plane by means of a cord which is wound on the cylinder. The cord is pulled in a direction parallel to the incline, under a constant tension of 200 lb. The cylinder does not slip. Calculate the total work done by the external forces, assuming that the frictional force between the cylinder and the plane does no work, while the center point moves 16 ft up the slope.

1061. A body weighing 75 lb rests on a horizontal plane. A helical spring having a modulus of 5 lb per in. is attached to the body in a horizontal position. The coefficient of friction is 0.25 for both static and kinetic conditions. A force is gradually applied to the spring until the body begins to move. The force then becomes constant. Calculate the total work done by all the forces up to the instant when the body has moved 15 in. *Ans.* +35.2 in-lb.

1062. A body weighing 250 lb is pulled along a horizontal plane by a constant horizontal force of 100 lb. The coefficient of kinetic friction is 0.2. A helical spring is attached horizontally to the rear end of the body. The rear end of the spring is attached to a post. The modulus of the spring is 2 lb per in., and the normal length is 18 in. Calculate the total work done on the body by the external forces during a displacement of 6 in., if the spring has an initial elongation of 8 in. at the beginning of the interval.

176. Work Done by a Constant Force. *The total work done during any interval by a force that remains constant in magnitude, inclination, and sense is equal to the magnitude of the force multiplied by the length of the straight line joining the initial and final positions of the particle on which the force acts, and by the cosine of the angle between the force and this line.*

This principle is valid, regardless of the nature of the path traversed by the particle in moving from the initial to the final position; therefore, the working component of the force is not necessarily constant.

Proof. Let A, in Fig. 498, represent a particle moving in the curved path, A_1AA_2. Let A_1 and A_2 represent the initial and final positions of the particle. Let P represent the constant force. Let the x-axis be placed, for convenience, parallel to P. Let x represent the x-coordinate of the moving particle at any instant. Let x_1 and x_2 represent the initial and final values of x. Let θ represent the angle between the line of action of P and the tangent to the path, at any instant. Throughout the motion P remains parallel to the x-axis; therefore,

$$\cos\theta = \frac{dx}{ds} \qquad [285]$$

Substituting this value of $\cos\theta$ in Eq. 283, of Art. 173,

$$U = \int P \frac{dx}{ds}\, ds = \int P\, dx = P \int_{x_1}^{x_2} dx = P\,(x_2 - x_1) \qquad [286]$$

The quantity $(x_2 - x_1)$ in Eq. 286 is equal to the distance A_1A_2 multiplied by the cosine of the angle between A_1A_2 and the line of

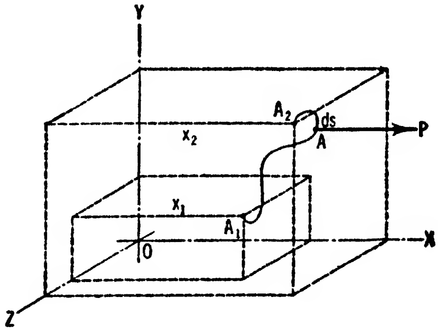

Fig. 498

action of P, as can be seen from a consideration of the figure. Thus, it is seen that the total work done by P for the interval during which the particle moves in any path from A_1 to A_2 is the same as if the particle moved in a straight line from A_1 to A_2.

The total work done by P, Fig. 498, is positive, because of the fact that the sense of the force is toward the right and the final position is to the right of the initial position. In general, if the final position is on that side of the initial position toward which the force acts, the work is positive, and if on the opposite side, the work is negative.

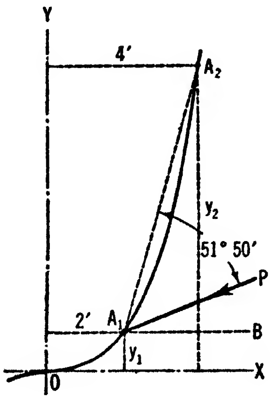

Fig. 499

Illustrative Problem

1063. Figure 499 represents a cubical parabola, whose equation is $4y = x^3$, in which x and y are expressed in feet. A particle moves upward along the curve. Throughout the motion a constant force, P, having a magnitude of 30 lb and inclined as shown, is applied to the particle. Calculate the work done by P while the particle moves from A_1 to A_2.

Solution. Substituting the values $x_1 = 2$ and $x_2 = 4$, in the equation of the curve,

$$4\,y = x^3 \qquad 4\,y_1 = (2)^3 \qquad y_1 = 2$$
$$4\,y_2 = (4)^3 \qquad y_2 = 16$$

From the figure,

$$\overline{A_1A_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\overline{A_1A_2} = \sqrt{(4 - 2)^2 + (16 - 2)^2} = \sqrt{200} = 14.1 \text{ ft}$$

By the principle stated at the beginning of the article,

$$U_P = -P\,(\overline{A_1A_2}) \cos \theta = -30 \times 14.1 \cos 51^\circ 30' = -261 \text{ ft-lb}$$

The work done by P is negative, because of the fact that the final position, A_2, is on the opposite side of A_1 from that toward which P acts.

PROBLEMS

1064. A certain force has a constant magnitude of 4 tons. The force is, and remains, parallel to the y-axis, and downward. During a certain interval the point of application of the force moves from the point $(+9', -12')$ to the point $(-6', +8')$, by any path. Calculate the work done by the force. How much work will be done if the force is parallel to the x-axis and toward the left? Also solve for the case in which the force is inclined upward and toward the left at a slope of 3 (horizontal) to 4 (vertical). *Ans.* -80 ft-tons; $+60$ ft-tons; $+100$ ft-tons.

1065. A certain constant force of 100 lb has a horizontal line of action and is directed toward the right. The point of application moves around a 20-ft circle in a vertical plane, clockwise. How much work is done by the force while the point of application moves from the "six o'clock" position on the circle to the "10 o'clock" position? How much work is done while the point moves from "1 o'clock" to "5 o'clock"?

1066. Calculate the total work done by the force in Prob. 1065 while the point of application makes $3\frac{7}{12}$ revolutions around the circle, clockwise, starting from the "4 o'clock" position. *Ans.* -1370 ft-lb.

1067. A certain constant force has a magnitude of 250 lb, and is vertical and downward. The path of the point of application is the cubical parabola whose equation is $4y = x^3$, in which x and y are in feet. During a certain interval the point of application moves from the origin to the point whose abscissa is $+8$ ft. Calculate the work done by the force.

1068. If the force in Prob. 1067 were inclined downward and toward the right, at a slope of 3 (horizontal) to 4 (vertical), how much work would it do? Let the other data of the problem remain unchanged. *Ans.* $-24{,}400$ ft-lb.

1069. A particle moves in a vertical plane, its path being the ellipse whose equation is $4x^2 + 16y^2 = 64$, in which x and y are in feet. Acting on the particle is a constant force of 200 lb, whose line of action is at a slope of 1 (horizontal) to 2 (vertical), and whose sense is downward and toward the left. Calculate the work done by this force while the particle moves from the point whose abscissa is $+4$ ft to the point whose ordinate is $+2$ ft. Also calculate the work done while the particle moves from the point whose ordinate is $+2$ ft to the point whose abscissa is -4 ft. The motion is counterclockwise in each case.

177. Work Done by Distributed Forces. All finite forces are more or less *distributed*, but are usually assumed to be *concentrated* when the area of contact between the two bodies is comparatively small. The work done by a force which is assumed to be concentrated is calculated by the methods of the preceding articles, the entire force being considered to be applied to a single particle within the area of contact.

A distributed force which is not to be assumed concentrated may be considered to consist of elementary forces, each of which is applied to a single particle. The work done by the distributed force during any interval is equal to the algebraic sum, or integral, of the quantities of work done by all the elementary forces of which the distributed force is composed. Frequently the elementary forces are parallel, and the particles move equal distances in paths which are alike. In such a case, the work done by the distributed force may be calculated as if the force were concentrated on any one of the particles.

178. Work Done by Gravity. The work done by the weight of a body is an important example of the work of a distributed force. In this case the elementary forces are parallel, but the paths of the particles are not necessarily alike. When a body moves, the earth-pull does work on all the particles, except those which may happen to move in horizontal planes.

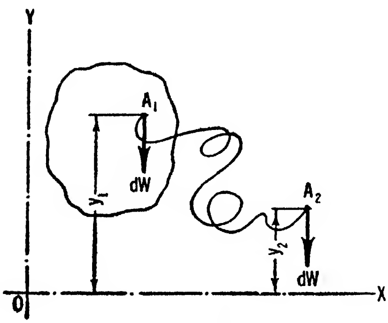

Fig. 500

The total work done by the weight of a body during any interval is equal to the total weight multiplied by the difference in elevation of the initial and final positions of the center of gravity of the body. If the final position of the center of gravity is at a lower elevation than the initial, the work done by the weight is positive; if higher, the work is negative. This principle applies to any group of particles or bodies, whether they constitute a rigid body or not.

Proof. Let Fig. 500 represent any body, whether rigid or non-rigid. Let A_1 represent the initial position of one of the particles, and let A_2 represent the final position of that particle. Let dW represent the weight of the particle. Let a set of fixed coordinate axes be selected, with the y-axis in a vertical position. It is not necessary to the proof to show the z-axis. By Art. 176, the work done by dW is equal to $dW(y_1 - y_2)$, no matter what path the particle traverses in moving from A_1 to A_2. The total work done by the weights of all the particles is, then, as follows:

$$U = \int dW\,(y_1 - y_2) = \int y_1\,dW - \int y_2\,dW \qquad [287]$$

In Eq. 287 the expression $\int y_1\,dW$ may be replaced by $\bar{y}_1 W$, and $\int y_2\,dW$ may be replaced by $\bar{y}_2 W$, in which $\bar{y}_1$ and $\bar{y}_2$ represent the y-coordinates of the center of gravity of the body in its initial and final positions. Equation 287 thus becomes,

$$U = W\,(\bar{y}_1 - \bar{y}_2) \qquad [288]$$

which proves the principle stated above.

The foregoing facts provide a very simple method for the calculation of the work done by the weight of a body, when the elevation of the center of gravity of the body is changed. For example, if a quantity of water is pumped from a reservoir into a standpipe, the total work done by gravity on the water can be easily calculated by multiplying the weight of the water pumped during any interval, by the difference between the elevation of the center of gravity of the water after it is in the standpipe and its elevation in the reservoir. In this case the center of gravity has been raised, and the work done on the water by gravity is negative. If the same body of water were permitted to flow back to its former position in the reservoir, gravity would do an amount of positive work numerically equal to the negative work done by it in the first case.

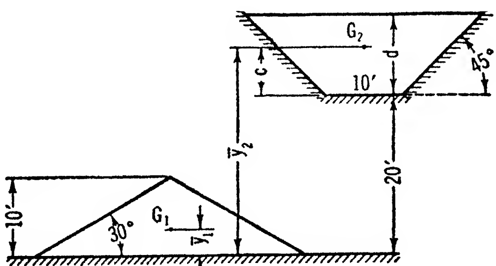

FIG. 501

ILLUSTRATIVE PROBLEM

1070. Figure 501 represents a quantity of coal, weighing 50 lb per cu ft, piled on the ground in a conical pile 10 ft high. The coal is elevated into a bin of trapezoidal cross section, the bottom of which is 20 ft above the ground. The bin is 15 ft long. Calculate the total work done by gravity on the coal, during the process.

Solution. From the figure, the radius of the base of the cone is equal to 10 cot 30° = 10 × 1.73 = 17.3 ft.

$$\text{Volume of cone} = \tfrac{1}{3}\pi\,(17.3)^2 10 = 3130 \text{ cu ft}$$

$$\text{Weight of coal} = 3130 \times 50 = 156{,}500 \text{ lb}$$

$$\text{Volume of coal in bin} = (10 + d)\,d \times 15 = 150\,d + 15\,d^2$$

Assuming that the coal occupies the same volume in the bin as it does on the ground,

$$150\,d + 15\,d^2 = 3130 \qquad d = 10.3 \text{ ft}$$

Let G_1 and G_2 represent the center of gravity of the body of coal in its initial and final positions. From Prob. 528, assuming the coal to be homogeneous, the depth of G_2 below the upper surface is as follows:

$$\frac{h\,(B + 2\,b)}{3\,(B + b)} = \frac{10.3\,(30.6 + 20)}{3\,(30.6 + 10)} = 4.28 \text{ ft}$$

$$c = 10.3 - 4.28 = 6.02 \text{ ft}$$
$$\bar{y}_2 = 20 + 6.02 = 26.0 \text{ ft}$$
$$\bar{y}_1 = \tfrac{1}{4} \times 10 = 2.5 \text{ ft}$$

By Eq. 288,

$$U = W\,(\bar{y}_1 - \bar{y}_2) = 156{,}500\,(2.5 - 26) = -3{,}680{,}000 \text{ ft-lb}$$

PROBLEMS

1071. Theoretically, the energy available from a waterfall is equal to the work that gravity does on the water as the water descends to a lower elevation. Calculate the theoretical energy available each second from a fall of 24 ft, if the discharge of the stream is 250,000 cu ft per sec. Assume that a cubic foot of water weighs 62.4 lb. *Ans.* 187,000 ft-tons.

1072. The hammer of a certain piledriver weighs 800 lb. It drops from a height of 18 ft and drives a pile through a distance of 1.5 in. Assuming that all the work done by gravity on the hammer is utilized in driving the pile, calculate the average force exerted by the hammer on the pile.

1073. A certain homogeneous cone weighs 80 lb. Its altitude is 12 in., and its base is 10 in. in diameter. It rests on its base on a horizontal surface. The cone is then tipped over onto its side without being lifted from the table. Calculate the positive work, the negative work, and the total work done by gravity. *Ans.* +189.5 in-lb; −226.5 in-lb; −37.0 in-lb.

1074. A rectangular hole, 8 by 10 ft, 6 ft deep, is dug in the earth. The dirt is elevated to a platform 6 ft above the surface, and is piled there in a conical pile having a slope of $1\frac{1}{2}$ (horizontal) to 1 (vertical). Calculate the total work done on the material, by gravity, during the process. The dirt weighs 80 lb per cu ft.

1075. A certain reservoir is rectangular in plan, 120 by 150 ft, with vertical sides. Water is pumped out of it into a cylindrical standpipe 18 ft in diameter. Before the pump is started the surface of the water in the standpipe is 100 ft above that in the reservoir. The pump is operated until the water level in the reservoir is lowered

1 ft. Calculate the work done by gravity during the operation. The water weighs 62.4 lb per cu ft. *Ans.* −76,300 ft-tons.

1076. A chain weighing 5 lb per lin ft hangs over a 3-ft pulley, the two ends of the chain being at the same elevation. Calculate the work done on the chain by gravity while the pulley is turned through 10 rev, assuming that the chain does not slip.

1077. Figure 502 represents a skip, *A*, being hoisted by a cable which passes over a pulley, *B*, and is wound on a drum, *C*. The skip weighs 1850 lb, and the cable weighs 1.44 lb per lin ft. Calculate the total work done on the skip and cable, by gravity, while the skip is raised from a point 3000 ft below *B*, to a point 500 ft below *B*. *Ans.* −5320 ft-tons.

1078. Two equal, homogeneous hemispheres rest "side by side" on a horizontal floor. The plane faces are uppermost, and horizontal. A man desires to move them through a short distance. He rolls the first one until it rests on its edge, and then lets it fall onto its plane face, in such a manner that no slipping occurs. He decides to drag the second one, without tipping it at all, and using a horizontal force. When he finishes, the hemispheres are again side by side, but one, of course, has been inverted. The amount of work that the man does against gravity in the one case is equal to the amount of work that he does against friction in the second case. Calculate the coefficient of kinetic friction.

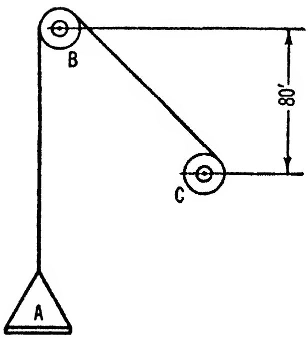

Fig. 502

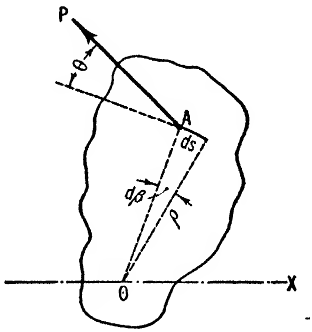

Fig. 503

179. Work in Terms of Torque. The work done by a force acting on a rotating body can be expressed in a simple manner in terms of the moment of the force about the axis of rotation, and the angular distance described by the body during any given interval.

Let Fig. 503 represent a body rotating about the *z*-axis, which passes through *O* at right angles to the plane of the paper. Let *A* represent one of the particles of the body, and let *P* represent a force which is applied to the body at *A* during the given interval. Let *ds* represent the distance through which *A* moves while the body rotates through a small angle, $d\beta$. Let ρ represent the radius of rotation of *A*.

By Eq. 283, Art. 173, $U = \int P \cos\theta \, ds$. From the figure, $ds = \rho \, d\beta$. Therefore, $U = \int P \cos\theta \, \rho \, d\beta$. The expression $(P \cos\theta)\,\rho$, in this equation, represents the moment of P about the axis of rotation. Let this be represented by M_z. Therefore,

$$U = \int M_z \, d\beta \qquad [289]$$

Frequently the torque, M_z, of the force, about the axis of rotation, remains constant throughout a given interval. In such a case, M_z may be placed outside the integral sign. Let β represent the total angular distance described by the body during any interval in which the force has a constant torque about the axis of rotation. It follows that

$$U = M_z\beta \qquad [290]$$

From the nature of the derivation of Eqs. 289 and 290, it is clear that the angle β must be expressed in radians before its value is substituted in the formulas.

PROBLEMS

1079. A certain automobile engine exerts a constant torque of 250 ft-lb on its crankshaft, at 2000 rpm. Calculate the work done on the crankshaft, per second. *Ans.* 52,400 ft-lb.

1080. A certain lineshaft is rotating at a speed of 360 rpm. The shaft is transmitting 100 ft-tons of work per second. Calculate the constant torque to which the shaft is subjected.

1081. A constant force of 60 lb acts tangentially on a pulley 30 in. in diameter. How much work will the force do on the pulley while the latter rotates through an angle of 325°? Solve by two methods. *Ans.* 425 ft-lb.

1082. A 36-in. pulley, partially encircled by a belt, runs at a constant speed of 360 rpm, and receives 600,000 ft-lb of work per minute from the belt. Calculate the difference between the tensions in the two parts of the belt.

1083. A certain shaft is clamped at one end, and a torque is applied gradually to the other end. The torque increases in accordance with the following law: $M_z = 1{,}260{,}000\beta$, in which M_z represents the torque about the axis of the shaft, in inch-pounds, and β represents the angle of twist, in radians. Calculate the work done on the shaft up to the instant when the angle of twist is 2°. *Ans.* +768 in-lb.

1084. A certain wheel is subjected to a torque that varies in accordance with the law $M_z = \sqrt{\beta}$, in which M_z is in foot-pounds and β is in radians. Calculate the work done on the wheel in the interval during which β varies from 0 to 9 radians.

180. Work Done by Frictional Forces. A static frictional force may, or may not, do work, depending on the conditions. It was explained in Art. 76 that static friction may exist between two bodies which are in motion, as long as there is no relative motion, or sliding, at the surface of contact. Therefore, since static friction may act on a body in motion, it may do work.

Ordinarily, when static frictional forces do work, that fact is quite obvious, and the method of calculating the work done is the same as for any other kind of force. For example, let a box be imagined resting on the level floor of a freight car, but not in contact with any object other than the floor of the car. As long as the car is at rest, or is moving with constant velocity, the car floor does not exert any frictional force on the box. If the car is given an acceleration whose magnitude is not sufficiently great to cause the box to slide on the floor, the floor will exert a static frictional force on the bottom of the box. The work done on the box by this force is calculated in the usual manner.

The Case of the Rolling Body. There is a certain case of static friction in which the frictional force does no work, but in which the conditions are such that it is very easy to make the mistake of concluding that it does. This is the case in which a wheel, cylinder, sphere, or other round object rolls with an accelerated motion, but without slipping, on a track, roadway or other surface. Under these conditions static friction exists between the rolling body and the supporting surface. *This static friction does no work on the rolling body.* Any given particle on the periphery of the rolling body is acted on by the static frictional force only for an instant as it comes into contact with the supporting surface, and at that instant the particle has zero velocity and is, therefore, stationary. The frictional force does not fulfil the condition under which work is done; that is, it does not have a component in the direction of the motion of the particle to which it is applied, during some interval of that motion.

The foregoing statements regarding the rolling body are made with the rigid body in mind. Actually a rolling body encounters more or less rolling resistance, and kinetic friction undoubtedly exists at certain points in the surface of contact. This kinetic friction does a certain amount of negative work on the rolling body, thus dissipating a portion, if not all, of the energy of that body. Naturally, this work is disregarded in all problems in which rolling resistance is disregarded.

Kinetic Friction. In the case of kinetic frictional forces, work is always done on one, and frequently on both, of the bodies between which these forces act. If a body A exerts a kinetic frictional force on a body B, body B must exert an equal and opposite kinetic frictional force on A. If the particles of body A remain stationary, work is done only by the force that A exerts on B. If the particles of both bodies move work may be done on both A and B.

As was indicated in the foregoing paragraph, if two bodies, A and B, are in sliding contact, they exert equal and opposite kinetic frictional forces on each other. It can be shown that the total work done by these two forces during any interval is equal to the magnitude of either, multiplied by the distance through which one body moves relative to the other, at the surface of contact. This distance is sometimes called the *distance of sliding*.

Illustrative Problem

1085. A certain automobile, equipped with four-wheel brakes, is brought to rest from a speed of 60 mi per hr, in a distance of 400 ft. It is estimated that the brakes must do 400,000 ft-lb of work in accomplishing the purpose. The tires are 29 in. in diameter, and the brake-drums are 12 in. in diameter. Assume that the four brakes share equally in the work of stopping the car, and calculate the average kinetic frictional force that each brake must exert on its drum.

Solution. The number of revolutions described by each wheel, while the car travels 400 ft, is as follows:

$$n = \frac{400}{\pi \frac{29}{12}} = 52.7$$

The distance described by each brake-shoe, relative to its drum, is,

$$s = 52.7 \left(\pi \tfrac{12}{12}\right) = 166 \text{ ft}$$

Let F represent the average kinetic frictional force exerted by each brake-shoe on its drum, during the interval.

$$166\, F = \tfrac{1}{4} \times 400{,}000 \qquad F = 602 \text{ lb}$$

PROBLEMS

1086. A body weighing 322 lb rests on the horizontal floor of a car. The car is drawn along a horizontal track with a constant acceleration of 1.5 ft per sec per sec. Calculate the work done on the body by the frictional force exerted on it by the car floor, in the interval during which the speed of the car changes from 10 to 30 ft per sec. Assume that the body does not slide on the car floor. *Ans.* +4000 ft-lb.

1087. In order to bring a certain rotating drum to rest, from a speed of 120 rpm, it is necessary to do 3000 ft-lb of work. The brake-wheel is 6 ft in diameter, and the axle shaft of the drum is 4 in. in diameter. The total frictional force acting on the axle is 25 lb. Calculate the frictional force that the brake-shoe must exert on the brake-wheel, in order to bring the drum to rest in 4 sec.

1088. A brake-shoe exerts a normal pressure of 2400 lb on the wheel of a car. The coefficient of kinetic friction for the shoe is 0.2. Calculate the work done on the wheel by the shoe while the car travels $\frac{1}{2}$ mi. Assume that no slipping occurs between the wheel and the track. *Ans.* −1,270,000 ft-lb.

1089. A cylinder 2 ft in diameter, weighing 500 lb, is placed on an inclined plane having a slope of 4 (horizontal) to 3 (vertical). A wire is wrapped around the cylinder, coming off near the highest point, in a direction parallel to the incline, and upward. A constant pull of 160 lb, parallel to the slope, is applied to the wire, causing the cylinder to roll up the slope, without slipping. Calculate the work done by each external force while the center of the cylinder moves 50 ft.

1090. A certain freight car, weighing 80,000 lb, has eight wheels, with one brake-shoe for each wheel. The kinetic energy of the car, at 30 mi per hr, is 1200 ft-tons.

In stopping the car the brake-shoes must do work equal in amount to the kinetic energy of the car, if other frictional resistances are disregarded. Assuming the coefficient of kinetic friction for the brake-shoes to be 0.15, calculate the average normal pressure necessary at each shoe, in order to stop the car in a distance of 1000 ft.

1091. A box weighing 644 lb rests on the floor at the front end of a car. The car is accelerated at the rate of 2 ft per sec per sec up a 4 per cent grade, starting from rest. While the car is moving a distance of 225 ft, the box slides to the rear end of the car. The distance, measured on the car floor, between the initial and final positions of the box, is 25 ft. Calculate the work done on the box by the kinetic frictional force, and by gravity. Calculate the total work done on the box. Calculate the coefficient of kinetic friction. *Ans.* $U_F = +12{,}300$ ft-lb; $U_W = -5150$ ft-lb; $U_T = +7150$ ft-lb; $\mu = 0.095$.

181. Work Done by Two Collinear, Equal, and Opposite Forces. The total work done by two forces that are collinear, equal in magnitude, and opposite in sense, during a given interval, can be ascertained by calculating separately the amount of work done by each force, and by

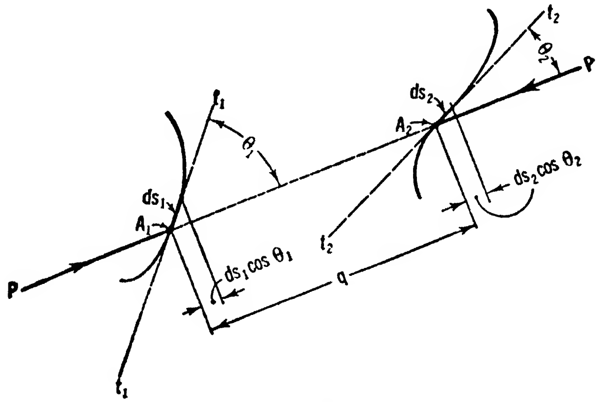

FIG. 504

adding these amounts algebraically. In most cases, however, the result can be obtained more readily by means of the formula to be derived below.

In Fig. 504, let PP represent the two collinear, equal, and opposite forces, applied to the particles A_1 and A_2, and directed toward each other, as shown. Let q represent the value, at any given instant, of the variable distance between the two particles. Let ds_1 and ds_2 represent elementary portions of the paths of the two particles. By Eq. 283, Art. 173, the total work done by the two forces during the elementary interval is as follows:

$$dU = P \cos\theta_1 \, ds_1 - P \cos\theta_2 \, ds_2 = P\,(ds_1 \cos\theta_1 - ds_2 \cos\theta_2) \quad [291]$$

It can be seen, from the figure, that the expression $(ds_1 \cos \theta_1 - ds_2 \cos \theta_2)$ represents the change that occurs in the value of q during the elementary interval. If the expression is positive, however, q is decreased, and if negative, q is increased. Therefore, the expression may be replaced by $(-\,dq)$. The formula for the total work done by the two forces during any finite interval may, then, be written as follows:

$$U = -\int P\,dq \tag{292}$$

Equation 292 is general for any pair of forces related to each other as specified above, no matter what paths the particles may traverse, or how the forces may vary. The formula was derived for two forces directed toward each other, and gives the correct sign for the work done in such a case, provided that the limits for the integration are correctly assigned.

For the case in which the two forces are directed away from each other the formula will be as follows:

$$U = +\int P\,dq \tag{293}$$

ILLUSTRATIVE PROBLEMS

1092. Figure 505 represents a helical spring whose modulus is 8 lb per in., and whose normal length is 16 in. The spring is subjected to a pair of equal,

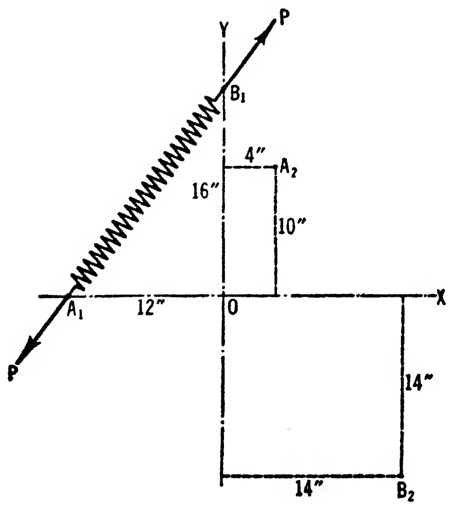

FIG. 505

opposite, and collinear forces, PP, acting along its axis, as shown. The figure shows the spring in its initial position, A_1B_1. The end A_1 moves to the final position A_2, and B_1 moves to B_2, the forces PP maintaining the same relation to each other, and to the axis of the spring, throughout the movement. Calculate the total work done on the spring by the two forces, regardless of the paths traversed by A and B in moving from their initial to their final positions. Assume that the elastic limit of the material is not exceeded at any point.

Solution. In the present problem the two forces are directed away from each other. Therefore, Eq. 293 applies.

$$U = +\int P\,dq$$

Since q represents the distance between the two particles to which the forces are applied, it also represents the length of the spring at any instant during the motion. Let q_1 and q_2 represent the initial and final values of q. From the figure,

$$q_1 = \sqrt{(12)^2 + (16)^2} = 20 \text{ in.}$$

$$q_2 = \sqrt{(14 - 4)^2 + (14 + 10)^2} = 26 \text{ in.}$$

The normal length of the spring is 16 in., and the modulus is 8 lb per in. Therefore,

$$P = 8\,(q - 16)$$

Substituting this value of P in the original equation, and integrating,

$$U = +\int_{q_1}^{q_2} 8\,(q - 16)\,dq = 8\left[\frac{q^2}{2}\right]_{20}^{26} - 8\left[16\,q\right]_{20}^{26} = +336 \text{ in-lb}$$

It is seen that in the foregoing solution it was not necessary to know what paths were traversed by the ends of the spring in moving from their initial to their final positions. This is true in any case in which Eq. 293 or Eq. 292 is used, provided that the magnitude of P depends only on q.

1093. Solve Prob. 1092 by the use of the average value of the force P.

Solution. In the solution of Prob. 1092 it was learned that the work done on the spring by the two forces did not depend on the paths followed by the ends of the spring in passing from the initial to the final positions. Therefore, the problem may be solved by the method of Prob. 1035, as though one end of the spring were held stationary, and the other end were moved along the axis of the spring. Using the values of q_1 and q_2 obtained in Prob. 1092,

$$U = +\frac{8\,(20 - 16) + 8\,(26 - 16)}{2}\,(26 - 20) = +336 \text{ in-lb}$$

PROBLEMS

1094. At a certain instant the points of application of two forces occupy the positions (+3″, +8″) and (−4″, −16″). At a later instant the points of application have shifted to the positions (−5″, +9″) and (+3″, −6″). The two forces are collinear, equal, and opposite, throughout the motion, and each has a constant magnitude of 12 lb. They are directed outward. Calculate the total work done by the pair during the interval. *Ans.* -96 in-lb.

1095. A helical spring whose modulus is 0.25 lb per in. has a normal length of 12 in. The ends of the spring are placed at the points whose coordinates are $(-3'', -8'')$ and $(-11'', +7'')$. The ends are then shifted to the points $(-3'', +6'')$ and $(+2'', +18'')$. Calculate the total work done on the spring by the pair of collinear and equal forces acting at its ends, during the interval.

1096. Solve Prob. 1037, Art. 173, by the method of the present article. *Ans.* $+124$ in-lb.

1097. Solve Prob. 1046, Art. 173, by the use of Eq. 293, and check by the method of Prob. 1093.

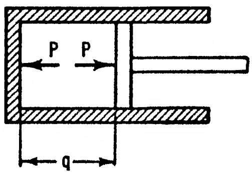

Fig. 506

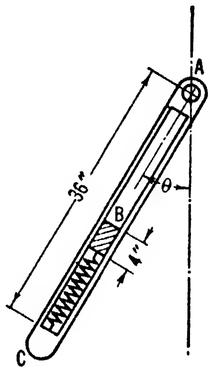

Fig. 507

1098. At a certain instant the points of application of two forces occupy the positions $(+3', +5')$ and $(-5', -10')$. At a later instant the two points have assumed the positions $(+15', -7')$ and $(+3', +9')$. The two forces are collinear, equal, and opposite, and are directed inward, throughout the interval. The magnitude of each force varies in accordance with the law $P = 20/q^2$, in which q represents the distance, in feet, between the points of application at any instant, and P represents the magnitude of each force, in pounds. Calculate the total work done by the pair of forces during the interval. *Ans.* -0.176 ft-lb.

1099. Figure 506 represents the cylinder and piston of a steam engine. The two equal forces, PP, represent the pressure of the steam on the head of the cylinder, and on the piston. Assume that the total pressure, as the piston moves along the cylinder, follows the law $P = 15{,}000/q$, in which P is in pounds and q is in inches. Calculate the total work done in the cylinder by the steam, while q varies from 1.5 to 13.5 in., regardless of any motion that the engine itself may have during the interval.

1100. Figure 507 represents a bar, AC, rotating about an axis through A at right angles to the plane of the paper. A block, B, weighing 8.05 lb, fits into a longitudinal slot, enabling it to slide lengthwise of the bar. The outer end of the slot contains a helical spring whose modulus is 4 lb per in., and whose normal length is 18 in. The bar rotates at a constant speed of 4 rad per sec, clockwise. Calculate the work done on the spring while θ varies from 0° to 90°. Disregard friction, and the weight of the spring. *Ans.* -22.4 in-lb.

1101. Solve Prob. 1100, Fig. 407, for the case in which θ varies from 0° to 120°.

CHAPTER XXIII

POWER

182. Power in General. *Power is the rate at which work is done, with respect to time.* Let p represent the power of a given force at any instant. Let U represent the total work done by the force, measured from some convenient initial instant. The general formula for the power of the force is, then, as follows:

$$p = \frac{dU}{dt} \qquad [294]$$

Equation 294 gives the power of any force at any instant, whether the rate at which the force does work is variable or constant. In the special case in which the force does work at a constant rate, the formula reduces to

$$p = \frac{U}{t} \qquad [295]$$

Equation 295 expresses the fact that *when the force does its work uniformly with respect to time the power of the force is equal to the total work done during any interval divided by the length of the interval.* In any case Eq. 295 gives the average power of the force for the entire interval.

Units. The fundamental unit of power, in the English system, is the foot-pound per second. The unit most commonly used in engineering, however, is the horsepower. In the English system, one horsepower is equal to 550 ft-lb per sec, or 33,000 ft-lb per min.

The Continental horsepower is equal to 75 kilogram-meters per second. The electrical unit of power is the watt, and is equal to 10,000,000 ergs per second. The English horsepower is equivalent to 746 watts.

The foregoing conception of power is in accordance with the technical and exact meaning of the term as used in mechanics. In non-technical literature the term is used rather broadly, even in connection with mechanical subjects. For example, a large locomotive that can pull a heavy train is frequently referred to as having great power. In the technical sense, however, the power of this locomotive may be comparatively small. Many electric and hydraulic motors, and even steam engines and steam turbines, develop much greater power than the locomotive. This is because they work at higher speeds, and are thus able to perform more work in a given time, although they may not be capable of exerting continuous forces as great as those obtainable from the locomotive.

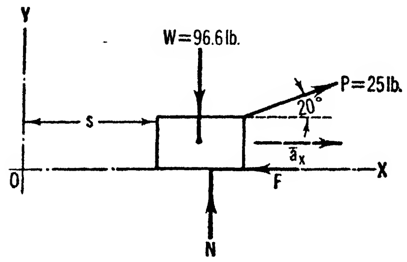

FIG. 508

ILLUSTRATIVE PROBLEMS

1102. Figure 508 represents a body, weighing 96.6 lb, being pulled along a horizontal plane by a constant force, P, of 25 lb, applied as shown. The coefficient of kinetic friction is 0.2. The initial velocity of the body is 4 ft per sec, toward the right. Calculate the power of each of the external forces acting on the body, at the initial instant. Calculate the power of each force at the instant when 10 sec have elapsed.

Solution. By Eqs. 184 and 185, Art. 126, and Eq. 32, Art. 81,

$$\Sigma F_x = \frac{W}{g}\bar{a}_x \qquad +25 \cos 20° - F = \frac{96.6}{32.2}(+\bar{a}_x)$$

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad +N + 25 \sin 20° - 96.6 = \frac{96.6}{32.2}(0)$$

$$F = \mu N \qquad F = 0.2\,N$$

The solution of these equations gives

$$\bar{a}_x = 1.96 \text{ ft/sec}^2 \qquad F = 17.6 \text{ lb} \qquad N = 88.0 \text{ lb}$$

Let s represent the distance traversed by the body since it passed O, a fixed point on the path. The work done by the force P since the body passed O is equal to $+(P \cos 20°)s$. By Eq. 294, the power of the force at any instant is as follows:

$$p_P = \frac{dU_P}{dt} = \frac{d\,(Ps \cos 20°)}{dt} = P \cos 20° \left(\frac{ds}{dt}\right) = (P \cos 20°)\,v$$

in which v represents the velocity of the body. At the instant when $v = 4$ ft/sec,

$$p_P = (25 \cos 20°)\,4 = 94 \text{ ft-lb/sec}$$

In a similar manner,

$$p_F = 17.6 \times 4 = 70.4 \text{ ft-lb/sec}$$

By Eq. 129, Art. 111, at the instant when 10 sec have elapsed, $v = v_0 + at = 4 + 1.96 \times 10 = 23.6$ ft/sec.

$$p_P = (25 \cos 20°)23.6 = 555 \text{ ft-lb/sec}$$
$$p_F = 17.6 \times 23.6 = 415 \text{ ft-lb/sec}$$

1103. The engine of a certain automobile delivers 80 hp at the flywheel, when the engine speed is 3000 rpm. Assume that the efficiency of the entire mechanism between the flywheel and the driving wheels, in high gear, is 90 per cent. The gear ratio at the differential is 4.5 to 1, and the tires are 29 in. in diameter. Calculate the constant force, or tractive effort, that the engine is capable of producing at the driving wheels, to propel the car under these conditions.

Solution. Let v represent the velocity of the car, in high gear. The value of v, with the engine rotating at 3000 rpm, can be calculated as follows:

$$\text{Angular velocity of wheels} = \frac{3000}{4.5} = 667 \text{ rpm} = 4190 \text{ rad/min}$$

By Eq. 211, Art. 150,

$$v = r\omega = \left(\frac{29}{2 \times 12}\right)4190 = 5060 \text{ ft/min}$$

$$\text{Power delivered at flywheel} = 80 \times 33{,}000 = 2{,}640{,}000 \text{ ft-lb/min}$$

$$\text{Power delivered at ground} = 0.90 \times 2{,}640{,}000 = 2{,}380{,}000 \text{ ft-lb/min}$$

Let P represent the tractive effort. In 1 min the work done by P is $P \times 5060$. Therefore,

$$P \times 5060 = 2{,}380{,}000 \qquad P = 470 \text{ lb}$$

PROBLEMS

1104. A certain electric locomotive exerts a drawbar pull of 30,000 lb on the train to which it is coupled, at a speed of 50 mi per hr. Calculate the horsepower being delivered to the train. *Ans.* 4000 hp.

1105. The discharge of a certain river is 160,000 cu ft per sec, and the height of a certain falls is 36 ft. Calculate the theoretical horsepower available. Assume that water weighs 62.4 lb per cu ft.

1106. The water passing through a certain hydraulic turbine has a total fall of 30 ft. The capacity of the turbine is 60 cu ft per sec, and the efficiency (Art. 191) is 80 per cent. Calculate the horsepower delivered by the turbine. Assume that water weighs 62.4 lb per cu ft. *Ans.* 163 hp.

1107. It is necessary to deliver 18 hp to a certain machine. The power is transmitted to the machine by a spur gear 12 in. in diameter, mounted on the shaft, and rotating at a speed of 120 rpm. Calculate the tangential force that must be exerted on the spur gear by the pinion.

1108. A given shaft transmits 80 hp at a speed of 360 rpm. Calculate the torque sustained by the shaft. *Ans.* 1170 ft-lb.

1109. A body weighing 0.5 ton is pulled up an inclined plane by a constant force parallel to the incline. The coefficient of kinetic friction is 0.25. The speed is 10 ft per sec, and is constant. It is found that it is necessary to deliver 4 hp to the body to satisfy the foregoing conditions. Calculate the angle of inclination of the plane.

1110. Power is supplied to a lineshaft by a belt running on a 24-in. pulley. The tension in the belt is 180 lb on the tight side and 12 lb on the slack side. The speed of the shaft is 360 rpm. Calculate the horsepower being transmitted by the belt to the shaft. *Ans.* 11.5 hp.

1111. In Prob. 878, Art. 155, calculate the horsepower being delivered to the centrifugal extractor by the motor at the instant just before the maximum speed is attained. Assuming, as in that problem, that the torque necessary to overcome the frictional resistances is 15 per cent of the total torque required during the accelerating period, calculate the horsepower that the motor must deliver to maintain the constant speed of 700 rpm. *Ans.* 15.2 hp; 2.28 hp.

1112. A cylinder 3 ft in diameter, weighing 250 lb, is rolled along a horizontal surface, without slipping. A wire is wrapped around the cylinder, coming off horizontally, at the top. A constant pull of 10 lb applied to the end of the wire serves to propel the cylinder. Calculate the horsepower being delivered to the cylinder at the instant when the velocity of the center point is 6 ft per sec.

1113. A certain internal-combustion engine has a bore of 10 in. and a stroke of 12 in. The average pressure on the piston during the power stroke is 100 lb per sq in, at a speed of 550 rpm. The engine is four-cycle, having one power stroke during two revolutions of the crankshaft. Calculate the horsepower developed in each cylinder by the expansion of the gas. *Ans.* 65.5 hp.

1114. In Prob. 453, Fig. 282, calculate the horsepower dissipated by the brake, if the drum is rotating at a speed of 240 rpm, assuming that the results obtained in the solution of Prob. 453 are valid for the present case.

1115. The normal pressure exerted on a certain car wheel by a brake-shoe is 5000 lb. The coefficient of kinetic friction for the rubbing surfaces is 0.15. The speed of the car is 60 mi per hr. Calculate the horsepower being dissipated by the brake at the given speed. *Ans.* 120 hp.

1116. A certain rectangular reservoir with vertical sides is 80 by 40 ft in plan, and contains water to a depth of 4 ft. This water is to be pumped into a cylindrical standpipe 16 ft in diameter. The bottom of the standpipe is at an elevation 100 ft above the bottom of the reservoir. It is desired to elevate the entire quantity of water in 15 min. Calculate the average horsepower necessary, disregarding friction losses.

1117. The skip of a certain mine hoist is raised to the surface of the ground, from a point 1000 ft below the surface, in 2 min. The cable by means of which the skip is raised weighs 4.62 lb per lin ft, and is wound on a drum whose axle shaft is 12 ft above ground. Calculate the work done solely in elevating the cable. Calculate the average horsepower expended for the same purpose. *Ans.* 1180 ft-tons; 35.8 hp.

1118. The S.A.E. formula for the brake horsepower of an automobile engine is as follows: bhp $= ND^2/2.489$, in which N represents the number of cylinders, and D represents the diameter of each cylinder, in inches. In the derivation of this formula the average effective pressure on each piston during its power stroke is assumed to be 90 lb per sq in., and the piston is assumed to travel at an average speed of 1000 ft per min. The efficiency of transmission between piston and flywheel (mechanical efficiency) is assumed to be 75 per cent. Check the derivation of the formula. It will be remembered that in the usual automobile engine each piston has one power stroke during two revolutions of the crankshaft.

1119. The air resistance against the front of a moving automobile may be calculated from the formula $P = CAv^2$, in which P represents the air resistance, in pounds, A represents the projected area of the front of the car, in square feet, and v represents the speed of the car, in miles per hour. The quantity C is a constant whose value depends on the type of car. In a certain car, $A = 12$ sq ft and $C = 0.002$. Calculate the horsepower utilized in overcoming air resistance at a speed of 20 mi per hr, for this car. Solve also for a speed of 60 mi per hr. Calculate the ratio of the larger value to the smaller. *Ans.* 0.512 hp; 13.8 hp; 27 to 1.

183. Measurement of Power. The rate at which work is done by the pressure of the steam, gas, or other fluid in the cylinders of an engine is usually expressed in horsepower, and is called the *indicated horsepower* of the engine. It is calculated from a diagram drawn on a card by a device called an *indicator*.

The horsepower that can be delivered by the crankshaft of an engine to a belt, gear train, or other device, or directly to a machine, is called the *brake horsepower* of the engine. Brake horsepower is measured by

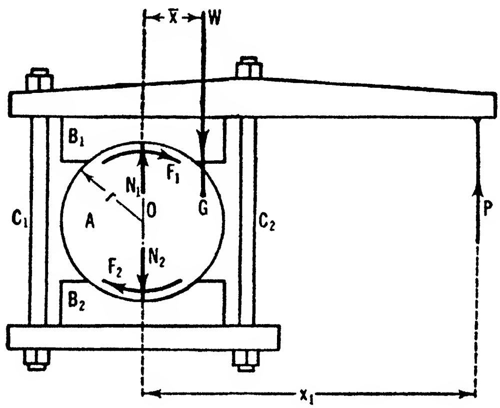

FIG. 509

means of a device called a *dynamometer*, of which there are several kinds. The most common form of dynamometer is the *Prony brake*, a simple type of which is represented in Fig. 509. In the figure, A represents the flywheel of the engine, or a pulley keyed to the crankshaft. B_1 and B_2 are wooden blocks, partially encircling the wheel. B_1 has a projecting arm, the outer end of which rests on a platform scale or other device for measuring the force exerted at the end of the arm. The pressure of the brake on the wheel can be increased or decreased at will by means of the bolts, C_1 and C_2.

Let G represent the center of gravity of the brake, and let W represent the weight of the brake. Let P represent the upward pressure at the end of the arm. Let it be assumed that the wheel is rotating in a clockwise direction. The forces F_1, F_2, N_1, and N_2 represent the kinetic frictional forces and normal pressures exerted by the wheel on the brake. When a test is being made the bolts are adjusted until the engine runs uniformly at the speed at which it is desired to measure the power.

The brake exerts frictional and normal forces on the wheel, equal and opposite to those shown in the figure. The normal forces do no work on the wheel, but the frictional forces do negative work. Let n represent the speed of the wheel in revolutions per minute. In 1 min the friction performs an amount of work on the wheel numerically equal to $(2\pi r)(F_1 + F_2)n$, which represents the power being delivered, in foot-pounds per minute, r being expressed in feet. The brake horsepower at the given speed is, then, as follows:

$$\text{hp} = \frac{(2\pi r)(F_1 + F_2)n}{33{,}000} \tag{296}$$

A relation between $(F_1 + F_2)$ and P will now be obtained by considering the brake itself, which is a body in equilibrium. This can best be done by taking the moment-sum of the external forces about O, the center of the wheel, as follows:

$$-(F_1 + F_2)r - W\bar{x} + Px_1 = 0 \tag{297}$$

Eliminating $(F_1 + F_2)$ between Eqs. 296 and 297,

$$\text{hp} = \frac{(Px_1 - W\bar{x})n}{5250} \tag{298}$$

The force P is measured by keeping the scale balanced during the progress of each test. The speed is measured at frequent intervals, usually by means of a watch and a revolution counter. For any given Prony brake, W, x_1, and $\bar{x}$ are constants.

In the case of the Prony brake, practically all the output of energy from the engine is absorbed by the brake, and passes to the atmosphere in the form of heat. For this reason brakes of this type are called *absorption dynamometers*. Another type, called *transmission dynamometers*, measure the power as it is transmitted from the crankshaft of the engine to another machine. Transmission dynamometers do not dissipate any considerable amount of energy. Electrical generators of known efficiency are sometimes used to measure the power being delivered by an engine.

PROBLEM

1120. The weight of a certain Prony brake is 36.5 lb. The distance x_1, in Fig. 509, is 30 in., and $\bar{x}$ is 6 in. The brake is attached to a pulley 16 in. in diameter. At a speed of 250 rpm it is found that the force P is equal to 175 lb. Calculate the brake horsepower of the engine at this speed. *Ans.* 20 hp.

CHAPTER XXIV

ENERGY

184. Energy in General. A force that opposes the motion, or the tendency toward motion, of the particle on which it acts, is called a *resistance*. If the particle moves against the opposition offered by the resistance, the particle, or the body to which the particle belongs, is said to *overcome* the resistance. A resistance does negative work on the body that overcomes it.

The situation that exists when a body is capable of overcoming resistances is described by saying that the body possesses energy. The energy possessed by the body is considered to be zero when the body is in a certain standard situation, or condition. This standard, or zero, energy condition is selected more or less arbitrarily, and depends on the form of energy under consideration. Energy is a scalar quantity, and is inherently positive.

Work is taken as the measure of energy. The amount of energy possessed by a particle at any instant is taken numerically equal to the total work that the forces acting on the particle would perform in bringing the particle from the condition obtaining at the given instant to the standard condition. The amount of energy possessed by a body at any instant is taken equal to the sum of the amounts of energy possessed by the particles of the body at that instant.

It is observed that bodies are able, within limits, to transfer their energy to other bodies, and that the energy is often changed in form during the transfer. *The Theory of the Conservation of Energy* teaches that *energy can neither be created nor destroyed*, or, in other words, that the total quantity of energy in the universe is constant. It follows that, if a given body gains a quantity of energy, some other body or bodies must lose an equal quantity, and that if the given body loses a quantity of energy, an equal quantity must be acquired by other bodies. Work is one of the means of transferring energy from one body to another.

Kinetic frictional forces tend to transform energy into heat. Heat is one of the forms of energy, but heat generated in such a manner usually represents energy wasted, in the sense that it does not result in the performance of work directly useful in the accomplishment of the desired purpose. Kinetic frictional forces constitute a portion of the resistances that bodies must overcome in practically all cases of motion and the energy that they transform into heat is usually conducted more or less directly to the atmosphere. It has been ascertained that a kinetic frictional force must do about 778 ft-lb of work in order to generate an amount of heat equal to 1 British thermal unit. This quantity of work is called the *mechanical equivalent* of the heat unit.

Since work is taken as the measure of energy, the latter is expressed in the same units as those in which work is expressed. There are many forms of energy, but the discussion in this book will be limited to the consideration of the two forms, *kinetic energy* and *potential energy*, that have the most direct bearing on the study of engineering mechanics. Kinetic energy and potential energy are sometimes grouped together under the name of *mechanical energy*.

185. Kinetic Energy. A body that is in motion is able to overcome resistances by virtue of its *motion*, alone. This form of energy is called *kinetic energy*. A reduction of the velocity of the body causes the body to give up a portion of its kinetic energy, and the energy thus delivered can frequently be made to serve some useful purpose.

For example, water issuing from a nozzle at high velocity can be made to drive an impulse wheel, together with an electric generator or other mechanism that may be connected to the wheel. The buckets of the wheel are so designed, and the speed of the wheel so regulated, that under average conditions the velocity of the water is reduced nearly to zero by the time the water leaves the wheel. Thus, it is seen that the water gives up nearly all its kinetic energy, a portion of the energy given up being transformed into heat by the action of kinetic frictional forces during the passage of the water across the buckets, and the remainder being delivered to the wheel. The change in velocity is virtually the only change in the condition of the water, and consequently the impulse type of hydraulic motor is peculiarly one that depends on the utilization of kinetic energy.

In the majority of engineering problems the velocity of a particle is considered to be zero when the particle is at rest relative to the earth. Naturally, the kinetic energy of a body is also considered to be zero when all the particles of the body are at rest relative to the earth. *Inherently, kinetic energy is always a positive quantity.*

The kinetic energy of a body at any instant is equal to the sum of the amounts of kinetic energy possessed by its parts. This fact is frequently convenient for use in calculating the kinetic energy of a complex body which can be divided into simpler parts.

In subsequent articles, formulas will be derived for the kinetic energy of a single particle, and for the kinetic energy of bodies having the various standard types of motion studied in previous articles. Relations will also be developed between work and kinetic energy which furnish a more direct and convenient method for the solution of certain types of problems than the methods heretofore used in this book.

186. Kinetic Energy of a Particle. Let A, in Fig. 510, represent one of the particles of a body that moves in any manner. Let it be imagined that the particle moves from left to right along the curved path shown in the figure. Let P_1, P_2, P_3, etc., represent the various forces acting on the particle during its motion, and let θ_1, θ_2, θ_3, etc., represent the angles between these forces and the tangent, tt, to the path of the particle. Let v_1 represent the velocity of the particle at the beginning of any interval, and let v_2 represent the velocity at the end of the interval. By Eq. 283, Art. 173, the total work performed on the particle during the interval can be expressed as follows:

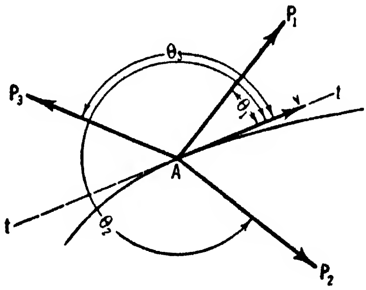

FIG. 510

$$U = \int P_1 \cos\theta_1\, ds + \int P_2 \cos\theta_2\, ds + \int P_3 \cos\theta_3\, ds \cdots, \text{etc.}$$

$$= \int (P_1 \cos\theta_1 + P_2 \cos\theta_2 + P_3 \cos\theta_3 \cdots, \text{etc.})\, ds \qquad [299]$$

In Eq. 299 the expression $(P_1 \cos\theta_1 + P_2 \cos\theta_2 + P_3 \cos\theta_3 \cdots$ etc.) represents the algebraic sum of the tangential components of all the forces acting on the particle at any instant. From Art. 121, it is seen that this expression may be replaced by $(dW/g)a_T$. Equation 299 then becomes

$$U = \int \frac{dW}{g} a_T\, ds \qquad [300]$$

But, $a_T = dv/dt$; therefore,

$$U = \frac{dW}{g} \int \frac{dv}{dt} ds = \frac{dW}{g} \int dv \frac{ds}{dt} \qquad [301]$$

Since $ds/dt = v$, Eq. 301 may be written, and integrated, as follows:

$$U = \frac{dW}{g} \int_{v_1}^{v_2} v\, dv = \frac{1}{2} \frac{dW}{g} (v_2^2 - v_1^2) \qquad [302]$$

In the foregoing analysis U represents the total work done on the particle in bringing it from the point at which its velocity is v_1 to the point at which its velocity is v_2. It will be assumed that the change in velocity is the only change in the condition of the particle. Equation 302, therefore, gives the change in the amount of kinetic energy possessed by the particle, for the given interval.

If v_1, in Eq. 302, is given the value zero, the formula then gives the kinetic energy acquired by the particle, starting from rest. If v_2 is given the value zero, the formula gives the kinetic energy given up by the particle in coming to rest. Since rest, or zero velocity relative to the earth, is the standard by which kinetic energy is usually evaluated, the formula for the kinetic energy possessed by a particle at any instant is as follows:

$$E = \frac{1}{2}\frac{dW}{g}v^2 \qquad [303]$$

If v_2 is given the value zero, in Eq. 302, the minus sign is obtained for U. This simply signifies that the work done in bringing the particle to rest is negative, but not that the kinetic energy of the particle is negative. Kinetic energy is inherently a positive quantity.

In subsequent articles Eq. 303 will be used as the basis for the derivation of specific formulas for the kinetic energy of finite bodies having the various standard types of motion treated in previous articles.

187. Kinetic Energy of a Translating Body. As was stated in Art. 184, the total energy possessed by a body at any given instant is taken equal to the sum of the amounts of energy possessed by the particles of the body at that instant. In Art. 186 it was shown that the kinetic energy of a single particle is given by the expression $\frac{1}{2}(dW/g)v^2$. The total amount of kinetic energy possessed by any body, having any type of motion, may be expressed, therefore, as follows:

$$E = \int \frac{1}{2}\frac{dW}{g}v^2 \qquad [304]$$

In the case of a body having a motion of translation, the velocities of all the particles are equal at any instant, and, therefore, v is constant for the various particles at a given instant. Equation 304, when applied particularly to a translating body, becomes

$$E = \frac{v^2}{2g}\int dW = \frac{1}{2}\frac{W}{g}v^2 \qquad [305]$$

in which v represents the velocity of the body and its particles at the given instant.

Units. In the great majority of engineering problems, under the English system, g is expressed in feet per second per second. On this account, v must be expressed in feet per second. W, representing the total weight of the body, may be expressed in terms of any convenient unit. Equation 305 will then give the kinetic energy of the body in foot-pounds, foot-tons, etc., depending on the unit used for W.

PROBLEMS

1121. It is believed that a shrapnel ball which has 58 ft-lb of kinetic energy will disable a man. If the ball weighs $\frac{1}{42}$ lb, calculate the necessary velocity. *Ans.* 396 ft/sec.

1122. In a certain accident in which several persons were killed, the car, with its passengers, weighed 3600 lb and was traveling at a speed of 100 mi per hr. Calculate the kinetic energy.

1123. Assuming that the risk of damage and injury, when an automobile is suddenly stopped in an accident, is proportional to the kinetic energy of the car, calculate the percentage increase in risk when the speed is increased from 40 to 50 mi per hr.

1124. Calculate the energy that must be delivered to a train weighing 2000 tons, to change its speed from 20 to 60 mi per hr, over and above that necessary to overcome frictional and other resistances. *Ans.* 214,000 ft-tons.

1125. The muzzle velocity of a certain cartridge for the Springfield military rifle is 2170 ft per sec. The bullet weighs 170 grains, and will penetrate twelve $\frac{7}{8}$-in. pine boards. Calculate the average retarding force (with respect to distance) acting on the bullet as it penetrates the wood. Disregard the rotational energy of the bullet. One pound is equal to 7000 grains.

1126. The driving wheels of a certain locomotive are 80 in. in diameter. The crankpin on each wheel is 12 in. from the center of the wheel. The side rod, which connects all the crankpins on one side of the locomotive, weighs 420 lb. Calculate the kinetic energy of the side rod, when the locomotive is running on a straight track at a speed of 60 mi per hr, and the rod is in its highest position. *Ans.* 85,400 ft-lb.

188. Kinetic Energy of a Rotating Body. Let Fig. 511 represent a body rotating about an axis through O at right angles to the plane of the paper. Let A represent any particle of the body. Let ρ represent the radius of the circular path in which A moves. Let v represent the linear velocity of the particle at a given instant, and let ω represent the angular velocity of the body at that instant. Let I_z represent the moment of inertia of the body with respect to the axis of rotation. Substituting the value $v = \rho\omega$ in Eq. 303, Art. 186, the following formula for the kinetic energy of particle A is obtained:

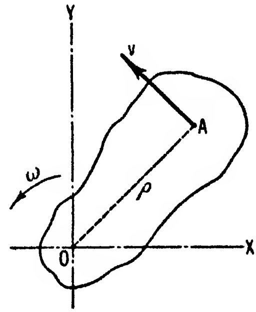

FIG. 511

$$E_A = \frac{1}{2}\frac{dW}{g}(\rho\omega)^2 \qquad [306]$$

The kinetic energy of the entire body at the given instant is the sum of the kinetic energy of all the particles. Since the summation is performed for the one instant, ω is constant. The formula for the kinetic energy of the body at the given instant is, then, as follows:

$$E = \frac{\omega^2}{2}\int \rho^2 \frac{dW}{g} = \frac{1}{2} I_z\omega^2 \qquad [307]$$

Units. If, in obtaining the value of I_z for use in Eq. 307, the customary value, 32.2, is used for g, it will be necessary to express ω in radians per second. The kinetic energy of the rotating body will then be obtained in foot-pounds, foot-tons, etc., depending on the unit in which W is expressed.

Illustrative Problem

1127. A certain power shears does 5000 ft-lb of work on the material placed between its jaws, during each cutting stroke. The flywheel of the machine is mounted on a shaft that rotates at a constant speed of 90 rpm, when idling. It is desired that the total decrease in the speed of the flywheel during a cutting stroke shall not exceed 8 per cent. Calculate the necessary moment of inertia of the flywheel, assuming that all the energy used in cutting the material is supplied by the flywheel. Disregard friction losses in the machine itself.

Solution. Let E_1 and E_2 represent the amounts of kinetic energy possessed by the flywheel at the beginning of, and end of, a cutting stroke. Let ω_1 and ω_2 represent the corresponding values of the angular velocity of the flywheel. From the problem, $\omega_2 = 0.92\,\omega_1$. By Eq. 307,

$$E_1 = \tfrac{1}{2} I_z\omega_1^2 = \tfrac{1}{2} I_z\left(\frac{90 \times 2\,\pi}{60}\right)^2 = 44.4 I_z$$

$$E_2 = \tfrac{1}{2} I_z\omega_2^2 = \tfrac{1}{2} I_z\left(\frac{0.92 \times 90 \times 2\,\pi}{60}\right)^2 = 37.6 I_z$$

$$E_1 - E_2 = 5000 \qquad 44.4 I_z - 37.6 I_z = 5000$$

$$I_z = 735 \text{ engineer's units}$$

PROBLEMS

1128. A solid, steel cylinder, 18 in. in diameter and 4 in. long, is keyed to a 3-in. steel shaft 6 ft long. The axis of the shaft coincides with the geometric axis of the cylinder. The entire assembly rotates at a speed of 600 rpm. Calculate its kinetic energy. The material weighs 490 lb per cu ft. *Ans.* 5060 ft-lb.

1129. The reduction in the kinetic energy of a certain flywheel, while the speed decreases from 60 to 50 rpm, is 100 ft-tons. How much would the kinetic energy be increased if the speed were increased from 60 to 70 rpm?

1130. A certain flywheel weighs 1500 lb, and its radius of gyration with respect to the axis of rotation is 2 ft. A brake is applied to the rim of the wheel, bringing the wheel to rest in 20 sec, from an initial speed of 300 rpm. Calculate the average horsepower of the brake, assuming that it absorbs all the energy lost by the wheel. *Ans.* 8.36 hp.

1131. The rim of a certain cast-iron flywheel is 16 in. wide and 6 in. thick. The outer diameter of the wheel is 12 ft. The peripheral speed of the wheel is 4000 ft per sec. Calculate the kinetic energy of the wheel. Disregard the spokes and hub, and assume that the radius of gyration is equal to the mean radius of the rim. The material weighs 450 lb per cu ft.

1132. A solid, cast-iron sphere 6 in. in diameter rotates about an axis 18 in. from the center of the sphere. The linear velocity of the center of the sphere is 15 ft per sec. Calculate the kinetic energy of the sphere. Also calculate the kinetic energy of the sphere, assuming a motion of translation, and compare with the first result. *Ans.* 104 ft-lb; 103 ft-lb.

1133. When the greater part of a body lies at a relatively large distance from the axis of rotation, an approximate value for the kinetic energy can be obtained by disregarding the rotation of the body, assuming that the body is translating with a velocity equal to the actual velocity of the center of gravity. This simplifies the calculations in many cases. Prove that the percentage error of a result obtained in this manner is equal to $100(1 - \bar{r}^2/k_z^2)$, in which $\bar{r}$ is the radius of rotation of the center of gravity, and k_z is the radius of gyration of the body with respect to the axis of rotation.

189. Kinetic Energy of a Body in Plane Motion. Let Fig. 512 represent a body in general plane motion, the plane of the motion being parallel to the plane of the paper. Let B represent any particle of the body, and let G represent the center of gravity. Let OX and OY represent any convenient pair of stationary axes parallel to the plane of the motion. Let q represent the distance from B to a line passing through G at right angles to the plane of the motion. Let v represent the velocity of B, and $\bar{v}$ the velocity of G, at the given instant. Let ω represent the angular velocity of the body at the given instant. Let $\bar{I}$ represent the moment of inertia of the body with respect to an axis through G at right angles to the plane of the motion.

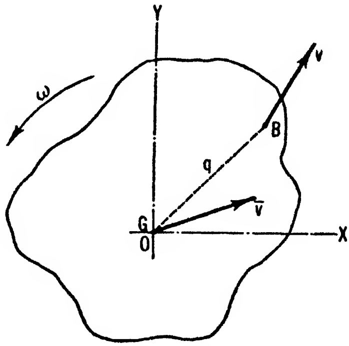

FIG. 512

Applying Eqs. 247 and 248, Art. 161, and choosing G as particle 1 and B as particle 2,

$$v_{2x} - v_{1x} = -y\omega \qquad v_x - \bar{v}_x = -y\omega \qquad v_x = \bar{v}_x - y\omega \tag{308}$$

$$v_{2y} - v_{1y} = x\omega \qquad v_y - \bar{v}_y = x\omega \qquad v_y = \bar{v}_y + x\omega \tag{309}$$

Obviously, $v^2 = v_x^2 + v_y^2$. Substituting in this equation the values of v_x and v_y given by Eqs. 308 and 309,

$$\begin{aligned} v^2 &= (\bar{v}_x - y\omega)^2 + (\bar{v}_y + x\omega)^2 \\ &= (\bar{v}_x^2 + \bar{v}_y^2) - 2\bar{v}_x y\omega + 2\bar{v}_y x\omega + \omega^2(y^2 + x^2) \\ &= \bar{v}^2 - 2\bar{v}_x y\omega + 2\bar{v}_y x\omega + q^2\omega^2 \end{aligned} \tag{310}$$

Substituting the foregoing value of v^2 in Eq. 303, Art. 186,

$$E_B = \frac{1}{2}\frac{dW}{g}(\bar{v}^2 - 2\bar{v}_x y\omega + 2\bar{v}_y x\omega + q^2\omega^2) \qquad [311]$$

The integral of Eq. 311 is the kinetic energy of the entire body. Therefore,

$$E = \frac{\bar{v}^2}{2g}\int dW - \frac{\bar{v}_x\omega}{g}\int y\,dW + \frac{\bar{v}_y\omega}{g}\int x\,dW + \frac{\omega^2}{2}\int q^2\frac{dW}{g} \qquad [312]$$

The expression $\int y\,dW$ represents the first moment of the weight of the body, with respect to a gravity axis, and is equal to zero (Art. 95). Likewise, $\int x\,dW = 0$. The expression $\int dW$ is equal to W, and the expression $\int q^2 \frac{dW}{g}$ represents the moment of inertia of the body with respect to a gravity axis at right angles to the plane of the motion. Equation 312 may now be written

$$E = \frac{1}{2}\frac{W}{g}\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 \qquad [313]$$

It will be noticed that, in a certain sense, Eq. 313 is a combination of Eq. 305, Art. 187, and Eq. 307, Art. 188. However, in the use of Eq. 313 it must be remembered particularly that $\bar{v}$ represents the velocity of the center of gravity of the body at the given instant, and that $\bar{I}$ represents the moment of inertia of the body with respect to a gravity axis at right angles to the plane of the motion. The statements in Arts. 187 and 188, regarding units, apply in the present case also.

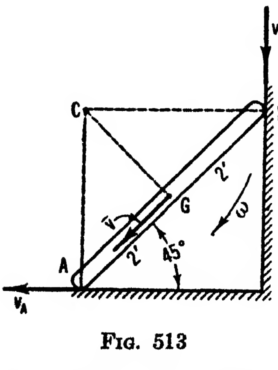

FIG. 513

ILLUSTRATIVE PROBLEM

1134. Figure 513 represents a straight slender bar, AB, 4 ft in length, and weighing 50 lb. The bar moves in a vertical plane, the lower end sliding along a horizontal floor, and the upper end sliding on a vertical wall. At the instant represented by the figure the linear velocity of the lower end is 3 ft per sec, toward the left. Calculate the kinetic energy of the bar at the given instant.

Solution. v_A is horizontal, and v_B is vertical, as shown. Therefore, the instantaneous axis is at C. $\overline{AC} = \overline{BC} = 4 \cos 45° = 2.83$ ft. By inspection,

$\overline{CG} = 2$ ft. By Art. 162,

$$\omega = \frac{v_A}{\overline{AC}} = \frac{3}{2.83} = 1.06 \text{ rad/sec}$$

$$\bar{v} = \overline{CG}\omega = 2 \times 1.06 = 2.12 \text{ ft/sec}$$

By Prob. 755, Art. 138,

$$\bar{I} = \frac{1}{12}\frac{W}{g}L^2 = \frac{50 \times (4)^2}{12 \times 32.2} = 2.07 \text{ engineer's units}$$

By Eq. 313,

$$E = \frac{1}{2}\frac{W}{g}\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2 = \frac{50\,(2.12)^2}{2 \times 32.2} + \frac{2.07\,(1.06)^2}{2} = 4.65 \text{ ft-lb}$$

PROBLEMS

1135. A solid cylinder 3 ft in diameter, weighing 322 lb, rolls on a plane surface, without slipping. Calculate the kinetic energy of the cylinder at the instant when the velocity of the center point is 6 ft per sec. *Ans.* 270 ft-lb.

1136. The cylinder of Prob. 1135 starts from rest on an inclined plane whose angle of inclination is 30°, and rolls without slipping. Calculate the kinetic energy of the cylinder at the instant when it is 10 ft from the starting point. Calculate the total work done on the cylinder up to that instant.

1137. Calculate the kinetic energy of the bar in Prob. 971, Fig. 452, under the conditions described in that problem. *Ans.* 16.7 ft-lb.

1138. Calculate the total kinetic energy of the plank and the two rollers in Prob. 973, Fig. 453, at the instant when the velocity of the center of each roller is 5 ft per sec.

1139. Calculate the kinetic energy of the connecting rod described in Prob. 977, Fig. 455, under the conditions stated in that problem. *Ans.* 6310 ft-lb.

1140. A homogeneous sphere rolls on a plane surface, without slipping. Prove that the right and left halves of the sphere possess equal amounts of kinetic energy. Prove that the kinetic energy of the upper half is $3\frac{4}{13}$ times that of the lower half.

1141. A wooden cylinder 4 ft in diameter and 1 ft long rolls on a horizontal surface, without slipping. The wood weighs 40 lb per cu ft. A cylindrical hole 1.5 ft in diameter extends through the cylinder. The axis of the hole is parallel to the geometric axis of the cylinder and is 1 ft therefrom. Calculate the kinetic energy of the body, for the instant when the velocity of the geometric center of the cylinder is 4 ft per sec, and the hole is in its highest position. *Ans.* 147 ft-lb.

1142. Solve Prob. 1141 for the case in which the center of the hole is level with the center of the cylinder.

190. Potential Energy. It is observed that bodies, or systems of bodies, sometimes possess energy which is attributable solely to the *configuration*, or relative position, of the particles of which the body or system is composed. This form of energy is called *potential energy.* Systems possessing potential energy tend to pass, of their own accord, to configurations in which their potential energy is less in amount. Thus they can readily be made to give up energy, a portion of which can frequently be turned to some useful purpose.

The amount of potential energy possessed by a system in a given configuration is taken numerically equal to the amount of work which would be done on the system in bringing it from the given configuration to some arbitrarily chosen zero, or standard, configuration. Therefore, the value assigned to the potential energy of a system in a given configuration depends on the choice of the standard configuration. The latter is usually chosen in such a manner that the potential energy of the system will have the positive sign in any configuration that is likely to be of importance in the particular problem at hand.

An elastic body, such as a steel spring, that has been stretched or deformed in any manner, will overcome resistances if it is permitted to return to its normal condition. Obviously, such a body, in its deformed state, possesses potential energy. The normal, or unstrained, condition is usually taken for the standard configuration. In most cases of elastic deformation the average value of any given force acting on the body is equal to the arithmetic mean of the initial and final values of that force, provided that the elastic limit of the material is not exceeded. The work done by each force can be found by means of this average value, and the potential energy calculated. Potential energy stored in elastic bodies by means of deformations is sometimes called *stress energy.*

The earth, together with any smaller body at an elevation above the earth's surface, constitute an important example of a system of distinct bodies possessing potential energy. By permitting the smaller body to descend to a lower elevation it is frequently possible to obtain energy for engineering purposes. Any convenient elevation for the smaller body is taken as the standard, and the potential energy is considered to be zero when the system is in this standard configuration.

The amount of energy thus possessed by *the system* is equal to the weight of the smaller body multiplied by the elevation of the latter above the standard, or datum. This datum is usually so selected that the potential energy will have the positive sign.

Care was taken in the foregoing discussion to refer to the potential energy stored up in the case of an elevated body as being *possessed by the system,* and *not by the elevated body.* In practice, however, the energy is frequently spoken of as the possession of the smaller body. The latter viewpoint leads to erroneous results in certain cases, and should be thought of merely as a convenient mode of expression, and not as an accurate statement of energy conditions.

The foregoing facts regarding potential energy of elevation are especially important in engineering problems involving the estimation of power available from reservoirs, streams, and other bodies of water.

191. Efficiency. Machines transmit energy from one place to another. Also, many machines convert energy from forms in which it cannot be directly utilized into forms in which it can be so utilized.

Energy is supplied to the machine at one or more points. This energy is called the *input.* The energy is then transmitted through the mechanism toward the point at which it is to be utilized. Frictional resistances and other conditions along the way always cause the escape of a certain amount of energy from the machine. Finally the point is reached at which the remaining energy can be applied to the purpose for which the machine has been designed. The energy thus utilized is called the *output.* The *efficiency* is the ratio of the output to the input during any given interval. This ratio is usually multiplied by 100, the efficiency thus being expressed as a percentage.

Inasmuch as losses are inevitable, the output is necessarily less than the input, and the efficiency is always less than unity. Sometimes a certain portion of the input is stored in the machine, for a time at least. In calculating efficiency in such cases it is necessary to make an allowance for any portion of the input that is stored in the machine and can ultimately be recovered and applied to the purpose for which the machine is intended. It should always be understood that efficiency is intended to be a measure of the degree to which a machine avoids waste of energy. Therefore, care should be exercised to make certain that the difference between the values used for the input and the output really represents energy wasted, or lost so far as the primary function of the machine is concerned.

The efficiency of an entire mechanism, or of any of its parts, may be calculated. The *overall efficiency* of a number of mechanisms placed in series is equal to the product of the individual efficiencies of the mechanisms in the series.

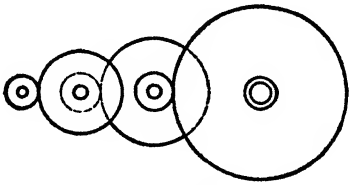

FIG. 514

PROBLEMS

1143. Calculate the efficiency of the train of spur gears shown in Fig. 514, assuming the efficiency of a single pair of such gears to be 96 per cent. *Ans.* 88.5%.

1144. A constant torque of 40 ft-lb is applied to the crank of a certain hand-power crane, in lifting a load of $\frac{1}{2}$ ton. The load is raised 2 in. during one revolution of the crank. Calculate the efficiency of the crane.

1145. Calculate the efficiency of the screw jack described in Prob. 452, Art. 84, when the load is being lifted by the jack. *Ans.* 20%.

1146. The available discharge of a certain stream is 200 cu ft per sec. The fall that can be utilized is 18 ft. The overall efficiency of the turbine installation is 72 per cent. Calculate the horsepower that the turbine can deliver.

1147. A certain elevator is to be operated at a speed of 200 ft per min. The motor must develop a pull of 1.4 tons in the cables, to drive the elevator at this speed. The overall efficiency of the mechanism is 50 per cent. Calculate the necessary horsepower of the motor. *Ans.* 33.9 hp.

1148. A jet of water 4 in. in diameter, having a velocity of 180 ft per sec, is used with a certain impulse wheel. The efficiency of the wheel is 85 per cent. Calculate the horsepower delivered by the wheel, assuming the weight of the water to be 62.4 lb per cu ft.

1149. A certain automobile develops 75 brake horsepower, at an engine speed of 2800 rpm, corresponding to a car speed of 50 mi per hr. Assuming that the efficiency of the transmission of power from the engine to the rear wheels is 85 per cent, calculate the tractive effort developed. *Ans.* 478 lb.

CHAPTER XXV

WORK AND ENERGY

192. The Work-Energy Method. The Theory of the Conservation of Energy can be used as the basis for a convenient method of solution of many problems in kinetics.

Energy is transferred from one body to another in various ways. *Work is one very important means of transferring energy.* When a force does positive work on a body the body gains energy, the amount of energy gained being equal to the work done. When a force does negative work on a body, the body loses an amount of energy numerically equal to the work done. Since energy can neither be created nor destroyed, it follows that *the total work done on a body, during any given interval, by the external forces, is equal to the difference between the amounts of energy possessed by the body at the beginning and at the end of the interval, provided that during the interval no energy is transferred to or from the body by agencies other than the external forces.* By means of this principle equations can be formed between work and energy, leading to the direct solution of many problems. In many cases the only energy change that occurs during the interval is a change in kinetic energy. This is true in problems in which the body is assumed to be rigid.

Whenever the work-energy method is applied to the solution of a problem, confusion in the matter of algebraic signs can be most easily avoided if the increment in the amount of energy possessed by the body is always calculated by the subtraction of the initial amount from the final amount. This procedure insures an agreement between the sign of the total work and the sign of the energy increment.

The Case of the Elevated Body. As was stated in Art. 190, it is customary to say that a body which is in an elevated position above the surface of the earth possesses potential energy by virtue of that elevation. This is a convenient mode of describing the situation, but it cannot be reconciled with the usual definition of potential energy. Potential energy is energy that is attributable to the configuration of the particles of the system possessing it. The configuration of the particles of a body is not changed by the mere raising or lowering of that body. A change does occur, however, in the configuration of the particles of the earth and the smaller body considered together as a system. Therefore, the potential energy that is stored up when a body is elevated should be assigned to the entire system, and not to the smaller body. Likewise, the potential energy released when a body is lowered should be considered as having come from the system.

The importance of the foregoing distinction lies in the application of the work-energy method to the solution of problems in which the center of gravity of the moving body at the end of the given interval is at an elevation different from that occupied at the beginning of the interval. The work-energy equation is usually written for the moving body only. *That part of the equation which expresses the increment in the energy possessed by the body should not contain any allowance for a change in potential energy unless the body itself is deformed elastically.*

If the work-energy equation were to be written for the entire system consisting of the earth and the moving body, the potential energy stored or released during the interval would, of course, be included.

Illustrative Problems

1150. A certain automobile weighs 3000 lb. The maximum retarding force that can be developed by the use of the brakes is equal to one-half the weight of the car. In what minimum distance can the car be brought to rest from a speed of 50 mi per hr? Disregard all resistances except that produced by the brakes. Assume that the entire car has a motion of translation.

Solution. Let U represent the total work done on the car. Let P represent the total retarding force developed by the brakes, and let s represent the distance in which the car is stopped. Let E_1 and E_2 represent the energy possessed by the car at the initial and final instants. The entire energy of the car is in the kinetic form at both instants. By the principle of work and energy,

$$U = E_2 - E_1 \qquad -Ps = 0 - \frac{1}{2}\frac{W}{g}v_1^2$$

$$-(0.5 \times 3000)\, s = -\frac{1}{2} \times \frac{3000}{32.2}\left(\frac{50 \times 5280}{3600}\right)^2$$

$$s = 167 \text{ ft}$$

1151. Figure 515 represents a helical spring whose modulus is 6 lb per in., and whose upper end is fastened to a ceiling. The normal length of the spring is 24 in. A ball weighing 12 lb is attached to the lower end of the spring, and is supported in the position A, so that the spring is unstretched. The support is suddenly removed and the ball descends, stretching the spring and momentarily coming to rest again at B, under the action of the spring. Calculate the downward distance traveled by the ball.

Solution. Let P represent the upward pull of the spring on the ball at any instant. The maximum value reached by P is $6s$, in which s represents the total elongation of the spring, in inches. The initial value of P is zero. The mean value is $(0 + 6s)/2 = 3s$. The initial energy possessed by the ball is zero, and the final energy is also zero. By the principle of work and energy,

$$U = E_2 - E_1 \qquad -(3\,s)\,s + 12s = 0 - 0$$
$$s = 4 \text{ in.}$$

It is instructive to observe that the spring is stretched twice as much as it would be if the ball were lowered gradually.

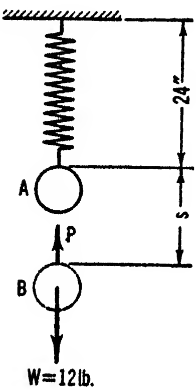

Fig. 515

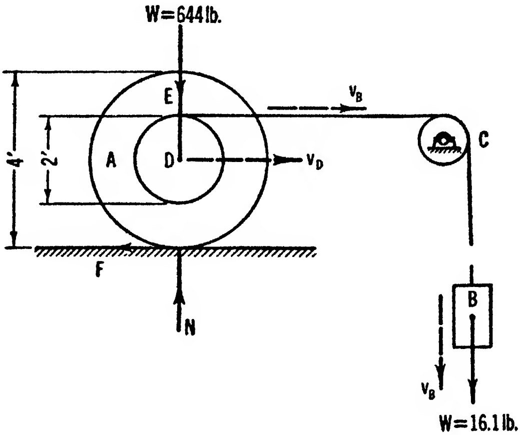

Fig. 516

1152. Figure 516 represents a wheel, A, and a drum, D, fastened rigidly together, and having a total weight of 644 lb. Their radius of gyration with respect to a gravity axis at right angles to the plane of motion is 1.2 ft. The wheel rolls without slipping, and, as it rolls, a cable is unwound from the drum and passes over the pulley, C. A body B, weighing 16.1 lb, is attached to the lower end of the cable. Calculate the velocity of the center of the wheel at the instant when the weight B has descended 12 ft, starting from rest. Disregard all frictional losses. Disregard the weight of the pulley, C.

Solution. Let v_B represent the velocity of B at the final instant. It is obvious that the particle E, at the highest point of the drum D, also has the velocity v_B. Let v_D represent the velocity of the center of the drum. By the method of Art. 161, or Art. 162, it is easily ascertained that

$$\omega = \frac{v_D}{r} = \frac{v_D}{2} \quad \text{and that} \quad v_D = \frac{2}{3}\,v_B$$

For the rolling body,

$$I = \frac{W}{g}k^2 = \frac{644}{32.2}(1.2)^2 = 28.8 \text{ engineer's units}$$

It is obvious that the weight of A and D does no work, and that N does no work. It was explained in Art. 180 that the static frictional force under a rolling body does no work. Therefore, considering the entire moving system, the only force doing work is the weight of body B. The work-energy equation for the present case is, then, as follows:

$$U = E_2 - E_1 \qquad +W_B(12) = \frac{1}{2}\frac{W_B}{g}v_B^2 + \left[\frac{1}{2}\frac{W_A + W_D}{g}v_D^2 + \frac{1}{2}I\omega^2\right] - 0$$

$$+16.1 \times 12 = \frac{16.1}{2 \times 32.2}v_B^2 + \frac{644}{2 \times 32.2}\left(\frac{2}{3}v_B\right)^2 + \frac{1}{2} \times 28.8\left(\frac{v_B}{3}\right)^2$$

$$v_B = 5.54 \text{ ft/sec} \qquad v_D = \tfrac{2}{3}v_B = 3.69 \text{ ft/sec}$$

PROBLEMS

1153. A ball weighing 6 lb is hung from a fixed point of support, by means of a fine wire. The center of gravity of the ball is 3 ft from the point of support. The ball is swung to a position level with the point of support, the wire being kept taut, and is then released. Calculate the velocity of the center of gravity of the ball when it reaches its lowest position. Disregard the rotational energy of the ball. *Ans.* 13.9 ft/sec.

1154. A certain railway train weighs 1000 tons. The total resistance attributable to friction is 10 lb per ton of weight, regardless of whether the train is on level track or on a grade. Calculate the drawbar pull necessary to increase the speed of the train from 20 to 40 mi per hr, on a 1 per cent upgrade, in a distance of 1 mi.

1155. A wheel 6 ft in diameter, weighing 3220 lb, is mounted on a shaft. The radius of gyration is 2.8 ft. A constant torque applied directly to the shaft gives the wheel a speed of 60 rpm after the wheel has described 20 rev, starting from rest. Calculate the torque. *Ans.* 123 ft-lb.

1156. Solve Prob. 871, Fig. 405, by the work-energy method.

1157. A body weighing 322 lb is propelled along a horizontal plane by a constant force of 100 lb. The force is inclined downward at a slope of 4 (horizontal) to 3 (vertical). The coefficient of kinetic friction is 0.15. How far will the body move while its speed changes from 25 to 50 ft per sec? *Ans.* 413 ft.

1158. A pulley 4 ft in diameter, weighing 644 lb, is mounted on a shaft. The radius of gyration is 1.8 ft. A belt passes halfway around the pulley, and is subjected to a constant pull of 50 lb on one side and 5 lb on the other. Calculate the angular velocity of the pulley at the instant when it has described 200 rev, starting from rest.

1159. A ball weighing 20 lb starts from rest and falls through a distance of 6 ft. At this point it strikes a helical spring whose axis is vertical and whose lower end is fixed. The spring is shortened 4 in. in bringing the ball momentarily to rest. Calculate the modulus of the spring in pounds per inch. *Ans.* 190 lb/in.

1160. A freight car weighing 80,000 lb, moving at a speed of 5 mi per hr, on a level track, strikes a spring buffer assembly whose modulus is 50 tons per inch. In what distance will the car be brought to rest? Disregard friction.

1161. A solid, homogeneous cylinder, 1 ft in diameter and weighing 161 lb, rolls up a 75 per cent grade, its center having an initial velocity of 30 ft per sec. Assuming that no slipping occurs, calculate the distance that the cylinder will travel up the incline before coming to rest. *Ans.* 34.9 ft.

1162. A body weighing 96.6 lb starts from rest and slides down an incline on a 75 per cent grade, through a distance of 200 ft. At this point the body moves onto a horizontal plane and continues to slide until brought to rest by friction. The coefficient of kinetic friction is 0.2. How far does the body move on the horizontal plane?

1163. A solid, homogeneous cylinder, 4 ft in diameter and weighing 1610 lb, is rolled along a horizontal plane, without slipping, by a constant force of 20 lb. The force is applied at the axis of the cylinder, at a downward slope of 4 (horizontal) to 3 (vertical). Calculate the velocity of the center point of the cylinder after it has traveled 60 ft, starting from rest. *Ans.* 5.06 ft/sec.

1164. A body weighing 20 lb starts from rest on a 75 per cent inclined plane, and slides downward through a distance of 100 ft. At this point the incline ends abruptly in a cliff 50 ft high, and the body is projected into space. The coefficient of kinetic friction for the incline is 0.25. Calculate the velocity of the body as it reaches the earth.

1165. A ball weighing 5 lb is hung from a fixed point of support, by means of a fine wire. The center of gravity of the ball is 6 ft from the point of support. The ball is swung out to a position in which the wire makes an angle of 60° with the vertical, and is released from rest at that point. Upon reaching its lowest position the ball strikes the end of a horizontal, helical spring whose modulus is 40 lb per in. The other end of the spring is fixed. Calculate the amount by which the spring is shortened. Assume that the ball moves horizontally after striking the spring. Disregard the rotational energy of the ball. *Ans.* 3 in.

1166. Figure 517 represents a solid, homogeneous cylinder 2 ft in diameter. A fine wire is wrapped around the cylinder. One end of the wire is fastened to a support. The cylinder is held momentarily in the position shown, and is then suddenly released. Calculate the velocity of the center point when it has descended 30 ft.

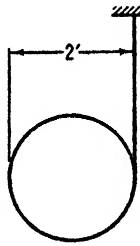

FIG. 517

1167. If the brake in Prob. 883 is applied when the load W has a velocity of 20 ft per sec, downward, and if the force P on the brake arm is 200 lb, how far will the load move before coming to rest? *Ans.* 43.1 ft.

1168. Calculate the velocity of the center point of the drum in Prob. 962, Fig. 450, when it has traveled 20 ft, starting from rest.

1169. What velocity would the centers of the drums in Prob. 970, Fig. 451, attain after traveling 10 ft, starting from rest? *Ans.* 6.68 ft/sec.

1170. If the plank and rollers in Prob. 973, Fig. 453, start from rest, what velocity will the centers of the rollers attain after traveling 4 ft?

1171. In a certain railway hump yard the grades, starting from the top of the hump, are as follows: 300 ft of 2.5 per cent grade; 100 ft of 1.5 per cent grade; 1000 ft of 1 per cent grade, the remainder of the yard being on a 0.5 per cent grade. A box car weighing 40,000 lb goes over the summit of the hump at a speed of 5 mi per hr, and coasts down into the yard. Assuming a constant resistance of 150 lb opposing the motion of the car, calculate the velocity of the car at a point $\frac{1}{2}$ mi from the summit. *Ans.* 22 mi/hr.

1172. Figure 518 represents a hoist of a type in common use. A and B are drums, fastened together so as to rotate as one body. Their total weight is 1000 lb, and their radius of gyration with respect to the axis of rotation is 26 in. The skips, C and D, weigh 1600 lb and 1200 lb, respectively. Calculate the torque that the motor must furnish in order to give C a velocity of 15 ft per sec, downward, in a distance of 20 ft, starting from rest. Disregard friction and the weight of the cable.

1173. A body weighing 50 lb starts from rest on an inclined plane, and slides downward through a distance of 10 ft. At this point it strikes the end of a helical spring placed with its axis parallel to the slope and with its lower end fixed. The modulus of the spring is 80 lb per in., the grade of the incline is 75 per cent, and the coefficient of friction is 0.25. How much will the spring be shortened in bringing the body to rest? The spring expands and projects the body back up the slope. By what amount will the body fail to reach its original starting position? *Ans.* 8 in.; 64 in.

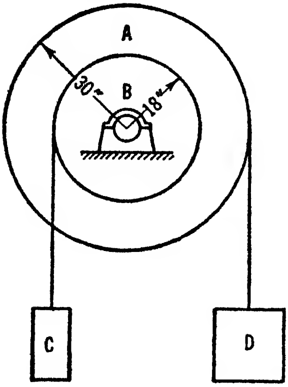

FIG. 518

1174. Figure 519 represents a solid, homogeneous cylinder 2 ft in diameter, weighing 161 lb, mounted on a shaft in frictionless bearings. *AB* represents a helical spring having a modulus of 2 lb per in. One end of the spring is attached to the rim of the wheel, and the other is fastened in a fixed position at *B*. The normal length of the spring is 2 ft. The wheel is turned until *A* occupies the position *A′*, and is then released, from rest. Find the angular velocity of the wheel at the instant when it reaches its original position. Disregard all frictional losses. Disregard the kinetic energy of the spring.

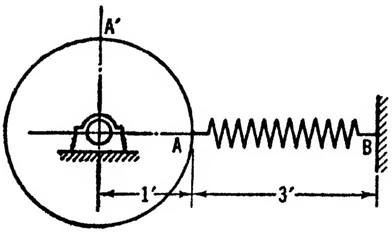

FIG. 519

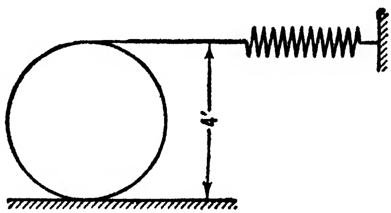

FIG. 520

1175. A straight, slender bar 5 ft long, weighing 32.2 lb, is placed with its upper end against a vertical wall, and its lower end resting on a horizontal floor at a point 3 ft from the wall. The bar is then released. Calculate the velocity of the lower end at the instant when it reaches a point 4 ft from the wall. Disregard friction. *Ans.* 5.88 ft/sec.

1176. Figure 520 represents a wheel 4 ft in diameter, weighing 644 lb, and having a radius of gyration of 1.5 ft. A fine wire is wrapped around the wheel, and is attached to a helical spring whose modulus is 60 lb per in. The other end of the spring is fixed. The wheel rolls toward the left, without slipping. At the instant when the wire tightens and the spring begins to act, the velocity of the center of the cylinder is 3 ft per sec. How far will the center point move before the wheel comes to rest?

CHAPTER XXVI

LINEAR IMPULSE AND LINEAR MOMENTUM

193. Elementary Linear Impulse. *The linear impulse of a force during an elementary interval of time is a vector quantity whose magnitude is equal to the magnitude of the force during the interval, multiplied by the length of the interval.* For example, if F represents the magnitude of a given force during a certain elementary interval, and dt represents the length of that interval, the magnitude of the linear impulse of the force during the interval is equal to $F\,dt$. Furthermore, linear impulse is a true vector quantity, with a line of action occupying a definite position in space. The line of action of a linear impulse coincides with the line of action of the force with which the impulse is associated. Also, the sense of the impulse agrees with the sense of the force.

Therefore, it is possible to resolve linear impulses into their components along any desired axes, to calculate their moments about axes, and to find their resultants, the procedure in all such cases being similar to that followed in the resolution and composition of forces.

194. Finite Linear Impulse; General Case. *The linear impulse of a force during a finite interval of time is the resultant of all the elementary impulses occurring during the interval.*

The process of calculating the resultant of a system of infinitesimal vectors is similar to that used in the case of finite vectors, such as a system of forces, and is called *vectorial integration.* The chief difference between the two processes lies in the fact that in the composition of infinitesimal vectors it is necessary, in general, to obtain the various component-sums and moment-sums by means of integrations, instead of finite summations.

In the general case, the magnitude and inclination of the force vary during the finite interval, and the line of action moves about in space. Thus it can be seen that the resultant impulse of the force for the interval may be a single vector, or a couple, or both, just as in the composition of force systems. A complete treatment of all these possible situations is not necessary to the purposes for which the conception of impulse is ordinarily used. The discussion in the present article will be limited to the derivation of a formula for the component, along any axis, of the linear impulse of a variable force during any finite interval of time.

The Formulas. The elementary linear impulses of a force whose magnitude and inclination vary during the interval, and whose line of action shifts about in space, constitute a general system of non-coplanar vectors. The resultant may answer any one of the three descriptions mentioned above, but in any event the component of the resultant along any axis is equal to the algebraic sum, or integral, of the components of the elementary impulses along that axis.

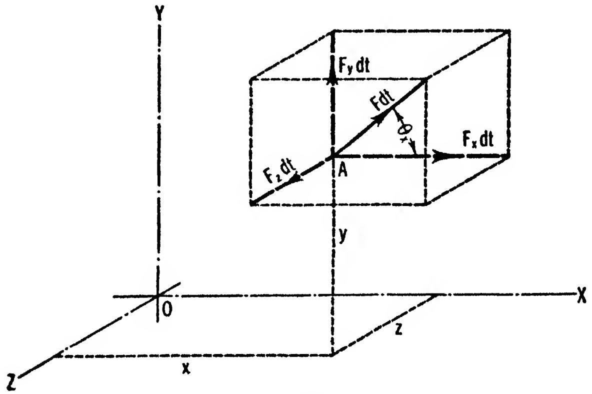

FIG. 521

Let the vector $F\,dt$, in Fig. 521, represent any one of the elementary linear impulses of a variable force, F. The line of action of $F\,dt$ coincides with the line of action of F. The force itself is not shown. Let θ_x represent the angle between $F\,dt$ and the x-axis. During the finite interval the force will produce a series of these elementary impulses, occupying various lines of action in space.

Let the x-component of the resultant impulse for the finite interval be represented by L_x. The following equation can now be written:

$$L_x = \int (F\,dt) \cos \theta_x = \int (F \cos \theta_x)\,dt \qquad [314]$$

The expression $(F \cos \theta_x)$, in Eq. 314, is equal to the x-component of the force F at the instant represented by the figure. Representing this component by F_x, the following formula results:

$$L_x = \int F_x\,dt \qquad [315]$$

Formulas for other components of L would be similar to Eq. 315. In order to perform the integration indicated in the formula it is necessary to know the law by which F_x varies with respect to t.

Units. There is no single name for the unit of linear impulse. It is designated by a combination of the names of the units of force and time used in the calculations. In the English system the pound-second is the most common unit of linear impulse.

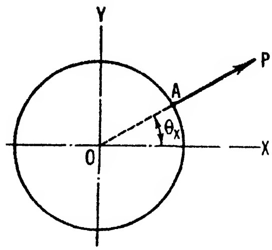

FIG. 522

ILLUSTRATIVE PROBLEM

1177. Figure 522 represents a particle, A, moving in a circular path in accordance with the law $\theta_x = t^{3/2}$, in which θ_x is expressed in radians and t in seconds. A force, P, acts radially on the particle, and outward, throughout the motion. The magnitude of P varies in accordance with the law $P = \sqrt{t}$, in which P is expressed in pounds and t in seconds. Find the magnitude and angle of inclination of the linear impulse of the force P, for the interval from $t = 0$ to $t = 36$ sec.

Solution. By Eq. 315,

$$L_x = \int F_x\,dt = \int P \cos\theta_x\,dt$$

From the problem, $P = t^{1/2}$ and $\theta = t^{3/2}$. Substituting these values in the foregoing equation, and integrating,

$$L_x = \int_0^{36} t^{1/2} \cos (t^{3/2})\,dt = \Big[\tfrac{2}{3} \sin (t^{3/2})\Big]_0^{36} = \tfrac{2}{3} \sin (216 \text{ rad}) - \tfrac{2}{3} \sin (0)$$

$$= \tfrac{2}{3} \sin 134^\circ\ 15' - 0 = \tfrac{2}{3}\,(0.716)$$

$$L_x = +\,0.477 \text{ lb-sec}$$

In a similar manner,

$$L_y = \int F_y\,dt = \int P \sin\theta_x\,dt$$

Substituting in this equation the values of P and θ_x given in the problem,

$$L_y = \int_0^{36} t^{1/2} \sin (t^{3/2})\,dt = \Big[-\tfrac{2}{3} \cos (t^{3/2})\Big]_0^{36} = -\tfrac{2}{3} \cos (216 \text{ rad}) + \tfrac{2}{3} \cos (0)$$

$$= -\tfrac{2}{3} \cos 134^\circ\ 15' + \tfrac{2}{3} = +0.465 + 0.667$$

$$L_y = +1.13 \text{ lb-sec}$$

$$L = \sqrt{L_x^2 + L_y^2} = \sqrt{(0.477)^2 + (1.13)^2} = 1.23 \text{ lb-sec}$$

$$\theta_x = \text{arc tan}\,\frac{L_y}{L_x} = \text{arc tan}\,\frac{1.13}{0.477} = \text{arc tan}\ 2.37 = 67^\circ\ 05'$$

Both L_x and L_y are positive; therefore, the sense of the linear impulse is upward and toward the right.

PROBLEMS

1178. The line of action of a certain force is horizontal, and the sense is toward the right. The magnitude varies in accordance with the law $P = 0.5t$, in which P is in pounds and t is in seconds. Calculate the linear impulse of the force for the interval between $t = 10$ sec and $t = 20$ sec. *Ans.* +75 lb-sec.

1179. The normal length of a certain helical spring is 48 in., and the modulus is 4 lb per in. One end is fastened to a fixed support. With the axis of the spring horizontal, a variable force is applied to the other end, of such a nature that the spring elongates at a constant rate of 12 in. per min. Calculate the linear impulse of the force for the interval during which the length of the spring changes from 54 to 66 in.

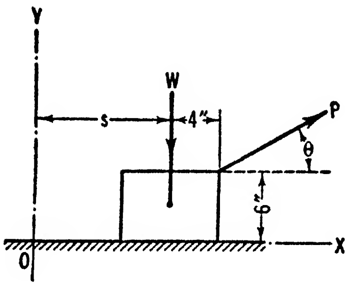

FIG. 523

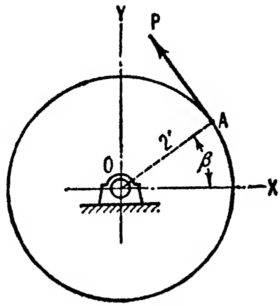

FIG. 524

1180. Figure 523 represents a body to which is applied a force P. This force has a constant magnitude of 10 lb, but its angle of inclination, θ, varies in accordance with the following law: $\theta = 0.05t$, in which θ is in radians and t is in seconds. Calculate the magnitude, and angle of inclination, of the linear impulse of P for the interval from $t = 0$ to $t = 30$ sec.

1181. The magnitude of the force P in Fig. 523 varies in accordance with the law $P = 2\sqrt{t^2 + 4}$, in which P is in pounds and t is in seconds. The angle of inclination, θ, varies in accordance with the law $\cos\theta = t/\sqrt{t^2 + 4}$, in which t is in seconds. Calculate the magnitude and angle of inclination of the linear impulse of P for the interval from $t = 0$ to $t = 8$ sec. *Ans.* 71.6 lb-sec, $\theta_x = 26° 35'$.

1182. Figure 524 represents a wheel rotating on a shaft. A is a point on the rim of the wheel. The force P is applied tangentially at A, following that point in its motion around the circle, and remaining tangent to the wheel at all times. The motion of the wheel obeys the following law: $\beta = 0.1t$, in which β is in radians and t is in seconds. The force has a constant magnitude of 12 lb. Calculate the magnitude and inclination of the linear impulse of the force during the interval in which β varies from 0 to $\pi/2$ rad.

1183. Calculate the magnitude and angle of inclination of the linear impulse of the force in Prob. 1182, for the interval during which β varies from 0 to π rad. *Ans.* 240 lb-sec, $\theta_x = 180°$.

1184. Calculate the linear impulse of the force in Prob. 1182, for the interval during which β varies from 0 to 2π rad.

1185. The magnitude of the force P in Fig. 524 varies in accordance with the law $P = 2\sin\beta$, in which P is in pounds and β is in radians. The other conditions are the same as in Prob. 1182. Calculate the magnitude and angle of inclination of the linear impulse of P for the interval from $t = 0$ to $t = 4$ sec. *Ans.* 1.57 lb-sec, $\theta_x = 105° 15'$.

195. Finite Linear Impulse; Special Cases. In many of the engineering problems in which the facts concerning impulse can be utilized to advantage, the force whose impulse is to be calculated remains constant in magnitude, or in inclination, or both, during the finite interval under consideration.

Force with Constant Inclination. A force whose inclination and sense remain constant, but whose magnitude varies, and whose line of action shifts about in space, will now be considered. In this case the series of elementary linear impulses constitutes a system of parallel vectors alike in sense but unequal in magnitude. It follows that the resultant linear impulse of the force for the finite interval is a single vector parallel to the line of action of the force.

Since the force is of variable magnitude, F_x is also variable, although θ_x is constant. The formula for L_x in the present case is, therefore, identical with Eq. 315, of Art. 194, which is,

$$L_x = \int F_x \, dt \qquad [316]$$

However, since all the elementary linear impulses are parallel, a simple formula can be written for the resultant linear impulse, as follows:

$$L = \int F \, dt \qquad [317]$$

Constant Force. Now let the magnitude of the force, as well as the inclination and sense, remain constant. In this case, it is readily seen that

$$L = Ft \qquad [318]$$

Thus, it is seen that *the finite linear impulse of a force that is constant in magnitude, inclination, and sense during the interval is equal simply to the product of the magnitude of the force and the length of the time interval.* Obviously, the inclination and sense of the resultant impulse, L, are the same as the inclination and sense of the force, F, producing the impulse.

Illustrative Problems

1186. A certain helical spring has a normal length of 12 in., and a modulus of 6 lb per in. One end of the spring is fastened in a fixed position. An axial force, P, is applied to the free end of the spring, in such a manner as to elongate the spring at a constant rate of 4 in. per min. Calculate the resultant linear impulse of the force P, for the interval during which the length of the spring changes from 12 in. to 14 in. Calculate the linear impulse of the force at the fixed end of the spring. Assume that the spring is weightless.

Solution. Let t represent the time that has elapsed since the spring had its normal length of 12 in. The value of t at the final instant, when the spring has a length of 14 in., is $\frac{14 - 12}{4} 60 = 30$ sec. The elongation of the spring, in inches, at any time, t, is given by the expression $4 \frac{t}{60}$, t being expressed in seconds. Therefore, the value of P at any instant is $P = 6\left(\frac{t}{15}\right) = 0.4\, t.$ By Eq. 317,

$$L = \int P\, dt = \int_0^{30} 0.4\, t\, dt = \Big[0.2\, t^2\Big]_0^{30} = 180 \text{ lb-sec}$$

The line of action of L coincides with the line of action of P, and the sense of L agrees with the sense of P.

Since the spring is considered to be weightless, the force at the fixed end of the spring will be equal to P at all times, but opposite in sense. Its linear impulse for the interval is 180 lb-sec, collinear with the linear impulse of P, but opposite in sense.

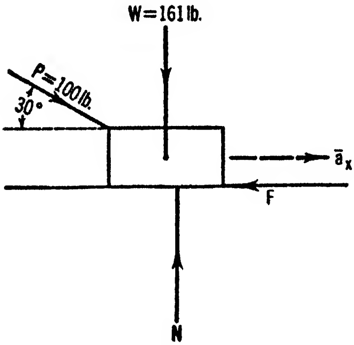

Fig. 525

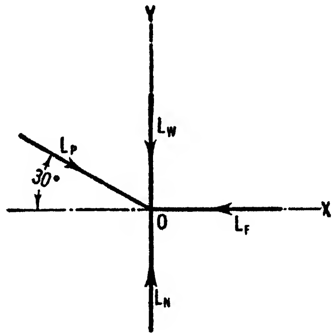

Fig. 526

1187. The body shown in Fig. 525 weighs 161 lb. It is pushed along a horizontal plane by a constant force, P, as shown. The coefficient of friction is 0.2. Calculate the linear impulse of each of the external forces acting on the body, for the interval during which the speed changes from 10 to 20 ft per sec. Find the resultant linear impulse for the entire system.

Solution. By Eqs. 184 and 185, Art. 126, and Eq. 32, Art. 81,

$$\Sigma F_x = \frac{W}{g}\bar{a}_x \qquad +100 \cos 30° - F = \frac{161}{32.2}(+\bar{a}_x)$$

$$\Sigma F_y = \frac{W}{g}\bar{a}_y \qquad -100 \sin 30° + N - 161 = \frac{161}{32.2}(0)$$

$$F = \mu N \qquad F = 0.2\, N$$

The solution of these equations gives

$$\bar{a}_x = 8.88 \text{ ft/sec}^2 \qquad F = 42.2 \text{ lb} \qquad N = 211 \text{ lb}$$

By Eq. 129, Art. 111, the time that elapses while the velocity of the body changes from 10 to 20 ft/sec is found as follows:

$$v = v_0 + at \qquad t = \frac{v - v_0}{a} = \frac{20 - 10}{8.88} = 1.13 \text{ sec}$$

By means of Eq. 318, the linear impulses of the four external forces, for the given interval, can now be calculated.

$$L = Ft \qquad \begin{aligned} L_W &= 161 \times 1.13 = 182 \text{ lb-sec} \\ L_P &= 100 \times 1.13 = 113 \text{ lb-sec} \\ L_F &= 42.2 \times 1.13 = 47.7 \text{ lb-sec} \\ L_N &= 211 \times 1.13 = 238 \text{ lb-sec} \end{aligned}$$

The vectors representing the four linear impulses are shown in Fig. 526. These vectors show only the correct inclinations and senses of the impulses; the exact positions of their lines of action in space need not be found.

The resultant linear impulse of the entire system of forces is found in the usual way. Let this resultant be represented by L_R.

$$L_{R_x} = \Sigma L_x \qquad L_{R_x} = +113 \cos 30° - 47.7 = +50.2 \text{ lb-sec}$$
$$L_{R_y} = \Sigma L_y \qquad L_{R_y} = -113 \sin 30° - 182 + 238 = 0$$

Therefore, the resultant linear impulse of the system has a magnitude of 50.2 lb-sec and is horizontal, and toward the right.

PROBLEMS

1188. A certain force acts at a constant, upward inclination of 4 (horizontal) to 3 (vertical). The magnitude of the force varies in accordance with the law $P = 10\sqrt{t}$, in which P is in pounds and t is in seconds. Calculate the linear impulse of the force for the interval from $t = 0$ to $t = 9$ sec. *Ans.* 180 lb-sec.

1189. A certain helical spring, whose modulus is 5 lb per in., is fixed at one end. A variable, axial force is applied to the other end, of such a nature that the elongation of the spring occurs in accordance with the law $e = \sqrt[3]{t}$. In this formula e is in inches and t is in seconds. Calculate the linear impulse of the force for the interval during which e varies from 0 to 4 in.

1190. A certain automobile, moving in still air, accelerates uniformly from rest and attains a speed of 60 mi per hr, in 40 sec. The air resistance for the particular car under consideration is given by the formula $P = 0.011v^2$, in which P is the resistance in pounds and v is the speed of the car in feet per second. Calculate the linear impulse of the air resistance for the period of acceleration. *Ans.* 1140 lb-sec.

1191. A ball weighing 10 lb is dropped from a height of 300 ft above the earth. Calculate the linear impulse of the weight of the ball for the interval during which the ball falls to the earth. Disregard air resistance.

1192. A body weighing 500 lb is placed on an incline whose slope is 4 (horizontal) to 3 (vertical). The coefficient of kinetic friction is 0.2. Calculate the linear impulse of each external force acting on the body, and the resultant linear impulse of all the forces, for the interval during which the body moves 300 ft down the incline, starting from rest. *Ans.* $L_W = 3250$ lb-sec, $\theta_x = 270°$; $L_N = 2600$ lb-sec, $\theta_x = 126° 50'$; $L_F = 520$ lb-sec, $\theta_x = 36° 50'$; $L = 1430$ lb-sec, $\theta_x = 216° 50'$.

1193. Calculate the linear impulse of each external force acting on the body in the first part of Prob. 669, Fig. 361, for the interval during which the velocity of the body changes from 10 to 20 ft per sec. Also calculate the resultant linear impulse of all the forces.

1194. Calculate the linear impulse of each external force acting on the cylinder in Prob. 954, Fig. 446, for the interval during which the velocity of the center point changes from 15 to 30 ft per sec. Also calculate the resultant linear impulse of all the forces. *Ans.* $L_P = +225$ lb-sec; $L_F = +75$ lb-sec; $L_W = -4830$ lb-sec; $L_N = +4830$ lb-sec; $L = 300$ lb-sec, $\theta_x = 0°$.

196. Linear Momentum of a Particle. *The linear momentum of a particle at any given instant is a vector quantity whose magnitude is equal to $(dW/g)v$, and whose inclination and sense are the same as the inclination and sense of the velocity of the particle.* In the foregoing expression for the magnitude of the linear momentum, dW represents the weight of the particle, and v represents the velocity of the particle at the given instant.

Linear momentum, like linear impulse, is treated as a *localized* vector quantity. It is considered that the line of action of the linear momentum of a particle passes through the particle.

197. Linear Momentum of a Body. *The linear momentum of a finite body at any given instant is the resultant of the linear momenta of all the particles of the body at that instant.*

Formulas will now be derived for the calculation of the linear momentum of a finite body at any given instant. These formulas will be valid for use in connection with either rigid or non-rigid bodies, moving in any manner.

The Formulas. Let Fig. 527 represent one of the particles of a rigid or non-rigid body having any kind of motion. The vector $(dW_1/g)v_1$ represents the linear momentum of this particle at the given instant. Let OX, OY, and OZ represent any convenient set of stationary coordinate axes. Let θ_{1x} represent the angle between the linear momentum of particle 1 and the x-axis. Let the other particles of the body be numbered 2, 3, 4, etc., their linear momenta, angles of inclination, and coordinates carrying the corresponding subscripts. The velocity of particle 1 has not been shown in the figure, but, in accordance with the definition given in Art. 196, the linear momentum of the particle must agree with it in inclination and sense.

Let the linear momentum of the entire body at the given instant be represented by T.

By the definition, the linear momentum of the body at the given instant is the resultant of the linear momenta of all the particles of the body at that instant. It follows that the component of the linear momentum of a finite body, along any axis, is equal to the algebraic sum of the components of the linear momenta of the particles along that axis. Therefore,

$$T_x = \frac{dW_1}{g} v_1 \cos \theta_1 + \frac{dW_2}{g} v_2 \cos \theta_2 + \frac{dW_3}{g} v_3 \cos \theta_3 \cdots, \text{ etc.} \qquad [319]$$

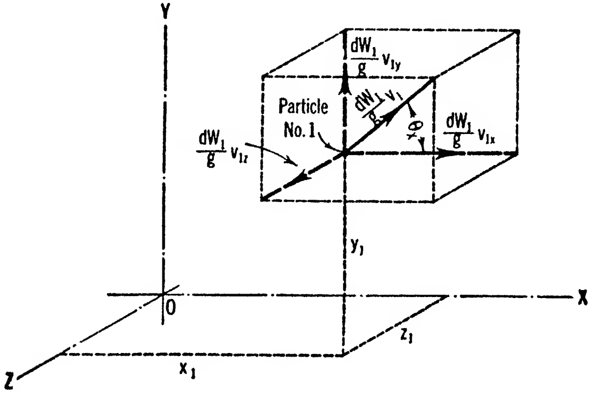

FIG. 527

The summation indicated by the right-hand side of Eq. 319 is to be understood to extend to all the particles of the body at the given instant. Since $v_1 \cos \theta_1 = v_{1x}$, $v_2 \cos \theta_2 = v_{2x}$, etc., Eq. 319 may be written as follows:

$$T_x = \frac{dW_1}{g} v_{1x} + \frac{dW_2}{g} v_{2x} + \frac{dW_3}{g} v_{3x} \cdots, \text{ etc.} \qquad [320]$$

Replacing v_{1x} by dx_1/dt, v_{2x} by dx_2/dt, etc., and multiplying through by $(g\,dt)$,

$$gT_x\,dt = dW_1\,dx_1 + dW_2\,dx_2 + dW_3\,dx_3 \cdots, \text{ etc.} \qquad [321]$$

Integrating Eq. 321,

$$\int gT_x\,dt = dW_1x_1 + dW_2x_2 + dW_3x_3 \cdots, \text{ etc.} \qquad [322]$$

The right-hand side of Eq. 322 is equivalent to $\int x\,dW$ for the entire body at the given instant, this being the first moment of the weight of the body with respect to OZ. The first moment may be represented by $W\bar{x}$, in which W represents the total weight of the body, and $\bar{x}$ represents the x-coordinate of the center of gravity of the body at the given instant. Equation 322 may now be written,

$$\int gT_x\,dt = W\bar{x} \qquad [323]$$

Now let Eq. 323 be differentiated with respect to t. Any constants of integration that might have been introduced in the integration performed above would be eliminated by the differentiation now to be performed. The result of the differentiation is

$$T_x = \frac{W}{g}\frac{d\bar{x}}{dt} \qquad [324]$$

The quantity $d\bar{x}/dt$, in Eq. 324, is the x-component of the velocity of the center of gravity of the body at the given instant. Let this be represented by $\bar{v}_x$. Therefore,

$$\boldsymbol{T}_x = \frac{\boldsymbol{W}}{\boldsymbol{g}}\bar{\boldsymbol{v}}_x \qquad [325]$$

It is obvious that similar formulas could be derived for the y- and z-components of T.

Furthermore, it can easily be shown that

$$\boldsymbol{T} = \frac{\boldsymbol{W}}{\boldsymbol{g}}\bar{\boldsymbol{v}} \qquad [326]$$

in which T represents the resultant linear momentum of the body. Equation 325 or Eq. 326 usually makes it possible to calculate the linear momentum of a body without the use of the calculus for that purpose.

Nothing in the foregoing analysis requires that the body be thought of as rigid. In fact, the results may be used in connection with a group of entirely disconnected bodies, considered collectively as a non-rigid body.

It is possible to conceive of a situation in which T_x, T_y, and T_z would all be equal to zero at the given instant. Such a condition could be interpreted only as meaning that the body, as a whole, had no linear momentum along any axis. The resultant momentum, however, might be of the nature of a couple, in which case the body would have angular momentum about certain axes. Angular momentum will be discussed in a later article.

Units. The unit of linear momentum is the same as the unit of linear impulse. In the English system the pound-second is used most frequently. Particular attention should be paid to the fact that the use of the usual value of 32.2, or thereabouts, for g, necessitates expressing the velocity in feet per second. The usual expression for linear momentum is made up as follows: $\dfrac{\text{lb}}{\text{ft/sec}^2}\dfrac{\text{ft}}{\text{sec}} = \text{lb-sec}.$

Illustrative Problem

1195. Figure 528 represents a homogeneous half-cylinder 4 ft in diameter and weighing 322 lb. At the instant represented by the figure the body has clockwise angular velocity of 1.6 rad per sec. It rolls, without slipping, on a horizontal surface. Find the linear momentum of the body at the given instant.

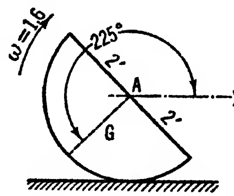

Fig. 528

Solution. Let G represent the center of gravity of the body. From Prob. 496, Art. 97,

$$\overline{AG} = \frac{4r}{3\pi} = \frac{4 \times 2}{3\pi} = 0.849 \text{ ft}$$

Let v_A represent the linear velocity of the point A, at the center of the plane face of the body.

$$v_A = r\omega = 2 \times 1.6 = 3.2 \text{ ft/sec}$$

Applying Eqs. 238 and 239, of Art. 161, to the points A and G, designating A as particle 1 and G as particle 2,

$$\begin{aligned} v_{2x} - v_{1x} &= -q\omega\sin\theta & \qquad \bar{v}_x - v_{Ax} &= -\overline{AG}\omega\sin 225° \\ & & \bar{v}_x - 3.2 &= -0.849\,(-1.6)\,(-0.707) \\ v_{2y} - v_{1y} &= q\omega\cos\theta & \bar{v}_y - v_{Ay} &= \overline{AG}\omega\cos 225° \\ & & \bar{v}_y - 0 &= 0.849\,(-1.6)\,(-0.707) \end{aligned}$$

The solution of the foregoing equations gives

$$\bar{v}_x = +2.24 \text{ ft/sec} \qquad \bar{v}_y = +0.96 \text{ ft/sec}$$

By Eq. 325,

$$T_x = \frac{W}{g}\bar{v}_x \qquad T_x = \frac{322}{32.2}(+2.24) = +22.4 \text{ lb-sec}$$

$$T_y = \frac{W}{g}\bar{v}_y \qquad T_y = \frac{322}{32.2}(+0.96) = +9.6 \text{ lb-sec}$$

$$T = \sqrt{T_x^2 + T_y^2} = \sqrt{(22.4)^2 + (9.6)^2} = 24.4 \text{ lb-sec}$$

$$\theta_x = \text{arc tan}\frac{T_y}{T_x} = \text{arc tan}\frac{9.6}{22.4} = \text{arc tan } 0.429 = 23°\ 15'$$

The sense of T is upward and toward the right.

PROBLEMS

1196. An automobile weighing 3600 lb has a speed of 70 mi per hr at a certain instant. Calculate the linear momentum of the car at that instant. *Ans.* 11,500 lb-sec.

1197. A solid, cast-iron cylinder 24 in. in diameter and 6 in. long rolls on a plane surface, without slipping. The cylinder rolls with a constant angular velocity of 90 rpm. Calculate the linear momentum of the cylinder at any instant. The material weighs 450 lb per cu ft.

1198. A wheel 3 ft in diameter, weighing 644 lb, rolls toward the right on a plane surface, without slipping. The center of gravity of the wheel is 6 in. from the geometric center. At a certain instant the center of gravity is level with the geometric center. The linear velocity of the geometric center is 6 ft per sec. Calculate the linear momentum of the wheel at the given instant. *Ans.* 126 lb-sec, $\theta_x = 341° 35'$.

1199. A straight, uniform bar 2.5 ft long, weighing 16.1 lb, rotates about an axis through one end. The axis of rotation is at a constant angle of 30° with the longitudinal axis of the bar. Calculate the linear momentum of the bar at the instant when the angular velocity is 600 rpm.

1200. The bar in Fig. 449, Art. 165, is 5 ft long and weighs 96.6 lb. The end A moves vertically and the end B moves horizontally. In the position shown by the figure the velocity of A is 9 ft per sec, downward. Calculate the linear momentum of the bar. *Ans.* 22.5 lb-sec, $\theta_x = 323° 10'$.

1201. Calculate the linear momentum of the wheel in Prob. 880, Fig. 411.

1202. Supply the missing half of the cylinder in Prob. 1195, Fig. 528. Calculate the linear momentum of the added half. Using this result, together with the result obtained in Prob. 1195, calculate the linear momentum of the entire cylinder. Check the result by calculating the linear momentum of the complete cylinder directly from Eq. 326. *Ans.* 64.0 lb-sec, $\theta_x = 0°$.

198. Relation between External Forces and Linear Momentum. *The algebraic sum of the components, along any axis, of the external forces acting on a body at any given instant, is equal to the time rate at which the component, along that axis, of the linear momentum of the body is changing at the given instant.*

The foregoing principle is valid for either rigid or non-rigid bodies, moving in any manner.

Proof. The following relation, applying either to rigid or to non-rigid bodies, moving in any manner, is found in Eq. 184, Art. 126:

$$\Sigma F_x = \frac{W}{g}\bar{a}_x \qquad [327]$$

By Eq. 172, Art. 118, $a_x = \dfrac{dv_x}{dt}$. Substituting in Eq. 327,

$$\Sigma F_x = \frac{W}{g}\frac{d\bar{v}_x}{dt} = \frac{d\left(\frac{W}{g}\bar{v}_x\right)}{dt} \qquad [328]$$

The quantity $(W/g)\bar{v}_x$, in Eq. 328, is equal to T_x, the x-component of the linear momentum of the body, as is shown by Eq. 325. Therefore,

$$\Sigma F_x = \frac{dT_x}{dt} \qquad [329]$$

Equation 329 is the algebraic expression of the principle stated at the beginning of the article, and the derivation of the equation constitutes a proof of the principle.

199. Conservation of Linear Momentum. *If the components of the external forces acting on a body, along any given axis, are balanced, and remain so during an interval of time, the component of the linear momentum of the body along the given axis is constant throughout the interval.*

The foregoing statement is called the *principle of the conservation of linear momentum.* Although it is merely a special case of the principle discussed in Art. 198, its usefulness in certain types of problems renders it worthy of special attention.

Proof. If the components of the external forces, along a given axis, are balanced, their algebraic sum is equal to zero. In such a case Eq. 329, Art. 198, becomes: $(dT_x/dt) = 0$. The expression (dT_x/dt) represents the time rate at which the x-component of the linear momentum of the body is changing at any instant. Since in the present case this rate of change is zero, and remains so during the interval, T_x must necessarily be constant throughout the interval.

The principle of the conservation of linear momentum is valid for either a rigid or a non-rigid body, having any type of motion.

Illustrative Problem

1203. Figure 529 represents a car, C, weighing 644 lb, and a block, B, weighing 161 lb. The car is coasting, toward the right, on a horizontal surface. When the forward end of the car is under the block the latter is dropped onto the horizontal floor of the car. At the instant of contact the absolute velocity of the car is 10 ft per sec, toward the right, and the absolute velocity of the block is 5 ft per sec, horizontal and toward the left. The block remains on the car floor, finally coming to rest relative to the car. The coefficient of kinetic friction for the surface of contact is 0.3. Assume that there are no resistances impeding the motion of the car except that caused by the block. Calculate the common velocity of the two bodies after the block has come to rest relative to the car. Calculate the time consumed by the block in sliding on the car floor. Calculate the minimum necessary length of the car.

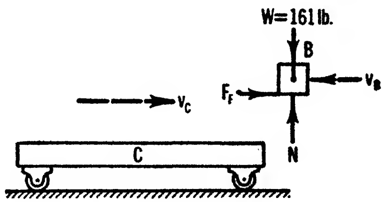

Fig. 529

Solution. Let v_C and v_B represent the initial absolute velocities of the car and the block, respectively. Let v represent the final velocity, common to the two bodies. Let T_C and T_B represent the initial linear momenta of the bodies, and let T represent the final linear momentum of the pair.

Considering the car and the block as one body after they are in contact, it can be seen that there are no horizontal external forces. Therefore, by the principle of the conservation of linear momentum, the total horizontal momentum of the two bodies is constant throughout the period of contact. Therefore,

$$T = T_C + T_B \qquad \frac{W_C + W_B}{g} v = \frac{W_C}{g} \bar{v}_C + \frac{W_B}{g} \bar{v}_B$$

$$\frac{644 + 161}{32.2} v = \frac{644}{32.2}(+10) + \frac{161}{32.2}(-5)$$

$$v = +7 \text{ ft/sec}$$

Considering the block as a free body, the external forces acting after contact with the car are W, F_F, and N, as shown in the figure. By inspection, $N = W = 161$ lb.

$$F = \mu N \qquad F_F = 0.3 \times 161 = 48.3 \text{ lb}$$

Let t represent the time consumed by the block in coming to rest relative to the car. By the principle of Art. 198, the component-sum of the external forces acting on the block, along the horizontal, is equal to the rate of change of the horizontal linear momentum of that body. This momentum changes uniformly with respect to time, the forces being constant. Therefore, for the block,

$$+F_F = \frac{\frac{W_B}{g} v - \frac{W_B}{g} v_B}{t}$$

$$+48.3 = \frac{\frac{161}{32.2}(+7) - \frac{161}{32.2}(-5)}{t}$$

$$t = 1.24 \text{ sec}$$

The velocity of the block relative to the car, at the first instant of contact, is, by inspection, -15 ft/sec. It is obvious that this relative velocity decreases uniformly, with respect to time. Let A represent the acceleration of the block relative to the car. Therefore, for the relative motion,

$$v = v_0 + at \qquad 0 = -15 + A\,(1.24) \qquad A = +12.1 \text{ ft/sec}^2$$

$$v^2 = v_0^2 + 2as \qquad 0 = (-15)^2 + 2 \times 12.1s$$

$$s = 9.3 \text{ ft}$$

Therefore, the length of the car must be equal to 9.3 ft plus the length of the block.

PROBLEMS

1204. A wooden block weighing 20 lb is suspended at the end of a long, fine wire. A rifle bullet weighing 140 grains is shot centrally and horizontally into the block, at a velocity of 3000 ft per sec, and remains embedded therein. Calculate the velocity imparted to the block. One pound equals 7000 grains. *Ans.* 3 ft/sec.

1205. A wooden block weighing 20 lb is suspended at the end of a long, fine wire, and is caused to swing in a vertical plane as a simple pendulum. At the lowest point of the swing the velocity is 10 ft per sec. At this instant a bullet weighing 140 grains is shot centrally and horizontally into the block, at a velocity of 3000 ft per sec, and remains embedded therein. The bullet and the block are moving in the same direction. Calculate the velocity of the block at the instant when the penetration ceases. *Ans.* 13 ft/sec.

1206. Solve Prob. 1205, if the block and bullet are moving in opposite directions when contact occurs.

1207. Solve Prob. 1205, if the bullet passes completely through the block and emerges with a residual velocity of 500 ft per sec. *Ans.* 12.5 ft/sec.

1208. The case of a certain shrapnel shell weighs 9 lb, and the balls contained therein have a total weight of 6 lb. Before bursting, the shell is moving with a velocity of 800 ft per sec. The bursting charge is designed to add 250 ft per sec to the velocity of the balls. Calculate the velocity of the shell case after the burst. The case remains intact, only the head being blown off.

1209. The recoiling portion of a certain field gun weighs 1000 lb. The shell weighs 15 lb, and the powder charge weighs 1.5 lb. The muzzle velocity is 1800 ft per sec. The velocity of the powder charge is equal to one-half the velocity of the shell. Calculate the velocity of recoil at the instant when the shell leaves the muzzle of the gun. Assume that the gun is fired horizontally, and that it recoils without resistance. *Ans.* 28.4 ft/sec.

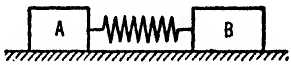

Fig. 530

1210. Two bodies are placed on a smooth, horizontal plane, with a helical spring between them, as shown in Fig. 530. A weighs 20 lb and B weighs 10 lb. The modulus of the spring is 40 lb per in. The normal length of the spring is 9 in. The bodies are moved nearer to each other until the clear distance between them is 6 in. They are then suddenly released, from rest. Calculate the velocity of each body at the instant when they are again 9 in. apart. Use the principle of work and energy, and the principle of the conservation of linear momentum.

1211. After the two bodies in Prob. 1210 have been pushed toward each other they are tied together, at a clear distance of 6 in., by means of a cord. The entire assembly is then pulled toward the left until a velocity of 10 ft per sec is reached. At this instant the propulsive force is removed, and the cord suddenly breaks. Calculate the velocity of each body at the instant when the spring regains its normal length. *Ans.* $v_A = -14.0$ ft/sec; $v_B = -1.98$ ft/sec.

200. Relation between Linear Impulse and Linear Momentum.

The algebraic sum of the components, along any axis, of the linear impulses of the external forces acting on a body, during any finite interval, is equal to the component, along that axis, of the linear momentum of the body at the instant ending the interval, minus the corresponding component at the instant beginning the interval.

The foregoing statement is called the principle of linear impulse and momentum, and applies to either rigid or non-rigid bodies, moving in any manner, under the influence of either variable or constant external forces.

Proof. Let F_1, F_2, F_3, etc., represent the various external forces acting on the body during the given finite interval. In Eq. 329, of Art. 198, the expression ΣF_x represents the algebraic sum of the x-components of the external forces at any given instant. Let ΣF_x be replaced by the equivalent expression, $(F_{1x} + F_{2x} + F_{3x} \cdots$, etc.), and let Eq. 329 be written as follows:

$$F_{1x}\,dt + F_{2x}\,dt + F_{3x}\,dt \cdots, \text{etc.} = dT_x \qquad [330]$$

Let T'_x and T''_x represent the x-components of the linear momentum of the body at the instants beginning and ending the interval, respectively. Integrating Eq. 330,

$$\int F_{1x}\,dt + \int F_{2x}\,dt + \int F_{3x}\,dt \cdots, \text{etc.} = \int_{T'_x}^{T''_x} dT_x = T''_x - T'_x \qquad [331]$$

By comparison with Eq. 315, of Art. 194, it is seen that $\int F_{1x}\,dt = L_{1x}$, $\int F_{2x}\,dt = L_{2x}$, etc., showing that the left-hand side of Eq. 331 represents the algebraic sum of the x-components of the linear impulses of the external forces for the given time interval. Denoting this sum by ΣL_x, Eq. 331 can then be written as follows:

$$\Sigma L_x = T''_x - T'_x \qquad [332]$$

which constitutes a proof of the principle of linear impulse and momentum.

In the application of the principle, or of Eq. 332, to a specific problem, strict attention must be paid to the algebraic signs of the impulse components, and also to the signs of the initial and final momentum components. If these precautions are taken, and the principle, or the formula, is applied exactly as it reads, there will be no confusion in the matter of signs.

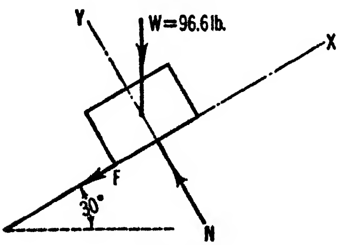

Fig. 531

The principle of linear impulse and momentum provides a rapid and convenient method of solution of problems in which force, time, and velocity are directly involved. This is especially true if the acceleration is not required.

Illustrative Problem

1212. The body shown in Fig. 531 weighs 96.6 lb. It is projected up the inclined plane with an initial velocity of 60 ft per sec. The coefficient of friction is 0.2 at all times. After a time the body comes to rest, and begins to descend the plane. Calculate the time consumed in moving up the incline. Calculate the time required for the body to attain a velocity of 60 ft per sec down the incline.

Solution.

$$\Sigma F_y = 0 \qquad N - W \cos 30° = 0 \qquad N = 96.6 \cos 30° = 83.7 \text{ lb}$$
$$F = \mu N \qquad F = 0.2 \times 83.7 = 16.7 \text{ lb}$$

Let t_1 represent the time consumed in moving up the incline, and let t_2 represent the time consumed in moving down the incline. By Eq. 332:

for the upward journey,

$$\Sigma L_x = T_x'' - T_x' \qquad -(96.6\, t_1) \sin 30° - 16.7\, t_1 = 0 - \left(+\frac{96.6}{32.2}\, 60\right)$$
$$t_1 = 2.77 \text{ sec}$$

for the downward journey,

$$-(96.6\, t_2) \sin 30° + 16.7\, t_2 = \left(-\frac{96.6}{32.2}\, 60\right) - 0$$
$$t_2 = 5.70 \text{ sec}$$

PROBLEMS

1213. A body falls from rest, without air resistance. Calculate the velocity at the end of 20 sec, by the method of linear impulse and momentum. Check by two other methods. *Ans.* 644 ft/sec.

1214. A ball is thrown vertically upward with an initial velocity of 60 ft per sec. How much time will elapse before the ball has a velocity of 40 ft per sec, downward? Solve by the method of linear impulse and momentum, writing the equation for the entire interval and solving directly for the required result. Check by another method. Disregard air resistance.

1215. It has been stated that a certain long-range cannon used during World War I fired a shell weighing 264 lb, with a muzzle velocity of 5500 ft per sec, and that the shell traversed the barrel of the gun in 0.02 sec. Assuming these values to be correct, calculate the average resultant force acting on the shell in its journey through the barrel of the gun. *Ans.* 1130 tons.

1216. A body weighing 100 lb starts from rest and slides down an inclined plane whose slope is 4 (horizontal) to 3 (vertical). The coefficient of kinetic friction is 0.3. Calculate the velocity of the body at the end of 30 sec.

1217. The engine of a certain automobile develops an average tractive effort of 400 lb, while the speed of the car is being increased from 10 to 40 mi per hr. The total wind and frictional resistance has an average value of 150 lb. Calculate the time required to accomplish the speed change, if the car is climbing a 6 per cent grade. The car weighs 3000 lb. *Ans.* 58.6 sec.

1218. Solve Prob. 675, Art. 126, by the method of linear impulse and momentum.

1219. A body weighing 200 lb is pushed up a 75 per cent grade by a constant force, P, applied horizontally. The coefficient of kinetic friction is 0.25. The speed increases from 50 to 100 ft per sec, in 25 sec. Calculate the magnitude of P. *Ans.* 265 lb.

201. Pressure Caused by the Diversion of a Jet or Stream. A jet or stream of water, or other fluid, exerts a force on any object that is placed in its path in such a manner as to cause any change in the velocity of the fluid. The principle of linear impulse and momentum is often used in the calculation of such forces. Two of the simpler cases will be discussed.

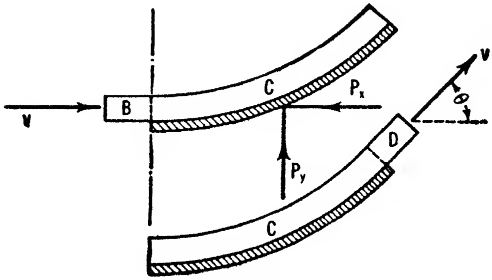

FIG. 532

Pressure of a Jet on a Stationary Vane. A jet is a stream discharged into space, from a pipe or reservoir. In Fig. 532 is represented a portion of a steady jet impinging on a stationary vane which deflects the jet through an angle, θ, in a horizontal plane. Let v represent the velocity of the jet. It will be assumed that the magnitude of v is not changed during the passage of the jet across the vane.

The body whose motion is to be studied is a portion of the jet, slightly longer than the vane, as shown in the figure. Let Δt represent the time that elapses while the body moves from the position shown in the upper portion of the figure to that shown in the lower portion. In the upper figure the body is represented as if divided, by an imaginary plane, into two parts, B and C. In the lower figure the body is divided into two parts, C and D. Let W_B, W_C, and W_D represent the weights of these parts. Let P_x and P_y represent the components of the force exerted on the body of fluid, by the vane.

Let T'_x and T''_x represent the x-components of the linear momentum of the body in the first and second positions, respectively. Let T_{Bx}, T_{Cx}, and T_{Dx} represent the x-components of the momenta of the portions B, C, and D.

Equation 332, of Art. 200, states that $\Sigma L_x = T''_x - T'_x$. In the present case, for the time interval Δt, $\Sigma L_x = -P_x\,\Delta t$. Also, $T''_x = T_{Cx} + T_{Dx}$, and $T'_x = T_{Bx} + T_{Cx}$. It follows that

$$-P_x\,\Delta t = T_{Cx} + T_{Dx} - T_{Bx} - T_{Cx} = T_{Dx} - T_{Bx} = \frac{W_D}{g}v\cos\theta - \frac{W_B}{g}v \qquad [333]$$

Let W_1 represent the weight of the fluid discharged by the jet per unit time. In steady flow W_1 is the same at all points of the jet. It follows that $W_B = W_D = W_1 \Delta t$. Equation 333 now reduces to

$$P_x = \frac{W_1}{g}(v - v\cos\theta) \qquad [334]$$

By means of a similar analysis it can be shown that

$$P_y = \frac{W_1}{g} v \sin\theta \qquad [335]$$

If the path of the jet across the vane is in a vertical plane, P_y will be increased by an amount equal to the weight of that portion of the jet in contact with the vane at any instant.

Equations 334 and 335 give the pressure exerted on the jet, by the vane. The pressure exerted on the vane by the jet is, of course, equal and opposite to that given by the equations.

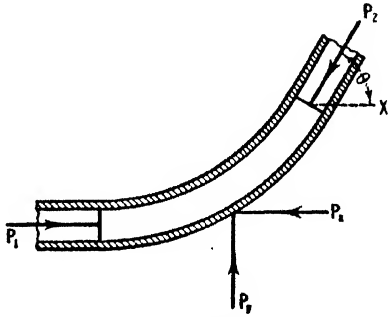

FIG. 533

Force Exerted at a Bend in a Pipe. The discussion will be limited to the usual case, in which the pipe is uniform in cross section. In Fig. 533, let P_x and P_y represent the x- and y-components of the force exerted on the body of fluid, by the pipe.

The analysis differs from that leading to Eqs. 334 and 335 in one respect, only. Those formulas were derived for a jet, surrounded by the atmosphere, and in the analysis no mention was made of the pressures exerted on the moving body of fluid by the bodies of fluid immediately preceding and following it in the jet. Such pressures exist, but since the jet is not enclosed they have values depending only on the pressure of the atmosphere. In the majority of hydraulic problems, atmospheric pressure acts in such a manner as to balance itself, and is thus of no effect on the values of the other forces concerned.

In the case now under discussion the fluid is enclosed in a pipe, and it is necessary to consider the forces acting on the ends of the body, since the pressures in a pipe are usually above atmospheric. These forces, in Fig. 533, are represented by P_1 and P_2.

The analysis is similar to that used in the case of the jet, and leads to the following formulas:

$$P_x = \frac{W_1}{g}(v - v\cos\theta) + P_1 - P_2\cos\theta \qquad [336]$$

$$P_y = \frac{W_1}{g} v\sin\theta + P_2\sin\theta \qquad [337]$$

If the bend is in a vertical plane, P_y will be increased by an amount equal to the weight of the fluid occupying the bend at any given instant.

Equations 336 and 337 give the pressure exerted on the fluid, by the pipe. The pressure exerted on the pipe, by the fluid, is equal and opposite to that given by the formulas. In a large pipe, placed above the ground, this pressure is often so large as to necessitate the use of heavy supports at the bends, to relieve the pipe of the excessive stress.

Illustrative Problem

1220. The pipe shown in Fig. 533 is 12 in. in diameter. The angle, θ, is 60°, and the velocity of the water in the pipe is 12 ft per sec. The unit pressure in the pipe is 20 lb per sq in., at both ends of the elbow. The elbow is in a horizontal plane. Calculate the x- and y-components of the resultant force exerted by the water, on the elbow.

Solution. The pressures, P_1 and P_2, in the formulas, represent total pressures on the entire cross section of the pipe. Therefore, in the present case,

$$P_1 = 20\,\pi(6)^2 = 2260 \text{ lb}$$

$$P_2 = 20\,\pi(6)^2 = 2260 \text{ lb}$$

W_1 represents the weight of the water discharged by the pipe, per second. This is equal to the weight of a cylindrical body of water 12 in. in diameter, whose length is equal to the distance traveled by the water in 1 sec. Therefore, assuming that the water weighs 62.4 lb per cu ft,

$$W_1 = \pi\left(\tfrac{1}{2}\right)^2 12 \times 62.4 = 588 \text{ lb/sec}$$

By Eq. 336,

$$P_x = \frac{W_1}{g}(v - v\cos\theta) + P_1 - P_2\cos\theta$$

$$P_x = \frac{588}{32.2}(12 - 12\cos 60°) + 2260 - 2260\cos 60°$$

$$P_x = +1240 \text{ lb}$$

By Eq. 337,

$$P_y = \frac{W_1}{g} v \sin\theta + P_2 \sin\theta$$

$$P_y = \frac{588}{32.2} 12 \sin 60° + 2260 \sin 60°$$

$$P_y = +2150 \text{ lb}$$

The pressures exerted on the pipe, by the water, are equal and opposite to the values of P_x and P_y found above.

PROBLEMS

1221. A cylindrical jet of water 3 in. in diameter, with a velocity of 80 ft per sec. impinges against a stationary, flat plate placed at right angles to the direction of the jet. Calculate the pressure exerted on the plate, in the direction of the jet. The water weighs 62.4 lb per cu ft. *Ans.* 609 lb.

1222. A cylindrical jet of water 3 in. in diameter, with a velocity of 80 ft per sec. impinges horizontally against a stationary vane. The vane deflects the jet through an angle of 60°, in a horizontal plane. Calculate the resultant horizontal pressure exerted on the vane.

1223. Solve Prob. 1222, if the jet is deflected 120°, all other data remaining unchanged. *Ans.* 1050 lb, $\theta_x = 30°$.

1224. Solve Prob. 1222, if the jet is deflected 180°, all other data remaining unchanged.

1225. A steady stream of bullets, each weighing 200 grains, is fired at the rate of 600 bullets per min, against a stationary target placed at right angles to the direction of fire. The velocity of the bullets is 2500 ft per sec. Calculate the average force exerted on the target. One pound equals 7000 grains. *Ans.* 22.2 lb.

1226. Solve Prob. 1220 for a 90° elbow, the other conditions being the same as in that problem.

1227. Solve Prob. 1220 for a 180° elbow, letting all the other conditions remain unchanged. *Ans.* $P_x = 4960$ lb; $P_y = 0$.

1228. Solve Prob. 1220 for a 120° elbow, letting all other conditions remain unchanged.

CHAPTER XXVII

ANGULAR IMPULSE AND ANGULAR MOMENTUM

202. Elementary Angular Impulse. *The angular impulse of a force about a given axis, during an elementary interval, is the moment, about that axis, of the linear impulse of the force during the given elementary interval.*

The moment of the elementary linear impulse about the given axis is calculated, and its sign determined, by the same methods as those used in calculating the moment of a force about an axis. If the axis about which the angular impulse is to be calculated is inclined to the line of action of the force, the elementary linear impulse is resolved into components, and the moment-sum of the components is calculated, just as is done in calculating the moment of a force about an inclined axis.

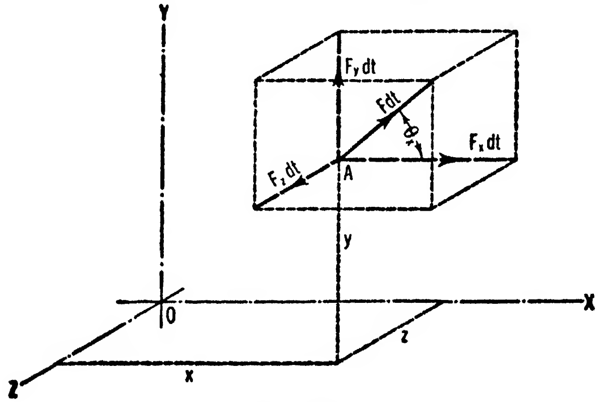

FIG. 534

203. Finite Angular Impulse; General Case. *The angular impulse of a force about a given axis, during a finite interval of time, is the algebraic sum, or integral, of all the elementary angular impulses of the force about the given axis, occurring during the finite interval.*

A formula for calculating the angular impulse of a force for a finite time interval will now be derived. The analysis will be made for the general case, in which the force is inclined to the given axis during the interval, and in which the moment of the force about the given axis varies.

The Formulas. Let the vector $F\,dt$, in Fig. 534, represent the linear impulse of a force, F, during an elementary interval, dt. Let θ_x represent the angle between $F\,dt$ and the x-axis. Let $F\,dt$ be resolved, at A, into its x-, y-, and z-components. The magnitude of the x-component is as follows: $(F\,dt)\cos\theta_x = (F\cos\theta_x)\,dt = F_x\,dt$. In a similar manner, the y- and z-components can be shown to be equal to $F_y\,dt$ and $F_z\,dt$, respectively.

Let the z-axis be the axis about which the moments are to be taken. Let λ_z represent the angular impulse of F about the z-axis for the finite interval under consideration. By the definition,

$$\lambda_z = \int(-F_x\,dt\,y + F_y\,dt\,x) = \int(-F_xy + F_yx)\,dt \qquad [338]$$

The integrations indicated in Eq. 338 are to be understood to extend throughout the finite time interval. The quantity $(-F_xy + F_yx)$ is equal to the moment of the force F about the z-axis. Representing this moment by M_z,

$$\lambda_z = \int M_z\,dt \qquad [339]$$

Thus it is seen that *the angular impulse of a force about a given axis, during a finite time interval, can be calculated by integrating the product of the moment of the force about the given axis, and the differential of the time.* It is necessary to know the law by which M_z varies with respect to t, in order to perform the integration. The integration extends throughout the finite time interval.

Units. There is no single name for the unit of angular impulse. The unit is designated by a combination of the names of the units used for the force, the moment-arm, and the time. The pound-foot-second is the unit most commonly used in the English system.

Illustrative Problem

1229. Figure 535 represents a wheel, 6 ft in diameter, mounted on a shaft and rotating in a counterclockwise direction in accordance with the law $\omega = 0.3t$, in which ω is expressed in radians per second, and t is expressed in seconds. A represents a particle on the rim of the wheel. The force P acts on the particle A, its line of action remaining tangent to the wheel at all times. The magnitude of P varies in accordance with the law $P = 10\,\omega$, in which P is expressed in pounds and ω is expressed in radians per second. Calculate the angular impulse of P, about the axis of rotation, for the interval from $t = 10$ sec to $t = 20$ sec.

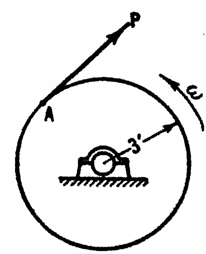

Fig. 535

Solution. Let M_z represent the moment of P about the axis of rotation, at any instant. From the figure, $M_z = -3\,P$. From the problem, $P = 10\,\omega$. Therefore, $M_z = -3\,(10\,\omega) = -30\,\omega$. From the problem, $\omega = 0.3t$. Therefore, $M_z = -9\,t$.

By Eq. 339,

$$\lambda_z = \int M_z\,dt \qquad \lambda_z = \int_{10}^{20} -9\,t\,dt = -9\left[\frac{t^2}{2}\right]_{10}^{20} = -1350 \text{ lb-ft-sec}$$

PROBLEMS

1230. A certain force, applied tangentially to the rim of a 4-ft pulley, varies in accordance with the law $P = 0.5\sqrt{t}$, in which P is in pounds and t is in seconds. Calculate the angular impulse of the force about the shaft of the pulley, for the interval during which t varies from 25 to 100 sec. *Ans.* 583 lb-ft-sec.

1231. The body shown in Fig. 523, Art. 194, weighs 100 lb. The force P has a constant magnitude of 20 lb, and the angle θ has a constant magnitude of 30°. The body is moving toward the right at a constant speed of 4 ft per sec. Calculate the angular impulse of the weight of the body, about a fixed axis through O at right angles to the plane of the figure, for the interval during which s varies from 0 to 50 ft.

1232. Calculate the angular impulse of the force P, in Prob. 1231, about the axis, and for the interval, designated in that problem. *Ans.* +3060 lb-ft-sec.

1233. The moment of a certain force about a given axis varies in accordance with the law $M_z = 10/\sqrt{t}$, in which M_z is in foot-pounds and t is in seconds. Calculate the angular impulse of the force about the given axis, for the interval during which t varies from 16 to 36 sec.

1234. A drum 4 ft in diameter is mounted on a shaft. One end of a wire is fastened to the drum, and the other end is attached to a helical spring whose normal length is 60 in., and whose modulus is 5 lb per in. The other end of the spring is fixed. The drum rotates at a constant speed of 0.01 rad per sec, the wire tightens, and the spring elongates. Calculate the angular impulse of the tension in the spring, about the shaft, for the interval during which the length of the spring changes from 70 to 80 in. *Ans.* 6250 lb-ft-sec.

204. Finite Angular Impulse; Special Case. In many of the engineering problems in which the facts concerning angular impulse can be utilized to advantage, the force whose angular impulse is to be calculated has a *constant moment* about the given axis, during the finite time interval. A formula will now be derived for the angular impulse of such a force.

The Formulas. Under the conditions stated above, the quantity M_z, in Eq. 339, of Art. 203, is constant during the interval, and may be placed outside the integral sign. Therefore,

$$\lambda_z = M_z \int dt = M_z t \qquad [340]$$

in which t represents the length of the finite time interval. Thus it is seen that *the angular impulse of a force whose moment is constant during the interval is equal to the product of the constant moment and the length of the time interval.*

PROBLEMS

1235. A cable which is being unwound tangentially from a 3-ft drum is under a constant tension of 500 lb. Calculate the angular impulse of the tension, about the shaft of the drum, for an interval of 6 min. *Ans.* 270,000 lb-ft-sec.

1236. A belt running on a 24-in. pulley has a tension of 200 lb on one side and 20 lb on the other. The pulley has a constant speed of 600 rpm. Calculate the resultant angular impulse of the two belt pulls, with respect to the axis of rotation of the pulley, for an interval during which the pulley describes 2400 rev.

1237. A motor delivers 20 hp to a machine, under constant load, at a constant speed of 1200 rpm. Calculate the angular impulse of the torque for an interval of 3 min. *Ans.* 15,800 lb-ft-sec.

1238. Calculate the constant force which, if applied tangentially to a 6-ft flywheel, would produce an angular impulse of 3,600,000 lb-ft-sec, about the axis of rotation, in an interval of 5 min.

1239. A belt running on a 30-in. pulley delivers 60 hp to the shaft, under constant load. The shaft rotates at a constant speed of 1500 rpm. Calculate the difference between the tensions in the two parts of the belt. Calculate the angular impulse of the torque, for an interval of 2 min. *Ans.* 25,200 lb-ft-sec.

205. Angular Momentum of a Particle. *The angular momentum of a particle about a given axis, at any instant, is the moment of the linear momentum of the particle at that instant, about the given axis.*

The moment of the momentum of a particle is calculated in accordance with the same rules as those followed in calculating the moment of a force.

Units. The unit of angular momentum is the same as the unit of angular impulse, which was described in Art. 203. The pound-foot-second is used most frequently.

206. Angular Momentum of a Rotating Body about the Axis of Rotation. *The angular momentum of any finite body about a given axis, at any instant, is the algebraic sum, or integral, of the angular momenta of all the particles of the body at that instant, about the given axis.*

It is possible to calculate the angular momentum of a translating body, and of a body having general plane motion. However, methods of solution involving angular momentum are used principally in connection with rotating bodies, and the discussion will be confined to this case. The axis of rotation is the axis about which the angular momentum of a rotating body is usually calculated, and the analysis will be further limited to this case.

The Formulas. Let the z-axis be placed so as to coincide with the axis of rotation. Let A, in Fig. 536, represent one of the particles of the rotating body. Let ρ represent the radius of the circle in which A moves. Let the vector $(dW/g)v$ represent the linear momentum of the particle. Obviously, ρ is also the moment-arm of the linear momentum, about the axis of rotation, OZ. Let τ_z represent the angular momentum of the body about the axis of rotation. By definition,

$$\tau_z = \int \rho \frac{dW}{g} v \tag{341}$$

In Eq. 341, v may be replaced by $\rho\omega$, in which ω represents the angular velocity of the body at the given instant. Then,

$$\tau_z = \int \rho \frac{dW}{g} \rho\omega = \omega \int \rho^2 \frac{dW}{g} \tag{342}$$

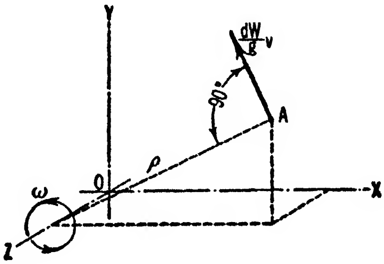

FIG. 536

The expression $\int \rho^2 (dW/g)$, in Eq. 342, is the moment of inertia of the body about the axis of rotation. Representing this by I_z, the equation becomes,

$$\tau_z = I_z\omega \tag{343}$$

Thus it is seen that *the angular momentum of a rotating body about the axis of rotation, at any instant, is equal to the moment of inertia of the body about that axis, multiplied by the angular velocity of the body at the given instant.*

Units. If the usual value of 32.2 is used for g in the calculation of I_z, the value of ω must be expressed in radians per second.

ILLUSTRATIVE PROBLEM

1240. A solid cylinder, 3 ft in diameter, is rotating about its geometric axis at a speed of 500 rpm. The weight of the cylinder is 300 lb. Calculate the angular momentum of the cylinder about the axis of rotation. A constant tangential force of 8 lb is applied to the periphery of the cylinder, bringing the latter to rest in 45.8 sec. Calculate the angular impulse of this force about the axis of rotation, for the period during which the cylinder is coming to rest.

Solution.

$$I_z = \frac{1}{2}\frac{W}{g}r^2 = \frac{300\,(1.5)^2}{2 \times 32.2} = 10.5 \text{ engineer's units}$$

$$\tau_z = I_z\omega = 10.5\left(\frac{500 \times 2\,\pi}{60}\right) = 550 \text{ lb-ft-sec}$$

$$\lambda_z = M_z t = (8 \times 1.5)\,45.8 = 550 \text{ lb-ft-sec}$$

PROBLEMS

1241. The moment of inertia of a certain flywheel about its axis of rotation is 5000 engineer's units. Calculate the angular momentum of the wheel about that axis at the instant when the angular velocity is 220 rpm. *Ans.* 115,000 lb-ft-sec.

1242. A straight, square steel bar 2 ft long, weighing 16.1 lb, rotates about an axis passing through its center at an angle of 60° with its longitudinal axis. The angular velocity is 480 rpm. Calculate the angular momentum of the bar. In calculating I_z assume the bar to be slender.

1243. A solid, cast-iron cylinder 3 ft in diameter and 6 in. long rotates about its geometric axis. The speed increases from 500 to 1000 rpm. Calculate the increase in angular momentum about the axis of rotation. The material weighs 450 lb per cu ft. *Ans.* 2910 lb-ft-sec.

1244. A solid, homogeneous sphere weighing 16.1 lb, and having a relatively small diameter, rotates about an axis at a constant angular velocity. The center of gravity of the sphere is 36 in. from the axis of rotation, and its linear velocity is 30 ft per sec. Calculate the angular momentum of the sphere about the axis of rotation. Solve by means of Eq. 343, then check by calculating the linear momentum of the sphere and its moment about the axis of rotation. Assume that the line of action of the linear momentum passes through the center of gravity of the sphere.

1245. The angular momentum of a certain pulley about the axis of rotation is 240 lb-ft-sec, at a speed of 200 rpm. Calculate the moment of inertia of the pulley about the axis of rotation.

207. Relation between External Forces and Angular Momentum of a Rotating Body. *The algebraic sum of the moments, about the axis of rotation, of the external forces acting on a rotating body at any instant, is equal to the time rate at which the angular momentum of the body, about that axis, is changing at the given instant.*

Proof. Let ΣM_z represent the algebraic sum of the moments, about the axis of rotation, of the external forces acting on the body at any given instant. In Art. 154 it was shown that $\Sigma M_z = I_z\alpha$, for a rotating body. This formula may be expressed as follows:

$$\Sigma M_z = I_z\frac{d\omega}{dt} = \frac{d(I_z\omega)}{dt} \qquad [344]$$

By Eq. 343, Art. 206, the quantity $I_z\omega$ is equal to τ_z, the angular momentum of the body about the axis of rotation. Therefore,

$$\Sigma M_z = \frac{d\tau_z}{dt} \qquad [345]$$

Equation 345 is the algebraic expression of the principle stated at the beginning of the article, and the derivation of the equation constitutes a proof of the principle.

208. Conservation of Angular Momentum of a Rotating Body. *If the moments of the external forces acting on a rotating body, about the axis of rotation, are balanced, and remain so during an interval of time, the angular momentum of the body about that axis is constant throughout the interval.*

The foregoing statement is called the *principle of the conservation of angular momentum.* Although it is merely a special case of the principle discussed in Art. 207, its usefulness in certain types of problems renders it worthy of special attention.

Proof. If the moments of the external forces about the axis of rotation are balanced, their algebraic sum is equal to zero. In such a case, Eq. 345 becomes $d\tau_z/dt = 0$. Thus the time rate of change of the angular momentum of the body about the axis of rotation is zero at all instants during the interval. It follows that τ_z remains constant throughout the interval.

Illustrative Problem

1246. Figure 537 represents a uniform, homogeneous timber, 6 ft in length, and weighing 125 lb. The timber rotates in a vertical plane, on a light, horizontal shaft passing through its center of gravity. The body is given a clockwise angular velocity of 0.6 rad per sec. At the instant when the timber is in a vertical position, a bullet weighing 0.04 lb is shot horizontally into the timber, toward the right, with a velocity of 2400 ft per sec. The bullet enters the timber at the point A, 1.5 ft from the axis of rotation, and remains embedded in the wood. Calculate the subsequent angular velocity of the timber. Disregard axle friction and air resistance.

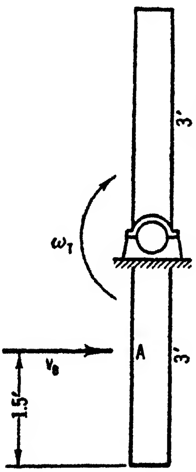

FIG. 537

Solution. Let I_T and I_B represent the moments of inertia of the timber and the bullet, respectively, with respect to the axis of rotation. By approximate methods,

$$I_T = \frac{1}{12}\frac{W}{g}L^2 = \frac{125\,(6)^2}{12 \times 32.2} = 11.6 \text{ engineer's units}$$

$$I_B = \frac{W}{g}r^2 = \frac{0.04\,(1.5)^2}{32.2} = 0.0028 \text{ engineer's unit}$$

Considering the timber and the bullet as one body, the external forces have practically no moments about the axis of rotation during the period of contact. Therefore, the principle of the conservation of angular momentum applies, for that axis. Let ω_T and ω_B represent the angular velocities of the two bodies at the first instant of contact, and let ω represent their common angular velocity after the penetration of the bullet ceases.

$$\omega_B = \frac{v_B}{r} = \frac{2400}{1.5} = +1600 \text{ rad/sec} \qquad \omega_T = -0.6 \text{ rad/sec}$$

$$I_T\omega_T + I_B\omega_B = (I_T + I_B)\,\omega$$

$$11.6\,(-0.6) + 0.0028\,(+1600) = (11.6 + 0.0028)\,\omega$$

$$\omega = -0.214 \text{ rad/sec}$$

PROBLEMS

1247. Reverse the direction of motion of the bullet in Prob. 1246, and solve. Calculate the kinetic energy of the system before, and after, the impact. How do you account for the reduction? *Ans.* -0.986 rad/sec.

1248. In Prob. 1246, how far from the axis of rotation would it be necessary to shoot the bullet into the timber in order to stop motion?

1249. Two pulleys are placed next to each other on the same shaft. The pulleys turn on the shaft, the shaft remaining stationary. One of the pulleys weighs 100 lb and has a radius of gyration of 12 in. about the axis of rotation. The other weighs 160 lb and has a radius of gyration of 15 in. The smaller pulley is rotating with an

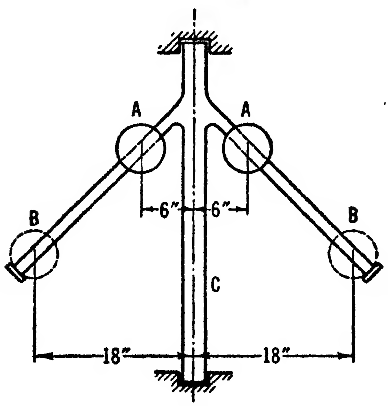

Fig. 538

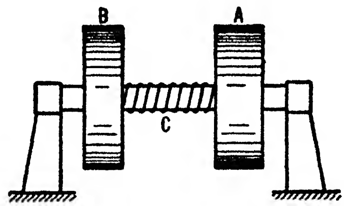

Fig. 539

angular velocity of 120 rpm, and the larger pulley has an angular velocity of 180 rpm in the same direction. The pulleys are provided with a mechanism by means of which they can be quickly clamped together while in motion. This is accomplished without introducing any forces external to the two pulleys. Calculate the angular velocity of the two pulleys after they have been joined. Disregard air resistance, and friction between the pulleys and the shaft. *Ans.* 163 rpm.

1250. Solve Prob. 1249 for the case in which the two pulleys are rotating in opposite directions before being clamped together.

1251. In Fig. 538 is shown a vertical shaft, C, equipped with two projecting arms. On each of these arms is placed a sphere weighing 8.05 lb, in the positions indicated by AA. The spheres are arranged in such a manner that they can be released and will slide along the arms to the positions BB. The moment of inertia of the shaft and arms, with respect to the axis of rotation, is 0.03 engineer's unit. The spheres may be considered to be small, relative to their radii of rotation. The entire system, with the spheres in their upper positions, is given an angular velocity of 60 rpm. Calculate the angular velocity of the system after one sphere has been permitted to drop to position B. Calculate the angular velocity after both spheres have dropped. Disregard friction. *Ans.* 14.2 rpm; 8.05 rpm.

1252. Figure 539 represents two disks mounted on a horizontal shaft. The disks rotate on the shaft, the shaft remaining stationary. Disk A has a moment of inertia of 4 engineer's units, and disk B has a moment of inertia of 1 engineer's unit. The spring, C, is attached to both disks. Disk B is held stationary, and disk A is turned in a clockwise direction until 10 ft-lb of energy has been "wound up" in the spring. Both disks are then held at rest for a moment, and released. Calculate the angular velocity of each disk at the instant when the spring has given up all its stress energy, assuming that the disks receive this energy. Disregard friction, and the weight of the spring. Let the directions of rotation be viewed from the right-hand end of the shaft.

1253. The spring in Prob. 1252 is "wound up" as there described, but the two disks are given a common angular velocity of 2 rad per sec, in a clockwise direction, before being released. Calculate the angular velocity of each disk at the instant when the spring has given up its energy. *Ans.* $\omega_A = -1$ rad/sec; $\omega_B = -6$ rad/sec.

1254. The spring in Prob. 1252 is "wound up" as there described, but the two disks are given a common angular velocity of 2 rad per sec, in a counterclockwise direction, before being released. Calculate the angular velocity of each disk at the instant when the spring has given up its energy.

209. Relation between Angular Impulse and Angular Momentum for a Rotating Body. *The algebraic sum of the angular impulses, about the axis of rotation, of the external forces acting on a rotating body, during any finite interval, is equal to the angular momentum of the body about that axis at the instant ending the interval, minus the corresponding angular momentum at the instant beginning the interval.*

The foregoing statement is called the principle of angular impulse and momentum.

Proof. Let the z-axis be taken as the axis of rotation. Let M_{1z}, M_{2z}, M_{3z}, etc., represent the moments of the various external forces about the axis of rotation, at any instant during the interval. In Eq. 345, of Art. 207, the expression ΣM_z represents the moment-sum of the external forces about the axis of rotation, at any instant. Let ΣM_z be replaced by the equivalent expression ($M_{1z} + M_{2z} + M_{3z} \cdots$, etc.), and let the relation expressed by Eq. 345 be written as follows:

$$M_{1z}\,dt + M_{2z}\,dt + M_{3z}\,dt \cdots, \text{etc.} = d\tau_z \qquad [346]$$

Let τ_z' and τ_z'' represent the angular momenta of the body about the axis of rotation at the instants beginning and ending the interval, respectively. Integrating, in Eq. 346,

$$\int M_{1z}\,dt + \int M_{2z}\,dt + \int M_{3z}\,dt \cdots, \text{etc.} = \int_{\tau_z'}^{\tau_z''} d\tau_z = \tau_z'' - \tau_z' \qquad [347]$$

Comparison with Eq. 339, of Art. 203, shows that $\int M_{1z}\,dt = \lambda_{1z}$, $\int M_{2z}\,dt = \lambda_{2z}$, etc., proving that the left-hand side of Eq. 347 represents the algebraic sum of the angular impulses of the external forces, about the axis of rotation, for the given time interval. Denoting this sum by $\Sigma\lambda_z$, Eq. 347 may now be written as follows:

$$\Sigma\lambda_z = \tau_z'' - \tau_z' \qquad [348]$$

which completes the proof of the principle of angular impulse and momentum of rotating bodies. If careful attention is paid to the algebraic sign of each angular impulse, and also to the signs of the initial and final angular momenta, there need be no confusion in applying the principle, or Eq. 348, in a specific problem.

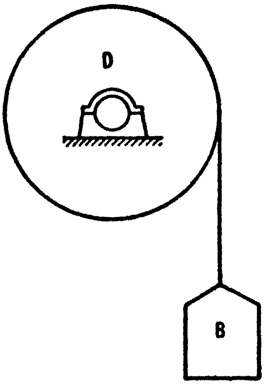

FIG. 540

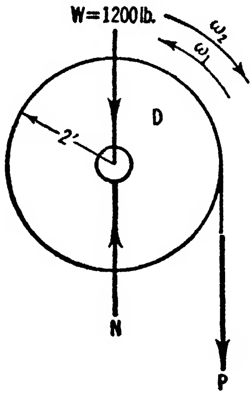

FIG. 541

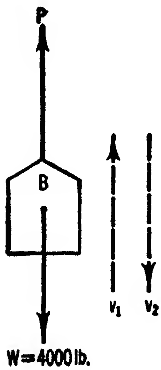

FIG. 542

ILLUSTRATIVE PROBLEM

1255. Figure 540 represents a drum, *D*, 4 ft in diameter, and weighing 1200 lb. The radius of gyration of the drum with respect to its axis of rotation is 22.5 in. A cable, a portion of which is wrapped around the drum, supports an elevator, *B*. The elevator weighs 2 tons. An angular velocity of 60 rpm is imparted to the drum, in a counterclockwise direction. The system is then permitted to coast. Eventually it comes to rest, reverses, and moves in the opposite direction. Calculate the total time elapsing until the drum attains an angular velocity of 90 rpm in a clockwise direction. Calculate the tension in the cable. Disregard friction, air resistance, and the weight of the cable.

Solution. Let I_z represent the moment of inertia of the drum, with respect to the axis of rotation.

$$I_z = \frac{W}{g}k^2 = \frac{1200}{32.2}\left(\frac{22.5}{12}\right)^2 = 131 \text{ engineer's units}$$

Let ω_1 and ω_2 represent the angular velocities of the drum at the initial and final instants, respectively. Let v_1 and v_2 represent the initial and final linear velocities of the elevator, B. Let P represent the tension in the cable. In Figs. 541 and 542 the drum and the elevator are represented as free bodies.

$$v_1 = r\omega_1 \qquad v_1 = 2\left(\frac{60 \times 2\pi}{60}\right) = 12.6 \text{ ft/sec, upward}$$

$$v_2 = r\omega_2 \qquad v_2 = 2\left(\frac{90 \times 2\pi}{60}\right) = 18.9 \text{ ft/sec, downward}$$

For the drum, by the principle of angular impulse and momentum,

$$\Sigma\lambda_z = \tau_z'' - \tau_z' \qquad \Sigma(M_z t) = I_z\omega_2 - I_z\omega_1$$

$$(-2P)t = 131\left(-\frac{90 \times 2\pi}{60}\right) - 131\left(+\frac{60 \times 2\pi}{60}\right)$$

For the elevator, by the principle of linear impulse and momentum,

$$\Sigma L_y = T_y'' - T_y' \qquad \Sigma(F_y t) = \frac{W}{g}v_2 - \frac{W}{g}v_1$$

$$+Pt - 4000\,t = \frac{4000}{32.2}(-18.9) - \frac{4000}{32.2}(+12.6)$$

The solution of the foregoing equations gives

$$t = 1.24 \text{ sec} \qquad P = 833 \text{ lb}$$

PROBLEMS

1256. A certain flywheel weighs 60,000 lb and has a radius of gyration of 8 ft. A constant torque of 5 ft-tons is applied. Calculate the time elapsing while the speed of the wheel changes from 50 to 100 rpm. Solve by the method of angular impulse and angular momentum, and check by two other methods. *Ans.* 62.4 sec.

1257. Solve Prob. 871, Fig. 405, by the method of angular impulse and angular momentum.

1258. Solve Prob. 876, Art. 155, by the method of angular impulse and angular momentum. *Ans.* $M_z = 3470$ ft-lb.

1259. A certain flywheel weighs 150,000 lb and has a radius of gyration of 10 ft. Calculate the constant torque required to increase the speed from 40 to 80 rpm in an interval of 3 min.

1260. Solve Prob. 877, Art. 155, by the method of angular impulse and angular momentum. *Ans.* 351 sec.

1261. Solve Prob. 878, Art. 155, by the method of angular impulse and angular momentum.

1262. A solid, homogeneous cylinder 1 ft in diameter, weighing 140 lb, is keyed to a short, horizontal shaft which is mounted in bearings in such a manner as to permit the system to rotate about its geometric axis. The shaft is 2 in. in diameter, and the total friction on the axle is 3 lb. The system is caused to rotate at a speed of 600 rpm, and is then permitted to coast until it comes to rest under the influence of the axle friction alone. How much time will be consumed in coming to rest? Disregard the inertia of the projecting portions of the shaft. *Ans.* 137 sec.

1263. A wheel 4 ft in diameter, weighing 644 lb, is mounted on a shaft. The radius of gyration is 1.8 ft. An initial speed of 300 rpm, clockwise, is imparted to the wheel. A constant force of 60 lb is applied to the rim, tangentially, in such a manner that it eventually brings the wheel to rest and causes it to rotate in the reverse direction. Calculate the total time elapsing up to the instant when the speed is 300 rpm, counterclockwise.

1264. In Prob. 1255, calculate the constant torque which, applied directly to the drum, would give the elevator a velocity of 20 ft per sec, upward, in an interval of 10 sec, starting from rest. *Ans.* 8630 ft-lb.

INDEX

The numbers refer to pages

A

Acceleration
  angular, 325, 352
  centripetal, 263
  complementary, 402
  "due to gravity," 273
  linear, 235, 251, 259, 261, 266
  of Coriolis, 402
  tangential and normal, 261, 330
  x- and y-components, 266
Acceleration-time curves, 236
Accuracy of calculations, 7
Algebraic signs of results, 19
Algebraic solutions, general features, 94
Algebraic sum
  its use in mechanics, 18
  significance of, 18
Angle
  of repose, 168
  of static friction, 168
Angles of inclination, 9, 63
Axes, selection of, 96
Axial stress, 126

B

Brake-bands, 185
British thermal unit, 448

C

Cables, suspended, 194, 195, 200
Catenary cable, 194, 200
Center of gravity
  applications, 217, 218, 222
  areas, special formulas, 212
  body, 206
  experimental methods, 228
  general formulas, 208
  graphic methods, 228
  line, area or volume, 207
  lines, special formulas, 210
Center of gravity
  motion of the, 278
  relation to symmetry, 209
  special formulas, 210, 212, 214
  volumes, special formulas, 214
Component of a force along a line, 16
Components, 6
  signs of, 17
Composition, 6, 91
Compressive stress, 127
Coriolis, acceleration of, 402
Couple, 32
  components of, 75
  moment of, 32
  representation by a single vector, 69
Couples, 32
  in parallel planes, 36
  in the same plane, 35
  permissible operations with, 37
Crank-and-connecting-rod mechanism, 245
Curvilinear motion, 251
  acceleration in, 251, 259
  linear velocity in, 251
  tangential and normal acceleration in, 259
  x- and y-components of acceleration in, 266
  x- and y-components of velocity in, 255

D

d'Alembert's principle, 277
Deceleration, 236
Distance of sliding, 436
Double rolling, 385
Dynamometer, 445

E

Effective force for a particle, 275
Effective forces and external forces, 277

Efficiency, 457
Energy, 447
  conservation of, 447, 459
  in general, 447
  kinetic, 448, 449, 450, 451, 453, 459
  mechanical, 448
  potential, 455
  stress, 456
  work as a measure of, 447
Equilibrant, 6, 289, 339, 374
Equilibrium
  equations of, 95
  general conditions for, 93, 94
  meaning and physical significance of, 93
  of the collinear system, 97
  of the coplanar concurrent system, 98, 101
  of the coplanar parallel system, 106, 108
  of the general coplanar system, 111, 114
  of the general non-coplanar system, 153
  of the non-coplanar concurrent system, 144
  of the non-coplanar parallel system, 149
  polygon, 40, 106, 111
  special condition for four forces, 111
  special condition for three forces, 99, 101, 106, 108, 111
  special condition for two forces, 97
Erg, 420

F

First moments, 210
  by finite summation, 216
Force, 1
  characteristics of a, 4
  effects of, 5
  kinds of, 2
  units of, 4
Force diagram, 20
Force polygon, 20
Force systems
  balanced, 6
  classification of, 6
  equivalence of, 5
Forces, external and internal, 2
  occurrence in pairs, 3
Formulas of statics and kinetics, differences, 277
Free-body diagrams, 95
Friction
  and normal pressure, 160, 162
  angle of kinetic, 176
  angle of static, 168
  coefficient of kinetic, 176
  coefficient of static, 150
  in general, 160
  kinetic, 161, 176
  limiting, 161
  static, 161, 162, 168, 173
Friction circle, 189
Funicular polygon, 40, 47, 106

G

Graphic solutions, general features, 94
Gravity axes, 210

H

Horsepower, 441
  brake, 445
  indicated, 445
Horsepower-hour, 420

I

Impending motion, 162
Impulse, angular
  elementary, 486
  finite, general case, 486
  finite, special case, 488
  relation to angular momentum, 494
Impulse, linear
  elementary, 465
  finite, general case, 465
  finite, special cases, 469
  relation to linear momentum, 479
Inclined plane, 177, 180
Independent equations, 95
Indicator, 388, 445
Inertia, 272, 274
  concept of mass, 274
Inertia-force method, 289
Input, 457

Instantaneous axis, 364
Instantaneous center, 364

J

Joule, 420
Journal bearings, 189

K

Kilowatt-hour, 420
Kinematics, definition of, 231
Kinetic energy, 448
  of a body in plane motion, 453
  of a particle, 449
  of a rotating body, 451
  of a translating body, 450
Kinetics
  definition of, 231
  of a body in general, 277
  of a particle, 271, 272

L

Laws of motion, Newton's, 271
Lever-arm, 27

M

Mass, 274
Mass center, 207
Mechanical equivalent of heat, 448
Members, 97
  two-force, 97
Moment
  about a line, 27, 66
  about a point, 27
  general case, 64
  physical significance of a, 28
  sign of, 27, 67
  special case, 26
Moment-arm, 27
Moment of inertia
  by finite summation, 314
  general formulas, 298, 299
  of irregular forms, 321
  of rods and plates, 309
  parallel axes, 301, 305
  physical significance of, 299
  polar, 304
  special formulas, 303, 305, 310, 314, 316, 319
  with respect to planes and points, 321
Moments
  axis of, 27, 67
  center of, 27
Momentum, angular
  conservation of, 492
  of a particle, 489
  of a rotating body, 489
  relation to angular impulse, 494
  relation to external forces, 491
Momentum, linear
  conservation of, 477
  of a finite body, 472
  of a particle, 472
  relation to external forces, 476
  relation to linear impulse. 479

N

Newton's laws of motion, 271
  third law, 3
Normal pressure, 160

O

Output, 457

P

Pappus and Guldinus, theorems of, 225
Parabolic cable, 195
Parallel-axis theorem, 301, 305
Parallelogram law, 11
Particle
  definition of, 231
  effective force for, 275
  kinetic formula for, 272
  kinetics of, 271, 272, 275
Plane motion
  angular acceleration in, 352
  angular velocity in, 351
  definition of, 351
  instantaneous axis in, 352
  kinematics of, 351, 352, 355, 364
  kinetics of, 371, 374
  motions of two particles in, 355
Power, 441, 445
  measurement of, 445
Pressure of a jet or stream, 482
Primary stresses, 128, 139
Principle
  d'Alembert's, 277

Principle
of angular impulse and momentum, 494
of components, 23, 42, 50, 64, 81, 91
general nature, 91
for bodies in equilibrium, 94
of linear impulse and momentum, 479
of moments, 30, 43, 50, 82, 90, 91
general nature, 91
for bodies in equilibrium, 94
for two concurrent forces, 30, 90
of the conservation of angular momentum, 492
of the conservation of linear momentum, 477
of the motion of the center of gravity, 278
of transmissibility, 7
Principles of composition, general nature, 91
Prony brake, 445

R

Radius of gyration, 300
Rectilinear motion
uniform, 239
uniformly accelerated, 240
with constant acceleration, 240
with variable acceleration, 225
Relative acceleration, 401
and absolute acceleration, 402
and absolute velocity, 391
Relative motion of two points, 415
Relative path, 388
Relative velocity, 390
Resolution, 6
of a couple, 75
of a force into a force and couple, 59
of a force into rectangular components, 16, 62
Resultant of a force system, 6
a coplanar couple and force, 58
coplanar couples, 56
external forces, in kinetics, 285, 288, 289, 336, 339, 371, 374
non-coplanar couples, 78
the collinear system, 19
the coplanar concurrent system, 20, 23
the coplanar parallel system, 37, 42
Resultant of a force system
the general coplanar system, 47, 50
the non-coplanar concurrent system, 64
the non-coplanar parallel system, 81
three couples at right angles, 71
three forces at right angles, 61
two concurrent forces, 11
two forces at right angles, 14
Retardation, 236
Rigid and non-rigid bodies, 5
Rollers, effect of, 9
Rolling bodies, kinematics of, 353
Rolling resistance, 383
coefficient of, 384, 386
Rotation
angular acceleration in, 325
angular velocity in, 323
axis of, 323
definition of, 323
kinematics of, 323
kinetics of, 336, 339
motions of particles of a body in, 329
normal acceleration in, 330
tangential acceleration in, 330
uniform, 332
uniformly accelerated, 334
with constant acceleration, 333
with constant velocity, 332
Rough surfaces, 160

S

Scotch crosshead, 249
Screw, the, 180
Screw jack, 183
Signs
conventions for, 19
of components, 69
of moments, 27, 67
of results, interpretation of, 19
Simple harmonic motion, 247
Smooth pulleys, 9
Smooth surfaces, 8
Solids of revolution, volume of, 226
Space diagram, 20
Speed, 251
Statically indeterminate cases, 96
Stress
axial, 126

Stress
  compressive, 127
  tensile, 127
Superelevation of curves, 349
Surfaces of revolution, area of, 225
Suspended cables, 194

T

Tensile stress, 127
Theory of the conservation of energy, 447, 459
Torque, 27
  work done by, 433
Translation
  definition of, 286
  kinematics of, 286
  kinetics of, 288, 289
Triangle law, 12
Trusses, 126, 127
Two-force members, 97

V

Varignon's theorem, 30
Vector of a couple, 69
Vector quantities, 9
Vectorial integration, 465
Velocity
  angular, 323, 351
  linear, 232, 251
  x- and y-components of, 255
Velocity-time curves, 233

W

Watt, 441
Weight, 4
Work
  and energy, 459
  as a means of transferring energy, 459
  a scalar quantity, 425
  done on a rolling body, 435
  in terms of torque, 433
  in the case of a constant working component, 424
  of a constant force, 427
  of a single concentrated force, 419
  of distributed forces, 430
  of frictional forces, 434
  of gravity, 430
  of kinetic friction, 435
  of two collinear forces, 437
  total, 425
Working component, 419