ELECTRICAL MEASUREMENTS

AND THE

CALCULATION OF THE ERRORS INVOLVED

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Part I

BY

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CONTENTS

ERRATA

Page 19.

For 'Example 5', read throughout 'Example 6'.

Page 22.

In last line, for first appearance of '=', read '+'.

Page 27.

Line 2.—Before second expression, add '±'.

Line 4.—For first '12' read '120'.

Lines 4 *and* 6.—For 0·00096 read 0·0096.

Page 30.

Line 12.—For 'nominator' read 'numerator'.

Line 17.—For third fraction read $\frac{\Delta_{c1}V'_1}{V'_1}\left(\frac{v'}{V'_1 \pm v'}\right)$.

Before final paragraph, add: 'When v is negative ($V = V_1 - v$), $\Delta_{c1}V'_1$ is not necessarily equal to $\Delta_c V'_1$, but must satisfy the equation $V = V_1 - v$, and therefore the equations

$\Delta_c V' = \pm \Delta_{c1}V'_1 + \Delta_c v'$ and $V' + \Delta_c V' = V'_1 \pm \Delta_{c1}V'_1 - v + \Delta_c v'$ . . . . (A)

$\frac{\Delta_{c1}V'_1}{V'_1}$ can be positive, negative, or zero, according to (A);

its sign is to be preserved in the equation, giving $\frac{\Delta R''}{R'}$.

Page 31.

Line 16.—After 'is negligible' read 'As $\Delta_c V' = \frac{0{\cdot}4 \times 8}{100} = 0{\cdot}032v$

and $\Delta_c V'_1 = \frac{0{\cdot}4 \times 2}{100} = 0{\cdot}008$, we get from (A),

$\Delta_{c1}v'_1 = 0{\cdot}032 - 0{\cdot}008 = 0{\cdot}024$; no other value would keep $\Delta_c V'$ within the limit of $0{\cdot}032v$. We have therefore $\frac{\Delta_{c1}V'_1}{V'_1} = \frac{0{\cdot}024 \times 100}{10} = 0{\cdot}24\%$'.

The constructional error on the voltmeter is therefore

$$\frac{\Delta_{c1}V'_1}{V'_1}\left(\frac{v'}{V'_1 \pm v'}\right) + \frac{\Delta_c v'}{v'}\left(\frac{v'}{V'_1 \pm v'}\right) = \pm\left[0{\cdot}24\left(\frac{2}{8}\right) + 0{\cdot}4\left(\frac{2}{8}\right)\right] = \pm\, 0{\cdot}16\%.$$

Delete lines 17–22.

Page 31—*continued.*

Line 26.—Delete 0·2% and substitute 0·16%

„ 0·435% „ „ 0·395%

Line 27.—Delete 0·435% „ „ 0·395%

Page 36.

Sub-heading (*c*).—For '$R_1\ R_4$' read '$R_1 \div R_4$'.

Page 48.

Line 4.—For 'resistance' read 'resistances'.

Line 5.—For '$R_1 + R$ have' read '$(R_1 + R)$ has'.

Page 49.

Equation (24a).—

Delete $\frac{\Delta R'_1}{R'_1}\left(\frac{+r'}{R'_1 \pm r'}\right)$

and substitute $\frac{\Delta_1 R'_1}{R'_1}\left(\frac{r'}{R'_1 \pm r'}\right)$

and add at end of equation 'when r is negative, $R_2 = R_1 - r$, also $\Delta_1 R'_1$ is not necessarily equal to $\Delta R'_1$, but must satisfy the equation $R_2 = R_1 - r$ and therefore also the equations $\Delta R'_2 = \pm \Delta R'_1 + \Delta r'$ and $R'_2 + \Delta R'_2 = R'_1 \pm \Delta R'_1 - r' + \Delta r'$ (B) $\frac{\Delta_1 R_1}{R_1}$ can be positive, negative, or zero, according to (B), its sign is to be preserved in the equation, giving $\frac{\Delta Rx}{R'x}$. (See errata for p. 31.)'

Page 56.

Line 19.—For a_0 substitute a'_0.

Page 57.

Line 4.—Insert 'as' in front of equation and 'decreases' after 'z'.

At end of line the last term in the numerator reads $\frac{1+k}{k}$

Page 75.

Lines 16 *and* 17.—In the equations delete g and substitute 9.

Page 76.

Line 19.—Add at end of line,

$$\text{and} \quad \frac{\Delta V'_1}{V'_1} = \frac{\Delta e'}{e'} + \frac{v'_D}{V'_1} + \frac{\Delta a'}{a'_A + a'_B} \quad \text{because}$$

$$e \cong V_{12} \cong V_1, \frac{\Delta e'}{e'} \cong \frac{\Delta V'_1}{V'_1}.$$

· *Equation* 35 (*a*).—Right-hand side of equation, numerators of second and fourth terms. For $\Delta V'_D$ read $\Delta v'_D$ and for V'_D read v'_D.

Add after the equation (35a),

$$\frac{\Delta V'_1}{V'_1} = \frac{\Delta e'}{e'} + \frac{\Delta v'_D}{V'_1} \cdot \frac{a'_A}{a'_A + a'_B} + \frac{v'_D}{V'_1} \cdot \frac{\Delta a'}{a'_A + a'_B}.$$

Page 78.

Line 20.—For $\frac{E}{r} + \frac{e}{R_1}$, substitute $\frac{E}{r} = \frac{e}{R_1}$.

Page 82.

Fig. 42 (*b*).—For right-hand resistance R*h* read Rh_1.

Page 84.

Line 20.—Delete the word 'between'.

Page 86.

Line 8.— Delete $\frac{de}{e} + \frac{dV_1}{V_1}\left(\frac{V_1}{V_n} - 1\right)$ and substitute $\frac{V_1}{V_n} \cdot \frac{dV_1}{V_1}$.*

Line 10.— Delete $\frac{de}{e} + \frac{dV_1}{V_1}\left(\frac{V_1 - V_n}{V_n}\right)$ and substitute $\frac{V_1}{V_n} \cdot \frac{dV_1}{V_1}$.

Line 13 *and last line*.—

Delete $\frac{\Delta e'}{e'} + \frac{\Delta V'_1}{V'_1}\left(\frac{V'_1 - V'_n}{V'_n}\right)$ and substitute

$$\frac{V'_1}{V'_n} \cdot \frac{\Delta_1 V'_1}{V'_1}.$$

*At bottom of page 86 add footnote:

As $e \cong v_1, \frac{de}{e} \cong \frac{dV_1}{V_1} \cdot \frac{de}{e} - \frac{V_1}{V_1} \cdot \frac{dV_1}{V_1}$ will cancel in line 6.

Page 87.

Line 1.—After the equation delete 'where' and the whole of line 2, and substitute 'when r_s and v_s are negative, $R_n = R_1 - r_s$, $V_n = V_1 - v_s$ then $\Delta_1 V'_1$ is not necessarily equal to $\Delta V'_1$ but must satisfy the equation $V_n = V_1 - v_s$ and therefore also satisfy the equations

$$\Delta V'_n = \pm\Delta_1 V'_1 + \Delta v_s$$

and $V_n + \Delta V'_n = V_1 \pm \Delta_1 V'_1 - v'_s + \Delta v'_s$ (C).

$\dfrac{\Delta_1 V'_n}{V'_n}$ can be positive, negative or zero, according to (C).

Its sign is to be preserved in the equation for $\dfrac{\Delta I'_n}{I'_n}$.

(See errata to p. 31.)'

Page 87.

Line 13.— At end of line add $\dfrac{dV_1}{V_1} = \dfrac{de}{e}$.

Line 14.— Delete $\dfrac{de}{e}$ at beginning of equation.

Line 15.— At beginning of line, for $\left(\dfrac{V_1}{V_n} - 1\right)$ substitute $\dfrac{V_1}{V_n}$.

Line 16.— At end of line, for $\dfrac{V_1 - V_n}{V_n}$ substitute $\dfrac{V_1}{V_n}$.

Page 88.

Line 2.— At beginning, delete $\dfrac{\Delta e'}{e'}$ and in the last expression take away $\dfrac{\Delta V'_1}{V'_1}.\dfrac{(V_1 - V'_n)}{V_n}$ and substitute $\dfrac{\Delta_1 V_1}{V'_1}\ \dfrac{V'_1}{V'_n}$.

Line 3.— Substitute, + for − between terms,.

Page 89.

Lines 16, 17, 18. *Equations* 39 *and* 39(a).—

Delete $\dfrac{\Delta e'}{e'}$ in each equation.

Delete also in each equation the factors $\dfrac{\Delta V'_1}{V'_1}.\dfrac{V'_1 - V'_n}{V'_n}$ and in their place put $\dfrac{\Delta_1 V'_1}{V'_1}.\dfrac{V'_1}{V'_n}$.

Page 95.

End of line 5.—For 'and' read 'inside'.

Page 98.

Last line.—First term of equation 43(a),

$$\text{for } \frac{d^2a}{dt} \text{ read } \frac{d^2a}{dt^2}.$$

Page 116.

Line 15.—Expression on left of equation, delete J, leaving a_2.

Page 121.

Equation (70).—For T substitute τ.

Page 123.

Line 11.—For '$t = o$ and ω_0 at t' read $t = t$ and ω_0 at $t = o$'.

Page 128.

Line 1.—Delete T and substitute τ.

Page 129.

Line 9.—Delete J on left-hand side of equation, also for result 6·83 read 8·34.

Line 10.—Replace 6·83 by 8·34 and change 0·92 sec. to 0·755 sec.

Page 139.

Line 1.—Add dt to each of the terms to be integrated. Right-hand side of equation reads

$$J\int_0^t \frac{d^2a}{dt^2}dt + D_1\int_0^t \frac{da}{dt}\cdot dt.$$

Page 141.

Lines 5 *and* 6.—Break the line dividing each of the expressions under the square root and add a $-$ sign, giving $-\dfrac{\Phi_0{}^2}{L_1J}$ in each case.

Page 143.

Line 6.—Second integral replace by l, thus giving $l\int_0^t di$.

Page 153.

Line 15.—For 'area' read 'cross-sectional area'.

Page 158.

Line 7.—For $\tau = 17 \times 10^{-4}$ read $\tau = 0{\cdot}17 \times 10^{-4}$.

Page 161.

Line 20.—For 'resistance' read 'resistances'.

Page 163.

Line 7.—For $\frac{dR_T}{R}$ read $\frac{dR_T}{R_T}$.

Page 164.

Fig. 84.—Mark cell as E and resistance as Rh.

Page 168.

Line 18.—For 'open' read 'closed'.

Page 171.

Fig. 90.—Mark left-hand resistance R_1 and that on the right R_2.

Page 189.

Lines 15 *and* 16.—For (119) read (120).

ELECTRICAL MEASUREMENTS AND THE CALCULATION OF THE ERRORS INVOLVED

PART I

CHAPTER I

ERRORS IN MEASUREMENTS

(1) Introductory

A MEASUREMENT of any kind is of use only when the limits of the maximum possible error are known.

The statement that the length of a certain bar is 3 ft. has no significance, and does not provide any information as to the usefulness of the bar, unless the length of the bar is precisely 3 ft. If, however, the length of the bar is stated to be 3 ft. $\pm$ 0·5 in., or, in other words, that the length of the bar is definitely between 35·5 in. and 36·5 in., we know that to whatever use the bar may be put it certainly cannot be used as a footrule; if the length were given as 3 ft. $\pm \frac{1}{64}$ in. we should know that the bar can be used as a footrule.

In the same way the statement that a condenser has a capacity of 1 μF is devoid of significance, but if the capacity were given as 1 μF $\pm$ 5%, that is, 1 μF $\pm$ 0·05 μF, we would know that this condenser can be used in such electrical applications where a capacity of 0·95 μF will do just as well as a capacity of 1·05 μF, but will not do as a laboratory standard. To use this condenser as a laboratory standard we would have to be sure that its capacity is definitely known to be 1 μF $\pm$ 0·1% or even 1 μF $\pm$ 0·01%; the importance of knowing the limits within which a certain value lies is, therefore, evident.

The calculation of the error or limits is more a matter of common sense than of strict mathematical exactitude, simply because the elements on which the calculations are based are known only approximately. What, however, we need to know is the maximum possible value of the error or limit, so as to be sure that the magnitude measured is without doubt contained within the specified limits.

(2) The Maximum Possible Error

If y is the exact value of the magnitude measured, an experiment might by chance give this exact value. Unfortunately, the experimenter would never know it. If, then, the value given by the experiment be y', the error will be $y - y' = \pm \delta y$. δy can be positive or negative, because y' can be smaller or greater than the real value y. If y' is smaller than y, $y - y' = + \delta y$ and the error is positive ; if y' is greater than y, $y - y' = - \delta y$, then the error is negative.

We never know the exact value of y or δy, but to make certain that the value of the magnitude measured lies within certain limits is to make sure that the error δy is definitely smaller than the maximum possible value of the error Δy, so that we can be sure that $y' - \Delta y < y < y' + \Delta y$.

Δy is the maximum possible error.

(3) The Relative Error

The maximum possible error Δy does not provide any information as to the quality of the measurement. What provides this information is the relative maximum error $\frac{\Delta y}{y}$.

Example 1. Two e.m.f.s, one of 500 volts, the other of 10 volts, were measured, and the error in each case was $\pm$ 1 volt. Which is the better measurement?

$\Delta y = \pm 1$ volt in each case, yet the 500-volt measurement is evidently far better in quality than the 10-volt measurement ; the relative errors are respectively

$$\frac{\Delta y}{y} = \pm \frac{1}{500} \text{ or } \pm 0{\cdot}2\% \text{ and } \frac{\Delta y}{y} = \pm \frac{1}{10} \text{ or } \pm 10\%.$$

The smaller Δy is, the better the measurement, and in order to diminish Δy the experimenter should :

(*a*) Have a good knowledge of the methods of measurements used
(*b*) Choose the most suitable method, or even improve the method
(*c*) Be familiar with the equipment used, and choose the most suitable
(*d*) Work in the best possible conditions
(*e*) Eliminate as far as possible all conditions tending to increase the error
(*f*) Understand and know how the error is calculated.

(4) Classification of Errors

The errors occurring in electrical measurements may be broadly divided into two main classes.

(*a*) *The Systematic Errors*. These are the errors inherent in the equipment and the method used. They are also dependent on some of the conditions under which the experiment is conducted ; conditions which are known, and the influence of which can be eliminated, minimised or calculated.

(*b*) *The Accidental Errors*. These are due to conditions over which we have only partial control or no control at all, such as noise or smoke in the laboratory, vibration of the building, physical conditions of the experimenter (such as fatigue, faulty and incompetent manipulation due to lack of knowledge or practice, etc.).

(5) The Systematic Errors

These errors can be subdivided into:

(*a*) THE CONSTRUCTIONAL ERROR. This error arises because the equipment used can be guaranteed only within certain limits. For instance, a resistance may be guaranteed to $\pm 0{\cdot}1\%$ or $\pm 0{\cdot}01\%$, so that when these guarantees apply to, say, resistances of 1000 Ω, their true values are between $1000 \pm 0{\cdot}1\%$; that is, $1000 + \frac{0{\cdot}1}{100} \times 1000 = 1001\ \Omega$ and $1000 - \frac{0{\cdot}1}{100} \times 1000 = 999\ \Omega$ or $1000 \pm 0{\cdot}01\%$; that is, $1000 + \frac{0{\cdot}01}{100} \times 1000 = 1000{\cdot}1\ \Omega$ and $1000 - \frac{0{\cdot}01}{100} \times 1000 = 999{\cdot}9\ \Omega$. If these resistances are used in a d.c. measurement and at the temperatures for which the guarantee of $\pm 0{\cdot}1\%$ or $\pm 0{\cdot}01\%$ holds, the constructional errors are as calculated. If, however, we use them for a.c., particularly at higher frequencies, their resistances will change, owing to skin effect, and the guarantee will not hold. The resistances will also have a certain amount of self-inductance and distributed capacity, and the values of these have to be known in order to calculate their effect on the measurement. All information should be provided by the manufacturer of these resistances, whose indications and figures have to be considered in the calculation of the error.

With multi-dial resistance boxes, the limits of error may be different for different dials; for instance, the constructional error for a five-dial resistance box is given by the manufacturers thus:

First	dial of $10 \times 0{\cdot}1\ \Omega$	guaranteed to	$\pm 0{\cdot}1\%$
Second	,, ,, $10 \times 1\ \Omega$	,, ,,	$\pm 0{\cdot}1\%$
Third	,, ,, $10 \times 10\ \Omega$	,, ,,	$\pm 0{\cdot}02\%$
Fourth	,, ,, $10 \times 100\ \Omega$	,, ,,	$\pm 0{\cdot}01\%$
Fifth	,, ,, $10 \times 1000\ \Omega$	,, ,,	$\pm 0{\cdot}01\%$

When using this box, the error has to be calculated accordingly.

Some resistances have their limits indicated in two forms, such as " $1{\cdot}5\%$ or $0{\cdot}01\ \Omega$, whichever is the greater "; here only the greater value of the two has to be taken as the constructional error.

Say we have such a guarantee, applicable to a resistance of 1 Ω, variable in steps of $0{\cdot}01\ \Omega$; if this resistance is partly in circuit, say $0{\cdot}5\ \Omega$, the error is $\pm 0{\cdot}01\ \Omega$, and not $\pm 0{\cdot}5 \times \frac{1{\cdot}5}{100} = \pm 0.0075\ \Omega$; the relative error is therefore $\pm \frac{0{\cdot}01}{0{\cdot}5} = \pm 0.02$ or $\pm 2\%$; if this resistance is totally in the circuit, then the error is not $\pm 0{\cdot}01\ \Omega$ but $\pm 1 \times \frac{1{\cdot}5}{100} = \pm 0.015\ \Omega$, and the relative error is $\pm \frac{0{\cdot}015}{1} = \pm 0{\cdot}015$ or $\pm 1{\cdot}5\%$. Note that if 0.01 Ω (that is, if one step only of this resistance were used) the error would be $\pm 0{\cdot}01\ \Omega$ or 100%. It stands to reason, therefore, that this resistance can be used only when an appreciable part of it is in circuit.

The effects of frequency, self-inductance, and distributed capacity should also be given by the manufacturer, either by a formula, graph or phase angle, from which the constructional error can be calculated.

Inductances, mutual inductances, and capacities, have their constructional error indicated in a similar way, and the same applies to tuning-forks, wavemeters, generators, instruments, etc.

With indicating instruments such as voltmeters, ammeters, etc., the constructional error is generally not uniform over all the scale, but is usually greatest in the first quarter or the first third of it. The manufacturer may therefore indicate the error as, say, $\pm 1\%$ in the first third of the scale, and $\pm 0{\cdot}5\%$ in the rest, or in any other manner. For instance, a manufacturer indicates the constructional error in a voltmeter, of 150 scale divisions corresponding to 150 volts, as $\pm 0{\cdot}4\%$ of the maximum reading in the first third of the scale and $\pm 0{\cdot}8\%$ of the reading in the rest; then at any reading in the first third of the scale the error will be: $\pm \frac{0{\cdot}4}{100} \times 150 = \pm 0{\cdot}6$ V., and the relative error can be enormous at small readings. After the first third of the scale, say at 100 V., the error will be $\Delta_c y = \pm \frac{0{\cdot}8}{100} \times 100 = \pm 0{\cdot}8$ V.

In some cases, especially in guarantee certificates, the constructional error may be indicated by significant figures, a significant figure being a digit which is thought to be nearer to the true value than is any other digit.

An inductance may be expressed as 1255oo mH or $1{\cdot}255 \times 10^5$ mH, which means that the true value is between 125400 mH and 125600 mH. The limits expressed as a percentage are here $\pm \frac{100}{125500} \times 100 = \pm 0{\cdot}0797 \cong \pm 0{\cdot}08\ \%$. The first four digits, 1255, are the significant figures, and the inductance is given to four significant figures. Again, when a resistance is given as 12222 Ω, or to five significant figures, it means that its value lies between 12221 Ω and 12223 Ω. Zeroes are significant figures only when other digits precede them in the number; thus 0·0125 has three significant figures and not five. The more significant figures, the greater the accuracy of the measurement.

In whatever manner the constructional error is given, it is best converted into a percentage for the calculation of the maximum possible error.

The constructional error can in some cases be eliminated or reduced.

(*b*) THE READING ERROR. The accuracy with which a scale reading, such as that of an ammeter, voltmeter, etc., can be made depends on the distance between consecutive divisions, on the thickness of the pointer, and on whether the instrument is provided with a mirror. Only instruments with mirrors and knife-edge pointers ought to be used in a laboratory.

Consider fig. 1.

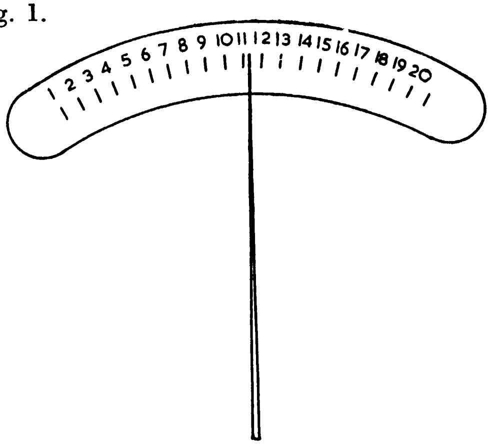

Fig. 1

The pointer reads more than 11·25; that is certain, but whether it is 11·27 or 11·3 or 11·37 is doubtful. The uncertainty in reading is about 0·1 or $\pm \frac{1}{20}$ of a division. Here the reading error is expressed as part of a division. The reading error of course depends on the instrument; in some cases it may be $\pm \frac{1}{10}$ of a division, in others much more.

The reading error is not confined to indicating instruments, but exists also wherever a scale has to be read against some indicator; it will therefore exist in the reading of continuously variable mutual inductances, continuously variable condensers and resistances, wavemeters, frequency meters, etc. In the more expensive laboratory equipment the scale is provided with a vernier and magnifying glass in order to diminish the reading error.

When reading a galvanometer deflection on a scale, the reading error depends on the thickness of the line image, which is generally $\frac{1}{2}$ mm. The movement of this line image by $\frac{1}{4}$ mm. is all that can really be distinguished, so that the reading error is here fairly great.

But the greatest error occurs when reading the deflection of a ballistic galvanometer. Here there is not only the thickness of the line to take into account, but also the difficulty of locating the position of the line whilst the galvanometer attains its greatest deflection. The error will depend very much on the care of the experimenter.

In instruments or any other apparatus provided with a uniform scale, that is, a scale having the same distance between consecutive divisions throughout, the relative error varies inversely with the indication of the instrument.

Say that we can distinguish $\frac{1}{a}$ of a division, and let each division correspond to b units of the measured magnitude, then the relative error on the first division will be $\frac{b}{ab}$, on the second division $\frac{b}{2ab}$, and on the x division $\frac{b}{xab} = \frac{1}{xa}$.

Example 2. A variable condenser has a uniform scale 0 to 100. The thickness of the reading index is such that $\pm\frac{1}{10}$ of a division can be distinguished. What is the percentage reading error at the fourth and last divisions?

(*a*) at the fourth division:

$$\frac{\Delta_r y}{y} = \pm\frac{1}{4 \times 10} = \pm\frac{1}{40}; \quad \% \frac{\Delta_r y}{y} = \pm\frac{100}{40} = \pm 2{\cdot}5\%.$$

(*b*) at the last division:

$$\frac{\Delta_r y}{y} = \pm\frac{1}{100 \times 10} = \pm\frac{1}{1000}; \quad \% \frac{\Delta_r y}{y} = \pm\frac{100}{1000} = \pm 0{\cdot}1\%.$$

Example 3. Voltages are measured by an instrument having a scale 0 to 100. In one measurement the instrument shows 5, in the other 90; what is the relative reading error in each case if $\pm\frac{1}{10}$ of a division can be distinguished?

(*a*) $$\frac{\Delta_r y}{y} = \pm\frac{1}{10 \times 5} = \pm\frac{1}{50}; \quad 100\frac{\Delta_r y}{y} = \pm\frac{100}{5 \times 10} = \pm 2\%.$$

(*b*) $$\frac{\Delta_r y}{y} = \pm\frac{1}{10 \times 90} = \pm\frac{1}{900}; \quad 100\frac{\Delta_r y}{y} = \pm\frac{100}{900} = \pm 0{\cdot}11\%.$$

It follows that an instrument provided with a direct reading scale is unreliable at the beginning of the scale, greatly so, since the calibration error usually is much greater in this part of the scale.

The answer to the question, from which part of the scale should the reading be taken, depends on the precision desired in the measurement. The error at the beginning of the scale is much greater in instruments with a scale which follows more or less a square law, because if we can distinguish $\frac{1}{a}$ of a division at division x of such an instrument, we can distinguish only $\frac{4}{a}$ at division $\frac{x}{2}$.

There are, however, scales where the distance between consecutive divisions is greatest at the beginning or in the middle of the scale. The question then arises: in which part of the scale is the reading error smallest?

Let z be the variable distance in mm. between consecutive divisions on the scale, and let the distance which can be distinguished be c mm.; c will be a constant for a given scale and pointer thickness; the relative reading error will be variable with z and equal to $\frac{c}{z}$; if one division corresponds to b units of the magnitude measured, then at division x the magnitude measured is bx and the relative error is

$$\frac{c}{z} \cdot b \div bx = c \cdot \frac{1}{zx}.$$

This error will be the smaller as the greater is the product zx. For minimum error the instrument has therefore to be read at those points of the scale where zx is greatest.

Example 4. In a certain instrument the distance between two consecutive divisions on the scale is:

2 mm. at 20 divisions	3 mm. at 80 divisions
3 mm. ,, 40 ,,	2 mm. ,, 90 ,,
4 mm. ,, 50 ,,	1·5 mm. ,, 100 ,,
4 mm. ,, 70 ,,	

Where should the scale be read in order that the reading error shall be minimum?

From the products zx

$2 \times 20 = 40$	$3 \times 80 = 240$
$3 \times 40 = 120$	$2 \times 90 = 180$
$4 \times 50 = 200$	$1{\cdot}5 \times 100 = 150.$
$4 \times 70 = 280$	

The minimum reading error is around division 70.

If, when reading an instrument over all its scale, that part of the scale is reached where the reading error is too great for the accuracy desired, either the sensitivity of the instrument or the instrument itself should be changed, so as to pass again into that part of the scale where the reading error is within the limits allowed.

Example 5. A current is read on an ammeter of 100 scale divisions, where $\frac{1}{10}$ of a division can be distinguished; the scale is uniform.

At what division should the ammeter (or its sensitivity) be changed if the maximum relative reading error is not to exceed $\pm$ 0·5%?

$$100\,\frac{\Delta_r y}{y} = \frac{100}{10\,x} = 0{\cdot}5\,; \quad x = \frac{100}{10 \times 0{\cdot}5} = 20 \text{ divisions.}$$

(*c*) THE DETERMINATION ERROR. This is the error due to any uncertainty in the final adjustment of the apparatus used for measurement. For instance, when measuring an inductance against a capacity on an a.c. bridge, it is found, in the final adjustment, that a variation of, say, $\pm$ 0·0001 μF in the capacity does not produce any change of sound in the headphone used as a detector. Owing to this uncertainty there will be an error in the calculation of the measured inductance; this is the determination error.

(*d*) THE ERROR INHERENT IN THE METHOD USED. This error is due to certain approximations inherent in the method used. The error can usually either be minimised or calculated.

(*e*) THE ERRORS DUE TO CONDITIONS IN WHICH EXPERIMENTS ARE CONDUCTED.

(i) Temperature changes which will affect resistances, frequency in tuning forks and generators, e.m.f.s in standard cells, indication of instruments, etc. This error can be eliminated by proper correction factors and information supplied by manufacturers.

(ii) Change of pressure affecting frequency standards such as tuning forks. Corrections can be applied according to manufacturer's indication.

(iii) Thermal e.m.f.s produced by a heated contact between two different metals. This can be avoided by a judicious disposition of the equipment, by keeping a uniform temperature in the space where the experiment is conducted and by avoiding overheating of components.

(iv) Resistance of plugs, switches and connecting wires. This resistance can be eliminated in certain experiments, in others it can be minimised or calculated.

(v) Capacity and inductance of plugs, contacts, wires, mutual inductance and inter-capacity between the different parts composing the experimental circuit. The effect of these can be minimised by a judicious disposition or eliminated by a suitable circuit.

(vi) Ageing of equipment. The calibration error given by the manufacturer has to be periodically checked, as it is in time found to vary.

(vii) Effect of external fields. This effect can be eliminated by suitable screening, by reversal of current, or by a suitable method of measurement, dependent on the experiment conducted.

(viii) False readings due to eddy currents in screens and additional capacities due to screens. These can be calculated if details of the equipment are known.

(ix) Residual tension in the suspension of instruments, and mechanical defects causing friction between mobile and immobile parts of instruments. These can be avoided by suitable maintenance and inspection.

In cases of residual tension, the difficulty can be overcome by interpolation.

(6) Accidental Errors

Over these we have very little or no control at all. Some causes of accidental errors, such as bad insulation, variation of the e.m.f. of supply, bad contacts, accidental overheating of resistances and inductances, can be minimised, or even completely eliminated, by adequate inspection. Other sources of error, such as noise and smoke, by co-operation and order in the laboratory; whilst many other causes of accidental errors, such as vibrations of the building, currents of air, operator's fatigue and false technique, are more or less beyond our control.

CHAPTER II

CALCULATION OF ERRORS

(1) Calculation of Accidental Errors

As the accidental errors are governed by chance, they are subject to the laws of probability.

Dealing for the moment with accidental errors alone, suppose a series of measurements be made, yielding the values: $y_1, y_2, y_3, y_4 \ldots y_n$. The accidental error on y_1 is, say, Δy_1, and this error can be positive or negative; the accidental error on y_2 is Δy_2, and Δy_2 can be positive or negative, and equal to, or greater or smaller than, Δy_1; the same applies to the errors Δy_3, Δy_4, Δy_n, on $y_2, y_3 \ldots y_n$.

It is clear that when n tends towards infinity, the number of positive errors of any magnitude will be equal to the number of negative errors of any magnitude; in other words, the average accidental error $\Delta_a y$ tends towards zero when the number, n, of experiments made tends towards infinity.

The true value of the magnitude measured is therefore

$$y = \frac{y_1 + y_2 + y_3 + \ldots y_n}{n}; \quad n \longrightarrow \infty$$

When n is not infinity, we get a probable value of the magnitude measured $y_p = \dfrac{y_1 + y_2 + y_3 + \ldots y_n}{n}$ . . . . (1)

In practice, the number of experiments, except in cases of very special importance, is very small. The best procedure is to make a small number of measurements, the number depending upon the time available, reject the results which are obviously wrong (the ones which are much smaller or much greater than the average), and calculate the most probable value according to formula (1).

Example 6. Several measurements of a capacity gave the following results: (1) 1·151 μF (2) 1·152 μF (3) 1·151 μF (4) 1·1525 μF (5) 1·152 μF (6) 1·25 μF (7) 1·01 μF (8) 1·174 μF (9) 1·153 μF (10) 1·3 μF (11) 1·151 μF (12) 1·152 μF.

The values of 6, 7, 8 and 10 are obviously wrong; the most probable value is

$$y_p = \frac{1{\cdot}151 + 1{\cdot}152 + 1{\cdot}151 + 1{\cdot}1525 + 1{\cdot}152 + 1{\cdot}153 + 1{\cdot}151 + 1{\cdot}152}{8}$$

$$= 1{\cdot}1518\,\mu F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Another method, which, however, requires a certain amount of skill, is to attach an importance coefficient to each measurement. The importance coefficient can be 0, 1, 2 or 3. When a result is obviously wrong, it has the coefficient 0 ; an average good measurement has the coefficient 1 ; a very good measurement, such as is achieved in very favourable circumstances, has the coefficient 2 ; and an excellent measurement, in which the experimenter has particular confidence, has the coefficient 3. The probable value of the magnitude measured is the sum of all the products of the values found by their coefficients divided by the sum of all the coefficients.

Example 7. Assuming that in the measurements of example 6 we attach the coefficient 1 to results 1, 2, 3, 5 and 9, the coefficient 2 to results 11 and 12, and the coefficient 3 to result 4. The results 6, 7, 8 and 10, being obviously wrong, get the coefficient 0. The most probable value is

$$y_p = \frac{1\times 1{\cdot}151 + 1\times 1{\cdot}152 + 1\times 1{\cdot}151 + 1\times 1{\cdot}152 + 1\times 1{\cdot}153 +}{1 + 1 + 1 + 1 + 1 +}$$

$$\frac{2\times 1{\cdot}151 + 2\times 1{\cdot}152 + 3\times 1{\cdot}1525}{2 + 2 + 3} = 1{\cdot}15187\,\mu\text{F}. \quad . \quad . \quad (3)$$

However, the result as given by (2) and (3) is still doubtful; what we require are the limits within which the magnitude tested certainly lies. The procedure is therefore to make several measurements, rejecting the obviously wrong results and accepting as limits the extreme good values. Thus the capacity will be taken as between $C=1{\cdot}151\ \mu F$ and $C=1{\cdot}153\ \mu F$.

The systematic errors are next calculated and the result given as: Magnitude measured $= y' \pm \Delta y$. If the systematic error in example 6 is $\pm$ 1%, the capacity will have a value between $C=1{\cdot}151 \pm 1\%$ and $C=1{\cdot}153 \pm 1\%$ or $C=1{\cdot}1518 \pm 1{\cdot}087\%$; $\pm$ 0·087% is the accidental error ; $\pm$ 1·087% is the maximum error.

The accidental error ought to be much smaller than in example 6, it should generally be negligible. In the examples given in this book the accidental error will be neglected.

(2) Calculation of Systematic Errors

The maximum possible error is most easily calculated by means of the logarithmic differential.

The Logarithmic Differential. The logarithmic differential of a function y is the ratio of its differential dy over the function y, or $\frac{dy}{y}$.

Note that the logarithmic differential is the relative increment of the function y.

The Logarithmic Differential of a Sum.

Let $y = u + v + z$

$$\frac{dy}{y} = \frac{d(u + v + z)}{y} = \frac{du}{y} + \frac{dv}{y} + \frac{dz}{y},$$

which can be written

$$\frac{dy}{y} = \frac{u}{y}\frac{du}{u} + \frac{v}{y}\frac{dv}{v} + \frac{z}{y}\frac{dz}{z};$$

that is, the relative increment, $\frac{dy}{y}$, equals the sum of the products formed by multiplying the relative increments of the terms of the function by the ratio of each term to the function.

The Logarithmic Differential of a Product.

Let $y = u.v.z$

$$\frac{dy}{y} = \frac{d(u.v.z)}{y} = \frac{z.v\,(du)}{y} + \frac{uz\,(dv)}{y} + \frac{uv\,(dz)}{y} =$$

$$\frac{uzv}{y}\cdot\frac{du}{u} + \frac{uzv}{y}\cdot\frac{dv}{v} + \frac{uzv}{y}\cdot\frac{dz}{z} = \frac{du}{u} + \frac{dv}{v} + \frac{dz}{z}$$

The relative increment of the function equals the sum of the relative increments of the terms.

Similarly, if $y = \frac{u.v.z}{wq}$, then $\frac{dy}{y} = \frac{du}{u} + \frac{dv}{v} + \frac{dz}{z} - \frac{dw}{w} - \frac{dq}{q}$.

The Logarithmic Differential of a Sum of Products.

Let $y = u.v + wz$

$$\frac{dy}{y} = \frac{d(u.v + wz)}{y} = \frac{d(u.v)}{y} + \frac{d(wz)}{y} = \frac{u.dv}{y} + \frac{v.du}{y} + \frac{w.dz}{y} + \frac{z.dw}{y} =$$

$$\frac{u.v}{y}\cdot\frac{dv}{v} + \frac{v.u}{y}\cdot\frac{du}{u} + \frac{wz}{y}\cdot\frac{dz}{z} + \frac{zw}{y}\frac{dw}{w}.$$

The Logarithmic Differential of a Power.

Let $y = u^n$

$$\frac{dy}{y} = \frac{d(u^n)}{y} = \frac{n(u^{n-1})du}{y} = \frac{n.u.u^{n-1}}{y}\frac{du}{u} = \frac{n.du}{u}.$$

The Logarithmic Differential of a Root.

Let $y = \sqrt[n]{u}$

$$\frac{dy}{y} = \frac{d(u^{1/n})}{y} = \frac{1/n(u^{1/n-1})}{y}du = \frac{1/n.u.u^{1/n-1}}{y}\frac{du}{u} = \frac{1}{n}\cdot\frac{du}{u}.$$

The practical calculation of the maximum possible error rests on the assumption that, as the errors are small, we can apply to them the same calculations as are applied to differentials.

(3) The Logarithmic Differential and the Systematic Relative Error

In electrical measurements it very seldom happens that a measured magnitude is determined directly. Generally it is determined by a relation to other magnitudes ; for instance, a voltage can be determined by measuring a current and a resistance, a capacity can be determined by measuring an inductance and a resistance, etc.

The measured magnitude can therefore be expressed as

$$y = f(u, v, z, \ldots) \qquad (4)$$

The quantities $u, v, z, \ldots$ are measured, and y is calculated from (4).

But when measuring $u, v, z, \ldots$ we commit errors, therefore the values we find are : $u', v', z', \ldots$ so that by using (4) we get

$$y' = f(u', v', z', \ldots).$$

The logarithmic differential of (4) is

$$\frac{dy}{y} = \frac{d\,f(u, v, z, \ldots)}{y}$$

and as $u' - u = \pm\, \Delta u'$, $v' - v = \pm\, \Delta v'$, $z' - z = \pm\, \Delta z'$, and $\Delta u'$, $\Delta v'$, $\Delta z'$, are very small, we can write :

$$\frac{\Delta y'}{y'} = \frac{\Delta f(u', v', z', \ldots)}{y'} \qquad (5)$$

$\frac{\Delta y'}{y'}$ is the maximum relative error on y', expressed by the same formula as the logarithmic differential of (4).

Example 8. In determining a voltage V across a resistance R in which a current I is flowing, I and R are measured. The voltage is given by

$$V = IR \qquad (6)$$

The logarithmic differential of (6) is

$$\frac{dV}{V} = \frac{dI}{I} + \frac{dR}{R}.$$

As we make errors when measuring I and R, we get the values I′ and R′. We have, therefore, $V' = I' \cdot R'$ (I′ and R′ are measured several times so as to be sure of all the digits in the numbers expressing their magnitude).

The systematic relative error on V′ is

$$\frac{\Delta V'}{V'} = \frac{\Delta I'}{I'} + \frac{\Delta R'}{R'}.$$

$\Delta I'$ is the error on the measurement of I′. This error is the sum of the constructional error $\Delta_c I'$ on the ammeter and the reading error $\Delta_r I'$; $\Delta_c I' + \Delta_r I' = \Delta I'$.

$\Delta R'$ is the constructional error on the resistance R′.

Let $\Delta_c I' = \pm 1\%$, $\Delta_r I' = \pm 1\%$, $\Delta R' = \pm 0{\cdot}1\%$;

then $\% \; \Delta V' = \pm 100 \dfrac{\Delta V'}{V'} = \pm (1 + 1 + 0{\cdot}1) = \pm 2{\cdot}1\%$.

Care has to be taken, when changing from the logarithmic differential (formula (4) to formula (5)), to change all negative signs between terms to positive signs, because the sign of the error is unknown.

Example 9. A current I is measured by means of a voltmeter connected across a resistance R; V and R are measured or known.

We have $I = \dfrac{V}{R}$.

The logarithmic differential is $\dfrac{dI}{I} = \dfrac{dV}{V} - \dfrac{dR}{R}$.

Changing the logarithmic differential into the error formula, we must write: $\dfrac{\Delta I'}{I'} = \dfrac{\Delta V'}{V'} + \dfrac{\Delta R'}{R'}$

simply because the sign of the error $\Delta R'$ is not known.

The change of the signs between terms has to be introduced only after all calculations and simplifications in formula (4) have been made; the non-observance of this rule may lead to grave mistakes, as can be seen from the following example.

Example 10. A magnitude y is calculated from the formula $y = w \cdot \dfrac{z + v}{z - v}$, after w, z and v have been measured.

We have $\dfrac{dy}{y} = \dfrac{dw}{w} + \dfrac{d(z+v)}{z+v} - \dfrac{d(z-v)}{z-v} =$

$$\frac{dw}{w} + \frac{dz}{z+v} + \frac{dv}{z+v} - \frac{dz}{z-v} + \frac{dv}{z-v} = \quad . \quad . \quad . \quad (7)$$

$$\frac{dw}{w} + dz \cdot \frac{(z-v) - (z+v)}{(z+v) \quad (z-v)} + dv \cdot \frac{(z-v) + (z+v)}{(z+v) \quad (z-v)} =$$

$$\frac{dw}{w} + \frac{dz\,(-2v)}{z^2 - v^2} + \frac{dv\,(2z)}{z^2 - v^2} = \frac{dw}{w} + \frac{2z dv}{z^2 - v^2} - \frac{2v dz}{z^2 - v^2} =$$

$$\frac{dw}{w} + \frac{2zv}{(z^2 - v^2)} \cdot \frac{dv}{v} - \frac{2zv}{(z^2 - v^2)} \frac{dz}{z}.$$

All simplifications having been made, we can write the error formula :

$$\frac{\Delta y'}{y'} = \frac{\Delta w'}{w'} + \frac{2z'v'}{[(z')^2 - (v')^2]} \cdot \frac{\Delta v'}{v'} + \frac{2z'v'}{[(z')^2 - (v')^2]} \cdot \frac{\Delta z'}{z'} \quad . \tag{8}$$

Suppose now that by mistake the change of signs from minus to plus was done immediately after (7), we then have :

$$\frac{\Delta y'}{y'} = \frac{\Delta w'}{w'} + \frac{\Delta z'}{z' + v'} + \frac{\Delta v'}{z' + v'} + \frac{\Delta z'}{z' - v'} + \frac{\Delta v'}{z' - v'} =$$

$$\frac{\Delta w'}{w'} + \frac{\Delta z' \cdot [(z' - v') + (z' + v')]}{(z')^2 - (v')^2} + \frac{\Delta v' \cdot [(z' - v') + (z' + v')]}{(z')^2 - (v')^2} =$$

$$\frac{\Delta w'}{w'} + \frac{2z' \, \Delta z'}{(z')^2 - (v')^2} + \frac{\Delta v' 2z'}{(z')^2 - (v')^2} =$$

$$\frac{\Delta w'}{w'} + \frac{2z'z'}{[(z')^2 - (v')^2]} \frac{\Delta z'}{z'} + \frac{2z'v'}{[(z')^2 - (v')^2]} \frac{\Delta v'}{v'},$$

which is quite different from (8).

The procedure in calculating the maximum possible error is therefore as follows :

(1) Measure the values $u, v, z, \ldots$ of the relation $y = f(u, v, z, \ldots)$.

(2) Calculate the logarithmic differential $\frac{dy}{y} = \frac{d\,[f(u, v, z, \ldots)]}{y}$.

(3) Simplify the logarithmic differential to a point where no ambiguity can arise.

(4) Write the relative error, changing all the minus signs between terms to positive signs

$$\frac{\Delta y'}{y'} = \frac{\Delta f(u', v', z', \ldots)}{y'}.$$

CHAPTER III

MEASUREMENT OF D.C. RESISTANCE

Measurement of Medium Resistances

(1) Measurement of Resistance by Voltmeter and Ammeter

A current passing through an unknown resistance R is measured by an ammeter reading I, a voltmeter connected across the resistance R reads V volts, and from the formula

$$R = \frac{V}{I} \text{ we calculate R.}$$

Two different connections are possible : fig. 2, (*a*) and (*b*).

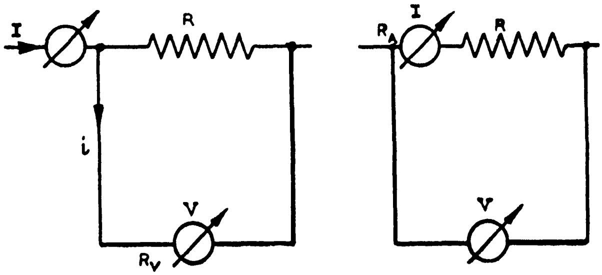

Fig. 2

Let R_A and R_V be respectively the resistances of the ammeter and the voltmeter. Using diagram (*a*), we have

The current in the resistance R is $I - i$, therefore

$$(I - i)\,R = iR_V = V\,;\;\; R = \frac{R_V i}{I - i} = \frac{V}{I - i}.$$

By assuming that $R = \frac{V}{I}$, the error due to the method (see *5d*, p. 16) is $\frac{V}{I - i} - \frac{V}{I} = \frac{Vi}{I\,(I - i)}$, but $R\,(I - i) = V$, therefore

$$\frac{Vi}{I\,(I - i)} = \frac{ViR}{IV} = \frac{ViR}{IR_V i} = \frac{V}{I} \cdot \frac{R}{R_V}.$$

The smaller R is, compared with R_V, the smaller the error due to the method ; a voltmeter of very high resistance ought therefore to be used, or (which amounts to the same thing) the current I ought to be much greater than the voltmeter current *i*.

The error $\frac{V}{I} \cdot \frac{R}{R_V}$ due to the method used can be calculated if we know R_V, or it can be neglected if $\frac{R}{R_V}$ is small enough not to affect appreciably the maximum relative error.

The value of R is given by $R = \frac{V}{I}\left(1 + \frac{R}{R_V}\right)$ because $\frac{VR}{IR_V}$ is positive.

Calculation of the Systematic Error.

From $R = \frac{V}{I}\left(1 + \frac{R}{R_V}\right)$ we get

$$\frac{dR}{R} = \frac{dV}{V} - \frac{dI}{I} + \frac{d\left(1 + \frac{R}{R_V}\right)}{1 + \frac{R}{R_V}} = \frac{dV}{V} - \frac{dI}{I} + \frac{d\left(\frac{R}{R_V}\right)}{\frac{R_V + R}{R_V}} =$$

$$\frac{dV}{V} - \frac{dI}{I} + \left[\frac{\frac{R_V dR - R dR_V}{R^2_V}}{\frac{R_V + R}{R_V}}\right] = \frac{dV}{V} - \frac{dI}{I} + \frac{R_V(R_V dR - R\,dR_V)}{(R_V + R)\,R^2_V} =$$

$$\frac{dV}{V} - \frac{dI}{I} + \frac{dR}{R_V + R} - \frac{R dR_V}{(R_V + R)\,R_V} =$$

$$\frac{dV}{V} - \frac{dI}{I} + \frac{R}{(R_V + R)}\frac{dR}{R} - \frac{R}{(R_V + R)}\frac{dR_V}{R_V}; \quad \frac{dR}{R}\left[1 - \frac{R}{(R_V + R)}\right] =$$

$$\frac{dV}{V} - \frac{dI}{I} - \frac{R\,dR_V}{(R_V + R)\,R_V}$$

writing $\left[1 - \frac{R}{(R_V + R)}\right] = a$;

$$\frac{dR}{R} = \frac{1}{a}\left[\frac{dV}{V} - \frac{dI}{I} - \frac{R}{(R_V + R)}\frac{dR_V}{R_V}\right].$$

The maximum relative error is

$$\frac{\Delta R'}{R'} = \frac{1}{a}\left[\frac{\Delta V'}{V'} + \frac{\Delta I'}{I'} + \frac{R'}{(R'_V + R')}\frac{\Delta R'_V}{R'_V}\right] \quad . \quad . \quad (9)$$

$\Delta V'$ is the constructional plus reading error due to the voltmeter.
$\Delta I'$ is the constructional plus reading error due to the ammeter.
$\Delta R'_V$ is the constructional error in the knowledge of R'_V.

(9) gives the exact calculation of the relative error $\frac{\Delta R'}{R'}$, but when $\frac{R}{R_V}$ is very small, the relative error will be

$$\frac{\Delta R'}{R'} = \frac{\Delta V'}{V'} + \frac{\Delta I'}{I'}$$

Example 11. A resistance R is to be measured, the connections being those of fig. 2 (*a*). The ammeter of 100 scale divisions, and uniform scale, shows 10 amps on division 100. The voltmeter of 150 scale divisions, and uniform scale, shows 120 volts on division 120. The resistance of the voltmeter is 100 ohms per volt. The scales of the ammeter and voltmeter are such that $\pm \frac{1}{10}$ of a division can be distinguished. The constructional error of the ammeter is $\pm 0{\cdot}5\%$ after the first third of the scale; the constructional error of the voltmeter is $\pm 0{\cdot}4\%$ after the first third of the scale. The resistance of the voltmeter is known to within $\pm 0{\cdot}1\%$. Calculate the value of R, and the systematic relative error.

The voltmeter reading error is

$$\Delta_r V' = \pm \frac{1}{10 \times 120} = \pm \frac{1}{1200};\ \Delta_r V'\ \% = \pm \frac{100}{1200} = \pm 0{\cdot}083\%.$$

The total systematic error on the voltmeter is

$$\pm (0{\cdot}083 + 0{\cdot}4) = \pm 0{\cdot}483\% = \Delta_r V' + \Delta_c V.$$

The ammeter reading error is

$$\%\ \Delta_r I' = \pm \frac{100}{10 \times 100} = \pm 0{\cdot}1\%.$$

The total systematic error on the ammeter is

$$\%\ \Delta_r I' + \%\ \Delta_c I' = \pm (0{\cdot}1 + 0{\cdot}5) = \pm 0{\cdot}6\%.$$

The erroneous value of the measured resistance is

$$R' = \frac{V'}{I'} = \frac{120}{10} = 12\ \Omega.$$

The total resistance of the voltmeter is

$$150 \times 100 = 15000\ \Omega, \text{ so that}$$

$$\frac{R'}{R'_V + R'} = \frac{12}{15000 + 12} \simeq 0{\cdot}0008 \text{ (negligible).}$$

The percentage error on the measured resistance is therefore

$$100\frac{\Delta R'}{R'} = \pm 100\left(\frac{\Delta V'}{V'} + \frac{\Delta I'}{I'}\right) = \pm\left(0{\cdot}483 + 0{\cdot}6\right) = \pm\, 1{\cdot}083\%.$$

The error due to the method is

$$\frac{V'}{I'}\cdot\frac{R'}{R'_V} = \frac{12 \times 12}{10 \times 15000} = 0{\cdot}00096\,\Omega.$$

The value of the tested resistance ought therefore to be given as

$$R = 12 + 0{\cdot}00096\,\Omega \pm 1{\cdot}083\%.$$

Consider diagram, fig. 2 (*b*).

Here we have $(R_A + R)\,I = V$; $R = \dfrac{V - R_A I}{I} = \dfrac{V}{I} - R_A$.

By taking the value as $R = \dfrac{V}{I}$, the error due to the method is

$$\frac{V}{I} - R_A - \frac{V}{I} = -\,R_A.$$

The error is negative, and equal to the ammeter resistance; the method is therefore suitable when R_A is much smaller than R.

Calculation of the constructional and reading error:

As $R = \dfrac{V}{I} - R_A = \dfrac{V - R_A I}{I}$, we have

$$\frac{dR}{R} = \frac{d\,(V - R_A I)}{V - R_A I} - \frac{dI}{I} = \frac{dV}{V - R_A I} - \frac{d\,(R_A I)}{V - R_A I} - \frac{dI}{I} =$$

$$\frac{dV}{V - R_A I} - \frac{R_A dI}{V - R_A I} - \frac{I\,dR_A}{V - R_A I} - \frac{dI}{I} =$$

$$\frac{V}{V - R_A I}\cdot\frac{dV}{V} - \frac{R_A I}{V - R_A I}\cdot\frac{dI}{I} - \frac{I R_A}{V - R_A I}\cdot\frac{dR_A}{R_A} - \frac{dI}{I}.$$

The maximum relative error is

$$\frac{\Delta R'}{R'} = \frac{V'}{V' - R'_A I'}\cdot\frac{\Delta V'}{V'} + \frac{R'_A I'}{V' - R'_A I'} \times \frac{\Delta I'}{I'} +$$

$$\frac{I' R'_A}{V' - R'_A I'} \times \frac{\Delta R'_A}{R'_A} + \frac{\Delta I'}{I'}.$$

If R_A is small, so that $R_A I$ can be neglected compared with V, the error will be

$$\frac{\Delta R'}{R'} = \frac{\Delta V'}{V'} + \frac{\Delta I'}{I'}.$$

Accidental errors most likely to occur in this measurement (fig. 2 (*a*) and (*b*)) are owing to

(*a*) Overheating the resistance. Care has to be taken to keep the temperature constant and normal.

(*b*) Variation of current between the reading of the ammeter and voltmeter.

As is obvious, the voltmeter and ammeter method of measuring resistance is rather a poor one, the reason being that two instruments, both direct reading, have to be used, introducing two reading and two constructional errors.

(2) The Voltmeter Comparison Method

The resistance R to be measured is connected in series, with a standard resistance R_s and rheostat Rh, to a source E (fig. 3). A voltmeter of resistance R_v is first connected across R, when it reads V, and then connected across R_s, when it reads V_1.

We have $V = RI$ across R

$V_1 = R_s I$ across R_s

$$\frac{V}{V_1} = \frac{R}{R_s};\ R = R_s \frac{V}{V_1}.$$

This calculation assumes that the voltmeter resistance R_v is infinite, which is of course not true, unless an electrostatic voltmeter is used.

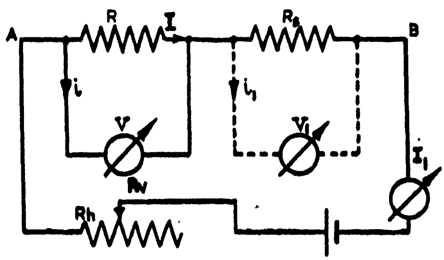

Fig. 3

In reality, when the voltmeter is connected across R, the total resistance between A and B in fig. 3 is

$$\frac{R\,R_v}{R + R_v} + R_s;$$

and when the voltmeter is across R_s, the resistance between A and B is

$$\frac{R_s\,R_v}{R_s + R_v} + R.$$

The current I has therefore changed; however, as R_v is much greater than R or R_s, the error introduced by the difference in the total resistance of the circuit of fig. 3 is negligible.

We have also to take into account the voltmeter current. When the instrument is across R, we have

$$RI = R_v i = \frac{R\,R_v}{R + R_v} \cdot I_1 = V\,; \quad (I_1 = I + i).$$

When the instrument is across R_s

$$R_s I = R_v i_1 = \frac{R_s R_v}{R_s + R_v} \cdot I_1 = V_1\,; \text{ therefore}$$

$$\frac{V}{V_1} = \frac{R\,R_v\,(R_s + R_v)}{R_s R_v\,(R + R_v)} = \frac{R\,(R_s + R_v)}{R_s\,(R + R_v)}.$$

$$R = \frac{R_s\,(R + R_v)}{(R_s + R_v)} \times \frac{V}{V_1}.$$

As V and V_1 are measured on the same voltmeter, we can write

$$V = V_1 \pm v$$

according to whether V is greater or smaller than V_1; therefore

$$R = \frac{R_s\,(R + R_v)}{(R_s + R_v)} \times \frac{V_1 \pm v}{V_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Calculation of the Systematic Error. The logarithmic differential of (10) is

$$\frac{dR}{R} = \frac{dR_s}{R_s} + \frac{d\,(R + R_v)}{R + R_v} - \frac{d\,(R_s + R_v)}{R_s + R_v} - \frac{dV_1}{V_1} + \frac{d\,(V_1 \pm v)}{V_1 \pm v}$$

$$= \frac{dR_s}{R_s} + \frac{dR}{R + R_v} + \frac{dR_v}{R + R_v} - \frac{dR_s}{R_s + R_v} - \frac{dR_v}{R_s + R_v} - \frac{dV_1}{V_1} +$$

$$\frac{dV_1}{V_1 \pm v} \pm \frac{dv}{V_1 \pm v}$$

$$= \frac{dR_s}{R_s} + \frac{R}{R + R_v}\,\frac{dR}{R} + \frac{R_v}{R + R_v}\,\frac{dR_v}{R_v} - \frac{R_s}{R_s + R_v}\,\frac{dR_s}{R_s} -$$

$$\frac{R_v}{R_s + R_v}\,\frac{dR_v}{R_v} - \frac{dV_1}{V_1} + \frac{V_1}{(V_1 \pm v)}\,\frac{dV_1}{V_1} \pm \frac{v}{V_1 \pm v}\,\frac{dv}{v}$$

$$= \frac{dR_s}{R_s} + \frac{R}{R + R_v}\,\frac{dR}{R} + \frac{R_v}{R + R_v}\,\frac{dR_v}{R_v} - \frac{R_s}{R_s + R_v}\,\frac{dR_s}{R_s} -$$

$$\frac{R_V}{R_S + R_V}\frac{dR_V}{R_V} + \frac{dV_1}{V_1}\left(-1 + \frac{V_1}{V_1 \pm v}\right) \pm \frac{v}{V_1 \pm v}\frac{dv}{v}$$

$$= \frac{dR_S}{R_S} + \frac{R}{R + R_V}\frac{dR}{R} + \frac{R_V}{R + R_V}\frac{dR_V}{R_V} - \frac{R_S}{R_S + R_V}\frac{dR_S}{R_S} -$$

$$\frac{R_V}{R_S + R_V}\frac{dR_V}{R_V} + \frac{dV_1}{V_1}\left(\frac{\mp v}{V_1 \pm v}\right) \pm \frac{v}{V_1 \pm v}\frac{dv}{v} \quad . \quad . \quad (10a)$$

As R_V is usually much greater than R or R_S, (10a) can be written

$$\frac{dR}{R} = \frac{dR_S}{R_S} + \frac{dV_1}{V_1}\left(\frac{\mp v}{V_1 \pm v}\right) \pm \frac{v}{V_1 \pm v}\frac{dv}{v} \quad . \quad . \quad (10b)$$

If it were certain that $\frac{dV_1}{V_1} = \frac{dv}{v}$, or that the voltmeter indication is absolutely proportional to its deflection, then (10b) would become

$$\frac{dR}{R} = \frac{dR_S}{R_S} + \frac{dV_1}{V_1}\left(\frac{\mp v}{V_1 \pm v} \pm \frac{v}{V_1 \pm v}\right) = \frac{dR_S}{R_S} \quad . \quad (10c)$$

It is safest, however, to use (10b).

The advantage of the method is obvious, but whilst the constructional error on the voltmeter is either reduced or entirely eliminated (according to (10b) or (10c)) owing to V_1 appearing in the ~~nominator~~ numerator and denominator of (10), the reading errors remain. Even if $v = 0$, that is, $V_1 = V$, we only know this within the limits of the reading error; the maximum relative error is therefore either

$$\frac{\Delta R'}{R'} = \frac{\Delta R'_S}{R'_S} + \frac{\Delta_r V'}{V'} + \frac{\Delta_r V'_1}{V'_1} \text{ or}$$

$$\frac{\Delta R'}{R'} = \frac{\Delta R'_S}{R'_S} + \frac{\Delta_c V'_1}{V'_1}\left(+\frac{v'}{V'_1 \pm v'}\right) + \frac{v'}{V'_1 \pm v}\frac{\Delta_c v'}{v'} + \frac{\Delta_r V'_1}{V'_1} + \frac{\Delta_r V'}{V'}.$$

The accidental error most likely to occur in this method, apart from overheating the resistances, is the variation of the current I, owing to the falling of the e.m.f. of the source E. If this is suspected, the best procedure is to take one reading V_{R1} across R, then the reading V_1 across R_S, and then another reading V_{R2} across R. Assuming the voltage variation of E to be linear during the time of these measurements, we shall have for the values of V and R

$$V = \frac{V_{R1} + V_{R2}}{2}$$

$$R = R_s \frac{(V_{R1} + V_{R2})}{2V_1}.$$

Example 12. Suppose, in fig. 3, $R_s = 1\,\Omega \pm 0{\cdot}01\%$, $V = 8$ volts, $V_1 = 10$ volts. The voltmeter resistance is 1000 Ω per volt, and its scale of 120 divisions, corresponding to 12 volts, is uniform, the constructional error on the voltmeter is $\pm\, 0{\cdot}4\%$ in the region where the readings are taken, and the reading error is $\pm \frac{1}{10}$, or $\frac{1}{8}$ of a division. Calculate the value of R and its limits.

The erroneous value of R is $R = 1 \times \dfrac{8}{10} = 0{\cdot}8\ \Omega.$

The total resistance of the voltmeter is $1000 \times 12 = 12000\ \Omega$.
The resistance of the part AB of the circuit of fig. 3 is

$$\frac{0{\cdot}8 \times 12000}{0{\cdot}8 + 12000} + 1 = 1{\cdot}799945\ \Omega$$

when the voltmeter is across R, and

$$\frac{1 \times 12000}{1 + 12000} + 0{\cdot}8 = 1{\cdot}799925\ \Omega$$

when the voltmeter is across R_s. The difference between the two resistances is therefore $0{\cdot}002\%$, and the variation of the current I due to this change in resistance is negligible.

The constructional error of the voltmeter is

$$\frac{\Delta_c V'_1}{V'_2}\left(+\frac{v'}{V'_1 \pm v'}\right) + \frac{v'}{V'_1 \pm v'}\,\frac{\Delta_c v'}{v'} = \pm\, 0{\cdot}4\left(\frac{2}{8}\right) \pm 0{\cdot}4\left(\frac{2}{8}\right)$$

$$= \pm\, 0{\cdot}2\%\,;\quad (v = -\,2)$$

because even if there is a doubt as to whether the sign of dv is the same as that of dV, we may assume that the magnitude of dv equals that of dV if $\Delta_c V = \Delta_c V_1$.

The reading error on V is $\%\ \Delta_r V' = \pm \dfrac{100}{10 \times 80} = \pm\, 0{\cdot}125\%.$

The reading error on V_1 is $\%\ \Delta_r V'_1 = \pm \dfrac{100}{10 \times 100} = \pm\, 0{\cdot}1\%.$

The maximum relative error on R is therefore

$$\%\ \Delta R' = \pm\,(0{\cdot}01 + 0{\cdot}125 + 0{\cdot}1 + 0{\cdot}2) = \pm\, 0{\cdot}435\%.$$

The value of R is therefore given as $R = 0{\cdot}8\,\Omega \pm 0{\cdot}435\%$.

(3) The Wheatstone Bridge

By the Wheatstone bridge we mean an arrangement of four arms and two diagonals, as shown in fig. 4.

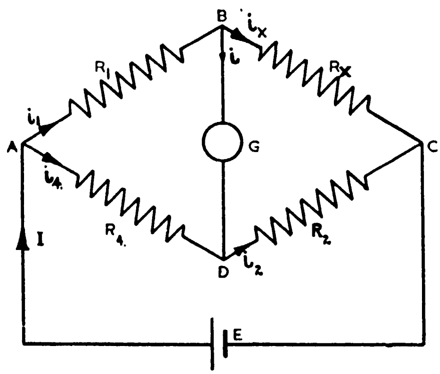

Fig. 4

AB, BC, CD, and DA are the bridge arms, BD is the galvanometer diagonal, and AC is the battery diagonal. G is the galvanometer, E the battery of suitable e.m.f.; the bridge arms are made up of resistances R_1, R_x, R_2, and R_4. The resistances R_1 and R_4 are the ratio arms ;

the ratio $\frac{R_1}{R_4}$ can be $\frac{1}{1}, \frac{1}{10}, \frac{1}{100}$; theoretically, the inverse ratios $\frac{10}{1}, \frac{100}{1}$ could be used, but, as will be seen later, the accuracy of the measurement would be insufficient.

The ratio $\frac{R_1}{R_4}$ being set, R_2 is varied until the galvanometer deflection is zero. The bridge is then said to be balanced.

Let the currents in the arms and diagonals of the bridge be as shown in fig. 4, at balance, the current in the galvanometer diagonal being zero, $i_1 = i_x$, $i_4 = i_2$, also the p.d. across arm BC equals the p.d. across arm CD, and the p.d. across arm AB equals the p.d. across arm AD.

$$R_1 i_1 = R_4 i_4 ; \quad R_x i_x = R_2 i_2$$
$$R_1 i_1 = R_4 i_4 ; \quad R_x i_1 = R_2 i_4 ; \text{ we can write:}$$

$$R_1 R_2 i_1 i_4 = R_4 R_x i_1 i_4 ; \; R_1 R_2 = R_4 R_x ; \; R_x = \frac{R_1}{R_4} R_2.$$

At balance the conditions prevailing in the galvanometer diagonal are independent of the e.m.f current and resistance in the battery diagonal ; also, the conditions in the battery diagonal are independent of the resistance, e.m.f., and current in the galvanometer diagonal.

(*a*) BRIDGE SENSITIVITY[1]. As, when balanced $R_1 R_2 = R_4 R_x$, it is evident that R_1 and R_2 cannot be the two greatest or the two smallest resistances, but must be one of the greater and one of the smaller resistances, the same is true of R_4 and R_x ; the only two possible arrangements of the bridge are therefore as in fig. 5(*a*) and (*b*).

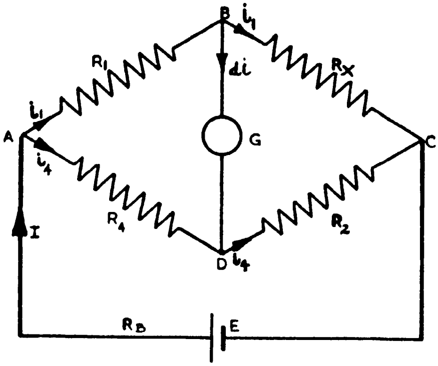

Fig. 5(*a*)

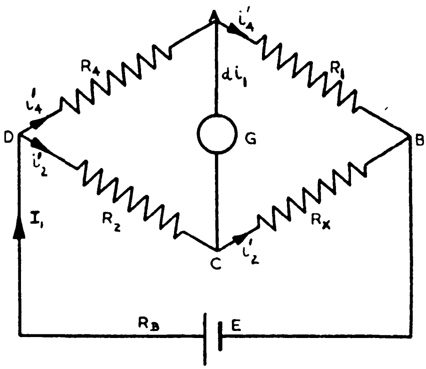

Fig. 5(*b*)

Suppose R_1 and R_x be the two smallest, and R_2 and R_4 the two greatest resistances. The question then arises : which of the two arrangements, (*a*) or (*b*), is the more sensitive ; that is, which of the two circuits will produce a greater current, *di*, in the galvanometer for a very small unbalance such that $R_1 R_2$ very nearly equals $R_x R_4$?

As, at balance, the conditions in one diagonal are independent of the resistance of the other diagonal, we may assume that this will also be very nearly true for a very small unbalance such as will produce a current *di* in the galvanometer in fig. 5(*a*), and a current di_1 in fig. 5(*b*). The resistance, measured between B and D in (*a*), is therefore very nearly the same whether the resistance of the battery diagonal is R_B or infinity ; the resistance measured between B and D is therefore (when the galvanometer is disconnected)

[1] Further details will be found in Vaschy: *Electricité et Magnétisme* (Baudry et Cie, Paris), and Bedeau: *Cours de Mesures Electriques* (S.F.E., E.S.E., Paris, Vol. V).

$$P = \frac{(R_1 + R_4)(R_x + R_2)}{R_1 + R_4 + R_x + R_2} = \frac{(R_1 + R_4)(R_x + R_2)}{K},$$

where $K = R_1 + R_4 + R_x + R_2$.

Suppose the p.d. between the points B and D in fig. 5(*a*) produced by the slight unbalance be V_{BD}, and let the resistance of the galvanometer diagonal be R_g; then, according to Thevenin's theorem, the current di is $di = \dfrac{V_{BD}}{P + R_g}$.

As $V_{BD} = R_1 \times i_1 - R_4 \times i_4$ and $(R_1 + R_x) i_1 = (R_4 + R_2) i_4$ (if di is very small), we can write: $(R_1 + R_x) i_1 = (R_4 + R_2)(I - i_1)$, because $I = i_1 + i_4$, therefore

$$i_1 = I\frac{(R_4 + R_2)}{K},\ i_4 = I\frac{(R_1 + R_x)}{K},$$

and

$$di = \frac{I}{K} \times \left[\frac{(R_4 + R_2) R_1 - (R_1 + R_x) R_4}{P + R_g}\right] = \frac{I}{K} \times \frac{(R_1 R_2 - R_x R_4)}{P + R_g}$$

the greater I, the greater di.

The conditions in the battery diagonal are at a small unbalance, also very nearly independent of the resistance in the galvanometer diagonal; when calculating the resistance between points A and C (fig. 5(*a*)), with battery diagonal disconnected, we may assume R_g to be infinity; the resistance between A and C is

$$P_1 = \frac{(R_1 + R_x)(R_4 + R_2)}{K},$$

the currents are therefore

$$I = \frac{E}{P_1 + R_B};\ di = \frac{E(R_1 R_2 - R_x R_4)}{K(P + R_g)(P_1 + R_B)}.$$

Considering fig. 5(*b*), it is evident that the resistance measured between A and C with galvanometer disconnected is

$$P_1 = \frac{(R_1 + R_x)(R_4 + R_2)}{K}$$

and the resistance between D and B battery diagonal disconnected is

$$P = \frac{(R_1 + R_4)(R_2 + R_x)}{K}.$$

The currents are therefore

$$I_1 = \frac{E}{P + R_B};\ di_1 = \frac{E(R_4 R_x - R_1 R_2)}{K(P_1 + R_g)(P + R_B)}.$$

If di is greater than di_1, then (*a*) is more sensitive than (*b*); di will be greater than di_1, if

$$K (P + R_g) (P_1 + R_B) < K (P_1 + R_g) (P + R_B)$$

or

$$(P + R_g) (P_1 + R_B) < (P_1 + R_g) (P + R_B)$$
$$P P_1 + P R_B + P_1 R_g + R_g R_B < P_1 P + R_B P_1 + P R_g + R_B R_g$$
$$P R_B + P_1 R_g < P_1 R_B + R_g P$$
$$P R_B + P_1 R_g - P_1 R_B - R_g P < 0$$

$(R_B - R_g) (P - P_1) < 0$; as R_g is always greater than R_B. $R_B - R_g$ is negative, therefore $(P - P_1)$ is positive, if $(R_B - R_g) (P - P_1) < 0$; $(P - P_1)$ being positive.

$$\frac{(R_1 + R_4) (R_x + R_2)}{K} > \frac{(R_1 + R_x) (R_4 + R_2)}{K}$$
$$(R_1 + R_4) (R_x + R_2) > (R_1 + R_x) (R_4 + R_2).$$

This is evidently true if R_1 and R_x are the two smallest resistances. Diagram (*a*) indicates, therefore, the more sensitive arrangement.

It follows that the galvanometer ought to be connected to the points where the two greatest and the two smallest resistances meet.

(*b*) Definition of Bridge Sensitivity. The bridge sensitivity is given by

$$S = \frac{di}{\frac{dR_2}{R_2}}$$

or the increase of the galvanometer current divided by the ratio of the small change dR_2 (producing di) to R_2.

$$S = R_2 \frac{di}{dR_2}.$$

As, at balance, we have $R_1 R_2 = R_x R_4$, a change in R_2 will unbalance the bridge so that

$$R_1 (R_2 + dR_2) \gtrless R_x R_4; \quad R_1 R_2 + R_1 dR_2 \gtrless R_x R_4.$$

In order to restore balance, we can change R_x or R_4. Then changing R_x by dR_x, we have

$$R_1 R_2 + R_1 dR_2 = (R_x + dR_x) R_4 = R_x R_4 + R_4 dR_x,$$

but, as $R_1 R_2 = R_x R_4$, we have $R_1 dR_2 = R_4 dR_x$.

$$\frac{R_1}{R_4} = \frac{R_x}{R_2} \text{ and } \frac{R_1}{R_4} = \frac{dR_x}{dR_2},$$

so that

$$\frac{R_x}{R_2} = \frac{dR_x}{dR_2} \text{ and } \frac{dR_x}{R_x} = \frac{dR_2}{R_2}.$$

In a similar way we can prove that

$$\frac{dR_x}{R_x} = \frac{dR_2}{R_2} = \frac{dR_1}{R_1} = \frac{dR_4}{R_4},$$

which means that the sensitivity of the bridge is unaltered whichever arm is varied for balance.

(c) EFFECT OF RATIO R_1 R_4 (RATIO ARMS VALUE) ON THE BRIDGE SENSITIVITY. Suppose there is a small unbalance dR_x in R_x; in order to restore balance we add dR_2 to R_2, so that

$$(R_x + dR_x) R_4 = R_1 (R_2 + dR_2)$$

$$R_x + dR_x = R_1 \frac{(R_2 + dR_2)}{R_4};$$

as $R_x = \frac{R_1}{R_4} R_2$, we have $dR_x = \frac{R_1}{R_4} dR_2$.

For a small unbalance dR_x, the greater $\frac{R_1}{R_4}$; the smaller dR_2 needed to restore balance, and as

$$S = \frac{di}{dR_2} R_2$$

the greater $\frac{R_1}{R_4}$, the greater the sensitivity; but the greater the value of $\frac{R_1}{R_4}$, the smaller the number of significant figures in the number giving R_x. For a given R_2, therefore (this applies to all measurements), *the greater the sensitivity, the smaller the accuracy.*

(*d*) INTERPOLATION. In order to determine R_x from $R_x = \frac{R_1}{R_4} R_2$, it is better to find a value of R_2 by interpolation instead of by the zero reading of the galvanometer. The reasons for this are as follows:

(i) It is difficult to see whether the galvanometer is exactly at zero.

(ii) The galvanometer can have a permanent small deflection (that is, a zero error), after having been deflected for a considerable time.

Let there be two values, R_{21} and R_{22}, of R_2, such that R_{21} produces a small deflection α_1 to the right and R_{22} a small deflection α_2 to the left. We know then that R_x is contained between

$$\frac{R_1}{R_4} R_{21} \text{ and } \frac{R_1}{R_4} R_{22}.$$

Consider fig. 6. We will assume that R_{21} and R_{22} are proportional to the deflections α_1 and α_2 (which is true enough if α_1 and α_2 are small).

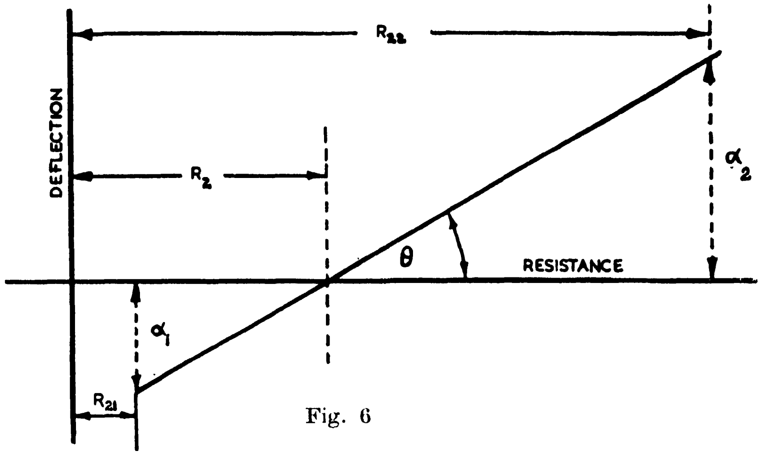

Fig. 6

From fig. 6. $\tan\theta = \dfrac{\alpha_1 + \alpha_2}{R_{22} - R_{21}} = \dfrac{\alpha_1}{R_2 - R_{21}}$; $\dfrac{R_2 - R_{21}}{R_{22} - R_{21}} = \dfrac{\alpha_1}{\alpha_1 + \alpha_2}$

$R_2 - R_{21} = (R_{22} - R_{21}) \times \dfrac{\alpha_1}{\alpha_1 + \alpha_2}$; writing
$R_{22} - R_{21} = \varrho$ we have

$$R_2 = R_{21} + \varrho \frac{\alpha_1}{\alpha_1 + \alpha_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

The value of R_2 from (11) is used in the formula $R_x = \dfrac{R_1}{R_4} \times R_2$, so that $R_x = \dfrac{R_1}{R_4}\left(R_{21} + \varrho \times \dfrac{\alpha_1}{\alpha_1 + \alpha_2}\right)$.

(*e*) Practical Manipulation. The arrangement of the bridge is shown in fig. 7.

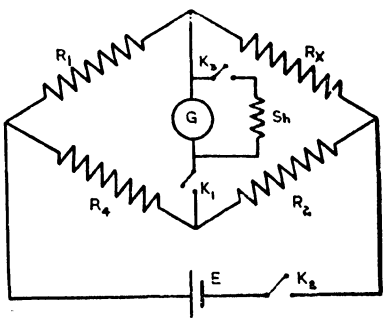

Fig. 7

E is the battery; Sh a variable resistance shunt; k_1, k_2, k_3 are switch-keys.

If the resistance to be measured is unknown, we start off with E = 1v. to 2v., and the galvanometer shunted by a suitably small resistance; $\frac{R_1}{R_4}$ is at first made equal to 1, R_2 is then varied till a rough balance is obtained; after that, the resistance of the shunt is increased or even k_3 opened (E can also be increased to about 4v.), $\frac{R_1}{R_4}$ is decreased, and R_2 increased as far as possible. The smaller $\frac{R_1}{R_4}$ and the greater R_2, the greater the accuracy of the measurement.

When interpolating, R_{21}, R_{22} should be such that the deflections a_1 and a_2 are not too small, say 10 to 15 mm., otherwise the reading error would be too great.

Switch k_2 should always be closed first, because if the resistances R_1, R_2, R_x, R_4 are inductive, the galvanometer may deflect violently even at balance, owing to the different rates of increase of the currents in the arms of the bridge.

Care must be taken not to overheat the bridge arms; the current should never exceed that indicated by the manufacturer, and even then switch k_2 should remain closed only for the time necessary for the measurement.

It is not always possible to arrange the bridge arms for maximum sensitivity because of the overloading of the bridge arms. Suppose we have (fig. 8) $R_1 = 10$, $R_4 = 1000$, $R_2 = 1500$, and $R_x = 15$, the battery voltage being 2 volts; the arms of the bridge should carry only 0·005 amps maximum.

In fig. 8, the current in the arms R_1 and R_x will equal (neglecting the battery resistance) $\frac{2}{10 + 15} = 0{\cdot}08$ amps. As this cannot be permitted, the bridge should be arranged, as in fig. 9, where the current

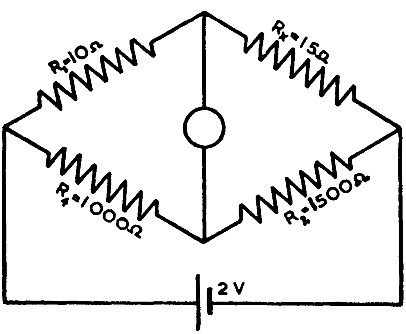

Fig. 8

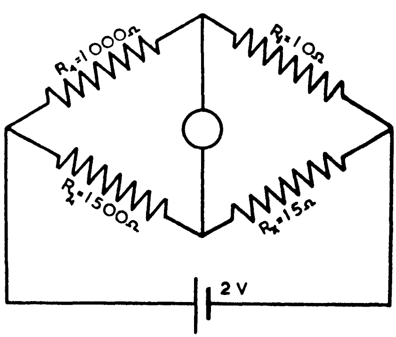

Fig. 9

in R_4 and R_1 will be $\frac{2}{1000 + 10}$; the arms of the bridge will now be safe, but the sensitivity will be diminished.

(*f*) Calculation of the Systematic Error.

We have

$$R_x = \frac{R_1}{R_4}\left(R_{21} + \varrho \times \frac{a_1}{a_1 + a_2}\right) = \frac{R_1}{R_4} \times R_2.$$

Let us first calculate the error on R_2.

$$dR_2 = dR_{21} + d\varrho\left(\frac{a_1}{a_1 + a_2}\right) + \varrho\,\frac{(a_1 + a_2)\,da_1 - a_1(da_1 + da_2)}{(a_1 + a_2)^2} =$$

$$dR_{21} + d\varrho \times \frac{a_1}{a_1 + a_2} + \varrho\,\frac{(da_1)\,a_1 + (da_1)\,a_2 - (da_1)\,a_1 - (da_2)a_1}{(a_1 + a_2)^2} =$$

$$dR_{21} + d\varrho \times \frac{a_1}{a_1 + a_2} + \varrho\,\frac{(da_1)\,a_2 - (da_2)\,a_1}{(a_1 + a_2)^2}.$$

As $da_1 = da_2 = da$, say, we can write :

$$dR_2 = dR_{21} + d\varrho \times \frac{a_1}{a_1 + a_2} + \varrho\,\frac{da\,(a_2 - a_1)}{(a_1 + a_2)^2}.$$

The error is therefore

$$\Delta R'_{21} + \Delta\varrho'\,\frac{a'_1}{a'_1 + a'_2} + \varrho'\,\frac{\Delta a'\,(a'_1 + a'_2)}{(a'_1 + a'_2)^2} =$$

$$\Delta R'_{21} + \Delta\varrho'\,\frac{a'_1}{a'_1 + a'_2} + \varrho'\,\frac{\Delta a'}{a'_1 + a'_2}.$$

If the resistances used are known to have the same error throughout, then

$$\frac{\Delta R'_{21}}{R'_{21}} = \frac{\Delta\varrho'}{\varrho'} = \text{m, say};\quad \Delta R'_{21} = R'_{21}\,\text{m};\quad \Delta\varrho' = \varrho'\text{m}.$$

Therefore

$$\Delta R'_2 = (R'_{21}\text{m}) + (\varrho'\text{m})\,\frac{a'_1}{a'_1 + a'_2} + \varrho'\,\frac{\Delta a'}{a'_1 + a'_2} =$$

$$\text{m}\left(R'_{21} + \varrho' \times \frac{a'_1}{a'_1 + a'_2}\right) + \varrho'\,\frac{\Delta a}{a'_1 + a'_2};$$

as $R_{21} + \varrho\,\frac{a_1}{a_1 + a_2} = R_2$.

$$\Delta R'_2 = \text{m}R'_2 + \varrho'\,\frac{\Delta a'}{a'_1 + a'_2},$$

and the relative error on R_2 is

$$\frac{\Delta R'_2}{R'_2} = m + \frac{\varrho'}{R'_2}\,\frac{\Delta' a}{a'_1 + a'_2} = \frac{\Delta R'_{21}}{R'_{21}} + \frac{\varrho'}{R'_2} \times \frac{\Delta a'}{a'_1 + a'_2}.$$

The relative error on R_x is then

$$\frac{\Delta R'_x}{R'_x} = \frac{\Delta R'_1}{R'_1} + \frac{\Delta R'_4}{R'_4} + \frac{\Delta R'_{21}}{R'_{21}} + \frac{\varrho'\,(\Delta a')}{R'_2(a'_1 + a'_2)} \text{ and as } \frac{\Delta R'_{21}}{R'_{21}} = \frac{\Delta R'_2}{R'_2}$$

$$\frac{\Delta R'_x}{R'_x} = \frac{\Delta R'_1}{R'_1} + \frac{\Delta R'_4}{R'_4} + \frac{\Delta R'_2}{R'_2} + \frac{\varrho'}{R'_2} \times \frac{\Delta a'}{a'_1 + a'_2}.$$

$\frac{\Delta R'_1}{R'_1}$, $\frac{\Delta R'_4}{R'_4}$, $\frac{\Delta R'_2}{R'_2}$ are the constructional errors ;

$\frac{\varrho'}{R'_2} \times \frac{\Delta a'}{a'_1 + a'_2}$ is the determination error.

If $\frac{\Delta R'_{21}}{R'_{21}}$ is not equal to $\frac{\Delta \varrho'}{\varrho'}$, then the relative error on R_2 is

$$\frac{\Delta R'_2}{R'_2} = \frac{\Delta R'_{21}}{R'_2} + \frac{\Delta \varrho'}{R'_2}\,\frac{a'_1}{a'_1 + a'_2} + \frac{\varrho'}{R'_2}\,\frac{\Delta a'}{a'_1 + a'_2},$$

and the relative error on R_x is

$$\frac{\Delta R'_x}{R'_x} = \frac{\Delta R'_1}{R'_1} + \frac{\Delta R'_4}{R'_4} + \frac{\Delta R'_{21}}{R'_2} + \frac{\Delta \varrho'}{R'_2}\,\frac{a'_1}{a'_1 + a'_2} + \frac{\varrho'}{R'_2}\,\frac{\Delta' a}{a'_1 + a'_2}.$$

Example 13. In a measurement with a Wheatstone bridge the ratio $\frac{R_1}{R_4}$ was $\frac{10}{1000}$. $R_{21} = 1499\ \Omega$ gave a deflection $a_1 = 10$ mm. to the right, $R_{22} = 1500\ \Omega$ gave a deflection $a_2 = 11$ mm. to the left. The constructional error on the bridge arms is $\pm \frac{1}{500}$; $\pm \frac{1}{8}$ of a millimeter can be distinguished on the galvanometer scale. Calculate the value of R_x and the limits of the error, assuming the resistances boxes to have the same error throughout.

The determination error is

$$100 \times \frac{\varrho'}{R'_{21}} \times \frac{\Delta a'}{a'_1 + a'_2} = \frac{1 \times 100}{1499} \times \frac{\pm\frac{0{\cdot}25}{2}}{10 + 11} =$$

$$\pm \frac{0{\cdot}125 \times 100}{1499 \times 21} = \pm\, 0{\cdot}00822\%.$$

The value of R_x is

$$R'_x = \frac{10}{1000} \times \left(1499 + 1 \times \frac{10}{10 + 11}\right) = \frac{1499{\cdot}476}{100} = 14{\cdot}99476\ \Omega.$$

The maximum possible percentage error on R_x is

$$\frac{\Delta R'_x}{R'_x} = \left(\frac{\Delta R'_1}{R'_1} + \frac{\Delta R'_4}{R'_4} + \frac{\Delta R'_2}{R'_2} + \frac{\varrho'}{R'_2} \times \frac{\Delta a'}{a'_1 + a'_2}\right) 100 = \frac{100}{500} +$$

$$\frac{100}{500} + \frac{100}{500} + 0{\cdot}00822 = \pm\, 0{\cdot}6082\%.$$

(*g*) THE ERROR DUE TO THE RESISTANCE OF THE CONNECTING WIRES, CONTACTS, AND PLUGS. Let the resistance of the plugs, contacts, and connecting wires in the arms R_1 R_x R_2 and R_4 be r_1, r_x, r_2 and r_4 respectively ; at balance we have

$$(R_x + r_x) = \frac{R_1 + r_1}{R_4 + r_4} \times (R_2 + r_2) = \frac{R_1 R_2 + R_1 r_2 + R_2 r_1 + r_1 r_2}{R_4 + r_4}.$$

We accept that $R_x = \frac{R_1}{R_4} \times R_2$, but the result will nevertheless be correct if

$$R_x + r_x - R_x = \frac{R_1 R_2 + R_1 r_2 + R_2 r_1 + r_1 r_2}{R_4 + r_4} - \frac{R_1}{R_4} R_2,$$

that is

$$r_x = \frac{R_1 R_2 + R_1 r_2 + R_2 r_1 + r_1 r_2}{R_4 + r_4} - \frac{R_1}{R_4} \times R_2.$$

As $\frac{R_1 R_2}{R_4 + r_4}$ very nearly equals $\frac{R_1 R_2}{R_4}$ and $\frac{r_1 r_2}{R_4 + r_4}$ is very small, we can write

$$r_x = \frac{R_1}{R_4 + r_4} \times r_2 + \frac{R_2}{R_4 + r_4} r_1 \cong \frac{R_1}{R_4} r_2 + \frac{R_2}{R_4} r_1 \quad . \quad . \qquad (12)$$

When condition (12) is realised (provided r_4 is small enough), the error due to the plug, contacts, and connecting wires resistance is zero. It would, however, be tedious, if not impossible, to comply with condition (12) in order to have reasonable accuracy when measuring small resistances ; the Wheatstone bridge is therefore not used for measuring resistances below 1 or 2 ohms, and even then, the error introduced by the plug, contacts, and wire resistance may be considerable.

Measurement of Small Resistances

(4) The Kelvin Bridge

The arrangement is as shown in fig. 10.

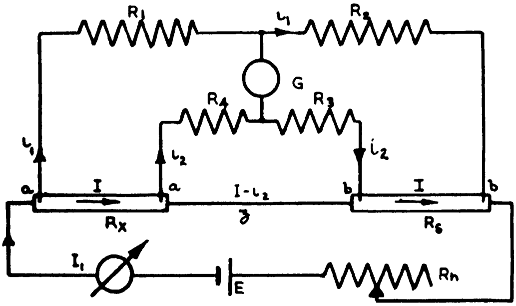

Fig. 10

R_x is the resistance to be measured ; R_s a standard resistance ; R_1, R_2, R_3 and R_4 standard resistances constituting the bridge proper ; G is a galvanometer.

The battery E supplies a current I_1 measured by an ammeter, in series with a regulating rheostat Rh. The currents are as shown in fig. 10, and I is very much greater than i_1 or i_2.

R_x and R_s are four terminal resistances ; the contacts *a,a, b,b* are taken from those points on R_x and R_s where the equipotential surfaces of the electric field are entirely uniform (fig. 11).

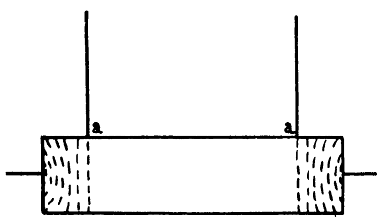

Fig. 11

If R_x is not a four-terminal resistance, contact is made by winding a clean, thin, bare copper wire at from 1 to 2 cm. from the ends of R_x and then soldering the wire ; the resistance measured will, of course, be that between the two soldered contacts.

The currents taken from the contacts *a,a, b,b* ought to be very small compared to the current I, otherwise the equipotential surfaces at *a,a, b,b* will be disturbed.

The bridge does not eliminate the contact and connecting-wire resistances from the measurement; it only greatly minimises their effect by putting them in series with much greater resistances R_1, R_2, R_3 and R_4.

The bridge is balanced when no current flows in the galvanometer; then i_1 flows in R_1 as well as in R_2, and i_2 in R_4 and in R_3; we also have

$$R_1 i_1 = R_x I + R_4 i_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$
$$R_2 i_1 = R_s I + R_3 i_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Now consider loop $R_4 R_3 z$, z is the resistance of the link between R_2 and R_4, so we have

$$(R_4 + R_3) i_2 = (I - i_2) z$$
$$(R_4 + R_3 + z) i_2 = (Iz);$$

therefore

$$i_2 = I \times \frac{z}{R_4 + R_3 + z}.$$

(13) and (14) can therefore be written

$$(15) \quad R_1 i_1 = \left(R_x + \frac{R_4 z}{R_4 + R_3 + z}\right) I$$

$$(16) \quad R_2 i_1 = \left(R_s + \frac{R_3 z}{R_4 + R_3 + z}\right) I; \quad \text{dividing (15) by (16)}$$

$$\frac{R_1}{R_2} = R_x + \frac{R_4 z}{R_4 + R_3 + z} \div R_s + \frac{R_3 z}{R_4 + R_3 + z}.$$

Therefore

$$R_2 R_x + \frac{R_2 R_4 z}{R_4 + R_3 + z} = R_1 R_s + \frac{R_1 R_3 z}{R_4 + R_3 + z};$$

$$R_x = \frac{R_1 R_s}{R_2} + \frac{z}{R_2} \times \frac{R_1 R_3 - R_2 R_4}{R_4 + R_3 + z}.$$

We make $R_1 R_3 = R_2 R_4$, then

$$R_x = \frac{R_1 R_s}{R_2}.$$

There is, of course, an error on $R_1 R_3 - R_2 R_4$, but if the resistance z is very small this error will be negligible.

Calculation of the Systematic Error. Writing $R_1 R_3 - R_2 R_4 = c$ (c will be the error on $R_1 R_3 - R_2 R_4$) and $R_4 + R_3 + z = g$, we have

$$R_x = \frac{R_1 R_8}{R_2} + \frac{zc}{R_2 g} = \frac{(R_1 R_8 g) + (z c)}{R_2 g}.$$

$$\frac{dR_x}{R_x} = \frac{d(R_1 R_8 g + z c)}{R_1 R_8 g + zc} - \frac{d(R_2 g)}{R_2 g} =$$

$$\frac{R_1 R_8 dg + gd(R_1 R_8) + zdc + cdz}{R_1 R_8 g + zc} - \frac{R_2 dg + g\, dR_2}{R_2 g} =$$

$$\frac{R_1 R_8 dg + gR_1 dR_8 + gR_8 dR_1 + zdc + cdz}{R_1 R_8 g + zc} - \frac{R_2 dg + gdR_2}{R_2 g} =$$

$$\frac{R_1 R_8 g}{(gR_1 R_8 + zc)} \frac{dg}{g} + \frac{gR_1 R_8}{(gR_1 R_8 + zc)} \frac{dR_8}{R_8} + \frac{gR_8 R_1}{(gR_1 R_8 + zc)} \frac{dR_1}{R_1} +$$

$$+ \frac{zc}{(gR_1 R_8 + zc)} \frac{dc}{c} + \frac{cz}{(gR_1 R_8 + zc)} \frac{dz}{z} - \frac{R_2 g}{R_2 g} \frac{dg}{g} - \frac{gR_2}{R_2 g} \frac{dR_2}{R_2} =$$

$$\frac{R_1 R_8 g}{(gR_1 R_8 + zc)} \frac{dg}{g} + \frac{gR_1 R_8}{(gR_1 R_8 + zc)} \frac{dR_8}{R_8} + \frac{gR_8 R_1}{(gR_1 R_8 + zc)} \frac{dR_1}{R_1} +$$

$$\frac{zc}{(gR_1 R_8 + zc)} \frac{dc}{c} + \frac{cz}{(gR_1 R_8 + zc)} \frac{dz}{z} - \frac{dg}{g} - \frac{dR_2}{R_2}.$$

As zc is much smaller than $gR_1 R_8$, we have $gR_1 R_8 + zc \simeq gR_1 R_8$, so that

$$\frac{dR_x}{R_x} = \frac{dg}{g} + \frac{dR_8}{R_8} + \frac{dR_1}{R_1} + \frac{zc}{gR_1 R_8} \frac{dc}{c} + \frac{zc}{gR_1 R_8} \frac{dz}{z} - \frac{dg}{g} - \frac{dR_2}{R_2} =$$

$$\frac{dR_8}{R_8} + \frac{dR_1}{R_1} + \frac{zc}{gR_1 R_8} \frac{dc}{c} + \frac{zc}{gR_1 R_8} \frac{dz}{z} - \frac{dR_2}{R_2};$$ the error is therefore

$$\frac{\Delta R'_x}{R'_x} = \frac{\Delta R'_8}{R'_8} + \frac{\Delta R'_1}{R'_1} + \frac{\Delta R'_2}{R'_2} + \frac{z'c'}{g'R'_1 R'_8} \frac{dc'}{c'} + \frac{z'c'}{g'R'_1 R'_8} \frac{dz'}{z'}$$

or very nearly

$$\frac{\Delta R'_x}{R'_x} = \frac{\Delta R'_8}{R'_8} + \frac{\Delta R'_1}{R'_1} + \frac{\Delta R'_2}{R'_2},$$ as zc is very small.

In the older Kelvin bridges, R_1, R_2, R_3, and R_4 were fixed, and R_8 was a calibrated bar with a sliding contact ; nowadays R_2 is made equal to R_3, both being fixed, whilst R_1 and R_4 are variable.

When interpolating, the determination error is calculated in the same way as in the Wheatstone bridge ; let, say, R_{11} be the value of R_1, giving a small deflection a_1 to the right, and R_{12} the value of R_1, giving a small deflection a_2 to the left ; we have then

$$R_1 = R_{11} + \varrho \frac{a_1}{a_2 + a_2}\,;\ R_x = \frac{R_s}{R_2}\left(R_{11} + \varrho \frac{a_1}{a_1 + a_2}\right),$$

where $\varrho = R_{12} - R_{11}$; the relative maximum error on R_x is then

$$\frac{\Delta R'_x}{R'_x} = \frac{\Delta R'_s}{R'_s} + \frac{\Delta R'_2}{R'_2} + \frac{\Delta R'_1}{R'_1} + \frac{\varrho' \,\Delta a'_1}{R'_1 (a'_1 + a'_2)}.$$

The greater the value of I, the greater the precision to be expected, but a very large current can give rise to thermo-electric effects. It is therefore advisable to make a second measurement with I reversed, and to take the mean of the two readings.

(5) The Differential Bridge (Mathieson's Method)

The arrangement is as shown in fig 12.

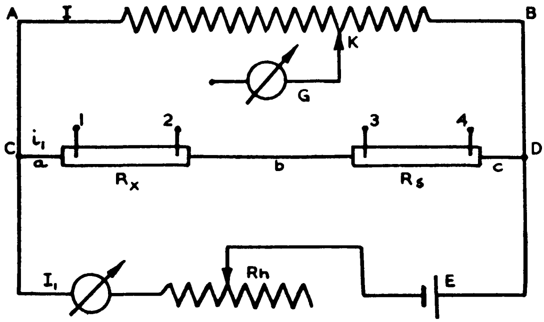

Fig. 12

The resistance AB is of known value and constant ; it includes variable resistance boxes and a slide wire for fine adjustment. Whatever changes are made in AB, its resistance has to be kept constant ; so that, when increasing the resistance between AK by, say, Z, the resistance between K and B is diminished by the same amount.

R_x and R_s are the unknown resistance and the standard resistance. A source E, in series, with the rheostat Rh, supplies a current I_1 indicated by an ammeter. R_x and R_s are four terminal resistances (see Kelvin bridge) ; the galvanometer G has one terminal connected to the slide K ; the other terminal can be connected to 1, 2, on R_x and 3, 4 on R_s. Suppose the resistance of the connecting wires be *a*, *b*, *c*. When the galvanometer is connected to 1, the position of K is varied till the galvanometer shows no deflection ; then let the resistance between

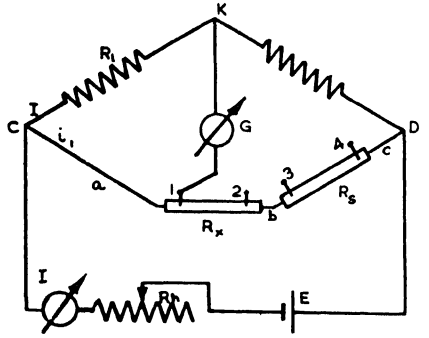

Fig. 13

C and K be R_1, and we have an arrangement which is a Wheatstone bridge, as can be seen from fig. 13.

In arm CK we have the resistance R_1 ; in arm KD the remaining resistance of AB ; in arm C1, the resistance *a* ; we can then write :

$$R_1 I = a i_1 \quad . \quad . \quad (17)$$

The galvanometer is then connected to 2 of R_x and the position of K and resistance of CK varied till new balance is obtained ; then, supposing the resistance between C and K to be R_2, the bridge arrangement is as shown in fig. 14.

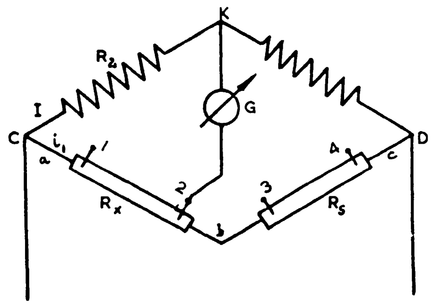

Fig. 14

We can write

$$R_2 I = (R_x + a)\, i_1 \quad . \ (18)$$

The galvanometer is then connected to 3 on R_s ; the bridge becomes as shown in fig. 15, and at balance

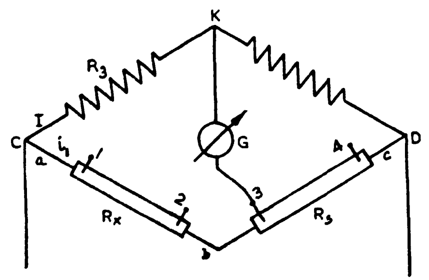

Fig. 15

$$R_3 I = (a + R_x + b)\, i_1 \quad .(19)$$

Finally, the galvanometer is connected to 4 on R_S, and the resistance between C and K is varied till new balance is obtained ; we then have a Wheatstone bridge, as shown in fig. 16 ; and we can write :

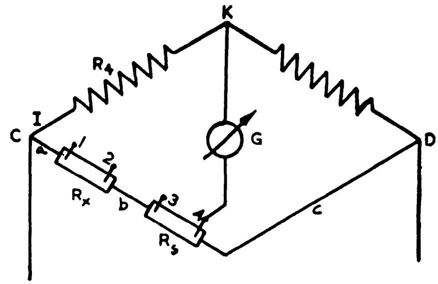

Fig. 16

$$R_4 I = (a + R_X + b + R_S)\, i_1 . \qquad (20)$$

Taking (17) of (18) and (19) of (20), we get

$$R_X i_1 = (R_2 - R_1)\, I \qquad (20a)$$

$$R_S i_1 = (R_4 - R_3)\, I \qquad (20b)$$

Dividing (20a) by (20b)

$$\frac{R_X}{R_S} = \frac{R_2 - R_1}{R_4 - R_3}; \text{ therefore } R_X = \frac{R_S (R_2 - R_1)}{R_4 - R_3}.$$

Calculation of the Systematic Error. As we can write $R_2 = R_1 + r_1$, $R_4 = R_3 + r_2$, we have

$$R_X = R_S \frac{r_1}{r_2}; \quad \frac{dR_X}{R_X} = \frac{dR_S}{R_S} + \frac{dr_1}{r_1} - \frac{dr_2}{r_2} = \frac{dR_S}{R_S} + \frac{dr_1}{R_2 - R_1} - \frac{dr_2}{R_4 - R_3},$$

passing to errors

$$\frac{\Delta R'_X}{R'_X} = \frac{\Delta R'_S}{R'_S} + \frac{\Delta r'_1}{r'_1} + \frac{\Delta r'_2}{r'_2}.$$

The method is a good one, because the contact and connecting-wire resistance are entirely accounted for, and there is no current taken, at balance, from contacts 1, 2, 3, 4 ; there is, therefore, no disturbance in the equipotential surfaces of R_X and R_S. The error will, however, be great if r_1 and r_2 are small.

(6) Opposition Method

The Mathieson method suggests a more convenient opposition method, as shown in fig. 17. E is a source of suitable e.m.f., supplying a heavy current I, indicated by an ammeter in series with a regulating rheostat Rh ; R_x is the resistance to be measured, and R_s a suitable standard resistance.

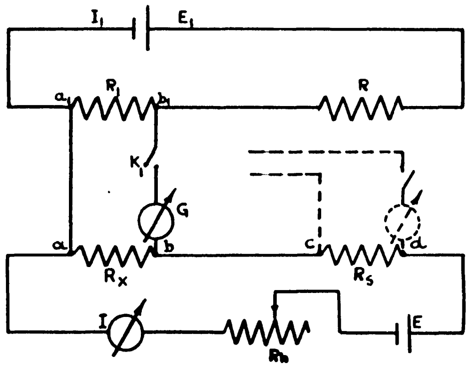

Fig. 17

The source E_1 supplies a small current I_1 to the resistance R_1 and R. $R_1 + R$ have to be kept constant, so that I should not vary. If R_1 is altered, say, increased, R has to be decreased by the same amount, and *vice versa.*

The points a, a_1 are joined by a conductor, while points b, b_1 are joined through the galvanometer G in series with switch k_1. When k_1 is closed and R_1 varied till the galvanometer shows no deflection, we have

$$R_1 I_1 = R_x I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Next, a_1, b_1 are joined to c, d, through G and K_1, and R_1 varied until the galvanometer again shows no deflection. Suppose, then, the value of R_1 is changed to that of R_2, we shall have

$$R_2 I_1 = R_s I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Dividing (21) by (22) we get

$$\frac{R_1}{R_2} = \frac{R_x}{R_s}; \ R_x = R_s \frac{R_1}{R_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

As we can write : $R_2 = R_1 \pm r$, where r is the change in the resistance box from R_1 to R_2 ; we have, therefore,

$$R_x = \frac{R_1}{R_1 \pm r} \cdot R_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Calculation of the Systematic Error. The logarithmic differential of R_x is

$$\frac{dR_x}{R_x} = \frac{dR_S}{R_S} + \frac{dR_1}{R_1} - \frac{d\,(R_1 \pm r)}{R_1 \pm r} =$$

$$\frac{dR_S}{R_S} + \frac{dR_1}{R_1} - \frac{dR_1}{R_1 \pm r} \mp \frac{dr}{R_1 \pm r} =$$

$$\frac{dR_S}{R_S} + \frac{dR_1}{R_1} - \frac{R_1}{R_1 \pm r}\,\frac{dR_1}{R_1} \mp \frac{r}{R_1 \pm r}\,\frac{dr}{r} =$$

$$\frac{dR_S}{R_S} + \frac{dR_1}{R_1}\left(1 - \frac{R_1}{R_1 \pm r}\right) \mp \frac{r}{R_1 \pm r}\,\frac{dr}{r} =$$

$$\frac{dR_S}{R_S} + \frac{dR_1}{R_1}\left(\frac{\pm r}{R_1 \pm r}\right) \mp \frac{r}{R_1 \pm r}\,\frac{dr}{r};$$

the relative error on R_x is therefore

$$\frac{\Delta R'_x}{R'_x} = \frac{\Delta R'_S}{R'_S} + \frac{\Delta R'_1}{R'_1}\left(\frac{+\,r'}{R'_1 \pm r'}\right) + \frac{r'}{R'_1 \pm r'}\,\frac{\Delta r'}{r'} \qquad (24a)$$

The error diminishes with r, that is, with $R_S - R_x$, and is smallest when $R_S = R_x$.

To the error given by (24a) we have to add the determination error due to interpolation. This error on R_1 and R_2 is calculated in the same way as in the Wheatstone bridge.

Other errors most likely to occur are due to

(*a*) Poor insulation, resulting in leakage between the two circuits R_1,R, and R_x,R_S ; the two circuits have to be well insulated.

(*b*) Variation of the currents I and I_1.

As the current I_1 is small, there is not much danger from variance of the e.m.f. of E_1 ; the same cannot, however, be said of the source E, since the current I has to be large. When variation of E is feared, the best procedure is to make one measurement on R_x with I, say, equal to I_A, and R_1 equal to, say, R_A, so that $R_xI_A = R_AI_1$. Next, a measurement is made on R_S, giving $R_SI = R_2I_1$; finally a third measurement is made on R_x, giving, say, $R_xI_B = R_BI_1$. Assuming the current I to have varied uniformly during the time of making these measurements, we have

$$I = \frac{I_A + I_B}{2} \text{ and } R_x = R_S \times \frac{R_A + R_B}{2R_2}.$$

The three measurements have to be made as speedily as possible.

Measurement of High Resistances

High resistances of the order of several MΩ to several hundred-thousand MΩ are rarely metallic. When non-metallic, they are generally dielectrics used as insulators. The measurement of metallic high resistances is made in the same way as the measurement of the resistance of dielectrics.

The resistance of a dielectric depends on very many factors, such as temperature, humidity, condition of the surface of the dielectric, shape and thickness, the time of flow of current or time of electrification, magnitude of e.m.f. applied, etc. Some dielectrics can have their resistance changed by as much as 100% after a short electrification time.

It follows that the measurement of high resistances, such as dielectrics, cannot be exact, owing to the many factors involved ; it also follows that when giving the result of such a measurement, the conditions in which it was conducted have to be specified.

(7) The Volume Resistance of a Dielectric

By volume resistance we mean the resistance as given by the relation $I = \frac{E}{\varrho l/s}$ where E is the voltage applied, ϱ = the resistivity, l = the length and s = the cross-section of the dielectric.

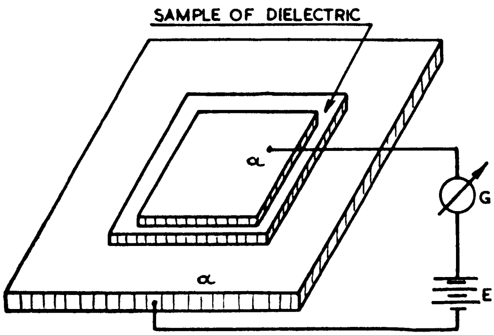

Fig. 18

If we were to measure the resistance of a sample of a dielectric in the manner shown in fig. 18, by putting the sample between two metal plates, *a*,*a*, and in series with a galvanometer, the instrument will be deflected by two currents: one limited by the volume resistance, and the other due to the leakage current ; that is, the current passing from plate to plate by way of the dielectric surface.

This leakage current is absolutely negligible when dealing with small or medium resistances because the leakage path is shunted by the much smaller volume resistance, but in a high resistance the leakage path resistance may be of the same order, or even smaller than, the volume resistance ; the leakage current has therefore to be eliminated from the measurement if any reasonable test is to be made.

This elimination is achieved by the aid of a metallic guard ring connected as shown in fig. 19.

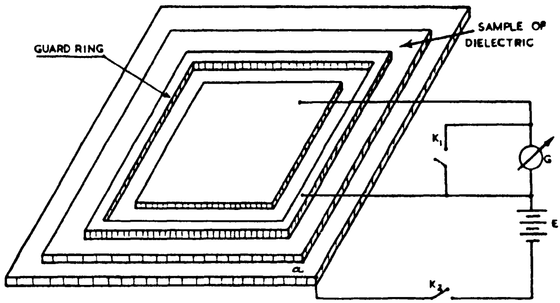

Fig. 19

The galvanometer is now deflected only by the current passing through the volume of the sample, while the leakage current passes from source to one plate, thence to the guard square, and back to the source, avoiding the galvanometer.

Because of the high resistance, even with E from 150 to 200 v., a direct reading instrument, such as a micro-ammeter, is unsuitable, and a galvanometer has to be used ; consequently a comparison method is indicated.

(8) The Comparison Method of Measuring High Resistance

The connections are as in fig. 19.

The switch k_1 is necessary only when the resistance tested has an appreciable capacity, such as a length of cable has ; in this case k_1 has to be closed (galvanometer shorted), when k_2 is first closed, so as to protect the galvanometer from the charging current.

When k_2 is closed and k_1 opened, the galvanometer will show a deflection, say, α. Suppose the current passing through the galvanometer to be

$$i = \frac{E}{R_x} \qquad (25)$$

R_x is the volume resistance of the tested sample ; the resistance of the source E and of the galvanometer can be neglected in this test.

The sample is then replaced by a known high resistance R_s, and the measurement repeated ; it is generally necessary to change the e.m.f. of the source to, say, E_1, instead of E ; the galvanometer deflection will be α_1, and the current

$$i_1 = \frac{E_1}{R_s} \qquad (26)$$

Dividing (25) over (26) we get $\frac{i}{i_1} = \frac{E \times R_s}{E_1 \times R_x}$, and assuming the galvanometer deflections to be directly proportional to the currents, we can write :

$$\frac{a}{a_1} = \frac{E \times R_s}{E_1 \times R_x}; \; R_x = \frac{E \times a_1 \times R_s}{E_1 a}.$$

The standard resistance R_s can be either in the form of a dielectric of known volume resistivity, in which case it has to be provided with a guard ring, or what is known as its resistance can include the volume resistance and the resistance of its leakage path in parallel. Care has to be taken to have the resistances clean, otherwise the leakage will be much greater than normal.

In order to ensure a good contact between the metal plates and the sample, the surfaces of the sample in contact with the plates are covered with thin tinfoil. The tinfoil should adhere well to the sample, and to ensure this the tinfoil is made slightly humid by one or two drops of water or oil. Too much water or oil might give a false result, or even spoil the dielectric.

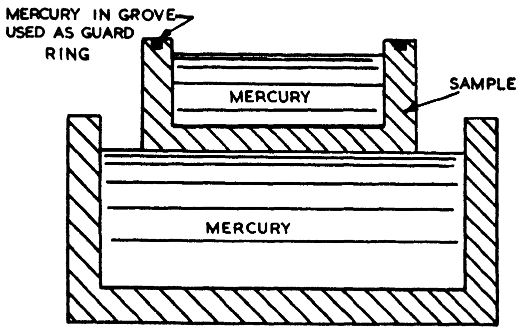

Fig. 20

The sample can be floated in mercury in the manner shown in fig. 20. The contact between sample and plates is here perfect ; nevertheless, this method is not recommended because the special machining of the sample is difficult, and there is great uncertainty as to the effective length and section of the sample for the calculation of the volume resistance.

There is no point in calculating the error in this measurement, considering that the accuracy obtained can never be better than $\pm 5\%$ to $\pm 10\%$.

Measurement of the Insulation Resistance of a Length of Cable by the Comparison Method. The insulation resistance of a cable is measured between the copper core and the outside sheath ; the sheath can be conductive or not. If the sheath is conductive, the arrangement is as shown in fig. 21. E is the source, G the galvanometer. Instead

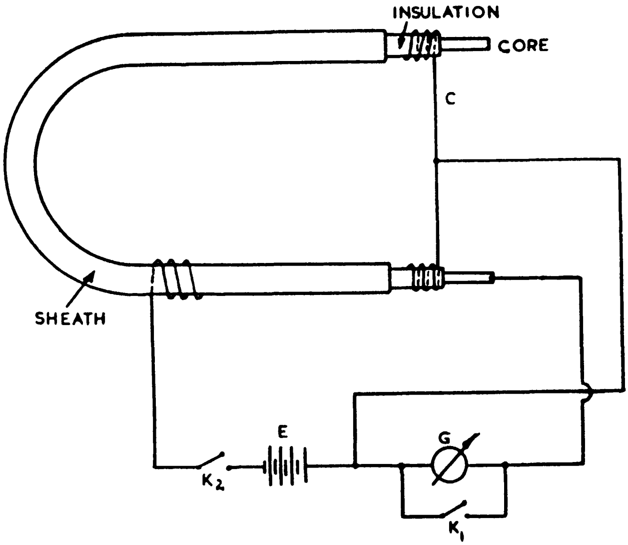

Fig. 21

of a guard ring, a bare copper wire is wound on the cable insulation near the core ; the insulation is bared and cleaned for this purpose for about 6 in. to 9 in., and the leakage current from core to sheath now passes through the wire c to the source and not through the galvanometer.

The cable is tested as in fig. 19, the galvanometer deflection being a ; then a known resistance R_S is substituted for the cable ; the e.m.f. being now, say, E_1, and the galvanometer deflection a_1. The insulation resistance of the cable will be given by

$$R_X = \frac{E a_1 R_S}{E_1 a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

As a length of cable has an appreciable capacity, the switch k_1 is essential ; the galvanometer must be shorted while the cable is being charged ; k_2 should be closed for a definite time, then opened, and the deflection a noted ; the time of electrification has to be stated in the result of the test, as R_X depends on this time.

When the cable sheath is not conductive, the arrangement for the measurement is as shown in fig. 22.

The cable is immersed in slightly salted water for about twenty-four hours, the temperature of the bath being kept at about 70° F. ;

then the measurement is taken as in fig. 21. First, the current through the cable is measured by the deflection α, then the cable is replaced by a known resistance R_s, the deflection being α_1; the insulation resistance of the cable is given by (27).

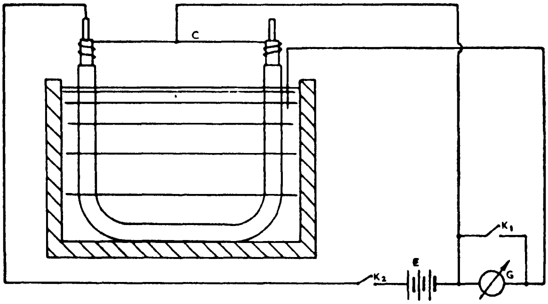

Fig. 22

It is preferable to connect the negative pole of E to the cable core, because if there is a defect in the insulation, the electrolysis will produce a smaller value of R_x and the measurement will be done under the worst possible conditions.

The main advantage of the comparison method is that the voltage applied is constant throughout the test; the result obtained gives a fair value of the resistance for the particular voltage applied.

(9) The Loss-of-charge Method of Measuring High Resistance

A condenser of capacity C charged to a potential V_0 is discharged through the resistance measured R_x and the time of discharge is noted. The potential across the condenser, which has fallen during the discharge to a value V_1 lower than V_0, is then measured.

R_x is calculated from the relation between R_x, V_0, V_1, and the time of discharge t.

Consider fig. 23; let v be the instantaneous value of the potential across the condenser C during the discharge; then the current in the circuit will be $i = \dfrac{v}{R_x}$; during the time dt the condenser will lose a charge $-dq = idt = \dfrac{v}{R_x}dt$.

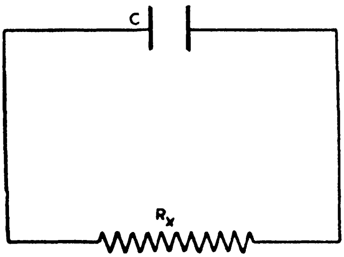

Fig. 23

dq appears with a minus sign because when v decreases the loss of charge increases.

As the charge on the condenser is $q = Cv$, and C is constant, $dq = C \cdot dv$, so that $\frac{v}{R_x} dt = -C\,dv$; $dt = \frac{-C R_x}{v} dv$; integrating between the time $t = O$ (start of discharge) and t (finish of discharge), we have

$$\int_0^t dt = -CR_x \int_{(v \text{ at } t = O) = V_0}^{(v \text{ at } t = t) = V_1} \frac{dv}{v} = -CR_x\Big[(log.v)\Big]_{V_0}^{V_1} \cdot \quad (28)$$

and reversing the limits of (28) we can write :

$$t = CR_x\Big[\;log.v\;\Big]_{V_1}^{V_0} = CR_x \cdot log.\frac{V_0}{V_1} ; \quad R_x = \frac{t}{C log.\frac{V_0}{V_1}}.$$

(*a*) Practical Manipulation. The circuit is arranged as in fig. 24.

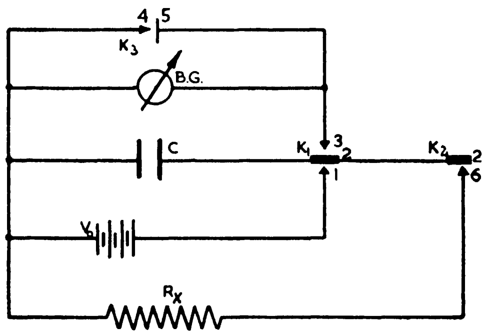

Fig. 24

B.G. is a ballistic galvanometer, V_0 the source of e.m.f., R_x the resistance to be measured, and C a standard capacity ; k_1, k_2, and k_3 are switches.

First, make 1 and 2 (key 1), charging the condenser to V_0 ; then break 1–2 and make 2–3, discharging the condenser through the ballistic galvanometer, which will be deflected by, say, a_0 ; the galvanometer is then damped by making 4–5. The condenser is again charged to V_0 by making 1–2 ; then 1–2 is broken and 2–6 made immediately after ; the time t is counted from the moment of making 2–6 to the moment of breaking it ; after the time t, break 2–6 and make 2–3, discharging C through B.G. ; the deflection being now, say, a_1.

Assuming the deflections a_0 and a_1 to be directly proportional to the p.d.s across the condenser, we can write :

$$R_x = \frac{t}{C log.\frac{a_0}{a_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

t is counted in seconds and C in farads ; if C is counted in μF, R_x will be given in MΩ.

The resistance R_x found from (29) comprises, of course, the volume resistance plus the leakage resistance in parallel.

There is no point whatever in calculating the error, because here the accuracy is even less than in the comparison method, mainly because the leakage current is not eliminated from the measurement, and the voltage applied to R_x varies from V_0 to V_1. We can, however, find the conditions which will give the minimum error.

The logarithmic differential of (29) is

$$\frac{dR_x}{R_x} = \frac{dt}{t} - \frac{dC}{C} - \frac{d\left(log.\frac{a_0}{a_1}\right)}{log.\frac{a_0}{a_1}} = \frac{dt}{t} - \frac{dC}{C} - \frac{d(log.a_0 - log.a_1)}{log.\frac{a_0}{a_1}} =$$

$$\frac{dt}{t} - \frac{dC}{C} - \frac{\left(\frac{da_0}{a_0}\right) + \left(\frac{da_1}{a_1}\right)}{log.\frac{a_0}{a_1}}$$; as da_0 and da_1 are the reading errors on the galvanometer (usually $\pm \frac{1}{8}$ to $\pm \frac{1}{4}$ mm. plus the indetermination, expressed in mm., of locating the exact position of the line image), and $\Delta a_0 = \Delta a_1$, we can write :

$$\frac{\Delta R'_x}{R'_x} = \frac{\Delta t'}{t'} + \frac{\Delta C'}{C'} + \frac{\Delta a'_0\left(\frac{1}{a'_0} + \frac{1}{a'_1}\right)}{log.\frac{a'_0}{a'_1}};$$

multiplying and dividing the numerator of the last term on the right-hand side by a_0, we get

$$\frac{\Delta R'_x}{R'_x} = \frac{\Delta t'}{t'} + \frac{\Delta C'}{C'} + \frac{\frac{\Delta a'_0}{a'_0}\left(1 + \frac{a'_0}{a'_1}\right)}{log.\frac{a'_0}{a'_1}}$$; and writing $\frac{a'_0}{a'_1} = k$, we have

$$\frac{\Delta R'_x}{R'_x} = \frac{\Delta t'}{t'} + \frac{\Delta C'}{C'} + \frac{\Delta a'_0 (1 + k)}{a'_0 \, log.k} \quad . \quad . \quad . \quad (30)$$

$\frac{\Delta t'}{t'}$ is the relative error on the determination of the time t. If t is small the error will be great, and if t is great the error will be small ; but then the variation of the p.d. from V_0 to V_1 will be great.

$\frac{\Delta C'}{C'}$ is the relative error on the condenser, and this can be diminished by using a good standard.

The last term on the right-hand side of (30) will be the smaller $\frac{1+k}{log.k} = z$. Differentiating z, we get $\frac{dz}{dk} = \frac{log.k - \frac{(1+k)}{k}}{log^2 k}$ equating to zero we have

$$log.k - \frac{(1+k)}{k} = 0\,;\; log.k = \frac{1+k}{k}\,;\; k = e^{\frac{1+k}{k}} \qquad . \qquad (30a)$$

(30a) will be satisfied when $k = \frac{a_0}{a^1} \simeq 3{\cdot}59$.

But when $\frac{a_0}{a_1} = 3{\cdot}59$, we also have $\frac{V_0}{V_1} = 3{\cdot}59$; and as the resistance of a dielectric varies with the voltage applied, this variation is too great; a ratio $\frac{V_0}{V_1} = 2$ is preferable.

(*b*) Errors Inherent in the Method. (i) As the insulation resistance of the condenser used in the measurement is not infinite, the condenser will discharge through the resistance R_x and also through its own resistance R_c; the circuit will therefore be as shown in fig. 25.

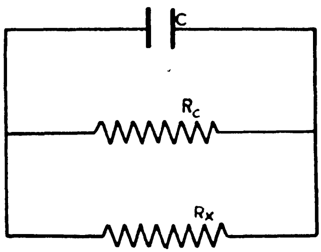

Fig. 25

The total resistance of the circuit is now $R_T = \frac{R_c R_x}{R_c + R_x}$, so that proceeding, as in fig. 24, we will have

$$R_T = \frac{t}{C log.\frac{a_0}{a_1}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

To find R_x, two measurements are made as in fig. 24, one with the condenser discharging through R_x and the other with the condenser discharging through its own resistance alone during a time, say, t_2; the deflections of the B.G. being a_{0c} and a_{1c}. Then

$$R_c = \frac{t_2}{C log.\frac{a_{0c}}{a_{1c}}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

From (31) and (32) we get R_x.

Instead of a ballistic galvanometer, an electrostatic voltmeter, of negligible capacitance compared with that of C, can be used to measure the p.d. across C.

Example 14. " A circuit consisting of a condenser, an electrostatic voltmeter, and a high resistance in parallel, is connected to a 230-volt d.c. supply. It is then disconnected and the reading on the voltmeter is found to fall from 200 volts to 100 volts in 203 seconds. The test is repeated without the high resistance, and the time for the same fall in voltage is now found to be 278 seconds. The capacitance of the condenser is 4 μF. Calculate the value of the high resistance and prove any formula employed, pointing out any assumptions made. Describe the experimental arrangements that would be required to make a measurement in this manner " (Univ. of London, B.Sc. Final Ext., Electr. and Meas. Instr. P. II. Q. 5, 1940).

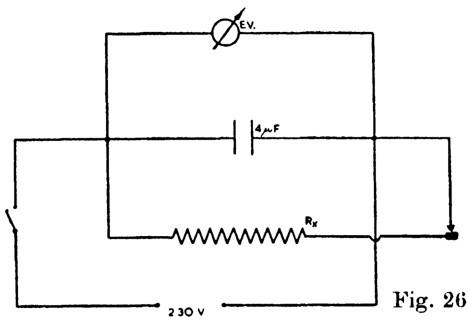

Fig. 26

The experimental arrangement will be that shown in fig. 26. The assumptions made are that the capacitance of the electrostatic voltmeter is very small compared to 4 μF, and, as will be seen later, that the resistance tested has no appreciable capacity.

Let the resistance tested be R_x, and the insulation resistance of the condenser R_c; then from the first test we have

$$R_T = \frac{R_c R_x}{R_c + R_x} = \frac{t}{C \, log. \dfrac{V_o}{V_1}} = \frac{203}{4 \times log. \dfrac{200}{100}} = 73{\cdot}4 \text{ M}\Omega.$$

In the second test we have

$$R_c = \frac{t_2}{C \, log. \dfrac{V_{oc}}{V_{1c}}} = \frac{278}{4 \times log. \dfrac{200}{100}} = 100 \text{ M}\Omega.$$

$$R_T (R_c + R_x) = R_c R_x \,; \quad R_x = \frac{R_T R_c}{R_c - R_T} = \frac{73{\cdot}4 \times 100}{100 - 73{\cdot}4} = 276 \text{ M}\Omega.$$

(ii) The resistance tested can itself have an appreciable capacity, say, C_X (for instance, a length of cable). In this case we have the condenser C in parallel with C_X, the two being in parallel with R_X and R_C; the circuit is therefore that of fig. 27.

If the measurement is made as in *b* (i), then, when the condenser C, fully charged to V_0, is connected to R_X, there will be an immediate transfer of charge from C to C_X, before C discharges appreciably through R_X and R_C; the p.d. across C will therefore immediately fall to a lower value, say, V_{01}, as soon as it is connected to R_X. What it amounts to is that we have a condenser $C_T = C + C_X$ charged to V_{01} and discharging through R_C and R_X in parallel.

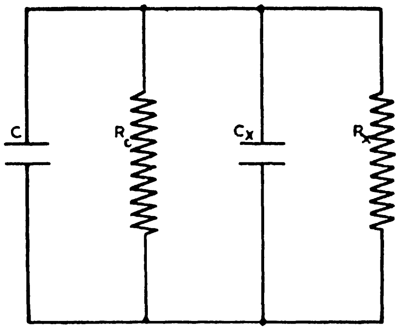

Fig. 27

Assuming the charge has not altered immediately after C is connected to R_X, we shall have

$$CV_0 = (C + C_X)\,V_{01} = C_T\,V_{01}\,;\quad V_{01} = \frac{C}{C + C_X} \times V_0.$$

The condenser C discharges then through R_X and R_C for a time t until its p.d. falls to V_1.

We can therefore proceed in the following manner. Charge C to V_0 and discharge through the galvanometer, the deflection being α_0; charge C again to V_0, and then connect C to R_X for a very short time; the p.d. will then drop to V_{01}. Discharge C through the galvanometer, the deflection being α_{01}; make sure that C_X is discharged; charge C again to V_0; connect to R_X, noting the time. After a time t, disconnect C from R_X, and discharge through the galvanometer, the deflection this time being α_1. We shall have

$$R_T = \frac{R_C\,R_X}{R_C + R_X} = \frac{t}{(C + C_X)\,log.\left(\frac{\alpha_{01}}{\alpha_1}\right)} \quad . \quad . \quad . \quad . \quad (33)$$

R_C is measured by discharging the condenser through its own resistance, as for C_X no separate measurement is necessary because

$$CV_0 = (C + C_X)\,V_{01}\,;\quad C_X = \frac{C\,(V_0 - V_{01})}{V_{01}},$$ and (33) becomes

$$R_T = \frac{t}{\left[C + \frac{C\,(\alpha_0 - \alpha_{01})}{\alpha_{01}}\right] log.\left(\frac{\alpha_{01}}{\alpha_1}\right)}$$

(c) **Measurement of the Insulation Resistance of a Length of Cable by the Loss-of-charge Method.** This method, particularly suitable to a length of cable, should be used whenever the resistance R_x has an appreciable capacity. The cable, previously charged, is then discharged through its own insulation resistance in the manner already explained.

The capacity of the cable can conveniently and easily be determined as shown in fig. 28 (case of a cable with conducting sheath).

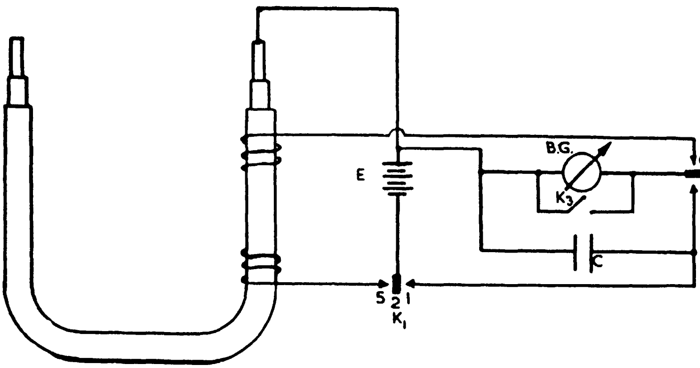

Fig. 28

E is a source of suitable e.m.f.; C a standard condenser; k_1, k_2, and k_3 switches; B.G. is the ballistic galvanometer.

The manipulation is as follows. Make 1–2 charging C to E, open 1–2 and make 3–4 discharging C through the B.G., noting deflection, say a; make 2–5 charging cable to E, break 2–5, making immediately 4–6 discharging the cable through the B.G., note the deflection a_1; assuming the deflections to be directly proportional to the charges on C and on the cable of capacity, say, C_c. We shall have

$C\,E = s_{cp}a$; $C_c\,E = s_{cp}a_1$, where s_{cp} is a constant; dividing $\frac{C}{C_c} = \frac{a}{a_1}$; $C_c = C\frac{a_1}{a}$; k_3 is for damping the galvanometer.

Knowing C, we connect as shown in fig. 29 (case of cable with conducting sheath), and manipulate as follows.

Make 1–2 charging cable to V_0. Then discharge through B.G., noting the deflection a_0, by making 2–3. Again make 1–2 charging the cable, then break 1–2 noting time, letting the cable discharge through its own insulation resistance for the time t. After the time

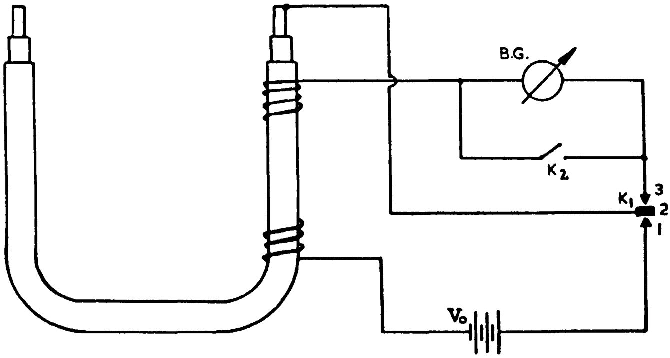

Fig. 29

interval t, make 2–3 discharging the cable and noting the deflection a_1. We have

$$R_x = \frac{t}{C \log.\frac{a_0}{a_1}}.$$

If the cable sheath is not conductive, the cable can be put in a bath in the manner shown in fig. 22.

(10) The Accumulation-of-charge Method of Measuring High Resistance

A standard condenser is charged to a p.d. V_0 through the resistance R_x to be measured for a time t; from the charge accumulated on the condenser we can determine R_x.

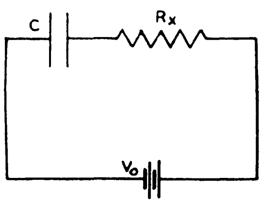

Fig. 30

Consider fig. 30.

Let the voltage of the source be V_0; the p.d. across the condenser is in opposition to V_0 and variable with t; let its instantaneous value be v; then the current in the circuit will be $i = \frac{V_0 - v}{R_x}$.

The charge accumulated on the condenser in a time dt is $idt = dq = \frac{V_0 - v}{R_x} dt$, and as $q = Cv$ and C is constant, $dq = Cdv$. Therefore

$$Cdv = \frac{V_0 - v}{R_x} \cdot dt \text{ and } dt = \frac{R_x C}{V_0 - v} \cdot dv.$$

Integrating between the time $t = O$ when the charging begins, to t when the charging ends, we have

$$\int_0^t dt = R_xC \int_{(v \text{ at } t = O) = O}^{(v \text{ at } t = t) = V_1} \frac{dv}{V_0 - v}; \; t = -R_xC \Big[log.\,(V_0 - v) \Big]_O^{V_1}$$

Reversing the limits we have

$$t = R_xC \Big[log.\,(V_0 - v) \Big]_{V_1}^{O} = R_xC\, log.\, \frac{V_0}{V_0 - V_1}.$$

$$R_x = \frac{t}{C\, log.\, \dfrac{V_0}{V_0 - V_1}}.$$

The practical arrangement is shown in fig. 31.

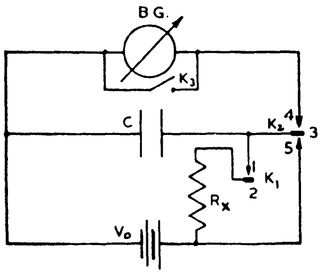

Fig. 31

Make 1–2, noting the time t during which the condenser is charged through the measured resistance R_x. Then break 1–2 and make 3–4 discharging the condenser through the ballistic galvanometer B.G. and noting the deflection α_1. Make 3–5 charging the condenser to V_0, then break 3–5 and make 3–4 discharging the condenser and noting the deflection α_0.

Assuming that α_1 and α_0 are directly proportional to the p.d.s across the condenser, we have $R_x = \dfrac{t}{C\, log.\, \dfrac{\alpha_0}{\alpha_0 - \alpha_1}}$.

It is necessary to get the deflection α_1 before α_0, because if the condenser is first fully charged, then after discharge there might be a residual charge left, and the deflection α_1 will not be a measure of the charge through R_x.

It can be proved that the error on α_0 and α_1 will be a minimum when $\dfrac{\alpha_0}{\alpha_1} \simeq 1{\cdot}87$. The accuracy of this measurement is, however, so poor that a different ratio $\dfrac{\alpha_0}{\alpha_1}$ will not much alter the error.

Similarly, as in the loss-of-charge method, the accumulation-of-charge method gives the combined resistance, leakage, and volume resistance. Its other chief disadvantage is that the voltage across R_x changes during the measurement.

ERRORS INHERENT IN THE METHOD.[1] The condenser insulation not being infinite, the circuit is in reality as shown in fig. 32.

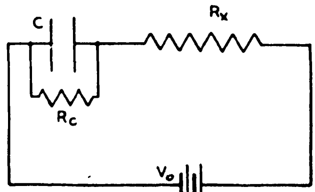

Fig. 32

Let R_c be the insulation resistance of the condenser C; as the p.d. across R_c is always equal to the p.d. across C, the current in R_c will be $\frac{v}{R_c}$ if v is the instantaneous voltage across C. The condenser therefore loses, during the time dt, a charge equal to $\frac{v}{R_c}\,dt$.

But as the condenser is being charged through R_x by a current $\frac{V_0 - v}{R_x}$, it therefore gains in the time dt a charge $\frac{V_0 - v}{R_x}\,dt$; the net charge gained is therefore

$$\frac{V_0 - v}{R_x}\,dt - \frac{v}{R_c}\,dt = C\,dv\,; \quad \frac{R_c\,(V_0 - v) - vR_x}{R_c\,R_x} \cdot dt = Cdv, \text{ and}$$

$$dt = \frac{C\,R_c\,R_x}{R_c\,V_0 - v\,(R_c + R_x)}\,dv.$$ Integrating between the time $t = 0$ when the charge begins, to the time t when the charge is finished, we have

$$\int_0^t dt = CR_xR_c \int_{(v \text{ at } t = 0) = 0}^{(v \text{ at } t = t) = V_1} \frac{dv}{R_cV_0 - v\,(R_c + R_x)};$$

$$t = -\frac{CR_xR_c}{R_c + R_x}\,log.\Big[R_cV_0 - v\,(R_c + R_x)\Big]_0^{V_1},$$

inverting the limits

$$t = \frac{C\,R_xR_c}{R_c + R_x}\,log.\Big[R_cV_0 - v\,(R_c + R_x)\Big]_{V_1}^{0} =$$

$$\frac{CR_xR_c}{R_c + R_x}\,log.\,\frac{R_cV_0}{R_cV_0 - V_1\,(R_c + R_x)}.$$

[1] Further details will be found in Chaumat: *Cours de Mesures Electriques* (S.F.E.; E.S.E., Vol. I).

If we know R_c, we can calculate R_x.

We also have

$$\frac{t\,(R_c + R_x)}{CR_xR_c} = log.\,\frac{R_cV_0}{R_cV_0 - V_1\,(R_c + R_x)};$$

$$e^{\frac{t\,(R_c + R_x)}{CR_xR_c}} = \frac{R_cV_0}{R_cV_0 - V_1\,(R_c + R_x)} \text{ and}$$

$$V_1 = \frac{R_cV_0}{R_c + R_x}\left[1 - e^{-\frac{t\,(R_c + R_x)}{CR_xR_c}}\right].$$

It follows that when the condenser insulation R_c is not infinity, the maximum p.d. to which it can be charged is $V_{1M} = \frac{R_cV_0}{R_c + R_x}$.

A careful check on the time t is therefore necessary, otherwise t can have any value whatever for the same voltage V_{1M}.

(*i*) *Practical Manipulation*: I. The diagram of connections is that of fig. 31.

Assuming R_c is known (it can be measured by any suitable method), the condenser is charged through R_x for a time t_1, then discharged through the B.G.; the deflection being a_1. The condenser is again charged through R_x, but for a shorter time t_{11}; discharged, and the deflection a_{11} noted. If $a_1 = a_{11}$, then the maximum voltage V_{1M} is reached. Another charge, for a time t_{111}, shorter than t_{11}, is necessary, giving on discharge a deflection a_{111}. If a_{111} is smaller, but very nearly equal to a_{11}, then t_{111} is the maximum time of charge. After every charge and discharge, the condenser C has to be shorted for a while, in order to eliminate any residual charge left.

Knowing t_{111} and a_{111}, the condenser is charged to the full voltage V_0, and then discharged; the deflection being a_0, we have

$$t = \frac{C\,R_xR_c}{R_x + R_c}\,log.\,\frac{a_0R_c}{a_0R_c - a_{111}\,(R_c + R_x)}.$$

When the resistance tested has an appreciable capacity, say C_x, the circuit will be as shown in fig. 33.

Immediately after the circuit is connected to the source of voltage V_0, the two condensers are charged electrostatically. The total capacity is $C_T = \frac{C\,C_x}{C + C_x}$; and the two condensers being in series, the

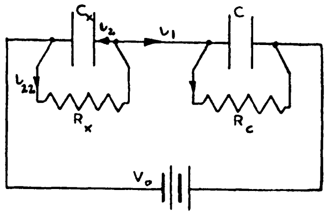

Fig. 33

charge Q is the same on both, so that $Q = C_T V_0 = C V_{01} = C_X V_X$, where V_{01} and V_X are the p.d.s across C and C_X respectively ; therefore

$$V_{01} = \frac{C_T V_0}{C} = \frac{C_X}{C + C_X} \cdot V_0 \, ; \; V_X = \frac{C_T V_0}{C_X} = \frac{C}{C + C_X} V_0.$$

After the electrostatic charge, when the p.d.s across the condensers are V_{01} and V_X, C will begin charging through R_X and losing charge through R_C. Let the values of the instantaneous currents be as shown in fig. 33.

The condenser C gains during dt a charge $i_1 \, dt$, and loses in the same time a charge $\frac{v}{R_C} dt$, where v is the instantaneous voltage across C. Starting from the value V_{01}, the net charge gained by C is therefore

$$q = i_1 dt - \frac{v}{R_C} dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The current in R_X is equal to $\frac{V_0 - v}{R_X} = i_{22}$, and the current i_2, which is the discharge current of C_X through R_X, is $i_2 = \frac{v_X}{R_X}$, where v_X is the instantaneous voltage across C_X starting from V_X.

The condenser C_X loses during the time dt a charge $i_2 \, dt = \frac{v_X}{R_X} dt$ and as $V_0 = v + v_X$; $v_X = V_0 - v$, we can write:

$$i_2 \, dt = \frac{v_X}{R_X} dt = \frac{V_0 - v}{R_X} dt = C_X \, d\,(V_0 - v) = - C_X \, dv.$$

Now $i_1 = i_{22} - i_2$; that is, $i_1 dt = i_{22} dt - i_2 dt$; so that (34) can be written : $q = C dv = \frac{V_0 - v}{R_X} dt + C_X dv - \frac{v}{R_C} dt$

$$\left(\frac{V_0 - v}{R_X} - \frac{v}{R_C}\right) dt = (C - C_X) \, dv = \frac{- R_X v + (V_0 - v) R_C}{R_X R_C} dt.$$

$dt = \frac{R_C R_X (C - C_X)}{R_C V_0 - v\,(R_C + R_X)} dv$. Integrating between the time $t = O$, when the charge through R_x begins, until the time t, when the charge is finished, we have

$$\int_o^t dt = R_x R_c (C - C_x) \int_{(v \text{ at } t = o) = V_{01}}^{(v \text{ at } t = t) = V_1} \frac{dv}{R_c V_o - v(R_c + R_x)},$$

$$t = -\frac{R_x R_c (C - C_x)}{R_c + R_x}\Big[log.\,[R_c V_o - v(R_c + R_x)]\Big]_{V_{01}}^{V_1}$$

and inverting the limits,

$$t = \frac{R_x R_c (C - C_x)}{R_c + R_x}\Big[log.\,[R_c V_o - v(R_c + R_x)]\Big]_{V_1}^{V_{01}} =$$

$$\frac{R_c R_x (C - C_x)}{R_c + R_x}\left[log.\,\frac{R_c V_o - V_{01}(R_c + R_x)}{R_c V_o - V_1 (R_c + R_x)}\right]$$

We have also

$$\frac{t(R_c + R_x)}{R_c R_x (C - C_x)} = log.\,\frac{R_c V_o - V_{01}(R_c + R_x)}{R_c V_o - V_1 (R_c + R_x)};$$

$$e^{\frac{t(R_c + R_x)}{R_c R_x (C - C_x)}} = \frac{R_c V_o - V_{01}(R_c + R_x)}{R_c V_o - V_1 (R_c + R_x)},$$

from which we get

$$V_1 = \frac{R_c V_o}{R_c + R_x}\left[1 - e^{-\frac{t(R_c + R_x)}{R_c R_x (C - C_x)}}\right] + V_{01}\left[e^{-\frac{t(R_c + R_x)}{R_c R_x (C - C_x)}}\right]$$

The maximum p.d. to which C can be charged is therefore

$$V_{1M} = \frac{R_c V_o}{R_c + R_x}.$$

A check on the time t is therefore necessary.

(*ii*) *Practical Manipulation :* II. The circuit is as in fig. 31.

Make 1–2 for a time t_1, and discharge by making 3–4, noting the deflection a_1. Again make 1–2 for a time $t_{11} < t_1$, and note the deflection. If $a_{11} = a_1$, charge again for a time $t_{111} < t_{11}$, discharge, and note the deflection a_{111}. If a_{111} is smaller but very nearly equal to a_{11} then t_{111} is the maximum time of charge. After each charge and discharge, C has to be shorted for a while in order to eliminate any residual charge. Having determined a_{111} and t_{111}, make again 1–2 for a very short time ; immediately after, 1–2 is broken and 3–4 is made, the deflection now being a_{01}, corresponding to V_{01}; finally 3–5 is made, charging C to V_0, then 3–4 is made, discharging C, the deflection being a_0. We have

$$t = \frac{R_c R_x (C - C_x)}{R + R_x}\left[log. \frac{R_c a_0 - a_{01}(R_c + R_x)}{R_c a_0 - a_{111}(R_c + R_x)}\right]$$

R_c has to be determined, but not C_x, because $C_x V_x = C V_{01}$; and $V_x = V_0 - V_{01}$; $C_x (V_0 - V_{01}) = CV_{01}$; so that $C_x = C\frac{V_{01}}{V_0 - V_{01}}$.

$$t = \frac{R_x R_c \left(-\frac{a_{01}}{a_0 - a_{01}} + 1\right) C}{R_c + R_x} log. \frac{R_c a_0 - a_{01}(R_c + R_x)}{R_c a_0 - a_{111}(R_c + R_x)}.$$

The accumulation-of-charge method is more difficult and even less reliable than the loss-of-charge method.

CHAPTER IV

CALIBRATION AND TESTING OF D.C. INSTRUMENTS. MEASUREMENT OF CURRENT, POTENTIAL DIFFERENCES AND E.M.F.

(1) The Comparison Method of Checking Ammeters and Voltmeters

Ammeters and voltmeters have to conform to certain specifications, and when a check for this purpose only is required, the comparison method is the best, because it is both simple and speedy.

(*a*) Calibration of an Ammeter. The instrument, which can be a permanent-magnet moving-coil, moving-iron, or hot-wire ammeter, is checked against a substandard ammeter. Any number of ammeters can be checked at the same time, provided there are sufficient reliable observers.

The connections are as shown in fig. 34, where two ammeters are compared with a substandard.

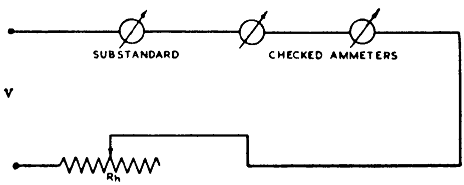

Fig. 34

A suitable source, of voltage V, supplies current to all the ammeters connected in series, the current being regulated by the rheostat R*h*.

The current is first set to such a value that the pointer of the sub-standard stands at $\frac{1}{10}$ of its maximum scale; then the readings of all the ammeters under test are noted. The current is now varied until the pointer of the substandard stands at $\frac{2}{10}$ of its maximum, and the readings of the test ammeters again noted. This is repeated for $\frac{3}{10}$, $\frac{4}{10}$, etc., of the substandard scale till its full indication is reached. The check is now repeated by going back on the substandard scale to $\frac{9}{10}$, $\frac{8}{10}$, $\frac{7}{10}$, etc., until zero is reached. With a moving-iron ammeter, this test will also give an idea of the instrument hysteresis. It is necessary in this case never to decrease the current when going up the scale, nor to increase it when going down.

When checking a hot-wire instrument, a certain time has to elapse between the setting of the substandard and the reading of the ammeter; this is required to allow the instrument to reach the temperature corresponding to the current flowing.

The readings of all the ammeters have to be taken simultaneously should it be found that the current is changing between the readings of separate instruments.

Each instrument zero has to be adjusted before starting the calibration, and it is important to note whether the pointer comes back to zero after the test.

The result of the test should be presented in a form which will at a glance show the reliability or otherwise of the instrument. The difference between the indications of each test ammeter and the substandard, given as a percentage of the substandard reading, is plotted against the substandard reading. When this difference is positive, it is plotted above the horizontal zero line, and if negative, below it, as shown in fig. 35(*a*).

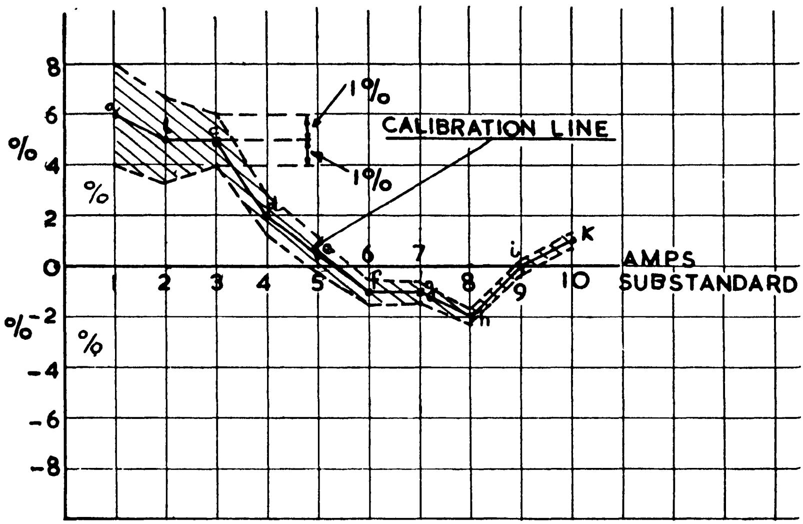

Fig. 35(*a*)

The points *a*, *b*, *c*, etc., are joined by straight lines ; the whole line *a*, *b*, *c*, . . . is the calibration line. If the true current is 3 A., we see that the test ammeter reads 3 + 5% or 3·15 A. Again, for 6 A., the checked ammeter reads 6 − 1% or 5·94 A.

As, however, the readings of the substandard are guaranteed only within certain limits, and there is also a reading error on the substandard, each point of the calibration line is subject to an error. The constructional error and the reading error of the substandard, expressed as a percentage of the substandard reading, is therefore plotted above or below each point of the calibration line. The points above and below the calibration line are also joined, as shown in fig. 35(*a*), by the dotted lines; the area between these lines is the uncertainty area. Considering a current of 3 A., we see that the test ammeter reading is 3·15 A. ± 1%, should the constructional plus reading error on the substandard be ± 1%.

(*i*) *Checking the Instrument Damping.* The damping is a very important characteristic of the instrument; two tests should be made.

(*a*) A test on the time it takes the pointer to come to rest. In order to avoid any ambiguity, and to perform the test under well-defined conditions, the time between the first deflection (greater than that corresponding to the current in the instrument, which is the steady reading) and that which differs from this steady indication by as near as possible $\frac{1}{10}$ of the steady value. Normally this time should not exceed 3 seconds.

(*b*) The ratio of the first deflection to the steady value. The test should be made with a current of about 60% of the maximum reading. This ratio should not exceed 1·33.

(*ii*) *Example of Ammeter Calibration.* A moving-iron and a hot-wire ammeter were checked against a substandard in the manner shown in fig. 34. The constructional error of the substandard can be taken as ± 0·5% over the whole of its scale, and the reading error is ± $\frac{1}{20}$ of a division. The following results were obtained.

Div.	*Substand. reading*	*Mov.-iron reading*	*Hot-wire reading*	*% diff. M.I.*	*% diff. H.W.*	*Substand. % read. error*	*Substand. tot. error %*
10	1 amp.	1	0·95	0	− 5	± 0·5	± 1
20	2 amps.	1·95	2	− 2·5	0	± 0·25	± 0·75
30	3 amps.	3·07	3·1	2·34	3·33	± 0·167	± 0·667
40	4 amps.	4·08	4·2	2·0	5	± 0·125	± 0·625
50	5 amps.	5·1	5·2	2	4	± 0·1	± 0·6
60	6 amps.	6·1	6·15	1·67	2·5	± 0·084	± 0·584
70	7 amps.	7·1	7·12	1·43	1·715	± 0·0715	± 0·571
80	8 amps.	8·25	8·18	3·13	2·25	± 0·0625	± 0·5625
90	9 amps.	9·20	9·25	2·22	2·78	± 0·0555	± 0·555
100	10 amps.	10·3	10·35	3	3·5	± 0·05	± 0·55

The calibration lines and uncertainty areas are shown in figs. 35(*b*) and (*c*)

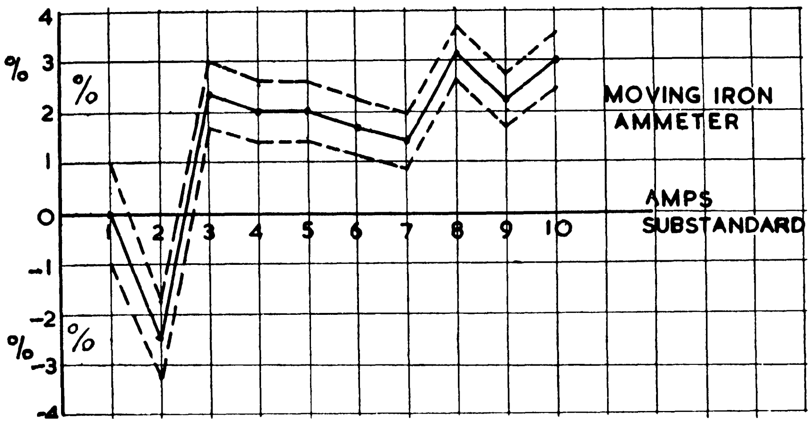

Fig. 35(*b*)

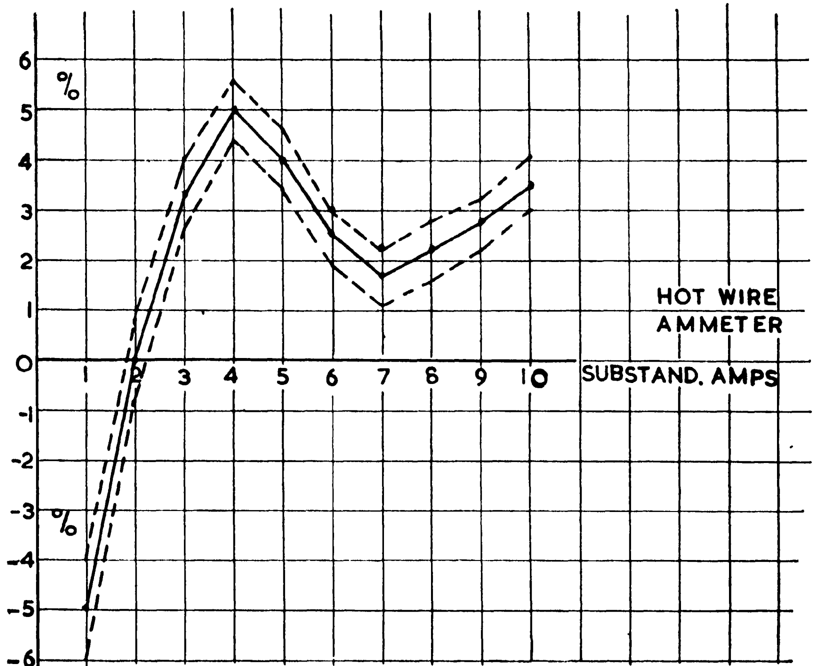

Fig. 35(*c*)

(*b*) Calibration of a Voltmeter. The method is similar to that of checking an ammeter. The voltmeter, or voltmeters, to be checked are all connected together with the substandard across a variable d.c. supply, or when a constant supply only is available, arranged in the manner shown in fig. 36.

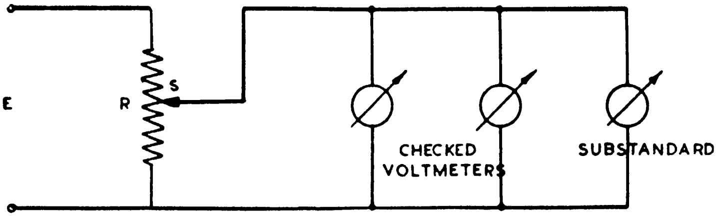

Fig. 36

E is the constant voltage supply; R a suitable resistance; S a sliding contact on R. S is moved till the substandard pointer is on $\frac{1}{10}$ of its scale maximum, then all the checked voltmeters are read and their indications noted. S is moved again till the pointer of the substandard is on $\frac{2}{10}$ of its maximum; the voltmeters are again read and their indications noted. This procedure is repeated, for every tenth of the scale, till the full scale indication of the substandard is reached. Then the checking is repeated from full-scale reading to zero.

The same precautions apply as when checking ammeters. The calibration lines and the uncertainty areas are plotted in a similar manner.

(2) D.C. Potentiometers

The opposition method described on p. 48 (fig. 17) is a simple potentiometer method. Considering the two resistances R_1 and R_2 in series with the rheostat Rh (fig. 37), the e.m.f. E will produce a current: $i = \dfrac{E}{R_b + R_1 + R_2 + R_h}$; R_h is the resistance of Rh, R_b that of E.

For a certain setting of Rh and with E constant, the current i is constant, if $R_1 + R_2$ is unaltered. If we vary R_1 while keeping $R_1 + R_2$ constant, the p.d. across R_1 will change, so that when connecting across R_1 an e.m.f. E_1 in series with a galvanometer (by connecting *b* to *a* in fig. 37), the galvanometer will show no deflection if $E_1 = iR_1$.

The arrangement shown in fig. 37 constitutes a potentiometer. In order to have a precise measurement of E_1, the p.d. across R_1, or across any part of it, has to be known with a high degree of accuracy before any useful measurement can be made. This determination of the p.d. across R_1 (or across part of it) is known as the setting, or the

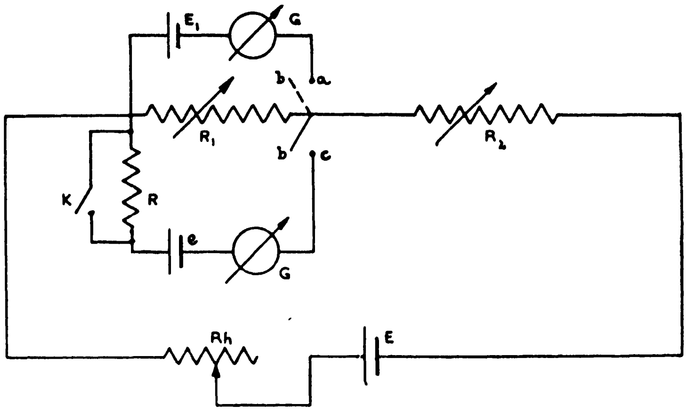

Fig. 37

standardisation, of the potentiometer, and it is best achieved by means of a standard cell.

The cell now officially recognised as a standard is the Weston cell, the e.m.f. of which, at a temperature of 20° C., is equal to $e = 1{\cdot}0183 \pm 0{\cdot}0001$ v., in international volts; or, as 1 international volt = 1·00036 absolute volts, we have $e = 1{\cdot}0186 \pm 0{\cdot}0001$ v., in absolute volts.

Its e.m.f. is therefore known within $0{\cdot}00982\% \simeq 0{\cdot}01\%$.

The temperature coefficient of the cell is negative; the e.m.f. of the cell as a function of its temperature is given by:

$$e = 1{\cdot}0183 - 0{\cdot}0000406\,(t - 20) - 0{\cdot}00000095\,(t - 20)^2 - 0{\cdot}00000001\,(t - 20)^3$$

where e is in international volts and t is the temperature in ° C. (e can be converted to absolute volts by the relation given above).

Extreme care has to be taken not to allow the cell to supply any appreciable current, even for a very short time; a voltmeter should never be used to measure the e.m.f. of the cell. The cell, in series with the galvanometer, and the resistance R, are connected across R_1 by joining b to c (fig. 37); and when the galvanometer shows no deflection, the drop across R_1 is equal to 1·0186 v., if the cell is at 20° C. If then R_1 is 10186 ohms, and variable by steps of 1 ohm (keeping, of course, $R_1 + R_2$ constant), the current in R_1 is : $i = \frac{1{\cdot}0186}{10186} = \frac{1}{10^4}$ amps, and a variation of e.m.f. of $\frac{1}{10^4}$ could be detected.

The resistance R is for the purpose of protecting the cell while standardising the potentiometer; after an approximate balance is obtained, R can be shorted by means of key k.

Once the potentiometer is standardised, let e be the e.m.f. of the cell; we have at balance $e = iR_1$.

Connecting b to a and varying R_1, so that it becomes R_{11}, and a new balance is obtained, we have $E_1 = iR_{11}$, therefore

$$\frac{R_1}{R_{11}} = \frac{e}{E_1}; \ E_1 = e\,\frac{R_{11}}{R_1}.$$

It is always preferable to interpolate in the same manner as with the Wheatstone bridge (p. 36) when standardising and when determining E_1.

The source E, supplying i, is usually between 2 and 4 volts, so that when using for R_1 and R_3 two resistances boxes of 11110 ohms each, the current i will be between $\frac{2}{22220}$ and $\frac{4}{22220}$ amps; a part of E is of course dropped in the connecting wires in Rh and R_b.

The current in the potentiometer should be low; $\frac{1}{10^4}$ to $\frac{1}{10^3}$ amps is usual (although there are potentiometers carrying $\frac{1}{100}$ of an amp), for if the current is high thermoelectric, e.m.f.s might arise.

The sensitivity of the arrangement shown in fig. 37 varies with the magnitude of the e.m.f. measured, because when we measure an e.m.f. of the order of a volt (R_1 being variable in steps of 1 ohm, and having been set by means of e to 10186 ohms) we can get a variation of $\frac{1}{10000}$ of a volt or $\frac{1}{10000}$, while if we measure an e.m.f. of say $\frac{1}{100}$ of a volt the smallest variation possible will be $\frac{1}{100}$.

(*a*) Industrial Potentiometers. These are potentiometers giving a precision of about $\frac{1}{1000}$, which is quite sufficient for such purposes as calibrating ammeters, voltmeters, and wattmeters, but not for high-precision measurements of current and e.m.f.

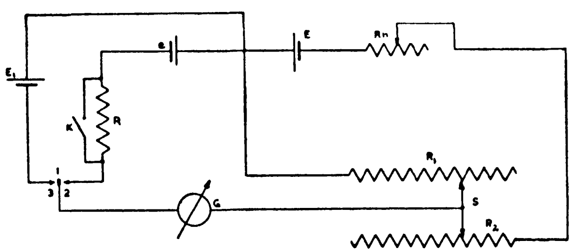

Fig. 38(*a*)

(*i*) ***First Type of Industrial Potentiometer.*** In the first type, which uses a galvanometer for determining balance, the circuit arrangement is as shown in fig. 38(*a*).

The source E supplies the current to R_1 and R_2, which are so arranged that their resistance can be varied by movement of the slider S, while $R_1 + R_2$ remains constant. The standard cell is *e*, by closing 1–2 the cell is put in circuit and the potentiometer can be standardised. R is the cell protecting resistance with its shortcircuiting key *k*. Making 1–3 brings the e.m.f. E_1, which is to be measured, in the circuit.

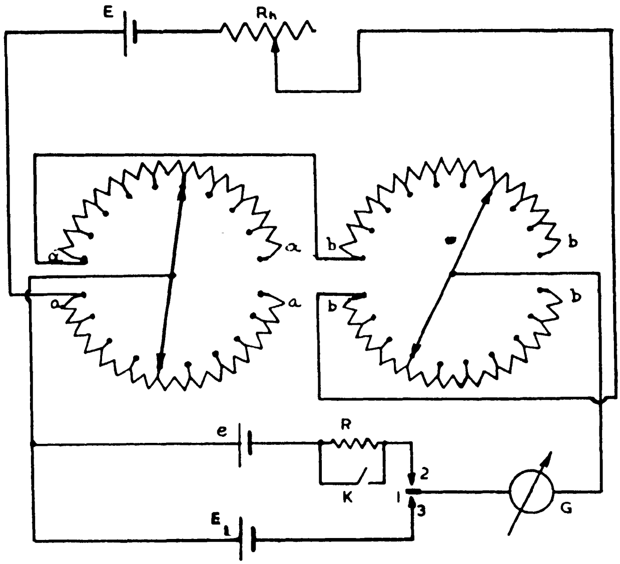

Fig. 38(*b*)

R_1 and R_2 are generally divided in sets of resistances in the manner shown in fig. 38(*b*); the total resistance between *a,a* being equal to one resistance of *b,b*.

The sliders move over dials marked directly in volts. If the resistance between *a,a* is equal to *r*, and is also equal to the resistance of one part of *b,b*, the total resistance of the potentiometer *a,a*, *b,b* is then $r + 9r = 10r$. Let the p.d. across *a,a*, *b,b* be V volts; then the smallest change in voltage we can have is

$$dV = \frac{V}{10r} \times \frac{r}{g}; \quad \text{say } V = 2 \text{ volts } r = 10\Omega, \text{ then}$$

$$dV = \frac{2}{10 \times 10} \times \frac{10}{g} = 0{\cdot}02222 \text{ volts.}$$

The potentiometer can of course have a different number of coils between *a,a* and *b,b*, than that shown in fig. 38(*b*).

There is a mark on the scales, corresponding to the standard cell setting, so that when standardising, the dials are set for the cell, and R*h* varied till balance is obtained. When this is obtained, the scales read directly in volts ; at balance with E_1, the value of E_1 is therefore read directly on the potentiometer scales.

When interpolating, while standardising, the dials are set for the standard cell and Rh varied till zero deflection of the galvanometer is found. Then R_1 should be decreased by a small amount (by means of the coils *a,a*) so that it becomes R_{12}, and a small deflection a_A is produced to the right, and then increased so that it becomes R_{13}, and a small deflection a_B is produced to the left. The value of R_1 is then (see Wheatstone bridge, p. 40)

$$R_1 = R_{12} + (R_{13} - R_{12}) \frac{a_A}{a_A + a_B} = \frac{e}{i}.$$

But as the dial readings are in volts and not in ohms, we can write :

$$V_1 = V_{12} + (V_{13} - V_{12}) \frac{a_A}{a_A + a_B} \quad . \quad . \quad . \quad . \quad (35)$$

V_1, V_{12} and V_{13} are the potentiometer dial readings corresponding to R_1, R_{12} and R_{13}.

The error on V_1 will be :

$$\frac{\Delta V'_1}{V'_1} = \frac{\Delta V'_{12}}{V'_1} + \frac{v'_D}{V'_1} + \frac{\Delta a'}{a'_A + a'_B}$$

where $v_D = V_{13} - V_{12}$, if the constructional error is known to be the same throughout the potentiometer, or

$$\frac{\Delta V'_1}{V'_1} = \frac{\Delta V'_{12}}{V'_1} + \frac{\Delta V'_D}{V'_1} \frac{a_A'}{a'_A + a'_B} + \frac{V'_D}{V'_1} \frac{\Delta a'}{a'_A + a'_B} \quad . \quad (35a)$$

if the constructional error is not the same through the potentiometer (see Wheatstone bridge, p. 40).

(*ii*) *Second Type of Industrial Potentiometer.* When calibrating an ammeter, voltmeter, or wattmeter, by means of a potentiometer provided with a galvanometer, the work involved in balancing and interpolating becomes tedious ; for the purpose of instrument calibration another type of a potentiometer, provided with a millivoltmeter instead of a galvanometer, is more suitable, although less accurate.

The general arrangement is shown in fig. 39.

The resistance of the potentiometer between AB is constant ; the division in sets of resistances and dials is the same as in fig. 42(*a*).

When we make 1–2 the standard cell *e*, in series with the millivoltmeter M.V. and the protecting resistance R, is in the circuit. The standardisation is done in the usual way : the sliding contacts (dials in reality) are set for e, and the value of Rh adjusted till the millivoltmeter reads zero. The potentiometer dials then read directly in volts. When interpolating, the procedure is as described in the first type of potentiometer.

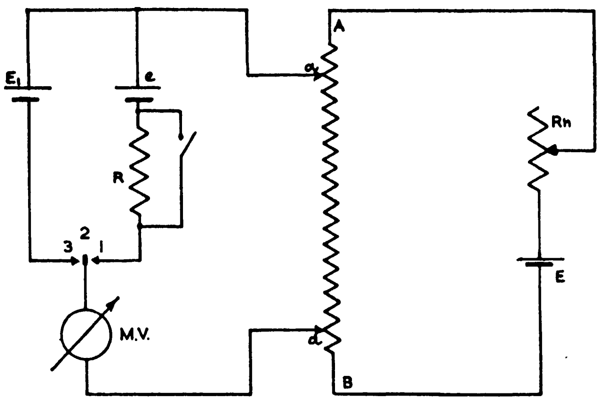

Fig. 39

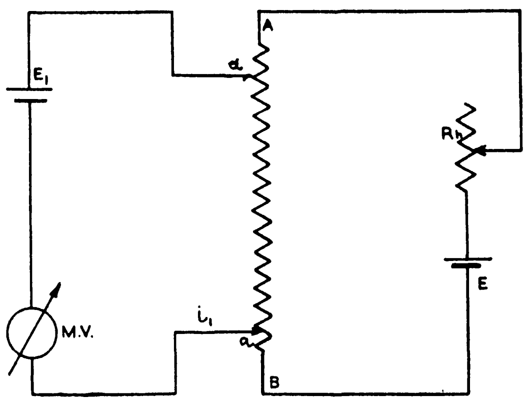

Fig. 40

To measure the e.m.f. of E_1, 2–3 is closed, and the sliders *a,a* (dials) moved till the millivoltmeter pointer *reads anywhere within its scale.* The value of E_1 is then given by the sum of the potentiometer dial reading and the millivoltmeter reading ; the potentiometer reading the unit, tenths and hundredths parts of the e.m.f. and the millivoltmeter the $\frac{1}{1000}$ and $\frac{1}{10000}$ parts.

That these readings are approximately the value of the e.m.f. measured can be seen by considering fig. 40.

Let R_1 be the resistance between the sliding contacts *a,a*, when the standard cell is in circuit; at balance we have

$$i\,R_1 = e\,;\; i = \frac{e}{R_1}.$$

Let the total resistance of the potentiometer circuit be r, then

$$ri = E\,;\; i = \frac{E}{r}\,;\ \text{therefore}\ \frac{E}{r} = \frac{e}{R_1}.$$

When making 2–3, E_1 is in the circuit; and as we let the millivoltmeter read within its scale, there will be a current through it. If this current be i_1, and in the direction as shown in fig. 40, then

$$E_1 = r_1\,(i_A + i_1) + r_2\,i_1 + r_V\,i_1 \quad . \quad . \quad . \quad . \quad (36)$$

where r_1 is the resistance between the sliding contacts *a,a* for the setting of E_1; r_2 is the resistance of E_1; and r_V the resistance of the millivoltmeter M.V. i_A is now the current in the parts A,*a*, *a*,B of the potentiometer (i_A is of course different from i, as the standard cell delivers no current).

The e.m.f. E being unaltered, we now have

$$E = r\,i_A + r_1\,i_1\,;\; i_A = \frac{E - r_1\,i_1}{r} \quad . \quad . \quad . \quad . \quad (36a)$$

Combining (36a) with (36) we get

$$E_1 = r_1\left[\frac{(E - r_1\,i_1)}{r} + i_1\right] + r_2\,i_1 + r_V\,i =$$

$$r_1\left(\frac{E - r_1\,i_1 + r\,i_1}{r}\right) + r_2\,i_1 + r_V\,i_1 =$$

$$\frac{r_1\,E}{r} + r_V\,i_1 + r_2\,i_1 + r_1\,i_1\,\frac{(r - r_1)}{r},\ \text{as}\ \frac{E}{r} + \frac{e}{R_1},\ \text{we have}$$

$$E_1 = \frac{r_1\,e}{R_1} + r_V\,i_1 + r_2\,r_1 + \frac{(r - r_1)}{r}\,r_1\,i_1.$$

$r_V\,i_1$ is of course the indication of the millivoltmeter, while the potentiometer dials can be arranged to read $\frac{r_1\,e}{R_1}$; so that when we assume that $E_1 = \frac{r_1\,e}{R_1} + r_V\,i_1$, the error due to the method is

$$r_1\,i_1\,\frac{(r - r_1)}{r} + r_2\,i_1.$$

$r_2\,i_1$ is the drop of volts in the source E_1, which, if small, will introduce a negligible error ; the term $i_1 r_1 \frac{(r - r_1)}{r}$ will be small if $r - r_1$ is small ; that is, if the measured e.m.f. E_1 is nearly equal to the e.m.f. E supplying the potentiometer.

The potentiometer is therefore unsuitable for small values of E_1, although the error can be made small by making i_1 small ; that is, restricting the reading of the millivoltmeter to very near its zero. Then the reading error on the millivoltmeter will be large, and the measurement will not be rapid. These are disadvantages which will defeat the main purpose of using this type of potentiometer.

When calibrating ammeters, voltmeters, and wattmeters, we can make E_1 fairly large, so diminishing the error, and with proper precautions the potentiometer is capable of giving an accuracy within $\frac{5}{10000}$ to $\frac{1}{1000}$.

There is no determination error here except when standardising ; we have, however, a reading error on the millivoltmeter.

(*iii*) *Third Type of Industrial Potentiometer.* The arrangement is shown in fig. 41.

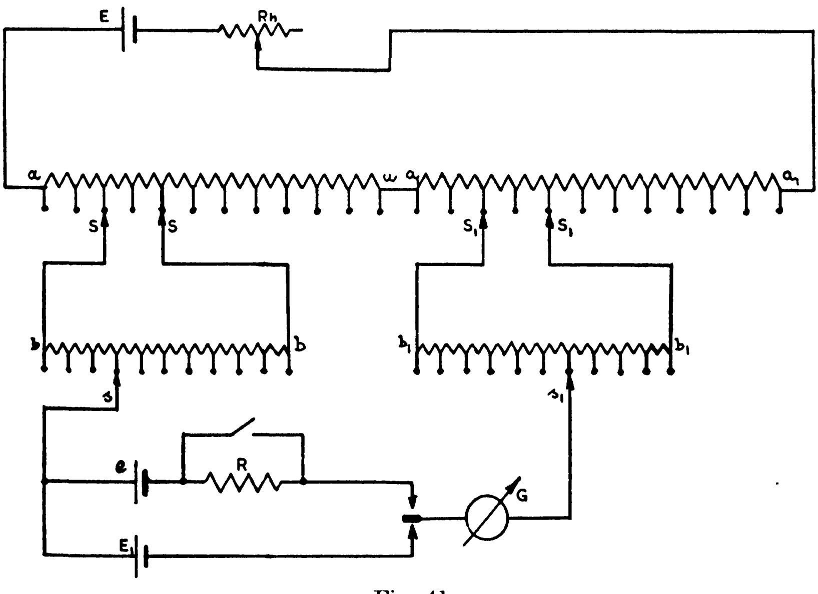

Fig. 41

The numbers relating to this type of potentiometer given below are for the purpose of an example only. They are not necessarily actual numbers.

Between	a,a	we have	11	resistances of	1000 ohms each.
,,	a_1,a_1	,, ,,	11	,, ,,	10 ,, ,,
,,	b,b	,, ,,	10	,, ,,	200 ,, ,,
,,	b_1,b_1	,, ,,	10	,, ,,	2 ,, ,,

The sliding contacts S,S and S_1,S_1 are always separated by two resistances, of a,a and a_1,a_1 respectively, so that the total resistance of a,a, b,b is

$$9 \times 1000 + \frac{(10 \times 200)\ (2 \times 1000)}{(10 \times 200) + (2 \times 1000)} = 10000\ \Omega$$

and that of a_1,a_1, b_1,b_1 is

$$9 \times 10 + \frac{(10 \times 2)\ (2 \times 10)}{(10 \times 2) + (2 \times 10)} = 100\ \Omega.$$

Assume for ease of calculation that the total p.d. across a,a, a_1,a_1 is 2·02 volts, the total resistance across a,a, a_1,a_1 being $10000 + 100 = 10100\ \Omega$; then the p.d. across a_1a_1 is $\frac{2{\cdot}02}{10100} \times 100 = 0{\cdot}02$ V.

The voltage change caused by the movement of the slider S_1,S_1 by one contact of a_1,a_1 is $\frac{0{\cdot}02}{10} = 0{\cdot}002$ V.

The movement of slider s_1 by one contact of b_1,b_1, which is the smallest change possible, will therefore be $\frac{0{\cdot}002}{10} = 0{\cdot}0002$ V.

The p.d. across a,a is $\frac{2{\cdot}02 \times 10000}{10100} = 2$ V.; the movement of the slider S,S by one contact produces a change of $\frac{2}{10} = 0{\cdot}2$ V.; and the movement of slider s by one contact of b,b produces therefore a change of $\frac{0{\cdot}2}{10} = 0{\cdot}02$ V., which is equal to the change produced by the slider S_1,S_1 over the whole of a_1,a_1.

(*b*) THE LABORATORY PRECISION POTENTIOMETER. The potentiometers mentioned so far, even those working with a galvanometer, are not precision instruments, because the resistance $R_1 + R_2$ cannot be kept absolutely constant. In fig. 37 there is no guarantee that, when changing the plugs in the resistance boxes, there will be no alteration in the total resistance, while in figs. 38(*a*) and 38(*b*) there is no guarantee that the sliding contact resistance is the same all over R_1 and R_2.

In potentiometers used for precision measurement of e.m.f. or potential differences, the sliding contacts are transferred to the galvanometer circuit ; their effect is therefore zero, because there is no current in the galvanometer circuit at balance (or the current there is extremely small when interpolating). The general arrangement of the potentiometer is as shown in fig. 42(*b*). The values given are for example only, and vary in different makes.

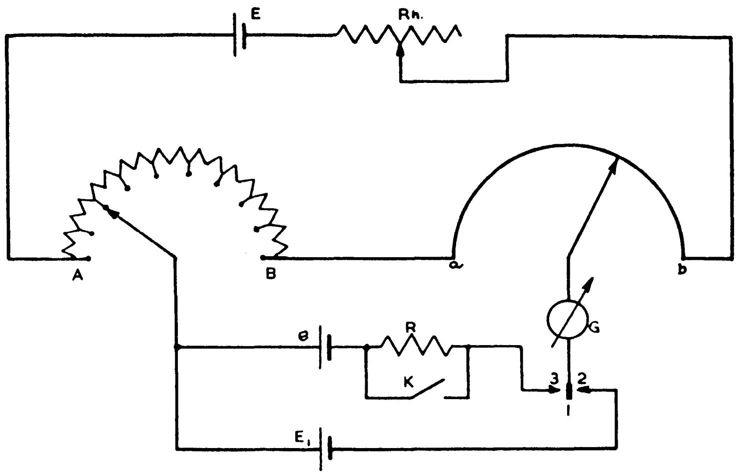

Fig. 42(*a*)

Consider first the simple arrangement of fig. 42(*a*). Between A and B we have a certain number of resistance coils (9 in the case of fig. 42(*a*)) ; *ab* is a resistance wire for fine adjustment; and the resistance of *ab* is equal to the resistance of one coil of AB.

The potentiometer is standardised in the usual way, by connecting the standard cell *e* in series with the galvanometer and the cell protecting resistance R to the terminals marked for the purpose, and varying Rh till the galvanometer gives no deflection. By making 1–2, E_1 can be measured, and its value read directly on the potentiometer, after a new balance has been achieved.

The current in the potentiometer circuit has to be small in order to avoid thermo-electric effects, and also to keep E constant during the measurement. If we allow a current of 20 m.amp, with a p.d. of 2 volts across A–*b*, the total resistance between A and *b* has to be

$$\frac{2 \times 1000}{20} = 100\ \Omega.$$

If now we have 9 resistance coils between A and B, then the resistance of each coil and also the resistance of the slide wire *ab* will be

$$\frac{100}{10} = 10\ \Omega.$$

The sliding contact on the wire ab is moved against a scale, and if we have 100 divisions on this scale, and the thickness of the index is such that we can distinguish $\pm \frac{1}{10}$ of a division, then as one division

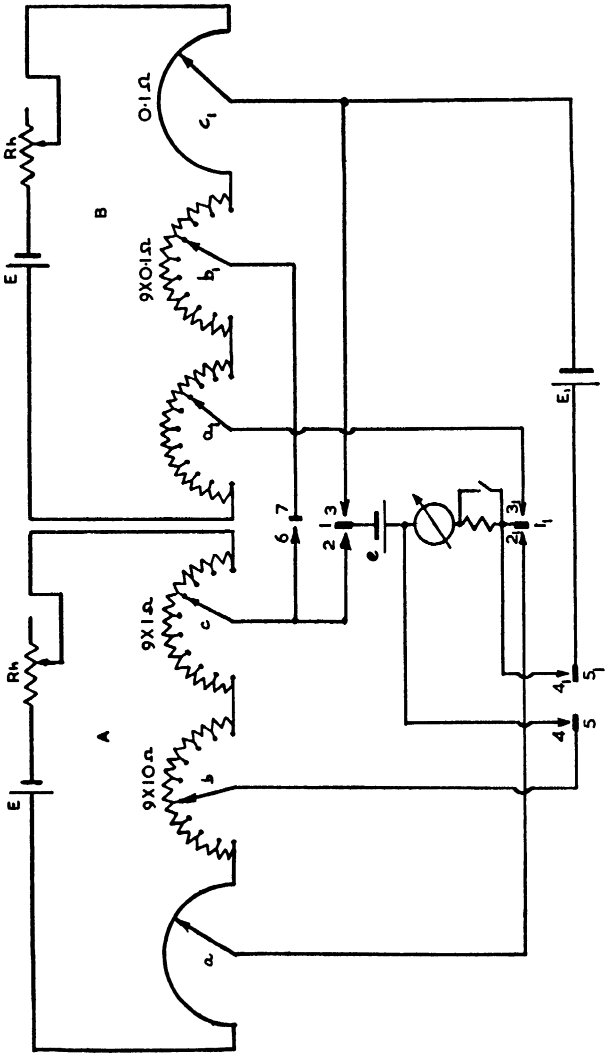

Fig. 42(b)

of the scale corresponds to $\frac{2}{10 \times 100} = \frac{2}{1000}$ V., the precision of the potentiometer will be $= \frac{2}{1000}\left(\pm \frac{1}{10}\right) = \pm \frac{2}{10^4}$ V.

This of course will not do, the precision being low, and the slide wire, having to be fairly thick in order to withstand wear, owing to the sliding contact, will be far too long, and very difficult to make uniform.

To increase the precision and have a reliable slide wire, its resistance should be about 0·1 ohm and its length no more than about 50 cm., but then, as the resistance of each coil of AB is equal to that of the slide wire, the number of resistance coils required with the arrangement of fig. 42(*a*) will be $\frac{100 - 0{\cdot}1}{0{\cdot}1} = 999$; this would of course make the potentiometer too cumbersome and costly.

Consider now fig. 42(*b*).

The potentiometer is divided into two parts A and B, the arrangement of the dials and resistances being as shown. The dials and resistances a and a_1 are for the special purpose of standardising the potentiometer, while b,c and b_1,c_1 are used for the measurement of the e.m.f. E_1. By joining 1–2 and 1_1–2_1, leaving all the other keys in their neutral positions, part A of the potentiometer is standardised; by making 1–3 and 1_1–3_1 (all other keys in neutral) part B is standardised. Joining 4–5, 4_1–5_1 and 6–7 (all other keys in neutral), the e.m.f. E_1 can be measured. The potentiometer dials are graded directly in volts.

The standardisation is done by varying Rh and Rh_1 till the galvanometer shows zero deflection and then interpolating by means of dial a and c_1 if desired.

When A and B are standardised, the current is exactly the same in A and B.

Let the sum of the p.d.'s across bc and b_1c_1 equal 2 V, then, with the resistances as shown in fig. 42(*b*), the p.d. across the slide wire is

$$\frac{2}{100} \times 0{\cdot}1 = 0{\cdot}002 \text{ V}.$$

If the slide wire dial has 100 divisions, and the index is such that $\pm \frac{1}{10}$ of a division can be distinguished, the potentiometer precision, or the smallest change of voltage available, will be

$$\frac{0{\cdot}002}{100}\left(\pm \frac{1}{10}\right) = \pm\ 2 \times 10^{-6}\text{ V}.$$

Besides a and a_1 we have here 27 resistances, plus a slide wire, while to get the same precision with the potentiometer shown in fig. 42(*a*) we should have had to have 999 resistances.

Triple potentiometers for still greater precision can also be made.

(3) Calibration of Meters by a Potentiometer

(*a*) Calibration of an Ammeter by a Potentiometer. The arrangement is that of fig. 43(*a*).

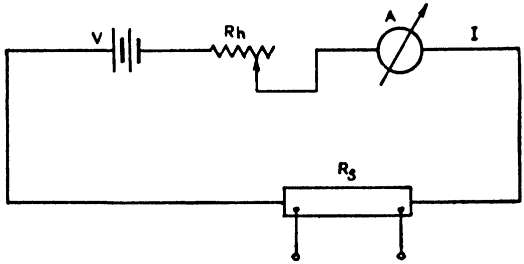

Fig. 43(*a*)

The ammeter A which is to be calibrated is in series with a source V, a rheostat Rh, and a standard resistance R_s ; the p.d. across the standard R_s is measured on the potentiometer in exactly the same way as the e.m.f. E_1 in figs. 42(*a*) and 39. The current I is varied so that the ammeter pointer indicates $\frac{1}{10}$, $\frac{2}{10}$, etc., of its scale to full scale reading. If the values of the p.d. across R_s are then V_1, V_2, V_3, . . . V_n, then the values of the current I are $\frac{V_1}{R_s}, \frac{V_2}{R_s}, \frac{V_3}{R_s} \ldots, \frac{V_n}{R_s}$.

The potentiometer is standardised before the calibration in the usual way, and the standardisation has to be checked during the calibration.

Calculation of the Systematic Error. For each value of the p.d. across R_s of value V_n we have

$$I_n = \frac{V_n}{R_s}, \text{ but } V_n = e\,\frac{R_n}{R_1}, \text{ so that } I_n = \frac{e\,R_n}{R_1\ R_s}$$

where R_n is the setting of the potentiometer for V_n and R_1 the setting for the standard cell.

But we can write $R_n = R_1 \pm r_s$ where r_s is the difference in the potentiometer setting between for the standard cell and for V_n, so that the interpolation is done on r_s and not on the common unaltered part of the potentiometer resistance. If therefore ${}_1r_s$ produces a deflection α_1 to the right and ${}_2r_s$ a deflection α_2 to the left, we shall have $r_s = {}_1r_s + ({}_2r_s - {}_1r_s)\,\frac{\alpha_1}{\alpha_1 + \alpha_2}$, and writing ${}_2r_s - {}_1r_s = \varrho$

$$r_s = {}_1r_s + \varrho\,\frac{\alpha_1}{\alpha_1 + \alpha_2}\ ;$$ R_n therefore becomes

$$R_n = R_1 \pm {}_1r_s \pm \varrho \frac{a_1}{a_1 + a_2}, \text{ so that}$$

$$I_n = \frac{e\left(R_1 \pm {}_1r_s \pm \varrho \frac{a_1}{a_1 + a_2}\right)}{R_1 R_s} \quad . \quad . \quad . \quad . \quad (37)$$

While changing the position of the dials of the potentiometer, we alter resistances ; the dials, however, read in volts, so that (37) ought to be written :

$$I_n = \frac{e\left(V_1 \pm {}_1v_s \pm v \frac{a_1}{a_1 + a_2}\right)}{V_1 R_s} = \frac{e V_n}{V_1 R_s} \quad . \quad . \quad . \quad (37a)$$

where V_1 corresponds to the resistance R_1 (setting for V_1). Also, v_s corresponds to the difference between V_n and V_1, and v to the difference between ${}_2v_s$ and ${}_1v_s$, which give respectively the deflections a_2 and a_1.

As $V_n = V_1 \pm {}_1v_s \pm v \frac{a_1}{a_1 + a_2}$, the differential of V_1 is

$$dV_n = dV_1 \pm d_1v_s \pm dv \frac{a_1}{a_1 + a_2} \pm v \frac{(a_1 + a_2)\, da_1 - a_1\,(da_1 + da_2)}{(a_1 + a_2)^2} =$$

$$dV_1 \pm d_1v_s \pm dv \frac{a_1}{a_1 + a_1} \pm v \frac{a_2da_1 - a_1da_2}{(a_1 + a_2)^2};$$

the logarithmic differential of V_n is therefore

$$\frac{dV_n}{V_n} = \frac{dV_1}{V_n} \pm \frac{d\,({}_1v_s)}{V_n} \pm \frac{dv}{V_n} \frac{a_1}{a_1 + a_2} \pm \frac{v}{V_n} \frac{a_2da_1 - a_1da_2}{(a_1 + a_2)^2}.$$

When standardising the potentiometer, we determined V_1, corresponding to R_1, by interpolation, and found that

$$V_1 = V_{12} + (V_{13} - V_{12}) \frac{a_A}{a_A + a_B} = V_{12} + v_D \frac{a_A}{a_A + a_B}, \quad \text{see (35)},$$

and the logarithmic differential of V_1 is

$$\frac{dV_1}{V_1} = \frac{dV_{12}}{V_1} + \frac{dv_D}{V_1} \frac{a_A}{a_A + a_B} + \frac{v_D}{V_1} \frac{a_Bda_A - a_Ada_B}{(a_A + a_B)^2}.$$

The logarithmic differential of (37a) will be

$$\frac{dI_n}{I_n} = \frac{de}{e} + \frac{dV_n}{V_n} - \frac{dV_1}{V_1} - \frac{dR_s}{R_s} =$$

$$\frac{de}{e} + \frac{dV_1}{V_n} \pm \frac{d\,({}_1v_s)}{V_n} \pm \frac{dv}{V_n}\frac{a_1}{a_1 + a_2} \pm \frac{v}{V_n}\frac{a_2da_1 - a_1da_2}{(a_1 + a_2)^2} - \frac{dV_{12}}{V_1} -$$

$$\frac{dv_D}{V_1}\frac{a_A}{a_A + a_B} - \frac{v_D\,(a_Bda_A - a_Ada_B)}{V_1\,(a_A + a_B)^2} - \frac{dR_s}{R_s} =$$

$$\frac{de}{e} + \frac{V_1}{V_n}\frac{dV_1}{V_1} \pm \frac{{}_1v_s}{V_n}\frac{d\,({}_1v_s)}{{}_1v_s} \pm \frac{v}{V_n}\frac{dv}{v}\frac{a_1}{a_1 + a_2} \pm \frac{v}{V_n}\frac{a_2da_1 - a_1da_2}{(a_1 + a_2)^2} -$$

$$\frac{V_{12}}{V_1}\frac{dV_{12}}{V_{12}} - \frac{v_D\,dv_D\,a_A}{V_1\,v_D\,(a_A + a_B)} - \frac{v_D}{V_1}\frac{a_Bda_A - a_Ada_B}{(a_A + a_B)^2} - \frac{dR_s}{R_s}.$$

As $V_1 \doteqdot V_{12}$ and ${}_1v_s \doteqdot v_s$, we have $\frac{dV_1}{V_1} = \frac{dV_{12}}{V_{12}}$ and $\frac{dv_s}{v_s} = \frac{d_1v_s}{{}_1v_s}$, so that

$$\frac{dI}{I} = \frac{de}{e} + \frac{V_1}{V_n}\frac{dV_1}{V_1} - \frac{V_1}{V_1}\frac{dV_1}{V_1} \pm \frac{v_s}{V_n}\frac{d\,(v_s)}{v_s} \pm \frac{v}{V_n}\frac{dv}{v}\frac{a_1}{a_1 + a_2} \pm$$

$$\frac{v}{V_n}\frac{a_2da_1 - a_1da_2}{(a_1 + a_2)^2} - \frac{v_D}{V_1}\frac{dv_D}{v_D}\frac{a_A}{a_A + a_B} - \frac{v_D}{V_1}\frac{a_Bda_A - a_Ada_B}{(a_A + a_B)^2} - \frac{dR_s}{R_s} =$$

$$\frac{de}{e} + \frac{dV_1}{V_1}\left(\frac{V_1}{V_n} - 1\right) \pm \frac{v_s}{V_n}\frac{dv_s}{v_s} \pm \frac{v}{V_n}\frac{dv}{v}\frac{a_1}{a_1 + a_2} \pm \frac{v}{V_n}\frac{a_2da_1 - a_1da_2}{(a_1 + a_2)^2} -$$

$$\frac{v_D}{V_1}\frac{dv_D}{v_D}\frac{a_A}{a_A + a_B} - \frac{v_D}{V_1}\frac{a_Bda_A - a_Ada_B}{(a_A + a_B)^2} - \frac{dR_s}{R_s} =$$

$$\frac{de}{e} + \frac{dV_1}{V_1}\left(\frac{V_1 - V_n}{V_n}\right) \pm \frac{v_s}{V_n}\frac{dv_s}{v_s} \pm \frac{v}{V_n}\frac{dv}{v}\frac{a_1}{a_1 + a_2} \pm \frac{v}{V_n}\frac{a_2da_1 - a_1da_2}{(a_1 + a_2)^2} -$$

$$\frac{v_D}{V_1}\frac{dv_D}{v_D}\frac{a_A}{a_A + a_B} - \frac{v_D}{V_1}\frac{a_Bda_A - a_Ada_B}{(a_A + a_B)^2} - \frac{dR_s}{R_s}.$$

The relative error on I_n is therefore

$$\frac{\Delta I'_n}{I'_n} = \frac{\Delta e'}{e'} + \frac{\Delta V'_1}{V'_1}\left(\frac{V'_1 - V'_n}{V'_n}\right) + \frac{v'_s}{V'_n}\frac{\Delta v'_s}{v'_s} + \frac{v'}{V'_n}\frac{\Delta v'}{v'}\frac{a'_1}{a'_1 + a'_2} +$$

$$\frac{v'}{V'_n}\frac{a'_2\Delta a_1 + a'_1\Delta a'_2}{(a'_1 + a'_2)^2} + \frac{v'_D}{V'_1}\frac{\Delta v'_D}{v'_D}\frac{a'_A}{a'_A + a'_B} +$$

$$\frac{v'_D}{V'_1}\frac{a_B\,\Delta a_A + a_A\,\Delta a_B}{(a'_A + a'_B)^2} + \frac{\Delta R'_s}{R'_s};$$

and as $\Delta a_1 = \Delta a_2 = \Delta a_A = \Delta a_B = \Delta a$ say, we have

$$\frac{\Delta I'_n}{I'_n} = \frac{\Delta e'}{e'} + \frac{\Delta V'_1}{V'_1}\left(\frac{V'_1 - V'_n}{V'_n}\right) + \frac{v'_s}{V'_n}\frac{\Delta v'_s}{v'_s} + \frac{v'}{V'_n}\frac{\Delta v'}{v'}\frac{a'_1}{a'_1 + a'_2} +$$

$$\frac{v'}{V'_n}\frac{\Delta a}{a'_1+a'_2}+\frac{v'_D}{V'_1}\frac{\Delta v'_D}{v'_D}\frac{a'_A}{a'_A+a'_B}+\frac{v'_D}{V'_1}\frac{\Delta a'}{a'_A+a'_B}+\frac{\Delta R'_s}{R'_s}, \text{ where}$$

$\frac{V_1 - V_n}{V_n}$ is taken as positive whether V_1 is greater or smaller than V_n.

If it were known that all the resistances of the potentiometer have the same constructional error, we could write $\frac{d\,({}_1v_s)}{{}_1v_s} = \frac{dv}{v} = k$ say, therefore

$$dV_n = dV_1 \pm k\,({}_1v_s) \pm kv\frac{a_1}{a_1+a_2} \pm v\frac{a_2da_1 - a_1da_2}{(a_1+a_2)^2};$$

and as $r_s = {}_1r_s + \varrho\frac{a_1}{a_1+a_2}$, we have $v_s = {}_1v_s + v\frac{a_1}{a_1+a_2}$, then $dV_n = dV_1 \pm kv_s \pm v\frac{a_2da_1 - a_1da_2}{(a_1+a_2)^2}$. The logarithmic differential of V_n is now

$$\frac{dV_n}{V_n} = \frac{dV_1}{V_n} \pm \frac{v_s}{V_n}\frac{dv_s}{v_s} \pm \frac{v}{V_n}\frac{a_2da_1 - a_1da_2}{(a_1+a_2)^2};\quad {}_1v_s \simeq v_s.$$

When standardising and interpolating we now have

$$\frac{dV_1}{V_1} = \frac{dV_{12}}{V_1} + \frac{v_D}{V_1}\frac{a_Bda_A - a_Ada_B}{(a_A+a_B)^2}, \text{ so that}$$

$$\frac{dI_n}{I_n} = \frac{de}{e} + \frac{V_1}{V_n}\frac{dV_1}{V_1} \pm \frac{v_s}{V_n}\frac{dvs}{v_s} \pm \frac{v}{V_n}\frac{a_2da_1 - a_1da_2}{(a_1+a_2)^2} - \frac{dV_{12}}{V_1} -$$

$$\frac{v_D}{V_1}\frac{a_Bda_A - a_Ada_B}{(a_A+a_B)^2} - \frac{dR_s}{R_s};\ \text{as}\ \frac{dV_{12}}{V_1} \simeq \frac{dV_1}{V_1},$$

$$\frac{dI_n}{I_n} = \frac{de}{e} \pm \frac{v}{V_n}\frac{(a_2da_1 - a_1da_2)}{(a_1+a_2)^2} - \frac{v_D}{V_1}\frac{a_Bda_A - a_Ada_B}{(a_A+a_B)^2} +$$

$$\frac{dV_1}{V_1}\left(\frac{V_1}{V_n} - 1\right) \pm \frac{v_s}{V_n}\frac{dvs}{v_s} - \frac{dR_s}{R_s} =$$

$$\frac{de}{e} \pm \frac{v}{V_n}\frac{a_2da_1 - a_1da_2}{(a_1+a_2)^2} - \frac{v_D}{V_1}\frac{a_Bda_A - a_Ada_B}{(a_A+a_B)^2} + \frac{dV_1}{V_1}\left(\frac{V_1 - V_n}{V_n}\right) \pm$$

$$\frac{v_s}{V_n}\frac{dvs}{v_s} - \frac{dR_s}{R_s}.$$

The relative error is now

$$\frac{\Delta I'_n}{I'_n} = \frac{\Delta e'}{e'} + \frac{v'}{V'_n}\frac{\Delta' a}{a'_1+a_2'} + \frac{v_D'}{V'_1}\frac{\Delta a'}{a'_A+a'_B} + \frac{\Delta V'_1}{V'_1}\left(\frac{V'_1 - V'_n}{V'_n}\right) + \frac{v'_s}{V'_n}\frac{\Delta v'_s}{v'_s} - \frac{\Delta R'_s}{R'_s}.$$

$\frac{V_1 - V_n}{V_n}$ has to be taken as positive whether V_1 is greater or smaller than V_n.

The smaller v_s, that is, the nearer V_n is to e, the smaller the error. The constructional error or errors of the resistances making up the potentiometer should be supplied by the manufacturer; $\frac{\Delta R'_s}{R'_s}$ is the constructional error on the standard resistance R_s, whilst $\frac{\Delta e'}{e'}$ is, as we have seen, $\pm\frac{0{\cdot}00982}{100}$, assuming the temperature correction to be properly applied.

(*b*) Calibration of a Voltmeter by a Potentiometer.

The arrangement is shown in figs. 43(*b*) and (*c*).

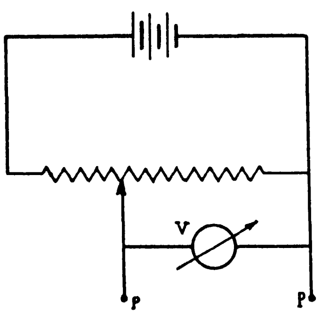

Fig. 43(*b*)

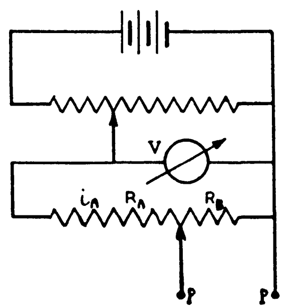

Fig. 43(*c*)

If the total voltmeter reading is not higher than the maximum voltage of the potentiometer, the arrangement is that of fig. 43(*b*). If the voltmeter reading is higher than that of the potentiometer voltage, the arrangement of fig. 43(*c*) should be used.

The potentiometer having been set or standardised with the aid of the standard cell, terminals *p*,*p* are connected to the potentiometer exactly as E_1 in figs. 39 or 42(*a*), and when balance is obtained, the potentiometer reading gives the p.d. across *p,p*.

The voltmeter is checked in the same manner as the ammeter for several positions of the pointer on the scale.

In fig. 43(*b*), the voltage read on the potentiometer is also the p.d. across the voltmeter, while in fig. 43(*c*), as there is no current from or to *p,p*, the p.d. across the voltmeter is $V = (R_A + R_B)\, i_A$, i_A being the current in R_A and R_B; the voltage read on the potentiometer is $V_n = R_B\, i_A$, so that

$$V = \frac{R_A + R_B}{R_B} \cdot V_n \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

The p.d. across the voltmeter equals the potentiometer indication times $\frac{R_A + R_B}{R_B}$. The arrangement in fig. 43(*c*) is known as a volt box.

Calculation of the Systematic Error. The error when using fig. 43(*b*), calculated in the same manner as for the ammeter, is

$$\frac{\Delta V'_n}{V'_n} = \frac{\Delta e'}{e'} + \frac{\Delta V'_1}{V'_1}\left(\frac{V'_1 - V'_n}{V'_n}\right) + \frac{v'_S}{V'_n}\frac{\Delta_S v'}{v'_S} + \frac{v'}{V'_n}\frac{\Delta v'}{v'}\frac{a'_1}{a'_1 + a'_2} +$$
$$\frac{v'}{V'_n}\frac{\Delta a}{a'_1 + a'_2} + \frac{v'_D}{V'_1}\frac{\Delta v'_D}{v'_D}\frac{a_A}{a_A + a_B} + \frac{v'_D}{V'_1}\frac{\Delta a}{a'_A + a'_B} \qquad . \qquad (39)$$

or

$$\frac{\Delta V'_n}{V'_n} = \frac{\Delta e'}{e'} + \frac{v'}{V'_n}\frac{\Delta a'}{(a'_1 + a'_2)} + \frac{v'_D}{V'_1}\frac{\Delta a'}{a'_A + 'a_B} + \frac{\Delta V'_1}{V'_1}\left(\frac{V'_1 - V'_n}{V'_n}\right) +$$
$$\frac{v'_s}{V'_w}\frac{\Delta v'_s}{v'_s} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (39a)$$

if all the potentiometer resistances are known to have the same constructional error.

The error when using fig. 43(*c*) is the same as (39) or (39a) plus the error on $\frac{R_A + R_B}{R_B}$; the logarithmic differential of (38) is

$$\frac{dV}{V} = \frac{dV_n}{V_n} + \frac{d\,(R_A + R_B)}{R_A + R_B} - \frac{dR_B}{R_B} =$$
$$\frac{dV_n}{V_n} + \frac{dR_A}{R_A + R_B} + \frac{dR_B}{R_A + R_B} - \frac{dR_B}{R_B} =$$
$$\frac{dV_n}{V_n} + \frac{R_A}{R_A + R_B}\frac{dR_A}{R_A} + \frac{R_B}{R_A + R_B}\frac{dR_B}{R_B} - \frac{dR_B}{R_B} =$$

$$\frac{dV_n}{V_n} + \frac{R_A}{R_A + R_B}\,\frac{dR_A}{R_A} + \frac{dR_B}{R_B}\left(\frac{R_B}{R_A + R_B} - 1\right) =$$

$$\frac{dV_n}{V_n} + \frac{R_A}{R_A + R_B}\,\frac{dR_A}{R_A} + \frac{dR_B}{R_B}\left(\frac{R_B - R_A - R_B}{R_A + R_B}\right) \quad . \qquad (40)$$

The relative error is therefore

$$\frac{\Delta V'}{V'} = \frac{\Delta V'_n}{V'_n} + \frac{R'_A}{R'_A + R'_B}\,\frac{\Delta R'_A}{R'_A} + \frac{R'_A}{R'_A + R'_B}\,\frac{\Delta R'_B}{R'_B}.$$

The error varies with R_A; if $\frac{\Delta R'_A}{R'_A}$ and $\frac{\Delta R'_B}{R'_B}$ are large, the precision of the measurement will be low even with the best potentiometer.

The results of the calibration should be presented as in the method of checking the instrument against a substandard, by drawing the calibration line and the uncertainty area.

(*c*) Calibration of a Wattmeter by a Potentiometer. The general arrangement, using a potentiometer provided with a millivoltmeter, is shown in fig. 44.

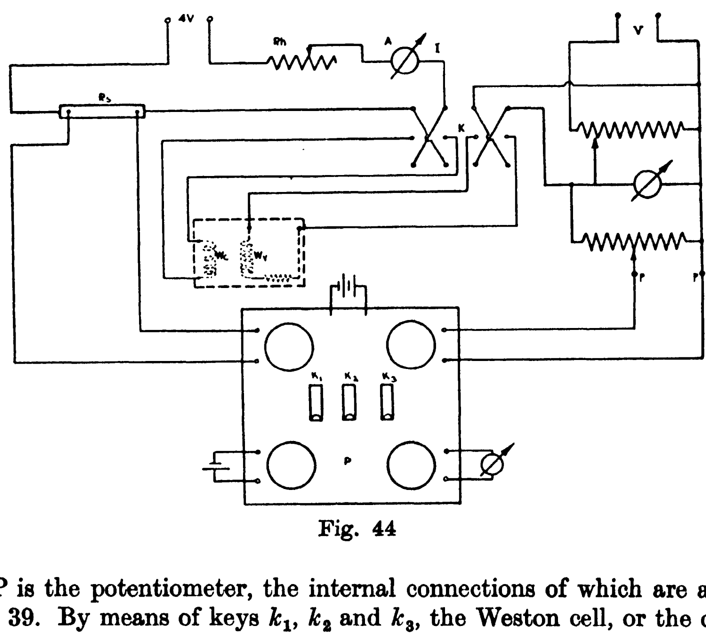

Fig. 44

P is the potentiometer, the internal connections of which are as in fig. 39. By means of keys k_1, k_2 and k_3, the Weston cell, or the drop across R_S, or the p.d. across p,p, is connected to the potentiometer.

A source of low voltage, about 4 volts, supplies current to an ammeter in series with the rheostat Rh, the standard resistance R_s, and the wattmeter current coil w_c, by way of the double-reversing switch K.

Another source of voltage, V, suitable for the wattmeter volt coil w_v, supplies this volt coil, the voltmeter and through the volt box, provides the p.d., across p,p, to the potentiometer. The wattmeter volt coil is also connected through the reversing switch K.

The switch K is used because two readings for each setting of the wattmeter pointer are necessary in order to eliminate the influence of any external fields. The mean of each pair of readings is taken as the true reading. The ammeter and voltmeter are used for giving a rough indication of the potentiometer setting which makes the calibration much speedier.

Calculation of the Systematic Error. The error is calculated in the usual way. There will be a determination error when standardising the potentiometer, and a reading error for each setting for R_s and for the p.d. across p,p.

As the wattmeter indicates $W = IV$, the error for each checked position of the pointer will be the sum of $\frac{\Delta I'}{I'}$ and $\frac{\Delta V'}{V'}$; $\frac{\Delta I'}{I'}$ and $\frac{\Delta V'}{V'}$ are calculated in the manner already explained.

The result of the calibration is given by drawing the calibration line and the uncertainty area.

CHAPTER V

D.C. GALVANOMETERS

(1) The Permanent-magnet Moving-coil Galvanometer

The essential part of the galvanometer is a rectangular coil wound on a non-conducting former and placed between the poles of a permanent magnet. The coil is suspended by a silver or phosphor-bronze wire, called the suspension, which provides the opposing torque and serves also as a conducting connection to the coil. The other connection is at the bottom of the coil, and is either a stretched wire similar to the suspension or a thin wire wound in a wide spiral. The spiral is practically torsionless.

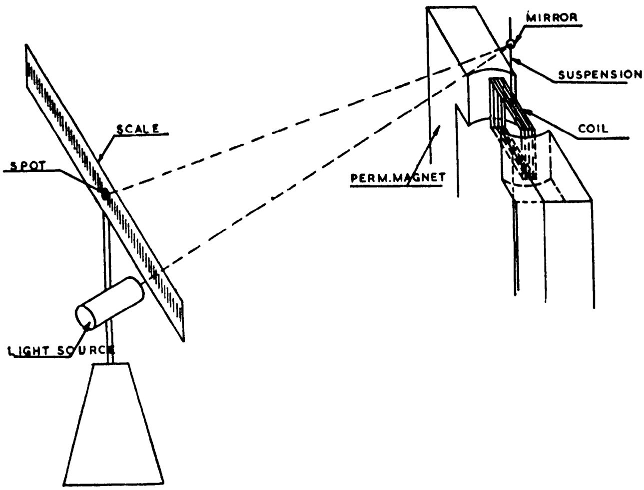

Fig. 45(a)

A small concave mirror is attached to the suspension ; light from a source provided with a suitable lens, passes through a round aperture and is projected on to the mirror. In the centre of the aperture, along its vertical diameter, there is a thin wire, so that when the light is reflected back from the mirror on to a transparent scale, often graded in millimetres, the reflected image is a bright circle with a dark vertical line in the centre ; this reflected image is called " the spot ".

The scale is usually placed at a distance of one metre from the mirror, with the zero of the scale at its centre. When the galvanometer coil is deflected or oscillates, the spot is deflected from the zero or oscillates.

The arrangement is shown on fig. 45(*a*).

Referring to fig. 45(*b*), the distance *ab* is a measure of the coil deflection. These deflections are small (the spot should not reach a deflection greater than 250 mm., which is half the total length of the scale), so that even when the deflection of the spot is 250 mm., the

angle *aob* is $2a = arc\ tan.\ \frac{250}{1000} = 14°\ 2'.$

It is easy to see from the figure that the deflection of the coil is half that of the beam of light, or that of the spot, so that the coil deflection is a, and when the spot deflection is $14°\ 2'$, $a = 7°\ 1'$.

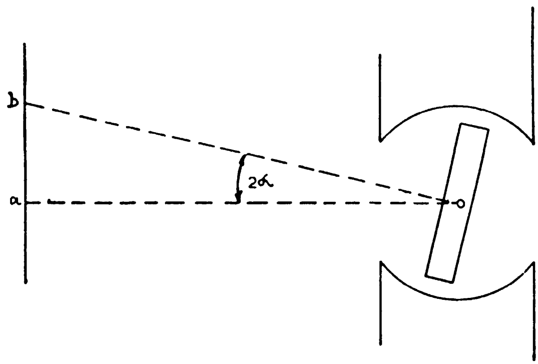

Fig. 45(*b*)

Permanent-magnet moving-coil instruments (ammeters, voltmeters) come under the category of galvanometers; the difference between the reflecting galvanometer and this instrument is that the latter is provided with a direct reading scale and a pointer, the moving parts rest on a pivot or pivots, and the opposing torque and conducting connection to the coil is provided by small springs; the opposing torque is therefore much greater in instruments than in reflecting galvanometers.

The arrangement of the essential parts of a moving-coil instrument is shown in fig. 46(*a*) and (*b*).

The coil *c* is placed between the two rectangles *a*,*a*, made of copper or aluminium, which are called the formers (the coil can also be wound on one rectangular frame, also called a former). When the coil deflects, the formers have currents induced in them by rotation

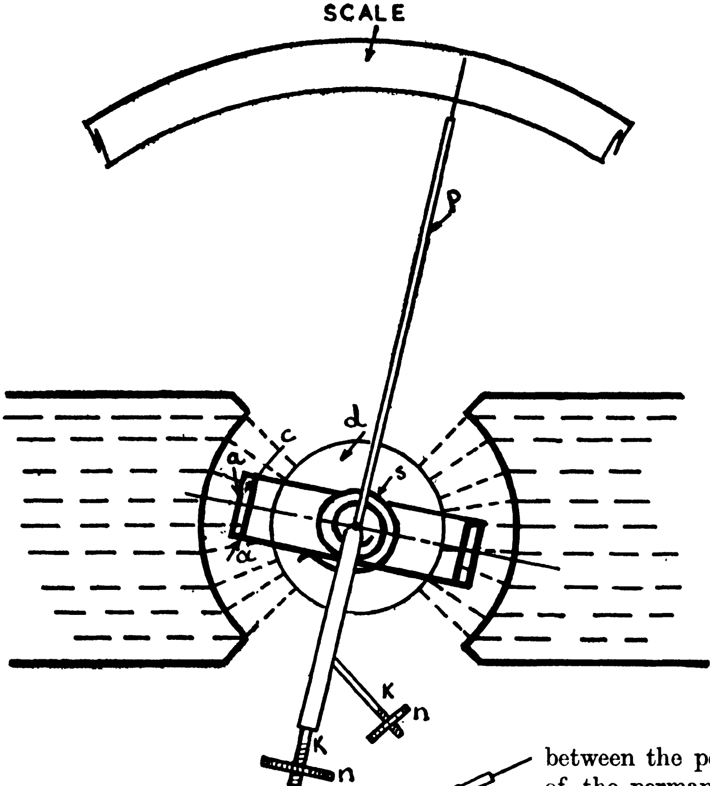

Fig. 46(*a*)

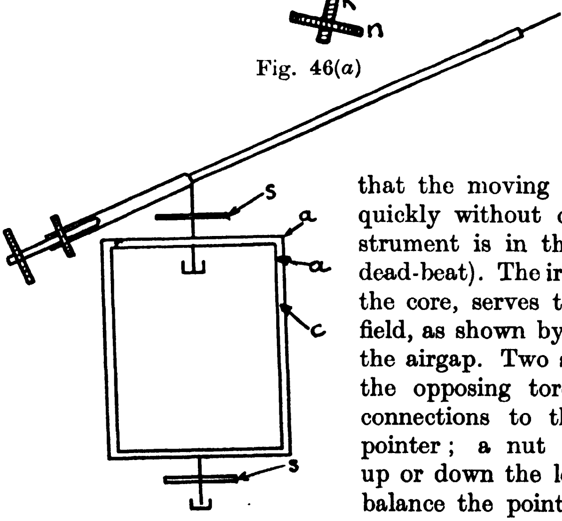

Fig. 46(*b*)

between the poles of the permanent magnet, and so provide a very effective damping, the result being that the moving part comes to rest quickly without oscillations (the instrument is in this case said to be dead-beat). The iron cylinder *d*, called the core, serves to produce a radial field, as shown by the dotted lines in the airgap. Two springs *s*,*s* provide the opposing torque, and also the connections to the coil; *p* is the pointer; a nut *n* can be screwed up or down the length *k* in order to balance the pointer around the axis of rotation; Sc is the scale.

There are two types of permanent-magnet moving-coil galvanometers; the uniform field type (fig. 47) and the radial field type (fig. 48).

The difference between fig. 47 and fig. 48 is that in the radial field type there is an iron cylinder, or core, between the magnet poles and

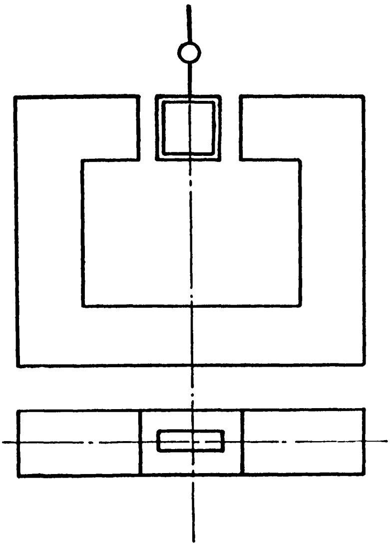

Fig. 47

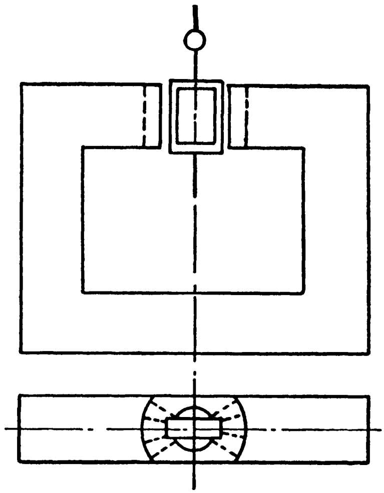

Fig. 48

the coil. The radial field thus produced results in the force on the coil being at 90° to the active coil sides, even if the coil is deflected (provided a is not too great); while in the uniform field type the torque on the active coil sides varies with the angle of coil deflection.

(2) Theory of the Permanent-magnet Moving-coil Galvanometer

Consider fig. 49(a). Suppose the current through the coil be I and the coil deflection due to this current a, the dimensions of the coil being as shown and the field in the airgap H. The force on one conductor of one side of the coil, of length l, will be $f = \mathrm{H}l\mathrm{I}$.

If the coil has n turns, the force on one side of the coil is $f_1 = \mathrm{H}l\mathrm{I}n$.

This is also the force on the other side of the coil, so that the total force is $\mathrm{F} = 2\mathrm{H}l\mathrm{I}n$.

The forces f_1 being at 90° to H, the torque on the coil is

$$\tau_1 = \frac{2\mathrm{H}Ilnw\,cos.a}{2} = \mathrm{H}lw\mathrm{I}n\,cos.a\,;$$

but as $lw = \mathrm{S}$ is the coil surface, we can write:

$\tau_1 = \text{HSI}n \cos.a = \Phi \text{ I}n \cos.a$, where $\Phi = \text{HS}$ is the flux through the coil. If we write $n\Phi = \Phi_0 =$ flux linkages, we have $\tau_1 = \Phi_0\text{I} \cos.a$. The torque τ_1 is balanced by the torsion of the suspension, if this torsion per radian deflection is τ, then for a deflection a we have $\tau a = \Phi_0\text{I} \cos.a$, and as a is usually very small, $\tau a = \Phi_0\text{I}$.

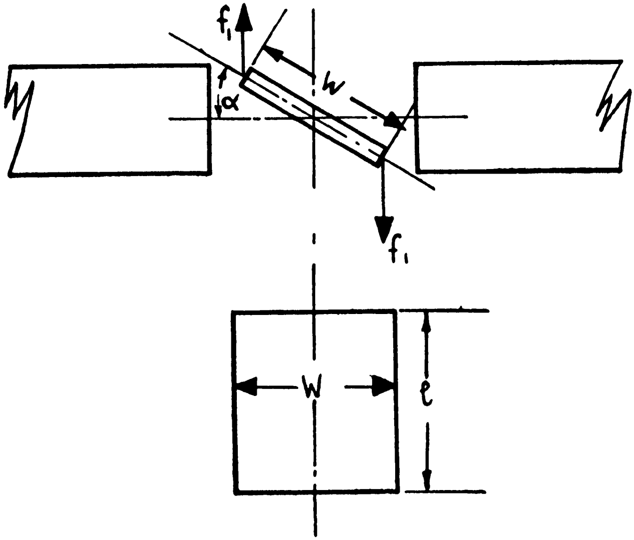

Fig. 49(*a*)

The same system of units has of course to be adhered to, so that if a is in radians, τ is in dyne cm. per radian, Φ_0 in Maxwell-turns, and I in e.m.c.g.s. units of current (in e.m.c.g.s. unit of current $=$ 10 amps).

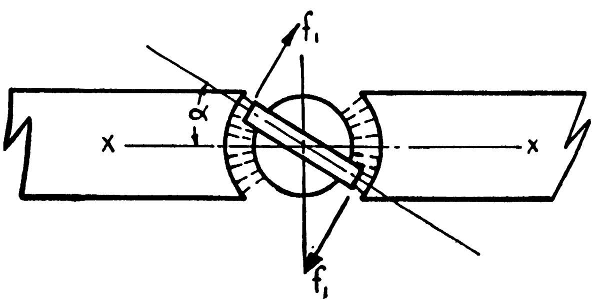

Fig. 49(*b*)

When the field is radial, fig. 49(*b*), the forces f_1 are always at 90° to the active coil sides (a small) even when the coil is deflected from the axis x–x (assuming the field in the airgap to be as shown); the torque is therefore $\tau_1 = \Phi\, n\text{I} = \Phi_0\, \text{I}$ . . . . . . (41)

Relation (41) gives the steady deflection state, but we have also to consider the behaviour of the galvanometer between the instant the current I (the injected current) starts flowing and the time the deflection corresponding to I is reached.

When a current is injected in the galvanometer, whether a steady current or a discharge, such as from a condenser, there is an electromagnetic torque acting on the coil, deflecting it from the zero position. Owing to the inertia of the moving parts, the coil can be deflected beyond its position corresponding to (41), after which the moving part will come to rest, and then start moving back towards zero in the opposite direction ; it might again pass its equilibrium position, come to rest, and again move in the first direction, etc. In other words, the mobile part of the galvanometer might oscillate around its position of equilibrium.

These oscillations would go on for ever were it not that the energy is used up in air friction (air damping), torsion of the suspension, or friction in the pivots in the case of instruments (frictional damping), and damping due to currents induced in the coil (or in the former on which the coil is wound in case of instruments), owing to the coil rotating between the poles of the permanent magnet (electrical damping).

The amplitude and character of the oscillations will depend on the current injected, on the inertia of the moving parts, on the total damping, and on the torsion of the suspension, or springs.

Let the damping moment relative to the suspension axis, resulting from all causes except the currents induced in the coil (that is, all damping on open circuit), be D_1, let the torsion constant (torsion moment per radian) be τ, and the moment of inertia of the moving part be J ; whatever the value of the instantaneous current in the coil, we shall have

$$J\frac{d^2a}{dt^2} + D_1\frac{da}{dt} + \tau a = \Phi_0 i \quad . \quad . \quad . \quad . \quad . \quad (42)$$

where a is the variable angle of deflection of the galvanometer coil, $\frac{da}{dt}$ the angular speed, and $\frac{d^2a}{dt^2}$ the angular acceleration.

In (42) we assume that the damping moment D_1 is proportional to the angular speed ; this is very nearly true for small speeds, and strictly true for electrical damping.

The current i must not be confused with I of (41) ; I is the current injected by an external p.d., while i is the sum of I and the current due to the movement of the coil between the poles of the magnet.

If v be the p.d. applied to the coil, and R and L its resistance and inductance respectively, we can write:

$$Ri + L\frac{di}{dt} + e_g = v.$$

e_g is the e.m.f. induced in the coil caused by its rotation between the poles of the magnet. As the sides of the coil move in a field H and cut lines at 90° in the radial field, or very nearly 90° in a uniform field if the angle a is small, the e.m.f. induced in one conductor of one side of the coil will be: $e_1 = Hlu$, where l is the length of the coil side and u is the linear speed of the conductor.

The e.m.f. induced in all the conductors of one coil side is therefore: $e_s = Hlun$, and the same e.m.f. being induced in the two sides of the coil the total e.m.f. is $e_g = 2\,Hlun$.

Now if $\frac{da}{dt}$ is the angular speed and w the width of the coil, we have

$$u = \frac{w}{2}\frac{da}{dt}, \text{ so that } e_g = \frac{Hln2w}{2}\frac{da}{dt}$$

but $lw = S =$ coil surface, and $SHn = \Phi_0$ so that

$$e_g = \Phi_0\frac{da}{dt}$$

$$\text{So that } Ri + L\frac{di}{dt} + \Phi_0\frac{da}{dt} = v, \text{ or } Ri + L\frac{di}{dt} = v - \Phi_0\frac{da}{dt}.$$

Now in permanent-magnet moving-coil galvanometers the term $L\frac{di}{dt}$ can be neglected, even if L is high, because $\frac{da}{dt}$ is small compared with the rate of increase of current; or, in other words, the current will arrive at its full value before the galvanometer coil has moved appreciably, so that we can write:

$$Ri = v - \Phi_0\frac{da}{dt} \text{ and } i = \frac{v}{R} - \frac{\Phi_0 da}{Rdt} \quad . \quad . \quad . \quad (43)$$

Putting the value of i from (43) in (42) we get

$$J\frac{d^2a}{dt^2} + D_1\frac{da}{dt} + \tau a = \Phi_0\frac{v}{R} - \frac{\Phi_0{}^2 da}{Rdt} \text{ or}$$

$$J\frac{d^2a}{dt} + \left(D_1 + \frac{\Phi^2{}_0}{R}\right)\frac{da}{dt} + \tau a = \frac{\Phi_0 v}{R} = \Phi_0 I \quad . \quad (43a)$$

I is the injected current, $\frac{\Phi^2{}_0}{R}$ is the damping constant due to the currents induced in the coil; writing $D_1 + \frac{\Phi_0{}^2}{R} = D$, we have

$$J\frac{d^2a}{dt} + D\frac{da}{dt} + \tau a = \Phi_0 I \qquad (44)$$

As we have a particular solution of (44), namely $\tau a = \Phi_0 I$, we need only solve the equation

$$J\frac{d^2a}{dt^2} + D\frac{da}{dt} + \tau a = 0 \qquad (45)$$

The characteristic equation of (45) is

$$Jp^2 + Dp + \tau = 0 \qquad (46)$$

the roots of which are

$$p_1 = -\frac{D}{2J} + \sqrt{\frac{D^2 - 4J\tau}{4J^2}} = -\frac{D}{2J} + \sqrt{\frac{D^2}{4J^2} - \frac{\tau}{J}}$$

$$p_2 = -\frac{D}{2J} - \sqrt{\frac{D^2 - 4J\tau}{4J^2}} = -\frac{D}{2J} - \sqrt{\frac{D^2}{4J} - \frac{\tau}{J}},$$

writing

$$a_1 = \frac{D}{2J};\ a_2 = \sqrt{\frac{D^2}{4J^2} - \frac{\tau}{J}},$$

which gives $p_1 = -a_1 + a_2$; $p_2 = -a_1 - a_2$.

We have three cases to consider:

Case I: $\frac{D^2}{4J^2} > \frac{\tau}{J}$.

The roots of (46) are real and unequal, the solution of (45) will be

$$a = Ae^{p_1 t} + Be^{p_2 t} = Ae^{(-a_1 + a_2)t} + Be^{(-a_1 - a_2)t}. \qquad (47)$$

where A and B are constants; differentiating (47) we get

$$\frac{da}{dt} = (-a_1 + a_2)Ae^{(-a_1 + a_2)t} + (-a_1 - a_2)Be^{(-a_1 - a_2)t} \qquad (48)$$

To determine A and B, let us count the time $t = 0$ when $a = 0$, the coil of the galvanometer having then an angular speed ω_0 (due to an impulse) ; then at $t = O$ (47) becomes

$$a = O = A + B \; ; \; - A = B \; ; \text{ and (48) becomes}$$
$$\omega_0 = (- a_1 + a_2) A + (- a_1 - a_2) B \; ; \text{ or}$$

$$\omega_0 = (-a_1 + a_2)(- B) + (- a_1 - a_2) B = - 2Ba_2 \; ; \text{ therefore}$$

$$B = - \frac{\omega_0}{2a_2}.$$

(47) and (48) can therefore be written

$$a = \frac{\omega_0}{2a_2} e^{(- a_1 + a_2) t} - \frac{\omega_0}{2a_2} e^{(- a_1 - a_2) t} =$$

$$\frac{\omega_0}{2a_2}\left[e^{(- a_1 + a_2) t} - e^{(- a_1 - a_2) t}\right] \quad . \quad . \quad . \quad (49)$$

$$\frac{da}{dt} = (-a_1 + a_2) \frac{\omega_0}{2a_2} e^{(- a_1 + a_2) t} - (-a_1 - a_2) \frac{\omega_0}{2a_2} e^{(- a_1 - a_2) t} =$$

$$\frac{\omega_0}{2a_2}\left[(- a_1 + a_2) e^{(- a_1 + a_2) t} - (- a_1 - a_2) e^{(- a_1 - a_2) t}\right] \quad (50)$$

When the maximum deflection is reached, the angular speed becomes zero, this will happen at a time t_m, such as will make (50) equal zero, or when

$$(- a_1 + a_2) e^{(- a_1 + a_2) t_m} - (- a_1 - a_2) e^{(- a_1 - a_2) t_m} = O \; ;$$

$$\text{or } (- a_1 + a_2) e^{(- a_1 + a_2) t_m} = (- a_1 - a_2) e^{(- a_1 - a_2) t_m}$$

$$(- a_1 + a_2) e^{a_2 t_m} = (- a_1 - a_2) e^{2a_2 t_m}$$

$$\frac{- a_1 - a_2}{- a_1 + a_2} = e^{2a_2 t_m} \; ;$$

the time t_m is therefore

$$t_m = \frac{1}{2a_2} log. \frac{- a_1 - a_2}{- a_1 + a_2} = \frac{1}{2a_2} log. \frac{p_2}{p_1} \quad . \quad . \quad . \quad (51)$$

The maximum deflection occurring at the time t_m is found by putting (51) in (49), which gives

$$a_m = \frac{\omega_0}{2a_2}\left[e^{\left(\frac{p_1}{2a_2}\right)\left(log.\frac{p_2}{p_1}\right)} - e^{\left(\frac{p_2}{2a_2}\right)\left(log.\frac{p_2}{p_1}\right)}\right]$$

As $\frac{p_1}{2a_2} \log.\frac{p_2}{p_1} = \log.\left(\frac{p_2}{p_1}\right)^{\frac{p_1}{2a_2}}$; $\frac{p_2}{2a_2} \log.\frac{p_2}{p_1} = \log.\left(\frac{p_2}{p_1}\right)^{\frac{p_2}{2a_2}}$;

$e^{\log.\left(\frac{p_2}{p_1}\right)^{\frac{p_1}{2a_2}}} = \left(\frac{p_2}{p_1}\right)^{\frac{p_1}{2a_2}}$; $e^{\log.\left(\frac{p_2}{p_1}\right)^{\frac{p_2}{2a_2}}} = \left(\frac{p_2}{p_1}\right)^{\frac{p_2}{2a_2}}$, we have

$$a_m = \frac{\omega_0}{2a_2}\left[\left(\frac{p_2}{p_1}\right)^{\frac{p_1}{2a_2}} - \left(\frac{p_2}{p_1}\right)^{\frac{p_2}{2a_2}}\right] =$$

$$\frac{\omega_0}{2a_2}\left\{\left(\frac{p_2}{p_1}\right)^{-\frac{a_1}{2a_2}}\left[\left(\frac{p_2}{p_1}\right)^{\frac{a_2}{2a_2}} - \left(\frac{p_2}{p_1}\right)^{-\frac{a_2}{2a_2}}\right]\right\} =$$

$$\frac{\omega_0}{2a_2}\left\{\left(\frac{-a_1-a_2}{-a_1+a_2}\right)^{-\frac{a_1}{2a_2}}\left[\left(\frac{-a_1-a_2}{-a_1+a_2}\right)^{\frac{1}{2}} - \left(\frac{-a_1-a_2}{-a_1+a_2}\right)^{-\frac{1}{2}}\right]\right\} =$$

$$\frac{\omega_0}{2a_2}\left(\frac{-a_1-a_2}{-a_1+a_2}\right)^{-\frac{a_1}{2a_2}}\left(-\frac{2a_2}{\sqrt{a_1^2 - a_2^2}}\right)$$

as $a_1^2 - a_2^2 = \frac{\tau}{J}$ we get $a_m = \left(\frac{p_2}{p_1}\right)^{-\frac{a_1}{2a_2}}\sqrt{\frac{J}{\tau}} \cdot (-\omega_0)$.

It follows that the maximum deflection depends on J, T and ω_0.

After the maximum deflection is reached, the moving part will start going back to zero, but at the maximum deflection the angular speed is zero, so that when putting t_m in (48) we can write:

$$0 = (-a_1 + a_2)\, Ae^{p_1 t_m} + (-a_1 - a_2)\, Be^{p_2 t_m} =$$

$$-a_1\left(Ae^{p_1 t_m} + Be^{p_2 t_m}\right) + a_2\left(Ae^{p_1 t_m} - Be^{p_2 t_m}\right).$$

As at t_m we have a_m, which according to (47) can be written

$a_m = Ae^{p_1 t_m} + Be^{p_2 t_m}$, we have

$O = -a_1 a_m + a_2 a_m - 2Ba_2 e^{p_2 t_m}$, hence the constant B is now

$B = \frac{(-a_1 + a_2)}{2a_2} a_m e^{-p_2 t_m}$, and putting this value of B in (47) we have

$a_m = Ae^{p_1 t_m} + \frac{(-a_1 + a_2)}{2a_2} a_m$, which determines the constant A.

$A = \frac{(a_1 + a_2)}{2a_2} a_m e^{-p_1 t_m}$, so that finally

$$a = \frac{(a_1 + a_2)}{2a_2} a_m e^{-p_1 t_m} e^{p_1 t} + \frac{a_2 - a_1}{2a_2} a_m e^{-p_2 t_m} e^{p_2 t};$$

writing $t = t_m + t_1$, t_1 is the time counted from t_m, we get

$$a = \frac{(a_1 + a_2)}{2a_2} a_m e^{p_1 t_1} + \frac{(a_2 - a_1)}{2a_2} a_m e^{p_2 t_1} =$$

$$a_m e^{(-a_1 + a_2) t_1} \left\{ \frac{a_1 + a_2}{2a_2} + \frac{a_2 - a_1}{2a_2} e^{-2a_2 t_1} \right\} \qquad (52)$$

As in this case $\frac{D_2}{4J_2} > \frac{\tau}{J}$, a_2 is positive, and therefore $-2a_2 t_1$ is negative; also as $a_1 = \frac{D}{2J}$ is greater than a_2, the terms of (52) tend towards zero; that is, a decreases from t_m onwards without theoretically ever reaching zero; practically, however, after a short time the deflection becomes nearly zero.

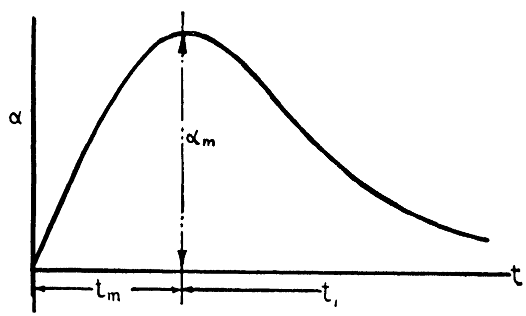

Fig. 50

There are no oscillations in this case, the curve $a = f(t)$ is shown in fig. 50.

Case II : $\frac{D^2}{4J^2} < \frac{\tau}{J}$.

Writing as in case I, $\frac{D}{2J} = a_1$; $\sqrt{\frac{D^2}{4J_2} - \frac{\tau}{J}} = a_2 \sqrt{-1}$, the two roots of the characteristic equation will now be

$$p_1 = -a_1 + a_2 \sqrt{-1} = -a_1 + ja_2;$$

$$p_2 = -a_1 - a_2 \sqrt{-1} = -a_1 - ja_2.$$

The solution of (45) is

$$a = A_1 e^{(-a_1 + ja_2)t} + Be^{(-a_1 - ja_2)t} =$$

$$e^{-a_1 t}\left(A_1 e^{ja_2 t} + B_1 e^{-ja_2 t}\right).$$

As $e^{ja_2 t} = cos.\, a_2 t + j\, sin.\, a_2 t$; $e^{-ja_2 t} = cos.\, a_2 t - j\, sin.\, a_2 t$ we have $a = e^{-a_1 t}[A_1(cos.a_2 t + j\, sin.a_2 t) + B_1(cos.a_2 t - j\, sin.a_2 t)] =$ $e^{-a_1 t}[(A_1 + B_1)\, cos.\, a_2 t + (A_1 - B_1)\, j\, sin.\, a_2 t]$; and writing $A_1 + B_1 = A$; $j\,(A_1 - B_1) = B$ we get

$$a = e^{-a_1 t}(A\, cos.\, a_2 t + B\, sin.\, a_2 t) \quad . \quad . \quad . \quad (53)$$

Differentiating (53), we have the angular speed

$$\frac{da}{dt} = -a_1 e^{-a_1 t}[A\, cos.\, a_2 t + B\, sin.\, a_2 t]$$

$$- a_2 e^{-a_1 t}[A\, sin.\, a_2 t - B\, cos.\, a_2 t].$$

To determine the constants A and B, count the time as zero when $a = 0$, the galvanometer having then an angular speed ω_0, (53) becomes $0 = A$, therefore

$$a = Be^{-a_1 t} sin.a_2 t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Differentiating (54), we get $Be^{-a_1 t}[a_2 cos.a_2 t - a_1 sin.a_2 t] = \frac{da}{dt}$, which for $t = 0$ gives $\omega_0 = Ba_2$, therefore $B = \frac{\omega_0}{a_2}$.

a and $\frac{da}{dt}$ can now be written (from (54))

$$a = \frac{\omega_0}{a_2} e^{-a_1 t} \sin. a_2 t \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$\frac{da}{dt} = \frac{\omega_0}{a_2} e^{-a_1 t} (a_2 \cos. a_2 t - a_1 \sin. a_2 t) \quad . \quad . \quad . \quad (55a)$$

Considering (55) we see that $\sin. a_2 t$ will make a alternately positive and negative, so that the moving part of the galvanometer will oscillate. Were it not for $e^{-a_1 t}$, the oscillations would be purely sinusoidal, of constant magnitude ; as it is, the amplitude of the oscillations decreases with time.

The times at which the moving part passes through zero are given by $\sin. a_2 t = 0$, that is when $a_2 t = 0, \pi, 2\pi, 3\pi, \ldots n\pi$, or at times $t = 0, \frac{\pi}{a_2}, \frac{2\pi}{a_2}, \frac{3\pi}{a_2}, \ldots \frac{n\pi}{a_2}$.

The time between two consecutive zeros is $\frac{n\pi}{a_2} - \frac{(n-1)\pi}{a_2} = \frac{\pi}{a_2}$, and the time of a complete oscillation (zero to maximum, back to zero, negative maximum, back to zero) is $T = \frac{2\pi}{a_2}$.

The maxima of the deflections are reached when $\frac{da}{dt} = 0$, that is, at times t_m such as will make (55a) equal to zero. This will happen when $a_2 \cos. a_2 t_m - a_1 \sin. a_2 t_m = 0$; $a_2 \cos. a_2 t_m = a_1 \sin. a_2 t_m$, or $\frac{a_1}{a_2} \tan. a_2 t_m = 1$; $t_m = \frac{1}{a_2} \arctan. \frac{a_2}{a_1}$.

Let β be the smallest positive arc (between 0 and $\frac{\pi}{2}$), whose tangent is $\frac{a_2}{a_1}$, then we can write

$$t_m = \frac{1}{a_2}(\beta + n\pi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Putting in (56) all integral numbers for n (from zero), we get the times t_m when the maxima deflections occur.

We can produce the following table:

Time t	Deflection	
O	O	First zero.
$\frac{\beta}{a_2}$	a_{m1}	First max. deflection, say positive.
$\frac{\pi}{a_2}=\frac{T}{2}$	O	Second zero, half period.
$\frac{\beta}{a_2}+\frac{\pi}{a_2}=\frac{T}{2}+\frac{\beta}{a_2}$	a_{m2}	Second max. deflection, negative.
$\frac{2\pi}{a_2}=\frac{2T}{2}$	O	One complete period.
$\frac{2\pi}{a_2}+\frac{\beta}{a_2}=\frac{2T}{2}+\frac{\beta}{a_2}$	a_{m3}	Third max. deflection, positive.
$\frac{3\pi}{a_2}=\frac{3T}{2}$	O	Fourth zero.
$\frac{\beta}{a_2}+\frac{3\pi}{a_2}=\frac{\beta}{a_2}+\frac{3T}{a_2}$	a_{m4}	Second negative deflection.
$\frac{4\pi}{a_2}=2T$	O	Two complete periods.

etc., for n periods.

We note from the table that the times t_{11} between the maxima deflections and the following zeros are greater than the times t_{22} between these zeroes and the following maxima deflections. We have

$$t_{11}=\frac{\pi}{a_2}-\frac{\beta}{a_2}=\frac{T}{2}-\frac{\beta}{a_2};\quad t_{22}=\frac{\beta}{a_2}+\frac{T}{2}-\frac{T}{2}=\frac{\beta}{a_2}$$

$$t_{11}-t_{22}=\frac{T}{2}-\frac{\beta}{a_2}-\frac{\beta}{a_2}=\frac{T}{2}-\frac{2\beta}{a_2};\text{ and as } a_2=\frac{2\pi}{T},$$

$$t_{11}-t_{22}=\frac{2\pi T-4T\beta}{4\pi}=\frac{T}{2}\times\frac{\pi-2\beta}{\pi};$$

as β is between O and $\frac{\pi}{2}$, t_{11} is greater than t_{22}.

The time between two consecutive maxima is

$$\frac{T}{2}+\frac{\beta}{a_2}-\frac{\beta}{a_2}=\frac{T}{2}$$

and the time between two consecutive maxima in the same direction is $\frac{2T}{2} = T =$ period.

The ratio between two successive maxima deflections in opposite directions is

$$\frac{a_{m1}}{a_{m2}} = \frac{\frac{\omega_0}{a_2} e^{-a_1 \frac{\beta}{a_2}} sin. a_2 \frac{\beta}{a_2}}{\frac{\omega_0}{a_2} e^{-a_1\left(\frac{\beta}{a_2} + \frac{\pi}{a_2}\right)} sin. a_2\left(\frac{\beta}{a_2} + \frac{\pi}{a_2}\right)} = -\frac{e^{-a_1 \frac{\beta}{a_2}}}{e^{-\frac{a_1}{a_2}(\beta + \pi)}}$$

$$= - e^{a_1 \frac{\pi}{a_2}}$$

and writing $\frac{a_1 \pi}{a_2} = \varrho$; $-\frac{a_{m1}}{a_{m2}} = e^{\varrho}$.

ϱ is constant for a given damping and is called the logarithmic decrement of the oscillations, and as $a_2 = \frac{2\pi}{T}$, we can write :

$$\frac{a_1 \pi}{\frac{2\pi}{T}} = \frac{a_1 T}{2} = \varrho.$$

The ratio between two successive maxima deflections in the same direction is $\frac{a_{m1}}{a_{m3}} = e^{2\varrho}$, between the first and the nth maxima deflection $\frac{a_{m1}}{a_{mn}} = e^{(n-1)\varrho}$.

The times t_{11} and t_{22} depend on ϱ because $\frac{t_{11}}{t_{22}} = \frac{T}{2} - \frac{\beta}{a_2} \div \frac{\beta}{a_2}$, and as $a_2 = \frac{2\pi}{T}$, $\frac{t_{11}}{t_{22}} = \frac{\pi - \beta}{\beta}$; as $\frac{\pi}{\varrho} = \frac{a_2}{a_1}$, we can write :

$$\beta = arc\,tan. \frac{a_2}{a_1} = arc\,tan. \frac{\pi}{\varrho}, \text{ so that } \frac{t_{11}}{t_{22}} = \frac{\pi - arc\,tan. \frac{\pi}{\varrho}}{arc\,tan. \frac{\pi}{\varrho}}.$$

For instance, when $\pi = \varrho$ we get $\frac{t_{11}}{t_{22}} = 3$.

Note that as $\frac{a_1}{a_2}\, tan.\, a_2 t_m = 1$, we have $tan.\, a_2 t_m = \frac{a_2}{a_1}$, which means that $sin.\, a_2 t_m = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}$; (55) can then be written :

$$\alpha_m = \frac{\omega_0}{a_2} e^{-a_1 t_m} \times \frac{a_2}{\sqrt{a_1^2 + a_2^2}}, \text{ and as } t_m = \frac{1}{a_2} \, arc\ tan. \frac{a_2}{a_1}$$

the maximum deflection is

$$\alpha_m = \frac{\omega_0}{\sqrt{a_1^2 + a_2^2}} e^{-\left(\frac{a_1}{a_2}\right) arc\ tan. \frac{a_2}{a_1}} \quad . \quad . \quad . \quad (56a)$$

Again, as $T = \frac{2\pi}{a_2}$, and $\varrho = \frac{a_1 T}{2}$, we can write :

$$\sqrt{a_1^2 + a_2^2} = \sqrt{\frac{(2\varrho)^2}{T^2} + \frac{(2\pi)^2}{T^2}} = \frac{2\sqrt{\varrho^2 + \pi^2}}{T}; \quad \frac{a_1}{a_2} = \frac{2\varrho}{T} \div \frac{2\pi}{T} = \frac{\varrho}{\pi}$$

therefore $\alpha_m = \dfrac{\omega_0 T}{2\sqrt{\varrho^2 + \pi^2}} e^{-\frac{\varrho}{\pi} arc\, tan. \frac{\pi}{\varrho}}$.

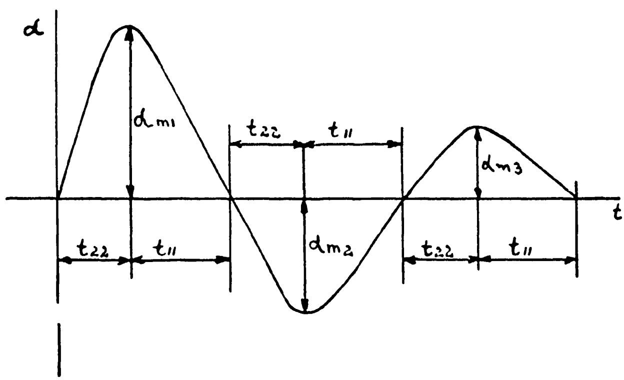

Fig. 51

The curve $\alpha = f(t)$ is shown in fig. 51.

Consider now the case when $\frac{D^2}{4J^2} = 0$; that is, $D = 0$. There is no damping; we have now $p_1 = ja_2$, $p_2 = -ja_2$; and as this is a case which comes under $\frac{D^2}{4J^2} < \frac{\tau}{J}$, the equations will be the same except that $D = 0$; we can therefore write:

$$\alpha = \frac{\omega_0}{a_2} sin. a_2 t = \frac{\omega_0}{\sqrt{\frac{\tau}{J}}} sin. \sqrt{\frac{\tau}{J}} \cdot t$$

$$\frac{da}{dt} = \frac{\omega_0}{a_2} a_2 cos. a_2 t = \omega_0 cos. a_2 t = \omega_0 cos. \sqrt{\frac{\tau}{J}} \cdot t.$$

The oscillations are now purely sinusoidal ; the amplitude is constant; and as $T = \frac{2\pi}{a_2}$, the period is $T = 2\pi\sqrt{\frac{J}{\tau}}$.

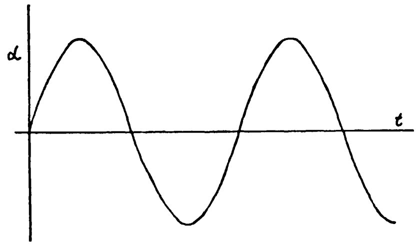

Fig. 52

The curve $a = f(t)$ is shown in fig. 52.

Case III : $\frac{D^2}{4J^2} = \frac{\tau}{J}$.

Here $a_1 = \frac{D}{2J}$ is a double root of the characteristic equation ; the solution of (45) is therefore

$$a = (A + Bt)\, e^{-a_1 t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

If, when $t = O$, $a = O$ and $\frac{da}{dt} = \omega_0$, then (57) becomes $O = A$ and

$$a = Bte^{-a_1 t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Differentiating (58), we get

$$\frac{da}{dt} = -a_1 Bte^{-a_1 t} + Be^{-a_1 t} = Be^{-a_1 t}(1 - a_1 t) \; . \quad (59)$$

which for $t = O$ becomes $\omega_0 = B$, so that (58) and (59) can be written :

$$a = \omega_0 te^{-a_1 t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

$$\frac{da}{dt} = \omega_0 e^{-a_1 t}(1 - a_1 t) \quad . \quad . \quad . \quad . \quad . \quad (61)$$

The maximum deflection a_m will be reached at a time t_m, such as will make (61) equal zero ; that is, when $1 - a_1 t_m = O$, $t_m = \frac{1}{a_1}$.

Putting this value of t_m in (60), we get the maximum deflection

$\alpha_m = \frac{\omega_0}{a_1} e^{-1}$; and as $a_1 = \frac{D}{2J}$; $\alpha_m = \frac{\omega_0 2J}{D} \cdot \frac{1}{e}$, again as $D^2 = 4J\tau$.

$\alpha_m = \omega_0 \sqrt{\frac{J}{\tau}} \cdot \frac{1}{e}$.

The relation giving α_m at t_m can be written (from (57))

$$\alpha_m = (A + Bt_m) \times e^{-a_1 t_m} \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Differentiating, we get

$\frac{d\alpha}{dt} = -a_1 [A + Bt_m] e^{-a_1 t_m} + Be^{-a_1 t_m}$; and as at $t_m = \frac{1}{a_1}$ the angular speed is zero, we have

$O = -a_1 \left(A + \frac{B}{a_1}\right) \frac{1}{e} + \frac{B}{e} = -a_1 \alpha_m + \frac{B}{e}$, from which we get the new value of B as $B = ea_1 \alpha_m$; and combining with (62)

$\alpha_m = (A + \alpha_m e) \frac{1}{e}$, so that $A = O$. With these values of A and B

we have $\alpha = a_1 \alpha_m \cdot et \cdot e^{-a_1 t}$. If t_1 is the time counted from t_m, we can write $t = t_m + t_1 = \frac{1}{a_1} + t_1$; and

$$\alpha = a_1 \alpha_m e \left(t_1 + \frac{1}{a_1}\right) e^{-a_1 \left(\frac{1}{a_1} + t_1\right)} =$$

$$\alpha_m (1 + a_1 t_1) e^{-a_1 t_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

differentiating $\frac{d\alpha}{dt} = \alpha_m \left[a_1 e^{-a_1 t_1} - a_1 e^{-a_1 t_1} (1 + a_1 t_1)\right] =$

$\alpha_m a_1 e^{-a_1 t_1} (1 - 1 - a_1 t_1) = -\alpha_m a_1^2 e^{-a_1 t_1} t_1$.

Comparing (63) with (52) of case I, we see that in case III α decreases more quickly than in case I ; there are no oscillations here either.

The curve $\alpha = f(t)$ is shown in fig. 53, where the curves of cases I and II are also shown for comparison.

Case III is the case of critical conditions, and the damping D for which it will occur is called the critical damping. To the damping on open circuit we have to add an electrical damping by currents induced in the coil. The value of external resistance, plus the coil resistance, which, together with the open circuit damping, will produce critical conditions, is called the critical resistance.

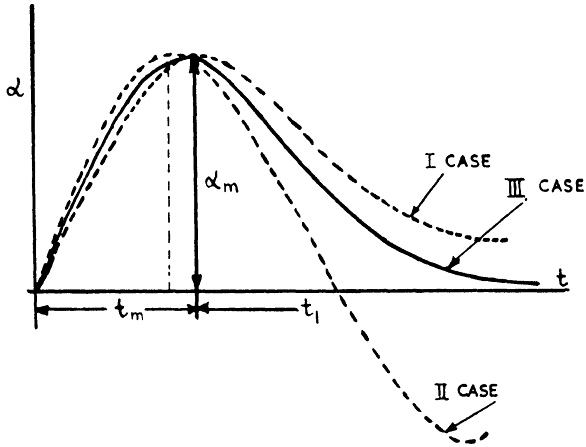

Fig. 53

It is obvious that a galvanometer should, if possible, be used in non-oscillatory conditions, as otherwise the galvanometer will oscillate and much time will be lost.

Of the two cases I and III, the latter, that is, the critical condition, is the better, because the galvanometer comes back to zero or rest more quickly.

It is, however, a fallacy, unfortunately maintained by many authors, to suggest that the critical conditions are the best; because if an additional damping occurs when working in critical conditions, it will not be easily noticeable, the spot coming to rest without passing the zero in any case. It is therefore better to use the galvanometer in oscillating conditions, preferably near the limit, giving the strong damping which occurs when the logarithmic decrement $\varrho = \pi$. Here the spot passes to the other side of the zero, so, when any additional damping occurs, it is immediately noticed; also, the damping being strong, the galvanometer comes to rest quickly. What, however, is more important still, is that the maximum deflection for a given impulse is greater in oscillatory conditions than when non-oscillatory.

COMPARISON BETWEEN NEAR LIMIT OF OSCILLATING CONDITIONS, AND THE CRITICAL CONDITIONS.[1] Near the limit of oscillating conditions we have $\varrho = \pi$; and as $\varrho = \frac{a_1}{a_2}\pi$, it follows that $a_1 = a_2$; that is,

$$\frac{D^2}{4J^2} = \left[\sqrt{\frac{D^2}{4J^2} - \frac{\tau}{J}}\right]^2 \times \frac{1}{(\sqrt{-1})^2} \quad . \quad . \quad . \quad . \quad (64)$$

As $\frac{\tau}{J}$ is not zero, (64) is only true if $\frac{\tau}{J} = 2\frac{D^2}{4J^2}$, so that

$$\sqrt{\frac{\tau}{J}} = \sqrt{2}\cdot\frac{D}{2J} = \sqrt{2}\cdot a_1.$$

If $\varrho = \pi$, then $a_1 = a_2$, and we are near oscillating conditions; if $\varrho < \pi$, then $a_1 < a_2$, and we have oscillating conditions with less damping than when $\varrho = \pi$.

When $\varrho = \pi$, the value of the maximum deflection becomes (from

(56a)) $a_m = \frac{\omega_0}{\sqrt{2}\cdot a_1} e^{-\frac{\pi}{4}} = \frac{\omega_0}{\sqrt{\frac{\tau}{J}}} e^{-\frac{\pi}{4}}$.

When working in critical conditions, the maximum deflection is $a_{mc} = \frac{\omega_0}{\sqrt{\frac{\tau}{J}}}\cdot e^{-1}$, so that the ratio of the maxima deflections with $\varrho = \pi$ and in critical conditions is, for the same impulse,

$$\frac{a_m}{a_{mc}} = \frac{e^{-\frac{\pi}{4}}}{e^{-1}} \simeq 1{\cdot}235.$$

For a given impulse the maximum deflection reached by the galvanometer when $\varrho = \pi$ is 1·235 times the deflection in critical conditions.

The ratio of the first maximum deflection a_{m1}, when $\varrho = \pi$, to the second deflection a_{m2} in the opposite direction, is

$$\frac{a_{m1}}{a_{m2}} = e^{\pi} = 23; \text{ and } a_{m2} = 0{\cdot}0435\, a_{m1}.$$

[1] For further details, see Bedeau: *Cours de Mesures Electriques* (S.F.E.; E.S.E., Paris; Vol. I).

The third deflection will be

$$\alpha_{m3} = \alpha_{m1} e^{-2\pi} = 0{\cdot}00189\, \alpha_{m1} \quad . \quad . \quad . \quad . \quad (65)$$

We see that α_{m3} will be quite imperceptible with the usual deflections, and the spot is practically at rest after the third deflection.

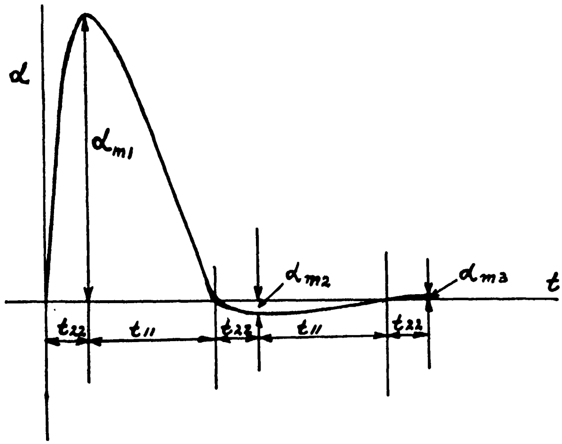

Fig. 54

The curve $\alpha = f(t)$ for $\varrho = \pi$ is shown in fig. 54.

Now let us see what the galvanometer deflection is, when used in critical conditions, after half a period $\frac{T_1}{2}$, and a whole period T_1, of the condition $\varrho = \pi$.

When $\varrho = \pi$, we have $\sqrt{\frac{\tau}{J}} = \sqrt{2}\, a_1$, and the period $T_1 = \frac{2\pi}{a_1}$ $(a_1 = a_2)$, the damping being greater in critical conditions where $a_{1c} = \sqrt{\frac{\tau}{J}}$. If we put a_{1c} in T_1, we get

$$T_1 = \frac{2\pi}{a_{1c}} \cdot \sqrt{2}\,;\; T_1\, a_{1c} = 2\pi\sqrt{2},$$

the deflection in critical conditions being

$$\alpha = \alpha_m (1 + a_1 t_1)\, e^{-a_1 t_1}\,;$$

we shall have at $\frac{T_1}{2}$, $\alpha = \alpha_m (1 + \pi\sqrt{2})\, e^{-\pi\sqrt{2}} \doteq 0{\cdot}0647\, \alpha_m$

and at T_1

$$a = a_m (1 + 2\pi \sqrt{2}) e^{-2\pi\sqrt{2}} = 0{\cdot}0014\, a_m \quad . \quad . \quad (66)$$

Comparing (65) with (66), we see that there is not much difference (in practice none) between the times taken by the spot to come to rest in critical conditions and when $\varrho = \pi$.

(3) The Sensitivity of the Galvanometer

(a) THE THEORETICAL CURRENT SENSITIVITY. This is defined as the ratio $\frac{a}{I} = s$, or the coil deflection in radians per 1 e.m.c.g.s. unit of current; s has only a theoretical importance.

(b) THE PRACTICAL CURRENT SENSITIVITY. This will be defined as the current in amps necessary to produce 1 mm. deflection of the spot when the scale is at 1 metre distance from the mirror. We will call it s_p.

(c) RELATION BETWEEN THE THEORETICAL AND PRACTICAL CURRENT SENSITIVITIES. As s_p is a current expressed in amps, we can write:

$s = \frac{a}{I} = \frac{a_p}{s_p}$, if a_p is the coil deflection corresponding to s_p; and as 1 mm. of spot deflection, at a distance of 1 metre, corresponds to s_p, we have

$a_p = \frac{1}{2\pi \times 1000} \times 2\pi \times \frac{1}{2} = \frac{1}{2000}$ radians, because the coil deflection is half of the spot deflection.

Remembering that s_p is in amps, and that 1 ampere $= \frac{1}{10}$ e.m.c.g.s. unit of current, we get

$$s = \frac{1}{2000} \div s_p \cdot \frac{1}{10} = \frac{1}{200\, s_p}.$$

(d) THE THEORETICAL POTENTIAL SENSITIVITY. This is defined as $s_1 = \frac{a}{v}$, or the coil deflection in radians when 1 e.m.c.g.s. unit of potential is applied to it.

(e) THE PRACTICAL VOLT SENSITIVITY. This is defined as the p.d. in volts across the coil, required to produce 1 mm. deflection of the spot, the scale being at 1 metre from the mirror. We will call it s_v.

(f) RELATION BETWEEN THEORETICAL POTENTIAL SENSITIVITY AND PRACTICAL VOLT SENSITIVITY. As s_v is expressed in volts, we can write:

$s_1 = \frac{a_v}{s_v}$, if a_v is the coil deflection for s_v. Remembering that the coil deflection is half the spot deflection, and that 1 volt equals 10^8 e.m.c.g.s. units of potential, we have

$$s_1 = \frac{1}{2000} \times \frac{1}{10^8} \times \frac{1}{s_V}.$$

The volt and ampere sensitivities are often expressed as the number of millimetres of spot deflection per 1 μV across and 1 μA in the coil respectively, the scale being at 1 metre distance from the mirror.

(*g*) THE MEGOHM CONSTANT. This is defined as the number of megohms that must be connected in series with the galvanometer in order that 1 volt applied to the terminals produces a deflection of 1 mm. of the spot when the scale is 1 metre away from the mirror.

(4) Measurement of the Galvanometer Characteristics

(*a*) MEASUREMENT OF THE OSCILLATION PERIOD. The period can be measured either between two successive maxima deflections in the same direction, or between two successive zeros.

If the angular speed of the galvanometer is great, it is difficult to determine the precise time when the spot passes through zero. In this case, it is better to determine the period between two successive maxima in the same direction.

If the angular speed is small, the spot will remain for an appreciable time near the maximum. In this case, it is better to determine the time between two zeros.

The galvanometer is given an impulse either by discharging a condenser through it (arrangement of fig. 57), or by injecting a current from a suitable source, in series with a sufficiently high resistance, for a short time. (Care has to be taken to inject a very small current, or when discharging a condenser, not to have too great a charge, otherwise the spot will overshoot the scale, and the galvanometer might even be permanently damaged.) The time is measured with the aid of a stop-watch.

For greater accuracy, it is preferable to count several periods, and to divide the total time by the number of periods counted. This method is, however, tedious when the period is long, and it is therefore often productive of error.

(*i*) *Gauss's Method of Determining the Period.* We determine first an approximate time T_1 of the period by counting several oscillations with the aid of a stop-watch. Having determined T_1, the galvanometer is left oscillating for a certain time, say, t, but *the periods are not counted.*

If T_1 were the exact value of the period T, then $\frac{t}{T_1}$ would be an integer; generally, however, $\frac{t}{T_1}$ will be an integer N, plus or minus a certain fraction, but $\frac{t}{N} = T_2$ will be a value nearer to T than is T_1.

Having determined T_2, we can repeat the experiment by letting the galvanometer oscillate for a time, say, t_2, without counting the periods, and calculating a time T_3 nearer to T than is T_2.

We must, however, make sure that $N = \frac{t}{T_1}$ differs from an integer by less than 0·5, preferably 0·3 at the most, otherwise there will be a doubt as to whether N or N + 1 is equal to $\frac{t}{T_1}$.

As the exact value of the period is $T = T_1 \pm \gamma$, where γ is the error in the measurement of time, we have $NT = NT_1 \pm N\gamma$; $\frac{NT}{T_1} = N \pm \frac{N\gamma}{T_1}$, we must therefore have $\frac{N\gamma}{T_1} \leqq 0{\cdot}3$.

Example 14. If we assume that $T_1 = 15$ secs. and that the experimenter has found or knows that the maximum error on T, that is, $\gamma = \frac{1}{4}$ or $\pm \frac{1}{8}$ secs., then

$$N\gamma \leqq 0{\cdot}3 \times 15 \quad N \leqq \frac{0{\cdot}3 \times 15}{\frac{1}{4}} = 18,$$

t should therefore not exceed

$$t \leqq NT_1 \leqq 15 \times 18 = 270 \text{ secs.} = 4 \text{ min. } 30 \text{ secs.}$$

(*ii*) *Calculation of the Systematic Error.* The only error here is in the determination of the time at the beginning and at the end of the count. The magnitude of this error $\pm\gamma$ ($\pm \frac{1}{2}\gamma$ at the start and $\pm \frac{1}{2}\gamma$ at the end of the count) depends mainly on the care of the experimenter. If one period only is counted, the error is of course $\pm \gamma$; if N periods are counted the error becomes $\frac{\pm\gamma}{N}$. When using Gauss's method, and stopping at T_2, the error will be $\frac{\pm\gamma}{N}$; and if we go on further to, say, T_3, with another N_2 counts the error becomes $\frac{\pm\gamma}{N \times N_2}$.

(*b*) Measurement of the Logarithmic Decrement ϱ. As the ratios of successive deflections in opposite directions are

$$\frac{a_1}{a_2} = e^{\varrho}; \ \frac{a_2}{a_3} = e^{\varrho}; \ldots \quad \frac{a_n}{a_{n+1}} = e^{\varrho}, \text{ we have}$$

$$\frac{a_1}{a_n} = e^{(n-1)\varrho}, \text{ therefore } n - 1\varrho = log.\frac{a_1}{a_n}; \ \varrho = \frac{1}{n-1} log.\frac{a_1}{a_n}.$$

The galvanometer is given an impulse, and the deflections a_1 and a_n noted. The damping has to be kept constant during the measurement, because $\varrho = \frac{a_1 T}{2}$ and $a_1 = \frac{D}{2J}$ depend on the damping.

Calculation of the Systematic Error. The logarithmic differential of ϱ is

$$\frac{d\varrho}{\varrho} = \frac{d\left(\frac{1}{n-1}\right)}{\frac{1}{n-1}} + \frac{d\left(log.\frac{a_1}{a_n}\right)}{log.\frac{a_1}{a_n}},$$

but n can be treated as a constant, because there should normally be no error on n, so that

$$\frac{d\varrho}{\varrho} = \frac{d\left(log.\ \frac{a_1}{a_n}\right)}{log.\ \frac{a_1}{a_n}} = \frac{d\,(log.\ a_1 - log.\ a_n)}{log.\ \frac{a_1}{a_n}} = \frac{\frac{(d\,a_1)}{a_1} - \frac{(d\,a_n)}{a_n}}{log.\ \frac{a_1}{a_n}};$$

and as $da_1 = da_n$ we can write

$$\frac{d\varrho}{\varrho} = \frac{(d\,a_1)\left(\frac{1}{a_1} - \frac{1}{a_n}\right)}{log.\frac{a_1}{a_n}} = \frac{d\,a_1\left(1 - \frac{a_1}{a_n}\right)}{a_1\ log.\ \frac{a_1}{a_n}}$$

The error is therefore $\frac{\Delta\varrho'}{\varrho'} = \frac{\Delta a'_1\left(1 - \frac{a'_1}{a'_n}\right)}{a'_1\,log.\frac{a'_1}{a'_n}}$ and as we have seen, in the loss of charge method of measuring high resistances (p. 57), this error will be minimum when $\frac{a_1}{a_n} \doteqdot 3{\cdot}59$.

$\Delta a'_1$ is the reading error on the galvanometer.

(*c*) Measurement of the Moment of Inertia J and the Torsion Constant τ. Having found T and ϱ, T being measured on open circuit, we have $Ja_2 = \sqrt{\frac{\tau}{J} - \frac{D_1^2}{4J^2}}$, because $\frac{\tau}{J} > \frac{D_1^2}{4J^2}$ (oscillating condition) and $a_1 = \frac{2\varrho}{T} = \frac{D_1}{2J}$, therefore $a^2_2 = \frac{(2\pi)^2}{T^2} = \frac{\tau}{J} - \frac{(2\varrho)^2}{T^2}$, and

$$J = \frac{T^2}{4\,(\pi^2 + \varrho^2)} \times \tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

Adding to the moving part of the galvanometer a small moment of inertia J_1, by putting on the coil a small ebonite strip with two depressions, in which two small, non-magnetic, non-metallic balls can be placed (see fig. 55), we repeat the measurement of the period and of the logarithmic decrement, getting now the values, say, T_1 and ϱ_1. We shall have $J + J_1 = \dfrac{T_1^2}{4(\pi^2 + \varrho_1^2)}\tau$; combining this with (67),

$$J = \frac{T^2(\pi^2 + \varrho_1^2)}{T_1^2(\pi + \varrho^2) - T^2(\pi^2 + \varrho_1^2)} \times J_1.$$

Knowing J, we get τ from (67).

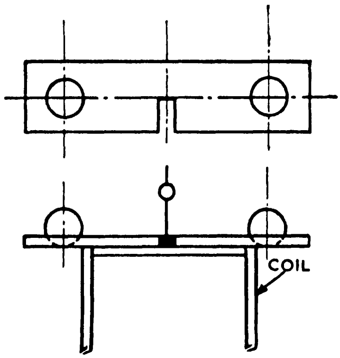

Fig. 55

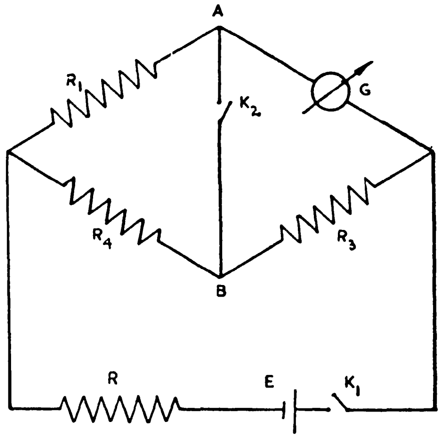

Fig. 56

As the damping on open circuit is usually small, except in instruments such as ammeters and voltmeters (an average value for ϱ and ϱ_1 is about 0·04), ϱ^2 and ϱ_1^2 can be neglected before π^2 and we have

$J = \dfrac{T^2}{4\pi^2}\tau$ and $J + J_1 = \dfrac{T_1^2}{4\pi^2}\tau$, from which we get

$$J = \frac{T^2}{T_1^2 - T^2}J_1\,;\ \tau = \frac{J_1\,4\pi^2}{T_1^2 - T^2}.$$

The errors on J and τ can be calculated in the usual way from the logarithmic differentials.

(*d*) Measurement of the Galvanometer Resistance. A suitable method is Kelvin's false zero method, the arrangement of which is shown in fig. 56. It is very similar to a Wheatstone bridge.

The galvanometer G, the resistance of which has to be measured, is in one arm of the bridge; the resistances R_1, R_3, R_4 make up the other arms, E is the source; R a suitable resistance to limit the current in the galvanometer; k_1 and k_2 are switches.

Manipulation. Close k_1, then there will be a current in the galvanometer, and the corresponding deflection will be, say, α. Close k_2, and vary the resistances R_1, R_3, R_4, until, with k_2 open or closed, the deflection remains α. When this happens, points A and B are at the same potential, and the balance conditions are the same as in the Wheatstone bridge; therefore $R_1R_3 = gR_4$; $g = \frac{R_1}{R_4} \cdot R_3$, where g is the galvanometer resistance.

It is preferable to interpolate around the deflection α instead of trying to keep α constant; the interpolation and calculation of the error are done in the same way as for the Wheatstone bridge.

(*e*) Measurement of the Critical Resistance R_c and the Resistance R_π Corresponding to Near Limit of Oscillating Conditions. The arrangement is shown in fig. 57.

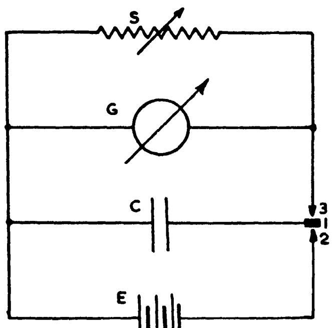

Fig. 57

(*i*) *Manipulation.* Make 1–2, charging the condenser C from the source E, then break 1–2 and make 1–3, discharging C through the galvanometer, and open 1–3 immediately after the discharge. The galvanometer, having received an impulse, starts with a speed ω_0, attains a maximum deflection α_m, and then comes back to zero, without passing it if the conditions are non-oscillatory. The galvanometer will start to oscillate if the conditions are oscillatory.

Start with the variable resistance S high enough to have oscillatory conditions, then gradually diminish S while the condenser C is being charged and discharged for each value of S, until the ratio of the first to the second deflection (opposite direction) is $e^{\pi} = 23$. When this happens, we have

$R_\pi = g + S_\pi$, where g is the galvanometer resistance and S_π the resistance of S for the condition $\varrho = \pi$.

Having determined R_π, diminish S still more (increasing the damping), until the spot just comes back to zero without passing it; then

$R_c = g + S_c$, where S_c is the resistance of S for critical conditions.

(*ii*) *Calculation of the Systematic Error.* The logarithmic differential of R_π or R_c is

$$\frac{dR}{R} = \frac{d(g+S)}{g+S} = \frac{dg}{g+S} + \frac{dS}{g+S} = \frac{g}{g+S}\frac{dg}{g} + \frac{S}{g+S}\frac{dS}{S},$$

and the error will be

$$\frac{\Delta R'}{R'} = \frac{g'}{g' + S'}\,\frac{\Delta g'}{g'} + \frac{S'}{g' + S'}\,\frac{\Delta S'}{S'}.$$

When calculating the error on R, ΔS is the known error of S (constructional error) plus the determination Δ_DS, which is the value, plus or minus, by which S can be changed without a noticeable change in the behaviour of the spot.

When calculating the error on R_π, Δ_DS is much more difficult to assess because of the reading error on the galvanometer, which here has to be translated into corresponding changes in S; very careful manipulation is necessary.

(*f*) MEASUREMENT OF THE FLUX LINKAGES, FLUX, FIELD IN THE AIRGAP AND THE CURRENT AND VOLT PRACTICAL SENSITIVITIES. The arrangement is shown in figs. 58(*a*) and (*b*).

G is the galvanometer, of resistance g; S a variable resistance shunting G; R a suitable variable resistance; E a suitable source of voltage E.

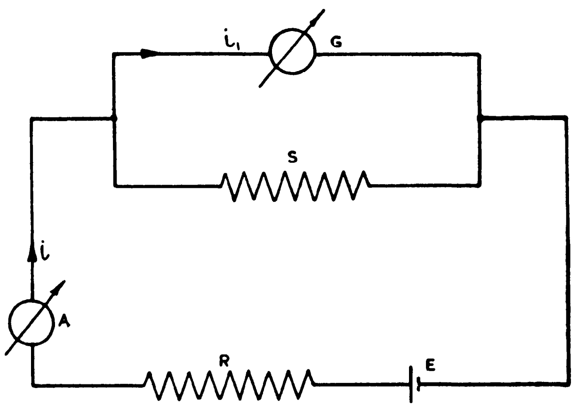

Fig. 58(*a*)

With the connections of fig. 58(*a*), we have

$$i = \frac{E}{R + \dfrac{gS}{g + S}}\,; \quad i_1 = \frac{S}{g + S}\cdot i\,;$$

E is measured by a suitable instrument.

If a be the deflection for i_1, we shall have

$$\tau a = \Phi_0 i_1\,; \quad \Phi_0 = \frac{\tau a}{i_1} = \tau a \times \frac{(g + S)}{Si} \qquad . \quad . \quad (68)$$

As τ has been determined, we have Φ_0.

If we know the number of turns on the galvanometer coil, say, n, then $\Phi = \dfrac{\Phi_0}{n}$; if we know the dimensions of the coil, say, l and w, $H = \dfrac{\Phi}{l \cdot w}$.

We can get the theoretical current sensitivity knowing that

$s = \frac{\Phi_0}{\tau} = \frac{a}{i_1}$, where a is in radians and i_1 in e.m.c.g.s. units, and the

practical current sensitivity from $s_p = \frac{i_1}{m}$, where m is the spot deflection in mm. and i_1 the current in amps.

Knowing the galvanometer resistance g, we get the practical volt sensitivity from $s_v = s_p \cdot g$.

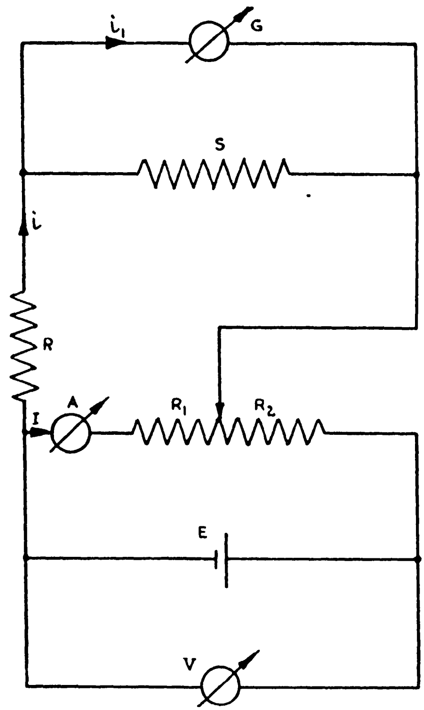

Fig. 58(*b*)

The arrangement of fig. 58(*a*) is not suitable for very sensitive galvanometers, since i_1 has to be very small for a suitable deflection, S has to be small and R very great; there can therefore be no precision in the measurement (as can be seen when giving the equation for the logarithmic differential of (68)). The arrangement of fig. 58(*b*) is preferable.

The resistances R_1 and R_2 can be varied by altering the position of the sliding contact; with the notations of fig. 58(*b*) we have

$E = I\,R_1 + (I + i)\,R_2$; $I\,R_1 = i\,R_T$ where $R_T = R + \frac{Sg}{S+g}$, so that

$E = i\,R_T + i\,R_2 + I\,R_2$; $i = \frac{E - I\,R_2}{R_T + R_2}$.

As $i_1 = \frac{S}{S+g}\,i$, we get $i_1 = \frac{S}{(S+g)}\,\frac{E - I\,R_2}{R_T + R_2}$.

The value of the flux linkages is

$$\Phi_0 = \frac{\tau a}{i_1} = \frac{\tau a\,(g + S)\,(R_T + R_2)}{S\,(E - I\,R_2)}.$$

As for the arrangement of fig. 58(a) we have also

$$\Phi = \frac{\Phi_0}{n};\quad H = \frac{\Phi}{l \cdot w};\quad s = \frac{\Phi_0}{\tau};\quad s_p = \frac{i_1}{m};\quad s_V = \frac{i_1}{m} \times g.$$

The source E in fig. 58(b) has to be an accumulator because care has to be taken that E does not vary during the measurement. The milliammeter A measuring I has its resistance included in R_1 ; the resistance of A need not be known. E is checked by the voltmeter V.

(g) Simple Determination of the Galvanometer Constants.[1] Generally the open-circuit damping is very small, and when this damping is neglected, the galvanometer constants can be determined very quickly from the measurement of the period T, the critical resistance R_c, and the practical current sensitivity s_p. We have

$$T = 2\pi\sqrt{\frac{J}{\tau}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

(at critical conditions $\frac{D^2}{4J^2} = \frac{\tau}{J}$).

All damping being by currents induced in the coil $a_1 = \frac{D}{2J} = \frac{\Phi_0{}^2}{2JR_c}$, it follows that $\frac{\tau}{J} = \frac{\Phi_0{}^4}{4J^2R_c{}^2}$; therefore

$$R_c = \frac{\Phi_0{}^2}{2\sqrt{TJ}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

As

$$s = \frac{a_p}{s_p} = \frac{1}{200} \times \frac{1}{s_p} = \frac{\Phi_0}{\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

[1] For further details, see Bedeau: *Cours de Mesures Electriques* (S.F.E.; E.S.E.; Vol. I).

Combining (69) and (70) we get

$$R_cT = \frac{\Phi_0^2}{2\sqrt{\tau J}} \times 2\pi\sqrt{\frac{J}{\tau}} = \frac{\pi\Phi_0^2}{\tau} \quad . \quad . \quad . \quad . \quad (72)$$

Combining (71) and (72) we get

$$R_cT = s\pi\Phi_0 ; \quad \Phi_0 = \frac{R_cT}{s\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (72a)$$

$$\text{As } \tau = \frac{I\,\Phi_0}{a} = \frac{\Phi}{s} = \frac{R_cT}{s^2\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

$$\text{As } T^2 = \frac{4\pi^2 J}{\tau} ; \quad J = \frac{T^2\tau}{4\pi^2} = \frac{R_cT^3}{4\pi^3 s^2} \quad . \quad . \quad . \quad . \quad (73a)$$

Example 15. A galvanometer on test gave the following results: period on open circuit T = 8 secs., critical resistance R_c = 2000 ohms, practical current sensitivity $s_p = 2 \times 10^{-9}$ amps. Find the galvanometer constants, neglecting the damping on open circuit.

The theoretical current sensitivity is

$$s = \frac{1}{200} \times \frac{1}{2 \times 10^{-9}} = 0{\cdot}25 \times 10^7 \text{ radians/e.m.c.g.s. unit of current.}$$

The critical resistance in e.m.c.g.s. units is

$$R_c = 2000 \times 10^9 = 2 \times 10^{12}.$$

The flux linkages are

$$\Phi_0 = \frac{R_cT}{s\pi} = \frac{2 \times 10^{12} \times 8}{0{\cdot}25 \times \pi 10^7} \doteqdot 20{\cdot}38 \times 10^5 \text{ Maxwell turns.}$$

The torsion constant is

$$\tau = \frac{R_cT}{s^2\pi} = \frac{2 \times 10^{12} \times 8}{(0{\cdot}25 \times 10^7)^2\pi} = 0{\cdot}815 \text{ dyne-cm.-radian.}$$

The moment of inertia is

$$J = \frac{R_cT^3}{4\pi^3 s^2} = \frac{2 \times 10^{12} \times 8^3}{4\pi^3(0{\cdot}25 \times 10^7)^2} = 1{\cdot}32 \text{ gm.-cm.}^2$$

(5) The Ballistic Galvanometer

A galvanometer is said to be ballistic when its period of oscillation is great compared with the time of a discharge through it, so that the galvanometer is practically still at rest when the discharge is finished. The galvanometer receives an impulse, and starts from rest with a speed ω_0. It follows that $T = 2\pi\sqrt{\frac{J}{\tau}}$ being great, $\frac{J}{\tau}$ has to be great; that is, the moment of inertia has to be great and the control constant small.

The general equation of the galvanometer being

$$J\frac{d^2a}{dt^2} + D\frac{da}{dt} + \tau a = \Phi_0 i;$$

i being the injected variable current of the discharge. If we integrate between the time $t = O$ at the beginning of the discharge, to $t = t$ at the end of the discharge, we have

$$J\int_O^t \frac{d^2a}{dt^2}\cdot dt + D\int_O^t \frac{da}{dt}\cdot dt + \tau\int_O^t a\,dt = \Phi_0\int_O^t idt \qquad (74)$$

Now, $J\int_O^t \frac{d^2a}{dt^2} \times dt = J\left(\frac{da}{dt}\right)_O^t = J\frac{da}{dt}$ at t, minus $J\frac{da}{dt}$ at $O = \omega_0$,

because the angular speed is zero at $t = O$ and ω_0 at t.

$D\int_O^t \frac{da}{dt} \times dt = Da$ at t, minus Da at $O = O$, because the galvanometer is at rest at $t = O$ and at $t = t$.

$\tau\int_O^t a\,dt = \tau at - \tau aO = O$, a, being zero at $t = O$ and at $t = t$.

$\int_O^t i\,dt = q$, where q is the charge received by the galvanometer.

(74) can therefore be written

$$J\omega_0 = \Phi_0 q \text{ and } \omega_0 = \frac{\Phi_0 q}{J}; \quad q = \frac{J\omega_0}{\Phi_0} \qquad (75)$$

When the conditions are critical, we get as the maximum deflection

$a_m = \frac{\omega_0}{\sqrt{\frac{\tau}{J}}} \times \frac{1}{e}$; and, combining with (75), we get $a_m = \frac{\Phi_0 q}{\sqrt{\tau J}} \times \frac{1}{e}$.

(*a*) THE THEORETICAL CHARGE SENSITIVITY. The theoretical charge sensitivity, which is defined as the number of radians by which the coil will be deflected per e.m.c.g.s. unit of charge, will be

$$\frac{a_m}{q} = \frac{\Phi_0}{\sqrt{\tau J}} \times \frac{1}{e} = s_c.$$

(*b*) THE PRACTICAL COULOMB SENSITIVITY. This is defined as the number of coulombs necessary to produce a deflection of 1 mm. of the spot, the scale being at 1 metre distance from the mirror. It will be denoted by s_{cp}.

(*c*) RELATION BETWEEN THE THEORETICAL CHARGE SENSITIVITY AND THE THEORETICAL CURRENT SENSITIVITY. As

$$s = \frac{\Phi_0}{\tau}, \text{ we have } \frac{s_c}{s} = \frac{\Phi_0}{\sqrt{\tau J}\, e} \div \frac{\Phi_0}{\tau} = \frac{\tau}{\sqrt{\tau J}\, e} = \sqrt{\frac{\tau}{J}} \times \frac{1}{e}.$$

(*d*) RELATION BETWEEN THE PRACTICAL SENSITIVITY IN COULOMBS AND THE PRACTICAL SENSITIVITY IN AMPS. As s_{cp} is a charge, we can write $s_c = \frac{a_{cp}}{s_{cp}}$, if a_{cp} is the coil deflection in radians corresponding to s_{cp}; we have seen that $s = \frac{a_p}{s_p}$, therefore $\frac{s_c}{s} = \frac{s_p}{s_{cp}} = \sqrt{\frac{\tau}{J}} \times \frac{1}{e}$.

We have also $T = 2\pi\sqrt{\frac{J}{\tau}}$, if the open-circuit damping is neglected, so that

$$s_{cp} = \frac{s_p T e}{2\pi} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (75a)$$

When working near the oscillation limit ($\varrho = \pi$), the maximum deflection is 1·235 times the deflection at critical conditions, so that when $\varrho = \pi$, we get

$$s_{c\pi} = \frac{a_{m\pi}}{q} = 1{\cdot}235 \frac{\Phi_0}{\sqrt{\frac{\tau}{J}}} \times \frac{1}{e},$$ where $s_{c\pi}$ is the charge sensitivity when $\pi = \varrho$.

(6) The Shunted Ballistic Galvanometer

Consider fig. 59. The ballistic galvanometer, of resistance g and inductance l, is shunted by a resistance R which has an inductance L; a discharge is produced across *a,a*.

Let the instantaneous values of the currents be as shown in fig. 59; then $Ri_1 + L\frac{di_1}{dt} = v$; $gi + l\frac{di}{dt} + n\Phi\frac{da}{dt} = v$, where v is the value of the instantaneous voltage across *a,a*; n is the number of turns on the galvanometer coil; and Φ the flux from the permanent magnet through the coil. Then $Ri_1 + L\frac{di_1}{dt} = gi + l\frac{di}{dt} + n\,\Phi\frac{da}{dt}$; and integrating between the time $t = 0$, beginning of the discharge, to $t = t$, end of the discharge, we have

$$R\int_0^t i_1 dt + L\int_0^t \frac{di_1}{dt}dt = g\int_0^t idt + l\int_0^t \frac{di}{dt} \times dt + n\int_0^t \Phi\frac{da}{dt} \times dt \quad (76)$$

The terms of (76) which are multiplied by L and l are zero, because the currents i_1 and i are zero at $t = 0$ and at $t = t$. The last term on the right-hand side of (76) is zero, because there is no flux cut by the coil at $t = 0$ as at $t = t$, considering that the coil did not move during t and the change $d\Phi$ is zero, so that

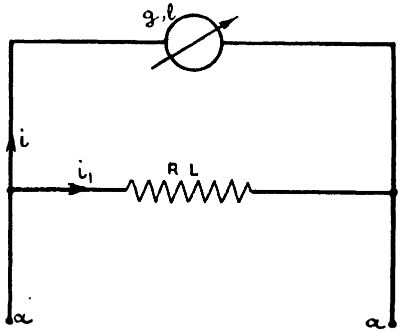

Fig. 59

$$R\int_0^t i_1 dt = g\int_0^t idt\,;\quad Rq_1 = gq \text{ and } \frac{q}{q_1} = \frac{R}{g},$$

or the charges in the galvanometer and the shunt, divide inversely as the resistance of the galvanometer and the shunt. The charges are independent of the inductances. As the total charge across a,a is $Q = q_1 + q$, we have

$$\frac{q}{Q - q} = \frac{R}{g}\,;\quad q = \frac{R}{R + g} \times Q \quad . \quad . \quad . \quad . \quad (77)$$

Measurement of the Practical Coulomb Sensitivity (Calibration). The arrangement is shown in fig. 60.

R_1 and R_2 are variable resistances; the sum of $R_1 + R_2$ has, however, to be kept constant, in order that the damping should be constant. C is a standard condenser, E a suitable source, and G is the galvanometer of resistance g.

(*i*) *Manipulation.* Set R_1 and R_2 so as to have the proper resistance for the required damping; the value of R_2 depends on the galvanometer sensitivity. Make 1–2, charging C to the e.m.f. E, then the charge on C is $Q = C\,E$.

Break 1–2 and make 1–3, discharging C through R_2, which, relative to C, is in parallel with $g + R$; from (77), the charge passing through the galvanometer is $q = \frac{R_2}{R_2 + g + R_1} \cdot C\,E = m\,s_{cp}$; m is the deflection of the spot in mm., so that $\frac{q}{m} = s_{cp}$, which is the practical galvanometer sensitivity in coulombs.

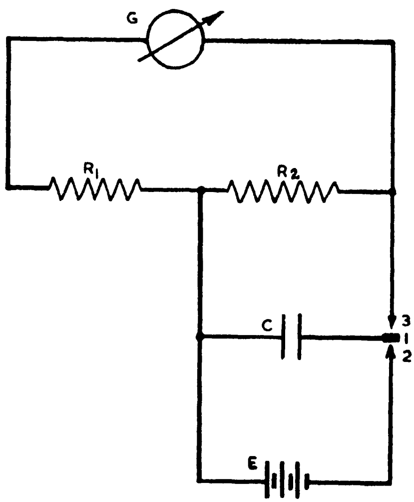

Fig. 60

The value s_{cp} is sometimes referred to as the galvanometer constant. It will, however, be found that s_{cp} is not always a constant throughout the whole length of the scale.

If C, E and the resistances are not known with sufficient accuracy, it is preferable to measure the practical current sensitivity and to calculate s_{cp} from (75a).

(*ii*) *Calculation of the Systematic Error.* The logarithmic differential of s_{cp} is

$$\frac{ds_{cp}}{s_{cp}} = \frac{dC}{C} + \frac{dE}{E} + \frac{dR_2}{R_2} - \frac{dR_2}{R_2 + g + R_1} - \frac{dg}{R_2 + g + R_1} - \frac{dR_1}{R_2 + g + R_1} - \frac{dm}{m} =$$

$$\frac{dC}{C} + \frac{dE}{E} + \frac{dR_2}{R_2} - \frac{R_2}{R_2 + g + R_1}\frac{dR_2}{R_2} - \frac{g}{R_2 + g + R_1}\frac{dg}{g} - \frac{R_1}{R_2 + g + R_1}\frac{dR_1}{R_1} - \frac{dm}{m} =$$

$$\frac{dC}{C} + \frac{dE}{E} + \frac{dR_2}{R_2}\left(\frac{g + R_1}{R_2 + g + R_1}\right) - \frac{g}{R_2 + g + R_1}\frac{dg}{g} - \frac{R_1}{R_2 + g + R_1}\frac{dR_1}{R_1} - \frac{dm}{m}.$$

The relative error is therefore

$$\frac{\Delta s'_{cp}}{s'_{cp}} = \frac{\Delta C'}{C'} + \frac{\Delta E'}{E'} + \frac{\Delta R'_2}{R'_2}\left(\frac{g' + R'_1}{R'_2 + g' + R'_1}\right) + \frac{g'}{R'_2 + g' + R'_1}\,\frac{\Delta g'}{g'} + \frac{R'_1}{R'_2 + g' + R'_1}\,\frac{\Delta R'_1}{R'_1} + \frac{\Delta m'}{m'}.$$

$\Delta E'$ is the error in the known value of E, which, if measured with a voltmeter, will include the constructional plus the reading error on the instrument.

Example 16. "Why is it advisable to short-circuit the terminals of a sensitive moving-coil instrument during transport?

"The coil of a moving coil meter is wound on a non-conducting former whose height and width are both 2 cm. It moves in a constant field of 1200 lines per cm.² The moment of inertia of its moving parts is 2·5 gm.-cm.², and the control spring exerts a torque of 300 dyne-cm. per radian. Calculate (*a*) how many turns must be wound on the coil to produce a deflection of 150° with a current of 10 mA.; (*b*) the resistance of the coil to produce critical damping, all damping being taken as electro-magnetic" (University of London, Electr. Measur., and Measur. Instruments, B.Sc. Final, 1948, paper 2, question 7).

If the instrument is not short-circuited, the only damping is the open-circuit damping, so that the moving part of the instrument will oscillate when the instrument is moved. These oscillations might become very violent, and damage might result. When shorted, the conditions will be non-oscillatory and there is much less danger of damage.

The surface of the coil is $S = 2 \times 2 = 4$ cm.², neglecting the thickness of the wire.

The flux through the coil is $\Phi = 4 \times 1200 = 4800$ lines per cm.²

From $\tau a = \Phi_0 I$ we have

$$\Phi_0 = \frac{300 \times 150}{57{\cdot}3} \times \frac{1}{\dfrac{10}{1000 \times 10}} = 0{\cdot}785 \times 10^6 \text{ linkages};$$

converting 150° to radians by dividing by 57·3 and $\frac{10}{1000}$ amps into e.m.c.g.s. units of current by dividing by 10.

The number of turns on the coil is

$$n = \frac{\Phi_0}{\Phi} = \frac{0{\cdot}785 \times 10^6}{4800} = 163{\cdot}5.$$

Open-circuit damping being neglected, we have

$$R_c = \frac{\Phi_0^2}{2\sqrt{JT}} = \frac{(0{\cdot}785 \times 10^6)^2}{2\sqrt{300 \times 2{\cdot}5} \times 10^9} = 11{\cdot}2\Omega$$

(dividing by 10^9 to get R_c in ohms).

Example 17. "The combined resistance of the coil and springs of a moving coil d.c. instrument is 1·1 ohms. The full-scale deflection of 90° is caused by a current of 12·5 mA. It is desired to adjust this movement so that its effective resistance measured at the terminals of the shunt is 5 ohms, and the full-scale deflection is produced by 15 mA. What resistances are required? Calculate the periodic time of this instrument on the assumption that no other form of electromagnetic damping is provided and air damping is negligible. The constant of the control springs is 313 dyne-cm. per radian and the moment of inertia of the movement 4·5 gm-cm.². The equation of motion for the instrument is given by

$$J\frac{d^2\theta}{dt^2} + \left(\frac{G^2}{R_g + R + R_s}\right)\frac{d\theta}{dt} + U\theta = Gi,$$

where Rg, R, and R_s are the movement resistance, series resistance and shunt resistance respectively" (University of London, Electr. Measur., and Measur. Instruments, B.Sc. Final, 1947, paper 1, question 3).

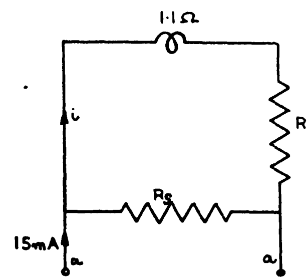

Fig. 61

One way of adjusting the movement is to add a resistance R in series with the moving coil and a shunt S across the instrument (fig. 61).

Before adjustment, a current of 12·5 mA. produces full-scale deflection, and this current is the coil current; after adjustment, as the coil itself is not changed, we must have the same current in the coil for full-scale deflection.

The p.d. across the coil is $\frac{1{\cdot}1 \times 12{\cdot}5}{1000} = \frac{13{\cdot}75}{1000}$ V.

As after adjustment the total resistance of the instrument and its shunt is 5 ohms, we have

$$\frac{5 \times 15}{1000} = \frac{(1{\cdot}1 + R)\,12{\cdot}5}{1000}; \quad 75 = 13{\cdot}75 + 12{\cdot}5R,$$

from which we get $R = \frac{75 - 13{\cdot}75}{12{\cdot}5} = \frac{61{\cdot}25}{12{\cdot}5} = 4{\cdot}9\Omega.$

The current in S being $\frac{15}{1000} - \frac{12 \cdot 5}{1000} = \frac{2 \cdot 5}{1000}$ mA., we have

$$S = \frac{5 \times 15}{1000} \div \frac{2 \cdot 5}{1000} = \frac{75}{2 \cdot 5} = 30\Omega.$$

The damping resistance is therefore $R_D = 1 \cdot 1 + 4 \cdot 9 + 30 = 36\Omega$. The flux linkages are

$$\Phi_0 = \frac{\tau a}{I} = \frac{\dfrac{313 \times 90}{57 \cdot 3}}{\dfrac{12 \cdot 5}{1000 \times 10}} = \frac{492 \times 10^4}{12 \cdot 5} = 39 \cdot 35 \times 10^4 \text{ Maxwell-turns.}$$

The letters G and U stand for Φ_0 and τ in our notation.

The damping constant will be

$$D = \frac{\Phi_0{}^2}{R_D} = \frac{(39 \cdot 35 \times 10^4)^2}{36 \times 10^9} = \frac{43 \times 10^8}{10^9} = 4 \cdot 3 \text{ ;}$$

we have therefore $Ja_2 = \sqrt{\frac{\tau}{J} - \frac{D^2}{4J^2}} = \sqrt{\frac{313}{4 \cdot 5} - \frac{(4 \cdot 3)^2}{4 \times (4 \cdot 5)^2}} = 6 \cdot 83$

and the period is $T = \frac{2\pi}{a_2} = \frac{2\pi}{6 \cdot 83} = 0 \cdot 92$ sec.

Example 18. " What is meant by critical damping in a measuring instrument? Deduce from the equation of motion of a moving-coil meter, an expression for the effective resistance of the coil necessary for critical damping, taking all damping to be electromagnetic.

" The coil of a moving-coil meter has 100 turns, wound on a non-conducting former, its width being 2 cm. and its height 3 cm. It works in a constant field of 1170 lines per cm.^{2} The moment of inertia of the moving parts is 5 gm.-cm.^{2}, and the control spring produces a torque of 500 dyne-cm. per radian. Calculate (*a*) the current in the coil to produce a deflection of 120°, and (*b*) the resistance of the coil to produce critical damping, assuming all damping to be electromagnetic " (University of London, Electr. Measur. and Measur. Instruments, B.Sc. Final, 1946, paper 1, question 8).

The first part of the question can be answered from the text.

The surface of the coil is $S = 2 \times 3 = 6$ cm.^{2}

The flux is $\Phi = 1170 \times 6 = 7020$ Maxwells (or lines).

The flux linkages are $\Phi_0 = 7020 \times 100 = 702000$ Maxwell turns (line turns).

From $\tau a = \Phi_0 I$, we get $I = \dfrac{500 \times 120}{57 \cdot 3 \times 702000} = 0 \cdot 00149$ e.m.c.g.s. units of current or 0·0149 amps.

As all damping is electromagnetic, and as the former is nonconductive, we have $R_c = \dfrac{\Phi_0^2}{2\sqrt{J\tau}} = \dfrac{702000^2}{2\sqrt{500 \times 5} \times 10^9} = 4 \cdot 93 \Omega$.

Example 19. " A moving coil reflecting galvanometer has a rectangular coil of 750 turns wound upon a former of non-conducting material. The mean breadth of the coil is 1·3 cm. The effective depth of the coil is 4·5 cm. and it is situated in a radial air-gap in which the flux density is 1200 lines per cm.^{2}

" The current sensitivity of the instrument is 0·001 micro-ampere per millimetre at 1 metre, and the undamped period is 7 seconds. Calculate the control torque of the suspension, the moment of inertia of the coil and the total circuit resistance which just renders the instrument dead beat. The effect of air damping can be neglected " (University of London Electr. Measur. and Measur. Instruments, B.Sc. Final, 1944, paper 1, question 4).

The question undoubtedly assumes that all damping, not only air damping, on open circuit is negligible. The expression " just dead beat " means critical conditions.

The coil surface is $S = 1 \cdot 3 \times 4 \cdot 5 = 5 \cdot 85$ cm.^{2}

The flux is $\Phi = 1200 \times 5 \cdot 85 = 7020$ Maxwells.

The flux linkages $\Phi_0 = 7020 \times 750 = 52 \cdot 6 \times 10^5$.

The theoretical current sensitivity is $s = \dfrac{1}{200} \times \dfrac{1}{s_p} = 0 \cdot 5 \times 10^7$ radians per e.m.c.g.s. units of current.

From (72a) we get $R_c = \dfrac{52 \cdot 6 \times 10^5 \times 0 \cdot 5 \times 10^7 \times \pi}{7} = 11 \cdot 80 \times 10^{12}$ e.m.c.g.s. units of resistance or $\dfrac{11 \cdot 80 \times 10^{12}}{10^9} = 11 \cdot 80 \times 10^3 \Omega$.

From (73) we get the torsion constant $\tau = \dfrac{11 \cdot 80 \times 10^{12} \times 7}{(0 \cdot 5 \times 10^7)^2 \pi} =$ 1·052 dyne-cm. per radian ; and from (73a) we have

$$J = \frac{11 \cdot 80 \times 10^{12} \times 7^3}{4\pi^3 \times 0 \cdot 25 \times 10^{14}} = 1 \cdot 3 \text{ gm./cm.}^2$$

Example 20. " Describe briefly one modern type of d.c. potentiometer suitable for precision measurements, and suggest a method of using the potentiometer to calibrate a 300-volt d.c. voltmeter over the whole range. Assuming that the resistance of the moving-coil galvanometer used to indicate balance is proportional to the square of the number of turns on the moving coil, deduce an expression for the resistance of the galvanometer to give maximum sensitivity when standardising the potentiometer with a standard cell of resistance R ohms " (University of London. Electr. Measur. and Measur. Instruments, B.Sc. Final, 1943, paper 2, question 2).

The first part of the question can be answered by reference to the text dealing with d.c. potentiometers.

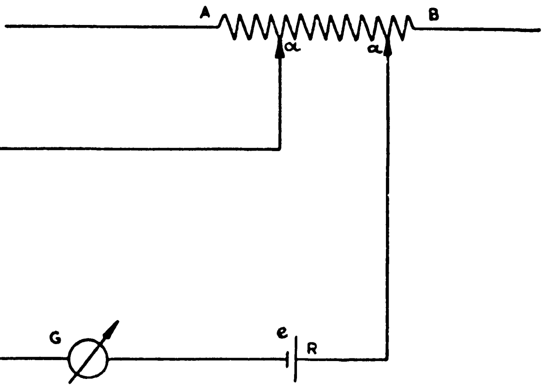

Fig. 62

Consider fig. 62. Let AB be the potentiometer resistance, and *a,a* that part of it which balances the standard cell of internal resistance R. If *i* is the current in *a,a,* due to the unbalance only, then the resultant e.m.f. in *e*, G, *a*, *a*, *e* is $e_1 = e - ir$, where *r* is the resistance between *a,a*. We can look upon the circuit as having an e.m.f. e_1 of resistance R in series with the galvanometer of resistance, say, $g\Omega$ per turn on the galvanometer coil. If the coil has *n* turns, the current in the circuit will be $i_1 = \frac{e_1}{R + gn^2}$. The torque on the coil is $\tau_1 = \tau a = \Phi n i_1$ which can be written $\tau_1 = \Phi n \frac{e_1}{R + gn^2}$.

Differentiating with respect to *n* and equating to zero we get

$$\frac{d\tau_1}{dn} = \frac{(R + gn^2)\,\Phi_1 e - \Phi n e_1\, 2gn}{(R + gn^2)^2} = 0$$

or: $R + gn^2 = 2gn^2$, that is $R = gn^2$. The torque, that is the sensitivity, is maximum when the galvanometer resistance is equal to the resistance of the standard cell.

Moving-magnet Galvanometers

(7) The Tangent Galvanometer

The principle is shown in fig. 63.

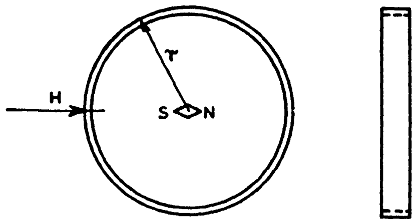

Fig. 63

A very small magnetic needle is placed in the centre of a circular coil of radius r. The suspension of raw silk (cocoon) is practically torsionless (torsion constant equal to zero). The plane of the coil is in the magnetic meridian, so that the restoring torque is produced by the action of the earth's field on the needle.

When a current I passes through the coil, the needle will be deflected due to the magnetic action of the coil.

The needle has to be small in order to have a magnetic field as constant as possible in the space occupied by the needle.

Let the horizontal component of the earth's field be H, and the magnetic moment of the needle M ; if a current I circulates through the coil of n turns, the field at the centre of the coil, that is the field acting on the needle, and due to the coil, is $F = \frac{2\pi n I}{r}$, and the field per unit current in the coil is $\frac{2\pi n}{r} = k$, say.

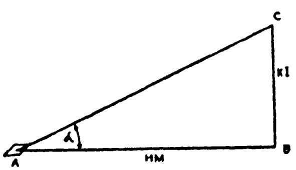

Fig. 64

Owing to the action of the earth's field and the field of the coil, the needle will deflect from its zero position AB to AC (fig. 64).

The torque on the needle due to the current I is IkM*cos.a*, the torque due to the earth's field HM*sin.a* ; and when the needle reaches the equilibrium position corresponding to I, we must have

$$\text{I}k\text{M}\cos.a = \text{HM}\sin.a, \text{ that is, } \text{HM}\tan.a = k\text{MI}.$$

As the deflections a are very small, we can write $\tan.a \doteqdot a$, so that $\text{HM}a = k\text{MI}$ and $\text{I} = \frac{\text{H}a}{k}$; $a = \frac{k\text{I}}{\text{H}}$.

When the needle is moving (before taking up its position corresponding to I) there will be a variation of flux through the coil, and therefore

an induced current in it; let this flux linkage variation be $d\Phi_0$ for a variation of the angle da, and let the induced current be i_1, i_1 reacting on the needle will produce a torque $kMi_1 cos.a$.

Again, as a is small, $cos.a \cong 1$, so that the torque is kMi_1.

When the needle turns by da the work done is $kMi_1\, da$, and this work has to be equal to the work of the electromagnetic forces $i_1\, d\Phi_0$, so that

$$kMI_1\, da = i_1\, d\Phi_0 \text{ and } kM\frac{da}{dt} = \frac{d\Phi_0}{dt} \qquad (78)$$

When a p.d. v is applied to the galvanometer coil of resistance R and inductance L we have

$$Ri + L\frac{di}{dt} + \frac{d\Phi_0}{dt} = v;$$

i is the instantaneous current in the coil; it is the sum of the injected current I and the current i_1 induced in the coil by the movement of the needle; $i = I + i_1$.

$\frac{di}{dt}$ being small compared with the angular speed $\frac{da}{dt}$, we can neglect the term $\frac{Ldi}{dt}$, and write

$$Ri = v - \frac{d\Phi_0}{dt};\; i = \frac{v}{R} - \frac{1}{R}\frac{d\Phi_0}{dt} \qquad (79)$$

Putting in (79) the value of $\frac{d\Phi_0}{dt}$ from (78), we get

$$i = \frac{v}{R} - \frac{1}{R}kM\frac{da}{dt} = I - \frac{1}{R}kM\frac{da}{dt}.$$

If the moment of inertia of the moving part relative to the suspension axis is J and the damping constant D_1, we have

$$J\frac{d^2a}{dt^2} + D_1\frac{da}{dt} + MHa = Mki,$$

and substituting for i from (79)

$$J\frac{d^2a}{dt^2} + D_1\frac{da}{dt} + MHa = Mk\frac{v}{R} - \frac{M^2k^2}{R}\frac{da}{dt}$$

$$J\frac{d^2a}{dt^2} + \left(D_1 + \frac{M^2k^2}{R}\right)\frac{da}{dt} + MHa = Mk\frac{v}{R} = MkI.$$

This equation is similar to (43a) of the permanent-magnet moving-coil galvanometer, and the same solutions therefore apply.

In the arrangement of fig. 63, the field created by the coil is not

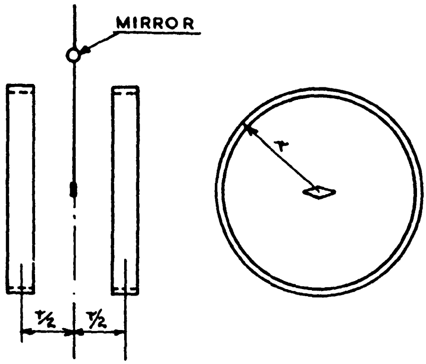

Fig. 65

uniform over all the space occupied by the needle; a better arrangement is shown in fig. 65.

Two coils of radius r each are used; the distance between the coils is exactly equal to their radius r, and the magnetic needle is in the centre between the coils. The field in the space occupied by the needle is much more uniform than in fig. 63.

It can be shown that the field in the centre o of fig. 65 is

$$H = \frac{32}{\sqrt{125}} \frac{\pi n I}{r},$$

and the current I for a deflection a is

$$I = \frac{\sqrt{125}\, r\, H}{32\, \pi n} \tan. a \simeq \frac{\sqrt{125}\, r H a}{32\, \pi n}.$$

(8) The Thomson or Kelvin Galvanometer

In order to increase the sensitivity of the moving-magnet galvanometer, and to make it independent of the earth's field, which is variable

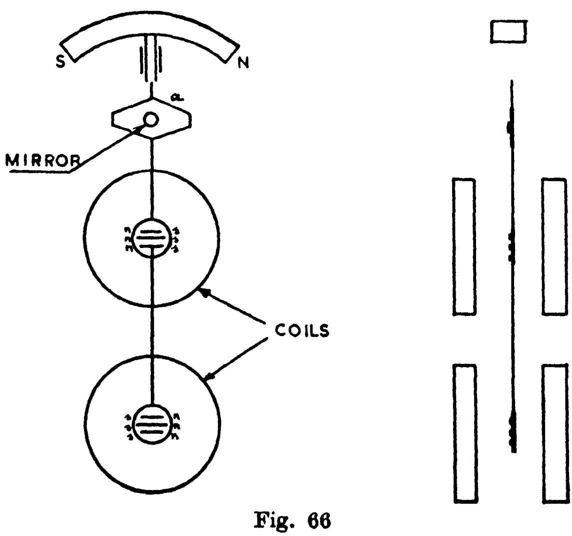

Fig. 66

in time and space, the arrangement shown in fig. 66, designed by Lord Kelvin, is used.

There are two magnets, or groups of magnets, disposed as shown; each group of magnets is placed in the centre between a pair of coils (one pair of coils for one group of magnets alone can also be used).

If the two groups of magnets are identical, the torque due to the earth's field will act in opposite directions on each group, and the resultant torque will be zero; the system is then said to be astatic.

The permanent magnet NS provides the restoring torque, and by changing the position of this magnet relative to the two groups of magnetic needles, the sensitivity, as well as the zero position of the galvanometer, can be altered.

The two groups of magnetic needles are stuck on discs of mica, damping is provided by a larger piece of mica.

(9) The Broca Galvanometer

The arrangement is shown in fig. 67.

In order to diminish the inertia of the moving system, two thin long magnetic needles are disposed as shown. The needles have consequent poles *ns*, *ns*, as shown, in the centre between a pair of coils. The earth's field acting on NS, N_1S_1 has no effect. The permanent magnet N_2S_2 acts on N_1S_1, providing the restoring torque, and by changing the position of N_2S_2 the sensitivity as well as the zero position of the galvanometer can be altered.

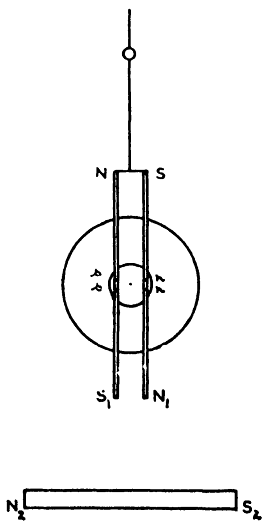

Fig. 67

CHAPTER VI

THE FLUXMETER

THE fluxmeter is a permanent-magnet moving-coil galvanometer, and in its essential parts similar to an ordinary permanent-magnet moving-coil galvanometer, except that its suspension has a torsion constant equal to zero, and the damping due to currents induced in the moving coil is very effective. The damping on open circuit is negligible.

There are two types of fluxmeters.

(1) The Industrial Type (Fig. 68)

The moving part rests on a pivot or pivots ; the current is brought to the coil by a very thin silver wire wound in a wide spiral, the torsion

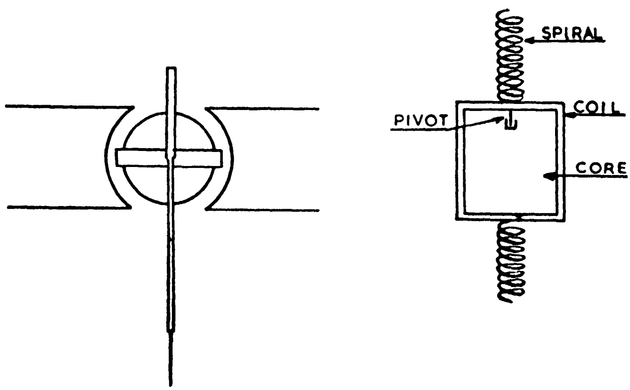

Fig. 68

torque of which is practically zero. There are no restoring springs. The instrument is provided with a pointer moving over a scale, provided with a mirror. The scale is graded in Maxwell-turns.

(2) The Laboratory Type (Fig. 69)

The coil is suspended by a thread of raw silk, of torsion constant zero. The current is brought to the coil by a silver wire wound in a spiral in the same way as in the industrial type. A mirror is attached to the suspension, and this mirror reflects light from a source on to a scale in the manner of the ordinary reflecting galvanometer.

A fluxmeter once deflected will not come back to zero (or at least not for a fair time) as there is no restoring torque ; a mechanical arrangement is therefore provided by means of which the pointer, or the spot, can be brought back to zero. A measurement can of course be made from any position of the pointer or the spot, because deflection can be measured from any part on the scale.

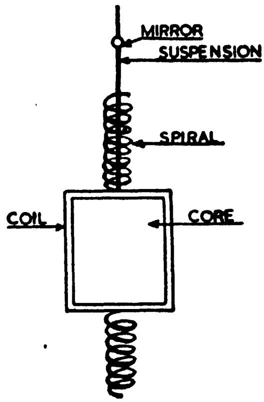

Fig. 69

(3) Theory of the Fluxmeter

The ballistic galvanometer is suitable for short time impulses, but for impulses lasting an appreciable time—a quarter of a second to several seconds—it is of course quite unsuitable. The fluxmeter, however, owing to the absence of a controlling torque and strong damping by induced currents, can measure long impulses with a high degree of precision.

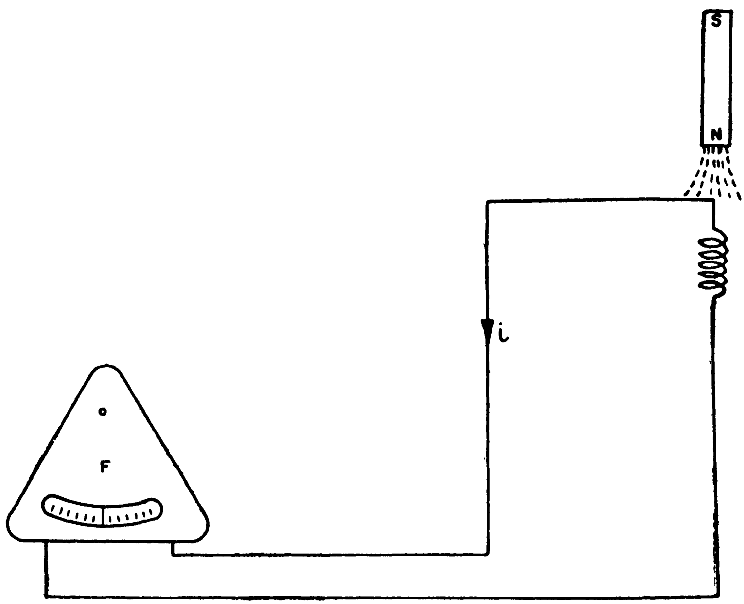

Fig. 70

The mechanical equation of the fluxmeter is

$$J \frac{d^2 a}{dt^2} + D_1 \frac{da}{dt} = \Phi_0 i \qquad (80)$$

J is the inertia, D_1 the damping constant on open circuit, and Φ_0 the flux linkages of the fluxmeter coil due to the permanent magnet within the poles of which the coil is placed. The term τa does not appear in the equation, because τ is zero.

Let us produce a discharge (impulse) in the fluxmeter by means of a change of flux in a coil connected to the fluxmeter, the arrangement being shown in fig. 70.

The change of flux $\Delta\Phi_1$ can be produced by moving a magnet near the coil. Let the fluxmeter resistance and inductance be r and l and that of the connected coil R and L, so that we can write :

$$(R + r)\, i + (L + l) \frac{di}{dt} + \Phi_0 \frac{da}{dt} = \frac{d\Phi_1}{dt} \qquad (80a)$$

As soon as the external field starts changing, the fluxmeter starts moving. It will stop practically instantaneously when the change stops, because of its very effective damping by induced currents. Integrating (80a) between the time $t = O$, beginning of the change of flux, to $t = t$, end of the change, we get

$$(R + r)\int_O^t i dt + (L + l)\int_O^t di + \int_O^t \Phi_0 \frac{da}{dt} = \int_O^t d\Phi_1.$$

The term $(L + l) \int_O^t di$ is zero because the current i is zero at $t = O$, and at $t = t$ (assuming that the fluxmeter stops instantaneously when the change stops); the term $(R + r)\int_O^t i dt = (R + r)\, q$, where q is the quantity of electricity which passed through the fluxmeter, so that

$$(R + r)\, q + \Phi_0 (a - a_0) = \Delta\Phi_1 \qquad (81)$$

$(a - a_0)$ is the deflection of the pointer or the spot from its position a_0 at $t = O$, to a at $t = t$.

From (81) we get $q = \dfrac{\Delta\Phi_1 - \Phi_0 (a - a_0)}{R + r}$ (82)

We see that, under the assumptions made, L and l do not have to be taken into account. Integrating (80) between $t = O$ and $t = t$, we have

$$J\int_0^t \frac{d^2a}{dt^2} + D_1\int_0^t \frac{da}{dt} - \Phi_0\int_0^t idt = J\left[\frac{da}{dt}\right]_0^t + D_1(a - a_0) = \Phi_0 q.$$

As $\frac{da}{dt} = 0$ at $t = 0$ and at $t = t$, the term containing J becomes zero,

and $D_1 (a - a_0) = \Phi_0 q$ . . . . (82a)

Substituting for q from (82)

$$D_1 (a - a_0) = \Phi_0 \left[\frac{\Delta\Phi_1 - \Phi_0 (a - a_0)}{R + r}\right];$$

$$(a - a_0)\left[\frac{D_1 (R + r) + \Phi_0^2}{R + r}\right] = \frac{\Phi_0\,\Delta\Phi_1}{R + r} \text{ and}$$

$$\Delta\Phi_1 = (a - a_0)\left[\frac{D_1 (R + r) + \Phi_0^2}{\Phi_0}\right] \quad . \quad . \quad . \quad (83)$$

The instrument deflection is therefore a measure of the change of flux produced.

The open-circuit damping can in a good fluxmeter be neglected, so that if $R + r$ is not too great the term $D_1 (R + r)$ is negligible and (83) becomes

$$\Delta\Phi_1 = \Phi_0 (a - a_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

As Φ_0 is constant (with usual deflections), the instrument deflection gives directly the change of flux produced.

a and a_0 are in radians. In a fluxmeter provided with scale and pointer, the scale is graded in Maxwell-turns, and if one division corresponds to k linkages (84) becomes

$$\Delta\Phi_1 = (m - m_0) k \quad . \quad . \quad . \quad . \quad . \quad . \quad (84a)$$

where m and m_0 are the divisions shown by the pointer at the time $t = 0$ and $t = t$.

It follows from 82 that the quantity of electricity q, which passed through the fluxmeter, is very small; in fact, if D_1 is negligible, we get from (82a) $\Phi_0 q = 0$; that is, the quantity of electricity passed through the fluxmeter is zero.

There is nothing surprising in this, for imagine a d.c. motor with no losses whatever and negligible inertia. As soon as an e.m.f. is applied to this motor it will immediately attain its normal speed, and as there are no losses, there will be no current flowing, the back e.m.f. $\Phi_0 \frac{da}{dt}$ exactly balancing the applied e.m.f.

If (84) is to be true, the fluxmeter has to comply with the following conditions:

(*a*) Having no restoring torque.

(*b*) Having a negligible open-circuit damping.

(*c*) Having a very strong damping by induced currents, the total resistance of the fluxmeter circuit (fluxmeter coil plus external circuit) should normally not exceed 20 to 30 ohms.

(*d*) The fluxmeter should follow all changes of impulse instantaneously ; its inertia should therefore be small.

As regards condition (*d*), it is evident that the fluxmeter cannot start and stop instantaneously, and it cannot immediately respond to all changes in the impulse received. The fluxmeter is used for measuring impulses such as the charge and discharge of a condenser, reversal or change of a magnetic field. It is assumed that what is lost by accelerating the fluxmeter at the beginning of the impulse is regained when the acceleration is negative and the moving part of the fluxmeter comes to rest.

Let us supply to the fluxmeter a small steady e.m.f. E, let the total fluxmeter circuit resistance be R_1 and its total inductance L_1. Let Φ_0 be the flux linkages of the moving coil.

Neglecting the open circuit damping, we can write :

$$J\frac{d^2a}{dt_2} = \Phi_0 i,$$

and writing $\frac{da}{dt} = \omega =$ angular speed of the coil, we get

$$J\frac{d\omega}{dt} = \Phi_0\, i\,;\quad i = \frac{J}{\Phi_0}\frac{d\omega}{dt};\ \frac{di}{dt} = \frac{J}{\Phi_0}\frac{d^2\omega}{dt^2}. \qquad (85)$$

We also have

$$R_1 i + L_1\frac{di}{dt} + \Phi_0\omega = E \qquad (86)$$

Combining (85) and (86), we get :

$$\frac{R_1 J}{\Phi_0}\frac{d\omega}{dt} + \frac{L_1 J}{\Phi_0}\frac{d^2\omega}{dt^2} + \Phi_0\omega = E\,;$$

$$(L_1 J)\frac{d^2\omega}{dt^2} + (R_1 J)\frac{d\omega}{dt} + \Phi^2{}_0\,\omega = E\,\Phi_0 \qquad (87)$$

When the fluxmeter coil is moving at constant speed ω, $\frac{d\omega}{dt}$ and $\frac{d^2\omega}{dt^2}$ are both zero, so that $\omega = \omega_c = \frac{E}{\Phi_0}$ is a particular solution of (87).

During the positive acceleration period, the solution of (87) is added to the solution of

$$(L_1 J)\frac{d^2\omega}{dt^2} + (R_1 J)\frac{d\omega}{dt} + \Phi_0{}^2\omega = O \quad . \quad . \quad . \quad (88)$$

The solution of (87) is

$$\omega = \omega_c + Ae^{p_1 t} + Be^{p_2 t} \quad . \quad . \quad . \quad . \quad . \quad (88a)$$

where

$$p_1 = -\frac{R_1 J}{2L_1 J} + \sqrt{\frac{(R_1 J)^2}{4(L_1 J)^2} - \frac{\Phi_0{}^2}{L_1 J}} = -\frac{R_1}{2L_1} + \sqrt{\frac{R_1{}^2}{4L_1{}^2} - \frac{\Phi_0{}^2}{L_1 J}}$$

$$p_2 = -\frac{R_1 J}{2L_1 J} - \sqrt{\frac{(R_1 J)^2}{4(L_1 J)^2} - \frac{\Phi_0{}^2}{L_1 J}} = -\frac{R_1}{2L_1} - \sqrt{\frac{R_1{}^2}{4L_1{}^2} - \frac{\Phi_0{}^2}{L_1 J}}$$

The constants in the fluxmeter circuit are such that the conditions are non-oscillatory ; that is, we have

$$\frac{R_1{}^2 J^2}{4L_1{}^2 J^2} > \frac{\Phi_0{}^2}{L_1 J};\ \frac{R_1{}^2}{4L_1} > \frac{\Phi_0{}^2}{J}.$$

At the time $t = O$ we have $\omega = O$ and $\frac{d\omega}{dt} = O$, so that

$$O = \omega_c + A + B\ ;\ A = -\omega_c - B\ ;\ B = -\omega_c - A \quad (89)$$

Differentiating (88a), we get : $\frac{d\omega}{dt} = p_1 Ae^{p_1 t} + p_2 Be^{p_2 t}$, which for $t = O$ becomes

$$O = p_1 A + p_2 B\ ;\ p_1 A = -p_2 B \quad . \quad . \quad . \quad . \quad (90)$$

Combining (89) and (90), we have

$$A = \frac{p_2}{p_1 - p_2}\omega_c\ ;\ B = -\frac{p_1}{p_1 - p_2}\omega_c$$

(88a) can therefore be written :

$$\omega = \omega_c + \frac{p_2}{p_1 - p_2}\omega_c e^{p_1 t} - \frac{p_1}{p_1 - p_2}\omega_c e^{p_2 t} \quad . \quad . \quad (91)$$

When the external e.m.f. E ceases to act, we have

$$L_1 J\frac{d^2\omega}{dt^2} + R_1 J\frac{d\omega}{dt} + \Phi^2{}_0\omega = O$$

the solution of which is

$$\omega = A_1 e^{p_1 t} + B_1 e^{p_2 t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Counting the time from the instant the e.m.f. E ceases, we shall have at $t = O$; $\omega = \omega_c$ and

$$\omega_c = A_1 + B_1 \; ; \; A_1 = \omega_c - B_1 \; ; \; B_1 = \omega_c - A_1 \quad . \quad . \quad (93)$$

Differentiating (92), we get

$$\frac{d\omega}{dt} = p_1 A_1 e^{p_1 t} + p_2 B_1 e^{p_2 t},$$

which for $t = O$ becomes

$$O = p_1 A_1 + p_2 B_1 \; ; \; p_1 A_1 = - p_2 B_1 \quad . \quad . \quad . \quad (94)$$

Combining (93) with (94) we have

$$A_1 = - \left(\frac{p_2}{p_1 - p_2}\right) \omega_c \; ; \; B_1 = \left(\frac{p_1}{p_1 - p_2}\right) \omega_c,$$

so that (92) can be written

$$\omega = - \frac{p_2}{p_1 - p_2} \omega_c e^{p_1 t} + \frac{p_1}{p_1 - p_2} \omega_c e^{p_2 t} \quad . \quad . \quad (95)$$

Considering (91) and (95), we see that what is lost in accelerating the fluxmeter is regained when the acceleration becomes negative and the fluxmeter coil stops moving.

When the time during which the e.m.f. is applied is very small, or when the impulse given to the fluxmeter is highly irregular, then, owing to the inductance of the fluxmeter circuit, the rate of increase and decrease of the fluxmeter current will affect the result. The loss in accelerating the fluxmeter will not be balanced by the gain when the acceleration is negative, so that, there is a lower limit, depending on the circuit constants, below which the fluxmeter is not true.

Again, the time of the impulse, due to the friction in the fluxmeter (open circuit damping), should not be too long. It may happen that the impulse is too weak (charge and discharge of a condenser through a high enough resistance), to move the fluxmeter, because of its friction. The error caused by friction is especially important when the impulse is irregular (resulting in stopping and producing violent changes in the angular speed during the impulse).

It has to be remembered that even if the impulse is regular and within the right time limits, we still have to calibrate and use the fluxmeter, in such conditions that we may neglect the open-circuit damping.

(4) The Shunted Fluxmeter

The arrangement is shown in fig. 71.

The fluxmeter F of resistance r and inductance l is shunted by the resistance R_1 ; L and R are the resistance and inductance of the coil through which a change of flux $\Delta\Phi_1$ is produced. We can write

$$R\,(i + i_1) + L\frac{d\,(i + i_1)}{dt} + R_1\,i_1 = \frac{d\Phi_1}{dt}$$

$$ri + \frac{ldi}{dt} + \Phi_0\frac{da}{dt} = R_1\,i_1.$$

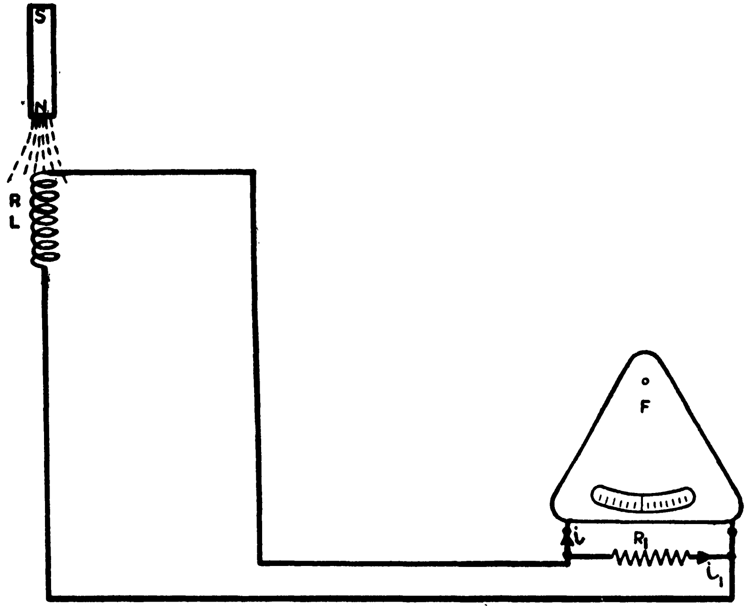

Fig. 71

Integrating between $t = 0$ beginning of change of flux to $t = t$ end of change, we have

$$R\int_0^t (i + i_1)\,dt + L\int_0^t d\,(i + i_1) + R_1\int_0^t i_1\,dt = \int_0^t d\Phi_1 = \Delta\Phi_1$$

$$r\int_0^t idt + l\int_0^t di + \Phi_0\,(a - a_0) = R_1\int_0^t i_1\,dt$$

The terms containing L and l are equal to zero, and writing

$$\int_0^t (i + i_1)\,dt = Q\,;\int_0^t i_1\,dt = q_1\,;\int_0^t idt = q, \text{ we get}$$

$$RQ + R_1\,q_1 = \Delta\Phi_1\,;\; rq + \Phi_0\,(a - a_0) = R_1\,q_1 \quad . \quad (96)$$

and as

$$Q = q_1 + q\,;\; q_1\,(R + R_1) + Rq = \Delta\Phi_1\,. \quad . \quad . \quad (97)$$

multiplying (96) by $R + R_1$ and (97) by R_1; $(R + R_1)\, rq + \Phi_0\, (a - a_0)\, (R + R_1) = R_1\, q_1\, (R + R_1)$

$R\, R_1\, q - R_1\, \Delta\Phi_1 = - R_1\, q_1\, (R + R_1)$.

Adding, we get $(R + R_1)\, rq + \Phi_0\, (a - a_0)\, (R + R_1) + R\, R_1\, q = R_1\, \Delta\Phi_1$.

$$q\left(\frac{Rr + R_1 r + R\, R_1}{R + R_1}\right) + \Phi_0\, (a - a_0) = \frac{R_1}{R + R_1}\, \Delta\Phi_1, \text{ or}$$

$$q\left[\frac{r\, (R + R_1)}{R + R_1} + \frac{R\, R_1}{R + R_1}\right] + \Phi_0\, (a - a_0) = \frac{R_1}{R + R_1}\, \Delta\Phi_1$$

The quantity in brackets is the resistance of the fluxmeter plus the resistance R_1 in shunt with R, relative to the fluxmeter.

Writing $r + \dfrac{R\, R_1}{R + R_1} = R_T$, we get

$$qR_T + \Phi_0\, (a - a_0) = \frac{R_1}{R + R_1}\, \Delta\Phi_1 \quad . \quad . \quad . \quad (98)$$

As we have $D\, (a - a_0) = \Phi_0\, q$, then $q = \dfrac{D\, (a - a_0)}{\Phi_0}$, substituting this value of q in (98), we have

$$\frac{DR_T}{\Phi_0}(a - a_0) + \Phi_0\, (a - a_0) = \Delta\Phi_1\, \frac{R_1}{R + R_1};$$

$$(a - a_0)\left(\frac{DR_T + \Phi_0^2}{\Phi_0}\right) = \frac{R_1}{R + R_1}\, \Delta\Phi_1.$$

Therefore $\Delta\Phi_1 = (a - a_0)\left(\dfrac{DR_T + \Phi_0^2}{\Phi}\right)\dfrac{R + R_1}{R_1}$.

As we can neglect DR_T, we have approximately

$$(a - a_0)\, \Phi_0\, \frac{R + R_1}{R_1} = \Delta\Phi_1 \quad . \quad . \quad . \quad . \quad . \quad (98a)$$

while, when the fluxmeter was not shunted, we had $(a - a_0)\, \Phi_0 = \Delta\Phi_1$.

(5) Calibration of the Fluxmeter and Determination of the Limit of the Fluxmeter Circuit Resistance

Two arrangements for the purpose are shown in figs. 72 and 73.

Fig. 72. The ring of non-magnetic material, of section S cm.² and mean diameter D cm., has a uniform primary winding of N_1 turns. The current is brought to the primary from the source E through the ammeter A, rheostat R*h* reversing switch k_2 and switch k_1. The secondary winding on the ring is of N_2 turns and resistance R; it is connected to the fluxmeter F, of resistance r, through the variable resistance R_1.

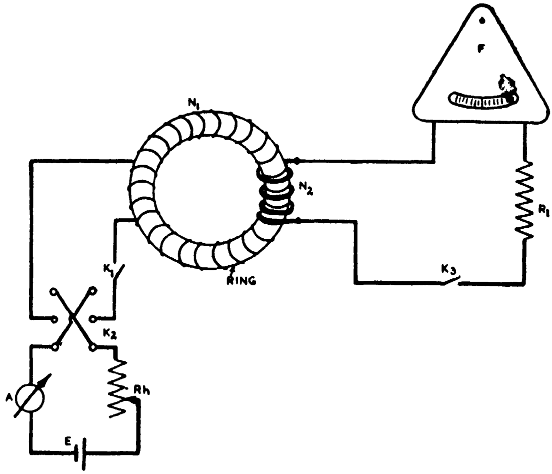

Fig. 72

When the current I_1 flows through the primary, the magneto-motive force is $Hl = \frac{4\pi N_1 I_1}{10}$, where $l = \pi D$.

The flux through the ring is therefore $\Phi = HS = \frac{4\pi N_1 I_1}{10l} \cdot S$.

Making or breaking k_1, we get a change of flux linkages through the secondary $(N_2 \Phi) = \frac{4\pi N_1 I_1 S N_2}{10l} = \Delta\Phi_1$.

We know that if the resistance of the fluxmeter circuit is not too great, we shall have

$$\Delta\Phi_1 = \frac{4\pi N_1 I_1 S N_2}{10l} = k(m - m_0),$$

or when starting from zero of the fluxmeter scale $(m_0 = O)$.

$$\Delta\Phi_1 = \frac{4\pi N_1 I_1 S N_2}{10l} = k\,m$$

where k is a constant and equal to the number of Maxwell turns per division of the fluxmeter scale.

Fig. 73. Here we use a standard mutual inductance M, the primary and secondary of which are connected as shown. The resistance of the secondary is R.

The flux linkages through the secondary caused by unit current in the primary are equal to M, and as M is given in Henrys, these flux linkages will be $M \times 10^9$ in the e.m.c.g.s. system, so that when a current I is established or stopped by means of k_1 in the primary the secondary flux linkages are $M \times 10^9 \times \frac{I_1}{10} = M I_1 \times 10^8$, where I_1 is in amps. We then have $k = \frac{\Delta\Phi_1}{m - m_0} = \frac{M I_1 10^8}{m - m_0}$, or, when starting from O, $k = \frac{M I_1 10^8}{m}$ . . . . (99)

When determining k by the arrangements shown, R should be zero.

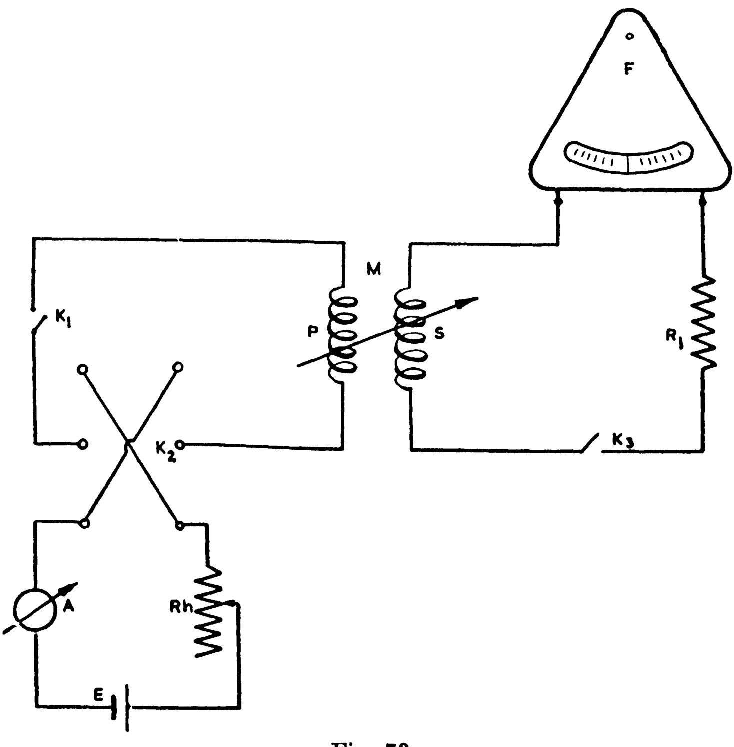

Fig. 73

The fluxmeter has to be calibrated over the whole scale on both sides of the zero. After the first making of k_1 (I_1 established, deflection arrived at m), the fluxmeter should not be brought to zero but disconnected by opening k_3. Then I_1 is interrupted by opening k_1. Then k_3 is closed, and the current again established, giving m_1 as the new deflection. For the same current in the primary we should find that $m_1 = m$. Having checked the whole of the scale, reverse the direction of I, by means of k_2, and repeat the experiment over the other half of the scale.

The resistance in the fluxmeter circuit should not exceed 20 *to* 30 *ohms.*

CALCULATION OF THE SYSTEMATIC ERROR. It is evident that the arrangement of fig. 73 is preferable to that of fig. 72, because its error should be smaller.

The logarithmic differential of (99) is

$$\frac{dk'}{k'} = \frac{dM'}{M'} + \frac{dI_1'}{I_1'} - \frac{dm'}{m'},$$

and the relative error

$$\frac{\Delta k'}{k'} = \frac{\Delta M'}{M'} + \frac{\Delta I'}{I'} + \frac{\Delta m'}{m'}.$$

I should be measured by a potentiometer, or at least by a substandard ammeter.

$\Delta M'$ is the constructional error on M, $\Delta I'_1$, the constructional plus the reading error on I_1, and $\Delta m'$ the reading error on the fluxmeter.

To find the value of the resistance $R + R_1$ beyond which the calibration will not hold, we use the arrangement of fig. 73 or 72; with a fixed current, say, I_1, giving when established a deflection m_1. We start with $R_1 = 0$ and increase it gradually, establishing the current I for each value of R_1. It will be found that above a certain value of R_1, say, R_{1L}, the deflection n will decrease. The limit of the fluxmeter circuit resistance, beyond which the instrument should not be used in subsequent applications, is $R_L = R_{1L} + r$.

(6) Charge and Discharge of a Condenser through the Fluxmeter : Measurement of a Quantity of Electricity or of a Capacity

The arrangement is shown in fig. 74.

F is the fluxmeter of resistance r and inductance l; F is connected across R_1; $R_1 + r$ should not exceed 20 to 30 ohms; E is the source of e.m.f. E; C a standard condenser; and R a variable resistance, high enough appreciably to change by its variation the time of charge or discharge of C.

Making 1–2, the condenser is charging; making 1–3, it is discharging. Each discharge will produce a deflection away from zero and each charge bring the pointer back to the zero position. When a charge or discharge starts, the fluxmeter begins to move; it stops when the charge or discharge finishes.

With the notations of fig. 74, we have $ri + \frac{ldi}{dt} + \Phi_0 \frac{da}{dt} = R_1(I - i)$, and integrating between the time $t = O$, beginning of charge or discharge, and $t = t$, end of charge or discharge,

$$r\int_{O}^{t} idt + l\int_{O}^{t} di + \int_{O}^{t} \Phi_0 \frac{da}{dt} = R_1\int_{O}^{t} (I - i)\, dt.$$

The term containing l being zero, we get

$rq + \Phi_0 (a - a_0) = R_1Q - R_1q$; $\Phi_0 (a - a_0) = R_1Q - q\,(R_1 + r)$, where Q is the total charge of the condenser and q the charge passed through the fluxmeter.

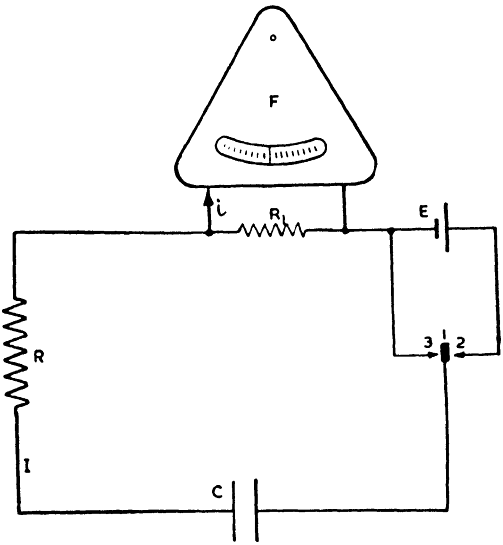

Fig. 74

We have seen, however, that in a good fluxmeter we have $q = O$; therefore we can write $\Phi_0 (a - a_0) = R_1Q$; $Q = \dfrac{\Phi_0 (a - a_0)}{R_1}$; or in terms of the fluxmeter scale constant (the number of Maxwell-turns per division) $Q = \dfrac{k\,(m - m_0)}{R_1}$.

But $Q = CE$, so that $CE = \dfrac{k\,(m - m_0)}{R_1}$; and as C is in farads, R in ohms, and E in volts, $CE = \dfrac{k\,(m - m_0)}{R_1 10^8}$.

We could therefore measure a charge Q, a capacity C, or an e.m.f. E, with the aid of the fluxmeter.

(7) Determination of the Effect of the Time of Charge or Discharge, or the Effect of Friction in the Fluxmeter

With the arrangement of fig. 74, starting from a small value of R, the condenser is charged and discharged while R is gradually increased. At a certain value of R, say, R_{L2}, the deflection will start decreasing.

If the condenser is not leaky this diminishing of the deflection is due mainly to $D_1 (R_1 + r)$.

If the condenser is discharging slowly, the impulse at the end of charge or discharge is too small to overcome the instrument friction. In fact, some energy is lost in ir^2 or heating the fluxmeter coil, but i being small this loss of energy is obviously negligible.

Having found R_{L2}, we can calculate the time constant $C (R_{L2} + R_1)$ for which the deflection starts altering. The experiment should be repeated for different values of C, so that in the subsequent use of the fluxmeter the time constant will be known for condensers of varying capacities.

Note that when the condenser is leaky, leakage will to a certain extent influence m, so that only very good condensers ought to be used for the check above.

R_1 should not be greater than the value R_{1L} found in the calibration experiment.

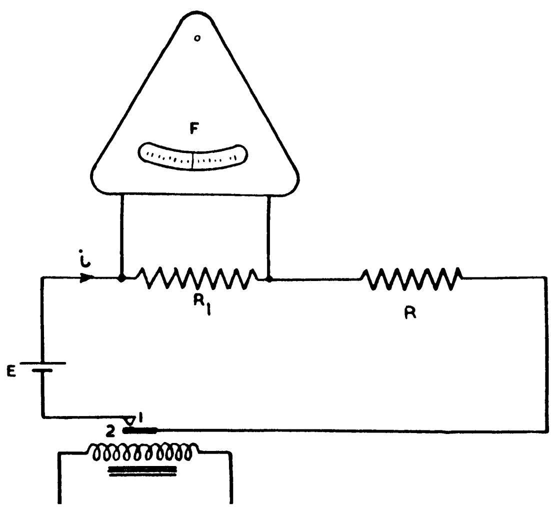

Fig. 75

(8) The Fluxmeter Used as a Clock

If we apply to the fluxmeter of resistance r and inductance l a constant e.m.f. E, for a time t, we can write

$$ri + l\frac{di}{dt} + \Phi_0\frac{da}{dt} = \mathrm{E} \text{ and } r\int_0^t idt + l\int_0^t di + \Phi_0(a - a_0) = \mathrm{E}\int_0^t dt.$$

Or $rq + \Phi_0(a - a_0) = \mathrm{E}t$; and as $q = 0$, we get $\Phi_0(a - a_0) = k(m - m_0) = \mathrm{E}t$.

If we know E, the fluxmeter deflection will give us the time t.

An arrangement whereby the time during which the contacts of a relay remain closed can be measured is shown in fig. 75.

1 and 2 are the relay contacts. The p.d. across R_1 should not exceed 1×10^{-6} to 5×10^{-4} volts, dependent on the fluxmeter used. The resistance R_1 plus the fluxmeter resistance should not exceed R_L (about 20 to 30 ohms), and R should be high enough to give the desired p.d. across R_1 with E between 1 to 2 volts.

The current in the fluxmeter being always zero, under the assumptions of the fluxmeter theory we can write

$$\mathrm{E} = (\mathrm{R} + \mathrm{R}_1)\,i\,;\ \ \mathrm{ER}_1 = (\mathrm{R} + \mathrm{R}_1)\,\mathrm{R}_1 i\,;\ \ i\mathrm{R}_1 = e = \frac{\mathrm{ER}_1}{\mathrm{R} + \mathrm{R}_1}$$

As soon as the relay contacts close, the fluxmeter starts moving. It stops when the relay contacts open, and we have

$$t = \frac{k(m - m_0)}{\dfrac{\mathrm{R}_1\mathrm{E}}{\mathrm{R} + \mathrm{R}_1}} \times 10^{-8} \text{ seconds.}$$

Example 21. "Derive an expression in terms of the angular movement of the suspended coil of a fluxmeter, for the change of magnetic flux through a search coil of T turns connected to the meter. The total resistance of the whole circuit, moving coil and search coil is R, and the total self inductance L. The moving coil consists of N turns each of area A suspended in a uniform field B" (University of London, Electr. Measur. and Measur. Instruments, B.Sc. Final, 1947, paper 1, question 1).

We have $\Phi_0 = \mathrm{ANB}$; the total change of flux through the search coil is $\Delta\Phi_1 = \Phi\mathrm{T}$, so that we can write :

$$\mathrm{R}i + \mathrm{L}\frac{di}{dt} + \mathrm{AN}\frac{d\mathrm{B}}{dt} = \mathrm{T}\frac{d\Phi}{dt}\,;\ \text{integrating,}$$

$$\mathrm{R}q + \mathrm{ANB}(a - a_0) = \mathrm{T}\,\Delta\Phi = \Delta\Phi_1 \quad . \quad . \quad . \quad (100)$$

We have also $D_1(a - a_0) = ANBq$; $q = \frac{D_1(a - a_0)}{ANB}$; and substituting for q in (100),

$$\frac{RD_1(a - a_0)}{ANB} + ANB(a - a_0) = \Delta\Phi T = \Delta\Phi_1;$$

$$(a - a_0)\left[\frac{RD_1 + (ANB)^2}{ANB}\right] = T\,\Delta\Phi = \Delta\Phi_1, \text{ therefore}$$

$$\Delta\Phi = \frac{(a - a_0)}{T}\left[\frac{RD_1 + (ANB)^2}{ANB}\right].$$

Example 22. "Describe the construction of a fluxmeter and find the relation between the deflection and the change of flux-linkages in the coil to which the fluxmeter is connected.

"An iron ring having a mean diameter of 25 cm. and a cross-sectional area of 3 cm.^{2} has a primary winding of 150 turns. The secondary winding of 50 turns is connected to a fluxmeter having a constant of 10,000 Maxwell-turns per division. A deflection of 30

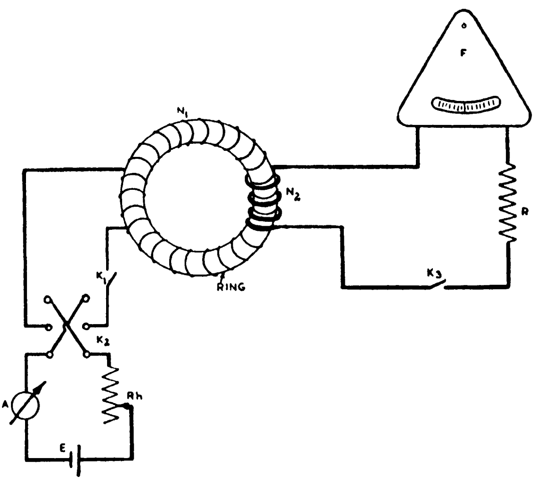

Fig. 76

divisions is obtained when a current of 1·5 amperes is reversed in the primary. Calculate the permeability of the iron. Is it essential that the windings should be wound uniformly around the ring?" (University of London, Electr. Measur. and Measur. Instruments, B.Sc. Final, 1946, Internal, paper 2, question 2).

The first part of the question can be answered by reference to the text.

The arrangement for the test as described in the question is shown in fig. 76.

The total change in flux linkages is $\Delta\Phi_1 = 10000 \times 30 = 300000$ Maxwell-turns. As $\Delta\Phi_1 = 2\text{SBN}$, because the current in the primary is reversed we have $\text{B} = \dfrac{300000}{2 \times 50 \times 3} = 1000$ Gauss.

The magnetomotive force in the ring is given by

$$\text{H}l = \frac{4\pi\text{NI}}{10} \text{ and } \text{H} = \frac{4\pi \times 150 \times 1{\cdot}5}{10 \times \pi \times 25} = 3{\cdot}6 \text{ oersted.}$$

The permeability of the ring is therefore

$$\mu = \frac{\text{B}}{\text{H}} = \frac{1000}{3{\cdot}6} = 278.$$

It is essential that N_1 should be wound uniformly round the ring, because then there is no leakage between any two points of the ring and all the flux is the same in all the sections of the ring. It is not essential that N_2 should be wound uniformly, because if the flux is the same in all sections of the ring to begin with, we start with an induction $+\text{B}$ and finish with an induction $-\text{B}$, so that the answer must be correct. The fluxmeter current, being negligible, cannot affect the flux in the ring.

Example 23. " Derive, from the equation of motion, an expression for the sensitivity of a fluxmeter.

" Explain, with the aid of the result obtained, why for low values of the search coil resistance the sensitivity of the instrument is substantially independent of the value of this resistance " (University of London, Electr. Measur. and Measur. Instruments, B.Sc. Final, 1944, Internal. paper 2, question 8).

As we have $(a - a_0)\,[\text{D}_1\,(\text{R} + r) + \Phi_0{}^2] = \Delta\Phi_1\Phi_0$ (from (83)). The sensitivity will be given by

$$\frac{a - a_0}{\Delta\Phi_1} = \frac{\Phi_0}{\text{D}_1\,(\text{R} + r) + \Phi_0{}^2} \simeq \frac{1}{\Phi_0}$$

because, D_1 being small, if r and R are small, the term $\text{D}_1\,(\text{R} + r)$ can be neglected, and the sensitivity is independent of R and inversely proportional to Φ_0.

This is understandable, because the smaller the number of flux linkages of the moving coil, the greater will be the deflection for a given impulse. As in a d.c. motor, the smaller the excitation provided the higher the speed of the motor.

CHAPTER VII

PERMANENT-MAGNET MOVING-COIL AMMETERS AND VOLTMETERS

THE arrangement of the essential parts of an ammeter or voltmeter is shown in fig. 46.

(1) Ammeters

An ammeter being connected in series in the circuit where the current is measured should produce as small a potential drop as possible ; its resistance ought therefore to be small.

The current sensitivity of a galvanometer is, as we have seen,

$$\frac{a}{I} = \frac{\Phi_0}{\tau} = \frac{HSn}{\tau},$$

H being the field in the airgap where the coil is located, S the coil section, and n the number of coil turns.

The coil resistance

$$R = \varrho\frac{l}{s}n \qquad (101)$$

s is the area of the wire the coil is made of, l the average length of one turn, and ϱ the resistivity.

The volume of the coil, without insulation, will be

$$v = nls \qquad (102)$$

Combining (101) and (102), we have $R = \frac{n^2l^2\varrho}{v}$.

Therefore, for a given volume and dimensions of the coil, that is, for a given airgap, and also for a given flux in the airgap, the sensitivity increases as the number of turns on the coil, while the resistance increases as the square of the number of turns. It follows that a galvanometer used as an ammeter ought to have a fairly high resistance. Naturally the coil has to be made of a metal of good conductivity (copper), because the number of turns will be greater for a given resistance than with a metal of low conductivity.

(*a*) THE PRACTICAL SENSITIVITY OF AN AMMETER. This will be defined as the current in amps necessary to deflect the pointer by one scale division.

Examples of galvanometers used as ammeters.

No. of scale divisions	*Current for total scale deflection in amps*	*Resistance of coil and springs ohms*	*p.d. across instr.*	*sensitivity pract.*
100	0·00002	800	0·016	2×10^{-7}
100	0·00005	160	0·008	5×10^{-7}
100	0·000075	100	0·0075	$7{\cdot}5 \times 10^{-7}$
100	0·001	10	0·01	1×10^{-5}
100	0·05	1	0·05	5×10^{-4}

In order to measure currents higher than the coil can carry and also not to introduce too high a potential drop in the circuit, shunts or resistances connected in parallel with the instrument are used.

(*b*) Ammeter Shunts. Shunts are four-terminal low resistances, usually made of strip manganin (84% copper, 4% nickel, 12% manganese) terminating in heavy copper blocks, as shown in fig. 77. The

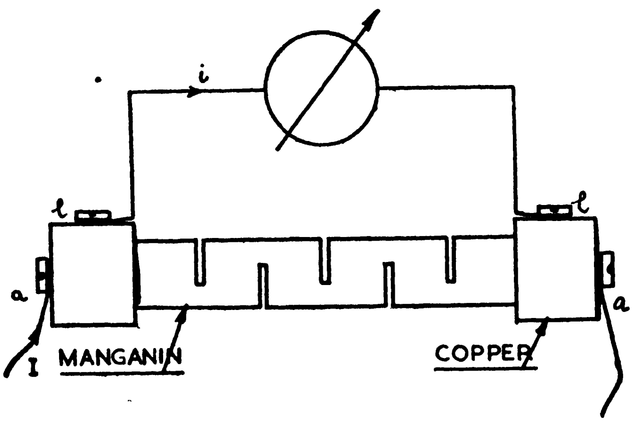

Fig. 77

current is brought in through the terminals *a*,*a* ; the instrument is connected to *b*,*b*. The total current to be measured divides into i and $I - i$, the major part of the current flowing through the shunt.

The shunt, if made of copper, would be too bulky. The reason it is made of manganin is that this alloy has an extremely small temperature coefficient (about 2×10^{-5} per °C), and its thermal e.m.f. with copper is only 1×10^{-6} volt per °C. The resistivity of manganin is about 42 μohm at 18°C.

Consider fig. 78. Let the instrument resistance be r and that of the shunt S. With the notation of the diagram we have

$$S(I - i) = ri\,;\ SI = i(r + S);\ I = \frac{r + S}{S} \cdot i.$$

The ratio of the total current I to the current in the instrument is

$$\frac{I}{i} = \frac{r + S}{S} = m.$$

m is called the multiplying factor of the shunt.

As $\frac{I - i}{i} = m - 1 = \frac{r}{S}$, we get $S(m - 1) = r$, $S = \frac{r}{m - 1}$.

A shunt should indicate the total current I for full-scale deflection of the instrument, the p.d. across the shunt for full-scale deflection of the instrument, or the resistance of the instrument with the shunt in parallel.

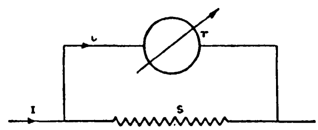

Fig. 78

Example 24. The resistance of an ammeter coil, springs and leads to shunt, is 0·8 ohm; the coil current for total scale deflection is 50 milli-amperes. What is the resistance and the particulars given on the shunt for a circuit current of 10 amps?

The p.d. across the shunt for full-scale deflection (marked on the shunt) is $\frac{50 \times 0{\cdot}8}{1000} = 0{\cdot}04$ v.

Or the indication on the shunt can be the total resistance of shunt and instrument, which is $\frac{0{\cdot}04}{10} = 0{\cdot}004\Omega$.

The other indication on the shunt will be 10 amps.

The shunt multiplying power is $m = \frac{I}{i} = 10 \div \frac{50}{1000} = 200$; the shunt resistance is therefore $S = \frac{r}{m - 1} = \frac{0{\cdot}8}{200 - 1} = 0{\cdot}00402\Omega$.

It is evident that the connecting wires, or leads, from shunt to instrument, are calibrated with the instrument and should not generally be interchanged with those belonging to other instruments.

(*c*) Ammeters of Several Sensitivities. When several sensitivities are required, for example full-scale deflections of 1, 10, 100 amps, several shunts are used ; one for each sensitivity. Let the circuit currents for full-scale deflection be I_1, I_2, I_3, and i the current in the ammeter coil ; the multiplying powers of the required shunts will be

$$\frac{I_1}{i} = \frac{r + S_1}{S_1} = m_1, \quad \frac{I_2}{i} = \frac{r + S_2}{S_2} = m_2, \quad \frac{I_3}{i} = \frac{r + S_3}{S_3} = m_3.$$

Considering the ratios of the multiplying powers

$$\frac{m_1}{m_2} = \frac{I_1}{I_2} = \frac{r + S_1}{S_1} \div \frac{r + S_2}{S_2} = \frac{r + S_1}{r + S_2} \times \frac{S_2}{S_1},$$

$$\frac{m_2}{m_3} = \frac{I_2}{I_3} = \frac{r + S_2}{S_2} \div \frac{r + S_3}{S_3} = \frac{r + S_2}{r + S_3} \times \frac{S_3}{S_2}$$

we see that these ratios depend on the ammeter resistance r.

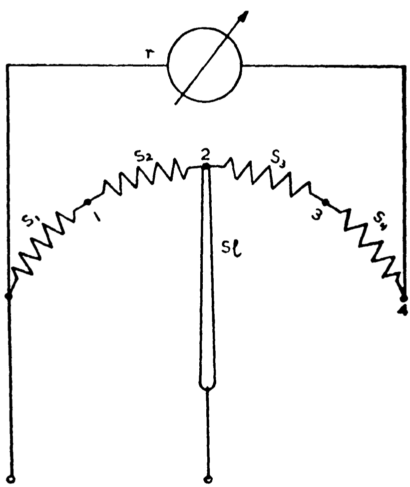

Fig. 79

(*d*) The Universal Shunt. The ratios of the multiplying powers can be made independent of r by the use of the universal shunt, the arrangement of which is shown in fig. 79.

The whole shunt is composed of several resistances S_1, S_2, S_3, S_4, . . . When the sliding contact Sl, is at 1, the total circuit current for full-scale deflection is, say, I_1 ; when at 2, it is I_2 ; when at 3, I_3 ; when at 4, I_4.

With the sliding contact at 1 the resistance in series with r is $S_2 + S_3 + S_4$ and $r + S_2 + S_3 + S_4$ is shunted by S_1, so that the multiplying power of the shunt is

$$m_1 = \frac{I_1}{i} = \frac{S_2 + S_3 + S_4 + r + (S_1)}{S_1}.$$

When the sliding contact is at 2, the resistance in series with r is $S_3 + S_4$, and $S_3 + S_4 + r$ is shunted by $S_1 + S_2$, so that

$$m_2 = \frac{I}{i} = \frac{S_3 + S_4 + r + (S_1 + S_2)}{S_1 + S_2}.$$

Similarly, at 3 we have

$$m_3 = \frac{I}{i} = \frac{S_4 + r + (S_1 + S_2 + S_3)}{S_1 + S_2 + S_3}.$$

And at 4

$$m_4 = \frac{I}{i} = \frac{r + (S_1 + S_2 + S_3 + S_4)}{S_1 + S_2 + S_3 + S_4}.$$

The ratios of the multiplying powers are

$$\frac{m_1}{m_2} = \frac{S_2 + S_3 + S_4 + r + (S_1)}{S_1} \div \frac{S_3 + S_4 + r + (S_1 + S_2)}{S_1 + S_2} = \frac{S_1 + S_2}{S_1}$$

$$\frac{m_2}{m_3} = \frac{S_1 + S_2 + S_3}{S_1 + S_2}; \quad \frac{m_3}{m_4} = \frac{S_1 + S_2 + S_3 + S_4}{S_1 + S_2 + S_3},$$

all these ratios being independent of r.

(*e*) Correction of Temperature Effect. The resistance of a conductor is given at some standard temperature, say, 0°C, or 20°C, or some other temperature. Over moderate ranges of temperature, say between 0°C and 100°C, the resistance is directly proportional to the change of temperature. If the resistance of the conductor is R_0 at 0°C, it will be, at a temperature t (within the limited range indicated)

$$R_t = R_0 [1 + \alpha_0 t] \qquad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

α_0 is the resistance temperature coefficient of the given conductor, referred to the temperature 0°C.

If the reference temperature is not zero, but some other temperature, say, t_1, then the resistance of the conductor at the temperature t is $R_t = R_{t1} [1 + \alpha_{t1} (t - t_1)]$, where α_{t1} is the temperature coefficient of resistance of the given conductor referred to the temperature t_1. α_{t1} is not the same as α_0, and is given by

$$\alpha_{t1} = \alpha_0 \frac{R_0}{R_{t1}} = \frac{\alpha_0 R_0}{R_0 (1 + \alpha_0 t_1)} = \frac{1}{\dfrac{1}{\alpha_0} + t_1}.$$

(103) takes no account of the linear expansion (change of dimensions of the conductor due to the change in temperature). If ρ_{t1} is the resistivity of a conductor at the temperature t_1, then its resistivity

at the temperature t will be, when taking into account the linear expansion, $\varrho_t = \varrho_{t1}[1 + \beta_{t1}(t - t_1)]$. β_{t1} is the temperature coefficient at constant mass.

We have $\beta_{t1} = \alpha_{t1} + \gamma$; γ being the coefficient of linear expansion, but as γ is generally very small compared with α_{t1} we have $\beta_{t1} \doteqdot \alpha_{t1}$; therefore (103) is very nearly true.

For industrial copper $\alpha_0 = 42{\cdot}7 \times 10^{-4}$, while $\gamma = 17 \times 10^{-4}$.

As α_{t1} for copper at 15°C, which is an average laboratory temperature, is about 0·004, if the temperature rises, say, by 1°C, the increase in resistance in the coil of an ammeter will be

$$R_{16} - R_{15} = R_{15}[1 + 0{\cdot}004] - R_{15} = 0{\cdot}004R_{15},$$

that is, 0·4% per °C, the current in the coil will also vary, and decrease by 0·4% °C (neglecting the temperature coefficient of the manganin shunt). This variation being far too great for laboratory instruments, special arrangements have to be made in order to lessen this variation.

(*f*) Arrangements for Temperature Effect Correction. The first arrangement is shown in fig. 80.

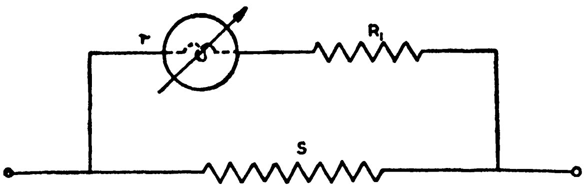

Fig. 80

A resistance R_1 of manganin is connected in series with the ammeter coil of resistance r (coil and springs), the two being connected across the shunt S.

Let $R_1 = kr$, where k is a constant (at the normal temperature). Neglecting the temperature coefficient of magnanin, for a temperature rise of 1°C from 15°C, the increase in resistance of $R_1 + r$ will be $kr + 1{\cdot}004\,r - r - kr = 0{\cdot}004r$ and the percentage increase will be $\dfrac{0{\cdot}004r}{r + kr} \times 100 = \dfrac{0{\cdot}4}{1 + k}\,\%$.

If, say, $r = R_1$; that is, $k = 1$; the percentage increase per °C will be $\dfrac{0{\cdot}4}{2} = 0{\cdot}2\%$ instead of 0·4% when R_1 is not present.

The resistance R_1 is sometimes called the swamping resistance ; k is very often equal to 1, the increase of resistance is then one-half of what it would be without R_1. We could get a better correction if

k were greater than 1, but then for the same ammeter sensitivity (same r and same number of turns on the coil), the p.d. across the instrument would be too great.

The arrangement of fig. 80 not being good enough for a precision ammeter, a better arrangement is shown in fig. 81.

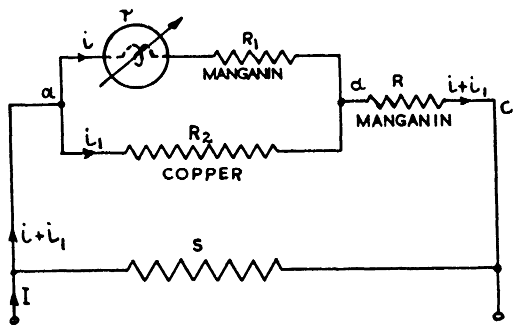

Fig. 81

R_1, R, and S are of manganin ; r is the resistance of the ammeter coil ; R_2 is of copper.

If the temperature varies, increasing, for example, then the resistance of r and R_2 increase ; the currents i and i_1 decrease ; the p.d. across R decreases ; and as the drop across S is practically constant, the p.d. across $r + R_1$ increases ; there is therefore a certain compensation for the increase in resistance.

To facilitate the calculation, assume $r = R_1$, $R_2 = R = 2r$ at 15°C ; then at this temperature the resistance across a,a (fig. 81) will be $\frac{2r \times 2r}{2r + 2r} = r$ and the resistance across a, c is $r + 2r = 3r$.

The current in R is $i + i_1 = I\frac{S}{S + 3r}$ and $i = i_1 = I\frac{S}{2(S + 3r)}$.

Suppose now that the temperature has risen by 1°C ; the coil resistance becomes $r_n = 1{\cdot}004r$; R_2 becomes $R_{n_2} = 2{\cdot}008r$; while R_1, R and S remain the same if we neglect the temperature coefficient of manganin.

The resistance across a,a is now $\frac{2{\cdot}004r \times 2{\cdot}008r}{2{\cdot}004r + 2{\cdot}008r} = 1{\cdot}003r$, the resistance across a,c becomes $2r + 1{\cdot}003r = 3{\cdot}003r$.

The currents i and i_1 now become i_n and i_{1n}, and assuming that the total current I has not changed, we have

$$i_n + i_{1n} = I \frac{S}{S + 3{\cdot}003r} \text{ and } i_n = (i_n + i_{n1}) \times \frac{2{\cdot}008r}{2{\cdot}008r + 2{\cdot}004r} = \frac{IS}{S + 3{\cdot}003r} \times \frac{2{\cdot}008r}{4{\cdot}012r}.$$

The ratio of the altered current i_n to the current i is therefore

$$\frac{i_n}{i} = \frac{IS \times 2{\cdot}008r}{(S + 3{\cdot}003r) \times 4{\cdot}012r} \div \frac{IS}{2\,(S + 3r)} \cong \frac{4{\cdot}016S + 12{\cdot}048r}{4{\cdot}012S + 12{\cdot}048r}.$$

If we had a multiplying power of the shunt

$$m = \frac{I}{i} = 10, \text{ that is, } S = \frac{r}{m - 1} = \frac{r}{9},$$

we have then

$$\frac{i_n}{i} = \frac{4{\cdot}016 + 9 \times 12{\cdot}048}{4{\cdot}012 + 9 \times 12{\cdot}048} = \frac{112{\cdot}448}{112{\cdot}444} \cong 1{\cdot}000035\,;$$

the change in current is therefore 0·0035% per °C, which is quite negligible if the temperature changes are normal.

(2) Voltmeters

(*a*) Practical Sensitivity of a Voltmeter. We define the practical sensitivity of the voltmeter as the voltage across the instrument required to deflect the pointer by one division of the scale.

We have seen that the theoretical volt sensitivity is

$$\frac{a}{v} = \frac{a}{IR} = \frac{HSn}{R\tau} \text{ and as } R = \frac{nl}{s}\varrho, \text{ we get } \frac{a}{v} = \frac{HSns}{\tau n l \varrho} = \frac{HS}{\tau} \times \frac{s}{l\varrho}.$$

The sensitivity is proportional to s and inversely proportional to l and ϱ whilst being of course proportional to n. A galvanometer used as a voltmeter has therefore to have a small resistance, and as a voltmeter is connected across a circuit (or part of a circuit), the voltage of which is to be measured, the voltmeter has to have a high resistance in order to take a small current. The voltmeter will therefore be a galvanometer of small resistance in series with a high resistance.

(*b*) Definition of the Quality of a Voltmeter with Respect to the Current Taken. The sensitivity, as well as the current taken and the power absorbed by the voltmeter will depend on its resistance, given in ohms per volt. The higher this number, the better the voltmeter. The ohms per volt vary from 10 to 1000 and beyond.

Example 25. A voltmeter of 1000 ohms per volt has a maximum scale indication of 150 volts. What is its total resistance?

The total resistance of the voltmeter is $150 \times 1000 = 150000\Omega$.

(*c*) VOLTMETERS OF SEVERAL SENSITIVITIES. An arrangement of a voltmeter of three sensitivities is shown in fig. 82.

The terminals are 0–1, 0–2, and 0–3. Let the resistance of the coil and springs be 10 ohms and the current for maximum deflection 1 milliamp.

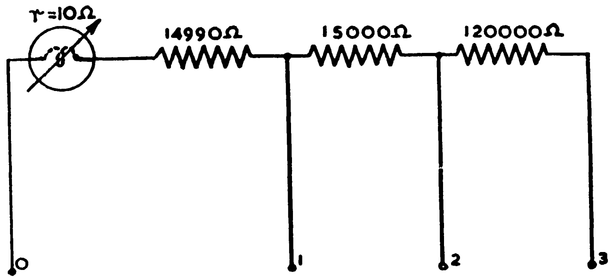

Fig. 82

The p.d. across the movement for full scale deflection is

$$10 \times 0{\cdot}001 = 0{\cdot}01 \text{ v.}$$

The ohms per volt are therefore $\frac{10}{0{\cdot}01} = 1000\Omega$.

Assume that the instrument has to have a full-scale deflection of

15 volts at the terminals 0–1,
30 volts at the terminals 0–2,
150 volts at the terminals 0–3.

The resistances in series with the coil are then calculated as follows :

Terminals 0–1 : total resistance $= 15 \times 1000 = 15000\Omega$, the series resistance is therefore $15000 - 10 = 14990\Omega$.
Terminals 0–2 : total resistance $= 30 \times 1000 = 30000\Omega$, series resistance $= 30000 - 15000 = 15000\Omega$.
Terminals 0–3 : total resistance $= 150 \times 1000 = 150000\Omega$, series resistance $= 150000 - 30000 = 120000\Omega$.

(*d*) EFFECT OF TEMPERATURE CHANGES. The resistance in series with the instrument movement are made of manganin, and as the temperature coefficient of manganin is negligible, the effect of temperature changes on the total voltmeter resistance is much smaller than in the case of an ammeter.

Suppose that in the voltmeter of three sensitivities given above there is a rise of 1°C. from the normal temperature of 15°C., the terminals 0–1 being used.

The resistance of the voltmeter will become

$$10\,(1 + 0{\cdot}004) + 14990 = 15000{\cdot}04\Omega$$

and the percentage change will be

$$100 \times \frac{15000{\cdot}04 - 15000}{15000} = 0{\cdot}000266\%,$$

which is quite negligible. The percentage change will be still less at the terminals 0–2 and 0–3.

(3) Measurement of the Resistance of an Ammeter

The resistance of an ammeter movement should be measured separately and not with its shunt. The ammeter resistance should not be calculated from its shunt resistance, and the ratio of the full-circuit current to the coil current (that is, the multiplying power of the shunt).

As $S = \dfrac{r}{m-1}$, we have $r = S\,(m - 1)$, and the logarithmic differential of r is $\dfrac{dr}{r} = \dfrac{dS}{S} + \dfrac{d\,(m-1)}{m-1} = \dfrac{dS}{S} + \dfrac{dm}{m}$, and the error on r will be $\dfrac{\Delta r'}{r'} = \dfrac{\Delta S'}{S'} + \dfrac{\Delta m'}{m'}$.

The error is therefore the error on S plus the error on m, while if the coil resistance is measured directly, we shall have one error only, which is of the order of the error on S.

The resistance can be measured on a Wheatstone bridge or by the opposition method. The movement ought to be clamped during the measurement and care has to be taken to limit the current to a suitable value.

(4) Measurement of the Resistance of a Voltmeter

The resistance of the movement of a voltmeter can be measured by the same methods as with the ammeter movement. The movement resistance should be measured alone, and not by measuring the movement resistance in series with one of the voltmeter external resistances, then measuring the external resistance alone and calculating the difference, because the error on the movement resistance can be very large.

Consider fig. 83.

Let r be the resistance of the movement and R the voltmeter external resistance. If a measurement first determines $R_T = r + R$ and then another determines R alone, we shall have $R_T - R = r$. The logarithmic differential will be

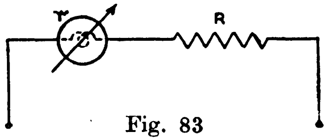
Fig. 83

$$\frac{dr}{r} = \frac{d(R_T - R)}{R_T - R} = \frac{dR_T}{R_T - R} - \frac{dR}{R_T - R} =$$

$$\frac{R_T}{R_T - R} \cdot \frac{dR_T}{R} - \frac{R}{R_T - R} \cdot \frac{dR}{R}.$$

Here, there is no partial elimination of the error due to the common part R. The error is therefore

$$\frac{\Delta r'}{r'} = \frac{R_T'}{R_T' - R'} \cdot \frac{\Delta R_T'}{R'} + \frac{R'}{R_T' - R'} \cdot \frac{\Delta R'}{R'},$$

which considering that R is much greater than r, can be very large.

Example 25. Let $R_T = 1010\ \Omega$, $R = 1000\ \Omega$, given by measurement, the errors on R_T and R being $\pm$ 0·1%.

The error on r is

$$\% \frac{\Delta r'}{r'} = \frac{1010 \times 0{\cdot}1}{10} + \frac{1000 \times 0{\cdot}1}{10} = \pm 20{\cdot}1\%,$$

which is definitely inadmissible.

For the same reason the resistance of an ammeter coil should not be calculated from the difference between the measurements of the coil plus swamping resistance and swamping resistance alone, even though the error will be much smaller than in the case of a voltmeter measured with its external resistance, because the swamping resistance is of the same order of resistance as the ammeter movement.

(5) Measurement of the Restoring Torque and Torsion Constant of an Ammeter or Voltmeter

In a permament-magnet moving-coil instrument we have $\tau a = \Phi_0 i$, where τ is the torsion constant, a the deflection corresponding to the current i in the moving coil, and Φ_0 the coil flux linkages.

As τ is supposedly constant, and Φ_0 is constant for small angles a, we ought to have a linear relation between a and i.

Let the deflection be a; if the coil is displaced by a very small angle da from its position a, then the work done is $\tau a da = \Phi_0 (da) i$.

But $\Phi_0 da = d\Phi_0 =$ flux cut, so that we have

$$\tau a da = d\Phi_0 \, i \text{ and } \tau a = \frac{d\Phi_0}{da} \cdot i.$$

If therefore we can determine $\frac{d\Phi_0}{da}$, we have τa and τ.

The arrangement for the test is shown in fig. 84.

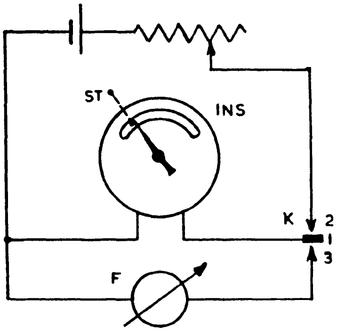

Fig. 84

The instrument tested is connected in series with a rheostat Rh to the source E through the contacts 1–2 of key k; in parallel with the instrument, through the contacts 1–3, we have a fluxmeter F.

In the case of an ammeter with a temperature correction arrangement as in fig. 80, the swamping resistance can be left in circuit, if the total resistance of the moving-coil, swamping resistance, and fluxmeter resistance, are low enough not to affect the fluxmeter reading. When the temperature correction arrangement is as in fig. 81 R_1 and R can be left if they are small enough not to affect the fluxmeter, but here we shall have a fluxmeter in series with R, shunted by R_2. (It is not advisable to disconnect R_2.)

In the case of a voltmeter the resistances external to the coil have to be out of the circuit, because, being fairly high, they will definitely affect the fluxmeter readings.

(*a*) Manipulation. With Rh at a safe value, make 1–2 and regulate Rh till the instrument pointer is at $\frac{1}{10}$ of the scale maximum. Break 1–2 and immediately make 1–3. The pointer will come back to zero, and while doing so the coil moving between the poles of the permanent magnet will have a current induced in it. The fluxmeter will start moving practically instantaneously with the pointer and will stop when the pointer stops. According to (84a) we shall have

$\Delta\Phi_0 = k\,(m - m_0)$; or, if the fluxmeter starts from zero, $\Delta\Phi_0 = km$.

The experiment is repeated for each tenth of the scale till full-scale deflection is reached.

Having secured the results of the test for the whole of the scale, $\Delta\Phi_0$ is plotted against a in radians. The result ought to be a straight line if Φ_0 is constant. The slope of this line is $\frac{d\Phi_0}{da}$ and $\frac{d\Phi_0}{da}$ multiplied by i will give the restoring torque for the corresponding values of i.

Plotting τa against i, we ought to get a straight line if τ is constant, the value of τ is $\frac{\tau a}{a}$ = the restoring torque for a given a divided by this value of a. The current i is the current in the coil for the particular position of the pointer.

(*b*) Errors Inherent in the Method. (*a*) When breaking 1–2 and making 1–3, there might be transferred to the fluxmeter part of the energy $\frac{1}{2}Li^2$, where L is the inductance of the instrument coil, and an erroneous reading will result. The part of $\frac{1}{2}Li^2$ transferred to the fluxmeter depends on the speed of switching over from 2 to 3. As i is usually small and the coil of the instrument is wound on a metal former, which acts like the secondary of a transformer, for any change of current in the coil, the error introduced is very small and practically negligible. Most of the energy $\frac{1}{2}Li^2$ will be used in heating the metal former.

(*b*) The switch k takes time to pass from 2 to 3, and during this time the pointer of the instrument coil will move while the fluxmeter will be stationary. Care has to be taken, therefore, to make the distance between 2 and 3 as small as possible, and to switch over quickly.

If a test free of the above errors is required, proceed as follows. With the connections as in fig. 84, we have a mechanical stop St. (dotted lines in fig. 84) by means of which the pointer can be held at, or released from any position, with no current in the coil (the glass over the scale has of course to be removed for the test). The pointer is brought to any desired position by making 1–2; St. is then set so that the pointer cannot go back to zero, and 1–2 is opened; 1–3 is made a short time after 1–2 is broken, and when 1–3 is made, St. is released, the pointer now coming back to zero.

With a temperature effect correction arrangement as in fig. 81 we shall have $(n - n_0)\, k\, \frac{(r + R_1) + R_2}{R_2} = \Delta\Phi_0$ (see formula 98a). The resistance R is taken as included in the resistance of the fluxmeter.

A ballistic galvanometer should not be used instead of the fluxmeter. The pointer takes an appreciable time to come back to zero, and an error will be introduced into the ballistic galvanometer deflection.

CHAPTER VIII

THE PERMANENT-MAGNET MOVING-COIL DIFFERENTIAL GALVANOMETER

(1) Principle

The differential galvanometer is identical with the ordinary permanent-magnet moving-coil galvanometer, except that it has two moving coils within the poles of the permanent magnet.

The coils have a common suspension to which the light reflecting mirror is attached.

The principle is shown in fig. 85.

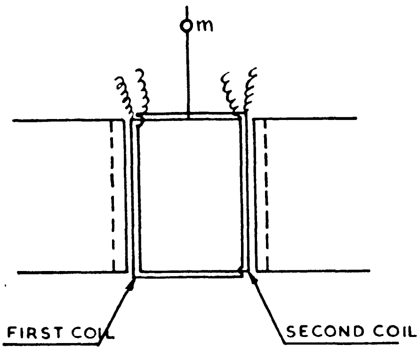

Fig. 85

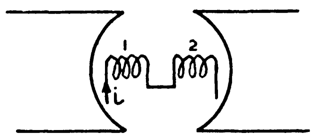

Fig. 86

The current is brought to the coils by four wide spirals, made of very thin silver wire. The torsion of these spirals is negligible, and the suspension, mirror, permanent magnet, etc., are identical with those of an ordinary reflecting galvanometer.

The two moving coils have to be as like as possible ; that is, they must have the same surface, number of turns, resistance, and flux linkages ; they have to be placed similarly within the poles of the permanent magnet, so that their flux linkages Φ_{01} and Φ_{02} will be the same.

If the coils were entirely alike geometrically and electrically, then when the same current flows in the two coils and in directions such that the torques on the two coils are in opposition (fig. 86), the resulting torque will be zero, that is, $\tau a = \Phi_{01}\, i - \Phi_{02}\, i = 0$, and the spot will remain at zero.

The differential galvanometer is mainly used in this way; that is, so that the torques on the two moving coils are in opposition; generally, however it is not possible to have the two coils entirely identical, and therefore $\tau a = \Phi_{01}\, i - \Phi_{02}\, i$ is not zero.

The differential galvanometer has many applications, such as the measurement of low, medium and high resistance, measurement of the difference or ratio of two currents, etc. Before it can be used, however, we have to eliminate the error which might result from the coils not being identical or identically placed.

When testing if the resulting torque is zero, care has to be taken to have a suitable resistance in series with the galvanometer coils, otherwise the current might be excessive, without the slightest visual indication (no deflection), when the resultant torque is zero.

(2) Check on the Differential Galvanometer

The arrangement is shown in fig. 87.

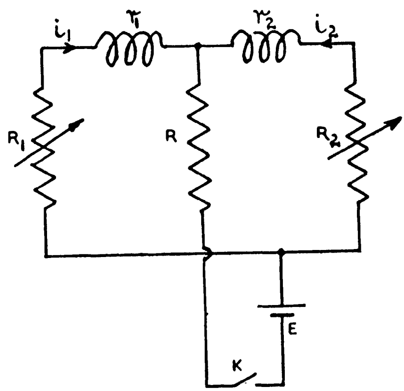

Fig. 87

Let r_1 and r_2 be the resistances of the two moving coils 1 and 2; R_1 and R_2 are two standard variable resistances; R is a fixed, suitable standard resistance. R, together with R_1 and R_2, should be high enough to make the currents i_1 and i_2 suitable for the coils. R_1 and R_2 are varied till the galvanometer shows no deflection, and when this happens we have $\Phi_{01}\, i_1 = \Phi_{02}\, i_2$; but

$$i_1 = \frac{E}{R + R_1 + r_1}; \; i_2 = \frac{E}{R + R_2 + r_2},$$

so that

$$\Phi_{01}\, i_1 = \frac{E\,\Phi_{01}}{R + R_1 + r_1}; \; \Phi_{02}\, i_2 = \frac{E\,\Phi_{02}}{R + R_2 + r_2}$$

and

$$\frac{i_1}{i_2} = \frac{R + R_2 + r_2}{R + R_1 + r_1} = p \text{ so that } i_1 = p \cdot i_2.$$

The nearer p is to unity, and the nearer r_1 is to r_2, the more alike are the coils and the better the galvanometer.

If we know that $r_1 = r_2$, and the resulting torque is not zero, then Φ_{01} is not equal to Φ_{02}; and as $\Phi_{01} i_1 = \Phi_{02} i_2$,

$$\frac{E\,\Phi_{01}}{R + R_1 + r_1} = \frac{E\,\Phi_{02}}{R + R_2 + r_2}; \quad \frac{\Phi_{01}}{\Phi_{02}} = \frac{R + R_1 + r_1}{R + R_2 + r_2} = \frac{1}{p}.$$

The resistance of the coils can be measured, one after the other, by Kelvin's false zero method.

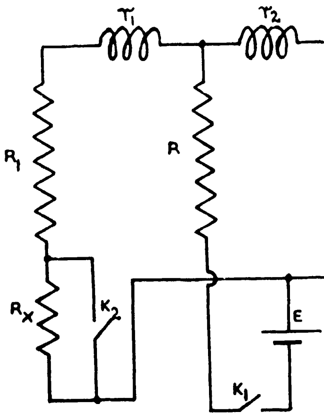

Fig. 88

(3) Measurement of Resistance Using the Differential Galvanometer

(*a*) Measurement of High Resistances. The arrangement is shown in fig. 88.

The resistance R_x which is to be measured can be shorted by a key k_2; R_1 and R_2 are variable standard resistances; R is a fixed standard resistance; and E is the source of current.

With k_2 open, we connect the source by closing k_1, and vary R_1 and R_2 till the deflection is zero; we then have

$$i_1 = \frac{E}{R + R_1 + r_1}; \quad i_2 = \frac{E}{R + R_2 + r_2}; \quad \frac{i_1}{i_2} = \frac{R + R_2 + r_2}{R + R_1 + r_1} = p.$$

Opening k_2, we have R_x in series with R_1. Without altering R_2, we diminish R_1 to, say, R_n, when the deflection is again zero. We have then

$$R_x = R_1 - R_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

The maximum value of the resistance which can be measured is equal to R_1.

Calculation of the Systematic Error. Let R_{11} be the value of R_1 which will give the smallest deflection, say, to the left, and R_{12} that which will give the smallest deflection to the right; in other words, $R_{12} - R_{11}$ is the indetermination on R_1. If, similarly, R_{1n} be the value of R_n giving the smallest deflection to the left, and R_{2n} that giving the smallest deflection to the right, the determination errors on R_1 and R_n will be

$$\frac{R_{12}-R_{11}}{R_1} \cong \frac{R_{12}-R_{11}}{R_{11}} = \frac{\varDelta_D R_1}{R_1};$$

$$\frac{R_{2n}-R_{1n}}{R_n} \cong \frac{R_{2n}-R_{1n}}{R_{1n}} = \frac{\varDelta_D R_n}{R_n}.$$

If $\varDelta_c R_1$ and $\varDelta_c R_n$ are the constructional errors on R_1 and R_n, then the total errors on R_1 and R_n are

$$\varDelta R_1 = \varDelta_c R_1 + \varDelta_D R_1; \quad \varDelta R_n = \varDelta_c R_n + \varDelta_D R_n.$$

The logarithmic differential of (104) is

$$\frac{dR_x}{R_x} = \frac{d(R_1 - R_n)}{R_1 - R_n} = \frac{dR_1}{R_1 - R_n} - \frac{dR_n}{R_1 - R_n} =$$

$$\frac{R_1}{R_1 - R_n} \cdot \frac{dR_1}{R_1} - \frac{R_n}{R_1 - R_n} \cdot \frac{dR_n}{R_n}$$

and the error on R_x is

$$\frac{\varDelta R'_x}{R'_x} = \frac{R'_1}{R'_1 - R'_n} \cdot \frac{\varDelta R'_1}{R'_1} + \frac{R'_n}{R'_1 - R'_n} \cdot \frac{\varDelta R'_n}{R'_n}.$$

The determination error can of course be made smaller by interpolation, as in the Wheatstone bridge. Let R_{11} be that value of R_1 which gives a small deflection α_1 to the left, and R_{12} that which gives a small deflection α_2 to the right; similarly R_{1n} is the value of R_n giving a small deflection α_{1n} to the left, and R_{2n} that which gives a small deflection α_{2n} to the right. We shall then have

$$R_1 = R_{11} + (R_{12} - R_{11}) \frac{\alpha_1}{\alpha_1 + \alpha_2};$$

$$R_n = R_{1n} + (R_{2n} - R_{1n}) \frac{\alpha_{1n}}{\alpha_{1n} + \alpha_{2n}}.$$

The determination errors on R_1 and R_n are now (calculated as in the Wheatstone bridge under the assumption that the constructional error is the same throughout R_1)

$$\frac{\varDelta_D R'_1}{R'_1} = \frac{R'_{12} - R'_{11}}{R'_1} \frac{\varDelta \alpha'}{\alpha'_1 + \alpha'_2}; \quad \frac{\varDelta_D R'_n}{R'_n} = \frac{R'_{2n} - R'_{1n}}{R'_n} \cdot \frac{\varDelta \alpha'}{\alpha'_n + \alpha'_{2n}}.$$

Again, as $\varDelta R'_1 = \varDelta_c R'_1 + \varDelta_D R'_1$ and $\varDelta R'_n = \varDelta_c R'_n + \varDelta_D R'_n$, we get

$$\frac{\varDelta R'_x}{R'_x} = \frac{R'_1}{R'_1 - R'_n} \frac{\varDelta R'_1}{R'_1} + \frac{R'_n}{R'_1 - R'_n} \frac{\varDelta R'_n}{R'_n}.$$

The greater $R_1 - R_n$, the smaller the error.

The above calculation does not apply to resistances such as dielectrics, which depend on too many outside factors.

Note that instead of decreasing R_1 to R_n, we could increase R_2 to, say, R_{2n}, and so get a new balance, with currents of i_{1n} and i_{2n}. We could then write

$$\frac{i_{1n}}{i_{2n}} = \frac{R + R_{2n} + r_2}{R + R_1 + R_x + r_1} = p, \text{ therefore}$$

$$\frac{R + R_{2n} + r_2}{R + R_1 + R_x + r_1} = \frac{R + R_2 + r_2}{R + R_1 + r_1},$$

from which we could get R_x.

To extend the range of measurement, R can be altered.

(*b*) MEASUREMENT OF MEDIUM RESISTANCES. The arrangement is shown in fig. 89.

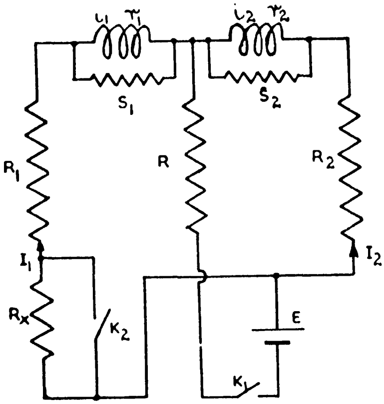

Fig. 89

The galvanometer coils are connected to shunts of resistances S_1 and S_2, otherwise the arrangement is the same as that of fig. 88. The resistance of coil 1 and shunt S_1 is $\varrho_1 = \frac{r_1S_1}{r_1 + S_1}$, that of coil 2 and shunt S_2 is $\varrho_2 = \frac{r_2S_2}{r_2 + S_2}$. With k_2 closed, we have

$$i_1 = \frac{E}{R + R_1 + \varrho_1}; \quad i_2 = \frac{E}{R + R_2 + \varrho_2},$$

R_1 and R_2 being varied till the galvanometer deflection is zero. Opening k_2, we have to diminish R_1 to, say, R_n in order to get zero deflection (leaving R_2 unchanged), and then have $R_x = R_1 - R_n$.

The only condition the shunts have to conform to is that their multiplying powers m_1 and m_2 should be such that $i_1 = \frac{I_1}{m_1}$ and $i_2 = \frac{I_2}{m_2}$ should be suitable for the galvanometer coils.

The error is calculated in the same way as for the measurement of high resistance.

(*c*) Measurement of Small Resistances. The arrangement is shown in fig. 90.

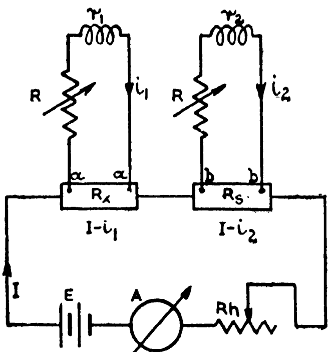

Fig. 90

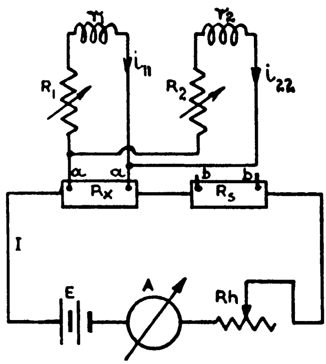

Fig. 91

R_X and R_S, the resistance to be measured and the standard resistance, are both four terminal. The current I is of a large value and supplied by the source E, measured by the ammeter A, and regulated by the rheostat Rh.

R_1 and R_2 are varied till there is no deflection.

The p.d. across *a,a,* is

$$(I - i_1) R_X = (r_1 + R_1) i_1 ; \quad i_1 = I \cdot \frac{R_X}{r_1 + R_1 + R_X}.$$

The p.d. across *b,b,* is

$$(I - i_2) R_S = (r_2 + R_2) i_2 ; \quad i_2 = I \cdot \frac{R_S}{r_2 + R_2 + R_S}$$

and

$$\frac{i_1}{i_2} = p = \frac{R_X}{R_S} \cdot \frac{r_2 + R_2 + R_S}{r_1 + R_1 + R_X} \quad . \quad . \quad . \quad . \quad . \quad (105)$$

Changing R_1 to, say, R_{11}, and R_2 to R_{22} till new balance is achieved, let the new currents be i_{11} and i_{22}. Then we now have

$$\frac{i_{11}}{i_{22}} = p = \frac{R_x(r_2 + R_{22} + R_s)}{R_s(r_1 + R_{11} + R_x)}, \text{ therefore}$$

$$\frac{r_2 + R_2 + R_s}{r_1 + R_1 + R_x} = \frac{r_2 + R_{22} + R_s}{r_1 + R_{11} + R_x}$$

from which we get

$$R_x = \frac{(r_2 + R_s)(R_1 - R_{11})}{R_2 - R_{22}} - r_1 + \frac{R_{22}R_1 - R_2R_{11}}{R_2 - R_{22}}.$$

A better way of proceeding is as follows: after having determined p (105), disconnect the terminals $b,b,$ from R_s and connect them to R_x, as shown in fig. 91.

Without altering R_1, change R_2 to, say, R_{22}, until there is no deflection. Let the new currents in the galvanometer coils then be i_{11} and i_{22}, giving $i_{11}(R_1 + r_1) = i_{22}(R_{22} + r_2)$, and therefore

$$\frac{i_{11}}{i_{22}} = p = \frac{R_{22} + r_2}{R_1 + r_1} \text{ and } \frac{R_x}{R_s} \times \frac{r_2 + R_2 + R_s}{r_1 + R_1 + R_x} = \frac{R_{22} + r_2}{R_1 + r_1}$$

$$\frac{R_x}{R_s} = \frac{R_{22} + r_2}{R_1 + r_1} \cdot \frac{r_1 + R_1 + R_x}{r_2 + R_2 + R_s}.$$

As R_x and R_s are very small, we have approximately

$$\frac{R_x}{R_s} \simeq \frac{R_{22} + r_2}{R_1 + r_1} \cdot \frac{r_1 + R_1}{r_2 + R_2} \simeq \frac{R_{22} + r_2}{R_2 + r_2} \text{ and } R_x = \frac{R_{22} + r_2}{R_2 + r_2} \cdot R_s.$$

Or, as we can write $R_{22} = R_2 \pm r$ because R_{22} and R_2 are in the same resistance box, $R_x = \dfrac{R_2 \pm r + r_2}{R_2 + r_2} \cdot R_s$.

Calculation of the Systematic Error. The logarithmic differential of R_x is

$$\frac{dR_x}{R_x} = \frac{d(R_2 \pm r + r_2)}{R_2 \pm r + r_2} - \frac{d(R_2 + r_2)}{R_2 + r_2} + \frac{dR_s}{R_s} =$$

$$\frac{dR_2}{R_2 \pm r + r_2} \pm \frac{dr}{R_2 \pm r + r_2} + \frac{dr_2}{R_2 \pm r + r_2} - \frac{dR_2}{R_2 + r_2} - \frac{dr_2}{R_2 + r_2} +$$

$$\frac{dR_s}{R_s} = \frac{R_2}{R_2 \pm r + r_2}\,\frac{dR_2}{R_2} \pm \frac{r}{R_2 \pm r + r_2}\,\frac{dr}{r} + \frac{r_2}{R_2 \pm r + r_2}\,\frac{dr_2}{r_2} -$$

$$\frac{R_2}{R_2 + r_2}\,\frac{dR_2}{R_2} - \frac{r_2}{R_2 + r_2}\,\frac{dr_2}{r_2} + \frac{dR_s}{R_s}.$$

The terms containing $\frac{dR_2}{R_2}$ and $\frac{dr_2}{r_2}$ are nearly equal because r is very small compared with $R_2 + r_2$ ($r = 0$ when $R_x = R_s$); the term containing $\frac{dr}{r}$ is also very small and for all practical purposes $\frac{dR_x}{R_x} = \frac{dR_s}{R_s}$, the error is therefore $\frac{\Delta R'_x}{R'_x} = \frac{\Delta R'_s}{R'_s}$.

A cause of error is the disturbing of the equipotential surfaces at *a,a*, and *b,b*. The currents i_1, i_2 and i_{11}, i_{22} ought therefore to be very small compared with I.

We will note for the present that the differential galvanometer when calibrated can be used for the measurement of the ratio or the difference of two currents.

One of the advantages of the differential galvanometer is that it can be used in non-oscillatory conditions and yet be connected in series with a high resistance. This is achieved by short-circuiting one moving coil and using the other coil as in an ordinary galvanometer.

CHAPTER IX

MEASUREMENT OF THE RESISTANCE OF BATTERIES

(1) Mance's Method

The arrangement is shown in fig. 92.

We have a bridge arrangement with the resistances R_1, R_3 and R_4 constituting three arms, while the battery E the resistance of which has to be measured form the fourth arm.

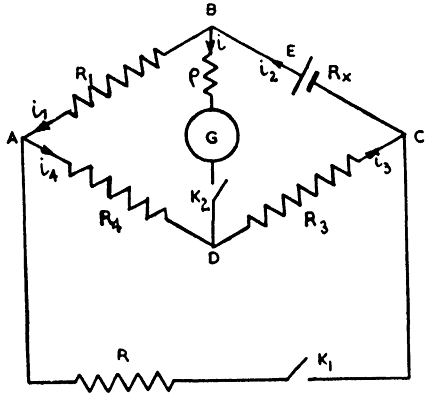

Fig. 92

The galvanometer diagonal contains a variable resistance ϱ for the galvanometer protection and the switch k_2; the second diagonal contains the resistance R and switch k_1.

With k_1 open and k_2 closed there will be a current in the galvanometer and this current is regulated by means of ϱ till the galvanometer deflection a is of suitable value. The bridge resistances are then varied till, when closing or opening k_1, the deflection a remains unchanged.

Considering the loop BCD (fig. 92): with the notation of the fig. we have

$$E - R_3 i_3 - (g + \varrho)\, i - R_x i_2 = 0\,;$$

$$(g + \varrho)\, i = E - R_3 i_3 - R_x i_2 \quad . \quad . \quad . \quad . \quad (106)$$

where g is the galvanometer resistance and R_x the resistance of E.

Considering loop ABD, we have

$$- R_1 i_1 - R_4 i_4 + (g + \varrho)\, i = 0\,;$$

$$(g + \varrho)\, i = R_1 i_1 + R_4 i_4 \quad . \quad . \quad . \quad . \quad . \quad (106a)$$

Differentiating (106) and (106a) we get

$$(g + \varrho)\, di = - R_3 di_3 - R_x di_2\,; \quad (g + \varrho)\, di = R_1 di_1 + R_4 di_4.$$

But if the closing and opening of k_1 does not alter i, then $di = 0$, and we can write $0 = - R_3 di_3 - R_x di_2 = R_1 di_1 + R_4 di_4$.

$- R_3 di_3 = R_x di_2\,; \quad R_1 di_1 = - R_4 di_4\,;$

$$-\frac{di_3}{di_2} = \frac{R_x}{R_3}; \quad -\frac{di_1}{di_4} = \frac{R_4}{R_1}.$$

Considering now the junction points B and D, we can write $i_2 = i + i_1$; $i_3 = i + i_4$, and differentiating $di_2 = di + di_1$; $di_3 = di + di_4$, and as $di = 0$, we have $di_2 = di_1$; $di_3 = di_4$; therefore

$$-\frac{di_3}{di_2} = -\frac{di_4}{di_1} = \frac{R_x}{R_3} = \frac{R_1}{R_4}; \; R_xR_4 = R_1R_4 \text{ and } R_x = \frac{R_1}{R_4} R_3;$$

the same relation as in a Wheatstone bridge.

A shunt across the galvanometer could be used if necessary. This is helpful because the galvanometer, being shunted, will work in non-oscillatory conditions. The resistances used should be great enough to prevent too great a current, which would polarise the source of e.m.f.

(2) Voltmeter Methods

The arrangement is shown in fig. 93.

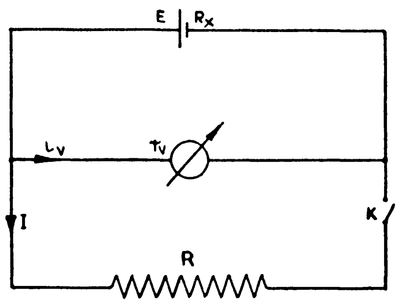

Fig. 93

The voltage across the battery is measured twice; first with k open (R disconnected), secondly with k closed (R connected). With k open, the voltmeter of resistance r_v is alone across the battery. It reads E_1, which is taken as the battery e.m.f., the real battery e.m.f. being E.

When k is closed, the battery delivers a current I through R (the voltmeter current i_v being neglected). The voltmeter will now read $V < E_1$.

Neglecting the resistance of the connections, we have

$$I = \frac{V}{R} = \frac{E_1}{R_x + R}; \; V(R_x + R) = E_1R; \; R_x = \frac{E_1 - V}{V} \cdot R \quad (107)$$

R_x is the battery resistance.

(*a*) Error Inherent in the Method.[1] With k open, we have in reality $i_v = \dfrac{E}{R_x + r_v}$; $i_v(R_x + r_v) = E$, while the voltmeter shows $E_1 = r_v i_v$. Dividing E by E_1, we get

$$\frac{E}{E_1} = \frac{R_x + r_v}{r_v}; \quad E_1 = \frac{Er_v}{R_x + r_v}; \quad E = \frac{E_1(R_x + r_v)}{r_v} \quad . \qquad (108)$$

With k closed, the voltmeter shows V, which is the p.d. across the battery or across R, if we neglect the resistance of the connections; the voltmeter current will now be i_{v1}, and we have

$$V = i_{v1} r_v = RI = (I + i_{v1}) \frac{R r_v}{R + r_v} \quad . \quad . \quad . \qquad (109)$$

but $(I + i_{v1}) = \dfrac{E}{R_x + \dfrac{R r_v}{R + r_v}}$; and substituting for $(I + i_{v1})$ in (109)

$$V = \frac{E}{R_x + \dfrac{Rr_v}{R + r_v}} \cdot \frac{Rr_v}{R + r_v} = \frac{ERr_v}{R_x(R + r_v) + Rr_v}.$$

We do not know E, but we have the relation (108), and substituting for E we get

$$V = \frac{E_1(R_x + r_v)Rr_v}{r_v[R_x(R + r_v) + Rr_v]} = \frac{E_1(R_x + r_v)R}{R_x(R + r_v) + Rr_v}, \text{ therefore}$$

$V[R_x(R + r_v) + Rr_v] = E_1R_xR + E_1Rr_v$, or
$R_x(VR + Vr_v - E_1R) = R(E_1r_v - Vr_v)$.

$$R_x = \frac{(E_1 - V)Rr_v}{V\left(R + r_v - E_1\dfrac{R}{V}\right)} = \frac{E_1 - V}{V} \cdot R \cdot \frac{r_v}{R + r_v - \dfrac{E_1R}{V}} =$$

$$\frac{E_1 - V}{V} \cdot R \cdot \frac{r_v}{R\left(1 - \dfrac{E_1}{V}\right) + r_v} \quad . \quad . \quad . \quad . \quad . \qquad (110)$$

Comparing (110) with (107), we see that the correction factor is $\dfrac{r_v}{R\left(1 - \dfrac{E_1}{V}\right) + r_v}$, and the greater r_v compared with R, the nearer the correction factor is to unity. We have therefore to use a volt-

[1] For further details, see Bedeau: *Cours de Mesures Electriques* (S.F.E.; E.S.E., Paris; Vol. I).

meter of great resistance, so that R should be negligible compared with r_V. If R is not negligible compared with r_V, the correction factor above has to be applied.

(*b*) CALCULATION OF THE SYSTEMATIC ERROR. Assuming that R is negligible compared with r_V, we have

$R_x = \frac{E_1 - V}{V} \cdot R$ and the log. differential of R_x is

$$\frac{dR_x}{R_x} = \frac{d(E_1 - V)}{E_1 - V} - \frac{dV}{V} + \frac{dR}{R} = \frac{dE_1}{E_1 - V} - \frac{dV}{E_1 - V} - \frac{dV}{V} + \frac{dR}{R} =$$

$$\frac{E_1}{E_1 - V} \cdot \frac{dE_1}{E_1} - \frac{V}{E_1 - V} \cdot \frac{dV}{V} - \frac{dV}{V} + \frac{dR}{R} =$$

$$\frac{E_1}{E_1 - V}\frac{dE_1}{E_1} - \frac{dV}{V}\left(\frac{V}{E_1 - V} + 1\right) + \frac{dR}{R} =$$

$$\frac{E_1}{E_1 - V}\frac{dE_1}{E_1} - \frac{dV}{V} \cdot \frac{E_1}{E_1 - V} + \frac{dR}{R}; \text{ the error is therefore}$$

$$\frac{\Delta R'_x}{R'_x} = \frac{E'_1}{E'_1 - V'}\frac{\Delta E'_1}{E'_1} + \frac{E'_1}{E'_1 - V}\frac{\Delta V'}{V'} + \frac{\Delta R'}{R'} =$$

$$\frac{E'_1}{E'_1 - V'}\left(\frac{\Delta E'_1}{E'} + \frac{\Delta V'}{V'}\right) + \frac{\Delta R'}{R'}.$$

$\Delta E'$ and $\Delta V'$ comprise the constructional plus reading errors on the voltmeter.

If the voltmeter indication is known to be absolutely proportional, then $\frac{dE_1}{E_1} = \frac{dV}{V}$, and it follows that the maximum relative error is

$$\frac{dR'_x}{R'_x} = \frac{E'_1}{E'_1 - V'}\left(\frac{\Delta_r E'_1}{E'} + \frac{\Delta_r V'}{V'}\right) + \frac{\Delta R'}{R'},$$ where $\Delta_r E'_1$ and $\Delta_r V'_1$ are the reading errors only.

(3) Measurement of the Resistance of a Battery of Small Internal Resistance

When dealing with batteries of small internal resistance, such as accumulators, we can neglect the difference between the e.m.f. and the reading of the voltmeter when *k* is open, if the voltmeter is of high resistance, so that $E_1 \doteqdot E$, but the resistance of the connections is no longer negligible compared with R and R_x.

Two measurements are now necessary after E_1 has been determined :

(*a*) With *k* closed (R connected) we measure the p.d. across the battery, fig. 94, and we have

$$V = E - R_x I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

(*b*) With *k* closed we measure the p.d. across R where $V_1 = R\,I$

$I = \frac{V_1}{R}$. Substituting for I in (111), we get $V = E - \frac{R_x V_1}{R}$,

therefore $R_x = \frac{(E - V)}{V_1} \cdot R$.

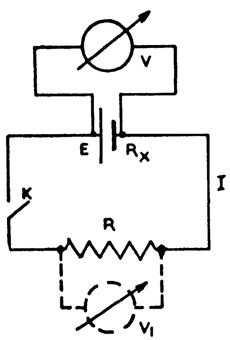

Fig. 94

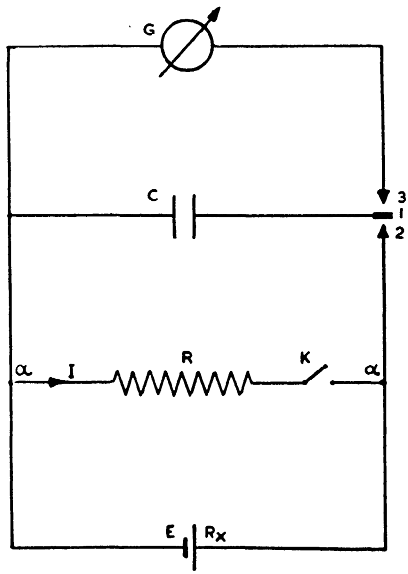

Fig. 95

In order that the error should be small, V ought to be an appreciable part of E ; this means that I has to be intense.

In all the voltmeter methods described, the choice of R is governed by the magnitude of the current supplied by the source ; I ought to be such that there is no danger of the source getting polarised.

(4) Munro's Method

The arrangement is shown in fig. 95.

G is a ballistic galvanometer, C a standard condenser, R a standard variable resistance, and E the battery; the resistance R_x of which has to be determined.

With k open, make 1–2, charging the condenser with a quantity of electricity $Q = C\,E$.

Breaking 1–2 and making 1–3, the condenser discharges through the ballistic galvanometer, the deflection being n; and if s_{cp} is the galvanometer constant (practical sensitivity in coulombs), we can write

$$C\,E = s_{cp}\,n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

Next, with k closed, the battery provides a current I through R so that the p.d. across a, a is $V = R\,I$; $I = \dfrac{E}{R + R_x}$; $V = \dfrac{R\,E}{R + R_x}$.

Making 1–2 again, the condenser is charged to $Q_1 = V\,C$; breaking 1–2 and making 1–3 the condenser discharges, the deflection being n_1, and

$$V\,C = s_{cp}\,n_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112a)$$

Dividing (112) by (112a) we have

$$\frac{E}{V} = \frac{n}{n_1};\ V = E\,\frac{n_1}{n} = \frac{R\,E}{R + R_x},\ \text{therefore}\ \frac{n_1}{n} = \frac{R}{R + R_x}$$

$$R_x = R \cdot \frac{(n - n_1)}{n_1}.$$

This method, although very simple, requires many precautions. The deflections n and n_1 have to differ considerably, otherwise the precision of the measurement will be small. On the other hand n_1 should not be too small, because if so the reading error will be great. Making $n - n_1$ large means a small resistance R, and therefore a large current I. The source can therefore polarise, and its characteristics will vary. What we need therefore is a measure of E at the same time as we measure V, and to achieve this we proceed as follows.

With k closed, we get a measure of V by charging and then discharging the condenser; then, with k open, we take several measurements of E (by charge and discharge of C) at regular intervals of time (say, at times t_1, t_2, t_3, . . ., t_n), the time being counted from the determination of V. Let the deflections corresponding to these times be n_1, n_2, n_3 . . ., n_4; we construct the curve $n = f(t)$ as shown in fig. 96, and by extrapolating this curve (part $n_1\ n$) we can approximately determine $n = o\text{–}a$ at time $t = O$; $o\text{–}a$ is an approximate value of the deflection which would have been obtained, had the time between the determination of V and E been zero.

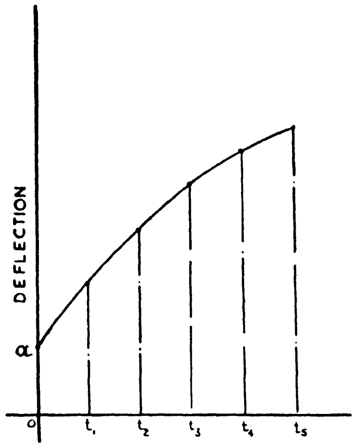

Fig. 96

CHAPTER X

MEASUREMENT OF THE INSULATION RESISTANCE OF INSTALLATIONS. DIRECT-READING OHMMETERS

THE insulation resistance of an installation, either to ground or between conductors, depends on so many uncontrollable factors, such as humidity, surface conditions, etc., that there can be no question of great accuracy. The calculation of the error in these measurements would therefore be meaningless.

When a line or a cable is tested, it should always be connected to the negative pole of the source supplying the testing current, because, if connected to the positive pole, oxidisation will be produced at the fault. This oxidisation will increase the fault resistance, and so give an erroneous, optimistic result. If connected to the negative pole, a disoxidisation will result, which will diminish the fault resistance, and it will be safer to accept the result obtained in this manner.

(1) The Comparison Method of Measuring the Insulation Resistance of a Main to Earth

The arrangement is shown in fig. 97.

The tested main is shown on the insulators 1, 2, 3.

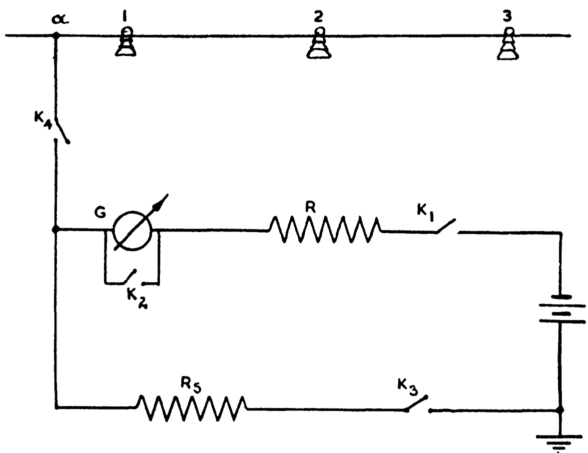

Fig. 97

The main, which is cut off from its supply, is connected at a, through the switch k_4 to the galvanometer G in series with a protecting resistance R, and switch k_1 to the negative pole of the source E; the positive pole of E is connected to earth. R_s is a known high resistance. The earth has to be a connection to a main water pipe, or made by burying in wet ground a clean metal sheet, of at least one yard square. The switch k_2 protects the galvanometer from the capacity current arising when switching on k_1.

The method is as follows. With k_3 open and k_2, k_4 closed, close k_1, then, after a short time, open k_2. The galvanometer current will be $i = \dfrac{E}{R + R_x + r}$, where R_x is the cable insulation resistance and r the galvanometer resistance (neglecting the resistance of the source). As r is small compared with R_x, we can write $i = \dfrac{E}{R + R_x}$ and if the galvanometer deflection is n, then

$$i = s_p n \qquad (113)$$

Now k_4 is opened and k_3 closed, cutting off the main and bringing into the circuit the resistance R_s. We have now a galvanometer current

$$i_1 = \frac{E}{R + R_s} = s_p n_1 \qquad (113a)$$

Dividing (113) by (113a), we have $\dfrac{s_p n}{s_p n_1} = \dfrac{R + R_s}{R + R_x}$, therefore

$$R_x = \frac{R(n_1 - n) + n_1 R_s}{n}.$$

The method is exactly the same as the comparison method of measuring high resistance; the resistance R is essential because the insulation resistance of the main may be very low.

(2) Measurement of the Insulation Resistance to Earth, and Between Conductors of Two Live Mains, by the Voltmeter Method

Consider fig. 98, where the insulation resistance to ground of the two mains + and − is represented by R_p and R_n.

A voltmeter of resistance R_v is first connected between the positive main and earth, where it reads, say, V_p; next between the negative main and earth, where it reads V_n; and finally between the two mains, the reading being the voltage V between the mains.

When the voltmeter is between − and earth, the current i through R_p divides into two parts, one part passing through the voltmeter, the other through R_n. Then

$$V = R_p i + \frac{R_n r_v}{R_n + r_v} \cdot i = i \left(R_p + \frac{R_n r_v}{R_n + r_v}\right) \quad . \quad . \quad (114)$$

When the voltmeter is between + and earth, we have

$$V = R_n i_1 + \frac{R_p r_v}{R_p + r_v} i_1 = i_1 \left(R_n + \frac{R_p r_v}{R_p + r_v}\right) \quad . \quad . \quad (114a)$$

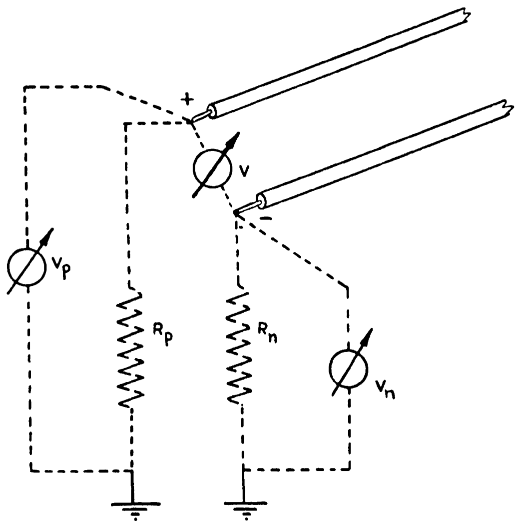

Fig. 98

When between − and earth, the voltmeter shows $V_n = i_v r_v$, where i_v is the voltmeter current, but

$$i_v = \frac{V_n}{r_v} = i \frac{R_n}{R_n + r_v},$$

and substituting for i from (114) we get

$$\frac{V_n}{r_v} = \frac{R_n}{R_n + r_v} \cdot \frac{V}{R_p + \frac{R_n r_v}{R_n + r_v}} =$$

$$\frac{V R_n}{r_v (R_p + R_n) + R_p R_n} \quad . \quad . \quad . \quad . \quad . \quad (115)$$

When the voltmeter is between + and earth, we have $V_p = i_{v1} \cdot r_v$, where i_{v1} is now the voltmeter current, but

$$i_{v_1} = \frac{V_p}{r_v} = \frac{R_p}{R_p + r_v} \cdot i_1,$$

and substituting for i_1 from (114a), we get

$$\frac{V_p}{r_v} = \frac{R_p}{R_p + r_v} \cdot \frac{V}{R_n + \dfrac{R_p r_v}{R_p + r_v}} =$$

$$\frac{V R_p}{r_v (R_p + R_n) + R_p R_n} \quad . \quad . \quad . \quad . \quad . \quad (115a)$$

From (115) and (115a) we have

$$r_v (R_p + R_n) + R_p R_n = \frac{V R_n r_v}{V_n} = \frac{V R_p r_v}{V_p},$$

therefore

$$\frac{R_n}{V_n} = \frac{R_p}{V_p}; \quad R_n = \frac{V_n R_p}{V_p}; \quad R_p = \frac{V_p R_n}{V_n}.$$

Substituting for R_p in (115) we get

$$\frac{V_n}{r_v} = \frac{R_n V}{r_v\left(R_n + \dfrac{V_p R_n}{V_n}\right) + \dfrac{R_n V_p R_n}{V_n}} = \frac{V V_n}{r_v V_n + r_v V_p + R_n V_p}, \text{ and}$$

$$R_n = \frac{V - V_p - V_n}{V_p} \cdot r_v \quad . \quad . \quad . \quad . \quad . \quad (116)$$

Similarly, substituting for R_n in (115a), we have

$$R_p = \frac{V - V_p - V_n}{V_n} \cdot r_v \quad . \quad . \quad . \quad . \quad . \quad (116a)$$

The insulation resistance between the two mains is simply

$$R_t = R_n + R_p$$

Example 25. "Describe a method by which the insulation resistance to earth of each of a pair of live mains can be measured by a voltmeter of known resistance. Discuss the limitations of the method.

"The following readings were taken with a 250-volt, 1000-ohms-per-volt voltmeter :

" Between two mains	218 volts
" Positive main to earth	188 volts
" Negative main to earth	10 volts.

" Calculate the insulation resistance of each main." (University of London, Electr. Measur. and Measur. Instruments, B.Sc. Final, 1947, paper 2, question 1.)

One method of measurement is that described above.

The limitations of the method are as follows :

(*a*) It cannot be used when one main is earthed.

(*b*) If the insulation resistance of the mains is high, the voltmeter current and its deflection will be very small, and as we have seen, an indicating instrument should not be used at the beginning of its scale. To measure high insulation resistance, the voltmeter resistance must be very high. The limit of the method is about 1 megohm with a very high quality voltmeter.

The voltmeter resistance is $r_v \doteq 250 \times 1000 = 250000\ \Omega$.

From (116) and (116a) we get

$$R_n = \frac{218 - 188 - 10}{188} \times 250000 = 26600\Omega.$$

$$R_p = \frac{218 - 188 - 10}{10} \times 250000 = 500000\Omega.$$

DIRECT-READING OHMMETERS

(3) Ohmmeter Voltmeters

These are very sensitive, permanent-magnet, moving-coil voltmeters, the scale of which is graded in ohms.

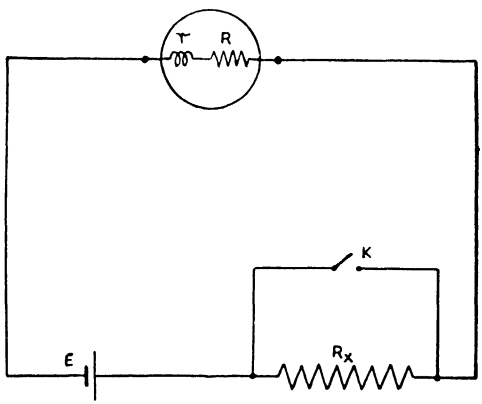

Fig. 99

Consider fig. 99, representing a voltmeter of coil resistance r, in series with its high resistance R ; R_x, the resistance to be measured, can be shorted by k ; the source has an e.m.f. E. With R_x shorted, let the full scale deflection be m for E volts ; the voltmeter current is then $i = \dfrac{E}{R + r} = s_p m$, s_p being the current per unit deflection.

We can write

$$E = (R + r)\, s_p m\ ;\ R + r = \frac{E}{s_p m} \quad . \quad . \quad . \quad . \quad . \quad (117)$$

If we know E and s_p we can calculate $R + r$.

With R_x in the circuit, we get a deflection m_x, and we have

$$(R + r) + R_x = \frac{E}{s_p m_x}.$$

Substituting for $\frac{E}{s_p}$, from (117) we get

$$(R + r) + R_x = \frac{(R + r)\, m}{m_x} \text{ and}$$

$$R_x = \frac{(R + r)(m - m_x)}{m_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

$$m_x = \frac{(R + r)\, m}{(R + r) + R_x}$$

The voltmeter scale can therefore be graded directly in ohms for the measurement of R_x.

If $R_x = 0$, we get $m_x = \frac{(R + r)\, m}{R + r} = m$; the scale of the instrument is therefore marked zero at full scale deflection.

If $R_x = \infty$, then $m_x = 0$; therefore zero deflection will be marked infinity.

Between 0 and ∞ the scale is graded in ohms.

If R_x is very great we can write $m_x \simeq \frac{m\,(R + r)}{R_x}$. The scale will be very crowded near zero deflection.

If $R + r$ is great compared with R_x, then the ohmmeter will not be very sensitive at small values of R_x, and to allow for a wide range of measurement the ohmmeter has to have several sensitivities ; that is, several resistances R.

As E does not enter into (118), it follows that it is not essential that E shall remain constant. For (118) to be true, all that is required is that the source voltage shall produce the full scale deflection m when R_x is shorted, and that it shall remain constant during a measurement. If therefore E has altered between two measurements, and there exists an arrangement by means of which m can be adjusted by variation of the field in the instrument airgap according to the variation of the source voltage, the instrument reading will remain true.

Construction of the Ohmmeter. The source and the instrument are contained in the same box. Before any test can be made, a check on the total deflection with $R_x = O$ is necessary. The terminals provided for R_x are therefore shorted, and m observed ; if the pointer is at zero no correction is necessary ; if not, the field in the airgap is altered by turning a knob or screw a, which in turn moves a magnetic shunt placed near the poles of the permanent magnet (fig. 100).

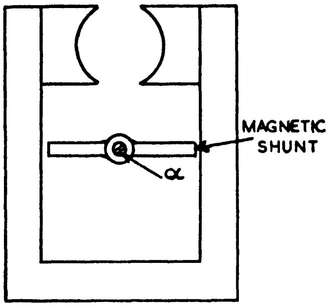

Fig. 100

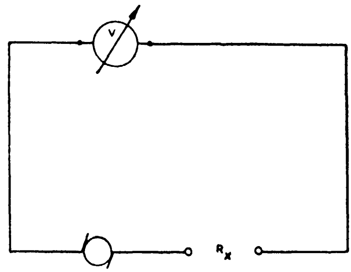

Fig. 101

The source E should be changed only if the pointer cannot be brought to zero by alteration of position of the magnetic shunt.

The ohmmeter can measure resistances up to 1 megohm, depending on the instrument sensitivity.

Another type of ohmmeter is shown in fig. 101.

Instead of a battery, we have a small hand-driven d.c. generator contained in the same box as is the instrument. The speed of rotation is about 2 revolutions per second, and a mechanical arrangement prevents any higher speed being attained, however quickly the handle is rotated. This type of ohmmeter can measure resistances up to 100 megohms.

(4) Ratio-meter Ohmmeters

A ratio-meter or logo-meter (from Greek *logos*) has two galvanometer coils, solidly attached, and at an angle $a°$ to one another, placed in the same magnetic field. The arrangement for an angle of 90° between the coils is shown in fig. 102.

The coils are free ; there is no restoring torque ; and in the absence of any current in the coils, they may be in any position. There is no fixed zero of scale to which pointer returns.

Consider fig. 103, where the coils 1 and 2 are placed in a uniform magnetic field H in the direction as shown, the angle between the coils being 90°. Let the currents in the coils be i_1 and i_2. The current i_1 in coil 1 will produce a magnetic field of, say, F_1, proportional to $n_1 S_1 i_1$; the current i_2 in coil 2 will produce a field F_2 proportional to $n_2 S_2 i_2$, where n_1, n_2, S_1, S_2 are the number of turns, and the areas of the coils 1 and 2.

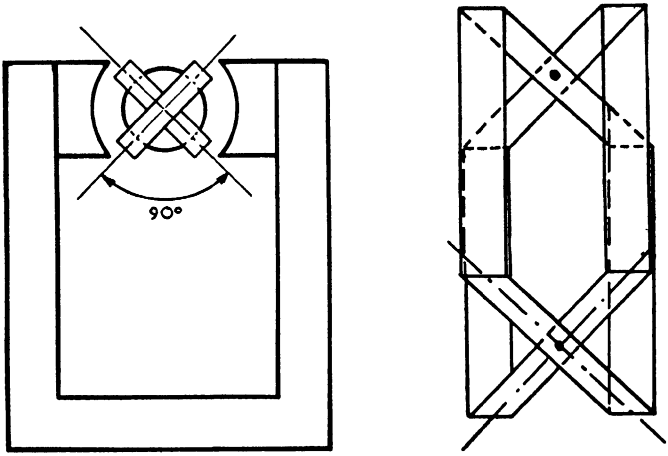

Fig. 102

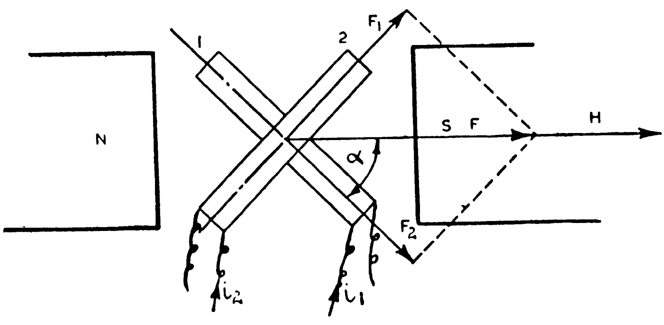

Fig. 103

The currents in the two coils are in such directions that the torques produced by the interaction of H with i_1 and i_2 are in opposite directions, and the only equilibrium position possible is when F, the resultant of F_1 and F_2, is parallel to H. The torques on coils 1 and 2 are then $T_1 = Kn_1\, i_1\, S_1\, H\, cos.a$; $T_2 = Kn_2\, i_2\, S_2\, H\, cos.\,(90 - a) = n_2\, i_2\, S_2\, H\, sin.a$ where K is a constant. As the coils are in the equilibrium position $n_1\, i_1\, S_1\, H\, cos.a = n_2\, i_2\, S_2\, H\, sin.a$, therefore

$$\frac{i_1}{i_2} = \frac{n_2\, S_2\, H}{n_1\, S_1\, H}\, tan.a = k_1\, tan.a,\ k_1 \text{ being a constant} \quad . \quad . \ (119)$$

If we know k_1, the scale can be graded directly in the ratio $\frac{i_1}{i_2}$.

As in practice the coils are not in a uniform field, but in a radial field (fig. 102), the equilibrium position is not such a simple function as that given by (119). Nevertheless we can write :

$\frac{i_1}{i_2} = f(\alpha)$; that is, the ratio of the two currents is a function of the deflection, and the instrument can be calibrated by comparison.

In order to improve the scale, the angle between the coils can be made other than 90°.

Consider the arrangement shown in fig. 104.

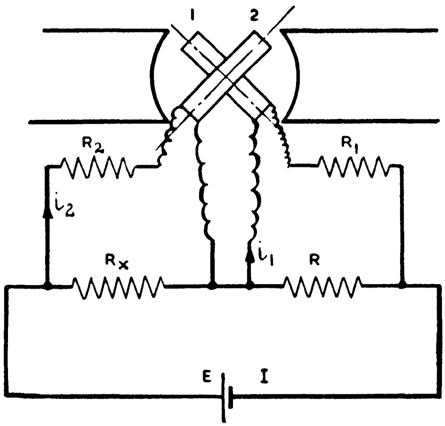

Fig. 104

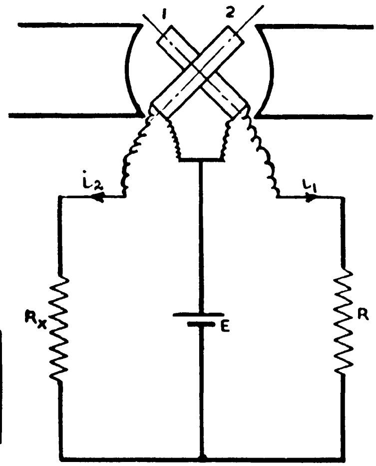

Fig. 105

A source E supplies the current I to a known resistance R in series with the resistance R_x which has to be measured. The moving coils of resistance r_1 and r_2, in series respectively with the resistances R_1 and R_2, are connected across R and R_x as shown. With the symbols used in the diagram we have

$$i_1 = I \cdot \frac{R}{R + R_1 + r_1} \; ; \; i_2 = I \cdot \frac{R_x}{R_x + R_2 + r_2} \text{ and}$$

$$\frac{i_1}{i_2} = \frac{R_x + R_2 + r_2}{R + R_1 + r_1} \cdot \frac{R}{R_x} = f(\alpha) \quad . \quad . \quad . \quad . \quad (119)$$

The scale can therefore be directly graduated in ohms. From (119) we get $R_x = \frac{R\,(R_2 + r_2)}{f(\alpha)\,(R + R_1 + r_1) - R}$, given directly on the scale.

A more practical arrangement is shown in fig. 105.

Coil 1 of resistance r_1 is in series with the resistance R; coil 2 of resistance r_2, in series with the resistance to be measured, R_x; E is the source of supply.

We have

$$i_1 = \frac{E}{R + r_1}; \quad i_2 = \frac{E}{R_x + r_2}; \quad \frac{i_1}{i_2} = \frac{R_x + r_2}{R + r_1} = f(a) \text{ and}$$

$$R_x = f(a)(R + r_1) - r_2 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (120)$$

R_x is read directly on the scale.

THE RATIOMETER OHMMETER WITH SEVERAL SENSITIVITIES. The arrangement is shown in fig. 106.

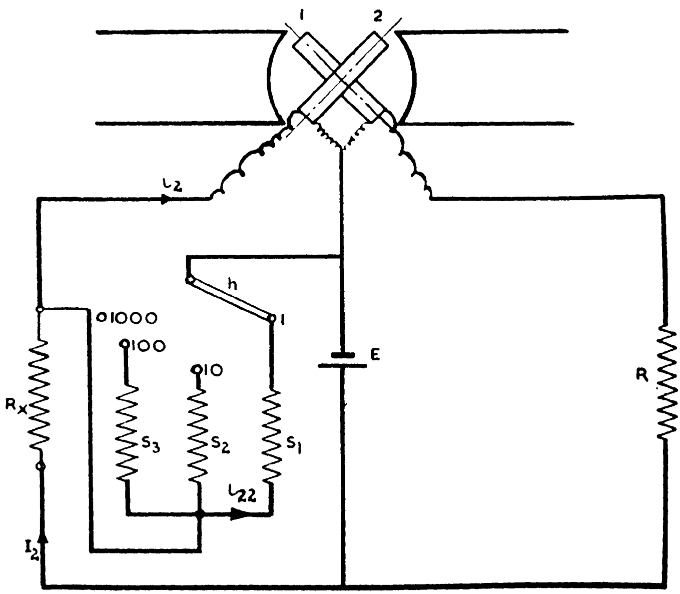

Fig. 106

Arm h can be connected to the contacts marked 1, 10, 100 and 1000. The shunts s_1, s_2, and s_3 have resistances respectively

$$s_1 = \frac{r_2}{999}, \; s_2 = \frac{r_2}{99}, \; s_3 = \frac{r_2}{9},$$

r_2 is the resistance of coil 2.

Let h be in contact with 1; the current I_2 in R_x divides into i_2 in coil 2 and i_{22} in the shunt s_1. We then have

$$i_2 = I_2 \cdot \frac{s_1}{s_1 + r_2} = I_2 \left(\frac{r_2}{999} \div \frac{r_2}{999} + r_2\right) = \frac{I_2 \cdot 999\, r_2}{1000 \times 999\, r_2} = \frac{I_2}{1000}.$$

When arm h is on contact 1000 (no shunt), the current in coil 2 is I_2, and if with a resistance R_x the deflection is a, when h is on 1 the current in coil 2 is $\frac{I_2}{1000}$, so that if we want the same deflection a as when h is on contact 1000, we have to replace R_x by a resistance R_{x1} such that

$$R_{x1} + \frac{s_1 r_2}{s_1 + r_2} = \frac{R_x + r_2}{1000}, \text{ that is}$$

$(R_x + r_2)(s_1 + r_2) = 1000\,[R_{x1}(s_1 + r_2) + s_1 r_2]$, from which we get

$$R_{x1} = \frac{R_x + r_2}{1000} - \frac{s_1 r_2}{s_1 + r_2}$$

$$\text{and as } s_1 = \frac{r_2}{999}$$

$$R_{x1} = \frac{R_x}{1000} + \frac{r_2}{1000} - \frac{r_2^2}{1000 r_2} = \frac{R_x}{1000}.$$

If the scale is such that we have to multiply the pointer readings by 1000, when h is on contact 1000, we have direct readings when h is on contact 1. Similarly the readings have to be multiplied by 10 and by 100, when h is respectively on contact 10 and contact 100.

Construction. The source of supply is either a small hand-driven d.c. generator or an a.c. generator with a rectifier. The generator is contained in the same box as the instrument.

The resistance to be measured is connected to the terminals marked for the purpose ; h is set in the proper position, and the handle turned ; the resistance is given directly on the scale.

From (120) it follows that the supply voltage need not be constant, the reading depending on the ratio of the curients alone.