A COURSE
OF
MATHEMATICAL ANALYSIS

BY

Shanti Narayan M. A.,

Professor of Mathematics

D. A. V. College, Lahore.

S. CHAND & CO.,

BOOK-SELLERS & PUBLISHERS

LAHORE & DELHI.

Price Rupees 12/-

Printed from page 1 to 88 at The Ripon Press and from page 89 to 304 at the Punjab Electric Press, Mohan Lal Road Lahore and Published by G. S. Sharma, Manager S. Chand & Co. Lahore Branch.

PREFACE

This book is designed for students who, after an introductory course on calculus based on the geometrically and intuitively perceived notion of the *Continuum*, desire to proceed to a course preparatory to a serious study of the purely arithmetical theory of functions of real variables in its completely general aspects and is based on the lectures which the author has, from year to year, been delivering to the students preparing for the M.A. (Mathematics) examination of the Punjab University.

The treatment of the various topics is essentially rigorous ; at the same time an attempt has been made to present the subject in a clear, lucid and intelligible manner.

The book contains a large number of examples some of which have been solved.

I have been, during my study of the subject influenced by the works of a large number of authors including Goursat, Carslaw, Pierpont, Courant, Landau, Kowalewski and Perron and I feel deeply grateful to them all.

I take this opportunity of also thanking the publishers and the printers who did all they could to give the book the best form possible under the present very difficult circumstances.

SHANTI NARAYAN

May, 1945.

CHAPTER I

AGGREGATE OF REAL NUMBERS

CHAPTER II

ELEMENTS OF THE THEORY OF AGGREGATES OF REAL NUMBERS

CHAPTER III

SEQUENCES

CHAPTER I

AGGREGATE OF REAL NUMBERS

1. The modern *Theory of Functions*, also known as *Analysis*, is a development of the notion of number. Its concepts and results along with the proofs of their correctness may be suggested by intuition or by some physical experience—in fact this suggestion is almost always there—but their actual description and their statements must be given strictly in terms of numbers. Some of the concepts which are so suggested, are *Length*, *area*, *between*, *inside*, etc., but before they can become the subject of discussion in analysis, they have to be adequately and properly defined in terms of numbers. In general, the same word is used to denote the two similar and corresponding concepts, one physical and the other analytical, but this ambiguity causes no confusion and rather proves helpful and suggestive.

1·1. In the course of development of mathematics, the notion of number has been subjected to a series of extensions. The system of numbers consisting of positive integers only has been extended so as to contain positive fractions, negative fractions including negative integers, and the system thus obtained is called the *system of rational numbers*—every rational number being of the form p/q, where p, q are integers, positive or negative ; p may be zero but not q. From the system of rational numbers, we pass on to the system of real numbers and then again to the system of complex numbers.

The treatment of rational numbers in *Elementary Arithmetic* proceeds along concrete lines and employs notions foreign to pure mathematics and does not, therefore, satisfy the demands of modern Analysis.

In Elementary Arithmetic a positive fraction p/q is introduced so as to denote p of the q equal parts in which any given unit *magnitude* is divided and a negative rational number, x, where x is positive, is introduced so as to denote a *loss* of x rupees as against a *profit* of x rupees which is denoted by x. The same concrete aspect is employed in Elementary Arithmetic in order to define and to determine the laws of the operations of addition, subtraction, multiplication, and division.

In this connection it is interesting to recall the manner in which the truth of the proposition

$$4 \times 5 = 5 \times 4,$$

is established in elementary arithmetic.

Consider 4 rows each consisting of 5 cross signs. Now 5×4, which denotes the number of things obtained by adding 5 things 4 times, denotes the number of cross signs if we count them horizontally in rows. Similarly, 4×5 denotes the number of cross signs if we count them vertically in columns. The number of signs being independent of the two different modes of counting, we see that, $4 \times 5 = 5 \times 4$.

×××××
×××××
×××××
×××××

*Analytical discussion of the theory of rational numbers is not, however, within the scope of this book. It will be *assumed* here that the reader is familiar with rational numbers and with the laws of manipulation with them. Some properties of the aggregate of rational numbers, reference to which is important from our present point of view, will be given in the next section.

The treatment of real numbers, as given in books on Elementary mathematics, is also far from satisfactory; in fact the satisfactory accounts of the theory of real numbers have only recently been given by Dedekind, Cantor and Weierstrass. The account as given by Dedekind, with some modifications of details, will be considered here.

The treatment of complex numbers is not included in the scope of this book, dealing as it does with functions of real variables only.

2. Some properties of the Aggregate of rational numbers. In the following a, b, c; etc., denote rational numbers.

I. *The aggregate is* **ordered.** This means that if the rational numbers a, b are different, then either $a > b$ or $a < b$.

Also, if $a < b < c$, then $a < c$. In this case b is said to lie between a and c.

II. *The aggregate is* **dense.** This means that between two different rational numbers there lie an infinite number of rational numbers.

III. *The aggregate is* **invariant** *with respect to the four operations of addition, subtraction, multiplication and division*; the division by zero being the only meaningless operation. This means that the four operations are always possible, so that if a, b be two rational numbers, then $a+b$, $a-b$, $a.b$, a/b are also rational.

[The student may easily see that the aggregate of positive integers is not invariant with respect to subtraction and division and the aggregate of positive rational numbers is not invariant with respect to subtraction.]

The four operations obey the following laws :—

Addition

(*i*) $a+b=b+a$, *Commutative law.*

(*ii*) $a+(b+c)=(a+b)+c$, *Associative law.*

(*iii*) If $a < b$, then $a+c < b+c$, *Law of monotony.*

Subtraction

(*iv*) $b+(a-b)=a$.

Multiplication

(*v*) $ab=ba$, *Commutative law.*

(*vi*) $a(bc)=(ab)c$, *Associative law.*

---

* For such a discussion the reader is referred to Chap. I of the Theory of functions of real variables, Vol. I, by Pierpont.

(*vii*) $(a+b)c=ac+bc$, *Distributive law.*

(*viii*) If $a>b$ and $c>0$, then $ac>bc$, *Law of monotony.*

Division

(*ix*) $b.\dfrac{a}{b}=a. \quad (b\neq 0).$

IV. *The system is* **Archimedean.** This means that if $a>b$, then there exists a positive integer n such that $nb>a$.

3. Sections of Rational Numbers. *Let the aggregate of rational numbers be divided into two classes L and R, in such a way that*

(*i*) *each class exists, i.e.*, the numbers do not all belong to the same class so that the other class contains no number;

(*ii*) *each number has a class, i.e.*, no number escapes classification;

(*iii*) *every member of L is less than every member of R.*

Such a division of rational numbers into two classes L and R is called a **section** of rational numbers and is denoted as (L, R); L is called the *lower*, and R, the *upper class* of the section.

Ex. Show that if (L, R) is a section, then any rational number which is less than a member of L is also a member of L and any rational number greater than a member of R is also a member of R.

3·1. Three types of sections.

(*i*) Let every rational number less than any rational number, say, 3, belong to L and every rational number $\geqslant 3$ belong to R. Clearly the two classes L and R constitute a section, satisfying as they do the three characteristics of a section given above in §3.

The class L has no greatest member but the class R has a least member, *viz.*, 3.

(*ii*) Let every rational number $\leqslant$ any rational number, say, 3 belong to L and every rational number >3 belong to R.

The class L of this section has a greatest member, *viz.*, 3, but the class R has no least member.

(*iii*) Let every negative rational number, zero, and every positive rational number whose square is less than 2 belong to L and every positive rational number whose square is >2 belong to R.

In order to be sure that no number escapes classification, it is necessary to prove that there is no rational number whose square is equal to 2.

If possible, let p/q be a rational number such that

$$(p/q)^2=2 \text{ or } p^2=2q^2. \qquad (i)$$

We suppose that p, q have no common factor, for, such factors, if any, can be cancelled to begin with.

From (*i*), we see that p^2 is an even number. Therefore p must also be even. Let, then, $p=2m$, where m is an integer. Therefore

$$4m^2=2q^2 \text{ or } q^2=2m^2.$$

Thus q^2 is also even and so q is even.

Hence p, q have a common factor 2 and this conclusion contradicts the hypothesis that p, q have no common factor.

It will now be shown that L has no greatest member and R has no least.

If possible, let K be the greatest member of L so that

$$0 < K \text{ and } K^2 < 2.$$

Consider, now the positive number $(4+3K)/(3+2K)$. We have

$$2-\left(\frac{4+3K}{3+2K}\right)^2=\frac{2-K^2}{(3+2K)^2}>0, \text{ so that } \left(\frac{4+3K}{3+2K}\right)^2<2;$$

$$\frac{4+3K}{3+2K}-K=\frac{2(2-K^2)}{3+2K}>0, \text{ so that } \frac{4+3K}{3+2K}>K.$$

Thus the positive number $(4+3K)/(3+2K)$ belongs to L and is greater than K so that we have a contradiction.

As above it may also be shown that if K is the least member of R so that $K^2>2$ then $(4+3K)/(3+2K)$ is a still smaller member of R so that we again have a contradiction.

Conclusion. Thus we see that a section (L, R) may be such that

(*i*) L has no greatest member, but R has a least;

(*ii*) L has a greatest member, but R has no least;

(*iii*) L has no greatest member, and R has no least.

In order to see that these are the only *three* possibilities, it is necessary to show that for no section (L, R) can L have a greatest member *and* also R a least. This fact is easily seen to be true if we observe that in such a case the infinite number of numbers lying between the greatest member of L and the least member of R will neither belong to L and nor to R and thus escape classification so that the classes L, R will not constitute a section.

3·2. Modification in the definition of a section. It will be seen that to each rational number there correspond **two** sections according as it is the greatest member of the lower class or the least member of the upper class. The development of the theory of real numbers is a good deal simplified if we so modify the definition of a section that to each rational number there corresponds only one section. Accordingly we modify the definition by insisting *that the lower class must not have a greatest member.* Thus we now say that any division of rational numbers into two classes L and R is a section if

(*i*) each class exists; (*ii*) each number has a class; (*iii*) every member of L is less than every member of R; (*iv*) L has no greatest member.

Note. An equally suitable modification could have been that R has no least member but what we have done above is more usual.

3·3. The following simple theorem which can be easily established will prove very convenient for the later developments :—

Any *given aggregate of rational numbers can form the lower class of a section if and only if it is such that*

(*i*) *all the rational numbers do not belong to it ;*

(*ii*) *it has no greatest member ;*

(*iii*) *a rational number which is less than any member of the aggregate is also a member of the aggregate.*

If these conditions are satisfied then all those numbers which do not belong to this aggregate form the upper class of the section in question.

Ex. What are the conditions which must be satisfied by an aggregate so that it may form the upper class of a section.

3·4. A property of sections. *Corresponding to any positive number k, however small, there exists a member x of L and a member y of R, such that*

$$y-x=k.$$

Let a and b be any two members of the classes L and R respectively. There exists a positive integer n such that (§ 2 IV),

$$nk>b-a, \text{ i.e., } a+nk>b.$$

Consider the set of numbers

$$a, a+k, a+2k......, a+nk.$$

The number a belongs to L and $a+nk$ to R. There must exist, therefore, two consecutive numbers $a+rk$, $a+(r+1)k$ of this set such that $a+rk$ belongs to L and $a+(r+1)k$ to R.

These are, then, the required numbers x and y.

4. Definitions.

1. **Real number.** *A section of rational numbers is called a real number.*

2. **Real rational number, irrational number.** *A section is said to be a real rational, or an irrational number according as the upper class of the section has or does not have a least member.*

If r be the least member of the upper class R of the real rational number (L, R), then we say that the rational number r corresponds to the real rational number (L, R).

Notation. *The real rational number corresponding to a rational number x will always be denoted as $\bar{x}$.*

Note. In view of the modified definition of a section, we see that to each rational number r there corresponds only one real rational number (L, R) where L consists of all those rational numbers which are $<r$ and R of those which are $\geqslant r$; also to each real rational number (L, R) corresponds only one rational number, *viz.*, the least member of R. Thus *there is a one-to-one correspondence between the aggregate of rational numbers and the aggregate of real rational numbers* which is the aggregate of those sections for which the upper class has a least member.

Note. To a beginner it might appear strange to call a *section*, which is only a division of rational numbers into two classes, a *real number*. There may be several reasons for this attitude on his part. One reason may be that the definition of a real number as a section is abstract and the idea of magnitude is foreign to it and the reader finds it difficult to disassociate *number* from any idea of magnitnde which is the very basis of the manner in which he is introduced to the notion of number in elementary Mathematics. Another reason for his difficulties may be that he is at a loss to understand as to how the notion of order and the four operations of addition, etc., can be extended to the new aggregate. This latter difficulty is only temporary and in the following sections it is shown as to how this can be removed.

5. Relationship of order between real numbers. Let
$$a_1 \equiv (L_1, R_1) \text{ and } a_2 \equiv (L_2, R_2),$$
be two real numbers.

The following are the three mutually exclusive possibilities :—

(*i*) *L_1 is a proper part of L_2, i.e.*, every member of L_1 is also a member of L_2 but every member of L_2 is not a member of L_1 ;

(*ii*) *L_2 is a proper part of L_1 ;*

(*iii*) *L_1 and L_2 are identical, i.e.*, every member of L_1 is a member of L_2 and every member of L_2 is a member of L_1.

In case (*i*) we write $a_1 < a_2$, in case (*ii*) $a_1 > a_2$, and in case (*iii*) $a_1 = a_2$.

Note. Since L_2 has no greatest member, we see that if L_1 is a proper part of L_2, there exist an *infinite* number of members of L_2 which do not belong to L_1, and which, therefore, belong to R_1.

Ex. Show that $\alpha_1 < \alpha_2$, if $\alpha_2 > \alpha_1$.

Ex. Show that $\alpha_1 > \alpha_2$ if and only if L_1 and R_2 have an infinite number of common members.

Some Simple Results.

5·1. *If α, β, γ are three real numbers such that $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.*

Let $\alpha \equiv (L_1, R_1)$, $\beta \equiv (L_2, R_2)$, $\gamma \equiv (L_3, R_3)$.

Since every member of L_1 is a member of L_2 and every member of L_2 is a member of L_3, therefore every member of L_1 is a member of L_3 ; also, there exists a member of L_3 which does not belong to L_2 and this, again, cannot belong to L_1, for, if it did belong to L_1, it will also have to belong to L_2. Hence L_1 is a proper part of L_3. Thus $\alpha < \gamma$.

Ex. If $\alpha = \beta$ and $\beta > \gamma$; show that $\alpha > \gamma$.

5·2. *If a_1 and a_2 are two rational numbers, then*
$$a_1 > a_2 \text{ or } a_1 < a_2 \text{ or } a_1 = a_2, \text{ according as } \bar{a}_1 > \bar{a}_2 \text{ or } \bar{a}_1 < \bar{a}_2 \text{ or } \bar{a}_1 = \bar{a}_2,$$
where $\bar{a}_1$, $\bar{a}_2$ are the real rational numbers corresponding to the rational numbers a_1, a_2 respectively.

Let $$\bar{a}_1 \equiv (L_1, R_1) \; ; \; \bar{a}_2 \equiv (L_2, R_2),$$
so that a_1, a_2 are the least members of R_1 and R_2 respectively.

Let $a_1 < a_2$.

A rational number which is $< a_1$ is also $< a_2$ so that every member of L_1 is a member of L_2 ; also, the rational numbers which lie between a_1 and a_2 belong to L_2 but not to L_1. Thus L_1 is a proper part of L_2 and, therefore $\bar{a}_1 < \bar{a}_2$.

Let $a_1 > a_2$ so that $a_2 < a_1$. From above $\bar{a}_2 < \bar{a}_1$, and, therefore $\bar{a}_1 > \bar{a}_2$.

The case of equality is obvious.

5.3. *If a_1, a_2 be any two members of the classes L, R respectively of the real number $\alpha \equiv (L, R)$, then*

$$\bar{a}_1 < \alpha \text{ and } \bar{a}_2 \geqslant \alpha.$$

Let $\bar{a}_1 \equiv (L_1, R_1)$ and $\bar{a}_2 \equiv (L_2, R_2)$,
so that a_1, a_2 are the least members of the classes R_1 and R_2 respectively.

Every member of L_1 which consists of the rational numbers $< a_1$ is also a member of L which contains a_1 and, therefore, also every number $< a_1$; also, a_1 is a member of L but not of L_1. Thus L_1 is a proper part of L and, $\therefore$ $\bar{a}_1 < \alpha$.

If a_2 be the least member of R, then L and L_2 are identical and, therefore, $\alpha = \bar{a}_2$.

If a_2 be not the least member of R, then it may also be easily shown that L is a proper part of L_2 so that $\alpha < \bar{a}_2$.

5·4. *Between two different real numbers, there lie an infinite number of real rational numbers.*

Let $\alpha_1 \equiv (L_1, R_1)$ and $\alpha_2 \equiv (L_2, R_2)$
be two different real numbers. For the sake of definiteness suppose that $\alpha_1 < \alpha_2$ so that L_1 is a proper part of L_2.

There exist an infinite number of rational numbers which belong to L_2 but not to L_1 and, therefore, belong to R_1. The real rational numbers corresponding to them are all less than α_2 and all, (excluding at the most one) greater than α_1. (§5·3).

6. Zero, positive and negative real numbers.

The real number, 0, corresponding to the rational number zero is called the real number zero. Thus the real number (L, R) is the real number zero if 0 is the least member of R.

A real number is said to be positive or negative according as it is greater or smaller than the real number zero, i.e., 0.

It is easy to show that (L, R) is positive, if L contains atleast one and, therefore, an infinite number of positive members; also it is negative if R contains at least one and, therefore, an infinite number of negative members.

Ex. 1. Show that every positive real number is greater than every negative real number.

Ex. 2. Show that $\bar{a}$ is positive or negative according as the rational number a is positive or negative.

7. Sum of two real numbers. Let $\alpha_1 \equiv (L_1, R_1)$ and $\alpha_2 \equiv (L_2, R_2)$, be any two real numbers.

Let a class L be formed of numbers obtained by adding every member of L_1 to every member of L_2.

Clearly, the class L exists and does not contain all the rational numbers.

In order to show that L can be the lower class of a section, we have to prove that any rational number which is less than any member of L is also a member of L. Let 'b' be a rational number less than any member 'a' of L which is obtained by adding the members a_1, a_2 of L_1, L_2 respectively so that $a=a_1+a_2$.

We write

$$b=a-x=a_1+a_2-x=(a_1-x)+a_2,$$

where x is a positive rational number.

The number a_1-x which is less than the member 'a_1' of L_1 must also be a member of L_1. Thus we see that the number 'b' can be obtained by adding the member a_1-x of L_1 to the member a_2 of L_2 and accordingly it must belong to L.

Since L_1 and L_2 have no greatest members, L, also, can have no greatest member.

Thus we find that L can be the lower class of a section (§3·3). The section (L, R), where R consists of all those rational numbers which do not belong to L, is called the sum of (L_1, R_1) and (L_2, R_2) and this relationship is exhibited as

$$(L, R)=(L_1, R_1)+(L_2, R_2)\equiv\alpha+\beta.$$

Ex. Prove that $\alpha+\beta$ is positive if α, β are both positive and negative if they are both negative.

Some simple results. In what follows, α, β, γ will denote real numbers. Also, $\alpha\equiv(L_1, R_1)$, $\beta\equiv(L_2, R_2)$, $\gamma=(L_3, R_3)$.

7·1. $\alpha+\beta=\beta+\alpha$,
so that the Commutative law holds for real numbers.

Let a_1, a_2 be any two members of L_1, L_2 respectively. The result now follows from the fact that $a_1+a_2=a_2+a_1$, so that the two classes formed of numbers obtained by adding the members of L_1 to the members of L_2 and the members of L_2 to the members of L_1 are identical.

7·2. $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$,
so that the Associative law holds for real numbers.

Let a_1, a_2, a_3 be any three members of L_1, L_2, L_3 respectively. The result now follows from the fact that the Associative law holds for the addition of rational numbers, *i.e.*,

$$a_1+(a_2+a_3)=(a_1+a_2)+a_3.$$

7·3 *If* $\alpha>\beta$, *then* $\alpha+\gamma>\beta+\gamma$.

We have $\alpha\equiv(L_1, R_1)$, $\beta\equiv(L_2, R_2)$, $\gamma\equiv(L_3, R_3)$.

Let $\alpha+\gamma\equiv(L', R')$, $\beta+\gamma\equiv(L'', R'')$,
so that the two classes L′, L″ are formed of numbers obtained by adding the members of L_1 to the members of L_3, and the members of L_2 to the members of L_3 respectively.

Since every member of L_2 is a member of L_1, therefore every member of L″ is also a member of L′.

Let a_1 be any member of L_1 which does not belong to L_2 so that it belongs to R_2. Let a_1' be any member of L_1 which is $>a_1$. Let $a_1'-a_1=\epsilon$, so that ϵ is a positive rational number.

There exists a member a_3 of L_3 and a member b_3 of R_3 such that $b_3-a_3=\epsilon$, (§ 3·4). We have $a_1'+a_3=a_1+\epsilon+b_3-\epsilon=a_1+b_3$.

Now, $a_1'+a_3$, which is obtained by adding the member a_1' of L_1 to the member a_3 of L_3 is a member of L'; also a_1+b_3 which is obtained by adding the member a_1 of R_2 to the member b_3 of R_3 is a member of R''. Thus there exists a common member of L' and R'' *i.e.*, there exists a member of L' which does not belong to L''.

Thus L'' is a proper part of L_1. Hence the result.

7·4. *To prove that* $\alpha+\overline{0}=\alpha$.

Let $$\overline{0}=(L', R'),$$
so that L' consists of all the negative rational numbers and 0 is the least member of R'.

Let $\alpha\equiv(L_1, R_1)$, and $\alpha+\overline{0}\equiv(L, R)$.

We have to show that the classes L, L_1 are identical.

A member of L, obtained, as it is, by adding some member a_1 of L_1 to a member of L' which is negative is essentially less than a_1 and must, therefore belong to L_1.

Let a_1 be any member of L_1. Since L_1, has no greatest member, there exists a member a_1+k, of L_1 greater than a_1, $(k>0)$. The member a_1 of L_1 which can be obtained by adding the member a_1+k of L_1 to the member $-k$ of L' is also a member of L.

Thus the classes L and L_1 are identical. Hence the result.

7·5. *If a, b be two rational numbers, then*
$$\overline{a}+\overline{b}=\overline{a+b},$$
i.e., the sum of two real rational numbers is a real rational number corresponding to the sum of the corresponding rational numbers.

Let $\overline{a}=(L_1, R_1)$, $\overline{b}=(L_2, R_2)$, $\overline{a}+\overline{b}=(L, R)$.

Let x, y be any members of L_1 and L_2 respectively so that $x+y$ is any member of L.

$\because$ $x<a$ and $y<b$, $\therefore$ $x+y<a+b$, so that *every member of L is* $<a+b$.

Again consider any rational number $a+b-k$, $(k>0)$, which is $<a+b$. We write
$$a+b-k=(a-\tfrac{1}{2}k)+(b-\tfrac{1}{2}k).$$

Since $a-\frac{1}{2}k$ belongs to L_1 and $b-\frac{1}{2}k$ to L_2, therefore $a+b-k$ belongs to L.

Thus *every rational number* $<(a+b)$ *is a member of L.*

Hence $(a+b)$ is the least member of R so that $\overline{a}+\overline{b}=\overline{a+b}$.

Ex. Show that the sum of a real rational and an irrational number is necessarily irrational.

8. The negative of a real number $\alpha\equiv(L_1, R_1)$. We form a class L of numbers which are the negatives of all the members of R_1, excepting that of the least member of R_1, if there be any.

Clearly the class L exists and has no greatest member.

In order to prove that L can be the lower class of a section, it has to be shown that a rational number which is less than any member of L is also a member of L.

Let a_1' be a rational number less than any member a_1 of L.

Since $-(-a_1)=a_1$, *i.e.*, a_1 is the negative of $-a_1$, therefore $-a_1$ belongs to R_1. Also

$$\because\ a_1'<a_1,\quad \therefore\ -a_1'>-a_1.$$

Thus $-a_1'$ is a member of R_1 and accordingly $-(-a_1')=a_1'$ is a member of L.

Since L is not to have the negative of the least member of R_1, therefore it cannot have a greatest member.

The section (L, R), where R consists of all those rational numbers which do not belong to L, is called the negative of (L_1, R_1) and is denoted by $-\alpha$ or by $-(L_1, R_1)$.

It is easy to show that R will consist of the negatives of L_1 and the negative of the least member of R_1, if there be any.

Some simple results.

8·1. *To prove that* $-(-\alpha)=\alpha$.

Let $\alpha\equiv(L_1, R_1)$, $-\alpha\equiv(L, R)$, $-(-\alpha)\equiv(L', R')$.

It has to be shown that L_1 and L' are identical.

The class R is composed of the negatives of the members of L_1 and the negative of the least member of R_1, if it exists, (this will be the least member of R also).

The class L' is composed of the negatives of the members of R excluding the negative of its least member, if it exists, *i.e.*, it consists of the members of L_1.

Thus L_1 and L' are identical. Hence the result.

8·2. *To prove that the real number* $-\alpha$ *is positive, negative or zero according as* α *is negative, positive or zero.*

Let $\alpha\equiv(L_1, R_1)$, $-\alpha\equiv(L, R)$.

Suppose that α is negative so that R_1 consists of an infinite number of negative members. Since the negative of a negative rational number is positive, we see that the class L consists of an infinite number of positive members. Hence $-\alpha$ is positive.

The remaining cases can be similarly disposed of.

8·3. *If* $\alpha>\beta$, *then* $-\alpha<-\beta$.

Let $\alpha\equiv(L_1, R_1)$, $\beta\equiv(L_2, R_2)$, $-\alpha\equiv(L', R')$, $-\beta\equiv(L'', R'')$.

Since L_2 is a proper part of L_1, therefore R'' is a proper part of R' and therefore L' is a proper part of L''. Hence the result.

8·4. *The negative of a real rational number* $\bar{a}$ *is also a real rational and* $-(\bar{a})=(\overline{-a})$.

Let $\bar{a} \equiv (L_1, R_1)$, $-\bar{a} \equiv (L, R) = \overline{(-a)} \equiv (L', R')$.

Since a is the least member of R_1, therefore $-a$ is the least member of R. (§ 8). Also, by def., $-a$ is the least member of R′. Thus R, R′ are identical.

9. The difference of two real numbers. The difference $a-\beta$ of two real numbers is defined by the equality

$$a-\beta=a+(-\beta),$$

so that to obtain $a-\beta$ we add, (§7), the negative of β, (§8), to a.

Ex. If $a<\beta$, show that $a-\gamma<\beta-\gamma$.

Some simple results.

9·1. *To prove that* $a-a=\bar{0}$.

Let $a\equiv(L_1, R_1)$, $-a\equiv(L, R)$, $a-a=a+(-a)\equiv(L', R')$.

The class L′ is formed of numbers obtained by adding to the members of L_1 the negatives of the members of R_1 excepting that of its least if there be any, these latter being the members of L.

Now, if a_1 be any member of L_1 and b_1, any member of R_1, (not the least), then L′ consists of the numbers of the type $a_1+(-b_1)$ *i.e.*, a_1-b_1, which are necessarily negative. *Thus every member of L′ is negative.*

Again, if k be any negative rational number, there exists a member b_1 of R_1 and a member a_1 of L_1, such that

$$b_1-a_1=-k, \text{ (§ 3·4)}$$

i.e.,
$$a_1+(-b_1)=k,$$

so that k belongs to L′. *Thus every negative rational number belongs to L′.*

From above we deduce that 0 is the least member of R′. Hence the result.

9·2. *To prove that* $a+(\beta-a)=\beta$.

We have

$$\begin{aligned} a+(\beta-a)&=a+[\beta+(-a)], && \text{§ 9}\\ &=a+[(-a)+\beta], && \text{§ 7·1}\\ &=[a+(-a)]+\beta, && \text{§ 7·2}\\ &=\bar{0}+\beta, && \text{§ 9·1}\\ &=\beta+\bar{0}, && \text{§ 7·1}\\ &=\beta. && \text{§ 7·4}\end{aligned}$$

Ex. Show that $-(a+\beta)=-a-\beta$, $-(a-\beta)=-a+\beta$.
Ex. If $a=\beta+\gamma$, show that $a-\beta=\gamma$.

9·3. *To prove that* $a-\beta$ *is positive or negative according as* $a>\beta$ *or* $a<\beta$.

Let $a\equiv(L_1, R_1)$, $\beta\equiv(L_2, R_2)$, $a-\beta\equiv(L, R)$.

Let $a>\beta$ so that L_2 is a proper part of L_1, or that L_1, R_2 have an infinite number of members in common. Let a_1, a_1' be any two such common members so selected that none of them is the least of R_2. Let $a_1<a_1'$.

Considering a_1' as a member of L_1 and a_1 as a member of R_2, we see that the number $a_1'-a_1$ which is positive belongs to L.

Hence $\alpha-\beta$ is positive.

The second case may be similarly discussed.

9·4. *If a, b be two rational numbers, then*

$$\bar{a}-\bar{b}=\overline{a-b},$$

i.e., the difference of two real rational numbers is a real rational number corresponding to the difference of the corresponding rational numbers.

Let $\bar{a}\equiv(L_1, R_1)$, $\bar{b}\equiv(L_2, R_2)$, $\bar{a}-\bar{b}\equiv(L, R)$.

Let x be any member of L_1 and $y\neq b$, any member of R_2.

$\because$ $x<a$ and $y>b$, $\therefore$ $x-y<a-b$, so that *every member of L is* $<a-b$.

Again, consider any rational number $a-b-k$, $(k>0)$ which is $<a-b$. We write $a-b-k=(a-\frac{1}{2}k)-(b+\frac{1}{2}k)$.

$\because$ $a-\frac{1}{2}k$ belongs to L_1 and $b+\frac{1}{2}k$ to R_2, $\therefore$ $a-b-k$ belongs to L.

Thus every rational number $<(a-b)$ *is a member of L.*

Hence $(a-b)$ is the least member of R so that

$$\bar{a}-\bar{b}=\overline{a-b}.$$

10. *Between two different real numbers there lie an infinite number of irrational numbers.*

Let α, β be two real numbers and let $\alpha<\beta$. Take γ any irrational number.

$$\because \qquad \alpha<\beta,$$

$$\therefore \qquad \alpha+(-\gamma)<\beta+(-\gamma) \text{ i.e., } \alpha-\gamma<\beta-\gamma.$$

Let $\bar{a}$ be any one of the infinite number of real rational numbers lying between $\alpha-\gamma$ and $\beta-\gamma$, (§ 5·4). We have

$$\alpha-\gamma<\bar{a}<\beta-\gamma.$$

or
$$\alpha+(-\gamma)+\gamma<\bar{a}+\gamma<\beta+(-\gamma)+\gamma$$

or
$$\alpha+\bar{0}<\bar{a}+\gamma<\beta+\bar{0}$$

or
$$\alpha<\bar{a}+\gamma<\beta,$$

so that the irrational number $\bar{a}+\gamma$ lies between α and β.

11. The product of two real numbers. Let

$$\alpha_1\equiv(L_1, R_1) \text{ and } \alpha_2\equiv(L_2, R_2),$$

be any two real numbers.

Firstly we suppose that α, β are both positive, so that L_1 and L_2 contain some positive rational numbers also.

Let a class L be formed of (*i*) all the negative rational numbers, (*ii*) the rational number zero, (*iii*) all those positive rational numbers which can be obtained by multiplying a positive member of L_1 with a positive member of L_2.

It will now be shown that L can be the lower class of a section.

Clearly the class L exists and does not contain all the rational numbers.

Let 'b' be any positive rational number which is smaller than a positive member 'a' of L. Let 'a' be obtained by multiplying the positive members a_1, a_2 of L_1, L_2 respectiveiy.

Let $b/a=x$ so that $0<x<1$.

We have $$b=ax=(a_1a_2)x=a_1(a_2x).$$
The number a_2x which is smaller than a_2 belongs to L_2. Thus we see that 'b' is the product of the members a_1 and a_2x of L_1 and L_2 respectively and accordingly it is a member of L.

Since L_1 and L_2 have no greatest members, we easily see that L also cannot have a greatest member.

Thus we see that L can be the lower class of a section, say, (L, R).

The section (L, R) is called the product of (L_1, R_1) and (L_2, R_2) and this relationship is exhibited as

$$(L, R)=(L_1, R_1).(L_2, R_2)\equiv\alpha.\beta$$

or $$(L, R)=(L_1, R_1)(L_2, R_2)\equiv\alpha\beta,$$ omitting the dot.

Let, now, α be positive and β negative so that $-\beta$ is positive. Then, by def.,

$$\alpha.\beta=-[\alpha.(-\beta)].$$

Let β be positive and α negative so that $-\alpha$ is positive. Then, by def.,

$$\alpha.\beta=-[(-\alpha).\beta].$$

Let α, β be both negative so that $-\alpha$, $-\beta$ are both positive. Then, by def.,

$$\alpha.\beta=(-\alpha).(-\beta).$$

Let either α or β or both be zero, then, by def.,

$$\alpha\beta=\bar{0}.$$

Ex. *Show that for all the real numbers* α, β

$$(-\alpha)(-\beta)=\alpha\beta,\ \alpha(-\beta),=-(\alpha\beta),\ (-\alpha)\beta=-(\alpha\beta).$$

By def., $(-\alpha)(-\beta)=\alpha\beta$, if $\alpha\beta$ be both negative.

Let α, β be both positive so that $-\alpha$, $-\beta$ are both negative. Therefore, by def.,

$$(-\alpha)(-\beta)=[-(-\alpha)][-(-\beta)],$$
$$=\alpha\beta. \quad \S\ 8{\cdot}1.$$

Let α be positive and β negative so that $-\beta$ is positive and $-\alpha$ negative. By def.,

$$(-\alpha)(-\beta)=-\{[-(-\alpha)][-\beta]\}$$
$$=-[(\alpha)(-\beta)]=\alpha\beta.$$

Let α be negative and β positive so that $-\alpha$ is positive and $-\beta$ negative. By def.,

$$(-\alpha)(-\beta)=-\{(-\alpha)[-(-\beta)]\}$$
$$=-[(-\alpha)\beta]=\alpha\beta.$$

The remaining two results may be similarly proved.

Ex. Prove that the product of two numbers, both positive or both negative is positive, but if one number is positive and the other negative then the product is negative.

Some important results.

11·1. *To prove that*

$$\alpha\beta=\beta\alpha. \quad (\textit{Commutative Law})$$

The proof is simple and depends upon the fact that the Commutative law of multiplication holds for rational numbers.

11·2. *To prove that*

$$(\alpha\beta)\gamma=\alpha(\beta\gamma). \quad (\textit{Associative Law})$$

The proof is simple.

11·3. *To prove that*

$$\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma. \quad (\textit{Distributive Law})$$

Let $\alpha\equiv(L_1, R_1)$, $\beta\equiv(L_2, R_2)$, $\gamma\equiv(L_3, R_3)$.

Let $\alpha(\beta+\gamma)\equiv(L, R)$ and $\alpha\beta+\alpha\gamma\equiv(L', R')$.

Firstly we consider the case when α, β, γ are all positive.

All the negative rational numbers and the rational number zero are necessarily members of L as well as of L$'$.

The positive members of L are of the type

$$a_1(a_2+a_3),$$

and the positive members of L$'$ are of the type

$$a_1a_2+a_1'a_3,$$

where a_1, a_1' are any positive members of L_1 and a_2, a_3 are any positive members of L_2, L_3 respectively.

Since $$a_1(a_2+a_3)=a_1a_2+a_1a_3,$$
therefore, on taking $a_1'=a_1$, we see that every positive member of L is also a member of L$'$.

Any member $a_1a_2+a_1'a_3$ of L$'$ is clearly a member of L, if $a_1'=a_1$. In general, let $a_1>a_1'$ so that $a_1'/a_1<1$.

We write $\quad a_1'a_3=a_1[(a_1'/a_1)a_3]=a_1a_3'$, say.

Now $\quad a_3'=(a_1'/a_1).a_3<1.a_3=a_3$,
and therefore a_3' belongs to L_3.

Since $\quad a_1a_2+a_1'a_3=a_1a_2+a_1a_3'=a_1(a_2+a_3')$,
we see that every positive member of L$'$ is also a member of L.

Thus L, L$'$ are identical. Hence the result.

Before proceeding to consider the other cases, we prove that

$$\alpha(\beta-\gamma)=\alpha\beta-\alpha\gamma,$$

where α, β, γ are all positive.

Case I. Let $\beta=\gamma$

L. H. S.$=\alpha(\beta-\beta)=\alpha.\bar{0}=\bar{0}$. § 11

R. H. S.$=\alpha\beta-\alpha\beta=\bar{0}$. § 9·1.

Case II. Let $\beta>\gamma$ so that $\beta-\gamma$ is positive. § 9·3.

We have $\quad \alpha\beta=\alpha[\gamma+(\beta-\gamma)]$, § 9·2.

$\quad =\alpha\gamma+\alpha(\beta-\gamma)$, proved above

$\therefore \quad \alpha\beta-\alpha\gamma=\alpha\gamma+[\alpha(\beta-\gamma)-\alpha\gamma]$

$\quad =\alpha(\beta-\gamma)$. § 9·2.

Case III. Let $\beta<\gamma$ so that $\beta-\gamma$ is negative and $\gamma-\beta$ positive. We have proved in case II that

$$\alpha(\gamma-\beta)=\alpha\gamma-\alpha\beta.$$

Now $\alpha(\beta-\gamma)=-\alpha\{[-(\beta-\gamma)]\}$, by def.,
$=-[\alpha(\gamma-\beta)]$. Ex. after §9·2.
$=-[\alpha\gamma-\alpha\beta]$, proved above
$=\ \alpha\beta-\alpha\gamma$. Ex. after §9 2

We now return to the main result.

Let α, β be negative and γ positive and let $\beta+\gamma$ be positive. We have $\alpha(\beta+\gamma)=-[(-\alpha)(\beta+\gamma)]$
$=-\{(-\alpha)[-(-\beta)+\gamma)]\}$. § 8·1
$=-\{(-\alpha)[\gamma-(-\beta)]\}$
$=-[(-\alpha)\gamma-(-\alpha)(-\beta)]$, proved above
$=-[-\alpha\gamma-\alpha\beta]=-(-\alpha\gamma)-(-\alpha\beta)$
$=\alpha\gamma+\alpha\beta=\alpha\beta+\alpha\gamma$.

Let α, β be negative and γ positive and let $\beta+\gamma$ be negative. We have $\alpha(\beta+\gamma)=(-\alpha)[-(\beta+\gamma)]$
$=(-\alpha)(-\beta-\gamma)$, Ex. after §9·2
$=(-\alpha)(-\beta)-(-\alpha)(\gamma)$, proved above
$=\alpha\beta+\alpha\gamma$.

The other possibilities may be similarly discussed.

A property of positive sections. Corresponding to any rational number $k>1$ and any positive real number (L, R), there exist positive members x, y, of L, R respectively such that $y/x=k$.

Let a, b be *any* positive members of L, R respectively.

We write $k=1+l$, so that l is positive.

There exists a positive integer n such that

$$n.al>(b-a) \text{ i.e., } a(1+nl)>b.$$

Consider the set of numbers

$$a,\ ak,\ ak^2, \ldots\ldots,\ ak^n.$$

Since $ak^n=a(1+l)^n>a(1+nl)>b$, we see that ak^n belongs to R.

There must exist two consecutive numbers ak^r, ak^{r+1} of this set such that ak^r belongs to L and ak^{r+1} to R. These, then, are the required numbers x and y.

11·4. *To prove that*
if $\alpha<\beta$ and γ is positive then $\alpha\gamma<\beta\gamma$.

Firstly suppose that α, β are both positive.

Let $\alpha\equiv(L_1, R_1)$, $\beta\equiv(L_2, R_2)$, $\gamma\equiv(L_3, R_3)$
$\alpha\gamma\equiv(L, R)$, $\beta\gamma\equiv(L', R')$.

All the negative rational numbers and the rational number zero are neeessarily contained in L as well as L$'$.

Since every member of L_1 is a member of L_2,, therefore every member of L is also a member of L$'$.

Let a_2 be any positive number which is a member of L_2 but not of L_1. Since L_2 has no greatest member, there exists a member of L_2, say a_2', which is $>a_2$.

The numbers a_2, a_2' both belong to R_1.

Let $a_2'/a_2=k$, which is greater than 1.

There exist positive members a_3, b_3 of L_3, R_3 respectively such that $b_3/a_3=k$.

We have $a_2'a_3=a_2ka_3=a_2b_3$.

Since a_2' is a positive member of L_2 and a_3 a positive member of L_3, therefore $a_2'a_3$ *i.e.*, a_2b_3 belongs to L′. Also since a_2 is a member of R_1 and b_3 of R_3, therefore a_2b_3 belongs to R and not to L. Thus every member of L′ is not a member of L. Therefore L is a proper part of L′. Hence the result.

Let, now, α, β be both negative so that $-\alpha$, $-\beta$ are both positive.

Since $\alpha<\beta$, therefore $-\alpha>-\beta$ or $-\beta<-\alpha$. §8·3

$\therefore$ $-\beta\gamma<-\alpha\gamma$, as proved above

or $-(-\beta\gamma)>-(-\alpha\gamma)$, §8·3

or $\beta\gamma>\alpha\gamma$,

If α be negative and β positive then $\alpha\gamma$ is negative and $\beta\gamma$ positive and accordingly $\alpha\gamma<\beta\gamma$.

11·5. *To prove that if $\alpha<\beta$ and γ is negative, then $\alpha\gamma>\beta\gamma$.*

Since γ is negative, therefore $-\gamma$ is positive.

$\therefore$ $\alpha(-\gamma)<\beta(-\gamma)$ or $-(\alpha\gamma)<-(\beta\gamma)$

or $-[-(\alpha\gamma)]>-[-(\beta\gamma)]$, *i.e.*, $\alpha\gamma>\beta\gamma$.

11·6. *To prove that $\alpha.\bar{1}=\alpha$.*

Let $\alpha\equiv(L_1, R_1)$, $\bar{1}\equiv(L_2, R_2)$, $\alpha.\bar{1}=(L, R)$.

Since 1 is the least member of R_2, L_2 is comprised of all those rational numbers which are <1.

Let α be positive.

If a_1 be any positive member of L_1 and a_2 of L_2 then a_1a_2 is a member of L.

Since $a_1a_2<a_1.1$, $=a_2$, we see that a_1a_2 is also a member of L_1. Thus every member of L is a member of L_1.

Let a_1 be any positive member of L_1. Let $a_2>a_1$ be also a member of L_1 Since $a_1/a_2<1$, therefore a_1/a_2 belongs to L_2. We have

$$a_1=a_2.(a_1/a_2)$$

so that a_1 appears as the product of a positive member of L_1 and of a positive member of L_2 and accordingly a_1 is a member of L. Thus every member of L_1 is a member of L.

Therefore L, L_1 are identical. Hence the result when α is positive.

Let, now, α be negative so that $-\alpha$ is positive. As proved above, we have

$$(-\alpha).\bar{1}=(-\alpha)$$

$\therefore$ $$-[(-\alpha).\bar{1}]=-(-\alpha)$$

or $$-[-(\alpha.\bar{1})]=-(-\alpha)$$

or $$\alpha.\bar{1}=\alpha.$$

11·7. *If a, b be two rational numbers, then*

$$\overline{a}\,.\,\overline{b}=\overline{a.b},$$

i.e., the product of two real rational numbers is also a real rational number which corresponds to the product of the corresponding rational numbers.

Let $\overline{a}\equiv(L_1, R_1)$, $\overline{b}\equiv(L_2, R_2)$, $\overline{a}\,.\,\overline{b}\equiv(L, R)$.

Let $\overline{a}$, $\overline{b}$ be both positive.

Let x, y be any positive members of L_1, L_2 respectively so that xy is a positive member of L.

$\because \quad x<a$ and $y<b$, $\therefore$ $xy<ab$.

Thus every member of L is $<ab$.

Again consider any positive rational number abk which is $<ab$; $k<1$.

We write $k=\dfrac{1+k}{2}\cdot\dfrac{2k}{1+k}$.

Since $k<1$, therefore $(1+k)/2$ and $2k/(1+k)$ are both less than 1.

We have $abk=\left(a.\dfrac{1+k}{2}\right).\left(b.\dfrac{2k}{1+k}\right)$.

Since $a.(1+k)/2<a$ and $b.[2k/(1+k)]<b$, therefore $a.(1+k)/2$ belongs to L_1 and $b.[2k(1+k)]$ belongs to L_2 and accordingly their product abk belongs to L.

Thus every rational number $<ab$ *is a member of L.*

Hence ab is the least member of R so that

$$\overline{a}\,.\,\overline{b}=\overline{ab}.$$

12. The reciprocal of a non-zero real number.

Let $\alpha\equiv(L_1, R_1)$ be any positive real number. Let a class L be formed of (*i*) all the negative rational numbers, (*ii*) zero and (*iii*) the reciprocals of all the members of the class R_1 excepting that of its least if it exists.

If a_1 be any positive member of L, then $1/a_1$ must be a member of R_1. Let $b_1<a_1$ be any positive rational number.

We have $1/b_1>1/a_1$ so that $1/b_1$ belongs to R_1 and accordingly $1/(1/b_1)$ *i.e.*, b_1 belongs to L.

It is now easy to see that L can be the lower class of a section, say, (L, R).

This section (L, R) is said to be the reciprocal of (L_1, R_1), *i.e.*, α and is denoted as $\overset{*}{\alpha}$.

If α be negative so that $-\alpha$ is positive, then, by def.;

$$\overset{*}{\alpha}=-(\overset{*}{-\alpha})$$

Ex. Show that $\overset{*}{\alpha}$ is positive or negative according as α is positive or negative.

The quotient of two real numbers. If α, β be two real numbers and $\beta \neq \bar{0}$, then the real number $\alpha.\overset{*}{\beta}$, which is the product of α and the reciprocal of β, is said to be obtained on dividing α by β and we write

$$\frac{\alpha}{\beta} = \alpha.\overset{*}{\beta}.$$

Some important results.

12·1. To prove that

$$\alpha.\overset{*}{\alpha} = \bar{1}, \text{ where } \alpha \neq \bar{0}.$$

To start with we suppose that α is positive.

Let $\alpha \equiv (L_1, R_1)$, $\overset{*}{\alpha} \equiv (L, R)$, $\alpha.\overset{*}{\alpha} \equiv (L', R')$.

Let a_1 be any positive member of L_1 and b_1 any, but not the least, member of R_1 so that $1/b_1$ is a positive member of L.

The positive members of L' are, therefore, of the type a_1/b_1. Since $a_1/b_1 < 1$, we see that *every member of L' is < 1.*

Again, let k be any positive rational number < 1.

There exist positive members a_1, b_1 of L_1, R_1, such that $a_1/b_1 = k$.

Since $k = a_1.(1/b_1)$, we see that *every rational number < 1 is a member of L'.*

Hence 1 is the least member of R' and accordingly

$$(L', R') \equiv \bar{1}.$$

Let, now, α be negative so that $-\alpha$ is positive.

Also, therefore, $\overset{*}{\alpha}$ is a negative and $(-\overset{*}{\alpha})$ is positive.
We have, by def.,

$$\alpha.\overset{*}{\alpha} = (-\alpha)(-\overset{*}{\alpha}) = \bar{1}.$$

12·2. *To prove that*

$$\overset{*}{\alpha} = \frac{\bar{1}}{\alpha}.$$

We have $$\frac{\bar{1}}{\alpha} = \bar{1}.\overset{*}{\alpha} = \overset{*}{\alpha}.$$

On account of this result the reciprocal $\overset{*}{\alpha}$ of a number α is denoted as $\dfrac{\bar{1}}{\alpha}$ or $\bar{1}/\alpha$.

12·3. *To prove that*

$$\alpha.\frac{\beta}{\alpha} = \beta, \text{ where } \alpha \neq \bar{0}.$$

We have

$$\alpha.\left(\frac{\beta}{\alpha}\right) = \alpha.\left(\beta.\frac{\bar{1}}{\alpha}\right) = \alpha.\left(\frac{\bar{1}}{\alpha}.\beta\right)$$
$$= \left(\alpha.\frac{\bar{1}}{\alpha}\right).\beta = \bar{1}.\beta = \beta.$$

12·4. *To prove that if α, β are two positive real numbers such that*

$$\alpha<\beta,$$

then $$\bar{1}/\alpha>\bar{1}/\beta.$$

The proof is simple.

12·5. *To prove that*

$$\frac{\bar{a}}{\bar{b}}=\overline{\left(\frac{a}{b}\right)},$$

i.e., the quotient of two real rational numbers is also a real rational number which corresponds to the quotient of the corresponding rational numbers.

Let $\bar{b}=(L_1, R_1)$, $\dfrac{\bar{1}}{\bar{b}}=(L, R)$.

Suppose that $\bar{b}$ is poistive.

Let $x\neq b$, be any positive member of R_1 so that $1/x$ is a member of L.

$\because$ $x>b$, $\because$ $1/x<1/b$.

so that every member of L is $<1/b$.

Let k/b, where $k<1$ be any positive number $<1/b$.

$\because$ $b/k>b$, $\therefore$ b/k is a member of R_1 and therefore its reciprocal k/b is a member of L. *Thus every number $<1/b$ is a number of L.*

Hence $1/b$ is the least member of L and accordingly $\dfrac{\bar{1}}{\bar{b}}=\overline{\left(\dfrac{1}{b}\right)}$.

If b be negative, we have

$$\frac{\bar{1}}{\bar{b}}=-\frac{\bar{1}}{\overline{-b}}=-\overline{\left(\frac{1}{-b}\right)}, \text{ as proved above}$$

$$=-\overline{\left(-\frac{1}{b}\right)}=\overline{\left(\frac{1}{b}\right)}.$$

Finally $$\frac{\bar{a}}{\bar{b}}=\bar{a}.\frac{\bar{1}}{\bar{b}}=\bar{a}.\overline{\left(\frac{1}{b}\right)}=\overline{\left(a.\frac{1}{b}\right)}=\overline{\left(\frac{a}{b}\right)}.$$

13. The modulus of a real number. Def. *By the modulus of a real number α is meant the real number α, $-\alpha$ or $\bar{0}$ according as α is positive, negative or zero* and is written as $|\alpha|$.

It will be seen that the modulus of a real number cannot be negative.

Ex. Show that $|\alpha-\beta|=|\beta-\alpha|$.

Some simple results.

13·1. *To prove that*

$$|\alpha+\beta|\leqslant|\alpha|+|\beta|.$$

Case I. Let α, β be both positive so that $\alpha+\beta$ is also positive. We have

$$|\alpha+\beta|=\alpha+\beta=|\alpha|+|\beta|.$$

Case II. Let α, β be both negative so that $\alpha+\beta$ is also negative. We have

$$\begin{aligned} |\alpha+\beta| &= -(\alpha+\beta) \\ &= -\alpha-\beta, \\ &= (-\alpha)+(-\beta) = |\alpha| + |\beta|. \end{aligned}$$

Case III. Let α be positive and β negative.

(*i*) Let $\alpha+\beta$ be positive, we have

$$|\alpha+\beta| = \alpha+\beta.$$

Since β is negative so that $-\beta$ is positive, we have

$$\beta < -\beta.$$

$\therefore \quad |\alpha+\beta| = \alpha+\beta < \alpha+(-\beta) = |\alpha| + |\beta|.$

i.e., $\quad |\alpha+\beta| < |\alpha| + |\beta|$

(*ii*) Let $\alpha+\beta$ be negative.

We have

$$\begin{aligned} |\alpha+\beta| &= -(\alpha+\beta) = -\alpha-\beta = -\alpha+(-\beta) \\ &= -|\alpha| + |\beta| < |\alpha| + |\beta|, \because |\alpha| > -|\alpha|. \end{aligned}$$

Case IV. *Let α be negative and β positive.*

This follows from Case III on interchanging α, β.

13·2. *To prove that*

$$|\alpha-\beta| \geqslant \big|\, |\alpha| - |\beta| \,\big|.$$

We have

$$\alpha = \alpha-\beta+\beta.$$

$\therefore \quad |\alpha| = |\alpha-\beta+\beta| \leqslant |\alpha-\beta| + |\beta|$

or $\quad |\alpha| - |\beta| \leqslant |\alpha-\beta| + |\beta| - |\beta| = |\alpha-\beta| + \bar{0}$
$\qquad = |\alpha-\beta|$

$\therefore \quad |\alpha-\beta| \geqslant |\alpha| - |\beta|.$

Also, $\therefore \quad |\alpha-\beta| = |\beta-\alpha| \geqslant |\beta| - |\alpha|.$

Now $\quad \big|\, |\alpha| - |\beta| \,\big|$ is

either $\quad = |\alpha| - |\beta|.$

or $\quad = -\{|\alpha| - |\beta|\}$
$\qquad - |\alpha| + |\beta| = |\beta| - |\alpha|$

$\therefore \quad |\alpha-\beta| \geqslant \big|\, |\alpha| - |\beta| \,\big|.$

13·3. If $\quad |\alpha-\beta| < \gamma,$

then $\quad \beta-\gamma < \alpha < \beta+\gamma.$

If $\quad \alpha-\beta$ be positive, we have

$$\alpha-\beta = |\alpha-\beta| < \gamma$$

$\therefore \quad \alpha-\beta+\beta < \gamma+\beta$

or $\quad \alpha+(-\beta)+\beta < \gamma+\beta$

or $\quad \alpha+[(\beta+(-\beta)] < \gamma+\beta$

or $\quad \alpha+\bar{0} < \gamma+\beta$, *i.e.*, $\alpha < \gamma+\beta = \beta+\gamma$

or $\quad \alpha < \beta+\gamma. \qquad$ (*i*)

Again, since $\alpha-\beta$ is positive, we have

$$\alpha > \beta.$$

$\therefore \quad \alpha+(-\gamma) > \beta+(-\gamma)$ or $\alpha-\gamma > \beta-\gamma.$

Again, $\because$ $\bar{0} > -\gamma$, γ being necessarily positive,

$\therefore$ $$\alpha = \alpha + \bar{0} > \alpha + (-\gamma) = \alpha - \gamma$$

$\therefore$ $$\alpha > \beta - \gamma. \qquad (ii)$$

Thus we have, from (*i*) and (*ii*)

$$\beta - \gamma < \alpha < \beta + \gamma.$$

Let $\alpha - \beta$ be negative. We have

$$|\alpha - \beta| = |\beta - \alpha|.$$

Since $$-(\alpha-\beta) = -[\alpha + (-\beta)]$$
$$= -[(-\beta)+\alpha] = -(-\beta) - \alpha = \beta - \alpha,$$

we see that $\beta - \alpha$ is positive.

Since $\beta - \alpha$ is positive and $|\beta - \alpha| < \gamma$, we have from above

$$\alpha - \gamma < \beta < \alpha + \gamma$$

$\because$ $$\beta < \alpha + \gamma, \therefore \beta - \gamma < \alpha + \gamma - \gamma = \alpha$$

or $$\alpha > \beta - \gamma. \qquad (iii)$$

Again, $\because$ $\beta > \alpha - \gamma$ $\therefore$ $\beta + \gamma > \alpha - \gamma + \gamma = \alpha$

or $$\alpha < \beta + \gamma. \qquad (iv)$$

From (*iii*) and (*iv*)

$$\beta - \gamma < \alpha < \beta + \gamma.$$

Hence the result.

13·4. *To prove that*

$$|\alpha\beta| = |\alpha| . |\beta|.$$

Case I. Let α, β be both positive so that $\alpha\beta$ is also positive.

We have $$|\alpha\beta| = \alpha\beta = |\alpha| . |\beta|.$$

Case II. Let α, β be both negative so that $\alpha\beta$ is positive.

We have

$$|\alpha\beta| = \alpha\beta = (-\alpha).(-\beta) = |\alpha| . |\beta|.$$

Case III. Let α be positive and β negative so that $\alpha\beta$ is negative.

We have

$$|\alpha\beta| = -(\alpha\beta) = -\{-[(\alpha).(-\beta)]\}$$
$$= (\alpha).(-\beta) = |\alpha| . |\beta|.$$

Case IV. Let α be negative and β positive.

We have

$$|\alpha\beta| = |\beta\alpha| = |\beta| . |\alpha|, \text{ Case III above.}$$
$$= |\alpha| . |\beta|.$$

13·5. *To prove that*

$$\left|\frac{\alpha}{\beta}\right| = \frac{|\alpha|}{|\beta|}, \text{ if } \beta \neq \bar{0}.$$

If β be positive, then $\bar{1}/\beta$ is also positive, so that we have

$$\left|\frac{\bar{1}}{\beta}\right| = \frac{\bar{1}}{\beta} = \frac{\bar{1}}{|\beta|};$$

if β be negative, then $\bar{1}/\beta$ is also negative, so that we have

$$\left|\frac{\bar{1}}{\beta}\right|=-\frac{\bar{1}}{\beta}=-\left[-\left(\frac{\bar{1}}{-\beta}\right)\right]=\frac{\bar{1}}{-\beta}=\frac{\bar{1}}{|\beta|}.$$

Finally, we have

$$\left|\frac{a}{\beta}\right|=\left|a.\frac{\bar{1}}{\beta}\right|=|a|.\left|\frac{\bar{1}}{\beta}\right|=|a|.\frac{\bar{1}}{|\beta|}=\frac{|a|}{|\beta|}.$$

14. Sections of real numbers Dedekind's theorem. *If all the real numbers be divided into two classes L and R such that*

(i) each class exists, (ii) each real number has a class, (iii) every member of L is less than every member of R, then

either the class L has a greatest member, or the class R has a least member.

We form two classes L_1, R_1 consisting of rational numbers which correspond to the real rational members of L, R respectively.

Let the class L_1 have a greatest member say, a. The real rational number $\bar{a}$ will belong to L. It will be shown that $\bar{a}$ is the greatest member of L.

If not, let α be a greater member of L.

Let $\bar{b}$ be any one of the infinite number of real rational numbers lying between $\bar{a}$ and α so that we have

$$\bar{a}<\bar{b}<\alpha.$$

Since $\bar{b}$ is less than a member α of L it must itself belong to L and accordingly b *is a member of L_1.*

Also, $\because \quad \bar{a}<\bar{b},$

$\therefore \quad a<b.$

Thus b, a member of L_1, is greater than the greatest member a of L_1.

This conclusion is absurd.

Hence in this case L must have a greatest member.

Let the class R_1 have a least member, say, b. In this case $\bar{b}$ will be the least member of R. This may be proved as above.

Let neither L_1 have a greatest member nor R_1, a least. In this case the section (L_1, R_1) is an irrational number α which must either belong to L or to R.

Let α belong to L. It will be shown that α is the greatest member of L. If not, let β be a greater member of L.

Let $\bar{a}$ be any real rational number such that

$$\alpha<\bar{a}<\beta.$$

The number $\bar{a}$ belongs to L and therefore a belongs to L_1.

Thus there exists a member a of the lower class L_1 of a real number $\alpha\equiv(L_1, R_1)$ such that

$$\bar{a}>\alpha$$

and this is absurd. §5·3.

It may similarly be shown that if α belongs to R then it is the least member of R.

Note 1. The theorem is sometimes stated in the following equivalent form :—If all the real numbers are divided into two classes L, R such that, (*i*) each class exists, (*ii*) each number has a class, (*iii*) every member of L is less than every member of R, then *there exists a real number α such that every real number less than α belongs to L and every real number greater than α belongs to R; α itself may belong to either class.*

This number α is the greatest member of L or the least of R which ever may exist. Also, this number α is said to determine the section.

Note 2. The theorem discussed above indicates a fundamental difference between the sections of rational numbers and the sections of real numbers in as much as we have seen, that if (L, R) be a section of rational numbers it is possible that neither L may have a greatest member nor R may have a least, but if it were a section of real numbers, then this cannot be the case. This difference is generally described by saying that there may be a gap between the classes L, R of rational numbers but there is no gap between the classes L, R of real numbers; the system of rational numbers has gaps while the system of real numbers has none.

Note 3. It is easy to show that any given aggregate of real numbers will form the lower class of a section if and only if (*i*) all the numbers do not belong to it (*ii*) a number which is less than any member of the aggregate is also a member of the aggregate.

15. Representation of real numbers by points along a straight line. Every thinking person possesses an intuitive idea of a straight line which, further, he can easily conceive as composed of points, even though this physical notion of a straight line and that of points on it has nothing to do with Analysis as such, yet it provides a very convenient and helpful *picture* of the aggregate of real numbers and is often employed in the course of study of Analysis to provide suitable language and suggest ideas. One danger, which is inherent in this use should, however, be avoided; it may be that we accept a proposition suggested by this picture, obvious as it may seem, as obviously true and this obviousness may blind us to the necessity of a rigorous proof.

We now proceed to see how a straight line can be employed to provide a picture of the aggregate of real numbers.

Representation of rational numbers by the points of a line. We consider any straight line and mark any two points O and A on it. The point O divides the line into two parts; the part containing the point A will be termed positive and the other negative.

According to the usual convention, the line in question is always drawn parallel to the printed lines of the page and the point A taken on the right of O. Representing the rational numbers 0 and 1 by the points O and A respectively, we find a point P of the line representing any rational number p/q, $(q>0)$ by marking from O, $|p|$ steps each equal to qth part of OA to the right or to the left of O according as p is positive or negative.

It is easy to see that if a, b be two rational numbers and $a<b$, then the point representing b lies to the right of the point reprerenting a.

If we call the points which represent rational numbers as rational points, we see that, since the aggregate of rational numbers is dense, an infinite number of rational points lie between every two different rational points.

Insufficiency of rational numbers to provide a picture of straight line. Even though, as we have seen above, a line can be covered with rational points as closely as we like, there exist points of the line which are not rational. For example, a point P such that OP is equal to the diagonal of the square with side OA is one such point (§ 3·1). Also a point L on the line such that OL is any rational multiple p/q of OP cannot be a rational point. For, if possible, let L represent a rational number m/n, so that we have

$$\frac{p}{q}.\mathrm{OP}=\mathrm{OL}=\frac{m}{n} \text{ or } \mathrm{OP}=mq/np,$$

which shows that OP is rational *i.e.*, P is a rational point and this is a contradiction.

Thus we see that the aggregate of rational numbers is not sufficient to provide us with a picture of complete straight line.

Real numbers.

Let $$\alpha\equiv(\mathrm{L}, \mathrm{R})$$
be any real number. The section (L, R) of rational numbers determines a section of the rational points of the line into two classes A and B such that A consists of rational points corresponding to the members of L and B of rational points corresponding to the members of R. Every point of the class A will lie to the left of every point of the class B.

From our intuitive picture of a straight line and its continuity, we can convince ourselves that there will exist a point P of the line separating the two classes in the sense that every point of the line lying to the left of P belongs to the class A and every point lying to the right of P belongs to the class B. *This point P, we say, denotes the real number (L, R).* Thus to every real number there corresponds a point of the line.

Conversely, let P be any point of the line. The point P divides the rational points of the line into classes A and B such that the points lying to the left of P belong to A and those to the right of P belong to B; the point P, if rational, belongs to B. The classes A, B of rational points determine a section (L, R) of rational numbers, which corresponds to the point P.

Thus to every point there corresponds a real number.

The aggregate of real numbers is called the *Arithmetical Continuum* and the aggregate of points on a straight line is called the *linear or Geometric Continuum.* In view of what has been shown above we see that there is a one-to-one correspondence between the two aggregates or continua and it may be found convenient to use the word 'point' for 'real number'.

16. Notation for real rational numbers. From above it will be seen that if (L, R) is a real rational number, then the point P which denotes this real number also denotes the rational number which is the least member of R and which, we know, is the rational number corresponding to (L, R). Thus we see that, according to the manner of representation explained above, a real rational number and the corresponding rational number are denoted by the same point of the line.

In view of this, *we agree to denote,* for future developments, *a real rational number by the same symbol which denotes the corresponding rational number* so that if 'a' is a rational number, then the same symbol 'a' will also, now, be used to denote the corresponding real rational number which has so far been denoted by $\bar{a}$.

The context in which the symbol may appear will fix the interpretation.

This use of a single symbol to denote two different concepts leads to no confusion, but is helpful, in as much as we have seen that a statement, which describes some relation between rational numbers, remains true when the symbols for rational numbers are interpreted as symbols for the corresponding real rational numbers.

In fact it has been shown that

$$\begin{aligned} &\text{if} \quad a>b, \text{ then} \quad \bar{a}>\bar{b}\,; \\ &\text{if } a\pm b=c, \text{ then} \quad \bar{a}\pm\bar{b}=\bar{c}\,; \\ &\text{if} \quad ab=c, \text{ then} \quad \bar{a}.\bar{b}=\bar{c}\,; \\ &\text{if} \quad a/b=c, \text{ then} \quad \bar{a}/\bar{b}=\bar{c}, \end{aligned}$$

where a, b, c denote rational numbers.

Thus, for example, the proposition

$$2+3=5,$$

where 2, 3, 5 denote rational numbers, remains true even when they denote corresponding real rational numbers which have so far been denoted by the symbols $\bar{2}$, $\bar{3}$, $\bar{5}$.

An important note: In the following chapters the word 'number' will always mean 'real number' and the word 'rational number will mean 'real rational number'.

EXAMPLES

1. Give an account of Dedekind's theory of real numbers. Show that there are gaps between rational numbers, but the continuum of real numbers, as postulated by Dedekind, is free from gaps. (P. U., M.A., 1939)

2. In order to generalize the conception of number, what are the essential requisites which must be satisfied. Develop Dedekind's theory of real numbers, and show how far this theory satisfies these requisites. (P. U., M.A., 1937.)

3. State and prove 'Dedekind's theorem.' (P. U., M.A., 1940.)

4. Explain briefly the theory of real numbers and establish their correspondence with the points of a line continuum.

Define addition of irrational numbers by the use of a partition of rational numbers, and show that the Associative law of addition holds. (P. U., M.A., 1936.)

5. Develop Dedekind's theory of real numbers. When are two numbers equal or unequal to, greater or less than, each other in accordance with this theory. (P. U., M.A., 1938.)

6. What is Arithmetic and what is Geometric continuum. Explain under which conditions they are equivalent.

CHAPTER II

ELEMENTS OF THE THEORY OF AGGREGATES

17. An **aggregate** or set S of numbers is defined, when there is given a law or laws which determine, without ambiguity, whether any given number does or does not belong to it. An aggregate of numbers may also be spoken of as an aggregate of points.

Intervals, closed and open. The aggregate of numbers x such that $a \leqslant x \leqslant b$, where a, b are any two numbers, is called a *closed interval* (a, b). The aggregate of numbers x such that $a < x < b$ is called an open interval $[a, b]$. The aggregates $a < x \leqslant b$, $a \leqslant x < b$ are called the intervals open on the left and open on the right respectively and are denoted as $[a, b)$ and $(a, b]$.

Sub-aggregates. An aggregate S_1 is said to be a sub-aggregate or a sub-set of another aggregate S, if every member of S_1 is also a member of S.

Finite and infinite aggregates. An aggregate is *finite* if there exists a positive integer n such that it contains just n members and otherwise the aggregate is *infinite.*

E. G. The aggregate of all the integers between—20 and 30 is finite, but the aggregate of all the rational numbers between—29 and 30 is infinite.

18. Greatest and least members of an aggregate. A number M is the greatest member of an aggregate S, if (*i*) M is a member of S; (*ii*) no member of S is greater than M. Again, a number m is the least member of an aggregate S, if (*i*) m is a member of S; (*ii*) no member of S is less than m. The greatest and least members are also respectively called the *maximum* and *minimum* members of the aggregate.

Every finite aggregate has necessarily a greatest and a least member, but an infinite aggregate may or may not have a greatest or a least member.

E. G. The aggregate of all the integers has neither a greatest nor a least member; the aggregate of positive integers has no greatest member but has a least *viz.*, 1; the aggregate of negative integers has no least member but has a greatest *viz.*, -1.

1 is the greatest and 0 is the least member of the closed interval (0, 1); the open interval [0, 1] has neither a greatest nor a least member; the semi-closed interval (0,1] has no greatest member but has a least *viz.*, (0); the interval [0, 1) has no least member but has a greatest member, *viz.*; 1.

19. Bounded and unbounded aggregates. If there exists a number K such that every member of an aggregate S is $\leqslant$K, then we say that S is *bounded above* or that it is *bounded on the right* and further say that K is a *rough upper bound* of S.

Similarly, if there exists a number k such that every member of an aggregate S is $\geqslant k$, then we say that S is *bounded below* or that it is *bounded on the left* and further say that k is a *rough lower bound* of S.

An aggregate is said to be *bounded* if it is bounded above as well as below.

E. G. The aggregate of all the integers is neither bounded above nor below; the aggregate of all the positive integers is bounded below but not above; the aggregate of all the negative integers is bounded above but not below.

The intervals (0, 1), [0, 1], (0, 1], [0, 1) are all bounded.

Ex. 1. Show that if K is a rough upper bound of an aggregate S and $K' > K$, then K' is also a rough upper bound of S; also give examples to show that if $K' < K$, then K' may or may not be a rough upper bound.

State a similar result concerning aggregates which are bounded below.

Ex. 2. Show that an aggregate with a greatest member is bounded above but that the converse is not necessarily true. State a similar result for aggregates which are bounded below.

Ex. 3. Show that every sub-aggregate of a bounded aggregate is bounded.

Ex. 4. *Show that for a bounded aggregate S there exists a positive number A such that* $|x| < A$, *where x is any member of S.*

Since S is bounded, there exist numbers k and K such that

$$k \leqslant x \leqslant K. \qquad (i)$$

Let A be any number greater than both $|k|$ and $|K|$ so that we have

$$-A < k \text{ and } K < A. \qquad (ii)$$

From (*i*) and (*ii*), we have

$$-A < k \leqslant x \leqslant K < A, \text{ i.e., } -A < x < A,$$

or

$$|x| < A.$$

Conversely, if $|x| < A$, then $-A < x < A$ so that the aggregate is bounded.

Ex. 5. x, y are any two members of the bounded aggregates S_1 and S_2 respectively; show that the aggregates of numbers, $x+y$, $x-y$, xy are also bounded.

Ex. 6. Show that the aggregate of rough upper bounds of an aggregate bounded above is bounded below. State a similar result for aggregates which are bounded below.

20. The upper and lower bounds. The two theorems obtained in this section are fundamental in the discussion of bounded aggregates.

20·1. Theorem. *For every aggregate S bounded above, there* **exists** *a number B such that*

(*i*) *every member of the aggregate is less than or equal to B;*

(*ii*) *every number less than B is smaller than at least one member of the aggregate, i.e., however small the positive number ϵ may be, there is a member of S greater than $B-\epsilon$.*

Divide all the numbers into two classes L and R, putting a number in L if it is smaller than atleast one member of S and otherwise in R.

Clearly each number has a class. Since a number less than any member of S belongs to L and K, any rough upper bound of S, belongs to R, we see that each class exists. Finally, a number which is less than a member of L is necessarily less than a member of the aggregate, and accordingly it belongs to L. Thus the two classes L, R determine a section of the real numbers. There exists, therefore, a number B separating the two classes such that every number less than B belongs to L and every number greater than B belongs to R. It will now be shown that this is the number B of the theorem.

Any number $B-\epsilon$, $(\epsilon>0)$, which is less than B belongs to L and is, therefore, smaller than at least one member of the aggregate.

Also no member of the aggregate is greater than B. For, if possible, let there be a member $B+\eta$, of the aggregate which is greater than B.$(\eta>0)$. The members of the open interval $[B, B+\eta]$ all belong to R, for each of them is greater than B; also they all belong to L, for each of them is smaller than a member $B+\eta$ of the aggregate. This is a contradiction.

Thus we have proved that the number B, which separates the two classes, possesses the two properties stated in the theorem.

This number B is said to be **the upper bound** of the aggregate S.

Remarks 1. It is easy to see that the upper bound B is a rough upper bound of S such that no number less than B is a rough upper bound, *i.e.*, B is the least of all the rough upper bounds. In other words, therefore, the theorem states that the *aggregate of rough upper bounds of an aggregate bounded above possesses a least member.*

2. The maximum *i.e.*, the greatest member of the aggregate, in case it exists, is also the upper bound, and we then say that the aggregate *attains* its upper bound.

20·2. Theorem. *For every aggregate S bounded below, there* **exists** *a number b such that*

(i) every member of the aggregate is greater than or equal to b ;

(ii) however small the positive number ϵ may be, there is a member of S less than $b+\epsilon$.

Its proof is similar to that of the previous theorem on upper bounds. To prove it, the real numbers will have to be divided into classes L and R such that a number will belong to R if it is greater than at least one member of the aggregate and otherwise to L.

This number b is said to **the lower bound** of S.

Remarks 1. It is easy to see that the lower bound b is the greatest of all the rough lower bounds of S so that, in other words, the theorem states that *the aggregate of rough lower bounds of an aggregate bounded below possesses a greatest member.*

2. The minimum, if it exists, is the lower bound, and we then say that the aggregate *attains* its lower bound.

3. Obviously $B \geqslant b$.

Ex. 1 is the upper bound and 0 is the lower bound of each of the four intervals (0, 1), [0, 1], (0, 1], [0, 1).

Ex The upper bound of the set of numbers

$$1, \frac{1}{10}, \frac{1}{10^2}, \ldots\ldots, \frac{1}{10^n}, \ldots\ldots$$

is 1 ; what is the lower bound ?

Ex. Construct examples to show that the bounds may or may not themselves be members of the aggregate.

Ex. Show that the greatest member of an aggregate, in case it exists, is the upper bound and the least member, if it exists, is the lower bound.

Ex. The members of a bounded aggregate are all positive ; show that the bounds cannot be negative.

20·3. Oscillation of a bounded aggregate. *The difference $B-b$, of the bounds B, b of a bounded aggregate is called its oscillation.*

EXAMPLES

1. *B, b are the bounds of an aggregate S and B_1, b_1 are the bounds of a sub-aggregate S_1 of S ; show that*

$$b \leqslant b_1 \leqslant B_1 \leqslant B.$$

Every member of S_1 is a member of S and, accordingly, it must be $\leqslant B$ and thus B is a rough upper bound of S_1. The upper bound B_1 of S_1 being the least of its rough upper bounds, we have

$$B_1 \leqslant B.$$

In the similar manner it may be proved that

$$b \leqslant b_1.$$

Also, obviously $\qquad b_1 \leqslant B_1.$

2. *x is any member of a bounded aggregate S_1 whose bounds are B_1, b_1 ; show that the bounds of the aggregate S of numbers $-x$ are $-b_1$, $-B_1$.*

Since $x \geqslant b_1$, therefore $-x \leqslant -b_1$, where $-x$ is any member of S. (*i*)

Let ϵ be any positive number, however small. There exists a member x of S_1 such that

$$x < b_1 + \epsilon,$$

which shows that there exists a member $-x$ of S such that

$$-x > -b_1 - \epsilon. \qquad (ii)$$

Hence $-b_1$ is the upper bound of S.

It may similarly be shown that $-B_1$ is the lower bound of S.

(In examples 3—6 below x, y denote any two members of the bounded aggregates S_1, S_2 and B_1, b_1 ; B_2, b_2, are respectively their bounds).

3. *Show that the bounds of the aggregate S of numbers $x+y$ are* B_1+B_2, b_1+b_2.

Since $\qquad x \leqslant B_1$ and $y \leqslant B_2$,

therefore $\qquad x+y \leqslant B_1+B_3, \qquad (i)$

where $x+y$ is *any* member of S.

Let ϵ be any positive number. There exist members x, y of S_1, S_2 respectively such that

$$x > B_1 - \tfrac{1}{2}\epsilon, \; y > B_2 - \tfrac{1}{2}\epsilon,$$

which show that there exists a member $x+y$ of S such that

$$x+y > (B_1+B_2) - \epsilon. \qquad (ii)$$

Thus (B_1+B_2) is the upper bound of S.

It may similarly be shown that b_1+b_2 is the lower bound of S.

4. Show that the bounds of the aggregate S of numbers $x-y$ are B_1-b_2, b_1-B_2.

5. *If the members of the aggregates S_1, S_2 are all positive, then show that the bounds of the aggregate S of numbers xy are $B_1 B_2$, $b_1 b_2$.*

The numbers B_1, B_2, b_1, b_2 must all be non-negative.

Since $\qquad x \leqslant B_1, y \leqslant B_2,$

therefore $\qquad xy \leqslant B_1 B_2, \qquad (i)$

where xy is *any* member of S.

Let ϵ be any given positive number, however small.

If ϵ_1, ϵ_2. are any two positive numbers, then there exist members x, y of S_1, S_2 respectively such that

$$x > B_1 - \epsilon_1, \; y > B_2 - \epsilon_2$$

whence we have $\qquad xy > (B_1 - \epsilon_1)(B_2 - \epsilon_2). \qquad (ii)$

It will now be shown that it is possible to choose ϵ_1, ϵ_2 in terms of ϵ such that

$$(B_1 - \epsilon_1)(B_2 - \epsilon_2) > B_1 B_2 - \epsilon. \qquad (iii)$$

so that it will be deduced from (*ii*) and (*iii*) that there exists a member xy of S such that

$$xy > B_1 B_2 - \epsilon. \qquad (iv)$$

Now, $$(B_1-\epsilon_1)(B_2-\epsilon_2)>B_1B_2-\epsilon,$$
if $$\epsilon_1B_2+\epsilon_2B_1<\epsilon+\epsilon_1\epsilon_2$$
or if $$\epsilon_1B_2+\epsilon_2B_1<\epsilon.$$
Taking $\epsilon_1=\epsilon/3B_2$, $\epsilon_2=\epsilon/3B_1$, if $B_1, \neq 0$, $B_2\neq 0$, we see that
$$\epsilon_1B_2+\epsilon_2B_1=\tfrac{2}{3}\epsilon<\epsilon.$$

The argument can be easily modified if either or both of B_1, B_2 are zero.
The case of lower bound may be similarly discussed.

6. *If the members of S_1, S_2 are all positive and $b_2\neq 0$, show that the bounds of the aggregate S of numbers x/y are B_1/b_2, b_1/B_2.*

The numbers B_1, B_2, b_1, b_2 must all be non-negative.

Since $$b_1\leqslant x\leqslant B_1,\ b_2\leqslant y\leqslant B_2 \text{ or } 1/B_2\leqslant 1/y\leqslant 1/b_2,$$
therefore $$\frac{b_1}{B_2}\leqslant\frac{x}{y}\leqslant\frac{B_1}{b_2}, \qquad (i)$$
where x/y is any member of S.

Let ϵ be any given positive number.

If ϵ_1, ϵ_2 be any two positive numbers, then there exist members x, y of S_1 and S_2 such that
$$x>B_1-\epsilon_1,\ y<b_2+\epsilon_2$$
whence we have
$$\frac{x}{y}>\frac{B_1-\epsilon_1}{b_2+\epsilon_2}. \qquad (ii)$$

It will now be shown that it is possible to choose ϵ_1, ϵ_2 in terms of ϵ such that
$$\frac{B_1-\epsilon_1}{b_2+\epsilon_2}>\frac{B_1}{b_2}-\epsilon \qquad (iii)$$
so that it will then be deduced from (*ii*) and (*iii*) that there exists a member x/y of S such that
$$\frac{x}{y}>\frac{B_1}{b_2}-\epsilon.$$

Now, (*iii*) will hold,

if $$(B_1-\epsilon_1)b_2>(b_2+\epsilon_2)(B_1-\epsilon b_2)$$
i.e., if $$\epsilon_1b_2+\epsilon_2B_1<\epsilon b_2^2+\epsilon\epsilon_2b_2$$
or if $$\epsilon_1b_2+\epsilon_2B_1<\epsilon b_2^2$$
or if $$\epsilon_1/b_2+\epsilon_2B_1/b_2^2<\epsilon.$$
Taking $\epsilon_1=\epsilon b_2/3$ and $\epsilon_2=\epsilon b_2^2/3B_1$ we see that
$$\frac{\epsilon_1}{b_2}+\frac{\epsilon_2B_1}{b_2^2}=\frac{2}{3}\epsilon<\epsilon.$$

The case of lower bound may be similarly discussed.

7. Show that the bounds of the aggregate S consisting of the members of both S_1 and S_2 are Max. $\{B_1, B_2\}$, min. $\{b_1, b_2\}$.

8. *x, y are any two members of a bounded aggregate S; show that the upper bound of the aggregate S_1 of numbers*
$$(i)\ x-y.\quad (ii)\ |x-y|,$$
is the oscillation of S.

Let B, b be the bounds of S.

(*i*) Since $$x\leqslant B \text{ and } y\geqslant b \text{ or } -y\leqslant -b,$$
therefore $$x-y\leqslant B-b \qquad (i)$$
where $x-y$ is any member of the aggregate S_1.

Let ϵ be any given positive number.

There exist two members x and y of S_1 such that
$$x>B-\tfrac{1}{2}\epsilon,\ y<b+\tfrac{1}{2}\epsilon \text{ or } -y>-b-\tfrac{1}{2}\epsilon$$
whence we see that there exists a member $x-y$ of S_1 such that
$$x-y>B-b-\epsilon. \qquad (ii)$$

Thus the oscillation $B-b$ is the upper bound of the aggregate of numbers $x-y$.

(*ii*) It is now obvious.

21. Limiting point of an aggregate. Def. *A number ξ is said to be a limiting point of an aggregate, if every interval* $(\xi-\epsilon, \xi+\epsilon')$, *which encloses ξ, contains an infinite number of members of the aggregate ;* ϵ, ϵ' are any positive numbers.

The limiting points of a set may or may not themselves be members of the set.

Obviously, a finite aggregate cannot have a limiting point; it is only infinite aggregates which may have one, more or even an infinite number of limiting points. Of course, even some infinite aggregates may have no limiting point.

The examples below illustrate the various possibilities.

Ex. Every real number is a limiting point of the aggregate of rational numbers. (Refer §5·4)

Rational limiting points are members of the aggregate but irrational limiting points are not.

Ex. Every real number is a limiting point of the aggregate of irrational numbers. (Refer §10)

Ex. The aggregate of integers, even though infinite, has no limiting point.

Ex. The aggregate of numbers

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots, \frac{1}{n}, \ldots\ldots$$

has only one limiting point, *viz.*; 0, and it does not belong to the aggregate.

Ex. The aggregate

$$1+\tfrac{1}{2}, -1-\tfrac{1}{2}, 1+\tfrac{1}{3}, -1-\tfrac{1}{3}, 1+\tfrac{1}{4}, -1-\tfrac{1}{4}, \ldots\ldots$$

has two limiting points, *viz.*; 1 and -1.

Ex. The aggregate of numbers

$$\frac{1}{m}+\frac{1}{n}+\frac{1}{p},$$

where m, n, p take up all integral values is the aggregate of numbers

$$\frac{1}{m}+\frac{1}{n},$$

and the number 0.

Ex. Limiting point of any sub-aggregate of an aggregate S is also a limiting point of S.

Ex. S_1, S_2 are two aggregates and an aggregate S consists of the members of S_1 and S_2; show that a limiting point of S must be a limiting point of either S_1 or of S_2 and conversely.

Ex. Show that the upper bound of an aggregate S which does not have a maximum is a limiting point of S; state and prove a similar result for lower bounds.

21·1. Weirstrass's theorem on the existence of limiting points.

Every infinite bounded aggregate has atleast one limiting point.

Let S be any infinite bounded aggregate.

Since S is bounded, there exist real numbers k, K such that every member of S belongs to the interval (k, K).

Divide all the numbers into two classes L, R, putting a number in L if it exceeds only a finite number of members of S and otherwise in R.

Clearly each number has a class. Also k belongs to L and, the aggregate being infinite, K belongs to R and therefore each class exists. Finally, any number which is less than a member of L can exceed only a finite number of members of S and accordingly it must belong to L. Thus the two classes L, R determine a section of real numbers. There exists, therefore, a number g separating the two classes. It will now be shown that g is a limiting point of S.

Consider *any* interval $(g-\epsilon, g+\epsilon')$ which encloses g. Now, $g-\epsilon$ which belongs to L can exceed only a finite number of members of S, and $g+\epsilon'$, which belongs to R, must exceed an infinite number of members of S and accordingly there must belong an infinite number of members of S to the interval $(g-\epsilon, g+\epsilon')$.

Hence g is a limiting point of S.

Ex. A number c is the only limiting point of a bounded aggregate S; I is *any* interval which encloses c; show that there can exist, at the most, a finite number of members of S not belonging to I. Construct an example to show that the result may not necessarily be true if S is not bounded.

22. Derived Aggregates. The aggregate consisting of the limiting points, if any, of another aggregate is called the *first derivative* or simply the *derivative* of S and is denoted by S′. The derivative of S′ is called the *second derivative* of S and is denoted by S″. Proceeding thus we may have a number of sets

$$S, S', S'', \ldots\ldots S^{(n)},$$

which are the successive derivative of S.

If the *nth derivative* $S^{(n)}$ contain only a finite number of members then it has no limiting points and this chain of successive derivatives ceases at $S^{(n)}$. In such a case *when an aggregate possesses only a finite number of derivatives, we say that the aggregate is of first species and otherwise of second species.*

Ex. Show that the aggregate of numbers $1/m+1/n+1/p$, where m, n, p take up all integral values is of first species.

Ex. Show that the aggregate of rational numbers is of second species.

22·1. Theorem. *The derived aggregate of a bounded aggregate is bounded and attains its bounds.*

Let S be any bounded aggregate and let its every member belong to an interval (k, K).

Clearly, no limiting point of S, *i.e.*, no member of the derivative S′ can be less than k or greater than K, and accordingly S′ is bounded. Let g, G be the bounds of S′. It will now be shown that g, G are themselves members of S′, *i.e.*, limiting points of S.

Let $(G-\epsilon, G+\epsilon')$ be any interval enclosing G.

Since G is the upper bound of S′, therefore there must exist a member ξ of S′ such that

$$G-\epsilon<\xi\leqslant G.$$

The interval $(G-\epsilon, G+\epsilon')$, which encloses the limiting point ξ of S, must contain an infinite number of members of S and accordingly G must be a limiting point of S.

It may similarly be shown that g is a limiting point of S.

Note. The theorem may also be stated thus : *The derived aggregate of a bounded aggregate is bounded and possesses greatest and least members.*

G, g are the greatest and least limiting points of S.

23. Def. Upper and lower limits. *The greatest limiting point G is called the upper limit and the lowest limiting point g the lower limit.*

23·1. Two characteristic properties of the upper limit G.

If ϵ be any positive number, however small, then

(*i*) an infinite number of members of the aggregate are greater than $G-\epsilon$;

(*ii*) a finite number, at the most, of members of the aggregate are greater than $G+\epsilon$.

23·2. Two characteristic properties of the lower limit g.

If ϵ be any positive number, however small, then

(*i*) an infinite number of members of the aggregate are smaller than $g+\epsilon$;

(*ii*) a finite number, at the most, of members of the aggregate are smaller than $g-\epsilon$.

These properties are easily deducible from the fact that G and g are the greatest and least limiting points.

Note 1. It is important to remember that for a bounded aggregate S,

$$b \leqslant g \leqslant G \leqslant B,$$

as may be easily seen.

2. Throughout this book, the maximum, the upper bound and the upper limit of an aggregate will be denoted by M, B, G, respectively similarly the minimum, the lower bound and the lower limit will be denoted by m, b, g respectively.

24. Theorem. For a bounded aggregate,

(*i*) M, B, G cannot all be different numbers ;

also (*ii*) m, b, g cannot all be different numbers.

(*i*) Let S be any bounded aggregate.

In case S has a maximum, *i.e.*, if M exists, then

$$M=B.$$

Now, let M not exist so that B is not a members of S. In this case, as will be shown,

$$G=B.$$

Consider any interval $(B-\epsilon, B+\epsilon)$ enclosing B.

No member of S is greater than B.

If possible, let only a *finite* number of members of S lie between $B-\epsilon$ and B so that there will be a greatest of them, say, λ.

Since $\lambda \neq B$, we see that no member of S is greater than λ which is less than the upper bound B and this is impossible. (§20·1)

Thus there belong an infinite number of members of S to $(B-\epsilon, B)$ and consequently to $(B-\epsilon, B+\epsilon)$ and accordingly B is a limiting point.

Thus $\qquad B \leqslant G.$

If possible, let $\qquad B < G.$

Since B is the upper bound, no member of S is greater than B and accordingly G cannot be a limiting point. This is a contradiction.

Thus $\qquad B = G.$

It may similarly be shown that m, b, g cannot all be different.

EXAMPLES

1. Construct bounded aggregates for which

(*i*) $b<g<G<B$. (*ii*) $b<g<G=B$.
(*iii*) $b<g=G<B$. (*iv*) $b<g=G=B$.
(*v*) $b=g<G<B$. (*vi*) $b=g<G=B$.
(*vii*) $b=g=G<B$. (*viii*) $b=g=G=B$.

The following aggregates exhibit the above possibilities :—

(*i*) $1+1/n$, $-1-1/n$. (*ii*) $-1-1/n$, $1/n$.
(*iii*) -2, 2, $-1/n$, $1/n$ (*iv*) -2, $-1/n$.
(*v*) $-1+1/n$, $1+1/n$. (*vi*) $-1+1/n$, $1-1/n$.
(*vii*) 2, $-1+1/n$.
(*viii*) This case is not possible.

Here n takes up different positive integral values.

The student may construct other aggregates exhibiting the various possibilities.]

2. Construct an aggregate whereof no element lies between its upper and lower limits.

3. Examine the existence and the values of M, B, G; m, b, g for the following aggregates :—

(*i*) $3, 2, (2^{n-1}+1)/2^n, (2^n-1)/2^n$.

(*ii*) $8, -3, 3, 4, 2\frac{1}{3}, 4\frac{1}{2}, 2\frac{1}{9}, 4\frac{3}{4}, 2\frac{1}{27}, 4\frac{5}{8}, 2\frac{1}{81}, 4\frac{7}{16}, \ldots\ldots$

(*iii*) $\frac{4}{3}, -\frac{1}{2}, \frac{6}{5}, -\frac{3}{4}, \frac{8}{7}, -\frac{5}{6}, \frac{10}{9}, -\frac{7}{8}, \frac{12}{11}, \ldots\ldots$

(*iv*) $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \ldots\ldots$

$$\left.\begin{array}{l} (v)\ a+n^{(-1)^n}, \\ (vi)\ a-n^{(-1)^n}, \end{array}\right\} n \text{ takes up all positive integral values.}$$

CHAPTER III

SEQUENCES

25. Sequences. Any set of numbers

$$a_1,\ a_2,\ a_3, \ldots\ldots,\ a_n, \ldots\ldots,$$

such that to each positive integer n there corresponds a number a_n, is called a sequence.

The notation $\{a_n\}$ is adopted to denote the sequence whose nth member is a_n. The integer n is known as the suffix of a_n.

Thus a *sequence is an aggregate* whose members are arranged so as to correspond to the set of positive integers.

E. G. The sets of numbers

(i) $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}, \ldots\ldots,\ 1/n, \ldots\ldots,$

(ii) $-1,\ 2,\ -3,\ 4, \ldots\ldots,\ (-1)^n n, \ldots\ldots,$

(iii) $-1,\ 1,\ -1,\ 1, \ldots\ldots,\ (-1)^n, \ldots\ldots,$

(iv) $1,\ 2,\ 3,\ 5,\ 7,\ \ldots\ldots,\ a_n, \ldots\ldots,$

where a_n denotes the nth prime,

(v) $-2,\ \frac{3}{2},\ -\frac{4}{3},\ \frac{5}{4},\ -\frac{6}{5},\ \frac{7}{6},\ -\frac{8}{7},\ \frac{9}{8}, \ldots\ldots,\ a_n, \ldots\ldots,$

where $a_n=(1+1/n)$ or $(-1-1/n)$ according as n is even or odd,

(vi) $\frac{1}{2},\ \frac{7}{12},\ \frac{37}{60}, \ldots\ldots,\ a_n, \ldots\ldots,$

where $a_n=\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\ldots\ldots+\dfrac{1}{n+n}\right),$

are all sequences.

26. The upper and lower limits of a bounded sequence.

The upper limit G of a bounded sequence $\{a_n\}$ has the following two characteristic properties :—(Refer §23·1)

If ϵ be any positive number however small, then

(i) *$G-\epsilon<a_n$ for an infinite number of values of n ;*

(ii) *there exists a positive integer m such that $a_n<G+\epsilon$, for every positive integral value of $n \geqslant m$.*

Here m is any integer greater than the suffix of every member of the sequence (which are finite in number) $>G+\epsilon$.

The lower limit g of a bounded sequence $\{a_n\}$ possesses the following two characteristic properties :—(Refer §23·2).

If ϵ be any positive number, however small, then

(i) *$a_n<g+\epsilon$ for an infinite number of valuee of n ;*

(ii) *there exists a positive integer m such that $g-\epsilon<a_n$, for every positive integral value of $n \geqslant m$.*

Here m is any integer greater than the suffix of every member of the sequence (which are finite in number) $<g-\epsilon$.

The above results follow from §23, or directly from the fact that G, g are the greatest and the least limiting points respectively of the bounded sequence $\{a_n\}$; their existence having already been established in §22.

Note. If an infinite number of members of a sequence are all equal to the same number, say, k *i.e.* ; if $a_n = k$ for an infinite number of values of n, then the number k will be considered as a limiting point of $\{a_n\}$. With this understanding it will be seen that every bounded sequence will have atleast one limiting point.

Ex. Find M, m ; B, b ; G, g, whichever may exist, for the following equences :

(*i*) $a_n = (-1)^n/n$. (*ii*) $a_n = 1 + (-1)^n$. (*iii*) $a_n = (-1)^n(1 + 1/n)$.

$$(iv)\ a_n = \begin{cases} (n+1)/n, & \text{when } n = 3m; \\ (n+2)/2n, & \text{when } n = 3m+1; \\ 1/(n^2+1, & \text{when } n = 3m+2, \end{cases}$$

m being a positive integer.

(*v*) $a_n = (4^n + 1)/4^n$ or $(1 - 4^n)/4^n$ according as n is even or odd.

(*vi*) $a_n = [n + (-1)^n]/n$.

Ex. Construct a sequence with ± 2 for its bounds and ± 1 for its upper and lower limits such that no member lies between ± 1.

27. Convergent sequences. Def. *A bounded sequence $\{a_n\}$ is said to be* **convergent** *if it has only one limiting point and this unique limiting point is called the limit of the sequence.*

If l be the limit of a convergent sequence $\{a_n\}$, then we say that $\{a_n\}$ converges to the limit l and symbolically write

$$\lim_{n \to \infty} a_n = l, \text{ or } a_n \to l \text{ as } n \to \infty.$$

It will be seen that a bounded sequence $\{a_n\}$ is convergent if and only if its upper and lower limits are equal.

An important note. It will be well to emphasize that the symbolic statement $\lim\limits_{n\to\infty} a =_n l$ is equivalent to the following two assertions :—

(*i*) The sequence $\{a_n\}$ is convergent.

(*ii*) The limit of the convergent sequence $\{a_n\}$ is l.

27·1. Theorem. *The necessary and sufficient condition for a* **sequence $\{a_n\}$ to converge to a limit** *l is that to every positive number ϵ, however small, there corresponds a positive integer m such that*

$$|\, a_n - l \,| < \epsilon, \text{ when } n \geqslant m.$$

Remarks. Before proceeding to prove this theorem, we observe that the condition implies that if I be *any* interval $[l-\epsilon, l+\epsilon]$ enclosing l, then a finite number of members of the sequence, at the most, can be outside I, *i.e.*, all the members excepting, at the most, a finite number of them belong to I. Here m denotes any integer greater than the suffix of every member which does not belong to I.

The condition is necessary. Let the sequence $\{a_n\}$ converge to a limit l so that it is bounded and l is its only limiting point.

Let ϵ be any positive number, however small. There lie, at the most, a *finite* number of members of $\{a_n\}$ outside $[l-\epsilon, l+\epsilon]$, for, if they were infinite, then the sequence, which is bounded, will have at least one more limiting point which is different from l.

Let m be any positive integer greater than the suffix of every member which lies outside $[l-\epsilon, l+\epsilon]$. Then we have

$$l-\epsilon < a_n < l+\epsilon, \text{ i.e., } |a_n - l| < \epsilon, \text{ when } n \geqslant m.$$

The condition is sufficient. It will firstly be shown that under this condition the sequence is bounded.

Consider any interval, say, $[l-1, l+1]$, which encloses l. There exists a positive integer p such that every member of the sequence excepting at the most $a_1, a_2, \ldots.. a_{p-1}$ belong to this interval. If k be the least and K be the greatest member of the finite set of numbers

$$a_1, a_2, \ldots\ldots, a_{p-1}, l-1, l+1,$$

we see that

$$k \leqslant a_n \leqslant K, \text{ for every value of } n,$$

so that $\{a_n\}$ is bounded.

Clearly l is a limiting point of $\{a_n\}$ and we have now to show that this is the only limiting point.

If possible, let $l' \neq l$, be anyother limiting point.

We enclose l in an interval I so small that l' does not belong to it. According to the given condition, a finite number of members, at the most, can lie outside I so that l' cannot be a limiting point.

Thus the condition is sufficient also.

Note. It should be carefully noted that a convergent sequence is necessarily bounded.

28. Non-convergent bounded sequences. *A bounded sequence $\{a_n\}$ which does not converge is said to* **oscillate finitely**.

29. Unbounded sequences. In the case of sequences, which are not bounded, we distinguish the following three behaviours.

29·1. Divergence to ∞. If to every positive number $\triangle$, however large, there corresponds a positive integer m such that

$$a_n > \triangle, \text{ when } n \geqslant m,$$

then we say that $\{a_n\}$ is divergent and that it tends (or diverges) to ∞ as n tends to infinity and, in symbols, write

$$a_n \to \infty \text{ as } n \to \infty.$$

29·2. If to every positive number $\triangle$, however large, there corresponds a positive integer m such that

$$a_n < -\triangle, \text{ when } n \geqslant m,$$

then we say that $\{a_n\}$ is divergent and that it tends (or diverges) to $-\infty$ as n tends to infinity and, in symbols, write

$$a_n \to -\infty \text{ as } n \to \infty.$$

29·3. If an unbounded sequence does not diverge, *i.e.*, when it neither tends to ∞ nor $-\infty$, then we say that it *oscillates infinitely*.

Ex. Show that a divergent sequence cannot have a limiting point.

Ex. $\{a_n\}$ and $\{b_n\}$ are two sequences; $\{a_n\}$ diverges to ∞ and $b_n > a_n$ for every n; show that b_n also diverges to infinity.

Ex. The sequence $\{a_n\}$ is divergent and the sequence $\{b_n\}$ is convergent; show that the sequence $\{a_n + b_n\}$ is also divergent.

Ex. Show that a sequence obtained on re-arranging the members of another convergent sequence is also convergent and that the limits are the same.

[A sequence $\{b_n\}$ is said to be obtained on re-arranging the members of the sequence $\{a_n\}$, if every member of either sequence is some member of the other.

The result follows from the fact that the aggregates consisting of the members of such sequences are identical.]

Ex. Prove that $\text{lt}\left(\frac{1}{n^2} + \frac{2}{n^2} + \ldots\ldots + \frac{n}{n^2}\right) = \frac{1}{2}$. $\quad \frac{n+1}{2n} = \frac{1}{2}$([illegible])

Ex. Show that the sequence $\{a_n\}$, where $a_n = (-1)^n$, does not converge.

Ex. *Determine the least value of m for which it is true that*

$$\left|\frac{n^2+n+1}{3n^2+1} - \frac{1}{3}\right| < \epsilon, \text{ when } n \geqslant m,$$

ϵ being any positive number.

Now,
$$\left|\frac{n^2+n+1}{3n^2+1} - \frac{1}{3}\right| = \frac{3n+2}{3(3n^2+1)} < \frac{3n+n}{9n^2}, \text{ if } n > 2$$
$$= \frac{4}{9n} < \epsilon, \text{ if } n > 4/9\epsilon.$$

The integer just greater than 2 and $4/9\epsilon$ is the required value of m.

For example $m=5$ if $\epsilon = \frac{1}{10}$, $m = 45$, if $\epsilon = \frac{1}{100}$.

It shows that $\text{lt}\,[(n^2+n+1)/(3n^2+1)] = 1/3$.

Ex. Show, with the help of §27·1, that

(*i*) $\text{lt}\,\frac{n}{n+1} = 1$. (*ii*) $\text{lt}\,\frac{n^2+2}{n^3-1} = 0$. (*iii*) $\text{lt}\,\frac{4n^3+6n-7}{n^3+2n^2+1} = 4$.

Ex. Show, with the help of §29, that

(*i*) $\text{lt}\,(n^2 - 2n) = \infty$. (*ii*) $\text{lt}\,[(n^2+1)/(n+1)] = \infty$.

(*iii*) $\text{lt}\,[n + (-1)^n] = \infty$. (*iv*) $\text{lt}\,[(2-n^2)/(n+1)] = -\infty$.

(*v*) $\{n(-1)^n\}$ oscillates infinitely.

30. Instrinsic tests of convergence. The condition, as obtained in §27·1, answers the question "Is any *given* number l the limit of a sequence $\{a_n\}$, or is it not." If, with the help of this condition, it be shown that a given number l is not the limit of $\{a_n\}$, then it will not follow that the sequence does not converge; there being another possibility also, *viz.*, that $\{a_n\}$ may converge to a number different from l. Thus this condition examines the question of '*Convergence to a number l*' and not that of '*Essential convergence*' which question, as will be seen, is of more frequent occurrence in the theoretical parts of the subject. To this purpose of examining the question of essential convergence are directed two tests which are developed in the following two sub-sections.

30·1. Cauchys' general principle of convergence. *The necessary and sufficient condition for the convergence of a sequence* $\{a_n\}$ *is that to every positive number* ϵ, *however small, there corresponds a positive integer* m *such that*

$$|a_{n+p}-a_n| < \epsilon,$$

when $n \geqslant m$ *and* p *has any positive integral value.*

Observation. The theorem may also be stated as follows :—

The necessary and sufficient condition for the convergence of $\{a_n\}$ is that to every positive number ϵ, however small, there corresponds a member a_m of the sequence such that the absolute value of the difference between any two members (not necessary consecutive) which come after a_m in the succession

$$a_1, a_2, \ldots a_m, a_{m+1} \ldots\ldots,$$

is less than ϵ]

The condition is necessary. Let the sequence converge and let its limit be l. If ϵ be any positive number, then there exists a positive integer m such that

$$|a_n-l| < \tfrac{1}{2}\epsilon, \text{ when } n \geqslant m;$$

from this we deduce that

$$|a_{n+p}-l| < \tfrac{1}{2}\epsilon, \text{ when } n \geqslant m \text{ and } p \geqslant 0.$$

$$\begin{aligned}\therefore \quad |a_{n+p}-a_n| &= |a_{n+p}-l+l-a_n| \\ &\leqslant |a_{n+p}-l| + |l-a_n| \\ &< \tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon=\epsilon\end{aligned}$$

i.e., $$|a_{n+p}-a_n| < \epsilon, \text{ when } n \geqslant m \text{ and } p \geqslant 0.$$

The condition is sufficient. It will firstly be shown that under this condition, the sequence is bounded.

We give any particular value to ϵ, say, 1. There exists, therefore, a positive integer r such that

$$|a_{n+p}-a_n| < \epsilon, \text{ when } n \geqslant r \text{ and } p \geqslant 0,$$

From this, taking $n=r$, see that

$$|a_{r+p}-a_r| < \epsilon, \text{ when } p \geqslant 0$$

i.e., $$a_r-\epsilon < a_{r+p} < a_r+\epsilon, \text{ when } p \geqslant 0.$$

This means that all the members of the sequence $\{a_n\}$, except, perhaps, the finite set of numbers

$$a_1, a_2, a_3, \ldots a_{r-1},$$

lie between two fixed numbers $a_r-\epsilon$ and $a_r+\epsilon$.

If k be the least and K the greatest of the finite set of numbers

$$a_1, a_2, a_3, \ldots\ldots a_{r-1}, a_r-\epsilon, a_r+\epsilon,$$

we see that

$$k \leqslant a_n \leqslant K, \text{ for every value of } n,$$

and accordingly $\{a_n\}$ is bounded.

The sequence $\{a_n\}$ has, therefore, at least one limiting point, say, l. If possible, let there be another limiting point l'.

Let ϵ be any positive number, however small. There exists a positive integer m such that

$$|a_{n+p}-a_n| < \tfrac{1}{3}\epsilon \text{ when } n \geqslant m \text{ and } p \geqslant 0. \qquad (1)$$

Also, since l, l' are the limiting points, there exist positive integers m_1, m_2 which are both $\geqslant m$ such that

$$|a_{m_1}-l| < \tfrac{1}{3}\epsilon, \; |a_{m_2}-l'| < \tfrac{1}{3}\epsilon. \qquad (2)$$

From (1), we have, in particular

$$| a_{m_2} - a_{m_1} | < \tfrac{1}{3}\epsilon. \qquad (3)$$

From (2) and (3), we obtain

$$| l' - l | = | l' - a_{m_2} + a_{m_2} - a_{m_1} + a_{m_1} - l |$$

$$\leqslant | l' - a_{m_2} | + | a_{m_2} - a_{m_1} | + | a_{m_1} - l |$$

$$< \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon = \epsilon$$

i.e., $\quad | l' - l | < \epsilon.$

Thus a non-negative number $| l' - l |$ is less than every positive number ϵ and accordingly it must be 0 so that $l' = l$.

Thus we prove that $\{a_n\}$ is convergent.

[**Alternatively,** the sufficiency may also be seen in another way as follows :—

Let $l' > l$ and let $l' - l = 3\epsilon$. The numbers $l + \epsilon$ and $' - \epsilon$ lie between l and l' and $l + \epsilon < l' - \epsilon$.

There exists a member a_m such that every two members of the sequence which appear after a_m differ from each other by a number which is less than ϵ. Since l, l' are the limiting points, there exist members which appear after a_m and lie in the intervals $(l - \epsilon, l + \epsilon)$ and $(l' - \epsilon, l' + \epsilon)$ and such members, obviously, differ from each other by a number greater than ϵ. This is a contradiction].

30.2. Monotonic Sequences and their convergence.

A sequence $\{a_n\}$ is said to be monotonically increasing if $\quad a_{n+1} \geqslant a_n$, for every value of n ;
it is said to be monotonically decreasing
if $\quad a_{n+1} \leqslant a_n$, for every value of n.

A sequence which is monotonically increasing or decreasing is known as a monotonic sequence.

E, G. The sequence $\{ a_n \}$, where

(*i*) $a_n = 1/n$, is monotonically decreasing. (*i*) $a_n = -1/n$, is monotonically decreasing. (*iii*) $a_n = (-1)^n/n$, is not monotonic.

30·21. Theorem. *The necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.*

The condition is necessary. This is obvious as a convergent sequence is necessarily bounded.

The condition is sufficient. Let a monotonic sequence $\{a_n\}$ be bounded.

Firstly let it be monotonically increasing and let B be its upper bound. It will be shown that B is the limit.

Let ϵ be any positive number, however small.

There exists a member say, a_m, of the sequence such that

$$B - \epsilon < a_m. (\S 20{\cdot}1) \qquad (1)$$

Also, since a_n is monotonically increasing,

$$a_m \leqslant a_n, \text{ when } n \geqslant m. \qquad (2)$$

From (1) and (2),

$$B-\epsilon < a_n, \text{ when } n \geqslant m. \tag{3}$$

Also, since $a_n \leqslant B$, therefore $a_n < B+\epsilon$, for every value of n, and in particular

$$a_n < B+\epsilon, \text{ when } n \geqslant m. \tag{4}$$

From (3) and (4),

$$B-\epsilon < a_n < B+\epsilon, \text{ when } n \geqslant m,$$

so that the sequence $\{a_n\}$ converges, its limit being the upper bound B.

It may similarly be shown that a bounded monotonically decreasing sequence is also convergent ; the limit, in this case, being the lower bound.

Cor. If K be a rough upper bound of a monotonically increasing sequence $\{a_n\}$, then

$$\text{lt } a_n \leqslant K\,;$$

if k be a rough lower bound of a monotonically decreasing sequence, $\{b_n\}$, then

$$\text{lt } b_n \geqslant k.$$

Ex. *Show that a monotonic sequence which is not bounded diverges to* ∞ *or* $-\infty$ *according as it is increasing or decreasing.*

Let $\{a_n\}$ be a monotonically increasing sequence which is not bounded.

Let $\triangle$ be any positive number, however large.

Since $\{a_n\}$ is not bounded, there exists a member a_m of the sequence such that

$$a_m > \triangle. \tag{1}$$

As $\{a_n\}$ is monotonically increasing,

$$a_n \geqslant a_m, \text{ when } n \geqslant m. \tag{2}$$

From (1) and (2), we deduce that

$$a_n > \triangle, \text{ when } n \geqslant m,$$

so that $\{a_n\}$ diverges to ∞.

The second part may be similarly proved.

Ex. $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences ; deduce from* §30.1 *that* $\{a_n+b_n\}$ *is also convergent.*

Let ϵ be any positive number.

Since $\{a_n\}$ and $\{b_n\}$ are convergent, there exist positive integers m_1, m_2 such that

$$|a_{n+p}-a_n| < \tfrac{1}{2}\epsilon, \text{ when } n \geqslant m_1 \text{ and } p \geqslant 0; \tag{i}$$

$$|b_{n+p}-b_n| < \tfrac{1}{2}\epsilon, \text{ when } n \geqslant m_2 \text{ and } p \geqslant 0. \tag{ii}$$

Let $m = Max(m_1, m_2)$. From (*i*) and (*ii*), we deduce that for every $n \geqslant m$ and $p \geqslant 0$,

$$|\overline{a_{n+p}+b_{n+p}}-\overline{a_n+b_n}| \leqslant |a_{n+p}-a_n| + |b_{n+p}-b_n| < \tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon=\epsilon.$$

Hence $\{a_n+b_n\}$ is convergent.

Ex. If $\{a_n\}$ and $\{b_n\}$ are convergent, then prove that $\{a_n.b_n\}$ is also convergent.

Ex. *Show, with the help of* §30·1, *that the sequence* $\{a_n\}$, *where*

$$a_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$$

not convergent.

Suppose that $\{a_n\}$ is convergent. Taking $\epsilon=\frac{1}{4}$, we see that there exists a positive integer m such that when $n \geqslant m$ and $p \geqslant 0$,

$$\left|a_{n+p}-a_n\right|=\left|\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+p}\right|<\frac{1}{4}.$$

In particular, taking $n=m$, we see that for every value of p, we must have

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{m+p}<\frac{1}{4}.$$

Taking $p=m$ we see that

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{m+p}>\frac{m}{m+m}=\frac{1}{2},$$

so that we arrive at a contradiction.

Ex. *Show that the sequence* $\{a_n\}$, *where*

$$a_n=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+n},$$

is convergent.

We have, $a_{n+1}=\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n+2}$,

$$a_{n+1}-a_n=\frac{1}{2n+1}+\frac{1}{2n+2}-\frac{1}{n+1}=\frac{1}{(2n+1)(2n+2)}>0,$$

so that $\{a_n\}$ is monotonically increasing.

We have $a_n=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+n}$

$$<\frac{1}{n+1}+\frac{1}{n+1}+\ldots+\frac{1}{n+1}=\frac{n}{n+1}=1-\frac{1}{n+1}<1.$$

Thus the monotonically increasing sequence $\{a_n\}$ is bounded and accordingly it is convergent.

Ex. *Show that the sequence* $\{a_n\}$, *where*

$$a_n=1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots+\frac{1}{\underline{|n}},$$

is convergent and that

$$2<\text{lt } a_n \leqslant 3.$$

Clearly $\{a_n\}$ is monotonically increasing.

Also $a_n<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}=1+2[1-(\frac{1}{2})^n]$

$$=3-(\tfrac{1}{2})^{n-1}<3, \text{ for all } n,$$

so that a_n is bounded above.

Hence the result. The second part is obvious.

Ex. *Show that the sequence x^n is convergent if and only if* $-1<x \leqslant 1$.

(*i*) *Let* $x>1$. We write $x=1+h$ so that h is positive.

By mathematical induction it may easily be shown that

$$x^n=(1+h)^n>1+nh.$$

Let Δ be any positive number, however large. We have

$$1+nh>\Delta, \text{ if } n>(\Delta-1)/h.$$

Taking m as any positive integer $>(\Delta-1)/h$, we see that

$$x^n>\Delta \text{ for } n \geqslant m, \text{ so that lt } x^n=\infty.$$

(*ii*) *Let* $x=1$. Clearly, in this case, lt $x^n=1$.

(*iii*) *Let* $0<x<1$. We write $x=1/(1+h)$ so that h is positive.

We have $0 < x^n = 1/(1+h)^n < 1/(1+nh)$.

Let ϵ be any positive number, however small. We have

$$1/(1+nh) < \epsilon, \text{ if } n > (1/\epsilon - 1)/h.$$

Taking m as any integer $> (1/\epsilon - 1)/h$, we see that

$$0 < x^n < \epsilon \text{ or } |x^n| < \epsilon, \text{ for } n \geqslant m,$$

so that $$\text{lt } x^n = 0.$$

(*iv*) *Let* $x=0$. Clearly, lt $x^n = 0$.

(*v*) *Let* $-1 < x < 0$. We write $x = -\alpha$ so that $0 < \alpha < 1$.

We have $$|x^n| = \alpha^n.$$

It now follows from (*iii*) that lt $x^n = 0$.

(*vi*) *Let* $x = -1$. Obviously x^n oscillates finitely.

(*vii*) *Let* $x < -1$. We write $x = -\alpha$ so that $\alpha > 1$.

Now $n \to \infty$ and, therefore, x^n takes values, both positive and negative greater than any assigned number. Hence x^n oscillates infinitely.

Ex. Show that x^n monotonically decreases if $0 < x < 1$ and monotonically increases if $x > 1$, and hence deduce the result of the previous example.

31. Some fundamental theorems on limits. *If* $\{a_n\}$, $\{b_n\}$ *be two sequences such that, when* $n \to \infty$,

$$\text{lt } a_n = A, \text{ lt } b_n = B,$$

then

$$(1)\ \ lt\{a_n + b_n\} = A + B\,;$$

$$(2)\ \ lt\{a_n - b_n\} = A - B\,;$$

$$(3)\ \ lt\{a_n . b_n\} = AB\,;$$

$$(4)\ \ lt\{a_n / b_n\} = A/B, \text{ if } B \neq 0.$$

(1), (2). Let ϵ be any positive number.

There exist positive integers m_1, m_2 such that

$$|a_n - \mathrm{A}| < \tfrac{1}{2}\epsilon, \text{ when } n \geqslant m_1,$$
$$|b_n - \mathrm{B}| < \tfrac{1}{2}\epsilon, \text{ when } n \geqslant m_2.$$

Let $m = \text{Max } (m_1, m_2)$. Then we see that for every $n \geqslant m$,

$$|a_n - \mathrm{A}| < \tfrac{1}{2}\epsilon,\ |b_n - \mathrm{B}| < \tfrac{1}{2}\epsilon.$$

Thus for every $n \geqslant m$,

$$|\overline{a_n + b_n} - \overline{\mathrm{A}+\mathrm{B}}| \leqslant |a_n - \mathrm{A}| + |b_n - \mathrm{B}| \leqslant \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon,$$

and $$|\overline{a_n - b_n} - \overline{\mathrm{A}-\mathrm{B}}| \leqslant |a_n - \mathrm{A}| + |\mathrm{B} - b_n| \leqslant \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon,$$

which show that

$$\{a_n + b_n\} \to (\mathrm{A}+\mathrm{B}) \text{ and } \{a_n - b_n\} \to (\mathrm{A}-\mathrm{B}).$$

(3). We have, for every value of n,

$$|a_n b_n - \mathrm{AB}| = |a_n(b_n - \mathrm{B}) + \mathrm{B}(a_n - \mathrm{A})|$$
$$\leqslant |a_n|\ |b_n - \mathrm{B}| + |\mathrm{B}|\ |a_n - \mathrm{A}|.$$

Since $\{a_n\}$ is convergent, there exists a number K such that

$$|a_n| < \mathrm{K}, \text{ for every value of } n.$$

Thus $|a_n b_n - \mathrm{AB}| \leqslant \mathrm{K}\,|b_n - \mathrm{B}| + \{|\mathrm{B}| + 1\}\,|a_n - \mathrm{A}|$, for every n. (*i*)

Let ϵ be any positive number. There exist positive integers m_1, m_2 such that

$$|b_n - B| < \epsilon/2K, \text{ for } n \geqslant m_1 \qquad (ii)$$

and $$|a_n - A| < \epsilon/2\{|B|+1\}, \text{ for } n \geqslant m_2 \qquad (iii)$$

Let $$m = \text{Max.}\ (m_1, m_2).$$

From (i), (ii) and (iii), we deduce that for every $n \geqslant m$,

$$|a_n b_n - AB| < \epsilon,$$

so that $$\{a_n b_n\} \to AB.$$

Note. If in i). we had not introduced $|B|+1$ in place of $|B|$, then in (iii), we would have to render $|a_n - A| < \epsilon/2|B|$, which will fail if $B=0$ and thus the proof, as given, will hold only if $B \neq 0$. It is to include this case that we had introduced this aitifice, for $|B|+1$ cannever be 0.

(4) We have, for every value of n,

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| = \left|\frac{B(a_n - A) - A(b_n - B)}{Bb_n}\right|$$

$$\leqslant \frac{|B|\ |a_n - A| + |A|\ |b_n - B|}{|B|\ |b_n|}, \qquad (iv)$$

Since $\{b_n\} \to B \neq 0$, there exists a positive integer m_1 such that when $n \geqslant m_1$,

$$|b_n - B| < \tfrac{1}{2}|B|,$$

or $$|B| - |b_n| \leqslant |b_n - B| < \tfrac{1}{2}|B|$$

i.e., $$\tfrac{1}{2}|B| < |b_n|. \qquad (v)$$

From (iv) and (v), we see that for every $n \geqslant m_1$,

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| \leqslant \frac{|B|\ |a_n - A| + |A|\ |b_n - B|}{\tfrac{1}{2}|B|^2}$$

$$= \frac{2}{|B|}|a_n - A + \frac{2|A|}{|B|^2}|b_n - B|$$

$$\leqslant \frac{2}{|B|}|a_n - A| + 2\frac{|A|+1|}{|B|^2}|b_n - B|. \quad (vi)$$

Let ϵ be any positive number, however small. There exist positive integers m_2 and m_3 such that

for every $n \geqslant m_2$, $|a_n - A| < \tfrac{1}{4}|B|\epsilon$, i.e., $\dfrac{2}{|B|}|a_n - A| < \dfrac{\epsilon}{2}$. (vii)

for every $n \geqslant m_3$, $|b_n - B| < \tfrac{1}{4}\dfrac{|B|^2\epsilon}{|A|+1}$,

$$\text{i.e., } \frac{2|A|+1}{|B|^2}|b_n - B| < \frac{\epsilon}{2}. \quad (viii)$$

Let $m = \text{Max.}\ (m_1, m_2, m_3)$.

From (vi), (vii) and $(viii)$, we deduce that for every $n \geqslant m$,

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| < \epsilon,$$

so that $\{a_n/b_n\} \to A/B$ as $n \to \infty$.

Note. As a particular case of the theorems proved above, we note that if the sequences $\{a_n\}$, $\{b_n\}$ be convergent, then the sequences $\{a_n \pm b_n\}$, $\{a_n . b_n\}$ are also convergent; further if lt $b_n \neq 0$, then $\{a_n/b_n\}$ is also convergent.

The converse however may not be true as the following examples show :

(*i*) Taking $a_n=(-1)^{n+1}$ and $b_n=(-1)^n$, we see that $\{a_n+b_n\}$ is convergent but $\{a_n\}$ and $\{b_n\}$ are not.

(*ii*) For $a_n=(-1)^n$, $b_n=(-1)^n$, $\{a_n-b_n\}$, $\{a_n . b_n\}$, $\{a_n/b_n\}$ all converge, but $\{a_n\}$ and $\{b_n\}$ do not.

32. Theorem. $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *are three sequences such that*

(*i*) $a_n \leqslant b_n \leqslant c_n$, *for every value of* n; (*i*)

(*ii*) lt $a_n=$ lt $c_n=l$;

Then lt $b_n=l$.

Let ϵ be any positive number. There exists a positive integer m such that, for every value of $n \geqslant m$,

$$l-\epsilon < a_n < l+\epsilon, \tag{ii}$$

$$l-\epsilon < c_n < l+\epsilon. \tag{iii}$$

From (*i*), (*ii*) and (*iii*), we deduce that

$$l-\epsilon < b_n < l+\epsilon, \text{ for } n \geqslant m.$$

Hence $\{b_n\} \to l$.

EXAMPLES

1. $\{a_n\} \to 0$ and $\{b_n\}$ oscillates finitely ; show that $\{a_n b_n\} \to 0$.

2. $\{a_n\}$ is convergent and $\{b_n\}$ divergent; show that $\{a_n/b_n\} \to 0$.

3. If $\{a_n\}$ is convergent and $\{b_n\}$ divergent, then $\{a_n+b_n\}$ is divergent.

4. If $\{a_n\} \to a$, then $\{|a_n|\} \to |a|$.

(It follows from the inequality $||a_n|-|a|| \leqslant |a_n-a|$).

5. If $\{a_n\} \to \infty$ and $b_n \geqslant a_n$ for all n, then $b_n \to \infty$.

6. lt $a_n=a$ and b is a number such that $a^n < b$, for all n, show that $a \leqslant b$.

7. lt $a_n=a$ and c is a number such that $c \leq a^n$, for all n ; show that $c \leqslant a$.

33. An important limit. The number e. *To show that the sequence* $(1+1/n)^n$ *is convergent.*

We have, by the binomial theorem,

$$a_n=(1+1/n)^n$$

$$=1+n.\frac{1}{n}+\frac{n(n-1)}{\lfloor 2}.\frac{1}{n^2}+\ldots\ldots+\frac{n(n-1)(n-2)\ldots\ldots1}{\lfloor n}.\frac{1}{n^n}$$

$$=1+1+\frac{1}{\lfloor 2}\left(1-\frac{1}{n}\right)+\ldots\ldots$$

$$+\frac{1}{\lfloor n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{n-1}{n}\right) \tag{1}$$

From this we easily deduce that $\{a_n\}$ is a monotonically increasing sequence. From (1),

$$a_n \leqslant 1+1+\frac{1}{\underline{|2}}+\ldots+\frac{1}{\underline{|n}}=b_n, \text{ say.} \tag{2}$$

As shown in an Ex. on P. 42,

$$b_n \leqslant 3, \text{ for all } n.$$

Thus $\{a_n\}$ is convergent.

The limit of this convergent sequence $(1+1/n)^n$ *is denoted by* e.

[It is interesting to note that lt $b_n=$ lt a_n. From (2),

$$e = \text{lt } a_n \leqslant \text{lt } b_n = b, \text{ say.} \tag{3}$$

Again, if m is any integer $>n$, we deduce from (1),

$$a_m > 1+1+\frac{1}{\underline{|2}}\left(1-\frac{1}{m}\right)\ldots+\frac{1}{\underline{|n}}\left(1-\frac{1}{m}\right)\ldots\left(1-\frac{n-1}{m}\right). \tag{4}$$

Keeping n fixed and letting $m\to\infty$, we obtain from (4),

$$e \geqslant b_n.$$

Therefore $\qquad b = \text{lt } b_n \leqslant e. \qquad (5)$

From (3) and (5), $\qquad e = b.$

34. Infinite series. Its convergence and sum. If $\{a_n\}$ be any given sequence, then a symbol of the form

$$\sum_{n=1}^{\infty} a_n,$$

i.e., $\qquad a_1+a_2+a_3+\ldots\ldots+a_n+\ldots\ldots$

is called an *infinite series*.

This infinite series is said to be convergent, if the sequence $\{S_n\}$, *where* S_n *denotes the sum*

$$a_1+a_2+a_3+\ldots\ldots+a_n,$$

is convergent, and lt S_n, *in case it exists, is said to be the sum of the series.*

The series is said to be divergent (or oscillatory) if the sequence $\{S_n\}$ is divergent (or oscillatory). The question of the sum of such a series does not arise.

Note. If we add the first two terms of an infinite series, and then add the sum so obtained to the third, and thus go on adding each term to the sum of the previous terms, we see that, as there is no last term of the series, the process will never be completed. In the case of a finite series, this process of addition will be completed at some stage, however large a number of terms the series may consist of. Thus, in the ordinary sense, the expression "Sum of an infinite series" has no meaning. The notion of limit has, therefore, been employed to give a meaning to this expression.

EXAMPLES

1. Show that the infinite geometrical series

$$\sum_{n=0}^{\infty} x^n,$$

is convergent, if and only if $|x| < 1$.

2. Show that the series Σa_n, where $a_n=(-1)^n$ is not convergent.

3. Show that the following series are convergent; also find their sum :—

(i) $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots+\frac{1}{n(n+1)}+\ldots\ldots$

(ii) $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots\ldots+\frac{1}{n(n+1)(n+2)}+\ldots\ldots$

(iii) $1+2x+3x^2+\ldots\ldots+nx^{n-1}+\ldots\ldots, \; |x|<1$.

4. Show that

$$\sum_{n=0}^{\infty}(-1)^n\frac{2n+3}{(n+1)(n+2)}=1$$

5. Show that

$$\sum_{1}^{\infty}\frac{n^2+9n+5}{(n+1)(2n+3)(2n+5)(n+4)}=\frac{5}{36}. \qquad (M.T.)$$

6. Show that the arithmetic series

$$a+(a+d)+(a+2d)+\ldots\ldots+(a+nd)+\ldots\ldots$$

is always divergent, except when a, b are both zero.

7. A series Σa_n is convergent and k is a constant; show that the series Σka_n is also convergent.

8. A series Σa_n is given; a sequence $\{b_n\}$ is defined such that

$$b_n=a_{m+n}\text{; } m \text{ being a given positive integer;}$$

show that the series Σa_n and Σb_n have the same behaviour in relation to convergence or otherwise.

9. Σa_n, Σb_n are two convergent series, S_1, S_2 being their sums; show that the series $\Sigma(a_n+b_n)$ is also convergent and its sum is equal to S_1+S_2.

10. Show that a sequence $\{a_n\}$ is convergent if and only if the series $\Sigma(a_{n+1}-a_n)$ is convergent.

35. Convergence of an infinite series.

35.1. Cauchy's general principle of convergence of a series. *The necessary and sufficient condition for the convergence of an infinite series Σa_n is that to every positive number ϵ, however small, there corresponds a positive integer m such that*

$$|a_{n+1}+a_{n+2}+\ldots\ldots+a_{n+p}|<\epsilon,$$

for every $n\geqslant m$ and every $p\geqslant 0$.

We write $\qquad S_n=a_1+a_2+\ldots\ldots+a_n$.

From §30.1, the necessary and sufficient condition for the convergence of $\{S_n\}$, *i.e.*, of Σa_n is that to every positive number ϵ, however small, there corresponds a positive integer m such that, for every $n\geqslant m$ and every $p\geqslant 0$,

$$|S_{n+p}-S_n|<\epsilon,$$

i.e., $$|a_{n+1}+a_{n+2}+\ldots\ldots+a_{n+p}|<\epsilon.$$

35.2. Convergence of a positive term series. *The necessary and sufficient condition for the convergence of a series Σa_n, whose terms a_n are all $\geqslant 0$, is that there exists a positive constant K such that*

$$S_n=a_1+a_2+\ldots\ldots+a_n<K,$$

for every n, i.e., S_n is bounded above.

The result follows from the fact that if every $a_n \geqslant 0$, then the sequence $\{S_n\}$ is monotonically increasing and will, therefore, be convergent if and only if it is bounded above. (§30·2).

Ex. *Show that the series*

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\ldots+\frac{(-1)^{n+1}}{n}+\ldots\ldots,$$

is convergent.

This is not a positive term series.

It is easy to see that

$$\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\frac{1}{n+4}+\ldots\ldots+\frac{(-1)^{p+1}}{n+p}$$

is positive and less than $1/(n+1)$.

Let ϵ be any positive number. We have

$$|S_{n+p}-S_n| = \left|\frac{1}{n+1}-\frac{1}{n+2}+\ldots\ldots+\frac{(-1)^{p+1}}{n+p}\right|$$

$$=\frac{1}{n+1}-\frac{1}{n+2}+\ldots\ldots+\frac{(-1)^{p+1}}{n+p}<\frac{1}{n+1}<\epsilon,$$

if $\qquad n>(1/\epsilon-1)$.

Let m be any integer greater than $(1/\epsilon-1)$. Then we have

$$|S_{n+p}-S_n| < \epsilon, \text{ for } n \geqslant m \text{ and } p \geqslant 0.$$

Hence the series converges.

Ex. *Show that the series*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots+\frac{1}{n}+\ldots\ldots$$

does not converge.

This is a positive term series.

Suppose that it converges. There exists a positive integer m such that for every $n \geqslant m$, and every $p \geqslant 0$,

$$\left|\frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots+\frac{1}{n+p}\right|<\frac{1}{4}. \quad (\text{Taking } \epsilon=1/4)$$

In particular taking $n=m$ and $p=m$, we see that

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots\ldots\frac{1}{2m}>\frac{m}{2m}=\frac{1}{2}.$$

Thus we have a contradiction. Hence the series does not converge.

Appendix

36. The meaning of a^x, when $a>0$, and x is any rational number. When x is a positive integer, the symbol a^x denotes the product

$$a.a.a.a\ldots\ldots a, \ (x \text{ times})$$

and when x is a negative integer so that $-x$ is a positive integer, we have

$$a^x = 1/a^{-x}.$$

Thus the concepts of multiplication and division (§§11, 12) are all that we require in order to define a^x, when x is any integer.

The theorem below is fundamental for giving a meaning to the symbol a^x when x is any rational number, and a is positive.

36·1. Theorem. *If m is a positive integer, and a, any given positive number, then the equation*

$$x^m = a, \qquad (i)$$

in x, has one and only one positive root.

Divide all the real numbers into two classes L and R, putting (*i*) all the negative numbers, (*ii*) zero, (*iii*) all the positive numbers x such that $x^m \leqslant a$, in L and all the others in R.

Clearly every number has a class The class R exists, for any number k which is greater than a as well as 1 belongs to R. Also if y be any positive number less than a member x of L, we have $y^m < x^m \leqslant a$ and accordingly y belongs to L. Thus the classes L, R determine a section of real numbers. Let B be the number which divides the two classes.

Clearly B cannot be negative. If possible, let B=0 so that $1/n$ which is $>$B=0, belongs to R, and accordingly

$$(1/n)^m > a, \qquad (ii)$$

n being any positive integer.

If $n \to \infty$, we obtain, from (1), $0 \geqslant a$, so that we have a contradiction.

Thus B is necessarily positive.

It will be shown that

$$B^m = a.$$

For every positive integral value of n, $B - 1/n$ belongs to L and $B + 1/n$ to R so that we have

$$(B - 1/n)^m \leqslant a < (B + 1/n)^m. \qquad (iii)$$

Let $n \to \infty$. We obtain, from (*iii*),

$$B^m \leqslant a \leqslant B^m$$

so that $$B^m = a.$$

Thus B is the positive root of $x^m = a$. If possible, let B′ be another positive root. From

$$B' \gtrless B,$$

we deduce $$B'^m \gtrless B^m,$$

and so B^m, B'^m cannot both be equal to the same number.

Def. *The unique positive root of the equation*

(*i*) $\qquad x^m = a, \qquad (a > 0,\ m,$ any positive integer)

is symbollically written as

$$a^{1/m} \text{ or } \sqrt[m]{a},$$

and is called the mth root of a.

Note. If $a > 0$, and m is even, the equation (*i*) possesses a negative root $-$B also ; but if m is odd, it cannot, obviously, have any negative root.

If $a < 0$, and m is even, the equation cannot, obviously, have any root, positive or negative, but if m is odd, it has no positive root but has a negative root $-$B, where B is the positive root of $x^m = -a$.

To avoid this ambiguity and indefiniteness, we will always take the base a positive, and the symbol $a^{1/m}$ will, then, always denote the *positive* root of (*i*).

Def. *If x is a rational number n/m, where m is positive, then by def.,*

$$a^{n/m} = (\sqrt[m]{a})^n ;$$

a being positive.

By def., $a^0=1$.

(The following examples are to be considered as a part of the text).

Ex. 1. Prove that $$(\sqrt[m]{a})^n=\sqrt[m]{(a^n)}.$$

Ex. 2. x, y are any rational numbers and a is positive; show that
$$(i)\ a^x.a^y=a^{x+y}.\quad (ii)\ (a^x)^y=a^{xy}.$$

(Reduce x, y to a common denominator).

Ex. 3. a, b are two positive real numbers and x is a rational number; show that
$$(ab)^x=a^x.b^x.$$

Ex. 4. x, y are two rational numbers such that $x>y$; show that
$$a^x \gtrless a^y, \text{ according as } a\gtrless 1.$$

Ex. 5. x is any positive rational number; show that

$$a^x \begin{matrix} > \\ = \\ < \end{matrix} 1, \text{ according as } a \begin{matrix} > \\ = \\ < \end{matrix} 1.$$

Ex. 6. a, b are positive numbers, and x is any positive rational number; show that
$$a^x \gtrless b^x \text{ according as } a\gtrless b.$$

Ex. 7. *Show that*
$$lt \sqrt[n]{a}=1, \textit{ when } n\to\infty. \quad (a>0)$$

For $a=1$, the result is obvious.

Let $a>1$. We write
$$\sqrt[n]{a}=1+h_n,$$
so that $h_n>0$. We know that
$$(1+h_n)^n>1+nh_n.$$
$$\therefore \qquad a=(1+h_n)^n>1+nh_n,$$
$$\text{or} \qquad 0<h_n<(a-1)/n.$$

Let ϵ be any positive number. There exists a positive integer m such that $(a-1)/n<\epsilon$, for $n\geqslant m$. Thus, we have
$$-\epsilon<0<h_n<(a-1)/n<\epsilon, \text{ for } n\geqslant m$$
$$\text{or} \qquad |\sqrt[n]{a}-1| = |h_n| <\epsilon, \text{ for } n\geqslant m.$$

Hence the result.

If $a<1$, we write $a=1/b$ so that $b>1$.

$\therefore$ $\sqrt[n]{a}=1/\sqrt[n]{b}$ and the result now follows.

Ex. 8. Show that $lt\ a^{-1/n}=1$.

Ex. 9. $\{x_n\}$ is any sequence of rational numbers such that $lt\ x_n=0$, when $n\to\infty$; show that
$$a^{x_n}\to 1, \text{ when } n\to\infty. \quad (a>0)$$

There exists a positive integer m such that for $n\geqslant m$, $a^{1/n}$ and $a^{-1/n}$ both lie between $1-\epsilon$ and $1+\epsilon$ and, in particular, $a^{1/m}$ and $a^{-1/m}$ lie between $1-\epsilon$ and $1+\epsilon$.

Since $lt\ x_n=0$, there exists a positive integer m_1 such that for $n \geqslant m_1$,

$$-1/m < x_n < 1/m.$$

$$\therefore \qquad a^{x_n} \text{ lies between } a^{1/m} \text{ and } a^{-1/m} \text{ for } n \geqslant m_1.$$

Thus we see that there exists a positive integer m_1 such that for $n \geqslant m_1$, a^{x_n} lies between $1-\epsilon$ and $1+\epsilon$.

Hence the result.

36.2. Powers with arbitrary real indices. *To define a^x, when, a, is any positive real number and, x, any real number.*

Let $\{x_n\}$ be *any monotonically increasing sequence of rational numbers such that

$$\text{lt } x_n = x.$$

If $a>1$, the sequence a^{x_n} is monotonically increasing and bounded above in as much as $a^{x_n} < a^k$, where k is any rational number greater than x. Thus the sequence $\{a^{x_n}\}$ is convergent.

If $a<1$, the sequence a^{x_n} is monotonically decreasing and bounded below in as much as $a^{x_n}>0$. Thus $\{a^{x_n}\}$ is convergent.

Let, now, $\{x'_n\}$ be *any* convergent sequence such that

$$\text{lt } x'_n = x.$$

The sequence $\{x'_n - x_n\} \to 0$ and, therefore, the sequence

$$a^{\{x'_n - x_n\}} \to 1. \qquad \text{(Ex. 9, Page 50).}$$

Now, $$a^{x'_n} = a^{x'_n - x_n} . a^{x_n}.$$

We have

$$\text{lt } a^{x'_n} = \text{lt } a^{x'_n - x_n} . \text{lt } a^{x_n} = 1 . \text{lt } a^{x_n} = \text{lt } a^{x_n}.$$

Thus we see that $\{a^{x'_n}\}$ is convergent and its limit is the same as that of the convergent sequence $\{a^{x_n}\}$.

This discussion justifies the following definition of a^x:—

If $a>0$, and x is any real number, then a^x is defined as the limit of $\{a^{x_n}\}$, where $\{x_n\}$ is any sequence of rational numbers with x as its limit.

Of course, it has already been shown that $\{a^{x_n}\}$ is convergent and its limit is the same whatever be the sequence $\{x_n\}$, provided $\text{lt } x_n = x$.

---

* One such sequence arises if we take x_n as any rational number such that

$$x - \frac{1}{n} < x_n < x - \frac{1}{n+1}.$$

Note. The laws of indices, *viz.*,

$$a^x.a^y=a^{x+y},\ (ab)^x=a^x.b^x,\ (a^x)^y=a^{xy},$$

may easily be shown to remain valid when x, y are any real numbers and $a>0$.

For example, let $\{x_n\}$, $\{y_n\}$ be any two sequences such that

$$\text{lt } x_n=x,\ \text{lt } y_n=y,$$

so that $$\text{lt } (x_n+y_n)=x+y,\ \text{lt } (x_ny_n)=xy.$$

We have $$a^{x_n}.a^{y_n}=a^{x_n+y_n}.$$

Taking limits, when $n\to\infty$, we obtain

$$a^x.a^y=a^{x+y}.$$

37. Theorem. *If x be any real number and, r, s, two rational numbers such that*

$$r<x<s,$$

then $$a^r \lesseqgtr a^x \lesseqgtr a^s,\ \text{if } a \gtrless 1.$$

Let $a>1$. Let $\{x_n\}$ be any sequence of rational numbers such that $\{x_n\}\to x$. Consider any pair r', s' of rational numbers such that

$$r<r'<x<s'<s.$$

As (r', s') is an interval which encloses x, there exists a positive integer m such that

$$r'<x_n<s',\ \text{when } n\geqslant m.$$

$\therefore$ $$a^{r'}<a^{x_n}<a^{s'} \text{ when } n\geqslant m. \quad \text{(Ex. 4, Page 50).}$$

Let $n\to\infty$, so that we obtain

$$a^{r'}\leqslant a^x\leqslant a^{s'}.$$

But we know that

$$a^r<a^{r'} \text{ and } a^{s'}<a^s$$

$\therefore$ $$a^r<a^x<a^s.$$

The case, when $a<1$, may be similarly discussed.

37.1. Cor. If x, y are two real numbers such that

$$x<y,$$

then $$a^x\lesseqgtr a^y,\ \text{if } a\gtrless 1.$$

Let r be any rational number such that

$$x<r<y.$$

Then $$a^x\lesseqgtr a^r\lesseqgtr a^y,\ \text{according as } a\gtrless 1,$$

or $$a^x\lesseqgtr a^y,\ \text{according as } a\gtrless 1.$$

38. Theorem. *If $\{a_n\}$ is any convergent sequence of* **real** *numbers such that*

$$lt\ \{a_n\}=a,$$

then $$lt\ a^{a_n}=a^{a}.\quad (a>0)$$

To each number a_n, we can associate a pair of rational numbers r_n and r_n+1/n such that

$$r_n<a_n<r_n+1/n, \qquad (i)$$

or $$0<a_n-r_n<1/n.$$

$\therefore$ $$\text{lt } (a_n-r_n)=0,$$

or $$\text{lt } r_n=\text{lt } a_n=a.$$

Also, $$\text{lt } (r_n+1/n)=a.$$

From (i), $\quad a^{r_n} \lessgtr a^{a_n} \lessgtr a^{r_n+1/n}$, according as $a \gtrless 1$.

Taking limits, we obtain

$$\text{lt } a^{a_n}=a^a.$$

39. Logarithms.

Theorem. *If a, b are any two real and positive numbers and $a \neq 1$, then there exists one and only one real number x such that $a^x=b$.*

Let $a>1$. We divide all the real numbers into two classes L and R putting any number x in L if $a^x \leqslant b$ and otherwise in R.

Clearly each number has a class; also each class has a number, for a negative integer $-k$ such that $a^{-k}<b$ belongs to L; and a positive integer m such that $a^m>b$ belongs to R. (The existence of k and m follow from the fact that $a^{-n} \to 0$ and $a^n \to \infty$, as $n \to \infty$). Also from § 37·1, it follows that each member of L is less than each member of R. Thus L, R determine a section of real numbers.

Let ξ be the number which separates the two classes. It will be shown that $a^{\xi}=b$.

Now $\xi-1/n$ belongs to L and $\xi+1/n$ to R, n being any positive integer We have

$$a^{\xi-1/n} \leqslant b < a^{\xi+1/n}.$$

Let $n \to \infty$, so that we obtain

$$a^{\xi} \leqslant b \leqslant a^{\xi}, \text{ i,e., } a^{\xi}=b.$$

Thus ξ satisfies the equation $a^x=b$.

If possible, let η be another root. We have

$$a^{\eta} \gtrless a^{\xi} \text{ according as } \eta \gtrless \xi,$$

and, accordingly, we cannot have $a^{\eta}=b=a^{\xi}$.

If $a<1$, we take, $\alpha=1/a$ so that $\alpha>1$. The number ξ is then obtained from $\alpha^x=1/b$.

Hence the theorem.

Def. *If a and b are any real positive numbers, then the number x, which is uniquely determined by $a^x=b$, is called the logarithm of b to the base a, and written as $\log_a b$.*

Ex. a, x, y are any real positive numbers; $x>y$; show that

$$\log_a x \gtrless \log_a y,$$

according as $a \gtrless 1$.

Ex. a, x, y are any real positive numbers; show that

(*i*). $\log_a(xy)=\log_a x+\log_a y$.

(*ii*). $\log_a(x/y)=\log_a x-\log_a y$.

(*iii*). $\log_a(x^y)=y\log_a x$.

Solved Examples

1. *Show that* $\text{lt}\ \sqrt[n]{n}=1$.

We write $$a_n=\sqrt[n]{n}.$$

Let $a_n=1+h_n$, where $h_n>0$.

We have
$$n=a_n^n=(1+h_n)^n$$
$$=1+nh_n+\tfrac{1}{2}n(n-1)h_n^2+\ldots\ldots+h_n^n$$
$$>\tfrac{1}{2}n(n-1)h_n^2.$$

$\therefore$ $$h_n^2<2/(n-1),$$

or $$0<h_n<\sqrt{2/(n-1)}.$$

The result now follows. To be rigorous, let ϵ be any positive number. Now

$$\sqrt{2/(n-1)}<\epsilon, \text{ if } n>1+2/\epsilon^2.$$

If m be any integer $>1+2/\epsilon^2$, we see that, for $n\geqslant m$,

$$-\epsilon<0<h_n<\sqrt{2/(n-1)}<\epsilon.$$

Hence $h_n\to 0$.

2. *$\{a_n\}$ is a sequence such that*

$$\text{lt}\ \{a_{n+1}/a_n\}=l,$$

where $|\,l\,|<1$; *show that* $\text{lt}\ a_n=0$.

Since $|\,l\,|<1$, we can choose a positive number ϵ so small that $|\,l\,|+\epsilon<1$.

There exists a positive integer m such that for $n\geqslant m$,

$$\left|\frac{a_{n+1}}{a_n}-l\right|<\epsilon,$$

or $$\left|\frac{a_{n+1}}{a_n}\right|-|\,l\,|\leqslant\left|\frac{a_{n+1}}{a_n}-l\right|<\epsilon,$$

or $$\left|\frac{a_{n+1}}{a_n}\right|<|\,l\,|+\epsilon=k, \text{ say, where } k<1.$$

Changing n to m, $m+1$, $m+2$,..., $(n-1)$ and multiplying, we get

$$\left|\frac{a_n}{a_m}\right|<k^{n-m} \text{ or } |\,a_n\,|<k^n.\,|\,a^m\,|\,/k^m.$$

Since $k^n\to 0$, we have the required result.

Note. The general result obtained here enables us to prove the following particular but important results on limits:—

(*i*) $\text{lt}\ (x^n/\underline{|n})=0$, x any number.

(*ii*) $\text{lt}\ (n^r/x^n)=0$, $|\,x\,|>1$.

(*iii*) $\text{lt}\ \dfrac{m(m-1)\ldots(m-n+1)x^n}{\underline{|n}}=0$, $|\,x\,|<1$.

3. *If $\{a_n\}$ is a sequence such that $a_n>0$ and $\text{lt}\ \{a_{n+1}/a_n\}=l>1$, then $\text{lt}\ a_n=\infty$.*

We choose a positive number ϵ such that $l-\epsilon>1$.

There exists a positive integer m such that for $n \geqslant m$

$$l-\epsilon < a_{n+1}/a_n < l+\epsilon,$$

Thus for $n \geqslant m$,

$$a_{n+1}/a_n > l-\epsilon = k, \text{ say.} \quad (k>1.)$$

From this we deduce that

$$a_n > k^n.(a^m/k^m).$$

Since $k^n \to \infty$, we have the required result.

4. *If the sequences $\{a_n\}$, $\{b_n\}$ tend to 0 and if $\{b_n\}$ is a strictly monotonically decreasing sequence so that $b_n > b_{n+1} > 0$, then*

$$lt\frac{a_n}{b_n} = lt\ \frac{a_n - a_{n+1}}{b_n - b_{n+1}},$$

provided that the limit on the right exists, whether finite or infinite.

Case I. Let $lt\ \dfrac{a_n - a_{n+1}}{b_n - b_{n+1}} = l$, where l is finite.

Let ϵ be any positive number. There exists a positive integer such that

$$l-\epsilon < \frac{a_n - a_{n+1}}{b_n - b_{n+1}} < l+\epsilon, \text{ when } n \geqslant m,$$

i.e., $(l-\epsilon)(b_n - b_{n+1}) < (a_n - a_{n+1}) < (l+\epsilon)(b_n - b_{n+1})$, when $n \geqslant m$, for $(b_n - b_{n+1})$ is positive.

Changing n to $n, n+1, n+2, \ldots, (n+p-1)$, in turn and adding we see that

$$(l-\epsilon)(b_n - b_{n+p}) < (a_n - a_{n+p}) < (l+\epsilon)(b_n - b_{n+p}),$$

for every $n \geqslant m$ and every $p \geqslant 0$.

Keep n fixed and let $p \to \infty$. Since $a_{n+p} \to 0$ and $b_{n+p} \to 0$, therefore we obtain

$$(l-\epsilon)b_n \leqslant a_n \leqslant (l+\epsilon)b_n,$$

or
$$l-\epsilon \leqslant (a_n/b_n) \leqslant l+\epsilon, \text{ for every } n \geqslant m.$$

Hence $a_n/b_n \to l$ as $n \to \infty$.

Case II. Let $lt\ \dfrac{a_n - a_{n+1}}{b_n - b_{n+1}} = \infty$.

Let Δ be any positive number. There exists a positive integer m such that

$$\frac{a_n - a_{n+1}}{b_n - b_{n+1}} > \Delta, \text{ when } n \geqslant m,$$

i.e., $(a_n - a_{n+1}) > \Delta(b_n - b_{n+1})$, when $n \geqslant m$, for $(b_n - b_{n+1})$ is positive.

As in Case I, we obtain

$$a_n - a_{n+p} > \Delta(b_n - b_{n+p})$$

or
$$a_n \geqslant \Delta b_n, \ i.e., \ a_n/b_n \geqslant \Delta, \text{ when } n \geqslant m.$$

Hence $a_n/b_n \to \infty$ as $n \to \infty$.

5. *If $\{b_n\}$ is a strictly monotonically increasing sequence so that $b_{n+1} > b_n$ and if $b_n \to \infty$ and $\{a_n\}$ be any sequence, then*

$$lt\ \frac{a_n}{b_n} = lt\ \frac{a_{n+1} - a_n}{b_{n+1} - b_n},$$

provided that the limit on the right exists, whether finite or infinite.

Case I. Let $lt\ \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} = l$, where l is finite.

Let ϵ be any positive number. There exists a positive integer m_1 such that

$$(l-\tfrac{1}{2}\epsilon)(b_{n+1}-b_n)<(a_{n+1}-a_n)<(l+\tfrac{1}{2}\epsilon)(b_{n+1}-b_n), \text{ when } n\geqslant m_1.$$

Changing n to n, $n+1$, $n+2$, .., $n+p-1$, in turn and adding, we see that

$$(l-\tfrac{1}{2}\epsilon)(b_{n+p}-b_n)<(a_{n+p}-a_n)<(l+\tfrac{1}{2}\epsilon)(b_{n+p}-b_n).$$

Dividing by b_{n+p} and adding a_n/b_{n+p}, we obtain

$$(l-\tfrac{1}{2}\epsilon)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}}<\frac{a_{n+p}}{b_{n+p}}<(l+\tfrac{1}{2}\epsilon)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}}$$

for every $n\geqslant m_1$, and $p\geqslant 0$. $\quad (i)$

We keep n fixed and let $p\to\infty$.

Since $$(l-\tfrac{1}{2}\epsilon)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}}\to l-\tfrac{1}{2}\epsilon,$$

and $$(l+\tfrac{1}{2}\epsilon)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}}\to l+\tfrac{1}{2}\epsilon,$$

we see that there exists a positive integer m_2 such that for every $p\geqslant m_2$ we have

$$l-\tfrac{1}{2}\epsilon-\tfrac{1}{2}\epsilon<(l-\tfrac{1}{2}\epsilon)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}}<l-\tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon, \qquad (ii)$$

and $$l+\tfrac{1}{2}\epsilon-\tfrac{1}{2}\epsilon<(l+\tfrac{1}{2}\epsilon)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}}<l+\tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon, \qquad (iii)$$

From i), (ii) and (iii), we obtain

$$l-\epsilon<\frac{a_{n+p}}{b_{n+p}}<l+\epsilon,$$

for every $n\geqslant m_1$ and $p\geqslant m_2$,

i.e., $$l-\epsilon<\frac{a_n}{b_n}<l+\epsilon, \text{ for every } n\geqslant(m_1+m_2).$$

Hence $a_n/b_n\to l$, as $n\to\infty$.

Case II. Let $lt\,\dfrac{a_{n+1}-a_n}{b_{n+1}-b_n}=\infty.$

Let k be any positive number, however large.

There exists a positive integer m_1 such that for every $n\geqslant m_1$,

$$a_{n+1}-a_n>(k+1)(b_{n+1}-b_n).$$

As in Case I, we obtain

$$a_{n+p}-a_n>(k+1)(b_{n+p}-b_n)$$

or $$\frac{a_{n+p}}{b_{n+p}}>(k+1)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}}. \qquad (i)$$

Keeping n fixed and letting $p\to\infty$, we see that

$$(k+1)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}}\to k+1,$$

so that there exists a positive integer m_2 such that for every $p\geqslant m_2$,

$$(k+1)-1<(k+1)\left(1-\frac{a_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}}<(k+1)+1, \qquad (ii)$$

From (i) and (ii), we obtain

$$\frac{a_{n+p}}{b_{n+p}}>k, \text{ when } n\geqslant m_1 \text{ and } p\geqslant m_2,$$

i.e., $$\frac{a_n}{b_n}>k, \text{ when } n\geqslant m_1+m_2$$

Hence $a_n/b_n\to\infty$ as $n\to\infty$.

6. *If* $\qquad lt\ a_n = l$, *when* $n \to \infty$,

then $$\underset{n\to\infty}{lt}\ \frac{a_1+a_2+a_3+\dots+a_n}{n}=l.$$

(This is known as Cauchy's first theorem on limits).

We write $\qquad a_n - l = b_n$

so that the sequence $\{b_n\} \to 0$.

We have $$\frac{a_1+a_2+\dots+a_n}{n}=l+\frac{b_1+b_2+\dots+b_n}{n},$$

so that we have to prove that $(b_1+b_2+\dots+b_n)/n \to 0$, when $b_n \to 0$.

Let ϵ be any positive number. There exists a positive integer μ such that

$$|\ b_n\ | < \tfrac{1}{2}\epsilon, \text{ when } n \geqslant \mu.$$

Also since $\{b_n\}$ is convergent, it is bounded and, therefore, there exists a number k such that

$$|\ b_n\ | < k, \text{ for all } n.$$

We write

$$\left|\frac{b_1+b_2+\dots+b_n}{n}\right| = \left|\frac{b_1+b_2+\dots+b_\mu}{n}+\frac{b_{\mu+1}+b_{\mu+2}+\dots+b_n}{n}\right|$$

$$\leqslant \frac{|\ b_1\ |+|\ b_2\ |+\dots+|\ b_\mu\ |}{n}+\frac{|\ b_{\mu+1}\ |+\dots+|\ b_n\ |}{n}$$

$$\leqslant \frac{k\mu}{n}+\frac{\epsilon(n-\mu)}{2n}<\frac{k\mu}{n}+\frac{\epsilon}{2}.$$

We keep μ fixed and see that

$$\frac{k\mu}{n}<\frac{\epsilon}{2}, \text{ if } n>\frac{2k\mu}{\epsilon}.$$

Let ν be any positive integer greater than $2k\mu/\epsilon$ so that for $n \geqslant \nu$, $k\mu/n < \epsilon/2$.

Let $m = \text{Max}\ (\mu, \nu)$.

Thus, for every $n \geqslant m$, we have

$$\left|\frac{b_1+b_2+\dots b_n}{n}\right| < \epsilon.$$

Hence the result.

Note. This result could also be deduced from Ex. 5, Page 55, by putting $(a_1+a_2+\dots+a_n)$ for a_n and n for b_n.

7. If $\underset{n\to\infty}{\text{lt}}\ (a_{n+1}-a_n)=l$, then $\underset{n\to\infty}{\text{lt}}\ (a_n/n)=l$.

8. *The sequence* $\{x_n\}$ *tends to the limit* l, *finite or infinite* ; $\{a_n\}$ *is another sequence of positive members such that the sequence* $\{a_1+a_2+\dots+a_n\}$ *diverges to* ∞ ; *prove that*

$$\frac{a_1x_1+a_2x_2+\dots+a_nx_n}{a_1+a_2+\dots+a_n}\to l \text{ as } n\to\infty.$$

Let $x_n - l = y_n$. Then we have to prove that $\sum\limits_{r=1}^{n} a_r y_r \Big/ \sum\limits_{r=1}^{n} a_r \to 0$,

when $\{y_n\} \to 0$ and $\sum\limits_{r=1}^{n} a_r \to \infty$.

Let ϵ be any positive number. There exists a positive integer μ such that $|y_n|<\frac{1}{2}\epsilon$ for every $n\geqslant\mu$. Also there exists a number k such that $|y_n|<k$, for all n.

We have

$$\left|\frac{\sum\limits_{r=1}^{n} a_r y_r}{\sum\limits_{r=1}^{n} a_r}\right| \leqslant \frac{\sum\limits_{r=1}^{\mu} a_r\,|y_r|}{\sum\limits_{r=1}^{n} a_r} + \frac{\sum\limits_{r=\mu+1}^{n} a_r\,|y_r|}{\sum\limits_{r=\mu+1}^{n} a_r}$$

$$\leqslant \frac{kc}{\sum\limits_{r=1}^{n} a_r} + \frac{\epsilon}{2},$$

where c is the constant $\sum\limits_{r=1}^{\mu} a_r$, μ having been fixed.

There exists a number ν such that for $n\geqslant\nu$,

$$\sum_{r=1}^{n} a_r > \frac{2kc}{\epsilon},\ i.e.,\ \frac{kc}{\sum\limits_{r=1}^{n} a_r} < \frac{\epsilon}{2}.$$

If $m=\text{Max}\ (\mu,\nu)$, then for $n\geqslant m$,

$$\left|\frac{\sum\limits_{r=1}^{r=n} a_r y_r}{\sum\limits_{r=1}^{n} a_r}\right| < \epsilon.$$

Hence the result.

9. *If $\{a_n\}$ and $\{b_n\}$ converge to A and B respectively, then*

$$\frac{a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1}{n} \to AB.$$

10. *If $\{a_n\}$ be a sequence of positive terms, prove that*

$$\underset{n\to\infty}{lt}\ [\sqrt[n]{a_n}] = \underset{n\to\infty}{lt}\ [a_{n+1}/a_n],$$

provided that the limit on the right exists, whether finite or infinite.

(This is known as Cauchy's second theorem on limits).

Case I. Let l be finite.

Let ϵ be any positive number. There exists a positive integer m_1 such that for every $n\geqslant m_1$,

$$l-\tfrac{1}{2}\epsilon < \frac{a_{n+1}}{a_n} < l+\tfrac{1}{2}\epsilon.$$

Changing n, n to $n+1, n+2, \ldots (n+p-1)$ and multiplying, we get

$$(l-\tfrac{1}{2}\epsilon)^p < \frac{a_{n+p}}{a_n} < (l+\tfrac{1}{2}\epsilon)^p$$

or $$(a_n)^{\frac{1}{n+p}}(l-\tfrac{1}{2}\epsilon)^{\frac{p}{n+p}} < (a_{n+p})^{\frac{1}{n+p}} < (l+\tfrac{1}{2}\epsilon)^{\frac{p}{n+p}}(a_n)^{\frac{1}{n+p}}.$$

Keeping n fixed, we let $p \to \infty$. Since

$$(a_n)^{\frac{1}{n+p}}(l-\tfrac{1}{2}\epsilon)^{\frac{p}{n+p}} \to (l-\tfrac{1}{2}\epsilon),$$

and $$(a_n)^{\frac{1}{n+p}}(l+\tfrac{1}{2}\epsilon)^{\frac{p}{n+p}} \to (l+\tfrac{1}{2}\epsilon),$$

we see, as in Ex. 5, Page 55, that there exists a positive integer m_2 such that

$$l-\epsilon < (a_{n+p})^{\frac{1}{n+p}} < l+\epsilon, \text{ when } n \geqslant m_1 \text{ and } p \geqslant m_2$$

or $$l-\epsilon < \sqrt[n]{a_n} < l+\epsilon, \text{ when } n \geqslant (m_1+m_2).$$

Hence the result.

Case II. Let l be infinite.

Let Δ be any positive number. There exists a positive integer m_1 such that for every $n \geqslant m_1$,

$$a_{n+1}/a_n > (\Delta+1).$$

Changing n to n, $(n+1)$, $(n+2)$,..., $(n+p-1)$ and multiplying, we get

$$(a_{n+p})^{\frac{1}{n+p}} > (a_n)^{\frac{1}{n+p}}.(\Delta+1)^{\frac{p}{n+p}}.$$

The right hand expression $\to (\Delta+1)$, as $p \to \infty$, keeping n fixed. There exists, therefore, a positive integer m_2 such that for every $p \geqslant m_2$,

$$\Delta = \Delta+1-1 < (a_n)^{\frac{1}{n+p}}.(\Delta+1)^{\frac{p}{n+p}} < \Delta+1+1 = \Delta+2.$$

From above, we deduce that, for every $n \geqslant (m_1+m_2)$,

$$\sqrt[n]{a_n} > \Delta,$$

i.e., $$\sqrt[n]{a_n} \to \infty.$$

11. *If x_1, x_2 are positive and $x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$, then the sequences*

$$x_1, x_3, x_5, \ldots\ldots;\ \textit{and}\ x_2, x_4, x_6, \ldots\ldots,$$

are one a decreasing and the other an increasing sequence, and they have the common limit $\frac{1}{3}(x_1+2x_2)$.

Let $x_1 > x_2$. On this account, we have

$$x_2 < x_3 < x_1.$$

Also, since $x_3 > x_2$, we have

$$x_2 < x_4 < x_3.$$

In this manner, we may easily see that

$$x_4 < x_5 < x_3;\ x_4 < x_6 < x_5;\ x_6 < x_7 < x_5;\ \text{and so on.}$$

Thus $$x_2 < x_4 < x_6 < \ldots < x_5 < x_3 < x_1.$$

Thus $x_1, x_3, x_5, \ldots$ is decreasing and $x_2, x_4, x_6, \ldots$ increasing and being bounded, they are both convergent.

We have $x_3 - x_2 = \frac{1}{2}(x_1 - x_2)$.

$$x_4 - x_2 = x_4 - x_3 + x_3 - x_2 = -\tfrac{1}{4}(x_1-x_2) + \tfrac{1}{2}(x_1-x_2)$$
$$= (x_1-x_2)(-\tfrac{1}{4}+\tfrac{1}{2}).$$
$$x_5 - x_2 = x_5 - x_4 + x_4 - x_2 = (x_1-x_2)(\tfrac{1}{8}-\tfrac{1}{4}+\tfrac{1}{2}).$$
$$x_6 - x_2 = x_6 - x_5 + x_5 - x_2 = (x_1-x_2)(-\tfrac{1}{16}+\tfrac{1}{8}-\tfrac{1}{4}+\tfrac{1}{2}).$$

In general, we get

$$x_n - x_2 = (x_1 - x_2)\left(\frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} \ldots\ldots \overline{n-2} \text{ terms}\right)$$

$$= \frac{1}{3}(x_1 - x_2)\left[1 - \left(-\frac{1}{2}\right)^{n-2}\right] \to \frac{1}{3}(x_1 - x_2),$$

as $n \to \infty$.

$\therefore$ $x_n \to \frac{1}{3}(x_1 + 2x_2)$, whether $n \to \infty$ through even or through odd integral values.

12 *If k is positive and a, $-\beta$ are the positive and negative roots of $x^2 - x - k = 0$, prove that*

if $u_n = \sqrt{(k + u_{n-1})}$ and $u_1 > 0$, then $u_n \to a$.

We have

$$u_n^2 - u_{n-1}^2 = (k + u_{n-1}) - (k + u_{n-2}) = u_{n-1} - u_{n-2},$$

so that $u_n \gtrless u_{n-1}$, according as $u_{n-1} \gtrless u_{n-2}$ and thus $\{u_n\}$ is a monotonic sequence; it is increasing or a decreasing sequence according as $u_2 >$ or $< u_1$.

Since $\qquad x^2 - x - k = (x - a)(x + \beta),$

therefore $\qquad u_1^2 - u_1 - k = (u_1 - a)(u_1 + \beta). \qquad (1)$

Let $u_1 > a$, then $u_1^2 - u_1 - k > 0$ so that

$$u_2 = \sqrt{(u_1 + k)} < u_1.$$

Therefore $\{u_n\}$ is a decreasing sequence.

Now, $\quad u_n^2 = u_{n-1} + k > u_n + k$, *i.e.*, $u_n^2 - u_n - k > 0$.

Therefore from (1), we deduce that $u_n > a$.

Hence $u_n \to a$ limit, say, l. Clearly $l \geqslant a$.

We have

$$(u_n - a)(u_n + \beta) = (u_n^2 - u_n - k) - (u_n^2 - u_{n-1} - k) = u_{n-1} - u_n.$$

In the limit, we get

$$(l - a)(l + \beta) = 0,$$

so that $l = a$, for l cannot be equal to $-\beta$ which is $< a$.

The case in which $u_1 < a$ can be similarly considered.

EXAMPLES.

1. (*i*) $\{a_n\}$ and $\{b_n\}$ are two sequences such that $a_n < b_n$, for every value of n. If $a_n \to a$ and $b_n \to b$, show that $a \leqslant b$.

Given an example to show that the equality is possible.

(*ii*) If $a_n \to a$ and $b_n \to b$ and if $c_n = \text{Max}(a_n, b_n)$, show that $c_n \to \text{Max}(a, b)$. (*M.T.*)

[Max $(a, b) = a$ or b according as $a \geqslant b$ or $b \geqslant a$.]

2. Find the limit of

$$\frac{2^n - 1}{2^n + 1} + \frac{(\frac{1}{3})^n - 1}{(\frac{1}{3})^n + 1},$$

as $n \to \infty$.

Find n such that for all values greater than this, the given sum differs from the limit by a number less than 1/1000.

3. If $\{a_n\}$ is a decreasing sequence, if $\{b_n\}$ is an increasing sequence, and if $\{a_n - b_n\} \to 0$ as $n \to \infty$, prove that both the sequences tend to the same finite limit as $n \to \infty$. (*M.T.*)

4. Prove that, when $n \to \infty$,

(*i*) $\text{lt}\left[\frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{(n^2+2)}} + \ldots + \frac{1}{\sqrt{(n^2+n)}}\right] = 1.$

(*ii*) $\text{lt}\left[\frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(2n)^2}\right] = 0.$

(*iii*) $\text{lt}\left[\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{(n+1)}} + \ldots \frac{1}{\sqrt{2n}}\right] = \infty.$

5. Prove that the sequence

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ldots,$$

converges to 2.

6. Prove that the sequence

$$\sqrt{7},\ \sqrt{7+\sqrt{7}},\ \sqrt{7+\sqrt{7+\sqrt{7}}},\ldots,$$

converges to the positive root of $x^2-x-7=0$.

7. Prove that

(i) $\text{lt}\ \dfrac{1}{n}\left(1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}\right)=0.$

(ii) $\text{lt}\ \dfrac{1+\sqrt{2}+\sqrt[3]{3}+\ldots+\sqrt[n]{n}}{n}=1.$

8. Show that

(i) $\text{lt}\ \{[(n+1)(n+2)\ldots(n+n)]^{1/n}/n\}=4/e.$

(ii) $\text{lt}\ \{(\underline{|n})^{1/n}/n\}=e^{-1}.$

9. If $k\neq 0$ and $\{a_n\}$ is a sequence such that $(a_{n+1}-a_n)\to k$, then $a_n\to\infty$ or $-\infty$ according as k is positive or negative.

10. If λ, μ are two given numbers and $\{a_n\}$ is a given sequence such that (i) $|\lambda|<1$, (ii) $(a_{n+1}+\lambda a_n+\mu)\to 0$, then show that $a_n\to-\mu/(1+\lambda)$.

11. $\{a_n\}$ and $\{b_n\}$ are two infinite sequences such that (i) $b_n>0$. (ii) $S_n=(b_1+b_2+\ldots+b_n)$ is divergent. (iii) $a_n/b_n\to s$. Show that

$$\text{lt}\ \frac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}=s.$$

12. If a sequence of intervals (a_n, b_n), any one of which is entirely contained within the preceeding one, is such that $\text{lt}\ (b_n-a_n)=0$, show that there is one and only one point common to all the intervals of the sequence.

Show that the sequence of intervals

$$\left(\frac{2^{n-1}-1}{2^n},\ 1-\frac{2^{n-1}-1}{2^n}\right)$$

satisfies the conditions of the above theorem and determine the point common to all of them.

13. If a, a_1 are positive and $a_n=a/(1+a_{n-1})$, then the sequence $\{a_n\}$ tends to α, the positive root of the equation $x^2+x=a$.

14. If k is positive and α, $-\beta$ are the positive and negative roots of $x^2+x-k=0$, prove that

if $v_n=\sqrt{(k-v_{n-1})}$ and $v_1<k$, then $v_n\to\beta$.

15. If u_1, v_1 are given unequal numbers and

$$u_n=\tfrac{1}{2}(u_{n-1}+v_{n-1}),\ v_n=\sqrt{(u_{n-1}v_{n-1})}\ \text{for}\ n\geqslant 2$$

prove that (i) u_n decreases, and v_n increases as n increases (ii) $\{u_n\}$, $\{v_n\}$ are both convergent and have the same limits.

16. If x_1, y_1 are positive and if, for $n\geqslant 1$

$$2x_{n+1}=x_n+y_n,\ 2/y_{n+1}=1/x_n+1/y_n,$$

then show that $\{x_n\}$ and $\{y_n\}$ are monotonic sequences and approach a common limit l, where $l^2=x_1y_1$.

17. If x_1, x_2 are positive and $x_{n+1}=\sqrt{(x_nx_{n-1})}$, then the sequences

$$x_1, x_3, x_5, \ldots\ldots;\ x_2, x_4, x_6, \ldots\ldots$$

are one a decreasing and the other an increasing sequence and they have the common limit $\sqrt[3]{(x_1x_2^2)}$.

18. $\{a_n\}$, $\{b_n\}$ are two bounded sequences ; prove that

(i) $\overline{\lim}\ a_n+\overline{\lim}\ b_n\geqslant\overline{\lim}\ (a_n+b_n)$.

(ii) $\underline{\lim}\ a_n+\underline{\lim}\ b_n\leqslant\underline{\lim}\ (a_n+b_n)$.

(iii) $\overline{\lim}\ (-a_n)=-\underline{\lim}\ (a_n)$.

[$\overline{\lim}\ a_n$ and $\underline{\lim}\ a_n$ denote the upper and lower limits of the sequence $\{a_n\}$.]

CHAPTER IV

FUNCTIONS AND THEIR CONTINUITY

Limit of a Function

40. Variable and its domain. If a symbol, x, denotes any member of a given aggregate, S, of numbers, then we say that the aggregate S is the *domain of variation* of the *variable x.*

If, now, to each value of the variable, x, there corresponds, according to any law whatsoever, a value of another symbol y, we say that y is a function of x defined for the aggregate, S, and symbolically write $y=f(x)$.

Also, x, is called the *independent variable* and, y, the *dependent variable.*

Sometimes, for a given function, the law of correspondence itself suggests the domain for which it is defined.

Illustrations.

1. If $y=0$, when x is rational and $y=1$, when x is irrational, then y is a function of x defined for $[-\infty, \infty]$, *i.e.*, for the whole continuum.

2. If $y=\underline{|x}$, then y is a function of x defined for the aggregate of positive integers.

3. If $y=[x]$, where $[x]$ denotes the greatest integer not greater than x, then y is a function of x defined for $[-\infty, \infty]$.

4. If $y=1/(1+x)$, then y is a function of x defined for the entire continuum excepting -1, since the determination of y for $x=-1$ involves the meaningless operation of division by 0.

5. If $y=\begin{cases}1/(1+x), & \text{when } x\neq-1,\\ 0 & \text{, when } x=-1,\end{cases}$

then the function y of x is defined for the entire continuum.

6. A sequence is a function; the domain of definition being the aggregate of positive integers.

Ex. Compare the domains of definition of the functions $(x^2-1)/(x-1)$ and $x+1$.

40·1. Classification of functions :

(*i*) *Algebraic.* (*ii*) *Transcendental.*

Before defining an algebraic function, we note that a function of the form

$$a_0+a_1x+a_2x^2+\ldots+a_mx^m,$$

where $a_0, a_1, \ldots, a_m$ are constants and m is an integer $\geqslant 0$, is called a *polynomial.*

(*i*) *A function $f(x)$ is called an algebraic function, if it satisfies an equation of the form*

$$P_0[f(x)]^n+P_1[f(x)]^{n-1}+\ldots+P_n=0,$$

where $P_0, P_1, \ldots, P_n$ are polynomials.

A polynomial itself is a particular case of an algebraic function as we may see on taking $n=1$ and $P_0=$ a constant.

The rational function, *i.e.*, a function of the form

$$\frac{a_0+a_1x+a_2x^2+\ldots+a_nx^n}{b_0+b_1x+b_2x^2+\ldots+b_mx^m},$$

is also an algebraic function.

(*ii*) *A function which is not algebraic is called a transcendental function.*

40·2. Bounded and unbounded functions. A function is said to be bounded if the aggregate of its values is bounded; the bounds of this aggregate, in case they exist, are said to be the bounds of the function.

Ex. Show that the function

$$f(x)=\begin{cases}1/x, & \text{when } x\neq 0,\\ 0\ , & \text{when } x=0,\end{cases}$$

is not bounded.

Ex. Show that the function $x/(x+1)$ is bounded in $(0, \infty]$. Find its bounds and show that it attains its lower bound but not the upper bound.

41. Limit of a function. *A function $f(x)$ is said to tend to the limit, l, as x approaches, a, or, symbolically*

$$\underset{x\to a}{lt}\ f(x)=l \text{ or } f(x)\to l, \text{ as } x\to a, \tag{1}$$

if, to every positive number ϵ, however small, there corresponds a positive number δ, such that

$$|f(x)-l|<\epsilon, \text{ when } 0<|x-a|\leqslant\delta,$$

i.e., for all those values of x, (except possibly a), which belong to the interval $(a-\delta, a+\delta)$, $f(x)$ lies between $l-\epsilon$ and $l+\epsilon$.

Note 1. In order that $f(x)$ may tend to a limit as $x\to a$, it is *necessary*, that $f(x)$ should be defined in a certain interval $(a-h, a+h)$ except possibly at a.

Any interval $(a-h, a+h)$ enclosing a is said to be a neighbourhood of a.

Note 2. The symbolic statement

$$\underset{x\to a}{lt}\ f(x)=l,$$

means two things:—(*i*) the limit of $f(x)$, as $x\to a$, exists; (*ii*) the limit is equal to l.

Ex. *Criticise the following statements*:

A function $f(x)$ is said to tend to the limit l as x tends to a,

(*i*) if $f(x)$ is nearly equal to l when x is nearly equal to a.

(*ii*) if as x approaches nearer and nearer a, then $f(x)$ approaches nearer and nearer l.

(*iii*) if the difference between $f(x)$ and l can be made as small as we like by taking x sufficiently near 'a'.

Ex. Show that a function $f(x)$ cannot tend to two different limits.

42. One sided limits.

42·1. Right handed limit. If to every positive number ϵ, there corresponds a positive number δ, such that

$$|f(x)-l|<\epsilon, \text{ when } a<x\leqslant a+\delta,$$

we say that $f(x)\to l$ as $x\to a$ through values greater than a, and mbolically write

$$\underset{x\to(a+0)}{lt}\ f(x)=l, \text{ or } f(a+0)=l.$$

42·2. Left handed limit. If to every positive number ϵ, there corresponds a positive number δ such that

$$|f(x)-l| < \epsilon, \text{ when } a-\delta \leqslant x < a,$$

we say that $f(x) \to l$ as $x \to a$ through values less than a, or symbolically

$$\underset{x \to (a-0)}{lt} f(x) = l \text{ or } f(a-0) = l.$$

Note. It is easy to see that

$$\underset{x \to a}{lt} f(x) = l,$$

if and only if

$$\underset{x \to (a+0)}{lt} f(x) = l = \underset{x \to (a-0)}{lt} f(x).$$

In case either or both the limits, *viz.*; $\underset{x \to (a+0)}{lt} f(x)$ and $\underset{x \to (a-0)}{lt} f(x)$ do not exist, or exist but are not equal, then $\underset{x \to a}{lt} f(x)$ does not exist.

Ex. If $y=[x]$, show that

$$\underset{x \to (2+0)}{lt} y = 2, \quad \underset{x \to (2-0)}{lt} y = 1, \text{ but } \underset{x \to 2}{lt} y \text{ does not exist.}$$

Ex. Show that $\underset{x \to 0}{lt} [\,|x|/x]$ does not exist.

Ex. A function $f(x) \to l$ as $x \to a$, and a sequence $\{x_n\} \to a$, show that the sequence $f(x_n) \to l$.

Ex. If $f(x) \to l$ as $x \to a$, then there exists a neighbourhood of a in which $f(x)$ is bounded.

Ex. Show that

(*i*) $lt\,(x^3+3x)=4$, as $x \to 1$.

(*ii*) $lt\,(2x^2+3)/(x+1)=3$, as $x \to 0$.

43·1. $\underset{x \to \infty}{\mathrm{lt}} f(x) = l$; $\underset{x \to -\infty}{\mathrm{lt}} f(x) = l$.

If, to every positive number ϵ, there corresponds a positive number Δ such that

$$|f(x)-l| < \epsilon, \text{ when } x \geqslant \Delta,$$

then we say that $f(x) \to l$ as $x \to -\infty$, or

$$\underset{x \to a}{lt} f(x) = \infty.$$

Similarly, if, to every positive number ϵ, there corresponds a positive number Δ such that

$$|f(x)-l| < \epsilon, \text{ when } x \leqslant -\Delta,$$

then we say that $f(x) \to l$, as $x \to -\infty$.

43·2. Infinite limits. The definitions, as given in §41 and §43·1 above, may be easily modified to give precise meanings to the following :—

$$\underset{x \to a}{lt} f(x) = \pm\infty, \quad \underset{x \to \infty}{lt} f(x) = \pm\infty, \quad \underset{x \to -\infty}{lt} f(x) = \pm\infty.$$

43·3. Finite and infinite oscillation. If a function $f(x)$ neither tends to a finite nor to an infinite limit as $x \to a$, (∞ or $-\infty$), we say that it *oscillates*; the oscillation is said to be finite or infinite according as $f(x)$ is bounded or not in a certain neighbourhood of a, (in a certain interval $[X, \infty]$ or $[-\infty, X]$).

44. Condition for the existence of finite limit. *The necessary and sufficient condition that $f(x)$ may tend to a finite limit, as x tends to a, is that, to every positive number ϵ, however small, there corresponds a positive number δ, such that*

$$|f(x_2)-f(x_1)|<\epsilon,$$

for every pair x_1, x_2 of values of x which satisfy the inequalities

$$0<|x_1-a|\leqslant\delta,\ 0<|x_2-a|\leqslant\delta,$$

i.e., for every pair x_1, x_2 of values, other than a, which belong to the interval $(a-\delta, a+\delta)$.

The condition is necessary. Let $f(x)\to l$, as $x\to a$.

Let ϵ be any positive number. There exists a positive number δ such that

$$|f(x)-l|<\tfrac{1}{2}\epsilon, \text{ when } 0<|x-a|\leqslant\delta,$$

so that if x_1, x_2 be *any* two numbers such that

$$0<|x_1-a|\leqslant\delta,\ 0<|x_2-a|\leqslant\delta,$$

we have

$$|f(x_1)-l|<\tfrac{1}{2}\epsilon,\ |f(x_2)-l|<\tfrac{1}{2}\epsilon,$$

and accordingly

$$\begin{aligned}|f(x_2)-f(x_1)| &= |f(x_2)-l+l-f(x_1)|\\ &\leqslant |f(x_2)-l|+|l-f(x_1)|\\ &<\tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon=\epsilon.\end{aligned}$$

The condition is sufficient. Let

$$\epsilon_1, \epsilon_2, \epsilon_3, \ldots\ldots, \epsilon_n, \ldots\ldots \tag{1}$$

be any monotonically decreasing sequence of positive numbers which converges to 0, as n tends to infinity.

To each positive number ϵ_n, there corresponds a positive number δ_n, such that

$$|f(x_2)-f(x_1)|<\epsilon_n, \text{ when } 0<|x_1-a|\leqslant\delta_n,\ 0<|x_2-a|\leqslant\delta_n.$$

Thus we obtain another sequence

$$\delta_1, \delta_2, \delta_3, \ldots\ldots, \delta_n, \ldots\ldots \tag{2}$$

of positive numbers corresponding to the sequence (1).

Obviously, we may suppose that this sequence $\{\delta_n\}$ is also monotonically decreasing.

Writing $a+\delta_n$ for x_1 and x for x_2, we see that

$$|f(x)-f(a+\delta_n)|<\epsilon_n, \text{ when } 0<|x-a|\leqslant\delta_n.$$

Thus for all values of x, other than a, which belong to the interval $(a-\delta_n, a+\delta_n)$, $f(x)$ belongs to the interval

$$[f(a+\delta_n)-\epsilon_n, f(a+\delta_n)+\epsilon_n],$$

which we call A_n and whose length is $2\epsilon_n$.

Since $\delta_{n+1}<\delta_n$, we may suppose that the interval A_{n+1} is contained in A_n.

*Thus we obtain a sequence of intervals

$$A_1, A_2, A_3, \ldots\ldots, A_n, \ldots\ldots \tag{3}$$

such that each member of the sequence is contained in the preceeding one. Also the length $2\epsilon_n$ of $A_n\to 0$ as $n\to\infty$.

*As the sequences $[f(a+\delta_n)-\epsilon_n]$, $[f(a+\delta_n)+\epsilon_n]$, formed of the end points of the intervals, are monotonic and bounded, and therefore convergent and, since $[f(a+\delta_n)+\epsilon_n]-[f(a+\delta_n)-\epsilon_n]=2\epsilon_n\to 0$, they converge to the same point which will be common to all the intervals.

There exists, therefore, one and only one point, say, l, common to all the intervals of the sequence (3).

Let ϵ be any positive number.

We choose n so large that $2\epsilon_n < \epsilon$. We have

$$f(a+\delta_n)-\epsilon_n \leqslant l \leqslant f(a+\delta_n)+\epsilon_n,$$

and $$f(a+\delta_n)-\epsilon_n < f(x) < f(a+\delta_n)+\epsilon_n,$$

when $$0 < |x-a| \leqslant \delta_n.$$

$\therefore$ $$|f(x)-l| < 2\epsilon_n < \epsilon, \text{ when } 0 < |x-a| \leqslant \delta_n.$$

Hence $$f(x) \to l, \text{ as } x \to a.$$

Ex. State and prove the corresponding theorems for the existence of the right handed and left handed limits.

Ex. Prove that the necessary and sufficient condition that $f(x)$ may tend to a finite limit as $x \to \infty$ is that to every positive number ϵ, there corresponds a positive number Δ such that

$$|f(x_2)-f(x_1)| < \epsilon,$$

for every pair x_1, x_2 of numbers which are both greater than or equal to Δ.

State and prove a similar condition for $f(x)$ to tend a finite limit as $x \to -\infty$.

45. Monotonic Functions. Let a function $f(x)$ be defined in an interval (a, b) and let x_1, x_2 be *any* two points of this interval such that $x_1 < x_2$. Then the function is said to be monotonically increasing if $f(x_1) \leqslant f(x_2)$ and monotonically decreasing if $f(x_1) \geqslant f(x_2)$.

A function $f(x)$ is said to be *strictly* monotonically increasing, if

$$f(x_2) > f(x_1), \text{ when } x_2 > x_1,$$

so that the sign of equality is not admissible.

For a strictly monotonically decreasing function $f(x)$,

$$f(x_2) < f(x_1), \text{ when } x_2 > x_1.$$

The properties of monotonic functions in regard to the existence of limits are quite similar to those of monotonic sequences and may be similarly proved.

We have the following results for monotonically increasing functions :—

(*i*) If $f(x)$ is a monotonically increasing function in $(a, \infty]$, and there exists a number k such that $f(x) \leqslant k$, when $x \geqslant a$, then

$$\underset{x\to\infty}{\text{lt}} f(x) \text{ exists and is } \leqslant k.$$

(*ii*) If $f(x)$ is monotonically increasing in $[-\infty, a)$ and there exists a number k such that $f(x) \geqslant k$ when $x \leqslant a$, then

$$\underset{x\to-\infty}{\text{lt}} f(x) \text{ exists and is } \geqslant k.$$

(*iii*) If $f(x)$ is a monotonically increasing function in the open interval $[a, b]$ and there exists a number k such that

(*a*) $f(x) < k$ in $[a, b]$, then $f(b-0)$, *i.e.*, lt $f(x)$, when $x \to (b-0)$, exists.

(*b*) $f(x) > k$ in $[a, b]$, then $f(a+0)$, *i.e.*, lt $f(x)$, when $x \to (a+0)$, exists.

Similar results are easily obtained for monotonically decreasing functions.

46. Fundamental theorems on limits. *If $f_1(x)$, $f_2(x)$ be two functions such that, when $x \to a$,*

$$lt\, f_1(x)=l_1,\ lt\, f_2(x)=l_2$$

then

(*i*) $lt[f_1(x)\pm f_2(x)]=lt\, f_1(x)\pm lt\, f_2(x)=l_1\pm l_2$.

(*ii*) $lt\,[f_1(x).f_2(x)]=lt\, f_1(x).lt\, f_2(x)=l_1.l_2$.

(*iii*) $lt\,[f_1(x)/f_2(x)]=lt\, f_1(x)/lt\, f_2(x)=l_1/l_2$, *when* $l_2\neq 0$.

The proofs are similar to those of the corresponding results on sequences. §31, p. 43.

Proof. (*i*) The proof is simple and is, therefore, left to the student.

(*ii*) Let ϵ be any positive number, however small.

We have

$$|f_1(x)f_2(x)-l_1l_2| = |f_2(x)[f_1(x)-l_1]+l_1[f_2(x)-l_2]|$$
$$\leqslant |f_2(x)|\ |f_1(x)-l_1| + |l_1|\ |f_2(x)-l_2|$$

There exists a positive number δ such that

$$|f_1(x)-l_1|<\epsilon',\quad |f_2(x)-l_2|<\epsilon'$$

when $\qquad 0<|x-a|\leqslant\delta,$

where ϵ' is *any* given positive number.

Since $\qquad |f_2(x)|-|l_2|\leqslant|f_2(x)-l_2|<\epsilon',$

$\therefore \qquad |f_2(x)|<|l_2|+\epsilon'.$

Therefore when $0<|x-a|\leqslant\delta$, we have

$$|f_1(x)f_2(x)-l_1l_2|\leqslant(|l_2|+\epsilon')\epsilon'+|l_1|\epsilon'=[|l_2|+|l_1|+\epsilon']\epsilon'.$$

Choosing ϵ' any positive number less than

$$1 \text{ and} <\epsilon/[l_2\,|\,l_2+|l_1|+1],$$

where ϵ is the given positive number, we see that

$$|f_1(x)f_2(x)-l_1l_2|<\epsilon, \text{ when } 0<|x-a|\leqslant\delta.$$

Hence the result.

(*ii*) Let ϵ be any positive number.

We have

$$\left|\frac{f_1(x)}{f_2(x)}-\frac{l_1}{l_2}\right|=\left|\frac{l_2(f_1(x)-l_1)-l_1(f_2(x)-l_2)}{l_2f_2(x)}\right|$$
$$\leqslant\frac{|l_2|\ |f_1(x)-l_1|+|l_1|\ |f_2(x)-l_2|}{|l_2|\ |f_2(x)|}. \qquad (i)$$

There exists a positive number δ_1, such that

$$|f_2(x)-l_2|<\tfrac{1}{2}|l_2|, \text{ when } 0<|x-a|\leqslant\delta_1, \text{ for } l_2\neq 0.$$

$\therefore \qquad |l_2|-|f_2(x)|\leqslant|f_2(x)-l_2|<\tfrac{1}{2}|l_2|,$

or $\qquad |f_2(x)|>\tfrac{1}{2}|l_2|$, when $0<|x-a|\leqslant\delta_1. \qquad (ii)$

There exists a positive number δ_2 such that

$$|f_1(x)-l_1|<\epsilon',\ |f_2(x)-l_2|<\epsilon', \text{ when } 0<|x-a|\leqslant\delta_2 \qquad (iii)$$

where ϵ' is any given positive number.

If $\delta=\min(\delta_1,\delta_2)$, we deduce from (*i*), (*ii*), (*iii*), that when $0<|x-a|\leqslant\delta$,

$$\left|\frac{f_1(x)}{f_2(x)}-\frac{l_1}{l_2}\right|\leqslant\frac{2}{|l_2|^2}[|l_2|+|l_1|]\epsilon',$$

Choosing ϵ' any positive number less than

$$\epsilon \mid l_2 \mid^2/2[\mid l_2 \mid + \mid l_1 \mid],$$

where ϵ is the given positive number, we see that

$$\left| \frac{f_1(x)}{f_2(x)} - \frac{l_1}{l_2} \right| < \epsilon, \text{ when } 0 < \mid x-a \mid \leqslant \delta.$$

Hence the result.

Ex. $f(x) \to l$ as $x \to a$ and k is a constant, show that

$$\text{lt} k f(x) = kl, \text{ as } x \to a.$$

Ex. If the functions $f_1(x)$, $f_2(x)$,......, $f_n(x)$ approach finite limits, when x approaches a, and k_1, k_2, k_3,......, k_n are constants then

(*i*) $\text{lt}\,(k_1 f_1 + k_2 f_2 + \ldots + k_n f_n) = k_1 \,\text{lt}\, f_1 + k_2 \,\text{lt}\, f_2 + \ldots + k_n \,\text{lt}\, f_n.$

(*ii*) $\text{lt} \prod_{r=1}^{n} f_r(x) = \prod_{r=1}^{n} \text{lt}\, f_r(x).$

Ex. If $\underset{x \to a}{\text{lt}} f(x) = 0$, show, by giving examples, that $1/f(x)$ may tend to $+\infty$, $-\infty$ or may oscillate infinitely.

Ex. If $\underset{x \to a}{\text{lt}} f(x) = 0$ and $f(x)$ is positive for values of x in a certain neighbourhood of a, show that $1/f(x) \to \infty$ as $x \to a$.

Ex. $f(x)$, $\phi(x)$, $\psi(x)$ are three functions such that for all values of x, (excepting a), which lie in a certain neighbourhood of a,

$$f(x) \leqslant \phi(x) \leqslant \psi(x)$$

and $\quad \text{lt}\, f(x) = \text{lt}\, \psi(x) = l$, as $x \to a$;

show that $\quad \phi(x) \to a$ as $x \to a$.

Ex. $f(x) \to l$ as $x \to a$; show that $\mid f(x) \mid \to \mid l \mid$, but the converse is not necessarily true except when $l = 0$.

Ex. *Show that*

$$e = \underset{x \to \infty}{\text{lt}} (1+1/x)^x = \underset{x \to -\infty}{\text{lt}} (1+1/x)^x = \underset{z \to 0}{\text{lt}} (1+z)^{1/z}.$$

(*i*) Let x be any real number greater than 1 and let the positive integer n be chosen so that

$$n \leqslant x < n+1.$$

$$\therefore \qquad 1+\frac{1}{n} \geqslant 1+\frac{1}{x} > 1+\frac{1}{n+1}$$

or

$$\left(1+\frac{1}{n}\right)^{n+1} \geqslant \left(1+\frac{1}{x}\right)^x > \left(1+\frac{1}{n+1}\right)^n.$$

Let $x \to \infty$; then n also $\to \infty$. We have

$$\text{lt}\left(1+\frac{1}{n}\right)^{n+1} = \text{lt}\left(1+\frac{1}{n}\right)^n . \text{lt}\left(1+\frac{1}{n}\right) = e.1 = e.$$

$$\text{lt}\left(1+\frac{1}{n+1}\right)^n = \text{lt}\left(1+\frac{1}{n+1}\right)^{n+1} \Big/ \text{lt}\left(1+\frac{1}{n+1}\right) = e/1 = e.$$

Hence lt $(1+1/x)^x=e$.

(*ii*) Let $x=-y$ so that $y\to+\infty$ as $x\to-\infty$. We have

$$\left(1+\frac{1}{x}\right)^x=\left(1-\frac{1}{y}\right)^{-y}=\left(1+\frac{1}{y-1}\right)^{y-1}\cdot\left(1+\frac{1}{y-1}\right)$$

$$\therefore \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)_x=e.1=e.$$

(*iii*) Putting $z=1/x$ so that $z\to(0+0)$ or $(0-0)$ according as $x\to+\infty$ or $-\infty$, we obtain this result.

47. Continuous Functions.

Let $f(x)$ be defined in an interval (a, b).

Continuity at an interior point. *The function $f(x)$ is said to be continuous at any interior point c, $a<c<b$, if*

$$lt. f(x)=f(c), \text{ when } x\to c.$$

i.e., if $lt\ f(x)$ exists and is equal to $f(c)$.

Continuity at an end point. *$f(x)$ is said to be continuous at the left-end a, if*

$$lt\ f(x)=f(a), \text{ when } x\to(a+0)$$

and is said to be continuous at the right end b; if

$$lt\ f(x)=f(b), \text{ when } x\to(b-0).$$

Continuity in an interval. $f(x)$ is said to be continuous in an interval (a, b), if it is continuous at every point of the interval.

In case $f(x)$ is continuous at every interior point but not at the end points a, b, we say that it is continuous in the open interval $[a, b]$.

Criticise the following statements :—

1. "A function $f(x)$ is said to be a continuous function of x between the limits a and b, when, to each value of x between these limits, there corresponds a finite value of the function and when an infinitely small change in the value of x produces only an infinitely small change in the function."

2. "The continuous function of a variable is a quantity that changes gradually and passes through every intermediate value from an initial to a final value as the variable that enters it passes through every intermediate value from its initial to its final value." (See Cor. to §50·1).

3. "$f(x)$ is continuous between $x=a$ and $x=b$, when the locus of $y=f(x)$ between the points $[a, f(a)]$ and $[b, f(b)]$ is an unbroken line, straight or curved."

Ex. Formally prove that the functions

(*i*) k, a constant; (*ii*) x,

are continuous for every value of x.

48. Classification of discontinuities. Let c be any point of the interval of definition (a, b) of $f(x)$. For continuity at c, it is necessary and sufficient that $lt\ f(x)$ should exist and be equal to $f(c)$.

Let, now, $f(x)$ be discontinuous at c.

The discontinuity at c is said to be of the **First kind**, *if $lt\ f(x)$ exists finitely but the limit $\neq f(c)$, and it is said to be of the* **second kind** *if $lt\ f(x)$ does not exist finitely.*

Further classification of the points of discontinuity of the second kind.

(*i*) If $\text{lt}\, f(x)=\infty$ or $-\infty$, c is said to be a point of *infinite discontinuity.*

(*ii*) If $f(x)$ oscillates as x tends to c, then c is said to be a point of *oscillatory discontinuity.*

Note. We may also sometimes distinguish between the two sides of c. Thus if $\underset{x\to(c+0)}{\text{lt}} f(x)$ exists and $=f(c)$ but $\underset{x\to(c-0)}{\text{lt}} f(x)$ does not exist finitely then c is a point of continuity on the right and of discontinuity of the second kind on the left.

Similarly we may describe the nature of the discontinuity at the point c in the other cases.

Ex. What is the nature of the discontinuity at
(*i*) $x=1$ of the function $[x]$. (*ii*) $x=0$ of $1/x$.

Ex. Obtain the points of discontinuity of the function $f(x)$ defined in $(0, 1)$ as follows:—

$f(x)=0$ when $x=0$, to $\frac{1}{2}-x$ in $0<x<\frac{1}{2}$, to $\frac{1}{2}$ when $x=\frac{1}{2}$, to $\frac{3}{2}-x$ when $\frac{1}{2}<x<1$ and 1 when $x=1$.

Examine also the nature of discontinuity.

Functions defined by means of limits.

Ex. Examine the nature of the points of discontinuity of the function $\phi(x)$ defined as follows, n tending to ∞;

(*i*) $\text{lt}\ (x^n-1)/(x^n+1)$.

(It is easy to see that $\phi(x)=1$ when $|x|>1$, $\phi(x)=-1$ when $|x|<1$, $\phi(x)=0$, when $x=1$ and $\phi(x)$ is not defined when $x=-1$).

(*ii*) $\text{lt}\ [1/(1+x^{2n})]$. (*iii*) $\text{lt}\ [x^n/(x^n+1)]$.

(*iv*) $\text{lt}\ [nx/(1+nx]$. (*iv*) $\text{lt}\ [(x^{2n}+8x^n+1)/(x^{2n}+x^n+1)]$.

49. *Continuity of functions which are combinations of continuous functions.* The following theorems follow easily from the definition of continuity and the theorems on limits proved in §46.

49·1. (*i*) *The sum, the difference, the product of two functions which are continuous at a point, (in an interval) are continuous at that point, (in that interval).*

(*ii*) *The quotient of two functions which are continuous at a point (in an interval) is continuous at that point (in that interval) provided that the denominator does not vanish at the point* (at any point of the interval).

As an illustration, we consider the case of product. Let $f_1(x)$, $f_2(x)$ be continuous at a point c so that

$$\underset{x\to c}{\text{lt}}\ f_1(x)=f_1(c);\ \underset{x\to c}{\text{lt}}\ f_2(x)=f_2(c).$$

By §46,

$$\underset{x\to c}{\text{lt}}\ [f_1(x)f_2(x)]=\underset{x\to c}{\text{lt}}\ f_1(x).\ \underset{x\to c}{\text{lt}}\ f_2(x)=f_1(c).f_2(c),$$

which is the value of $f_1(x)f_2(x)$ at c.

Hence $f_1(x)f_2(x)$ is continuous at c.

49·2. Continuity of a function of a function. Let $f(x)$ be a function defined in an interval (a, b) and $\phi(t)$ a function defined in (α, β); and let every value of $\phi(t)$ belong to theinterval (a, b).

Writing

$$x=\phi(t),\ y=f(x),$$

we see that y is a function of t defined in (α, β) and which we may write as

$$y=f[\phi(t)].$$

Here y is a function of a function of t.

Theorem. *If $x=\phi(t)$ be a continuous function of t at a point t_0 of (α, β) and $y=f(x)$, a continuous function of x at the corresponding point $x_0=\phi(t_0)$, then $y=f[\phi(t)]$, is a continuous function of t at t_0.*

Let ϵ be *any* positive number. Let $y_0=f(x_0)$.

Since $y=f(x)$ is continuous at x_0, there exists a positive number δ_0 such that

$$|y-y_0| < \epsilon, \text{ when } |x-x_0| \leqslant \delta_0. \tag{1}$$

Again, since $x=\phi(t)$ is continuous at t_0, there exists a positive number δ such that

$$|x-x_0| < \delta_0, \text{ when } |t-t_0| \leqslant \delta. \tag{2}$$

From (1) and (2), we see that there exists a positive number δ such that

$$|y-y_0|\ \epsilon, \text{ when } |t-t_0| \leqslant \delta.$$

Hence the result.

Ex. 1. Show that a polynomial is continuous for every value of x.

Ex. 2. Show that an algebraic rational function of x is continuous for every value of x which is not a zero of the denominator.

Ex. 3. Show that

$$f(x)=(x^3-7x^2+3x-1)/(x^2-3x)$$

is continuous at $x=2$ and hence find $lt\, f(x)$ when $x \to 2$.

50. Properties of functions which are continuous in any closed finite interval.

50·1. Theorem. *If $f(x)$ is continuous in a closed interval (a, b) and $f(a)$, $f(b)$ have opposite signs, then $f(x)$ vanishes for atleast one point of the interval.*

Lemma 1. If $f(x)$ is continuous at any interior point c of (a, b) and $f(c)\neq 0$, then there exists an interval $(c-\delta, c+\delta)$ enclosing c such that for every point x of this interval $f(x)$ has the sign of $f(c)$.

If ϵ be *any* positive number, whatsoever, there exists a positive number δ such that

$$|f(x)-f(c)| < \epsilon,\ i.e.,\ f(c)-\epsilon < f(x) < f(c)+\epsilon,$$

for every point, x, of the interval $(c-\delta, c+\delta)$.

Let $f(c)$ be positive. If we take for ϵ any positive number less than $f(c)$, we see that for every point x of $(c-\delta, c+\delta)$, $f(x)$ is positive, lying as it does between two positive numbers $f(c)-\epsilon$ and $f(c)+\epsilon$.

Let $f(c)$ be negative. If we take for ϵ any positive number less than $-f(c)$, we see that for every point x of $(c-\delta, c+\delta)$, $f(x)$ is negative, lying as it does between two negative numbers $f(c)-\epsilon$ and $f(c)+\epsilon$.

Lemma II. If $f(x)$ is continuous at the end point a of (a, b) and $f(a) \neq 0$, then there exists an interval $(a, a+\delta)$ such that for every point x of this interval $f(x)$ has the sign of $f(a)$.

A similar result holds for continuity at b.

The proof is exactly similar to that of Lemma I.

Proof of the main theorem. Take the number $c = \frac{1}{2}(a+b)$, the mid-point of (a, b). In case $f(c)=0$, we have finished.

If $f(c) \neq 0$, then either $f(a), f(c)$ or $f(c), f(b)$ have opposite signs. Of the two intervals (a, c) and (c, b), the one at the ends of which $f(x)$ has opposite signs, we re-name as (a_1, b_1).

Tnus in every case we have

$$a \leqslant a_1 < b_1 \leqslant b;$$
$$b_1 - a_1 = \tfrac{1}{2}(b-a);$$
$$f(a_1), f(b_1) \text{ have opposite signs.}$$

We now bisect (a_1, b_1) and proceeding as above, see that either $f(x)=0$ at the mid-point $c_1 = \frac{1}{2}(a_1+b_1)$ of (a_1, b_1) or otherwise we obtain an interval (a_2, b_2) such that

$$a_1 \leqslant a_2 < b_2 \leqslant b_1$$
$$b_2 - a_2 = \tfrac{1}{2}(b_1 - a_1) = (\tfrac{1}{2})^2(b-a)$$
$$f(a_2), f(b_2) \text{ have opposite signs.}$$

Proceeding as above we see that either, after a finite number of steps, we will arrive at a point at which the function vanishes, or we will obtain an infinite sequence of intervals

$$(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n) \ldots$$

such that

$$a \leqslant a_1 \leqslant a_2 \ldots \leqslant a_n < b_n \leqslant \ldots \leqslant b_2 \leqslant b_1 < b; \qquad (i)$$
$$b_n - a_n = (b-a)/2^n \qquad (ii)$$
$$f(a_n), f(b_n) \text{ have opposite signs.} \qquad (iii)$$

From (i), we see that the sequence $\{a_n\}$ is monotonically increasing and bounded above and the sequence $\{b_n\}$ is monotonically decreasing and bounded below and accordingly we see that the sequences $\{a_n\}$, $\{b_n\}$ are both convergent.

From (ii), we see that

$$\text{lt}\,(b_n - a_n) = 0.$$

Now, since $\{a_n\}$ and $\{b_n\}$ are convergent, therefore

$$0 = \text{lt}\,(b_n - a_n) = \text{lt}\, b_n - \text{lt}\, a_n$$

or $$\text{lt}\, b_n = \text{lt}\, a_n = \xi, \text{ say.}$$

The point ξ may be either an interior or an end-point of (a, b). It will be shown that $f(\xi)=0$. If possible, let $f(\xi) \neq 0$.

Let ξ be an interior point. There exists, by lemma I, an interval $(\xi-\delta, \xi+\delta)$ such that for every point x of this interval $f(x)$ and $f(\xi)$ have the same sign.

Also, since $a_n \to \xi$ from below and $b_n \to \xi$, from above, there exists an integer m such that (a_m, b_m) lies within $(\xi-\delta, \xi+\delta)$ and accordingly $f(a_m)$, $f(b_m)$ have the same sign so that we arrive at a contradiction of (iii). Hence $f(\xi)=0$.

Let ξ coincide with a. In this case $a_n=a$, for every n. Since $f(a)\neq 0$, there exists an interval $(a, a+\delta)$ such that for every point x of this interval $f(x)$ and $f(a)$ have the same sign. Also, since $b_n \to \xi = a$, there exists a positive integer m such that $a<b_m<\xi+\delta$, and accordingly $f(a_m)=f(a)$ and $f(b_m)$ have the same sign so that again we have a contradiction.

Hence ξ cannot coincide with a.

It may similarly be shown that ξ cannot coincide with b.

Another proof. For the sake of definiteness, we suppose that $f(a)>0$ and $f(b)<0$.

Since $f(x)$ is continuous at a and $f(a)>0$, there exists an interval $(a, a+\delta)$, $(\delta>0)$, such that for every point x of this interval $f(x)$ is positive. (Lemma II).

Consider an aggregate S defined as follows :—

Any point x of (a, b) belongs to S, if $f(x)$ is positive for every point of the closed interval (a, x).

Clearly S exists in as much as $a+\delta$ belongs to it. Also S is bounded above, b being a rough upper bound.

Let c be the upper bound of $f(x)$. Suppose that c is an interior point of (a, b). It will be shown that $f(c)=0$. If possible, let $f(c)\neq 0$.

There exists an interval $(c-h, c+h)$ such that for every point x of this interval $f(x)$ and $f(c)$ have the same sign.

Since c is the upper bound of S, there exists a member η of S such that

$$c-h<\eta\leqslant c.$$

As η belongs to S, $f(x)$ is positive for every point x of (a, η) and, in particular, $f(\eta)$ is positive. Also since η is a member of $(c-h, c+h)$, we deduce that for every point x of $(c-h, c+h)$, $f(x)$ is positive.

Thus we see that $f(x)$ is positive for every point x of $(a, c+h)$ so that $c+h$ belongs to S and this contradicts the fact that c is the upper bound of S.

Hence $f(c)=0$.

If possible, let c coincide with b. There exists an interval $(b-k, b)$ for every point x of which $f(x)$ is negative. Also since b is the upper bound of S, there exists a member α of $[(b-k), b]$ such that $f(x)$ is positive in (a, α) and, in particular $f(\alpha)$ is positive. Thus we have a contradiction so that c cannot coincide with b.

Cor. *If $f(x)$ is continuous in a closed interval (a, b) and $f(a)\neq f(b)$, then as x changes from a to b, $f(x)$ assumes atleast once every value between $f(a)$ and $f(b)$.*

Let k be any number between $f(a)$ and $f(b)$. The function

$$\phi(x)=f(x)-k,$$

is continuous in (a, b) and $\phi(a)=f(a)-k$ and $\phi(b)=f(b)-k$ have opposite signs. There exists, therefore, a point c of (a, b) such that $0=\phi(c)=f-k$, (c) or $f(c)=k$.

50·2. Theorem. *If $f(x)$ is continuous in a closed interval (a, b), and ϵ is any positive number, however small, then there exists a division of (a, b) into a finite number of sub-intervals such that*

$$|f(x_2)-f(x_1)|<\epsilon,$$

where x_1, x_2 are any two numbers belonging to the same sub-interval.

We *assume* that the theorem is false, *i.e.*, we suppose that it is not possible to divide (a, b) into a finite number of sub-intervals which possess the required property.

Take $c=\frac{1}{2}(a+b)$.

The theorem must be false for atleast one of the two sub-intervals (a, c) or $(c, b$, for, otherwise, the theorem would be true for (a, b). Let the sub-interval in which the theorem is false be re-named as (a_1, b_1). If there be a choice, which will happen if the theorem is false for both, we may, for the sake of definiteness, consider the left-hand interval (a, c). In every case, we have

$$a \leqslant a_1 < b_1 \leqslant b,$$
$$b_1 - a_1 = \tfrac{1}{2}(b-a),$$

the theorem is false in (a_1, b_1).

We now bisect (a_1, b_1) and proceeding, as above, obtain another interval (a_2, b_2) such that

$$a_1 \leqslant a_2 < b_2 \leqslant b_1,$$
$$b_2 - a_2 = \tfrac{1}{2}(b_1 - a_1) = (\tfrac{1}{2})^2 (b-a),$$

the theorem is false in (a_2, b_2).

Proceeding as above, we will obtain an infinite sequence of intervals

$$(a_1, b_1), (a_2, b_2) \ldots\ldots, (a_n, b_n),$$

such that

(*i*) $a \leqslant a_1 \leqslant a_2 \ldots\ldots \leqslant a_n < b_n \leqslant b_{n-1} \ldots\ldots \leqslant b_2 \leqslant b_1 < b$,

(*ii*) $b_n - a_n = (b-a)/2^n$.

(*iii*) the theorem is false in (a_n, b_n).

From (*i*) and (*ii*) we easily deduce, as in §50.1 that the sequences $\{a_n\}$, $\{b_n\}$ converge to the same limit. Let this common limit be c. The point c may be an interior or an end point of (a, b).

Let c be an interior point.

Since $f(x)$ is continuous at c, there exists a positive number δ such that

$$| f(x) - f(c) | < \tfrac{1}{2}\epsilon, \text{ when } | x-c) | \leqslant \delta.$$

If x_1, x_2 be any two members of $(c-\delta, c+\delta)$, we have

$$| f(x_1) - f(c) | < \tfrac{1}{2}\epsilon, \quad | f(x_2) - f(c) | < \tfrac{1}{2}\epsilon,$$

and, accordingly,

$$\begin{aligned} | f(x_2) - f(x_1) | &= | f(x_2) - c + c - f(x_1) | \\ &\leqslant | f(x_2) - c | + | c - f(x_1) | < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon, \end{aligned}$$

so that the theorem is true for $(c-\delta, c+\delta)$.

Let m be a positive integer so chosen that (a_m, b_m) lies within $(c-\delta, c+\delta)$. The theorem being true in $(c-\delta, c+\delta)$, we see that it is also true in (a_m, b_m) which is a part of $(c-\delta, c+\delta)$ and thus we arrive at a contradiction. Hence the assumption that the theorem is false for (a, b) is not true. Thus the theorem is true.

Let c coincide with the end point a. There exists a positive number δ such that

$$| f(x) - f(a) | < \tfrac{1}{2}\epsilon, \text{ when } a \leqslant x \leqslant a+\delta.$$

Now, if x_1, x_2 be any two members of $(a, a+\delta)$ we prove, as above, that

$$|f(x_2)-f(x_1)| < \epsilon,$$

so that the theorem is true for $(a, a+\delta)$.

If m be a positive integer so chosen that $b_m < a+\delta$, then (a_m, b_m) lies within $(a, a+\delta)$ and accordingly the theorem is true in (a_m, b_m) which is only a part of $(a, a+\delta)$.

Here, again, we have a contradiction and so on.

The case when c coincides with b may be similarly disposed of.

50·3. Theorem. *If a function $f(x)$ is continuous in a closed interval (a, b), then it is bounded in the interval.*

Let ϵ be any positive number.

We can divide (a, b) into a finite number of sub-intervals such that

$$|f(x_2)-f(x_1)| < \epsilon,$$

when x_1, x_2 are any two members of the same sub-interval.

Let $a, t_1, t_2, t_3, t_{r-1}, t_r, t_{r+1}, \ldots\ldots t_{n-1}, b$
be the points of division.

If x be any point of (a, t_1), we have

$$|f(x)-f(a)| < \epsilon, \text{ i.e., } f(a)-\epsilon < f(x) < f(a)+\epsilon;$$

if x be any point of (t_1, t_2), we have

$$|f(x)-f(t_1)| < \epsilon, \text{ i.e., } f(t_1)-\epsilon < f(x) < f(t_1)+\epsilon;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

if x be any point of (t_r, t_{r+1}). we have

$$|f(x)-f(t_r)| < \epsilon, \text{ i.e., } f(t_r)-\epsilon < f(x) < f(t_r)+\epsilon;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

finally, if x be any point of (t_{n-1}, b), we have

$$|f(x)-f(t_{n-1})| < \epsilon, \text{ i.e., } f(t_{n-1})-\epsilon < f(x) < f(t_{n-1})+\epsilon.$$

If k be the least member of the finite set of numbers

$$f(a)-\epsilon, f(t_1)-\epsilon, \ldots\ldots, f(t_r)-\epsilon, \ldots\ldots, f(t_{n-1})-\epsilon,$$

and K be the greatest member of the finite set of numbers

$$f(a)+\epsilon, f(t_1)+\epsilon, \ldots\ldots, f(t_r)+\epsilon, f(t_{n-1})+\epsilon,$$

we see that for every x of (a, b),

$$k < f(x) < \mathrm{K},$$

i.e., $f(x)$ is bounded.

Another proof. If ϵ be any positive number, then, because of the continuity of $f(x)$ at a, there exists a positive number δ such that for every point x of $(a, a+\delta)$,

$$|f(x-f(a)| < \epsilon, \text{ i.e., } f(a)-\epsilon < f(x) < f(a)+\epsilon,$$

so that we see that $f(x)$ is bounded in $(a, a+\delta)$

Consider, now, an aggregate S defined as follows :—

Any point x of (a, b) belongs to S, if $f(x)$ is bounded in (a, x). The aggregate S exists in as much as $a+\delta$ belongs to S and, as no number of S is $> b$, it is bounded also.

Let c be the upper bound of S. Now $c \leqslant b$.

If possible, let c be an interior point of S.

There exists an interval $(c-h, c+h)$ such that for every point x of this interval, $f(x)$ lies between $f(c)-\epsilon$ and $f(c)+\epsilon$ so that we see that $f(x)$ is bounded in $(c-h, c+h)$.

Since c is the upper bound of S, there exists a member η of S such that

$$c-h<\eta \leqslant c.$$

As η belongs to S, $f(x)$ is bounded in (a, η).

As η is an interior point of $(c-h, c+h)$, we deduce from above that $f(x)$ is bounded in $(a, c+h)$ and accordingly $c+h$ is a member of S and this plainly is a contradiction. Thus c cannot be an interior point so that we have $c=b$.

As $f(x)$ is continuous at b there exists an interval $(b-k, b)$ such that for every point x of this interval $f(x)$ lies between $f(b)-\epsilon$ and $f(b)+\epsilon$ *i.e.*, $f(x)$ is bounded in this interval.

There exists a member μ of S such that

$$b-k<\mu \leqslant b.$$

Now, $f(x)$ is bounded in (a, μ) and in $(b-k, b)$ and, therefore, we deduce that it is bounded in (a, b).

Hence the result.

Another proof. Suppose that $f(x)$ is not bounded. On this account, to every positive integer n, there corresponds a point t_n of (a, b) such that

$$|f(t_n)|>n.$$

Now $\{t_n\}$ is a bounded infinite sequence and has, therefore, at least one limiting point. Let t be any limiting point of $\{t_n\}$.

As $f(x)$ is continuous at t, there exists a positive number δ such that.

$$|f(x)-f(t)|<1, \text{ when } |x-t| \leqslant \delta, \qquad [\text{taking } \epsilon=1].$$

or

$$|f(x)|<1+|f(t)|,$$

for any member x of the interval $(t-\delta, t+\delta)$.

The interval $(t-\delta, t+\delta)$ contains an infinite number of members of the sequence $\{t_n\}$. Thus for an infinite number of positive integral values of n,

$$n<|f(t_n)|<1+|f(t)|,$$

which is clearly not true.

Hence $f(x)$ must be bounded in (a, b).

Ex. Show, by a process of continued bisection, that if a function $f(x)$ is defined in any interval (a, b),

(*i*) and is not bounded, then there exists a point c of (a, b) such that $f(x)$ is not bounded in any neighbourhood of c. Deduce theorem above.

(*ii*) and is bounded, then there exists a point c of (a, b) such that in any neighbourhood of c, the upper bound of $f(x)$ is the same as the upper bound of $f(x)$ in the whole interval. (Weirstrasas' Theorem).

50·4. Theorem. *If a function $f(x)$ is continuous in a closed interval (a, b), then it has greatest and least values, i.e., it attains its bounds at least once in the interval.*

It has been shown above in §50·3, that $f(x)$ is bounded in (a, b). Let M, m be the bounds of $f(x)$.

We have to show that there exist members α, β of (a, b) such that

$$f(\alpha)=M, f(\beta)=m.$$

We consider the case of upper bound. Suppose that $f(x)$ does not attain the value M for any value of x so that $M-f(x)$ does not vanish for any point x of (a, b).

From §49·1, we deduce that $M-f(x)$ and therefore, $1/[M-f(x)]$ is continuous in (a, b) and accordingly $1/[M-f(x)]$ is bounded.

Let k be any positive number, however large.

Since M is the upper bound of $f(x)$, there exists a value $f(c)$ of $f(x)$ such that

$$f(c)>M-1/k \text{ or } M-f(c)<1/k \text{ or } 1/[M-f(c)]>k.$$

There exists, therefore, a value of the function $1/[M-f(x)]$ which is greater than any positive, number k, however large, *i.e.*, $1/[M-f(x)]$ is not bounded. This is a contradiction.

Hence the theorem.

The case of lower bound may be similarly disposed of.

Cor. I. The function $f(x)$ which is continuous in (a, b) must also be continuous in (α, β) and so must assume every value between $f(\alpha)=M$ and $f(\beta)=m$.

Thus a function which is continuous in a closed interval must assume every value between its upper and lower bounds.

Cor. II. If a function $f(x)$ is continuous in a closed interval (a, b), and ϵ is any given positive number, then there exists a division of (a, b) into a finite number of sub-intervals such that the oscillation of $f(x)$ in every subinterval is less than ϵ.

This follows from §50·2 and the cor. I, above.

50·5. Uniform continuity.

Theorem. *If a function $f(x)$ is continuous in a closed interval (a, b) and ϵ is any given positive number, then there exists a positive number δ such that the oscillation of $f(x)$ in every sub-interval of length less than δ is less than ϵ.*

We can divide (a, b) into a finite number of sub-intervals such that

$$|f(x_2)-f(x_1)|<\tfrac{1}{2}\epsilon,$$

where x_1, x_2 are any two members of the same sub-interval.

Now every one of these sub-intervals has a length. Let δ be the least of these lengths. Clearly δ is positive.

Consider any pair of points x_1, x_2 such that $|x_2-x_1|\leqslant\delta$. The points x_1, x_2 either belong to the same or to two consecutive sub-intervals of the division obtained above.

In case they belong to the same sub-interval, we have

$$|f(x_2)-f(x_1)|<\tfrac{1}{2}\epsilon<\epsilon;$$

If they belong to two consecutive sub-intervals, let t_r be the common end point. We have

$$\begin{aligned} | f(x_2)-f(x_1) | &= | f(x_2)-f(t_r)+f(t_r)-f(x_1) | \\ &\leqslant | f(x_2)-f(t_r) | + | f(t_r)-f(x_1) | \\ &< \tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon=\epsilon. \end{aligned}$$

Now, consider any arbitrary sub-interval δ_r of (a, b) whose length $\leqslant \delta$. If M_r, m_r be the bounds of $f(x)$ in δ_r, there exists points ξ_r, η_r of this interval such that

$$f(\xi_r)=M_r, f(\eta_r)=m_r.$$

Since $| \xi_r-\eta_r | \leqslant \delta$, we have

$$\begin{aligned} M_r-m_r &= f(\xi_r)-f(\eta_r) \\ &= | f(\xi_r)-f(\eta_r | < \epsilon. \end{aligned}$$

Hence the theorem.

Note. The property of continuous functions proved above is known by the name of *uniform continuity* so that the theorem may be restated as follows :—

A function which is continuous in a closed interval is also uniformly continuous in that interval.

We now consider the *justification* for the name 'Uniform continuity.'

The continuity of $f(x)$ at a point x' implies that there exists a positive number δ such that

$$| f(x)-f(x') | < \epsilon, \text{ when } | x-x' | \leqslant \delta,$$

ϵ being any given positive number. Corresponding to the same ϵ, there exists a number δ for every point x of (a, b). The question arises, *does there exist a positive number δ which holds uniformly for every point of (a, b).* The theorem proved above shows that such a choice of δ is possible. In fact it has been shown there that corresponding to any positive number ϵ, there exists a positive number δ, such that

$$| f(x_2)-f(x_1) | < \epsilon,$$

where x_1, x_2 are any two numbers such that

$$| x_2-x_1 | \leqslant \delta.$$

Ex. *If* $f(x)=x^2+2x$ *in* $(-1, 1)$, *find a* δ *so that*

$$| f(x_2)-f(x_1) | < \epsilon \textit{ whenever } | x_2-x_1 | < \delta.$$

We have

$$\begin{aligned} | f(x_2)-f(x_1) | &= | x_2-x_1 | \; | x_2+x_1+8 | \leqslant | x_2-x_1 | [| x_2 | + | x_1 | +3] \\ &\leqslant 5 | x_2-x_1 | < \epsilon, \text{ if } | x_2-x_1 | < \epsilon/5. \end{aligned}$$

Thus $\delta = \epsilon/5$.

In particular $\delta = \frac{1}{50}$, if $\epsilon=\frac{1}{10}$: $\delta = \frac{1}{500}$ if $\epsilon=\frac{1}{100}$ etc,

Ex. Do the same, as in the Ex. above, for

(*i*) $f(x)=x^3+3x^2-2x+7$ in $(-3, 2)$

(*ii*) $f(x)=(x+2)/(2x+3)$ in $(-1, 5)$

51. Inverse functions. Theorem. *If a function $f(x)$ be continuous and strictly monotonic in (a, b), then there exists* **one and only one** *value of x which satisfies the equation*

$$f(x)=y, \tag{i}$$

where y is any number lying between $f(a)$ and $f(b)$.

The existence of at least one value of x follows from the fact of continuity of $f(x)$, (cor. to §50 1). Also since the function is strictly monotonic there cannot exist more than one such value. Hence the result.

51·1. Continuity of Inverse Functions.

To each value of y, lying between $f(a)$ and $f(b)$, there corresponds one and only one value of x, as determined from (i), and thus (i) determines x as a function of y, say, $\phi(y)$, defined in the interval $[f(a), f(b)]$ or $[f(b), f(a)]$.

This function $\phi(y)$ is said to be the *inverse* of $f(x)$. We now show that $\phi(y)$ is continuous in its interval of definition.

Let y_1 be any value of y and let $x_1=\phi(y_1)$ so that, we have
$$f(x_1)=y_1.$$

Let ϵ be any positive number.

Firstly suppose that $f(x)$ is strictly *increasing.*

Let $\quad f(x_1-\epsilon)=y_1-\delta_1, f(x_1+\epsilon)=y_1+\delta_2,$

δ_1, δ_2 being necessarily positive.

Since $f(x)$ is strictly monotonic, y will lie in the interval $(y_1-\delta_1, y_1+\delta_2)$ when x lies in $(x_1-\epsilon, x_1+\epsilon)$.

If $\delta=\min(\delta_1, \delta_2)$, we see that
$$|x-x_1|<\epsilon, \text{ i.e., } |\phi(y)-\phi(y_1)|<\epsilon, \text{ when } |y-y_1|\leqslant\delta$$

Hence $\phi(y)$ is continuous at y, and, therefore, in $[f(a), f(b)]$.

The continuity of $\phi(y)$ can be similarly established if $f(x)$ is strictly *decreasing.*

It should be understood that the *inverse function $\phi(y)$ exists and is continuous if and only if $f(x)$ is continuous and strictly monotonic.*

52. The continuity of a^x, $(a>0)$. *To show that a^x is continuous for every value of x.*

Let c be any value of x.

Suppose that a^x is not continuous for $x=c$.

This implies that there exists a positive number ϵ, such that whatever be the interval $(c-\delta, c+\delta)$ around c, for at least one point x of this interval,
$$|a^x-a^c|>\epsilon.$$

Consider a sequence of intervals $(c-1/n, c+1/n)$ around c. There exists a point x_n of $(c-1/n, c+1/n)$ such that
$$|a^{x_n}-a^c|>\epsilon. \tag{1}$$

Since $\quad c-1/n<x_n<c+1/n,$

we see that the sequence $\{x_n\}$ converges to c, and hence
$$lt\ a^{x_n}=a^c \quad (\S 33, \text{P. } 52)$$
and this conclusion contradicts the statement (1).

Hence the result.

Cor. The exponential function e^x is continuous for every value of x.

Cor. Since a^x is continuous and strictly monotonic, its inverse function, *viz.*, the logarithmic function $\log_a y$ is continuous for every positive value of y.

It should be remembered that the logarithmic function $\log_e x$ is defined for positive values of x only.

Cor. (*i*) If $f(x)$ is continuous in any interval, then $e^{f(x)}$ is also continuous in the same interval.

(*ii*) If $f(x)$ is positive and continuous in any interval, then $\log f(x)$ is also continuous in the same.

These results follow from §49·2, P. 70.

Cor. If $f(x)$, $\phi(x)$ are continuous in any interval and $f(x)$ is also positive then $f(x)^{\phi(x)}$ is continuous in the same interval.

This result is seen to be true on writing

$$[f(x)]^{\phi(x)}=e^{\phi(x)\log f(x)}$$

Examples

Ex. 1. *Consider the continuity for $x=0$ of*

$$f(x)=1/(1-e^{1/x}) \text{ when } x\neq 0 \text{ and } f(0)=0.$$

Now $1/x\to\infty$ or $-\infty$ according as $x\to(0+0)$ or $(0-0)$.

Thus
$$\lim_{x\to(0+0)} f(x)=0,$$

and
$$\lim_{x\to(0-0)} f(x)=1, \text{ for } \lim_{x\to(0-0)} e^{1/x}=0,$$

Since these two limits are different, $\lim f(x)$ as $x\to 0$, does not exist. Thus $f(x)$ is discontinuous for $x=0$ and the point of discontinuity is of the second kind.

Of course the function is continuous on the right and has a discontinuity of the second kind on the left of $x=0$.

Ex. Show that

$$f(x)=(x-1)/[1+e^{1/(x-1)}],\ x\neq 1,\ f(1)=0,$$

is continuous for every value of x.

Ex. Show that

$$f(x)=(e^{1/x}-1)/(e^{1/x}+1), \text{ when } x\neq 0;\ f(0)=0,$$

is continuous for every value of x except $x=0$. What is the nature of the discontinuity at $x=0$.

Ex. Consider the continuity of

$$f(x)=e^{1/x^2}/(e^{1/x^2}-1), \text{ when } x\neq 0;\ f(0)=1.$$

53. *To prove that*

(*a*) $\displaystyle\lim_{x\to\infty} a^x=\infty$ or 0 *according as* $a>$ or <1;

(*b*) $\displaystyle\lim_{x\to-\infty} a^x=0$ or ∞ *according as* $a>$ or <1.

The result (*a*) follows from the facts that (*i*) the sequence a^n tends to ∞ or 0 according as $a>1$ or $0<a<1$, (*ii*), a^x is positive and monotonically increasing or decreasing according as $a>1$ or $0<a<1$.

The result (*b*) can be deduced from (*a*) by putting

$$x=-y \text{ so that } y\to\infty \text{ as } x\to-\infty.$$

53·1. Theorem. To prove that

$$\underset{x\to\infty}{lt} \log_a x = \infty \text{ or } -\infty \text{ according as } a> \text{ or } <1,$$

$$\underset{x\to(0+0)}{lt} \log_a x = -\infty \text{ or } \infty \text{ according as } a> \text{ or } <1.$$

This can be deduced from above on writing

$$y=\log_a x \text{ i.e., } x=a^y.$$

54. Infinitesimals. *A variable which tends to zero is called an infinitesimal,* In order to know whether a function is an infinitesimal or not, we must know the independent variable and its limit. For example (x^2-a^2), $e^{-1/(x-a)^2}$ are infinitesimals only when $x\to a$ and $1/x$ is an infinitesimal when $x\to\infty$. An independent variable which tends to zero is also an infinitesimal.

Comparison of infinitesimals. Let $f(x)$, $\phi(x)$ be two infinitesimals. The following cases arise :—

(*i*) $f/\phi\to l$, $l\neq 0$. In this case we say that $f(x)$ is an infinitesimal of the same order as $\phi(x)$ and symbolically write

$$f(x)=O[\phi(x)].$$

In case $l=1$, we say that the two infinitesimals are equivalent and write

$$f(x)\sim\phi(x)$$

If $f(x)=O[\phi^r(x)]$ so that $f(x)/\phi^r(x)\to$a finite non zero limit, we say that $f(x)$ is an infinitesimal of order r with respect to $\phi(x)$.

(*ii*) $f/\phi\to 0$. In this case we say that $f(x)$ is an infinitesimal of higher order and $\phi(x)$ of lower order and write

$$f(x)=0[\phi(x)].$$

(*iii*) $f/\phi\to\infty$ or $-\infty$ so that $\phi/f\to 0$. In this case we have

$$\phi(x)=0[f(x)].$$

(*iv*) f/ϕ oscillates ; we write

$$f(x)=O[\phi(x)].$$

if the oscillation is finite.

Thus

$$2x+x^3=O(x), \text{ if } x\to 0,\ 2x+x^3=O(x^3), \text{ if } x\to\infty,$$

Principal part of an infinitesimal. If an infinitesimal be expressible as a sum of a number of infinitesimals of different orders, then the one of lowest order is called the principal part.

Note. It should be carefully noted that a constant number, however small, is not an infinitesimal. A great deal of confusion has arisen because of the assumption in the older forms of presentation of analysis that there existed numbers so small that they can be neglected.

CHAPTER V

THE DERIVATIVE

55. Derivability. Derivative. Let $f(x)$ be a function defined in an interval (a, b).

55·1. *Derivability at an interior point.* Let c be any interior point of (a, b). We take $c+h$, any other point of the same interval. Then $f(x)$ is said to be derivable at $x=c$, if

$$\lim_{h\to 0}\left\{\frac{f(c+h)-f(c)}{h}\right\},$$

exists; the limit, which is called the derivative of $f(x)$ at $x=c$, is symbolically denoted by $f'(c)$.

One sided derivatives. The

$$\lim_{h\to(0+0)}\{[f(c+h)-f(c)]/h\},$$

in case it exists, is called the *right hand derivative* of $f(x)$ at $x=c$ and is denoted by $\mathrm{R}f'(c)$.

Similarly

$$\lim_{h\to(0-0)}\{[f(c+h)-f(c)]/h\},$$

in case it exists, is called the *left hand derivative* of $f(x)$ at $x=c$ and is denoted by $\mathrm{L}f'(c)$.

55·2. *Derivability at end points.* $f(x)$ is said to be derivable at a, if $\mathrm{R}f'(a)$ exists and $f'(a)$, then, means $\mathrm{R}f'(a)$; it is said to be derivable at b, if $\mathrm{L}f'(b)$ exists and $f'(b)$, then, means $\mathrm{L}f'(b)$.

Note. It is obvious that for an interior point c, $f'(c)$ exists if and only if $\mathrm{R}f'(c)$, $\mathrm{L}f'(c)$ both exist and are equal, and conversely; also, then

$$f'(c)=\mathrm{L}f'(c)=\mathrm{R}f'(c).$$

55·3. *Finitely derivable functions.* A function is said to be finitely derivable at a point, if its derivative at that point exists and is finite.

55·4. *Derivability in an interval. Derived function.* A function is said to be derivable in an interval, if it is derivable at every point thereof. The function determined by the values of the derivatives of $f(x)$ for points of (a, b) is called the derived function of $f(x)$ and is denoted by

$$f'(x),\ df/dx \text{ or } \mathrm{D}_x f(x).$$

Ex. Show that the functions

(*i*) k, a constant. (*ii*) x,

are derivable for every value of x.

56. Theorem. *If a function is finitely derivable at a point, it is also continuous at that point.*

Let $f(x)$ be finitely derivable for $x=c$ so that $\{[f(c+h)-f(c)]/h\}$ tends to a finite limit denoted by $f'(c)$ as $h\to 0$. We have

$$f(c+h)-f(c)=\frac{f(c+h)-f(c)}{h}.h$$

$$\therefore \quad \text{lt}\,[f(c+h)-f(c)]=\text{lt}\frac{f(c+h)-f(c)}{h}.\text{lt}\,h=f'(c).0=0$$

$$\therefore \quad \text{lt}\,f(c+h)=f(c),\text{ when } h\to 0,$$

i.e., $$\text{lt}\,f(x)=f(c),\text{ when } x\to c.$$

Hence $f(x)$ is continuous at $x=c$.

Note. This theorem may also be stated as follows: *The necessary condition for a function to be finitely derivable at a point is that it is continuous at that point.*

The converse of this theorem is not necessarily true *i.e.*, the condition of continuity is not sufficient for derivability.

Consider the derivability of

$$f(x)=|x|.$$

for $x=0$. We have

$$\frac{f(0+h)-f(0)}{h}=\frac{f(h)}{h}=\frac{|h|}{h}=\begin{cases}1, \text{ if } h>0,\\ -1, \text{ if } h<0,\end{cases}$$

$$\therefore \quad Rf'(0)=1 \text{ and } Lf'(0)=-1.$$

Hence $f'(0)$ does not exist.

To examine the continuity, we take any positive number ϵ. We have

$$|f(x)-f(0)|=|x|<\epsilon,\text{ when } |x|\leqslant\delta,$$

δ being any number $<\epsilon$,
and thus $f(x)$ is continuous for $x=0$.

Hence the result.

Ex. Show that $|x+1|+|x|+|x-1|$ is continuous but not derivable for $x=-1$, 0 and 1.

Note. There exist functions which are continuous in an interval but are not derivable for *any* point thereof, but the consideration of such functions is not easy.

Note. The student would do well to remember that the statement

$$f'(c)=l,$$

is equivalent to two distinct statements, *viz.*, (*i*) that $f(x)$ is derivable at c and (*ii*) that the derivative is l.

Note. The existence of the derivative of $f(x)$ for $x=c$ implies that

(*i*) $f(x)$ is defined in a certain neighbourhood of c;

(*ii*) $f(x)$ is continuous at c.

57. Differentiability and differentials. *A function $f(x)$ is said to be differentiable at a point x of the interval of definition (a, b) of $f(x)$, if the change, $[f(x+\delta x)-f(x)]$, in the function which corresponds to the change δx in x is capable of being expressed in the form*

$$\delta f=f(x+\delta x)-f(x)=A\delta x+\epsilon\delta x,$$

where A is independent of δx and ϵ is a function of δx which $\to 0$ as $\delta x\to 0$.

Taking δx as the principal infinitesimal, we see that the principal part of the infinitesimal δf is $A\delta x$. This principal part is called the differential of $f(x)$ and is denoted by $df(x)$ or simply df. If y represents $f(x)$, then the differential of $f(x)$ is denoted by dy.

Theorem. *The necessary and sufficient condition for $f(x)$ to be differentiable at a given point is that it possesses a finite derivative at that point.*

Let $f(x)$ be differentiable at x. We have

$$f(x+\delta x)-f(x)=A\,\delta x+\epsilon\delta x$$

or

$$[f(x+\delta x)-f(x)]/\delta x=A+\epsilon.$$

Let $\delta x\to 0$. In the limit, we see that

$$f'(x)=A,$$

so that $f(x)$ is finitely derivable at x, the derivative being A.

Let $f(x)$ possess a finite derivative $f'(x)$ at x so that

$$\text{lt }\{[f(x+\delta x)-f(x)]/\delta x\}=f'(x) \text{ as } \delta x\to 0.$$

We write

$[f(x+\delta x)-f(x)]/\delta x-f'(x)=\epsilon$, so that $\epsilon\to 0$ as $\delta x\to 0$, and obtain

$$f(x+\delta x)-f(x)=\delta(x)f'(x)+\epsilon\delta x.$$

Thus $f(x)$ is differentiable at x.

By definition, we have

$$dy=A\delta x=f'(x)\delta x, \text{ for } A=f'(x).$$

Taking $y=x$, we see that

$$dx=dy=1.\delta x \text{ or } dx=\delta x,$$

i.e., the differential of an independent variable may be taken equal to the arbitrary increment δx in x Thus we have

$$dy=f'(x)\,dx.$$

Since the derivative $f'(x)$ appears as the coefficient of the differential dx, the derivative $f'(x)$ is often called the *differential co-efficient* of $f(x)$ and the process of obtaining the derivative is called *differentiation.*

58. **Fundamental theorems on Derivation.** If $f_1(x)$, $f_2(x)$ are derivable for $x=c$, then

(*i*) $\phi'(c)=f_1'(c)\pm f_2'(c)$, where $\phi(x)=f_1(x)\pm f_2(x)$.

(*ii*) $\phi'(c)=f_1'(c)f_2(c)+f_2'(c)f_1(c)$. where $\phi(x)=f_1(x).f_2(x)$.

(*iii*) $\phi'(c)=[f_1'(c)f_2(c)-f_2'(c)f_1(c)]/[f_2(c)]^2$, where $\phi(x)=f_1(x)/f_2(x)$ and $f_2(c)\neq 0$.

(*i*) We have

$$\frac{\phi(c+h)-\phi(c)}{h}=\frac{\overline{f_1(c+h)\pm f_2(c+h)}-\overline{f_1(c)\pm f_2(c)}}{h}$$

$$=\frac{f_1(c+h)-f_1(c)}{h}\pm\frac{f_2(c+h)-f_2(c)}{h},$$

The result now follows from §46.

(*ii*) We have

$$\frac{\phi(c+h)-\phi(c)}{h}=\frac{f_1(c+h)f_2(c+h)-f_1(c)f_2(c)}{h}$$

$$=f_2(c+h)\frac{f_1(c+h)-f_1(c)}{h}+f_1(c)\ \frac{f_2(c+h)-f_2(c)}{h}.$$

Since $f_2'(c)$ exists, $f_2(x)$ is continuous for $x=c$, *i.e.*,

$$f_2(c+h)\to f_2(c) \text{ when } h\to 0.$$

The result now follows from §46.

(*iii*) As $f_2(c)$ exists, therefore $f_2(x)$ is continuous at c. Also $f_2(c)\neq 0$. There exists, therefore, an interval $(c-\delta, c+\delta)$, such that $f_2(x)\neq 0$ for any point x of this interval.

Let $(c+h)$ be any point of this interval so that $f_2(c+h)\neq 0$. We have

$$\frac{\phi(c+h)-\phi(c)}{h}=\left[\frac{f_1(c+h)}{f_2(c+h)}-\frac{f_1(c)}{f_2(c)}\right]\Big/h$$
$$=\frac{1}{f_2(c)f_2(c+h)}\left[f_2(c)\frac{f_1(c+h)-f_1(c)}{h}\right.$$
$$\left.-\frac{f_2(c+h)-f_2(c)}{h}f_1(c)\right].$$

The result now follows from §46.

Note. As a particular case of the above we see that if two functions be derivable at a point (or in an interval), then (*i*) their sum, difference and product are also derivable at that point (or in that interval), (*ii*) their quotient is also derivable at that point (or in that interval) provided that the denominator of the quotient is not zero at that point (or at any point of that interval).

59·1. Derivation of function of a function. *If $\phi(t)$ possesses a finite derivative $\phi'(t)$ at a certain point t and $f(x)$ possesses a finite derivative $f'(x)$ at the corresponding point $x=\phi(t)$, then the function $\psi(t)=f[\phi(t)]$ also possesses a derivative at t and*

$$\psi'(t)=f'(x).\phi'(t).$$

We write $\qquad y=\psi(t)=f[\phi(t)].$

With usual notation since $x=\phi(t)$ and $y=f(x)$ possess finite derivatives, therefore

$$\delta x=\phi'(t)\,\delta t+\epsilon_1\delta t,\quad \text{where } \epsilon_1\to 0 \text{ as } \delta t\to 0\ ;$$
$$\delta y=f'(x)\delta x+\epsilon_2\delta x,\ \text{where } \epsilon_2\to 0 \text{ as } \delta x\to 0.$$

From these we obtain

$$\delta y=f'(x)\phi'(t)\delta t+[\epsilon_1 f'(x)+\epsilon_2\phi'(t)+\epsilon_1\epsilon_2]\delta t$$
$$=f'(x)\phi'(x)\delta t+\epsilon_3\,\delta t,\ \text{where } \epsilon_3\to 0 \text{ as } \delta t\to 0$$

Thus y is a differentiable function of t and

$$\psi'(t)=f'(x).\phi'(t),$$

Note. The proof given in elementary text books, which is based on the equality

$$\frac{\delta y}{\delta t}=\frac{\delta y}{\delta x}\cdot\frac{\delta x}{\delta t},$$

is incomplete in as much as no heed is paid to the case which arises when $\delta x=0$ for some point in every neighbourhood of the point t. For examples of such functions refer to Chapter VII.

59·2. Derivation of inverse functions. Let $y=f(x)$ be continuous and strictly monotonic and let $x=g(y)$ be its inverse and let $f(x)$ possess a finite non-zero derivative $f'(x)$, at a point x, then $g(y)$ also possesses a finite derivative at the corresponding point y such that

$$g'(y)=1/f'(x).$$

With usual notation, we have

$$\frac{\delta x}{\delta y}=1\Big/\left(\frac{\delta y}{\delta x}\right).$$

Proceeding to the limit, we obtain the required result.

60. Derivatives of $\log_a x$, a^x, x^n.

60·1. *The function $\log_a x$ is derivable for every value of x for which it is defined, i.e., for positive values of x and its derivative is $\log_a e/x$.* $(a>0)$.

Let
$$y=\log_a x.$$
$$\therefore \qquad \frac{\delta y}{\delta x}=\frac{1}{x}.\log_a\left(1+\frac{\delta x}{x}\right)^{(x/\delta x)}.$$

When $\delta x \to 0$, $\left(1+\frac{\delta x}{x}\right)^{x/\delta x} \to e$ and, since $\log_a t$ is a continuous function of t, therefore
$$lt \log_a\left(1+\frac{\delta x}{x}\right)^{(x/\delta x)}=\log_a e.$$

Thus $lt\ (\delta y/\delta x)$ exists and we have
$$\frac{dy}{dx}=\frac{1}{x}\log_a e.$$

If $y=\log x=\log_e x$, then $dy/dx=1/x$

Cor. If $f(x)$ is positive and derivable, then $\log f(x)$ is also derivable and its derivative is $f'(x)/f(x)$.

Follows from §59·1.

60·2. *The function a^x is derivable for every value of x and its derivative is $a^x \log_e a$. (a 70)*

The function $y=a^x$ is the inverse of the derivable function $x=\log_a y$. Hence y is derivable (§59·2). Also
$$\frac{dy}{dx}=1\Big/\left(\frac{dx}{dy}\right)=1\Big/(\log_a e/y)=y\log_e a=a^x\log_e a.$$

Cor. If $f(x)$ is derivable, then $a^{f(x)}$ is also derivable, $(a>0)$, and its derivative is
$$a^{f(x)}.f'(x)\log_e a. \quad (§59·1)$$

Cor. If $f(x)$, $\phi(x)$ are derivable and $f(x)$ is positive, then $f(x)^{\phi(x)}$ is also derivable.

{Write $f(x)^{\phi(x)}=e^{[\phi(x)\log f(x)]}$}.

60·3. *The function x^n is derivable for every positive value of x and its derivative is nx^{n-1}, n being any real number.*

We have
$$y=x^n=e^{n\log x}$$
and, therefore, by Cor. to §60·2, y is derivable, and
$$\frac{dy}{dx}=e^{n\log x}.\frac{n}{x}=nx^{n-1}.$$

Cor. If $f(x)$ is positive and derivable, then $[f(x)]^n$ is also derivable and its derivative is $n[f(x)]^{n-1}f'(x)$.

Note. If n is a positive integer, then x^n is derivable for every value of x; and if n is a negative integer, then x^n is derivable for every value of x except zero. The proof based on the binomial theorem for a positive integral index, as given in elementary books, is satisfactory for this case.

61. Meaning of the sign of derivative.

Let c be any *interior* point of the interval of definition (a, b) of a function $f(x)$. *Let* $f'(c)>0$.

To each positive number ϵ, there corresponds a positive number δ such that

$$\left| \frac{f(c+h)-f(c)}{h} - f'(c) \right| < \epsilon, \text{ when } 0< |h| \leqslant \delta.$$

i.e., $f'(c)-\epsilon < \{[f(c+h)-f(c)]/h\} < f'(c)+\epsilon$, when $0< |h| \leqslant \delta$.

Giving to ϵ any positive value smaller than the positive number $f'(c)$, we find that

$$[f(c+h)-f(c)]/h>0, \text{ when } 0< |h| \leqslant \delta,$$

i.e., $\quad f(c+h)>f(c)$, when $0<h\leqslant\delta$,

and $\quad f(c+h)<f(c)$, when $-\delta\leqslant h<0$.

Thus we conclude that if $f'(c)$ be positive, then there exists a neighbourhood $(c-\delta, c+\delta)$ of c such that

$$f(x)>f(c), \text{ for any point } x \text{ of } [c, c+\delta),$$

$$f(x)<f(c), \text{ for any point } x \text{ of } (c-\delta, c].$$

Let $f'(c)<0$. We write

$$\phi(x)=-f(x) \text{ so that } \phi'(c)=-f'(c)>0.$$

From above we see that there exists a neighbourhood $(c-\delta, c+\delta)$ of c such that

$$\phi(x)>\phi(c), \textit{ i.e., } f(x)<f(c) \text{ for any point } x \text{ of } [c, c+\delta),$$

$$\phi(x)<\phi(c), \textit{ i.e., } f(x)>f(c) \text{ for any point } x \text{ of } (c-\delta, c].$$

These results may also be obtained independently exactly in the manner in which the first case has been treated.

We now consider *end-points*. It is easy to show that if

(*i*) $f'(a)$ is positive, (negative), there exists an interval $(a, a+\delta)$ such that $f(x)>f(a)$, $[(f(x)<f(a)]$, for any point x of $[a, a+\delta)$.

(*ii*) $f'(b)$ is positive, (negative), there exists an interval $(b-\delta, b)$ such that $f(x)<f(b)$ $[f(x)>f(b)]$ for any point x of $(b-\delta, b]$.

62. Darboux's theorem. *If $f(x)$ is derivable in a closed interval (a, b) and $f'(a)$, $f'(b)$ are of opposite signs, then there exists at least one point 'c' of the interval such that $f'(c)=0$.*

For the sake of definiteness, we suppose that $f'(a)$ is positive and $f'(b)$ negative. On this account there exist intervals $(a, a+\delta)$, $(b-\delta, b)$, $(\delta>0)$, such that

for every point x of $[a, a+\delta)$, $f(x)>f(a)$; $\quad$ (*i*)

and $\quad$ for every point x of $(b-\delta, b]$, $f(x)>f(b)$. $\quad$ (*ii*)

Again $f(x)$, being derivable, is continuous in (a, b). Therefore it is bounded and attains its bounds. Thus if M be the upper bound, there exists a point c such that $f(c)=\text{M}$.

From (*i*) and (*ii*), we see that the upper bound is not attained at the end points a and b so that c is an interior point of (a, b).

If $f'(c)$ be positive, then there exists an interval $(c, c+\eta)$, $(\eta>0)$ such that for every point x of this interval $f(x)>f(c)=\text{M}$ and this is a contradiction;

if $f'(c)$ be negative, then there exists an interval $(c-\eta, c]$, $(\eta>0)$ such that for every point x of this interval $f(x)>f(c)=\text{M}$ and this is a contradiction.

Hence $$f'(c)=0.$$

Cor. If $f(x)$ is derivable in a closed interval (a, b) and $f'(a)\neq f'(b)$ and k is any number lying between $f'(a)$ and $f'(b)$, then there exists at least one point c of the interval such that
$$f'(c)=k.$$

We write $$\phi(x)=f(x)-kx.$$

The function $\phi(x)$ is derivable in (a, b) and $\phi'(a)=f'(a)-k$; $\phi'(b)=f'(b)-k$ are of opposite signs. Therefore there exists at least one point c of (a, b) such that $\phi'(c)=0$, *i.e.*, $f'(c)-k=0$.

63. Rolle's theorem. *If a function $f(x)$ is*

(i) continuous in a closed interval $[a, b]$;

(ii) derivable in the open interval (a, b);

and (iii) $f(a)=f(b)$,

then there exists at least one point c of the open interval (a, b) such that
$$\mathbf{f'(c)=0.}$$

As $f(x)$ is continuous in $[a, b]$, it is bounded and attains its bounds, so that if M, m be the bounds, there exist points c, d such that
$$f(c)=\text{M}, f(d)=m.$$

Now,
$$\text{either M}=m \text{ or M}\neq m.$$

If $\text{M}=m$, the function $f(x)$ is clearly constant throughout $[a, b]$ and its derivative $f'(x)$, therefore, is equal to 0 for every value of x in $[a, b]$. Hence the theorem is proved for this case.

If $\text{M}\neq m$, then at least one of them must be different from the equal values $f(a)$, $f(b)$. Let $\text{M}=f(c)$ be different from them. The number 'c', being different from a and b belongs to the open interval (a, b).

The function $f(x)$ which is derivable in the open interval (a, b) is, in particular, derivable at $x=c$, *i.e.*, $f'(c)$ exists.

If $f'(c)$ be positive, then there exists an interval $[c, c+\delta]$ such that for every point x of this interval $f(x)>f(c)=\text{M}$ and this is a contradiction.

If $f'(c)$ be negative, then there exists an interval $[c-\delta, c)$ such that for every point x of this interval $f(x)>f(c)=\text{M}$ and this is a contradiction.

Hence $$f'(c)=0.$$

[The vanishing of $f'(c)$ may also be shown as follows :—

We have $$\text{R}f'(c)=\text{L}f'(c)=f'(c).$$

Also $$f(c+h)-f(c)\leqslant 0,$$
for every point $(c+h)$ of (a, b).

$\therefore$ $$[f(c+h)-f(c)]/h\leqslant 0, \text{ when } h>0 \qquad \text{I}$$

and $$[f(c+h)-f(c)]/h\geqslant 0, \text{ when } h<0. \qquad \text{II}$$

From I, we have, $\text{R}f'(c)\leqslant 0$, *i.e.*, $f'(c)\leqslant 0$

From II, we have, $\text{L}f'(c)\geqslant 0$, *i.e.*, $f'(c)\geqslant 0$.

Hence $$f'(c)=0.$$

64. Lagrange's mean value theorem. *If a function $f(x)$ is*
(i) continuous in a closed interval (a, b);
and (ii) derivable in the open interval $[a, b]$;
then there exists at least one point 'c' of the open interval $[a, b]$ such that

$$\frac{f(b)-f(a)}{b-a}=f'(c).$$

Let a function $\phi(x)$ be defined by

$$\phi(x)=f(x)+Ax,$$

where A is a constant to be determined such that

$$\phi(a)=\phi(b).$$

This requires

$$f(a)+Aa=f(b)+Ab.$$

$$\therefore \quad A=-[f(b)-f(a)]/(b-a).$$

The function $\phi(x)$ is continuous in (a, b), derivable in $[a, b]$, and $\phi(a)=\phi(b)$. Hence, by Rolle's theorem, there exists at least one point 'c' of $[a, b]$ such that $\phi'(c)=0$.

But
$$\phi'(x)=f'(x)+A.$$

$$\therefore \quad 0=\phi'(c)=f'(c)+A,$$

or
$$f'(c)=-A=\frac{f(b)-f(a)}{b-a}. \quad a<c<b.$$

Another form of statement. If a function $f(x)$ is, (*i*) continuous in $(a, a+h)$, (*ii*) derivable in $[a, a+h]$, then there exists atleast one number θ between 0 and 1 such that

$$f(a+h)-f(a)=hf'(a+\theta h). \quad (0<\theta<1).$$

65. Theorem. *If $f(x)$ is continuous at a point 'c' and*

$$\lim_{x\to c} f'(x)=l,$$

then
$$f'(c)=l.$$

The condition that $f'(x)\to l$ as $x\to(c+0)$ implies that there exists an interval $[c, c+h)$, $(h>0)$ for every point x of which $f'(x)$ exists and therefore, $f(x)$ is continuous. Since $f(x)$ is given to be continuous at c also, we see that $f(x)$ is continuous in the closed interval $(c, c+h)$. If x be any point of this interval, we have

$$f(x)-f(c)=(x-c)f'(\xi), \quad c<\xi<x$$

or
$$\frac{f(x)-f(c)}{x-c}=f'(\xi).$$

Let $x\to(c+0)$. Then we have

$$Rf'(c)=\lim_{x\to(c+0)} f'(\xi)=\lim_{x\to(c+0)} f'(x)=l.$$

It may similarly be shown that

$$Lf'(c)=l.$$

Thus $f'(c)$ exists and is equal to l.

66. Some elementary deductions from the mean value theorem. It will be assumed that the function $f(x)$ is continuous in the closed interval (a, b) and derivable in the open interval $[a, b]$ so that the mean value theorem is applicable to the interval (a, b) and, therefore, also to any sub-interval thereof.

66·1. Theorem. *If $f'(x)=0$ for every point x of $[a, b]$, then $f(x)$ is a constant in (a, b).*

Let x be any number such that

$$a<x\leqslant b.$$

Applying the mean value theorem to (a, x), we get

$$f(x)-f(a)=(x-a)f'(\xi) \qquad \text{where } a<\xi<x$$
$$=0,$$

or $$f(x)=f(a).$$

Hence the result.

Cor. If $f'(x)=k$, for every point x of $[a, b]$, k being a constant, then $f(x)$ is of the form $kx+l$, where l is a constant.

If x be any number such that $a<x\leqslant b$, then we have

$$f(x)-f(a)=(x-a)f'(\xi), \text{ where } a<\xi<x$$
$$=k(x-a)$$

or $$f(x)=kx+l, \text{ where } l=f(a)-ak.$$

As $f(x)$ is continuous at a,

$$f(a)=\lim_{x\to a} f(x)=ka+l,$$

so that the result is true for $x=a$ also.

Cor. If two functions $f(x)$, $F(x)$ are (*i*) continuous in (a, b) (*ii*) derivable in $[a, b]$ and (*iii*) $f'(x)=F'(x)$ in $[a, b]$; then $f(x)$ and $F(x)$ differ by a constant.

Let $$\phi(x)=f(x)-F(x).$$
Then $$\phi'(x)=f'(x)-F'(x)=0.$$
$\therefore$ $$\phi(x)=\text{a constant}.$$

66·2. Theorem. *If $f'(x)>0$ for every point x of $[a, b]$, then $f(x)$ is strictly increasing in (a, b).*

Let x_1, x_2 be any two numbers such that

$$a\leqslant x_1<x_2\leqslant b.$$

We have

$$f(x_2)-f(x_1)=(x_2-x_1)f'(\xi), \qquad x_1<\xi<x_2$$
$$>0$$

$\therefore$ $$f(x_2)>f(x_1).$$

Hence the result.

Cor. If $f'(x)\geqslant 0$ in $[a, b]$ and does not vanish throughout any sub-interval of (a, b) then $f(x)$ is *strictly* increasing in (a, b).

If x_1, x_2 be *any* two numbers such that

$$a\leqslant x_1<x_2\leqslant b,$$

we have

$$f(x_2)-f(x_1)=(x_2-x_1)f'(\xi)\geqslant 0.$$

$\therefore$ $$f(x_2)\geqslant f(x_1),$$

so that $f(x)$ is a *monotonically* increasing function. We have now to show that no two values of the function can be equal. If possible, let

$$f(\alpha)=f(\beta)$$

where $$a\leqslant\alpha<\beta\leqslant b.$$

For any x in (α, β), we have

$$f(\alpha)\leqslant f(x)\leqslant f(\beta)=f(\alpha), \text{ or, } f(x)=f(\alpha)$$

i. e., $f(x)$ remains constant in (α, β). Therefore $f'(x)$ vanishes throughout (α, β) and this is a contradiction.

66·3. Theorem. *If $f'(x)<0$ for every point x of $[a, b]$, or if $f'(x)\leqslant 0$ in $[a, b]$ and does not vanish in any sub-interval of (a, b) then $f(x)$ is strictly increasing in (a, b).*

The proof is similar to that of the last case.

67. Cauchy's mean value theorem. *If two functions $f(x)$ and $F(x)$ are*

(i) continuous in a closed interval (a, b) ;

(ii) derivable in the open interval $[a, b]$;

and (iii) $F'(x)\neq 0$ for any point x in the open interval $[a, b]$,

then

there exists atleast one point 'c' of the open interval $[a, b]$ such that

$$\frac{f(b)-f(a)}{F(b)-F(a)}=\frac{f'(c)}{F'(c)}. \qquad a<c<b.$$

Let a function $\phi(x)$ be defined by

$$\phi(x)=f(x)+AF(x),$$

where A is a constant to be determined such that

$$\phi(a)=\phi(b).$$

This requires

$$[F(b)-F(a)]A= -[f(b)-f(a)]. \tag{1}$$

Now, $[F(b)-F(a)]\neq 0$, for, if it were 0, then $F(x)$ would satisfy all the conditions of Rolle's theorem, and its derivative would, therefore, vanish atleast once in $[a, b]$ and the condition (*iii*) would be contradicted. On this account, we have from (1),

$$A= -[f(b)-f(a)]/[F(b)-F(a)].$$

The function $\phi(x)$ is continuous in (a, b), derivable in $[a, b]$, and $\phi(a)=\phi(b)$. Hence, by Rolle's theorem, there exists atleast one point 'c' of $[a, b]$ such that $\phi'(c)=0$.

But $$\phi'(x)=f'(x)+AF'(x).$$

$\therefore$ $$0=\phi'(c)=f'(c)+AF'(c),$$

or $$\frac{f'(c)}{F'(c)}= -A=\frac{f(b)-f(a)}{F(b)-F(a)}, \qquad \because F'(c)\neq 0.$$

Another form of statement. If two functions $f(x)$, $F(x)$ are continuous in $(a, a+h)$, derivable in $[a, a+h]$ and $F'(x)\neq 0$ in $[a, a+h]$, then there exists at least one number θ, between 0 and 1 such that

$$\frac{f(a+h)-f(a)}{F(a+h)-F(a)}=\frac{f'(a+\theta h)}{F'(a+\theta h)}, \qquad 0<\theta<1.$$

Note: Taking $F(x)=x$, we may see that Lagrange's mean value theorem is only a particular case of Cauchys' theorem.

68. Higher derivatives. Let $f(x)$ be derivable *i. e.*, let $f'(x)$ exist in a certain neighbourhood of c. This implies that $f(x)$ is defined and continuous in a neighbourhood of c. If the function $f'(x)$ has a derivative at c, then this derivative is called the second derivative of $f(x)$ at c, and is denoted by $f''(c)$. In this case, $f'(x)$ is necessarily continuous at c.

In general, if $f^{n-1}(x)$ exists in a neighbourhood of c, then the derivative of $f^{n-1}(x)$ at c, in case it exists, is called the nth derivative of $f(x)$ at c and is denoted by $f^n(c)$.

69. Taylor's theorem. *If a function $f(x)$ is such that*

(*i*) *the $(n-1)$th derivative $f^{n-1}(x)$ is continuous in a closed interval $(a, a+h)$,*

(*ii*) *the nth derivative $f^n(x)$ exists in the open interval $[a, a+h]$, and* (*iii*) *p is any given positive integer,*

then there exists at least one number θ between 0 *and* 1 *such that*

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$
$$+\frac{h^n(1-\theta)^{n-p}}{(n-1)!\,.\,p}.f^n(a+\theta h). \quad \ldots\ldots(1)$$

The condition (*i*) implies the continuity of

$$f(x), f'(x), f''(x), \ldots\ldots f^{n-2}(x) \text{ in } (a, a+h).$$

Let a function $\phi(x)$ be defined by

$$\phi(x)=f(x)+(a+h-x)f'(x)+\frac{(a+h-x)^2}{2!}f''(x)+\ldots\frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x)$$
$$+A(a+h-x)^p,$$

where A is a constant to be determined such that

$$\phi(a)=\phi(a+h).$$

Thus A is given by

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+Ah^p \quad \ldots(2).$$

The function $\phi(x)$ is continuous in $(a, a+h)$, derivable in $[a, a+h]$ and $\phi(a)=\phi(a+h)$. Hence, by Rolle's theorem, there exists atleast one number θ between 0 and 1 such that

$$\phi'(a+\theta h)=0.$$

But $$\phi'(x)=\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)-pA(a+h-x)^{p-1}.$$

$$\therefore \quad 0=\phi'(a+\theta h)=\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)-pA(1-\theta)^{p-1}.h^{p-1}$$

or $$A=\frac{h^{n-p}(1-\theta)^{n-p}}{p.(n-1)!}.f^n(a+\theta h), \text{ for } (1-\theta)\neq 0 \text{ and } h\neq 0.$$

Substituting this value of f in (2), we get the required result (1).

(*i*) *Remainder after n terms.* The term

$$R_n=\frac{h^n(1-\theta)^{n-p}}{p.(n-1)!}.f^n(a+\theta h),$$

is known as the Taylor's remainder R_n after n terms and is due to *Schlomilch and Roche.*

(*ii*) Putting $p=1$, we obtain

$$R_n=\frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h),$$

which form of remainder is due to *Cauchy.*

(*iii*) Putting $p=n$, we obtain

$$R_n=\frac{h^n}{n!}f^n(a+\theta h),$$

which is due to *Lagrange.*

Cor. Let x be *any* point of the interval $(a, a+h)$.

Let $f(x)$ satisfy the conditions of Taylor's theorem for $(a, a+h)$. Then it satisfies the conditions for (a, x) also.

Changing $a+h$ to x *i. e.*, h to $x-a$, in (1), we obtain

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\dots\dots\dots$$

$$+\dots\dots+\frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{(x-a)^n(1-\theta)^{n-p}}{p.(n-1)!}f^n[a+\theta(x-a)],$$

$$0<\theta<1.$$

This result holds for every point x of $(a, a+h)$. Of course, θ, may be different for different points x.

Cor. Maclaurin's theorem. Putting $a=0$, we see that if x is any point of the interval $(0, h)$, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)\quad+\frac{x^n(1-\theta)^{n-p}}{p.(n-1)!}f^n(\theta x),$$

which holds when

(*i*) $f^{n-1}(x)$ is continuous in $(0, h)$,

and (*ii*) $f^n(x)$ exists in $[0, h]$.

Putting $n=1$ and $n=p$, respectively in the Schlomilch form of remainder

$$\frac{x^n(1-\theta)^{n-p}f^n(\theta x)}{p.(n-1)!},$$

we see that Cauchy's and Lagrange's forms are respectively

$$\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)\ \text{ and }\ \frac{x^n}{n!}f^n(\theta x).$$

70. Taylor's infinite series. Suppose that a given function $f(x)$ possesses a continuous derivative of *every* order in $(a, a+h)$.

Then to every positive integer n, however large it may be, there corresponds a result of the form

$$f(a+h)=f(a)+hf'(a)+\dots\dots\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+R_n,$$

where R_n denotes Taylor's remainder after n terms.

We write

$$S_n=f(a)+hf'(a)+\dots\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a),$$

so that
$$f(a+h)=S_n+R_n.$$

If $R_n\to 0$, as $n\to\infty$, we have

$$\text{lt } S_n=f(a+h),$$

so that the series

$$f(a)+hf'(a)+\dots\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}f^n(a)+\dots\dots\dots$$

converges and its sum is equal to $f(a+h)$.

Thus we have proved that if

(*i*) *$f(x)$ possesses continuous derivatives of every order in* $(a, a+h)$, *and* (*ii*) *Taylor's remainder* $R_n \to 0$ *as* $n \to \infty$, *then*

$$f(a+h)=f(a)+hf'(a)+\ldots\ldots+\frac{h^n}{n\,!}f^n(a)+\ldots\ldots$$

Note. The infinite series

$$f(a)+hf'(a)+\ldots\ldots+\frac{h^n}{n\,!}f^n(a)+\ldots\ldots$$

is known as *Taylor's series.*

It should be clearly understood that the *mere convergence of this series does not mean that its sum is equal to* $f(a+h)$. (See § 80, Page 104)

70.1. Maclaurin's infinite series. From above we deduce that *if* $f(x)$ *possesses a continuous derivative of every order in* $(0, x)$ *and* $R_n \to 0$ *as* $n \to \infty$, *then*

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\ldots\ldots+\frac{x^n}{n\,!}f^n(0)+\ldots\ldots$$

71. Maclaurin's expansions of e^x, log $(1+x)$, $(1+x)^n$.

71.1. *Let* $f(x)=e^x$. We know that e^x possesses continuous derivatives of every order for every value of x. In fact, $f^n(x)=e^x$.

If R_n denotes Lagrange's remainder, we have

$$R_n=\frac{x^n}{n\,!}\;e^{\theta x}.$$

If x be positive, then $e^{\theta x}<e^x$; and if x be negative, $e^{\theta x}<1$. Thus

$$|R_n|=\left|\frac{x^n}{n\,!}\right|\cdot e^{\theta x}<\begin{cases}(x^n/n\,!)\,.\,e^x, & \text{if } x>0.\\ (x^n/n\,!) & \text{, if } x<0.\end{cases}$$

Since $|x^n/n\,!|\to 0$ as $n\to\infty$, we see that $R_n\to 0$ as $n\to\infty$, for every value of x.

The validity of Maclaurin's infinite expansion for e^x has thus been established for every value of x; making substitutions, we obtain

$$e^x=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots\ldots+\frac{x^n}{n\,!}+\ldots\ldots$$

71.2. *Let* $f(x)=\log\,(1+x)$.

We know that log $(1+x)$ possesses continuous derivatives of every order for every value of x such that $(1+x)$ is positive, *i. e.*, $x>-1$. In fact, when $x>-1$, we have

$$f^n(x)=\frac{(-1)^{n-1}\,(n-1)\,!}{(1+x)^n}.$$

If R_n denotes Lagrange's form of remainder, we have

$$R_n=\frac{x^n}{n\,!}f^n\,(\theta x)=(-1)^{n-1}.\frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n.$$

Let $0\leqslant x\leqslant 1$ so that $x/(1+\theta x)$ is positive and <1. We have

$$|R_n|<\tfrac{1}{n}$$

Therefore $R_n\to 0$, when $0\leqslant x\leqslant 1$.

Let $-1<x<0$. In this case $x/(1+\theta x)$ may not be numerically less than unity so that we fail to draw any conclusion from Lagranges' R_n in this case. Taking Cauchy's form of remainder, we have

$$R_n=\frac{x^n}{(n-1)\,!}(1-\theta)^{n-1}\,f^n(\theta x)=(-1)^{n-1}x^n.\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\cdot\frac{1}{1+\theta x}$$

Since $|x|<1$, $[(1-\theta)/(1+\theta x)]$ is positive and less than unity and hence

$$0<\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}<1.$$

Also $$\frac{1}{1+\theta x}<\frac{1}{1-|x|}.$$

Finally $$x^n\to 0 \text{ as } n\to\infty.$$

Thus $R_n\to 0$, as $n\to\infty$.

We have thus established that the Maclaurin's expansion for $\log(1+x)$ is valid when $-1<x\leqslant 1$; making substitutions, we obtain

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\cdots\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots\cdots$$

71·3. *Let* $f(x)=(1+x)^m$.

If m is a positive integer, then we know that $f^{m+1}(x)$ and the following successive derivatives are identically zero for every value of x, so that we may easily show that, *when m is a positive integer,*

$$(1+x)^m=1+mx+\frac{m(m-1)}{2\,!}x^2+\cdots\cdots+x^m,$$

for every value of x.

If m is any real number, not necessarily a positive integer, then we know that $(1+x)^m$ possesses continuous derivatives of every order for values of x such that $(1+x)$ is positive, *i.e.*, $x>-1$. We have

$$f^n(x)=\frac{m(m-1)\cdots\cdots(m-n+1)}{n\,!}(1+x)^{m-n}$$

If R_n denotes Cauchy's form of remainder, we have

$$R_n=\frac{x^n}{(n-1)\,!}(1-\theta)^{n-1}f^n(\theta x)$$

$$=\frac{m(m-1)\ldots(m-n+1)}{(n-1)\,!}x^n\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}(1+\theta x)^{m-1}$$

If $|x|<1$,

$$\frac{m(m-1)\ldots(m-n+1)}{(n-1)\,!}x^n\to 0,$$

$$0<\left(\frac{1-\theta}{1-\theta x}\right)^{n-1}<1,$$

$$(1+\theta x)^{m-1}<\begin{cases}(1+1)^{m-1}=2^{m-1}, & \text{if } (m-1) \text{ is positive}\\ (1-|x|)^{m-1}, & \text{if } (m-1) \text{ is negative.}\end{cases}$$

Thus $R_n \to 0$ as $n \to \infty$, when $|x| < 1$.

Maclaurin's infinite expansion for $(1+x)^m$ being thus valid when $|x| < 1$, we see that

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots\ldots + \frac{m(m-1)\ldots(m-n+1)}{n!}x^n + \ldots,$$

when $|x| < 1$.

Note. It is easy to show that we cannot prove that the *Lagranges'* form of $R_n \to 0$, when $-1 < x < 0$.

74. Young's form of Taylor's theorem. *If a function $f(x)$ be such that $f^n(a)$ exists and M be defined as a function of h by the equation*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}M,$$

then $$M \to f^n(a) \text{ as } h \to 0.$$

The fact that $f^n(a)$ exists implies that $f(x), f'(x), f''(x), \ldots, f^{n-1}(x)$ all exist in a certain neighbourhood $(a-\delta, a+\delta)$ of a.

The result holds good whether $f^n(a)$ be finite or infinite.

Case I. *Let $f^n(a)$ be finite.*

Let ϵ be any positive number.

Firstly we take $h > 0$. We define a function $\phi(h)$ as follows :—

$$\phi(h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots\ldots\ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}[f^n(a)+\epsilon] - f(a+h).$$

We easily see that

$\phi(0) = 0, \ \phi'(0) = 0, \ldots\ldots, \ \phi^{n-1}(0) = 0$ and $\phi^n(0) = \epsilon > 0$.

Since $\phi^n(0)$ is positive and $\phi^{n-1}(0) = 0$, we see that there exists an interval $[0, \delta_1]$ such that for every point h of this interval $\phi^{n-1}(h)$ is positive. (§ 61, Page 87).

Since $\phi^{n-1}(h)$ is positive in $[0, \delta_1]$ and $\phi^{n-2}(0) = 0$, we see that $\phi^{n-2}(h)$ is positive in $[0, \delta_1]$ (§ 66·2, Page 90).

Now successively applying the theorem of § 66·2, we deduce that $\phi(h)$ is positive in $[0, \delta_1]$.

Thus we see that there exists a positive number δ_1 such that for $0 < h < \delta_1$,

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}[f^n(a)+\epsilon] - f(a+h) > 0$$

Similarly we may prove that there exists a positive number δ_2 such that for $0 < h < \delta_2$

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}[f^n(a)-\epsilon] - f(a+h) < 0$$

But, as given,

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}M - f(a+h) = 0.$$

Let $\eta = \min(\delta_1, \delta_2)$

From above we deduce that corresponding to every positive number ϵ there exists a positive number η, such that

$$f^n(a)-\epsilon < M < f^n(a)+\epsilon, \text{ i.e., } |M-f^n(a)| < \epsilon, \text{ when } 0<h<\eta.$$

$$\therefore \quad \lim_{h\to(0+0)} M = f^n(a).$$

Taking negative values of h, we may similarly show that

$$\lim_{h\to(0-0)} M = f^n(a).$$

Hence $\quad M \to f^n(a)$ as $h\to 0$.

Case II. *Let $f^n(a)$ be infinite.*

Let Δ be any positive number, however large. Firstly we take $h>0$, and define a function $\phi(h)$ as follows :—

$$\phi(h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}\Delta-f(a+h)$$

We easily see that

$$\phi(0)=0,\ \phi'(0)=0, \ldots\ldots \phi^{n-1}(0)=0.$$

Also since

$$\phi^{n-1}(h)=f^{n-1}(a)+h\Delta-f^{n-1}(a+h),$$
$$=h\left[\Delta-\frac{f^{n-1}(a+h)-f^{n-1}(a)}{h}\right]$$

and $[f^{n-1}(a+h)-f^{n-1}(a)]/h \to\infty$ as $h\to 0$, we see that there exists a positive number δ such that for $0<h\leqslant\delta$, $\phi^{n-1}(h)$ is negative.

Now, proceeding as in case I, we prove that there exists a positive number δ_1 such that for every point h of $(0, \delta_1)$.

$$\phi(h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}\Delta-f(a+h)<0$$

Also, as given,

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}M-f(a+h)=0.$$

From above we deduce that corresponding to every positive number Δ, however large, there exists a positive number δ_1 such that for every point h of $[0, \delta_1)$,

$$M>\Delta \quad \text{i.e.,} \quad \lim_{h\to(0+0)} M=\infty$$

It may similarly be shown that

$$\lim_{h\to(0-0)} M=\infty.$$

Hence $\quad M\to\infty$ as $h\to 0$.

75. The application of Taylor's theorem.

The application of Taylor's theorem in a finite form to the problem of *Extreme* values of a function and to the problem of the evaluation of a certain special kind of limits popularly known as *Indeterminate forms* is given in the following three sections.

76 Extreme values of a function. Maxima and minima.

If c be any *interior* point of the interval of definition (a, b) of a function $f(x)$, then

(i) $f(c)$ is said to be a *maximum* value of $f(x)$, if there exists some neighbourhood $(c-\delta, c+\delta)$ of c such that for every point x of this neighbourhood, other than c,

$$f(c)>f(x);$$

(*ii*) $f(c)$ is said to be a ***minimum*** value of $f(x)$, if for every point x, other than c, of some neighbourhood $(c-\delta, c+\delta)$ of c,

$$f(c)<f(x);$$

(*iii*) $f(c)$ is said to be an *extreme* value of $f(x)$, if it is either a maximum or a minimum value.

For $f(c)$ to be an extreme value,

$$f(c)-f(x)$$

must keep the same sign for every point x, other than c, of some neighbourhood $(c-\delta, c+\delta)$ of c.

76·1 Theorem. *If $f(c)$ be an extreme value of a function $f(x)$, then $f'(c)$, in case it exists, is zero.*

If $f'(c)$ be not 0, then in every neighbourbood of c there exist points x for which $f(x)>f(c)$ and points x for which $f(x)<f(c)$, (§61. Page 87) so that $f(c)$ can not be an extreme value.

Note 1. The theorem may also be stated a little differently as follows :—

The necessary condition for $f(c)$ to be an extreme value is that $f'(c)=0$, in case it exists.

To show that this condition is only necessary and not sufficient, we consider the function $f(x)=x^3$ when $x=0$.

Clearly

$$f'(0)=0.$$

Also, when $x>0, f(x)>f(0)$, and when $x<0, f(x)<f(0)$, so that $f(0)$ is not an extreme value even though $f'(0)=0$.

Note. 2. If $f(x)=|x|$, then clearly $f(0)$ is a minimum value and $f'(0)$ does not exist. (Note § 56, Page 83).

This example shows that $f(c)$ may be an extreme value even when $f'(c)$ does not exist.

76·2 Criteria for extreme values. *Let c be an interior point of the interval of definition (a, b) of a function $f(x)$. Let*

(*i*) *$f^n(c)$ exist and be not zero,*

and (*ii*) *$f'(c)=f''(c)=f'''(c)=.........=f^{n-1}(c)=0$;*

then $f(c)$ is not an extreme value if n is odd; and if n be even, $f(c)$ is a maximum or a minimum value according as $f^n(c)$ is negative or positive.

The condition (*i*) implies that $f'(x), f''(x),......, f^{n-1}(x)$ all exist in a certain neighbourhood, $(c-\delta_1, c+\delta_1)$, of c. I

As $f^n(c)$ exists and $\neq 0$, there exists a neighbourhood $(c-\delta, c+\delta)$ of c, $(0<\delta<\delta_1)$ such that

$$\left.\begin{array}{l} f^{n-1}(x)<f^{n-1}(c)=0, \text{ when } c-\delta\leq x<c \\ f^{n-1}(x)>f^{n-1}(c)=0, \text{ when } c<x\leq c+\delta, \end{array}\right\} (\S 61.\text{ P. } 87), \text{II}$$

in case $f^n(c)$ is positive;

$$\text{and}\quad \left.\begin{array}{l} f^{n-1}(x)>f^{n-1}(c)=0, \text{ when } c-\delta\leq x<c \\ f^{n-1}(x)<f^{n-1}(c)=0, \text{ when } c<x\leq c+\delta, \end{array}\right\} (\S 61.\text{P. } 87), \text{III}$$

in case $f^n(c)$ is negative.

Because of I, we have, by Taylor's theorem, when $|h|\leq\delta$,

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{2!}f''(c)+......+\frac{h^{n-1}}{(n-1)!}f^{n-1}(c+\theta h),$$

which, by virtue of (*ii*), gives

$$f(c+h)-f(c)=\frac{h^{n-1}}{(n-1)!}f^{n-1}(c+\theta h) \qquad \text{IV}$$

where $c+\theta h$ belongs to the interval $(c-\delta, c+\delta)$.

Let n be even. From II and IV, we deduce that if $f^n(c)$ be >0, then for every point $x=c+h$ of $(c-\delta, c+\delta)$, other than c,

$$f(c+h)>f(c),$$

i.e. $f(c)$ is a minimum.

From III and IV, it may similarly be shown that $f(c)$ is a maximum if $f^n(c)$ be <0.

Let n be odd. From II and IV, we deduce that if $f^n(c)>0$, then

$$f(c+h)>f(c), \text{ when } c<x=c+h\leqslant c+\delta$$

and $$f(c+h)<f(c), \text{ when } c-\delta\leqslant x=c+h<c$$

so that $f(c)$ is no extreme value.

It may similarly be shown that $f(c)$ is not an extreme value when $f^n(c)<0$.

Another proof. We now have another proof which is dependent upon the Young's form of Taylor's theorem.

We have

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{2!}f''(c)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(c)+\frac{h^n}{n!}M,$$

where $$M\to f^n(c) \text{ as } h\to 0.$$

With the help of (*ii*), it becomes

$$f(c+h)-f(c)=\frac{h^n}{n!}M. \qquad \text{V}$$

Since $M\to f^n(c)$ as $h\to 0$, there exists a positive number δ, such that for $0<|h|\leqslant\delta$, M has the same sign as $f^n(c)$.

From V, we now deduce that

when n is even $f(c+h)-f(c)$ has the same sign as $f^n(c)$ for $0<|h|\leqslant\delta$, so that $f(c)$ is a maximum or minimum according as $f^n(c)$ is negative or positive, and

when n is odd, $f(c+h)-f(c)$ changes sign with the change in the sign of h so that $f(c)$ is not an extreme value.

Ex. If c be an interior point of the interval of definition of a function $f(x)$ then

$f(c)$ is a maximum value of $f(x)$, if

$$Rf'(c)=-\infty,\ Lf'(c)=\infty,$$

and $f(c)$ is a minimum value of $f(x)$, if

$$Rf'(c)=\infty,\ Lf'(c)=-\infty.$$

77. The Indeterminate form, 0/0.

77·1 Theorem. *Let $f(x)$, $g(x)$ be two functions such that*

(*i*) $$\lim_{x\to c} f(x)=0, \quad \lim_{x\to c} g(x)=0,$$

and (*ii*) *$f'(c)$, $g'(c)$ both exist and $g'(c)\neq 0$,*

then $$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{f'(c)}{g'(c)}.$$

Since $f(x)$, $g(x)$ are derivable at c, therefore they are continuous at c and accordingly $f(c)=g(c)=0$.

We have

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=\lim_{h\to 0}\frac{f(c+h)}{h},$$

$$g'(c)=\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}=\lim_{h\to 0}\frac{g(c+h)}{c}.$$

$$\therefore \quad \frac{f'(c)}{g(c)} = \lim_{h\to 0} \frac{f(c+h)/h}{g(c+h)/h} = \lim_{h\to 0} \frac{f(c+h)}{g(c+h)} = \lim_{x\to c} \frac{f(x)}{g(x)}.$$

Note. This result may also be stated as follows :—If $f(c)=g(c)=0$, and $f'(c)$, $g'(c)$ exist but $g'(c)\neq 0$, then $\lim [f(x)/g(x)]=f'(c)/g'(c)$, when $x\to c$.

77·2. *Theorem. Let $f(x)$, $g(x)$ be two functions such that*

$$(i) \qquad \lim_{x\to c} f(x)=0, \quad \lim_{x\to c} g(x)=0 \qquad (i)$$

and (ii) $\lim [f'(x)/g'(x)]=l$, *when* $x\to c$. (ii)

then $\lim [f(x)/g(x)]=l$, *when* $x\to c$.

The condition (ii) implies that $f'(x)$ and $g'(x)$ exist and $g'(x)\neq 0$ at every point x, other than c, of a certain neighbourhood $(c-\delta, c+\delta)$ of c.

We *suppose* that $f(c)=g(c)=0$, for this change in the definition of $f(x)$ and $g(x)$ influences neither the hypothesis nor the conclusion of the theorem.

If x be any point of $(c-\delta, c+\delta)$, we have, by Cauchy's theorem,

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(c)}{g(x)-g(c)} = \frac{f'(\xi)}{g'(\xi)},$$

where ξ lies between c and x and also depends upon x.

By virtue of (ii), $f'(\xi)/g'(\xi)\to l$ as $x\to c$.

$\therefore$ $f(x)/g(x)\to l$ as $x\to c$.

Note. The reader will find it useful to compare the hypotheses and the conclusions of the two preceding theorems. In §77·1, the derivability of $f(x)$ and $g(x)$ is assumed at $x=c$ *only* whereas in §77·2, while assuming the derivability of $f(x)$, $g(x)$ in the neighbourhood of $x=c$, it is exactly at $x=c$ where we have not needed it. (For illustrations of each of these theorems, see examples 2 and 5 at the end of ch. VII.)

77·3 Theorem. *Let $f(x)$, $g(x)$ be two functions such that, when $x\to c$,*

$$(i) \quad \begin{cases} \lim f(x)=\lim f'(x)=\ldots\ldots=\lim f^{n-1}(x)=0, \\ \lim g(x)=\lim g'(x)=\ldots\ldots=\lim g^{n-1}(x)=0, \end{cases}$$

and (ii) $\lim [f^n(x)/g^n(x)]=l$,

then $\lim [f(x)/g(x)]=l$, *when* $x\to c$.

Since $\lim f^{n-1}(x)=\lim g^{n-1}(x)=0$ and $\lim [f^n(x)/g^n(x)]=l$,

therefore $\lim [f^{n-1}(x)/g^{n-1}(x)]=l$, §77·2 above

Again, since $\lim f^{n-2}(x)=\lim g^{n-2}(x)=0$

and $\lim [f^{n-1}(x)/g^{n-1}(x)]=l$,

therefore $\lim [f^{n-2}(x)/g^{n-2}(x)]=l$.

Proceeding in this manner, we finally prove that

$$\lim [f(x)/g(x)]=l,$$

when $x\to c$.

77·4. Theorem. *Let $f(x)$, $g(x)$ be two functions such that when $x\to c$*

$$(i) \quad \begin{cases} \lim f(x)=\lim f'(x)=\ldots\ldots=\lim f^{n-1}(x)=0 \\ \lim g(x)=\lim g'(x)=\ldots\ldots=\lim g^{n-1}(x)=0 \end{cases}$$

and (ii) *$f^n(c)$, $g^n(c)$ exist and $g^n(c)\neq 0$,*

then $\lim [f(x)/g(x)]=f^n(c)/g^n(c)$, *when* $x\to c$.

By virtue of (ii), on applying the theorem of §77·1, we see that

$$\lim \frac{f^{n-1}(x)}{g^{n-1}(x)} = \frac{f^n(c)}{g^n(c)}.$$

Also from theorem of §77·3, changing n to $(n-1)$, we see that

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f^{n-1}(x)}{g^{n-1}(x)} = \frac{f^n(c)}{g^n(c)}.$$

77·5. Theorem. *Let $f(x)$, $g(x)$ be two functions such that*

(*i*) $f(c)=f'(c)=f''(c)=\ldots\ldots=f^{n-1}(c)=0,$

$g(c)=g'(c)=g''(c)=\ldots\ldots=g^{n-1}(c)=0$

and (*ii*) $f^n(c)$, $g^n(c)$ *exist but* $g^n(c)\neq 0$,

then $\text{lt}\ [f(x)/g(x)]=f^n(c)/g^n(c)$, *when* $x\to c$.

Employing Young's form of Taylor's theorem, we have

$$f(c+h)=f(c)+hf'(c)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(c)+\frac{h^n}{n\,!}M=\frac{h^n}{n\,!}M$$

$$g(c+h)=g(c)+hg'(c)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}g^{n-1}(c)+\frac{h^n}{n\,!}M'=\frac{h^n}{n\,!}M'$$

$$\therefore \qquad \frac{f(c+h)}{g(c+h)}=\frac{M}{M'}$$

$$\therefore \qquad \underset{x\to c}{\text{lt}}\frac{f(x)}{g(x)}=\underset{h\to 0}{\text{lt}}\frac{f(c+h)}{g(c+h)}=\underset{h\to 0}{\text{lt}}\frac{M}{M'}=\frac{f^n(c)}{g^n(c)}.$$

Note. The theorems of this and the preceding section are really identical, even though they appear to have been stated differently.

77·6. Theorem. *Let $f(x)$, $g(x)$ be two functions such that*

(*i*) $\underset{x\to\infty}{lt} f(x)=0,\quad \underset{x\to\infty}{lt}\ g(x)=0,$

and (*ii*) $\underset{x\to\infty}{lt}\ [f'(x)/g'(x)]=l,$

then $\underset{x\to\infty}{lt}\ [f(x)/g(x)]=l.$

We write $x=1/z$ so that $x\to\infty$, when $z\to(0+0)$.

Let $\quad F(z)=f(1/z),\ G(z)=g(1/z),$

so that $\underset{z\to(0+0)}{\text{lt}} F(z)=0,\quad \underset{z\to(0+0)}{\text{lt}} G(z)=0.$ (1)

we have

$$F'(z)=-f'\left(\frac{1}{z}\right)\cdot\frac{1}{z^2};\quad G'(z)=-g'\left(\frac{1}{z}\right)\frac{1}{z^2}$$

$$\therefore \qquad \frac{F'(z)}{G'(z)}=\frac{f'(1/z)}{g'(1/z)}$$

so that, by (*ii*), $F'(z)/G'(z)\to l$, when $z\to(0+0)$. (2)

Hence, from (1) and (2),

$$\underset{z\to(0+0)}{\text{lt}}\ [F(z)/G(z)]=l,$$

i.e.,

$$\underset{x\to\infty}{\text{lt}}\ [f(x)/g(x)]=l.$$

78. The indeterminate form (∞/∞).

78·1 Theorem. *If $f(x)$, $g(x)$ be two functions such that when $x\to c$,*

(*i*) $lt\ |g(x)|=\infty,$

and (*ii*) $lt\ \dfrac{f'(x)}{g'(x)}=0,$

then $lt\ \dfrac{f(x)}{g(x)}=0$, *when* $x\to c$.

The condition (*ii*) implies that there exists a neighbourhood $(c-\delta, c+\delta)$ of c such that for every point x, other than c, of this neighbourhood, $f'(x)$, $g'(x)$ exist and $g'(x)\neq 0$.

From Darboux's theorem of §62. P. 87, it follows that $g'(x)$ keeps the same sign, positive or negative for every point x of $[c, c+\delta]$, and the same thing is true for $[c-\delta, c]$ also.

Firstly we consider $[c, c+\delta]$. Let $g'(x)$ remain positive in it.

Let ϵ be any positive number, however small. There exists by virtue of (*ii*), a positive number $\delta_1 < \delta$ such that for every point x of $[c, c+\delta_1]$,

$$\left|\frac{f'(x)}{g'(x)} - 0\right| < \frac{\epsilon}{2} \text{ i.e., } \left| f'(x) \right| < \tfrac{1}{2}\epsilon \left|g'(x)\right| = \tfrac{1}{2}\epsilon g'(x),$$

i.e., $\quad -\tfrac{1}{2}\epsilon .\, g'(x) < f'(x) < \tfrac{1}{2}\epsilon .\, g'(x).$

By virtue of the theorem of § 66. P. 89, we have

$$-\tfrac{1}{2}\epsilon\, [g(c+\delta)-g(x)] < [f(c+\delta)-f(x)] < \tfrac{1}{2}\epsilon .\, [g(c+\delta)-g(x)]$$

i.e., $\quad | f(c+\delta)-f(x) | < \tfrac{1}{2}\epsilon\, | g(c+\delta)-g(x) |$

or $\quad | f(x) | - | f(c+\delta) | < \tfrac{1}{2}\epsilon .\, | g(c+\delta) | + \tfrac{1}{2}\epsilon\, | g(x) |$

or $\quad | f(x) | < \tfrac{1}{2}\epsilon\, | g(x) | + \tfrac{1}{2}\epsilon .\, | g(c+\delta) | + | f(c+\delta |,$

or $\quad | f(x) | < \tfrac{1}{2}\epsilon .\, | g(x) | + k$, for any point x of $[c, c+\delta_1]$, k being free of x,

or
$$\left|\frac{f(x)}{g(x)}\right| < \tfrac{1}{2}\epsilon + \frac{k}{| g(x) |}.$$

There exists a positive number $\delta_2 < \delta_1$, such that

$$\frac{k}{| g(x) |} < \tfrac{1}{2}\epsilon,$$

for any point x of $[c, c+\delta_2]$.

Thus we see that corresponding to every positive number ϵ there exists a positive number δ_2, such that for every point x of $[c, c+\delta_2]$,

$$\left|\frac{f(x)}{g(x)}\right| < \epsilon.$$

Hence
$$\operatorname*{lt}_{x\to(c+0)} \frac{f(x)}{g(x)} = 0$$

It may similarly be shown that

$$\operatorname*{lt}_{x\to(c-0)} \frac{f(x)}{g(x)} = 0.$$

Thus $\quad f(x)/g(x) \to 0$ as $x \to c$.

78·2. Theorem.

If $f(x)$, $g(x)$ be two functions such that when $x \to c$,
(*i*) $\operatorname{lt} | g(x) | = \infty$ $\qquad$ (*ii*) $\operatorname{lt} [f'(x)/g'(x)] = l$,

then $\quad \operatorname{lt} f(x)/g(x) = l$, *when* $x \to c$

We write $\quad \phi(x) = f(x) - lg(x).$

We have

$$\operatorname*{lt}_{x\to c} \frac{\phi'(x)}{g'(x)} = \operatorname*{lt}_{x\to 0} \left[\frac{f'(x)}{g'(x)} - l\right] = 0, \text{ (by } ii)$$

$$\therefore \quad \operatorname*{lt}_{x\to 0} \frac{\phi(x)}{g(x)} = 0, \text{ (§78·1)}$$

i. e., $\lim_{x \to c} \left[\frac{f(x)}{g(x)} - l \right] = 0$

or $\lim_{x \to c} \frac{f(x)}{g(x)} = l.$

Note. As in §77 6, it may be shown that this result remains true even when $x \to \infty$.

Note. It should be specially noted that in the preceding theorem nothing whatsoever has been said about the limit of $f(x)$ as $x \to c$ so that the result holds good independently of the behaviour of $f(x)$. In particular, therefore, the result holds good when $f(x) \to \infty$ as $x \to c$; this being the form in which the theorem is usually stated.

79. Two important special cases of limits.

79·1. Theorem. *To prove that*

$$\lim_{x \to \infty} \left(\frac{x^m}{e^{ax}} \right) = 0,$$

where a, m are any positive numbers whatsoever.

Let $f(x) = x^m$, $g(x) = e^{ax}$,
so that $f(x) \to \infty$ and $g(x) \to \infty$, as $x \to \infty$

We can write

$$m = n + p,$$

where n is some positive integer and p is a number such that $0 \leqslant p < 1$.

It is easy to see that $f'(x)$, $f''(x), \ldots, f^n(x)$ all tend to ∞ and $f^{n+1}(x) \to 0$ as $x \to \infty$.

Also $g'(x)$, $g''(x)$, $g'''(x), \ldots\ldots$ all tend to ∞.

Thus $\lim_{x \to \infty} \left\{ f^{n+1}(x) / g^{n+1}(x) \right\} = 0$

Therefore from the theorem of §78·2, we deduce that

$$\lim\, [f(x)/g(x)] = 0, \text{ when } x \to \infty.$$

Note. The result of this theorem is roughly expressed by saying that e^{ax} tends to ∞ more rapidly than any positive power of x, when $x \to \infty$.

79·2. Theorem. *To prove that*

$$\lim_{x \to \infty} \frac{(\log x)^m}{x^a} = 0,$$

where a, m are any positive numbers whatsoever.

The independent proof is similar to that of the preceding result. Otherwise if we write $\log x = y$, we see that

$$\lim_{x \to \infty} \frac{(\log x)^m}{x^a} = \lim_{y \to \infty} \frac{y^m}{e^{ay}} = 0.$$

Note. This shows that a positive power of x tends to infinity more rapidly than any positive power of $\log x$, as $x \to \infty$.

Cor. $\lim_{x \to (0+0)} [x^a (\log x)^m] = 0$, a, m being any positive numbers.

Putting $x = 1/y$, we may easily obtain it.

80. Note on a special function. The function $f(x)$, defined as

$$f(x)=e^{-1/x^2} \text{ when } x\neq 0\text{ ; } f(0)=0,$$

possesses a remarkable property, viz, that $f^n(0)=0$ for every value of n.

We have

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{e^{-1/x^2}}{x}=\lim_{y\to\infty}\frac{y}{e^{y^2}}=0, \ (y=1/x)$$

$$f'(x)=(2/x^3)\,e^{-1/x^2}, \text{ when } x\neq 0.$$

$$\therefore\ f''(0)=\lim_{x\to 0}\frac{f'(x)-f'(0)}{x}=\lim_{x\to 0}\frac{2e^{-1/x^2}}{x^4}=\lim_{y\to\infty}\frac{2y^4}{e^{y^2}}=0.$$

$$f''(x)=\left(\frac{4}{x^6}-\frac{6}{x^4}\right)e^{-1/x^2}. \text{ when } x\neq 0,$$

$$\therefore\ f'''(0)=\lim_{x\to 0}\left(\frac{4}{x^6}-\frac{6}{x^4}\right)e^{-1/x^2}=\lim_{y\to\infty}\frac{4y^6-6y^4}{e^{y^2}}=0.$$

If we form the higher derivatives for $x\neq 0$, we shall obviously always obtain the product of e^{-1/x^2} and a polynomial in $1/x$. Thus we see that all the higher derivatives will vanish at the point $x=0$.

It is thus seen that this function possesses a continuous derivative for every value of x. Maclaurin's infinite series for this function is

$$0+x.0+\frac{x^2}{2!}\,.0+\frac{x^3}{3!}\,.0+\ldots\ldots+\frac{x^n}{n!}\,.0+\ldots\ldots$$

which is certainly not equal to $f(x)$ for every value of x.

Thus we see that there exist functions which possess continuous derivatives for every value of x and yet cannot be expanded in terms of Maclaurin's series.

Examples

1. If

$$f(x)=x\,\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, \text{ when } x\neq 0, \text{ and } f(0)=0,$$

show that $f(x)$ is continuous but not derivable for $x=0$; show that

$$Rf'(0)=+1,\ Lf'(0)=-1.$$

2. Verify the Rolle's theorem in (a, b) for
(*i*) $\log[(x^2+ab)/(a+b)x]$. (*ii*) $x(x-2)e^{-x/2}$.
(*iii*) $(x-a)^m(x-b)^n$, m, n being positive integers.

3. Verify Lagrange's mean value theorem for
(*i*) lx^2+mx+n in (a, b), (*ii*) $x(x-1)(x-2)$ in $(0, \frac{1}{2})$.

4. Discuss the applicability of Rolle's theorem to $f(x)=|x|$ in $(-1,1)$.

5. Applying Lagranges' mean value theorem in turn to the functions $\log x$ and e^x determine the corresponding values of θ in terms of a and h.
Deduce that
(*i*) $0<[\log(1+x)]^{-1}-x^{-1}<1$, (*ii*) $0<x^{-1}\log[(e^x-1)/x]<1$.

6. Show that
(*i*) $x^2>(1+x)[\log(1+x)]^2$ for $x>0$.
(*ii*) $x<\log[1/(1-x)]<x/(1-x)$, where $0<x<1$.

7. *Prove that e is an irrational number.*
If possible, let $e=p/q$, where p, q are integers.
Let n be any integer greater than q.
By Maclaurin's theorem, with Lagrange's form of remainder after n terms, we have

$$e=2+\frac{1}{2!}+\ldots\ldots+\frac{1}{n!}+\frac{1}{(n+1)!}e^{\theta}, \quad \text{where } 0<\theta<1$$

$$\therefore\ n!\,.e=2\,n!+\frac{n!}{2!}+\ldots\ldots1+\frac{1}{n+1}e^{\theta}.$$

Now, $n!\,.\,e=n!(p/q)$ must be an integer, and since $e^{\theta}<e<3$, we have $0<e^{\theta}/(n+1)<1$.

Hence, we see that

an integer=an integer plus a non-zero proper fraction, and this is impossible.

7. *Prove that e^x is not a rational function*

If possible, let

$$e^x=\frac{a_0x^m+a_1x^{m-1}+\dots\dots+a_m}{b_0x^n+b_1x^{n-1}+\dots\dots+b_n}=f(x),\text{ say}$$

where $a_0\neq0$, $b_0\neq0$.

Obviously $f(x)/x^{m-n}\to a_0/b_0$ as $x\to\infty$ and this is impossible if $f(x)=e^x$. (Refer theorem of § 79·1, P. 103.).

8. *Show that 'θ' which occurs in the Lagrange's form of remainder, viz., $(h^n/n!)f^n(a+\theta h)$ tends to the limit $1/(n+1)$, when $h\to0$, provided that $f^{n+1}(x)$ is continuous at a and $f^{n+1}(a)\neq0$.*

Since $f^{n+1}(x)$ is continuous at a, there exists an interval $(a-\delta,a+\delta)$ at every point of which $f^{n+1}(x)$ exists. Also, therefore, $f(x)$, $f'(x)$,......$f^n(x)$ are all continuous in $(a-\delta,a+\delta)$. If $(a+h)$ be any point of this interval, we obtain, the necessary conditions being satisfied,

$$f(a+h)=f(a)+hf'(a)+\dots\dots+\frac{h^n}{n!}f^n(a+\theta h)$$

and $$f(a+h)=f(a)+hf'(a)+\dots\dots+\frac{h^n}{n!}f^n(a)+\frac{h^{n+1}}{(n+1)!}f^{n+1}(a+\theta' h).$$

$$\therefore\quad f^n(a+\theta h)-f^n(a)=\frac{h}{n+1}f^{n+1}(a+\theta' h).$$

Again, applying Lagrange's mean value theorem to the expression on the left, we see that

$$\theta hf^{n+1}(a+\theta\theta'' h)=\frac{h}{n+1}f^{n+1}(a+\theta' h)$$

i. e., $$\theta f^{n+1}(a+\theta\theta'' h)=\frac{1}{n+1}f^{n+1}(a+\theta' h).$$

Let $h\to0$. Then, we obtain

$$\text{lt }\theta\cdot f^{n+1}(a)=\frac{1}{n+1}f^{n+1}(a),\qquad\because f^{n+1}(x)\text{ is continuous.}$$

or $$\text{lt }\theta=\frac{1}{n+1},\qquad\because f^{n+1}(a)\neq0.$$

9. If $\phi''(x)\geqslant0$ for every value of x, then
$$\phi[\tfrac{1}{2}(x_1+x_2)]\leqslant\tfrac{1}{2}[\phi(x_1)+\phi(x_2)].$$

10. Assuming $f''(x)$ continuous in (a,b), show that

$$f(c)-f(a)\,\frac{b-c}{b-a}-f(b)\,\frac{c-a}{b-a}=\tfrac{1}{2}(c-a)(c-b)f''(\xi)$$

where c and ξ both lie in (a,b).

11. If $f^n(x)=0$ for every value of x in (a,b), then there are numbers a_0, a_1......a_{n-1} such that

$$f(x)=\sum_{r=0}^{n-1}a_rx^r,$$

in (a,b).

12. If a derivable function $f(x)$ satisfies the equation
$$f(x+y)=f(x)\,f(y),$$
then either $f(x)\equiv0$ or else $f(x)=e^{\alpha x}$.

13. If a derivable function $f(x)$ satisfies the equation
$$f(xy)=f(x)+f(y)$$
then $f(x)=\alpha\log x$.

CHAPTER VI

RIEMANN THEORY OF DEFINITE INTEGRAL

81. Riemann integrability and the integral of a bounded function for a finite range. Let $f(x)$ be a *bounded* function defined for some *finite* interval (a, b).

Divide (a, b) into a finite number of sub-intervals by means of any *arbitrary* set of points $x_0, x_1, x_2, \ldots x_{r-1}, x_r, \ldots x_n$, where

$$a=x_0<x_1<x_2\ldots\ldots<x_{r-1}<x_r<\ldots\ldots<x_{n-1}<x_n=b.$$

Thus

$$(x_0, x_1), (x_1, x_2), \ldots\ldots (x_{r-1}, x_r) \ldots\ldots (x_{n-1}, x_n)$$

are the sub-intervals in which (a, b) is divided.

Let the sub-interval (x_{r-1}, x_r) and also its length $x_r - x_{r-1}$ be both denoted by δ_r.

The function $f(x)$ which is bounded in (a, b) is also necessarily bounded in each sub-interval δ_r.

Let M_r, m_r be the bounds of $f(x)$ in δ_r.

Set up the two sums S and s defined as follows :—

$$S = M_1\delta_1 + \ldots\ldots\ldots + M_r\delta_r + \ldots\ldots + M_n\delta_n = \sum_{r=1}^{r=n} M_r\delta_r,$$

$$s = m_1\delta_1 + \ldots\ldots\ldots + m_r\delta_r + \ldots\ldots + m_n\delta_n = \sum_{r=1}^{r=n} m_r\delta_r.$$

If M, m be the bounds of $f(x)$ in (a, b), we have for every value of r,

$$m \leqslant m_r \leqslant M_r \leqslant M,$$

or

$$m\delta_r \leqslant m_r\delta_r \leqslant M_r\delta_r \leqslant M\delta_r.$$

Putting $r=1, 2, \ldots\ldots\ldots, n$, and adding, we deduce that

$$m(b-a) \leqslant s \leqslant S \leqslant M(b-a) \qquad (1).$$

Now, a pair of sums S, s correspond to each mode of division of (a, b) into sub-intervals and from (1) we see that the aggregates of the sums S, s, obtained by considering all possible modes of division, are bounded.

Def. *The lower bound of the aggregate of the sums 'S' is called the* **upper integral** *of $f(x)$ over (a, b) and is denoted by*

$$U = \overline{\int_a^b} f(x)dx.$$

The upper bound of the aggregate of the sums 's' is called the **lower integral** *of $f(x)$ over (a, b) and is denoted by*

$$L = \underline{\int_a^b} f(x)dx.$$

[Clearly the upper bound of the sums 'S' is $M(b-a)$ and the lower bound of the sums 's' is $m(b-a)$ and they are attained.]

A bounded function $f(x)$ is said to be Riemann integrable, or simply integrable, (for the purposes of this book), *over (a,b) if its upper and lower integrals are equal ; the common value of these integrals which is called the* **Riemann integral** *or simply the* **integral** *is denoted by the symbol*

$$I=\int_a^b f(x)dx.$$

Note 1. The numbers a, b are respectively called the *lower* and the *upper limits of integration.*

Note 2. The definition of integrability given above is based on the *notion of bounds.* Another equivalent definition based on the notion of limits is given in § 83.

Note 3. It should be clearly understood that *every bounded function is not necessarily integrable, i.e.* there may exist a bounded function $f(x)$ for which

$$\overline{\int_a^b} f(x)dx \neq \underline{\int_a^b} f(x)dx.$$

The necessary and sufficient condition for the existence of the symbol $\int_a^b f(x)dx$ *i.e.* for the integrability of $f(x)$ over (a, b) is obtained in § 84.

Note 4. The concept of integrability of a function over an interval as introduced here, is subject to two very important limitations, viz. (*i*) *the function is bounded* (*ii*) *the interval is finite* so that neither end point is infinite. In chapter VIII we shall see how these limitations can be removed and the concept generalised so as to be applicable to cases where the function is not bounded or where one or both the limits of integration are infinite.

Note 5. The statement that "$\int_a^b f(x)dx$" exists, implies that the function $f(x)$ is bounded and integrable over (a, b).

Note 6. The symbol

$$D(a=x_0, x_1, x_2, \ldots\ldots x_{r-1}, x_r, \ldots\ldots x_n=b),$$

will be used to denote the division obtained by inserting the points $x_0, x_1, \ldots\ldots x_{r-1}, x_r \ldots. x_n$ between a and b. The numbers

$$x_0, x_1, x_2, \ldots\ldots x_{r-1}\ x_r, \ldots\ldots, x_n$$

will be called the points of the division D, and the sub-intervals

$$(x_0, x_1), \ldots\ldots (x_{r-1}, x_r), \ldots\ldots (x_{n-1}, x_n)$$

will be called the intervals of the division D.

The length of the greatest of all the intervals (x_{r-1}, x_r) of D will be called the **norm** of the division D.

Note 7. The sums S, s corresponding to a division D are sometimes written as S_D, s_D respectively. Clearly

$$S_D \geq s_D.$$

Note 8. Oscillatory Sum. We have

$$S_D - s_D = \Sigma M_r \delta_r - \Sigma m_r \delta_r = \Sigma (M_r - m_r)\delta_r = \Sigma O_r \delta_r,$$

where O_r denotes the oscillation of the function in δ_r. The sum $\Sigma O_r \delta_r$ is called the oscillatory sum and is denoted by ω_D.

As $O_r \geq 0$, the oscillatory sum ω_D consists of the sum of a finite number of non-negative terms.

Examples.

1. If

$$f(x)=\begin{cases}0, \text{ where } x \text{ is rational,}\\ 1, \text{ where } x \text{ is irrational,}\end{cases}$$

show that $f(x)$ is not integrable in any interval.

2. Show that

$$\overline{\int_a^b} k\,dx=\underline{\int_a^b} k\,dx=k(b-a),$$

where k is a constant.

(This proves that every function which is a constant is integrable.)

3. A function $f(x)$ is bounded in (a, b); show that

$$\text{(i)}\quad \overline{\int_a^b} kf(x)dx=k\overline{\int_a^b} f(x)dx,\quad \underline{\int_a^b} kf(x)dx=k\underline{\int_a^b} f(x)dx,$$

where k is a positive constant.

$$\text{(ii)}\quad \overline{\int_a^b} kf(x)dx=k\underline{\int_a^b} f(x)dx,\quad \underline{\int_a^b} kf(x)dx=k\overline{\int_a^b} f(x)dx,$$

where k is a negative constant.

Deduce that if $f(x)$ is bounded and integrable over (a, b) then so is $kf(x)$, where k is any constant, and that

$$\int_a^b kf(x)\,dx=k\int_a^b f(x)dx$$

[If M_r, m_r be the bounds of $f(x)$ in δ_r, then kM_r, km_r, (km_r, kM_r) are the bounds of $kf(x)$ in δ_r, when k is positive, (k is negative).

4. A bounded function $f(x)$ is integrable over (a, b) and M, m are the bounds of $f(x)$, show that

$$m(b-a)\leqslant\int_a^b f(x)dx\leqslant M(b-a).$$

82. Darboux's theorem. I. *To every positive number ϵ, there corresponds a positive number δ such that*

$$S<\overline{\int_a^b} f(x)dx+\epsilon$$

for every division whose norm is less than or equal to δ.

Lemma. *Let $|f(x)|<k$ in (a, b) and D_1 any division of (a, b) and let δ be a positive number such that the length of every interval of D_1 is $<\delta$. Then if D_2 be any other division of (a, b) consisting of, as its points of division, all the points of D_1 and at the most p more then,*

$$S_{D_1}\geqslant S_{D_2}\geqslant S_{D_1}-2pk\delta.$$

Firstly suppose that $p=1$ so that ~~only~~ only one interval, say δ_r, of D_1 is divided into intervals, say δ'_r and δ''_r. Let M_r, M'_r, M''_r be the upper bounds of $f(x)$ in δ_r, δ'_r, δ''_r respectively.

We have

$$S_{D_1}-S_{D_2}=M_r\delta_r-(M'_r\delta'_r+M''_r\delta_r)$$
$$=(M_r-M'_r)\delta'_r+(M_r-M''_r)\delta''_r, \text{ for } \delta_r=\delta'_r+\delta''_r.$$

Now, since $|f(x)|<k$, therefore

$$-k<M'_r<M_r<k, \text{ i.e., } 0<M_r-M'_r<2k$$

Similarly $$0<M_r-M''_r<2k.$$

$$\therefore \quad S_{D_1}-S_{D_2}\left\}\begin{array}{l}\geqslant 0\\ \leqslant 2k\delta_r<2k\delta\end{array}\right.$$

Now supposing that each additional point is introduced one by one, we obtain the result.

We now prove the main theorem.

As $f(x)$ is bounded, there exists a positive number such that

$$|f(x)|<k \text{ in } a, b).$$

Since

$$\overline{\int_a^b} f(x)dx$$

is the lower bound of the aggregate of the sums S, there exists a division,

$$D_1(a=x_0, x_1, x_2.........x_{p-1}\ x_p, =b)$$

such that for the corresponding sum S_{D_1}, we have

$$S_{D_1}<\overline{\int_a^b} f(x)dx+\frac{\epsilon}{2}.$$

The points of D_1 are $(p+1)$ in number.

We determine a positive number δ such that

$$2k(p-1)\delta=\tfrac{1}{2}\epsilon,$$

Consider any division D whose norm is less than or equal to δ. consider a division D_2 consisting of, as its points of division, the points of D_1 as well as of D.

We have

$$\frac{\epsilon}{2}+\overline{\int_a^b} f(x)\ dx>S_{D_1}\geqslant S_{D_2}\geqslant S_D-2(p-1)k\delta \quad \text{(Lemma)}$$

or

$$S_D\leqslant 2(p-1)k\delta+\frac{\epsilon}{2}+\overline{\int_a^b} f(x)dx$$

$$=\overline{\int_a^b} f(x)dx+\epsilon.$$

Darboux's theorem II. *To every positive number ϵ, there corresponds a positive number δ, such that*

$$s > \underline{\int_a^b} f(x)dx - \epsilon,$$

for every division whose norm is less than or equal to δ.

The proof is similar to that of the corresponding result on the upper integral.

Note. Darboux's theorem may also be symbolically exhibited as follows :

$$\text{lt } S_D = \overline{\int_a^b} f(x)dx, \quad \text{lt } s_D = \underline{\int_a^b} f(x)dx,$$

when δ, the norm of the division D, tends to zero.

Cor.

$$\overline{\int_a^b} f(x)\,dx \geqslant \underline{\int_a^b} f(x)dx.$$

If possible, let

$$\overline{\int_a^b} f(x)\,dx < \underline{\int_a^b} f(x)dx.$$

Let k be any number lying between the upper and lower integrals.

There exists, by Darboux's theorem, a positive number δ_1 such that for every division whose norm is $\leqslant \delta_1$,

$$S < k.$$

Also, there exists a positive number δ_2 such that for every division whose norm is $\leqslant \delta_2$,

$$s > k.$$

If δ be any positive number smaller than both δ_1 and δ_2, then for every division whose norm is $\leqslant \delta$,

$$S < k < s, \text{ i.e., } S < s,$$

which is absurd.

Hence the result.

Ex. Show that

$$S_{D_1} \geqslant s_{D_2}$$

even when D_1, D_2 are two different divisions

(This atonce follows from the cor. above).

83. Another equivalent definition of integrability and integral. Let $f(x)$ be a function defined in (a, b).

Let

$$D(a = x_0 < x_1 < x_2 \ldots\ldots < x_{r-1} < x_r < \ldots\ldots x_n = b)$$

be any division of (a, b) and ξ_r any arbitrary point of $\delta_r \equiv (x_{r-1}, x_r)$.

Form the sum

$$\sum_{r=1}^{r=n} f(\xi_r)\delta_r = f(\xi_1)\delta_1 + \ldots + f(\xi_r)\delta_r + \ldots + f(\xi_n)\delta_n.$$

Def. *The function $f(x)$ is said to be integrable over (a, b), if there exists a number I, such that to every positive number ϵ, however small, there corresponds a positive number δ such that for every division*

$$D(a=x_0<x_1,\ldots,x_{r-1}, x_r,\ldots x_n=b)$$

of norm $\leqslant \delta$, and for every arbitrary choice of ξ_r in (x_{r-1}, x_r),

$$\left|\sum_{r=1}^{r=n} f(\xi_r)(x_r-x_{r-1})-I\right|<\epsilon,$$

Also, then, I is said to be the integral of $f(x)$ over (a, b).

In a more concise but less precise manner the definition may be stated a little differently as follows :—*A function $f(x)$ is integrable if*

$$\lim_{\delta\to 0} \sum_{r=1}^{r=n} f(\xi_r)\delta_r,$$

exists and is independent of the choice of the interval δ_r and of the point ξ_r of δ_r; the limit I, if it exists, is said to be the integral of $f(x)$ over (a, b).

The equivalence of the two definitions will now be established.

Let a bounded function $f(x)$ be integrable according to the former definition so that

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx = \int_a^b f(x)dx.$$

Let ϵ be any positive number.

By Darboux's theorem, there exists a positive number δ such that for every division D whose norm is $\leqslant \delta$,

$$S_D < \overline{\int_a^b} f(x)dx+\epsilon = \int_a^b f(x)dx+\epsilon, \tag{1}$$

$$s_D > \underline{\int_a^b} f(x)dx-\epsilon = \int_a^b f(x)dx-\epsilon. \tag{2}$$

If ξ_r be *any* point of the interval δ_r of D, we have

$$s_D \leqslant \sum_{r=1}^{r=n} f(\xi_r)\delta_r \leqslant S_D \tag{3}$$

From (1), (2) and (3), we deduce that every division D, whose norm is $\leqslant \delta$,

$$\int_a^b f(x)dx-\epsilon < \sum_{r=1}^{r=n} f(\xi_r)\delta_r < \int_a^b f(x)dx+\epsilon$$

i.e.,

$$\left|\sum_{r=1}^{r=n} f(\xi_r)\,\delta_r - \int_a^b f(x)dx\right| < \epsilon,$$

and accordingly $f(x)$ is integrable according to the latter definition also and

$$I=\int_a^b f(x)dx.$$

Now, *let $f(x)$ be integrable according to the latter definition* so that

$$\operatorname{lt} \Sigma f(\xi_r)\delta_r,$$

exists, as the norm $\delta \to 0$.

It will firstly be deduced that $f(x)$ is bounded in (a, b).

If possible, let $f(x)$ be not bounded.

There exists a division D such that for every choice of ξ_r in δ_r,

$$|\Sigma f(\xi_r)\delta_r - I| < 1.$$

or

$$|\Sigma f(\xi_r)\delta_r| < |I| + 1.$$

As $f(x)$ is not bounded in (a,b), it must also be so in atleast one δ_r say in δ_m.

We take $\xi_r = x_r$, when $r \neq m$ so that every number ξ_r, except ξ_m, is fixed and, accordingly, every term of $\Sigma f(\xi_r)\delta_r$ except the term $f(\xi_m)\delta_m$ is also fixed. Since $f(x)$ is not bounded in δ_m, we can choose a point ξ_m in δ_m such that

$$|\Sigma f(\xi_r)\delta_r| > |I| + 1$$

and thus we arrive at a contradiction. Hence $f(x)$ is bounded in (a, b).

Now, let ϵ be any positive number. There exists a positive number δ such that for every division whose norm is $\leqslant \delta$,

$$|\Sigma f(\xi_r)\delta_r - I| < \tfrac{1}{2}\epsilon,$$

i.e.,

$$I - \tfrac{1}{2}\epsilon < \Sigma f(\xi_r)\delta_r < I + \tfrac{1}{2}\epsilon, \tag{1}$$

for every choice of the point ξ_r in δ_r.

If M_r, m_r be the bounds of $f(x)$ in δ_r, there exist points α_r, β_r of δ_r such that

$$f(\alpha_r) > M_r - \epsilon/2(b-a),$$

$$f(\beta_r) < m_r + \epsilon/2(b-a).$$

From these we deduce that

$$\Sigma f(\alpha_r)\delta_r > S - \tfrac{1}{2}\epsilon \text{ or } S < \Sigma f(\alpha_r)\delta_r + \tfrac{1}{2}\epsilon, \tag{2}$$

$$\Sigma f(\beta_r)\delta_r < s + \tfrac{1}{2}\epsilon \text{ or } s > \Sigma f(\beta_r)\delta_r - \tfrac{1}{2}\epsilon. \tag{3}$$

From (1), (2) and (3) we deduce taking $\xi_r = \alpha_r$ and β_r that

$$I - \epsilon < s < S < I + \epsilon, \tag{4}$$

for every division whose norm is $\leqslant \delta$.

Also we know that

$$s \leqslant \underline{\int_a^b} f(x)dx \leqslant \overline{\int_a^b} f(x)dx \leqslant S \tag{5}$$

From (4) and (5), we have

$$I - \epsilon < \underline{\int_a^b} f(x)dx \leqslant \overline{\int_a^b} f(x)dx < I + \epsilon, \tag{6}$$

or
$$0 \leqslant \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < 2\epsilon$$

so that the non-negative number

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx$$

is less than every positive number ; ϵ being arbitrary, and hence

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx = 0, \tag{7}$$

so that the function is integrable according to the former definition also.

From (6) and (7), we have

$$\mathrm{I} - \epsilon < \int_a^b f(x)dx < \mathrm{I} + \epsilon,$$

and since, ϵ is arbitrary, this gives

$$\mathrm{I} = \int_a^b f(x)dx.$$

Thus the equivalence is completely established.

84. The condition for integrability.

84.1. First form. *The necessary and sufficient condition for the integrability of a bounded function $f(x)$ is, that to every positive number ϵ, there corresponds a positive number δ, such that for every division D whose norm is $\leqslant \delta$, the oscillatory sum w_{D} is $< \epsilon$.*

The condition is necessary. The bounded function $f(x)$ being integrable,

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx = \int_a^b f(x)dx.$$

Let ϵ be any positive number. By Darboux's theorem, there exists a positive number δ such that for every division D whose norm is $\leqslant \delta$,

$$\mathrm{S}_{\mathrm{D}} < \int_a^b f(x)dx + \frac{\epsilon}{2} = \int_a^b f(x)dx + \frac{\epsilon}{2},$$

and $$s_D > \int_{\underline{a}}^{b} f(x)dx - \frac{\epsilon}{2} = \int_a^b f(x)dx - \frac{\epsilon}{2}\,.$$

$$\therefore \int_a^b f(x)dx - \frac{\epsilon}{2} < s_D \leqslant S_D < \int_a^b f(x)dx + \frac{\epsilon}{2}$$

or $$w_D = S_D - s_D < \epsilon,$$

for every division D whose norm is $\leqslant \delta$.

The condition is sufficient. Let ϵ be any positive number. There exists a division D such that

$$S_D - s_D = \left\{ S_D - \int_a^{\bar{b}} f(x)dx \right\} + \left\{ \int_a^{\bar{b}} f(x)dx - \int_{\underline{a}}^{b} f(x)dx \right\} + \left\{ \int_{\underline{a}}^{b} f(x)dx - s_D \right\}$$

$$< \epsilon.$$

Since each one of the three numbers

$$S_D - \int_a^{\bar{b}} f(x)dx, \quad \int_a^{b} f(x)dx - \int_a^b f(x)dx, \quad \int_{\underline{a}}^{b} f(x)dx - s_D$$

is non-negative, we see that

$$0 \leqslant \int_a^{\bar{b}} f(x)dx - \int_{\underline{a}}^{b} f(x)dx < \epsilon,$$

As ϵ is an arbitrary positive number, we see that the non-negative number

$$\int_a^{\bar{b}} f(x)dx - \int_{\underline{a}}^{b} f(x)dx,$$

is less than *every* positive number, and hence

$$\int_a^{\bar{b}} f(x)dx - \int_a^b f(x)dx = 0 \text{ or } \int_a^{\bar{b}} f(x)dx = \int_a^b f(x)dx,$$

so that $f(x)$ is integrable.

Thus the theorem is established.

Note. This theorem is sometimes stated differently as follows :—

The necessary and sufficient condition for a bounded function $f(x)$ to be integrable in (a, b) is that

$$\lim \omega_D = 0,$$

where δ, the norm of the division D, tends to 0.

84.2. The condition of integrability. Second form. *The necessary and sufficient condition for the integrability of a bounded function $f(x)$ is that to every positive number ϵ, there corresponds a division D such that the corresponding oscillatory sum $\omega_D < \epsilon$.*

That this condition is necessary follows atonce from the first part of §84.1.

The condition is sufficient. As in the second part of §84.1, we write

$$S_D - s_D = \left\{ S_D - \overline{\int_a^b} f(x)dx \right\} + \left\{ \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \right\} + \left\{ \int_a^b f(x)dx - s_D \right\} < \epsilon,$$

and see that

$$0 \leqslant \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \epsilon.$$

From this relation, which holds for every positive ϵ, we deduce that

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx = 0, \text{ i.e., } \int_a^b f(x)dx = \underline{\int_a^b} f(x)dx.$$

Note. On comparing the two forms of conditions, the reader will easily see that, from the point of view of *necessity*, the first form is more valuable than the second but, from the point of view of sufficiency, the second form is more valuable than the first.

85. Particular classes of bounded integrable functions.

85.1. Every continuous function is integrable.

Suppose that $f(x)$ is continuous in (a, b).

Since $f(x)$ is continuous, it is bounded.

Let ϵ be any positive number.

We divide (a, b) into a finite number of sub-intervals 'δ_r' such that the oscillation of $f(x)$ in each sub-interval is $<\epsilon/(b-a)$; this being possible as proved in §50.5.

If ω_D be the oscillatory sum for this division,

$$\omega_D = \Sigma(M_r - m_r)\delta_r < \Sigma[\epsilon/(b-a)]\delta_r = [\epsilon/(b-a)]\Sigma\delta_r$$

$$= [\epsilon/(b-a)] \cdot (b-a) = \epsilon$$

i.e., $$\omega_D < \epsilon.$$

Hence $f(x)$ is integrable in (a, b). [§84.2].

85.2. *A bounded function $f(x)$ which has only a finite number of points of discontinuity in (a, b) is integrable in (a, b).*

Let

$$a_1\ a_2,\ a_3, \ldots\ldots, a_p,$$

be the finite number of points of discontinuity of $f(x)$.

Let ϵ be any positive number.

We enclose the points $a_1, a_2, \ldots\ldots, a_p$, in p non-overlapping intervals

$$(a_1', a_1''),\ (a_2', a_2''), \ldots\ldots, (a_p', a_p'')$$

such that the sum of their lengths is $<\epsilon/2\,(M-m)$, M, m being the bounds of $f(x)$ in (a, b). The oscillation of $f(x)$ in each of these intervals is $<(M-m)$ and accordingly the oscillatory sum for these is

$$<[\epsilon/2(M-m)] \cdot (M-m) = \epsilon/2.$$

Now, $f(x)$ is continuous in the $(p+1)$ sub-intervals

$$(a, a_1'),\ (a_1'', a_2'), \ldots\ldots, (a_p'', b).$$

As in §85.1 above each of these $(p+1)$ sub-intervals can be further sub-divided so that the part of the oscillatory sum arising from the sub-intervals of each of them separately is $<\epsilon/2(p+1)$.

Thus there exists a division of (a, b) such that the corresponding oscillatory sum is

$$<\frac{\epsilon}{2} + \frac{\epsilon}{2(p+1)} \cdot (p+1) = \epsilon.$$

Hence $f(x)$ is integrable in (a, b).

85.3. *If a function $f(x)$ is bounded in (a, b) and the aggregate of its points of discontinuity has only a finite number of limiting points, then $f(x)$ is integrable in (a, b).*

Let

$$a_1,\ a_2,\ a_3, \ldots\ldots,\ a_p,$$

be the limiting points of the aggregate of the points of discontinuity of $f(x)$.

We enclose them in p non-overlapping intervals

$$(a_1', a_1''),\ (a_2', a_2''), \ldots\ldots, (a_p', a_p'')$$

such that the sum of their lengths is $<\epsilon/2(M-m)$; M, m being the bounds of $f(x)$ in (a, b). The oscillatory sum for these intervals is

$$<\epsilon/2.$$

Only a *finite* number of points of discontinuity of $f(x)$ can lie in each of the $(p+1)$ intervals

$$(a, a_1'), (a_1'', a_2'), \ldots\ldots (a_p'', b),$$

so that, as in §85.2, they can be so sub-divided that the part of the oscillatory sum arising from the sub-intervals of each of these $(p+1)$ intervals is separately $<\epsilon/2(p+1)$.

Thus there exists a division of (a, b) such that the corresponding oscillatory sum is

$$<\frac{\epsilon}{2}+\frac{\epsilon}{2(p+1)}\cdot(p+1)=\epsilon.$$

Hence $f(x)$ is integrable in (a, b).

85.4. *If $f(x)$ is monotonic in (a, b), then it is integrable in (a, b).*

Clearly $f(x)$ is bounded and $f(a)$, $f(b)$ are its two bounds.

Let ϵ be any positive number.

For the sake of definiteness, we suppose that $f(x)$ is monotonically increasing.

We divide (a, b) so that the length of each sub-interval is

$$<\epsilon/[f(b)-f(a)+1].$$

Let

$$D(a=x_0, x_1, x_2, \ldots\ldots x_{r-1}, x_r, \ldots\ldots, x_n=b)$$

be any one such division.

Let $\qquad \delta_r=x_r-x_{r-1}.$

Here $\qquad M_r=f(x_r),\ m_r=f(x_{r-1}).$

$$\therefore \quad \omega_D=\Sigma(M_r-m_r)\delta_r=\Sigma[f(x_r)-f(x_{r-1})]\delta_r$$

$$<\frac{\epsilon}{f(b)-f(a)+1}\cdot\Sigma[f(x_r)-f(x_{r-1})]$$

$$=\frac{\epsilon}{f(b)-f(a)+1}\cdot[f(b)-f(a)]<\epsilon.$$

Hence $f(x)$ is integrable in (a, b).

Note. If we had taken $[f(b)-f(a)]$ instead of $[f(b)-f(a)+1]$, the proof would not have been valid for the case when $f(b)-f(a)=0$, *i.e.*, when $f(x)$ is a constant. The artifice of taking $[f(b)-f(a)+k]$ where k is positive, or, in particular $f(b)-f(a)+1$ serves to make the proof applicable even to this case.

Ex. 1. *A function $f(x)$ is defined in $(0, 1)$ as follows :—*

$$f(0)=0,$$
$$f(x)=1, \text{ when } \tfrac{1}{2}<x\leqslant 1,$$
$$f(x)=\tfrac{1}{2}, \text{ when } (\tfrac{1}{2})^2<x\leqslant\tfrac{1}{2},$$
$$\ldots \quad \ldots \quad \ldots$$
$$f(x)=(\tfrac{1}{2})^{n-1}, \text{ when } (\tfrac{1}{2})^n<x\leqslant(\tfrac{1}{2})^{n-1},$$
$$\ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots$$

show that $f(x)$ is integrable.

Since $f(x)$ is monotonically increasing in $(0, 1)$, it is integrable. (§ 85·4)

Or, we notice that $f(x)$ is continuous in $(0, 1)$ except at the set of points

$$0, \tfrac{1}{2}, (\tfrac{1}{2})^2, (\tfrac{1}{2})^3, \ldots\ldots, (\tfrac{1}{2})^n, \ldots\ldots$$

which has only one limiting point, *viz.*, 0 and hence $f(x)$ is integrable. (§ 85·3).

Ex. 2. A function $f(x)$ is integrable in (a, b); show that

(i)
$$\operatorname{lt} \sum_{r=1}^{r=n} hf(a+rh) = \int_a^b f(x)dx,$$

when $h \to 0$, $n \to \infty$, $nh = b-a$.

(ii)
$$\operatorname{lt} \sum_{p=1}^{p=n} f(ar^{p-1})\,(ar^p - ar^{p-1}) = \int_a^b f(x)dx,$$

when $r \to 1$, $n \to \infty$, $r^n = b/a$.

86. Properties of integrable functions.

86·1. *If a bounded function $f(x)$ is integrable in (a, b), then it is also integrable in (a, c) and (c, b), where c is any point of (a, b).*

Conversely, if $f(x)$ is bounded and integrable in (a, c), (c, b), then it is also integrable in (a, b).

Also in either case

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \qquad a<c<b.$$

Suppose that $f(x)$ is bounded and integrable in (a, b).

Let ϵ be any positive number.

There exists a positive number δ such that for each division of (a, b) whose norm is $\leqslant \delta$, the oscillatory sum is $< \epsilon$. Let D be a division of (a, b) such that 'c' is a point of D and that the norm of D is $\leqslant \delta$.

The oscillatory sum for the division D breaks itself into two parts, respectively consisting of the terms which arise from the sub-intervals of (a, c), and (c, b). Since the terms of an oscillatory sum are all positive, each part must itself be $<\epsilon$. Hence $f(x)$ is integrable both in (a, c) and (c, b).

Let, now, $f(x)$ be bounded and integrable in (a, c) and (c, b). Let ϵ be any positive number. There exist divisions of (a, c) and (c, b) such that the corresponding oscillatory sums are $< \epsilon/2$. The divisions of (a, c) and (c, b) give rise to a division of (a, b) for which the oscillatory sum is $<(\epsilon/2+\epsilon/2)=\epsilon$. Hence $f(x)$ is integrable in (a, b).

The relationship of equality is to be proved now. Let ϵ be any positive number.

As $f(x)$ is simultaneously integrable in (a, c), (c, b), and (a, b), there exists a positive number δ such that for divisions of norm $\leqslant \delta$, and of which c is a point, we have

$$\left| \sum_{(a,\, c)} f(\xi_r)\,\delta_r - \int_a^c f(x)dx \right| < \frac{\epsilon}{3}, \quad \left| \sum_{(c,\, b)} f(\xi_r)\delta_r - \int_c^b f(x)dx \right| < \frac{\epsilon}{3}.$$

$$\left| \sum_{(a,\, b)} f(\xi_r)\delta_r - \int_a^b f(x)dx \right| < \frac{\epsilon}{3};$$

where the meanings of the symbols $\sum\limits_{(a,\, c)} f(\xi_r)\delta_r$, etc., are obvious.

Since $\sum\limits_{(a,\, c)} f(\xi_r)\delta_r + \sum\limits_{(c,\, b)} f(\xi_r)\delta_r = \sum\limits_{(a,\, b)} f(\xi_r)\delta_r,$

we deduce that

$$\left| \int_a^b f(x)\,dx - \int_a^c f(x)\,dx - \int_c^b f(x)\,dx \right| < \epsilon,$$

from which it follows that

$$\int_a^b f(x)\,dx - \int_a^c f(x)\,dx - \int_c^b f(x)dx = 0,$$

ϵ being an arbitrary positive number.

Cor. *If $f(x)$ is bounded and integrable in (a, b), then it is also bounded and integrable in (α, β) where $a < \alpha < \beta < b$.*

As $f(x)$ is integrable in (a, b), therefore it is integrable in (a, β) and hence also in (α, β).

86·2. *If $f(x)$ and $g(x)$ are two functions, both bounded and integrable in (a, b), then $f(x) \pm g(x)$ are also bounded and integrable in (a, b), and*

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

Let ϵ be *any* positive number.

Since $f(x)$, $g(x)$ are integrable, there exists a positive number δ, such that for every division of norm $\leqslant \delta$ and for every choice of ξ_r in δ_r,

$$\left| \sum f(\xi_r)\delta_r - \int_a^b f(x)dx \right| < \frac{\epsilon}{2}, \tag{1}$$

$$\left| \sum g(\xi_r)\delta_r - \int_a^b g(x)dx \right| < \frac{\epsilon}{2}. \tag{2}$$

Therefore

$$\left| \sum \left\{ f(\xi_r) \pm g(\xi_r) \right\} \delta_r - \left\{ \int_a^b f(x)dx \pm \int_a^b g(x)dx \right\} \right| < \epsilon.$$

Hence $f(x) \pm g(x)$ are integrable in (a, b) and

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx. \text{ (§83)}$$

Cor. The result can be extended, by Mathematical induction, to the case of the algebraic sum of any finite number of functions.

86·3. *If $f(x)$, $g(x)$ are two functions, both bounded and integrable in (a, b), then $f(x)\, g(x)$ is also bounded and integrable in (a, b).*

Since $f(x)$, $g(x)$ are bounded, there exists a number k, such that for every x in (a, b).

$$|f(x)| < k, \quad |g(x)| < k.$$

Therefore $|f(x)\, g(x)| < k^2$ in (a, b) so that $f(x)\, g(x)$ is bounded.

Let

$$D(a = x_0, x_1, x_2, \ldots\ldots x_{r-1}, x_r, \ldots\ldots, x_n = b)$$

be any division of (a, b).

Let $\qquad M'_r, m'_r\ ;\ M''_r, m''_r\ ;\ M_r, m_r$

be the bounds of $f(x)$, $g(x)$ and $f(x)\, g(x)$ in $\delta_r \equiv (x_{r-1}, x_r)$. If α_1, α_2 be any two points of δ_r,

$$f(\alpha_2)\, g(\alpha_2) - f(\alpha_1)\, g(\alpha_1) = g(\alpha_2)[f(\alpha_2) - f(\alpha_1)] + f(\alpha_1)[g(\alpha_2) - g(\alpha_1)]$$

or $\quad |f(\alpha_2) g(\alpha_2) - f(\alpha_1)\, g(\alpha_1)| \leqslant |g(\alpha_2)|\ |f(\alpha_2) - f(\alpha_1)|$

$$+ |f(\alpha_1)|\ |g(\alpha_2) - g(\alpha_1)|$$

$$\leqslant k(M'_r - m'_r) + k(M''_r - m''_r).$$

$$\therefore \qquad (M_r - m_r) \leqslant k(M'_r - m'_r) + k(M''_r - m''_r) \qquad (1)$$

Now let ϵ be *any* positive number.

Since $f(x)$, $g(x)$ are integrable, there exists a positive number δ such that for every division of norm $\leqslant \delta$, the oscillatory sums of $f(x)$ and $g(x)$ are both $< \epsilon/2k$.

We now suppose that D is any division of norm $\leqslant \delta$, so that for D, we have, from (1),

$$\Sigma(M_r - m_r)\delta_r \leqslant k\, \Sigma(M'_r - m'_r)\delta_r + k\Sigma(M_r'' - m_r'')\delta_r$$
$$< k.(\epsilon/2k) + k.(\epsilon/2k) = \epsilon,$$

i. e., the oscillatory sum $\Sigma(M_r - m_r)\delta_r < \epsilon$.

Hence $f(x)g(x)$ is integrable in (a, b),

Cor. The result can, by Mathematical induction, be extended to the product of a *finite* number of bounded and integrable functions.

86·4. *If $f(x)$, $g(x)$ are two functions, both bounded and integrable in (a, b), and there exists a positive number 't' such that $|g(x)| > t$. in (a, b), then $f(x)/g(x)$ is bounded and integrable in (a, b).*

Since there exist positive numbers g and t such that in (a, b)

$$|f(x)| < k, \quad |g(x)| < k, \quad |g(x)| > t,$$

therefore

$$|f(x)/g(x)| < k/t$$

in (a, b). **Hence $f(x)/g(x)$ is bounded.**

Let $\qquad D(a=x_0, x_1, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots b x_n=b)$
be any division of (a, b) and let M'_r, m'_r ; M''_r, m''_r ; M_r, m_r be the bounds of $f(x), g(x), f(x)/g(x)$ in $\delta_r \equiv (x_{r-1}, x_r)$. If α_1, α_2 be any two points of δ_r

$$\left| \frac{f(\alpha_2)}{g(\alpha_2)} - \frac{f(\alpha_1)}{g(\alpha_1)} \right| = \left| \frac{g(\alpha_1)[f(\alpha_2)-f(\alpha_1)]-f(\alpha_1)[g(\alpha_2)-g(\alpha_1)]}{g(\alpha_1)\,g(\alpha_2)} \right|$$

$$\leqslant (k/t^2) \mid (f(\alpha_2)-f(\alpha_1) \mid + (k/t^2) \mid g(\alpha_2)-g(\alpha_1) \mid$$

$$\leqslant (k/t^2)(M'_r-m'_r)+(k/t^2)(M''_r-m''_r) \quad \text{(Ex. 8. Page 30)}$$

$$\therefore \quad (M_r-m_r) \leqslant (k/t^2)(M'_r-m'_r)+(k/t^2)(M''_r-m''_r) \qquad (1)$$

Let, now, ϵ be *any* positive number.

Since $f(x)$, $g(x)$ are integrable, there exists a positive number δ such that for every division D of norm $\leqslant \delta$, the oscillatory sums for $f(x)$, $g(x)$ are both less than $t^2\epsilon/2k$.

We now suppose that D is any division of norm $\leqslant \delta$ so that for D we have, from (1),

$$\Sigma(M_r-m_r)\delta_r \leqslant (k/t^2)\,\Sigma(M'_r-m'_r)\delta_r+(k/t^2)\,\Sigma(M''_r-m''_r)\delta_r$$

$$< (k/t^2)(t^2\epsilon/2k)+(k/t^2)(t^2\epsilon/2k)=\epsilon.$$

Hence $f(x)/g(x)$ is bounded and integrable in (a, b).

86·6. *If $f(x)$ is bounded and integrable in (a, b), then $\mid f(x) \mid$ is also bounded and integrable in (a, b).*

There exists a positive number k such that $\mid f(x) \mid < k$ so that $\mid f(x) \mid$ is bounded.

Let ϵ be *any* positive number

Since $f(x)$ is integrable, there exists a division

$$D(a=x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n=b)$$

such that the corresponding oscillatory sum for $f(x)$ is $< \epsilon$.

Let M'_r, m'_r ; M_r, m_r be the bounds of $f(x)$ and $\mid f(x) \mid$ in $\delta_r \equiv (x_{r-1}, x_r)$.

If α_1, α_2 be any two points of δ_r,

$$\big| \mid f(\alpha_2) \mid - \mid f(\alpha_1) \mid \big| \leqslant \mid f(\alpha_2)-f(\alpha_1) \mid$$

$$\leqslant M'_r-m'_r$$

$$\therefore \qquad M_r-m_r \leqslant M'_r-m'_r \qquad \text{(Ex. 8. Page 30).}$$

This gives $\qquad \Sigma(M_r-m_r)\delta_r \leqslant \Sigma(M'_r-m'_r)\delta_r < \epsilon$

i.e., $\qquad \Sigma(M_r-m_r)\delta_r < \epsilon.$

Hence $\mid f(x) \mid$ is integrable in (a, b).

Note. The converse of this theorem is not true. If we take

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational,} \\ -1, & \text{when } x \text{ is irrational,} \end{cases}$$

then

$$\overline{\int_a^b} f(x)\,dx = (b-a), \quad \underline{\int_a^b} f(x)\,dx = -(b-a)$$

so that

$$\int_a^b f(x)\,dx$$

does not exist,

But since $|f(x)| = 1$, for all x, therefore

$$\int_a^b |f(x)|\,dx \text{ exists and is equal to } (b-a)$$

87. Definition. *The meaning of*

$$\int_a^b f(x)dx.$$

when $b \leqslant a$.

If $f(x)$ be bounded and integrable in (b, a) where $a > b$, then, by def.,

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Also, by def., $$\int_a^a f(x)dx = 0.$$

It is easy to show that the results about integrals obtained in §§85, 86 hold true even when the upper limit $\leqslant$ the lower limit.

Note. The reader may carefully note that the statement

$$\text{“}\int_a^b f(x)dx \text{ exists,”}$$

means that $f(x)$ is bounded and integrable in (a, b).

88. Theorem. *If* $$\int_a^b f(x)dx$$

exists and M, m are the bounds of $f(x)$ in (a, b), then

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a), \qquad \text{if } b > a,$$

$$m(b-a) \geqslant \int_a^b f(x)dx \geqslant M(b-a), \qquad \text{if } b \leqslant a.$$

For $a = b$, the result is trivial.

If $b > a$, then for any division D, we have

$$m(b-a) \leqslant s_D \leqslant \int_a^b f(x)dx \leqslant S_D \leqslant M(b-a). \qquad (\S 81)$$

i.e., $$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a).$$

If $b<a$, *i.e.*, $a>b$, then, as proved above,

$$m(a-b) \leqslant \int_b^a f(x)dx \leqslant M(a-b).$$

$\therefore$ $$-m(a-b) \geqslant -\int_b^a f(x)dx \geqslant -M(a-b)$$

i.e., $$m(b-a) \geqslant \int_a^b f(x)dx \geqslant M(b-a).$$

Hence the result.

Cor. 1. *If $\int_a^b f(x)\,dx$ exists, then there exists a number μ, lying between the bounds of $f(x)$ such that*

$$\int_a^b f(x)dx = \mu(b-a).$$

Cor. 2. *If $f(x)$ is continuous in (a,b), then there exists a number 'c' lying between a and b such that*

$$\int_a^b f(x)dx = (b-a)\, f(c).$$

Note. We may write $c=a+\theta(b-a)$, where $0 \leqslant \theta \leqslant 1$.

Cor. 3. *If $\int_a^b f(x)dx$ exists, and k is a number such that, for all x.*

$$|f(x)| < k,$$

then $$\left| \int_a^b f(x)\,dx \right| \leqslant k\,|\,b-a\,|\,.$$

For $a=b$, the result is trivial.

We have

$$-k < f(x) < k,$$

so that if M, m be the bounds of $f(x)$ in (a, b),

$$-k \leqslant m \leqslant f(x) \leqslant M \leqslant k. \qquad (1)$$

Let $b>a$. Therefore, from 1,

$$-k(b-a)\leqslant m(b-a)\leqslant \int_a^b f(x)dx \leqslant M(b-a)\leqslant k(b-a)$$

or

$$\left|\int_a^b f(x)\,dx\right| \leqslant k\,|(b-a)|$$

Let $b<a$. We have, from above,

$$\left|\int_b^a f(x)\,dx\right| \leqslant k\,|a-b|,$$

i.e.,

$$\left|\int_a^b f(x)\,dx\right| \leqslant k\,|b-a|.$$

Cor. 4. *If* $\int_a^b f(x)dx$ *exists and* $f(x)\geqslant 0$, *then*

$$\int_a^b f(x)\,dx \begin{cases} \geqslant 0, \text{ when } b\geqslant a; \\ \leqslant 0, \text{ when } b\leqslant a. \end{cases}$$

For $b=a$, the result is trivial.

Since $f(x)\geqslant 0$, therefore, $m\geqslant 0$.

Let $b>a$. We have

$$\int_a^b f(x)dx \geqslant m(b-a)\geqslant 0.\ (b-a)=0.$$

Let $b<a$. We have, as proved above,

$$\int_b^a f(x)dx \geqslant 0.$$

$\therefore$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \leqslant 0.$$

Cor. 5. *If*

$$\int_a^b f(x)\,dx,\ \int_a^b g(x)dx$$

exist and $f(x) \geqslant g(x)$, *then*

$$\int_a^b f(x)dx \geqslant \int_a^b g(x)dx, \text{ when } b \geqslant a,$$

$$\int_a^b f(x)dx \leqslant \int_a^b g(x)dx, \text{ when } b \leqslant a.$$

Under the given conditions $[f(x)-g(x)]$ is integrable and $\geqslant 0$. Therefore

$$\int_a^b [f(x)-g(x)]dx \geqslant 0 \text{ or } \leqslant 0,$$

according as $b \geqslant a$ or $b \leqslant a$,

i.e., $$\int_a^b f(x)dx - \int_a^b g(x)dx \geqslant 0 \text{ or} \leqslant 0,$$

according as $b \geqslant a$ or $b \leqslant a$.

Hence the result.

Cor. 6. *If*

$$\int_a^b f(x)dx,$$

exists, then

$$\left|\int_a^b f(x)dx\right| \leqslant \left|\int_a^b |f(x)|\, dx\right|.$$

It has been shown in §86·6, that

$$\int_a^b |f(x)|\, dx$$

exists. We have

$$-|f(x)| \leqslant f(x) \leqslant |f(x)|.$$

If $b \geqslant a$, we have

$\therefore$ $$-\int_a^b |f(x)|\, dx \leqslant \int_a^b f(x)dx \leqslant \int_a^b |f(x)|\, dx, \qquad \text{cor. 5}$$

or $$\left|\int_a^b f(x)\, dx\right| \leqslant \int_a^b |f(x)|\, dx = \left|\int_a^b |f(x)|\, dx\right|$$

If $b \leqslant a$, we have, as proved above,

$$\left| \int_b^a f(x)\,dx \right| \leqslant \left| \int_b^a |f(x)|\,dx \right|$$

i.e.,
$$\left| \int_a^b f(x)\,dx \right| \leqslant \left| \int_a^b |f(x)|\,dx \right|.$$

89. Functions defined by definite integrals. If

$$\int_a^b f(x)dx$$

exists, then the function $\phi(t)$ of t, where

$$\phi(t) = \int_a^t f(x)dx,$$

is defined in (a, b). It is now proposed to study the properties of $\phi(t)$, in relation to continuity and derivability.

The function $\phi(x)$ may be called the integral function of $f(x)$.

89.1. *If* $\int_a^b f(x)dx$ *exists, then*

$$\phi(t) = \int_a^b f(x)dx$$

is continuous in (a, b).

There exists a number k such that $|f(x)| < k$ in (a, b).

Let c be any point of (a, b).

Let ϵ be *any* positive number. We have

$$\phi(c) = \int_a^c f(x)dx, \quad \phi(c+h) = \int_a^{c+h} f(x)dx.$$

$$\therefore \quad |\phi(c+h) - \phi(c)| = \left| \int_c^{c+h} f(x)dx \right| \leqslant |h|.k \qquad \text{(cor. 3. §88)}$$

$$< \epsilon, \text{ if } |h| < \epsilon/k.$$

Hence $\phi(t)$ is continuous at any point c of (a, b) and so in the interval (a, b),

Ex. If $\int_a^b f(x)dx$ exists, prove that

$$\psi(t)=\int_t^b f(x)dx$$

is continuous in (a, b),

89.2. *If $f(x)$ is continuous in (a, b), then*

$$\phi(t)=\int_a^t f(x)dx,$$

is derivable in (a, b) and

$$\phi'(t)=f(t).$$

Let c be any point of (a, b). We have

$$\phi(c+h)-\phi(c)=\int_c^{c+h} f(x)dx=hf(c+\theta h). \ (0\leqslant\theta\leqslant1). \text{ (cor. 2. §88)}$$

Since $f(x)$ is continuous at c, therefore, when $h\to0$,

$$\text{lt } f(c+\theta h)=f(c).$$

Hence
$$\underset{h\to0}{\text{lt}} \frac{\phi(c+h)-\phi(c)}{h}=\underset{h\to0}{\text{lt}} f(c+\theta h)=f(c).$$

Thus
$$\phi'(c)=f(c).$$

As c is any point of (a, b), we have in (a, b),

$$\phi'(t)=f(t).$$

Ex. If $f(x)$ is continuous, prove that

$$\psi(t)=\int_t^b f(x)dx$$

is derivable in (a, b), and $\psi'(t)=-f(t)$.

Note. Primitive. If there exists a derivable function $\phi(x)$ such that $\phi'(x)$ is equal to a given function $f(x)$ in (a, b) then we say that $\phi(x)$ is the primitive of $f(x)$. The theorem above shows that *Every continuous function possesses a primitive.*

Cor. *If a function $\phi(x)$ possesses a continuous derivative $\phi'(x)$, then*

$$\int_a^b \phi'(x)dx=\phi(b)-\phi(a)$$

The continuous function $\phi'(x)$ is integrable. Let

$$\psi(t)=\int_a^t \phi'(x)dx,$$

Since $\phi'(x)$ is continuous, we have $\psi'(t)=\phi'(t)$. Therefore the functions $\phi(t)$, $\psi(t)$ differ by a constant. Let

$$\phi(t)=\psi(t)+k=\int_a^t \phi'(x)dx+k$$

where k is a constant.

$$\therefore \qquad \phi(a)=0+k=k$$

$$\phi(b)=\int_a^b \phi'(x)dx+k$$

Hence
$$\phi(b)-\phi(a)=\int_a^b \phi'(x)dx.$$

90. The Fundamental theorem of the Integral Calculus. *If*

$$\int_a^b f(x)dx$$

exists and there exists a function $\phi(x)$ *such that*

$$\phi'(x)=f(x) \text{ in } (a, b),$$

then
$$\int_a^b f(x)dx=\phi(b)-\phi(a).$$

Let ϵ be any positive number. Since $\phi'(x)=f(x)$ is integrable in (a, b), there exists a division

$$D(a=x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n=b)$$

such that

$$\left| \sum_{r=1}^{r=n} \phi'(\xi_r)\delta_r - \int_a^b \phi'(x)dx \right| < \epsilon,$$

We particularise the *arbitrary* point ξ_r of $\delta_r \equiv (x_{r-1}, x_r)$, in the following manner :—

By the Lagrange's mean value theorem of differential calculus, there exists a point ξ_r of δ_r such that

$$\phi(x_r)-\phi(x_{r-1})=\phi'(\xi_r)\delta_r$$

$$\therefore \qquad \Sigma\phi'(\xi_r)\delta_r=\Sigma[\phi(x_r)-\phi(x_{r-1})]=\phi(b)-\phi(a).$$

$$\therefore \qquad \left| \phi(b)-\phi(a)-\int_a^b \phi'(x)dx \right| < \epsilon,$$

As ϵ is an arbitrary positive number, we have

$$\phi(b)-\phi(a)-\int_a^b \phi'(x)dx=0.$$

Hence the result

Note. In the Cor. to the preceding section and in the present section, we have established the truth of the same equality, *viz.*,

$$\int_a^b \phi'(x)dx=\phi(b)-\phi(a),$$

but the proofs are different, depending as they do upon the different conditions imposed upon the function in question.

For the validity of the proof given in the Cor., $\phi'(x)$ has to be assumed *continuous* but for the validity of the proof in the present section we require $\phi'(x)$ to be *merely bounded and integrable.* Thus the present proof seems to be more valuable, but the reader would do well to notice that the value of the theorem in the preceding section lies in the fact that it sets down a *sufficient* condition for the *existence* of a function $\phi(x)$ whose derivative is a given function $f(x)$; such a function $\phi(x)$ is usually known as the primitive or the indefinite integral of $f(x)$. Thus the theorem of the preceding section may be stated a little differently as follows :—

The sufficient condition for a function $f(x)$ to possess a primitive is that $f(x)$ is continuous and the primitive is, then, given by

$$\int_a^x f(x)dx+c,$$

where c is any constant whatsover.

The present theorem is quite unconcerned with the question of the *existence* of a primitive of $f(x)$; it simply states that if a bounded and integrable function $f(x)$ does possess a primitive $\phi(x)$, then

$$\int_a^b f(x)dx=\int_a^b \phi'(x)dx=\phi(b)-\phi(a).$$

91. Mean value theorems of the Integral Calculus.

91·1. First mean value theorem. *If*

$$\int_a^b f(x)dx \text{ and } \int_a^b \phi(x)dx$$

both exist and $\phi(x)$ keeps the same sign, positive or negative, throughout the interval of integration, then there exists a number μ, lying between the upper and lower bounds of $f(x)$ such that

$$\int_a^b f(x)\phi(x)dx=\mu \int_a^b \phi(x)dx. \tag{1}$$

Firstly suppose that $\phi(x)$ is positive. If M, m be the bounds of $f(x)$,

$$m\leqslant f(x)\leqslant \text{M}.$$

$$\therefore \quad m\phi(x)\leqslant f(x)\phi(x)\leqslant \text{M}\phi(x), \text{ for } \phi(x)>0.$$

$$\left.\begin{array}{ll} m\int_a^b \phi(x)dx\leqslant \int_a^b f(x)\phi(x)dx\leqslant \text{M}\int_a^b \phi(x)dx, & \text{if } b>a\,; \\ m\int_a^b \phi(x)dx\geqslant \int_a^b f(x)\phi(x)dx\geqslant \text{M}\int_a^b \phi(x)dx, & \text{if } b\leqslant a. \end{array}\right\} \text{(§88 Cor.5 P.124)}$$

In either case we see that there exists a number μ, lying between M and m, such that (1) is true. Hence the result.

The case when $\phi(x)$ is negative may be similarly disposed of.

Cor. In addition to the conditions of the theorem, if $f(x)$ is continuous also then there exists a number ξ belonging to the range of integration such that

$$\int_a^b f(x)\phi(x)dx=f(\xi)\int_a^b \phi(x)dx.$$

91.2. Second mean value theorem. *If*

$$\int_a^b f(x)dx \text{ and } \int_a^b \phi(x)dx$$

both exist and $\phi(x)$ is monotonic in (a, b), then there exists a point ξ of (a, b) such that

$$\int_a^b f(x)\phi(x)dx=\phi(a)\int_a^\xi f(x)dx+\phi(b)\int_\xi^b f(x)dx.$$

(This theorem is due to *Weierstrass*).

Abel's Lemma. The proof of the theorem depends upon a lemma which is due to Abel and which we now state and prove.

If

(*i*) $a_1, a_2, \ldots\ldots, a_n$ *is a monotonically decreasing set of* n *positive members,*

(*ii*) $v_1, v_2, \ldots\ldots, v_n$ *is a set of any* n *numbers,*

and (*iii*) k, K *are two numbers such that*

$$k < v_1 + v_2 + \ldots\ldots + v_p < K,$$

for all $p \leqslant n$,

then
$$a_1 k < \sum_{r=1}^{r=n} a_r v_r < a_1 K.$$

We write
$$S_p = v_1 + v_2 + \ldots\ldots + v_p.$$

We have

$$\sum_{r=1}^{r=n} a_r v_r = a_1 S_1 + a_2(S_2 - S_1) + \ldots + a_r(S_r - S_{r-1}) + \ldots + a_n(S_n - S_{n-1})$$
$$= (a_1 - a_2)S_1 + (a_2 - a_3)S_2 + \ldots\ldots + (a_{n-1} - a_n)S_{n-1} + a_n S_n.$$

Now, by (*i*), $(a_1 - a_2)$, $(a_2 - a_3), \ldots\ldots, (a_{n-1} - a_n)$, a_n are all positive. Also, by (*iii*), $k < S_p < K$, for all $p \leqslant n$.

Therefore

$$\sum_{r=1}^{r=n} a_r v_r < (a_1 - a_2)K + (a_2 - a_3)K + \ldots\ldots + (a_{n-1} - a_n)K + a_n K = a_1 K,$$

$$\sum_{r=1}^{r=n} a_r v_r > (a_1 - a_2)k + (a_2 - a_3)k + \ldots\ldots + (a_{n-1} - a_n)k + a_n k = a_1 k.$$

Hence the lemma.

Proof of the theorem. Firstly we prove the following :—

If $\int_a^b f(x)dx$ *and* $\int_a^b \psi(x)dx$ *both exist and* $\psi(x)$ *is monotonically* **decreasing** *and* **positive** *in* (a, b), *then there exists a point* ξ *of* (a, b) *such that*

$$\int_a^b f(x)\psi(x)dx = \psi(a) \int_a^\xi f(x)dx.$$

(This result is due to *Bonnett*).

Let
$$D(a = x_0, x_1, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n = b)$$

be any division of (a, b). Let M_r, m_r be the bounds of $f(x)$ in $\delta_r \equiv (x_{r-1}, x_r)$. Let $\xi_1 = a$ and let ξ_r, when $r \neq 1$, be any arbitrary point of δ_r.

We have

$$\left.\begin{aligned} m_r\,\delta_r \leqslant \int_{x_{r-1}}^{x_r} f(x)dx \leqslant M_r\,\delta_r \\ m_r\,\delta_r \leqslant f(\xi_r)\delta_r \leqslant M_r\,\delta_r \end{aligned}\right\}$$

Putting $r=1, 2, 3, \ldots\ldots, p$ where $p \leqslant n$, and adding we obtain

$$\left.\begin{aligned} \sum_{r=1}^{r=p} m_r\,\delta_r \leqslant \int_a^{x_p} f(x)dx \leqslant \sum_{r=1}^{r=p} M_r\,\delta_r \\ \sum_{r=1}^{r=p} m_r\,\delta_r \leqslant \sum_{r=1}^{r=p} f(\xi_r)\,\delta_r \leqslant \sum_{r=1}^{r=p} M_r\,\delta_r \end{aligned}\right\}$$

This gives

$$\left| \int_a^{x_p} f(x)dx - \sum_{r=1}^{r=p} f(\xi_r)\,\delta_r \right| \leqslant \sum_{r=1}^{r=p} (M_r - m_r)\delta_r \leqslant \sum_{r=1}^{r=n} (M_r - m_r)\delta_r,$$

$$\text{or } \int_a^{x_p} f(x)dx - \sum_{r=1}^{r=n} O_r\,\delta_r \leqslant \sum_{r=1}^{r=p} f(\xi_r)\delta_r \leqslant \int_a^{x_p} f(x)dx + \sum_{r=1}^{r=n} O_r\,\delta_r,$$

where $O_r = (M_r - m_r)$ is the oscillation of $f(x)$ in δ_r.

Now, $\int_a^t f(x)dx$, being a continuous function of t, (§§89·1, 50·3) is bounded. Let C, D be the bounds. Therefore we have

$$C - \sum_{r=1}^{r=n} O_r\,\delta_r \leqslant \sum_{r=1}^{r=p} f(\xi_r)\delta_r \leqslant D + \sum_{r=1}^{r=n} O_r\,\delta_r.$$

In the Abel's lemma, we put, as is justifiable,

$$v_r = f(\xi_r)\delta_r \quad a_r = \psi(\xi_r);$$
$$k = C - \Sigma\, O_r\,\delta_r, \qquad K = D + \Sigma\, O_r\,\delta_r,$$

and obtain

$$\psi(a)\left[C - \sum_{r=1}^{r=n} O_r\,\delta_r \right] \leqslant \sum_{r=1}^{r=n} f(\xi_r)\psi(\xi_r)\delta_r \leqslant$$
$$\psi(a)\left[D + \sum_{r=1}^{r=n} O_r\,\delta_r \right].$$

Let the norm of the division tend to 0. We then obtain, in the limit,

$$C\,\psi(a) \leqslant \int_a^b f(x)\,\psi(x)dx \leqslant D\,\psi(a),$$

or

$$\int_a^b f(x)\,\psi(x)dx = \mu.\psi(a),$$

where μ is some number between C and D.

The continuous function

$$\int_a^t f(x)dx$$

of t must obtain, for some value ξ of t, the value μ which lies between its bounds C, D. (Cor. 1 to §50·4). Thus we obtain

$$\int_a^b f(x)\,\psi(x)dx = \psi(a)\int_a^\xi f(x)dx \tag{1}$$

We now return to the theorem proper.

Let $\phi(x)$ be monotonically decreasing so that

$$\psi(x) = \phi(x) - \phi(b)$$

is monotonically decreasing and positive.

There exists, therefore, a number ξ, between a and b, such that

$$\int_a^b f(x)\,[\phi(x) - \phi(b)]dx = [\phi(a) - \phi(b)]\int_a^\xi f(x)dx$$

or

$$\int_a^b f(x)\,\phi(x)dx = \phi(a)\int_a^\xi f(x)dx + \phi(b)\left\{\int_a^b f(x)dx - \int_a^\xi f(x)dx\right\}$$

$$= \phi(a)\int_a^\xi f(x)dx + \phi(b)\int_\xi^b f(x)dx.$$

Let $\phi(x)$ be monotonically increasing so that

$$\psi(x) = \phi(b) - \phi(x)$$

is monotonically decreasing and positive.

There exists, therefore, a number ξ, between a and b, such that

$$\int_a^b f(x)\,[\phi(b)-\phi(x)]dx=[\phi(b)-\phi(a)]\int_a^\xi f(x)dx$$

or
$$\int_a^b f(x)\,\phi(x)dx=\phi(a)\int_a^\xi f(x)dx+\phi(b)\left\{\int_a^b f(x)dx-\int_a^\xi f(x)dx\right\}$$

$$=\phi(a)\int_a^\xi f(x)dx+\phi(b)\int_\xi^b f(x)dx.$$

Thus we have completely established the theorem.

Note. The reader may easily show that the theorem holds good even if $a>b$.

Ex. Taking $f(x)=x$ and $\phi(x)=e^x$ verify the two mean value theorems for the interval $(-1, 1)$.

Ex. Show that the second mean value theorem does not hold good in the interval $(-1, 1)$ for $f(x)=\phi(x)=x^2$. What about the validity of the first mean value theorem in the same case.

92. Change of variable in an integral. *If*

(*i*) $\int_a^b f(x)dx$ *exists,*

(*ii*) $x=\phi(t)$ *is a derivable function in* (α, β) *and* $\phi'(t)\neq 0$ *for any value of* t, *and* $\phi(\alpha)=a$, $\phi(\beta)=b$,

(*iii*) $f[\phi(t)]$ *and* $\phi'(t)$ *are bounded and integrable in* (α, β),

then
$$\int_a^b f(x)\,dx=\int_\alpha^\beta f[\phi(t)]\;\phi'(t)dt.$$

Since $\phi'(t)\neq 0$, it follows by Darboux's theorem (§62, P. 87) that $\phi'(t)$ must always have the same sign and therefore $\phi(t)$ must be strictly monotonic in (α, β) (§66).

let $\qquad D(\alpha=t_0, t_1, t_2, \ldots\ldots, t_{r-1}, t_r, \ldots\ldots, t_{n-1}, t_n=\beta)$
be any division of (α, β) and

let $\qquad D'(a=x_0, x_1, x_2, \ldots\ldots x_{r-1}, x_r, \ldots\ldots, x_{n-1}, x_n=b)$
be the corresponding division of (a, b), $\phi(t_r)$ being equal to x_r.

By the Lagrange's mean value theorem, we have

$$x_r-x_{r-1}=\phi(t_r)-\phi(t_{r-1})=(t_r-t_{r-1})\,\phi'(\eta_r),$$

where η_r lies between t_{r-1} and t_r.

Let $\qquad \phi(\eta_r)=\xi_r$.

We have

$$\sum_{r=1}^{r=n} f(\xi_r)(x_r-x_{r-1})=\sum_{r=1}^{r=n} f[\phi(\eta_r)]\,\phi'(\eta_r)(t_r-t_{r-1}) \qquad (1)$$

Now $f(x)$ is integrable in (a, b);

also $f[\phi(t)]$, $\phi'(t)$ are integrable in (α, β) so that $f[\phi(t)]\ \phi'(t)$ is also integrable in (α, β).

As the norm of the division $D \to 0$, the norm of D' also $\to 0$. From (1), therefore, we obtain in the limit

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\phi(t)]\ \phi'(t)dt.$$

Note. *The theorem holds even if $\phi'(t)=0$ for a* **finite** *number of values of t belonging to* (α, β) In this case we can divide the range (α, β) into a finite number of ranges in each of which $\phi(t)$ is strictly increasing or decreasing and repeat the argument for each interval in turn and add the results.

93. Integration by parts. *If*

$$\int_a^b f(x)dx,\ \int_a^b g(x)dx$$

both exist and

$$F(x)=A+\int_a^x f(x)dx,\ G(x)=B+\int_a^x g(x)dx,$$

where A, B are two constants, then

$$\int_a^b F(x)\ g(x)dx = \Big|\ F(x)\ G(x)\ \Big|_a^b - \int_a^b G(x)\ f(x)dx.$$

[Here $\Big|\ F(x)\ G(x)\ \Big|_a^b$ denotes the difference $[F(b)\ G(b)-F(a)\ G(a)]$.

Proof. Let

$$D(a=x_0,\ x_1,\ x_2,\ x_3, \ldots\ldots, x_{r-1},\ x_r, \ldots\ldots, x_n=b)$$

be any division of (a, b). We have

$$\Big|\ F(x)\ G(x)\ \Big|_a^b = \sum_{r=1}^{r=n} [F(x_r)\ G(x_r)-F(x_{r-1})\ G(x_{r-1})]$$

$$=\Sigma F(x_r)\ [G(x_r)-G(x_{r-1})]+\Sigma G(x_{r-1})[F(x_r)-F(x_{r-1})]$$

$$= \Sigma F(x_r) \int_{x_{r-1}}^{x_r} g(x)dx+\Sigma G(x_{r-1}) \int_{x_{r-1}}^{x_r} f(x)\ dx \qquad (1)$$

Let M_r, m_r, O_r denote the bounds and oscillation of $f(x)$ and M'_r, m'_r, O'_r those of $g(x)$ in $\delta_r \equiv (x_{r-1}, x_r)$.

For every x in δ_r,

$$|\ g(x)-g(x_r)\ | \leqslant O'_r,\ |\ f(x)-f(x_{r-1})\ | \leqslant O_r,$$

i. e.,
$$\left.\begin{aligned} g(x_r)-O'_r\leqslant g(x)\leqslant g(x_r)+O'_r\,;\\ f(x_{r-1})-O_r\leqslant f(x)\leqslant f(x_{r-1})+O_r.\end{aligned}\right\}$$

$$\therefore\left.\begin{aligned}[g(x_r)-O'_r]\delta_r\leqslant\int_{x_{r-1}}^{x_r} g(x)dx\leqslant[g(x_r)+O'_r]\delta_r\,;\\ [f(x_{r-1})-O_r]\,\delta_r\leqslant\int_{x_{r-1}}^{x_r} f(x)dx\leqslant[f(x_{r-1})+O_r]\delta_r.\end{aligned}\right\}$$

These give

$$\int_{x_{r-1}}^{x_r} g(x)dx=[g(x_r)+\theta'_r.O'_r]\delta_r \tag{2}$$

$$\int_{x_{r-1}}^{x_r} f(x)dx=[f(x_{r-1})+\theta_r.O_r]\delta_r, \tag{3}$$

where $-1\leqslant\theta_r,\ \theta'_r\leqslant 1$.

From (1), (2) and (3) we obtain

$$\Big|\,F(x)\;G(x)\,\Big|_a^b=\Sigma\,F(x_r)\;g(x_r)\delta_r+\Sigma\,G(x_{r-1})\;f(x_{r-1})\delta_r+\sigma, \tag{4}$$

where $\sigma=\Sigma[F(x_r)\theta'_r O'_r+G(x_{r-1})\theta_r\,O_r]\delta_r$.

Sine $F(x)$, $G(x)$ are continuous, therefore they are bounded. Let k be a number such that

$$|\,F(x)\,|<k,\quad |\,G(x)\,|<k.$$

$$\therefore\qquad |\,\sigma\,|\leqslant k[\,\Sigma\,O_r+\Sigma O'_r]\delta_r.$$

Let the norm of the division $D\to0$ so that $\sigma\to0$.

From (4), we now obtain

$$\Big|\,F(x)\;G(x)\,\Big|_a^b=\int_a^b F(x)\;g(x)dx+\int_a^b G(x)\;f(x)\;dx.$$

Hence the result.

Cor. If a function $g(x)$ is bounded and integrable in (a, b) and a function $f(x)$ is derivable in (a, b) and the derivative $f'(x)$ is bounded and integrable, then

$$\int_a^b f(x)g(x)dx=\Big|\,f(x)\int_a^x g(x)dx\,\Big|_a^b-\int_a^b\left\{f'(x)\int_a^x g(x)dx\right\}dx$$

$$=f(b)\int_a^b g(x)dx-\int\left\{f'(x)\int_a^x g(x)dx\right\}dx$$

Examples

1. (*i*) Show that if $f(x)$ is a continuous function of x in $a \leqslant x \leqslant b$, then

$$\frac{d}{dx}\int_a^x f(y)dy = f(x).$$

(*ii*) If $G(x, \xi) = \begin{cases} x(\xi - 1), \text{ when } x \leqslant \xi \\ \xi(x-1), \text{ when } \xi \leqslant x, \end{cases}$

and if $f(x)$ is a continuous function of x in $0 \leqslant x \leqslant 1$,

and if $g(x) = \int_0^1 f(\xi)\, G(x, \xi) d\xi$, show that $g''(x) = f(x)$,

and find $g(0)$ and $g(1)$.

(*ii*) *Sol.* [$g(x) = \int_0^x f(\xi)\, \xi\, (x-1) d\xi + \int_x^1 f(\xi) x (\xi - 1) d\xi$

$$= (x-1)\int_0^x \xi f(\xi) d\xi + x\int_x^1 (\xi - 1) f(\xi) d\xi$$

$$= x\int_0^1 \xi f(\xi) d\xi - \int_0^x \xi f(\xi) d\xi - x\int_x^1 f(\xi) d\xi$$

$$\therefore\ g'(x) = \int_0^1 \xi f(\xi) d\xi - x f(x) + x f(x) - \int_x^1 f(\xi) d\xi.$$

$\therefore\ g''(x) = f(x)$.

$g(0)$ and $g(1)$ are both zero.]

2. If $f(y, x) = 1 + 2x$, for y rational
$f(y, x) = 0$, for y irrational;

calculate

$$F(y) = \int_0^1 f(y, x) dx.$$

3. Integrate in $(0, 2)$ the function $x[x]$, where $[x]$ denotes the greatest integer not greater than x.

4. Evaluate

$$\int_0^{\frac{\ }{2}} f(x) dx,$$

where

$f(x)=0$, when $x=n/(n+1),(n+1)/n$, $(n=1, 2, 3,\ldots\ldots)$.
$f(x)=1$, elsewhere.
Is $f(x)$ integrable in $(0, 2)$?

Examine for continuity the function so defined at the point $x=1$.

5. A function $f(x)$ is defined, for $x \geqslant 0$, by

$$f(x)=\int_{-1}^{1} \frac{dt}{\sqrt{(1-2tx+x^2)}}.$$

Prove that if $0 \leqslant x \leqslant 1, f(x)=2$. What is the value of $f(x)$ if $x>1$? Has the function $f(x)$ a differential co efficient for $x=1$.

[For $x>1$, $f(x)=2/x$, $f(x)$ is not derivable for $x=1$ even though it is continuous thereat.]

6. Prove that if the functions $f(x)$ and $\phi(x)$ are bounded and integrable in (a, b), then

$$\left[\int_a^b f(x)\phi(x)dx\right]^2 \leqslant \int_a^b [f(x)]^2 dx. \int_a^b [\phi(x)]^2 dx.$$

Under what conditions does the sign of equality hold ?

Sol. We have

$$\left[\int_a^b f(x)\phi(x)dx\right]^2=[\text{lt } \Sigma(x_r-x_{r-1})\, f(\xi_r)\phi(\xi_r)]^2$$

$$\int_a^b f^2(x)\, dx=\text{lt } \Sigma[\sqrt{(x_r-x_{r-1})}\, f(\xi_r)]^2$$

$$\int_a^b \phi^2(x)dx=\text{lt } \Sigma[(x_r-x_{r-1})\phi(\xi_r)]^2.$$

Now putting

$$a_r=\sqrt{(x_r-x_{r-1})}f(\xi_r),\ b_r=\sqrt{(x_r-x_{r-1})}\phi(\xi_r)$$

in the Cauchy's inequality

$$\Sigma a^2_r\, \Sigma b^2_r \geqslant (\Sigma a_r b_r)^2,$$

we get the required result.

The sign of equality holds when

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots\ldots$$

i. e.,
$$\frac{f(\xi_1)}{\phi(\xi_1)}=\frac{f(\xi_2)}{\phi(\xi_2)}=\ldots\ldots$$

i. e., when $f(x)$, $\phi(x)$ are both constants

7. *Show that*

$$\lim_{m\to\infty}\int_0^x \frac{t^m}{t+1}dt=0,\ for -1<x\leqslant 1.$$

[Let $0 \leqslant x \leqslant 1$. Then

$$0 \leqslant \int_0^x \frac{t^m dt}{1+t} < \int_0^x t^m dt = \frac{x^{m+1}}{m+1} \leqslant \frac{1}{m+1}.$$

Let $-1 < x < 0$. Putting $t = -u$, we obtain

$$\left| \int_0^x \frac{t^m}{1+t} dt \right| = \left| \int_0^{-x} \frac{u^m du}{1-u} \right| < \frac{1}{1+x} \int_0^{-x} u^m du < \frac{1}{(m+1)(x+1)}.$$

8. *Show that, when* $|x| \leqslant 1$,

$$\int_0^x \frac{dt}{1+t^4} = x - \frac{1}{5} x^5 + \frac{1}{9} x^9 - \frac{1}{13} x^{13} + \ldots\ldots$$

We have

$$\frac{1}{1+t^4} = 1 - t^4 + t^8 - t^{12} + \ldots\ldots + (-1)^{n-1} t^{4n-4} + \frac{(-1)^n t^{4n}}{1+t^4}$$

$$\therefore \int_0^x \frac{dt}{1+t^4} = x - \frac{1}{5} x^5 + \frac{1}{9} x^9 - \frac{1}{13} x^{13} + \ldots\ldots + \frac{(-1)^{n-1}}{4n} x^{4n}$$

$$+ (-1)^n \int_0^x \frac{t^{4n}}{1+t^4} dt.$$

Now, we have

$$0 \leqslant \left| \int_0^x \frac{t^{4n}}{1+t^4} dt \right| < \left| \int_0^x t^{4n} dt \right| = \left| \frac{x^{4n+1}}{4n+1} \right| < \frac{1}{4n+1}$$

so that

$$\lim_{n \to \infty} \int_0^x \frac{t^{4n}}{1+t^4} dt = 0.$$

9. Prove that

$$\lim_{x \to \infty} e^{-x^2} \int_0^x e^{t^2} dt = 0.$$

10. If $f(x)$ is positive and monotonically decreasing in $(1, \infty]$, show that the sequence $\{ A_n \}$, where

$$A_n = \left\{ f(1) + f(2) + \ldots\ldots + f(n) - \int_1^n f(x) dx \right\},$$

is convergent.

Deduce the convergence of $(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} - \log n)$.

CHAPTER VII

UNIFORM CONVERGENCE

Trigonometric Functions

94. Limit function of a convergent sequence of Functions. Let $[S_n(x)]$, *i.e.*,

$$S_1(x),\ S_2(x), \ldots\ldots\ldots,\ S_n(x), \ldots\ldots$$

be a sequence, every member of which is a function of x defined in some interval (a, b).

To each point c of (a, b), there corresponds a sequence

$$S_1(c),\ S_2(c), \ldots\ldots, S_n(c), \ldots\ldots\ldots$$

of constant numbers. We suppose that all such sequences obtained by taking different points of (a, b) are convergent. The limiting values of these sequences define a function, say, $S(x)$, such that the value $S(c)$ of this function for a value c of x is the limiting value of the convergent sequence $\{ S_n(c) \}$. This function $S(x)$ is said to be the limit function or the limit of the convergent sequence $\{ S_n(x) \}$.

Again, let $\{ f_n(x) \}$ be any given sequence of functions.

Consider the infinite series

$$f_1(x)+f_2(x)+\ldots\ldots+f_n(x)+\ldots\ldots\ldots\ldots \qquad (1)$$

This series gives rise to a sequence of functions $\{ S_n(x) \}$, where

$$S_n(x)=f_1(x)+f_2(x)+\ldots\ldots+f_n(x).$$

The series (1) is said to be convergent, if the sequence $\{ S_n(x) \}$ is convergent and the limit $S(x)$ of the sequence is said to be the sum function or the sum of the series.

95. In the chapters, IV, V, VI, it has been shown that the algebraic sum of a *finite* number of continuous (derivable, integrable) functions is itself continuous (derivable, integrable). Also if

$$S_n(x)=f_1(x)+f_2(x)+\ldots\ldots+f_n(x),$$

then

$$S'_n(x)=f'_1(x)+f'_2(x)+\ldots\ldots+f'_n(x)$$

i.e., $$\frac{d}{dx}\Sigma f_r(x)=\Sigma \frac{d}{dx}f_r(x),$$

where each, $f_r(x)$, is derivable, and

$$\int_a^b S_n(x)dx=\int_a^b f_1(x)dx+\int_a^b f_2(x)dx+\ldots+\int_a^b f_n(x)dx,$$

i.e., $$\int_a^b \Sigma f_r(x)dx=\Sigma\int_a^b f_r(x)dx,$$

where each $f_r(x)$ is integrable in (a, b).

We now consider some examples which illustrate that these results may not hold good in case the number of functions is *infinite.*

95·1. Let $f_n(x)=x^2(1-x^2)^{n-1}$.

The series $\Sigma f_n(x)$, *i.e.*,

$$x^2+x^2(1-x^2)+x^2(1-x^2)^2+\dots\dots+x^2(1-x^2)^{n-1}+\dots\dots$$

is a geometrical progression whose common ratio is $(1-x^2)$. The sum $S_n(x)$ of the n terms of this series is given by

$$S_n(x)=\frac{x^2[1-(1-x^2)^n]}{1-(1-x^2)}=1-(1-x^2)^n \text{ ; if } x\neq 0,$$

and $\quad S_n(x)=0$, if $x=0$.

We know that if $x\neq 0$, then lt $(1-x^2)^n$ exists finitely and $=0$, if and only if

$$-1<1-x^2<1, \text{ i.e., if } |x|<\sqrt{2}.$$

Thus we see that if $S(x)$ denotes the sum to infinity of the given series, we have

$$S(x)=\begin{cases}0, \text{ when } x=0\\ 1, \text{ when } 0<|x|<\sqrt{2}.\end{cases}$$

This shows that the sum function $S(x)$ is not continuous for $x=0$ even though every term $f_n(x)$ is continuous for $x=0$.

95·2. Let $\qquad f_n(x)=nxe^{-nx^2}-(n-1)\,xe^{-(n-1)x^2}$.

It will now be shown that

$$\int_0^1 [f_1(x)+f_2(x)+\dots\dots+f_n(x)+\dots\dots]dx$$

$$\neq\int_0^1 f_1(x)dx+\int_0^1 f_2(x)dx+\dots+\int_0^1 f_n(x)dx+\dots, \quad (1)$$

i.e., the integral of the sum $\neq$ the sum of the integrals.

The sum $S_n(x)$ of the first n terms of the infinite series is given by

$$S_n(x)=nxe^{-nx^2}.$$

We know from §71·1, P. 94 that for every value of x

$$e^{nx^2}>(nx^2)^2/2=n^2x^4/2$$

so that $$\left|nxe^{-nx^2}\right|<\frac{n\,|x|}{n^2x^4/2}=\frac{2}{n\,|x|^3}, \text{ if } x\neq 0.$$

Thus we see that as $n\to\infty$, $S_n(x)\to 0$ for every value of x, *i.e.*,

$$S(x)=0, \qquad \text{for every value of } x.$$

Therefore the left hand side of $(1)=0$.

The sum ρ of the infinite series on the right hand side of (1) is the limit of ρ_n where ρ_n denotes the sum of the first n integrals. Since

$$\int_0^1 nx\, e^{-nx^2}\, dx = -\tfrac{1}{2} \left| e^{-nx^2} \right|_0^1 = \tfrac{1}{2}(1-e^{-n}),$$

we have

$$\int_0^1 f_n(x)dx = \tfrac{1}{2}(1-e^{-n}) - \tfrac{1}{2}\left\{1-e^{-(n-1)}\right\} = \tfrac{1}{2}\left\{e^{-(n-1)} - e^{-n}\right\},$$

and accordingly

$$\rho_n = \tfrac{1}{2}(1-e^{-n}).$$

$$\therefore \qquad \rho = \text{lt}\, \rho_n = \tfrac{1}{2}, \text{ as } n \to \infty.$$

Since $0 \neq \frac{1}{2}$, we have the required result.

95·3. Let $$f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}.$$

It will be shown that

the derivative for $x=0$ of the sum $S(x)$, *i.e.*, $S'(0)$

$\neq$ the sum of the derivatives, *i.e.*, $\Sigma f'_n(0)$.

We have

$$S_n(x) = \frac{nx}{1+n^2x^2}.$$

$$\therefore \qquad S(x) = 0, \text{ for every value of } x,$$

$$\therefore \qquad S'(0) = 0.$$

Also it is easy to see that $f'_n(0)=1$, for every value of n, so that $\Sigma f'_n(0)$ is a divergent series. Hence

$$\frac{d}{dx} \Sigma f_n(x) \neq \Sigma \frac{d}{dx} f_n(x),$$

for $x=0$.

Note. It will thus be seen that the *inversion of the operations of addition and integration* as implied by the equality

$$\int_a^b \Sigma f_n(x) d = x\Sigma \int_a^b f_n'x) dx$$

(integral of the sum = the sum of the integrals)

and the *inversion of the operations of addition and differentiation* as implied by the equality

$$\frac{d}{dx} \Sigma f_n'x) = \Sigma \frac{d}{dx} f_n(x)$$

(derivative of the sum = the sum of the derivatives)

(which are certainly valid when the summation extends only to a finite number of terms) may not be valid when the summation extends to an infinite number of terms. The concept of *Uniform convergence* which is introduced in the following section enables us to obtain sufficient conditions for the validity of the inversions in the case of an infinite number of terms.

96. The uniform convergence. The condition that $S(x)$ may be the limit of a convergent sequence of functions $\{S_n(x)\}$ in (a, b), is that to each positive number ϵ, there corresponds a positive integer m such that

$$|S_n(x)-S(x)|<\epsilon, \text{ when } n \geqslant m.$$

Obviously the value of m will depend upon ϵ as well as x and as such it may be written as $m(\epsilon,x)$.

Suppose, now, that we fix ϵ and vary x. To each value of x will correspond a value of m. The infinite aggregate of these values of m may or may not be bounded above. In case this aggregate is bounded above there exists a value m_0, (the upper bound of the aggregate of the values of m), such that

$$|S_n(x)-S(x)|<\epsilon,$$

when $n \geqslant m_0$ and x *has any value whatsoever.* In such a case we say that the sequence uniformly converges.

Def. *A sequence $\{S_n(x)\}$ is said to converge uniformly to a function $S(x)$ in (a, b), if, given any positive number ϵ, there exists a positive integer m such that*

$$|S_n(x)-S(x)|<\epsilon,$$

for every value of $n \geqslant m$ and every value of x in (a, b).

Also a series $\Sigma f_n(x)$ is said to converge uniformly if the sequence $\{S_n(x)\}$, where

$$S_n(x)=f_1(x)+f_2(x)+\ldots\ldots+f_n(x),$$

uniformly converges.

We now illustrate the notion of uniform convergence by considering some particular cases.

1. *Let* $S_n(x)=n/(x+n)$. $(x \geqslant 0)$.

We have, when $n \to \infty$,

$$S(x)=\text{lt } [n/(x+n)]=1, \text{ for every value of } x.$$

If ϵ be any given positive number, we have

$$|S_n(x)-S(x)|=x/(x+n)<\epsilon, \text{ if } n>x(1/\epsilon-1).$$

Here $m(\epsilon, x)$=the integer just greater then $x(1/\epsilon-1)$.

Obviously $m(\epsilon, x)$ increases as x increases and $\to \infty$ as $x \to \infty$ so that it is not possible to choose a number m_0 such that

$$|S_n(x)-S(x)|<\epsilon,$$

for every $n \geqslant m_0$ and every value of x in $(0, \infty)$.

Thus the convergence is not uniform in $(0, \infty)$.

If, however, we consider the interval $(0, k)$, where k is any fixed number, however large, we see that the maximum value of $x(1/\epsilon-1)$ is $k(1/\epsilon-1)$ so that, taking m_0=any integer greater than $k(1/\epsilon-1)$, we have

$$|S_n(x)-S(x)|<\epsilon,$$

for every $n \geqslant m_0$ and every x in $(0, k)$.

Thus we see that the sequence $\{ S_n(x) \}$ converges uniformly in the interval $(0, k)$ where k is any fixed positive number, however large.

Note. It may similarly be shown that the sequence converges uniformly in $(-k, 0)$, where k is any fixed positive number, however large.

2. *Let* $S_n(x)=x^n$. $\qquad (0 \leqslant x \leqslant 1)$.

We have, when $n \to \infty$,

$$S(x)=\text{lt } S_n(x)=\begin{cases} 0, \text{ when } 0 \leqslant x < 1 \\ 1, \text{ when } x=1. \end{cases}$$

We consider $0 \leqslant x < 1$ *i.e.*, the interval $(0, 1]$.

Let ϵ be any positive number. We have

$$| S_n(x)-S(x) | = x^n < \epsilon,$$

$$\left.\begin{array}{ll} \text{if} & (1/x)^n > 1/\epsilon, \ i.e., \text{ if } n \log(1/x) > \log(1/\epsilon), \\ \text{or if} & n > \log(1/\epsilon)/\log(1/x), \end{array}\right\} \text{ when } x \neq 0,$$

Thus, when $x \neq 0$, $m(\epsilon, x)=$the integer just greater than $\log(1/\epsilon)/\log(1/x)$

Also obviously $m(\epsilon, x)=1$, when $x=0$.

We now see that $m(\epsilon, x)$ increases and $\to \infty$, when x, starting from 0 increases and tends to '1' so that it is not possible to choose m_0 such that

$$| S_n(x)-S(x) | < \epsilon,$$

for every $n \geqslant m$, and every value of x in $(0, 1]$. Thus the convergence is not uniform in the interval $(0, 1]$.

If, however, we consider any number k such that $0 \leqslant k < 1$, we see that the maximum value of $\log(1/\epsilon)/\log(1/x)$ is $\log(1/\epsilon)/\log(1/k)$ so that if we take $m_0=$any integer greater than this maximum value, we have

$$| S_n(x)-S(x) | < \epsilon,$$

for every $n \geqslant m_0$ and every x in $(0, k)$.

Thus the convergence is uniform in $(0, k)$.

Note. The point $x=1$, which is such that the sequence does not converge uniformly in any neighbourhood of $x=1$, however small it may be, is said to be a **point of non-uniform convergence** of the sequence.

3. *Let* $S_n(x)=e^{-nx}$ $\qquad x \geqslant 0$.

We have, when $n \to \infty$,

$$S(x)=\text{lt } S_n(x)=\begin{cases} 1, \text{ when } x=0, \\ 0, \text{ when } x>0. \end{cases}$$

We consider $x>0$.

Let ϵ be any positive number. We have

$$| S_n(x)-S(x) | = e^{-nx} < \epsilon,$$

if $\qquad e^{nx} > 1/\epsilon$, *i.e.*, if $nx > \log(1/\epsilon)$,

or if $\qquad n > \log(1/\epsilon)/x$.

Here $m(\epsilon, x)=$the integer just greater than $\log(1/\epsilon)/x$. Arguing as before, we see that the convergence is not uniform in $[0, \infty]$, but it is uniform in $(k, \infty]$, where k is any positive number whatsoever.

The point $x=0$ is a point of non-uniform convergence of the sequence.

Ex. *Show that '0' is a point of non-uniform convergence of the sequence* $\{S_n(x)\}$, *where* $S_n(x)=1-(1-x^2)^n$.

It is easily seen that

$$S(x)=\begin{cases} 0, \text{ when } x=0, \\ 1, \text{ when } 0<|x|<\sqrt{2}. \end{cases}$$

Let, if possible, the sequence be uniformly convergent in a neighbourhood $[0, k)$ of '0' where k is a number such that $0<|k|<\sqrt{2}$. There exists, therefore, a positive integer m such that taking $\epsilon=\frac{1}{2}$,

$$|S_m(x)-S(x)|=(1-x^2)^m<\tfrac{1}{2}, \tag{1}$$

for every value of x in $[0, k)$.

Since $(1-x^2)^m\to 1$ as $x\to 0$, we see that the inequality (1) cannot hold true in the neighbourhood of $x=0$ Hence the supposition is wrong.

Ex. *Show that 'o' is a point of non-uniform convergence of the sequence* $\{S_n(x)\}$, *where* $S_n(x)=nxe^{-nx^2}$.

It is easily seen that $S(x)=0$ for every value of x.

If possible, let the sequence be uniformly convergent in a neighbourhood $(0, k)$ of 0, where k is any positive number.

There exists, therefore, a positive integer m such that, taking $\epsilon=1$,

$$|S_n(x)-S(x)|=nxe^{-nx^2}<1. \tag{1}$$

for every value of (x) in $(0, k)$ and for $n\geqslant m$.

In particular, the inequality (1) must be true for $x=1/\sqrt{n}$ where n is any integer $>1/k^2$ as well as m, so that we have

$$nxe^{-nx^2}=\sqrt{n}/e<1. \tag{2}$$

Since $x\to 0$ when $n\to\infty$, we see that on taking x sufficiently near 0 we can take n so large that $\sqrt{n}/e>1$ and thus we have a contradiction of (2).

Ex. Show that $x=0$ is a point of non-uniform convergence of the sequence $[nx/(1+n^2x^2)]$.

Ex. Show that the series

$$\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+\cdots$$

is uniformly convergent in $(k, \infty]$, where k is any positive number and that $x=0$ is a point of non-uniform convergence.

Ex. Show that the series H.S.

$$(1-x)^2+x(1-x)^2+x^2(1-x)^2+\ldots\ldots\ldots\ldots$$

is not uniformly convergent in $(0, 1)$.

Ex. Show that the series

$$\Sigma\frac{x}{n(n+1)}$$

is uniformly convergent in $(0,k)$ where k is any positive number whatsoever but that it does not converge uniformly in $(0, \infty]$.

97. Test for the uniform convergence of a series.

97·1. General principle of convergence. *The necessary and sufficient condition for the uniform convergence in (a, b) of a series $\Sigma f_n(x)$ is that to every positive number ϵ there corresponds a positive integer m such that*

$$|f_{n+1}(x)+f_{n+2}(x)+\ldots\ldots+f_{n+p}(x)|<\epsilon,$$

for every $n \geqslant m$, every positive integer p and every value of x in (a, b).

The condition is necessary. Let ϵ be any positive number.

Let $\qquad S_n(x)=f_1(x)+f_2(x)+\ldots\ldots+f_n(x),$

and $\qquad S(x)=\text{lt } S_n(x)$, when $n \to \infty$.

Since the series is uniformly convergent, there exists a positive integer m such that

$$|S_n(x)-S(x)|<\tfrac{1}{2}\epsilon, \tag{1}$$

for every $n \geqslant m$ and for every value of x in (a, b).

Also, therefore,

$$|S_{n+p}(x)-S(x)|<\tfrac{1}{2}\epsilon, \tag{2}$$

for every $n \geqslant m$, every positive integer p and every x in (a, b),

From (1) and (2), we see that

$$|f_{n+1}(x)+f_{n+2}(x)+\ldots\ldots+f_{n+p}(x)|=|S_{n+p}(x)-S_n(x)|<\epsilon,$$

for every $n \geqslant m$, every positive integer p and every x in (a, b).

The condition is sufficient. We know that when this condition is satisfied the series is convergent. (§35·1, P. 47). All that we have now to show is that the convergence is uniform. Let $S(x)$ be the sum of the series.

Let ϵ be any positive number. There exists a positive integer m such that

$$|S_{n+p}(x)-S_n(x)|<\tfrac{1}{2}\epsilon,$$

i. e. $$S_n(x)-\tfrac{1}{2}\epsilon<S_{n+p}(x)<S_n(x)+\tfrac{1}{2}\epsilon, \tag{3}$$

for every $n \geqslant m$, every $p \geqslant 0$ and every x in (a, b).

Keeping n fixed, let $p \to \infty$ so that $S_{n+p}(x) \to S(x)$.

Therefore we have, from (3),

$$S_n(x)-\tfrac{1}{2}\epsilon \leqslant S(x) \leqslant S_n(x)+\tfrac{1}{2}\epsilon$$

i. e. $$|S_n(x)-S(x)| \leqslant \tfrac{1}{2}\epsilon<\epsilon,$$

for every $n \geqslant m$ and every x in (a, b).

Hence the series is uniformly convergent.

97·2. Weierstrass's M-test for uniform convergence. *A series $\Sigma f_n(x)$ will converge uniformly in (a, b), if there exists a convergent series ΣM_n of positive constants such that*

$$|f_n(x)| \leqslant M_n, \tag{i}$$

for every value of n and every value of x in (a, b).

Let ϵ be any positive number. Since ΣM_n is convergent, there exists a positive integer m such that

$$|M_{n+1}+M_{n+2}+\ldots\ldots\ldots+M_{n+p}|<\epsilon \tag{ii}$$

for every $n \geqslant m$ and every $p \geqslant 0$. (§35·1, P. 47).

From (*i*) and (*ii*), we obtain

$$|f_{n+1}(x)+f_{n+2}(x)+\ldots\ldots+f_{n+p}(x)|$$
$$\leqslant [M_{n+1}+M_{n+2}+\ldots\ldots+M_{n+p}]<\epsilon,$$

for every $n \geqslant m$, every $p \geqslant 0$ and every x in (a, b).

Hence $\Sigma f_n(x)$ is uniformly convergent in (a, b).

98. Properties of uniformly convergent series.

98·1. Continuity of the sum. *The sum function of a uniformly convergent series of continuous functions is itself continuous.*

Let $\Sigma f_n(x)$ be a series which is uniformly convergent in (a, b) and let each term $f_n(x)$ be continuous in (a, b). It will be shown that the sum function $S(x)$ of the series is also continuous in (a, b).

Let $$S_n(x)=f_1(x)+f_2(x)+\ldots\ldots\ldots+f_n(x).$$

Let c be *any* point of (a, b).

Since the series is uniformly convergent, there exists a positive integer m such that

$$|S_n(x)-S(x)|<\tfrac{1}{3}\epsilon,$$

for every $n \geqslant m$ and every x in (a,b).

In particular, this gives

$$|S_m(x)-S(x)|<\tfrac{1}{3}\epsilon, \text{ for every } x \text{ in } (a, b), \qquad (i)$$

and $$|S_m(c)-S(c)|<\tfrac{1}{3}\epsilon. \qquad (ii)$$

Now $S_m(x)$, being the sum of a finite number m of continuous functions, is also continuous. (§ 4·1, P. 70)

There exists, therefore, a positive number δ such that

$$|S_m(x)-S_m(c)|<\tfrac{1}{3}\epsilon, \text{ when } |x-c| \leqslant \delta. \qquad (iii)$$

From (*i*), (*ii*), (*iii*), we deduce that when $|x-c| \leqslant \delta$,

$$|S(x)-S(c)| = |S(x)-S_m(x)+S_m(x)-S_m(c)+S_m(c)-S(c)|$$
$$<\epsilon.$$

Hence $S(x)$ is continuous at any point c and, therefore, in (a, b).

Note. This theorem shows that if the sum function of a series of continuous functions is discontinuous at any point, then that point must be a point of non-uniform convergence of the series. On the other hand, as the following example shows, the condition of uniform convergence is *only sufficient but not necessary* for the continuity of the sum function.

Let $$f_n(x)=\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}.$$

We have

$$S_n(x)=nx/(1+n^2x^2).$$

$\therefore$ $$S(x)=0, \text{ for every value of } x$$

so that $S(x)$ is continuous at $x=0$ even though, as may be easily shown, $x=0$ is a point of non-uniform convergence of the series.

98·2. Term-by-term integration. *If*

(*i*) *$\Sigma f_n(x)$ is uniformly convergent in (a,b)*

and (*ii*) *each $f_n(x)$ is bounded and integrable in (a, b),* *then the sum function $S(x)$ of the series is also bounded and integrable in (a,b), and*

$$\int_a^b \Sigma f_n(x)dx = \Sigma \int_a^b f_n(x)dx.$$

Firstly, we will show that the sum function $S(x)$ is integrable in (a, b). Let ϵ be any positive number.

Since the series is uniformly convergent, there exists a positive integer m such that

$$| S_n(x)-S(x) | < \epsilon/4(b-a),$$

for every $n \geqslant m$ and every x in (a, b).

In particular, we have

$$| S_m(x)-S(x) | < \epsilon/4(b-a),$$

for every x in (a, b).

We write

$$S(x)-S_m(x)=R_m(x), \text{ i. e., } S(x)=S_m(x)+R_m(x).$$

Now the function $S_m(x)$, being the sum of a finite number m of integrable functions, is itself integrable. (§ 86 2,P.119). There exists, therefore, a division D of (a, b) such that the corresponding *oscillatory sum* for $S_m(x)$ is $<\frac{1}{2}\epsilon$.

Since for every x in (a, b), $| R_m(x) | \leqslant \epsilon/4(b-a)$, the oscillation of $R_m(x)$ in every sub-interval of (a, b) is $<2. \epsilon/4(b-a)=\epsilon/2(b-a)$. Thus the oscillatory sum corresponding to any division and, in particular, the division D is

$$<(b-a).\epsilon/2(b-a)=\tfrac{1}{2}\epsilon.$$

Now corresponding to any division, the oscillatory sum for the sum of two functions is $\leqslant$ the sum of the oscillatory sums for the two functions. (Use the result proved in Ex. 8, P. 30). Thus there exists a division D such that the corresponding oscillatory sum for $S(x)$ is $<(\frac{1}{2}\epsilon+\frac{1}{2}\epsilon)=\epsilon$. Hence $S(x)$ is integrable in (a, b) This proves the first part.

Again, let ϵ be any positive number.

Since the series is uniformly convergent, there exists a positive integer m such that

$$| S_n(x)-S(x) | < \epsilon/(b-a),$$

i. e., $$-\epsilon/(b-a)<S_n(x)-S(x)<\epsilon/(b-a), \tag{1}$$

for every $n \geqslant m$ and every x in (a, b)

Now $S_n(x)$, being the sum of a finite number n of integrable functions, is itself integrable. Also $S(x)$ is integrable. Therefore $S_n(x)-S(x)$ is integrable. From (1), we have, therefore,

$$-\epsilon<\int_a^b S_n(x)dx-\int_a^b S(x)dx<\epsilon$$

i. e.,
$$\left|\int_a^b S_n(x)dx - \int_a^b S(x)dx\right| < \epsilon, \tag{2}$$

for every $n \geq m$.

Also, we have

$$\int_a^b S_n(x)dx = \int_a^b \sum_{r=1}^{r=n} f_r(x)dx = \sum_{r=1}^{r=n} \int_a^b f_r(x)dx.$$ (cor. to §86·2, P. 120)

so that the relation (2) means that

$$\lim_{n\to\infty} \sum_{r=1}^{r=n} \int_a^b f_r(x)dx = \int_a^b S(x)dx = \int_a^b \lim_{n\to\infty} \sum_{r=1}^{r=n} f_r(x)dx,$$

i. e.,
$$\Sigma \int_a^b f_n(x)dx = \int_a^b \Sigma f_n(x)dx.$$

Note. From (1), we have

$$\left| \int_a^x S_n(x)dx - \int_a^x S(x)dx \right| < [\epsilon/(b-a)](x-a) < \epsilon,$$

i. e.,
$$\left| \sum_{r=1}^{r=n} \int_a^x f_r(x)dx - \int_a^x S(x)dx \right| < \epsilon,$$

for every $n \geq m$ and every x in (a, b).

This shows that the series $\Sigma \int_a^x f_n(x)dx$, i. e.,

$$\int_a^x f_1(x)dx + \int_a^x f_2(x)dx + \ldots + \int_a^x f_n(x)dx + \ldots\ldots$$

is also uniformly convergent in (a, b). Also since each $\int_a^x f_r(x)dx$ is integrable, (in fact continuous), we deduce that the operation of successive integration may be carried out any number of times.

Note. The condition of uniform convergence is only sufficient but not necessary for the validity of term by term integration, as is shown by considering the series $\Sigma f_n(x)$, for which the sum $S_n(x)$ of the first n terms is given by

$$S_n(x) = nx/(1+n^2x^2) \qquad \text{(See note after § 98. 1)}$$

We have

$$\int_0^1 \Sigma f_n(x)dx = \int_0^1 S(x)dx = \int_0^1 0\, dx = 0,$$

and $$\sum_{r=1}^{r=n} \int_0^1 f_r(x)dx = \frac{1}{2n} \log(1+n^2) \text{ which} \to 0 \text{ as } n \to \infty,$$

Thus $$\int_0^1 \Sigma f_n(x)dx = \Sigma \int_0^1 f_n(x)dx,$$

even though 0 is a point of non-uniform convergence of the series.

98·3. Term-by-term differentiation. *If*

(*i*) *the series* $S(x)=\Sigma f_n(x)$ *is convergent in* (a, b),

(*ii*) *each term* $f_n(x)$ *is derivable in* (a, b),

and (*iii*) *the series* $\sigma(x)=\Sigma f_n'(x)$ *is uniformly convergent in* (a, b) *then*

$$\sigma(x)=S'(x).$$

Let c be any *fixed* point of (a, b),

Let y be a variable which varies in the interval $(a-c, b-c)$ so that $c+y$ is also a variable which varies in the interval (a, b).

We define, as follows, a sequence $\{\phi_n(y)\}$ of functions of y.

$$\phi_n(y)=\begin{cases} \dfrac{f_n(c+y)-f_n(c)}{y}, \text{ when } y\neq 0 \\ f'_n(c), \text{ when } y=0. \end{cases}$$

By the Lagrange's mean value theorem, there exists a number η lying between c and $c+y$, *i. e.*, between a and b such that

$$[f_n(c+y)-f_n(c)]/y=f'_n(\eta).$$

Thus we see that for *any* value of y in $(a-c, b-c)$, there exists a value of x in (a, b) such that

$$\phi_n(y)=f'_n(x).$$

Since the series $\Sigma f'_n(x)$ is uniformly convergent in (a, b), there exists a positive integer m such tnat

$$| f'_{n+1}(x)+f'_{n+2}(x)+\ldots\ldots+f'_{n+p}(x) | < \epsilon,$$

for every $n \geqslant m$, every $p \geqslant 0$ and every x in (a, b).

Therefore we have

$$| \phi_{n+1}(y)+\phi_{n+2}(y)+\ldots\ldots+\phi_{n+p}(y) | < \epsilon,$$

for every $n \geqslant m$, every $p \geqslant 0$ and every y in $(a-c, b-c)$, so that the series $\Sigma\phi_n(y)$ is uniformly convergent in $(a-c, b-c)$. From the definition of $\phi_n(y)$, we see that $\phi_n(y)$ is continuous for $y=0$. Therefore from § 98·1, we deduce that the sum function $G(y)=\Sigma\phi_n(y)$ is also continuous for $y=0$.

We have

$$\sigma(c)=\Sigma f'_n(c)=\Sigma\phi_n(0)=G(0)$$
$$= \underset{y\to 0}{\text{lt}}\ G(y) = \underset{y\to 0}{\text{lt}}\ \Sigma\phi_n(y)$$

$$=\underset{y\to 0}{\text{lt}}\ \Sigma\ \frac{f_n(c+y)-f_n(c)}{y}$$

$$=\underset{y\to 0}{\text{lt}}\ \frac{S(c+y)-S(c)}{y}=S'(c).$$

Since c is any point of (a, b), we have

$$\sigma(x)=S'(x).$$

A simple case of term by term differentiation. If we assume that *each function* $f'_n(x)$ *is continuous* in (a, b), then the proof becomes much simplified.

Since $\Sigma f'_n(x)$ is a uniformly convergent series of continuous and, therefore, integrable functions, the term by term integration is valid, so that we have

$$\int_a^x \sigma(x)dx = \int_a^x f'_1(x)dx+\int_a^x f'_2(x)dx+\ldots+\int_a^x f'_n(x)dx+\ldots\ldots$$

$$=\sum_{r=1}^{\infty}[f_r(x)-f_r(a)]=\sum_{r=1}^{\infty} f_r(x)-\sum_{r=1}^{\infty} f_r(a)$$

$$=S(x)-S(a).$$

Since $\sigma(x)$ is continuous, (§ 98·1), we have, on differentiating with respect to x, (§89·2, P. 127).,

$$\sigma(x)=S'(x).$$

This is the form in which the theorem is generally proved.

Note. The reader should note that for the validity of term-by-term differentiation, it is the derived series which must be uniformly convergent and that the original series need only be simply convergent.

99. Analytical theory of Trigonometrical Functions. The functions sin x and cos x.

99·1. Theorem. *The two series*

(A) $$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)\,!},\quad i.\ e.,\ x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\ldots\ldots\ldots$$

(B) $$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2n\,!},\quad i.\ e.,\ 1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\ldots\ldots\ldots$$

are uniformly convergent in every interval (a, b).

Let M be any positive number greater than $|a|$ as well as $|b|$ so that if x denotes any number in (a, b), we have

$$\left|\frac{(-1)^n x^{2n+1}}{(2n+1)\,!}\right|\leqslant\frac{M^{2n+1}}{(2n+1)\,!},$$

and

$$\left|\frac{(-1)^n x^{2n}}{2n\,!}\right|\leqslant\frac{M^{2n}}{2n\,!}.$$

Consider now the two series

(C) $M+\frac{M^3}{3!}+\frac{M^5}{5!}+\dots\dots\dots$

(D) $1+\frac{M^2}{2!}+\frac{M^4}{4!}+\dots\dots\dots$

We know that, (§71·1, P. 94.)

$$e^{M}=1+M+\frac{M^2}{2!}+\frac{M^3}{3!}+\dots\dots$$

$$e^{-M}=1-M+\frac{M^2}{2!}-\frac{M^3}{3!}+\dots\dots$$

$$\therefore\ \tfrac{1}{2}\left\{e^{M}+e^{-M}\right\}=1+\frac{M^2}{2!}+\frac{M^4}{4!}+\dots\dots\dots$$

$$\tfrac{1}{2}\left\{e^{M}-e^{-M}\right\}=M+\frac{M^3}{3!}+\frac{M^5}{5!}+\dots\dots\dots$$

This shows that the series (C), (D) are convergent, whatever the positive constant M may be

Therefore, by Weierstrass's M test, we prove that the series (A), (B) are uniformly convergent in (a, b).

Definitions. This theorem justifies the following two definitions:—

(A) $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}\dots\dots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\dots\dots,$

(B) $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}\dots\dots+\frac{(-1)^n x^{2n}}{2n!}+\dots\dots,$

99·2. *The functions sin x, cos x are defined and continuous for every value of x.* This follows from §98·1.

99 3. *The functions sin x, cos x are derivable for every value of x and*

$$\frac{d(\sin x)}{dx}=\cos x \quad \textit{and} \quad \frac{d(\cos x)}{dx}=-\sin x.$$

If we differentiate term by term the convergent series (A), we get the uniformly convergent series (B) and hence, (§98·3), $\sin x$ is derivable and $(\sin x)'=\cos x$.

Again, on differentiating term by term the convergent series (B), we get the uniformly convergent series

$$-x+\frac{x^3}{3!}-\frac{x^5}{5!}+\dots\dots$$

and hence $\cos x$ is derivable and $(\cos x)'=-\sin x$.

99·4. *$\sin 0=0$ and $\cos 0=1$.* The proof is obvious.

99·5. *$\sin(-x)=-\sin x$ and $\cos(-x)=\cos x$.* The proof is obvious.

100. The addition theorems. *If x, y are any numbers, then*

$\sin(x+y)=\sin x\cos y+\cos x\sin y$;

$\cos(x+y)=\cos x\cos y-\sin x\sin y$.

Giving any value to y and then keeping it fixed, we write

$f(x)=\sin(x+y)-\sin x\cos y-\cos x\sin y$;

$g(x)=\cos(x+y)-\cos x\cos y+\sin x\sin y$.

We have

$$f'(x)=\cos(x+y)-\cos x\cos y+\sin x\sin y=g(x),$$
$$g'(x)=-\sin(x+y)+\sin x\cos y+\cos x\sin y=-f(x)$$
$$\therefore\quad \frac{d[f^2(x)+g^2(x)]}{dx}=2f(x)\,f'(x)+2g(x)\,g'(x)$$
$$=2f(x)\,g(x)-2g(x)\,f(x)=0,$$

so that

$$f^2(x)+g^2(x) \text{ is a constant.}$$

Hence for every value of x,

$$f^2(x)+g^2(x)=f^2(0)+g^2(0)=0$$
$$\therefore\quad f(x)=0,\ g(x)=0.$$

Hence thetheorems.

Cor. 1. *$\cos^2 x+\sin^2 x=1$.* We have

$$1=\cos 0=\cos(x-x)=\cos x\cos(-x)-\sin x\sin(-x)$$
$$=\cos x\cos x+\sin x\sin x=\cos^2 x+\sin^2 x.$$

Cor. 2. $|\sin x|\leqslant 1,\quad |\cos x|\leqslant 1.$

This follows from the preceding corollary.

Cor. 3. Changing y to x, we have

$$\sin 2x=2\sin x\cos x,\ \cos 2x=\cos^2 x-\sin^2 x.$$

Cor. 4. Changing y to $-y$, we have

$$\sin(x-y)=\sin x\cos y-\cos x\sin y\ ;$$
$$\cos(x-y)=\cos x\cos y+\sin x\sin y.$$

101. The number π. Smallest positive root of the equation $\cos x=0$. Theorem. *To prove that there exists a positive number 'π' such that*

$$\cos(\pi/2)=0,$$

and $\quad \cos x>0,$ *for* $0\leqslant x<\pi/2.$

Consider the interval $(0, 2)$.

We know that $\cos 0$, $(=1)$, is positive and we will now show that $\cos 2$ is negative. We have

$$\cos 2=1-\frac{2^2}{2\,!}+\frac{2^4}{4\,!}-\frac{2^6}{6\,!}+\ldots\ldots$$
$$=1-\frac{2^2}{2\,!}\left(1-\frac{2^2}{3\cdot4}\right)-\frac{2^6}{6\,!}\left(1-\frac{2^2}{7\cdot8}\right)\ldots\ldots\ldots$$

Since the brackets are all positive, we have

$$\cos 2<1-\frac{2^2}{2\,!}\left(1-\frac{2^2}{3\cdot4}\right)=-\frac{1}{3},$$

so that $\cos 2$ is negative.

There exists, therefore, atleast, one number α between 0 and 2, such that $\cos\alpha=0$. Also there cannot exist more than one such value, for, if possible let β be another so that

$$\cos\alpha=0,\ \cos\beta=0,\qquad 0<\alpha,\ \beta<2.$$

By Rolle's theorem, there exists atleast one number λ between α and β such that the derivative, $-\sin x$, of $\cos x$ vanishes for $x=\lambda$, *i. e.*,

$$\sin\lambda=0\qquad 0<\lambda<2.$$

But $\quad \sin\lambda=\dfrac{\lambda}{1\,!}\left(1-\dfrac{\lambda^2}{2\cdot3}\right)+\dfrac{\lambda^5}{5\,!}\left(1-\dfrac{\lambda^2}{6\cdot7}\right)+\ldots\ldots$

which is clearly positive.

Thus there exists one and only one root of the equation $\cos x=0$ lying between 0 and 2. Denoting twice this value by π, we see that $\pi/2$ is the *least positive* root of the equation $\cos x=0$. Also, therefore, we have $\cos x>0$, when $0\leqslant x<\pi/2$.

102. *Sin* $x>0$, *when* $0<x\leqslant\frac{1}{2}\pi$.

Since the derivative $\cos x$ of $\sin x$ is positive in $(0, \frac{1}{2}\pi]$, therefore by §66·2, P. 90, $\sin x$ is strictly increasing. Also, since $\sin 0=0$, we see that $\sin x$ is positive when $0<x\leqslant\frac{1}{2}\pi$.

103·1. *Sin*$(\pi/2)=1$.

Since $\sin^2(\pi/2)+\cos^2(\pi/2)=1$,

therefore $\sin^2(\pi/2)=1$ or $\sin(\pi/2)=\pm 1$.

But, by the Lagrange's mean value theorem,

$$\sin(\pi/2)=\sin(\pi/2)-\sin 0=(\pi/2)\cos\alpha>0, \qquad 0<\alpha<\pi/2.$$

$$\therefore \quad \sin(\pi/2)=1.$$

103·2. *Cos*$\pi=-1$, $\sin\pi=0$.

$\cos\pi=2\cos^2(\pi/2)-1=-1$ and $\sin\pi=2\sin\pi/2\cos\pi/2=0$.

103·3. *Cos* $2\pi=1$, $\sin 2\pi=0$.

$\cos 2\pi=2\cos^2\pi-1=1$ and $\sin 2\pi=2\sin\pi\cos\pi=0$.

103·4. *Cos*$(\pi/4)=\sin(\pi/4)=1/\sqrt{2}$

We have $0=\cos(\pi/2)=2\cos^2(\pi/4)-1$,

$$\therefore \quad \cos(\pi/4)=1/\sqrt{2},$$

rejecting the negative sign as $\cos\pi/4$ is necessarily positive. §101.

Also $1=\sin(\pi/2)=2\sin(\pi/4)\cos(\pi/4)$

$$\therefore \quad \sin(\pi/4)=1/\sqrt{2}.$$

103·5.

$$\begin{array}{llr} Sin\ (\tfrac{1}{2}\pi-x)=\ \ \cos x, & \cos(\tfrac{1}{2}\pi-x)=\ \ \sin x. & (i)\\ Sin\ (\tfrac{1}{2}\pi+x)=\ \ \cos x, & \cos(\tfrac{1}{2}\pi+x)=-\sin x. & (ii)\\ Sin\ (\pi+x)=-\sin x, & \cos(\pi+x)=-\cos x. & (iii)\\ Sin\ (\pi-x)=\ \ \sin x, & \cos(\pi-x)=-\cos x. & (iv)\\ Sin\ (2\pi+x)=\ \ \sin x, & \cos(2\pi+x)=\cos x. & (v)\end{array}$$

These are easily proved with the help of the addition formulae.

Note. Because of the formulae (v), $\sin x$, $\cos x$ are said to be *periodic functions* with 2π as their *period*.

104.1. *Cos* $x>0$, *when* $0\leqslant x<\frac{1}{2}\pi$ or $\frac{3}{2}\pi<x\leqslant 2\pi$;
Cos $x<0$, *when* $\frac{1}{2}\pi<x\leqslant\pi$ or $\pi\leqslant x<\frac{3}{2}\pi$.

When $0\leqslant x<\frac{1}{2}\pi$, we know from §101 that $\cos x>0$.

When $\frac{1}{2}\pi<x\leqslant\pi$, we write $x=\frac{1}{2}\pi+y$ so that $0<y\leqslant\frac{1}{2}\pi$.

$\therefore \quad \cos x=\cos(\frac{1}{2}\pi+y)=-\sin y<0.$ (§102).

When $\pi\leqslant x<\frac{3}{2}\pi$, we write $x=\pi+y$ so that $0\leqslant y<\frac{1}{2}\pi$.

$\therefore \quad \cos x=\cos(\pi+y)=-\cos y<0.$ (§101).

When $\frac{3}{2}\pi<x\leqslant 2\pi$, we write $x=\pi+y$ so that $\frac{1}{2}\pi<y\leqslant\pi$.

$\therefore \quad \cos x=\cos(\pi+y)=-\cos y>0.$

104·2. *sin* $x>0$, *when* $0<x<\pi$; *sin* $x<0$, *when* $\pi<x<2\pi$.

The proof is similar to that of the preceding theorem.

Ex *Discuss how* $\sin x$ *and* $\cos x$ *vary, (monotonically increase or decrease,) as* x *varies in the interval* $(0, 2\pi)$.

Ex. Show that when n is any integer,

$$\sin n\pi=0, \qquad \cos\tfrac{1}{2}(2n+1)\pi=0;$$

and

$$\sin\tfrac{1}{2}(2n+1)\pi=(-1)^n, \qquad \cos n\pi=(-1)^n,$$

105. The function tan x. Tan x is defined by the relation

$$\tan x = \frac{\sin x}{\cos x}.$$

Clearly tan x is defined, continuous and derivable for all values of x except those for which the denominator cos x vanishes which is the case for $x=\frac{1}{2}(2n+1)\pi$, n being any integer, positive, negative or zero.

From the formula (*iii*) of §103·5, we have

$$\tan(\pi+x)=\tan x,$$

so that we see that tan x is a periodic function whose period is π.

Also we may easily show that when $x \neq \frac{1}{2}(2n+1)\pi$,

$$\frac{d(\tan x)}{dx}=\frac{d(\sin x/\cos x)}{dx}=\frac{1}{\cos^2 x}.$$

105·1. *To show that*

$$\lim_{x\to(\frac{1}{2}\pi-0)} \tan x=\infty, \qquad \lim_{x\to(\frac{1}{2}\pi+0)} \tan x=-\infty.$$

Let G be any positive number, however large.

Since $\sin x \to \sin \frac{1}{2}\pi=1$ as $x\to\frac{1}{2}\pi$ there exists a positive number δ_1 such that, (taking $\epsilon=\frac{1}{2}$),

$$\tfrac{1}{2}< \sin x, \quad \text{when } \tfrac{1}{2}\pi-\delta_1 \leqslant x \leqslant \tfrac{1}{2}\pi+\delta_1. \tag{1}$$

Since $\cos x \to 0$ as $x\to\frac{1}{2}\pi$, there exists a positive number δ_2 such that, $-1/2G<\cos x<1/2G$, when $\frac{1}{2}\pi-\delta_2\leqslant x\leqslant\frac{1}{2}\pi+\delta_2$.

As cos x is positive in $(0, \frac{1}{2}\pi]$ and negative in $[\frac{1}{2}\pi, \pi)$, we have

$$0<\cos x<1/2G, \quad \text{when } \tfrac{1}{2}\pi-\delta_2\leqslant x<\tfrac{1}{2}\pi. \tag{2}$$

and

$$-1/2G<\cos x<0, \quad \text{when } \tfrac{1}{2}\pi<x\leqslant\tfrac{1}{2}\pi+\delta_2. \tag{3}$$

From (1) and (2), we have, if $\delta=\min(\delta_1, \delta_2)$,

$$\tan x=\frac{\sin x}{\cos x}>G. \quad \text{when } \tfrac{1}{2}\pi-\delta\leqslant x<\tfrac{1}{2}\pi$$

and from (1) and (3), we have

$$\tan x=\frac{\sin x}{\cos x}<-G, \text{ when } \tfrac{1}{2}\pi<x\leqslant\tfrac{1}{2}\pi+\delta.$$

Hence the results.

106. The inverse trigonometrical functions, $\sin^{-1}y$, $\cos^{-1}y$, $\tan^{-1}y$.

106·1. $\sin^{-1}y$. Since, as may easily be seen, $y=\sin x$ strictly increases from -1 to 1 as x increases from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, we have x as an inverse function of y, (§51 P. 78), known as inverse sine of y. In symbols, we write

$$x=\sin^{-1}y.$$

Thus $\sin^{-1}y$ is the number x lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose sine is y.

Clearly $\sin^{-1}y$ is defined in the interval $(-1, 1)$ only.

To prove that $\sin^{-1} y$ is derivable in the open range $[-1, 1]$ and that its derivative with respect to y is $1/\sqrt{(1-y^2)}$.

Let $x=\sin^{-1}y$ so that $y=\sin x$.

We have $dy/dx=\cos x$, which is 0 only when $x=-\frac{1}{2}\pi$ or $\frac{1}{2}\pi$ *i. e.*, when $y=-1$ or 1.

$\therefore$ when $y\neq 1$ or -1, we have

$$\frac{dx}{dy}=\frac{1}{\cos x}=\frac{1}{\sqrt{(1-\sin^2 x)}}=\frac{1}{\sqrt{(1-y^2)}},$$

as $\cos x$ is necessarily positive in $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$,

106·2. Cos^{-1}y. Since $y=\cos x$ strictly decreases from 1 to -1 as x increases from 0 to π, we have x as an inverse function of y and write

$$x=\cos^{-1}y.$$

Thus $\cos^{-1}y$ is the number x, lying between 0 and π, whose cosine is x and is defined in the interval $(-1, 1)$ only.

It is easy to show that $\cos^{-1}y$ is derivable in the open interval $[-1, 1]$ and that its derivative is $-1/\sqrt{(1-y^2)}$.

106·3. Tan^{-1}y. Since, as may easily be seen, $y=\tan x$ strictly increases from $-\infty$ to ∞ as x increases from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, we have x as an inverse function of y and write

$$x=\tan^{-1}y.$$

Thus $\tan^{-1}y$ is the number x lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose tangent is y and is defined for the entire aggregate of real numbers.

It may easily be shown that $\tan^{-1}_{1}y$ is derivable for every value of y and that its derivative is $1/(1+y^2)$.

Examples

1. (*i*) Show that $\text{lt} \sin(1/x)$, when $x\to 0$, does not exist.

(*ii*) If $f(x)=x\sin(1/x)$, when $x\neq 0$ and $f(0)=0$, show that $f(x)$ is continuous but not derivable for $x=0$.

(*iii*) If $f(x)=x^2\sin(1/x)$, when $x\neq 0$ and $f(0)=0$ show that $f'(x)$ exists for every value of x but is not continuous for $x=0$.

2. If

$$f(x)=\begin{cases} x^2\sin(1/x), & \text{when } x\neq 0 \\ 0, & \text{when } x=0, \end{cases}$$

and $$g(x)=x,$$

show that $\text{lt}\,[f'(x)/g'(x)]$, as $x\to 0$, does not exist, but $\text{lt}\,[f(x)/g(x)]$ exists and is equal to $f'(0)/g'(0)$.

3. If $f(x)=|x|$, $g(x)=2|x|$, show that $f'(0)$ and $g'(0)$ do not exist but $\text{lt}[f(x)/g(x)]$ exists and is equal to $\text{lt}\,[f'(x)/g'(x)]$, when $x\to 0$.

4. *If*

$$f(x)=e^{-1/x^2}\sin(1/x), \text{ for } x\neq 0, ;\ f(0)=0,$$

show that (i) this function has at every point a differential co-efficient and this is continuous at $x=0$, (ii), the differential co-efficient vanishes at $x=0$, and at an infinite number of points in the neighbourhood of $x=0$.

$$\text{When } x \neq 0, f'(x) = \frac{e^{-1/x^2}}{x^3}\left(2\sin\frac{1}{x} - x\cos\frac{1}{x}\right),$$

$$\text{and when } x=0, f'(0) = \underset{x\to 0}{\text{lt}} \frac{e^{-1/x^2}\sin(1/x) - 0}{x}$$

Since $e^{1/x^2} > 1/x^2$, we see that

$$\left|\frac{e^{-1/x^2}\sin(1/x)}{x}\right| < \frac{x^2.1}{|x|} = |x|.$$

$$\therefore \qquad f'(0)=0.$$

Also, when $x \neq 0$,

$$\left|f'(x) - f'(0)\right| = \left|\frac{e^{-1/x^2}}{x^3}\left(2\sin\frac{1}{x} - x\cos\frac{1}{x}\right)\right|$$

$$\leqslant \frac{e^{-1/x^2}}{|x|^3}(2 + |x|).$$

Since $e^{1/x^2} > 1/2x^4$, we see that

$$|f'(x) - f'(0)| \leqslant 2|x|(2 + |x|),$$

so that $[f'(x) - f'(0)] \to 0$, as $x \to 0$, *i. e.*, $f'(x)$ is continuous for $x=0$.

We have now to show that $f'(x)$ vanishes at a point in *every* neighbourhood of $x=0$.

Let δ be any positive number however small.

There surely exists a positive integer n such that

$$0 < \frac{2}{(2n+1)\pi} < \frac{1}{n\pi} < \delta.$$

It is easy to see that,

for $x = 1/n\pi$, $f'(x)$ is $-$ve or $+$ve according as n is even or odd,

for $x = 2/(2n+1)\pi$, $f'(x)$ is $+$ve or $-$ve according as n is even or odd.

Therefore $f'(x)$, which is continuous, must vanish atleast once between $2/(2n+1)\pi$ and $1/n\pi$.

We may similarly dispose of left-handed neighbourhood of $x=0$.

6. If

$$f(x) = \sqrt{x}(1 + x\sin 1/x), \text{ for } x > 0;$$
$$f(x) = -\sqrt{-x}(1 + x\sin 1/x), \text{ for } x < 0;$$
$$f(0) = 0,$$

show that $f'(x)$ exists everywhere and is finite except at $x=0$, in the neighbourhood of which it oscillates between indefinitely great positive and negative values.

7. Show that the function

$$f(x) = x[1 + \tfrac{1}{3}\sin(\log x^2)], \text{ when } x \neq 0,$$
$$f(0) = 0,$$

is everywhere continuous but has no differential co-efficient at $x=0$.

8. Show that the function

$$f(x) = 4 + 7x + x^2(8 + x\sin 1/x),$$

where $x\sin(1/x)$ is zero for $x=0$ has a first derivative but no second derivative at the origin.

9. Find the points of discontinuity of

$$f(x)=\lim_{m\to\infty}(\cos \pi x)^{2m}\,;\ \phi(x)=\lim_{n\to\infty}\lim_{m\to\infty}(\cos \pi n\,!\,x)^{2m}]$$

[$f(x)=1$ when x is an integer and otherwise $f(x)=0$. Since, when x is rational $n\,!\,x$ is an integer for sufficiently large values of n therefore $\phi(x)=1$ when x is rational and 0 when x is irrational].

10. Find the points of discontinuity of

$$(i)\ f(x)=\lim_{t\to\infty}\frac{(1+\sin \pi x)^t-1}{(1+\sin \pi x)^t+1}\qquad (ii)\ f(x)=\lim_{n\to\infty}\lim_{t\to 0}\frac{\sin^2(n\,!\,\pi x)}{\sin^2(n\,!\,\pi x)+t^2}$$

11. If

$$f(x)=\sin x \sin (1/\sin x),\ \text{when } 0<x<\pi<x<2\pi,$$

and $f(x)=0$ when $x=0,\ \pi,\ 2\pi$;

show that $f(x)$ is continuous but not derivable for $x=0,\ \pi,\ 2\pi$.

12. If $a_0,\ a_1,\ a_2 \ldots\ldots a_n$ are real and

$$|\,a_0\,|+|\,a_1\,|+|\,a_2\,|+\ldots\ldots+|\,a_{n-1}\,|<a_n,$$

show that

$$u(x)=a_0+a_1\cos x+a_2\cos 2x+\ldots\ldots+a_n\cos nx$$

has atleast $2n$ zeros in the interval $0<x<2\pi$. Show also that $u'(x)$ has atleast $2n$ zeros in the interval $\alpha\leqslant x<2\pi+\alpha$ for every real α.

[$u(0)$, $u(\pi/n)$, $u(2\pi/n)$.......,$u(2n\pi/n)$ have positive and negative signs alternately)

13. By use of the inequality $\sin x<x$, or ortherwise, prove that, if $0<c<1$, then

$$2\int_0^1 \sqrt{(1-c^2\sin^2 x)}\,dx>\sqrt{(1-c^2)}+1.$$

14. If

$$f(x)=\lim_{n\to\infty}\frac{\log (2+x)-x^{2n}\sin x}{1+x^{2n}},$$

explain, why the function does not vanish anywhere in the interval $(0,\ \frac{1}{2}\pi)$, although $f(0)$ and $f(\frac{1}{2}\pi)$ differ in sign.

15. Determine the differential co-efficient, if any, of

$$f(x)=(x-a)\,\frac{Ae^{1/(x-a)}+Be^{-1/(x-a)}}{e^{1/(x-a)}+e^{-1/(x-a)}}\sin\frac{\pi}{2(x-a)},\ \text{when } x\neq a,$$

$$f(a)=0,$$

when $x=a$.

16. A function $f(x)$ is defined as follows :—

$$f(x)=(x-w_1)(x-w_2)^2(x-w_3)^3\sin\frac{1}{(x-w)}\ \sin\frac{1}{(x-w_2)}\ \sin\frac{1}{(x-w_3)},$$

for all values of x except $w_1,\ w_2,\ w_3$ in a domain $(a,\ b)$, and $f(x)=0$ when $x=w_1$ or $w_2,\ w_3$. Show that

(*i*) df/dx does not exist at the point $x=w_1$,

(*ii*) df/dx exists but has a discontinuity of the second kind at $x=w_2$.

(*iii*) df/dx exists and is continuous at $x=w_3$.

17. If a is a fixed positive number, prove that

$$\lim_{h\to 0}\int_{-a}^{a}\frac{h\,dx}{h^2+x^2}=\pi,$$

18. Assuming that $|n| \neq |m|$, prove that

$$\lim_{y\to\infty} \frac{1}{y}\int_0^y \sin nx \sin mx \, dx=0.$$

19. *If $f(x)$ is continuous in $(0, 1)$, show that*

$$\lim_{n\to\infty} \int_0^1 \frac{nf(x)}{1+n^2x^2}\,dx=\frac{\pi}{2}\,f(0).$$

We write

$$\int_0^1 \frac{nf(x)}{1+n^2x^2}dx=\int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}\,dx+\int_{1/\sqrt{n}}^1 \frac{nf(x)dx}{1+n^2x^2}.$$

By the first mean value theorem, we have

$$\int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}dx=f(\alpha_n)\int_0^{1/\sqrt{n}} \frac{ndx}{1+n^2x^2}, \text{ where } 0\leqslant \alpha_n\leqslant 1/\sqrt{n},$$

$$=f(\alpha_n).\tan^{-1}\sqrt{n}, \text{ which } \to f(0).\tfrac{1}{2}\pi \text{ as } n\to\infty.$$

Again

$$\left|\int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}\,dx\right|=\left|f(\beta_n)\int_{1/\sqrt{n}}^1 \frac{ndx}{1+n^2x^2}\right|, \text{ where } 1/\sqrt{n}\leqslant\beta_n\leqslant 1$$

$$=|f(\beta_n)(\tan^{-1}n-\tan^{-1}\sqrt{n})|$$
$$\leqslant M(\tan^{-1}n-\tan^{-1}\sqrt{n}), \text{ which } \to 0, \text{ as } n\to\infty$$

M, being the upper bound of $|f(x)|$.

20. If $f(x)$ is continuous in the interval $(-1, 1)$, prove that

$$\lim_{h\to 0}\int_{-1}^1 \frac{hf(x)}{1+h^2x^2}dx=\pi f(0)$$

[Split up the range $(-1, 1)$ into three ranges $(-1, -\sqrt{h})$, $(-\sqrt{h}, \sqrt{h})$, $(\sqrt{h}, 1)$].

21. In the second mean value theorem of Integral Calculus, show that $\phi(x)$ must be necessarily monotonic by proving that the theorem does not hold if $\phi(x)=\cos x, f(x)=x^2$.

22. *If (x) is bounded and integrable in the range (a, b), show that*

$$\lim_{n\to\infty}\int_a^b f(x)\cos nx\,dx=0.$$

We write $I_n=\int_a^b f(x)\cos nx\,dx.$

Let ϵ be any positive number. Since $f(x)$ is bounded and integrable in (a, b), there exists a division

$$D(a=x_0<x_1<x_2<\ldots\ldots<x_{r-1}<x_r<\ldots\ldots<x_n=b)$$

such that the corresponding oscillatory sum

$$\Sigma(x_r-x_{r-1})O_r<\tfrac{1}{2}\epsilon,$$

O_r being the oscillation of $f(x)$ in (x_{r-1}, x_r).

We have

$$I_n=\Sigma\int_{x_{r-1}}^{x_r} f(x)\cos nx\,dx$$

$$=\Sigma f(x_{r-1})\int_{x_{r-1}}^{x_r}\cos nx\,dx+\Sigma\int_{x_{r-1}}^{x_r}[f(x)-f(x_{r-1})]\cos nx\,dx.$$

$$\therefore\quad |I_n|\leqslant\Sigma\,|f(x_{r-1})|\left|\int_{x_{r-1}}^{x_r}\cos nx\,dx\right|+$$

$$\Sigma\int_{x_{r-1}}^{x_r}\left|\left\{f(x)-f(x_{r-1})\right\}\cos nx\right|dx$$

As x varies in (x_{r-1}, x_r), we have

$$|f(x)-f(x_{r-1})|\leqslant O_r$$

and $\quad\therefore\quad |[f(x)-f(x_{r-1})]\cos nx|\leqslant O_r.$

Also $\left|\int_{x_{r-1}}^{x_r}\cos nx\,dx\right|\leqslant\frac{1}{n}\left\{|\sin nx_r|+|\sin nx_{r-1}|\right\}\leqslant\frac{2}{n}.$

$$\therefore\quad |I_n|\leqslant\frac{2}{n}\Sigma\,|f(x_{r-1})|+\Sigma(x_r-x_{r-1})O_r$$

$$\leqslant\frac{2}{n}\Sigma\,|f(x_{r-1})|+\frac{\epsilon}{2}.$$

Keeping the division D fixed, we see that $\Sigma\,|f(x_{r-1})|$ is fixed. We now choose a positive integer m such that

$$\frac{2}{n}\Sigma\,|f(x_{r-1})|<\frac{\epsilon}{2},\text{ where } n\geqslant m$$

Thus for $n\geqslant m$, $|I_n|<\epsilon$.

Hence the result.

It may similarly be shown that

$$\lim_{n\to\infty}\int_a^b f(x)\sin nx\,dx=0.$$

23. If $f(x)$ is bounded and integrable in (a, b), show that

$$\lim_{n\to\infty} \int_a^b f(x) \frac{\sin nx}{x} dx = 0, \text{ when } 0 < a < b.$$

24. Show that, for $0 < a < b$,

$$(i) \left| \int_a^b \frac{\sin x}{x} dx \right| \leqslant \frac{2}{a}, \quad (ii) \left| \int_a^b \sin x^2 \, dx \right| \leqslant \frac{1}{a}.$$

25. *Show that* $\lim I_n$, *where*

$$I_n = \int_0^h \frac{\sin nx}{x} dx.$$

exists when $n \to \infty$ *through positive integral values and that the limit is equal to* $\pi/2$.

The integrand becomes continuous for every value of x, if we assign to it the value n for $x=0$. The result will be proved in *three* steps.

I. Firstly, it will be proved that $\{ I_n \}$ is convergent. Putting $nx=t$, we have

$$I_n = \int_0^{nh} \frac{\sin t}{t} dt.$$

$$\therefore \quad \left| I_{n+p} - I_n \right| = \left| \int_{nh}^{(n+p)h} \frac{\sin t}{t} dt \right|$$

As $1/t$ is positive and monotonically decreasing in $[nh, (n+p)h]$, we have, by the Bonnett's form of the second mean value theorem,

$$\left| I_{n+p} - I_n \right| = \frac{1}{nh} \left| \int_{nh}^{\alpha} \sin t \, dt \right| \leqslant \frac{2}{nh} < \epsilon, \text{ if } n > 2/\epsilon h$$

Hence, by Cauchy's principle of convergence, $\{ I_n \}$ converges. (§30·1, P. 39)

II. It will now be proved that, when $n \to \infty$,

$$\lim I_n = \lim \int_0^{\frac{1}{2}\pi} \frac{\sin nx}{\sin x} dx.$$

We write

$$\int_0^{\frac{1}{2}\pi} \frac{\sin nx}{x} dx = \int_0^h \frac{\sin nx}{x} dx + \int_h^{\frac{1}{2}\pi} \frac{\sin nx}{x} dx.$$

As proved in Ex. 22 above, $\displaystyle\int_h^{\frac{1}{2}\pi} \frac{\sin nx}{x}\,dx \to 0$ as $n \to \infty$.

$$\therefore \qquad \text{lt } I_n = \text{lt} \int_0^{\frac{1}{2}\pi} \frac{\sin nx}{x}\,dx.$$

Again, taking $f(x) = (1/x - 1/\sin x)$ in Ex. 22 above, we have

$$\text{lt} \int_0^{\frac{1}{2}\pi} \left(\frac{1}{x} - \frac{1}{\sin x}\right) \sin nx\,dx = 0,$$

for $f(x)$ is continuous in $(0, \frac{1}{2}\pi)$, if we set $f(0) = 0$.

$$\therefore \qquad \text{lt} \int_0^{\frac{1}{2}\pi} \frac{\sin nx}{x}\,dx = \text{lt} \int_0^{\frac{1}{2}\pi} \frac{\sin nx}{\sin x}\,dx$$

$$\therefore \qquad \text{lt } I_n = \text{lt} \int_0^{\frac{1}{2}\pi} \frac{\sin nx}{x}\,dx = \text{lt} \int_0^{\frac{1}{2}\pi} \frac{\sin nx}{\sin x}\,dx.$$

To determine the actual value of the limit we proceed by making $n \to \infty$ through odd integral values.

III. We have, as may be easily shown,

$$\frac{\sin (2n+1)x}{\sin x} = 2[\tfrac{1}{2} + \cos 2x + \cos 4x + \ldots + \cos 2nx]$$

$$\therefore \qquad \int_0^{\frac{1}{2}\pi} \frac{\sin (2n+1)x}{\sin x}\,dx = \frac{\pi}{2}.$$

Hence the result.

26. Show that

$$\sum_{n=-\infty}^{\infty} e^{-(x-n)^2}$$

converges uniformly in any fixed interval (a, b).

27. Show that the series

$$S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$$

is uniformly convergent for all values of x; and that $S'(x)$ is given by term-by-term differentiation.

28. Show that

$$S(x) = \sum_{n=0}^{\infty} \frac{1}{1 + n^2 + n^4 x^2}$$

converges uniformly for all values of x, examine whether $S'(0)$ can be found term by term differentiation.

29. Show that the series $\sum n^{-x}$ is uniformly convergent in $(1+\delta, \infty]$ where δ is any positive number. Show also that term-by term differentiation is valid in the same interval.

30. Show that the following series converge uniformly in the intervals ndicated.

$$x - x^2 + x^3 - x^4 + x^5 - x^6 + \ldots\ldots\ldots (-\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2})$$

$$e^x + e^{2x} + e^{3x} + e^{4x} + \ldots\ldots\ldots\ldots\ldots (-2 \leqslant x \leqslant -\tfrac{1}{2}).$$

CHAPTER VIII

IMPROPER INTEGRALS

107. The theory of Riemann integration, as developed in Chapter VI, expressly requires that the range of integration is finite and that the integrand is bounded in that range. It is possible, however, to so extend the theory that the symbol

$$\int_a^b f(x)dx$$

may sometimes have a meaning (*i. e.*, denote a number) even when $f(x)$ is not bounded or when either a or b or both are infinite. In case $f(x)$ is unbounded or the limits a or b are infinite, the symbol

$$\int_a^b f(x)dx$$

is called an **improper** (or **generalised** or **infinite**) integral. Thus

$$\int_0^1 \frac{dx}{x^3}, \quad \int_1^2 \frac{dx}{(1-x)(2-x)}, \quad \int_{-\infty}^{\infty} \frac{dx}{1+x^2},$$

are examples of improper integrals.

For the sake of distinction an integral which is not improper will be called a **proper** integral.

We know (Ex. after, § 50 3, P. 76) that if a function $f(x)$ is not bounded in a finite interval (a, b), then there exists atleast one point 'c' of the interval such that in every neighbourhood of c, however small it may be, $f(x)$ is not bounded. Such a point 'c' is a point of **infinite discontinuity** of the function $f(x)$. It will always be assumed that the function $f(x)$ is such that its points of infinite discontinuity, which lie in any interval, finite or infinite, are finite in number; the consideration of functions having an infinite number of points of infinite discontinuity being beyond the scope of the book.

In a finite interval which encloses no point of infinite discontinuity, the function is always bounded and we *assume*, once for all in order to avoid tedious repetition, that it is also integrable in such an interval.

108. Definitions.

108·1. Convergence at the left-end. Let 'a' be the *only* point of infinite discontinuity of a function $f(x)$ in a finite interval (a, b) so that, according to the assumption made in the last paragraph, the integral

$$\int_{a+\epsilon}^{b} f(x)dx, \qquad \text{where } 0<\epsilon<(b-a),$$

exists and is a function of ϵ, say $\phi(\epsilon)$.

If, when $\epsilon \to (0+0)$, $\phi(\epsilon)$ tends to a finite limit, say I, we say that the improper integral

$$\int_a^b f(x)dx, \tag{1}$$

exists, or converges at a, and use the symbol (1) to denote the number I. Thus

$$\int_a^b f(x)dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f(x)dx,$$

provided that the limit on the right exists. In case the limit does not exist we say that $\int_a^b f(x)dx$ does not exist or that it does not converge.

Ex. 1. Show that if $\int_a^b f(x)dx$ converges at a, then $\int_a^c f(x)dx$, $(a<c<b)$ also converges at a.

Ex. 2. The improper integral $\int_a^b f(x)dx$ converges at a and k is any constant; show that $\int_a^b kf(x)dx$ also converges at a and conversely.

108·2. Convergence at the right-end. Let b be the *only* point of infinite discontinuity of $f(x)$ in a finite interval (a, b). If then the proper integral

$$\int_a^{b-\epsilon} f(x)dx, \quad \text{where } 0<\epsilon<(b-a) \tag{1}$$

which is a function of ϵ, tends to a finite limit, as $\epsilon \to (0+0)$, we say that the improper integral

$$\int_a^b f(x)dx \tag{2}$$

exists or converges at b and use the symbol (2) to denote the limit of (1).

Ex. *Examine the existence of the improper integrals*

$$(i) \int_0^1 \frac{dx}{x^2}, \qquad (ii) \int_0^1 \frac{dx}{\sqrt{(1-x^2)}}.$$

(*i*) The left-end point '0' is the only point of infinite discontinuity of the integrand $1/x^2$ in (0, 1). We have, when $0<\epsilon\leqslant 1$,

$$\phi(\epsilon)=\int_{\epsilon}^{1}\frac{dx}{x^2}=\left|-\frac{1}{x}\right|_{\epsilon}^{1}=\frac{1}{\epsilon}-1, \text{ which}\rightarrow+\infty \text{ as } \epsilon\rightarrow(0+0).$$

Thus we see that $\phi(\epsilon)$ does not tend to a finite limit. Hence the improper integral, in question, does not exist.

(*ii*) The right-end point '1', is the only point of infinite discontinuity of $1/\sqrt{(1-x^2)}$ in (0, 1). We have

$$\phi(\epsilon)=\int_{0}^{1-\epsilon}\frac{dx}{\sqrt{(1-x^2)}}=\sin^{-1}(1-\epsilon), \text{ which }\rightarrow\sin^{-1}1=\tfrac{1}{2}\pi, \text{ as } \epsilon\rightarrow 0.$$

Thus the improper integral exists and is equal to $\frac{1}{2}\pi$.

Note. The reader will note that the integrand $1/x^2$ is not defined for $x=0$ and $1/\sqrt{(1-x^2)}$ is not defined for $x=1$. We may assign to the integrand at such points any value we please without affecting the existence and the value of the corresponding improper integral.

108 3. *Let the end points a and b be the only two points of infinite discontinuity of $f(x)$.*

We take any point c within (a, b).

If the improper integrals

$$\int_{a}^{c} f(x)dx \text{ and } \int_{c}^{b} f(x)dx$$

converge at the left-end a and at the right-end b respectively,

we say that the improper integral $\int_{a}^{b} f(x)\,dx$ exists and write

$$\int_{a}^{b} f(x)dx=\int_{a}^{c} f(x)dx+\int_{c}^{b} f(x)dx.$$

It is easy to show that the existence and the value of the improper integral, in question, is independent of the position of c. If d be any point of (a, b), we have

$$\int_{a+\epsilon}^{d} f(x)dx=\int_{a+\epsilon}^{c} f(x)dx+\int_{c}^{d} f(x)dx.$$

If $\epsilon\rightarrow 0$, we see that $\int_{a+\epsilon}^{d} f(x)dx$ tends to a finite limit *i. e.*, $\int_{a}^{d} f(x)dx$

exists if, and only if, $\int_{a+\epsilon}^{c} f(x)dx$ tends to a finite limit *i. e.*, $\int_{a}^{c} f(x)dx$ exists.
Also, in case they exist finitely,

$$\int_{a}^{d} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{d} f(x)dx.$$

It may similarly be shown that $\int_{d}^{b} f(x)dx$ exists if, and only if, $\int_{c}^{b} f(x)dx$ exist and in case they exist,

$$\int_{d}^{b} f(x)dx = \int_{d}^{c} f(x)dx + \int_{c}^{b} f(x)dx.$$

Adding, we get

$$\int_{a}^{d} f(x)dx + \int_{d}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx = \int_{a}^{b} f(x)dx.$$

Ex. Examine the existence of the improper integrals

$$(i) \int_{0}^{1} \frac{dx}{\sqrt{x-x^2}} \qquad (ii) \int_{0}^{2} \frac{dx}{x(2-x)}, \qquad (iii) \int_{0}^{\pi} \frac{dx}{\sin x}.$$

108·4. *Let $c_1, c_2, c_3, \ldots\ldots c_n$ be any* **finite** *number of points of infinite discontinuity of $f(x)$ lying in (a, b), where*

$$a \leqslant c_1 < c_2 \ldots\ldots < c_{n-1} < c_n \leqslant b.$$

If the improper integrals

$$\int_{a}^{c_1} f(x)dx, \int_{c_1}^{c_2} f(x)dx, \ldots\ldots\ldots, \int_{c_{n-1}}^{c_n} f(x)dx, \int_{c_n}^{b} f(x)dx$$

all exist in accordance with the definitions given above and $\int_{a}^{c_1} f(x)dx=0$ if $c_1=a$ and $\int_{c_n}^{b} f(x)dx=0$ if $b=c_n$, then we say that $\int_{a}^{b} f(x)dx$ exists and write

$$\int_{a}^{b} f(x)dx = \int_{a}^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots\ldots + \int_{c_n}^{b} f(x)dx.$$

108·5. Infinite Range of Integration. Convergence at ∞. Let $f(x)$ be bounded and integrable in (a, X) where X is any number $> a$ so that the proper integral

$$\int_a^X f(x)dx$$

exists and is a function of X, say, $\phi(X)$.

If $\phi(X)$ tends to a finite limit, I, as $X \to \infty$, we say that the improper integral

$$\int_a^\infty f(x)dx \tag{1}$$

exists or that it converges at ∞ and regard the symbol (1) as denoting the number I. Thus

$$\int_a^\infty f(x)\,dx = \lim_{X\to\infty} \int_a^X f(x)dx,$$

provided the limit exists.

Ex. Examine the convergence of

$$(i) \int_0^\infty \frac{dx}{1+x^2}, \qquad (ii) \int_1^\infty \frac{dx}{x^2}.$$

108·6. Convergence at $-\infty$. If $f(x)$ be bounded and integrable in (X, b) where $X \leqslant b$, and

$$\int_X^b f(x)dx \tag{1}$$

tends to a finite limit as $X \to -\infty$, we say that

$$\int_{-\infty}^b f(x)dx \tag{2}$$

exists and regard the symbol (2) as denoting the limit of (1).

108·7. *Let the range of integration be* $(-\infty, \infty)$.

If c be any number and

$$\int_{-\infty}^c f(x)dx, \quad \int_c^\infty f(x)dx,$$

both exist in accordance with the definitions already given, then we say that $\int_{-\infty}^\infty f(x)dx$ exists and write

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx.$$

It is easy to show that the existence and the value of

$$\int_{-\infty}^{\infty} f(x)dx$$

is independent of the choice of c.

107·8. If an infinite range of integration includes a finite number of points of infinite discontinuity, then we arbitrarily consider an interval (a, b) which embraces all the points of infinite discontinuity of $f(x)$ and examine the existence of the three improper integrals

$$\int_{-\infty}^{a} f(x)dx,\quad \int_{a}^{b} f(x)dx,\quad \int_{b}^{\infty} f(x)dx$$

in accordance with the definitions given above, and in case they all exist we say that

$$\int_{-\infty}^{\infty} f(x)dx$$

exists and write

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx.$$

Note. It may be seen that in order to examine the existence of any given improper integral, we have to examine the convergence of a system of improper integrals of the four types which have been considered in the sub-sections 108·1, 108·2, 108·5, 108·6. Also we see that the case §108·2 is analogous to that of §108·1 and the case of §108·6 is analogous to that of §1085.

109. From the foregoing it will be seen that it is enough to consider tests for the convergence of

(*i*) $\int_{a}^{b} f(x)dx$ at a, where $f(x)$ is bounded and integrable in $(a+\epsilon, b)$, $0<\epsilon\leqslant(b-a)$.

(*ii*) $\int_{a}^{\infty} f(x)dx$ at ∞, where $f(x)$ is bounded and integrable in (a, X), $X>a$.

Ex. Examine the existence of the following improper integrals and evaluate those which exist:

(*i*) $\int_1^\infty \frac{dx}{x(1+x)}$. (*ii*) $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$. (*iii*) $\int_0^\pi \frac{dx}{1+\cos x}$. (*iv*) $\int_{-1}^\infty \frac{dx}{\sqrt{x}(1+x)}$.

(*v*) $\int_{-\infty}^\infty \frac{dx}{x(1+x^2)}$. (*vi*) $\int_0^1 \log x\, dx$. (*vii*) $\int_{-2}^2 \frac{dx}{x^2(1-x)}$ (*viii*) $\int_0^\infty x^2 e^{-x}\, dx$.

Note. In the examples given above the improper integrals are such that the integrands admit of primitives in terms of elementary functions. In such cases the examination of the existence is generally easy but more advanced methods are necessary when the integrand does not possess a primitive in terms of elementary functions.

110. Test for convergence at 'a.' Positive integrand. Let a be the only point of infinite discontinuity of $f(x)$ in (a, b). The case where the integrand $f(x)$ is positive in a certain neighbourhood (a,c) of 'a' is particularly simple and important and covers a large class of improper integrals

Since

$$\int_{a+\epsilon}^{b} f(x)dx = \int_{a+\epsilon}^{c} f(x)dx + \int_{c}^{b} f(x)dx,$$

it follows that either

$$\int_a^c f(x)dx \text{ and } \int_a^b f(x)dx$$

are both convergent at 'a' or both non-convergent. It is, therefore, no loss of generality to suppose that $f(x)$ is positive in (a, b).

The question of the existence of the integral is, in such a case, decided by comparison with another suitably chosen integral whose existence or otherwise is already known.

110·1 *The necessary and sufficient condition for the convergence of the improper integral.*

$$\int_a^b f(x)dx$$

at 'a,' where, $f(x)$ is positive in (a, b), is that there exists a positive number 'M', *independent of* ϵ, *such that*

$$\int_{a+\epsilon}^{b} f(x)dx < M, \text{ where } 0<\epsilon<(b-a).$$

The proof follows from the fact that, since $f(x)$ is positive in $[a, b)$, the integral

$$\int_{a+\epsilon}^{b} f(x)dx$$

monotonically increases as ϵ decreases and will, therefore, tend to a finite limit if, and only if, it is bounded above.

Note. In case $\phi(\epsilon)=\int_{a+\epsilon}^{b} f(x)dx$ is not bounded above, then $\phi(\epsilon)\to+\infty$ as $\epsilon\to(0+0)$ and we say that the improper integral $\int_{a}^{b} f(x)dx$ diverges to ∞.

110·2. Comparison of two integrals. *Let $f(x)$ and $\phi(x)$ be two functions such that in $[a, b)$, they are both positive and $f(x)\leqslant\phi(x)$; then*

(*i*) $\int_{a}^{b} f(x)dx$ *converges if* $\int_{a}^{b} \phi(x)dx$ *converges,*

and (*ii*) $\int_{a}^{b} \phi(x)dx$ *does not converge if* $\int_{a}^{b} f(x)dx$ *does not.*

It is assumed that $f(x)$ and $\phi(x)$ are both bounded and integrable in $(a+\epsilon, b)$, $0<\epsilon\leqslant(b-a)$, and 'a' is the only point of infinite discontinuity.

We have

$$\int_{a+\epsilon}^{b} f(x)dx \leqslant \int_{a+\epsilon}^{b} \phi(x)dx. \quad \text{(cor. 5, § 88, P. 124)} \quad (1)$$

Let $\int_{a}^{b} \phi(x)dx$ converge so that there exists a number M such that

$$\int_{a+\epsilon}^{b} \phi(x)dx<M \text{ for } 0<\epsilon\leqslant(b-a) \quad (2)$$

From (1) and (2),

$$\int_{a+\epsilon}^{b} f(x)dx<M \text{ for } 0<\epsilon\leqslant(b-a),$$

therefore $\int_a^b f(x)dx$ converges at 'a'

For the second part we see that if $\int_a^b f(x)dx$ does not converge at a, then $\int_{a+\epsilon}^b f(x)dx$ is not bounded above and consequently, from (1), $\int_{a+\epsilon}^b \phi(x)dx$ is also not bounded above so that $\int_a^b \phi(x)dx$ does not converge.

110·8. *If $f(x)/\phi(x) \to l$ when $x \to a$, and l is neither 0 nor infinite, then the two integrals*

$$\int_a^b f(x)dx \quad \textit{and} \int_a^b \phi(x)dx$$

either both converge or both do not converge.

Since $f(x)/\phi(x)$ is positive, l cannot be negative.

Let δ be any positive number less than l.

There exists a number c, $(a<c<b)$, such that

$$l-\delta< f(x)/\phi(x)<l+\delta \text{ for } a<x\leqslant c,$$

i.e., in $[a, c)$, we have

$$(l-\delta)\phi(x)< f(x) \qquad (i)$$

and $$f(x)<(l+\delta)\phi(x) \qquad (ii)$$

Let $$\int_a^b f(x)dx \text{ converge ;}$$

$\therefore$ $$\int_a^c f(x)dx \text{ converges ;}$$

$\therefore$ from (*i*), $$(l-\delta)\int_a^c \phi(x)dx \text{ converges ;}$$

$\therefore$ $$\int_a^c \phi(x)dx \text{ converges ;}$$

hence $$\int_a^b \phi(x)dx \text{ converges.}$$

From (*i*), it may similarly be shown that if $\int_a^b \phi(x)\,dx$ does not converge, then $\int_a^b f(x)dx$, also, does not. Also, from (*ii*), we may prove that if $\int_a^b \phi(x)dx$ converges, then $\int_a^b f(x)dx$ converges, and if, $\int_a^b f(x)dx$ does not converge then $\int_a^b \phi(x)dx$ also does not.

Ex. Prove that if $f(x)/\phi(x)\to 0$, then $\int_a^b f(x)dx$ converges if $\int_a^b \phi(x)dx$ converges and that if $f(x)/\phi(x)\to\infty$, then $\int_a^b \phi(x)dx$ converges if $\int_a^b f(x)dx$ converges.

110·4. An important comparison integral, $\int_a^b \frac{dx}{(x-a)^\mu}$.

The improper integral

$$\int_a^b \frac{dx}{(x-a)^\mu} \textit{ converges if, and only if, } \mu<1.$$

We have, if $\mu\neq 1$,

$$\int_{a+\epsilon}^{b} \frac{dx}{(x-a)^\mu}=\left|\frac{1}{(1-\mu)(x-a)^{\mu-1}}\right|_{a+\epsilon}^{b}$$

$$=\frac{1}{(1-\mu)}\left\{\frac{1}{(b-a)^{\mu-1}}-\frac{1}{\epsilon^{\mu-1}}\right\},$$

which tends to $1/(1-\mu)(b-a)^{\mu-1}$ or $+\infty$ according as $\mu<1$ or >1.

Again, if $\mu=1$,

$$\int_{a+\epsilon}^{b} \frac{dx}{x-a}=\log(b-a)-\log\epsilon, \text{ which}\to+\infty \text{ as } \epsilon\to(0+0).$$

Hence the theorem.

Note. The integral $\int_a^b \frac{dx}{(x-a)^\mu}$ is proper if $\mu \leqslant 0$. The notion of the convergence of improper integrals has enabled us to give a meaning to the symbol $\int_a^b \frac{dx}{(x-a)^\mu}$ even for those values of μ which are positive but < 1. For $\mu \geqslant 1$, the symbol does not represent any number.

111. With the help of § 110, we now deduce two important *practical* tests for the convergence at 'a' of

$$\int_a^b f(x)dx. \tag{1}$$

I. *Let $f(x)$ be positive in (a, b). Then the integral* (1) *converges at 'a' if there exists a positive number μ less than* 1 *and a fixed positive number M such that $f(x) \leqslant M/(x-a)^\mu$ in the interval $a < x \leqslant b$.*

Also, the integral (1) *does not converge if there exists a number $\mu \geqslant 1$ and a fixed positive number M such that $f(x) \geqslant M/(x-a)^\mu$ in the interval $a < x \leqslant b$.*

II. *If* $\lim_{x \to a} [f(x)(x-a)^\mu]$ *exists and is equal to 'l' and 'l' is neither* 0 *nor infinite, then the integral* (1) *converges if, and only if, $\mu < 1$.*

Examples

1. *Examine the convergence of*

$$(i) \int_0^1 \frac{dx}{x^{\frac{1}{3}}(1+x^2)}, \quad (ii) \int_0^1 \frac{dx}{x^2(1+x)}, \quad (iii) \int_0^1 \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{3}}}$$

The integrands are all positive.

(*i*) Here, '0' is the only point of infinite discontinuity of the integrand. Let

$$f(x) = 1/x^{\frac{1}{3}}(1+x^2).$$

Now, $\quad (x-0)^{\frac{1}{3}} f(x) \to 1$ as $x \to 0$ so that $\mu = \frac{1}{3} < 1$.

Therefore $\int_0^1 f(x)dx$ converges.

(*ii*) Here '0' is the only point of infinite discontinuity of the integrand.

Let $\quad f(x) = 1/x^2(1+x)$.

Now $\quad x^2 f(x) \to 1$ as $x \to 0$, so that $\mu = 2 > 1$.

Therefore $\int_0^1 f(x)dx$ does not converge.

(*iii*) Here, '0' and '1' are the two points of infinite discontinuity of the integrand.

Let $$f(x)=1/x^{\frac{1}{2}}(1-x)^{\frac{1}{3}}.$$

We take any number between 0 and 1, say, $\frac{1}{2}$, and examine the convergence of the improper integrals,

$$\int_0^{\frac{1}{2}} f(x)dx \text{ and } \int_{\frac{1}{2}}^1 f(x)dx$$

at '0' and '1' respectively. We have

$$(x-0)^{\frac{1}{2}}\cdot f(x) \to 1 \text{ as } x \to 0 \text{ so that } \mu=\tfrac{1}{2}<1. \therefore \int_0^{\frac{1}{2}} f(x)dx \text{ exists.}$$

$$(1-x)^{\frac{1}{3}}\cdot f(x) \to 1 \text{ as } x\to 1 \text{ so that } \mu=\tfrac{1}{3}<1. \therefore \int_{\frac{1}{2}}^1 f(x)dx \text{ exists.}$$

Hence $\int_0^1 f(x)dx$ converges.

2. *Show that* $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ *exists if, and only if, m, n are both positive.*

The integral is proper if $m\geq 1$ and $n\geq 1$.

The number '0' is a point of infinite discontinuity if $m<1$ and the number '1' is a point of infinite discontinuity if $n<1$.

Let $m<1$ and $n<1$.

We take any number, say $\frac{1}{2}$, between 0 and 1 and examine the convergence of the improper integrals

$$\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx \text{ and } \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\, dx$$

at '0' and '1' respectively.

Let $$f(x)=x^{m-1}(1-x)^{n-1}.$$

We have $\left[(x-0)^{1-m}\cdot f(x)\right]\to 1$, as $x\to 0$. Here $\mu=1-m$.

Therefore $\int_0^{\frac{1}{2}} f(x)dx$ converges if, and only if, $\mu=(1-m)<1$,

i. e., if, and only if, $m>0$.

Also $(1-x)^{1-n} f(x) \to 1$ as $x \to 1$. Here $\mu = 1-n$ so that $\int_{\frac{1}{2}}^{1} f(x)dx$ converges if, and only if, $\mu = (1-n) < 1$ *i. e.*, if, and only if, $n > 0$.

Thus $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ *exists for positive values of* m, n *only*. It is a function of m, n and is called **Beta function** denoted by $\mathbf{B(m, n)}$.

3. *Show that*

$$\int_0^{\frac{1}{2}\pi} \frac{x^m\, dx}{(\sin x)^n} \qquad (i)$$

exists if, and only if, $n < (m+1)$.

Writing,

$$f(x) = \frac{x^m}{(\sin x)^n} = \left(\frac{x}{\sin x}\right)^n . x^{m-n},$$

we see that if $x \to 0$ then, $f(x) \to 0$ if $(m-n) \geqslant 0$ and $\to \infty$ if $(m-n) < 0$.

Thus (i) is a proper integral if $(m-n) \geqslant 0$; and improper if $(m-n) < 0$, '0' being the only point of infinite discontinuity of the integrand in this case. Let $(m-n) < 0$.

Now, $[x^m/(\sin x)^n].(x-0)^{n-m} \to 1$, as $x \to 0$, and therefore the integral converges if, and only if, $(n-m) < 1$ *i. e.*, if, and only if, $n < (m+1)$, which also includes the case $n \leqslant m$ when the integral is proper.

4. *Examine the convergence of*

$$\int_0^1 x^{n-1}(\log x)dx.$$

The integrand is negative in the interval (0, 1) and we, therefore, consider $\int_0^1 -x^{n-1} \log x\, dx$.

The integral is proper if $(n-1) > 0$, in as much as the integrand, then, $\to 0$ as $x \to 0$. (Cor. to §79·2, p. 103)

Let $(n-1) \leqslant 0$ so that we have now to examine the convergence at '0'. Let μ be a positive number such that

$$(\mu + n - 1) \text{ is } > 0.$$

We have

$$\lim_{x \to 0} \left[-x^{\mu+n-1} \log x \right] = 0,$$

so that for values of x, sufficiently near 0,

$$-x^{\mu+n-1}\log x<\epsilon,$$

where ϵ is a given positive number.

or $$-x^{n-1}\log x<\epsilon/x^{\mu}$$

Now, the integral of ϵ/x^{μ} converges at '0' if, and only if, $\mu<1$.

It is possible to choose a number $\mu<1$ such that $(\mu+n-1)>0$ if, and only if, $n>0$.

Thus the integral converges if $n>0$.

When $n=0$, the integrand becomes $x^{-1}\log x$. We have

$$\int_{\epsilon}^{1} x^{-1}\log x\,dx=-\frac{(\log\epsilon)^2}{2} \text{ which } \to -\infty \text{ as } \epsilon\to 0.$$

When $n<0$, we have

$$x^{n-1}>x^{-1} \text{ for } x \text{ in } (0,1),$$

so that in this case also the integral does not converge. *Thus the integral converges if, and only if,* $n>0$.

4. Test the convergence of the following infinite integrals :—

(i) $\displaystyle\int_0^{\pi/2}\frac{\cos x}{x^n}\,dx$. (ii) $\displaystyle\int_0^{\pi/2}\frac{\sin x}{x^n}\,dx$ (iii) $\displaystyle\int_0^1\frac{x^{a-1}}{1+x}\,dx$.

(iv) $\displaystyle\int_0^1\frac{x^{a-1}}{1-x}\,dx$. (v) $\displaystyle\int_0^2\frac{\log x}{\sqrt{2-x}}\,dx$, (vi) $\displaystyle\int_0^{\pi/4}\frac{dx}{\sqrt{\tan x}}$.

(vii) $\displaystyle\int_0^{\pi/2}\sin^{m-1}x\cos^{n-1}x\,dx$. (viii) $\displaystyle\int_0^1 x^{a-1}e^{-x}\,dx$.

(ix) $\displaystyle\int_0^3\frac{dx}{[(x-1)^2(x-2)^3]^{\frac{1}{5}}}$ (x) $\displaystyle\int_0^1\frac{x^p\log x\,dx}{(1+x)^2}$

(xi) $\displaystyle\int_0^{\pi}\frac{\sqrt{x}}{\sin x}\,dx$. (xii) $\displaystyle\int_0^{\pi}\frac{dx}{\cos\alpha-\cos x}$

5. *Show that*

$$\int_0^{\frac{1}{2}\pi}\sin x\log\sin x\,dx$$

is convergent and its value is $(\log 2-1)$, Integrating by parts,

$$\int_{\epsilon}^{\frac{1}{2}\pi} \sin x \log \sin x \, dx = \Big| -\cos x \log \sin x + \log \tan \tfrac{1}{2}x + \cos x \Big|_{\epsilon}^{\frac{1}{2}\pi}$$

$$= -(1-\cos \epsilon) \log \sin \tfrac{1}{2}\epsilon + \cos \epsilon \log 2 \cos \tfrac{1}{2}\epsilon - \cos \epsilon + \log \cos \tfrac{1}{2}\epsilon.$$

Now, when $\epsilon \to 0$,

$$\text{lt } (1-\cos \epsilon) \log \sin \tfrac{1}{2}\epsilon = \text{lt } 2t^2 \log t; \text{ when } t \to 0, \ (t = \sin \tfrac{1}{2}\epsilon)$$
$$= 0$$

$$\therefore \qquad \int_0^{\frac{1}{2}\pi} \sin x \log \sin x \, dx = \log 2 - 1.$$

6. Show that $\int_0^{\frac{1}{2}\pi} \log \sin x \, dx$ converges.

112. f(x), not necessarily positive. We now obtain a *general* test for convergence at 'a' of the infinite integral

$$\int_a^b f(x)dx. \tag{1}$$

The necessary and sufficient condition for the convergence of the improper integral (1) *at 'a' is that corresponding to every positive number η there exists a positive number δ such that*

$$\left| \int_{a+\epsilon_1}^{a+\epsilon_2} f(x) \, dx \right| < \eta,$$

where ϵ_1, ϵ_2 are any two positive numbers less than or equal to δ.

We write

$$\phi(\epsilon) = \int_{a+\epsilon}^{b} f(x)dx.$$

From § 44, p. 65, we know that the necessary and sufficient condition for $lt \ \phi(\epsilon)$ to exist finitely is that corresponding to every positive number η, there exists a number $\delta > 0$ such that when $0 < \epsilon_1, \epsilon_2 < \delta$, then

$$| \phi(\epsilon_1) - \phi(\epsilon_2) | < \eta,$$

$$i.e. \quad \left| \int_{a+\epsilon_1}^{b} f(x)dx - \int_{a+\epsilon_2}^{b} f(x) \, dx \right| < \eta \quad \text{or} \quad \left| \int_{a+\epsilon_1}^{a+\epsilon_2} f(x) \, dx \right| < \eta.$$

112.1. *From above we deduce an important sufficient test for convergence viz* ;

If $\int_a^b |f(x)|\,dx$ *exists, then* $\int_a^b f(x)dx$ *also exists.*

(It is assumed that the proper integral $\int_{a+\epsilon}^b f(x)dx$ exists).

This theorem follows from the general condition of convergence above and the inequality

$$\left|\int_{a+\epsilon_1}^{a+\epsilon_2} f(x)\,dx\right| < \left|\int_{a+\epsilon_1}^{a+\epsilon_2} |f(x)|\,dx\right| \quad \text{(Cor. 6 to § 88, p. 125)}$$

Def. Absolute convergence. *The improper integral* $\int_a^b f(x)\,dx$ *is said to be absolutely convergent if* $\int_a^b |f(x)|\,dx$ *is convergent.*

From the result proved above it follows that every absolutely convergent integral is also convergent

Ex. 1. *Test the convergence of* $\int_0^1 \frac{\sin(1/x)}{\sqrt{x}}\,dx.$

Let $f(x)=\sin(1/x)/\sqrt{x}$. Here there is no neighbourhood of the point '0' in which $f(x)$ constantly keeps the same sign.

In [0, 1), we have

$$\left|\frac{\sin 1/x}{\sqrt{x}}\right| = \frac{|\sin 1/x|}{\sqrt{x}} \leqslant \frac{1}{\sqrt{x}}.$$

Also $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent.

$\therefore \int_0^1 \left|\frac{\sin 1/x}{\sqrt{x}}\right| dx$ is convergent so that $\int_0^1 \frac{\sin 1/x}{\sqrt{x}}\,dx$ is absolutely convergent.

118. Convergence at ∞. *Convergence of*

$$\int_a^\infty f(x)\,dx,$$

where $f(x)$ *is bounded and integrable in* (a, X) *for every* $X \geqslant a$.

Positive integrand. *Let $f(x)$ be positive in (a, X). The necessary and sufficient condition for $\int_a^\infty f(x)dx$ to be convergent is that there exists a positive number M, independent of X, such that*

$$\int_a^X f(x)dx < M \text{ for every } X \geqslant a.$$

The proof follows immediately from the fact that, since $f(x)$ is positive, the integral $\int_a^X f(x)dx$ *monotonically increases as X increases* and will, therefore, tend to a finite limit or to ∞ according as it is bounded above or not.

From this we immediately deduce that *if $f(x)$ and $\phi(x)$ are both positive and $f(x) \geqslant \phi(x)$ in (a, X), then*

$$\int_a^\infty \phi(x)\, dx \quad \text{converges if } \int_a^\infty f(x)dx \text{ converges.}$$

Also, it may be shown that *if $f(x)$ and $\phi(x)$ are positive and, when $x \to \infty$, then lt (f/ϕ) exists and is equal to l and 'l' is neither 0 nor ∞, then*

$$\int_a^\infty f(x)dx \text{ and } \int_a^\infty \phi\,(x)\, dx \text{ either both converge or both do not converge.}$$

Ex. What conclusion can be drawn if l is zero or infinite.

113·1. An important comparison integral. *To prove that*

$$\int_a^\infty \frac{dx}{x^\mu}, \qquad (a>0) \tag{1}$$

converges if and only if $\mu > 1$.

We have, if $\mu \neq 1$,

$$\int_a^X \frac{dx}{x^\mu} = \frac{1}{1-\mu} \left| \frac{1}{x^{\mu-1}} \right|_a^X = \frac{1}{1-\mu} \left[\frac{1}{X^{\mu-1}} - \frac{1}{a^{\mu-1}} \right]$$

which $\to 1/(\mu-1)a^{\mu-1}$ or ∞ according as $\mu > 1$ or $\mu < 1$.

For $\mu = 1$, we have

$$\int_a^X \frac{dx}{x} = \log \frac{X}{a}, \text{ which} \to \infty \text{ as } X \to \infty.$$

Hence the result.

Adopting (1) as the comparison integral and employing the test of §113, we may now easily obtain the following practical tests for convergence at ∞.

113·2. *If $f(x)$ is positive in (a, X), then the integral converges if there exists a positive number μ greater than 1 and a fixed positive number M such that*

$$f(x) \leqslant M/x^{\mu} \text{ for every } x \geqslant a.$$

Again, the integral does not converge if there exists a positive number $\mu \leqslant 1$ and a fixed positive number M such that

$$f(x) > M/x^{\mu} \text{ for every } x \geqslant a.$$

113·3. *If, when $x \to \infty$, $\mathrm{lt}\ [f(x)\,x^{\mu}]$ exists finitely, the limit being neither 0 nor infinite, then the integral converges if, and only if, $\mu > 1$.*

Examples

1. *Examine the convergence of*

(*i*) $\displaystyle\int_0^{\infty} \frac{x\,dx}{(1+x)^3}$. (*ii*) $\displaystyle\int_1^{\infty} \frac{dx}{(1+x)\sqrt{x}}$.

(*iii*) $\displaystyle\int_1^{\infty} \frac{dx}{x^{\frac{1}{3}}(1+x)^{\frac{1}{2}}}$ (*iv*) $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx$.

(*i*) Let $f(x) = x/(1+x)^3$.

We have $\displaystyle\lim_{x\to\infty} x^2 f(x) = 1$ so that $\mu = 2 > 1$.

$$\therefore \quad \int_0^{\infty} \frac{x\,dx}{(1+x)^3} \text{ is convergent.}$$

(*ii*) Let $f(x) = 1/(1+x)\sqrt{x}$.

We have $\displaystyle\lim_{x\to\infty} x^{\frac{3}{2}} f(x) = 1$, so that $\mu = \frac{3}{2} > 1$.

$$\therefore \quad \int_1^{\infty} f(x)\,dx \text{ is convergent.}$$

(*iii*) Let $f(x) = 1/x^{\frac{1}{3}}(1+x)^{\frac{1}{2}}$.

Now $x^{\frac{5}{6}} f(x) \to 1$ as $x \to \infty$. Here $\mu = \frac{5}{6} < 1$.

$$\therefore \quad \int_1^{\infty} f(x)\,dx \text{ is divergent.}$$

(*iv*) Let $f(x) = \sin^2 x/x^2$,

Here $x^2 f(x)=\sin^2 x$, which does not tend to a limit but is bounded.

But $\dfrac{\sin^2 x}{x^2} \leqslant \dfrac{1}{x^2}$ and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges.

$\therefore$ $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges. Also $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is a proper integral for lt $(\sin^2 x/x^2)=1$ when $x \to 0$, so that '0' not a point of infinite discontinuity. Therefore $\displaystyle\int_0^\infty f(x)dx$ is convergent.

2. *Show that* $\displaystyle\int_0^\infty x^{a-1}\, e^{-x}\,dx$ *is convergent if, and only if,* $a>0$.

Let $f(x)=x^{a-1}\, e^{-x}$.

Here, '0' is a point of infinite discontinuity of $f(x)$ if $a<1$. Thus we have to examine the convergence at ∞ as well as at '0'.

We consider any positive number >0, say 1, and examine the convergence of

$$\int_0^1 f(x)dx \text{ and } \int_1^\infty f(x)dx$$

at 0 and ∞ respectively

(*i*) Let $a<1$.

$\lim\limits_{x\to 0} x^{1-a} f(x)=1$ so that $\displaystyle\int_0^1 f(x)dx$ converges if, and only if, $(1-a)<1$, *i.e.*, if $0<a$.

(*ii*) We know that $e^x>x^{a+1}$ whatever value a may have

$$\therefore\ x^{a-1}\, e^{-x}<1/x^2.$$

But $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges. Therefore $\displaystyle\int_1^\infty x^{a-1}\, e^{-x}dx$ also converges for every value of a.

Thus $\displaystyle\int_0^\infty x^{a-1}\, e^{-x}\,dx$ converges if, and only if, $a>0$.

The integral $\int_0^\infty x^{a-1} e^{-x}\, dx$, which is a function of a, is called a **Gamma Function** and is denoted by Γa

3. Discuss the convergence of the following :—

(i) $\int_0^\infty \frac{x^m(1+x^n)}{1+x^p} dx$. (ii) $\int_0^\infty \frac{x^{2m}}{1+x^{2n}} dx$. $(m>0,\ n>0)$

(iii) $\int_0^\infty \frac{x^{m-1}-x^{n-1}}{1-x} dx$. (iv) $\int_1^\infty x^{m-1} e^{-nx}\, dx$.

(v) $\int_2^\infty x^m(\log x)^n\, dx$. (vi) $\int_0^\infty \frac{xa}{(1+x)^b\ [1+(\log x)^2]} dx$.

4. Show that the improper integral

$$\int_0^\infty \log(1+2 \operatorname{sech} x)dx$$

converges.

[$\log(1+2 \operatorname{sech} x)<2 \operatorname{sech} x$, for $\log (1+x)<x$, if $x>0$

$$=\frac{2\cdot 2}{e^x+e^{-x}}<\frac{4}{e^x}=4e^{-x}].$$

5. Show that

$$\int_0^\infty \frac{\cosh bt}{\cosh at} dt, \qquad a>0,\ b>0,$$

converges if, and only if, $b<a$.

[If $b<a$, we have

$$\frac{\cosh bt}{\cosh at}=\frac{e^{bt}+e^{-bt}}{e^{at}+e^{-at}}<\frac{e^{bt}+e^{bt}}{e^{at}}=2e^{-(a-b)t},$$

and if $b>a$, we have

$$\frac{\cosh bt}{\cosh at}=\frac{e^{bt}+e^{-bt}}{e^{at}+e^{-at}}>\frac{e^{bt}}{e^{at}+e^{at}}=\tfrac{1}{2}e^{(b-a)t}].$$

6. Show that

$$\int_0^\infty \frac{\sinh bx}{\sinh ax} dx, \qquad a>0,\ b>0,$$

converges if, and only if $a>b$.

[If $a>b$, we write

$$\frac{\sinh bx}{\sinh ax}=\frac{e^{bx}-e^{-bx}}{e^{ax}-e^{-ax}}<\frac{e^{bx}}{e^{ax}-1};$$

and if $a<b$, we write

$$\frac{\sinh bx}{\sinh ax}=\frac{e^{bx}-e^{-bx}}{e^{ax}-e^{-ax}}>\frac{e^{bx}-1}{e^{ax}}\Bigg].$$

7. *Show that*

$$\int_0^\infty \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{dx}{x}$$

is convergent.

The point '0' is not a point of infinite discontinuity of the integrand in as much as the integrand $\to \frac{1}{6}$ as $x \to 0$. We have, therefore, to examine the convergence at ∞ only. We have

$$\left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{1}{x} = \frac{e^x - e^{-x} - 2x}{x^2(e^x - e^{-x})} < \frac{e^x}{x^2(e^x - e^{-x})}, \text{ for } x > 0$$

$$= \frac{e^{2x}}{e^{2x} - 1} \cdot \frac{1}{x^2}.$$

Since $e^{2x}/(e^{2x} - 1) \to 1$ as $x \to \infty$, we can find a number X such that for $x \geqslant X$

$$e^{2x}/(e^{2x} - 1) < \tfrac{3}{2}.$$

Thus for $x \geqslant X$,

$$\left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{1}{x} < \frac{3}{2} \cdot \frac{1}{x^2},$$

so that $\int_X \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{1}{x}\,dx$ is convergent, and, also, therefore the given integral is convergent.

8. Show that the integrals

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-(x-a/x)^2}\,dx$$

converge.

9. Show that the integral

$$\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x}$$

is convergent.

[The integrand, $\dfrac{e^x - 1 - x}{x(1+x)e^x}$, is clearly positive, when $x > 0$.

Also, '0' is not a point of infinite discontinuity, for the integrand tends to $\frac{1}{2}$ as $x \to 0$. For the convergence at ∞, we rewrite the integral as

$$\frac{1}{x^2} \cdot \left(\frac{x}{1+x} - \frac{x}{e^x}\right) \text{ and note that } \left(\frac{x}{1+x} - \frac{x}{e^x}\right) \to 1 \text{ as } x \to \infty.$$

114. Convergence of $\int_a^\infty f(x)dx$ at ∞, when f(x) is not necessarily positive. General test for convergence.

The necessary and sufficient condition for the convergence of

$$\int_a^\infty f(x)dx$$

at ∞ is that, corresponding to every positive number η, there exists a number X, such that

$$\left| \int_{X_1}^{X_2} f(x)dx \right| < \eta,$$

when X_1, X_2 are any two numbers $\geqslant X$.

We write

$$\phi(X) = \int_a^X f(x)dx.$$

From § 44, we know that the necessary and sufficient condition for $lt\ \phi(x)$ to exist finitely is that corresponding to every positive number η there exists a number X, such that when X_1, X_2 are $\geqslant X$,

$$| \phi(X_2) - \phi(X_1) | < \eta,$$

i. e., $$\left| \int_a^{X_2} f(x)dx - \int_a^{X_1} f(x)dx \right| < \eta \quad \text{or} \quad \left| \int_{X_1}^{X_2} f(x)dx \right| < \eta$$

115. Theorem. If $\int_a^\infty | f(x) | dx$ *converges, then* $\int_a^\infty f(x)dx$ *also converges.*

Let η be any positive number. Since $\int_a^\infty | f(x) | dx$ converges, there exists a number X, such that

$$\left| \int_{X_1}^{X_2} | f(x) | dx \right| < \eta, \text{ where } X_1, X_2 \text{ are both} \geqslant X. \tag{1}$$

Also, $$\left| \int_{X_1}^{X_2} f(x)dx \right| \leqslant \left| \int_{X_1}^{X_2} | f(x) | dx \right| \tag{2}$$

From (1) and (2), the result follows.

Def. Absolute convergence. *The improper integral* $\int_a^\infty f(x)dx$ *is* *said to be absolutely convergent if* $\int_a^\infty |f(x)|\,dx$ *is convergent.*

From the theorem above it follows that *every absolute convergent improper integral is convergent.*

It will later on be seen that the converse is not necessarily true. (See Ex. 4 p. 190, at the end of this chapter).

115·1. Test for the absolute convergence of the integral of a product. *Let $\phi(x)$ be bounded in $(a, \infty]$ and integrable in (a, X) where X is any number $> a$. Let* $\int_a^\infty f(x)dx$ *converge absolutely at ∞. Then*

$$\int_a^\infty f(x)\,\phi(x)dx$$

is absolutely convergent.

Since $\phi(x)$ is bounded in $(a, \infty]$ there exists a positive constant A such that

$$|\phi(x)| \leqslant A, \text{ for every } x > a. \tag{1}$$

Since $\int_a^\infty |f(x)|\,dx$ is convergent, there exists a positive number B such that

$$\int_a^X |f(x)|\,dx < B, \text{ for every } X > a, \tag{2}$$

the integrand, $|f(x)|$, being positive. (§ 113).

We have, from (1),

$$|f(x)\,\phi(x)| \leqslant A\,|f(x)|, \text{ for every } x > a$$

$$\therefore \int_a^X |f(x)\phi(x)|\,dx \leqslant A\int_a^X |f(x)|\,dx \leqslant AB, \text{ for every } X > a.$$

so that $\int_a^X |f(x)\,\phi(x)|\,dx$ is bounded above for $X > a$,

Hence $\int_a^\infty |f(x)\,\phi(x)|\,dx$ is convergent,

i. e., $\int_a^\infty f(x)\,\phi(x)dx$ is absolutely convergent.

Ex. Discuss the convergence of the following integrals :

(i) $\int_1^\infty \frac{\sin x}{x^3} dx$. (ii) $\int_0^\infty e^{-ax} \cos x \, dx$.

116. Tests for conditional convergence.

116·1. Abel's theorem for the convergence of the integral of a product. *Let $\phi(x)$ be bounded and monotonic in $(a, \infty]$ and*

let $$\int_a^\infty f(x)dx \text{ be convergent.}$$

Then $$\int_a^\infty f(x)\, \phi(x)dx \text{ is convergent.}$$

The bounded function $\phi(x)$, which is monotonic in $x > a$, is integrable in (a, X) where X is any number $> a$.

Applying the second mean value theorem, we have

$$\int_{X_1}^{X_2} f(x)\, \phi(x)dx = \phi(X_1) \int_{X_1}^{\xi} f(x)dx + \phi(X_2) \int_{\xi}^{X_2} f(x)dx, \tag{1}$$

where $a < X_1 \leqslant \xi \leqslant X_2$.

Let η be any positive number.

Since $\phi(x)$ is bounded in $(a, \infty]$, there exists a positive number A such that

$$| \phi(x) | \leqslant A, \text{ for every } x > a.$$

In particular, ∴, $$| \phi(X_1) | \leqslant A,\ | \phi(X_2) | \leqslant A. \tag{2}$$

Also, since $\int_a^\infty f(x)dx$ is convergent, there exists, by §114, a number X_0 such that,

$$\left| \int_{X_1}^{X_2} f(x)dx \right| < \frac{\eta}{2A}, \text{ for } X_1, X_2 > X_0.$$

We now suppose that in (1), X_1, X_2 are numbers $> X_0$ so that ξ, which lies between X_1 and X_2, is also $> X_0$.

$$\therefore \quad \left| \int_{X_1}^{\xi} f(x)dx \right| < \frac{\eta}{2A} \text{ and } \left| \int_{\xi}^{X_2} f(x)dx \right| < \frac{\eta}{2A} \tag{3}$$

From (1) and (2) and (3) we deduce that there exists a number X_0 such that for any pair of numbers $X_1, X_2 > X_0$,

$$\left|\int_{X_1}^{X_2} f(x)\,\phi(x)dx\right| \leqslant |\phi(X_1)| \left|\int_{X_1}^{\xi} f(x)dx\right| + |\phi(X_2)| \left|\int_{\xi}^{X_2} f(x)dx\right|$$

$$< A.\frac{\eta}{2A}+A.\frac{\eta}{2A}=\eta,$$

where η is any positive number assigned arbitrarily.

Hence $\int_a^{\infty} f(x)\,\phi(x)dx$ converges at ∞.

116·2. Drichilet's theorem for the convergence of the integral of a product. *Let $\phi(x)$ be bounded and monotonic in $(a, \infty]$ and let $\phi(x)\to 0$, when $x\to\infty$. Let $\int_a^X f(x)dx$ be bounded when $X>a$.*

Then $\int_a^{\infty} f(x)\,\phi(x)dx$ is convergent.

The function $\phi(x)$, which is monotonic in $(a, \infty]$, is integrable in (a, X) where X is any number $>a$.

Applying the second mean value theorem, we have

$$\int_{X_1}^{X_2} f(x)\,\phi(x)dx=\phi(X_1)\int_{X_1}^{\xi} f(x)dx+\phi(X_2)\int_{\xi}^{X_2} f(x)dx, \tag{1}$$

where $a<X_1\leqslant \xi \leqslant X_2$.

Since $\int_a^X f(x)dx$ is bounded when $X>a$, there exists a number A such that

$$\left|\int_a^X f(x)dx\right| \leqslant A, \text{ for every } X > a.$$

$$\therefore \quad \left|\int_{X_1}^{\xi} f(x)dx\right| = \left|\int_a^{\xi} f(x)dx-\int_a^{X_1} f(x)dx\right|$$

$$\leqslant \left|\int_a^{\xi} f(x)dx\right|+\left|\int_a^{X_1} f(x)dx\right| \leqslant A+A=2A. \tag{2}$$

Similarly $$\left|\int_{\xi}^{X_2} f(x)dx\right| \leqslant 2A \tag{3}$$

Let η be any positive number.

Since $\phi(x) \to 0$ as $x \to \infty$, there exists a number X_0, such that
$$|\phi(x)| < \eta/4A, \text{ when } x > X_0 \tag{4}$$

We now suppose that X_1, X_2 are any two numbers $> X_0$, so that from (4),
$$|\phi(X_1)| < \eta/4A, \qquad |\phi(X_2)| < \eta/4A \tag{5}$$

From (1), (2), (3) and (5), we deduce that there exists a number X_0, such that for any pair of numbers X_1, $X_2 > X_0$,
$$\left|\int_{X_1}^{X_2} f(x)\phi(x)dx\right| \leqslant |\phi(X_1)| \left|\int_{X_1}^{\xi} f(x)dx\right| + |\phi(X_2)| \left|\int_{\xi}^{X_2} f(x)dx\right|$$
$$\leqslant (\eta/4A).2A + (\eta/4A).2A = \eta,$$
where η is any positive number arbitrarily assigned.

Hence $\int_a^{\infty} f(x)\,\phi(x)dx$ converges at ∞.

Examples

1. *Show that*
$$\int_0^{\infty} \frac{\sin x}{x}\,dx \tag{1}$$
is convergent.

Since the integrand $\to 1$, as $x \to 0$, therefore '0' is not a point of infinite discontinuity.

Now, consider the improper integral
$$\int_1^{\infty} \frac{\sin x}{x}\,dx. \tag{2}$$

The factor $1/x$ of the integrand is monotonic and $\to 0$ as $x \to \infty$. Also
$$\left|\int_1^{X} \sin x\,dx\right| = \left|-\cos X + \cos 1\right| \leqslant |\cos X| + |\cos 1| \leqslant 2,$$

so that

$$\int_1^X \sin x \, dx \text{ is bounded above for every } X > 1.$$

Here by §116·2, the integral (2) is convergent. Now since $\int_0^1 \frac{\sin x}{x} dx$ is only a proper integral, we see that the integral (1) is convergent.

2. *Show that* $\int_0^\infty \sin x^2 \, dx$ *is convergent.*

We write

$$\sin x^2 = \frac{1}{2x} \cdot 2x \sin x^2,$$

and consider the improper integral

$$\int_1^\infty \sin x^2 \, dx, \ \textit{i.e.}, \int_1^\infty \frac{1}{2x} \cdot 2x \sin x^2 \, dx.$$

Now, $1/2x$ is monotonic and $\to 0$ as $x \to \infty$. Also

$$\left| \int_1^X 2x \sin x^2 \, dx \right| = \left| -\cos X^2 + \cos 1 \right| \leqslant 2,$$

so that

$$\int_1^X 2x \sin x^2 \, dx \text{ is bounded for } X > 1.$$

Hence

$$\int_1^\infty \frac{1}{2x} . 2x \sin x^2 dx, \ i.e. \int_1^\infty \sin x^2 dx$$

is convergent.

Since $\int_0^1 \sin x^2 \, dx$ is only a proper integral, we see that the given integral is convergent.

3. Show that

$$\int_0^\infty e^{-ax} \frac{\sin x}{x} dx, \quad a \geqslant 0,$$

is convergent.

[e^{-ax} is monotonic and $\int_0^\infty (\sin x/x)dx$ is convergent.

4. **Test the following for convergence**

(i) $\int_0^\infty \frac{\sin kx}{x} dx$ (ii) $\int_0^\infty \frac{\sin x}{\sqrt{x}} dx$

(iii) $\int_0^\infty e^{-a^2x^2} \sin 2bx \frac{dx}{x}$. (iv) $\int_0^\infty \frac{\cos ax \cos bx}{x} dx$

(v) $\int_0^\infty \frac{\sin x^m}{x^n} dx$ (vi) $\int_0^\infty \frac{\sin x(1-\cos x)}{x^a} dx$

(vii) $\int_0^\infty \frac{\sin (x+x^2)}{x^n} dx$ (viii) $\int_0^\infty \frac{x^m \cos ax}{1+x^n} dx$

Examples

1. If $\phi(x)$ is bounded and integrable in an interval (a, b) and $\int_a^b f(x)\, dx$ converges absolutely at a, then $\int_a^b f(x)\phi(x)dx$ also converges absolutely at a. [Analogue of §115].

2. If $\phi(x)$ is monotonic in an interval (a, b) and $\int_a^b f(x)dx$ converges at a, then $\int_a^b f(x)\phi(x)dx$ also converges at a. [Analogue of §116·1].

3. If $\phi(x)$ is monotonic in (a,b) and $\rightarrow 0$, as $x \rightarrow (a+0)$, and $\int_X^b f(x)\, dx$ is bounded for all X such that $a < X \leqslant b$, then $\int_a^b f(x)\phi(x)\, dx$ converges at a. [Analogue of §116.2].

4. *Show that the improper integral*

$$\int_0^\infty \frac{\sin x}{x} dx$$

is not absolutely convergent.

We have to show that

$$\int_0^\infty \frac{|\sin x|}{x}\,dx$$

is not convergent.

Consider the proper integral

$$\int_0^{n\pi} \frac{|\sin x|}{x}\,dx,$$

where n is a positive integer. We have

$$\int_0^{n\pi} \frac{|\sin x|}{x}\,dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\,dx.$$

We put $x=(r-1)\pi+y$ so that y varies in $(0, \pi)$. We have
$|\sin[(r-1)\pi+y]| = |(-1)^{r-1}\sin y| = \sin y.$

$$\therefore \quad \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\,dx = \int_0^{\pi} \frac{\sin y}{(r-1)\pi+y}\,dy.$$

Since $r\pi$ is the max. value of $[(r-1)\pi+y]$ in $(0, \pi)$, we have

$$\int_0^{\pi} \frac{\sin y}{(r-1)\pi+y}\,dy > \frac{1}{r\pi}\int_0^{\pi} \sin y\,dy = \frac{2}{r\pi}.$$

$$\therefore \quad \int_0^{n\pi} \frac{|\sin x|}{x}\,dx > \sum_1^n \frac{2}{r\pi} = \frac{2}{\pi}\sum_1^n \frac{1}{r}.$$

Since $\sum_1^n \frac{1}{r} \to \infty$ as $n \to \infty$, we see that

$$\int_0^{n\pi} \frac{|\sin x|}{x}\,dx \to \infty, \text{ as } n \to \infty.$$

Let, now, X be any real number. There exists a positive integer n such that

$$n\pi < X \leqslant (n+1)\pi.$$

We have

$$\int_0^{X} \frac{|\sin x|}{x}\,dx > \int_0^{n\pi} \frac{|\sin x|}{x}\,dx$$

Let $X \to \infty$ so that n also $\to \infty$. Thus we see that

$$\int_0^X \frac{|\sin x|}{x} dx \to \infty \text{ as } X \to \infty.$$

so that $$\int_0^\infty \frac{|\sin x|}{x} dx \text{ does not converge.}$$

5. *Show that*

$$\int_0^\infty \frac{x dx}{1+x^6 \sin^2 x}$$

is convergent.

The integrand is positive for positive values of x but the test obtained in §113 does not enable us to establish the convergence.

In order to show that the integral converges we proceed as follows. Consider the proper integral

$$\int_0^{n\pi} \frac{x\, dx}{1+x^6 \sin^2 x},$$

and write

$$\int_0^{n\pi} \frac{x\, dx}{1+x^6 \sin^2 x} = \sum_{r=}^{n} \int_{(r-1)\pi}^{r\pi} \frac{x\, dx}{1+x^6 \sin^2 x}$$

Now, if x varies in $(\overline{r-1}\,\pi, r\pi)$, we have

$$\frac{x}{1+x^6 \sin^2 x} < \frac{r\pi}{1+(r-1)^6\pi^6 \sin^2 x}.$$

$$\therefore \int_{(r-1)\pi}^{r\pi} \frac{x\, dx}{1+x^6 \sin^2 x} < \int_{(r-1)\pi}^{r\pi} \frac{r\pi\, dx}{1+(r-1)^6\pi^6 \sin^2 x} = a_r, \text{ say}$$

Putting $x=(r-1)\pi + y$, we see that

$$a_r = \int_0^{\pi} \frac{r\pi dy}{1+(r-1)^6 \pi^6 \sin^2 y} = 2\int_0^{\pi/2} \frac{r\pi dy}{1+(r-1)^6\pi^6 \sin^2 y}.$$

If $A>0$, we have

$$\int_0^{\pi/2} \frac{dy}{1+A \sin^2 y} = \int_0^{\pi/2} \frac{\operatorname{cosec}^2 y\, dy}{A+1+\cot^2 y} = -\frac{1}{\sqrt{(A+1)}} \left| \tan^{-1} \frac{\cot y}{\sqrt{(A+1)}} \right|_0^{\pi/2}$$

$$= \frac{\pi}{2} \cdot \frac{1}{\sqrt{(A+1)}}$$

$$\therefore\ a_r = 2.r\pi.\ \frac{\pi}{2} \cdot \frac{1}{\sqrt{[(r-1)^6\pi^6+1]}} = \frac{r\pi^2}{\sqrt{[(r-1)^6\pi^6+1]}}.$$

$$\therefore\ \int_{(r-1)\pi}^{r\pi} \frac{x}{1+x^6 \sin^2 x}\, dx < \frac{r\pi^2}{\sqrt{[1+(r-1)^6\ \pi^6]}} < \frac{r}{(r-1)^3} \cdot \frac{1}{\pi},\ (r \neq 1)$$

Now
$$\frac{r}{(r-1)^3} = \frac{1}{(r-1)^2} + \frac{1}{(r-1)^3},$$

and, $\sum_{r=2}^{\infty} \frac{1}{(r-1)^2}$ and $\sum_{r=2}^{\infty} \frac{1}{(r-1)^3}$ are both convergent.

$$\therefore\ \sum_{r=2}^{\infty} \frac{r}{(r-1)^3} \frac{1}{\pi} \text{ is convergent.}$$

$$\therefore\ \int_{\pi}^{n\pi} \frac{x\, dx}{1+x^6 \sin^2 x} \to \text{a finite limit as } n \to \infty.$$

$$\therefore\ \int_{0}^{\infty} \frac{x\, dx}{1+x^6 \sin^2 x} \text{ converges.}$$

6. *Show that* $\int_0^\infty \frac{x\, dx}{1+x^4 \sin^2 x}$ *is divergent.*

We write
$$\int_0^{n\pi} \frac{x\, dx}{1+x^4 \sin^2 x} = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{x\, dx}{1+x^4 \sin^2 x}$$

If x varies in $(\overline{r-1}\ \pi, r\pi)$, we have
$$\frac{(r-1)\pi}{1+(r\pi)^4 \sin^2 x} < \frac{x}{1+x^4 \sin^2 x}$$

or
$$\int_{(r-1)\pi}^{r\pi} \frac{(r-1)\pi}{1+(r\pi)^4 \sin^2 x}\, dx < \int_{(r-1)\pi}^{r\pi} \frac{x}{1+x^4 \sin^2 x} dx$$

Now,
$$\int_{(r-1)\pi}^{r\pi} \frac{(r-1)\pi}{1+(r\pi)^4 \sin^2 x}\, dx = (r-1)\pi \int_0^{\pi} \frac{dy}{1+(r\pi)^4 \sin^2 y}$$

$$= 2(r-1)\pi \int_0^{\pi/2} \frac{dy}{1+(r\pi)^4 \sin^2 y} = 2(r-1)\pi.\ \frac{\pi}{2} \cdot \frac{1}{\sqrt{[1+(r\pi)^4]}}$$

$$= \frac{(r-1)\pi^2}{\sqrt{[1+(r\pi)^4]}}$$

The infinite series $\Sigma \dfrac{(r-1)\pi^2}{\sqrt{[1+(r\pi)^4]}}$ diverges as we see on comparision with $\Sigma(1/r)$. Hence the integral does not converge.

7. Show that

$$\int_0^\infty \frac{dx}{1+x^4\sin^2 x} \text{ converges but } \int_0^\infty \frac{dx}{1+x^2\sin^2 x} \text{ does not.}$$

8. Show that the following improper integrals converge :—

$$(i) \int_0^1 \log x \log(1+x)dx. \quad (ii) \int_0^1 \frac{\log x}{\sqrt[3]{(1-x)}}dx. \quad (iii) \int_0^\infty \frac{x}{(1+x^2)\sqrt[5]{\sin x}}dx$$

$$(iv) \int_0^\infty e^{-a^2x} \log\cos^2 x\, dx, \ (a \neq 0) \qquad (v) \int_0^\infty \frac{e^{-ax}}{\sqrt[3]{\sin x}}dx. \qquad (a>0)$$

$$(vi) \int_0^\infty \frac{x^m+x^{-m}}{x}\log(1+x)dx. \ |m|<1 \quad (vii) \int_0^\infty x^n e^{-a^2x^2-b^2/x^2}dx. \quad (n>-1)$$

$$(viii) \int_0^\infty e^{-x} x^{n-1}(\log x)^m dx. \quad (n>0 \text{ and } m \text{ is a positive integer}).$$

$$(ix) \int_0^\infty \frac{x\cosh ax}{\sinh x}dx. \ |a|<1. \qquad (x) \int_0^\infty \frac{e^x}{e^{4x}\sin^2 x+\cos^2 x}dx.$$

$$(xi) \int_0^\infty \frac{\cosh ax \cosh bx}{\cosh x}dx. \qquad |a|+|b|<1.$$

9. *Show that the integral*

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx \qquad (a>0,\ b>0)$$

converges and is equal to $\log(b/a)$.

Since, when $x \to 0$,

$$\text{lt}\ \frac{e^{-ax}-e^{-bx}}{x} = -a+b,$$

we see that $x=0$ is not a point of infinite discontinuity.

Again, since $e^{ax}>a^3x^3/6$, *i.e.*, $(e^{-ax}/x)<(6/a^3x^4)$, therefore

$$\int_0^\infty \frac{e^{-ax}dx}{x}$$

converges at ∞. Similarly

$$\int_0^{\infty} \frac{e^{-bx}dx}{x}$$

converges at ∞. The convergence follows by Abel's test §(116·1) also

Now, we have

$$\int_0^{\infty} \frac{e^{-ax}-e^{-bx}}{x}\, dx = \underset{\lambda \to 0}{\text{lt}} \int_{\lambda}^{\infty} \frac{e^{-ax}-e^{-bx}}{x}\, dx,$$

the integral being a continuous function of λ.

Putting $ax=t$, $$\int_{\lambda}^{\infty} \frac{e^{-ax}}{x}\, dx = \int_{a\lambda}^{\infty} \frac{e^{-t}}{t}\, dt\,;$$

putting $bx=t$, $$\int_{\lambda}^{\infty} \frac{e^{-bx}}{x}\, dx = \int_{b\lambda}^{\infty} \frac{e^{-t}}{t}\, dt.$$

$$\therefore \quad \int_{\lambda}^{\infty} \frac{e^{-ax}-e^{-bx}}{x}\, dx = \int_{a\lambda}^{\infty} \frac{e^{-t}dt}{t} - \int_{b\lambda}^{\infty} \frac{e^{-t}dt}{t} = \int_{a\lambda}^{b\lambda} \frac{e^{-t}dt}{t}.$$

$$= \int_a^b \frac{e^{-\lambda y}}{y}\, dy, \text{ where } y = \frac{t}{\lambda}$$

Since $e^{-\lambda y} \to 1$ as $\lambda y \to 0$, there exists a positive number δ, corresponding to any positive number ϵ, such that

$$|e^{-\lambda y} - 1| < \epsilon, \text{ when } 0 < \lambda y < \delta.$$

If y lies in (a, b), $\lambda y \leqslant \lambda b$, so that we see that

$$|e^{-\lambda y} - 1| < \epsilon, \text{ when } \lambda < \delta/b,$$

for every value of y in (a, b).

This gives

$$\frac{1-\epsilon}{y} < \frac{e^{-\lambda y}}{y} < \frac{1+\epsilon}{y},$$

or $$\left| \int_a^b \frac{e^{-\lambda y}}{y}\, dy - \log \frac{b}{a} \right| < \epsilon \log \frac{b}{a}, \text{ when } 0 < \lambda < \frac{\delta}{b}.$$

$$\therefore \quad \underset{\lambda \to 0}{\text{lt}} \int_a^b \frac{e^{-\lambda y}}{y}\, dy = \log \frac{b}{a}.$$

10. Frullani's Integrals.

(*a*) If

$$\int_0^\infty \phi(x)dx$$

converges or oscillates between finite limits at ∞ and $\phi(x)$ tends to a definite limit ϕ_0 as x tends to 0, prove that

$$\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx=\phi_0 \log\frac{b}{a} .$$

[The integral considered in the previous example is only a particular case of the one considered here.

(*b*) If $\phi(x)$ tends to a definite number ϕ_1 as $x \to \infty$ and to a definite number ϕ_0 as $x \to 0$ then

$$\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = (\phi_0-\phi_1)\log\frac{b}{a}.$$

[In this case the integral of $\phi(x)$ diverges at ∞ unless $\phi_1=0$].

(*c*) Evaluate

$$\int_0^\infty \frac{\tan^{-1} ax-\tan^{-1} bx}{x}dx.$$

11. Show that a, b being positive,

$$\int_0^\infty \frac{\cos ax-\cos bx}{x}dx=\log\frac{b}{a}$$

and deduce that

$$\int_0^\infty \frac{\sin ax \sin bx}{x}dx=\tfrac{1}{2}\log\frac{a+b}{a-b}; \qquad (a>b>0).$$

12. Show that

$$\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$$

$$\left[(i) \quad \int_0^\infty \frac{\sin}{x}dx = \underset{n\to\infty}{\text{lt}} \int_0^{\frac{1}{2}(2n+1)\pi} \frac{\sin x}{x}dx = \underset{n\to\infty}{\text{lt}} \int_0^{\frac{1}{2}\pi} \frac{\sin (2n+1)x}{x}dx \right.$$

(*ii*) Integrating by parts or otherwise, (see **Ex. 22**, p. 159 and **Ex. 25**, p. 161) show that

$$\underset{n\to\infty}{\text{lt}} \int_0^{\frac{1}{2}\pi} \sin (2n+1)x \left(\frac{1}{\sin x}-\frac{1}{x}\right)dx=0.$$

(*iii*) Show that

$$\left. \int_0^{\frac{1}{2}\pi} \frac{\sin (2n+1)x}{\sin x}dx=\frac{\pi}{2}\cdot \right]$$

13. Show that

$$\int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2},\ 0 \text{ or } -\frac{\pi}{2},$$

according as a is positive, zero or negative.

14. Show that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

(Integrate by parts)

15. Show that

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}(b-a). \qquad (a>0,\ b>0)$$

(Integrate by parts).

16. Show that

$$\int_0^\infty \frac{\sin ax \sin x}{x^2}\,dx = \tfrac{1}{2}\pi a \text{ when } (0 \leqslant a \leqslant 1)\ ;\ \tfrac{1}{2}\pi \text{ when } (a \geqslant 1).$$

$$\int_0^\infty \frac{\sin ax \sin^2 x}{x^3}\,dx = \tfrac{1}{8}\pi a(4-a) \text{ when } (0 \leqslant a \leqslant 2)\ ;\ \tfrac{1}{2}\pi \text{ when } (a \geqslant 2).$$

17. Find the value of the definite integral

$$\int_0^\pi \frac{\sin x\,dx}{\sqrt{(1-2a\cos x + a^2)}},$$

where a is positive.

18. Evaluate the integral

$$\int_{-1}^{+1} \frac{\sin a\,dx}{1-2x\cos a + x^2}\ ;$$

for what value of a is the integral a discontinuous function of a.

19. Show that

$$\int_0^1 \frac{dx}{x^2+2x\cos a+1} = \frac{a}{2\sin a},$$

if $-\pi < a < \pi$, except when $a=0$, when the value of the integral is $\tfrac{1}{2}$.

20. Prove that

$$\int_0^\infty \log(1+a^2x^{-2})\,dx = \pi a, \text{ if } a>0,$$

21. Discuss the convergence of

$$\int_0^{\frac{1}{2}\pi} \cos 2nx \log \sin x\,dx,$$

and evaluate it when n is a positive integer.

[0 is the only point of infinite discontinuity. Now, since, when $x \to 0$, $\text{lt}\,(\sqrt{x}\log\sin x \,.\, \cos 2nx)=0$, therefore for values of x sufficiently near 0,

$$|\cos 2nx \log \sin x| < \epsilon/\sqrt{x}.$$

For evaluation, proceed integrating by parts].

22. From the preceding example, deduce that

(i)
$$\int_0^{\frac{1}{2}\pi} \cos 2nx \log \cos x \, dx = (-1)^{n+1} \cdot \frac{\pi}{4n}$$

(ii)
$$\int_0^{\pi} \cos nx \log 2(1-\cos x) dx = -\frac{\pi}{n}$$

(iii)
$$\int_0^{\pi} \cos nx \log 2(1+\cos x) dx = (-1)^{n+1} \frac{\pi}{n}.$$

23. Prove that if $g(x)$ is bounded and integrable then

$$\int_a^b g(x) \sin nx \, dx \to 0 \text{ as } n \to \infty,$$

where (a, b) is any finite interval.

If further $\int_{-\infty}^{+\infty} |g(x)|\, dx$ is convergent, prove that

$$\int_{-\infty}^{+\infty} g(x) \sin nx \, dx \to 0 \text{ as } n \to \infty.$$

24. *Prove that*

$$\underset{k\to\infty}{\text{lt}} \int_0^{\infty} \frac{dx}{1+kx^{10}} = 0.$$

We write

$$\int_0^{\infty} \frac{dx}{1+kx^{10}} = \int_0^{\epsilon} \frac{dx}{1+kx^{10}} + \int_{\epsilon}^{\infty} \frac{dx}{1+kx^{10}},$$

where ϵ is any given positive number. Now, we have

$$\int_0^{\epsilon} \frac{dx}{1+kx^{10}} < \epsilon,$$

and
$$\int_{\epsilon}^{\infty} \frac{dx}{1+kx^{10}} < \int_{\epsilon}^{\infty} \frac{dx}{kx^{10}} = \frac{1}{9k\epsilon^9} < \epsilon, \text{ if } k > \frac{1}{9\epsilon^{10}}.$$

$$\therefore \qquad 0 < \int_0^{\infty} \frac{dx}{1+kx^{10}} < 2\epsilon, \text{ if } k > \frac{1}{9\epsilon^{10}}.$$

Hence the result.

25. *Prove that as $p \to 0$ through positive values, then*

$$p \sum_{r=1}^{\infty} \frac{1}{r^{1+p}} \to 1.$$

Since $(1+p)>0$, therefore the infinite series $\Sigma(1/r^{1+p})$ is convergent.

Let $$s_n = \sum_{r=1}^{n} \frac{1}{r^{1+p}}; \qquad s = \sum_{r=1}^{\infty} \frac{1}{r^{1+p}}.$$

We have $$\frac{1}{2^{1+p}} \leqslant \int_1^2 \frac{1}{x^{1+p}}\, dx \leqslant \frac{1}{1^{1+p}};$$

$$\frac{1}{3^{1+p}} \leqslant \int_2^3 \frac{1}{x^{1+p}}\, dx \leqslant \frac{1}{2^{1+p}};$$

....

....

$$\frac{1}{n^{1+p}} \leqslant \int_{n-1}^{n} \frac{1}{x^{1+p}}\, dx \leqslant \frac{1}{(n-1)^{1+p}}$$

$$\therefore \quad \left[s_n - 1 \right] \leqslant \int_1^n \frac{dx}{x^{1+p}} = \frac{1}{p}\left(1 - \frac{1}{n^p}\right) \leqslant \left[s_n - \frac{1}{n^{1+p}} \right]$$

Let $n \to \infty$.

$\therefore \quad (s-1) \leqslant 1/p \leqslant s \quad$ or $\quad 1 \leqslant ps \leqslant (p+1)$

$\therefore \quad \text{lt } ps = 1$ as $p \to (0+0)$.

26. Prove that

$$\underset{a \to (0+0)}{\text{lt}} \ \underset{n \to \infty}{\text{lt}} \left(a \sum_{1}^{n} \frac{1}{r^{1+a}} \right) = 1.$$

27. If $f(x)$ is positive and decreases for $x \geqslant 1$, prove that

$$u_n = \sum_{r=1}^{n} f(r) - \int_1^n f(x)\,dx$$

tends to a limit l as $n \to \infty$ and that $0 \leqslant l \leqslant f(1)$.

[Show that (u_n) is a monotonic bounded sequence].

Prove that if $s>1$, then

$$\sum_{r=1}^{n} \left(\frac{s}{r} - \frac{1}{r^s} \right) - s \log n,$$

tends to a limit as $n \to \infty$ (s being fixed) and that if this limit is $\phi(s)$ then,

$$0 \leqslant [\phi(s) + (s-1)^{-1}] \leqslant (s-1).$$

CHAPTER IX

FUNCTIONS OF SEVERAL VARIABLES

Differentiation

117. Upto this point we have concerned ourselves with functions of one independent variable only but in the following part of the book the functions of *several variables* will be considered.

It is usually sufficient to consider the case of *two* independent variables only, for the extension to three or more variables can, in general, be made without introducing any essentially new ideas. In order to avoid complicating our statements, notations and proofs, we shall, therefore, mainly confine ourselves to functions of two variables only.

118. A function of two variables defined in a certain domain. Let x, y be two variables. One of these variables, say x, may have any value belonging to a certain given interval and corresponding to this value of x, y may have any value belonging to any given interval or a set of given intervals. In this way we obtain a system of *ordered* pairs of numbers (x, y). If, now, to each possible pair (x, y), there is associated, in any manner whatsoever, a value of another variable z, then we say that z is a *function* of x, y, and the aggregate of the pairs of numbers (x, y) is said to be the *domain or region of definition* of the function.

The simplest domain arises when the range of x is an interval (a, b) and for *each* value of x, y has the *same* range (c, d). Such a domain is called *rectangular* and will be symbolically denoted as $\mathrm{R}(a, b\ ;\ c, d)$.

Note. In the theory of functions of *two* variables, an ordered pair of numbers (x, y) is called a *point*. The use of the word 'point' is suggested by 'Plane analytical geometry' where, by choosing a pair of co-ordinate axes, a point is represented by a pair of numbers. Obviously, a domain $\mathrm{R}\ (a, b\ ;\ c, d)$ corresponds, when geometrically interpreted, to the geometrical rectangle bounded by the lines $x=a$, $x=b$, $y=c$, $y=d$.

Illustration. For the domain of variation of a point (x, y) defined by the inequalities $x\geqslant 0$, $y\geqslant 0$, $x+y\leqslant 1$, x varies in the interval $(0, 1)$ and to each value of x in this interval y varies in the variable interval $(0, 1-x)$. Geometrically this domain consists of the points lying in the interior and on the boundary of the triangle formed by the co ordinate axes and the line $x+y=1$.

Analytically a domain is always given by means of relationships of inequality between x and y.

Ex. Locate geometrically the domains of definition of the following functions :

(*i*) $z=\sqrt{(1-x^2-y^2)}$. (*ii*) $z=\sqrt{[(x-y)/(x+y)]}$. (*iii*) $z=[\log(xy)]^{-1}$.

118·1. The neighbourhood of a point. The square

$$(a-\delta,\ a+\delta\ ;\ b-\delta,\ b+\delta),$$

where δ has any positive value whatsoever, is said to be a neighbourhood of the point (a, b).

119. Simultaneous limit. $\lim\limits_{(x, y)\to(a, b)} f(x, y)$.

A function $f(x, y)$ is said to tend to the limit l, as a point (x, y) tends to the point (a, b), if, to every positive number ϵ, there corresponds a positive number δ, such that

$$|f(x,y)-l| < \epsilon,$$

for every point (x, y), [different from (a, b) itself], which belongs to the domain of definition of the function and which is such that

$$|x-a| \leqslant \delta, \quad |y-b| \leqslant \delta.$$

This means that for every point (x, y), other than $(a\ b)$, which is common to the domain of definition of the function and to the square $(a-\delta, a+\delta\ ;\ b-\delta, b+\delta)$, $f(x, y)$ differs from l numerically by a number which is less than ϵ.

Note. Instead of saying that '$(x, y)\to(a, b)$'. we also sometimes say that '$x\to a$ and $y\to b$' and write

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y)=l.$$

Note. In the definition of limit, we can replace the domain "$|x-a| \leqslant \delta$, $|y-b| \leqslant \delta$" by the domain "$\sqrt{[(x-a)^2+(y-b)^2]} \leqslant \delta$."

Ex. If $\lim\limits_{(x, y)\to(a, b)} f(x, y)=l$, then $\lim\limits_{x\to a} f(x, b)= \lim\limits_{y\to b} f(a, y)=l.$

119·1. Non-Existence of a limit In general, to determine whether a simultaneous limit exists or not, is a difficult matter but a simple consideration, as we now describe, sometimes enables us to show that the limit does not exist.

It is easy to see that if

$$\lim_{(x, y)\to(a, b)} f(x, y)=l,$$

and if $y=\phi(x)$ is *any function whatsoever* such that

$$\phi(x)\to b, \text{ when } x\to a,$$

then, when $x\to a$

$$\lim f[x, \phi(x)]$$

must exist and be equal to l.

Thus if we can determine two functions $\phi_1(x)$, $\phi_2(x)$ such that the limits of $f[x, \phi_1(x)]$ and $f[x, \phi_2(x)]$ are different, then we can certainly say that the simultaneous limit, in question, does not exist.

(The reader is advised to geometrically interpret the consideration outlined here.)

Ex. *Show that*

$$\lim\ [2xy/(x^2+y^2)], \text{ when } (x, y)\to(0, 0)$$

does not exist.

Taking $y=mx$, we see that when $x\to 0$,

$$\lim \frac{2x.mx}{x^2+m^2x^2}=\frac{2m}{1+m^2},$$

which is different for different values of m. Hence the limit does not exist.

Ex. Show that

$$\lim \frac{xy^3}{x^2+y^6}, \text{ when } (x, y)\to(0, 0).$$

does not exist.

[Consider the relation $x=my^3$]

Ex. Evaluate the following limits or show that the limits do not exist :—

(*i*) $\text{lt}\ \dfrac{xy^2}{x^2+y^4}$ (*ii*) $\text{lt}\ (x+y)\ \dfrac{y+(x+y)^2}{y-(x+y)^2}$ (*iii*) $\text{lt}\ \dfrac{xy}{\sqrt{(x^2+y^2)}}$

(*iv*) $\text{lt}\ (y \sin 1/x + x \sin 1/y)$

when, in each case, $(x, y) \to (0, 0)$.

120. Theorem. *If, when* $(x, y) \to (a, b)$,

$$lt\, f(x, y) = l_1,\ lt\, g(x, y) = l_2$$

then

(*i*) $lt\ [f(x, y) \pm g(x, y)] = l_1 \pm l_2$ (*ii*) $lt\ [f(x, y)\ g(x, y)] = l_1 l_2$

(*iii*) $lt\ [f(x, y)/g(x, y)] = l_1/l_2$, if $l_2 \neq 0$.

The proofs are exactly similar to those of the corresponding results in the case of functions of one variable.

121. Repeated limits. Let $f(x, y)$ be defined in a certain neighbourhood of (a, b). Then

$$\underset{x \to a}{\text{lt}}\ f(x, y),$$

if it exists, is a function of y, say $\phi(y)$. If

$$\underset{y \to b}{\text{lt}}\ \phi(y),$$

exists and is equal to λ, we write

$$\underset{y \to b}{\text{lt}}\ \underset{x \to a}{\text{lt}}\ f(x, y) = \lambda,$$

and say that λ is a *repeated limit* of $f(x, y)$ as $x \to a$ and $y \to b$. A change in the order of passing to limits may produce a change in the final result. Thus

$$\underset{x \to a}{\text{lt}}\ \underset{y \to b}{\text{lt}}\ f(x, y),$$

where $y \to b$ and then $x \to a$ may be different from λ.

Ex. We have

$$\underset{y \to 0}{\text{lt}}\ \underset{x \to 0}{\text{lt}}\ \frac{x-y}{x+y} = \underset{y \to 0}{\text{lt}}\ \frac{-y}{y} = -1,$$

$$\underset{x \to 0}{\text{lt}}\ \underset{y \to 0}{\text{lt}}\ \frac{x-y}{x+y} = \underset{x \to 0}{\text{lt}}\ \frac{x}{x} = 1,$$

so that the two repeated limits are different. The simultaneous limit, as may easily be seen, does not even exist.

122. Continuity. *A function* $f(x, y)$ *is said to be continuous at a point* (a, b) *of its domain of definition, if*

$$\underset{(x, y) \to (a, b)}{lt}\ f(x, y) = f(a, b).$$

Again $f(x, y)$ *is said to be continuous in a domain if it is continuous at every point of the domain.*

It can now be easily shown that (*i*) the sum, difference, and product of continuous functions are also continuous. (*ii*) The quotient of two continuous functions is continuous except where the denominator vanishes. (*iii*) Continuous functions of continuous functions are themselves continuous.

Note. It is easy to show that if $f(x, y)$ is a continuous function of two variables at (a, b), then $f(x, b)$ is a continuous function of one variable x for $x = a$ and $f(a, y)$ is a continuous function of one variable y for $y = b$.

The converse of this result is not necessarily true as may be seen by considering a function $f(x, y)$ which is such that $f(x, y)=0$ when x is 0 or when y is 0 and $f(x, y)=1$ elsewhere.

Note. The continuous functions of two variables have properties analogous to those of a single variable and attention to some of these properties will be paid in Chap. XI. One property whose proof is simple may, however, be stated here :—

If $f(x, y)$ is continuous at (a, b) and $f(a, b)\neq 0$, then there exists a neighbourhood of (a, b) such that for every point of this neighbourhood $f(x, y)$ has the sign of $f(a, b)$.

Ex. Discuss the continuity and discontinuity of the following functions :—

(*i*) $f(x, y)=2xy^2/(x^2+y^2)$ when $(x, y)\neq(0, 0)$ and $f(0, 0)=0$.

(*ii*) $\phi(x, y)=2xy/\sqrt{(x^2+y^2)}$ when $(x, y)\neq(0,0)$ and $\phi(0,0)=0$.

Ex. Show that the following function is continuous at (0 0).

$$f(x, y)=e^{-|x-y|/(x^2-2xy+y^2)}, \text{ when } (x, y)\neq(0, 0) \text{ and } f(0, 0)=0.$$

Ex. Show that the functions

(*i*) $f(x, y)=x^4y^4/(x^2+y^4)^3$, when $(x, y)\neq(0, 0)$ and $f(0, 0)=0$

(*ii*) $\phi(x, y)=x^2/(x^2+y^2-x)$ when $(x, y)\neq(0, 0)$ and $\phi(0, 0)=0$.

tend to 0 if (x, y) approaches the origin along any straight line, but that they are discontinuous at the origin.

[**Hint.** lt $f(x, y)=\frac{1}{8}$, lt $\phi(x, y)=1$ if $(x, y)\rightarrow(0, 0)$ along $y^2=x$].

123. Partial Derivatives. Let (a, b) be any point of the domain of definition of a function $f(x, y)$. Then

$$\lim_{h\rightarrow 0}\frac{f(a+h, b)-f(a, b)}{h}$$

if it exists, is called the partial derivative of $f(x, y)$ with respect to x at (a, b) and is symbolically denoted by $f_x(a, b)$ or $\partial f(a, b)/\partial x$.

Similarly

$$\lim_{k\rightarrow 0}\frac{f(a, b+k)-f(a, b)}{k},$$

if it exists, is called the partial derivative with respect to y at (a, b) and is denoted by $f_y(a, b)$ or $\partial f(a, b)/\partial y$.

Now, if, $f(x, y)$ possesses a partial derivative w. r. to x at every point of its domain of definition, then the function determined by the aggregate of the values of the derivative, is called the partial derivative of $f(x, y)$ w. r. to x and is denoted by $f_x(x, y)$ or $\partial f/\partial x$.

Similarly the partial derivative w. r. to y may be defined.

Note. If $(a-\delta, a+\delta\ ;\ b-\delta, b+\delta)$ be a neigtbourhood of (a, b), then the question of the existence and values of the partial derivatives of $f(x, y)$ at (a, b) depends only on the values of the function at those points of the neighbourhood which lie along the lines $x=a$, $y=b$ and is absolutely independent of the values of the function at the remaining points of the neighbourhood. The question of continuity at (a, b), however, takes into account the values of the function at *every* point of the neighbourhood. There is, therefore, nothing surprising in the fact, which we will now illustrate by means of an example, that the *partial derivatives of a function may exist at a point at which the function is not even continuous.* Consider a function $f(x, y)$ such that

$f(x, y)=0$ when either x or y is 0, *i.e*, along the lines $x=0, y=0$ and $f(x, y)=1$ elsewhere.

Clearly $f(x, y)$ is not continuous at (0, 0) even though $f_x(0, 0)$, $f_y(0, 0)$ both exist and are zero.

Ex. *If*

$$\phi(x, y)=(x^3+y^3)/(x-y) \qquad \textit{when } x \neq y,$$
$$\phi(x, y)=0 \qquad \textit{when } x=y,$$

show that this function is discontinuous at the origin, but that the partial derivatives exist at that point.

Putting $y=x-mx^3$, we see that

$$\lim_{x \to 0} \phi(x, y)=2/m,$$

so that this limit is different for different values of m.

Thus $\lim \phi(x, y)$ when $(x, y) \to (0, 0)$ does not exist and therefore the function is necessarily discontinuous at $(0, 0)$.

Again, we have

$$\phi_x(0, 0)= \lim_{h \to 0} \frac{\phi(0+h, 0)-\phi(0, 0)}{h}= \lim_{h \to 0} \frac{h^3}{h}=0 ;$$

$$\phi_y(0, 0)= \lim_{k \to 0} \frac{\phi(0, 0+k)-\phi(0, 0)}{k}= \lim_{k \to 0} \frac{-k^3}{k}=0,$$

Ex. If $f(x, y)=xy(x^2-y^2)/(x^2+y^2)$ when x and y are not simultaneously zero and $f(0, 0)=0$, show that $f_x(x, 0)=0$ and $f_y(0, y)=0$.

Ex. Show that the function

$$f(x,y)=xy/\sqrt{(x^2+y^2)} \text{ if } (x^2+y^2)\neq 0 \text{ and } f(0, 0)=0$$

possesses partial derivatives f_x and f_y at every point (x, y) but they are not continuous at $(0, 0)$.

Ex. Show that $f(x, y)=\sqrt{(x^2+y^2)}$ possesses partial derivatives f_x and f_y at all points different from the origin.

124. Differentiability and Differentials. Let (a, b), $(a+h, b+k)$ be any two points of the domain of definition of a function $f(x, y)$. Now $f(a+h, b+k)-f(a, b)$ is the change in the function as the point (x, y) changes from (a, b) to $(a+h, b+k)$.

The function $f(x, y)$ is said to be differentiable at (a, b) if, as (x, y) changes from (a, b) to $(a+h, b+k)$, the change in the value of the function can be expressed in the form

$$f(a+h, b+k)-f(a, b)=Ah+Bk+h\phi(h, k)+k\psi(h, k),$$

where A, B are constants independent of h and k and $\phi(h, k)$, $\psi(h, k)$ are functions of h and k such that

$$\lim_{(h, k)\to(0, 0)} \phi(h, k)=0= \lim_{(h, k)\to(0, 0)} \psi(h, k).$$

Also then $Ah+Bk$ is called the differential of $f(x, y)$ at (a, b) and is denoted as $df(a, b)$.

Ex. Show that $f(x, y)=xy$ is differentiable at every point (x, y).

124·1. Theorem. *If $f(x, y)$ is differentiable at (a, b) then it is also continuous at (a, b).*

Since

$$f(a+h, b+k)-f(a, b)=Ah+Bk+h\phi(h, k)+k\psi(h, k),$$

we have, when $(h, k)\to(0, 0)$,

$$[f(a+h, b+k)-f(a, b)]\to 0,$$

i.e., $f(x, y)$ is continuous at (a, b).

The converse of this result is not true. (See Ex. 1, below).

124·2. Theorem. *If $f(x, y)$ is differentiable at (a, b), then it also possesses the partial derivatives $f_x(a, b)$ and $f_y(a, b)$.*

Putting $k=0$ in the relation

$$f(a+h, b+k)-f(a, b)=Ah+Bk+h\phi(h, k)+k\psi(h, k),$$

we have $$[f(a+h, b)-f(a, b)]/h=A+\phi(h, 0),$$

so that when $h\to 0$, we obtain

$$f_x(a, b)=A.$$

Similarly $$f_y(a, b)=B.$$

The demonstration also shows that if $f(x, y)$ is differentiable at (a, b) then A, B are uniquely defined in as much as they are the partial derivatives of $f(x, y)$ at (a, b).

The converse of this theorem is also not true. (See Ex. 1, below)

Ex. 1. *Given*

$$f(x, y)=xy/\sqrt{(x^2+y^2)} \text{ when } (x, y)\neq(0, 0),$$
$$f(x, y)=0 \qquad \text{ when } (x, y)=(0, 0),$$

show that $f(x, y)$ is continuous, possesses partial derivatives but is not differentiable at $(0, 0)$.

We have

$$\left| \frac{xy}{\sqrt{(x^2+y^2)}} -0 \right| = \frac{| xy |}{\sqrt{(x^2+y^2)}}$$

Putting $x=r \cos \theta, y=r \sin \theta$, we see that

$$\frac{| xy |}{\sqrt{(x^2+y^2)}}=r \mid \cos \theta \ \sin \ \theta \mid \leqslant r= \sqrt{x^2+y^2},$$

Again

$\sqrt{(x^2+y^2)}<\epsilon$; if $x^2<\frac{1}{2}\epsilon^2, y^2<\frac{1}{2}\epsilon^2$, *i.e.*, if $| x | <\epsilon/\sqrt{2}, | y | <\epsilon/\sqrt{2}$.

Thus $f(x, y)$ is continuous at $(0, 0)$.

Also it is easy to see that $f_x(0, 0)=0=f_y(0, 0)$ so that if $f(x, y)$ were differentiable at $(0, 0)$, we should have, by definition,

$$\frac{hk}{\sqrt{(h^2+k^2)}}=0h+0k+h\phi(h, k)+k\psi(h, k) \tag{1}$$

where $\phi(h, k)$ and $\psi(h, k)\to 0$ as $(h, k)\to(0, 0)$.

Putting $k=mh$ and letting $h\to 0$, we obtain from (1),

$$m/\sqrt{(1+m^2)}=0$$

which is absurd, as m may have any value whatsoever. Hence $f(x, y)$ is not differentiable at $(0, 0)$.

Ex. 2. Show that $f(x, y)= | x | + | y |$ is continuous but not differentiable at (0, 0).

Ex. 3. Discuss the differentiability of the following functions at (0, 0)

(*i*) $f(x, y)= \sqrt{| xy |}$.

(*ii*) $f(x, y)=xy^2/(x^2+y^2)$ when $(x, y)\neq(0, 0)$ and $f(0, 0)=0$.

Examine them for continuity also.

Ex. 4. If $f(x, y)=x \sin 1/x+y \sin 1/y$ when $x\neq 0, y\neq 0$; $f(x, 0)=x \sin 1/x$ when $x\neq 0, f(0, y)=y \sin 1/y$ when $y\neq 0$; $f(0, 0)=0$, show that $f(x, y)$ is continuous but not differentiable at (0, 0).

124·3. A sufficient condition for differentiability. Theorem. *If (a, b) be a point of the domain of definition of a function $f(x, y)$ such that*

(*i*) $f_x(a, b)$ *exists,*
(*ii*) $f_y(x, y)$ *is continuous at* (a, b),
then $f(x, y)$ *is differentiable at* (a, b).

The condition (*ii*) implies that $f_y(x, y)$ exists in a certain neighbourhood $(a-\delta, a+\delta\ ;\ b-\delta, b+\delta)$ of (a, b). Let $(a+h, b+k)$ be any point of this neighbourhood. We have

$$f(a+h, b+k)-f(a, b)=f(a+h, b+k)-f(a+h, b)+f(a+h,b)-f(a, b). \quad (1)$$

The function $f(a+h, y)$ of y is derivable w. r. to y in the interval $(b, b+k)$. Therefore, by mean value theorem,

$$f(a+h, b+k)-f(a+h, b)=k\,f_y(a+h, b+\theta k), \quad (2)$$

where θ, which lies between 0 and 1, is a function of h and k.

Now, if we write,

$$f_y(a+h, b+\theta k)-f_y(a, b)=\psi(h, k), \quad (3)$$

we see that, because of the condition (*ii*), $\psi(h, k)\to 0$ as $(h, k)\to(0, 0)$.

Again, because of the condition, (*i*) we have, when $h\to 0$,

$$\text{lt}\,\frac{f(a+h, b)-f(a, b)}{h}=f_x(a, b),$$

so that if we write

$$[f(a+h, b)-f(a, b)]/h-f_x(a, b)=\phi(h) \quad (4)$$

then $\phi(h)\to 0$ as $h\to 0$.

From (1), (2), (3), (4), we obtain

$$f(a+h, b+k)-f(a, b)=hf_x(a, b)+kf_y(a, b)+h\phi(h)+k\psi(h, k).$$

Hence the result.

Note. It may similarly be shown that if $f_y(a, b)$ exists and $f_x(x, y)$ is continuous at (a, b) then $f(x, y)$ is differentiable at (a, b).

Note. We have shown above that the *mere existence of one* partial derivative and the *continuity of the other* is *sufficient* for the differentiability of the function but, by considering an example, we now show that the condition of continuity is *not necessary* so that a function may be differentiable even though neither partial derivative is continuous.

Let $f(x, y)=x^2 \sin(1/x)+y^2 \sin(1/y)$ when $x\neq 0, y\neq 0$,
$f(x, 0)=x^2 \sin(1/x)$ when $x\neq 0, y=0$
$f(0, y)=y^2 \sin(1/y)$ when $y\neq 0$, $x=0$
$f(0, 0)=0$.

It is easy to see that

$$f_x(x, y)=2x \sin(1/x)-\cos(1/x) \text{ when } (x\neq 0) \text{ and } f_x(0, y)=0,$$

$$f_y(x, y)=2y \sin(1/y)-\cos(1/y) \text{ when } y\neq 0 \text{ and } f_y(x, 0)=0,$$

so that neither $f_x(x, y)$ nor $f_y(x, y)$ is continuous at the origin.

Again, we have

$$f(h, k)-f(0, 0)=h^2 \sin(1/h)+k^2 \sin(1/k)$$
$$=0h+0k+h\,(h \sin 1/h)+k.(k \sin 1/k)$$

so that, since $(h \sin 1/h)$ and $(k \sin 1/k)$ both$\to 0$ as $(h, k)\to(0, 0)$, $f(x, y)$ is differentiable at $(0, 0)$.

125. Theorem. *If* $f(x, y)$ *and* $g(x, y)$ *are differentiable at* (a, b), *then*

$f(x, y)\pm g(x, y)$, $f(x, y)\,g(x, y)$ *are also differentiable at* (a, b)
and $d(f\pm g)=df\pm dg\ ;\ d(fg)=fdg+gdf\ ;$

also *if* $g(a, b) \neq 0$, *then* $f(x, y)/g(x, y)$ *is differentiable at* (a, b) *and*

$$d(f/g) = (g\,df - f\,dg)/g^2$$

The proof is simple.

126. Partial derivatives of the second and higher orders. Suppose that a function $f(x, y)$ possesses partial derivatives $f_x(x, y)$, $f_y(x, y)$ of the first order in a certain neighbourhood of (a, b).

Then we write

$$\underset{h \to 0}{\mathrm{lt}} \frac{f_x(a+h, b) - f_x(a, b)}{h} = f_{x^2}(a, b),$$

$$\underset{k \to 0}{\mathrm{lt}} \frac{f_x(a, b+k) - f_x(a, b)}{k} = f_{yx}(a, b) ;$$

$$\underset{h \to 0}{\mathrm{lt}} \frac{f_y(a+h, b) - f_y(a, b)}{h} = f_{xy}(a, b),$$

$$\underset{k \to 0}{\mathrm{lt}} \frac{f_y(a, b+k) - f_y(a, b)}{k} = f_{y^2}(a, b),$$

in case the limits exist.

$f_{x^2}(a, b)$, $f_{yx}(a, b)$, $f_{xy}(a, b)$, $f_{y^2}(a, b)$ are known as partial derivatives of second order at (a, b) and are also sometimes written as $(\partial^2 f/\partial x^2)(a, b)$, $(\partial^2 f/\partial y \partial x)(a, b)$, $(\partial^2 f/\partial x \partial y)(a, b,)$ $(\partial^2 f/\partial y^2)(a, b)$ respectively.

The reader should carefully note the difference in the meanings of $f_{yx}(a, b)$ and $f_{xy}(a, b)$.

Partial derivatives of the third and higher orders can be similarly defined.

126·1. Change in the order of derivation. In general, a partial derivative has the same value in whatever order the different operations are performed. Thus, for example, in general, we have

$$f_{xy} = f_{yx},\ f_{x^2y} = f_{xyx},\ f_{x^2y^3} = f_{xy^2xy^2}$$

That this is not always the case is shown below by considering an example.

127. It is easy to see *a priori* why $f_{yx}(a, b)$ may be different from $f_{xy}(a, b)$.

We have

$$f_{xy}(a, b) = \underset{h \to 0}{\mathrm{lt}} \frac{f_y(a+h, b) - f_y(a, b)}{h}$$

Also $$f_y(a+h, b) = \underset{k \to 0}{\mathrm{lt}} \frac{f(a+h, b+k) - f(a+h, b)}{h},$$

and $$f_y(a, b) = \underset{k \to 0}{\mathrm{lt}} \frac{f(a, b+k) - f(a, b)}{k}$$

$$\therefore\ f_{xy}(a, b) = \underset{h \to 0}{\mathrm{lt}}\ \underset{k \to 0}{\mathrm{lt}} \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}$$

$$= \underset{h \to 0}{\mathrm{lt}}\ \underset{k \to 0}{\mathrm{lt}} \frac{\phi(h, k)}{hk},\ \text{say.}$$

It may similarly be shown that

$$f_{yx}(a, b) = \underset{k \to 0}{\mathrm{lt}}\ \underset{h \to 0}{\mathrm{lt}} \frac{\phi(h, k)}{hk}.$$

Thut we see that $f_{yx}(a, b)$ and $f_{xy}(a, b)$ are repeated limits of the same expression taken in different orders.

Ex. 1. *If* $f(x, y)=xy(x^2-y^2)/(x^2+y^2)$ *when* $(x, y)\neq(0, 0)$ *and* $f(0, 0)=0$,
show that $f_{xy}(0, 0)\neq f_{yx}(0, 0)$.

We have

$$f_{xy}(0, 0)=\underset{h\to 0}{\mathrm{lt}}\ \frac{f_y(0+h, 0)-f_y(0, 0)}{h}$$

But $$f_y(0, 0) = \underset{k\to 0}{\mathrm{lt}}\ \frac{f(0, 0+k)-f(0, 0)}{k}=\underset{k\to 0}{\mathrm{lt}}\ \frac{0}{k}=0,$$

and $$f_y(h, 0) = \underset{k\to 0}{\mathrm{lt}}\ \frac{f(h, 0+k)-f(h, 0)}{k}=\underset{k\to 0}{\mathrm{lt}}\ \frac{hk(h^2-k^2)}{k(h^2+k^2)}=h$$

$$\therefore\quad f_{xy}(0, 0)=\underset{h\to 0}{\mathrm{lt}}\ \frac{h-0}{h}=1.$$

Again $$f_{yx}(0, 0)=\underset{h\to 0}{\mathrm{lt}}\ \frac{f_x(0, 0+k)-f_x(0, 0)}{k}$$

But $$f_x(0, 0) = \underset{h\to 0}{\mathrm{lt}}\ \frac{f(0+h, 0)-f(0, 0)}{h}=0$$

and $$f_x(0, k) = \underset{h\to 0}{\mathrm{lt}}\ \frac{f(0+h, k)-f(0, k)}{h}=\underset{h\to 0}{\mathrm{lt}}\ \frac{hk(h^2-k^2)}{h(h^2+k^2)}=-k$$

$$\therefore\quad f_{yx}(0, 0) = \underset{k\to 0}{\mathrm{lt}}\ \frac{-k-0}{k}=-1$$

$$\therefore\quad f_{xy}(0, 0)\neq f_{yx}(0, 0).$$

Ex. 2. Given that
$f(x, y)=xy$ if $|y|\leqslant|x|$ and $f(x, y)=-xy$ if $|y|>|x|$
show that $f_{xy}(0, 0)\neq f_{yx}(0, 0)$.

Ex. 3. Examine for the change in the order of derivation at the origin :—

(*i*) $f(x, y)=|x^2-y^2|$

(*ii*) $f(x, y)=x^2\tan^{-1}(y/x)-y^2\tan^{-1}(x/y)$, $x\neq 0$, $y\neq 0$
$f(x, y)=0$, elsewhere.

127. We now prove two theorems which lay down sufficient conditions for the validity of the statement $f_{xy}=f_{yx}$.

127·1. Schawarz's theorem. *If* (a, b) *be a point of the domain of definition of a function* $f(x, y)$ *such that*

(*i*) $f_x(x, y)$ *exists in a certain neighbourhood of* (a, b) ;

(*ii*) $f_{xy}(x, y)$ *is continuous at* (a, b),

then

$f_{yx}(a, b)$ *exists and is equal to* $f_{xy}(a, b)$.

The given conditions imply that there exists a certain neighbourhood of (a, b) at every point (x, y) of which $f_x(x, y)$, $f_y(x, y)$ and $f_{xy}(x, y)$ exist. Let $(a+h, b+k)$ be any point of this neighbourhood. We write

$$\phi(h, k)=f(a+h, b+k)-f(a+h, b)-f(a, b+k)+f(a, b),$$
$$g(y)=f(a+h, y)-f(a, y),$$

so that $\phi(h, k)=g(b+k)-g(b)$ (1)

Since f_y exists in a neighbourhood of (a, b), the function $g(y)$ is derivable in $(b, b+k)$, and therefore, by applying the mean value theorem to the expression on the right of (1), we have

$$\phi(h, k)=kg'(b+\theta k) \qquad (0<\theta<1)$$
$$=k[f_y(a+h, b+\theta k)-f_y(a, b+\theta k) \qquad (2)$$

Again since f_{xy} exists in a neighbourhood of (a, b), the function $f_y(x, b+\theta k)$ of x is derivable w. r. to x in $(a, a+h)$ and therefore by applying the mean value theorem to the right of (2), we have

$$\phi(h, k)=hk\, f_{xy}(a+\theta' h, b+\theta k) \qquad (0<\theta'<1)$$

or
$$\frac{1}{k}\left[\frac{f(a+h, b+k)-f(a, b+k)}{h}-\frac{f(a+h, b)-f(a, b)}{h}\right]$$
$$=f_x\,(a+\theta' h, b+\theta k)$$

Since $f_x(x, y)$ exists in a neighbourhood of (a, b) this gives, when $h\to 0$,

$$\frac{f_x(a, b+k)-f_x(a, b)}{k}=\lim_{h\to 0} f_{x}(a+\theta' h, b+\theta k)$$

Let, now, $k\to 0$ Since $f_{xy}(x, y)$ is continuous at (a, b), we obtain

$$f_{yx}(a, b)=\lim_{k\to 0}\ \lim_{h\to 0} f_x\,(a+\theta' h, b+\theta k)=f_x\,(a, b)$$

Cor. 1. If $f_{xy}(x, y)$ and $f_{x}(x, y)$ are both continuous at (a, b), then $f_{xy}(a, b)=f_{yx}(a, b)$.

128·2. Young's theorem. *If (a, b) be a point of the domain of definition of a function $f(x, y)$ such that $f_x(x, y)$ and $f_y(x, y)$ are both differentiable at (a, b), then*

$$f_{xy}(a, b)=f_{yx}(a, b).$$

The differentiability of f_x and f_y at (a, b) implies that they exist in a certain neigbourhood of (a, b) and

$$f_{x^2},\ f_{yx},\ f_{xy},\ f_{y^2}$$

exist at (a, b). Let $(a+h, b+h)$ be any point of this neighbourhood.

We write

$$\phi(h, h)=f(a+h, b+h)-f(a+h, b)-f(a, b+h)+f(a, b)$$
$$g(h)=f(a+h, y)-f(a, y)$$

so that $\phi(h, h)=g(b+h)-g(b)$ (1)

Since f_y exists in a neighbourhood of (a, b), the function $g(y)$ is derivable in $(b, b+h)$ and therefore by applying the mean value theorem to the expression on the right of (1), we obtain

$$\phi(h, h)=hg'(b+\theta h) \qquad (0<\theta<1)$$
$$=h[f_y(a+h, b+\theta h)-f_y(a, b+\theta h)] \qquad (2)$$

Since $f_y(x, y)$ is differentiable at (a, b), we have, by definition,

$$f_y\,(a+h, b+\theta h)-f_y(a, b)=hf_{xy}\,(a, b)+\theta h\, f_{y^2}\,(a, b)+h\phi_1(h, h)$$
$$+\theta k\psi_1(h, h), \quad (3)$$

and $f_y\,(a, b+\theta h)-f_y(a, b)=\theta hf_{y^2}\,(a, b)+\theta h\ \psi_2(h, h),$ (4)

where ϕ_1, ψ_1, ψ_2 all$\to 0$ as $h\to 0$

From (2), (3) and (4), we obtain

$$\phi(h, h)/h^2=f_{xy}(a, b)+\phi_1(h, h)+\theta\psi_1(h, h)-\theta\psi_2(h, h) \qquad (5)$$

By a similar argument and on considering

$$H(x)=f(x, b+h)-f(x, b),$$

we can show that

$$\phi(h, h)/h^2=f_{yx}(a, b)+\psi_3(h, h)+\theta'\phi_2(h, h)-\theta'\phi_3(h, h) \tag{6}$$

where ϕ_2, ϕ_3, ψ_3 all $\to 0$ as $h\to 0$

Equating the right hand sides of (5) and (6) and making $h\to 0$, we obtain

$$f_{xy}(a, b)=f_{yx}(a, b).$$

Ex. In view of the Schwarz's and Young's theorems, explain the inequality $f_{xy}(0, 0)\neq f_{yx}(0, 0)$ for the function considered in Ex. 1 after §127, p. 208.

[Show that neither $f_{xy}(x, y)$ nor $f_{yx}(x, y)$ is continuous at (0, 0) and that $f_x(x, y)$ and $f_y(x, y)$ are not differentiable at (0, 0)].

Explain the same inequality for the functions considered in Ex 2 and Ex. 3 also.

128. Theorem. *If $f(x, y)$ possesses continuous partial derivatives of the nth order at a point (a, b), then*

$$f_{p_n p_{n-1}\cdots p_2 p_1}(a, b)=f_{q_n q_{n-1}\cdots q_2 q_1}(a, b)$$

where each p and q is either x or y and the number of x's among p's is the same as the number of x's among q's and similarly about the y's.

The proof which is simple, may be obtained by the principle of mathematical induction from the cor. of §127·1.

For example, the theorem shows that when $n=3$,

$$f_{x^2y}=f_{xyx}=f_{yx^2}.$$

Ex. $f(x, y)\equiv(x^2+y^2)\log(x^2+y^2)$ for $(x, y)\neq(0, 0)$ and $f(0, 0)=0$. Show that f_{xy} and f_{yx} are not continuous at (0, 0), but $f_{xy}(0, 0)=f_{yx}(0, 0)$.

129. Differentials of second and higher orders. Let $z=f(x, y)$ be defined in a domain E and let it be differentiable at every point (x, y) of the domain. The differential dz of the function z at *any* point (x, y) is given by

$$dz=\frac{\partial z}{\partial x}\,\delta x+\frac{\partial z}{\partial x}\,\delta y. \tag{1}$$

where we have taken δx, δy for h, k respectively and $\partial z/\partial x$, $\partial z/\partial y$ denote the partial derivatives of the function z at (x, y).

Taking $z=x$, we obtain from (1)

$$dx=dz=1\,.\,\delta x+0\,.\,\delta y=\delta x$$

Similarly taking $z=y$, we obtain $dy=\delta y$.

Thus we see that the differentials dx, dy of the independent variables x, y are δx and δy respectively so that we can write

$$dz=\frac{\partial z}{\partial x}\,dx+\frac{\partial z}{\partial y}\,dy. \tag{2}$$

Regarding dx and dy as constants, we see that the differential dz is a function of two variables x and y and is itself differentiable in the domain E, if $\partial z/\partial x$, $\partial z/\partial y$ are both differentiable in E. (§125, p. 206) and also then

$$d(dz)=d\left(\frac{\partial z}{\partial x}\right)dx+d\left(\frac{\partial z}{\partial x}\right)dy$$

Replacing z by $\partial z/\partial x$ and $\partial z/\partial y$ in (2), we obtain

$$d\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}\,dx+\frac{\partial^2 z}{\partial y\partial x}\,dy\,;$$

$$d\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}\,dx+\frac{\partial^2 z}{\partial y^2}\,dy.$$

Since $\partial z/\partial x$, $\partial z/\partial y$ are differentiable in E, we have, by Young's theorem, at every point of the domain,

$$\partial^2 z/\partial x\partial y=\partial^2 z/\partial y\partial x.$$

Thus denoting $d(dz)$ by d^2z, we obtain

$$d^2z=\frac{\partial^2 z}{\partial x^2}\,dx^2+2\,\frac{\partial^2 z}{\partial x\partial y}\,dxdy+\frac{\partial^2 z}{\partial y^2}\,dy^2,$$

where $\qquad dx^2=(dx)^2,\ dy^2=(dy)^2.$

For the sake of brevity this is usually written as

$$d^2z=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^2 z,$$

whose meaning is self evident.

Again, d^2z is differentiable in the domain E if the second order partial derivatives $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x\partial y$, $\partial^2 z/\partial y^2$ are all differentiable in E. This condition ensures the legitimacy of the inversion of the order of derivation w. r. to x and w. r. to y in the partial derivatives of the third order. Thus we have

$$d^3z=\frac{\partial^3 z}{\partial x^3}\,dx^3+3\frac{\partial^3 z}{\partial x^2\partial y}dx^2dy+3\,\frac{\partial^3 z}{\partial x\partial y^2}\,dxdy^2+\frac{\partial^3 z}{\partial y^3}\,dy^3$$

$$=\left(dx\,\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^3 z.$$

Proceeding in this manner we see that d^nz exists if $d^{n-1}z$ is differentiable which is the case when all the partial derivatives of the $(n-1)$th order are differentiable. This condition also ensures the legitimacy of inverting the order of derivation w. r. to x and w. r. to y in the partial derivatives of the nth order. The expression for d^nz in terms of partial derivatives of the nth order, as may be shown by *Mathematical induction,* is given by

$$d^nz=\frac{\partial^n z}{\partial x^n}\,dx^n+n\,\frac{\partial^n z}{\partial y\partial x^{n-1}}\,dydx^{n-1}+\,\frac{n(n-1)}{2\,!}\,\frac{\partial^n z}{\partial y^2\partial x^{n-2}}\,dy^2dx^{n-2}+\ldots\ldots$$

$$\ldots\ldots+\frac{\partial^n z}{\partial y^n}\,dy^n$$

i.e., $$d^nz=\left[dx\,\frac{\partial}{\partial x}+dy\,\frac{\partial}{\partial y}\right]^n z.$$

130. Differentiation of Functions of Functions. If

$$x=\phi(u,\,v),\ y=\psi(u,\,v)$$

be two functions of $(u,\,v)$ defined in a domain E of the point $(u,\,v)$, then the domain E_1 of the point $(x,\,y)$ as $(u,\,v)$ varies in E is said to be the *image* of E.

Theorem. *If*

(*i*) $x=\phi(u, v), y=\psi(u, v)$ *are two differentiable functions of* (u, v) *in a domain* E,

(*ii*) $z=f(x, y)$ *is a differentiable function of* (x, y) *in a domain* E_1,

(*iii*) E_1 *is the image of* E,

then

z, *regarded as a function of* u, v, *is differentiable in* E *and*

$$dz=\frac{\partial z}{\partial x}\,dx+\frac{\partial z}{\partial y}\,dy.$$

[This theorem shows that the expression for dz is the same whether the variables x, y are independent or dependent upon some other variables. Of course dx, dy are not constants when x, y are dependent variables].

Let (u, v) and $(u+\delta u, v+\delta v)$ be any two points of E, and let (x, y) and $(x+\delta x, y+\delta y)$ be the two corresponding points of E_1 so that

$$\delta x=\phi(u+\delta u, v+\delta v)-\phi(u, v),\ \delta y=\psi(u+\delta u, v+\delta v)-\psi(u, v).$$

Because of the differentiability, $\phi(u, v)$ and $\psi(u, v)$ are continuous functions of u and v, *i. e.*, δx and $\delta y \to 0$ as $(\delta u, \delta v)\to(0, 0)$.

Since $x=\phi(u, v)$, $y=\psi(u, v)$ are differentiable at (u, v), therefore

$$\delta x=\phi_u\delta u+\phi_v\delta v+\phi_1\delta u+\phi_2\delta v, \tag{1}$$

$$\delta y=\psi_u\delta u+\psi_v\delta v+\psi_1\delta u+\psi_2\delta v, \tag{2}$$

where $\phi_1, \phi_2, \psi_1, \psi_2$ are functions of $\delta u, \delta v$ and $\to 0$ as $(\delta u, \delta v)\to(0, 0)$.

Also since $z=f(x, y)$ is differentiable at (x, y), therefore

$$\delta z=f_x\delta x+f_y\delta y+f_1\delta x+f_2\delta y, \tag{3}$$

where f_1, f_2 are functions of $\delta x, \delta y$ and $\to 0$, as $(\delta x, \delta y)\to(0, 0)$.

From (1), (2), (3), we obtain

$$\delta z=(f_x\phi_u+f_y\psi_u)\delta u+(f_x\phi_v+f_y\psi_v)\delta v+F_1\delta u+F_2\delta v,$$

where $F_1=(f_x\phi_1+f_y\psi_1+f_1\phi_u+f_1\phi_1+f_2\psi_u+f_2\psi_1)$

and $F_2=(f_x\phi_2+f_y\psi_2+f_1\phi_v+f_1\phi_2+f_2\psi_v+f_2\psi_2)$

Since the co-efficients F_1 and F_2 of δu and $\delta v\to 0$ as $(\delta u, \delta v)\to(0, 0)$ we see that z is a differentiable function of (u, v) and

$$\begin{aligned} dz&=(f_x\phi_u+f_y\psi_u)du+(f_x\phi_v+f_y\psi_v)dv\\ &=f_x(\phi_u du+\phi_v dv)+f_y(\psi_u du+\psi_v dv)\\ &=\frac{\partial z}{\partial x}\,dx+\frac{\partial z}{\partial y}\,dy.\end{aligned}$$

130·1. Differentials of higher orders of functions of functions. From the preceding theorem we deduce that if $\partial z/\partial x, \partial z/\partial y$ are differentiable functions of x, y (so that they are also differentiable functions of u, v) and dx, dy are differentiable functions of u, v (*i. e.*, d^2x, d^2y exist, then dz is a differentiable function of u, v and we have

$$d^2z=d(dz)=d\left(\frac{\partial z}{\partial x}\right)dx+\frac{\partial z}{\partial x}d(dx)+d\left(\frac{\partial z}{\partial y}\right)dy+\frac{\partial z}{\partial y}d(dy)$$

(§125, p. 206)

$$=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\partial y}dxdy+\frac{\partial^2 z}{\partial y^2}dy^2+\frac{\partial z}{\partial x}d^2x+\frac{\partial z}{\partial y}d^2y$$

Proceeding in this manner we see that $d^{n-1}z$ is a differentiable function of u, v, *i. e.*, $d^n z$ exists, if the $(n-1)$th order partial derivatives of z are differentiable functions of x and y and the nth differentials $d^n x$, $d^n y$ of x, y exist. But the formation of differentials of higher orders becomes more and more complicated and no simple general formula exists for $d^n z$ in this case.

Note In case x, y are *linear functions of u and v i. e.*, x, y are of the form $x=a+bu+cv$, $y=d+eu+fv$, then d^2x, d^2y and all higher differentials of x and y are 0 and, therefore, we have

$$d^n z=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^n z,$$

so that the form is the same as if x and y are independent.

131. Derivation of functions of functions. *If (i) $x=\phi(u, v)$, $y=\psi(u, v)$ possess continuous first order partial derivatives in a domain E of the point (u, v) (ii) $z=f(x, y)$ possesses continuous first order partial derivatives in a domain E_1 of (x, y) and (iii) E_1 is the image of E, then z possesses continuous first order partial derivatives w, r. to u, and v in E ; also*

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u};$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

Because of (*i*) x, y are differentiable functions of u, v and because of (*ii*) z is a differentiable function of x, y. Hence from §130. z is a differentiable function of (u, v). Therefore $\partial z/\partial u$, $\partial z/\partial v$ exist and

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv \tag{1}$$

Also from §130, we have

$$\begin{aligned}dz&=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy\\&=\frac{\partial z}{\partial x}\cdot\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)+\frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv\right)\\&=\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right)du+\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)dv\end{aligned} \tag{2}$$

From (1) and (2), on comparison, we obtain the values of $\partial z/\partial u$ and $\partial z/\partial v$.

Again, because of the conditions (*i*) and (*ii*), we see that $\partial z/\partial u$ and $\partial z/\partial v$ are continuous functions of (u, v).

Cor. If (*i*) $x=\phi(u, v)$, $y=\psi(u, v)$ possess continuous nth order partial derivatives in the domain E of (u, v), (*ii*) $z=f(x, y)$ possesses continuous nth order partial derivatives in the domain E_1 of (x, y) and (*iii*) E_1 is the image of E, then z possesses continuous nth order partial derivatives w. r. to u and v in the domain E.

Because of (*i*) and (*ii*), $d^{n-1}x$, $d^{n-1}y$ are differentiable in E and $d^{n-1}z$ in E_1 so that the result now follows from §130·1.

Cor. A particular case. *If $z=f(x, y)$ possesses nth order partial derivatives and $x=a+ht$, $y=b+kt$, where a, b, h, k are constants, then*

$$\frac{d^n z}{dt^n}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n z.$$

We have

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=h\frac{\partial z}{\partial x}+k\frac{\partial z}{\partial y}$$

$$\frac{d^2 z}{dt^2}=h\left[\frac{\partial^2 z}{\partial x^2}\frac{dx}{dt}+\frac{\partial^2 z}{\partial y\partial x}\frac{d^2 y}{dt}\right]+k\left[\frac{\partial^2 z}{\partial x\partial y}\frac{dx}{dt}+\frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}\right]$$

$$=h^2\frac{\partial^2 z}{\partial^2 x}+2hk\frac{\partial^2 z}{\partial x\partial y}+k^2\frac{\partial^2 z}{\partial y^2}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 z.$$

By Mathematical induction, we may now obtain the value of $d^n z/dt^n$.

132. Taylor's theorem for a function of two variables. *If $f(x, y)$ possesses continuous partial derivatives of the nth order in any neighbourhood of a point (a, b) and if $(a+h, b+k)$ be any point of this neighbourhood then, there exists a positive number θ which is less than 1 such that*

$$f(a+h, b+k)=f(a, b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a, b)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)+\dots\dots\dots+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a, b)$$

$$+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k), \qquad (0<\theta<1).$$

We write

$$x=a+th,\ y=b+tk \qquad (0\leqslant t\leqslant 1)$$

Let

$$f(x, y)=f(a+th, b+tk)=g(t).$$

Since all the partial derivatives of $f(x, y)$ of order n are continuous, $g^n(t)$ is continuous in $(0, 1)$; also, as proved in § 131, we have

$$g^n(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x, y)$$

Applying Maclaurin's theorem to $g(t)$, we obtain

$$g(t)=g(0)+tg'(0)+\frac{t^2}{2!}g''(0)+\dots\dots\dots+\frac{t^{n-1}}{(n-1)!}g^{n-1}(0)+\frac{t^n}{n!}g^n(\theta t)$$
$$(0<\theta<1)$$

For $t=1$, this becomes

$$g(1)=g(0)+g'(0)+\frac{1}{2!}g''(0)+\dots\dots\dots+\frac{1}{(n-1)!}g^{n-1}(0)+\frac{1}{n!}g^n(\theta)$$

Since $g(1)=f(a+h, b+k)$; $g(0)=f(a, b)$

$$g'(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a, b)\ ;\ g''(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

... ...

$$g^n(\theta)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k),$$

We get the result as stated.

[It is easy to see that Taylor's theorem can also be written more compactly as

$$f(a+h,\ b+k)=f(a,\ b)+df(a,\ b)+\frac{1}{2\,!}\,d^2f(a,\ b))+\ldots\ldots$$

$$+\ldots\ldots\ldots+\frac{1}{(n-1)\,!}d^{n-1}\,f(a,\ b)\ +\frac{1}{n\,!}\ d^nf(a+\theta h,\ b+\theta k)$$

Ex. Show that

(*i*) $\sin x \sin y=xy-\frac{1}{6}[(x^3+3xy^2)\ \cos\theta x \sin\theta y+(y^3+3x^2y)\sin\theta x\sin\theta y]$ $(0<\theta<1)$

(*ii*) $e^{ax}\sin by=by+abxy+\frac{1}{6}[(a^3x^3-3ab^2xy^2)\sin v+(3a^2bx^2y-b^3y^3)\cos v]e^u$, where $u=a\theta x,\ v=b\theta y,\ (0<\theta<1)$.

133. Maxima and Minima for functions of two variables. Let $(a,\ b)$ be any* inner point of the domain of definition of a function $f(x,\ y)$. *Then $f(a,\ b)$ is said to be an extreme value of $f(x,\ y)$, if for every point $(x,\ y)$, [other than $(a,\ b)$], of some neighbourhood of $(a,\ b)$, the difference*

$$f(x,\ y)-f(a,\ b)$$

is of the same sign ; the extreme value $f(a,\ b)$ being called a maximum or a minimum value according as this difference is negative or positive.

133 1. The necessary conditions *for $f(a,\ b)$ to be an extreme value of $f(x,\ y)$ are that*

$$f_x(a,\ b)=0=f_y(a,\ b),$$

provided that these partial derivatives exist.

If $f(a,\ b)$ is an extreme value of the function $f(x,\ y)$ of two variables then, clearly, it is also an extreme value of the function $f(x,\ b)$ of one variable x for $x=a$ and therefore its derivative $f_x(a,\ b)$ for $x=a$ must necessarily be 0. Similarly we have $f_y(a,\ b)=0$.

Note. If $f(x,\ y)=0$ when $x=0$ or $y=0$ and $f(x,\ y)=1$ elsewhere then $f_x(0,\ 0)=0=f_y\ (0,\ 0)$ but $f\ (0,\ 0)$ is not an extreme value so that we see that the conditions obtained above are only necessary and not sufficient.

Ex. Show that $f(x,y)=\ |\,x\,|\ +\ |\,y\,|$ has an extreme value at $(0,\ 0)$, even though $f_x(0,\ 0), f_y(0,\ 0)$ do not even exist.

133·2. *To determine* **sufficient conditions** *for $f(a,\ b)$ to be an extreme value of $f(x,\ y)$.*

We suppose that

$$f_x(a,\ b)=0=f_y(a,\ b).$$

Also we suppose that $f(x,y)$ possesses continuous second order partial derivatives in a certain neighbourhood of $(a,\ b)$ and that the derivatives at $(a,\ b)$, *viz.*,

$$f_{x^2}\,(a,\ b),\ f_{xy}(a,\ b),\ f_{y^2}\ (a,\ b) \text{ are not all zero.}$$

We write

$$\mathrm{A}=f_{x^2}\,(a,\ b),\ \ \mathrm{B}=f_{xy}(a,\ b),\ \ \mathrm{C}=f_{y^2}\,(a,\ b).$$

By Taylor's theorem we have

$f(a+h,\ b+k)=f(a,\ b)+hf_x(a,\ b)+kf_y(a,\ b)$

---

*A point $(a,\ b)$ is said to be an *inner point* of a domain if every point of some neighbourhood of $(a,\ b)$ is a point of the domain.

$$+\frac{1}{2!}\left[h^2 f_{x^2}(\alpha,\beta)+2hk f_{xy}(\alpha,\beta)+k^2 f_{y^2}(\alpha,\beta)\right]$$

where $\alpha=a+\theta h$, $\beta=b+\theta k$ and $0<\theta<1$.

We write

$$f_{x^2}(a+\theta h,\ b+\theta k)-f_{x^2}(a,b)=\rho_1,$$
$$f_{xy}(a+\theta h,\ b+\theta k)-f_{xy}(a,b)=\rho_2,$$
$$f_{y^2}(a+\theta h,\ b+\theta k)-f_{y^2}(a,b)=\rho_3,$$

so that ρ_1, ρ_2, ρ_3 are functions of h and k and $\to 0$ as $(h, k)\to 0$.

$\therefore f(a+h,\ b+k)-f(a,b)=\frac{1}{2}[Ah^2+2Bhk+Ck^2+\rho(h^2+k^2)]$.

where ρ is a function of h, k defined by

$$\rho(h^2+k^2)=h^2\rho_1+2hk\rho_2+k^2\rho_3.$$

Since $$|\rho|\leqslant\frac{h^2}{h^2+k^2}|\rho_1|+2\frac{|hk|}{h^2+k^2}|\rho_2|+\frac{k^2}{h^2+k^2}|\rho_3|$$
$$\leqslant|\rho_1|+|\rho_2|+|\rho_3|,$$

we see that $\rho\to 0$ as $(h, k)\to(0, 0)$.

Writing

$$h=r\cos\phi,\ k=r\sin\phi,$$

we obtain

$$f(a+h,\ b+k)-f(a,b)=\tfrac{1}{2}r^2[A\cos^2\phi+2B\sin\phi\cos\phi+C\sin^2\phi+\rho] \quad (1)$$

We now determine conditions for the function

$$G(\phi)=A\cos^2\phi+2B\sin\phi\cos\phi+C\sin^2\phi$$

of ϕ to be *definite, semi-definite,* or *indefinite.*

(The definitions of definite, semi-definite and indefinite functions are given in a note at the end of this section.)

Let $A\neq 0$. Then

$$G(\phi)=[(A\cos\phi+B\sin\phi)^2+(AC-B^2)\sin^2\phi]/A$$

If $(AC-B^2)>0$, *then* $G(\phi)$ *has always the sign of* A *and is, therefore, definite.*

The only possibility of the vanishing of $G(\phi)$ arises when $\sin\phi=0$ as well as $A\cos\phi+B\sin\phi=0$, *i.e.*, when $\sin\phi=0$ and $\cos\phi=0$ but this is impossible.

If $AC-B^2=0$ *then the function is semi-definite,* for it vanishes when $A\cos\phi+B\sin\phi=0$ and has for every other value of ϕ the sign of A.

If $(AC-B^2)<0$, *then the function is indefinite,* for it assumes value of different signs when $\sin\phi=0$ and when $A\cos\phi+B\sin\phi=0$.

Let $A=0$ *but* $B\neq 0$, $C\neq 0$. Then

$$G(\phi)=2B\sin\phi\cos\phi+C\sin^2\phi=[(C\sin\phi+B\cos\phi)^2-B^2\cos^2\phi]/C$$

is clearly indefinite.

Let $A=0$, $B\neq 0$, $C=0$. Then

$$G(\phi)=2B\sin\phi\cos\phi=B\sin 2\phi$$

is clearly indefinite.

Let $A=0$, $B=0$, $C\neq 0$. Then

$$G(\phi)=C\sin^2\phi$$

is semi-definite.

Thus we see that $G(\phi)$ *is definite if and only if* $(AC-B^2)>0$; *also, then, it is positively or negatively definite according as* A (*or* C) *is positive or negative.*

We now return to the equality (1), *viz.*,

$$f(a+h, b+k)-f(a, b)=\tfrac{1}{2}r^2[G(\phi)+\rho].$$

Let $G(\phi)$ *be positively definite.* There exists, in this case, a positive number m, the lower bound of the continuous function $G(\phi)$, such that for every value of ϕ,

$$m \leqslant G(\phi).$$

Also, since $\rho \to 0$ as $(h, k)\to(0, 0)$, there exists a positive number δ such that, when $|h| \leqslant \delta$, $|k| \leqslant \delta$, we have

$$|\rho| \leqslant \tfrac{1}{2}m, \textit{ i.e., } -\tfrac{1}{2}m \leqslant \rho \leqslant \tfrac{1}{2}m.$$

Thus when $|h| \leqslant \delta$, $|k| \leqslant \delta$, we have

$$G(\phi)+\rho \geqslant m-\tfrac{1}{2}m=\tfrac{1}{2}m>0.$$

Hence $f(a, b)$ is a minimum value of $f(x, y)$ in this case.

Let $G(\phi)$ *be negatively definite.* There exists in this case a negative number M, the upper bound of the continuous function $G(\phi)$, such that for every value of ϕ,

$$G(\phi) \leqslant M.$$

Also there exists a positive number δ such that when $|h| \leqslant \delta$, $|k| \leqslant \delta$,

$$|\rho| \leqslant -\tfrac{1}{2}M, \textit{ i.e., } \tfrac{1}{2}M \leqslant \rho \leqslant -\tfrac{1}{2}M.$$

Thus when $|h| \leqslant \delta$, $|k| \leqslant \delta$,

$$G(\phi)+\rho \leqslant \tfrac{1}{2}M<0.$$

Hence $f(a, b)$ is a maximum value of $f(x, y)$ in this case.

Let $G(\phi)$ *be semi-definite.* This case is doubtful for the sign of $f(a+h, b+k)-f(a, b)$ depends upon ρ.

Let $G(\phi)$ *be indefinite.* Let ϕ_1 and ϕ_2 be any two numbers such that

$$G(\phi_1)>0, \; G(\phi_2)<0.$$

There surely exists a positive number m such that

$$G(\phi_1)>m, \; G(\phi_2)< -m.$$

We choose a positive number δ such that when $|h| \leqslant \delta$ and $|k| \leqslant \delta$,

$$|\rho| < \tfrac{1}{2}m, \textit{ i.e., } -\tfrac{1}{2}m<\rho<\tfrac{1}{2}m.$$

Thus when $|h| \leqslant \delta$, $|k| \leqslant \delta$,

$$G(\phi_1)+\rho>\tfrac{1}{2}m>0 \text{ and } G(\phi_2)+\rho< -\tfrac{1}{2}m<0,$$

so that in every neighbourhood of (a, b) there exist points $(a+h, b+k)$ for which the difference $f(a+h, b+k)-f(a, b)$ has different signs. Hence $f(a, b)$ is not an extreme value in this case.

Rule. *If at a point* (a, b)

$$f_x(a, b)=0=f_y(a, b)$$

and

$$f_{x^2}(a, b)\, f_{y^2}(a, b)-f^2_{yx}(a, b)>0,$$

then $f(a, b)$ *is an extreme value which is a maximum or a minimum according as* $A<0$ *or* >0 (*and consequently* $C<0$ *or* >0).

Note. The case when $A=B=C=0$ or the doubtful case which arises when $AC-B^2=0$ or when $A=B=0$ and $C\neq 0$ require more elaborate consideration where we have to employ Taylor's theorem with remainder after three or more terms but this consideration is beyond the scope of this book.

Note. Any function $G(\phi)$ of ϕ is said to be *definite* if for every value of ϕ it assumes values which have always one sign ; also a definite function is said to be positively or negatively definitely negative according as the sign is positive or negative.

2. The function is said to be *semi-definite* if it can vanish for some value or values of ϕ and yet, when not 0, has always one sign.

3. The function is said to be *indefinite* if it can assume values which are of different signs.

Note. Since $df(a, b)=hf_x(a, b)+kf_y(a, b)$

and
$$d^2f(a, b)=h^2f_{x^2}(a, b)+2hkf_{xy}(a, b)+k^2f_{y^2}(a, b)$$
$$=\rho^2[\cos^2\phi f_{x^2}(a, b)+2\sin\phi\cos\phi f_{xy}(a, b)$$
$$+\sin^2\phi f_{y^2}(a, b)],$$

we see that $f(a, b)$ *will be an extreme value of* (x, y) *if at* (a, b) *the first differential* $df=0$ *and the second differential* d^2f *is of invariable sign (i.e., is definite) for all values of* h *and* k. Stated in this form the result is quite general and applies to a function of any number of variables as can be proved by an obvious extension of the method for two variables.

Examples

1. Examine the following for extreme values :—

(*i*) $y^2+4xy+3x^2+x^3$. (*ii*) $3x^4y^2-6x^2y^2+3x^4+2y^3-6x^2-3y^2+1$.

(*iii*) $y^2+x^2y+ax^4$ (*iv*) $x^2+xy+y^2+ax+by$.

(*v*) $x^2y^2(12-3x-4y)$. (*vi*) $xy(a-x-y)$.

(*vii*) $2xyz+x^2+y^2+z^2$. (*viii*) $2x^2+3y^2+4z^2-3xy+8z$.

2. Show that
$$f(x, y)=(y-x)^4+(x-2)^4$$
has a min. at (2, 2) even though at (2, 2),
$$f_{x^2}f_{y^2}-f^2_{xy}=0.$$

3. Show that
$$f(x, y)=y^2+x^2y+x^4$$
has a min. at (0, 0) even though at (0, 0)
$$f_{x^2}f_{y^2}-f^2_{xy}=0.$$

4. Show that
$$f(x, y)=2x^4-3x^2y+y^2$$
has neither a max nor a min. at (0, 0) where
$$f_{x^2}f_{y^2}-f^2_{xy}=0.$$

5. Find the maximum and minimum values of the function
$$(x^2+y^2)e^{6x+2x^2}$$
(*I. C. S.*)

6. If r is the distance between two points P and Q situated respectively on the curves
$$y^2=4ax \text{ and } (x-3a)^2+y^2=12a^2,$$
find the positions of P and Q when r is a maximum or a minimum and interpret the results geometrically.

CHAPTER X

IMPLICIT FUNCTIONS

134. If $f(x, y)$ be a function of two variables and $y=\phi(x)$ be a function of x such that for every value of x for which $\phi(x)$ is defined, $f[x, \phi(x)]$ vanishes identically, then we say that $y=\phi(x)$ is an *implicit function* defined by the functional equation $f(x, y)=0$.

A functional equation may not define any implicit function or it may define one or more than one such function; for example, the functional equation $x^2+y^2-1=0$ determines two implicit functions, *viz.*, $y=\sqrt{(1-x^2)}$ and $y=-\sqrt{(1-x^2)}$, whereas the equation $x^2+y^2+1=0$ determines no such function.

It is only in elementary cases, such as those given above, that we may be able to determine the implicit functions, (in case they exist), defined by a functional equation. For more complicated functional equations such as $x^2y+\sin y+\log x=0$, no such *determination* is possible; this impossibility of determination, however, does not rule out the possibility of the *existence* of the implicit function or functions defined by an equation. The question of the *existence* of implicit functions, (apart from their actual determination) will be investigated in this chapter.

135. Implicit functions determined by a functional equation. *Let $f(x, y)$ be a function of two variables and let (a, b) be a point of its domain of definition such that*

(*i*) $f(a, b)=0$,

(*ii*) *the function possesses continuous derivatives f_x and f_y in a certain neighbourhood of (a, b), and*

(*iii*) $f_y(a, b)\neq 0$;

then *there exists a rectangle $(a-h, a+h, b-k, b+k)$ about (a, b) such that for every value of x in the interval $(a-h, a+h)$, the equation $f(x, y)=0$ determines one and only one value $y=\phi(x)$ lying in the interval $(b-k, b+k)$ and this function $\phi(x)$ is such that*

(1) $b=\phi(a)$.

(2) $f[x, \phi(x)]=0$ *for every value of x in $(a-h, a+h)$.*

(3) $\phi(x)$ *is derivable and both $\phi(x)$ as well as $\phi'(x)$ are continuous in $(a-h, a+h)$.*

(The reader is advised to follow the following line of argument by means of the diagram which he may easily construct for himself).

Without any loss of generality, we suppose that $f_y(a, b)>0$, for, otherwise we should only have to replace $f(x, y)$ by $-f(x, y)$ and this change would leave the equation $f(x, y)=0$ unaltered.

Unique Existence. Let f_x, f_y be continuous in the neighbourhood $R_1(a-h_1, a+h_1 ; b-k_1, b+k_1)$ of (a, b).

Since f_x, f_y are continuous in R_1, therefore $f(x, y)$ is also continuous in R_1.

Since $f_y(x, y)$ is continuous at (a, b) and $f_y(a, b)>0$, there exists a rectangle

$$R_2(a-h_2, a+h_2 ; b-k, b+k) \qquad h_2<h_1, k<k_1$$

such that for every point (x, y) of this rectangle R_2, $f_y(x, y)>0$.

Now, since $f_y(x, y)>0$ in R_2, therefore for every value of x in $(a-h_2, a+h_2)$, the function $f(x, y)$ of y strictly increases as y increases from $b-k$ to $b+k$. In particular, since $f(a, b)=0$, we have

$$f(a, b-k)<0, \ f(a, b+k)>0.$$

In view of this and the fact that $f(x, y)$ is continuous, there exists an interval $(a-h, a+h)$, $(h<h_2)$ such that for every x of this interval, we have

$$f(x, b-k)<0, \ f(x, b+k)>0.$$

Now for every fixed value of x in $(a-h, a+h)$ the continuous function $f(x, y)$ of y *strictly* increases from a negative to a positive value as y increases from $b-k$ to $b+k$ and therefore there exists *one and only one* value of y for which the function $f(x, y)$ vanishes.

Hence for each value of x in $(a-h, a+h)$ there is a uniquely determined value of y for which $f(x, y)=0$; this value of y is a function of x, say $\phi(x)$ such that the properties (1) and (2) are true.

This completes the proof of the *existence* and the *uniqueness* of the implicit function $\phi(x)$.

Continuity. We now prove that $\phi(x)$ is continuous in $(a-h, a+h)$. Let x_0 be any point of this interval and let $y_0=\phi(x_0)$. Let ϵ be any given positive number. Let

$$R'(x_0-\delta_1, x_0+\delta_1 ; y_0-\epsilon, y_0+\epsilon)$$

be a rectangle entirely lying within the rectangle

$$R(a-h, a+h ; b-k, b+k)$$

found above. For this rectangle we can carry out exactly the same process as before in order to obtain a solution of $f(x, y)=0$. Since the solution was uniquely determined in R which encloses R′, we see that the same function *viz.*, $y=\phi(x)$ is the solution in R′ also. Thus there exists an interval $(x_0-\delta, x_0-\delta)$, $(\delta \leqslant \delta_1)$, such that for every value of x in this interval $y=\phi(x)$ lies between $y_0-\epsilon$ and $y_0+\epsilon$, *i. e.*,

$$|y-y_0| = |\phi(x)-\phi(x_0)| < \epsilon, \text{ when } |x-x_0| \leqslant \delta.$$

Hence $\phi(x)$ is *continuous* at x_0 and, therefore, in $(a-h, a+h)$.

Derivability. Let x be a point of the interval $(a-h, a+h)$ and let $x+\delta x$ be another point of the same interval. Let

$$y=\phi(x), \ y+\delta y=\phi(x+\delta x),$$

so that

$$f(x, y)=0, \ f(x+\delta x, y+\delta y)=0.$$

$$\begin{aligned}\therefore \quad 0&=f(x+\delta x, y+\delta y)-f(x, y)\\ &=f(x+\delta x, y+\delta y)-f(x+\delta x, y)+f(x+\delta x, y)-f(x, y)\\ &=\delta y f_y(x+\delta x, y+\theta_1\delta y)+\delta x f_x(x+\theta_2\delta x, y)\end{aligned}$$

(By mean value theorem)

Since $f_y \neq 0$ in R and $(x+\delta x, y+\theta_1 \delta y)$ is a point of R we have

$$\frac{\delta y}{\delta x} = -\frac{f_x(x+\theta_2 \theta x, y)}{f_y(x+\delta x, y+\theta_1 \delta y)}.$$

Since $\phi(x)$ is continuous, $\delta y \to 0$ as $\delta x \to 0$. Therefore, f_x and f_y being continuous, we obtain from above, when $(\delta x, \delta y) \to (0, 0)$,

$$\phi'(x) = \frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}.$$

Thus $\phi'(x)$ is *derivable* and $\phi'(x) = -f_x(x, y)/f_y(x, y)$. Also this formula for $\phi'(x)$ shows that it is continuous.

Note. The function $y=\phi(x)$ is said to be the unique solution of $f(x, y)=0$ *near* (a, b) or the unique implicit function determined by $f(x, y)=0$ *near* (a, b).

Examples

1. If $f(x, y)$ is a continuous function of each variable x and y separately in a certain neighbourhood of (a, b), $f(a, b)=0$, $f_y(x, y)$ is continuous at (a, b) and $f_y(a, b) \neq 0$, then the equation $f(x, y)=0$ determines a *unique* continuous implicit function $y=\phi(x)$ near (a, b).

2. If $f(x, y)$ is a continuous function of each variable x and y separately in a neighbourhood of (a, b), $f(a, b)=0$ and $f_y(a, b)$ exists and $\neq 0$, then the equation $f(x, y)=0$ determines *atleast one* continuous implicit function $y=\phi(x)$ near (a, b).

3. $f(x, y)$ is a continuous function of x and y in the neighbourhood of a, b such that $f(a, b)=0$; and further $f(x, y)$ is, for all x in the neighbourhood of a, a strictly increasing function of y. Prove that there exists a unique function $y=\phi(x)$, which when substituted in the equation $f(x, y)=0$ satisfies it identically for all values of x in the neighbourhood of a, and that $\phi(x)$ is continuous for all values of x in the neighbourhood of a.

If $f(x, y)=y^4-y^2+x^2$, discuss the existence of the function $y=\phi(x)$ in the neighbourhood of $x=0$, $y=0$.

4. Show that the following equations determine unique implicit functions near the points indicated; find also the first derivatives of the solutions.

(*i*) $x^3+y^3-3xy+y=0$, $(1, 1)$. (*ii*) $xy \sin x+\cos y=0$, $(0, \frac{1}{2}\pi)$.

(*iii*) $y^3 \cos x+y^2 \sin^2 x=7$, $(\frac{1}{3}\pi, 2)$ (*iv*) $2xy-\log xy=2$, $(1, 1)$.

5. Examine the following equations for the existence of unique implicit functions near the points indicated and verify by direct calculation.

(*i*) $y^2-yx^2-2x^5=0$ near $(0, 0)$ and $(1, -1)$.

(*ii*) $y^4+y^2x^2-2x^5=0$ near $(1, 1)$.

(*iii*) $y^2+2x^2y+x^6=0$ near $(1, -1)$.

(*iv*) $y^2+yx^2+x^2=0$ near $(0, 0)$.

6. Show that the least positive root of $xy=\tan y$ is a continuous function of x throughout the interval $(1, \infty]$ and increases steadily from 0 to $\frac{1}{2}\pi$ as x increases from 1 towards ∞.

(Use §51, p. 78).

136. Before considering the question of the existence of implicit functions determined by a system of functional equations containing any number of variables, we give, in this section, a few necessary definitions. These definitions constitute an obvious extension of those given in the preceding chapter for functions of two variables.

A point in a space of n dimensions. *Any ordered set of n numbers* $(a_1, a_2, a_3, \ldots, a_n)$ *is called a point in a space of n dimensions*

A set of points $(x_1, x_2, \ldots\ldots, x_n)$ such that

$$|x_1-a_1| \leqslant h_1,\ |x_2-a_2| \leqslant h_2, \ldots\ldots\ |x_n-a_n| \leqslant h_n$$

is said to be a rectangle with the point $(a_1, a_2, \ldots\ldots a_n)$ at its centre and will be denoted by

$$(a_1-h_1,\ a_1+h_1\ ;\ a_2-h_2,\ a_2+h_2\ ;\ldots\ldots\ ;\ a_n-h_n,\ a_n+h_n).$$

If to each point $(x_1, x_2, \ldots\ldots, x_n)$ of anv given set of points in the space of n dimensions, be associated a value of another variable u, then we say that u is a functiou of the n variables $x_1, x_2, \ldots\ldots, x_n$ and the given set of points is said to be the domain of definition of this function.

Ex. $u=x_1+x_2+\ldots\ldots+x_n$ is a function of the n variables $x_1, x_2, \ldots\ldots, x_n$.

Neighbourhood of a point. The rectangle

$$(a_1-h_1,\ a_1+h_1\ ;\ a_2-h_2,\ a_2+h_2\ ;\ldots;a_n-h_n,\ a_n+h_n)\ ;$$

when $h_1, h_2, \ldots\ldots h_n$ are any positive numbers whatsoever is said to be a neighbourhood of $(a_1, a_2, \ldots, a_n)$

Continuity. A function $f(x_1, x_2, \ldots, x_n)$ is said to be continuous at a point $P(a_1, a_2, \ldots, a_n)$, if to every positive number ϵ, there corresponds a neighbourhood of P such that for every point $(x_1, x_2, \ldots, x_n)$ of this neighbourhood

$$|f(x_1, x_2, \ldots, x_n)-f(a_1, a_2, \ldots, a_n)| < \epsilon.$$

Partial Derivatives. The limit

$$\underset{h_1 \to 0}{\text{lt}} \frac{f(a_1+h_1, a_2, \ldots, a_n)-f(a_1, a_2, \ldots, a_n)}{h_1},$$

if it exists, is called the partial derivative of the function f w. r. to x_1 at $(a_1, a_2, \ldots, a_n)$ and is denoted as

$$f_{x_1}(a_1, a_2, \ldots, a_n).$$

We may similarly define other partial derivatives of the first and of the second and higher orders.

137. Implicit function determined by a functional equation. General Theorem. *If $f(x_1, x_2, \ldots, x_n, y)$ be a function of $(n+1)$ variables and $(a_1, a_2, \ldots, a_n, b)$ be any point of its domain of definition such that*

(*i*) $f(a_1, a_2, \ldots, a_n, b)=0$,

(*ii*) *the function possesses continuous first order partial derivatives w. r. to the $(n+1)$ variables in a certain neighbourhood of the point $(a_1, a_2, \ldots, a_n, b)$, and*

(*iii*) $f_y(a_1, a_2, \ldots, a_n, b) \neq 0$,

then *there exists a rectangle*

$$(a_1-h_1,\ a_1+h_1\ ;\ a_2-h_2,\ a_2+h_2\ ;\ \ldots\ ;\ a_n-h_n,\ a_n+h_n\ ;\ b-k,\ b+k)$$

such that for every point $(x_1, x_2, \ldots, x_n)$ of the rectangle

$$R(a_1-h_1,\ a_1+h_1\ ;\ a_2-h_2,\ a_2+h_2\ ;\ \ldots\ ;\ a_n-h_n,\ a_n+h_n)$$

the equation $f(x_1, x_2, \ldots, x_n, y)=0$ *determines one and only one value* $y=\phi(x_1, x_2, \ldots, x_n)$ *lying in* $(b-k,\ b+k)$ *and this function* $\phi(x)$ *is such that*

(1) $b=\phi(a_1, a_2\ a_3, \ldots, a_n)$

(2) $f[x_1, x_2, \ldots, x_n, \phi]=0$ *for every point* $(x_1, x_2, \ldots, x_n)$ *in* R

(3) ϕ *is continuous and possesses continuous partial derivatives of the first order w. r. to* $x_1, x_2, \ldots, x_n$ *in* R.

The proof of this general theorem follows exactly on the same lines as the proof of the preceding theorem (§ 135) and offers no fresh difficulties.

138. Jacobians. If $u_1,\ u_2,\ u_3, \ldots, u_n$ be n functions of n variables $x_1, x_2, \ldots, x_n$ possessing partial derivatives of the first order at every point of the domain of definition of the functions, then the determinant

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial x_1}, & \dfrac{\partial u_1}{\partial x_2}, & \ldots\ldots, & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1}, & \dfrac{\partial u_2}{\partial x_2}, & \ldots\ldots, & \dfrac{\partial u_2}{\partial x_n} \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{\partial u_n}{\partial x_1}, & \dfrac{\partial u_n}{\partial x_2}, & \ldots\ldots, & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \ldots, u_n$ w. r. to $x_1,\ x_2, \ldots,\ x_n$ and is denoted by

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} \text{ or } J\left(\frac{u_1, u_2, \ldots, u_n}{x_1, x_2, \ldots, x_n}\right)$$

139. Implicit functions determined by a system of functional Equations. Theorem. *Let* $f_1(x, y, z, u, v)$ *and* $f_2(x, y, z, u, v)$ *be two functions of five variables and let* $(a_1, a_2, a_3, b_1, b_2)$ *be a point of their domain of definition such that*

(*i*) $f_1(a_1, a_2, a_3, b_1, b_2)=0=f_2(a_1, a_2, a_3, b_1, b_2)$,

(*ii*) *the functions possess continuous first order partial derivatives w. r. to the five variables in a certain neighbourhood of the point* $(a_1, a_2, a_3, b_1, b_2)$, *and*

(*iii*) $\partial(f_1, f_2)/\partial(u, v)\neq 0$ *at* $(a_1, a_2, a_3, b_1, b_2)$,

then *there exist one and only one pair of functions*

$$u=\phi_1(x, y, z), \qquad v=\phi_2(x, y, z)$$

of x, y, z *defined in a certain neighbourhood of* (a_1, a_2, a_3) *such that*

(1) $\phi_1(a_1, a_2, a_3)=b_1 \qquad \phi_2(a_1, a_2, a_3)=b_2$

(2) $f_1(x, y, z, \phi_1, \phi_2)=0=f_2(x, y, z, \phi_1, \phi_2)$ *for every point* x, y, z *of the domain of definition of* ϕ_1 *and* ϕ_2.

(3) ϕ_1, ϕ_2 *are continuous and possess continuous first order partial derivatives w. r. to* x, y, z.

Proof. Since $\partial(f_1, f_2)/\partial(u, v)\neq 0$ at $(a_1, a_2, a_3, b_1, b_2)$, one atleast of the partial derivatives $\partial f_1/\partial v$, $\partial f_2/\partial v$ must not vanish at this point. Suppose that $\partial f_1/\partial v\neq 0$.

Then, by the preceding theorem (§ 136) the equation

$$f_1(x, y, z, u, v)=0$$

is satisfied by one and only one function

$$v=g(x, y, z, u)$$

defined in a certain neighbourhood of (a_1, a_2, a_3, b_1) and such that $b_2=g(a_1, a_2, a_3, b_1)$; the function g is continuous and possesses continuous first order partial derivatives. Replacing v by $g(x, y, z, u)$ in the function $f_2(x, y, z, u, v)$, we write

$$h(x, y, z, u)=f_2(x, y, z, u, g),$$

so that

$$h(a_1, a_2, a_3, b_1)=0$$

We have, at (a_1, a_2, a_3, b_1)

$$\frac{\partial h}{\partial u}=\frac{\partial f_2}{\partial u}+\frac{\partial f_2}{\partial v}\cdot\frac{\partial g}{\partial u}$$

Also, since $f_1(x, y, z, u, g)=0$, we have at (a_1, a_2, a_3, b_1).

$$\frac{\partial f_1}{\partial u}+\frac{\partial f_1}{\partial v}\cdot\frac{\partial g}{\partial u}=0.$$

From these we obtain

$$\frac{\partial h}{\partial u}=-\frac{\partial(f_1, f_2)/\partial(u, v)}{\partial f_1/\partial v}$$

so that $\partial h/\partial u\neq 0$ at (a_1, a_2, a_3, b_1).

Thus, again, by the preceeding theorem the equation

$$h(x, y, z, u)=0$$

is satisfied by one and only one function

$$u=\phi_1(x, y, z)$$

defined in a certain neighbourhood of (a_1, a_2, a_3) and such that $b_1=\phi_1(a_1, a_2, a_3)$. Replacing u by $\phi_1(x, y, z)$ in $g(x, y, z, u)$ we obtain

$$v=g(x, y, z, \phi_1)=\phi_2(x, y, z)$$

where $\phi_2(a_1, a_2, a_3)=g(a_1, a_2, a_3, b_1)=b_2$.

It is easy to see that ϕ_1, ϕ_2 are continuous and possess continuous first order partial derivatives.

Hence the theorem.

Cor. *To determine* $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y$.

Applying the theorem on the derivation of functions of functions to $f_1(x, y, z, u, v)$ and $f_2(x, y, z, u, v)$ where $u=\phi_1(x, y, z)$ and $v=\phi_2(x, y, z)$, we have, f_1, f_2 being indentically zero, we have

$$\frac{\partial f_1}{\partial x}+\frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x}=0$$

$$\frac{\partial f_2}{\partial x}+\frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial x}=0$$

which give

$$\frac{\partial u}{\partial x}=-\frac{\partial(f_1, f_2)/\partial(x, v)}{\partial(f_1, f_2)/\partial(u, v)},\ \frac{\partial v}{\partial x}=\frac{\partial(f_1, f_2)/\partial(x, u)}{\partial(f_1, f_2)/\partial(u, v)}$$

We may similarly determine $\partial u/\partial y$ and $\partial v/\partial y$.

140. Change of variables. *If $z=f(x, y)$ be a function of two independent variables x and y, and if u, v be two new variables connected with x and y by the relations*

$$x=\phi(u, v),\ y=\psi(u, v),$$

to express the first and second order partial derivative of z with respect to x and y in terms of u, v and the partial derivatives of z with respect to u and v.

By the rule of §131, we have

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v},$$

whence we obtain

$$\frac{\partial z}{\partial x}=A\frac{\partial z}{\partial u}+B\frac{\partial z}{\partial v},$$

$$\frac{\partial z}{\partial y}=C\frac{\partial z}{\partial u}+D\frac{\partial z}{\partial v},$$

where $A=\dfrac{\partial y}{\partial v}\Big/\dfrac{\partial(x, y)}{\partial(u, v)}$, $B=-\dfrac{\partial y}{\partial u}\Big/\dfrac{\partial(x, y)}{\partial(u, v)}$;

$C=-\dfrac{\partial x}{\partial v}\Big/\dfrac{\partial(x, y)}{\partial(u, v)}$, $D=\dfrac{\partial x}{\partial v}\Big/\dfrac{\partial(x, y)}{\partial(u, v)}$.

are known functions of u and v. The determinant $\partial(x, y)/\partial(u, v)$ cannot vanish, for, if it did, the change of variables performed would have no meaning. (§139) Thus we see that the *derivative of z with respect to x is the sum of the two products formed by multiplying the derivatives of z with respect to u and v by the known functions A, B respectively.* Similar rule holds good for $\partial z/\partial y$. These rules may be expressed as follows :

$$\frac{\partial}{\partial x}=A\frac{\partial}{\partial u}+B\frac{\partial}{\partial v},\quad \frac{\partial}{\partial y}=C\frac{\partial}{\partial u}+D\frac{\partial}{\partial v}.$$

In order to calculate the second derivatives, we apply the above rule to the first derivatives and obtain

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(A\frac{\partial z}{\partial u}+B\frac{\partial z}{\partial v}\right)$$

$$=\left(A\frac{\partial}{\partial u}+B\frac{\partial}{\partial v}\right)\left(A\frac{\partial z}{\partial u}+B\frac{\partial z}{\partial v}\right)$$

$$=A\frac{\partial}{\partial u}\left(A\frac{\partial z}{\partial u}+B\frac{\partial z}{\partial v}\right)+B\frac{\partial}{\partial v}\left(A\frac{\partial z}{\partial u}+B\frac{\partial z}{\partial v}\right)$$

$$=A^2\frac{\partial^2 z}{\partial u^2}+2AB\frac{\partial^2 z}{\partial u\partial v}+B^2\frac{\partial^2 z}{\partial v^2}+\left(A\frac{\partial A}{\partial u}+B\frac{\partial A}{\partial v}\right)\frac{\partial z}{\partial u}$$

$$+\left(A\frac{\partial B}{\partial u}+B\frac{\partial B}{\partial v}\right)\frac{\partial z}{\partial v}.$$

$\partial^2 z/\partial x\partial y$, $\partial^2 z/\partial y^2$ can be similarly obtained.

Examples

1. *If z be a function of the two variables x, y and*

$$x=r\cos\theta,\ y=r\sin\theta,$$

prove that

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}$$

We have

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}=\cos\theta\frac{\partial z}{\partial x}+\sin\theta\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}=-r\sin\theta\frac{\partial z}{\partial x}+r\cos\theta\cdot\frac{\partial z}{\partial y}$$

These give

$$\frac{\partial z}{\partial x}=\cos\theta\cdot\frac{\partial z}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial z}{\partial \theta}\ i.e.,\ \frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}$$

$$\frac{\partial z}{\partial y}=\sin\theta\cdot\frac{\partial z}{\partial r}+\frac{\cos\theta}{r}\cdot\frac{\partial z}{\partial \theta},\ i.e.,\ \frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta}$$

Hence $$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial z}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial z}{\partial \theta}\right)$$

$$=\cos^2\theta\cdot\frac{\partial^2 z}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 z}{\partial r\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 z}{\partial \theta^2}$$

$$+\frac{2\cos\theta\sin\theta}{r^2}\frac{\partial z}{\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial z}{\partial r}$$

Similarly $$\frac{\partial^2 z}{\partial y^2}=\sin^2\theta\frac{\partial^2 z}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 z}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 z}{\partial \theta^2}$$

$$-\frac{2\cos\theta\sin\theta}{r^2}\frac{\partial z}{\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial z}{\partial r}$$

On adding, we get the result as given.

2. If z be a function of two variables x and y and

$$x=c\cosh u\cos v,\ y=c\sinh u\sin v,$$

prove that

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\frac{1}{2}c^2\left(\cosh 2u-\cos 2v\right)\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right).$$

3. Express $(x^2+y^2)^{-1}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)$ in terms of the derivatives of f with respect to u and v where $u=x^2-y^2$, $v=2xy$, $f(u,v)=\phi(x,y)$.

Deduce that the most general function of x, y satisfying

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0,$$

is $axy+b$, where a and b are constants and find the most general function of $xy(x^2-y^2)$ satisfying the same equation.

4. A function $f(x, y)$, when expressed in terms of the new variables u, v defined by the equations

$$x=\tfrac{1}{2}(u+v),\ y^2=uv,$$

becomes $g(u, v)$; prove that

$$\frac{\partial^2 g}{\partial u \partial v}=\frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2}+2\frac{x}{y}\frac{\partial^2 f}{\partial x \partial y}+\frac{\partial^2 f}{\partial y^2}+\frac{1}{y}\frac{\partial f}{\partial y}\right).$$

5. If $F(u, v)$ is a twice differentiable function of (u, v) and if $u=x^2-y^2$, $v=2xy$, preve that

$$4(u^2+v^2)F_{uv}+2uF_v+2vF_u=xy(f_{x^2}-f_{y^2})+(x^2-y^2)f_{xy}$$

6. Show that if $x=u^2v$, $y=v^2u$, then

$$2x^2\frac{\partial^2 f}{\partial x^2}+2y^2\frac{\partial^2 f}{\partial y^2}+5xy\frac{\partial^2 f}{\partial x \partial y}=uv\frac{\partial^2 f}{\partial u \partial v}-\frac{2}{3}\left(u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}\right).$$

141. Dependence of Functions. Let $u_1, u_2, u_3, \ldots, u_n$ be n functions of n variables $(x_1, x_2, x_3, \ldots, x_n)$. The functions are said to be *dependent* if they satisfy one or more equations in which the variables $x_1, x_2, \ldots, x_n$ do not appear explicitly and otherwise they are said to be independent. Thus if

$$u=x+y-z,\ v=x-y+z,\ w=x^2+y^2+z^2-2yz$$

then u, v, w are dependent in as much as we have the relation $u^2+v^2=2w$ which does not contain x, y, z explicitly, and, which, therefore, is satisfied by every set of values of x, y, z.

Theorem. *The necessary and sufficient condition that a functional relation independent of x, y, z should exist between the three functions u, v, w (i.e., u, v, w be dependent) of the three independent variables x, y, z is that the Jacobian*

$$\partial(u, v, w)/\partial(x, y, z),$$

vanishes identically.

The condition is necessary. There exists a function $\phi(u, v, w)$ which vanishes for every set of values of x, y, z. The partial derivatives, of $\phi(u, v, w)$ must vanish identically, so that we have

$$\frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial \phi}{\partial w}\cdot\frac{\partial w}{\partial x}=0,$$

$$\frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial \phi}{\partial w}\cdot\frac{\partial w}{\partial y}=0,$$

$$\frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial z}+\frac{\partial \phi}{\partial w}\cdot\frac{\partial w}{\partial z}=0.$$

Eliminating the partial derivatives $\partial\phi/\partial u$, $\partial\phi/\partial v$, $\partial\phi/\partial w$, we get

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}=0.$$

Thus the condition is necessary.

The condition is sufficient.

Let $\quad u=f_1(x, y, z),\ v=f_2(x, y, z),\ w=f_3(x, y, z) \quad (1)$

The Jacobian $\partial(u, v, w)/\partial(x, y, z)=0$,

i.e.,

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial x}, & \dfrac{\partial f_1}{\partial y}, & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x}, & \dfrac{\partial f_2}{\partial y}, & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x}, & \dfrac{\partial f_3}{\partial y}, & \dfrac{\partial f_3}{\partial z} \end{vmatrix}=0 \tag{2}$$

Suppose that one atleast of the first minors of the Jacobian, say $\partial(u, v)/\partial(x, y)\neq 0$. By theorem §139, the functional equations

$$u-f_1(x, y, z)=0,\ v-f_2(x, y, z)=0 \tag{3}$$

determine implicit functions

$$x=g_1(u, v, z),\ y=g_2(u, v, z). \tag{4}$$

These give

$$w=f_3(x, y, z)=f_3(g_1, g_2, z)=h(u, v, z) \tag{5}$$

We will now prove that $\partial h/\partial z=0$.

In the following discussion, u, v, z will be looked upon as independent variables and x, y, w as dependent.

From (5), differentiatiating with respect to z,

$$\frac{\partial h}{\partial z}=\frac{\partial f_3}{\partial x}\cdot\frac{\partial g_1}{\partial z}+\frac{\partial f_3}{\partial y}\cdot\frac{\partial g_2}{\partial z}+\frac{\partial f_3}{\partial z}$$

From (3), differenting with respect to z,

$$0=\frac{\partial f_1}{\partial x}\cdot\frac{\partial g_1}{\partial z}+\frac{\partial f_1}{\partial y}\cdot\frac{\partial g_2}{\partial z}+\frac{\partial f_1}{\partial z}$$

$$0=\frac{\partial f_2}{\partial x}\cdot\frac{\partial g_1}{\partial z}+\frac{\partial f_2}{\partial y}\cdot\frac{\partial g_2}{\partial z}+\frac{\partial f_2}{\partial z}.$$

Multiplying the elements of the I, II, columns by $\partial g_1/\partial z$, $\partial g_2/\partial z$ respectively and adding to those of the third in the determinant (2) we obtain

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial x}, & \dfrac{\partial f_1}{\partial y}, & 0 \\ \dfrac{\partial f_2}{\partial x}, & \dfrac{\partial f_2}{\partial y}, & 0 \\ \dfrac{\partial f_3}{\partial x}, & \dfrac{\partial f_3}{\partial y}, & \dfrac{\partial h}{\partial z} \end{vmatrix}=0$$

i.e., $(\partial h/\partial z)\partial(f_1, f_2)/\partial(x, y)=0$,

Therefore $\partial h/\partial z=0$ so that h does not depend on z and $w=h(u, v)$ is the relation sought.

Now, suppose that all the first minors of the Jacobian are zero, but one of the second minors, say $\partial f_1/\partial u\neq 0$. Therefore, by theorem §137 the functional equation $u-f_1(x, y, z)=0$ determines an implicit function

$$x=g(u, v, z)$$

so that we get

$$v=f_2(x, y, z)=f_2(g, y, z)=h_1(u, y, z)$$
$$w=f_3(x, y, z)=f_3(g, y, z)=h_2(u, y, z)$$

For the following discussion, u, y, z will be looked upon as independent variables and x, v, w as dependent.

We have

$$\frac{\partial h_1}{\partial y}=\frac{\partial f_2}{\partial x}\cdot\frac{\partial g}{\partial y}+\frac{\partial f_2}{\partial y}, \quad \frac{\partial h_1}{\partial z}=\frac{\partial f_2}{\partial x}\cdot\frac{\partial g}{\partial z}+\frac{\partial f_2}{\partial z}$$

$$\frac{\partial h_2}{\partial y}=\frac{\partial f_3}{\partial x}\cdot\frac{\partial g}{\partial y}+\frac{\partial f_3}{\partial y}, \quad \frac{\partial h_2}{\partial z}=\frac{\partial f_3}{\partial x}\cdot\frac{\partial g}{\partial z}+\frac{\partial f_3}{\partial z}$$

Also
$$0=\frac{\partial f_1}{\partial x}\cdot\frac{\partial g}{\partial y}+\frac{\partial f_1}{\partial y}, \quad 0=\frac{\partial f_1}{\partial x}\cdot\frac{\partial g}{\partial z}+\frac{\partial f_1}{\partial z}$$

Since

$$\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{vmatrix}=0, \quad \begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial z}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial z}\end{vmatrix}=0; \quad \begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y}\end{vmatrix}=0, \quad \begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial z}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial z}\end{vmatrix}=0$$

we see on making use of the relations obtained above

$$0=\partial h_1/\partial y=\partial h_1/\partial z=\partial h_2/\partial y=\partial h_2/\partial z$$

so that the functions h_1 and h_2 do not contain y and z. There are thus two relations in this case.

Note. The theorem above is capable of generalisation to n functions of n variables.

Ex. If
$$u=(ayz+by+bz+c)/(y-z),$$
$$v=(azx+bz+bx+c)/(z-x),$$
$$w=(axy+bx+by+c)/(x-y),$$
show that u, v, w are connected by a functional relation and find it.

Ex. Show that
$$u=3x+2y-z,\ v=x-2y+z,\ w=x(x+2y-z)$$
are connected by a functional equation and and find it.

Ex. Show that the quadratic forms $ax^2+2hxy+by^2$ and $Ax^2+2Hxy+By^2$ are independent unless

$$\frac{a}{A}=\frac{h}{H}=\frac{b}{B}.$$

141. Extreme values of a function of n variables. Stationary points and stationary values. As in the case of functions of one and two variables, we say that a function $f(x_1, x_2, \ldots\ldots, x_n)$ has a maximum (or a minimum) at a point$(a_1, a_2, \ldots\ldots, a_n)$ if at every point in a certain neighbourhood of $(a_1, a_2, \ldots\ldots a_n)$ the function assumes a smaller value (or a larger value) than at the point itself; in either case we say that $(a_1, a_2, \ldots\ldots, a_n)$ is an extreme point and that $f(a_1, a_2, \ldots\ldots, a_n)$ is an extreme value of the function.

If $f(a_1, a_2, \ldots\ldots a_n)$ is an extreme value of the function $f(x_1, x_2, \ldots\ldots x_n)$, then it is also an extreme value of the function $f(x_1, a_2, \ldots\ldots, a_n)$ of one variable x_1 for $x_1=a_1$ and therefore the derivative $f_{x_1}(a_1, a_2, \ldots\ldots a_n)$, in case it exists, must be zero.

Thus we see that the **necessary conditions** *for $f(a_1, a_2, \ldots\ldots, a_n)$ to be an extreme value of the function $f(x_1, x_2, \ldots\ldots, x_n)$ are the equations*

$$f_{x_1}(a_1, a_2, a_3, \ldots\ldots, a_n)=0,$$

$$f_{x_2}(a_1, a_2, a_3, \ldots\ldots, a_n)=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f_{x_n}(a_1, a_2, a_3, \ldots\ldots, a_n)=0\ ;$$

incase the function possesses partial derivatives, in question, at the point $(a_1, a_2, \ldots\ldots, a_n)$.

To find the position of the extreme points of a function $f(x_1, x_2, \ldots\ldots, x_n)$, which possesses partial derivatives at every point of its domain of definition, we have to find the points $(x_1, x_2, \ldots, x_n)$ which satisfy the n equations, obtained by equation to zero the partial derivatives of the function. Since there are as many unknowns as there are equations we can calculate the positions of the extreme points by means of them. But a point obtained in this way may not necessarily be an extreme point ; further investigation is necessary to decide whether it is really an extreme point or not (See note below).

It proves useful to have a special name for the point at which the necessary conditions for the extreme position are satisfied, irrespective of whether it is really an extreme point or not. Thus we say *that a point* $(a_1, a_2, a_3, \ldots\ldots, a_n)$ *is a* **stationary point** *if all the first order partial derivatives of the function vanish at that point.*

The conditions for a stationary point may also be given an other more compact form. Thus if $(a_1, a_2, \ldots, a_n)$ is a stationary point then

$$df(a_1, a_2, \ldots\ldots, a_n)=0,$$

i.e., the differential of a function vanishes at a stationary point, no matter what values may be assigned to the differentials $dx_1, dx_2, \ldots dx_n$ of the independent variables. For, we have

$$df(a_1, a_2, \ldots, a_n)=f_{x_1}(a_1, a_2, \ldots, a_n)dx_1+f_{x_2}(a_1, a_2, \ldots, a_n)dx_2+\ldots=0$$

Conversely, if the differential df is 0 for arbitrary values of the differentials dx_1, dx_2, etc., of the independent voriables, then separately taking all but one of these differentials equal to zero, we can show that all the partial derivatives are zero.

Note.—It is easy to see that the results obtained in §133, concerning the extreme values of a function $f(x, y)$ of two variables may be restated in the following form :—

If at (a, b), *the first differential* $df=0$, *then*

(*i*) $f(a, b)$ *is a minimum or a maximum according as* d^2f *is a positive or a negative form* ; (*ii*) $f(a, b)$ *is not an extreme value if* d^2f *is indefinite and finally* (*iii*) *the case is doubtful if* d^2f *is a semi-definite form.*

Similar results hold true for a function of any number of variables also but the details of the proof will not be given here.

Thus in the case of a function $f(x, y, z)$ of three variables sufficient conditions for (a, b, c) to be an extreme value are that

$$df(a, b, c)=f_x dx+f_y dy+f_z dz=0$$

for arbitrary values of dx, dy, dz, *i. e.*,

$$f_x=f_y=f_z=0.$$

and (*ii*) $d^2f(a, b, c) = f_{x^2}(dx)^2 + f_{y^2}(dy)^2 + f_{z^2}(dz)^2 + 2f_{xy}\,dxdy + 2f_{yz}\,dydz + 2f_{zx}\,dxdz$

is definite, *i. e.*, assumes values of the same sign for abitrary values of dx, dy, dz.

Also if the conditions are satisfied, $f(a, b, c)$ is a maximum or minimum according as d^2f is a negative or a positive definite form.

It will thus be seen that in order to find out whether a stationary point is also an extreme point or not one has to examine the character of the second differential.

143. Stationary points under subsidiary conditions. The problem. *To find the stationary points of the function*

$$f(x_1, x_2, \ldots\ldots x_n;\ u_1, u_2, \ldots\ldots u_m) \tag{1}$$

of $(n+m)$ *variables which are connected by the* m *equations*

$$\phi_r(x_1, x_2, \ldots\ldots, x_n\ ;\ u_1, u_2; \ldots\ldots u_m)=0. \quad (r=1, 2, \ldots\ldots m) \tag{2}$$

If we assume that the system of equations (2) is such so as to determine the m variables $u_1, u_2, \ldots\ldots, u_m$ as functions of $x_1, x_2, \ldots, x_n$, then the functions f in (1) is essentially a function of n independent variables $x_1, x_2, \ldots\ldots, x_n$.

For a stationary point of this function, we must have $df=0$.

$$0=df=\frac{\partial f}{\partial x_1}dx_1+\ldots\ldots+\frac{\partial f}{\partial x_n}.dx_n+\frac{\partial f}{\partial u_1}du_1,+\ldots.+\frac{\partial f}{\partial u_m}du_m \quad (\S 130), \tag{3}$$

Again differentiating the system of equations (2), we obtain

$$\left.\begin{array}{l}\dfrac{\partial \phi_1}{\partial x_1}dx_1+\ldots\ldots+\dfrac{\partial \phi_1}{\partial x_n}dx_n+\dfrac{\partial \phi_1}{\partial u_1}du_1+\ldots\ldots+\dfrac{\partial \phi_1}{\partial u_m}du_m=0\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ \dfrac{\partial \phi_m}{\partial x_1}dx_1+\ldots\ldots+\dfrac{\partial \phi_m}{\partial x_n}dx_n+\dfrac{\partial \phi_m}{\partial u_1}du_1+\ldots\ldots+\dfrac{\partial \phi_m}{\partial u_m}dx_m=0.\end{array}\right\} \tag{4}$$

Solving the system (4) of m equations for the m differentials $du_1, du_2 \ldots\ldots du_m$ of the dependent variables in terms of the n differentials $dx_1, dx_2, \ldots\ldots, dx_n$ of the independent variables and substituting their values in (3) we will express df in terms of the differentials of the independent variables only. Since $df=0$, the co-efficients of each of these n differentials must separately vanish. These n equations together with the system of equations (2) constitute a system of $(m+n)$ equations for determining the $(m+n)$ co-ordinates of the stationary points.

Ex. Find the stationary points of the function xy where x, y are connected by the relation $x^2+y^2-a^2=0$.

Ex. Find the stationary points of the function $x^2y^2z^2$, where x, y, z are connected by the relation $x^2+y^2+z^2-a^2=0$.

144. Lagrange's method of multipliers. Lagrange has given a method of forming the system of equations for the determination of stationary points, which is often useful. It rests on the introduction of certain *undetermined multipliers*

$$\lambda_1, \lambda_2, \ldots\ldots, \lambda_m.$$

Multiplying the system of equations (4) of §143, by $\lambda_1, \lambda_2, \ldots\ldots \lambda_m$ and adding to the results to the equation (3), we obtain

$$0=df=\left(\frac{\partial f}{\partial x_1}+\Sigma\lambda_r\frac{\partial \phi_r}{\partial x_1}\right)dx_1+\ldots\ldots+\left(\frac{\partial f}{\partial u_1}+\Sigma\lambda_r\frac{\partial \phi_r}{\partial u_1}\right)du_1$$
$$+\ldots\ldots=0 \quad (5)$$

We now assume that the m multipliers $\lambda_1, \lambda_2, \ldots\ldots, \lambda_m$ have been so chosen that the m co-efficients of the differentials $du_1, du_2 \ldots du_m$ all vanish, *i. e.*,

$$\frac{\partial f}{\partial u_1}+\Sigma\lambda_r\frac{\partial \phi_r}{\partial u_1}=0, \ldots\ldots, \frac{\partial f}{\partial u_m}+\Sigma\lambda_r\frac{\partial \phi_r}{\partial u_m}=0 \quad (6)$$

Then (5) gives

$$0=df=\left(\frac{\partial f}{\partial x_1}+\Sigma\lambda_r\frac{\partial \phi_r}{\partial x_1}\right)dx_1+\ldots\ldots+\left(\frac{\partial f}{\partial x_n}+\Sigma\lambda_r\frac{\partial \phi_r}{\partial x_n}\right)dx_n$$

so that the differential df which vanishes is expressed in terms of the differentials of the independent variables only. Hence

$$\frac{\partial f}{\partial x_1}+\Sigma\lambda_r\frac{\partial \phi_r}{\partial x_1}=0, \ldots\ldots, \frac{\partial f}{\partial x_n}+\Sigma\lambda_r\frac{\partial \phi_r}{\partial x_n}=0 \quad (7)$$

The systems (3), (6) and (7) of $(m+m+n)$ *i. e.*, $(n+2m)$ equations suffice to determine the values of the m multipliers and of the $(m+n)$ co-ordinates of the stationary points of the function f.

An important note. We define a function

$$g=f+\lambda_1\phi_1+\lambda_2\phi_2+\ldots\ldots+\lambda_m\phi_m$$

and observe that, considering $x_1, x_2, \ldots\ldots x_n, u_1, u_2 \ldots u_m$ as independent variables,

$$\frac{\partial g}{\partial x_1}=0, \ldots\ldots, \frac{\partial g}{\partial x_n}=0, \frac{\partial g}{\partial u_1}=0, \ldots\ldots, \frac{\partial g}{\partial u_m}=0$$

are exactly the systems of equations (7) and (6).

In practice, therefore, the system of equations (6) and (7) may be conveniently obtained by first forming the function 'g' and then equating its first order partial derivatives to zero, considering **all** the variables as independent.

To determine whether a stationary point is really an extreme point or not it is necessary to consider the second order differential d^2F where F denotes the function f considered as a function of $x_1, x_2, \ldots, x_n$ only. In this connection it is generally found convenient to make use of the fact *that at a stationary point $d^2F=d^2g$, where the differential d^2g is calculated on the supposition that* **all** *the variables are independent.*

Let f and g considered as functions of $x_1, x_2, \ldots\ldots, x_n$ alone be denoted as F and G. Since, at the stationary points 'ϕs' vanish, we have

$$F(x_1, x_2, \ldots\ldots, x_n)=G(x_1, x_2, \ldots\ldots, x_n),$$

and $$d^2F=d^2G$$

But $d^2G = d^2g + \Sigma\left(\frac{\partial g}{\partial x_1}d^2x_1 + \ldots\ldots + \frac{\partial g}{\partial u_1}d^2u_1 + \ldots\right)$,(§130.1, P.212)

$= d^2g$

Hence the result.

Examples

Ex. 1. *If* $u = a^3x^2 + b^3y^2 + c^3z^2$ *where* $1/x + 1/y + 1/z = 1$, *show that a stationary value is given by* $ax = by = cz$ *and this gives a true maximum or a minimum if* $abc(a+b+c)$ *is positive.*

Let $$g(x,y,z) = a^3x^2 + b^3y^2 + c^3z^2 + \lambda(1/x + 1/y + 1/z - 1).$$

Equating to zero, the first order partial derivatives of the function g, we obtain

$$2a^3x - \lambda/x^2 = 0,\ 2b^3y - \lambda/y^2 = 0,\ 2c^3z - \lambda/z^2 = 0, \tag{i}$$

which give $$ax = by = cz. \tag{ii}$$

To determine whether the stationary value, in question, is an extreme value or not, we find d^2g. We have

$$d^2g = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)^2 g$$

$$= \left(2a^3 + \frac{2\lambda}{x^3}\right)\left(dx\right)^2 + \left(2b^3 + \frac{2\lambda}{y^3}\right)\left(dy\right)^2 + \left(2c^3 + \frac{2\lambda}{z^3}\right)\left(dz\right)^2$$

$$= 6[a^3(dx)^2 + b^3(dy)^2 + c^3(dz)^2], \text{ from } (i).$$

There being only two independent variables, we shall express d^2g in terms of dx and dy alone. From the relation $\Sigma 1/x = 1$, we have

$$\Sigma dx/x^2 = 0 \text{ or } \Sigma a^2dx = 0, \text{ from } (ii)$$

$$\therefore\ d^2g = \frac{6}{c}\left\{(c+a)a^3(dx)^2 + 2a^2b^2dxdy + (c+b)b^3(dy)^2\right\}$$

We have seen in §133.2, P. 215, that $Ah^2 + 2Bhk + Ck^2$ is definite if $B^2 - AC$ is negative. $\therefore$ d^2g is definite if

$$\frac{a^4b^4}{c^2} - \frac{(c+a)(c+b)a^3b^3}{c^2} = \frac{-(a+b+c)a^3b^3}{c} = \frac{-(a+b+c)(abc)^3}{c^4},$$

is negative, *i. e.*, $(a+b+c)abc$ is positive.

Ex. 2. *Find the rectangular parallelopiped of greatest volume that can be inscribed in the ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

The problem is to find the greatest value of $V = 8xyz$ where x, y, z are all positive and subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1. \tag{1}$$

Let $$g(x,y,z) = 8xyz + \lambda(x^2/a^2 + y^2/b^2 + z^2/c^2 - 1).$$

For extreme points, we have

$$g_x = 8yz + 2\lambda x/a^2 = 0, \tag{2}$$

$$g_y = 8zx + 2\lambda y/b^2 = 0, \tag{3}$$

$$g_z = 8xy + 2\lambda z/c^2 = 0. \tag{4}$$

Multiplying (2), (3), (4) by x, y, z respectively and adding, we get

$$24xyz + 2\lambda = 0, \text{ or } \lambda = -12xyz, \text{ using (1)}. \tag{5}$$

Thus we obtain, from (2) and (5), $x = a/\sqrt{3}$.

Similarly $y = b/\sqrt{3}$, $z = c/\sqrt{3}$.

Thus the stationary value of $V = 8abc/3\sqrt{3}$.

Again

$$d^2g = 2\lambda\Sigma(dx)^2/a^2 + 16\Sigma ydzdx = -(8abc/\sqrt{3})\Sigma(dx)^2/a^2 + (16/\sqrt{3})\Sigma bdzdx \tag{6}$$

Now, we have

$$xdx/a^2+ydy/b^2+zdz/c^2=0,$$

i. e.,
$$dx/a+dy/b+dz/c=0,$$

for the stationary point in question.

These give

$$2dxdy/ab=(dz)^2/c^2-(dx)^2/a^2-(dy)^2/b^2$$

and two similar results.

Substituting these values of $dxdy$, etc, in (6), we see that

$$d^2g=(-16abc/\sqrt{3})[\Sigma(dx)^2/a^2],$$

which is a definite negative form.

Hence the stationary value, in question, is a maximum.

Note. As in the preceding example, the question could also be completed by expressing d^2g in terms of dx and dy alone.

Ex. 3. *If* $\phi(a)=k\neq 0, \phi'(a)\neq 0$ and x, y, z *satisfy the relation*

$$\phi(x)\phi(y)\phi(z)=k^3,$$

prove that the function

$$f(x)+f(y)+f(z)$$

has a maximum when $x=y=z=a$ *provided that*

$$f'(a)\left[\frac{\phi''(a)}{\phi'(a)}-\frac{\phi'(a)}{\phi(a)}\right]>f''(a)$$

Let $g(x, y, z)=f(x)+f(y)+f(z)+\lambda[\phi(x)\phi(y)\phi(z)-k^3]$

For stationary points, we have

$$g_x=f'(x)+\lambda\phi'(x)\phi(y)\phi(z)=0$$
$$g_y=f'(x)+\lambda\phi(x)\phi'(y)\phi(z)=0$$
$$g_z=f'(z)+\lambda\phi(x)\phi(y)\phi'(z)=0.$$

For the function to be a minimum at (a, a, a), we must necessarily have

$$f'(a)+\lambda\phi'(a)\phi(a)\phi(a)=0,$$

i. e., $\lambda=-f'(a)/\phi'(a)\phi^2(a)$, for $\phi'(a)\neq 0$, $\phi(a)\neq 0$.

Now,
$$d^2g=\Sigma[f''(x)+\lambda\phi''(x)\phi(y)\phi(z)](dx)^2$$
$$+2\lambda\Sigma\phi'(x)\phi'(y)\phi(z)dxdy$$
$$=[f''(a)+\lambda k^2\phi''(a)]\Sigma(dx)^2$$
$$+2\lambda k[\phi'(a)]^2\Sigma\, dxdy, \text{ at } (a, a, a).$$

From the given equation of condition, we have

$$\Sigma\phi'(x)\phi(y)\phi(z)\, dx=0$$

so that for (a, a, a), we have $\Sigma\, dx=0$,

This gives

$$2\Sigma\, dxdy=-\Sigma(dx)^2.$$

$\therefore$ we have

$$d^2g=[f''(a)-f'(a)\phi''(a)/\phi'(a)]\Sigma(dx)^2+[f'(a)\phi'(a)/\phi(a)]\Sigma(dx)^2$$
$$=\left\{f''(a)-f'(a)\left[\frac{\phi''(a)}{\phi'(a)}-\frac{\phi'(a)}{\phi(a)}\right]\right\}\Sigma(dx)^2,$$

which under the given condition is a definite negative form. Hence the result.

4. Find the minimum value of $x^2+y^2+z^2$, when

(i) $x+y+z=3a$. (ii) $xy+yz+zx=3a^2$. (iii) $xyz=a^3$

5. Find the extreme values of xy when $x^2+xy+y^2=a^2$.

6. Find the shortest distance of the point (a, b, c) from the plane $lx+my+nz=0$.

7. Find the shortest distance between the lines

$$(x-x_1)/l_1=(y-y_1)/m_1=(z-z_1)/n_1\,;\ (x-x_2)/l_2=(y-y_2)/m_2=(z-z_2)/n_2$$

8. Find the point of the ellipse $5x^2-6xy+5y^2=4$ for which the tangent is at the greatest distance from the origin.

9. Which point of the sphere $x^2+y^2+z^2=1$ is at the maximum distance from the point (2, 1, 3).

10. Determine the maximum and minimum values of $u=(x+1)(y+1)(z+1)$ where x, y, z are connected by the relation $a^x b^y c^z = k$.

11. Given, x_0, y_0, z_0 are constants and x_1, y_1, z_1 are variables such that $\Sigma x_0 = \Sigma x_1 = 1$, $lx_1+my_1+nz_1=1$; find the minimum value of the function

$$a^2(y_0-y_1)(z_0-z_1)+b^2(z_0-z_1)(x_0-x_1)+c^2(x_0-x_1)(y_0-y_1),$$

where a, b, c and l, m, n are constants.

12. Find the shortest distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, if P_1 lies on the plane $x+y+z=2a$ and P_2 lies on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

13. Show that the lengths of the axes of the section of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ by the plane $lx+my+nz=0$ are the roots of the equation

$$l^2a^2/(r^2-a^2)+m^2b^2/(r^2-b^2)+n^2c^2/(r^2-c^2)=0.$$

14. If (x_1, y_1, z_1), (x_2, y_2, z_2) are two points on the conic

$$lx+my+nz=0=ax^2+by^2+cz^2-1$$

the distance between these points is stationary when

$$l^2/(1-ar^2)+m^2/(1-br^2)+n^2/(1-cr^2)=0.$$

15. Show that if

$$x_1^2+y_1^2+z_1^2=a^2,\ x_2^2+y_2^2+z_2^2=b^2,\ x_3^2+y_3^2+z_3^2=c^2,$$

then the maximum and minimum values of the determinant

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}$$

are respectively $+abc$ and $-abc$.

16. If x, y, z are subject to the condition $ax+by+cz=1$, show that, in general, $x^3+y^3+z^3-3xyz$ has the two stationary values 0 and $(a^3+b^3+c^3-3abc)^{-1}$ of which the first is a maximum or a minimum according as $a+b+c \gtrless 0$, but the second is not an extreme value. Discuss, in particular, the cases in which (*i*) $a+b+c=0$, (*ii*) $a=b=c$.

17. Obtain the stationary values of

$$x^3+y^3+z^3+3mxyz, \quad (m \neq 2)$$

when x, y, z are subject to the condition

$$x+y+z=1,$$

and show that the symmetrical stationary value is a maximum or minimum according as $m \gtrless 2$. Show also that the other stationary values are not extreme values.

Show that $x^3+y^3+z^3+6xyz$ has only one stationary value and no extreme value.

18. Establish Lagrange's method of undetermined multipliers for finding the stationary values of a function $f(x, y, z, \ldots\ldots)$ of n variables connected by $m(<n)$ independent equations $\phi_r(x, y, z, \ldots\ldots)=0$, $(r=1, 2, \ldots\ldots m)$, f and ϕ being supposed to have continuous derivatives for all values of the variables concerned.

The sum of 12 edges of a rectangular block is a; the sum of the areas of the 6 faces is $a^2/25$. Prove that, when the excess of the volume of the block over that of a cube whose edge is equal to least edge of the block is greatest, the least edge is $a/20$ and find the other edges.

19. If x, y and z are connected by the relation

$$f(x)+\phi(y)+\psi(z)=f(a)+\phi(b)+\psi(c),$$

show that the following conditions are sufficient for $x^2+y^2+z^2$ to have a maximum value at (a, b, c);

$$A\psi_1^2(c)+Cf_1^2(a) < 0 < AB\psi_1^2(c)+BCf_1^2(a)+CA\phi_1^2(b)$$

where

$$A=1-af_2(a)/f_1(a), B=1-b\phi_2(b)/\phi_1(b), C=1-c\psi_2(c)/\psi_1(c),$$

provided none of the derivatives $f_1(a)$, $\phi_1(b)$, $\psi_1(c)$ is zero and

$$a/f_1(a)=b/\phi_1(b)=c/\psi_1(c),$$

CHAPTER XI

PROPERTIES OF CONTINUOUS FUNCTIONS OF TWO VARIABLES

Definite Integrals as Functions of a Parameter

145. Theorem. *If $f(x, y)$ is continuous in a rectangle R $(a, b; c, d)$ and ϵ is any positive number, then there exists a division of R into a finite number of sub-rectangles such that*

$$|f(x_2, y_2)-f(x_1, y_1)| < \epsilon,$$

where (x_1, y_1) and (x_2, y_2) are points belonging to the same sub-rectangle.

We *assume* that the theorem is false in R. By drawing lines parallel to the co-ordinate axes, divide R into four sub-rectangles

$$[a, \tfrac{1}{2}(a+b) ; c, \tfrac{1}{2}(c+d)], [\tfrac{1}{2}(a+b), b ; c, \tfrac{1}{2}(c+d)]$$
$$[a, \tfrac{1}{2}(a+b) ; \tfrac{1}{2}(c+d), d], [\tfrac{1}{2}(a+b), b ; \tfrac{1}{2}(c+d), d].$$

Because of the assumption, the theorem must be false in atleast one of these sub-rectangles, say R_1, which we would rename as $(a_1, b_1 ; c_1, d_1)$. Subdividing R_1 in a similar manner and continuing the process indefinitely, we would obtain a sequence of rectangles

$$R_1, R_2, R_3, \ldots\ldots, R_n, \tag{1}$$

where R_n is the rectangle $(a_n, b_n ; c_n, d_n)$,

The theorem is false in every rectangle R_n. As in §50 2, p. 73, it can be shown that the sequences $\{ a_n \}$ and $\{ b_n \}$ approach a common limit ξ and $\{ c_n \}$ and $\{ d_n \}$ approach a common limit η. This point (ξ, η) belongs to every R_n.

Since $f(x, y)$ is continuous at (ξ, η), there exists a positive number δ such that

$$|f(x, y)-f(\xi, \eta)| < \tfrac{1}{2}\epsilon, \tag{2}$$

for every point (x, y) of R which lies in the rectangle

$$(\xi-\delta, \xi+\delta ; \eta-\delta, \eta+\delta) \tag{3}$$

If (x_1, y_1) and (x_2, y_2) be any two points of (3), we deduce from (2) that

$$|f(x_2, y_2)-f(x_1, y_1)| < \epsilon.$$

Now, since there exists a positive integer m such that a rectangle R_m of the sequence (1) lies wholly within the rectangle (3), we see that the theorem is true for R_m. Thus we arrive at a contradiction. Hence the theorem.

Cor. 1. *If $f(x, y)$ is continuous in R, then it is necessarily bounded in R.* We divide R into a finite number, say 'p', of sub-rectangles such that for every pair of points (x_1, y_1) and (x_2, y_2) of the same sub-rectangle

$$|f(x_2, y_2)-f(x_1, y_1)| < \epsilon,$$

so that if (α_r, β_r) be any *fixed* point of the rth sub-rectangle and (x, y) any *variable* point of the same, we have

$$f(\alpha_r, \beta_r)-\epsilon < f(x, y) < f(\alpha_r, \beta_r)+\epsilon. \tag{4}$$

From p inequalities, similar to (4), obtained for the p sub-rectangles we deduce as in (§50·3,P. 75) that $f(x, y)$ is bounded.

Cor. 2. *If $f(x, y)$ is continuous in R, then it must attain its bounds.*

The proof is exactly similar to the proof of the corresponding theorem for functions of a single variable.

Cor. 3. *If $f(x, y)$ is continuous in R, then it is possible to divide R into a finite number of sub-rectangles such that the oscillation of $f(x, y)$ in every sub-rectangle is less than a given positive number.*

Follows from the main theorem and the Cor. 2.

Cor. 4. Uniform Continuity. *If $f(x, y)$ is continuous in R, and ϵ is any given positive number, then there exists a positive number δ such that*

$$|f(x_2, y_2)-f(x_1, y_1)| < \epsilon,$$

when (x_1, y_1), (x_2, y_2) are any two points such that

$$|x_2-x_1| \leqslant \delta, \quad |y_2-y_1| \leqslant \delta.$$

We divide R into a finite number of sub-rectangles such that for every pair of points (x_1, y_1), (x_2, y_2) of the same subrectangle

$$|f(x_2, y_2)-f(x_1, y_1)| < \tfrac{1}{2}\epsilon.$$

It will now be shown that a positive number δ which is smaller than every side of every sub-rectangle is the requisite one. Now, a pair of points (x_1, y_1) and (x_2, y_2) such that

$$|x_2-x_1| \leqslant \delta, \quad |y_2-y_1| \leqslant \delta,$$

either belong to the same sub-rectangle or to two such sub-rectangles as have a side or a part of side in common. In the former case,

$$|f(x_2, y_2)-f(x_1, y_1)| < \tfrac{1}{2}\epsilon < \epsilon,$$

and in the latter, if (x, y) be a point of the common side, we have

$$|f(x_2, y_2)-f(x_1, y_1)| \leqslant |f(x_2, y_2)-f(x, y)| + |f(x, y)-f(x_1, y_1)| < \tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon=\epsilon.$$

Hence the result.

Cor. 5. *If (x, y) is continuous in R, then it must assume every value between its upper and lower bounds.*

Let M, m be the bounds and let

$$f(\alpha, \beta)=\text{M}, \quad f(\gamma, \delta)=m.$$

The points (α, β), (γ, δ) can be joined by any number of continuous curves lying inside R. Along such a curve $f(x, y)$ is a continuous function of only one variable and, therefore, assumes every value between M and m.

146. Definite integral as function of a parameter. Let $f(x, y)$ be a continuous function of two variables defined in $\text{R}(a, b\,;\, c, d)$. For a *fixed* value of y in (c, d), the function $f(x, y)$ of x is continuous and therefore the integral

$$\int_a^b f(x, y)\,dx$$

exists and defines a function of y ,say $\phi(y)$, in (c, d).

It is now proposed to investigate the nature of the function $\phi(y)$ in relation to continuity and differentiability.

146·1. Theorem. *The function $\phi(y)$ is continuous in (c, d).*

Let $y, y+\Delta y$ be any two points of (c, d). We have

$$\phi(y+\Delta y)-\phi(y) = \int_a^b [f(x, y+\Delta y)-f(x, y)]dx.$$

Let ϵ be any positive number.

Since $f(x, y)$ is continuous in R therefore there exists a positive number δ such that

$$|f(x_2, y_2)-f(x_1, y_1)| < \epsilon/(b-a), \text{ when } |x_2-x_1| \leqslant \delta, |y_2-y_1| \leqslant \delta$$

(Cor. 4, P. 237)

In particular we see that *when $|\Delta y| \leqslant \delta$, and x has any value,* we have

$$|f(x, y+\Delta y)-f(x, y)| < \epsilon/(b-a).$$

$$\therefore \quad |\phi(y+\Delta y)-\phi(y)| = \left|\int_a^b [f(x, y+\Delta y)-f(x, y)]dx\right|$$

$$\leqslant \int_a^b |f(x, y+\Delta y)-f(x, y)|\, dx$$

$$\leqslant [\epsilon/(b-a).](b-a)=\epsilon, \text{ when } |\Delta y| \leqslant \delta.$$

Hence $\phi(y)$ is a continuous function of y.

Ex. If $f(x, y)$ is continuous in $R(a, b\,; c, d)$ and $F(x)$ is a function of x which is bounded and integrable in (a, b) then $\int_a^b f(x, y)\,F(x)dx$ is a continuous function of y in (c, d).

146·2. Theorem. *If, in addition to the continuity of $f(x, y)$, $f_y(x, y)$ also exists and is continuous in $R(a, b\,; c, d)$, then $\phi(y)$ is derivable in (c, d) and*

$$\phi'(y) = \int_a^b \frac{\partial f(x, y)}{\partial y}\, dx.$$

$$i.e., \qquad \frac{d}{dy}\left\{\int_a^b f(x, y)\, dx\right\} = \int_a^b \frac{\partial f(x, y)}{\partial y}\, dx,$$

so that the inversion of the operations of differentiation and integration is valid.

We have

$$\phi(y+\Delta y)-\phi(y)= \int_a^b [f(x, y+\Delta y)-f(x, y)]dx.$$

By Lagrange's mean value theorem, we have

$$f(x, y+\Delta y)-f(x, y)=\Delta y f_y\ (x, y+\theta\Delta y)$$
$$=\Delta y[f_y(x, y+\theta\Delta y)-f_y(x, y)+f_y(x, y)]$$

$\therefore$ we obtain

$$\frac{\phi(y+\Delta y)-\phi(y)}{\Delta y}-\int_a^b f_y(x, y)dx=\int_a^b [f_y(x, y+\theta\Delta y)-f_y(x, y)]dx.$$

Let ϵ be any positive number. Because of the continuity of $f_y(x, y)$, there exists a positive number δ, such that when $|\Delta y| \leqslant \delta$ and x has any value, then

$$|f_y(x, y+\theta\delta y)-f_y(x, y)| < \epsilon/(b-a).$$

Thus there exists a positive number δ such that

$$\left|\frac{\phi(y+\Delta y)-\phi(y)}{\Delta y}-\int_a^b f_y(x, y)dx\right| < \epsilon, \text{ when } |\Delta y| \leqslant \delta.$$

Hence the result.

Ex. If $f(x, y)$ and $f_y(x, y)$ be continuous in R and $F(x)$ be bounded and integrable in (a, b), then $\int_a^b f(x, y)\, F(x)dx$ is derivable in (c, d) and its derivative is $\int_a^b f_y(x, y)\, F(x)dx$.

Note. In the investigation above, the limits (a, b) of integration were constants independent of y. The case where the limits are themselves functions of y is considered below.

146·3. *Let $f(x, y), f_y(x, y)$ be continuous in $R(a, b\ ;\ c, d)$; and let $g_1(y), g_2(y)$ be two functions of y derivable in (c, d) such that the points*

$$[g_1(y), y] \text{ and } [g_2(y), y]$$

belong to the rectangle R for every value of y in (c, d), then

$$\phi(y)=\int_{g_1(y)}^{g_2(y)} f(x, y)dx$$

is derivable in (c, d) and

$$\phi'(y)=\int_{g_1(y)}^{g_2(y)} f_y(x, y)dx-g_1'(y)f[g_1(y), y]+g_2'(y)f[g_2(y), y].$$

We have

$$\phi(y+\Delta y)-\phi(y)=$$
$$\int_{g_1(y)}^{g_2(y)} [f(x, y+\Delta y)-f(x, y)]dx-\int_{g_1(y)}^{g_1(y+\Delta y)} f(x, y+\Delta y)dx\ \int_{g_2(y)}^{g_2(y+\Delta y)} f(x, y+\Delta y)dx$$

Applying the result of Cor. 2 to §88, p. 123 to each of the last two integrals and dividing by Δy, we get

$$\frac{\phi(y+\Delta y)-\phi(y)}{\Delta y}=\int_{g_1(y)}^{g_2(y)} f_y(x, y+\theta\Delta y)dx=$$

$$\frac{g_1(y+\Delta y)-g_1(y)}{\Delta y} f(\xi, y+\Delta y) + \frac{g_2(y+\Delta y)-g_2(y)}{\Delta y} f(\eta, y+\Delta y),$$

where ξ, η lie between $g_1(y)$, $g_1(y+\Delta y)$ and $g_2(y)$, $g_2(y+\Delta y)$ respectively. As in the previous section it can be shown that when $\Delta y \to 0$ then

$$\int_{g_1(y)}^{g_2(y)} f_y(x, y+\theta\Delta y)dx \to \int_{g_1(y)}^{g_2(y)} f_y(x, y)dx.$$

Thus on taking limits, when $\Delta y \to 0$, we deduce, that

$$\phi'(y)=\int_{g_1(y)}^{g_2(y)} f_y(x, y)dx - g_1'(y)f[g_1(y), y]+g_2'(y) f[g_2(y), y].$$

146·4. Inversion of the order of Integration. *If $f(x, y)$ be continuous in $R(a, b\ ;\ c, d)$, then*

$$\int_c^d \left\{ \int_a^b f(x, y)dx \right\} dy = \int_a^b \left\{ \int_c^d f(x, y)dy \right\} dx,$$

i.e., the two repeated integrals are equal.

Because of the continuity of $f(x, y)$, the integrals of $f(x, y)$ over (a, b) with respect to x and over (c, d) with respect to y exist (§85 1, P. 115) and are continuous functions of y and x respectively (§146·1, P. 238) and therefore both the repeated integrals exist.

We consider two functions of t defined as follows :—

$$\phi(t)=\int_c^t \left\{ \int_c^b f(x, y)dx \right\} dy\ ;\ \psi(t)=\int_a^b \left\{ \int_c^t f(x, y)dy \right\} dx,$$

so that $$\phi(0)=\psi(0)=0.$$

Now,

$$\phi'(t)=\int_a^b f(x, t)\, dx \qquad (\S 89{\cdot}2,\ P.\ 127)$$

Also $$\psi'(t)=\int_a^b \left\{ \frac{\partial}{\partial t}\int_c^t f(x, y)dy \right\} dx \qquad (\S\ 146\cdot2\ \text{P. } 238)$$

$$=\int_a^b f(x, t)dx. \qquad (\S\ 89\cdot2,\ \text{P. } 127).$$

Since, $\phi'(t)=\psi'(t)$, therefore $\phi(t)$ and $\psi(t)$ differ by a constant. Also, since $\phi(0)=\psi(0)$, we see that for every vaule of t, $\phi(t)=\psi(t)$. Putting $t=b$, we obtain the required equality.

Ex. If $f(x, y)$ is continuous in $R(a, b\,;\,c, d)$ and

$$F(x, y)=\int_c^y \left\{ \int_a^x f(x, y)dx \right\} dy,$$

then show that

$$F_{xy}=F_{yx}=f(x, y).$$

Deduce that

$$\int_a^b \left\{ \int_c^d f(x, y)dy \right\} dx = F(a, c)+F(b, d)-F(a, d)-F(b, c)$$

$$=\int_c^d \left\{ \int_a^b f(x, y)dx \right\} dy.$$

Note. The theorems above enable us to evaluate certain definite integrals without the knowledge of the corresponding primitives. A few examples of such evaluation are given below.

Examples

1. Since $\displaystyle\int_0^b \frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{b}{a}$, it follows on differentiating w. r. to a under the integral sign that

$$-2a\int_0^b \frac{dx}{(x^2+a^2)^2}=-\frac{1}{a^2}\tan^{-1}\frac{b}{a}-\frac{b}{a(b^2+a^2)}$$

i.e., $$\int_0^b \frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^3}\tan^{-1}\frac{b}{a}+\frac{b}{2a^2(b^2+a^2)},$$

if b is independent of a.

Let $b\to\infty$. We see that

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2}=\frac{\pi}{4a^3}. \qquad (a>0).$$

2. Since $\int_0^a \frac{dx}{x^2+a^2}=\frac{\pi}{4a}$, $(a>0)$, on applying the result of §, 146·3, P. 239 it follows that

$$\int_0^a -\frac{2a}{(x^2+a^2)^2}\,dx+\frac{1}{2a^2}=-\frac{\pi}{4a^2}\text{ i.e., }\int_0^a \frac{dx}{(x^2+a^2)^2}=\frac{\pi+2}{8a^3}.$$

3. From

$$\int_0^\pi \frac{dx}{a+b\cos x}, \qquad a>0,\ |b|<a,$$

deduce that

$$\int_0^\pi \frac{dx}{(a+b\cos x)^2}=\frac{\pi a}{(a^2-b^2)^{\frac{3}{2}}}\text{ and }\int_0^\pi \frac{\cos x\,dx}{(a+b\cos x)^2}=-\frac{\pi b}{(a^2-b^2)^{\frac{3}{2}}}.$$

4. If

$$f(x)=\int_0^{\frac{1}{2}\pi} \log(1-x^2\cos^2\theta)d\theta,$$

prove that $f(x)$ is finite if $x^2\leqslant 1$, and that if $x^2<1$,

$$\frac{d}{dx}f(x)=\int_0^{\frac{1}{2}\pi}\frac{d}{dx}\left\{\log(1-x^2\cos^2\theta)\right\}d\theta,$$

and hence show that

$$f(x)=\pi\log(1+\sqrt{1-x^2})-\pi\log 2.$$

Show also that the result is true even when $x^2=1$,

5. If $|a|<1$, show that

$$\int_0^\pi \frac{\log(1+a\cos x)}{\cos x}\,dx=\pi\sin^{-1}a.$$

6. If $|a|\leqslant 1$, show that

$$\int_0^\pi \log(1+a\cos x)dx=\pi\log[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1-a^2)}]$$

7. Show that

$$\int_0^a \frac{\log(1+ax)}{1+x^2}\,dx=\tfrac{1}{2}\log(1+a^2)\tan^{-1}a.$$

Hence, deduce that

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx=\frac{\pi\log 2}{8}.$$

8. Find $\frac{d\phi}{dx}$, if $\phi(x)=\int_0^{x^2}\sqrt{x}\,dx.$

9. Verify that

$$y=\frac{1}{k}\int_0^x f(t)\sin k(x-t)dt,$$

satisfies the differential equation

$$\frac{d^2y}{dx^2}+k^2y=f(x),$$

where k is a constant.

147. Uniform convergence of improper integrals.

Definitions 1. Range Infinite. *The convergent improper integral*

$$\phi(y)=\int_a^\infty f(x\ y)dx \tag{1}$$

is said to converge uniformly w. r. to y in the interval $c\leqslant y\leqslant d$, if corresponding to any positive number η there exists a positive number X_1 **which does not depend on y**, *such that*

$$\left|\int_a^X f(x,y)dx-\phi(y)\right|=\left|\int_X^\infty f(x,y)dx\right|<\eta,$$

for $X>X_1$.

2. The integrand unbounded. Let $f(x,y)$ tend to ∞ as $x\to a$. *The convergent improper integral*

$$\phi(y)=\int_a^b f(x,y)\,dx \tag{2}$$

is said to converge uniformly in the interval $c\leqslant y\leqslant d$, if corresponding to any positive number η, there exists a positive number δ, independent of y, such that

$$\left|\int_{a+\epsilon}^b f(x,y)dx-\phi(y)\right|<\eta,$$

for $0<\epsilon\leqslant\delta$.

148. The tests for uniform convergence. The following are the straightforward analogues of the tests for the uniform convergence of the series and may be proved in a similar manner.

148·1. General test. *The necessary and sufficient condition for the uniform convergence of the improper integral* (1) *is that corresponding to any positive number η there exists a positive number X,* **independent of** *y, such that*

$$\left|\int_{X_1}^{X_2} f(x,y)dx\right|<\eta,\ \textit{when}\ X_1,\ X_2>X.$$

148·2. Weierstrass's M Test. *If a function $M(x)$ of x is positive such that*

$$\int_a^\infty M(x)dx$$

converges and if $\quad |f(x, y)| \leqslant M(x)$
for $c \leqslant y \leqslant d$ and every value of x belonging to the interval under consideration then the integral, (1) *of* §147 *is uniformly convergent.*

Note. Similar theorems hold for the uniform convergence of

$$\int_a^b f(x, y)dx.$$

Ex. Since $\left| e^{-x^2} \cos yx \right| \leqslant e^{-x^2}$ for every value of y, and

$$\int_0^\infty e^{-x^2} dx$$

is convergent, therefore

$$\int_0^\infty e^{-x^2} \cos yx \, dx$$

is uniformly convergent in y in the interval $[-\infty, \infty]$.

Ex. Since $\left| \cos yx/\sqrt{(1-x^2)} \right| \leqslant 1/\sqrt{(1-x^2)}$

and
$$\int_{-1}^{+1} \frac{1}{\sqrt{(1-x^2)}} dx$$

is convergent, therefore

$$\int_{-1}^{+1} \frac{\cos xy}{\sqrt{(1-x^4)}} dx$$

is uniformly convergent in y in the interval $[-\infty, \infty]$.

Ex. The uniformity of convergence of

$$\int_0^\infty \frac{x \sin xy}{1+x^2} dx$$

cannot be established by M-test.

For $x > 1$, $x/(1+x^2)$ is monotonically decreasing and tends to 0 as $x \to \infty$ so that employing Bonnett's form of second mean value theorem we have

$$\left| \int_{X_1}^{X_2} \frac{x \sin xy}{1+x^2} dx \right| = \left| \frac{X_1}{1+X_1^2} \int_{X_1}^{\xi} \sin xy \, dx \right| < \frac{2X_1}{k(1+X_1^2)},$$

if $y > k$, where k is any fixed positive number.

Thus we see that the integral is uniformly convergent for $y \geqslant k > 0$. (§148·1).

Ex. **Establish the uniform convergence of the following integrals :—**

(i) $\int_0^\infty \frac{y\,dx}{x^2+y^2}$, $(0 < c \leqslant y \leqslant d)$. (ii) $\int_0^\infty e^{-yx} \sin x\,dx, (y \geqslant k > 0)$

(iii) $\int_0^\infty e^{-xy} \frac{\sin x}{x}\,dx$, $(y \geqslant 0)$ (iv) $\int_0^\infty e^{(-x^2-y^2/x^2)}\,dx$

(v) $\int_0^\infty e^{-xy}\,dx$, $(y \geqslant k > 0)$.

149·1. Theorems. *Uniformly convergent improper integral of a continuous function is itself a continuous function.*

Consider the improper integral (1) of § 147, P. 243. We choose a number X such that

$$\left| \int_X^\infty f(x,y)dx \right| = \left| \int_a^X f(x,y)dx - \phi(y) \right| < \frac{\epsilon}{3}.$$

We have

$$| \phi(y+\Delta y) - \phi(y) | \leqslant \left| \int_a^X [f(x, y+\Delta y) - f(x,y)]dx \right| + \frac{2\epsilon}{3}.$$

The range (a, X) being finite and $f(x, y)$ being continuous, we can choose a positive number δ, as in § 146·1, P. 238, so that the finite integral on the right is less than $\epsilon/3$ when $| \Delta y | \leqslant \delta$. Thus $\phi(y)$ is continuous.

149·2. *If $f(x, y)$ be a continuous function of the point (x, y) when $c \leqslant y \leqslant d$ and $x \geqslant a$ and the integral*

$$\phi(y) = \int_a^\infty f(x,y)dx$$

is uniformly convergent, then $\phi(y)$ can be integrated under the integral sign, i. e.,

$$\int_c^d \left\{ \int_a^\infty f(x,y)dx \right\} dy = \int_c^d \phi(y)dy = \int_a^\infty \left\{ \int_c^d f(x,y)dy \right\} dx.$$

We choose a number X_1 such that

$$\left| \int_X^\infty f(x,y)dx \right| < \epsilon/(d-c), \text{ when } X \geqslant X_1$$

We have

$$\int_c^d \phi(y)dy = \int_c^d \left\{ \int_a^X f(x, y)dx \right\} dy + \int_c^d \left\{ \int_X^\infty f(x, y)dx \right\} dy$$

In the first integral on the right, the order can be interchanged. We, therefore, have

$$\left| \int_c^d \phi(y)dy - \int_a^X \left\{ \int_c^d f(x, y)dy \right\} dx \right| \leqslant \epsilon, \text{ when } X \geqslant X_1,$$

Hence the result.

149·3. Theorem. *If, in addition to the conditions of the preceding theorem, the integral* $\int_a^\infty f_y(x, y)dx$ *also converges uniformly in y in* (c, d), *then*

$$\phi'(y) = \int_a^\infty f_y(x, y)dx.$$

Let

$$\psi(y) = \int_a^\infty f_y(x, y)dx.$$

As proved above the order of integration can be changed so that we have

$$\int_c^y \psi(y)dy = \int_c^y dy \int_a^\infty f_y(x, y)dx = \int_a^\infty dx \int_c^y f_y(x, y)dy$$

$$= \int_a^\infty [f(x, y) - f(x, c)]dy = \phi(y) - \phi(c)$$

Differentiating, we get $\psi(y) = \phi'(y)$, as was to be proved.

Note. Results similar to those above hold also when the range of integration is finite but the integrand has a point of finite discontinuity.

150. Evaluation of some improper definite integrals.

1. *Evaluate*

$$f(\alpha, \beta) = \int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x} dx, \text{ where } \alpha > 0,$$

and deduce that

$$\int_0^\infty \frac{\sin \beta x}{x} dx = \begin{cases} +\frac{1}{2}\pi, \text{ if } \beta > 0, \\ 0, \text{ if } \beta = 0, \\ -\frac{1}{2}\pi, \text{ if } \beta < 0. \end{cases}$$

When $x>0$, we have $|e^{-\alpha x}(\sin \beta x)/x| \leqslant e^{-\alpha x}/x$ and the integral of $(e^{-\alpha x}/x)$ is convergent at ∞ if $\alpha>0$. Thus we see that the integral is uniformly convergent with respect to β as parameter varying in $[-\infty, \infty]$; α being any fixed positive number.

Again, the derivative of the integrand with respect to β is

$$e^{-\alpha x} \cos \beta x.$$

For a fixed $\alpha>0$, we have

$$| e^{-\alpha x} \cos \beta x | \leqslant e^{-\alpha x},$$

which is independent of β.

Since, when $\alpha>0$, the integral of $e^{-\alpha x}$ converges over the range $(0, \infty]$, therefore, for a fixed positive α, the integral

$$\int_0^\infty e^{-\alpha x} \cos \beta x \, dx$$

is uniformly convergent w. r. to β varying in $[-\infty, \infty]$.

Thus the differentiation under the integral sign is valid and we have

$$f_\beta(\alpha, \beta) = \int_0^\infty e^{-\alpha x} \cos \beta x \, dx = \left| e^{-\alpha x} \frac{\beta \sin \beta x - \alpha \cos \beta x}{\alpha^2+\beta^2} \right|_0^\infty,$$

$$= \alpha/(\alpha^2+\beta^2).$$

$\therefore$ $f(\alpha, \beta)=\tan^{-1}(\beta/\alpha)+C$, where C is a constant.

To determine C we have

$$0=f(\alpha, 0)=0+C, \text{ i. e., } C=0$$

It will now be shown that $f(\alpha, \beta)$ is a continuous function of α also (for $\alpha \geqslant 0$ and for a fixed β). We have

$$\left| \int_{X_1}^{X_2} e^{-\alpha x} \frac{\sin \beta x}{x} dx \right| = \left| e^{-\alpha X_1} \int_{X_1}^{\xi} \frac{\sin \beta x}{x} dx \right| \leqslant \left| \int_{X_1}^{\xi} \frac{\sin \beta x}{x} dx \right|,$$

Now, since

$$\int_0^\infty \frac{\sin \beta x}{x} dx$$

is known to be convergent, we deduce that the integral is uniformly convergent with α as parameter ; α being $\geqslant 0$.

$$\therefore \quad f(0, \beta)= \text{lt } f(\alpha, \beta)=\tfrac{1}{2}\pi, \text{ when } \alpha \to (0+0). \quad (\beta>0)$$

or

$$\int_0^\infty \frac{\sin \beta x}{x} dx = \tfrac{1}{2}\pi. \qquad (\beta>0)$$

The other results, now, at once follow.

Ex. *Evaluate*

$$f(\alpha) = \int_0^\infty e^{-x^2} \cos \alpha x \, dx.$$

Since $\quad | xe^{-x^2} \sin \alpha x | \leqslant xe^{-x^2},$
we see that the differentiation under integral sign is justified.

$$\therefore \quad f'(\alpha) = -\int_0^\infty xe^{-x^2} \sin \alpha x \, dx.$$

Integrating by parts, we have

$$f'(\alpha) = \left| \tfrac{1}{2} e^{-x^2} \sin \alpha x \right|_0^\infty - \frac{\alpha}{2} \int_0^\infty e^{-x^2} \cos \alpha x \, dx = -\frac{\alpha}{2} f(\alpha)$$

$$\therefore \quad \log f(\alpha) = -\tfrac{1}{4}\alpha^2 + C_1 \quad \text{or } f(\alpha) = Ce^{-\frac{1}{4}\alpha^2}$$

Now, $\quad f(0) = \displaystyle\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ (See Ch. XII)

$$\therefore \quad C = f(0) = \sqrt{\pi}/2$$

$$\therefore \quad f(\alpha) = \tfrac{1}{2}\sqrt{\pi}\, e^{-\frac{1}{4}\alpha^2}.$$

Ex. *Evaluate*

$$f(y) = \int_0^\infty \frac{\cos xy}{1+x^2} dx. \qquad \ldots \quad (i)$$

It is easy to see that differentiation and integration under integral sign are valid. We, therefore, have

$$f'(y) = -\int_0^\infty \frac{x \sin xy}{1+x^2} dx = -\int_0^\infty \frac{\sin xy}{x} dx + \int_0^\infty \frac{\sin xy}{x(1+x^2)} dx,$$

Again

$$\int_0^y f(y) dy = \int_0^y dy \int_0^\infty \frac{\cos xy}{1+x^2} dx = \int_0^\infty dx \int_0^y \frac{\cos xy}{1+x^2} dy = \int_0^\infty \frac{\sin xy}{x(1+x^2)} dx$$

$$\therefore \quad f'(y) = -\frac{\pi}{2} + \int_0^y f(y) dy. \qquad y>0) \qquad (ii)$$

Since $f(y)$ is continuous, we have, on differentiation,

$$f''(y) = f(y).$$

$\therefore \quad f(y) = Ae^{y} + Be^{-y}$, where A, B are constants.

Since $f(0)$ is easily seen to be $\pi/2$, we have, on making $y \to 0$,

$$\tfrac{1}{2}\pi = A + B.$$

Also, since from (*ii*), $f'(0) = -\pi/2$, we see that

$$-\tfrac{1}{2}\pi = A - B.$$

Thus $A=0$, and $B=\frac{1}{2}\pi$ so that we have, when $y>0$,

$$\int_0^\infty \frac{\cos xy}{1+x^2}\,dx = \tfrac{1}{2}\pi\, e^{-y} \text{ and } \int_0^\infty \frac{\sin xy}{x(1+x^2)}\,dx = \tfrac{1}{2}\pi(1-e^{-y}).$$

The necessary modifications can easily be made when y is negative.

Ex. Show, by differentiating under integral sign, that

$$\int_0^\infty e^{-(x^2-a^2/x^2)}\,dx = \tfrac{1}{2}\sqrt{\pi}\, e^{-2|a|}.$$

Ex. Establish the right to integrate under integral sign

$$\int_0^\infty e^{-xy}\cos mx\,dx$$

and deduce that

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\cos mx\,dx = \tfrac{1}{2}\log\frac{b^2+m^2}{a^2+m^2} \qquad a,\ b>0.$$

Ex. By considering the identity

$$\frac{1}{y} = \int_0^\infty e^{-xy}\,dx, \qquad y>0,$$

deduce that

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \log\frac{b}{a}, \qquad a>0,\ b>0.$$

Ex. Show that

$$\int_0^\infty \frac{\operatorname{sech} ax - \operatorname{sech} bx}{x}\,dx = \log\frac{b}{a} \qquad (a>0;\ b>0)$$

[Change the order of integration in

$$\int_a^b dy \int_0^\infty \operatorname{sech} xy \tanh xy\,dx\Big].$$

Ex. Under certain conditions show that

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = f(0)\log\frac{b}{a}.$$

CHAPTER XII

LINE INTEGRALS. DOUBLE INTEGRALS

151. The concept of a plane curve. Let $x=\phi(t)$, $y=\psi(t)$ be two functions of t defined in an interval (α, β). Then the aggregate of points (x, y) obtained by giving different values to t is called a curve. The curve is said to be closed if the points corresponding to the end points α, β of the interval of t coincide, *i.e.*, if

$$\phi(\alpha)=\phi(\beta)\ ;\ \psi(\alpha)=\psi(\beta).$$

If $f(x)$ be a function defined in an interval (a, b), then the aggregate of points (x, y) where $y=f(x)$ is also a curve as we may see on setting

$$x=t,\ y=f(t).$$

Similarly $x=\psi(y)$ is a curve.

152. Line integral. Let

$$x=\phi(t),\ y=\psi(t)$$

be a curve C, where $\phi(t)$ and $\psi(t)$ are functions of t defined in an interval (α, β).

Let $f(x, y)$ be a function of x and y defined in a region containing the curve C.

To define the line integral, $\int_C f(x, y)dx$.

Let $\quad D(\alpha=t_0<t_1<t_2\ldots\ldots<t_{r-1}<t_r<\ldots\ldots<t_{n-1}<t_n=\beta)$ be any division of (α, β).

Let $\quad x_r=\phi(t_r),\ y_r=\psi(t_r).$

Let ξ_r be any value of t belonging to the interval (t_{r-1}, t_r).

We form the sum

$$S=\Sigma f[\phi(\xi_r),\ \psi(\xi_r)](x_r-x_{r-1}). \tag{1}$$

If, as the norm of D tends to zero, this sum, S, tends to a finite limit which is independent of the choice of the points ξ_r, then we denote this limit by the symbol

$$\int_C f(x, y)dx$$

and call it a *line integral* of the function $f(x, y)$ along the curve C.

152·1. A sufficient condition for the existence of the line integral. *Assuming* that $f(x, y)$, $\phi(t)$ and $\psi(t)$ are continuous and $\phi(t)$ possesses a continuous derivative $\phi'(t)$, we now show that this limit does exist.

There exists a point η_r of (t_{r-1}, t_r) such that

$$(x_r-x_{r-1})=\phi(t_r)-\phi(t_{r-1})=(t_r-t_{r-1})\phi'(\eta_r)=\phi'(\eta_r)\delta_r$$

We have

$$S=\Sigma f[\phi(\xi_r),\ \psi(\xi_r)](x_r-x_{r-1})$$
$$=\Sigma f[\phi(\xi_r),\ \psi(\xi_r)]\phi'(\xi_r)\delta_r+\Sigma f[\phi(\xi_r),\ \psi(\xi_r)][\phi'(\eta_r)-\phi'(\xi_r)]\delta_r$$
$$=S_1 \qquad\qquad\qquad\qquad +S_2$$

Since $\phi'(t)$ and $f[\phi(t),\ \psi(t)]$ are continuous, it follows that the sum $S_1 \to a$ finite limit, *viz.*,

$$\int_{\alpha}^{\beta} f[\phi(t),\ \psi(t)]\phi'(t)dt,$$

as the norm of $D \to 0$.

Since $f[\phi(t),\ \psi(t)]$ is continuous, it is bounded. Let A be a positive number such that

$$|\ f[\phi(t),\ \psi(t)]\ | \leqslant A, \text{ for } t \text{ in } (\alpha,\ \beta).$$

Since $\phi'(t)$ is continuous, there exists a positive number δ such that for every division D of norm less than or equal to δ, the oscillatory sum of $\phi'(t)$ is less than the arbitrarily small positive number ϵ/A.

$$\therefore\ |\ S_2\ | \leqslant A\Sigma\ |\ \phi'(\eta_r)-\phi'(\xi_r)\ |\ \delta_r \leqslant A\Sigma(M_r-m_r)\delta_r < A(\epsilon/A)=\epsilon.$$

where M_r, m_r are the bounds of $\phi'(t)$ in $\delta_r \equiv (t_{r-1},\ t_r)$.

Thus $S_2 \to 0$ as the norm of $D \to 0$.

Hence we see that the line integral, in question, does exist and further we have the equality

$$\int_C f(x,y)\ dx = \int_{\alpha}^{\beta} f[\phi(t),\ \psi(t)]\phi'(t)dt,$$

where we have an ordinary integral on the right.

Note. We may similarly define and examine the existence of the line integral

$$\int_C f(x,y)dy.$$

Note. It is easy to see that the line integral

$$\int_C f(x,y)dy$$

along the curve $C, y=\phi(x),\ a \leqslant x \leqslant b$, is equal to the ordinary integral

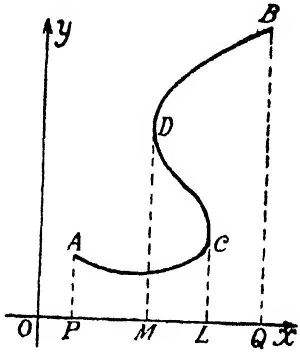
$$\int_a^b f[x,\ \phi(x)]\phi'(x)dx.$$

Note. The ordinary integral is a special case of a line integral, where we take an interval of the x or y axis as the path of integration.

152·2. *It can be easily proved that*

(*i*) *line integrals are additive for arcs, i.e.,*

$$\int_{AB} f(x, y)dx = \int_{AC} f(x, y)dx + \int_{CD} f(x, y)dx + \int_{DB} f(x, y)dx$$



and (*ii*) $$\int_{AC} f(x, y)dx = -\int_{CA} f(x, y)dx$$

Examples

1. Evaluate

$$\int_C (x^2+y^2)dx \text{ and } \int_C (x^2+y^2)dy$$

where C is the arc of the Parabola $y^2=4ax$ between $(0, 0)$ and $(a, 2a)$.

2. Show that

$$\int_C [(x-y)^3dx+(x-y)^3dy]=3\pi a^4,$$

taken along the Circle $x^2+y^2=a^2$ in the counter clockwise sense.

3. Evaluate

$$\int_C \frac{dx}{x+y},$$

where C is the curve $x=at^2, y=2at, \quad 0 \leqslant t \leqslant 2$.

4 Show that the value of the line integral

$$\int (xy^2\,dy - x^2y\,dx)$$

taken in the counter-clockwise sense along the Cardioide $r=a(1+\cos\theta)$ is $35a^4\pi/16$.

5. Find the value of

$$\int [(x+y^2)dx+(x^2-y)dy]$$

taken in the clockwise sense along the closed curve C formed by $y^3=x^2$ and $y=x$ between $(0, 0)$ and $(1, 1)$.

6. Find the value of

$$\int (x^2ydx+y^2xdy)$$

taken in the clockwise sense along the hexagons whose vertices are

(*i*) $(\pm 3a, 0), (\pm 2a, \pm\sqrt{3}a)$. (*ii*) $(0, \pm 3a), (\pm\sqrt{3}a, \pm 2a)$.

153. The area of a plane region.

The area of a rectangle $R(a, b\,;\,c, d)$. As suggested by intuitive considerations we *define* the area of a rectangle $R(a, b\,;\,c, d)$ to be the number $(b-a)(d-c)$.

The area of any bounded plane region E. Since E is bounded, there exists a rectangle R which completely encloses E. We divide R into sub-rectangles by drawing parallels to the sides of R. Let s_D denote the sum of the areas of those sub-rectangles which consist entirely of points of E and let S_D denote the sum of the areas of the sub-rectangles which have atleast one point in common with E. Clearly $s_D \leqslant S_D$. To each mode of division D of R will correspond a pair of sums S_D and s_D. Clearly the aggregates of these sums are both bounded.

The upper bound of the sums s_D will be called the *inner area* of E and the lower bound of the sums S_D will be called the *outer area* of E.

The region E will be said to possess an area if the inner and outer areas are equal and the common value will be called the area of E.

In §160, P. 261 will be obtained a sufficient condition for a region to possess an area.

154. Integrability of a bounded function over a rectangle. Let $f(x, y)$ be a bounded function of x, y defined in a rectangle $R(a, b\,;\,c, d)$.

Let $\quad D\,(a=x_0<x_1<x_2\ldots\ldots<x_{r-1}<x_r<\ldots\ldots<x_n=b),$

and $\quad D'(c=y_0<y_1<y_2\ldots\ldots<y_{s-1}<y_s<\ldots\ldots<y_m=d),$

be any divisions of the intervals (a, b) and (c, d) respectively. These divisions of the intervals give rise to a division of the rectangle R into mn sub-rectangles $(x_{r-1}, x_r\,;\,y_{s-1}, y_s)$ where r, s take up all positive integral values from 1 to n and 1 to m respectively. We will denote the rectangle $(x_{r-1}, x_r\,;\,y_{s-1}, y_s)$ as well as its area $(x_r-x_{r-1})(y_s-y_{s-1})$ by the same symbol, w_{rs}. Let M_{rs}, m_{rs} denote the bounds of $f(x, y)$ in w_{rs} Consider the two sums

$$S=\sum_{s=1}^{s=m}\sum_{r=1}^{r=n} M_{rs}\,w_{rs},\quad s=\sum_{s=1}^{s=m}\sum_{r=1}^{r=n} m_{rs}\,w_{rs}$$

It is easy to see that for every mode of division of R into sub-rectangles, we have

$$m(b-a)(d-c)\leqslant s\leqslant S\leqslant M(b-a)(d-c),$$

where M, m are the bounds of $f(x, y)$ in $(a, b\,;\,c, d)$.

Thus the two aggregates of the sum S and s are bounded.

Def. The lower bound of the aggregate of the upper sums S is called the *upper integral* and the upper bound of the aggregate of the lower sums s is called the *lower integral* of $f(x, y)$ over R and are denoted by the symbols

$$\mathrm{U}=\overline{\int\int_{\mathrm{R}}} f(x, y)dxdy, \quad \mathrm{L}=\underline{\int\int_{\mathrm{R}}} f(x, y)dxdy$$

respectively.

In case these two upper and lower integrals are equal then $f(x, y)$ is said to be integrable and the common value, which is denoted by the symbol

$$\mathrm{I}=\int\int_{\mathrm{R}} f(x, y)dxdy,$$

is said to be the *double integral* of $f(x, y)$ over R.

Note. In the same manner, we may define the *triple* integral of a function of $f(x, y, z)$ over a rectangular parallelopiped $\mathrm{R}(a, b\,;\, c, d\,;\, e, f)$.

Note. Norm of a division of a rectangle. By the norm of a division of a rectangle into sub-rectangles $(x_{r-1}, x_r\,;\, y_{s-1}, y_s)$ is meant the greatest member of the set of numbers (x_r-x_{r-1}) and (y_s-y_{s-1}) ; $r=1, 2, \ldots\ldots, n$; $s=1, 2, \ldots\ldots, m$.

155. Darboux's theorem. To every pre-assigned positive number ϵ, there corresponds a positive number δ, such that for every division whose norm is $\leqslant \delta$,

$$\mathrm{S}<\mathrm{U}+\epsilon\,;\quad s>\mathrm{L}-\epsilon.$$

The proof is exactly similar to the corresponding proof for functions of a single variable.

Cor. *The upper integral $\geqslant$ the lower integral.*

156. Condition for Integrability.

156·1. First form. *The necessary and sufficient condition for the integrability of a bounded function $f(x, y)$ over a rectangle* R *is that to every positive number ϵ, there corresponds a positive number δ, such that for every division of R whose norm $\leqslant \delta$, the oscillatory sum* $(\mathrm{S}-s)$ *is less than ϵ.*

The condition is necessary. The bounded function $f(x, y)$ being integrable,

$$\mathrm{U}=\mathrm{L}=\mathrm{I}.$$

If ϵ be any positive number, then, by Darboux's theorem, there exists a positive number δ such that for every division of norm $\leqslant \delta$,

$$\mathrm{S}<\mathrm{U}+\tfrac{1}{2}\epsilon=\mathrm{I}+\tfrac{1}{2}\epsilon\;;\; s>\mathrm{L}-\tfrac{1}{2}\epsilon=\mathrm{I}-\tfrac{1}{2}\epsilon$$

i. e.,

$$\mathrm{I}-\tfrac{1}{2}\epsilon<s\leqslant\mathrm{S}<\mathrm{I}+\tfrac{1}{2}\epsilon$$

or

$$\mathrm{S}-s<\epsilon.$$

The condition is sufficient. There exists a division such that if S, s be the corresponding upper and lower sums, then

$$\mathrm{S}-s=(\mathrm{S}-\mathrm{U})+(\mathrm{U}-\mathrm{L})+(\mathrm{L}-s)<\epsilon$$

Since each one of the three numbers $(S-U),(U-L),(L-s)$, is non-negative, it follows that

$$0 \leqslant (U-L) < \epsilon.$$

As ϵ is an arbitrary positive number, we deduce that

$$U-L=0, \text{ i. e., } U=L.$$

Therefore the function is integrable.

156·2. Second form. *The necessary and sufficient condition for the integrability of a bounded function $f(x, y)$ is that to every pre-assigned positive number ϵ, there corresponds a division for which the oscillatory sum is less than ϵ.*

The proof which is quite similar to that of the first form is left to the reader.

157. Particular classes of bounded integrable functions.

157·1. Every continuous function is integrable. Let $f(x, y)$ be continuous in a rectangle $R(a, b ; c, d)$. Let ϵ be any positive number. There exists (Cor 3 to §145, P. 237) a division such that the oscillation of $f(x, y)$ in every sub-rectangle of the division $< \epsilon/(b-a)(d-c)$. For such a division, the oscillatory sum

$$S-s=\Sigma\,\Sigma(M_{rs}-m_{rs})w_{rs} \leqslant [\epsilon/(b-a)(d-c)].\ \Sigma\Sigma w_{rs}=\epsilon.$$

Hence $f(x, y)$ is integrable in the rectangle R.

157·2. *If a function $f(x, y)$ is bounded in $R(a, b ; c, d)$ and is such that its points of discontinuity can be enclosed in a finite number of rectangles the sum of whose areas is less than a given positive number, then $f(x, y)$ is integrable in R.*

Let ϵ be any positive number. We enclose the points of discontinuity in rectangles the sum of whose areas is $<\epsilon/2(M-m)$. The part of the oscillatory sum $(S-s)$, arising from these rectangles or from the sub-rectangles into which they may be further divided is $<\epsilon/2$.

On producing the sides of the rectangles which enclose the points of discontinuity, we obtain a division of R into sub-rectangles. These sub-rectangles are of two types ; (*i*) those which include points of discontinuity and (*ii*) those which do not include any point of discontinuity. The sub-rectangles of the latter type can be further sub-divided such that the part of $(S-s)$ arising from them is $<\epsilon/2$.

Thus we have a division of R such that the corresponding oscillatory sum is less than the given positive number ϵ.

Hence the result. (§156·2)

Cor. *If a function $f(x, y)$ is bounded in $R(a, b ; c, d)$ and its points of discontinuity lie on a finite number of curves of the form $y=\phi(x)$, $x=\psi(y)$, etc., where $\phi(x), \psi(y)$ etc., are continuous, then $f(x, y)$ is integrable in R.*

Let p be the number of the curves in question. Let ϵ be any positive number.

Since $\phi(x)$ is continuous in (a, b), there exists a division

$$D(a=x_0<x_1<x_2\ldots\ldots<x_{r-1}<x_r\ldots\ldots<x_n=b) \text{ of } (a, b)$$

such that the oscillation of $\phi(x)$ in every sub-interval is $<\epsilon/p(b-a)$.

The points of $y=\phi(x)$ which correspond to the values of x of the sub-interval (x_{r-1}, x_r) of the division D clearly belong to the rectangle $(x_{r-1}, x_r \,;\, m_r, M_r)$ of area $(x_r-x_{r-1})(M_r-m_r)$; M_r, m_r being the bounds of $\phi(x)$ in (x_{r-1}, x_r). Thus all the points of $y=\phi(x)$ have been enclosed in a system of rectangles of total area $\Sigma(x_r-x_{r-1})(M_r-m_r)$ which is clearly less than ϵ/p.

Applying the same process to the other curves we see that they can all be enclosed in a finite number of rectangles of total area less any given positive nnmber ϵ. Hence the result. (§157·2)

158. The calculation of a double integral. Equivalence of a double with a repeated integral. Theorem. *If the double integral*

$$\iint_R f(x, y)dxdy$$

exists where R is the rectangle $(a, b \,;\, c, d)$ and if also

$$\int_a^b f(x, y)dx$$

exists for each value of y in (c, d), then the repeated integral

$$\int_c^d \left\{ \int_a^b f(x, y)\, dx \right\} dy$$

exists and is equal to the double integral.

[**Observation.** The proof depends upon a simple consideration, *viz.*, that if D be any division of (a, b) and K_r, k_r be a rough upper and a rough lower bound of a function $\phi(x)$ in any sub-interval δ_r of the division, then

$$\overline{\int_a^b} \phi(x)dx \leqslant S_D \,\Sigma \leqslant K_r\delta_r \text{ and } \underline{\int_a^b} \phi(x)dx \geqslant s_D \geqslant \Sigma k_r\delta_r. \Big]$$

Let U and L denote the upper and lower integrals of $f(x, y)$ over R. Let ϵ be any positive number.

There exists a division of R into sub-rectangles $(x_{r-1}, x_r \,;\, y_{s-1}, y_s)$ such that

$$\Sigma\Sigma M_{rs}\,(x_r-x_{r-1})(y_s-y_{s-1})<U+\epsilon. \quad (\S155, \text{ P. } 254) \qquad (1)$$

Since for every fixed value of y in (y_{s-1}, y_s), M_{rs} is a rough upper bound of $f(x, y)$ in (x_{r-1}, x_r), therefore

$$\overline{\int_a^b} f(x, y)dx < \sum_{r=1}^{r=n} M_{rs}(x_r-x_{r-1}), \text{ when } y_{s-1} \leqslant y \leqslant y_s. \qquad (2)$$

Since, from (2), $\sum_{r=1}^{r=n} M_{rs}\,(x_r - x_{r-1})$

is a rough upper bound of the function

$$\overline{\int_a^b} f(x, y)\,dx, \text{ of } y \text{ in } (y_{s-1}, y_s),$$

we have, by an application of the same reasoning,

$$\overline{\int_c^d}\left\{\overline{\int_a^b} f(x, y)dx\right\}dy \leqslant \sum_{s=1}^{m}\sum_{r=1}^{n} M_{rs}(x_r - x_{r-1})(y_s - y_{s-1}) < U + \epsilon, \tag{3}$$

As ϵ is an arbitrary positive number, and by hypothesis

$$\overline{\int_a^b} f(x, y)dx = \int_a^b f(x, y)dx = \underline{\int_a^b} f(x, y)dx,$$

therefore we have

$$\overline{\int_c^d}\left\{\int_a^b f(x, y)dx\right\}dy = \overline{\int_c^d}\left\{\overline{\int_a^b} f(x, y)dx\right\}dy \leqslant U = \overline{\iint_R} f(x, y)dxdy \tag{4}$$

We can similarly prove that

$$\underline{\int_c^d}\left\{\int_a^b f(x, y)dy\right\}dy = \underline{\int_c^d}\left\{\int_a^b f(x, y)dx\right\}dy \geqslant L = \underline{\iint_R} f(x, y)dxdy \tag{5}$$

Therefore

$$\underline{\iint_R} f(x, y)dxdy \leqslant \underline{\int_c^d}\left\{\int_a^b f(x, y)dx\right\}dy$$

$$\leqslant \overline{\int_c^d}\left\{\int_a^b f(x, y)dx\right\}dy \leqslant \overline{\iint_R} f(x, y)dxdy. \tag{6}$$

But by hypothesis

$$\overline{\iint_R} f(x, y)dxdy = \underline{\iint_R} f(x, y)dxdy \tag{7}$$

Now, from (5), (6), (7), we have

$$\iint_R f(x, y)dxdy = \int_c^d \left\{ \int_a^b f(x, y)dx \right\} dy$$

$$= \int_{\underline{c}}^d \left\{ \int_a^b f(x, y)dx \right\} dy = \int_c^d \left\{ \int_a^b f(x, y)dx \right\} dy.$$

Cor. *If a double integral exists then the two repeated integrals cannot exist without being equal.*

Ex. A function $f(x, y)$, is defined in R(0, 1 ; 0, 1) as follows:—
$f(x, y)=\frac{1}{2}$, when y is rational, $f(x, y)=x^2$, when y is irrational.

Show that $\int_0^1 \left\{ \int_0^1 f(x, y)dx \right\} dy$ exists and is equal to $\frac{1}{3}$, but the double integral and the second repeated integral do not exist.

Note. If $f(x, y, z)$ be continuous in R(a, b ; c, d ; e, f) so that the triple integral exists, then

$$\iiint_R f(x, y, z)dxdydz = \int_a^b dx \int_c^d dy \int_e^f f(x, y, z)dz$$

Examples

1. Evaluate the following integrals :—

(*i*) $\iint xy(x^2+y^2)dxdy$ over R(0, a ; 0, b).

(*ii*) $\iint ye^{xy}\, dxdy$ over R(0, a ; 0, b)

(*iii*) $\iint \frac{x-y}{x+y}\, dxdy$ over R(0, 1 ; 0, 1).

[The integrand is bounded and (0, 0) is its only point of discontinuity in the square (0, 1 ; 0, 1)].

(*iv*) $\iint \frac{dxdy}{\sqrt{[c^2+(x-y)^2]}}$ over the square (0, a ; 0, a)

(*v*) $\iiint xyz\, dxdydz$ over the rectangular parallelopiped
(−1, 1 ; −1, 1 ; −1, 1).

2. *Prove that*

$$\int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3}\, dy \right\} dx = \tfrac{1}{2} \neq -\tfrac{1}{2} = \int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3}\, dx \right\} dy,$$

We suppose (x, y) to be any point of the square (0, 1 ; 0, 1). For any fixed value of $x \neq 0$, the function $(x-y)/(x+y)^3$ is a bounded function of y and if $x=0$ then $y=0$ is a point of infinite discontinuity.

$$\text{If } x \neq 0,\ \phi(x) = \int_0^1 \frac{x-y}{(x+y)^3}\,dy = \int_0^1 \left\{ \frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right\} dy = \frac{1}{(1+x)^2}$$

and $\phi(0)$ does not exist.

$$\text{Again, } \int_0^1 \phi(x)dx = \underset{\epsilon \to 0}{\text{lt}} \int_\epsilon^1 \frac{1}{(1+x)^2}\,dx = \underset{\epsilon \to 0}{\text{lt}} \left(-\frac{1}{2} + \frac{1}{1+\epsilon} \right) = \tfrac{1}{2}.$$

$$\therefore \quad \int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3}\,dy \right\} dx = \tfrac{1}{2}.$$

$$\text{If } y \neq 0,\ \psi(y) = \int_0^1 \frac{x-y}{(x+y)^3}\,dx = \int_0^1 \left\{ \frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right\} dx = -\frac{1}{(1+y)^2}$$

and $\psi(0)$ does not exist.

$$\therefore \text{ as before, } \quad \int_0^1 \psi(y)dy = -\tfrac{1}{2}.$$

The function $(x-y)/(x+y)^3$ is not bounded in the neighbourhood of the origin $(0, 0)$ and therefore the double integral does not exist.

3. Show that

$$\int_0^1 \left[\int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dy \right] dx \neq \int_0^1 \left[\int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dx \right] dy.$$

4. Show that

$$\iint_R \phi(x)\ \psi(y)\,dxdy = \left[\int_a^b \phi(x)dx \right] \left[\int_c^d \psi(y)dy \right]$$

where R is the rectangle $(a, b\ ;\ c, d)$.

159. Integrability and integral of a bounded function f(x, y) over any finite region E. Since the given region E is finite, there must exist a rectangle R which completely encloses E. We define a function $F(x, y)$ over R as follows :—

$$F(x, y) = \begin{cases} f(x, y), & \text{at all points of E,} \\ 0, & \text{elsewhere.} \end{cases}$$

Def. *A function $f(x, y)$ is said to be integrable over E, if $F(x, y)$ is integrable over the rectangle R and*

$$\iint_E f(x, y)\,dxdy = \iint_R F(x,y)\,dxdy.$$

159·1. An important case of integrability. *If $f(x, y)$ is continuous in a region E which is bounded by a finite number of continuous curves of the form $y=\phi(x)$, $x=\psi(y)$, etc., then*

$$\iint_E f(x, y)\, dxdy$$

exists.

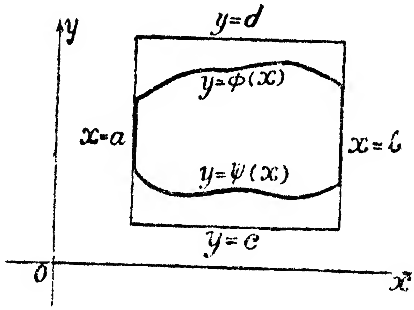
The result follows from the fact that the only possible points of discontinuity of $F(x, y)$, defined as in §159 above, are the points of the curves $y=\phi(x)$, etc., (Cor. to §157 2, p. 255).

159·2. Calculation of double integrals. *If $f(x, y)$ is continuous in a region E which is enclosed by the curves*

$$y=\phi(x),\ y=\psi(x)\ ;\ x=a,\ x=b\ ;$$

where $\phi(x)$, $\psi(x)$ are continuous and $\phi(x) \geqslant \psi(x)$ in (a, b), then

$$\iint_E f(x, y) dxdy = \int_a^b \left\{ \int_{\psi(x)}^{\phi(x)} f(x, y) dy \right\} dx.$$



Let $R(a, b\ ;\ c, d)$ enclose the region E and let $F(x, y)$ be defined over R as in §159. We have

$$\iint_E f(x, y) dxdy = \iint_R F(x, y)\ dxdy = \int_a^b \left\{ \int_c^d f(x, y) dy \right\} dx$$

$$= \int_a^b \left\{ \int_c^{\psi(x)} F dy + \int_{\psi(x)}^{\phi(x)} F dy + \int_{\phi(x)}^{d} F dy \right\} dx = \int_a^b \left\{ \int_{\psi(x)}^{\phi(x)} F dy \right\} dx$$

each of the remaining two integrals being equal to zero.

Note. If a function $f(x, y)$ is continuous in a region E which is bounded by the curves $x=\phi(y)$, $x=\psi(y)$; $y=c$, $y=d$, where $\phi(y)$, $\psi(y)$ are continuous and $\psi(y) \leqslant \phi(y)$, then

$$\iint_E f(x, y) dxdy = \int_c^d \left[\int_{\psi(y)}^{\phi(y)} f(x, y) dx \right] dy.$$

160. Area of a region. *A sufficient condition for a region E to possess an area is that it is bounded by a finite number of continuous curves of the form $y=\phi(x)$, $x=\psi(y)$, etc., and the area is given by the double integral*

$$\iint_E dxdy.$$

We enclose E in a rectangle R and define a function $F(x, y)$ in R as follows :--

$$F(x, y)=1 \text{ at points of } E \text{ and } F(x, y)=0 \text{ elsewhere.}$$

Corresponding to any division D of R into sub-rectangles, the upper sum S_D of $F(x, y)$ is the sum of the areas of those sub-rectangles which contain atleast one point of E and the lower sum s_D of $F(x, y)$ is the sum of the areas of those sub-rectangles which consist entirely of points of E. From this we conclude that the outer and inner areas of E are the upper and lower integrals of $F(x, y)$ over R. Since these two integrals are equal, we see that E possesses an area and its area is given by

$$\iint_R F(x, y)dxdy$$

But, by def.,

$$\iint_E dxdy=\iint_E 1.dxdy=\iint_R F(x, y)\, dxdy$$

Hence the result.

Examples

1. The double integral

$$\iint f(x, y)dxdy$$

where the field of integration is the circle $x^2+y^2=a^2$, is equivalent to the repeated integral

$$\int_{-a}^{+a} dx \int_{-\sqrt{(a^2-x^2)}}^{+\sqrt{(a^2-x^2)}} f(x, y)dy$$

2. The triple integral

$$\iiint f(x, y, z)dxdydz,$$

where the field of integration is the sphere $x^2+y^2+z^2=a^2$ is equivalent to the repeated integral

$$\int_{-a}^{+a} dx \int_{-\sqrt{(a^2-x^2)}}^{+\sqrt{(a^2-x^2)}} dy \int_{-\sqrt{(a^2-x^2-y^2)}}^{+\sqrt{(a^2-x^2-y^2)}} f(x, y, z)dz.$$

3. *Evaluate*

$$\int\int\int (x+y+z+1)^2 \, dxdydz$$

throughout the region defined by

$$x \geqslant 0, y \geqslant 0, z \geqslant 0, x+y+z \leqslant 1.$$

The given integral is equal to the triple integral

$$\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} (x+y+z+1)^2 \, dz = \tfrac{31}{60}.$$

4. Express, as a repeated integral, the double integral

$$\int\int f(x, y) dxdy$$

taken over the quadrilateral bounded by the lines

$$x+y=0,\ x-y=0,\ 2x-y=1,\ 2x-3y+5=0,$$

taken in order.

5. Evaluate $\int\int x^2y^2 \, dxdy$ over the region defined by

$$x \geqslant 0, y \geqslant 0, (x^2+y^2) \leqslant 1.$$

6. Show that the value of

$$\int\int \sqrt{(4y-x^2)} \, dxdy,$$

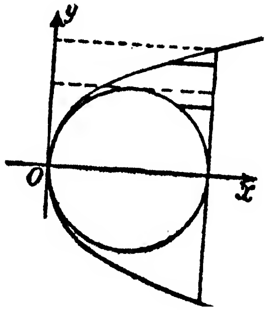
taken over the interior of the circle $x^2+y^2=2y$ is $\pi+\frac{8}{3}$.

7. Express the integral

$$\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} V(x, y) \, dx \, dy$$

by changing its order of integration.

Ans.



$$\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} V \, dxdy = \int_0^{a} \int_{y^2/2a}^{a-\sqrt{(a^2-y^2)}} V \, dydx$$

$$+ \int_0^{a} \int_{a+\sqrt{(a^2-y^2)}}^{2a} V \, dydx + \int_a^{2a} \int_{y^2/2a}^{2a} V \, dydx$$

8. Change the order of integration in

$$\int_0^{2a} \int_{x^2/4a}^{3a-x} \phi(x, y) dxdy.$$

9. Show that

$$\int_a^b \int_{a^2/x}^{x} F \, dx \, dy = \int_{a^2/b}^{a} \int_{a^2/y}^{b} F \, dx \, dy + \int_a^b \int_y^b F \, dx \, dy.$$

10. In the integral

$$\int_2^4 \int_{4/x}^{(20-4x)/(8-x)} (y-4) dy \, dx,$$

change the order of integration, and evaluate the integral.

11. By changing the order of integration, prove that

$$\int_0^{2a}\int_0^{\sqrt{(2ax-x^2)}} \frac{(x^2+y^2)x\,\phi'(y)\,dxdy}{\sqrt{[4a^2x^2-(x^2+y^2)^2]}}=\pi a^2[\phi(a)-\phi(0)]$$

12. Change the order of integration in the integral

$$\int_0^1 dx\int_0^{\sqrt{(1-x^2)}}\frac{dy}{(1+e^y)\sqrt{(1-x^2-y^2)}}$$

and hence evaluate it.

13. Prove that

$$\int_0^1 dx\int_x^{1/x}\frac{y\,dy}{(1+xy)^2(1+y^2)}=\tfrac{1}{4}(\pi-1)$$

14. Evaluate $\int\int(x^2+y^2)dxdy$ over the region bounded by $xy=1$, $y=0$, $y=x$, $x=2$.

15. Evaluate $\int\int x^2\,dxdy$ over the area bounded by $x^2-y^2=1$, $x^2+y^2=4$ which contains the origin.

16. Show that

$$\int\int\int\frac{dxdydz}{(x+y+z+1)^3}=\frac{1}{16}\log\frac{256}{e^5}$$

taken throughout the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=1$.

17. Show that

$$\int\int\int(lx+my+nz)^2\,dxdydz=\tfrac{4}{15}\,\pi(l^2+m^2+n^2),$$

taken throughout the sphere $x^2+y^2+z^2=1$.

18. Prove that the value of

$$\int\int\int\frac{xyz}{\sqrt{(x^2+y^2+z^2)}}\,dxdydz$$

taken through the positive octant of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ is

$$a^2b^2c^2(bc+ca+ab)/15(b+c)(c+a)(a+b).$$

161. Def. A domain E will be said to be *Quadratic* with respect to y-axis, if it is bounded by the curves of the form

$$y=\phi(x),\ y=\psi(x)\ ;\ x=a,\ x=b,$$

where $\phi(x)$, $\psi(x)$ are continuous and $\phi(x)\geqslant\psi(x)$ in (a, b).

Thus a domain which is quadratic with respect to y-axis is such that a line parallel to y-axis and lying between $x=a$, $x=b$ meets the boundary of E in just two points.

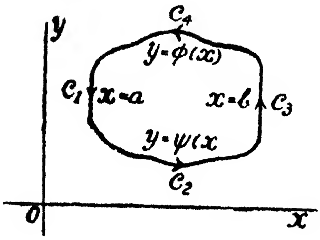
Similarly we may have regions which are quadratic with respect to x-axis.

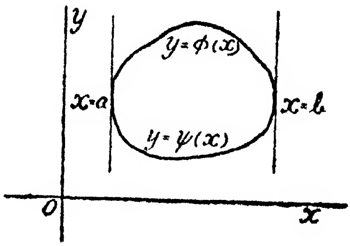
Imp **162. Green's Theorem. Connection between double and line integrals.** Let $f(x, y)$ and $f_1(x, y)$ be continuous in any region E which is quadratic with respect to y-axis, and let C denote the contour of E.

Let the region E be bounded by the curves $x=a$, $x=b$; $y=\psi(x)$, $y=\phi(x)$. We have

$$\iint_E f_y(x, y)\, dx\, dy = \int_a^b \left\{ \int_{\psi(x)}^{\phi(x)} f_y(x, y)dy \right\} dx$$

$$= \int_a^b f[x, \phi(x)]dx - \int_a^b f[x, \psi(x)]dx$$





Let C_1, C_2, C_3, C_4 denote the four parts of C taken in the positive sense, *i.e.*, in such a way that the interior of the region lies to the left as the contour is described in the counter-clockwise. We have

$$\int_C f(x, y)dx = \int_{C_1} f(x, y)dx + \int_{C_2} f(x, y)dx + \int_{C_3} f(x, y)dx + \int_{C_4} f(x, y)dx.$$

But

$$\int_{C_1} f(x, y)dx = 0 = \int_{C_3} f(x, y)dx,$$

$$\int_{C_2} f(x, y)dx = \int_a^b f[x, \psi(x)]dx,$$

and

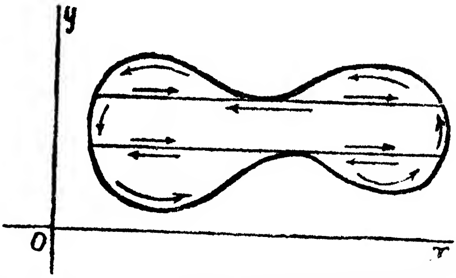
$$\int_{C_4} f(x, y)\, dx = \int_b^a f[x, \phi(x)]dx = -\int_a^b f[x, \phi(x)]dx.$$

$$\therefore \qquad \iint_D f_y(x, y)dx\, dy = -\int_C f(x, y)dx$$

We, now, *generalise* the above result.

Let a region E be such as can be divided into a finite number of sub-regions E_1, $E_2, \ldots,$ E_n each of which is quadratic with respect to **y-axis.**

The contour C of the region E and the contours C_1, C_2, $C_3, \ldots C_n$ of E_1, E , $E_3, \ldots E_n$ are to be so described that the corresponding region constantly lie on the left hand side. The theorem holds for each separate region, and on addition, the parts of the line integrals along the connecting lines cancel one another since each of these is described twice, once in each direction, and we arrive at the theorem for the whole region.



We have thus proved that *if $f(x, y)$ and $f_y(x, y)$ are continuous in a region E which can be split up into a finite number of regions quadratic with respect to y-axis, and C is the contour of E, described in the positive sense, then*

$$\int\int_E f_y(x, y)dx\, dy = -\int_C f(x, y)dx. \qquad (i)$$

It may similarly be proved that *if $g(x, y)$ and $g_x(x, y)$ are continuous in a region E which can be split up into a finite number of regions quadratic with respect to x-axis and C is the contour of E, then*

$$\int\int_E g_x(x, y)dx\, dy = \int_C g(x, y)dy. \qquad (ii)$$

Finally, on subtracting (*i*) from (*ii*), we see that

If we suppose that $f(x, y)$, $g(x, y)$, $f_y(x, y)$ and $g_x(x, y)$ are continuous in a domain E which can be split up into a finite number of regions quadratic with respect to either axis, then

$$\int_C [f(x, y)dx + g(x, y)dy] = \int_E [g_x(x, y) - f_y(x, y)]dxdy,$$

where the integral on the left is a line integral round the contour C of the region taken in such a way that the interior of the region remains on the left as the boundary is described.

This may be written as

$$\int_C (fdx + gdy) = \int_E \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right]dxdy$$

A particular case. Taking $f(x, y) = y$, we see from (*i*) that the area of a domain which can be split up into a finite number of domains quadratic with respect to Y-axis $= -\int_C ydx$.

Similarly putting $g(x, y) = x$, we see from (*ii*) that the area of a domain which can be split up into a finite number of domains quadratic with respect to X-axis $= \int_C xdy$.

Thus the area of a region E, which can be divided into a finite number of regions quadratic w. r. to either axis, is given by

$$\frac{1}{2}\int_{C}(xdy-ydx)$$

where the contour C of E is described in the positive sense.

Ex. Verify the Green's theorem by evaluating in two ways the following line integrals :—

(*i*) $$\int (x^2y\,dx+xy^2dy),$$
taken along the closed path formed by $y=x$ and $x^2=y^3$ from $(0, 0)$ to $(1, 1)$.

(*ii*) $$\int [(x^3+y^2)dx+(x^2+y^3)dy],$$
taken along the boundary of the pentagon whose vertices are $(0, 0)$, $(1, 0)$, $(2, 1)$, $(1, 2)$, $(0, 1)$.

163. Double integral as a limit. Let $f(x, y)$ be continuous in a region E. Let the region E be divided into sub-regions E_1, $E_2, \ldots, E_n$ with areas $A_1, A_2, \ldots, A_n$. Let (ξ_r, η_r) be any point of the sub-region E_r. Form the sum

$$\Sigma f(\xi_r, \eta_r)A_r. \tag{1}$$

By the diameter of a sub-region E_r will be meant the upper bound of the aggregate of the distances between pairs of points on its boundary and the greatest of the diameters of all the sub-regions will be called the norm of the division.

It can now be proved that *as the norm* $\delta\to 0$, *the limit of the sum* (1) *is the integral of* $f(x, y)$ *over* E, *i.e.*,

$$\underset{\delta\to 0}{lt}\ \Sigma f(\xi_r, \eta_r)A_r=\iint_{E} f(x, y)dxdy.$$

The details of the proof will not be given here.

Cor *If* $f(x, y)$ *be continuous in a region E with area A, then there exists a point* (ξ, η) *of E such that*

$$\iint_{E} f(x, y)dxdy=Af(\xi, \eta).$$

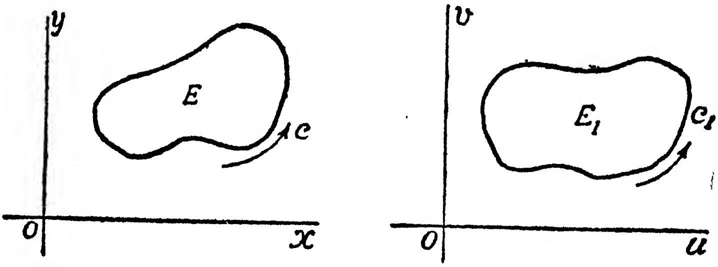
164. Change of variables in a double integral.

Lemma. Let

$$x=\phi(u, v),\ y=\psi(u, v) \tag{1}$$

be two functions of u, v defined in a certain region E_1 of the uv plane bounded by a curve C_1. We suppose that the two functions possess continuous first order partial derivatives at all points of E_1 and C_1. Further, we suppose that the equations (1) transform the region E_1 bounded by C_1 into a region E of the xy plane bounded by a curve C in such a way that a *one-to-one* correspondence exists between the two regions and their contours. Finally, we suppose that the Jacobian $\partial(x, y)/\partial(u, v)$ does not change sign at any point of E_1, though it may vanish at certain points of C_1.

As the point (u, v) describes the contour C_1 in the positive sense then the point (x, y) may describe C in the positive or else in the negative sense without ever changing the direction of motion. The transformation will be said to be *direct* or *inverse* respectively in the two cases.



We will now obtain a formula connecting the areas A and A_1 of the regions E, E_1.

We have

$$A=\int_{C} x\,dy,$$

taken along C in the positive sense.

Changing the variables (x, y) to (u, v) in this line integral, we obtain

$$A=\pm\int_{C_1}\phi(u, v)\left(\frac{\partial\psi}{\partial u}\cdot du+\frac{\partial\psi}{\partial v}\,dv\right),$$

where the new integral is to be taken along the positive sense of C_1 and the sign is $+$ or $-$ according as the transformation is direct or inverse

Applying Green's theorem, we have

$$\int_{C_1}(\phi\psi_u\,du+\phi\psi_v\,dv)=\int\int_{E_1}\left(\frac{\partial(\phi\psi_v)}{du}-\frac{\partial(\phi\psi_u)}{dv}\right)dudv$$

$$\overset{*}{=}\int\int_{E_1}\frac{\partial(\phi, \psi)}{\partial(u, v)}\,dudv=A_1\left[\frac{\partial(\phi, \psi)}{\partial(u, v)}\right]_{\xi,\eta}, \quad \text{(Cor. to §163)}$$

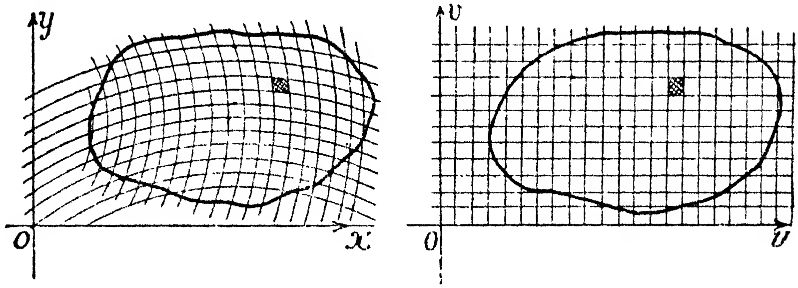
where (ξ, η) is a point of E_1.

$$\therefore\ A=\pm A_1\left[\frac{\partial(\phi, \psi)}{\partial(u, v)}\right]_{\xi,\eta}=\pm A_1\,[J]_{\xi,\eta}=A_1\,|J|_{\xi,\eta}.$$

Since A, A_1 are essentially positive, we see that the sign $+$ or $-$ should be taken according as J is positive or negative. This shows that *the transformation is direct or inverse according as the Jacobian J is positive or negative*

*It is assumed that $\psi_{uv}=\psi_{vu}$.

Main theorem Let $f(x, y)$ be continuous in the region E. We divide the region E_1 by lines parallel to the u-axis and v-axis. This



division of E_1 gives rises to a curvilinear division of the region E. Let E'_{rs} be any sub-region of E_1 and E_{rs} the corresponding sub-region of E and let w'_{rs}, w_{rs} be their areas.

By the lemma, we have

$$w_{rs}=w'_{rs} \mid J \mid_{\xi_{rs},\, \eta_{rs}}$$

where (ξ_{rs}, η_{rs}) is some point of E'_{rs}. Let (x_{rs}, y_{rs}) be the corresponding point of E_{rs}. We have

$$f(x_{rs}, y_{rs})\, w_{rs}=f\,[\phi(\xi_{rs}, \eta_{rs}), \psi(\xi_{rs}, \eta_{rs})] \mid J \mid_{(\xi_{rs},\, \eta_{rs})} \cdot w'_{rs}$$

A similar equality will be obtained for each pair of corresponding sub-regions. Adding them and letting the norm of the divisions tend to zero, we see that

$$\iint\limits_{E} f(x, y)\,dxdy=\iint\limits_{E_1} f[\phi(u, v), \psi(u, v)] \mid J \mid dudv,$$

which gives the rule for change of variables in a double integral.

Examples

1. *Evaluate* $\displaystyle\iint \frac{\sqrt{(a^2b^2-b^2x^2-a^2y^2)}}{\sqrt{(a^2b^2+b^2x^2+a^2y^2)}}\,dxdy$, *the field of integration being the positive quadrant of the ellipse* $x^2/a^2+y^2/b^2=1$.

Changing the variables x, y to X, Y where

$$x=aX,\ y=bY$$

we see that, since $\partial(x, y)/\partial(X, Y)=ab$, the integral

$$=ab\iint \sqrt{\frac{1-X^2-Y^2}{1+X^2+Y^2}}\,dXdY,$$

the new field of integration being the positive quadrant of the circle $X^2+Y^2=1$.

Changing X, Y to r, θ where

$$X=r\cos\theta,\ Y=r\sin\theta,$$

so that $\partial(X, Y)/\partial(r, \theta)=r$, we see that the integral

$$=ab\iint \frac{\sqrt{(1-r^2)}}{\sqrt{(1+r^2)}}\,r\,drd\theta.$$

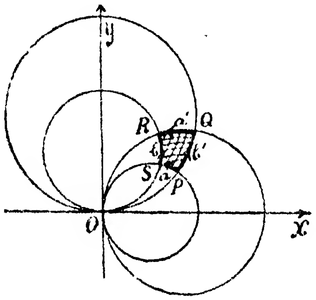
It is easily seen that the positive quadrant of the circle $X^2+Y^2=1$, will be described if θ varies from 0 to $\pi/2$ and corresponding to each value of θ between 0 and $\pi/2$, r varies from 0 to 1. This new field of integration, therefore, is the rectangle $(0, 1\ ;\ 0, \frac{1}{2}\pi)$. Thus,

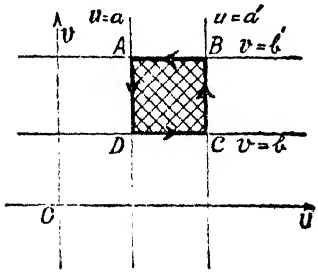
$$\text{the integral}=ab\int_0^{\frac{1}{2}\pi} d\theta\int_0^1 \frac{\sqrt{(1-r^2)}}{\sqrt{(1+r^2)}}\,r\,dr=\frac{\pi}{2}\,ab\int_0^1 \frac{\sqrt{(1-r^2)}}{\sqrt{(1+r^2)}}\,r\,dr$$

$$=\tfrac{1}{8}\pi(\pi-2)ab,$$

where the integral has been evaluated on putting $r^2=\cos t$.

2. *Integrate the function $1/xy$ over the area bounded by the four circles $x^2+y^2=ax$, $a'x$, by, $b'y$ where a, a', b, b' are all positive.*





The integration is to be carried over the shaded area shown in figure.

We change the variables to u, v where

$$u=(x^2+y^2)/x,\quad v=(x^2+y^2)/y.$$

It is easy to see that $\partial(u, v)/\partial(x, y)=-(x^2+y^2)^2/x^2y^2$.

$\therefore$ $\qquad$ * $\partial(x, y)/\partial(u, v)=-x^2y^2/(x^2+y^2)^2$.

Since the Jacobian is negative, the transformation is inverse. This fact, may also be directly verified. The new field of integration is determined by the boundaries $u=a$, $u=a'$, $v=b$, $v=b'$, and is, therefore, the rectangle $(a, a'\ ;\ b, b')$. Thus we see that

$$\text{the integral}=\int\!\!\int \frac{1}{xy}\,|J|\,du\,dv=\int\!\!\int \frac{xy}{(x^2+y^2)^2}\,du\,dv$$

$$=\int\!\!\int \frac{1}{uv}\,du\,dv=\int_a^{a'}\frac{1}{u}\,du\int_b^{b'}\frac{1}{v}\,dv=\log\frac{a'}{a}\cdot\log\frac{b'}{b}.$$

---

*If $x=\phi(u, v)$ and $y=\psi(u, v)$ and we look upon u, v as functions of x and y, then

$$1=\frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial x},\qquad 0=\frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$0=\frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x},\qquad 1=\frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

$$\therefore\ \frac{\partial(x, y)}{\partial(u, v)}\cdot\frac{\partial(u, v)}{\partial(x, y)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1.$$

8. *By substituting $x+y=u$, $x=uv$, prove that the value of*

$$\int\int \sqrt{[xy(1-x-y)]}\, dxdy$$

taken over the area of the triangle bounded by the lines $x=0$, $y=0$, $x+y-1=0$ is $2\pi/105$.

Since $x=uv$, $y=u(1-v)$, we have

$$\partial(x, y)/\partial(u, v)=-u,$$

which is negative.

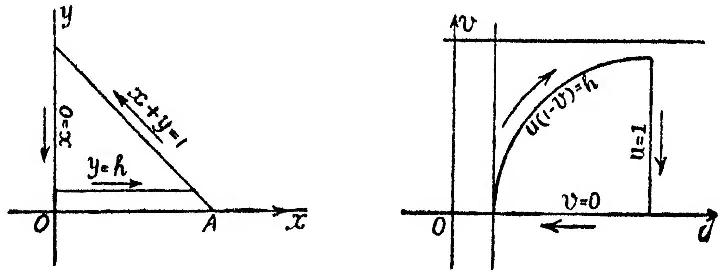
The Jacobian vanishes when $u=0$, *i.e.*, when $x=y=0$, but not otherwise. It is easy to see that to the origin of the xy plane corresponds the whole line $u=0$ of the uv plane so that the correspondence ceases to be *one to one* In order to exclude $x=0$, $y=0$, we look upon the given integral, which certainly exists, as the limit, when $h \to 0$, of the integral over the region bounded by

$$x+y=1,\ x=0,\ y=h, \qquad (h>0).$$

The transformed region is, then, bounded by the lines

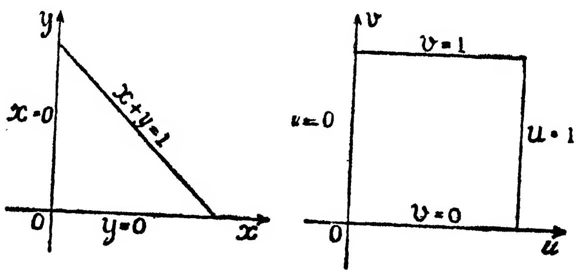
$$u=1,\ v=0,\ u\ (1-v)=h,$$

which correspond to the three boundaries of the region in the xy plane.



When $h \to 0$, this new region of the uv plane tends, as its limit, to the square bounded by the lines

$$u=1,\ v=0,\ u=0,\ v=1.$$



Thus

$$\text{the integral}=\int\int \sqrt{uv.u(1-v)(1-u)}\,.\,u\, dudv$$

$$=\int_0^1 u^2\sqrt{(1-u)}du \int_0^1 \sqrt{v(1-v)}\, dv$$

Putting $u=\sin^2\theta$ and $v=\sin^2\psi$, we see that

$$\int_0^1 u^3\sqrt{(1-u)}\,du=2\int_0^{\frac{1}{2}\pi}\sin^6\theta\cos^2\theta\,d\theta=\frac{2.4.2.1}{7.5.3.1}=\frac{16}{105}$$

$$\int_0^1 \sqrt{v(1-v)}\;dv=2\int_0^{\frac{1}{2}\pi}\sin^2\psi\cos^2\psi\,d\psi=\frac{2.1.1}{4.2}\cdot\frac{\pi}{2}=\frac{\pi}{8}\cdot$$

Hence the result.

4. *Prove that*

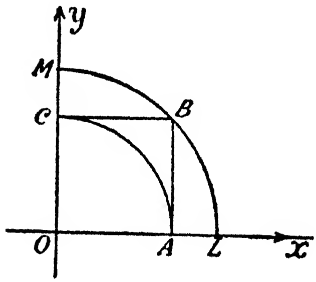
$$\beta(m,n)=\frac{\Gamma(m)\;\Gamma(n)}{\Gamma(m+n)}\cdot \qquad (m>0,\ n>0)$$

We have

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx=2\int_0^{\frac{1}{2}\pi}\cos^{2m-1}\theta\sin^{2n-1}\theta\,d\theta, \qquad (1)$$

(putting $x=\cos^2\theta$)

$$\Gamma(m)=\int_0^\infty t^{m-1}e^{-t}\,dt=2\int_0^\infty r^{2m-1}e^{-r^2}\,dr,\ \text{(putting } t=r^2\text{)}. \qquad (2)$$



From (2), the integral

$$4\iint_E x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dxdy,$$

where E is the square (0, R ; 0, R) tends, as its limit, to $\Gamma(m)\,\Gamma(n)$ as $R\to\infty$.

The positive quadrant of the circle $x^2+y^2=R^2$ is a part of the square E which, again, is a part of the positive quadrant of the circle $x^2+y^2=2R^2$. We denote these positive quadrants by E_1, E_2 respectively. The integrand being positive, we have,

$$4\iint_{E_1} x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dxdy\leqslant 4\iint_E x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dxdy$$

$$\leqslant 4\iint_{E_2} x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dxdy$$

But changing the variables to r, θ where $x=r\cos\theta$, $y=r\sin\theta$, we have

$$4\int\!\!\int_{E_1} x^{2m-1} y^{2n-1}\, e^{-x^2-y^2}\, dxdy$$

$$=4\int_0^{\frac{1}{2}\pi} \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, d\theta \int_0^R e^{-r^2}\, r^{2m+2n-1}\, dr$$

$$=2\beta(m,\ n)\int_0^R e^{-r^2}\, r^{2m+2n-1}\, dr.$$

Similarly

$$4\int\!\!\int_{E_2} x^{2m-1}\, y^{2n-1} e^{-x^2-y^2}\, dxdy=2\beta(m,\ n)\int_0^{\sqrt{2}R} e^{-r^2}\, r^{2m+2n-1}\, dr.$$

$$\therefore\ 2\beta(m,\ n)\int_0^R e^{-r^2}\, r^{2m+2n-1}\, dr\leqslant 4\int\!\!\int_E x^{2m-1}\, y^{2n-1}\, e^{-x^2-y^2}\, dxdy$$

$$\leqslant 2\beta(m,\ n)\int_0^{\sqrt{2}R} e^{-r^2}\, r^{2m+2n-1}\, dr$$

Letting $R\to\infty$, we obtain the required result.

5. *Evaluate*

$$I=\int\!\!\int\!\!\int \sqrt{(a^2b^2c^2-b^2c^2x^2-c^2a^2y^2-a^2b^2z^2)}\,dxdydz$$

taken throughout the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

Changing the variables $x,\ y,\ z$ to X, Y, Z where

$$x=aX,\ y=bY,\ z=cZ,$$

we see that since $\partial(x,\ y,\ z)/\partial(X,\ Y,\ Z)=abc$, therefore

$$I=a^2b^2c^2\int\!\!\int\!\!\int \sqrt{(1-X^2-Y^2-Z^2)}\,dXdYdZ,$$

taken throughout the sphere $X^2+Y^2+Z^2=1$.

Changing X, Y, Z to polar coordinates $r,\ \theta,\ \phi$ so that

$$X=r\sin\theta\cos\phi,\ Y=r\sin\theta\sin\phi,\ Z=r\cos\theta,$$

we have, since $\partial(X,\ Y,\ Z)/\partial(r,\ \theta,\ \phi)=r^2\sin\theta$,

$$I=a^2b^2c^2\int\!\!\int\!\!\int \sqrt{(1-r^2)}\,r^2\sin\theta\, drd\theta d\phi.$$

It is easily seen that to describe the whole sphere $X^2+Y^2+Z^2=1$, r varies from 0 to 1, θ from 0 to π and ϕ from 0 to 2π so that the new field of integration is the rectangular parallelopiped $(0,\ 1\ ;\ 0,\ \pi\ ;\ 0,\ 2\pi)$.

$$\therefore\quad I=a^2b^2c^2\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\ d\theta\int_0^1 \sqrt{(1-r^2)}\,r^2dr=\tfrac{1}{4}a^2b^2c^2\pi^2.$$

6. Evaluate

$$\int\!\!\int \sqrt{[x(2a-x)+y(2b-y)]}\ dxdy,$$

over the circle $x^2+y^2-2ax-2by=0$.

7. Evaluate

$$\int\!\!\int \sqrt{(a^2-x^2-y^2)}\,dxdy$$

over the circle $x^2+y^2=ax$.

8. Show that

$$\int\int x^{\frac{1}{2}}y^{\frac{1}{3}}(1-x-y)^{\frac{1}{4}}\,dxdy=\beta(\tfrac{17}{6},\tfrac{5}{4})\beta(\tfrac{3}{2},\tfrac{4}{3}),$$

over the triangle bounded by the lines $x=0, y=0,\ x+y=1$.

9. Show that

$$\int\int \sqrt{[x^3y^4(1-x^2-y^2)]}\,dxdy=\tfrac{1}{4}\,\beta(\tfrac{11}{4},\tfrac{3}{2})\beta(\tfrac{5}{4},\tfrac{3}{2}),$$

over the positive quadrant of the circle $x^2+y^2=1$.

10. Show that

$$\int_0^2\int_0^x \left\{(x+y+1)^2-4xy\right\}^{-\frac{1}{2}}dxdy=\tfrac{1}{2}\log(16/e).$$

by means of the transformation

$$x=u(1+v),\ y=v(1+u).$$

[$\partial(x,y)/\partial(u,v)=1+u+v$, which is positive for non-negative values of u, v. The field of integration in the xy plane is bounded by $y=0$, $x=2$, $y=x$. Taking into consideration only non-negative values of u and v we see that the corresponding region of the uv plane lies in the positive quadrant and is bounded by $v=0$, $u(1+v)=2$, $u=v$, Therefore, the integral

$$=\int_0^1\int_0^u dudv+\int_1^2\int_0^{(2-u)/u} dudv.]$$

11. Transform the integral

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sqrt{\frac{\sin\phi}{\sin\theta}}\,d\phi d\theta$$

by the substitution $x=\sin\phi\cos\theta$, $y=\sin\phi\sin\theta$ and find its value.

12. Change the variables in the integral

$$\int_0^{2a}dx\int_{\sqrt{(2ax-x^2)}}^{\sqrt{(4ax-x^2)}}\left(1+\frac{y^2}{x^2}\right)dy,$$

to r and θ, where $x=r\cos^2\theta$, $y=r\sin\theta\cos\theta$ and show that the value of the integral is $(\pi+\frac{8}{3})a^2$.

13. Transform the integral

$$\int_0^c\int_0^{c-x} V(x,y)\,dxdy,$$

by the substitution $x+y=u$, $y=uv$

14. By using the transformation

$$x=u^2-v^2,\ y=2uv,$$

or otherwise, evaluate

$$\int\int\frac{dxdy}{\sqrt{(x^2+y^2)}}$$

taken over the region enclosed by arcs, of the confocal parabolas

$$y^2=4a_r(x+a_r),\ (r=1, 2, 3)$$

where $a_1>a_2>0$, $a_3<0$.

15. Find the area of the curvilinear quadrilateral bounded by the four confocal conics of the system

$$x^2/\lambda+y^2/(\lambda-c^2)=1$$

which are determined by giving λ the values $\frac{1}{3}c^2$, $\frac{2}{3}c^2$, $\frac{4}{3}c^2$, $\frac{5}{3}c^2$ respectively.

(Transform into confocal co-ordinates, *i.e.*, express x and y in terms of λ, μ, the parameters of the two confocals which pass through (x, y).

16. Express the integral

$$\int_0^3 dx \int_0^{\sqrt{(5-5x^2/9)}} dy$$

in terms of the variables λ and μ defined by the equations

$$\lambda+\mu=\sqrt{[(x+2)^2+y^2]},\ \lambda-\mu=\sqrt{[(x-2)^2+y^2]}$$

and thus verify that the value of the integral is $\frac{3}{4}\pi\sqrt{5}$.

17. Evaluate $\int\int xy\,dxdy$; the field of integration being the area common to the circles $x^2+y^2=x$, $x^2+y^2=y$.

[Change x, y to u, v where $u=(x^2+y^2)/x$, $v=(x^2+y^2)/y$. The new field in the uv plane is the square $(0, 1 ; 0, 1)$].

18 Evaluate

$$\int\int(xy^3+x^3y)\,dxdy,$$

where the field of integration lies in the first quadrant and is bounded by the central conics

$$ax^2+by^2=l,\ ax^2+by^2=m,\ ax^2-by^2=n,\ ax^2-by^2=p, \qquad (l>m>0;\ n>p>0),$$

19. Show that

$$\int\int\int z\,dxdydz=\frac{\pi}{4}h^4\cot\theta\cot\phi,$$

taken throughout the volume bounded by the cone $z^2=x^2\tan^2\theta+y^2\tan^2\phi$ and the planes $z=0$, $z=h$.

20. Show that

$$\int\int\int z^2\,dxdydz=\frac{30\pi-32}{225}a^5,$$

the field of integration being the region common to the surfaces $x^2+y^2+z^2=a^2$ and $x^2+y^2=ax$.

21. Show that the integral of the function

$$e^{\sqrt{(x^2/a^2+y^2/b^2+z^2/c^2)}}$$

taken throughout the volume of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ is

$$4\pi abc(e-2).$$

22. Evaluate

$$\int\int\int\sqrt{\left(\frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}\right)}dxdydz,$$

integral being taken over all positive values of x, y, z such that $x^2+y^2+z^2\leqslant 1$.

23. Show that

$$\int\int\int\frac{dxdydz}{\sqrt{(1-x^2-y^2-z^2)}}=\frac{\pi^2}{8},$$

integral being extended to all positive values of the variables for which the expression is real.

24. Integrate $1/xyz$ throughout the volume bounded by the six spheres $x^2+y^2+z^2=ax$, $a'x$, by, $b'y$, cz, $c'z$, where a, a', b, b', c, c' are positive.

(Take $u=(x^2+y^2+z^2)/x$, $v=(x^2+y^2+z^2)/y$, $w=(x^2+y^2+z^2)/z$.

25. Prove that $\int\int\int xyz\,dxdydz$ taken through the volume common to the three spheres $x^2+y^2+z^2=2ax$, $x^2+y^2+z^2=2by$, $x^2+y^2+z^2=2cz$ is

$$\tfrac{4}{15}(a^{-2}+b^{-2}+c^{-2})^{-6}.$$

26. Prove that

$$\iiint \frac{1}{z^3}\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)}\,dxdydz$$

taken through the volume common to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ and the part of the cone $\alpha^2x^2+\beta^2y^2=z^2$ for which z is positive is $\pi/\alpha\beta$.

27. Evaluate

$$\iiint x^{l-1}\ y^{m-1}\ z^{n-1}\,dxdydz$$

when the variables are all positive and $x+y+z\leqslant 1$.

(Change to u, v, w where

$$x+y+z=u,\ y+z=uv,\ z=uvw$$

28. Show that

$$\iiint (x+y+z)x^2y^2z^2\,dxdydz=\tfrac{1}{50400}$$

throughout the region $x+y+z\leqslant 1$, $x\geqslant 0, y\geqslant 0, z\geqslant 0$.

29. Evaluate

$$\iiint xyz^2\,dxdydz \text{ for } 0\leqslant 4(x-2)^2+(y-1)^2+9z^2\leqslant 36.$$

30. Evaluate $\iiint (y^2z^2+z^2x^2+x^2y^2)dxdydz$ taken through the volume of the cylinder $x^2+y^2-2ax=0$ between the sheets of the cone $z^2=k^2(x^2+y^2)$.

165. Evaluation of volumes.

By triple integration. The triple integral

$$\iiint_E dxdydz$$

carried throughout a region E in a space of three dimensions gives the volume of E.

By double integration.

Let C be the boundary of a region E of the xy plane and let a cylinder be constructed by lines through the points of C parallel to z—axis. Then the volume of the cylinder enclosed between the

surface $z=\phi(x, y)$, $z=\psi(x, y)$, $[\phi(x, y)\geqslant\psi(x, y)]$

is, as can be easily seen, given by the double integral

$$\iint_E (\phi-\psi)\,dx\,dy.$$

1. Find the volume included between the co-ordinate planes, the part of the right cylinder standing on the quadrant $y=\sqrt{(9-x^2)}$, x and y both positive and the plane $z=3-\frac{1}{3}x-\frac{2}{3}y$.

2. Show that the volume included between the elliptical paraboloid $2z=x^2/p+y^2/q$, the cylinder $x^2+y^2=a^2$ and the x, y plane is $\pi a^4(p+q)/8pq$.

3. Calculate the volume bounded by the surface $z=kxy$, the plane OXY and the first quarter of the cylinder $x^2+y^2=a^2$.

4. Show that the volume of the solid bounded by the cylinder $x^2+y^2=2ax$ and the paraboloid $y^2+z^2=4ax$ is $\frac{2}{3}a^3(3\pi+8)$.

5. Find the volume of the region bounded by the plane $z=x+y$ and the paraboloid $cz=x^2+y^2$.

6. Show that the volume of the region bounded by the hyperboloid of one sheet $x^2/a^2+y^2/b^2-z^2/c^2=1$, its asymptotic cone $x^2/a^2+y^2/b^2-z^2/c^2=0$ and the planes $z=z_1$, $z=z_2$, $(z_2>z_1)$ is $\pi ab(z_2-z_1)$.

7. Prove that the volume in the positive octant bounded by the planes $x=0, y=0, z=h$ and the surface $z/c=(x/a)^m+(y/b)^m$ is equal to

$$\frac{\frac{1}{2}abh(h/c)^{2/m}\ \Gamma(1/m)\ \Gamma(1/m)}{(m+2)\ \Gamma(2/m)} \qquad (I.C.S.)$$

8. Show that the volume enclosed by the surfaces defined by the equations $x^2+y^2=cz$, $x^2+y^2=ax$, $z=0$ is $3\pi a^4/32c$.

9. Show that the entire volume bounded by the positive side of the three co-ordinate planes and the surface $(x/a)^{\frac{1}{2}}+(y/b)^{\frac{1}{2}}+(z/c)^{\frac{1}{2}}=1$, is $abc/90$.

[Change the variables to r, θ, ϕ where

$$x/a=r^4\sin^4\theta\cos^4\phi,\ y/b=r^4\sin^4\theta\sin^4\phi,\ z/c=r^4\cos^4\theta;$$

r varies from 0 to 1, θ from 0 to $\frac{1}{2}\pi$ and ϕ from 0 to $\frac{1}{2}\pi$.]

10. Show that the entire volume of the solid $(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}+(z/c)^{\frac{2}{3}}=1$ is $4\pi abc/35$.

11. Show that the volume of the solid bounded by the cylinders

$$bz^2=c^2y,\ bz^2=2c^2y,\ cx^2=a^2z,\ cx^2=2a^2z,\ ay^2=b^2x,\ ay^2=2b^2x \text{ is } \tfrac{1}{4}abc.$$

12. The area lying in the first quadrant which is enclosed by the curves

$$y=ax^3,\ y=bx^3\ ;\ x=cy^3,\ x=dy^3,\ (a>b,\ c>d)$$

revolves about X-axis ; obtain the volume of the solid generated.

[The required volume is

$$\int\int 2\pi\, y\,dx\,dy$$

change the variables to u, v where

$$u=y/x^3,\ v=x/y^3].$$

CHAPTER XIII

FOURIER SERIES

166. Fourier Series. A trigonometric series of the form

$$\tfrac{1}{2}a_0+(a_1 \cos x+b_1 \sin x)+\ldots+(a_n \cos nx+b_n \sin nx)+\ldots$$

is said to be a Fourier series. Here a_n, b_n are constants independent of x and are known as Fourier constants. In this chapter we propose to find a set of sufficient conditions for a function $f(x)$ to be represented as a Fourier series and to find the constants a_n and b_n if $f(x)$ can be so represented.

Since every term of the Fourier series is periodic with the period 2π, it is obvious that the sum function $f(x)$ of a Fourier series of the above form must also be necessarily periodic with period 2π. It is not necessary, however, that $f(x)$ should be a trigonometric function. A function with 2π as its period will arise if it is *arbitrarily* defined in an interval of length 2π and then periodically extended beyond this interval to the left and to the right so as to satisfy the functional equation $f(x\pm 2\pi)=f(x)$.

Firstly, we proceed to show that the constants a_n, b_n can be easily determined if we *assume* that a given function $f(x)$ can actually be represented as a Fourier series and that the series is uniformly convergent. It should, however, be clearly understood that since there is nothing to prove, *a priori*, that these two assumptions are justifiable, this determination is *purely formal.*

This determination depends upon the following simple integrals.

$$\int_{-\pi}^{+\pi} \sin nx \; dx=0=\int_{-\pi}^{+\pi} \cos nx \, dx; \quad \int_{-\pi}^{+\pi} \sin mx \cos nx \; dx=0.$$

$$\int_{-\pi}^{+\pi} \cos mx \cos nx \; dx=\begin{cases}0, \text{ if } m\neq n \\ \pi, \text{ if } m=n\end{cases}; \quad \int_{-\pi}^{+\pi} \sin mx \sin nx \; dx=\begin{cases}0, \text{ if } m\neq n \\ \pi, \text{ if } m=n.\end{cases}$$

Integrating term by term from $-\pi$ to $+\pi$ the hypothetical equality,

$$f(x)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx), \tag{1}$$

we obtain
$$a_0=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx.$$

Multiplying the equality (1) by $\cos nx$ and integrating term by term from $-\pi$ to $+\pi$, we obtain

$$a_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \cos nx \; dx \tag{2}$$

which is seen to be true for $n=0$ also.

Finally on multiplying with $\sin nx$, we obtain

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin nx \, dx. \tag{3}$$

Note. The series (1) with the co-efficients (2) and (3) will be called the *Fourier series* **corresponding** *to $f(x)$ in the interval* $(-\pi, \pi)$.

Of course the mere fact that a series corresponding to a function can be written does not ensure its convergence, or if convergent, that its sum will be $f(x)$. Complete result in regard to this is given in §169, P. 281.

167. Theorem. *Let $f(x)$ be bounded and integrable in* $(0, a)$ *and let it be monotonic in* $(0, h)$ *where h is any positive number less than a. Then, as $n \to \infty$,*

$$\int_0^a f(x) \frac{\sin nx}{x} \, dx \to f(+0) \int_0^\infty \frac{\sin x}{x} dx$$

where $f(+0)$ denotes the limit of $f(x)$ as x tends to zero through positive values.

(Since $f(x)$ is monotonic in $(0, h)$, therefore $f(+0)$ must exist).

Firstly we suppose that $f(+0)=0$. Also, without affecting the result in any way whatsoever we can suppose that $f(0)=0$.

By second mean value theorem, we have

$$\int_0^h f(x) \frac{\sin nx}{x} \, dx = f(0) \int_0^{h_1} \frac{\sin nx}{x} \, dx + f(h) \int_{h_1}^h \frac{\sin nx}{x} \, dx = f(h) \int_{h_1}^h \frac{\sin nx}{x} \, dx$$

$$= f(h) \int_{nh_1}^{nh} \frac{\sin y}{y} \, dy = f(h) \int_{nh_1}^{nh} \frac{\sin x}{x} \, dx. \tag{1}$$

Since

$$\int_0^\infty \frac{\sin x}{x} \, dx$$

is convergent, there exists a positive constant k such that

$$\left| \int_0^X \frac{\sin x}{x} \, dx \right| \leqslant k, \text{ for all } X \geqslant 0.$$

$$\therefore \left| \int_{nh_1}^{nh} \frac{\sin x}{x} \, dx \right| = \left| \int_0^{nh} \frac{\sin nx}{x} dx - \int_0^{nh_1} \frac{\sin x}{x} \, dx \right| \leqslant 2k. \tag{2}$$

Also $\qquad f(h) \to 0$ as $h \to (0+0)$ $\qquad$ (3)

From (1), (2) and (3) we deduce that, there exists a positive number δ such that

$$\left|\int_0^h f(x)\frac{\sin nx}{x}\,dx\right| \leqslant 2k\,|f(h)|<\tfrac{1}{2}\epsilon, \text{ when } 0<h\leqslant\delta.$$

Now, we have

$$\int_0^a f(x)\frac{\sin nx}{x}\,dx=\int_0^\delta f(x)\frac{\sin nx}{x}\,dx+\int_\delta^a f(x)\frac{\sin nx}{x}\,dx$$

*Since, as proved in Ex. 22, P. 159, the second integral on the right$\to 0$, as $n\to\infty$, we see that there exists a positive integer m such that for $n\geqslant m$.

$$\left|\int_0^a f(x)\frac{\sin nx}{x}\,dx\right|<\tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon=\epsilon.$$

Thus the theorem is proved in this case.

In the general case, changing $f(x)$ to $[f(x)-f(+0)]$, we see that as $n\to\infty$,

$$\int_0^a [f(x)-f(+0)]\,\frac{\sin nx}{x}\,dx\to 0,$$

Now

$$\int_0^a \frac{\sin nx}{x}\,dx=\int_0^{na}\frac{\sin y}{y}\,dy\to\int_0^\infty \frac{\sin y}{y}\,dy, \text{ as } n\to\infty.$$

$$\therefore\qquad \int_0^a f(x)\frac{\sin nx}{x}\,dx\to f(+0)\int_0^\infty \frac{\sin x}{x}\,dx$$

Cor. *If $f(x)$ satisfies the conditions of the theorem above and $0<a<\pi$, then, as $n\to\infty$,*

$$\int_0^a f(x)\frac{\sin nx}{\sin x}\,dx\to f(+0)\int_0^\infty \frac{\sin x}{x}\,dx$$

We write

$$f(x)\frac{\sin nx}{\sin x}=f(x)\frac{x}{\sin x}\cdot\frac{\sin nx}{x}=G(x)\frac{\sin nx}{x},$$

where we assign the values 1 and n to $(x/\sin x)$ and $(\sin nx/x)$ respectively for $x=0$. Now we know that $(x/\sin x)$ is positive and monotonically increasing in $(0, \frac{1}{2}\pi)$. If $f(x)$ be monotonically *increasing* in $(0, h)$, $(h<\frac{1}{2}\pi)$, then $G(x)$ is also monotonically increasing in $(0, h)$.

*$f(x)/x$ being bounded and integrable in (δ, a).

Therefore, as $n \to \infty$,

$$\int_0^a f(x) \frac{\sin nx}{\sin x} dx = \int_0^a G(x) \frac{\sin nx}{x} dx \to G(+0) \int_0^\infty \frac{\sin x}{x} dx = f(+0) \int_0^\infty \frac{\sin x}{x} dx$$

If $f(x)$ be decreasing, so that $-f(x)$ is increasing, we see that, as $n \to \infty$

$$\int_0^a [-f(x)] \frac{\sin nx}{\sin x} dx \to -f(+0) \int_0^\infty \frac{\sin x}{x} dx,$$

i.e.,
$$\int_0^a f(x) \frac{\sin nx}{\sin x} dx \to f(+0) \int_0^\infty \frac{\sin x}{x} dx.$$

168. *Let $f(x)$ be bounded and integrable in $(-\pi, \pi)$ and let it be monotonic in $(-h, 0)$ and $(0, h)$ (not necessarily in the same sense), where h is any positive number less than π. Then*

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n = \frac{f(+0)+f(-0)}{2},$$

where $f(+0)$, $f(-0)$ denote the limits of $f(x)$ as x tends to 0 through positive and negative values respectively.

We have

$$\frac{1}{2} a_0 + \sum_{n=1}^{m} a_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx + \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sum_{n=1}^{m} \cos nx \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)[1 + 2 \sum_{\pi=1}^{m} \cos nx] dx = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) \frac{\sin (m+\frac{1}{2})x}{\sin \frac{1}{2}x} dx$$

$$= \frac{1}{2\pi} \int_0^{\pi} f(-x) \frac{\sin (m+\frac{1}{2})x}{\sin \frac{1}{2}x} dx + \frac{1}{2\pi} \int_0^{\pi} f(x) \frac{\sin (m+\frac{1}{2})x}{\sin \frac{1}{2}x} dx$$

$$= \frac{1}{2\pi} . 2 \int_0^{\frac{1}{2}\pi} f(-2x) \frac{\sin(2m+1)x}{\sin x} dx + \frac{1}{2\pi} . 2 \int_0^{\frac{1}{2}\pi} f(2x) \frac{\sin (2m+1)x}{\sin x} dx,$$

$$= \frac{1}{\pi} f(-0) + f(+0)] \int_0^{\infty} \frac{\sin x}{x} dx \text{ as } m \to \infty.$$

Taking $f(x)=1$, we see that

$$\frac{1}{2}a_0=\frac{1}{2\pi}\int_{-\pi}^{+\pi}1\,.\,dx=1 \text{ and } a_n=\frac{1}{\pi}\int_{-\pi}^{+\pi}\cos nx\,dx=0.$$

$$\therefore \qquad 1=\frac{1+1}{\pi}\int_0^\infty \frac{\sin x}{x}\,dx, \quad i.e., \int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}.$$

Hence the theorem.

Note. The value of the integral of $(\sin x/x)$ over $(0, \infty)$ has also been already obtained in two ways in Ex. 12. P. 196 and §150, P. 246.

169. Let f(x) be bounded and integrable in $(-\pi, \pi)$ and let it be possible to divide $(-\pi, \pi)$ into a finite number of open sub-intervals, in each of which f(x) is monotonic. Then

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\xi+b_n\sin n\xi)=\begin{cases}\tfrac{1}{2}[f(\xi-0)+f(\xi+0)], & \text{for } -\pi<\xi<\pi\\ \tfrac{1}{2}[f(\pi-0)+f(-\pi+0)], & \text{for } \xi=-\pi \text{ or } \xi=\pi.\end{cases}$$

Here $f(\xi-0)$ and $f(\xi+0)$ stand for the limits of $f(x)$ as x tends to ξ from values smaller than and greater than ξ respectively. Under the given conditions $f(\xi-0)$ and $f(\xi+0)$ necessarily exist.

Lemma. If $f(x)$ is bounded and integrable in every interval and is periodic with 2π as its period, then

$$\int_{-\pi}^{+\pi}f(x)dx=\int_{-\pi}^{+\pi}f(a+x)dx,$$

a being any number whatsoever.

Putting $a+x=y$, we have

$$\int_{-\pi}^{+\pi}f(a+x)dx=\int_{a-\pi}^{a+\pi}f(y)dy$$

$$=\int_{a-\pi}^{-\pi}f(y)dy+\int_{-\pi}^{+\pi}f(y)dy+\int_{+\pi}^{a+\pi}f(y)dy.$$

Putting $y=z-2\pi$, we see that

$$\int_{a-\pi}^{-\pi}f(y)dy=\int_{a+\pi}^{\pi}f(z-2\pi)dz=-\int_{\pi}^{a+\pi}f(z)dz=-\int_{\pi}^{a+\pi}f(y)dy$$

Hence the result.

Main theorem. We have

$$\tfrac{1}{2}a_0+\sum_{n=1}^{m}(a_n\cos n\xi+b_n\sin n\xi)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{+\pi}f(x)dx+\sum_{n=1}^{m}\frac{1}{\pi}\int_{-\pi}^{+\pi}f(x)(\cos nx\cos n\xi+\sin nx\sin n\xi)dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{+\pi}f(x)[1+2\sum_{n=1}^{m}\cos n(x-\xi)]dx=\frac{1}{2\pi}\int_{-\pi}^{+\pi}f(x+\xi)[1+2\sum\cos nx]dx$$

(lemma).

$$=\frac{1}{2\pi}\int_{-\pi}^{+\pi}f(x+\xi)\frac{\sin(m+\frac{1}{2})x}{\sin\frac{1}{2}x}dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{0}f(x+\xi)\frac{\sin(m+\frac{1}{2})x}{\sin\frac{1}{2}x}dx+\frac{1}{2\pi}\int_{0}^{\pi}f(x+\xi)\frac{\sin(m+\frac{1}{2})x}{\sin\frac{1}{2}x}dx$$

$$=\frac{1}{\pi}\int_{0}^{\frac{1}{2}\pi}f(-2y+\xi)\frac{\sin(2m+1)y}{\sin y}dy+\frac{1}{\pi}\int_{0}^{\frac{1}{2}\pi}f(2y+\xi)\frac{\sin(2m+1)y}{\sin y}dy$$

$$\to\frac{1}{\pi}\left\{\frac{\pi}{2}\cdot f(\xi-0)+\frac{\pi}{2}\cdot f(\xi+0)\right\}=\frac{f(\xi+0)+f(\xi-0)}{2},\text{ as } m\to\infty.$$

Note. The result arrived at above may be restated as follows :— *If $f(x)$ be bounded and integrable in $(-\pi, \pi)$ and if it be possible to divide (a, b) into a finite number of open sub-intervals in each of which $f(x)$ is monotonic, then the Fourier series corresponding to $f(x)$ converges for every value of x and if $S(x)$ denote the sum function of the series, then*

$$S(x)=\tfrac{1}{2}[f(x+0)+f(x-0)] \text{ when } -\pi<x<\pi$$

$$S(x)=\tfrac{1}{2}[f(\pi-0)+(-\pi+0)] \text{ when } x=\pm\pi.$$

and $$S(x+2\pi)=S(x).$$

The relation $S(x+2\pi)=S(x)$ enables us to determine the value of the sum function at a point which does not belong to $(-\pi, \pi)$.

Cor. At a point of continuity ξ of $f(x)$,

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\xi+b_n\sin n\xi)=\frac{f(\xi+0)+f(\xi-0)}{2}$$

$$=\frac{f(\xi)+f(\xi)}{2}=f(\xi).$$

Thus we see that if $f(x)$ satisfies the conditions of the theorem of §169, in $(-\pi,\pi)$ then the sum of a Fourier series *corresponding* to $f(x)$ is actually $f(x)$ at all such points x of $(-\pi,\pi)$ where $f(x)$ is continuous; at points of discontinuity the sum of the series is $\frac{1}{2}[f(x+0)+f(x-0)]$.

Note. It is not necessary that $f(x)$ be given by a single analytical expression.

Ex. *Expand in a series of sines and cosines of multiples of x a function*

$$f(x)=x-\pi, \textit{ when } -\pi<x<0 \;;\; f(x)=\pi-x, \textit{ when } 0<x<\pi.$$

What is the sum of the series for $x=\pm\pi$ and $x=0$.

The given function $f(x)$ satisfies the conditions of the theorem of §169. We have

$$a_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(x-\pi)\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\cos nx\,dx.$$

Integrating by parts, we obtain

$$a_n=\frac{2[1-(-1)^n]}{n^2\pi}, \; n\neq 0.$$

Also, $\quad a_0=-\pi.$

Similarly

$$b_n=\frac{1}{\pi}\int_{-\pi}^{0}(x-\pi)\sin nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\sin nx\,dx=\frac{2[1-(-1)^n]}{n}.$$

The co-efficients a_n and b_n are zero when n is even.

In $(-\pi,\pi)$, the points $x=0$ and $x=\pm\pi$ are the only points of discontinuity of $f(x)$. Therefore when x is different from 0 and $\pm\pi$, we have

$$f(x)=-\frac{1}{2}\pi+\frac{4}{\pi}\cdot\frac{1}{1^2}\cos x+\frac{4}{1}\sin x+\frac{4}{\pi}\cdot\frac{1}{3^2}\cos 3x+\frac{4}{3}\sin 3x+\ldots\ldots$$

$$=-\frac{1}{2}\pi+\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\ldots\right]+4\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\ldots\ldots\right].$$

Since $x=0$ is a point of discontinuity of $f(x)$, therefore, the sum of the series for $x=0$

$$=\tfrac{1}{2}[f(+0)+f(-0)]=\tfrac{1}{2}[(\pi)+(-\pi)]=0.$$

Thus we obtain a well known result that

$$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots \tag{1}$$

For $x=\pm\pi$, the sum of the series

$$=\tfrac{1}{2}[f(\pi-0)+f(-\pi+0)]=\tfrac{1}{2}[0+(-2\pi)]=-\pi.$$

Putting $x=\pm\pi$, we obtain the same result, *viz.*, 1.

170. Fourier Series for Odd and Even Functions.

The Fourier series with sines only. If $f(x)$ be an odd function, *i. e.*, if $f(-x)=-f(x)$, then $f(x)$ cos nx is an odd function and $f(x)$ sin nx is even and hence

$$a_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \cos nx\, dx=0,$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \sin nx\, dx=\frac{2}{\pi}\int_{0}^{+\pi} f(x) \sin nx\, dx. \tag{2}$$

so that we see that the Fourier series corresponding to an odd function consists of terms with sines only and the co-efficients (b_n) may be computed by (2).

The Fourier series with cosines only. If $f(x)$ is an even function, *i.e.*, if $f(-x)=f(x)$ then $f(x)$ cos nx is even and $f(x)$ sin nx is odd and hence

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx,\ b_n=0.$$

so that we see that the Fourier series corresponding to an even function consists of terms with cosines only.

171. Half range series. From the preceding we will now deduce the following two results :—

171·1. *If $f(x)$ satisfies the conditions of the theorem of* §169 *in* $(0, \pi)$, *then the sum of the sine series*

$$\Sigma\, b_n \sin nx,\ \textit{where}\ b_n=\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx,$$

is equal to $\tfrac{1}{2}[f(x+0)+f(x-0)]$ *at every point* x *between* 0 *and* π *and is equal to* 0 *when* $x=0$ *and* $x=\pi$.

To see the truth of this result, we define an odd function $F(x)$ in $(-\pi, \pi)$ which is identical with $f(x)$ in $(0, \pi)$. Thus

$F(x)=f(x)$ in $(0, \pi)$ and $F(x)=-F(-x)=-f(-x)$ in $(-\pi, 0)$
Clearly $F(x)$ will satisfy the conditions of the theorem of § 169 in $(-\pi, \pi)$ if $f(x)$ does so in $(0, \pi)$. Thus we see that sum of the series

$$\Sigma\, b_n \sin nx,\ \textit{where}\ b_n=\frac{2}{\pi}\int_0^{\pi} F(x) \sin nx\, dx=\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx,$$

is equal to

$$\tfrac{1}{2}[F(x+0)+F(x-0)]=\tfrac{1}{2}[f(x+0)+f(x-0)]$$

at every point x between 0 and π.

At $x=0$, the sum of the series$=\tfrac{1}{2}[F(+0)+F(-0)]=0$, for $F(x)$ is odd. Similarly we see that the sum of the series is 0 for $x=\pi$.

171·2. *If $f(x)$ satisfies the conditions of the theorem of* § 169, *in $(0, \pi)$ then the sum of the series*

$$\tfrac{1}{2}a_0+\Sigma\, a_n \cos nx, \text{ where } a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx,$$

is equal to $\tfrac{1}{2}[f(x+0)+f(x-0)]$ at every point x between 0 and π and is equal to $f(+0)$ for $x=0$ and $f(\pi-0)$ for $x=0$.

To prove this result we have to consider an even function $F(x)$ defined in $(-\pi, \pi)$ which is identical with $f(x)$ in $(0, \pi)$.

Note. The sum functions of the Half-range sine and cosine series are periodic with periodic 2π

Ex. *Find (a) the Fourier sine series, and (b) the Fourier cosine series which represents $f(x)=(\pi-x)$ in $0<x<\pi$.*

To find sine series. We have

$$b_n=\frac{2}{\pi}\int_0^{\pi}(\pi-x)\sin nx\, dx=\frac{2}{n}.$$

Since $f(x)$ is continuous in $0<x<\pi$, we have

$$\pi-x=2\,\Sigma\,\tfrac{1}{n}\sin nx=2\left(\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\frac{\sin 4x}{4}+\ldots\ldots\right)$$

According to § 171.1, the sum of series must be 0 for $x=0$ and $x=\pi$ and this fact can also be directly verified. The representation holds for $x=\pi$ but not for $x=0$.

To find cosine series. We have

$$a_n=\frac{2}{\pi}\int_0^{\pi}(\pi-x)\cos nx\, dx=\frac{2[1-(-1)^n]}{\pi n^2},\ n\neq 0.$$

Also, $a_0=\pi$.

Since $f(x)$ is continuous in $0<x<\pi$, we have

$$\pi-x=\frac{1}{2}\pi+\Sigma\,\frac{2[1-(-1)^n]}{\pi n^2}\cos nx$$

$$=\frac{1}{2}\pi+\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\ldots\right)$$

According to the theorem of § 171.2, the sum of the series must be $f(+0)=\pi$ for $x=0$ and $f(\pi-0)=0$ for $x=\pi$. This gives the result

$$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\ldots,$$

which was obtained on P. 283 also.

Observation. The series in the Ex. on page 283 and the two series abtained above have identical sums for values of x which belong to the open interval $[0, \pi]$ but for values of x belonging to the interval $[-\pi, 0]$ their sums are different. Thus for $-\pi<x<0$, the sum of the series in the example on page 283, is $(x-\pi)$ whereas the sums of the series above are $-[\pi-(-x)]=-(\pi+x)$ and $[\pi-(-x)]=(\pi+x)$ respectively.

Similar differences in the sums exist for values of x outside the interval $(-\pi, \pi)$.

The sum functions of all the three series are periodic with period 2π.

172. Other forms of Fourier series. The particular interval $(-\pi, \pi)$ which we have so far considered had been introduced only as a matter of convenience We shall now see that it is easy to change to any other finite interval.

172·1 The interval $(0, 2\pi)$. We write $x=y+\pi$ so that y varies in $(-\pi, \pi)$ as x varies in $(0, 2\pi)$. Let

$$f(x)=f(y+\pi)=F(y).$$

Let $f(x)$ satisfy the conditions of the theorem of §169 in $(0, 2\pi)$ so that $F(y)$ satisfies the same conditions in $(-\pi, \pi)$. Thus we see that the sum of the series

$$\tfrac{1}{2}a'_0+\Sigma\ (a'_n \cos ny+b'_n \sin ny),$$

where $a'_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi} F(y) \cos ny\, dy,\ b'_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi} F(y) \sin ny dy$

is $\frac{1}{2}[F(y+0)+F(y-0)]$ at any point y between $-\pi$ and π and is $\frac{1}{2}[F(\pi-0)+F(-\pi+0)]$ at $y=\pm\pi$ and is periodic with period 2π.

Changing the variables, we see that

$$a'_n=\frac{1}{\pi}\int_0^{2\pi} F(x-\pi)\ \cos\ n(x-\pi)dx=\frac{(-1)^n}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$$

$$b'_n=\frac{1}{\pi}\int_0^{2\pi} F(x-\pi) \sin n\ (x-\pi)dx=\frac{(-1)^n}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx$$

Also $\quad \cos ny=(-1)^n \cos nx,\ \sin ny=(-1)^n \sin nx.$

Finally, we have

$\frac{1}{2}[F(y+0]+F(y-0)]=\frac{1}{2}[f(y+\pi+0)+f(y+\pi-0)]=\frac{1}{2}[f(x+0)+f(x-0)]$

and $\quad \frac{1}{2}[F(\pi-0)+F(-\pi+0)]=\frac{1}{2}[f(2\pi-0)+f(+0)]$

Thus we see that *if $f(x)$ satisfies the conditions of the theorem of §169, in $(0, 2\pi)$, then the sum of the series*

$$\tfrac{1}{2}a_0+\Sigma(a_n \cos nx+b_n \sin nx),$$

where $a_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x) \cos nx\, dx,\ b_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x) \sin nx\, dx$

is $\frac{1}{2}[f(x+0)\ f(x-0)]$ *at every point x between* 0 *and* 2π *and is* $\frac{1}{2}[f(2\pi-0)+f(+0)]$ *at* $x=0$ *and* $x=2\pi$ *and is periodic with period* 2π.

172.2. The interval $(-l, l)$, *where l is any real number.* By means of the substitution $y=\pi x/l$ and considering a function $F(y)$ such that

$$f(x)=f(ly/\pi)=F(y)$$

so that y varies in $(-\pi, \pi)$ as x varies in $(-l, l)$, we can show as before that *if* $f(x)$ *satisfies the conditions of the* **theorem** of §169 *in* $(-l, l)$ *then the sum of the series*

$$\tfrac{1}{2}a_0+\Sigma\ [a_n \cos(n\pi x/l)+b_n \sin (n\pi x/l)],$$

$$\text{where } a_n=\frac{1}{l}\int_{-l}^{+l} f(x)\ \cos\frac{n\pi x}{l}\,dx,\quad b_n=\frac{1}{l}\int_{-l}^{+l} f(x)\frac{\sin n\pi x}{l}\,dx$$

is $\frac{1}{2}[f(x+0)+f(x-0)]$ *at every point x between* $-l$ *and* l *and is*

$$\tfrac{1}{2}[f(l-0)+f(-l+0)]$$

for $x=-l$ *and* $x=l$ *and is periodic with period* $2l$.

Note, As in § 171, we can have half range sine and cosine series at will for a function $f(x)$ given in $(0, l)$.

172.3 Any interval (a, b). *If* $f(x)$ *satisfies the conditions of the* **theorem** of § 169 *in* (a, b) *then the sum of the series*

$$\tfrac{1}{2}a_0+\Sigma\ \{ a_n \cos[2n\pi x/(b-a)]+b_n \sin [2n\pi x/(b-a)] \}$$

$$\text{where } a_n=\frac{2}{b-a}\int_a^b f(x)\ \cos\left(\frac{2n\pi x}{b-a}\right)dx,\quad b_n=\frac{2}{b-a}\int_a^b f(x)\ \sin\left(\frac{2n\pi x}{b-a}\right)dx$$

is $\frac{1}{2}[f(x+0)+f(x-0)]$ *at every point x between a and b and is* $\frac{1}{2}[f(a+0)+f(b-0)]$ *at* $x=a$ *and at* $x=b$ *and is periodic with period* $(b-a)$.

This result follows on writting

$$^*y=\frac{2\pi}{b-a}x-\frac{b+a}{b-a}\pi$$

so that y varies in $(-\pi, \pi)$ as x varies in (a, b).

Ex. *Show that when* $0<x<\pi$,

$$\pi-x=\tfrac{1}{2}\pi+\frac{\sin 2x}{1}+\frac{\sin 4x}{2}+\frac{\sin 6x}{3}+\ldots\ldots$$

Each term of the series is periodic with period π. Taking $a=0$ and $b=\pi$ in § 172.3, we see that $a_0=\pi$

$$a_n=\frac{2}{\pi}\int_0^\pi (\pi-x)\cos 2nx\,dx=0,(n\neq 1);b_n=\frac{2}{\pi}\int_0^\pi(\pi-x)\sin 2nx\,dx=\frac{1}{n}.$$

Hence the result.

Note. It will now be seen that we have obtained *four* different series which represent $(\pi-x)$ in $0<x<\pi$ and it will naturally prove of great interest to examine the differences in the sums of these series for values of x other than those which belong to $[0, \pi]$ To do so, student would do well to draw the graphs of the four sum functions.

---

*This transformation is obtained if we determine the two constants l and m such that the relation $y=lx+m$ gives $y=-\pi$ when $x=a$ and $y=\pi$ when $x=b$.

One important difference which must be emphasised is that the sum function of the series obtained above in § 172·3, is periodic with period π whereas the former three sum functions were periodic with period 2π.

Examples

1. If $f(x)=-\frac{1}{4}\pi$ when $-\pi<x<0$, $f(x)=\frac{1}{4}\pi$ when $0<x<\pi$, $f(-\pi)=f(0)=f(\pi)=0$ and $f(x+2\pi)=f(x)$ for all x, show that
$$f(x)=\sin x+\tfrac{1}{3}\sin 3x+\tfrac{1}{5}\sin 5x+\ldots\ldots$$
for all values of x. Deduce that
$$\tfrac{1}{4}\pi=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots\ldots$$

2. If $f(x)=cx$ when $-\pi<x<\pi$, $f(-\pi)=f(\pi)=0$, and $f(x+2\pi)=f(x)$ for all x, show that
$$f(x)=2c[\sin x-\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x-\tfrac{1}{4}\sin 4x+\ldots\ldots]$$
for all values of x. Draw the graph of $f(x)$.

3. Show that the Fourier series which converges to $f(x)$ in $-\pi\leqslant x\leqslant\pi$, where
$$f(x)=x+x^2 \text{ when } -\pi<x<\pi \text{ and } f(x)=\pi^2 \text{ when } x=\pm\pi, \text{ is}$$
$$\frac{\pi^2}{3}+4\sum(-1)^n\left(\frac{\cos nx}{n^2}-\frac{\sin nx}{2n}\right).$$
Deduce that
$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots$$

4. Obtain Fourier series which will be equal to $-\pi-x$ when $-\pi\leqslant x<-\frac{1}{2}\pi$; equal to x when $-\frac{1}{2}\pi\leqslant x<\frac{1}{2}\pi$ and equal to $\pi-x$ when $\frac{1}{2}\pi\leqslant x\leqslant\pi$. Explain graphically how to obtain the sum of the series for any value of x.

5. Obtain Fourier series whose sum is equal to $f(x)$ where
$$f(x)=0 \text{ when } -\pi\leqslant x<-\tfrac{1}{2}\pi,\ f(-\tfrac{1}{2}\pi)=-\tfrac{1}{4}\pi,$$
$$f(x)=x \text{ when } -\tfrac{1}{2}\pi<x<\tfrac{1}{2}\pi,\ f(\tfrac{1}{2}\pi)=\tfrac{1}{4}\pi$$
$$f(x)=0 \text{ when } \tfrac{1}{2}\pi<x\leqslant\pi.$$

6. $f(x)=\cos x$ for $0<x<\pi$ and $f(x)=-\cos x$ for $-\pi<x<0$; show that the Fourier series which converges to $f(x)$ is
$$\frac{4}{\pi}\left(\frac{2}{1\cdot 3}\sin 2x+\frac{4}{3\cdot 5}\sin 4x+\frac{6}{5\cdot 7}\sin 6x+\ldots\ldots\right)$$
Draw the graph of the sum function of the series for $-2\pi\leqslant x\leqslant 2\pi$.

7. Find the Fourier series which represents $|\sin x|$ in $-\pi\leqslant x\leqslant\pi$.

8. Find a series of sines of multiples of x, which will represent $f(x)$ in the interval $0<x<\pi$, where
$$f(x)=\tfrac{1}{3}\pi, \text{ when } 0<x<\tfrac{1}{3}\pi;$$
$$f(x)=0, \text{ when } \tfrac{1}{3}\pi<x<\tfrac{2}{3}\pi;$$
$$f(x)=-\tfrac{1}{3}\pi, \text{ when } \tfrac{2}{3}\pi<x<\pi.$$
Find the sums of the series at the points $2\pi/3$ and π,

9. (*a*) Show that for $-\pi<x<\pi$,
$$e^x=\frac{e^{\pi}-e^{-\pi}}{\pi}\left[\frac{1}{2}+\sum_1^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx-n\sin nx)\right].$$
What is the sum of the series for $x=\pm\pi$.

(*b*) Show that for $0<x<\pi$
$$e^x=\begin{cases}\dfrac{2}{\pi}\displaystyle\sum_1^{\infty}\left(1-(-1)^n e^{\pi}\right)\frac{n\sin nx}{n^2+1}\\[2ex] \dfrac{e^{\pi}-1}{\pi}-\dfrac{2}{\pi}\displaystyle\sum_1^{\infty}\left(1-(-1)^n e^{\pi}\right)\frac{\cos nx}{n^2+1}\end{cases}$$
What are the sums of these series for $-\pi\leqslant x\leqslant 0$.

10 Show that when $-\pi \leqslant x \leqslant \pi$,

$$\cos kx = \frac{\sin k\pi}{\pi}\left(\frac{1}{k} - \frac{2k\cos x}{k^2-1^2} + \frac{2k\cos 2x}{k^2-2^2} - \dots\right),$$

k being non-integral. Deduce that

$$\pi \cot k\pi = \frac{1}{k} + 2k\,\Sigma \frac{1}{k^2-n^2},$$

and

$$\frac{\pi}{\sin k\pi} = \Sigma(-1)^n\left(\frac{1}{n+k} + \frac{1}{n+1-k}\right).$$

11. Obtain a sine series which will be equal to x^2 for $0 \leqslant x < \pi$. Draw a graph of the sum function of the series for $-2\pi \leqslant x \leqslant 2\pi$.

12 Show that in $(0, \pi)$

$$f(x) = \frac{\pi^2}{16} - 2\sum_1^\infty \frac{\cos(4n-2)x}{(4n-2)^2},$$

where $f(x) = \frac{1}{4}\pi x$ when $0 \leqslant x \leqslant \frac{1}{2}\pi$ and $f(x) = \frac{1}{4}\pi(\pi - x)$, when $\frac{1}{2}\pi < x \leqslant \pi$.

13. Prove that when $0 < \theta < 2\pi$,

$$\tfrac{1}{2}(\pi - \theta) = \sum_{n=1}^\infty \frac{\sin n\theta}{n}.$$

What is the sum of the series for $\theta = 0$ and $\theta = 2\pi$.

14. Prove that when $-1 < x < 1$,

$$x + x^2 = \tfrac{1}{3} + \frac{2}{\pi}\sum_1^\infty (-1)^n\left[\frac{2\cos n\pi x}{n^2\pi} - \frac{\sin n\pi x}{n}\right]$$

15. Find (*i*) the series of sines, (*ii*) the series of cosines, which will represent $f(x)$ in $(0, l)$, where

$f(x) = kx$, when $0 \leqslant x \leqslant \frac{1}{2}l$, $f(x) = k(l-x)$ when $\frac{1}{2}l < x \leqslant l$

16. If $\phi(x)$ be a periodic function of period 4 and such that $\phi(x) = 0$ when $0 < x < 1$ and $\phi(x) = x - 1$ when $1 \leqslant x < 2$, and $\phi(2) = 0$, express $\phi(x)$ as a series of sines of multiples of x. Draw the graph of $\phi(x)$.

17. Show that for $0 \leqslant x \leqslant 1$,

$$x - \tfrac{1}{2} = -\frac{1}{\pi}\left\{\sum_{n=1}^\infty \frac{\sin 2n\pi x}{n}\right\},\quad x^2 - x + \tfrac{1}{6} = \frac{1}{\pi^2}\left\{\sum_{n=1}^\infty \frac{\cos 2n\pi x}{n^2}\right\},$$

$$x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x = \frac{3}{2\pi^3}\left\{\sum_{n=1}^\infty \frac{\sin 2n\pi x}{n^3}\right\}.$$

(use §172·3, taking $a = 0$, $b = 1$)

18. If $f(x)$ be a periodic function of period $\frac{1}{2}\pi$ and such that $f(x) = \sin x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$ and $f(x) = \cos x$ for $\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{2}\pi$, express $f(x)$ as a Fourier series.

19. If $f(x) = 1$ when $0 < x < 1$, $f(x) = 2$ when $1 < x < 3$ and $f(x) = \frac{3}{2}$ when $x = 0$, 1 and 3 and $f(x+3) = f(x)$ for all x; show that

$$f(x) = \tfrac{5}{3} - \frac{1}{\pi}\sum_1^\infty \frac{2\sin\frac{1}{3}n\pi}{n}\cos\frac{n\pi(2x-1)}{3}$$

for all x.

20. If $f(x) = \frac{1}{4}a - x$ when $0 \leqslant x \leqslant \frac{1}{2}a$ and $f(x) = x - \frac{3}{4}a$ when $\frac{1}{2}a \leqslant x \leqslant a$, show that for all x in $0 \leqslant x \leqslant a$,

$$f(x) = \frac{2a}{\pi^2}\sum_1^\infty \frac{1}{(2n-1)^2}\cos\frac{(4n-2)\pi x}{a}$$

21. Show that if a function $f(x)$ is bounded in $(0, \frac{1}{2}\pi)$ and is such that it is possible to divide $(0, \frac{1}{2}\pi)$ into a finite number of open sub-intervals in each of which $f(x)$ is monotonic, then $f(x)$ can be expressed in the form

$$b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x + \dots\dots\dots$$

for $0 < x < \frac{1}{2}\pi$ and obtain the co-efficients b_n and the sum of the series outside $[0, \frac{1}{2}\pi]$.

If $f(x)=\sin x$ when $0\leqslant x\leqslant\frac{1}{4}\pi$ and $f(x)=\cos x$ when $\frac{1}{4}\pi\leqslant x\leqslant\frac{1}{2}\pi$, show that

$$f(x)=\frac{\sin x}{2}+\frac{2}{\pi}\left(\frac{\sin 3x}{1.2}-\frac{\sin 5x}{2.3}+\frac{\sin 11x}{5.6}-\frac{\sin 13x}{6.7}+\dots\dots\dots\right)$$

22. Expand in a Fourier series of the form $a_n \sin n\pi x$ a function $f(x)$ given by

$$f(x)=\sinh \pi x, \text{ for } 0\leqslant x<\tfrac{1}{2},\ f(0)=0, \text{ for } \tfrac{1}{2}<x\leqslant 1.$$

Deduce that

$$\sum_{n=1}^{\infty}\frac{1}{n^2+(n+1)^2}=\frac{\pi}{2}\tanh\frac{\pi}{2}.$$

23. Show that the graph of the equation

$$y^2=\frac{a^2}{3}+\frac{16a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\cos\frac{(2n-1)\pi x}{2a}+\frac{8a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)^2}\cos\frac{n\pi x}{a}$$

is a series of circles of radius a connected by straight lines of length $2a$, the origin being the centre of one of the circles.

[Expand $f(x)$ in $-2a$, $2a$ where $f(x)=0$ when $-2a\leqslant x\leqslant -a$, $f(x)=a^2-x^2$, when $-a\leqslant x\leqslant a$; $f(x)=0$ when $a\leqslant x\leqslant 2a$].

Miscellaneous Examples.

1. Show that an algebraical equation

$$x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n=0,$$

with integral co-efficients, cannot have a rational but non-integral root.

Deduce that $\sqrt{3}$ and $\sqrt[3]{2}$ are not rational.

2. If m/n is a good approximation to $\sqrt{2}$, prove that

$$(m+2n)/(m+n)$$

is a better one, and that the errors in the two cases are in opposite directions. Apply this result to show that the limit of the sequence

$$\tfrac{1}{1},\ \tfrac{3}{2},\ \tfrac{7}{5},\ \tfrac{17}{12},\ \tfrac{41}{29},\ \tfrac{99}{70},\ \tfrac{239}{169},\ \tfrac{577}{408},\dots \text{is } \sqrt{2}.$$

3. Prove that, as $n\to\infty$,

$$[(2n)!/(n!)^2]^{\frac{1}{n}}\to\tfrac{1}{4}. \qquad \text{(Use Ex. 10, P.58)}$$

4. Prove that, as $n\to\infty$,

$$(n!)(a/n)^n\to 0 \text{ or } +\infty,$$

according as $a<e$ or $a>e$. (Use Ex. 2, Ex. 3, P.54).

5. Show that, as $n\to\infty$,

$$(i)\ (n^5/2^n)\to 0,\quad (ii)\ (n^2)^{\frac{1}{n}}\to 1.$$

6. Denoting

$$lt\left[1+\tfrac{1}{2}+\tfrac{1}{3}+\dots+\tfrac{1}{n}-\log n\right], \qquad \text{(Ex. 10, P. 139).}$$

by γ, known as Euler's constant, show that

$$lt\,\frac{1}{n}\left[\frac{n}{1}+\frac{n-1}{2}+\frac{n-2}{3}+\dots+\frac{1}{n}-\log(n!)\right]=\gamma. \qquad \text{(Use Ex. 5, P.55)}$$

7. Prove that if a and b are positive, then

$$\tfrac{1}{2}\left\{\left(a^{\frac{1}{n}}+b^{\frac{1}{n}}\right)\right\}^n\to\sqrt{(ab)}, \text{ as } n\to\infty.$$

8. Prove that if $x>0$, then

$$n\log\left\{\tfrac{1}{2}\left(1+x^{\frac{1}{n}}\right)\right\}\to -\tfrac{1}{2}\log x, \text{ when } n\to\infty,$$

9. Prove that, if $f'(c)$ exists and $\neq 0$, then, as $h \to 0$,

$$lt \frac{f(c+h)+f(c-h)-2f(c)}{h} \text{ and } lt \frac{f(c+h)-f(c)}{f(c-h)-f(c)}$$

exist and find their values. State the theorems on limits you require in the proof.

Examine the existence of the limits if $f(x)=|x|$.

10. State with reasons for your conclusion which of the following sets of circumstances are sufficient and which of them are not sufficient to determine the value of $f(0)$ for a function $f(x)$ giving the value of $f(0)$ where possible :—

(*i*) $f(x)$ is continuous at $x=0$ and takes in any neighbourhood of $x=0$ both positive and negative values.

(*ii*) For any arbitrary positive ϵ, there is a δ such that

$$|f(x)|<\epsilon \text{ for } 0<|x|<\delta.$$

(*iii*) $[f(h)+f(-h)-2f(0)]/h \to l$ and $f(h) \to a$ as $h \to 0$.

11. If $f(0)=0$ and $f''(x)$ exists in $(0, \infty]$, show that

$$f'(x)-f(x)/x=\tfrac{1}{2}x f''(\xi),\ 0<\xi<x$$

and deduce that if $f(0)=0$ and $f''(x)$ is positive for positive values of x, then $f(x)/x$ is a strictly increasing function of x.

12. Let $f(x)$ be continuous in $(-1, 1)$ and assume rational values only and let $f(0)=0$. Prove that $f(x)=0$ everywhere.

13. Show that the function defined as follows :—

$\phi(x)=x$, when x rational, and $\phi(x)=1-x$, when x is irrational, assumes every value between 0 and 1 once and only once as x increases from 0 to 1, but is discontinuous for every value of x except $x=\frac{1}{2}$.

14. A function $f(x)$ is defined for all values of x in the following manner :

$f(x)=0$, when x is irrational ;
$f(x)=1/q$, when x is a rational number p/q,

when p/q is a fraction in its lowest terms. Prove that $f(x)$ is continuous for all irrational values of x and discontinuous for all rational values of x.

15. If $0<x<\frac{1}{4}\pi$, show that $\tan x<4x/\pi$.

16. Prove that $a^x>x^a$ if $x>a \geq e$.

[Let $f(x)=x \log a - a \log x$; show that $f(a)=0$ and $f'(x)>0$].

17. If

$$f(x, y)=(xy^5-x^2y^3)/(x^2+xy^2+y^4), \text{ when } (x, y)\neq(0, 0)$$

and $\quad f(0, 0)=0$,

show that at the origin $f_{xy} \neq f_{yx}$ but that f_x is continuous and f_y differentiable at the origin. Discuss the applicability of the conditions of §. 127.

18 Discuss the existence and the equality at the origin of f_{xy} and f_{yx} for

$$f(x, y)=(ax^2+2bxy+cy^2)^3/(x^2+y^2)^3 \text{ when } (x, y)\neq(0, 0)$$

and $\quad f(0, 0)=0.$

19. If at all points of the plane $f(x, y)$ is continuous with respect to x and has a partial derivative $f_y(x, y)$ and if $f_y(x, y)$ is bounded in the whole plane, prove that $f(x, y)$ is continuous with respect to two variables at all points of the plane.

20. The number θ is defined by the Taylor's formula

$$f(a+h,\ b+k)=f(a,\ b)+hf_x(a+\theta h,\ b+\theta k)+kf_y(a+\theta h,\ b+\theta k).$$

Prove that if f_x and f_y are differentiable at $(a,\ b)$ and if $(h,\ k)\to(0,\ 0)$ so that

$$(h^2+k^2)/\{\ h^2f_{x^2}(a,\ b)+2hk\,f_{xy}(a,\ b)\ +k^2f_{y^2}(a,\ b)\ \}$$

is bounded, then $\theta\to\frac{1}{2}$.

21. Discuss at (0, 0) the existence of the first order partial derivatives and the continuity and differentiability of $f(x,\ y)$ itself when $f(x, y)$ has the following forms :—

(*i*) $x^2y^2 \log(x^2+y^2)$, (*ii*) $xy \log(x^2+y^2)$,

$f(0,\ 0)$ being equal to zero in either case.

22. Show that, if $f_x(x,\ y),\ f_y(x,\ y)$ are differentiable at a point, then $f(x, y)$ is also differentiable at the point.

23. Prove that the functions

$$ax^2+by^2+cz^2,\ Ax+By+Cz,$$

$$a^2x^2(B^2c+C^2b)+b^2y^2(C^2a+A^2c)+c^2z^2(A^2b+B^2a)$$
$$-2abc(BCyz+CAzx+ABxy)$$

are not independent and find the relation between them.

24. Examine for extreme values the function

$$2xyz-4zx-2yz+x^2+y^2+z^2-2x-4y+4z.$$

25. Show that the function

$$3 \log(x^2+y^2+z^2)-2x^3-2y^3-2z^3$$

has only one extreme value and find the same.

26. Discuss for maximum and minimum values the function

$$3x^4y^2-6x^2y^2+3x^4+2y^3-6x^2-3y^2+1.$$

27. For

$$u=(x+y+z)^3-3(x+y+z)-24xyz+a^3,$$

investigate the existence or otherwise of maximum and minimum values at each of the following points

$$(1,\ 1,\ 1),\ (\pm1,\ 0,\ 0),\ (0,\ \pm1,\ 0),\ (0,\ 0,\pm1),\ (-1,\ -1,\ -1.)$$

28. Find the values of x, y, z for which the function $e^{-u}(x-y+2z)$ is a maximum where $u=x^2+y^2+2z^2-2yz+2zx-xy$.

29. Prove that the maximum value of

$$f(x,\ y,\ z)=(ax+by+cz)e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$$

is $$\sqrt{[\tfrac{1}{2}(a^2\alpha^{-2}+b^2\beta^{-2}+c^2\gamma^{-2})(e^{-1})]}.$$

30. Prove that if $x+y+z=3c$, then $f(x)\,f(y)\,f(z)$ will be a maximum or minimum for $x=y=z=c$ according as

$$[f(c)]f''(c)<\text{ or }>[f'(c)]^2$$

31. Prove that if all the symbols denote positive quantities, the stationary value of $lx+my+nz$ subject to the condition

$$x^p+y^p+z^p=c^p$$

is given by

$$c(l^q+m^q+n^q)^{1/q},$$

where $q=p/(p-1)$. Show further that this value is a maximum or a minimum according as $p>$ or <1.

32. Find the volume of the largest rectangular parallelopiped which has three faces in the co-ordinate planes and one vertex in the plane $x/a+y/b+z/c=1$.

33. Find the dimensions of the rectangular box, without a top, of maximum capacity whose surface is a^2.

34. Given n points P_i whose co-ordinates are

$$(x_i, y_i, z_i), \ (i=1, 2, ..., n);$$

show that the co-ordinates of the point $P(x, y, z)$ such that the sum of the squares of the distances from P to the points P_i is a minimum are given by

$$[\Sigma x_i/n, \ \Sigma y_i/n, \ \Sigma z_i/n].$$

35. Find the maximum value of $x^2y^2z^2$ subject to the condition $x^2+y^2+z^2=c^2$. Interpret the result.

36. If $u=x^2+y^2+z^2$ where $x^p+y^p+z^p=3c^p$, show that a stationary value of the function u is equal to $3c^2$ and prove that this is a minimum or a maximum according as $p<$ or >2.

37. Find the greatest and least distances from the origin of a point of the surface

$$(x/a)^p+(y/b)^p+(z/c)^p=1;$$

where a, b, c are fixed positive numbers and p is a fixed even integer greater than 2.

38. By using the transformation

$$x=uv, \ y=u(1-v),$$

prove that

$$\int\int \frac{f(x+y)}{\sqrt{(xy)}}\,dxdy=\pi\int_0^a f(u)du,$$

where the double integral extends over all positive values of x and y subject to $x+y \nless a$.

39. Show that the volume of the wedge intercepted between the cylinder $x^2+y^2=2ax$ and the planes $z=x\tan\alpha$, $z=x\tan\beta$ is $\pi(\tan\beta-\tan\alpha)a^3$.

40. Find the volume contained between the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

and the cylinder $x^2/a^2+y^2/b^2=x/a$.

41. Show that the volume obtained by one revolution of the loop of the Folium $x^3+y^3=3axy$ about OX is $4\pi^2a^3/3\sqrt{3}$.

(Change the variables to u, v where $u=x^2/y$, $v=y^2/x$)

42. If $x=r\cos\theta$, $y=r\sin\theta$, find the volume included by the surfaces whose equations are $r=a$, $z=0$, $\theta=0$, $z=mr\cos\theta$.

43. Using the transformation $x=c\cosh u\cos v$, $y=c\sinh u\sin v$, where $c^2=a^2-b^2$, or otherwise, show that the mean value of the product of the focal distances of a point inside the ellipse $x^2/a^2+y^2/b^2=1$ is

$$\frac{1}{4}\left[a^2+b^2+\frac{(a^2-b^2)^2}{ab}\log\frac{a+b}{a-b}\right]$$

(The mean value$=[\int\int SP.S'P\,dxdy]/[\int\int dxdy]$.

44. Evaluate

$$\int\int (2x^2+y^2)/xy\;dxdy$$

taken over the area in the positive quadrant of the xy plane bounded by the curves

$$x^2+y^2=h^2,\; x^2+y^2=k^2,\; y^2=4ax,\; y^2=4bx.$$

45. The axes of two equal cylinders intersect at right angles. If a be their radius, show that the volume common to the cylinders is $\frac{16}{3}a^3$.

[If $x^2+y^2=a^2$ and $y^2+z^2=a^2$ be the two cylinders, then the required volume

$$=8\int_0^a dx\int_0^{\sqrt{(a^2-x^2)}} dy\int_0^{\sqrt{(a^2-y^2)}} dz$$

46. By employing the first mean value theorem of integral calculus, show that, if $k^2<1$,

$$\frac{\pi}{6}<\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{[(1-x^2)(1-k^2x^2)]}}<\frac{\pi}{6}\cdot\frac{1}{\sqrt{(1-\frac{1}{4}k^2)}}$$

47. Prove that

$$\tfrac{1}{2}\left(1-\frac{1}{e}\right)<\int_0^1 e^{-x^2}dx<1 \text{ and } 0<\int_1^\infty e^{-x^2}dx<\frac{1}{2e}$$

and deduce that

$$\tfrac{1}{2}\left(1-\frac{1}{e}\right)<\int_0^\infty e^{-x^2}dx<1+\frac{1}{2e}.$$

48. $f(x)$ is bounded and integrable in (a, b) ; show that

$$\int_a^b [f(x)]^2 dx=0,$$

if, and only if, $f(c)=0$ at every point c of continuity of $f(x)$.

49. Prove that the graph of

$$y=f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin^3 t\cos xt}{t}\,dt,$$

consists of parts of the lines $y=-\frac{1}{4}$, $y=0$ and $y=\frac{1}{2}$ together with four isolated points.

(Use the result obtained in § 150, P. 246).

50. If $f(x, y)=xy^4e^{-xy}+x^2y/(1+y)$ and a, b are positive, show that

$$\lim_{y\to\infty}\int_a^b f(x, y)dx=\int_a^b\left(\lim_{y\to\infty} f(x, y)\right)dx$$

Explain why the values are not equal when $a=0$.

51. Show that, when $n\to\infty$,

$$\int_a^b f(x)\,|\sin nx|\,dx\to\frac{2}{\pi}\int_a^b f(x)dx$$

52. If a, b are positive and p is a positive integer, show that, as $n\to\infty$,

(*i*) $\displaystyle\sum_{r=1}^{pn}\frac{1}{na+r}\to\log\left(1+\frac{p}{a}\right)$

(*ii*) $\displaystyle\sum_{r=1}^{pn}\frac{1}{na+rb}\to\frac{1}{b}\log\left(1+\frac{pb}{a}\right).$

53. Evaluate the limit of the sequence $\{u_n\}$, where

$$u_n=\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots\dots+\frac{1}{\sqrt{n}}\right),\text{ when } n\to\infty$$

54. Show that, as $n\to\infty$,

$$\frac{2.2.4.4.6.6\ldots(2n)(2n)}{1.3.3.5.5.7\ldots(2n-1)(2n+1)}\to\frac{\pi}{2}.$$

(Walli's formula).

Deduce that

$$\{[(n!)^2\,2^{2n}]/(2n)!\,\sqrt{n}\}\to\sqrt{\pi}\text{ as }n\to\infty.$$

55. $f(x)$ is a non-negative function admitting an elementary infinite integral and another function $f_n(x)$ is defined thus :—

$$f_n(x)=\begin{cases} f(x), & \text{if } f(x)<n\\ n, & \text{if } f(x)\geq n,\end{cases}$$

show that

(*i*) if the interval of integration is bounded, then as $n\to\infty$

$$\int_a^b f_n(x)\,dx\to\int_a^b f(x)dx.$$

(*ii*) if the interval of integration is infinite, then as $n \to \infty$

$$\int_{-n}^{+n} f_n(x)dx \to \int_{-\infty}^{+\infty} f(x)\,dx.$$

56. If $f(x)$, $f'(x)$ are continuous in (a, b) and

$$S_n = \sum_{r=1}^{n} hf(a+rh),\ I = \int_a^b f(x)\,dx,$$

where $h=(b-a)/n$, then as $n \to \infty$,

$$n(S_n - I) \to \tfrac{1}{2}(b-a)\,[f(b)-f(a)].$$

57. If I, h have the same meaning as in the Ex. above, but if now $f''(x)$ is also continuous in (a, b) and

$$S_n = \sum_{r=1}^{n} h\,f[a+\tfrac{1}{2}(2r-1)h]$$

show that, as $n \to \infty$,

$$n^2(I-S_n) \to \tfrac{1}{24}(b-a)^2[\,f'(b)-f'(a)].$$

58. If $f(t, x) = \pi t \sin \pi tx$ when $0 \leqslant x \leqslant 1/t$ and $f(t, x)=0$ when $1/t < x \leqslant 1$, show that

$$\int_0^1 \left\{ \operatorname*{lt}_{t \to \infty} f(t, x) \right\} dx = 0 \neq 2 = \operatorname*{lt}_{t \to \infty} \left\{ \int_0^1 f(t, x)dx. \right\}$$

59. If $f(t, x)=tx/(1+t^2x^4)$, show that

$$\int_0^1 \left\{ \operatorname*{lt}_{t \to \infty} f(t, x) \right\} dx = 0 \neq \frac{\pi}{4} = \operatorname*{lt}_{t \to \infty} \left\{ \int_0^1 f(t, x)dx \right\}$$

60. Prove that if

(*i*) a is positive ;

(*ii*) $f(x)$ is continuous except perhaps at the origin ;

(*iii*) $\displaystyle\int_0^a f(x)dx = \operatorname*{lt}_{\epsilon \to 0} \int_\epsilon^a f(x)dx$, where the limit exists ;

(*iv*) $\displaystyle g(x) = \int_x^a \frac{f(t)}{t}\,dt,$

then

$$\int_0^a g(x)dx = \int_0^a f(x)dx$$

[**Hint** :—Change the order of integration.]

61. If $0<a<b$ and $f(x)$ is continuous in $(0, b)$, prove that

$$\lim_{h\to(0+0)} \int_h^1 \frac{f(at)-f(bt)}{t}\,dt$$

exists and is equal to

$$f(0)\log\frac{b}{a} - \int_a^b \frac{f(x)}{x}\,dx.$$

62. Show that

$$\int_{-1}^{+1} \frac{\sqrt{(1-x^2)}\,dx}{(1+\alpha^2+2\alpha x)(1+\beta^2+2\beta x)} = \begin{cases} \pi/2(1-\alpha\beta), & \text{if } \alpha^2<1, \beta^2<1, \\ \pi/2\alpha(\alpha-\beta), & \text{if } \alpha^2>1>\beta^2. \end{cases}$$

63. A is a fixed point on the axis OX and an ellipse is drawn with OA as its minor axis. Calculate the value of $\int (x^2dy-y^2dx)$ taken along the ellipic arc from O to A, y negative and show that if it has its max. value, the eccentricity is $\sqrt{(1-64/9\pi^2)}$.

64. If, for $x \geqslant 0$, $\phi(x)$ is defined as

$$\lim_{n\to\infty} \frac{x^n+2}{x^n+1},$$

and

$$f(x)=\int_0^x \phi(t)\,dt,$$

prove that $f(x)$ is continuous but not differentiable for $x=1$.

65. By repeatedly employing the method of integration by parts to the integral

$$\int_0^x e^{-t}dt,$$

show that

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots+\frac{x^n}{n!}+e^x\int_0^x \frac{t^n}{n!}e^{-t}\,dt,$$

and deduce the Maclaurin's infinite series for e^x.

[$\int_0^x \frac{t^n}{n!}e^{-t}\,dt$ is positive and $<\frac{x^{n+1}}{n!}$, which $\to 0$ as $n\to\infty$, for every value of x].

66. Obtain, by integration, from the identity,

$$\frac{1}{1+t}=1-t+t^2-t^3+\ldots\ldots+(-1)^{n-1}t^{n-1}+(-1)^n\frac{t^n}{1+t},$$

the Maclaurin's infinite series for $\log(1+x)$ in $[-1,1)$.

67. Show that, $(a, b, >0)$

$$\frac{t^{a-1}}{1+t^b}=t^{a-1}-t^{a+b-1}+\ldots+(-1)^{n-1}t^{a+\overline{n-1}\,b-1}+(-1)^n\frac{t^{a+nb-1}}{1+t^b},$$

and justify the equation

$$\int_0^1\frac{t^{a-1}}{1+t^b}\,dt=\frac{1}{a}-\frac{1}{a+b}+\frac{1}{a+2b}-\frac{1}{a+3b}+\ldots,$$

Deduce that $\frac{1}{1}-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\ldots=\frac{1}{3}\left(\frac{\pi}{\sqrt{3}}+\log 2\right)$.

68. Show that

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots+\frac{(-1)^{n-1}}{2n-1}x^{2n-1}+(-1)^n\int_0^x\frac{t^{2n}}{1+t^2}\,dt,$$

and deduce the Maclaurins' infinite power series for $\tan^{-1}x$ in $[-1,1]$

69. *Show that, when $n\to\infty$,*

$$\int_0^1\frac{x^{n+1}}{1-x}\log x\,dx\to 0.$$

We write, $(0<a<1)$

$$\int_0^1\frac{x^{n+1}}{1-x}\log x\,dx=\int_0^a\frac{x^{n+1}}{1-x}\log x\,dx=\int_a^1\frac{x^{n+1}}{1-x}\log x\,dx=\mathrm{I}_1+\mathrm{I}_2,$$

$$\text{Now, } |\,\mathrm{I}_1\,|\leqslant a^{n+1}\int_0^a\left|\frac{\log x}{1-x}\right|dx, \text{ which }\to 0 \text{ as } n\to\infty : a \text{ being} <1.$$

Again, if we assign to the function $\log x/(1-x)$, the value 1 for $x=1$, it becomes continuous in $(a, 1)$. Thus $\log x/(1-x)$ is bounded in $(a, 1)$. Therefore

$$|\,\mathrm{I}_2\,|\leqslant \mathrm{M}\int_a^1 x^{n+1}\,dx, \text{ which }\to 0, \text{ as } n\to\infty$$

Hence the result.

70. From the identity $\frac{1}{1-x}=1+x+x^2+\ldots+x^n+\frac{x^{n+1}}{1-x}$,
show by integration in the interval $(0, 1)$ that

$$\int_0^1\frac{\log x}{1-x}\,dx=-\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

71. If p, q are positive, prove that

$$\int_0^\infty\left\{e^{-px}\,[1+(p+r)x]-e^{-qx}[1+(q+r)x]\right\}\frac{dx}{x^2}=q-p+r\log\frac{q}{p}$$

72. Show that $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx=\frac{\pi}{\sin p\pi}$, $(0< p < 1)$

It is easy to see that the given improper integral converges, when $0< p< 1$. We have

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx=\int_0^1 \frac{x^{p-1}}{1+x}\,dx+\int_1^\infty \frac{x^{p-1}}{1+x}\,dx$$

By means of the substitution $y=1/x$, we see that

$$\int_1^\infty \frac{x^{p-1}}{1+x}\,dx=\int_0^1 \frac{y^{-p}}{1+y}\,dy=\int_0^1 \frac{x^{-p}}{1+x}\,dx$$

$$\therefore \int_0^\infty \frac{x^{p-1}}{1+x}\,dx=\int_0^1 \left\{\frac{x^{p-1}+x^{-p}}{1+x}\right\}dx$$

$$=\int_0^1 (x^{p-1}+x^{-p})\left(1-x+x^2-\ldots\ldots\ldots\ldots+(-1)^n x^n+(-1)^{n+1}\frac{x^{n+1}}{1+x}\right)dx,$$

$$=\sum_{k=0}^{n} (-1)^k\left(\frac{1}{k+p}+\frac{1}{k+1-p}\right)+\mathrm{R}_n,$$

where $$|\,\mathrm{R}_n\,|=\int_0^1 \left(x^{p-1}+x^{-p}\right)\frac{x^{n+1}}{1+x}\,dx<\int_0^1 \left(x^{p-1}+x^{-p}\right)x^{n+1}\,dx$$

$$=\left\{\frac{1}{n+p+1}+\frac{1}{n+2-p}\right\}\to 0 \text{ as } n\to\infty.$$

$$\therefore \int_0^\infty \frac{x^{p-1}}{1+x}\,dx=\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{k+p}+\frac{1}{k+1-p}\right)=\frac{\pi}{\sin p\,\pi}.$$ (See Ex. 10, P. 289)

73. Prove that $\Gamma(x+1)=x\Gamma(x)$.

74. *Show that* $\Gamma(a)\Gamma(1-a)=\pi/\sin a\pi$.

As proved in Ex. 4, P. 271, we have

$$\beta(a,\ 1-a)=\frac{\Gamma(a)\Gamma(1-a)}{\Gamma(1)}=\Gamma(a)\Gamma(1-a).$$

Also $$\beta(a,\ 1-a)=\int_0^1 x^{a-1}(1-x)^{-a}dx.$$

By means of the substitution $x=y/(1+y)$ we may now show that

$$\int_0^1 x^{a-1}(1-x)^{-a}dx=\int_0^\infty \frac{y^{a-1}}{1+y}\,dy=\frac{\pi}{\sin a\,\pi}$$

Hence the result

Taking $a=\frac{1}{2}$, we deduce that

$$\frac{1}{2}\int_0^\infty e^{-y^2}dy=[\Gamma(\tfrac{1}{2})]^2=\pi, \ i.e., \int_0^\infty e^{-y^2}\,dy=\sqrt{\left(\frac{\pi}{2}\right)}$$

75. Show that

$$\Gamma\left(\frac{1}{a}\right)\Gamma\left(\frac{2}{a}\right)\Gamma\left(\frac{3}{a}\right)\ldots\ldots\Gamma\left(\frac{a-1}{a}\right)=\left\{\frac{(2\pi)^{a-1}}{a}\right\}^{1/2}$$

76. Prove that

$$\sum_{k=1}^{n}\log\Gamma\left(\frac{k}{n}\right)=\tfrac{1}{2}(n-1)\log(2\pi)-\tfrac{1}{2}\log n.$$

77. *Show that* $\int_0^1 \log\Gamma(x)dx=\frac{1}{2}\log(2\pi)$.

The point '0' is a point of infinite discontinuity of the integral.

We have $\Gamma(x)=\Gamma(x+1)/x$,

so that $\log\Gamma(x)=\log\Gamma(x+1)-\log x$.

We know that the integral of $\log x$ converges at 0 and the integral of $\log\Gamma(x+1)$ is proper, and hence the integral of $\log\Gamma(x)$ is convergent. By means of the substitution $x=1-y$, we see that

$$\int_0^1 \log\Gamma(x)dx=\int_0^1\log\Gamma(1-y)dy=\int_0^1\log\Gamma(1-x)dx$$

$$=\tfrac{1}{2}\int_0^1\log[\Gamma(x)\Gamma(1-x)]dx=\tfrac{1}{2}\int_0^1\log\frac{\pi}{\sin\pi x}=\tfrac{1}{2}\log(2\pi)$$

78. *Prove that* $(\sqrt{\pi})\Gamma(2x)=2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2})$.

We have $$\frac{\Gamma(x)\Gamma(\frac{1}{2})}{\Gamma(x+\frac{1}{2})}=\beta(x,\tfrac{1}{2})=\int_0^1\frac{t^{x-1}dt}{\sqrt{(1-t)}},$$

and $$\frac{\Gamma(x)\Gamma(x)}{\Gamma(2x)}=\beta(x,x)=\int_0^1 t^{x-1}(1-t)^{x-1}dt.$$

In the latter integral, put $v=2t-1$ so that we have

$$\beta(x,x)=\frac{1}{2^{2x-1}}\int_{-1}^{+1}(1-v^2)^{x-1}dv=\frac{1}{2^{2x-1}}\int_0^1(1-w)^{x-1}w^{-\frac{1}{2}}dw,\ (v^2=w)$$

$$=\frac{1}{2^{2x-1}}\beta(\tfrac{1}{2},x)=\frac{1}{2^{2x-1}}\beta(x,\tfrac{1}{2}).$$

Since $\Gamma(\frac{1}{2})=\sqrt{\pi}$, we can now obtain the given result.

ANSWERS

Pages 137-139.

2. $F(y)$ is 2 for y rational and is 0 for y irrational.

3. $\frac{3}{2}$. 4. $f(x)$ is integrable ; 1 being the only limiting point of the aggregate of the points of discontinuity ; the integral is 2.

9. Use §78·2, p. 102.

10. $A_{n+1}-A_n=[f(n+1)-\int_n^{n+1} f(x)\,dx,]$ which is <0 by § 88 :

Also $A_n=\{ [f(1)-\int_1^2 f(x)dx]+\ldots\ldots[f(n-1)-\int_{n-1}^{n} f(x)dx]\}+f(n)>0.$

$\therefore \{A_n\}$ is monotonically decreasing and bounded below.

Page 158.

10 (*i*) $x=n$, n being any integer (*ii*) Every point x.

15. Does not exist.

Page 169.

(*i*) log 2. (*ii*) π. (*vi*) -1. (*viii*) 2. The integrals (*iii*), (*iv*), (*v*) and (*vii*) do not exist.

Page 176.

Ex. 4. C stands for *Convergent* and N.C. for *Not convergent.*

(*i*) C for $n<1$. (*ii*) C for $n<2$. (*iii*) C for $a>0$.
(*iv*) N. C. (*v*) C. (*vi*) C.
(*vii*) C for $m>0$ and $n>0$· (*viii*) C for $a>0$. (*ix*) C.
(*x*) C for $p>-1$. (*xi*) N. C. (*xii*) N. C.

Page 182.

Ex. 3. (*i*) C for $p>(1+m+n)$; m, n, being >0. Consider also the case when either m or n or both are negative

(*ii*) C for $(n-m)>\frac{1}{2}$. (*iii*) C for $0<m, n<1$.
(*iv*) C for $n>0$. (*v*) C for $m<-1$.
(*vi*) C for $0<(a+1)<b$.

Page 186. **Ex.** (*i*) C (*ii*) C for $a>0$.

Page 190. **Ex.** 4. (*i*) C. (*ii*) C. (*iii*) C. (*iv*) N. C.
(*v*) C for $(1-m)<n<(1+m)$. (*vi*) C for $0<a<4$.
(*vii*) C for $-1<n<2$.
(*viii*) C for $n>0$, $-1<m<n$ and for $n<0$, $0>m>(n-1)$.

Page 197.

17. The integral is $2/a$, $-2/a$ or 2 according as $a>1$, $a<-1$ or, $-1\leqslant a\leqslant 1$.

18. $\pi/2$ if $2m\pi<a<(2m+1)\pi$;
$-\pi/2$ if $(2m+1)\pi<a<(2m+2)\pi$ and 0 if $a=2m\pi$ or $(2m+1)\pi$; m being any integer.

21. $-\pi/4n$.

Page 202.

Ex. (*i*) Does not exist. (*ii*) does not exist. (*iii*) 0. (*iv*) 0.

Page 203. **Ex.** (*i*) Discontinuous. (*ii*) continuous.

Page 205.

Ex. 3. (*i*) Not differentiable. (*ii*) not differentiable.

Page 208. **Ex.** 3. (*i*) equal. (*ii*) unequal.

Page 218.

(*i*) min. at $(\frac{2}{3}, -\frac{4}{3})$. (*ii*) min. at (1, 2) and (−1, 2) ; max. at (0, 0)

(*iii*) min. at $(0,0)$ if $a>\frac{1}{4}$.
(*iv*) min. at $[\frac{1}{3}(b-2a), \frac{1}{3}(a-2b)]$. (*v*) max. at $(2, 1)$.
(*vi*) max. or min. at $(\frac{1}{3}a, \frac{1}{3}a)$ according as $a>0$ or $a<0$.
(*vii*) min. at $(0, 0, 0)$. (*viii*) min. at $(0, 0, -1)$.

5. Extremes at $(0, 0)$, $(-1, 0)$, $(-\frac{1}{2}, 0)$.

6. The positions of the corresponding points P, Q are
$(3a, \pm\sqrt{12a})$, $(3a, \pm\sqrt{12a})$; $(0, 0)$, $(\pm\sqrt{12a}+3a, 0)$;
$(a, 2a)$, $(3a\mp\sqrt{6a}, \pm\sqrt{6a})$; $(a, -2a)$, $(3a\pm\sqrt{6a}, \pm\sqrt{6a})$.

Page 221.

3. $\phi(x)$ is not unique.

5. (*i*) $\phi(x)$ unique near $(1, -1)$ but not near $(0, 0)$
(*ii*) $\phi(x)$ unique near $(1, 1)$. (*iii*) $\phi(x)$ not unique but the equation possesses a unique continuous solution $x=\psi(y)$ near $(1, -1)$
(*iv*) Does not exist.

Page 226.

3. $4(f_{u^2}+f_{v^2})$, $axy(x^2-y^2)$.

Page 229.

Ex. 1 $uv+vw+wu=ac-b^2$. **Ex. 2** $u^2-v^2=8w$.

Page 234.

4. (*i*) $3a^2$ for (a, a, a). (*ii*) $3a^2$ for (a, a, a), $(-a, -a, -a)$.
(*iii*) $3a^2$ for (a, a, a), $(-a, -a, a)$, $(-a, a, -a)$, $(a, -a, -a)$.

5. $-a^2$ (min.) at $(a, -a)$ and $(-a, a)$; $\frac{1}{3}a^2$ (max.) at $(a/\sqrt{3}, a/\sqrt{3})$ and $(-a/\sqrt{3}, -a/\sqrt{3})$.

8. $(1, 1)$ and $(-1, -1)$. **9.** $(-2/\sqrt{14}, -1/\sqrt{14}, -3/\sqrt{14})$.

10. The extreme value is

$$\frac{1}{27}(\log abck)^3/(\log a \log b \log c)$$

and is a max. or min. according as $\log(abck)/(\log a \log b \log c)$ is positive or negative.

12. $(\sqrt{\Sigma a^2}-2a)/\sqrt{3}$.

17. The symmetrical stationary value is $\frac{1}{9}(1+m)$ and there are three unsymmetrical values, each being equal to $(m^2-m+1)/(m+1)^2$.

18. $(a/10, a/10)$.

Page 242. **8.** $\frac{2}{3}x^3$

Page 252. **1.** $\frac{7}{3}a^3$, $\frac{48}{15}a^3$. **3.** $2\log 2$. **5.** $\frac{1}{84}$. **6.** (*i*) $38a^4/\sqrt{3}$

Page 258.

1. (*i*) $\frac{1}{8}a^2b^2(a^2+b^2)$. (*ii*) $(e^{ab}-1)/a-b$.
(*iii*) 0. (*iv*) $2[a\sinh^{-1}(a/c)+c-\sqrt{(a^2+c^2)}]$. (*v*) 0.

Page 262.

4. $$\int_{-1}^{0}dx\int_{-x}^{\frac{1}{3}(2x+5)}f\,dy+\int_{0}^{1}dx\int_{x}^{\frac{1}{3}(2x+5)}f\,dy+\int_{1}^{2}dx\int_{(2x-1)}^{\frac{1}{3}(2x+5)}f\,dy$$

5. $\pi/96$. **8.** $$\int_0^a dy\int_0^{\sqrt{(4ay)}}\phi\,dx+\int_a^{3a}dy\int_0^{3a-y}\phi\,dx$$

10. $12-16\log 2$. **12.** $\frac{1}{2}\pi\log[2e/(1+e)]$. **14.** $47/24$.

15. $8\sin^{-1}(\sqrt{10}/4)+\frac{1}{2}\log[(\sqrt{3}+\sqrt{5})/\sqrt{2}]-3\sqrt{15}/4$.

Page 272.

6. $\frac{2}{3}\pi\sqrt{(a^2+b^2)^3}$. **7.** $\frac{1}{9}a^3(3\pi-4)$. **11.** $\frac{1}{4}\beta(\frac{3}{4}, \frac{1}{2})\,\beta(\frac{1}{2}, \frac{1}{4})$

12. In the $r\theta$ plane the field of integration is bounded by the lines $r=2a$, $r=4a$; $\theta=\frac{1}{2}\pi$, $r=2a\sec^2\theta$,

13. $\int_0^1 \int_0^c V[u(1-v), uv]u \; dudv$

14. $8(\sqrt{a_1}-\sqrt{a_2})\sqrt{-a_3}$.

15. If λ, μ denote the values of the parameter for the confocal ellipse and hyperbola respectively through (x, y), then

$$\lambda+\mu=(x^2+y^2+c^2, \quad \lambda\mu=c^2x^2,$$

so that $x=\sqrt{\lambda\mu}/c, \qquad y=\sqrt{(\lambda-c^2)(c^2-\mu)}/c.$

$\therefore$ the required area

$$\int\int dxdy=\int\int\frac{\partial(x, y)}{\partial(\lambda, \mu)} \, d\lambda d\mu = \tfrac{1}{4}\int\int\frac{\mu-\lambda}{\sqrt{\lambda\mu(\lambda-c^2)(c^2-\mu)}} \, d\lambda d\mu,$$

$$=\tfrac{1}{6}c^2(\sqrt{10}-2)\sin^{-1}\tfrac{1}{3}$$

where for the double integral $\frac{1}{3}c^2 \leqslant \mu \leqslant \frac{2}{3}c^2$; $\frac{4}{3}c^2 \leqslant \lambda \leqslant \frac{5}{3}c^2$.

16. The field of integration in the xy plane is the positive quadrant of the ellipse $x^2/9+y^2/5=1$. As (x, y) moves along x-axis from (0, 0) to (2, 0) the point (λ, μ) moves in the $\lambda\mu$ plane along the line $\lambda=2$ from (2, 0) to (2, 2) ; as (x, y) moves along x-axis from (2, 0) to (3, 0), (λ, μ) moves along $\mu=2$ from (2, 2) to (3, 2) ; as (x, y) moves along the arc of the ellipse from (3, 0) to (0, $\sqrt{5}$), (λ, μ) moves along $\lambda=3$ from (3, 2) to (3, 0) ; finally as (x, y) moves along $x=0$ from (0, $\sqrt{5}$) to (0, 0), (λ, μ) moves along $\mu=0$ from (3, 0) to (2, 0). Thus the region in $\lambda\mu$ plane is the rectangle (2, 3 ; 0, 2).

Since, as may easily be seen,

$$x=\tfrac{1}{2}\lambda\mu, \; y=\tfrac{1}{2}\sqrt{(\lambda^2-4)(4-\mu^2)},$$

$\partial(x, y)/\partial(\lambda, \mu)$ can be calculated.

17. $\frac{1}{78}$. Do this question by the substitution $x=r\cos\theta$, $y=r\sin\theta$ also.

18. $(n-p)(l-m)[(l+m)(a+b)-(n+p)(a-b)]/32a^2b^2$.

22. $\frac{1}{2}\pi[\beta(\frac{3}{4}, \frac{1}{2})-\beta(\frac{5}{4}, \frac{1}{2})]$. 24. $\log(a'/a)\log(b'/b)\log(c'/c)$.

27. $\Gamma(l)\,\Gamma(m)\,\Gamma(n)/\Gamma(l+m+n+1)$. 29. $384\pi/5$.

30. $\frac{8192}{785}\,a^7\,k(k^2+\frac{8}{33})$.

Page 275. 1. $9(3\pi-5)/4$. 3. $\frac{1}{6}ka^4$. 5. $\frac{1}{8}\pi c^3$.

12. $\frac{16}{35}\pi\,(b^{-\frac{5}{8}}-a^{-\frac{5}{8}})(d^{-\frac{7}{8}}-c^{-\frac{7}{8}})$.

Page 288.

4. $\dfrac{4}{\pi}\,\Sigma\,\dfrac{\sin\frac{1}{2}n\pi\,\sin nx}{n^2}$.

5. $\dfrac{2}{\pi}\,\Sigma\left(\dfrac{\sin\frac{1}{2}n\pi}{n^2}-\dfrac{\pi\cos\frac{1}{2}n\pi}{2n}\right)\sin nx$.

7. $\dfrac{2}{\pi}-\Sigma\,\dfrac{4\cos 2nx}{(4n^2-1)\pi}$.

8. $\dfrac{\sin 2x}{1}+\dfrac{\sin 4x}{2}+\dfrac{\sin 8x}{4}+\dfrac{\sin 10x}{5}+\dfrac{\sin 14x}{7}+\ldots\ldots$

9. (*a*) $\cosh\pi$. (*b*) $-e^{-x}$ for $-\pi<x<0$ and 0 for $x=0, \pm\pi$, e^{-x} for $-\pi\leqslant x\leqslant 0$.

11. $\Sigma\left[\dfrac{-2(-1)^n\pi}{n}-\dfrac{4}{\pi n^3}\,[1-(-1)^n]\right]\sin nx$. 13. 0

15. (i) $\frac{4lk}{\pi^2}\Sigma\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$.

(ii) $\frac{kl}{4}+\frac{4kl}{\pi^2}\Sigma\frac{1}{n^2}\left(\cos\frac{n\pi}{2}-\cos^2\frac{n\pi}{2}\right)\cos\frac{n\pi x}{l}$.

16. $\Sigma\left[\frac{2(-1)^{n-1}}{n\pi}-\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\sin\frac{n\pi x}{2}$.

18. $\frac{4-2\sqrt{2}}{\pi}+\frac{4\sqrt{2}}{\pi}\Sigma\left[\frac{(-1)^n-\sqrt{2}}{16n^2-1}\right]\cos 4nx$.

Pages 290—300.

9. 0, -1. For $f(x)=|x|$ and $c=0$ the first limit does not exist but the second exists and is equal to 1. For $c\neq 0$, both the limits exist.

10. (i) $f(0)=0$. (ii) $f(0)$ is not determinable. (iii) $f(0)=a$.

11. By Taylor's theorem, $f(0)-f(x)=-xf'(x)+1/2!\ x^2f''(\xi)$ so that $f'(x)-f(x)/x>0$ and $\therefore$, $d[f(x)/x]/dx$ is positive.

12. Use Cor. to §50·1 on p. 73. 17. $f_{yx}(0, 0)=1\neq 0=f_{xy}(0, 0)$.

18. $f_{xy}(0, 0)=6a^2b\neq 6c^2b=f_{yx}(0, 0)$.

21. $f(x, y)$ is differentiable.

23. $w+abc\,v^2=(bc\,A^2+ca\,B^2+ab\,C^2)u$.

24. min. at (1, 2, 0). 25. $\log(3/e^2)$.

26. max. at (0, 0) and min. at (1, 2) and $(-1, 2)$.

27. min. at (1, 1, 1) ; max. at $(-1, -1, -1)$.

28. $(0, 0, \frac{1}{2})$. 32. $\frac{1}{27}abc$.

33. $(a/\sqrt{3}, a/\sqrt{3}, a/2\sqrt{3})$. 34. $(\frac{1}{n}\Sigma x_i, \frac{1}{n}\Sigma y_i, \frac{1}{n}\Sigma z_i)$. 35. $\frac{1}{27}C^6$.

37. The extreme distance, d, is given by $\Sigma a^{2p/(p-2)}=d^{2p/(p-2)}$

40. $\frac{2}{9}abc(3\pi-4)$. 42. $\frac{2}{3}ma^3$.

44. $\frac{1}{2}(k^2-h^2)\log(b/a)$.

53. If $S_n=\left(\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots\ldots+\frac{1}{\sqrt{n}}\right)$, then since

$$1/\sqrt{r}<\int_{r-1}^{r}(1/\sqrt{x})dx<1/\sqrt{(r-1)},$$

we have, on putting $r=2,\ldots, n$,

$$S_n-1<2(\sqrt{n}-1)<S_n-1/\sqrt{n}$$

i.e., $$2+\frac{1}{n}-\frac{2}{\sqrt{n}}<\frac{S_n}{\sqrt{n}}<2-\frac{1}{\sqrt{n}}$$ so that $S_n/\sqrt{n}\to 2$.

54. The formula follows from the following 3 results :—

(i) $$\frac{\pi}{2}=\frac{2.2.4.4.6.6\ldots(2n)(2n)}{1.3.3.5.5.7\ldots(2n-1)(2n+1)}\cdot\frac{\int_0^{\frac{1}{2}\pi}\sin^{2n}x\,dx}{\int_0^{\frac{1}{2}\pi}\sin^{2n+1}x\,dx}$$

(ii) $0\leqslant\sin^{2n+1}x\leqslant\sin^{2n}x\leqslant\sin^{2n-1}x$.

(iii) $$\frac{\int_0^{\frac{1}{2}\pi}\sin^{2n-1}x\,dx}{\int_0^{\frac{1}{2}\pi}\sin^{2n+1}x\,dx}=\frac{2n+1}{2n}.$$