ADVANCED SURVEYING

ADVANCED SURVEYING

A TEXT-BOOK FOR STUDENTS

BY

ALEX. H. JAMESON, M.Sc., M.Inst.C.E.

PROFESSOR OF CIVIL ENGINEERING, UNIVERSITY OF LONDON, KING'S COLLEGE

AUTHOR OF "CONTOUR GEOMETRY AND ITS APPLICATIONS TO EARTH-WORK DESIGN AND QUANTITIES"

AND JOINT AUTHOR OF "MATHEMATICAL GEOGRAPHY"

SECOND EDITION

LONDON

SIR ISAAC PITMAN AND SONS LIMITED

First Edition 1934
Reprinted 1944
Second Edition 1948
Reprinted 1950

SIR ISAAC PITMAN & SONS, LTD
PITMAN HOUSE, PARKER STREET, KINGSWAY, LONDON, W.C.2
THE PITMAN PRESS, BATH
PITMAN HOUSE, LITTLE COLLINS STREET, MELBOURNE
27 BECKETTS BUILDINGS, PRESIDENT STREET, JOHANNESBURG

ASSOCIATED COMPANIES
PITMAN PUBLISHING CORPORATION
2 WEST 45TH STREET, NEW YORK

SIR ISAAC PITMAN & SONS (CANADA), LTD.
(INCORPORATING THE COMMERCIAL TEXT BOOK COMPANY)
PITMAN HOUSE, 381 383 CHURCH STREET, TORONTO

MADE IN GREAT BRITAIN AT THE PITMAN PRESS, BATH
Eo-(T.767)

PREFACE
TO THE SECOND EDITION

A NUMBER of corrections have been made in the text and a Table of Adjustments for the angles of a Quadrilateral with a Central Point has been added, as the method of Successive Approximations is unsuitable when the errors are large. A chapter has been added as a brief introduction to Survey from the Air, for which the Author frankly acknowledges his indebtedness to Brigadier J. M. Hotine, C.B.E., R.E., for his great work *Surveying from Air Photographs* (Constable).

PREFACE

A COURSE in Surveying, like those for other subjects in Final Degree examinations, should extend over two years. A student is only ready to appreciate the higher parts of the subject when he has been through an elementary course such as is given in Professor M. T. Ormsby's *Elementary Principles of Surveying*, illustrated by practical work in Chain Surveying, Levelling, Theodolite Traversing, Tacheometry, and Curve Ranging. It is assumed that the student has undergone such a training and has an elementary knowledge of the Calculus and of Geometrical Optics.

This is a textbook, not a treatise, which a student can reasonably be expected to master in one session, along with the other subjects required for his Final examination. It is thus necessarily brief, and the Author's aim has been to give clear explanations of the principles of the chief modern instruments and methods of Surveying, with the additional

Mathematics involved, and he must refer the student, for fuller descriptions and illustrations of these and other instruments and methods of surveying, to much larger books, such as the late Professor David Clark's *Plane and Geodetic Surveying*. A large number of fully worked out illustrative examples are given in the text. Many of these, marked "L.U.," and modified if necessary to suit the Author's purpose, are taken from the Examination Papers for the B.Sc. (Engineering) Degree of London University, with the kind permission of the University.

Many students find great difficulty in assimilating Astronomical ideas. The best way to overcome this is to practise oneself in *approximate* estimations of the position of stars in different latitudes and at various hours of the day and times of the year, as explained in Chapter II, with the aid of the simple model suggested: at a later stage the model can be dispensed with and a "clock diagram" used instead. The student should be thoroughly at home in the astronomical part of *Whitaker's Almanack* before being introduced to the much more complicated *Nautical Almanac*.

The use of Surveys for the location, design, and quantities of earthwork is dealt with in the Author's companion volume on *Contour Geometry*. In conclusion, the Author gratefully acknowledges the help he has received from Professor J. B. Dale, of King's College, London, in his treatment of the "Method of Least Squares" and its applications to surveying problems.

CONTENTS

CHAPTER IX

ADVANCED SURVEYING

CHAPTER I

MATHEMATICS

Spherical Trigonometry—Approximations, Common Catenary—Method of Least Squares

Spherical Trigonometry

A sphere is a surface every point on which is at the same distance from a point called the "Centre"—this constant distance being the "Radius." All sections of a Sphere by a Plane are Circles—when the plane passes through the centre of the sphere the radius of the circle is a maximum and equals the radius of the sphere—such circles are, therefore, called "Great Circles."

The arc of a great circle passing through any two points on a sphere is the shortest distance between them; in this respect a great circle on a sphere corresponds to a straight line on a plane.

The figure contained by the arcs of three great circles is called a "Spherical Triangle," and the intersections of the great circles are the Vertices of the triangle. The Sides of the spherical triangle are the lengths of the arcs of the great circles between the vertices and are, of course, proportional to the angles subtended at the centre of the sphere by these arcs and to the radius of the sphere. In Fig. 1, if ABC is a spherical triangle on the sphere of radius R, and if the angles at the centre in circular measure are as follows: $BOC = a$, $COA = b$, $AOB = c$; then the lengths of the sides are $BC = Ra$, $CA = Rb$, $AB = Rc$. We shall, in general, consider the radius of the sphere as equal to unity, so that $BC = a$, $CA = b$, $AB = c$, i.e. the *sides of the spherical triangle are the angles subtended by the sides at the centre of the sphere.*

The Angles of the spherical triangle are the angles between the planes of the sides, e.g. the angle A or BAC is the angle

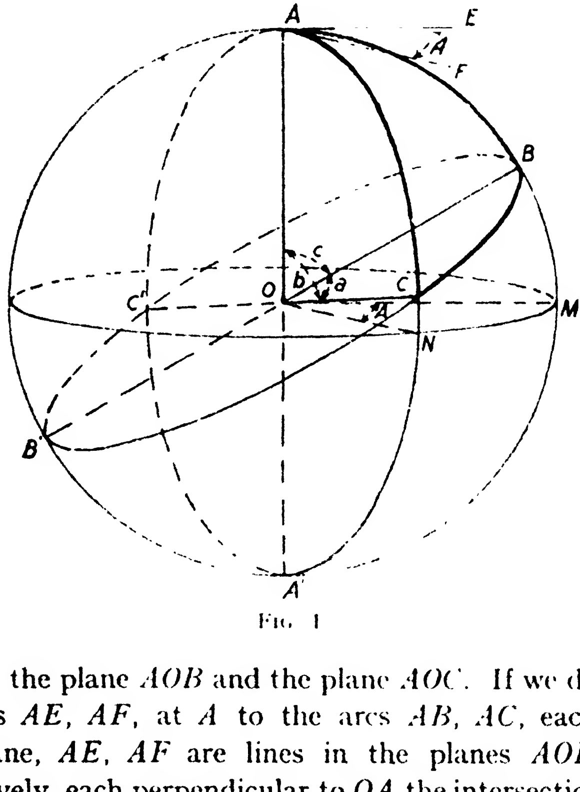


FIG. 1

between the plane AOB and the plane AOC. If we draw the tangents AE, AF, at A to the arcs AB, AC, each in its own plane, AE, AF are lines in the planes AOB, AOC respectively, each perpendicular to OA the intersection of the two planes. The angle EAF is, therefore, the angle between the planes AOB, AOC. Consequently, the angles A, B, and C of the spherical triangle can also be defined as the angles between the tangents to the sides at the respective vertices.

Or, again, if we draw a great circle in a plane perpendicular to AO, cutting AB and AC in M and N, the angle MON or MN, also = angle A.

Spherical Trigonometry deals with the relations between the sides and angles of a spherical triangle or with the relations of the angles between any three intersecting lines AO, BO, CO with the angles between the three planes AOB, BOC, COA which contain each pair of such lines.

Spherical Excess. The sum of the three angles of a spherical triangle is always greater than π or than two right angles; in fact, A, B, and C may each be a right angle or more, the maximum value of the sum being six right angles, or 3π. When two angles of a triangle are given, therefore, it is not possible, as in Plane Trigonometry, to find the value of the third angle without other data being given. This "Spherical excess," i.e. the amount by which the sum of the three angles exceeds π depends on the ratio of the area of the triangle to the area of the hemisphere, as may be readily shown.

Produce the radii AO to A', BO to B', CO to C' on the opposite surface of the sphere, then the sides of the triangle intersect again at the points A', B', and C' and the triangle $A'B'C$ is identically equal to triangle ABC' The figure $ABA'C$ between two great circles is called a "lune"—its area is obviously proportional to the angle A, and when $A = 180° = \pi$ it becomes a hemisphere of area $2\pi R^2$. Therefore we can write

$$ABC + A'BC = ABA'C$$

$$= \frac{A}{\pi} \cdot 2\pi R^2 = 2AR^2$$

Similarly,

$$ABC + AB'C = BAB'C$$

$$= \frac{B}{\pi} \cdot 2\pi R^2 = 2BR^2$$

and $ABC + ABC' = ABC + A'B'C = CAC'B$

$$= \frac{C}{\pi} \cdot 2\pi R^2 = 2CR^2$$

Adding together, we get

$$2ABC + \text{area of hemisphere} = 2R^2(A + B + C)$$

$$\therefore 2 \times \text{area } ABC = 2R^2(A + B + C) - 2\pi R^2$$

or, $\quad$ area of triangle $ABC = R^2(A + B + C - \pi)$

$$\therefore A + B + C = \pi + \frac{\text{area of triangle}}{R^2}$$

Therefore, the spherical excess is $\dfrac{\text{area of triangle}}{R^2}$

The maximum area of a spherical triangle is the hemisphere; the maximum spherical excess is therefore $\dfrac{2\pi R^2}{R^2}$ $= 2\pi$, and the maximum value therefore for the sum of the three angles of a spherical triangle is, therefore, 3π.

When one of the angles of a spherical triangle is a right angle, the triangle is said to be "Right-angled." If *two* of the angles are right angles, as in triangle AMN in Fig. 1, then the opposite sides are also right angles and the remaining side equals the remaining angle; if all three angles are right angles then all three sides are right angles. In the latter case the spherical excess is 90° or $\frac{\pi}{2}$. The area of the triangle is, therefore, $\frac{\pi}{2} R^2$ or $\frac{1}{4} 2\pi R^2$, i.e. one-quarter of a hemisphere as, of course, is obvious.

The fundamental formula in Spherical Trigonometry is $\cos A = \dfrac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$. To prove this (Fig. 2), let ABC be any spherical triangle on a sphere of radius unity. At A draw the tangents AD, AE to the sides AB, AC

respectively meeting the radii OB, OC, at D and E respectively. Then $AD = \tan c$, $OD = \sec c$, $AE = \tan b$, $OE = \sec b$.

Then in triangle DAE, $\cos A = \dfrac{\tan^2 b + \tan^2 c - DE^2}{2 \tan b \, . \tan c}$

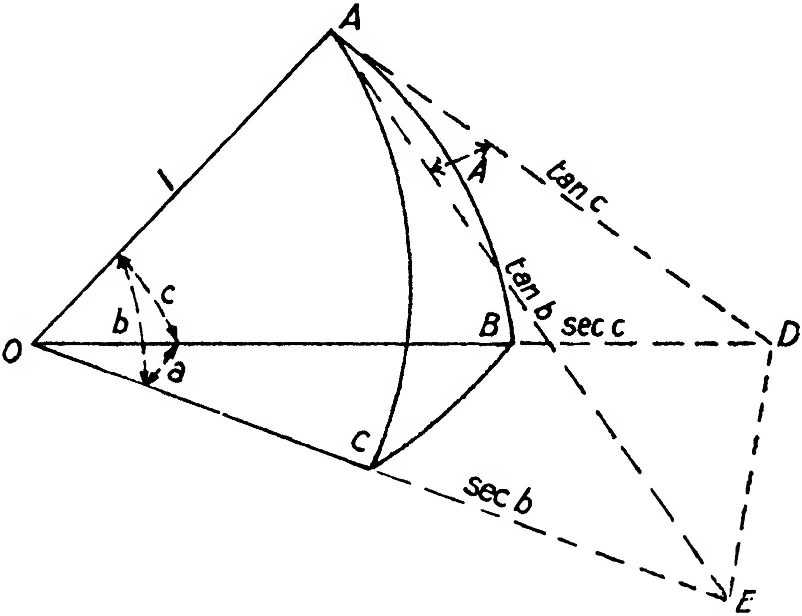


FIG. 2

but in triangle DOE, $DE^2 = \sec^2 b + \sec^2 c - 2 \sec b \, . \sec c \cos a$.

$\therefore \cos A$

$$= \frac{\tan^2 b - \sec^2 b + \tan^2 c - \sec^2 c + 2 \sec b \, . \sec c \, . \cos a}{2 \tan b \, . \tan c}$$

$$= \frac{2 \sec b \, . \sec c \, . \cos a - 2}{2 \tan b \, . \tan c}$$

$$= \frac{\sec b \, . \sec c \, . \cos a - 1}{\tan b \, . \tan c}$$ (multiply both Numerator and

Denominator by $\cos b \cdot \cos c$)

$$= \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$$, with similar formulae for $\cos B$ and $\cos C$ . . . . (1)

Although this is the fundamental formula, it is not well suited for use with logarithms, so we proceed to transform it by Plane Trigonometry, thus—

$$\cos A = 1 - 2\sin^2\frac{A}{2} = \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$$

$$\therefore 2\sin^2\frac{A}{2} = 1 - \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$$

$$= \frac{\sin b \cdot \sin c + \cos b \cdot \cos c - \cos a}{\sin b \cdot \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \cdot \sin c}$$

$$= \frac{2\sin\frac{a + b - c}{2} \cdot \sin\frac{a - b + c}{2}}{\sin b \cdot \sin c}$$

Now let $s = \frac{a + b + c}{2}$, and we have

$$\sin\frac{A}{2} = \sqrt{\frac{\sin (s - b) \cdot \sin (s - c)}{\sin b \cdot \sin c}}$$ with similar formulae for $\sin\frac{B}{2}$ and $\sin\frac{C}{2}$ . . . . (2)

$$\text{Also } \cos A = 2\cos^2\frac{A}{2} - 1 = \frac{\cos a - \cos b \cdot \cos c}{\sin b \cdot \sin c}$$

$$\therefore\ 2\cos^2\frac{A}{2} = \frac{\sin b\,.\,\sin c - \cos b\,.\,\cos c + \cos a}{\sin b\,.\,\sin c}$$

$$= \frac{\cos a - \cos(b+c)}{\sin b\,.\,\sin c}$$

$$= \frac{2\sin\dfrac{a+b+c}{2}\,.\,\sin\dfrac{b+c-a}{2}}{\sin b\,.\,\sin c}$$

$$\therefore\quad \cos\frac{A}{2} = \sqrt{\frac{\sin s\,.\,\sin(s-a)}{\sin b\,.\,\sin c}}$$ with similar formulae for $\cos\dfrac{B}{2}$ and $\cos\dfrac{C}{2}$ . . . . . (3)

Dividing (2) by (3) we get

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\,.\,\sin(s-c)}{\sin s\,.\,\sin(s-a)}}$$ and similarly for $\tan\dfrac{B}{2}$ and $\tan\dfrac{C}{2}$ . . . . . (4)

Again, $\sin A = 2\sin\dfrac{A}{2}\,.\,\cos\dfrac{A}{2}$

$$= 2\,\frac{\sqrt{\sin s\,.\,\sin(s-a)\,.\,\sin(s-b)\ \ \sin(s-c)}}{\sin b\,.\,\sin c}$$

$$\therefore\ \frac{\sin A}{\sin a} = 2\,\frac{\sqrt{\sin s\,.\,\sin(s-a)\,.\,\sin(s-b)\,.\,\sin(s-c)}}{\sin a\,.\,\sin b\,.\,\sin c}$$

Now this result is quite symmetrical in a, b, and c, so that we should get the same result for $\dfrac{\sin B}{\sin b}$ and $\dfrac{\sin C}{\sin c}$

Therefore $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$, or, the *sines of the angles are proportional to the sines of the opposite sides* . . . . (5)

This last formula is, however, not nearly so useful as the corresponding "rule of sines" for a plane triangle as the sum of the three angles of a spherical triangle is not a constant quantity.

Polar Triangles and their Reciprocal Properties. The ends of the diameter of the sphere at right angles to any great circle are called the "Poles" of that great circle as they occupy the same relation to it as the Poles to the Equator on the earth. The distance along the sphere of either pole from the great circle is the same in all directions, subtending 90° at the centre of the sphere. If ABC is a spherical triangle (Fig. 3) and A is the pole of the great circle BC, B' is the pole of the great circle CA and C' is the pole of the great circle AB, the spherical triangle $A'B'C'$ is called the "polar triangle" of the triangle ABC. Then since A' is the pole of BC, CA' is a quadrant of the sphere, also since B' is the pole of CA, CB' is a quadrant. $\therefore$ $CA' = CB' =$ a quadrant and therefore C is the pole of $A'B'$ Similarly, B is the pole of $C'A'$ and A is the pole of $B'C'$ *Therefore, the original triangle ABC is the polar triangle of its polar triangle A'B'C'*

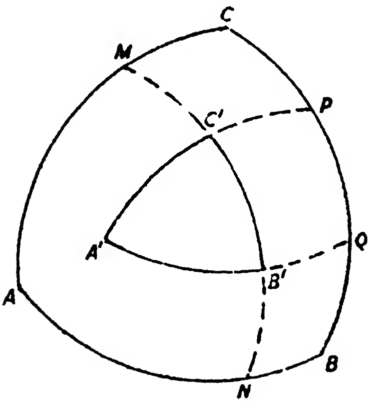


FIG. 3

Now, produce side $B'C'$ both ways to M and N on AC and AB respectively. Then $B'M$ and $C'N$ are both quadrants, $\therefore$ $B'M + C'N = MN + C'B' = 180°$ But MN is the arc measured along a great circle 90° from A and is therefore equal to A. $\therefore$ $B'C' = 180° - A$. Thus, *the sides* *of the polar triangle are the supplements of the corresponding angles of the original triangle.*

Now, produce $A'C'$ and $A'B'$ to meet BC in P and Q Then BP and CQ are both quadrants.

$$\therefore BP + CQ = BC + PQ = 180^\circ$$

But PQ is the arc measured along a great circle 90° from A' and is therefore equal to A'

$$\therefore \qquad A' = 180^\circ - BC$$

Thus, *the angles of the polar triangle are the supplements of the corresponding sides of the original triangle.*

This supplemental property of the polar triangle is a very useful one, as it enables us to duplicate the formulae already proved, obtaining corresponding ones in which the sides take the place of angles and *vice versa*. For example, take the formula $\cos A = \dfrac{\cos a - \cos b \quad \cos c}{\sin b \, . \sin c}$, which we have proved for the triangle ABC whose sides are a, b, and c respectively. Let $A'B'C'$ be the polar triangle of ABC and let a', b', c' be its corresponding sides.

Then $A = 180^\circ - a'$, $a = 180^\circ - A'$, $b = 180^\circ - B'$, $c = 180^\circ - C'$

Then $\cos (180^\circ - a')$

$$= \frac{\cos (180^\circ - A') - \cos (180^\circ - B') \, . \cos (180^\circ - C')}{\sin (180^\circ - B') \quad \sin (180^\circ - C')}$$

$$\therefore -\cos a' = \frac{-\cos A' - (-\cos B') \quad (-\cos C')}{\sin B' \, . \sin C'}$$

$$\therefore \quad \cos a' = \frac{\cos A' + \cos B' \, . \cos C'}{\sin B' \, . \sin C'}$$

or, discarding the dashes,

$$\cos a = \frac{\cos A + \cos B \, . \cos C}{\sin B \, . \sin C} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where we now have a formula which gives us a side when the three angles are known instead of the original formula, which gave us an angle when the three sides were known.

Or, taking the formula $\tan\frac{C}{2} = \sqrt{\frac{\sin(s-a)\,.\,\sin(s-b)}{\sin s\,.\,\sin(s-c)}}$,

we have $C = 180° - c'$, and let $S' = \frac{A' + B' + C'}{2}$

$$\text{Then } s = \frac{a+b+c}{2} = \frac{180° - A' + 180° - B' + 180° - C'}{2}$$
$$= 270° - S'$$

And $s - a = 270° - S' - (180° - A') = 90° - (S' - A')$, similarly $(s - b) = 90° - (S' - B')$, $s - c = 90° - (S' - C')$

$$\therefore \tan\frac{180° - c'}{2} = \sqrt{\frac{\cos(S' - A')\,.\,\cos(S' - B')}{-\cos S'\,.\,\cos(S' - C')}},$$ or, discarding the dashes,

$$\cot\frac{c}{2} = \sqrt{\frac{\cos(S - A)\,.\,\cos(S - B)}{-\cos S\,.\,\cos(S - C)}} \quad . \quad . \quad . \quad (7)$$

Now, reverting to formula (6),

$$\cos a = \frac{\cos A + \cos B\,.\,\cos C}{\sin B\,.\,\sin C},$$

$$\text{let } k = \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin A + \sin B}{\sin a + \sin b}$$

$$\therefore \cos A + \cos B\,.\,\cos C = \sin B\,.\,\sin C\,.\,\cos a$$
$$= k\,.\,\sin b \sin C\,.\,\cos a$$

Similarly, $\cos B + \cos C\,.\,\cos A = \sin C\,.\,\sin A\,.\,\cos b$
$$= k\,.\,\sin a\,.\,\sin C\,.\,\cos b$$

$\therefore$ adding, $(\cos A + \cos B)(1 + \cos C) = k\,.\,\sin C\,.\,\sin(a+b)$

$\therefore \sin A + \sin B = k(\sin a + \sin b)$

$$\text{and } \cos A + \cos B = \frac{k \cdot \sin C \cdot \sin (a+b)}{1+\cos C}$$

$$\therefore \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin a + \sin b}{\sin (a+b)} \cdot \frac{1+\cos C}{\sin C}$$

$$\text{Now } \tan \frac{A+B}{2} = \frac{\sin \frac{A+B}{2}}{\cos \frac{A+B}{2}} = \frac{\dfrac{\sin A + \sin B}{2 \cos \frac{A-B}{2}}}{\dfrac{\cos A + \cos B}{2 \cos \frac{A-B}{2}}}$$

$$= \frac{\sin A + \sin B}{\cos A + \cos B}$$

$$= \frac{\sin a + \sin b}{\sin (a+b)} \cdot \frac{1+\cos C}{\sin C}$$

$$\text{while } \frac{1+\cos C}{\sin C} = \frac{2\cos^2 \frac{C}{2}}{2 \sin \frac{C}{2} \cdot \cos \frac{C}{2}} = \frac{\cos \frac{C}{2}}{\sin \frac{C}{2}} = \cot \frac{C}{2}$$

$$\therefore \tan \frac{A+B}{2} = \frac{\sin a + \sin b}{\sin (a+b)} \cot \frac{C}{2}$$

$$= \frac{2 \sin \frac{a+b}{2} \cdot \cos \frac{a-b}{2}}{2 \sin \frac{a+b}{2} \cdot \cos \frac{a+b}{2}} \cot \frac{C}{2}$$

$$= \frac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2}} \cdot \cot \frac{C}{2} \qquad . \quad . \quad . \quad (8)$$

$$\text{Also, } \tan\frac{A-B}{2} = \frac{\sin\frac{A-B}{2}}{\cos\frac{A-B}{2}} = \frac{\dfrac{\sin A - \sin B}{2\cos\frac{A+B}{2}}}{\dfrac{\cos A + \cos B}{2\cos\frac{A+B}{2}}}$$

$$= \frac{\sin A - \sin B}{\cos A + \cos B} = \frac{\sin a - \sin b}{\sin (a+b)} \cot\frac{C}{2}$$

$$= \frac{2\cos\frac{a+b}{2} \cdot \sin\frac{a-b}{2}}{2\sin\frac{a+b}{2} \cdot \cos\frac{a+b}{2}} \cot\frac{C}{2}$$

$$= \frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}} \cdot \cot\frac{C}{2} \quad . \quad . \quad . \quad (9)$$

Formulae (8) and (9) are two of "Napier's Analogies," and are useful in finding the remaining angles of a triangle, when two sides and the included angle are given.

By the use of the supplementary properties of polar triangles, the remaining two of Napier's Analogies can be readily proved, viz.—

$$\tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}} \cdot \tan\frac{c}{2} \quad . \quad . \quad . \quad (10)$$

$$\tan\frac{a-b}{2} = \frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}} \tan\frac{c}{2} \quad . \quad . \quad . \quad (11)$$

(The student should prove these two formulae for practice.)

It is interesting to compare these formulae with the corresponding ones in Plane Trigonometry—

SPHERICAL FORMULA		PLANE FORMULA
$\cos C = \dfrac{\cos c - \cos a \cdot \cos b}{\sin a \cdot \sin b}$ . .	(1)	$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$
$\tan \dfrac{C}{2} = \sqrt{\dfrac{\sin (s-a) \cdot \sin (s-b)}{\sin s \cdot \sin (s-c)}}$.	(4)	$\tan \dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$
$\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$	(5)	$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$
$\tan \dfrac{A+B}{2} = \dfrac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2}} \cot \dfrac{C}{2}$ . .	(8)	$\tan \dfrac{A+B}{2} = \cot \dfrac{C}{2}$, as $A + B = 180° - C$
$\tan \dfrac{A-B}{2} = \dfrac{\sin \frac{a-b}{2}}{\sin \frac{a+b}{2}} \cot \dfrac{C}{2}$ . .	(9)	$\tan \dfrac{A-B}{2} = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$

This table should assist the student in memorizing the above five important formulae.

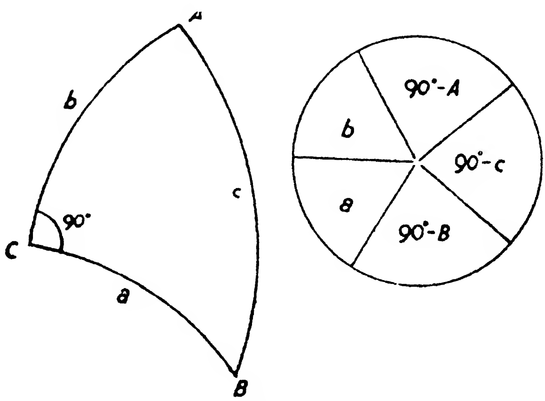


FIG. 4

Right-angled Spherical Triangles (Fig. 4). When $C = 90°$ we have

$$\cos C = \frac{\cos c - \cos a \cdot \cos b}{\sin a \cdot \sin b} = 0$$

$$\therefore \cos c = \cos a \cdot \cos b \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Also $\cos A = \dfrac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c} = \dfrac{\dfrac{\cos c}{\cos b} - \cos b \,.\, \cos c}{\sin b \,.\, \sin c}$

$$= \frac{\cos c \,.\, \sin^2 b}{\cos b \sin b \,.\, \sin c} = \frac{\tan b}{\tan c} \qquad . \quad . \quad (13)$$

Similarly, $\cos B = \dfrac{\tan a}{\tan c}$ . . . . . . (14)

Again, $\sin A = \dfrac{\sin a}{\sin c} \,.\, \sin C$

$$= \frac{\sin a}{\sin c}, \text{ as } \sin C = 1 \quad . \quad . \quad . \quad (15)$$

Similarly, $\sin B = \dfrac{\sin b}{\sin c}$ . . . . . . (16)

Dividing (15) by (13) we have

$$\tan A = \frac{\sin a}{\sin c} \,.\, \frac{\tan c}{\tan b} = \frac{\sin a}{\cos c \,.\, \tan b}$$

$$= \frac{\sin a \,.\, \cos b}{\cos a \,.\, \cos b \,.\, \sin b} = \frac{\tan a}{\sin b} \qquad . \quad . \quad (17)$$

Similarly, we have $\tan B = \dfrac{\tan b}{\sin a}$ . . . . (18)

Also, from (17) and (18),

$$\cot A \,.\, \cot B = \frac{\sin b \,.\, \sin a}{\tan a \,.\, \tan b} = \cos a \,.\, \cos b = \cos c \qquad (19)$$

Again, $\cos A = \dfrac{\tan b}{\tan c} = \dfrac{\sin b}{\sin c} \,.\, \dfrac{\cos c}{\cos b}$

$$= \sin B \,.\, \frac{\cos a \,.\, \cos b}{\cos b} = \sin B \,.\, \cos a \quad . \quad (20)$$

Similarly, $\cos B = \sin A \,.\, \cos b$ . . . . . (21)

These ten simple (but confusing) formulae may be written down by Napier's "5-part Circle Rule," viz.: Sketch a circle (Fig. 4) and divide it into 5 parts; in these write down *in order* the two sides adjacent to the right angle and the complements of the three remaining parts (excluding the right angle). Then if any "part" is taken as "middle part" we have two "adjacent parts" and two "opposite parts," relative to it. The rule then is "*Sine of middle part = product of tangents of adjacent parts = product of cosines of opposite parts.*" Note that "sine" and "middle," "tangents" and "adjacent," "cosines" and "opposite" have similar vowels,

e.g. $\sin(90° - c) = \tan(90° - A) \,.\, \tan(90° - B)$,

i.e. $\cos c = \cot A \,.\, \cot B$ . . . (19)

and $\sin(90° - c) = \cos b \,.\, \cos a$

i.e. $\cos c = \cos b \,.\, \cos a$ . . . . (12)

(As there are 5 parts we can thus write down all the 10 formulae. The student should apply this rule to find the other 8 formulae.)

Example 1. To find the shortest distance (for an airship route) of Winnipeg 49° 55′ N., 97° 06′ W. from Prague 50° 05′ N., 14° 25′ E., the direction of Winnipeg from Prague and of Prague from Winnipeg, assuming the earth a sphere of radius 3,957 miles (Fig. 5).

Here we have a spherical triangle with its angles at Prague (A), Winnipeg (B), and the North Pole (C); the lengths of the sides are: $a = 90° - 49°\ 55' = 40°\ 05'$, $b = 90° - 50°\ 05' = 39°\ 55'$, and the included angle at the pole is

14° 25′ + 97° 06′ = 111° 31 = C. For the angles A and B we use the formulae

$$\tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}} \cdot \cot\frac{C}{2}$$

$$\tan\frac{A-B}{2} = \frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}} \cot\frac{C}{2}$$

where $\frac{a-b}{2} = 0° \; 5'$ $\frac{a+b}{2} = 40° \; 0'$, $\frac{C}{2} = 55° \; 45\frac{1}{2}'$

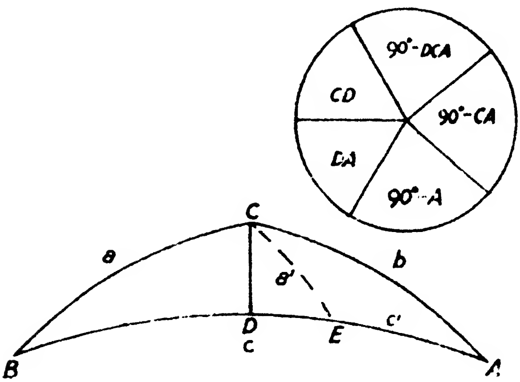


FIG. 5

	logarithm		logarithm
cos 0° 05′	$\bar{1}$·999,9995	sin 0° 05′	$\bar{3}$·162,6960
sec 40° 0′	0·115,7460	cosec 40° 0′	0·191,9325
cot 55° 45½′	$\bar{1}$·832,9321	cot 55° 45½′	$\bar{1}$·832,9321
$\tan\frac{A+B}{2}$	$\bar{1}$·948,6776	$\tan\frac{A-B}{2}$	$\bar{3}$·187,5606

$$\frac{A+B}{2} = 41° \; 37' \; 21''. \qquad \frac{A-B}{2} = 0° \; 05' \; 19''$$

$\therefore A = 41° 42' 40''$, i.e. direction of Winnipeg from Prague
$= N. 41° 42' 40'' W.$

$B = 41° 32' 02''$, i.e. direction of Prague from Winnipeg
$= N. 41° 32' 02'' E.$

For the distance c we use the formula,

$$\sin c = \sin C \cdot \frac{\sin a}{\sin A}$$

sin $\left\{\begin{matrix}111° 31' \\ 68° 29'\end{matrix}\right.$	$\bar{1}$·968,6281
sin 40° 05′	$\bar{1}$·808,8192
cosec 41° 42′ 40″	0·176,9334
sin c	$\bar{1}$·954,3807

$c = 64° 11' 45'' = 1{\cdot}1204287$ radians.

$\therefore$ distance $= 3{,}851\frac{3}{4}$ sea miles, which subtend 1′ at centre of earth

$= 1{\cdot}1204287 \times 3957 = 4{,}433\frac{1}{2}$ statute miles.

[The distance due east and west along the parallel of latitude of 50°, the radius of which is 3,957 cos 50°, would be: circular measure of 111° 31′ × 3,957 cos 50°

1·946,3330	0·289,2172
3957	3·597,3661
cos 50°	$\bar{1}$·808,0675
	3·694,6508

$\therefore$ distance along parallel would be $4{,}950\frac{1}{2}$ statute miles, or $11\frac{1}{2}$ per cent greater.]

N.B. As the side c was found from its sine, there is a possible ambiguity here, as c might be $180° - 64° 11' 45'' = 115° 48' 15''$. This value is, however, inadmissible here, as the distance must be less than the distance along the parallel of latitude.

EXAMPLE 2. To find the most northerly point reached on the course in Example 1. Where the great circular course reaches its most northerly point D (Fig. 5), it crosses the meridian at right angles. Thus CDA is a right-angled triangle of which we know the side $CA = 39° \, 55'$ and the angle $A = 41° \, 42' \, 40''$. Then $\sin CD = \sin CA \, . \, \sin A$.

sin 39° 55′	$\bar{1}$·807,3136
sin 41° 42′ 40″	$\bar{1}$·823,0666
sin CD	$\bar{1}$·630,3802

$$\therefore \quad CD = 25° \, 16' \, 28''.$$

$$\therefore \text{ Latitude of } D = 64° \, 43' \, 32'' \text{ N.}$$

$$\text{Also } \cos CA = \cot DCA \, . \cot . \, A$$

$$\therefore \tan DCA = \cot A \, . \sec CA$$

cot 41° 42′ 40″	0·049,9686
sec 39° 55′	0·115,2168
tan DCA	0·165,1854

$$\therefore \; DCA = 55° \, 38' \, 34''$$

$$\therefore \text{ Longitude of } D = 14° \, 25' \text{ E.} - 55° \, 38' \, 34'' = 41° \, 13' \, 34'' \text{ W.}$$

This point D is on the South-east coast of Greenland, and just here the course is due west.

Example 3. To find an intermediate point on the course in Example 1 and the direction to pursue at this point. Let the point be E (Fig. 5) and its longitude 3° W.

In the triangle ECA the angle C is 17° 25′, $CA = 39° \, 55'$, $A = 41° \, 42' \, 40''$. Let $EA = c'$. To find $a' = CE$ the co-latitude of E, we use formulae (10) and (11), viz.—

$$\tan \frac{a+c}{2} = \frac{\cos \frac{A-C}{2}}{\cos \frac{A+C}{2}} \, . \tan \frac{CA}{2}$$

$$\tan\frac{a-c}{2} = \frac{\sin\frac{A-C}{2}}{\sin\frac{A+C}{2}}\tan\frac{CA}{2}$$

$$\text{where } \frac{A-C}{2} = \frac{24^\circ\ 17'\ 40''}{2} = 12^\circ\ 08'\ 50''$$

$$\text{and } \frac{A+C}{2} = \frac{59^\circ\ 07'\ 40''}{2} = 29^\circ\ 33'\ 50''$$

$$\frac{CA}{2} = 19^\circ\ 57'\ 30''$$

cos	12° 08′ 50″	$\bar{1}$·990,1657
sec	29° 33′ 50″	0·060,5776
tan	19° 57′ 30″	$\bar{1}$·560,0823
tan	$\frac{a+c}{2}$	$\bar{1}$·610,8256

sin	12° 08′ 50″	$\bar{1}$·323,0959
cosec	29° 33′ 50″	0·306,8063
tan	19° 57′ 30″	$\bar{1}$·560,0823
tan	$\frac{a-c}{2}$	$\bar{1}$·189,9845

$$\therefore \frac{a+c}{2} = 22^\circ\ 12'\ 11'' \qquad \frac{a-c}{2} = 8^\circ\ 48'\ 14''$$

$\therefore a' = 31^\circ\ 00'\ 25''$. $\therefore$ latitude of $E = 58^\circ\ 59'\ 35''$ N. at longitude 3° W. (in the Orkney Islands).

For the direction at E, we use the formula $\dfrac{\sin E}{\sin CA} = \dfrac{\sin A}{\sin a}$

sin	41° 42′ 40″	$\bar{1}$·823,0666
cosec	31° 00′ 25″	0·288,0731
sin	39° 55′ 00″	$\bar{1}$·807,3136
sin E		$\bar{1}$·918,4533

$\therefore E = 55^\circ\ 58'\ 35''$ or $124^\circ\ 01'\ 25''$.

The angle AEC must be greater than 180° as the most northerly point has not yet been reached.

$\therefore$ angle $AEC = 124^\circ\ 01'\ 25''$, the angle $CED = 55^\circ\ 58'\ 35''$,

$\therefore$ direction of flight = N. 55° 58′ 35″ W.

EXAMPLE 4. The inclined angle at a point B between two points A and C is measured with a sextant and found to be 58° 16′ 20″. The elevation of A is 29° 32′, and the depression of C is 11° 42′. Find the horizontal angle ABC.

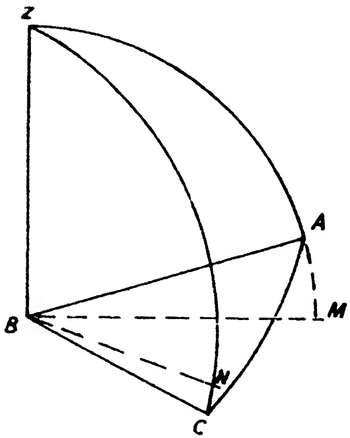


FIG. 6

Here (Fig. 6), if Z is the zenith or point vertically over B we have a spherical triangle ZAC where the three sides are

$$ZA = 90° - 29° 32' = 60° 28'$$

$$ZC = 90° + 11°42' = 101° 42'$$

$$AC = 58° 16' 20''$$ and we require the angle Z.

We use the formula $\tan \frac{Z}{2} = \sqrt{\frac{\sin (s - ZA) \,.\, \sin (s - ZC)}{\sin s \,.\, \sin (s - AC)}}$ and proceed as follows—

Side			Log. sine	Sums
60° 28′	$s - ZA$ =	49° 45′ 10″	$\bar{1}$·882,6746	$\bar{1}$·053,3616
101° 42′	$s - ZC$ =	8° 31′ 10″	$\bar{1}$·170,6870	
58° 16′ 20″	$s - AC$ =	51° 56′ 50″	$\bar{1}$·896,2193	$\bar{1}$·868,5960
2)220° 26′ 20″	s =	110° 13′ 10″	$\bar{1}$·972,3767	
s = 110° 13′ 10″		69° 46′ 50″		
				2) $\bar{1}$·184,7656
				$\bar{1}$·592,3828
				= log tan $\frac{Z}{2}$

$$\therefore \frac{Z}{2} = 21° 21' 53''$$

$$\therefore Z = 42° 43' 46'' = \text{horizontal angle } ABC.$$

EXAMPLE 5 A pipe has an inclination from A to B of 10° to the horizontal. At B it turns a plan angle of 30° and the inclination of the length BC is 25° to the horizontal. Find the deflection (inclined) angle between AB and BC.

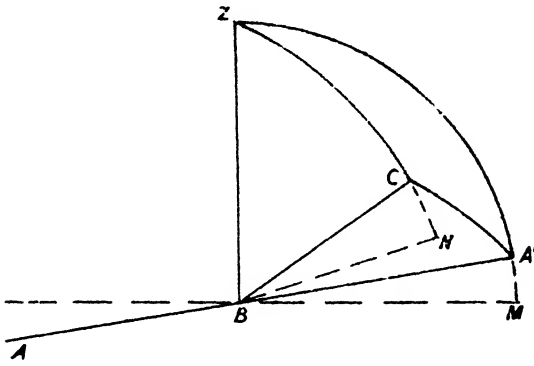


Fig. 7

Here, if we produce the length AB forward to A' (Fig. 7) we have a spherical triangle $ZA'C$, where the sides are $ZA' = 80°$, $ZC = 65°$, and the angle $Z = 30°$. We require the side $A'C$.

Here four-figure tables are sufficiently accurate and we shall use the formula

$$\cos Z = \frac{\cos A'C - \cos ZA' \,.\, \cos ZC}{\sin ZA' \,.\, \sin ZC}$$

or $$\cos A'C = \cos ZA' \cos ZC + \sin ZA' \sin ZC \,.\, \cos Z$$

cos 80°	$\bar{1}$·2397	sin 80°	$\bar{1}$·9934
cos 65°	$\bar{1}$·6259	sin 65°	$\bar{1}$·9573
		cos 30°	$\bar{1}$·9375
0·07338	$\bar{2}$·8656	0·7731	$\bar{1}$·8882

$$\therefore \cos A'C = 0{\cdot}0734 + 0{\cdot}7731 = 0{\cdot}8465$$
$$\therefore \quad A'C = 32° \, 10'$$

We can derive a formula for these two useful cases, viz. if α and β are the inclinations of the legs of the angle to the horizontal, ϕ is the (deflection) angle on plan and θ is the actual (deflection) angle between the legs, then

$$\cos\phi = \frac{\cos\theta - \cos(90° - a) . \cos(90° - \beta)}{\sin(90° - a) . \sin(90° - \beta)}$$

or $$\cos\phi = \frac{\cos\theta - \sin a . \sin\beta}{\cos a . \cos\beta},$$

or $$\cos\theta = \sin a . \sin\beta + \cos a . \cos\beta . \cos\phi$$

When one inclination a is zero, this simplifies to

$$\cos\theta = \cos\beta . \cos\phi$$

If one inclination a is negative, these formulae become

$$\cos\phi = \frac{\cos\theta + \sin a . \sin\beta}{\cos a . \cos\beta}$$

and $$\cos\theta = \cos a . \cos\beta \cos\phi - \sin a . \sin\beta$$

EXAMPLE 6. Two roots are inclined to the horizontal at angles whose tangents are $\frac{1}{2}$ and $\frac{2}{3}$, and the wall plates on which they rest make an angle of 120° with each other. Find the angle between the planes of the roofs, and the angles between their intersection and the wall plates.

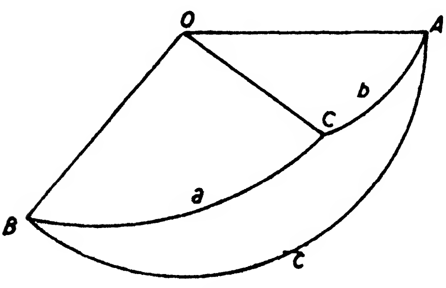


FIG. 8

Let OA and OB be the wall plates, OC the intersection of the roofs (Fig. 8). Then in the spherical triangle ABC,

$B = \tan^{-1} \cdot 5000 = 26° 34'$, $A = \tan^{-1} \cdot 6667 = 33° 41\frac{1}{2}'$,

and the side $c = 120°$; we have to find C, b, and a.

We use the supplementary formulae

$$\tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2}$$

$$\tan\frac{a-b}{2} = \frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\cdot\tan\frac{c}{2}$$

$\cos\frac{A-B}{2} = \cos\frac{7^\circ\,7\frac{1}{2}'}{2}$	$\bar{1}\cdot99916$; $\sin\frac{A-B}{2} = \sin 3^\circ\,33\frac{3}{4}'$		$\bar{2}\cdot79335$
$\sec\frac{A+B}{2} = \sec\frac{60^\circ\,15\frac{1}{2}'}{2}$	$0\cdot06304$; $\operatorname{cosec}\frac{A+B}{2} = \operatorname{cosec} 30^\circ\,07\frac{3}{4}''$		$0\cdot29934$
$\tan\frac{c}{2} = \tan 60^\circ$	$0\cdot23856$; $\tan\frac{c}{2} = \tan 60^\circ$		$0\cdot23856$
$\tan\frac{a+b}{2}$	$0\cdot30076$	$\tan\frac{a-b}{2}$	$\bar{1}\cdot33125$

$$\therefore \frac{a+b}{2} = 63^\circ\,25\frac{1}{4}' \qquad \frac{a-b}{2} = 12^\circ\,06'$$

$\therefore a = 75^\circ\,31\frac{1}{4}'$, $b = 51^\circ\,19\frac{1}{4}'$ are the angles of the intersection with the wall plates.

For the angle C we have $\sin C = \sin c\,.\,\frac{\sin A}{\sin a}$

$\sin c$	$= \sin 120^\circ$	$\bar{1}\cdot93753$
$\sin A$	$= \sin 33^\circ\,41\frac{1}{4}'$	$\bar{1}\cdot74408$
$\operatorname{cosec} a$	$= \operatorname{cosec} 75^\circ\,31\frac{1}{4}'$	$0\cdot01402$
	$\sin C$	$\bar{1}\cdot69563$

$$\therefore C = 29^\circ\,44\frac{3}{4}' \text{ or } 150^\circ\,15\frac{1}{4}'$$

Here we choose the value $150^\circ\,15\frac{1}{4}'$ as the greater angle must be opposite the greater side.

Approximations

The use of series, e.g. the Binomial Theorem that $(1 \pm x)^2 = 1 \pm \frac{nx}{1} + \frac{n \,.\, (n-1)}{1 \,.\, 2} x^2 + \ldots$ where $x < 1$, enables some useful short cuts to be made in computation by omitting terms that will be negligible when x is small, in view of the limits of accuracy fixed by that of the data. For example, if the accuracy of computation is to be $\frac{1}{100{,}000}$ the omitted terms must not exceed $\frac{1}{200{,}000}$ of the result. If the series is rapidly convergent, i.e. if each term is very small compared to the preceding term, we need only consider the value of the *next* term in deciding how many terms of the series to employ Taking first $(1 \pm x)^2 = 1 \pm 2x + x^2$, if we wish to omit the last term (x^2) and the result is to be correct to the nearest $\frac{1}{m}$ part, we write $x^2 = \frac{1}{2m}(1 \pm 2x + x^2)$, whence $(2m-1)\,x^2 \mp 2x - 1 = 0$.

$$\therefore\ x = \frac{\pm 2 + \sqrt{4 + 4(2m-1)}}{2(2m-1)} = \frac{\pm 2 + \sqrt{8m}}{4m-2}$$

$$\text{If } m = 1000,\ x = \frac{\pm 2 + \sqrt{8000}}{3998} = 0{\cdot}0229 \text{ and } 0{\cdot}0219.$$

Next, taking $(1 \pm x)^3 = 1 \pm 3x + 3x^2 \pm x^3$, if we wish to omit the last two terms and the result is to be correct to the nearest $\frac{1}{m}$ part, we write $3x^2 = \frac{1}{2m}(1 \pm 3x + 3x^2)$, whence $3(2m-1)x^2 \mp 3x - 1 = 0$.

$$\therefore\ x = \frac{\pm 3 + \sqrt{9 + 12(2m-1)}}{6(2m-1)} = \frac{\pm 3 + \sqrt{24m-3}}{12m-6}$$

$$\text{If } m = 1000,\ x = \frac{\pm 3 + \sqrt{23997}}{11994} = 0{\cdot}0132 \text{ and } 0{\cdot}0127.$$

The following table will be found useful—

TABLE I

Accuracy	$(1+x)^2 = 1+2x$	$(1-x)^2 = 1-2x$	$(1+x)^3 = 1+3x$	$(1-x)^3 = 1-3x$
To the nearest—				
$\frac{1}{1000}$ part	If $x \not> 0{\cdot}0229$	If $x \not> 0{\cdot}0219$	If $x \not> 0{\cdot}0132$	If $x \not> 0{\cdot}0127$
$\frac{1}{10,000}$ part	If $x \not> 0{\cdot}0071$	If $x \not> 0{\cdot}0070$	If $x \not> 0{\cdot}0041$	If $x \not> 0{\cdot}0041$
$\frac{1}{100,000}$ part	If $x \not> 0{\cdot}0022$	If $x \not> 0{\cdot}0022$	If $x \not> 0{\cdot}0013$	If $x \not> 0{\cdot}0013$
$\frac{1}{1,000,000}$ part	If $x \not> 0{\cdot}0007$	If $x \not> 0{\cdot}0007$	If $x \not> 0{\cdot}0004$	If $x \not> 0{\cdot}0004$

EXAMPLE 7. Find $(100{\cdot}129)^3$ and $(99{\cdot}871)^3$ to the nearest $\frac{1}{100,000}$ part.

$$(100{\cdot}129)^3 = 100^3 \times 1{\cdot}00129^3 = 10^6 \times (1 + {\cdot}00129)^3$$
$$= 10^6 \times 1{\cdot}00387 = 1,003,870.$$

$$(99{\cdot}871)^3 = 100^3 \times 0{\cdot}99871 = 10^6 \times (1 - {\cdot}00129)^3$$
$$= 10^6 \times (1 - {\cdot}00387) = 10^6 \times 0{\cdot}99613$$
$$= 996,130.$$

Take next

$$\sqrt{1 \pm x} = (1 \pm x)^{\frac{1}{2}} = 1 \pm \frac{x}{2} + \frac{\frac{1}{2}(-\frac{1}{2})}{2} x^2 \pm \ldots$$
$$= 1 \pm \frac{x}{2} - \frac{x^2}{8} \pm \ldots$$

A useful case of this expansion is $\sqrt{l^2 \pm h^2}$ where $h < l$.

We have

$$\sqrt{l^2 \pm h^2} = l\sqrt{1 \pm \frac{h^2}{l^2}} = l\left(1 \pm \frac{h^2}{2l^2} - \frac{h^4}{8l^4} \pm \ldots\right)$$

$$= l \pm \frac{h^2}{2l} - \frac{h^4}{8l^3} \pm \ldots \quad . \quad . \quad . \quad (22)$$

The first two terms will be accurate enough provided

$$\frac{h^4}{8l^3} = \frac{1}{2m}\left(l \pm \frac{h^2}{2l} - \frac{h^4}{8l^3}\right)$$

where the result is to be correct to the nearest $\frac{1}{m}$ part.

We have

$$\frac{h^4}{8l^4} = \frac{1}{2m}\left(1 \pm \frac{h^2}{2l^2} - \frac{h^4}{8l^4}\right)$$

or $$\frac{y^4}{8} = \frac{1}{2m}\left(1 \pm \frac{y^2}{2} - \frac{y^4}{8}\right) \text{ where } y = \frac{h}{l}$$

Therefore, $(2m + 1)y^4 \mp 4y^2 - 8 = 0$. From which the following table is calculated—

TABLE II

Accuracy	$\sqrt{l^2 + h^2} = l + \frac{h^2}{2l}$	$\sqrt{l^2 - h^2} = l - \frac{h^2}{2l}$
To the nearest—		
$\frac{1}{1000}$ part . . .	If $\frac{h}{l} \not> 0{\cdot}2535$	If $\frac{h}{l} \not> 0{\cdot}2495$
$\frac{1}{10,000}$ part . . .	If $\frac{h}{l} \not> 0{\cdot}1418$	If $\frac{h}{l} \not> 0{\cdot}1411$
$\frac{1}{100,000}$ part . . .	If $\frac{h}{l} \not> 0{\cdot}0796$	If $\frac{h}{l} \not> 0{\cdot}0795$
$\frac{1}{1,000,000}$ part . . .	If $\frac{h}{l} \not> 0{\cdot}0447$	If $\frac{h}{l} \not> 0{\cdot}0447$

EXAMPLE 8. To find the horizontal projection of a length of 100 ft. measured on the slope, the rise being 10 ft. in this length, i.e. to correct the length for slope.

The correction is $-\frac{h^2}{2l}-\frac{h^4}{8l^3} = -0{\cdot}500 - 0{\cdot}00125$

Therefore, if the result is required to two decimal places, the answer is 99·50 ft. If the result is required to three decimal places the answer is 99·499 ft.; if to four decimal places, 99·4988 ft. (The next term in the expansion, viz. $\frac{h^6}{16l^5}$, is only 0·000006,25.)

EXAMPLE 9. A reading of 12·13 ft. is taken on a level staff which is 2·15 ft. out of plumb in 14 ft. Find the correct reading if the staff were vertical.

In 12·13 ft. the staff is $\frac{12{\cdot}13}{14} \times 2{\cdot}15 = 1{\cdot}86$ ft. out of plumb.

$\therefore$ corrected staff reading is $12{\cdot}13 - \frac{(1{\cdot}86)^2}{24{\cdot}26} = 12{\cdot}13 - 0{\cdot}14$ $= 11{\cdot}99$ ft.

EXAMPLE 9A. A point B is 656·7 ft. north of A and 91·3 ft. west of A. Find the distance AB.

Here we require $\sqrt{l^2 + h^2}$, where $l = 656{\cdot}7$, $h = 91{\cdot}3$

$$\therefore AB = 656{\cdot}7 + \frac{(91{\cdot}3)^2}{1313{\cdot}4} = 656{\cdot}7 + 6{\cdot}3 = 663{\cdot}0 \text{ ft.}$$

$\left(\text{As } \frac{91{\cdot}3}{656{\cdot}7} = 0{\cdot}139\text{, this result will be correct to } \frac{1}{10{,}000} \text{ part.}\right)$

A somewhat similar application of the use of approximations is to find the amount δl by which we must correct a length l measured on a slope, rising a height h in that distance, in order that the horizontal distance may be l (Fig. 9).

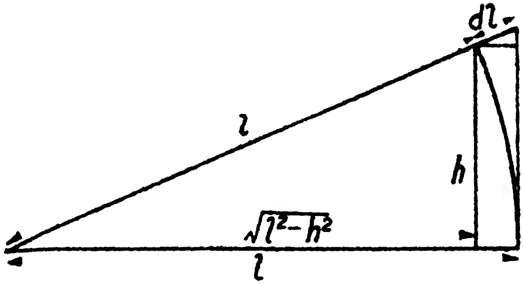


Fig. 9

$$\text{Here}\quad \delta l = (l - \sqrt{l^2 - h^2})\frac{l}{\sqrt{l^2-h^2}} = \frac{l^2}{\sqrt{l^2-h^2}} - l$$

$$= l\left(\frac{1}{\sqrt{1-\frac{h^2}{l^2}}} - 1\right)$$

$$= l\left(\left(1-\frac{h^2}{l^2}\right)^{-\frac{1}{2}} - 1\right)$$

$$= l\left(1 + \frac{h^2}{2l^2} - \frac{\frac{1}{2}(-\frac{1}{2}-1)}{1\,.\,2}\frac{h^4}{l^4} + \ldots - 1\right)$$

$$= \frac{h^2}{2l} + \frac{3}{8}\frac{h^4}{l^3} + \ldots \qquad (23)$$

The correction δl along the slope can, therefore, be taken as $\frac{h^2}{2l}$ provided the equation $\frac{3}{8}\frac{h^4}{l^3} = \frac{1}{2000}\left(l + \frac{h^2}{2l} + \frac{3}{8}\frac{h^4}{l^3}\right)$ gives a value of $\frac{h}{l}$ which is $\not<$ the given value of $\frac{h}{l}$, for an accuracy of $\frac{1}{1000}$ part, as is sufficient for good ordinary chaining. The solution of this equation is $\frac{h}{l} \doteq 0{\cdot}1920$ for this accuracy, i.e. a slope of 11° 04′ or 1 in 5·11[illegible].

This is a very convenient way of applying the correction for slope in ordinary chaining. The chain is first stretched on the slope and the leading arrow inserted. The difference of level of the ends is then ascertained to 0·5 by a hand-level and a ranging rod marked in feet (or in links if a Gunter's chain is used). The correction $\frac{h^2}{2l}$ is then calculated, where the denominator is usually 200, and the leading arrow is then advanced along the slope by that amount.

Other useful approximations are afforded by the Trigonometrical Series for Angles, viz.

$$\sin x = x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \ldots$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots$$

$$\cos x = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \ldots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots$$

$$\tan x = x + \frac{2x^3}{1 \cdot 2 \cdot 3} + \frac{16x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

where x is the circular measure of the angle, or "arc" of the angle, i.e. arc subtended at unit radius.

By a similar method to the preceding the following table has been prepared—

TABLE III

Accuracy	sin x $= x$	cos x $= 1$	tan x $= x$	sin x $= x - \frac{x^3}{6}$	cos x $= 1 - \frac{x^2}{2}$	tan x $= x + \frac{x^3}{3}$
To the nearest						
$\frac{1}{1000}$ part	3° 08′	1° 49′	2° 13′	28° 04′	18° 42′	14° 15′
$\frac{1}{10,000}$ part	1° 00′	0° 34′	0° 42′	15° 54′	10° 37′	7° 59′
$\frac{1}{100,000}$ part	0° 19′	0° 11′	0° 13′	8° 58′	5° 59′	4° 29′
$\frac{1}{1,000,000}$ part	0° 06′	0° 03′	0° 05′	5° 02′	3° 22′	2° 31′

Similarly, if x is a "small" angle in view of the required accuracy so that the first term only of the expansions need be used, we may write

$$\sin(\pi - x) = x,\ \cos(\pi - x) = -1,\ \tan(\pi - x) = -x$$

for angles close to 180°.

$$\sin\left(\frac{\pi}{2} - x\right) = 1,\ \cos\left(\frac{\pi}{2} - x\right) = x,$$

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{1}{\tan x} = \frac{1}{x},$$ for angles close to 90°.

For angles, therefore, not exceeding a few minutes in magnitude we can say $\sin A = \tan A = \text{arc } A = \frac{A''}{206,264\cdot8}$ to a very high degree of accuracy, as there are 206,264·8″ in one radian. The quantity $\frac{1}{206,264\cdot8}$ is usually written as

$\sin 1''$, and we can write $\sin A = A''.\sin 1'' = \tan A = \text{arc } A$ for such angles.

When the logarithms of $\sin A$, $\tan A$, and arc A are required for computation, it is necessary to adopt this form, as the logarithms of very small fractions increase very rapidly (negatively) as the fraction diminishes, the logarithm of zero being $-\infty$, and though the values of log sine and log tan are given for every $1'$ in, say, *Chambers' Seven-Figure Mathematical Tables*, it would be inaccurate to interpolate proportionally among these for the intermediate seconds as the increments from minute to minute are so variable. The value of $\log \sin 1'' = \bar{6}{\cdot}685{,}5749$ and for such small angles we write $\log \sin A = \log \tan A = \log \text{arc } A = \log A'' + \bar{6}{\cdot}685{,}5749$.

Thus, if we have to apply the "Rule of Sines" to a plane triangle in which one angle C, and consequently the opposite side c, is very small, instead of writing $\frac{c}{\sin C} = \frac{b}{\sin B} = \frac{a}{\sin A}$ we write $\frac{c}{C'' \sin 1''} = \frac{b}{\sin B}$, etc., or in logarithms

$$\log c = \log C'' + \log \sin 1'' + \log b + \log \text{cosec } B$$

Similarly, in a spherical triangle, if one angle C, and consequently the opposite side c, is very small, instead of writing $\frac{\sin c}{\sin C} = \frac{\sin b}{\sin B} = \frac{\sin a}{\sin A}$, we write $\frac{c'' \sin 1''}{C'' \sin 1''} = \frac{\sin b}{\sin B}$, etc., i.e. $\frac{c''}{C''} = \frac{\sin b}{\sin B}$, or in logarithms

$$\log c'' = \log C'' + \log \sin b + \log \text{cosec } B$$

In measuring vertical heights with the theodolite, if A'' is the vertical angle in seconds (corrected for the curvature of the earth and atmospheric refraction) and is small and D

is the horizontal distance, then the height H is not written as $D \cdot \tan A''$ but as $D \cdot A'' \cdot \tan 1''$, or in logarithms

$$\log H = \log D + \log A'' + \log \tan 1''$$
$$= \log D + \log A'' + \bar{6}{\cdot}685{,}5749$$

THE COMMON CATENARY

This is the curve assumed by a uniform, inextensible, but flexible, chain, wire, tape, or cord, hanging freely. We shall

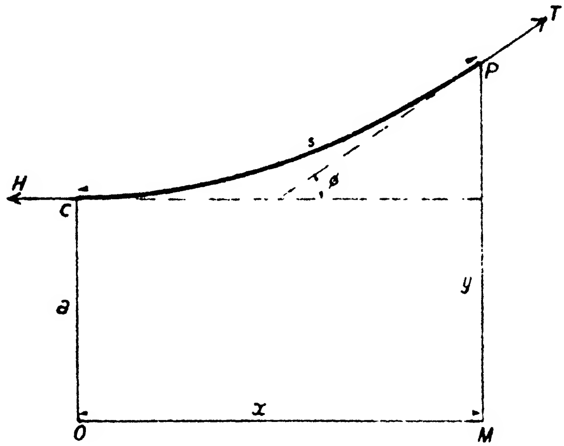


FIG. 10

first consider an arc CP of the catenary, commencing at its lowest point, or vertex C, where the curve is horizontal (Fig. 10).

Let s be the length of the arc CP, w its weight per unit length, $W = ws =$ weight of arc CP, $H = wa =$ the horizontal tension at C (so that a is the length of catenary whose weight $= H$), and T the tension at P acting along the tangent at P at an inclination ϕ to the horizontal.

Take the origin at O, vertically below C and so that $CO = a$, and take horizontal and vertical axes of x and y through O, so that $OM = x$, $PM = y$. Then, considering the equilibrium of the arc CP we have

$$T \sin \phi = ws,\ T \cos \phi = wa, \text{ and } \therefore s = a \tan \phi \qquad (24)$$

$$\text{while } \frac{dx}{ds} = \cos \phi,\ \frac{dv}{ds} = \sin \phi,\ \frac{ds}{d\phi} = a \sec^2 \phi = \frac{a}{\cos^2 \phi}$$

$$\text{Then } \frac{dy}{d\phi} = \frac{dy}{ds}\,\frac{ds}{d\phi} = \frac{\sin \phi \,.\, a}{\cos^2 \phi}$$

$$\therefore y = \int dy = a \int \frac{\sin \phi \,.\, d\phi}{\cos^2 \phi} = -a \int \frac{d(\cos \phi)}{\cos^2 \phi}$$

$$= \frac{a}{\cos \phi} = a \sec \phi$$

There is no constant of integration as at C, where $\phi = 0$, $y = a$.

$$\text{But } T = wa \sec \phi, \quad \therefore T = wy \qquad (25)$$

$$\text{Also } \frac{dx}{d\phi} = \frac{dx}{ds}\,\frac{ds}{d\phi} = \frac{\cos \phi \,.\, a}{\cos^2 \phi}$$

$$\therefore x = \int dx = a \int \frac{\cos \phi}{\cos^2 \phi}\, d\phi = a \int \frac{d(\sin \phi)}{1 - \sin^2 \phi}$$

$$= \frac{a}{2}\left(\int \frac{d(\sin \phi)}{1 + \sin \phi} + \int \frac{d(\sin \phi)}{1 - \sin \phi} \right)$$

$$= \frac{a}{2}\left(\log_e (1 + \sin \phi) - \log_e (1 - \sin \phi)\right)$$

$$= \frac{a}{2} \log_e \frac{1 + \sin \phi}{1 - \sin \phi} = \frac{a}{2} \log_e \frac{1 + \frac{ws}{T}}{1 - \frac{ws}{T}}$$

$$= \frac{a}{2} \log_e \frac{T + W}{T - W} \qquad (26)$$

Here, again, there is no constant of integration as $\phi = 0$, when $x = 0$.

If the supports of a catenary are at the same level, it is obvious that the catenary will be symmetrical about its

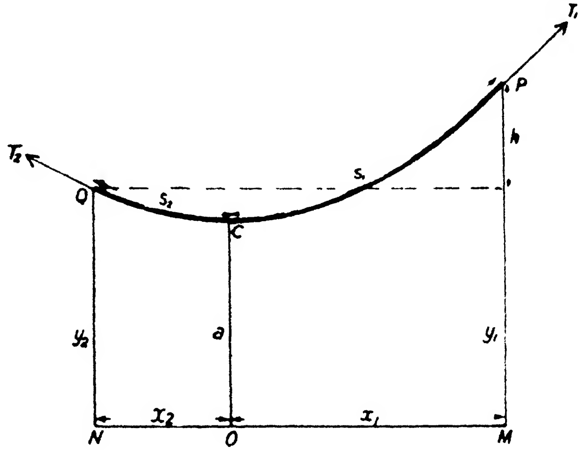


FIG. 11

vertex and the span, or horizontal distance between the supports, will be $2x$ or $a \log_e \frac{T + W}{T - W}$, where W is half the weight of the catenary of length $2s$.

If, however, the supports P and Q are at different levels there will be two cases—

(*a*) When the vertex lies between the supports (Fig. 11).

Let the lengths into which the catenary is divided by the vertex C be s_1, s_2, and let the total length be s, and the difference of level of P and Q be h.

At any point we have

$$T^2 = H^2 + W^2 \quad \therefore w^2y^2 = w^2a^2 + w^2s^2 \quad \therefore y^2 = a^2 + s^2$$

Then for P, $y_1^2 = a^2 + s_1^2$; for Q, $y_2^2 = a^2 + s_2^2$

$$\therefore y_1^2 - y_2^2 = s_1^2 - s_2^2$$

$$\therefore\ (y_1 - y_2)(y_1 + y_2) = (s_1 - s_2)(s_1 + s_2)$$

$$\therefore\ h(y_1 + y_2) = s(s_1 - s_2)$$

$$\therefore\ s_1 - s_2 = \frac{h}{s}(y_1 + y_2)$$

Also, $s_1 + s_2 = s$

$$\therefore\ s_1 = \frac{s}{2} + \frac{h}{2s}(y_1 + y_2) = \frac{s}{2} + \frac{h}{W}\cdot\frac{T_1 + T_2}{2}$$

and $$s_2 = \frac{s}{2} - \frac{h}{2s}\cdot(y_1 + y_2) = \frac{s}{2} - \frac{h}{W}\cdot\frac{T_1 + T_2}{2} \qquad (27)$$

where $W = ws =$ the total weight of the catenary.

(*b*) When the vertex lies outside the supports (Fig. 12).

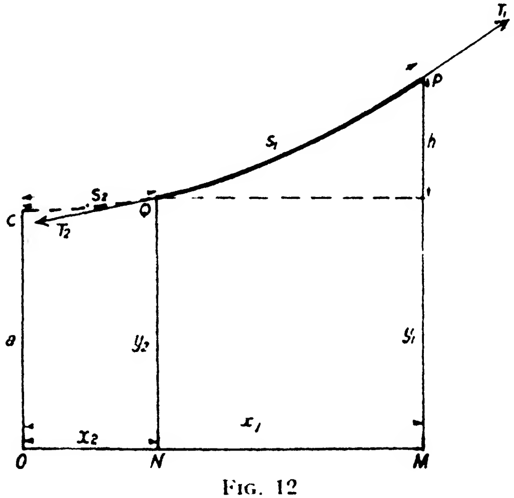


FIG. 12

As before we have $y_1^2 - y_2^2 = s_1^2 - s_2^2$, but here $s_1 - s_2 = s$.

$$\therefore\ s_1 + s_2 = \frac{h}{s}(y_1 + y_2)$$

$$\therefore\ s_1 = \frac{h}{2s}(y_1 + y_2) + \frac{s}{2} = \frac{h}{W}\cdot\frac{T_1 + T_2}{2} + \frac{s}{2}$$

$$\therefore \quad s_2 = \frac{h}{2s}(y_1 + y_2) - \frac{s}{2} = \frac{h}{W} \cdot \frac{T_1 + T_2}{2} - \frac{s}{2} \quad . \quad . \quad (28)$$

In both cases the distance of the vertex from the centre of the catenary is $\frac{h}{W}\frac{T_1 + T_2}{2}$, measured along the catenary.

Having found s_1 and s_2, we can find $W_1 = w_1 s_1$ and $W_2 = w_2 s_2$, while $a = \frac{H}{w} = \frac{\sqrt{T_1^2 - W_1^2}}{w} = \frac{\sqrt{T_2^2 - W_2^2}}{w}$, and thence the span

$$x_1 \pm x_2 = \frac{a}{2}\left(\log_e \frac{T_1 + W_1}{T_1 - W_1} \pm \log_e \frac{T_2 + W_2}{T_2 - W_2}\right)$$

Note that, if we know the difference of level, h, of the supports, only T_1 or T_2 need be known, as $T_1 = wy_1 = w(y_2 + h) = T_2 + wh$.

We can, therefore, find the horizontal span in all cases, if we know w, s, h, and the tension at one end of a catenary. The process is, however, somewhat lengthy. For the "taut" catenaries which are used in base-line measurement, where the tension is great compared to the weight the horizontal span can be found from the measured length much more quickly by an approximate correction, which we shall now proceed to find.

From equation (26) we have

$$x = \frac{a}{2}\log_e \frac{1 + \sin\phi}{1 - \sin\phi} = \frac{a}{2}\log_e \frac{(1 + \sin\phi)^2}{\cos^2\phi}$$

$$= a \log_e \frac{1 + \sin\phi}{\cos\phi} = a \log_e (\sec\phi + \tan\phi)$$

$$\therefore \qquad \sec\phi + \tan\phi = e^{\frac{x}{a}}$$

but $\sec\phi - \tan\phi = \frac{1}{\sec\phi + \tan\phi}$, as $\sec^2\phi = 1 + \tan^2\phi$

$$\therefore \qquad \sec\phi - \tan\phi = e^{-\frac{x}{a}}$$

$$\therefore\ \sec\phi = \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2} \text{ and } \tan\phi = \frac{e^{\frac{x}{a}} - e^{-\frac{x}{a}}}{2}$$

$$\therefore \qquad y = a\sec\phi = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

$$\text{and} \qquad s = a\tan\phi = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$$

Expanding these we get

$$y = \frac{a}{2}\left\{\left(1 + \frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{6a^3} + \frac{x^4}{24a^4} + \ldots\right) + \left(1 - \frac{x}{a} + \frac{x^2}{2a^2} - \frac{x^3}{6a^3} + \frac{x^4}{24a^4} + \ldots\right)\right\}$$

$$= a + \frac{x^2}{2a} + \frac{x^4}{24a^3} + \ldots \qquad (29)$$

$$s = \frac{a}{2}\left\{\left(1 + \frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{6a^3} + \frac{x^4}{24a^4} + \frac{x^5}{120a^5} + \ldots\right) - \left(1 - \frac{x}{a} + \frac{x^2}{2a^2} - \frac{x^3}{6a^3} + \frac{x^4}{24a^4} - \frac{x^5}{120a^5} + \ldots\right)\right\}$$

$$= x + \frac{x^3}{6a^2} + \frac{x^5}{120a^4} + \ldots \qquad (30)$$

From equation (29) we see that the catenary (Fig. 10) is very approximately a parabola with its vertex at C, if the third term of the expansion can be neglected in comparison with the second, i.e. if the ratio $\frac{x^2}{12a^2}$ is very small. This ratio is approximately $\frac{s^2}{12a^2} = \frac{w^2s^2}{12w^2a^2} = \frac{W^2}{12H^2} = \frac{W^2}{12T^2}$ nearly. As the tension in baselines will at least be 10 times the weight of the tape or $T = 20W$, this ratio will not be greater than $\frac{1}{4800}$, so that the approximation to a parabola will be very close. If the span were 100 ft., so that $x = 50$ ft., the dip $= y - a = \frac{x^2}{2a}$ nearly $= \frac{50 \cdot s}{2a} = \frac{50W}{2H} = \frac{50}{40} = 1{\cdot}25$ ft. The maximum difference in ordinate from a parabola would only, therefore, be $\frac{1{\cdot}25}{4800}$ or about $\frac{1}{4000}$ ft.

From equation (30) we can take $s = x + \frac{x^3}{6a^2}$, ignoring the third term of the expansion as its ratio to the second term is only $\frac{x^2}{20a^2}$ or about $\frac{1}{8000}$, and write the correction of length as $s - x = \frac{x^3}{6a^2}$ on each half of a catenary with level supports.

The correction for the whole catenary will be $2(s-x) = \frac{x^3}{3a^2} = \frac{l^3}{24a^2}$ where l is the horizontal span $= \frac{w^2l^3}{24H^2}$. As, however, it is the length $S = 2s$ which we actually measure, and the tension T at the supports, we write the "*Sag Correction*" for a *catenary with level supports* as

$$\frac{w^2S^3}{24T^2} \text{ or as } \frac{S}{24} \cdot \frac{W^2}{T^2} \quad . \quad . \quad . \quad (31)$$

where *in this case* $W =$ the weight of the tape between supports $= wS$. (Note that we have slightly increased the numerator and denominator, changing l^3 to S^3 and H^2 to T^2.)

EXAMPLE 10. A steel tape is supported on two knife edges at the same level, and the length of the tape between supports is 100·0000 ft. when corrected for temperature and tension. Find the horizontal distance between the supports if the tension there is 30 lb. and the weight of the tape is ·025 lb. per ft.—also find the dip of the tape.

(*a*) Horizontal distance by the exact formula (26)

$$= a \log_e \frac{T + W}{T - W} \text{ where } W = 1{\cdot}25 \text{ lb.}$$

$$H = \sqrt{30^2 - 1{\cdot}25^2} = 30 - \frac{1{\cdot}25^2}{60} = 30 - \frac{1{\cdot}563}{60}$$

$$= 30 - {\cdot}02605 = 29{\cdot}97395 \text{ lb.}$$

$$\therefore \; a = \frac{H}{w} = 29{\cdot}97395 \times 40 = 1198{\cdot}9580 \text{ ft.}$$

$$\text{Then } l = 1198{\cdot}9580 \times 2{\cdot}302585 \log \frac{31{\cdot}25}{28{\cdot}75}$$

$$= 99{\cdot}97112 \text{ ft.}$$

31·25	1·494,8500
28·75	1·458,6378
	0·036,2122
0·036,2122	$\bar{2}$·558,8549
2·302,585	0·362,2156
1198·9580	3·078,8040
99·97112	1·999,8745

(*b*) The approximate "Sag Correction" is $\frac{S}{24} \cdot \frac{W^2}{T^2}$ where $W = 2{\cdot}5$ lb.

$$= \frac{100}{24}\left(\frac{2{\cdot}5}{30}\right)^2 = \frac{100}{24 \times 144} = \frac{100}{3456} = {\cdot}028935$$

Then $l = 99{\cdot}971065$ ft., so that the error is only ${\cdot}000055$ ft. or $\frac{1}{2,000,000}$ part.

(*c*) For the dip, we have $T = wy$, $H = wa$,

$$\therefore\ T - H = w(y - a).$$

$$\therefore\ \text{dip} = y - a = \frac{T - H}{w} = \frac{{\cdot}02605}{{\cdot}025} = 1{\cdot}042 \text{ ft.}$$

In practice, however, the supports will rarely be at the same level and for an exact calculation we must find the lengths s_1, s_2 of the arcs from the vertex to the supports by the formulae (27), (28), and from them calculate x_1 and x_2, and hence $x_1 \pm x_2$ for the span. To avoid this labour we can use an approximate correction when the parabola is a "taut" one as above defined, which will, when applied, give the horizontal span with great accuracy. This correction will be $\frac{x_1^3}{6a^2} \pm \frac{x_2^3}{6a^2} = \frac{w^2(x_1^3 \pm x_2^3)}{6H^2}$, but as the measured quantities are $S = s_1 \pm s_2$ and T_1, T_2 the tensions at the supports, we shall write this correction as $\frac{w^2(s_1^3 \pm s_2^3)}{6T^2}$ where $T = \frac{T_1 + T_2}{2}$ = average tension at the supports.

Substituting the values of s_1, s_2 from formulae (27), (28), we get

$$\frac{w^2}{6T^2}\left\{\left(\frac{S}{2} + \frac{hT}{W}\right)^3 + \left(\frac{S}{2} - \frac{hT}{W}\right)^3\right\}$$

for vertex between supports, and

$$\frac{w^2}{6T^2}\left\{\left(\frac{hT}{W} + \frac{S}{2}\right)^3 - \left(\frac{hT}{W} - \frac{S}{2}\right)^3\right\}$$

for vertex outside supports, which, however, are identical.

In both cases, therefore, the correction is

$$\frac{w^2}{6T^2}\left\{\left(\frac{S^3}{8} + 3\frac{S^2}{4}\cdot\frac{hT}{W} + 3\frac{S}{2}\cdot\frac{h^2T^2}{W^2} + \frac{h^3T^3}{W^3}\right)\right.$$
$$\left. + \left(\frac{S^3}{8} - 3\cdot\frac{S^2}{4}\cdot\frac{hT}{W} + 3\cdot\frac{S}{2}\cdot\frac{h^2T^2}{W^2} - \frac{h^3T^3}{W^3}\right)\right\}$$
$$= \frac{w^2S^3}{24T^2} + \frac{w^2S\cdot h^2}{2W^2} = \frac{w^2S^3}{24T^2} + \frac{h^2}{2S} = \frac{S}{24}\cdot\frac{W^2}{T^2} + \frac{h^2}{2S} \quad (32)$$

= *sag correction for level supports + slope correction.*

EXAMPLE 11. A steel tape is supported on two knife-edges whose difference of level is 10 ft. The length of the tape between supports is 100·0000 ft. when corrected for temperature and tension, the weight per foot is 0·025 lb. and the tension at the lower support is 30 lb. Find the horizontal length between the supports.

(*a*) Here $T_2 = 30$ lb., $T_1 = 30$ lb. $+ 10 \times 0{\cdot}025 = 30{\cdot}25$ lb. $W = 100 \times 0{\cdot}025$ lb. $= 2{\cdot}50$ lb.

$$\therefore \quad s_1 = \frac{100}{2} + \frac{10}{5}(30 + 30{\cdot}25) = 50 + 120{\cdot}50 = 170{\cdot}50 \text{ ft.}$$

$$\therefore W_1 = \frac{170{\cdot}50}{40} = 4{\cdot}2625 \text{ lb.}$$

$$s_2 = \frac{100}{2} - \frac{10}{5}(30 + 30{\cdot}25) = 50 - 120{\cdot}50 = -70{\cdot}50 \text{ ft.}$$

$$\therefore W_2 = \frac{70{\cdot}50}{40} = 1{\cdot}7625 \text{ lb.}$$

This shows that the vertex of the catenary is *outside* the supports.

$$a = \frac{\sqrt{30^2 - (1{\cdot}7625)^2}}{0{\cdot}025} = 40\left(30 - \frac{3{\cdot}1064}{60} - \frac{3{\cdot}1064^2}{216000}\right)$$
$$= (30 - {\cdot}051773 - {\cdot}000045)40 = 29{\cdot}94818 \times 40$$
$$= 1197{\cdot}9272 \text{ ft.}$$

$$\therefore\ x_1 = 598{\cdot}9636 \times \log_e \frac{30{\cdot}25 + 4{\cdot}2625}{30{\cdot}25 - 4{\cdot}2625}$$

$$= 598{\cdot}9636 \times 2{\cdot}302585 \log \frac{34{\cdot}5125}{25{\cdot}9875}$$

$$= 169{\cdot}9295 \text{ ft.}$$

34·5125	1·537,9764
25·9875	1·414,7645
	0·123,2119
0·123,2119	$\bar{1}$·090,6537
2·302585	0·362,2156
598·9636	2·777,4004
169·9295	2·230,2697

$$x_2 = 598{\cdot}9636 \log_e \frac{30{\cdot}00 + 1{\cdot}7625}{30{\cdot}00 - 1{\cdot}7625}$$

$$= 598{\cdot}9636 \times 2{\cdot}302585 \log \frac{31{\cdot}7625}{28{\cdot}2375}$$

$$= 70{\cdot}4593 \text{ ft.}$$

31·7625	1·501,9147
28·2375	1·450,8263
	0·051,0884
0·051,0884	$\bar{2}$·708,3223
2·302585	0·362,2156
598·9636	2·777,4004
70·4593	1·847,9383

$$\therefore \text{ Horizontal distance } = x_1 - x_2 = 99{\cdot}4702 \text{ ft.}$$

$$(b) \text{ Sag correction } = \frac{100}{24}\left(\frac{2{\cdot}5}{30{\cdot}125}\right)^2 = \frac{100}{24 \times 145{\cdot}2}$$

$$= \frac{100}{3485} = 0{\cdot}0287 \text{ ft.}$$

$$\text{Slope correction } = \frac{h^2}{2S} = \frac{100}{200} = 0{\cdot}500 \text{ ft.}$$

$$\text{Total correction } = 0{\cdot}5287 \text{ ft.}$$

$$\therefore \text{ Horizontal distance } = 99{\cdot}4713 \text{ ft.}$$

The approximate method, therefore, is only in error by $\frac{1}{90,000}$ for a slope of 1 in 10. To show the accuracy of this simple method of correcting for sag and slope in a baseline, the following table has been prepared; in all cases the length of the catenary between supports is assumed to be

100·0000 ft. after correcting for temperature and tension, the tape weighs 0·025 lb. per ft. and the tension at the lower support is 30 lb.

TABLE IV

Rise	$\frac{a}{2}$	x_1	x_2	Horizontal Distance	$\frac{S}{24}\ \frac{W^2}{T^2}$	$\frac{h^2}{2S}$	Total Correction	Corrected Length	Error
Ft.	Ft.	Ft	Ft.	Ft.					Ft.
0	599·4790	49·98556	49·98556	99·97112	·028935	0	·028935	99·971065	− ·00005
2	599·8594	73·97315	25·97799	99·95114	·02889	·02	·04889	99·95111	− ·00003
4	599·9992	97·97122	1·92008	99·89130	·02884	·08	·10884	99·89116	− ·00014
6	599·8975	121·9699	− 22·1787	99·7912	·02879	·18	·20879	99·7912	·0000
8	599·5528	145·9594	− 46·3085	99·6509	·02874	·32	·34874	99·6513	+ ·0004
10	598·9636	169·9295	− 70·4593	99·4702	·02870	·50	·52870	99·4713	+ ·0011

It seems, therefore, that the approximate correction may be safely used with an accuracy of $\frac{1}{100,000}$ up to a rise of 8 ft. in 100 ft., which it may be noted is the limit for this accuracy in Table II. The student should work out one of the above cases as an exercise in the catenary and in the use of 7-figure logarithms.

Correction of Errors by the Method of Least Squares

In scientific terminology we discriminate between "errors" and "mistakes." The latter are due to carelessness of observation or calculation, and should be eliminated by a thorough system of checking results. By "errors" we mean the small residual differences from the correct values due to unavoidable defects in the instruments or accuracy of observation and these errors are of two kinds: (*a*) *Systematic* (or Cumulative) errors, which are always in the same direction, i.e. always positive or always negative; and (*b*) *Accidental* (or Compensating) errors, which are equally likely to be positive or negative.

For example, if a mile on level ground is chained with absolute accuracy with a Gunter's chain, which is 0·1 link short of 66 ft. in length, the error in the mile will be $+ 80 \times 0{\cdot}1 = + 8{\cdot}0$ links. This would be a systematic error, and the total error would be proportional to the number of observations. On the other hand, if the chain was quite correct in length but the arrow was sometimes inserted 0·1 links in front of the end of the chain and sometimes 0·1 links behind the end, in both cases quite unsystematically, it can be shown mathematically that the error in the mile would probably be $\sqrt{\text{sum of (errors)}^2}$, i.e. $\sqrt{80.(\pm 0{\cdot}1)^2} = \sqrt{80} \times (\pm 0{\cdot}1) = \pm 8{\cdot}9443 \times 0{\cdot}1 = \pm 0{\cdot}89443$ links. The total error is therefore much smaller, being of a compensating nature and proportional to $\sqrt{\text{number of observations}}$. Here the observations may be said to be "in series."

Again, if we average a number n of observations of the same quantity on each of which there is an error $+ e$, the error of the average will be $+ \dfrac{ne}{n} = + e$, but if the errors are compensating the error of the average will be $\dfrac{\sqrt{\text{sum of (errors)}^2}}{n} = \dfrac{\sqrt{ne^2}}{n} = \pm \dfrac{e}{\sqrt{n}}$, so that the error is reduced by averaging. Here the observations may be said to be "in parallel."

The Method of Least Squares is a methbd of dealing with slightly discrepant observations where the discrepancies or errors are assumed to be of an accidental nature only. It assumes that the individual errors will most probably be such that the *Sum of the Squares of the Errors will be a Minimum,* and by this means enables us to reduce the observations to their most probable consistent values. A numerical example will first be worked out from first principles—

EXAMPLE 12. The following results were obtained in the rating tests of a current meter—

n = revolutions per second	v = velocity in feet per second
0·30	1·40
1·00	4·60
1·50	6·85
2·00	9·00

Find the constants a and b in the rating formula $v = an + b$.

The observations are shown plotted in Fig. 13, and are found to be not quite in one straight line. If we "average" the observations graphically by eye by drawing a straight line and measure the intercept on the v-axis and the ordinate q and abscissa p to some point on the line, we have b = intercept and $a = \frac{q - b}{p}$, but it is doubtful if any two people will draw quite the same "average line" and obtain quite the same values for a and b. The Method of Least Squares will give definite values for a and b which will be obtained by all computers and which will be, in addition, the most probable values, providing, of course, that the law is really of the form $v = an + b$. Also, as will be shown later, it is equally applicable to the law $v = an^2 + bn + c$ (which is a parabola and which it would be very laborious to draw so as to "average" the points), or to any other mathematical expression, and the criterion as to which formula is the most probable will be that it is the one for which the sum of the squares of the errors is least.

If a and b are the correct values of those quantities, the

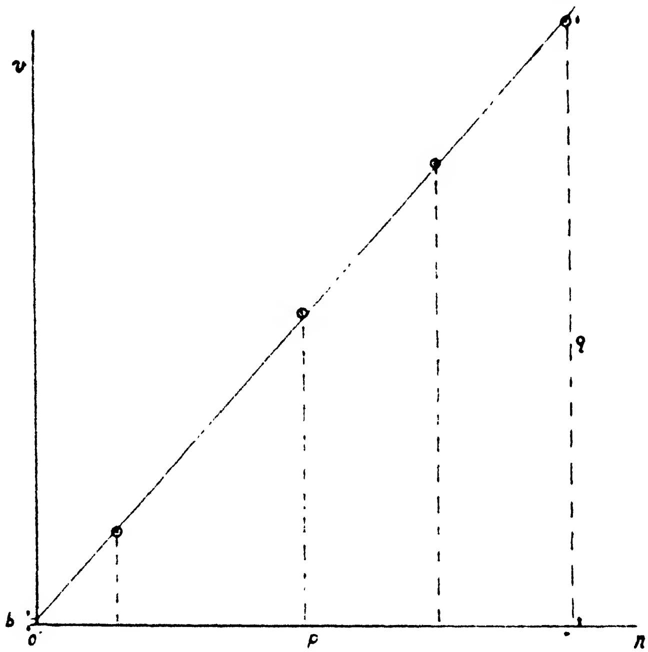


FIG. 13

errors of the four observations are $1{\cdot}40 - 0{\cdot}30a - b$, $4{\cdot}60 - a - b$, $6{\cdot}85 - 1{\cdot}50a - b$, and $9{\cdot}00 - 2{\cdot}00a - b$.

Then $(1{\cdot}4 - 0{\cdot}3a - b)^2 + (4{\cdot}6 - a - b)^2 + (6{\cdot}85 - 1{\cdot}5a - b)^2 + (9 - 2a - b)^2$ is to be a minimum.

Differentiating with regard to a and equating to zero we get

$$-0{\cdot}3(1{\cdot}4 - 0{\cdot}3a - b) - 1(4{\cdot}6 - a - b) \quad 1{\cdot}5(6{\cdot}85 - 1{\cdot}5a - b)$$
$$-2(9 - 2a - b) = 0$$

Differentiating with regard to b and equating to zero we get

$$-1(1{\cdot}4-0{\cdot}3a-b)-1(4{\cdot}6-a-b)-1(6{\cdot}85-1{\cdot}5a-b)$$
$$-1(9-2a-b)=0$$

These equations reduce to

$$\begin{cases} 7{\cdot}34a+4{\cdot}8b-33{\cdot}295=0 \\ 4{\cdot}80a+4{\cdot}0b-21{\cdot}85\ \ =0 \end{cases}$$

$$\therefore \quad \begin{cases} 29{\cdot}36a+4{\cdot}8\times 4b=133{\cdot}18 \\ 23{\cdot}04a+4{\cdot}8\times 4b=104{\cdot}88 \end{cases}$$

$$\therefore \quad a=\frac{28{\cdot}30}{6{\cdot}32}=4{\cdot}478$$

$$b=\frac{21{\cdot}85-21{\cdot}4944}{4}=0{\cdot}089$$

$\therefore$ the formula is, $v=4{\cdot}478n+0{\cdot}089$

The errors of the observed points are, therefore,

	(Error)²
$1{\cdot}40-1{\cdot}343-0{\cdot}089=-0{\cdot}032$	0·001024
$4{\cdot}60-4{\cdot}478-0{\cdot}089=+0{\cdot}033$	0·001089
$6{\cdot}85-6{\cdot}717-0{\cdot}089=+0{\cdot}044$	0·001936
$9{\cdot}00-8{\cdot}956-0{\cdot}089=-0{\cdot}045$	0·002025
$\therefore\ \Sigma\,(\text{error})^2=$	0·006074

We can now generalize the method taking the formula $y=ax^2+bx+c$ for illustration, and let suffixes distinguish the various observations x_1y_1, x_2y_2, x_3y_3, etc. The errors are therefore, $y_1-ax_1^2-bx_1-c$, $y_2-ax_2^2-bx_2-c$, $y_3-ax_3^2-bx_3-c$ etc. The condition is that

$$\Sigma(y-ax^2-bx-c)^2=(y_1-ax_1^2-bx_1-c)^2+(y_2-ax_2^2-bx_2-c)^2$$
$$+(y_3-ax_3^2-bx_3-c)^2+\text{etc.}$$

shall be a minimum. Differentiating this in turn with regard to a, b, and c, and equating each differentiation to zero we get (dividing by 2 in each case)

$$\begin{cases} -x_1^2(y_1 - ax_1^2 - bx_1 - c) - x_2^2(y_2 - ax_2^2 - bx_2 - c) \\ \qquad - x_3^2(y_3 - ax_3^2 - bx_3 - c) - \ldots = 0 \\ -x_1(y_1 - ax_1^2 - bx_1 - c) - x_2(y_2 - ax_2^2 - bx_2 - c) \\ \qquad - x_3(y_3 - ax_3^2 - bx_3 - c) - \ldots = 0 \\ -(y_1 - ax_1^2 - bx_1 - c) - (y_2 - ax_2^2 - bx_2 - c) \\ \qquad - (y_3 - ax_3^2 - bx_3 - c) \ldots = 0 \end{cases}$$

$$\text{or,} \begin{cases} a\Sigma(x^4) + b\Sigma(x^3) + c\Sigma(x^2) - \Sigma(x^2y) = 0 \\ a\Sigma(x^3) + b\Sigma(x^2) + c\Sigma(x) - \Sigma(xy) = 0 \\ a\Sigma(x^2) + b\Sigma(x) + n \cdot c - \Sigma(y) = 0 \end{cases}$$

where n is the number of observations.

These are the simultaneous equations for a, b, and c, and are called the "Normal Equations."

We notice that the first equation is formed by multiplying each error by the coefficient of a (viz. $-x_1^2, -x_2^2, -x_3^2, \ldots$) in that error, adding the products together and equating to zero; similarly, that the second equation is formed by multiplying each error by the coefficient of b (viz. $-x_1, -x_2, -x_3, \ldots$) in that error, adding the products and equating to zero, while the third equation is formed by multiplying each error by the coefficient of c (viz. $-1, -1, -1, \ldots$) in that error, adding the products and equating to zero.

We do not, therefore, need to go through the process of differentiation each time, but, instead, we prepare a table of coefficients of a, b, c, and of the term not involving them, thus –

a	*b*	*c*	*N*
$-x_1^2$	$-x_1$	-1	y_1
$-x_2^2$	$-x_2$	-1	y_2
$-x_3^2$	$-x_3$	-1	y_3
—	—	—	—
—	—	—	—

Then, multiplying by coefficients of *a*, we get

$$\begin{aligned} & a(x_1^4 + x_2^4 + x_3^4 + \ldots) \\ & + b(x_1^3 + x_2^3 + x_3^3 + \ldots) \\ & + c(x_1^2 + x_2^2 + x_3^2 + \ldots) \\ & - (x_1^2y_1 + x_2^2y_2 + x_3^2y_3 + \ldots) = 0 \end{aligned}$$

Multiplying by the coefficients of *b*,

$$\begin{aligned} & a(x_1^3 + x_2^3 + x_3^3 + \ldots) \\ & + b(x_1^2 + x_2^2 + x_3^2 + \ldots) \\ & + c(x_1 + x_2 + x_3 + \ldots) \\ & - (x_1y_1 + x_2y_2 + x_3y_3 + \ldots) = 0 \end{aligned}$$

Multiplying by the coefficients of *c*,

$$\begin{aligned} & a(x_1^2 + x_2^2 + x_3^2 + \ldots) \\ & + b(x_1 + x_2 + x_3 + \ldots) + nc \\ & - (y_1 + y_2 + y_3 + \ldots) = 0 \end{aligned}$$

Which are the same Normal Equations as previously obtained.

Applying this tabular method to Example 12, we get the following table :—

a	*b*	*N*
−·30	−1	1·40
−1	−1	4·60
−1·5	−1	6·85
−2	−1	9·00

and the normal equations

$$(0{\cdot}09 + 1{\cdot}0 + 2{\cdot}25 + 4)a + (0{\cdot}30 + 1{\cdot}0 + 1{\cdot}5 + 2{\cdot}0)b - (0{\cdot}42 + 4{\cdot}60 + 10{\cdot}275 + 18) = 0$$

$$(0{\cdot}30 + 1{\cdot}0 + 1{\cdot}5 + 2{\cdot}0)a + (1{\cdot}0 + 1{\cdot}0 + 1{\cdot}0 + 1{\cdot}0)b - (1{\cdot}40 + 4{\cdot}60 + 6{\cdot}85 + 9{\cdot}0) = 0$$

which reduce to $\begin{cases} 7{\cdot}34a + 4{\cdot}80b = 33{\cdot}295 \\ 4{\cdot}80a + 4{\cdot}00b = 21{\cdot}85 \end{cases}$

the Normal Equations found before.

EXAMPLE 13. The same data as Example 12, but assume the law is $v = an^2 + bn + c$. The errors are, therefore, $1{\cdot}40 - 0{\cdot}09a - 0{\cdot}3b - c$, $4{\cdot}60 - a - b - c$, $6{\cdot}85 - 2{\cdot}25a - 1{\cdot}5b - c$, $9{\cdot}0 - 4a - 2b - c$, and our table is

a	*b*	*c*	*N*
−0·09	−0·3	−1	1·40
−1	−1	−1	4·60
−2·25	−1·5	−1	6·85
−4	−2	−1	9·00

Hence our three equations are

$$({\cdot}0081 + 1 + 5{\cdot}0625 + 16)a + ({\cdot}027 + 1 + 3{\cdot}375 + 8)b + ({\cdot}09 + 1 + 2{\cdot}25 + 4)c = {\cdot}126 + 4{\cdot}60 + 15{\cdot}4125 + 36{\cdot}0.$$

$$({\cdot}027 + 1 + 3{\cdot}375 + 8)a + ({\cdot}09 + 1 + 2{\cdot}25 + 4)b + ({\cdot}3 + 1 + 1{\cdot}5 + 2)c = {\cdot}42 + 4{\cdot}60 + 10{\cdot}275 + 18.$$

$$({\cdot}09 + 1 + 2{\cdot}25 + 4)a + ({\cdot}3 + 1 + 1{\cdot}5 + 2)b + 4c = 1{\cdot}40 + 4{\cdot}60 + 6{\cdot}85 + 9{\cdot}00$$

which reduce to

$$\begin{cases} 22{\cdot}0706a + 12{\cdot}402b + 7{\cdot}340c = 56{\cdot}1385 \\ 12{\cdot}402a + 7{\cdot}340b + 4{\cdot}800c = 33{\cdot}295 \\ 7{\cdot}340a + 4{\cdot}800b + 4{\cdot}000c = 21{\cdot}85 \end{cases}$$

$$\therefore \begin{cases} 14{\cdot}9082a + 5{\cdot}6540b = 25{\cdot}0795 \\ 14{\cdot}376a + 6{\cdot}32b = 28{\cdot}30 \end{cases}$$

$$\therefore a = -0{\cdot}116383, \quad \therefore b = \frac{28{\cdot}30 + 1{\cdot}67312}{6{\cdot}32} = 4{\cdot}7426$$

$$\therefore c = \frac{21{\cdot}85 + 0{\cdot}8542 - 22{\cdot}7644}{4} = \frac{-0{\cdot}0602}{4} = -0{\cdot}01505$$

Therefore, the formula is $v = -0{\cdot}1164n^2 + 4{\cdot}743n - 0{\cdot}0150$

The errors of the observed points will then be—

	(Error)²
$1{\cdot}40 + 0{\cdot}0105 - 1{\cdot}4229 + 0{\cdot}0150 = 0{\cdot}0026$	0·0000,0676
$4{\cdot}60 + 0{\cdot}1164 - 4{\cdot}743 + 0{\cdot}0150 = -0{\cdot}0116$	0·0001,3460
$6{\cdot}85 + 0{\cdot}2619 - 7{\cdot}1145 + 0{\cdot}0150 = 0{\cdot}0124$	0·0001,5380
$9{\cdot}00 + 0{\cdot}4656 - 9{\cdot}486 + 0{\cdot}0150 = -0{\cdot}0054$	0·0000,2916
Σ (error)² =	0·0003,2432

Obviously this new formula fits the observations much more closely than the linear equation, $v = 4{\cdot}478n + 0{\cdot}089$, but its additional complication is probably not warranted by the accuracy of the observations.

This method is not restricted to ordinary algebraical functions, e.g. $ax + b$, $ax^2 + bx + c$, but may be used to find the coefficients in any expression, e.g. $y = af(x) + b\phi(x) + c$ where $f(x)$, $\phi(x)$ are such transcendental functions as $\sin x$, $\log x$, e^x, etc.

EXAMPLE 14. Given the observed values

x	y
0·5	0·80
1·0	1·36
1·5	2·22
2·0	3·70

Find a if $y = ae^x$.

Here the error in each case is $y - ae^x$ and our table is

N	a
0·80	−1·649
1·36	−2·718
2·22	−4·482
3·70	−7·389

the figures in the "a" column being the values of e^x for $x = 0·5$, 1·0, 1·5, and 2·0.

$$\text{Then} \quad (2·720 + 7·388 + 20·088 + 54·597)a$$
$$= 1·319 + 3·696 + 9·950 + 27·339$$

$\therefore$ $84·793a = 42·304$, $\therefore$ $a = 0·499$ and the law is

$$y = 0·499e^x$$

Example 15. In levelling a round of levels *ABCDA* it was found that *B* was 4·71 ft. above *A*, *C* 3·59 ft. above *B*. *D* 1·48 ft. above *C* and 9·72 ft. above *A*. The accuracy of all the four levellings is to be assumed equal. Find the probable heights of *B*, *C*, and *D* above *A*.

We note that from the first three results *D* appears to be 9·78 ft. above *A*, while the last result makes it 9·72 ft. above *A*; a discrepancy of 0·06 ft. Calling the correct heights of *B*, *C*, and *D* above *A*, *b*, *c*, *d* respectively, we have the errors, $4·71 - b$, $3·59 - (c - b)$, $1·48 - (d - c)$, $9·72 - d$. The table is, therefore.

b	c	d	N
−1	0	0	4·71
+1	−1	0	3·59
0	+1	−1	1·48
0	0	−1	9·72

$$\therefore \quad 2b - c = 4{\cdot}71 - 3{\cdot}59 = 1{\cdot}12$$
$$-b + 2c - d = 3{\cdot}59 - 1{\cdot}48 = 2{\cdot}11$$
$$-c + 2d = 1{\cdot}48 + 9{\cdot}72 = 11{\cdot}20$$

$\therefore 2b - 2d = -10{\cdot}08$. Also $4b - 2c = 2{\cdot}24$

$$-b + 2c - d = 2{\cdot}11$$

$\therefore \; 3b - d = 4{\cdot}35$. $\quad \therefore \; 2b - 2d = -10{\cdot}08$

$$6b - 2d = 8{\cdot}70$$

$\therefore 4b = 18{\cdot}78$. $\therefore b = 4{\cdot}695 \quad \therefore c = 9{\cdot}39 - 1{\cdot}12 = 8{\cdot}27$

and $d = \dfrac{11{\cdot}20 + 8{\cdot}27}{2} = 9{\cdot}735$

Therefore, *B* is 4·695 ft. above *A*, *C* is 3·575 ft. above *B*, *D* is 1·465 ft. above *C* and 9·735 ft. above *A*. The errors in levelling were, therefore, probably + 0·015, + 0·015 + 0·015, and − 0·015 ft. respectively, so that the total error of 0·06 has been equally divided among the four level-differences.

Example 16 (L.U.). The same data as in Example 15, but add "that *D* was found to be 5·12 ft. above *B*," the accuracy of this levelling being assumed equal to that of the others. The errors are now $4{\cdot}71 - b$, $3{\cdot}59 - (c - b)$, $1{\cdot}48 - (d - c)$, $9{\cdot}72 - d$, and $5{\cdot}12 - (d - b)$. The table is now—

b	*c*	*d*	*N*
− 1	0	0	4·71
+ 1	− 1	0	3·59
0	+ 1	− 1	1·48
0	0	− 1	9·72
+ 1	0	− 1	5·12

$$\therefore \quad 3b - c - d = 4{\cdot}71 - 3{\cdot}59 - 5{\cdot}12 = -4$$
$$-b + 2c - d = 3{\cdot}59 - 1{\cdot}48 = 2{\cdot}11$$
$$-b - c + 3d = 1{\cdot}48 + 9{\cdot}72 + 5{\cdot}12 = 16{\cdot}32$$

$$\therefore \quad 3b - c - d = -4 \qquad \therefore 5c - 4d = 2{\cdot}33$$
$$-3b + 6c - 3d = 6{\cdot}33 \qquad 3c - 4d = -14{\cdot}21$$

$$\therefore c = 8{\cdot}27\,;\ d = \frac{41{\cdot}35 - 2{\cdot}33}{4} = 9{\cdot}755$$

$$\therefore b = \frac{-4 + 8{\cdot}27 + 9{\cdot}755}{3} = 4{\cdot}675$$

Therefore B is 4·675 ft. above A, C is 3·595 ft. above B, D is 1·485 ft. above C, 5·08 ft. above B, 9·755 ft. above A, and the errors were + 0·035, − 0·005, − 0·005, + 0·04, and − 0·035 respectively.

Example 17 The same data as in Example 16, but add "C is *known* to be 8·28 ft. above A." Here we are said to have an "Equation of Condition," c is now 8·28 and is not subject to correction. The errors are now $4{\cdot}71 - b$, $3{\cdot}59 - (8{\cdot}28 - b) = -4{\cdot}69 + b$, $1{\cdot}48 - (d - 8{\cdot}28) = 9{\cdot}76 - d$, $9{\cdot}72 - d$, and $5{\cdot}12 - d + b$.

The table is therefore

b	d	N
− 1	0	4·71
+ 1	0	− 4·69
0	− 1	9·76
0	− 1	9·72
+ 1	− 1	5·12

$$\therefore \quad 3b - d = 4{\cdot}71 + 4{\cdot}69 - 5{\cdot}12 = 4{\cdot}28$$
$$-b + 3d = 9{\cdot}76 + 9{\cdot}72 + 5{\cdot}12 = 24{\cdot}60$$
$$\therefore \quad 3b - d = 4{\cdot}28$$
$$-3b + 9d = 73{\cdot}80. \quad \therefore d = 9{\cdot}76 \text{ ft.} \quad \therefore b = 4{\cdot}68 \text{ ft.}$$

Therefore B is 4·68 ft. above A, C is 3·60 ft. above B, D is 1·48 ft. above C, 5·08 ft. above B, 9·76 ft. above A, and the errors were $+0{\cdot}03$, $-0{\cdot}01$, $0{\cdot}00$, $+0{\cdot}04$, and $-0{\cdot}04$ respectively.

Weighting the Observations. Hitherto we have assumed that all the observations are of equal accuracy, but this is often not the case. If one of the observations has been repeated n times and the average of the results taken its inaccuracy should be reduced in the ratio $\frac{1}{\sqrt{n}}$, i.e. its "observation equation" should be multiplied by $\sqrt{n}$. For example, in Example 15, where $c-b$ was observed to be 3·59, we call $c-b=3{\cdot}59$ an "observation equation," and if it is the average of n observations of equal accuracy we treat it as equivalent to $\sqrt{n}$ observations, all of which gave $c-b=3{\cdot}59$. We, therefore, treat its error in the summation of squares as $\sqrt{n}(3{\cdot}59-c+b)$. The (error)^2 is then $n(3{\cdot}59-c+b)^2$, and when we differentiate this with regard to b and c we get $n(3{\cdot}59-c+b)$, and $-n(3{\cdot}59-c+b)$ respectively, so that as far as this observation is concerned we multiply by an additional factor n when preparing the normal equations. The factor n is called the *weight* of the observation.

On the other hand, if one of the observations is the sum of a series of observations each of the same accuracy as all the other observations, its inaccuracy will be increased in the ratio $\sqrt{n}:1$, or its accuracy decreased in the ratio $\frac{1}{\sqrt{n}}$, so that its "observation equation" should be multiplied by $\frac{1}{\sqrt{n}}$. This means that we must multiply, or "weight," its (error)^2 by $\frac{1}{n}$, when deducing the normal equations. *The weights are inversely proportional to the squares of the probable errors.*

But in addition to cases of repeated observations as above, whether "in parallel" or "in series," there are other cases where weighting should be employed. The observations may have been made by different observers of greater or less accuracy or the conditions under which the observations were made may not have been equally favourable, in which case the observations which are thought to be more accurate should be weighted with a higher number than those considered to be less accurate. This is largely a matter of judgment, though as regards the relative accuracy of particular observers this can be deduced mathematically from the consistency of their results when repeating the same observations.

EXAMPLE 18. The same data as for Example 15, but let the weights be 1 for AB and DA and 2 for BC and CD.

Our table then becomes

w	b	c	d	N
1	−1	0	0	4·71
2	+1	−1	0	3·59
2	0	+1	−1	1·48
1	0	0	−1	9·72

with an additional column for weight (w), and our normal equations are

$$3b - 2c = 4{\cdot}71 - 7{\cdot}18 = -2{\cdot}47 \quad . \quad . \quad (1)$$

$$-2b + 4c - 2d = 7{\cdot}18 - 2{\cdot}96 = 4{\cdot}22 \quad . \quad . \quad (2)$$

$$-2c + 3d = 2{\cdot}96 + 9{\cdot}72 = 12{\cdot}68 \quad . \quad . \quad (3)$$

From (1) and (3), $3b - 3d = -15{\cdot}15$.

From (1) and (2), $4b - 2d = -0{\cdot}72$.

$$\therefore \left.\begin{aligned} b - d &= -5{\cdot}05 \\ 2b - d &= -0{\cdot}36 \end{aligned}\right\} \therefore b = 4{\cdot}69. \quad \therefore d = 9{\cdot}74. \quad \therefore c = 8{\cdot}27.$$

Therefore B is 4·69 ft. above A, C is 3·58 ft. above B, D is 1·47 ft. above C and 9·74 ft. above A, so that the errors were + 0·02, + 0·01, + 0·01, − 0·02 respectively, so that the errors were distributed inversely as the weights of the various levellings.

EXAMPLE 19. The same data as for Example 16, but with weights 1 for AB and DA and 2 for BC, CD, and DB. The table is then—

w	b	c	d	N
1	− 1	0	0	4·71
2	+ 1	− 1	0	3·59
2	0	+ 1	− 1	1·48
2	+ 1	0	− 1	5·12
1	0	0	− 1	9·72

$$\therefore\ 5b - 2c - 2d = 4{\cdot}71 - 7{\cdot}18 - 10{\cdot}24 = -12{\cdot}71 \qquad (1)$$

$$-2b + 4c - 2d = 7{\cdot}18 - 2{\cdot}96 = 4{\cdot}22 \qquad (2)$$

$$-2b - 2c + 5d = 2{\cdot}96 + 10{\cdot}24 + 9{\cdot}72 = 22{\cdot}92 \qquad (3)$$

From (1) and (2), $7b - 6c = -16{\cdot}93$

From (2) and (3), $-7b + 8c = 33{\cdot}47$

$\therefore\ c = 8{\cdot}27$ ft. $\therefore\ b = 4{\cdot}67$ ft. $\therefore\ d = 9{\cdot}76$ ft.

Therefore B is 4·67 ft. above A, C is 3·60 ft. above B, D is 1·49 ft. above C, 5·09 ft. above B, 9·76 ft. above A, so that the errors were + 0·04, − 0·01, − 0·01, + 0·03, − 0·04 respectively.

Alternative Method of "Correlates." In Example 17 we saw that, when there was an "Equation of Condition" (i.e. any fixed relationship between the variables) to be satisfied, it had the effect of eliminating one of the variables. Similarly, if we have m Equations of Condition we must first eliminate m of the variables, putting them in terms of the other variables by means of these m equations. When there are a number of such equations, the following method provides a quicker solution. We shall take the case of the four levels as an illustration. Let the required "corrections" in the four

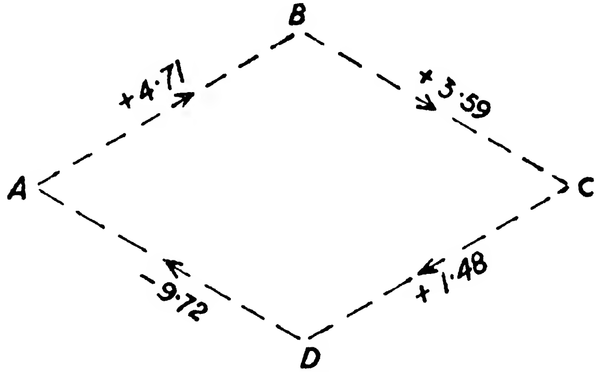


FIG. 14

level differences AB, BC, CD, and DA be e_1, e_2, e_3, and e_4 respectively, and the weights be w_1, w_2, w_3, w_4 respectively, in the simple round of four levels of Examples 15 and 18 (see Fig. 14). "Correction," of course, means error with changed sign: a + error means a − correction, and *vice versa*.

The arrows in Fig. 14 indicate the directions in which the rises or falls given are to be reckoned. Here the total correction required on the round is −0·06 ft. We have, therefore, one equation of condition, viz.

$$\Sigma(e) = e_1 + e_2 + e_3 + e_4 = -0{\cdot}06$$

Also for the Least Square condition we have
$\Sigma(we^2) = w_1e_1^2 + w_2e_2^2 + w_3e_3^2 + w_4e_4^2 =$ a minimum. If we vary e_1, e_2, e_3, e_4 by amounts δe_1, δe_2, δe_3, δe_4 we must

have $\Sigma(\delta e) = \delta e_1 + \delta e_2 + \delta e_3 + \delta e_4 = 0$, also (differentiating) $\Sigma(we\delta e) = w_1e_1\delta e_1 + w_2e_2\delta e_2 + w_3e_3\delta e_3 + w_4e_4\delta e_4 = 0$.

Multiply the first of these equations by $-\lambda$ and add it to the second; then $(w_1e_1 - \lambda)\delta e_1 + (w_2e_2 - \lambda)\delta e_2 + (w_3e_3 - \lambda)\delta e_3 + (w_4e_4 - \lambda)\delta e_4 = 0$. Now, as δe_1, δe_2, δe_3, and δe_4 are independent quantities, each of the coefficients of the δe's must vanish independently, or $w_1e_1 = \lambda = w_2e_2 = w_3e_3 = w_4e_4$, i.e. $e_1 = \frac{\lambda}{w_1}$, $e_2 = \frac{\lambda}{w_2}$, $e_3 = \frac{\lambda}{w_3}$, $e_4 = \frac{\lambda}{w_4}$, i.e. the corrections are inversely proportional to the weights. If we substitute these values in the original equation of condition, we have $\lambda\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4}\right) = -0{\cdot}06$, an equation which gives us λ and hence gives the values of e_1, e_2, e_3, and e_4.

In Example 15, all the weights are equal, say 1, and we have $\lambda = \frac{-0{\cdot}06}{4} = -0{\cdot}015 = e_1 = e_2 = e_3 = e_4$. The corrected level-differences are, therefore, $+4{\cdot}695$, $+3{\cdot}575$, $+1{\cdot}465$, and $-9{\cdot}735$, as found before.

In Example 18, the weights are $w_1 = 1$, $w_2 = 2$, $w_3 = 2$, $w_4 = 1$. $\therefore\ \lambda(1 + \frac{1}{2} + \frac{1}{2} + 1) = -0{\cdot}06$. $\therefore\ \lambda = -0{\cdot}02$. So that we have $e_1 = -0{\cdot}02$, $e_2 = -0{\cdot}01$, $e_3 = -0{\cdot}01$, $e_4 = -0{\cdot}02$, so that the corrected level-differences are $+4{\cdot}69$, $+3{\cdot}58$, $+1{\cdot}47$, $-9{\cdot}74$, as found previously.

In Examples 16 and 19 we have two rounds of levelling (Fig. 15), *ABD* and *BDC*, with total corrections required of $-0{\cdot}11$ and $-0{\cdot}05$ respectively. Calling e_5 the correction in *BD* we have two equations of condition, viz.

$$e_1 + e_5 + e_4 = -0{\cdot}11\,;\quad e_5 + e_3 + e_2 = -0{\cdot}05$$

$$\therefore\quad \delta e_1 + \delta e_5 + \delta e_4 = 0\,;\quad \delta e_5 + \delta e_3 + \delta e_2 = 0$$

while $w_1e_1\delta e_1 + w_2e_2\delta e_2 + w_3e_3\delta e_3 + w_4e_4\delta e_4 + w_5e_5\delta e_5 = 0$

Multiplying the first two of these equations by $-\lambda_1$, $-\lambda_2$ respectively, adding all three equations and equating the coefficients of each δe to zero, we get $w_1e_1 = \lambda_1$, $w'_2e_2 = \lambda_2$, $w'_3e_3 = \lambda_2$, $w'_4e_4 = \lambda_1$, $w'_5e_5 = \lambda_1 + \lambda_2$.

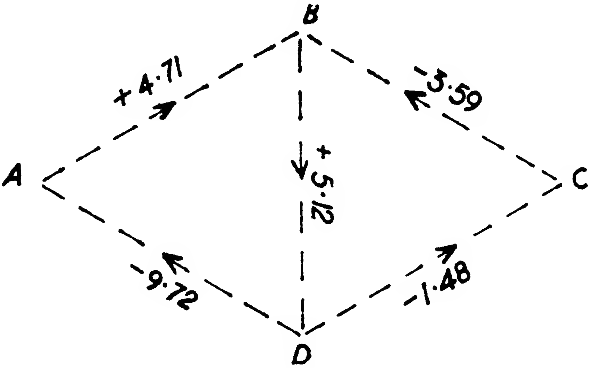


Fig. 15

Substituting these values of e_1, e_2, etc. in the original equations we get

$$\begin{cases} \lambda_1\left(\dfrac{1}{w'_1} + \dfrac{1}{w'_4} + \dfrac{1}{w'_5}\right) + \dfrac{\lambda_2}{w'_5} = -0{\cdot}11 \\ \dfrac{\lambda_1}{w'_5} + \lambda_2\left(\dfrac{1}{w'_2} + \dfrac{1}{w'_3} + \dfrac{1}{w'_5}\right) = -0{\cdot}05 \end{cases}$$

which two simultaneous equations give us λ_1, λ_2, and hence e_1, e_2, e_3, e_4, and e_5.

In Example 16, the weights are equal, say 1, and the equations are

$$\begin{cases} 3\lambda_1 + \lambda_2 = -0{\cdot}11 \\ \lambda_1 + 3\lambda_2 = -0{\cdot}05 \end{cases}$$

Hence $8\lambda_1 = -0{\cdot}28$, $\therefore \lambda_1 = -0{\cdot}035$, $\lambda_2 = -0{\cdot}005$, so that we have $e_1 = -0{\cdot}035$, $e_2 = -0{\cdot}005$, $e_3 = -0{\cdot}005$, $e_4 = -0{\cdot}035$,

$e_5 = -0{\cdot}04$, which give corrected level-differences of $+4{\cdot}675$, $-3{\cdot}595$, $-1{\cdot}485$, $-9{\cdot}755$, and $+5{\cdot}08$, as found before.

In Example 19, where $w_1 = w_4 = 1$, $w_2 = w_3 = w_5 = 2$, the equations are—

$$\left.\begin{aligned} \lambda_1(1 + 1 + \tfrac{1}{2}) + \frac{\lambda_2}{2} &= -0{\cdot}11 \\ \frac{\lambda_1}{2} + \lambda_2(\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2}) &= -0{\cdot}05 \end{aligned}\right\}$$

$$\text{i.e.}\quad \begin{cases} 5\lambda_1 + \lambda_2 = -0{\cdot}22 & \therefore\ 14\lambda_1 = -0{\cdot}56 \\ \lambda_1 + 3\lambda_2 = -0{\cdot}10 & \therefore\ \lambda_1 = -0{\cdot}04,\ \lambda_2 = -0{\cdot}02 \end{cases}$$

so that the corrections are $e_1 = -0{\cdot}04$, $e_2 = -0{\cdot}01$, $e_3 = -0{\cdot}01$, $e_4 = -0{\cdot}04$, $e_5 = -0{\cdot}03$, and the corrected level differences are $+4{\cdot}67$, $-3{\cdot}60$, $-1{\cdot}49$, $-9{\cdot}76$, $+5{\cdot}09$, as found before.

In Example 17, all the weights are equal and unity, but we have now three equations of condition, viz.

$e_1 + e_5 + e_4 = -0{\cdot}11$, $e_5 + e_3 + e_2 = -0{\cdot}05$, $e_1 - e_2 = -0{\cdot}02$,

as we require a $-$ correction on AB and a $+$ correction on CB to diminish the rise (from A to C) of $8{\cdot}30$ to $8{\cdot}28$, the known value. [Instead of $e_1 - e_2 = -0{\cdot}02$ we could have put

$$e_4 - e_3 = -0{\cdot}04,$$

as the fall from C to A is the *difference* of the falls from D to A and from D to C, and a $-$ correction is required to increase the fall from $-8{\cdot}24$ to $-8{\cdot}28$. But we must not use *both* of these equations, as one is derivable from the other, together with the first two equations.] We have, therefore,

$$\delta e_1 + \delta e_5 + \delta e_4 = 0,\ \ \delta e_5 + \delta e_3 + \delta e_2 = 0,\ \ \delta e_1 - \delta e_2 = 0,$$

also $$e_1\delta e_1 + e_2\delta e_2 + e_3\delta e_3 + e_4\delta e_4 + e_5\delta e_5 = 0.$$

Multiplying the first three equations by $-\lambda_1$, $-\lambda_2$, $-\lambda_3$

respectively, adding the four equations together, and equating the coefficients of each δe to zero, we have $e_1 = \lambda_1 + \lambda_3$, $e_2 = \lambda_2 - \lambda_3$, $e_3 = \lambda_2$, $e_4 = \lambda_1$, $e_5 = \lambda_1 + \lambda_2$.

Substituting these values in the original equations we have—

$$\begin{cases} 3\lambda_1 + \lambda_2 + \lambda_3 = -0{\cdot}11 \\ \lambda_1 + 3\lambda_2 - \lambda_3 = -0{\cdot}05 \\ \lambda_1 - \lambda_2 + 2\lambda_3 = -0{\cdot}02 \end{cdases}$$

$$\therefore\ 4\lambda_1 + 4\lambda_2 = -0{\cdot}16$$

$$\therefore\ \left\{\begin{array}{l} \lambda_1 + \lambda_2 = -0{\cdot}04 \\ 3\lambda_1 + 5\lambda_2 = -0{\cdot}12 \end{array}\right.\ \text{Also}$$

$$\therefore\ 2\lambda_1 = -0{\cdot}08$$

$$\therefore\ \lambda_1 = -0{\cdot}04$$

$$\therefore\ \lambda_2 = 0,\quad \therefore\ \lambda_3 = \frac{-0{\cdot}02 + 0{\cdot}04}{2} = +0{\cdot}01$$

Therefore $e_1 = -0{\cdot}03$, $e_2 = -0{\cdot}01$, $e_3 = 0$, $e_4 = -0{\cdot}04$, $e_5 = -0{\cdot}04$, and the corrected level-differences are $+4{\cdot}68$, $-3{\cdot}60$, $-1{\cdot}48$, $-9{\cdot}76$, and $+5{\cdot}08$ as found before.

The reader should repeat this example, substituting the equation $e_4 - e_3 = -0{\cdot}04$ for the equation $e_1 - e_2 = -0{\cdot}02$, and he will find that he will get the same values of the corrections.

CHAPTER II

ELEMENTARY ASTRONOMY

ASTRONOMICAL DEFINITIONS—SIDEREAL AND MEAN TIME—CORRECTIONS OF ALTITUDE

ASTRONOMICAL DEFINITIONS

AT any place on the earth's surface the fixed stars (i.e. excluding the sun, moon, and planets, which are parts of the solar system) appear to revolve in small circles round a fixed point in the heavens called the "Celestial Pole," in a period of 23h. 56m. 4·09s. of mean time (i.e. of ordinary clock time). This is the actual period of rotation of the earth and is called a "Sidereal Day." Fig. 1 shows the earth P_N, E, P_S, E; $P_N P_S$ being the North and South Poles and EE the equator, while at A and B are shown the celestial hemispheres visible to an observer in the northern and southern hemispheres respectively. The fixed point or celestial pole is the point p_N, p_S where a line Ap_N, Bp_S parallel to the earth's axis cuts the hemisphere: its *altitude above the horizon NS is easily seen to be equal to* ϕ, *the latitude of the place.* A star X makes an angle XAp_N with the celestial pole, which, as the star is infinitely distant, remains constant as the observer and his celestial hemisphere rotate into the position A', so that the star appears to describe a "small circle" about P_N. The star Y is not visible from A'. It is visible from B, but not from B', as it is below the horizon $N'S$.

In the northern hemisphere the fixed point is the North Celestial Pole, and it lies north of the Zenith (or point immediately overhead); in the southern hemisphere the fixed point is the South Celestial Pole, and it lies south of the zenith. As we go north from *A* the North Celestial Pole will rise until at the North Pole, latitude 90° N., the North Pole will be at the zenith and the stars will describe horizontal

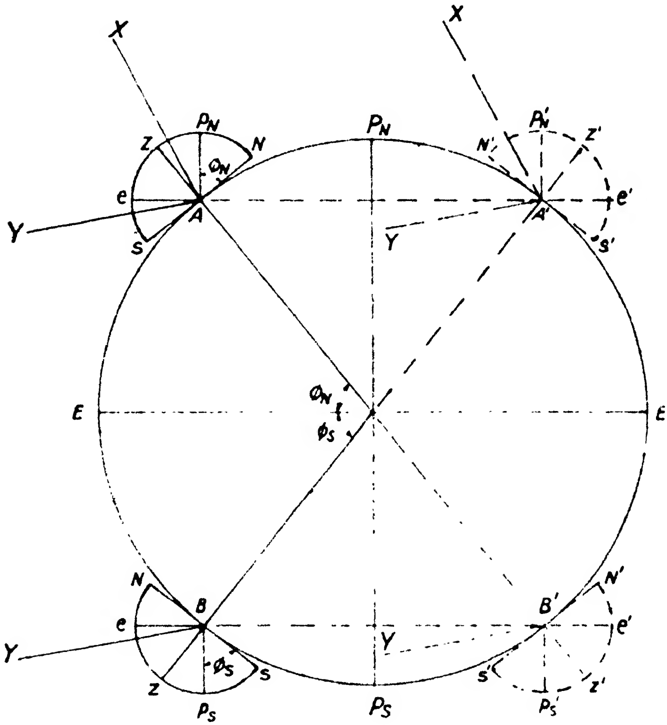


FIG. [illegible]

circles round it. As we go south from *A* the North Celestial Pole will fall until, when the place is on the equator, latitude 0°, the North Celestial Pole is on the north horizon, the South Celestial Pole on the south horizon, and the stars now describe semi-circles across the sky. Proceeding into the southern hemisphere, the South Celestial Pole rises in the sky until finally, at the South Pole, the stars again revolve in horizontal circles about the zenith.

The sidereal day is shorter than the solar day because the earth travels round the sun once in a year in the same direction as it rotates on its axis. In Fig. 2, if the earth E, the sun S, and a star X are in line one day, and on the next day the earth has moved to E', it is obvious that a point on the earth's surface will pass the star before it passes the sun, and that the angle $X'E'S$, which measures this difference between the sidereal and the solar day, is equal to the angle $E'SE$ as $E'X'$ is parallel to EX. In a year, therefore, there will be one more sidereal day than there are solar days, i.e. there are 366·24 sidereal days. One sidereal day, therefore, $= \frac{365·24}{366·24}$ solar days $= 1 - \frac{1}{366·24}$ solar days $=$ 1 solar day $-$ 235·91 solar seconds $=$ 23h. 56m. 4·09s. of mean solar time, as above stated. The sidereal day is divided into 24 hours, etc., and 1 *sidereal hour* $=$ 1 solar hour $- \frac{235·91}{24}$ seconds $=$ 1 *solar hour* $-$ 9·830 *solar seconds*. Similarly, 1 solar day $= \frac{366·24}{365·24}$ sidereal days $= 1 + \frac{1}{365·24}$ sidereal days $=$ 1 $+$ 236·56 sidereal seconds $=$ 24h. 3m. 56·56s. of sidereal time. Therefore, 1 *solar hour* $=$ 1 sidereal hour $+ \frac{236·56}{24}$ sidereal seconds $=$ 1 *sidereal hour* $+$ 9·857 *sidereal seconds*.

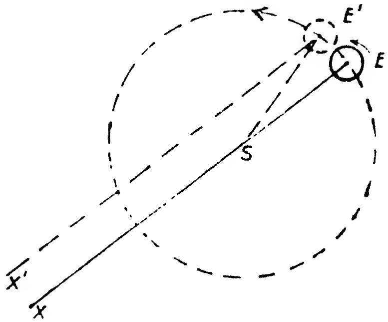


Fig. 2

By solar time we here mean *mean* solar time, because, as we shall see later, solar days—from noon to noon—are not quite equal in length and have to be averaged to give a uniform time interval.

In the case of the North Celestial Pole, there is a fairly

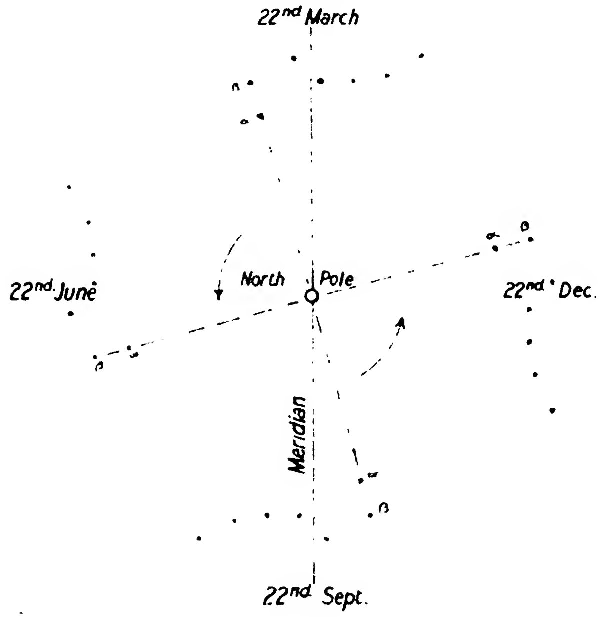


FIG. 3

bright star—the Pole Star, or "Polaris"—within about 1° from it, and this is easily found from the well-known constellation of the "Plough" (Ursa Major = Great Bear), as shown in Fig. 3. By producing the line of the two stars α, β of the Plough to a distance about $5 \times \alpha\beta$, we find Polaris. Actually Polaris describes a small circle of about 1° round the Celestial Pole. The Plough is shown in four different positions, as it appears at midnight on four dates at London. There is no corresponding bright star near the South Celestial Pole.

As the fixed stars remain very approximately in the same positions relative to each other we require a system of spherical co-ordinates to define their positions, similar to latitude and longitude on the earth. The great circle (or, rather, semicircle) on the observer's celestial hemisphere, parallel to the earth's equator and, therefore, at 90° to the celestial pole, we call the *Celestial Equator.* Its inclination to the horizon is, of course, $90° - \phi$ = the *co-latitude*, where ϕ is the latitude of the place. The great circle through any star and the celestial pole we call the star's *Declination Circle.* The great circle through the zenith and the celestial pole at any place we call the *meridian* of the place—it is, of course, the vertical plane running due north and south. Then the co-ordinate of a star corresponding to latitude on the earth is its *Declination,* which is the *angle from the celestial equator to the star,* measured north or south of the equator along the declination circle of the star. The declination, δ, of a star is, in fact, the latitude of the point on the earth immediately below it, at which point the star appears to be in the zenith when it crosses the meridian (see Fig. 4).

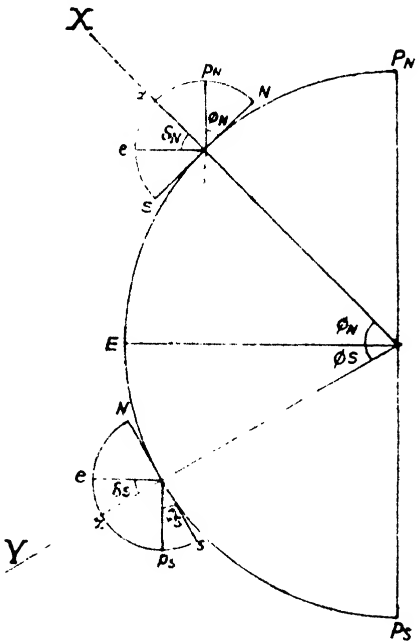


FIG. 4

The co-ordinate of a star corresponding to longitude is its *Right Ascension* (or R.A.), which is defined as the angle, measured in the opposite direction to that in which the stars revolve round the pole, *from a certain fixed point on the celestial equator, called the "First Point of Aries" (denoted by the symbol* ♈*), to the star's declination circle.* The angle is, however, measured from 0° to 360°, not from 0° to 180° E. and 0° to 180° W. as on the earth, or, rather, it is measured in hours, minutes, and seconds from 0 hours to 24 hours, at the rate of 15° to 1 hour. The "First Point of Aries," which corresponds to Greenwich, is the point where the sun, in its annual path among the stars, crosses the equator from south to north at the spring equinox, on or about the 22nd March.

But, as each star is revolving round the celestial pole at the rate of 15° per sidereal hour, we require a third co-ordinate to define its position relative to a place on the earth at any instant, viz. the star's *Hour Angle*, which is the *angle measured from the meridian of the place to the star's declination circle*, measured in the direction in which the stars revolve, from 0° to 360° or, more usually, in hours, minutes, and seconds, at the rate of 15° to an hour. It is measured from the side of the meridian *opposite* to the pole.

The hour angle of any star, therefore, increases 15° per sidereal hour, and any fixed star could be considered as the hour-hand of a 24-hour sidereal clock. Actually, ♈, the first point of Aries, is utilized for this purpose. The instant when it crosses the meridian of a place is called 0h. sidereal time at that place, or 0h. local sidereal time, or even "sidereal noon," and we have the definition: *Local sidereal time*, at any place and instant, is the *hour angle of* ♈.

Fig. 5 shows the celestial hemispheres of two points, the one with a latitude ϕ north, and the other with a latitude

ϕ south, and in both cases the same star X with a declination δ north is shown, it being above the celestial equator in the northern hemisphere and below it in the southern hemisphere. The hour angle is ZPX in both cases, and is measured from the south in the northern hemisphere and

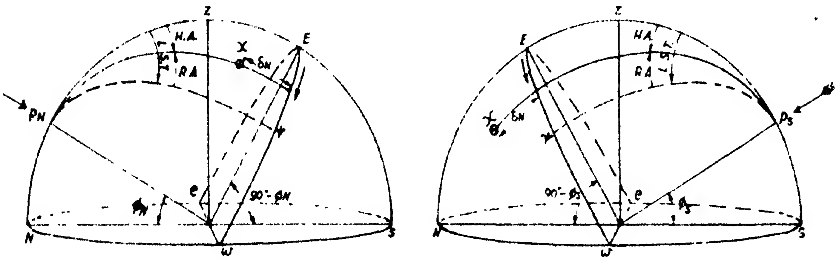


FIG. 5

from the north in the southern hemisphere. The R.A. of the star is ♈PX in both cases and the local sidereal time is ZP♈ in both cases, measured from the south in the northern hemisphere and from the north in the southern hemisphere. As ♈ is always considered ahead of the star, we have the equation: *The right ascension of a star + its hour angle = local sidereal time* at the instant. If the two places are on the same meridian, the hour angles of the same star will be the same at the same instant, and the local sidereal times will be the same at the same instant. When the star is crossing the meridian *above* the pole it is said to be at *Upper Transit* or *Upper Culmination*, as its altitude is a *maximum* and its hour angle is zero. *The right ascension of the star then equals the local sidereal time.*

It will be noted that *facing the equator* the stars move clockwise in the northern hemisphere (from east to west through south) and anti-clockwise in the southern hemisphere (from east to west through north). If we face the *pole*, of course, these directions are reversed in each case. The angular distance of a star from the nearest pole is $90° - \delta$, which is called the *Co-declination* or *Polar Distance*. When this is less than the latitude of the place, the star will cross the meridian again 12 sidereal hours later, above the horizon and *below* the pole. It is then said to be at *Lower Transit* or *Lower Culmination*, as its altitude is a *minimum*. Of course, owing to daylight it may not be

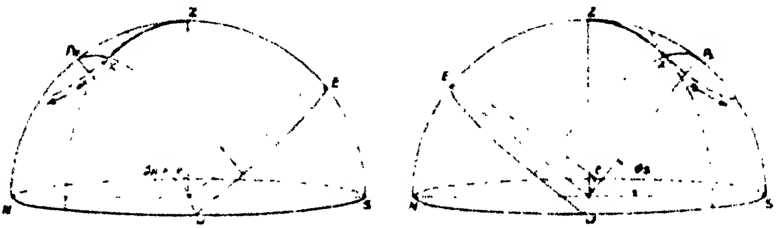


FIG. 6

visible at one of its two transits, or even at both of them when the nights are short.

When the co-declination is less than the co-latitude (or $PX < PZ$), or the star's declination is greater than the latitude of the place, the star's upper transit will be between the pole and the zenith, and the star will never be seen south in the northern hemisphere or north in the southern hemisphere. Consequently, when viewed with a theodolite, such a star will attain a maximum easterly bearing, then move north again (or south in the southern hemisphere), then attain a maximum westerly bearing, then again move north (or south in the southern hemisphere). When at such maximum easterly or westerly bearings it is said to be at eastern or western *elongation* respectively, and the angle ZXP is a right angle (Fig. 6).

In any position of a star X we have a spherical triangle

ZXP formed by the zenith, the pole, and the star. As before defined, the side ZP is the *co-latitude* $= 90° -$ latitude $= 90° - \phi$, the side PX is the *co-declination* $= 90° -$ declination $= 90° - \delta$, while the third side ZX is the *co-altitude* $= 90° -$ altitude $= 90° - a =$ *zenith distance* (Fig. 7).

If the declination is of opposite sign to the latitude, the co-declination $PX = 90° + \delta$.

The angle at P is the *hour angle* if the star is *west* of the meridian, otherwise it is $360° -$ *hour angle*. The angle at Z is the *azimuth* of the star if the star is east of the meridian (as azimuths are reckoned eastwards from north) in the northern hemisphere, and $360° -$ azimuth if it is west of the meridian. If azimuths are still reckoned eastwards from the north in the southern hemisphere, the angle at Z will be $180° -$ azimuth when the star is east of the meridian, and azimuth $- 180°$ when the star is west of the meridian. The angle at Z will be a right angle when the star is on the *Prime Vertical*, that is, the *great circle eZw through the zenith at right angles to the meridian*, and, therefore, passing through the east and west points of the horizon. It is shown on the R.H. diagram.

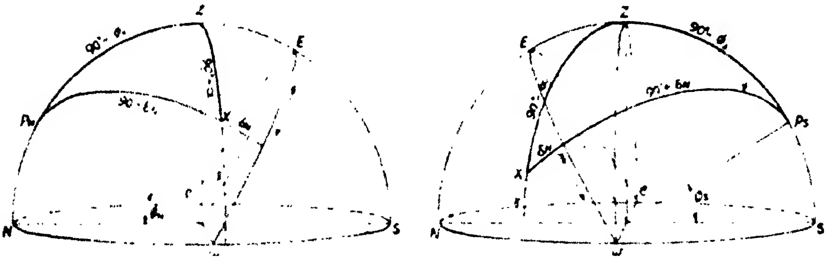


FIG. 7

Altitudes of Stars Crossing the Meridian. When the star is crossing the meridian this spherical triangle becomes the great circle PZX (or PXZ or XPZ) and the calculations are much simplified. The hour angle is 0° or 180°, the azimuth 0° or 180°, while the star's declination, its altitude, and the latitude of the place are so simply related that if any two

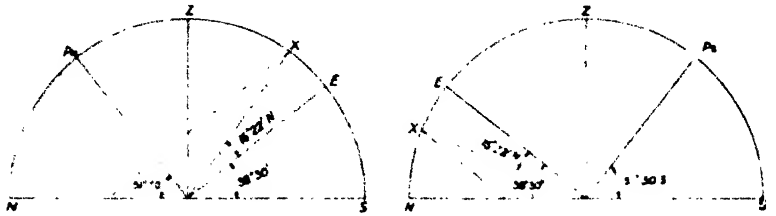


Fig. 8

of these are known, the third can be found by mere addition and subtraction. For this purpose we sketch a section through Z and P, along the meridian of the place.

Example 1. To find the altitudes of Aldebaran (α Tauri) at its upper transit at places in latitudes 51° 30′ N. and 51° 30′ S., given the declination of Aldebaran is 16° 22′ N. (Fig. 8). The inclination of the equator is 90° − 51° 30′ = 38° 30′ S. in the northern, 38° 30′ N. in the southern hemisphere. Draw OE at these inclinations on the two diagrams. Draw OX at an angle of 16° 22′ on the northern side of the equator in each case, and we see that its altitude at transit is 38° 30′ + 16° 22′ = 54° 52′ S. in the first case, and 38° 30′ − 16° 22′ = 22° 08′ N. in the second case.

Conversely, if we have found the altitude of Aldebaran at upper transit to be 54° 52′ S. and subtract its north declination, we know the inclination of the equator is 38° 30′ S., and, therefore, that the latitude is 90° − 38° 30′ = 51° 30′ N. On the other hand, if its altitude is 22° 08′ N., we add the declination 16° 22′ and find the inclination of the equator to be 38° 30′ N. Therefore the latitude of the place is 90° − 38° 30′ S. = 51° 30′ S.

Example 2. To find the altitudes of the upper and lower transits of *α* Ursae Majoris (declination 62° 07′ N.) in latitude

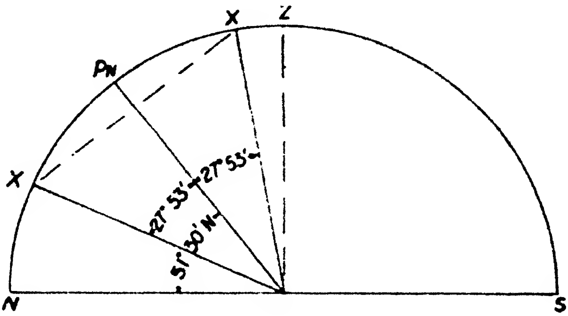


Fig. 9

51° 30′ N. (Fig. 9). The co-declination or polar distance of the star is 90° − 62° 07′ = 27° 53′. Draw the pole at an

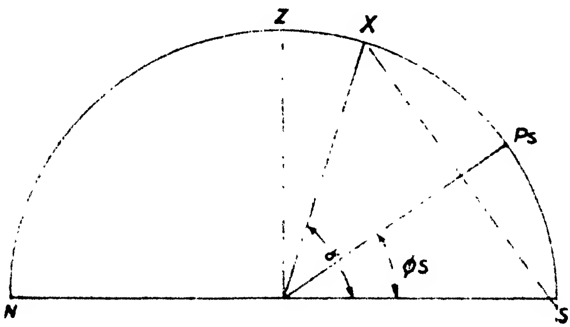


Fig. 10

altitude of 51° 30′ N. and draw lines at 27° 53′ on each side of OP_N. The altitudes are, therefore, 79° 23′ N. and 23° 37′ N.

Example 3. If the (corrected) altitude of Canopus (*α* Argus) (declination 52° 39′ 30″ S.) at upper transit is found to be 72° 29′ 15″ S., what is the latitude? (Fig. 10.) Draw the star X at this altitude, 72° 29′ 15″ S. Then subtract its polar distance 37° 20′ 30″ (as at upper transit) and we find the latitude as 35° 08′ 45″ S. At lower transit the star will be 37° 20′ 30″ − 35° 08′ 45″ = 2° 11′ 45″ *below* the horizon.

SIDEREAL TIME

Approximate Determination of Time when a Star Crosses the Meridian. For this purpose it is convenient to sketch

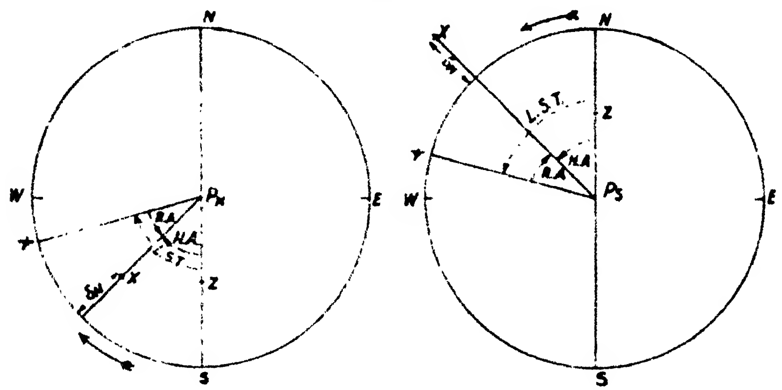


FIG. 11

a "clock diagram" or diagrammatic plan looking down the earth's axis towards the celestial equator in the direction of the arrow of Fig. 5 (page 69). Fig. 11 shows such diagrams for the northern and southern hemispheres respectively. The pole is in the centre and the circle can be considered to represent the equator. The zenith is south of the North Pole P_N and north of the South Pole P_S, while the straight line ZP represents the meridian. The same star X is shown on both diagrams with a north declination, and therefore, above (inside) the equator in the northern hemisphere and below (outside) it in the southern hemisphere. By taking the *radius* of the equator as 90° to scale we could plot the star's declination radially from the equator towards, or away from, the pole. In both cases the star's hour angle is ZPX, its right ascension is φPX, and the local sidereal time is $ZP\varphi$.

When the star is at upper transit, the position of φ is known, the local sidereal time $ZP\varphi$ being the star's right ascension φPX (Fig. 12).

Now, just as we consider φ as the hour-hand of a 24-hours

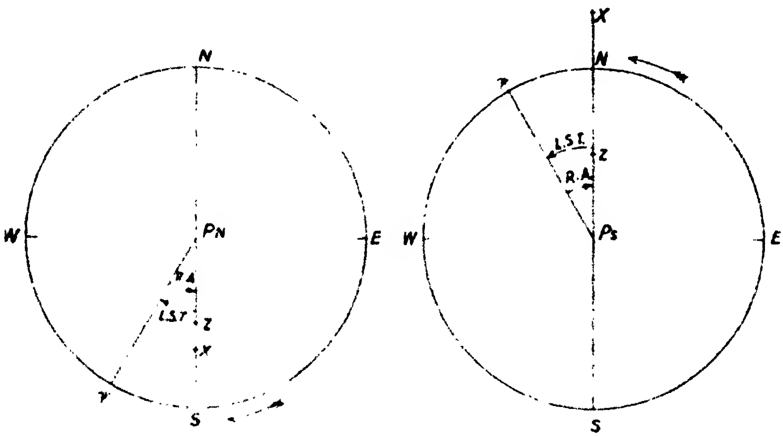


Fig. 12

sidereal time clock, we can consider a point M, which we call the "Mean Sun," and which moves at a *uniform* speed round the *equator*, as the hour hand of a 24-hour mean time clock, and if we know the angle between M and φ on any day we can plot the position of M at the instant of transit, and so find the mean time.

In Fig. 13 the four circles represent the faces of a 24-hour clock, with a mean time hour hand M, and a sidereal hour hand φ at the two equinoxes (spring and autumn) and the two solstices (summer and winter): in each case at mean noon. As there is one more sidereal day in the year than there are mean days, the sidereal hour hand goes 6 hours fast every 3 months, or, approximately, 4 minutes per day. The angle between the two hour hands is the *sidereal time*

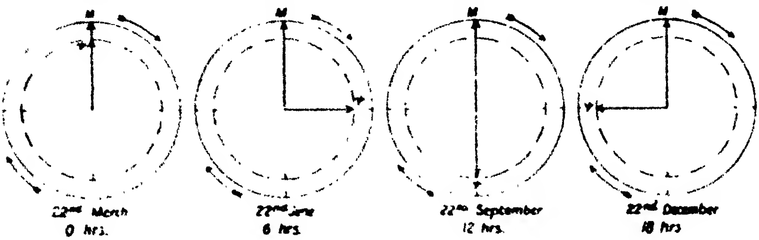


FIG. 13

at mean noon, or the *right ascension of the mean sun at mean noon*, or it is the angle between ♈ and the mean sun on the equator, and we see that this is 0 hours on 22nd

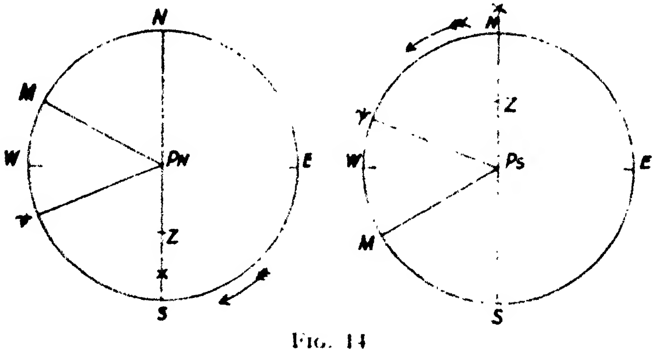


FIG. 14

March, 6 hours on 22nd June, 12 hours on 22nd September, 18 hours on 22nd December. The exact value of this quantity is given for every day at Greenwich mean noon in the *Nautical Almanac* and (to the nearest second) in *Whitaker's Almanack*, but we can find it approximately (to within some 5 minutes) by adding (or subtracting) 4 minutes per day to (or from) its value on the nearest of the above four dates.

EXAMPLE 4. To find the approximate mean time of transit of Aldebaran (α Tauri) (R.A. 4h. 32m.) on 31st January.

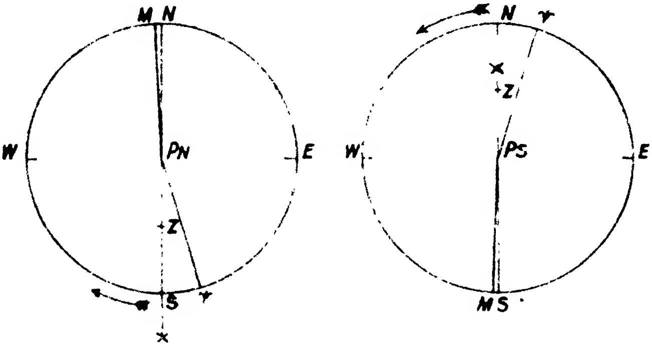


FIG. 15

(Fig. 14.) The approximate R.A. of the mean sun is 18h. + 40 (days) × 4m. = 20h. 40m. We now mark γ at 4h. 32m. from Z towards the west, then lay off 20h. 40m. in the

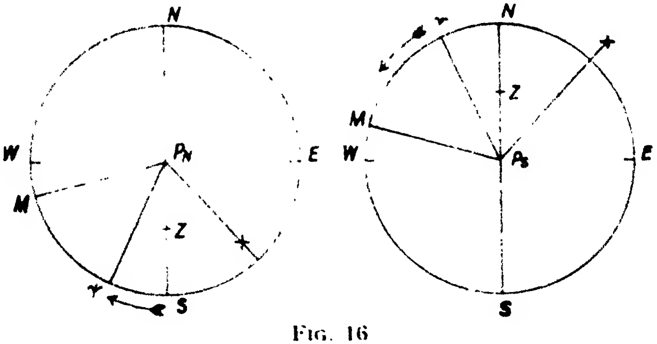


FIG. 16

opposite direction from γ to find M, which gives the hour angle of M as 28h. 32m. − 20h. 40m. = 7h. 52 m. p.m., which is the local mean time of transit approximately on that date.

EXAMPLE 5. At what hour (approximately) will Formalhaut (α Pisces Australis) transit on 7th September if its R.A. is 22h. 54m.? (Fig. 15.) The R.A. of the mean sun is 12h. − 15 (days) × 4m. = 11h. 0m. Mark ♈ at 22h. 54m. from Z, then mark *M* at 22h. 54m. − 11h. 0m. = 11h. 54m., and we see that the mean time of the transit is 11h. 54m. p.m.

EXAMPLE 6. What will be the hour angle of Aldebaran (R.A. = 4h. 32m.) at 5·0 p.m. on 31st January? (Fig. 16.) As in Example 4, the R.A. of the mean sun on this date = 20h. 40m. (approx.). Mark the mean sun at 5·0h. Then ♈ is at 5·0h. + 20h. 40m. = 25h. 40m. = 1h. 40m. West and the star at 1h. 40m. − 4h. 32 m. = 2h. 52m. East, or 25h. 40m. − 4h. 32m. = 21h. 08m. West. This is in accordance with Example 4, as it will transit at 5h. 0m. + 2m. 52s. = 7h. 52m. p.m. (as before).

Model for Illustrating the Positions of the Stars (Fig. 17). The student will find this helpful in familiarizing himself with the above astronomical definitions, and even in locating bright stars from the Almanac.

Divide a circle, drawn on cardboard, 6 inches diameter, into 24 parts of 15° each and number these from 0 to 24 (clockwise for the northern hemisphere, anti-clockwise for the southern hemisphere). This we shall call the "hour circle." Draw a square touching the circle at the points 6, 12, 18, and 24, and cut out the square. Cut a right-angled triangle out of cardboard, with its hypotenuse 6 inches long and its other angles equal to the latitude ϕ of the place and the co-latitude $90° - \phi$. Fasten the triangle with its hypotenuse to the back of the square along the line 12–24, so as to tilt the square to the horizontal at an inclination equal to the co-latitude, with the 24 mark at the top.

Divide a circle, drawn on cardboard, 5 inches in diameter, into 24 parts and number them 0–24 (anti-clockwise for the northern hemisphere, clockwise for the southern hemisphere). Cut out the circle, which we shall call the "right ascension circle." Mark ♈ at the 24 mark. Then divide a quadrant, drawn on cardboard, 2½ inch radius, into 9 parts of 10° each and number these as shown in Fig. 17 from 0° to 90°.

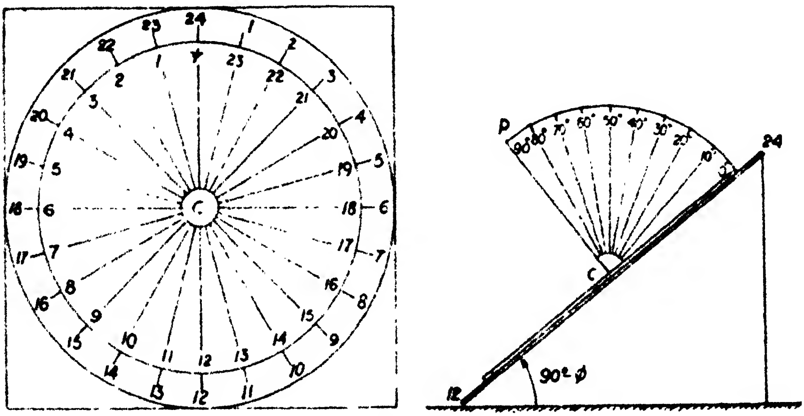


FIG. 17

Cut out the quadrant, which is our "declination quadrant." Pin the centre of the right ascension circle to the centre of the hour circle, place the model on a horizontal table with the 24 mark of the hour circle facing south in the northern hemisphere, north in the southern hemisphere. The hour circle is then in the plane of the celestial equator. If the quadrant is then placed with its centre *C* at the centre of the hour circle, the edge *CP* of the quadrant will then point to the celestial pole.

To illustrate Example 4 with the model: Bring the star's R.A. (4h. 32m.) on the R.A. circle up to 24h. on the hour circle; φ will then be in its right position and 20h. 40m. on the R.A. circle will then be the position of the mean sun, which will be found to read 7h. 52m. on the hour circle. Example 5 can be illustrated in the same way. For Example 6 the mean sun is at 5h. on the hour circle and its R.A. is 20h. 40m. Therefore, bring the 20h. 40m. on the R.A. circle opposite 5h. on the hour circle. φ will then be correctly placed and Aldebaran will be at 4h. 32m. on the R.A. circle, which will be found at the hour angle 21h. 08m. If the declination quadrant be placed perpendicularly on the R.A. circle at this mark, and a knitting needle placed on it from C to 16° 22′ on the quadrant, the needle will point to Aldebaran if the place is in the northern hemisphere.

By this means the student should be able to find any of the brighter stars given in the list on pages 140, 141 of *Whitaker's Almanack*, provided they have a north declination in the northern hemisphere and, conversely, a south declination in the southern hemisphere. Even when the declination is of the opposite sign to the latitude, the declination can be *estimated* below the plane of the equator. A star chart will also be of great use in this connection, and will give the shapes and relative positions of the various constellations. The horizon, of course, is an imaginary horizontal plane through the centre of the model.

The model would serve as a sun-dial if the knitting needle is inserted at C, perpendicular to the plane of the hour circle, its shadow recording the (apparent) time in hours from midnight, but for this, of course, the sun must be above the equator. In practice, therefore, sun-dials are usually constructed with their hour circles horizontal, and the hour intervals projected thereon are, therefore, unequal in size.

Fig. 18 shows how this is effected; PNH is a spherical triangle right-angled at N, in which the side $PN = \phi$.

$\therefore \sin\phi = \tan NH \, . \cot P \quad \therefore \tan NH = \sin\phi \, . \tan P.$

Choosing values 15°, 30°, 45° . . . for P, this gives the value of the angle NH to set out for each hour on each side of noon.

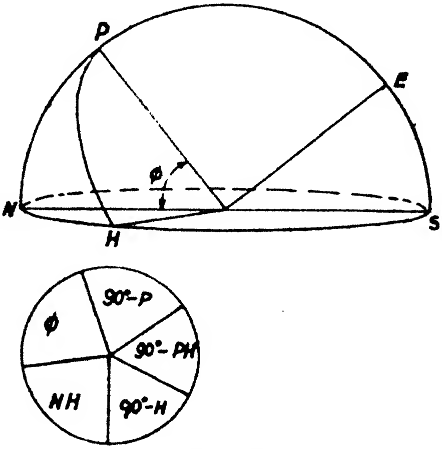


FIG. 18

Conversion of Sidereal Time into Mean Time. This is a calculation which must frequently be made and must be thoroughly mastered. When a star crosses the meridian at upper transit the local sidereal time is the R.A. of the star: at lower transit it is the R.A. + 12 hours. In other positions of the star the local sidereal time = R.A. of star + its hour angle. If we know the star's declination, the latitude of the place, and measure its altitude, we know the three sides of the spherical triangle ZPS and can calculate the hour angle at P, and hence we find the local sidereal time.

The *Nautical Almanac* gives the sidereal time of midnight *at Greenwich* for each day very exactly, and *Whitaker's Almanack* gives the sidereal time of mean noon there each day to the nearest second, and we must be able to find this quantity for a place of any given longitude L. Fig. 19 represents the faces of two 24-hour clocks *at the same instant*, each with a mean time, and a sidereal time, hour hand, one at Greenwich, the other at a place L° west of Greenwich. It is mean noon at Greenwich. Then at the place in longitude $L°$ west, the mean hour hand is $\frac{L°}{15}$ hours before noon, because it will be $\frac{L°}{15}$ mean hours before the place comes opposite the mean sun, while the sidereal hour hand is also $\frac{L°}{15}$ hours behind its Greenwich position, because it will take $\frac{L°}{15}$ sidereal hours to attain that position—in other words, *both* hour hands are

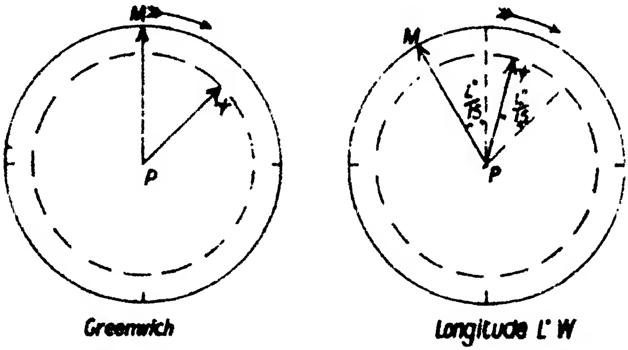


FIG. 19

always slow by the same amount, $\frac{L°}{15}$ hours, owing to the difference of longitude of $L°$. On the other hand, the sidereal hour hand in both clocks is going *fast* on the mean time hand by the same amount, viz. 9·857s. per mean hour (or 3m. 56·56s. per day or 24h. per year). By the time, therefore, that it is mean noon at the place in longitude $L°$ *west*, the angle between M and ψ has *increased* by $\frac{L°}{15} \times 9{\cdot}857$ sec. Similarly, if the place is in longitude $L°$ *east*, mean noon occurs when it is $\frac{L°}{15}$ mean hours *before* mean noon at Greenwich and the angle is $\frac{L^\circ}{15} \times 9{\cdot}857$ sec. *less* than at G.M.N. We have, therefore, the rule—

$$\textit{Local sidereal time of local mean}\begin{Bmatrix}\textit{Noon}\\ \textit{Midnight}\end{Bmatrix}$$

$$= \textit{Greenwich sidereal time of Greenwich Mean}\begin{Bmatrix}\textit{Noon}\\ \textit{Midnight}\end{Bmatrix}$$

$$\pm\ 9{\cdot}857 \textit{ sec. for every hour of longitude}\begin{Bmatrix}\textit{West}\\ \textit{East}\end{Bmatrix}$$

Having found the local sidereal time of the *previous* local mean $\begin{Bmatrix}\text{noon}\\ \text{midnight}\end{Bmatrix}\left(\text{L.S.T. of }\begin{matrix}\text{L.M.N.}\\ \text{0h. M.T.}\end{matrix}\right)$ we deduct it from the given local sidereal time, and get the "sidereal interval," since local mean $\begin{Bmatrix}\text{noon}\\ \text{midnight}\end{Bmatrix}$. As this is in sidereal units. we turn it into mean units by deducting 9·830 sec. per hour. These calculations are facilitated by tables in *Whitaker's Almanack* and in *Chambers' Seven-Figure Mathematical Tables,* which give the "acceleration" at 9·857 sec. per hour, and the "retardation" at 9·830 sec. per hour for each hour of the 24, each of the 60 minutes, and each of the 60 seconds. Picking out the figures for the required hour, minute, and second, we have only to add them together to get the correction in the first case for the longitude, in the second case for the sidereal interval. Similar tables in *Chambers' Tables* facilitate the process of turning longitude in angle (arc) into longitude into hours, minutes, and seconds of time at the rate of 15° = 1h., 1° = 4m., 1′ = 4s., and, conversely, tables for converting time into angle (or arc) at the rate of 1h. = 15°, 1m. = 15′, 1s. = 15″.

Failing these tables, it is, however, quite a simple matter to calculate these corrections on the slide-rule.

EXAMPLE 7. Find the local mean time of transit of Aldebaran (R.A. 4h. 32m. 0s.) on the 31st January, 1933, in a place of longitude 64° 16′ E.

(*a*) Longitude = 64 × 4m. + 16 × 4s.
= 4h. 16m. + 1m. 4s. = 4h. 17m. 4s. E.

G.S.T. of G.M.N. (from *Whitaker's Almanack*)
= 20h. 41m. 1s.

Deduct $4 \times 9{\cdot}857\text{s.} + 17 \times \frac{9{\cdot}857}{60} + 4 \times \frac{9{\cdot}857}{3600}$

= 39·43 + 2·79 + ·01 = 42·23s.—say, 42·2s.
(as only required to nearest second)

∴ L.S.T. of L.M.N. = 20h. 40m. 18·8s.

(*b*) Local sidereal time = Right ascension of star

	= 4h. 32m. 0s.	= 28h. 32m. 0s.
	Deduct L.S.T. of L.M.N.	= 20h. 40m. 18·8s.
	Sidereal interval since L.M.N. =	7h. 51m. 41·2s.
Deduct $7 \times 9{\cdot}83\text{s.} + 51 \times \frac{9{\cdot}83}{60}\text{s.} + 41{\cdot}2 \times \frac{9{\cdot}83}{3600}\text{s.}$		
	= 68·81 + 8·35 + 0·11 = 77·27s.	= 1m. 17·3s.
	∴ Local mean time	= 7h. 50m. 23·9s.
	say	7h. 50m. 24s. p.m.

EXAMPLE 8. Find the local mean time of transit of Formalhaut (R.A. 22h. 53m. 57s.) on 7th September, 1933, in a place of longitude 157° 51′ W.

(*a*) Longitude = 10h. 28m. + 3m. 24s. = 10h. 31m. 24s. W.
G.S.T. of G.M.N. = 11h. 4m. 27s. (from *Whitaker's Almanack*)

L.S.T. of L.M.N. = 11h. 4m. 27s. + 1m. 38·56s. + 5·09s.
+ 0·07s. = 11h. 4m. 27s. + 1m. 43·72s. = 11h. 6m. 10·7s.

(*b*) Local sidereal time = Right ascension of star

	= 22h. 53m. 57s.
Deduct L.S.T. of L.M N.	= 11h. 6m. 10·7s.
Sidereal interval since L.M.N.	= 11h. 47m. 46·3s.
Deduct 1m. 48·13s. + 7·70s. + 0·13s. = 1m. 55·96s.	= 1m. 56·0s.
∴ Local mean time	= 11h. 45m. 50s.
	p.m. to nearest second

The converse operation—of converting local mean time into local sidereal time—is best executed by reversing the process, e.g. in Example 8.

Local mean time	= 11h. 45m. 50s.
Add $11 \times 9{\cdot}857\text{s.} + \frac{45}{60} 9{\cdot}857\text{s.} + \frac{50}{3600} 9{\cdot}857\text{s.}$	
= 108·42s. + 7·39s. + 0·14s. = 115·95	= 1m. 56s.
Sidereal interval since mean noon	11h. 47m. 46s.
Add L.S.T. of L.M N (as above)	= 11h. 6m. 11s.
∴ Local sidereal time	= 22h. 53m. 57s.

Alternative Method of Conversion of Sidereal Time into Mean Time. The *Nautical Almanac* and *Whitaker's Almanack*

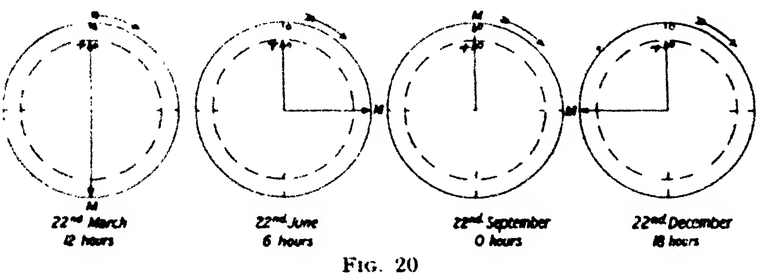


FIG. 20

also give daily the mean time of transit of ♈ at Greenwich, or "mean time of 0h. sidereal time at Greenwich," the mean time being reckoned *from the previous midnight* as is usual when 24-hour time is used. Fig. 20 shows the 24-hour clocks again at the equinoxes and solstices, but with ♈ at

0h. The mean time hour hand goes *slow* 6h. in 3 months, or 3m. 55·91s. per sidereal day, or 9·830s. per sidereal hour. Therefore, as the transit of ψ occurs 1 sidereal hour $\begin{cases}\text{earlier}\\\text{later}\end{cases}$ for each hour (15°) of longitude $\begin{cases}\text{East}\\\text{West}\end{cases}$, we have the rule—

Local mean time of transit of ψ

$$= \textit{Mean time of transit of } \psi \textit{ at Greenwich}$$

$$\pm\ 9{\cdot}830 \textit{ sec. per hour of longitude} \begin{cases}\textit{East}\\\textit{West}\end{cases}$$

Having found the local mean time of transit of ψ, we turn the given local sidereal time into a mean time interval since transit of ψ, by deducting 9·830 per hour and then add the two together to get the local mean time, reckoning from the previous midnight.

EXAMPLE 9. Taking the data for Example 7, but working by this alternative method. Mean time of 0h. sidereal time at Greenwich = 15h. 18m. 26s. on 31st January, 1933, from *Whitaker's Almanack.*

(*a*) Longitude 64° 16′ E. = 4h. 17m. 4s. E. (as before).

$$\text{Add } 4 \times 9{\cdot}83\text{s.} + 17 \times \frac{9{\cdot}83}{60} + \frac{4}{3600} \times 9{\cdot}83$$

$$= 39{\cdot}32 + 2{\cdot}78\text{s.} + 0{\cdot}01\text{s.} = 42{\cdot}11\text{s.}$$

Local mean time of transit of ψ = 15h. 19m. 08·1s.

(*b*) Local sidereal time = 4h. 32m. 0s.

$$\text{Deduct } 4 \times 9{\cdot}83\text{s.} + \frac{32}{60} \times 9{\cdot}83\text{s.} = 39{\cdot}32\text{s.} + 5{\cdot}24\text{s.}$$

$$= 44{\cdot}56\text{s.}$$

∴ Mean time interval since transit of ψ	= 4h. 31m. 15·4s.
Add L.M.T. of transit of ψ	= 15h. 19m. 08·1s.
∴ Local mean time	= 19h. 50m. 23·5s.
	= 7h. 50m. 23·5s. p.m.

Our previous value was 7h. 50m. 23·9s., the difference being due to the quantities from *Whitaker's Almanack* being only given to the nearest second.

To turn local mean time into local sidereal time by this method, reverse the process, viz. deduct the local mean time of transit of ψ found by the above rule. This gives the mean time interval since the transit of ψ, then convert this into sidereal time by adding 9·857 sec. per hour.

EXAMPLE 10. To find the local sidereal time at 3·30 a.m. on 1st February, 1933, in a place of longitude 64° 16′ E.

3h. 30m. a.m. 1st February	= 27h. 30m. 00s.	Local mean time on 31st Jan.
Deduct Local mean time of transit of ψ	= 15h. 19m. 08·1s.	as found above for 31st Jan.
Mean time interval since transit of ψ	= 12h. 10m. 51·9s.	
Add $9{\cdot}857\text{s.} \times 12 + 9{\cdot}857 \times \frac{10}{60} + 9{\cdot}857 \times \frac{51{\cdot}9}{3600}$ = 1m. 58·28s. + 1·64s. + 0·14s. = 2m. 0·06s.	= 2m. 0·1s.	
Local sidereal time	= 12h. 12m. 52s.	

MEAN AND APPARENT SOLAR TIME

Apparent Solar Time is the Hour Angle of the Centre of the Actual Sun. It is the time given by a sun-dial, correctly graduated and with the gnome or style pointing to the celestial pole. But the intervals of time between transits of the sun across the meridian are not quite equal, and as a measure of solar time it is necessary to define ***mean time as the hour angle of the mean sun,*** where the Mean Sun is defined as a point in the ***equator*** which moves at a ***uniform*** speed and the hour angle of which agrees with that of the actual sun at least once a year, while it makes, of course, the same number of revolutions in a year as the actual sun. It makes, therefore, one revolution among the fixed stars in a year, its right ascension increasing uniformly by 3m. 56·56s. per day.

The actual sun, denoted by the symbol ☉, moves among the stars in a great circle called the "Ecliptic," inclined to the equator at an angle of 23° 27′, the "obliquity" of the earth's axis, which cuts the equator at the First Point of Aries (♈) and at a point diametrically opposite called the First Point of Libra (♎) these being the sun's positions at the equinoxes, on or about the 22nd March and 22nd September respectively. The declination of the sun is, therefore, constantly changing, being 0° at the equinoxes, 23° 27′ N. on or about the 22nd June (summer solstice in the northern hemisphere) and 23° 27′ S. on or about the 22nd December (winter solstice in the northern hemisphere), while its right ascension does not increase uniformly for two reasons.

First, the earth moves round the sun in an ellipse (eccentricity = $\frac{1}{60}$) with the sun at one of the foci, not a circle, and consequently its speed is not uniform, but varies in such a way that the line joining the earth to the sun sweeps out equal areas in equal times as required by the law of gravitation. Consequently, the speed increases from the 1st July, when the earth is farthest from the sun to the 31st December, when it is nearest to the sun, and then decreases again till the 1st July. If a point moved at a uniform speed in the ecliptic among the fixed stars, the actual sun would be ahead of such point from 31st December to 1st July, and behind it from 1st July to 31st December, the point and the sun agreeing on these two dates. The difference between the right ascension of the point and the sun between these dates has a maximum value of about ± 7 minutes, due to *eccentricity*—this correction being shown in a broken line on Fig. 21.

Secondly, even if the point moved uniformly in the ecliptic its right ascension would not increase uniformly, for right ascensions are measured along the equator about the pole *P*, not along the ecliptic about its "pole" *Q* (Fig. 22). In each quarter between 22nd March, 22nd June, 22nd September,

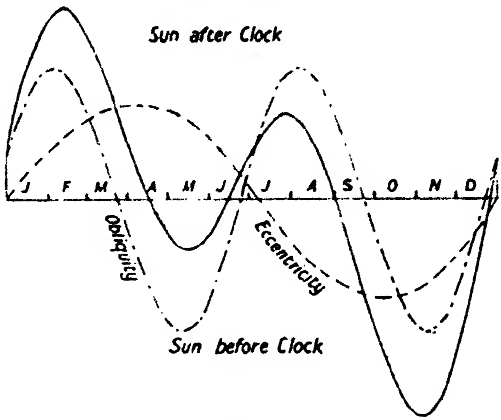


FIG. 21

and 22nd December, 90° is described by both the mean sun in the equator and the steadily moving point in the ecliptic, so that the two imaginary points would have the same Right Ascension on these four dates, but at intervening dates the right ascension of the point in the ecliptic would differ from the right ascension of the mean sun in the equator by amounts whose maximum value is about ± 2½° or ± 10 minutes. This is the correction for *obliquity* and is shown by the chain line in Fig. 21.

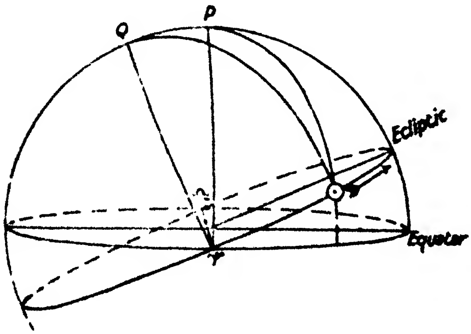


FIG. 22

Compounding these two corrections, we get the total correction for the right ascension of the actual sun to obtain the right ascension of the mean sun, which is shown by the full line in Fig. 21. This curve has been drawn to show the ± corrections to the right ascension of the mean sun to obtain the right ascension of the actual sun, but as hour angle is measured in the opposite direction to right ascension, it represents also the ± *correction on the hour angle of the actual sun (apparent time) to obtain the hour angle of the mean sun (mean time)*.

This correction is called the "Equation of Time" (we should now call it the "Correction" of Time). We may consider it positive when the sun is "after the clock," negative when the sun is "before the clock." We have, therefore the equation (Fig. 23)—

Apparent time (A.T.) + Equation of time (e) = Local mean time (L.M.T.).

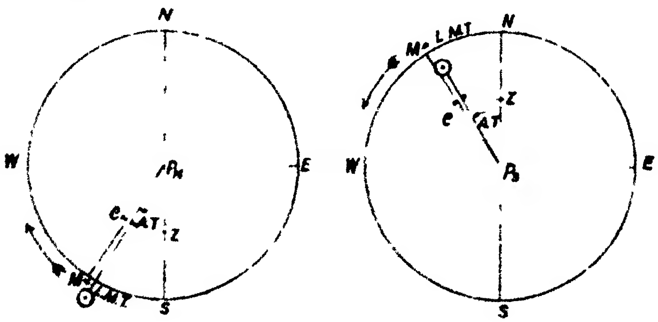


FIG. 23

At apparent noon when the sun is crossing the meridian, the apparent time is zero and, therefore, the *equation of time is the local mean time of apparent noon.*

The maximum and minimum values of the equation of time are—

12th February	+ 14m. 23s.	14th May	− 3m. 48s.
15th April	0m. 0s.	14th June	0m. 0s.
26th July	+ 6m. 21s.	3rd November	− 16m. 22s.
1st September	0m. 0s.	25th December	0m. 0s.

Longitude. The importance of the determination of time to surveyors is that difference of local time between two places, whether mean, apparent, or sidereal, at the same instant, is convertible into difference of longitude. If, for example, the Greenwich mean (or sidereal) time is known by a chronometer at the instant when a star is observed crossing the meridian of a place and the local sidereal time is, therefore, known, viz. the right ascension of the star, the difference of local sidereal (or mean) time and the Greenwich sidereal (or mean) time, converted into angle is the longitude of the place. If the local time is more than the Greenwich time, the place is east of Greenwich, and conversely. In the case of a star transit being observed and a Greenwich mean time chronometer used, the G.M.T. should be converted into G.S.T. and compared with the L.S.T. actually determined. If the sun's transit is observed, the L.M.T. at the instant of transit is the Equation of Time for the Greenwich mean time recorded on the chronometer (obtained by interpolation from the Almanac) the difference of longitude being the difference of this L.M.T. and the G.M.T. converted into angle. The altitude of the sun's centre when crossing the meridian can be used to determine the latitude, but as the declination is not constant we must know the Greenwich time in order to find the declination.

Standard Time. It would be quite impracticable to use local mean time for ordinary purposes, as this varies 4 min.

for every 1° of longitude, e.g. for every 43·1 miles east and west in latitude 51° 30′ N. (that of London). In practice, therefore, the local mean time of some one meridian is kept as the "standard time" for the whole of a country, e.g. Greenwich mean time for the whole of the British Isles, even though it differs by 40 minutes from the local mean time at 10° W. in the West of Ireland. For convenience, standard time usually varies from country to country in zones, differing by 1 hour; a large area like the United States being divided into several such zones.

Apparent Right Ascension and Apparent Declination. Hitherto we have assumed that the right ascension and declination of fixed stars are absolutely constant; in reality they vary by a small amount throughout the year and from year to year. The reasons for this variation are as follows: *Precession.*—The First Point of Aries moves backward along the ecliptic 50·22″ on the average per annum, which means that the celestial pole describes a circle of angular radius 23° 27′ about the "pole" of the ecliptic in a period of 25,800 years. This is called the "Precession of the Equinoxes." The First Point of Aries has moved backwards into the next constellation, "Pisces," since the term was first used. *Nutation.*—The obliquity of the earth's axis is not constant but varies over a range of 18″ in a period of 18⅔ years, so that the celestial pole describes not a circle but a wavy path, averaging a circle. This is called "Nutation." *Parallax.*—The distances of the fixed stars are not quite infinite, and the passage of the earth in 6 months from one extremity of its orbit to the other (some 186,000,000 miles) causes a slight annual change in the position of some of the nearer stars called "Heliocentric Parallax." *Proper Motion.*—The motion of the whole solar system through space, and those of the fixed stars, cause slight progressive changes in the positions of the stars. *Aberration of Light.*—The velocity of the earth in its orbit, and of rotation about its axis, are not quite negligible compared to the velocity of light (186,330 miles per second), so that the direction in which the light of a star appears to reach the earth is not quite the direction of the star, being compounded of the two velocities. This is called the "Aberration of Light" and varies throughout the year.

As corrections for all these effects would be very complicated, the *apparent* right ascension and declination of a number of bright stars have been calculated and are given in the *Nautical Almanac* at 1 day or 10 day intervals, so that only a correction for atmospheric refraction needs to be applied to the observed altitude of the fixed stars. *Whitaker's Almanack* only gives the "Mean Places" of the fixed stars on the 1st January and the annual variation of their right ascension and declination, except in the case of the Pole Star, where the apparent right ascension and declination are given every 20 days, and of the sun, whose apparent right ascension and declination are given daily. For accurate determinations of position and direction on the earth from star observations, recourse must, therefore, be had to the *Nautical Almanac.*

Whitaker's Almanack. The student should now be able to understand the extracts from this Almanack (1933), which refer, of course, to Greenwich. (See page 94.)

He will note that the "Sidereal Time at Mean Noon" increases by 3m. 56·56s. per day, while the "Mean Time at 0h. Sidereal Time" decreases by 3m. 55·91s. per day, that the R.A. of the sun at mean noon = sidereal time at mean noon (i.e. R.A. of the mean sun at mean noon) + equation of

Date	Equation of Time		The Sun (Mean Noon)				Sidereal Time at Mean Noon	Mean Time at 0h. Sidereal Time
	Subtract from Appar. Time	Hourly Variation	Apparent Right Ascension	Hourly Variation of R.A.	Apparent Declination	Hourly Variation of Declin.		
	M. S.	S.	H. M. S.	S.			H. M. S.	H. M. S.
Dec. 1st	10 59	0·93	16 28 35	10·78	21° 47·3'S.	0·39'	16 39 34	19 19 14
Dec. 2nd	10 36	0·96	16 32 51	10·81	21° 56·4'	0·37'	16 43 31	19 15 18
Dec. 3rd	10 13	0·98	16 37 14	10·84	22° 5·2'	0·36'	16 47 27	19 11 22

time, the equation of time being *negative* here, and that the hourly variation of the R.A. of the sun = hourly variation

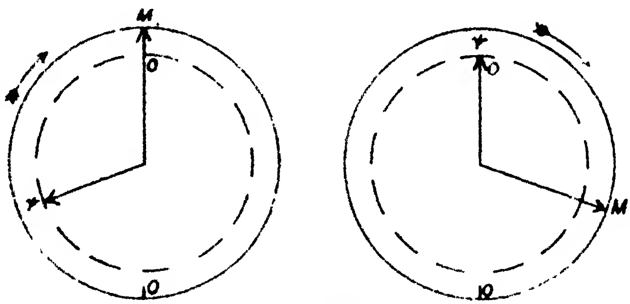


Fig. 24

of the R.A. of the mean sun (viz. 9·857s.) + hourly variation of the equation of time (in this case positive, as it is a decrease of a negative quantity.) To understand the connection between the last two columns we shall work an example—

Example 11. If the sidereal time at mean noon is 16h. 39m. 34s., what is the mean time of transit of ψ? (See data for 1st December above.) (Fig. 24.) ψ will transit in

24h. – 16h. 39m. 34s. = 7h. 20m. 26s. sidereal time
= 7h. 20m. 26s. – (1m. 8·81s. + 3·28s. + 0·07s.)
= 7h. 20m. 26s. – 1m. 12·16s.
= 7h. 19m. 13·84s. mean time,

i.e. at 7h. 19m. 14s. after mean noon or 19h. 19m. 14s. local mean time, reckoned from midnight.

As an exercise on the data for the sun we shall work the following example—

EXAMPLE 12. The centre of the sun is observed to cross the meridian of a place at a (corrected) altitude of 43° 02′ 12″ N., when the time on a Greenwich mean time chronometer

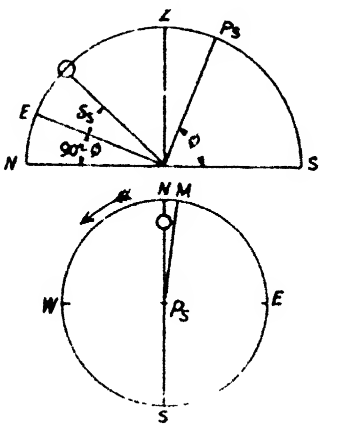


FIG. 25

is 7h. 20m. a.m. on 2nd December, 1933. Find the latitude and longitude of the place (Fig. 25).

(*a*) The declination of the sun is 21° 56′ 24″ S. − 4⅔ × 0·37′
= 21° 56′ 24″ − 1′ 44″ = 21° 54′ 40″ S.

As the sun is to the north, the latitude is *south*.
∴ $90° - \phi$ = 43° 02′ 12″ − 21° 54′ 40″ = 21° 07′ 32″
∴ ϕ = latitude = 90° − 21° 07′ 32″ = 68° 52′ 28″ S.

(*b*) The equation of time = 10m. 36s. + 4⅔ × ·96s
= 10m. 36s. + 4·48s. = 10m. 40·48s. and is *negative*.
∴ Local mean time = 12h. − 10m. 40·48s.
= 11h. 49m. 19·52s. a.m.

Greenwich mean time = 7h. 20m. 00s. a.m.

∴ Longitude = 4h. 29m. 19·52s. east

$= 60° + 29 \times 15' + 19{\cdot}52 \times 15'' = 60° + 7° 15' + 4' 53''$

$= 67° 19' 53''$ east

N.B. The above use of the hourly variation is not quite correct. If the hourly variation of the equation of time, for example, was 0·96s. at Greenwich mean noon on 2nd December and 0·93s. at Greenwich mean noon on 1st December, it would be $0{\cdot}96\text{s.} - \frac{2\frac{1}{3}}{24} 0{\cdot}03 = 0{\cdot}96\text{s.} - 0{\cdot}003 = 0{\cdot}957\text{s.}$ *half way* between 7h. 20m. a.m. and G.M.N. on 2nd December. This will be the *average* hourly variation over the period of $4\frac{2}{3}$ hours, and the correction is $4\frac{2}{3} \times {\cdot}957\text{s.} = 4{\cdot}47\text{s.}$

Corrections of Altitude

(*a*) *Atmospheric Refraction.* When a ray of light passes from a rarer to a denser medium it is bent, in the plane of the incident ray and the normal at the point of incidence, closer to the normal according to the law

$$\frac{\text{sine of angle of incidence}}{\text{sine of angle of refraction}} = \mu$$

where μ, the coefficient of refraction, depends on the two media. Also when a ray of light passes through a succession of media in *parallel* layers its path in any layer is parallel to the path it would have followed in that layer if it had passed directly into that layer without passing through the intervening ones. The path of a ray of light from space to the earth's surface is, therefore, a curved line through the earth's atmosphere, and a star appears at a greater altitude than it actually possesses. We shall assume for the moment that the air is arranged in *parallel* layers. The ray of light is displaced laterally by an amount, insignificant in comparison with the distance of the stars, but its real altitude $a-r$ is altered to appear as a, the observed altitude, and

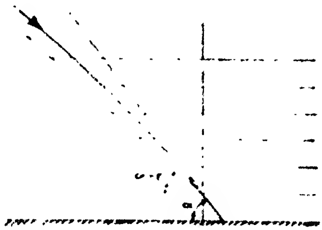


FIG. 26

all we need consider is the final densest layer of air in contact with the earth's surface.

The angle of incidence is $90^\circ-(a-r)$, the angle of refraction is $90^\circ-a$ (Fig. 26). Then

$$\frac{\sin(90^\circ-(a-r))}{\sin(90^\circ-a)}=\frac{\cos(a-r)}{\cos a}$$

$$=\frac{\cos a \,.\, \cos r+\sin a \,.\, \sin r}{\cos a}=\mu$$

where μ depends on the air pressure and temperature.

$$\therefore \quad \cos r+\tan a \,.\, \sin r=\mu$$

Now, r is a small angle, and we can take $\cos r=1$, $\sin r=r$.

$$\therefore \quad r=(\mu-1)\cot a \text{ in circular measure}$$

$$=206265\,(\mu-1)\,.\,\cot a \text{ seconds}=A\cot a$$

where A is about 58″ at a pressure of 29·6 inches of mercury and temperature of 49° F.

For approximate purposes we may, therefore, take the *correction for refraction* as 58″ × *cot . apparent altitude*, when the altitude is over 20°. This obviously should not be used for small altitudes as it would make the refraction correction *infinite* when $a = 0°$, whereas "horizontal refraction" is about 33′. For small angles, the above approximations do not hold, as r becomes comparatively large, and for small altitudes the assumption that the layers of air should be *parallel* becomes very far from true owing to the curvature of the earth.

For more accurate work, we must use Bessel's Tables of Refractions, as given in *Chambers' Tables*. These are for use in the formula $r'' = (Bt)^M T^N \, . \, A \, . \cot a$, where B is a factor depending on the height of the barometer, t one depending on the temperature of the barometer, T one depending on the temperature of the air, A is a factor, and M and N are indices, depending on the observed altitude a, the values of these factors and indices being tabulated in *Chambers' Tables*. Now, when $a > 20°$, M and N are very nearly equal to 1, while B, t, and T all vary from values a little less than 1 to a little more than 1. For altitudes greater than 20° it is, therefore, usually accurate enough to write the formula as $r'' = B \, . \, t \, . \, T \, . \, A \cot a$. If the reading of the barometer was 29·6 inches and its temperature 32° F., while the air temperature was 49° F., the factors B, t, and T would each be unity and the refraction would then be simply $r'' = A \, . \cot a$. This value is called the *Mean Refraction* for the altitude: it is given in the above tables, and for an accurate result must be multiplied by the factors B, t, and T, depending on the barometer, attached thermometer, and air thermometer. A varies from 57·24″ at 20°, to 57·75″ at 90°.

(*b*) *Geocentric Parallax.* This is an additional correction which we must apply to the *sun's* altitude, as its distance cannot be taken as infinite compared with the earth's radius, so that its apparent position is slightly affected by the observer's position on the earth: all altitudes of the *sun* must, therefore, be corrected to the *earth's centre.*

Let the sun's altitude (corrected for refraction) at a point A on the earth be $a = SAS'$ (Fig. 27). Then its

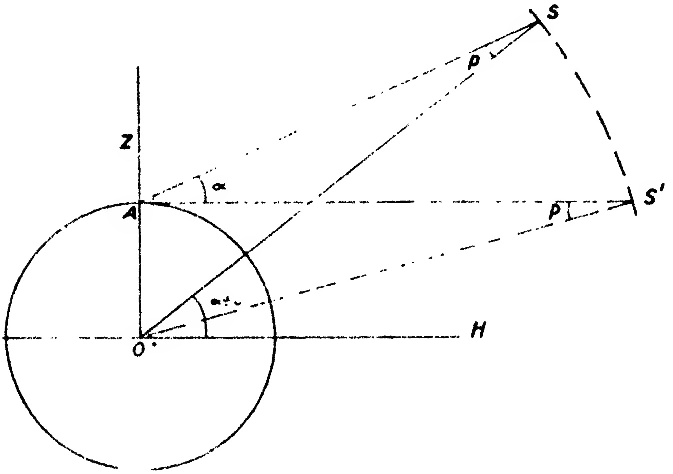


FIG. 27

geocentric altitude $= SOH = a + p$, where p is the angle subtended by the earth's radius AO at the sun's centre, S.

$$\text{Now, } \frac{\sin p}{\sin (90^\circ + a)} = \frac{AO}{OS} \quad \therefore \sin p = \frac{AO}{OS} . \cos a.$$

When the sun is at S' on the horizon, the angle subtended by AO is $P =$ "horizontal parallax," where

$$\sin P = \frac{AO}{OS'} = \frac{AO}{OS}$$

so that we have $\sin p = \sin P . \cos a$. As both p and P are very small angles, we have the formula, *correction for parallax* $= + p = + P . \cos a$.

The average value of

$$P = \frac{3960}{92,800,000} \text{ miles} \times 206,265'' = 8{\cdot}80'',$$

but P varies inversely with the distance of the earth from the sun, being 8·95″ on 31st December and 8·65″ on 1st July. Its value, i.e. "Sun's Horizontal Parallax," is given daily in the *Nautical Almanac* and every 10 days in *Whitaker's Almanack*. It is, of course, always a *positive* correction.

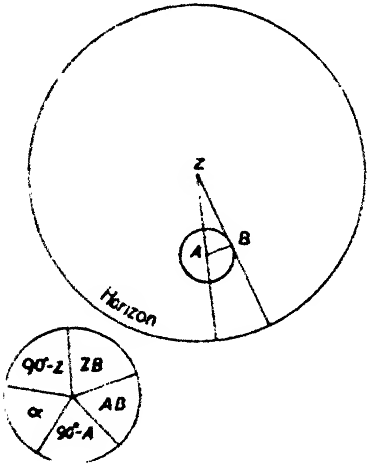


FIG. 28

(*c*) *Sun's Semi-Diameter.* When observing the sun it is impossible to set the cross-hairs of the theodolite on its *centre*, for which the apparent right ascension, apparent declination, and equation of time are given in the Almanac, and we must observe the altitude of its upper or lower "limb" (i.e. edge) and then subtract or add its angular "semi-diameter." This latter varies inversely with the sun's distance, being 16′ 18″ on 31st December and 15′ 45″ on 1st July. It is given daily in the *Nautical Almanac* and every 10 days in *Whitaker's Almanack*.

Similarly, when measuring a horizontal angle to the sun, we set the vertical hair of the theodolite to the east or west limb of the sun and apply a correction "in azimuth" in order to get the horizontal angle to the sun's centre. This correction depends, however, on the altitude of the sun (see Fig. 28, where two vertical circles are shown, one through the sun's centre ZA and the other touching the sun at B). We have then a right-angled spherical triangle ZAB, right-angled at B, where AB = semi-diameter, $ZA = 90° - a$, and Z is the correction in azimuth.

$\therefore \sin AB = \cos(90° - Z) . \cos a = \sin Z . \cos a,$

but as AB and Z are small, we have $Z = AB . \sec a$, or

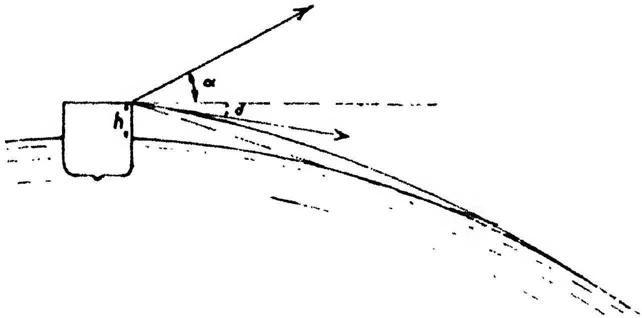


FIG. 29

correction for semi-diameter in azimuth = semi-diameter × *secant altitude.*

(*d*) *Dip of Horizon.* When the sextant is used to find the altitude of the sun or of a star at sea (Fig. 29), the altitude is measured from the visible horizon and a correction *d* for the dip of the horizon must be *subtracted*, depending upon the *height h* of the observer above the sea. This correction has to be determined by experiment, as the line of sight which touches the sea's surface is curved downwards by refraction. A table of this correction is given in *Chambers' Tables*.

CHAPTER III

THE THEODOLITE AND LEVEL

Instrumental Errors and Adjustments—Modern Developments

The Theodolite

The theodolite is an instrument for measuring horizontal and vertical angles. A horizontal angle is one between two vertical planes intersecting in a vertical line through the centre of the instrument; a vertical angle is one measured in a vertical plane, upwards or downwards, from a horizontal plane through the centre of the instrument. If the theodolite is in adjustment, the collimation line of the telescope, i.e. the line joining the intersection of the cross-hairs to the optical centre of the object glass, should sweep out a vertical plane as the telescope rotates about the trunnion axis. In order that it should sweep out a *plane*, the line of collimation must be perpendicular to the trunnion axis: in order that this plane should be a *vertical* plane, the trunnion axis must be horizontal. For the trunnion axis to be horizontal when turned in all directions, two conditions are necessary: (*a*) the trunnion axis must be perpendicular to the vertical axis, and (*b*) the vertical axis must be vertical. If condition (*b*) were satisfied but not condition (*a*), the trunnion axis, when rotated round the vertical axis, would make a constant angle (*i*) with the horizontal, and the planes generated by the line of collimation would make a constant angle (*i*) with the vertical. If condition (*a*) were fulfilled but not condition (*b*), so that the vertical axis made an angle (*v*) with the vertical, the inclination of the trunnion axis to the horizontal (and, therefore, the inclination of the generated planes to the vertical) would vary from v to zero, according as the trunnion axis was in the plane of the inclination of the vertical axis to the vertical, or at right angles to that plane.

In order that vertical angles should be read correctly, there should be no "index-error of the vertical circle," i.e. the vertical circle verniers should read zero when the line of collimation is horizontal. It is also important that the line of collimation should not vary in position in the telescope when focusing objects at different distances from the instrument, but this defect has been minimized in modern instruments with "Internal Focusing" of the telescope. We shall first consider the errors caused in horizontal and vertical angles by lack of adjustment in each of the above particulars taken separately: all the other adjustments being assumed correct.

1. *Horizontal Collimation Error.* Let the line of collimation make a small angle (c) with the perpendicular to the trunnion axis, in the plane of the trunnion axis and collimation line: the line of collimation now sweeps out a very flat cone of semi-angle $90° - c$ with the trunnion axis as axis, viz. the small circle XK in broken line in Fig. 1 instead of the great circle ZM. When the line of collimation is directed to a point X, the reading of the horizontal circle is as if X were in the vertical circle ZM, whereas it is actually in the vertical circle ZN. The error in the horizontal circle reading is, therefore, MN or δZ. Draw XY, a great circle perpendicular to ZM, then $XY = c$. In the right-angled triangle XYZ, we have $\sin c = \sin \delta Z \cdot \sin ZX = \sin \delta Z \cdot \cos a$ where a is the true altitude XN. As c and δZ are very small angles, we have, therefore,

$$c = \delta Z \cdot \cos a, \text{ or } \delta Z = c \cdot \sec a = \textit{error in azimuth}.$$

If the face of the theodolite were changed, and the line of collimation again directed to X, X would now appear to be on the vertical circle ZM', making an equal angle δZ on the other side of the true vertical circle ZN, so that the

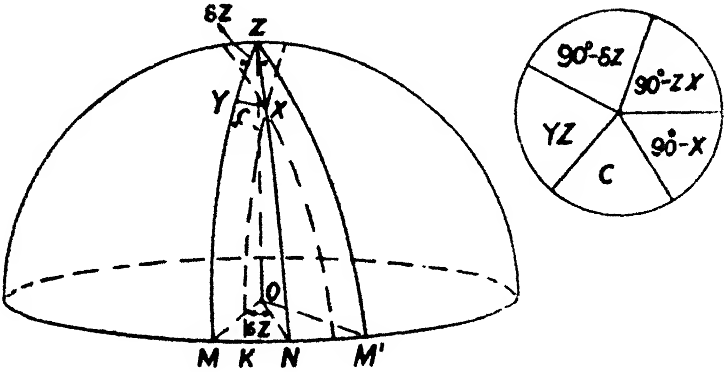


FIG. 1

average of two observations with changed face eliminates this error on horizontal angles.

Again, $\sin(90 - ZX) = \cos YZ \cdot \cos c$

$$\therefore \qquad \sin a = \sin YM \cdot \cos c = \sin a' \cdot \cos c$$

where a' is the apparent altitude YM. As, however, c is a very small angle, never more than a few minutes, we may say $a = a'$ for all practical purposes.

2. *Trunnion Axis not Perpendicular to Vertical Axis.* If the trunnion axis is inclined to the vertical axis at an angle $90° - i$ (Fig. 2), the line of sight now describes the broken great circle XM, making an angle i with the vertical great circle ZM. The point X appears to be on the vertical circle ZM, whereas it is actually on the vertical circle ZN. The error in horizontal circle reading is $MN = \delta Z$. In the right-angled triangle XNM we have $\sin MN = \tan i \cdot \tan XN$,

or $\sin \delta Z = \tan i \cdot \tan \alpha$, but as i and δZ are very small, we have $\delta Z = i \cdot$ *tan* $=$ *error in azimuth.*

If the face of the theodolite were changed and the line of sight again directed to X, X would now appear to be on

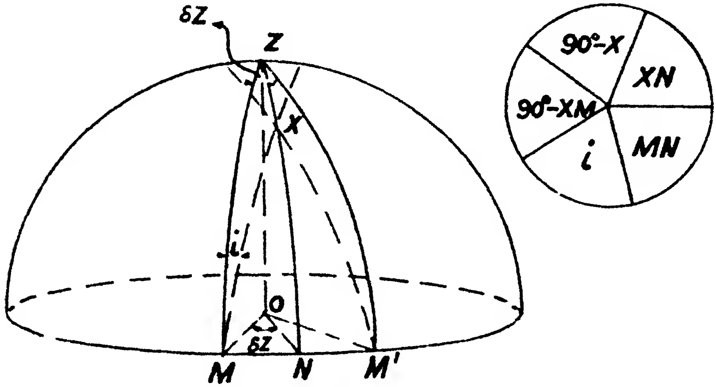


FIG. 2

the vertical circle ZM', making an equal angle δZ on the other side of the true vertical circle ZN, so that *the average of two observations with changed face eliminates this error on horizontal angles.*

Again, $\sin XN = \sin XM \cdot \cos i$, or $\sin \alpha = \sin \alpha' \cdot \cos i$, as XN is the true altitude, XM the false one, but as i is a very small angle, a few minutes at most, we may say $\alpha = \alpha'$ for all practical purposes.

3. *Vertical Axis not Vertical*, but inclined at a small angle (v) to the vertical, in the plane of the paper (Fig. 3). OZ is the true vertical, OZ' the vertical axis, and the angle $ZOZ' = v$. The line of collimation through an object X describes the great circle $Z'XY$, where Y is on the great circle RST perpendicular to OZ' described by the trunnion axis, and inclined at an angle v to the horizontal.

Draw ZXU, the true vertical circle through X, then $XU = a =$ true altitude, $XY = a' =$ apparent altitude of X. Also the error in azimuth of the horizontal circle reading is $SZX - SZ'X$ or $Z - Z'$.

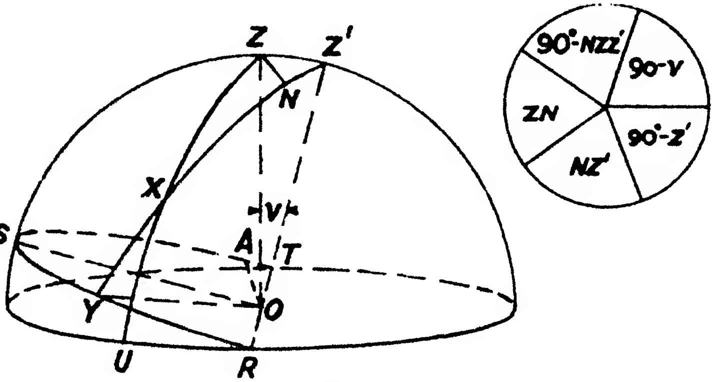


Fig. 3

In the triangle XZZ', we have

$$\tan\frac{180° - Z + Z'}{2} = \frac{\cos\frac{XZ' - XZ}{2}}{\cos\frac{XZ' + XZ}{2}}\cot\frac{X}{2}$$

$$= \frac{\cos\frac{90° - a' - (90° - a)}{2}}{\cos\frac{90° - a' + 90° - a}{2}}\cot\frac{X}{2}$$

$$= \frac{\cos\frac{a - a'}{2}}{\sin\frac{a + a'}{2}}\cot\frac{X}{2}$$

$$\therefore \qquad \cot \frac{Z - Z'}{2} = \frac{\cos \frac{a - a'}{2}}{\sin \frac{a + a'}{2}} \cdot \cot \frac{X}{2}$$

$$\therefore \qquad \tan \frac{Z - Z'}{2} = \frac{\sin \frac{a + a'}{2}}{\cos \frac{a - a'}{2}} \cdot \tan \frac{X}{2}$$

Now $\frac{Z - Z'}{2}$, $\frac{a - a'}{2}$, and $\frac{X}{2}$ are small angles and $\frac{a + a'}{2} = a$ (nearly). $\therefore$ we may say $Z - Z' = X \cdot \sin a$.

From Z draw a great circle ZN, perpendicular to $Z'X$. In the right-angled triangle ZNX we have

$$\frac{\sin X}{\sin 90^\circ} = \frac{\sin ZN}{\sin ZX}$$

$\therefore \ \sin X = \frac{\sin ZN}{\cos a}$, and as X and ZN are small quantities, we may write $X = \frac{ZN}{\cos a}$.

Substituting this value of X in the first formula, we have

$$Z - Z' = ZN \cdot \tan a.$$

Now ZN is the inclination of the plane described by the collimation line to the vertical or the inclination of the trunnion axis OA to the horizontal when sighting on X, so that, just as in case (2), we have ***error in azimuth = inclination of trunnion axis × tangent of the altitude***, the difference being that now the inclination of the trunnion axis, OA, varies with the direction of the line of sight. This error is obviously *not* eliminated by change of face; it can, however, be corrected by measuring the inclination of the trunnion axis by means of a "striding level" *for each reading*.

In the right-angled triangle ZNZ' we have

$$\frac{\sin ZN}{\sin Z'} = \frac{\sin ZZ'}{\sin 90^\circ} \quad \therefore \sin ZN = \sin v \,.\, \sin Z'$$

and as ZN and v are small angles, we have $ZN = v \,.\, \sin Z'$, so that, as stated above, the inclination of the trunnion axis varies from zero to v as the instrument is rotated. As OA is perpendicular to OY, $ZN = v \,.\, \sin Z'$ is the inclination of the plane RST in the direction of the circumference at Y, and if Z' is increased by 180° this inclination will be the same but of changed sign.

The error in altitude of the point X is $XU - XY = (90^\circ - XZ) - (90^\circ - XZ') = XZ' - XZ = Z'N$ (nearly).

In the right-angled triangle ZNZ' we have (Fig. 3), $\cos Z' = \tan NZ' \,.\, \cot v$.

$\therefore \tan NZ' = \tan v \,.\, \cos Z'$, or, as NZ' and v are small quantities, error in altitude $= NZ' = v \,.\, \cos Z'$.

In practice, however, the use of the "altitude level" on the vernier arm of the vertical circle enables any error from this cause to be corrected, as will be explained later.

The Spirit Level. This is a sealed glass tube, generally circular in cross-section, with its interior surface very accurately ground longitudinally to a circular arc of large radius, and containing a bubble of air floating in spirit, the centre of the bubble always coming to rest at the highest point of the tube. The outer surface of the tube is marked in equal divisions, usually 0·10 in. or 2 mm. long, which should be numbered *outwards* from a zero at the centre of the tube. It is obvious that if the bubble is in its central position, and the tube is then tilted longitudinally, the bubble will move through a number of divisions proportional to the inclination of the tube. One division of the tube has, therefore, an "angular value," which usually varies from 10″ to 20″ per division of 0·10 in. The radius of the longitudinal curvature of the tube, therefore, varies from

$$\frac{0 \cdot 10}{10} \times 206265 = 2062 \cdot 65 \text{ in.} = 171 \cdot 9 \text{ ft.}$$

to one-half that amount or 85·95 ft. The maker determines the angular value by placing the bubble tube on a "level trier," which is a frame hinged at one end and elevated by a micrometer screw at the other, so that the change of inclination required to move the bubble through a certain number of divisions can be found very accurately. Of course, the position of the *centre* of the bubble cannot be read directly, but if l is the reading of the L.-H. end of the bubble and r is the reading of the R.-H. end, then $\frac{l - r}{2}$ is the reading for the centre of the bubble, provided, of course, that the bubble is large enough to overlap the zero of the graduations in its extreme position, as it should be. If $\frac{l - r}{2}$ is positive, the L.-H. end of the tube is the higher, and *vice-versa*. It is not enough to read one end of the bubble only, as the length of the bubble decreases with a rise of temperature, and conversely.

EXAMPLE 1. In order to determine the angular value of one division of the altitude level of a theodolite, readings are taken on a vertical levelling staff 50 ft. away, first with the bubble near its extreme L.-H. position, secondly with the bubble near its extreme R.-H. position. The vertical circle is clamped and the position of the bubble was altered by using the plate screws. The following were the readings—

Altitude Level		Staff Reading
L.H.	R.H.	Feet
14·6	2·8	4·457
3·2	14·2	4·496

Find the angular value of a division.

The movement of the centre of the bubble is from $\frac{14{\cdot}6-2{\cdot}8}{2}$ to $\frac{3{\cdot}2-14{\cdot}2}{2}$, i.e. from $+5{\cdot}9$ to $-5{\cdot}5$, a total of 11·4 divisions. The change of inclination is

$$\frac{4{\cdot}496-4{\cdot}457}{50}\times 206265'' = 160{\cdot}9''$$

$$\therefore \text{ angular value of one division } = \frac{160{\cdot}9''}{11{\cdot}4} = 14{\cdot}1''$$

The "axis" of the level tube is the tangent to the curve of the tube at its zero (central) division.

The glass tube is, of course, encased in metal and if the spirit level is a separate detachable instrument, e.g. the "striding level" used for determining the inclination of the trunnion axis (Fig. 4), the plane containing the points on which the instrument rests may be called the "base" of the instrument. The base of the spirit level should be parallel to its axis, so that when the bubble is central the base is horizontal; if not, the "error" of the level is the inclination of the axis to the base or the inclination of the base when the bubble is central.

By "reversing" the level, i.e. by turning it through 180°, end for end, we can find not only the error (e) of the level but also the inclination (i) of the surface on which it rests. Let l_1, r_1 be the readings of the L.-H. and R.-H. ends of the

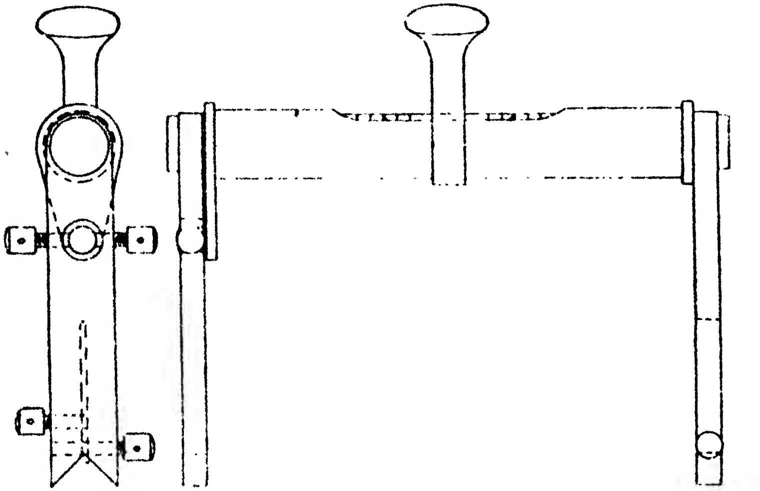


FIG. 4

bubble in the first or "direct" position (Fig. 5); l_2, r_2 the L.-H and R.-H. readings in the second or "reversed" position,

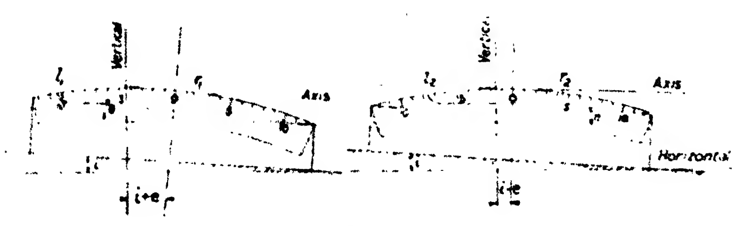


FIG. 5

L.-H. and R.-H. being reckoned from the observer's position, assumed unchanged. Let θ be the angular value of a division.

Then we have $\frac{l_1 - r_1}{2}\theta = i + e$; $\frac{l_2 - r_2}{2}\theta = i - e$.

$$\text{And } i = \frac{1}{2}\left(\frac{l_1 - r_1}{2} + \frac{l_2 - r_2}{2}\right)\theta = \frac{\Sigma(l) - \Sigma(r)}{4}\,\theta$$

= inclination of the base of the level.

$$e = \frac{1}{2}\left(\frac{l_1 - r_1}{2} - \frac{l_2 - r_2}{2}\right)\theta = \text{error of the level.}$$

This latter is zero if the readings are unchanged; if the readings are reversed the base surface is level. We can thus eliminate the error of a spirit level by reversing it and averaging the readings.

EXAMPLE 2. The following readings were taken on a reference mark and on a star

Object	Striding Level on Trunnion Axis				Altitude	Horizontal Circle		Average
	Direct L.	Direct R.	Reversed L.	Reversed R.		Vernier *A*	Vernier *B*	
R.M.	11·2	3·4	9·8	4·8	3° 14′ 00″	66° 14′ 10″	246° 14′ 20″	66° 14′ 15″
•	10·9	3·7	9·5	5·1	60° 46′ 20″	112° 31′ 50″	292° 32′ 00″	112° 31′ 55″

The value of a bubble division is 12″. The altitude has been corrected for the altitude level. Find the corrected horizontal angle between the reference mark and the star, and the error of the striding level.

The inclination of the trunnion axis when the reference mark is sighted is

$$\frac{11{\cdot}2 + 9{\cdot}8 - 3{\cdot}4 - 4{\cdot}8}{4} \times 12'' = 38{\cdot}4''$$

The correction of the horizontal circle is

$$i \tan a = 38{\cdot}4'' \tan 3^\circ\ 14' = 2''.$$

The left-hand end of the trunnion axis being high, the correction is positive, therefore corrected reading = 66° 14′ 17″.

The inclination of the trunnion axis when the star is sighted is

$$\frac{10{\cdot}9 + 9{\cdot}5 - 3{\cdot}7 - 5{\cdot}1}{4} \times 12'' = 34{\cdot}8''$$

The correction of the horizontal circle is 34·8″ tan 60° 46′ 20″ = 62″. This correction is also positive and the corrected horizontal circle reading is 112° 32′ 57″.

The horizontal angle from reference mark to star is then 112° 32′ 57″ − 66° 14′ 17″ = 46° 18′ 40″.

The error of the level is

$$\frac{1}{2}\left(\frac{11{\cdot}2 - 3{\cdot}4}{2} - \frac{9{\cdot}8 - 4{\cdot}8}{2}\right) = 0{\cdot}7 \text{ divisions} = 8{\cdot}4''$$

$$\left(\text{or } \frac{1}{2}\left(\frac{10{\cdot}9 - 3{\cdot}7}{2} - \frac{9{\cdot}5 - 5{\cdot}1}{2}\right) = 0{\cdot}7 \text{ divisions} = 8{\cdot}4''\right.$$

from the second set of readings).

Spirit Level on the Upper Horizontal Plate, parallel to the trunnion axis. The object of this level is, of course, to enable the vertical axis to be made vertical when setting up the instrument. As it is connected rigidly to the vertical axis it may be considered as attached to a plane perpendicular to the vertical axis and as giving the inclination of this imaginary plane in various circumferential directions. The "error" of the level is then the inclination of its axis to this plane. It has already been shown that at points on this plane diametrically opposite the circumferential inclinations are equal but reversed in sign: the effect, therefore, on the spirit level of turning the level through 180° is exactly the same as the reversing of a detachable level. The object, however, is not now to measure the inclination of the vertical axis but to render it vertical, and the procedure is as follows: Turn the upper plate until the spirit level is approximately parallel to two of the three levelling screws (or to two opposite screws if there are four of them). Bring the bubble central by raising one screw and lowering the other, so that $l_1 = r_1$. The inclination of the perpendicular plane in this direction has now been made equal and opposite to the error of the level (Fig. 6*a*). Now turn the upper plate through 180°; the inclination of the level axis is now *twice* the error of the level (Fig. 6*b*). Bring the bubble half-way back to its central position by the adjusting screws provided; the error of the spirit level is now corrected (Fig. 6*c*) and the bubble is then made central by the two levelling

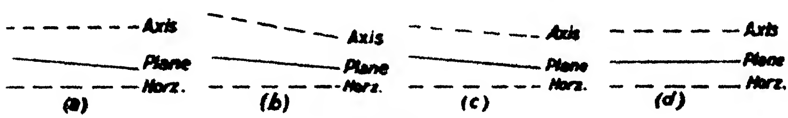


Fig. 6

screws which makes the perpendicular plane horizontal in the direction of the spirit level (Fig. 6*d*). Now turn the upper plate through 90°, so that the spirit level is perpendicular to its former direction, and turn the third levelling screw (or the two remaining screws in a four-screw instrument) until the bubble is central. The imaginary perpendicular plane is now horizontal and, therefore, the vertical axis is vertical, but, in practice, the whole operation is one of "trial and error" and has to be repeated several times. It is impossible to set the spirit level exactly parallel to the plate screws at the beginning, and it is not worth the trouble to rotate the plate through exactly 180° and 90° by reading the verniers. As a preliminary the bubble should be brought to the centre in two directions at right angles in order to make the axis approximately vertical before any reversing is effected. If the altitude level is more sensitive than the level on the upper plate, the process may be repeated with this level for a finer adjustment of the vertical axis, the clip screw being employed for correcting half the bubble movement on reversal.

The Altitude Level on the Vernier Arm of the Vertical Circle. The object of this level is to provide a horizontal datum from which altitudes may be measured even if the vertical axis is not truly vertical. By means of the clip screw the bubble can be brought central, the zero line of the two verniers being rotated through the same angle. When the clamp and tangent screw of the vertical circle are on the same side of the telescope as the clip screw, turning the clip screw alters the pointing of the telescope but leaves the readings of the vertical circle unaltered, as it rotates the vertical circle and the verniers together. In more modern practice the clamp and tangent screw are placed on the opposite side of the telescope to the vertical circle and clip screw; turning the clip screw now alters the reading of the vertical circle but does not alter the pointing of the telescope.

In Fig. 7*a*, let *SS* be the line of collimation inclined at an angle α to the horizontal *HH*, *OO* the line of zeros of the vertical circle showing a small vertical collimation error δ, and *VV* the line of the vernier zeros inclined at an angle ϵ to *HH* ***when the bubble of the altitude level AB has been brought central by the clip screw.*** Let β_1 be the reading of the vertical circle, then $\alpha = \beta_1 + \delta + \epsilon$. In Fig. 7*b* the face of the theodolite has been changed, the line of sight is directed to the same object, and the observer views the vertical circle from the opposite side of the theodolite. ***The bubble of the altitude level AB has again been brought central by the clip screw,*** so that *VV* has the same inclination ϵ to the horizontal as before. If β_2 is now the reading of the vertical circle,

$\alpha = \beta_2 - \delta - \varepsilon$. Averaging the two values of α we have $\alpha = \frac{\beta_1 + \beta_2}{2}$, i.e. *if the bubble of the altitude level is centred by the clip screw for each reading, the average of the vertical circle readings with changed face is the correct altitude.* $\delta + \varepsilon$ is called the "*Index Error of the Vertical Circle,*" and is thus eliminated.

Now suppose by turning the clip screw in Fig. 7*a* the

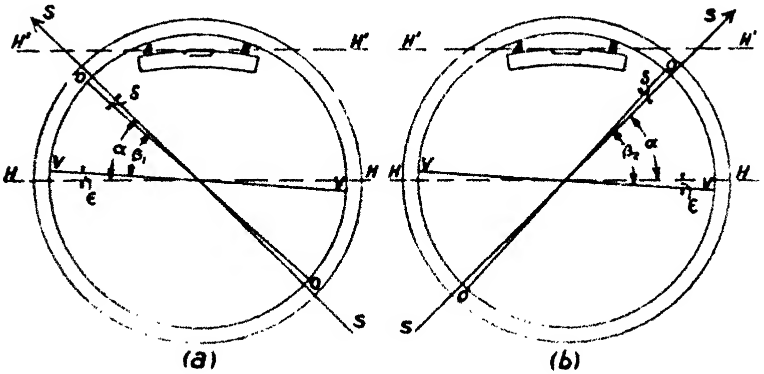


FIG. 7

"object" end of the level is raised so that the centre of the bubble reads $\frac{o_1 - e_1}{2}$, where $\left\{\begin{matrix} o \\ e \end{matrix}\right.$ is the reading of the $\left\{\begin{matrix} \text{object} \\ \text{eye} \end{matrix}\right.$ end of the bubble, the angle ε between VV and HH will be increased by $\frac{o_1 - e_1}{2}\theta$, where θ = angular value of a division, and if the line of sight SS is directed to the same object the vertical circle reading will be decreased by the same amount. Let the vertical circle reading now be γ_1, then

$$\alpha = \gamma_1 + \delta + \varepsilon + \frac{o_1 - e_1}{2}\theta$$

Similarly in Fig. 7*b*, if the "object" end of the level is raised by turning the clip screw so that the centre of the bubble reads $\frac{o_2 - e_2}{2}$, the angle ε between VV and HH will be decreased by $\frac{o_2 - e_2}{2}\theta$, and if the line of sight SS is still directed to the same object the vertical circle reading will be decreased by the same amount. If the reading of the vertical circle is now γ_2, then

$$\alpha = \gamma_2 - \delta - \left(\varepsilon - \frac{o_2 - e_2}{2}\theta\right) = \gamma_2 - \delta - \varepsilon + \frac{o_2 - e_2}{2}\theta$$

Averaging the two values of α we have

$$\alpha = \frac{\gamma_1 + \gamma_2}{2} + \left(\frac{o_1 - e_1}{2} + \frac{o_2 - e_2}{2}\right)\frac{\theta}{2}$$

$$= \frac{\gamma_1 + \gamma_2}{2} + \frac{\Sigma o - \Sigma e}{4}\theta$$

the Index Error, $\delta + \varepsilon$, of the vertical circle being again eliminated.

Therefore, if the bubble of the altitude level is not brought central for each reading, the altitude is the average of the vertical circle readings with changed face + average reading of centre of bubble × value of a division. Turning the clip screw, therefore, merely transfers part of the measurement of the altitude from the vertical circle to the altitude level. It is advisable, however, to keep the bubble of the altitude level near the centre of its run, as the value of a level division is rather liable to be affected by change of temperature.

EXAMPLE 3. The readings shown in the table on page 118 are taken to determine the elevation of an object.

The value of a level division is 15″. Find the elevation of the object.

Face	Altitude Level		Vertical Circle		Average
	O	E	Vernier C	Vernier D	
R	8·6	7·4	193° 14′ 25″	13° 14′ 30″	13° 14′ 27·5″
L	7·8	8·2	346° 45′ 25″	166° 45′ 30″	13° 14′ 32·5″

The average vertical angle from the vertical circle is 13° 14′ 30″.

The level correction is

$$\frac{8{\cdot}6 + 7{\cdot}8 - (7{\cdot}4 + 8{\cdot}2)}{4} \times 15'' = +3''$$

∴ the *elevation is* 13° 14′ 33″

In precise work, therefore, we must not assume that the vertical axis is vertical, no matter how carefully the instrument was set up, and, in addition to changing face and averaging the angles read on the horizontal and vertical circles, we must correct each reading of the horizontal circle by adding $i \tan a$, where $i = \dfrac{\Sigma l - \Sigma r}{4} \times$ angular value of a division, l and r being the readings of the striding level on the trunnion axis taken direct and reversed for each reading, before subtracting such readings to obtain the horizontal angle, and we must correct the average angle of elevation read on the vertical circle by adding $\dfrac{\Sigma o - \Sigma e}{4} \times$ angular value of a division, where o and e are the object and eye-end readings of the altitude level. If the angle is a *depression* we must *subtract* $\dfrac{\Sigma o - \Sigma e}{4} \times$ value of a division.

But although the instrumental errors can be thus eliminated, it is advisable to keep them as small as possible by adjusting them from time to time.

To Adjust the Line of Collimation Perpendicular to the Trunnion Axis (Fig. 8). Choose a piece of level ground about 600 ft. long. Set up the theodolite midway at B and sight on a fine mark at A, say 300 ft. away. Transit the telescope and make a fine mark C_1 on the line of sight at about 300 ft. on the other side of B to A, then transit back on to A to ensure that there is no slackness in

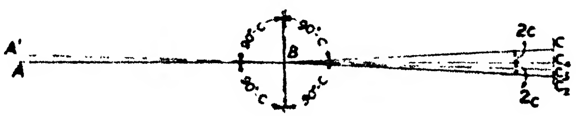


FIG. 8

the instrument. Turn the upper plate through 180°, and transit the telescope so as to change face. Again sight on A, transit the telescope, and if C_1 is not now on the line of sight, make a second fine mark C_2, opposite to C_1, then again check back on to A. From Fig. 8 it is obvious that if the line of sight makes an angle of $90° - c$ with the trunnion axis, the angle C_1BC_2 is $4c$. To correct the adjustment mark a point C_3 between C_2 and C_1 so that $C_2C_3 = \frac{C_2C_1}{4}$ and after sighting on C_2 with the same face as before, move the diaphragm by its horizontal adjusting screws until C_3 is on the line of sight. To check the result mark a fourth point C_4, bisecting C_1C_2, bring the line of sight on to C_4 by the horizontal tangent screw, transit the telescope, when A should be on the line of sight. The amount of the error $C = \frac{C_1C_2}{4BC_4} \times 206265''$. This adjustment should also be tested at a much shorter distance, say 30 ft. for BC_4, to test for "draw tube error." The ground should be level, or at least of uniform slope, so that non-perpendicularity of the trunnion axis to the vertical axis may not affect the result. This will be the case if A, B, and C are in a straight line, vertically as well as horizontally, the two planes generated by the line of collimation, if slightly inclined to the vertical on opposite sides of it intersecting in the line ABC.

FIG. 9

To Adjust the Trunnion Axis Perpendicular to the Vertical Axis (Fig. 9). Set up the theodolite at, say, 30 ft. from a vertical wall, sight on a fine mark A on the wall immediately opposite and at an elevation of about 45°, depress the telescope and make a fine mark B_1 on the wall, level with the centre of the instrument. Check back on to A. Change face, again sight on A, depress the telescope and, if B_1 is not on the line of sight, make a second mark, B_2, at the same level as B_1. Bisect $B_1 B_2$ at B_3. (AB_3 is a vertical line if the vertical axis is vertical.) If the inclination of the trunnion axis to the vertical axis is $90° - i$, the angles B_2AB_3, B_1AB_3 each $= i$, and when sighting B_2 it is evident that the R.-H. end of the trunnion axis is low. Now sight on B_3 and by means of the adjustment under one end of the trunnion axis raise the R.-H. (or lower the L.-H.) end until the line of sight when the telescope is raised passes through both B_3 and A. The amount of the error is $\dfrac{B_1B_2}{2.AB_3} \times 206265''$.

If the collimation error had not been first adjusted, the traces of the lines of sight on the wall would have been hyperbolas, as shown in broken lines in Fig. 9, and it would have been impossible to disentangle the two maladjustments.

If a striding level is available, the adjustment can be performed more readily by setting up the theodolite with ***truly vertical*** axis and then applying the striding level to the trunnion axis—direct and reversed. If $\frac{l_2 - r_2}{2} = -\frac{l_1 - r_1}{2}$ the trunnion axis is level; if not, alter the trunnion axis

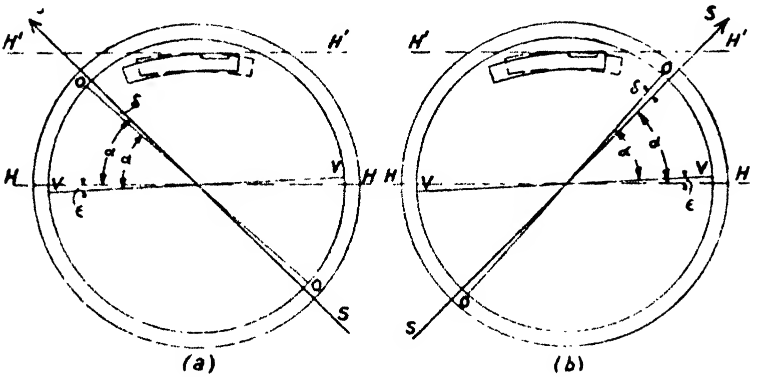


FIG. 10

adjustment until the bubble takes up positions equally to left and right of the centre.

To Adjust the Index Error of the Vertical Circle (Fig. 10). Bring the altitude bubble carefully central and sight on a fine mark at some distance away, which is distinctly above (or below) the instrument, read the two vertical verniers and average them to get the angle of elevation (or depression). Change face, again bring the altitude bubble carefully central, sight on the same mark again, read the two vertical verniers and average them for the angle of elevation (or depression). Average the two values of the elevation thus obtained. (*N.B.* The difference of the two elevations will be 2 × index-error.) Alter the readings of the vertical circle to read this average elevation by the tangent screw in the old form, by the clip screw in the new form. In the former it will now be necessary to re-sight on the mark, using the clip screw for the purpose. In both cases the effect is to make $\varepsilon = -\delta$, so that $\beta_1 = \beta_2 = a$. Then bring the bubble central by its own adjusting screws, which alter the inclination of the bubble axis to VV, as it will be out of centre owing to turning the clip screw.

Internal Focusing. Modern theodolites and levels have their telescopes constructed with the diaphragm at a fixed distance from the object glass, the image of the object being focused on to the cross-hairs by the movement of a double-concave lens between the object glass and its principal focus. There are great advantages in this: (1) Damp and dust are not admitted to the telescope by the movement of a draw-tube carrying the diaphragm or the object glass. (2) The balance of the telescope is practically unaffected by alteration of focus. (3) A difference of focus is much less likely to cause a variation in the line of collimation, partly because the internal lens, being near the centre of the telescope, can be made to travel more rigidly in a line through the optical centre of the object glass, and partly because any displacement of the internal lens from such line produces a much less angular error than would an equal displacement of the object glass or diaphragm with a simple draw-tube, as will be illustrated in Example 5. (4) The telescope can be made very nearly anallatic for use in Tacheometry, as will be illustrated in Examples 4 and 5.*

EXAMPLE 4. The fixed distance of the diaphragm from the optical centre of the object glass of a theodolite is 7 in., the focal length of the object glass is 6 in., and that of the internal focusing lens is 12 in. Find (*a*) the position of the internal lens for an infinite distance; (*b*) the interval between the stadia marks for a multiplying constant of 100 at a distance of 500 ft.; and (*c*) the errors that will be caused with this constant at distances of 100 ft. and 50 ft.

(*a*) *For Infinite Distance.* The rays would be focused by the object glass at 6 in. from its centre, i.e. at 1 in. from the

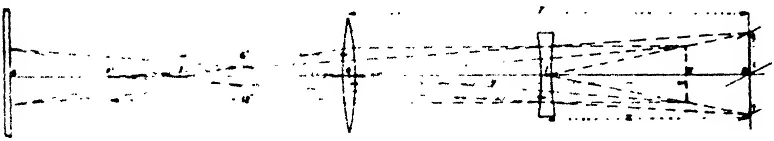


FIG. 11

diaphragm. If x is the distance of the optical centre of the internal lens from the diaphragm, we have

$$\frac{1}{x} - \frac{1}{x-1} = -\frac{1}{12}$$

$\therefore\ 12(x-1) - 12x = -x^2 + x. \quad \therefore\ x^2 - x - 12 = 0$

$\therefore\ x = 4$ in.

(*b*) *For* 500 *ft.* = 6,000 in. from centre of instrument, or 5996·5 in. from centre of object glass. (Fig. 11.)

Let y be the distance of the virtual image V from the object glass and i the interval between the stadia marks.

Then $\dfrac{1}{5996{\cdot}5} + \dfrac{1}{y} = \dfrac{1}{6}. \quad \therefore\ \dfrac{1}{y} = \dfrac{5990{\cdot}5}{35979}$

$\therefore\ y = 6{\cdot}0060$ in. $\therefore\ VD = 0{\cdot}9940$ in.

Also $\dfrac{1}{x} - \dfrac{1}{x - {\cdot}9940} = -\dfrac{1}{12}. \quad \therefore\ x^2 - {\cdot}9940x - 11{\cdot}928 = 0$

$$\therefore\ x = \frac{{\cdot}9940 \pm \sqrt{{\cdot}9880 + 47{\cdot}712}}{2} = 3{\cdot}9863 \text{ in.}$$

$\therefore\ IV = 3{\cdot}9863 - {\cdot}9940 = 2{\cdot}9923$ in.

As the distance is 6,000 in. and the constant is 100, the intercept on the staff is to be 60 in.

$$\therefore\ i = 60 \times \frac{6{\cdot}0060}{5996{\cdot}5} \times \frac{3{\cdot}9863}{2{\cdot}9923} = {\cdot}08006 \text{ in.}$$

(c) *For* 100 *ft.* = 1,200 in. from centre of instrument:

$$\frac{1}{1196{\cdot}5} + \frac{1}{y} = \frac{1}{6} \quad \therefore\ y = \frac{7179}{1190{\cdot}5} = 6{\cdot}0302 \text{ in.}$$

$$\therefore\ VD = {\cdot}9698 \text{ in.} \quad \therefore\ \frac{1}{x} - \frac{1}{x - {\cdot}9698} = -\frac{1}{12}$$

$$\therefore\ x^2 - {\cdot}9698 - 11{\cdot}6376 = 0$$

$$\therefore\ x = \frac{{\cdot}9698 \pm \sqrt{{\cdot}9405 + 46{\cdot}5504}}{2} = 3{\cdot}9306 \text{ in.}$$

$$\therefore\ IV = 3{\cdot}9306 - {\cdot}9698 = 2{\cdot}9608 \text{ in.}$$

$$\therefore\ \text{Staff reading} = {\cdot}08006 \times \frac{2{\cdot}9608}{3{\cdot}9306} \times \frac{1196{\cdot}5}{6{\cdot}0302}$$

$$= 11{\cdot}966 \text{ in.} = {\cdot}997 \text{ ft.}$$

The reading should, of course, be 12·00 in. The error is, therefore, 1200 – 1196·6 = 3·4 in. or 0·28 ft.

For 50 *ft.* = 600 in. from centre of instrument:

$$\frac{1}{596{\cdot}5} + \frac{1}{y} = \frac{1}{6} \quad \therefore\ y = \frac{3579}{590{\cdot}5} = 6{\cdot}0610 \text{ in.}$$

$$\therefore\ VD = {\cdot}9390 \text{ in.} \quad \therefore\ \frac{1}{x} - \frac{1}{x - {\cdot}9390} = -\frac{1}{12}$$

$$\therefore\ x^2 - {\cdot}9390x - 11{\cdot}2680 = 0$$

$$\therefore\ x = \frac{{\cdot}9390 \pm \sqrt{{\cdot}8817 + 45{\cdot}0720}}{2} = 3{\cdot}8590 \text{ in.}$$

$$\therefore\ IV = 3{\cdot}8590 - {\cdot}9390 = 2{\cdot}9200 \text{ in.}$$

$$\therefore\ \text{Staff reading} = {\cdot}08006 \times \frac{2{\cdot}9200}{3{\cdot}8590} \times \frac{596{\cdot}5}{6{\cdot}0610}$$

$$= 5{\cdot}962 \text{ in.} = 0{\cdot}497 \text{ ft.}$$

The reading should, of course, be 6·000 in. Therefore, the error is 600 – 596·2 = 3·8 in. = 0·32 ft.

Therefore, over a range of distances from 50 ft. to 500 ft. the distances will be correct to the nearest foot.

[*N.B.* If, on the other hand, the stadia interval were reduced to give the distance of the staff *from the object glass*, the approximation would be still closer. For 500 ft. or 5996·5 in. from the object glass, the stadia interval should be

$$\frac{59\cdot965}{60} \times \cdot08006 = \cdot08001 \text{ in.}$$

For 100 ft. the reading would be

$$\frac{\cdot08001}{\cdot08006} \times 11\cdot966 = 11\cdot958 \text{ in.}$$

Error 1196·5 – 1195·8 = 0·7 in. = ·06 ft.

For 50 ft. the reading would be

$$\frac{\cdot08001}{\cdot08006} \times 5\cdot962 = 5\cdot958 \text{ in.}$$

Error 596·5 – 595·8 = 0·7 in. = ·06 ft.

However, in this case we should have to *add* 3·5 in. = 0·29 ft. for *all distances*.]

Example 5. If, in the above instrument, the internal lens has its optical centre correctly in line between the intersection of the cross-hairs and the optical centre of the object glass when focused at 500 ft., but its centre deviates 0·001 in. laterally from that line when focused at 50 ft., find the angular error introduced in measuring the horizontal angle between two points at these distances respectively from the instrument. (Fig. 12.)

When sighting the farther point *A*, the line of sight would be *DOA*, but when sighting the nearer point *B*, its image would have to be at *U* in the line *ID* at a distance

0·939 in. from the diaphragm, so that the line of sight would be UOB, where

$$VU = \frac{0{\cdot}939}{3{\cdot}859} \times {\cdot}001 \text{ in.} = {\cdot}000243 \text{ in.}$$

The angular error caused by this change in the line of collimation would be

$$UOV = \frac{{\cdot}000243}{6{\cdot}0610} \times 206265'' = 8{\cdot}3''$$

If B lay to the right of A, the horizontal angle read would be too small by this amount. Changing face would eliminate

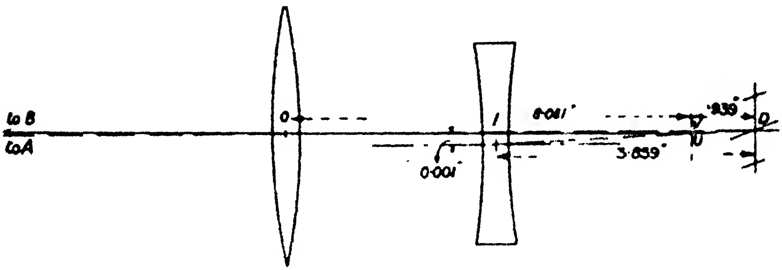


Fig. 12

this error, of course, on the average reading, as the line of sight would be deviated equally in the opposite direction, provided the displacement of the internal lens remained unaltered and was not due to mere looseness. If the object glass, or diaphragm, moved in a *draw-tube*, as in the older instruments, and were ·001 in. out of its former line at the nearer distance, the angular error would have been

$$\frac{{\cdot}001}{6{\cdot}061} \times 206265'' = 34{\cdot}0''$$

or more than four times greater.

Estimating Microscopes. Instead of verniers some theodolites are fitted with microscopes having a glass diaphragm at the common focus of the objective and eye-piece on which is engraved a scale of ten divisions, the length of the scale agreeing with that of the image of the smallest division of the graduations on the plate and its zero forming the index mark. Fig. 13 illustrates a reading by this method. The observer estimates that the reading of the 97° 40′ mark on the small scale is 3·3 divisions. The full reading is, therefore,

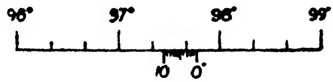


FIG. 13

$$97^\circ\,40' + \frac{3{\cdot}3}{10} \times 20' = 97^\circ\,46{\cdot}6' = 97^\circ\,46'\,40''$$

to the nearest 10″. If he estimates 3·5 divisions on the opposite microscope, the average reading is

$$97^\circ\,40' + \frac{3{\cdot}3 + 3{\cdot}5}{2} \times \frac{20}{10} = 97^\circ\,46{\cdot}8' = 97^\circ\,46'\,50''$$

to the nearest 10″, the two scale readings being simply added together to give the addition in minutes to the last division on the plate.

Micrometer Theodolites. Much greater accuracy of reading is obtainable when the horizontal and vertical circles are read by micrometers instead of verniers. Each micrometer consists of a microscope with two cross-hairs, close together, at the common focus of the eye-piece and objective, which can be moved laterally by a micrometer screw with a graduated drum attached.

For a 5-in. theodolite the circles are subdivided to 10′ and the pitch of the screw is such that one complete turn of the screw moves the cross-hairs across the image of the 10′ graduation. The drum is divided into ten divisions, numbered as single minutes, and each division is subdivided into six parts, each representing 10″. The index mark for the reading is the central position of the cross-hair when the drum reads 0, and this position is marked approximately by a V-notch on a plate just in front of the hairs. (Fig. 14.)

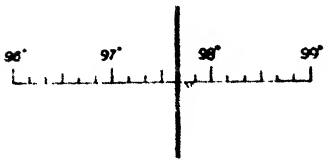


Fig. 14

To take a reading the micrometer is turned till the cross-hairs are so placed that the next lower division on the scale lies equally between the cross-hairs, the drum is then read at its index mark. If this latter reading is 6′ 40″, the total reading in Fig. 14 is 97° 46′ 40″, as the notch shows that it lies between 97° 40′ and 97° 50′. It is advisable also to bring the cross-hairs over the next higher division as a precaution against back-lash. If the readings differ slightly the mean should be taken. If the two readings differ appreciably the distance between the cross-hairs and the object glass will need adjustment.

It is important to ensure that there is no parallax, so the cross-hairs should first be focused, by moving the eye-piece, on a piece of white paper placed on the scale until they appear as clear as possible. The piece of paper is then removed and the scale graduations are carefully focused by moving the whole microscope towards or away from the scale and clamping it in the position in which the graduations are clearest, and in which the cross-hairs appear fixed when the eye is moved.

Parallel Plate Micrometer (Fig. 15). If a ray of light falls normally on a plate of glass with truly parallel surfaces it passes through it unchanged in direction. If the plate is rotated the emergent ray is parallel to the entering ray, but shifted by an amount (x) depending on the angle of rotation (θ). If the angle is small, the shift is proportional to the

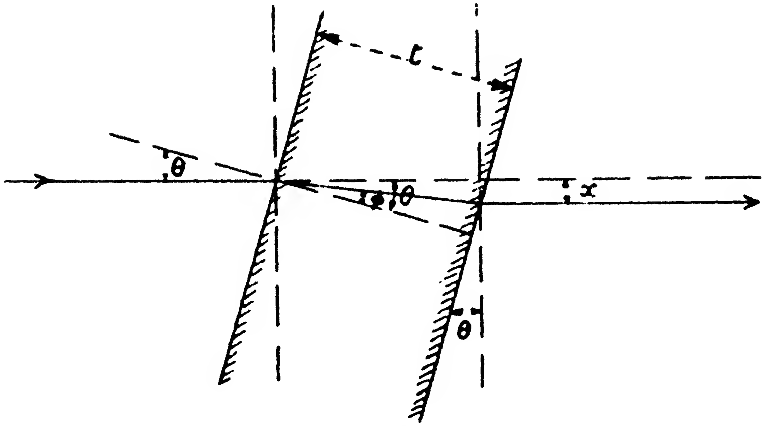


FIG. 15

angle. In the figure we have $\frac{\sin \theta}{\sin \phi} = \mu =$ index of refraction from air to glass. Then

$$x = t(\tan \theta - \tan \phi) \cos \theta = t \sin \theta \left(1 - \frac{\cos \theta}{\sin \theta} \cdot \frac{\sin \phi}{\sqrt{1 - \sin^2 \phi}} \right)$$

$$= t \sin \theta \left(1 - \frac{\cos \theta}{\sin \theta} \cdot \frac{\frac{\sin \theta}{\mu}}{\sqrt{1 - \frac{\sin^2 \theta}{\mu^2}}} \right)$$

$$= t \sin \theta \left(1 - \frac{\cos \theta}{\sqrt{\mu^2 - \sin^2 \theta}} \right)$$

$$= t \sin \theta \left(1 - \frac{\sqrt{1 - \sin^2 \theta}}{\sqrt{\mu^2 - \sin^2 \theta}} \right)$$

If θ be small, $\sin \theta =$ arc θ, and $\sin^2 \theta$ is negligible, and $x = t\theta \left(1 - \frac{1}{\mu} \right)$, so that, approximately, x is proportional to θ and can be measured by a micrometer drum on the axis of rotation. The following table shows the closeness of the approximation, taking t as 1, μ as 1·6.

θ	x	$\theta\left(1-\frac{1}{\mu}\right)$
1°	0·00655	0·00655
2°	0·01310	0·01309
3°	0·01965	0·01963
4°	0·02622	0·02618
5°	0·03281	0·03272
10°	0·06613	0·06545
15°	0·10048	0·09818
20°	0·13640	0·13090
25°	0·17441	0·16363
30°	0·21510	0·19635

When the ratio $\frac{x}{t}$ is so large that the degree of approximation is insufficient for the required accuracy, the divisions on the drum must be placed closer together, to correspond with the exact values of θ for the required values of x. The range for any required accuracy *with equal graduations* can be doubled by rotating the drum through small angles on both sides of the normal, e.g. if θ varies from $+3°$ to $-3°$ the error is less than $\frac{1}{1000}$th part.

Double Reading Theodolites. This is the most modern development, viz. by means of a system of prisms, to bring the images of the opposite parts of the horizontal circle, or of the vertical circle, into the field of view of the same microscope. This saves time in reading and possible disturbance of the theodolite by walking round it. This may be represented diagrammatically in Fig. 16*a*. As the images of the opposite readings are read at the same index mark, the average reading is $70° + \frac{a+b}{2} = 70° + \frac{x}{2}$ where x is the distance between the corresponding degree divisions (250° and 70°) which have not passed each other.

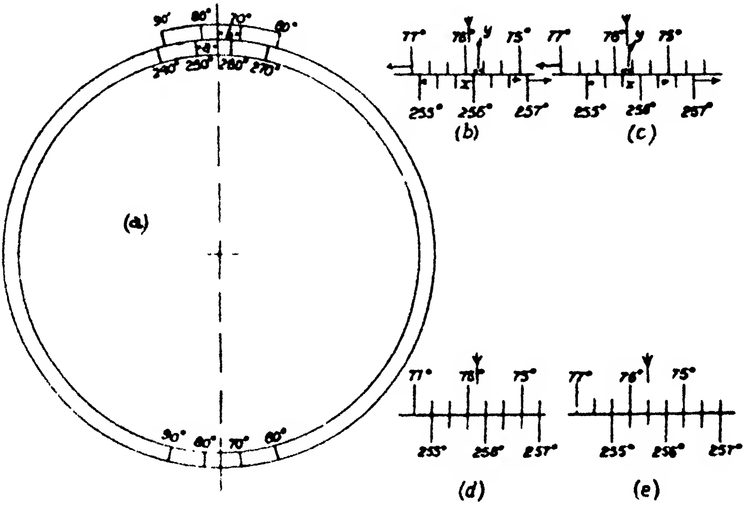


FIG. 16

Similarly, in the enlarged views, Fig. 16*b* and Fig. 16*c*, the average readings are $75° + \frac{x}{2}$. In Fig. 16*b*, counting the 20′ divisions and estimating the fraction of a division, we find $x = 5{\cdot}6$ divisions, therefore the average reading is

$$75° + \frac{5{\cdot}6 \times 20'}{2} = 75° \ 56'$$

while in Fig. 16*c* we find $x = 4{\cdot}6$ divisions, therefore the average reading is $75° + \frac{4{\cdot}6 \times 20'}{2} = 75° \ 46'$. The index line is shown thus ↓

To read to greater accuracy, Estimating Microscopes may be used with a *double* scale, thus: 10—0—10, the scale readings to the last divisions, upper and lower, are thus estimated and added together for the average reading.

Messrs. C. F. Casella & Co. employ a micrometer which reads zero when its cross-hairs are over the index mark. With the micrometer, the distances (a and b) from the index mark to the next lower division on both scales would be read successively, the drum having complementary numbering to its 1′ divisions, thus: $\begin{matrix} 7 & 6 & 5 & 4 \\ 3 & 4 & 5 & 6 \end{matrix}$ for the two readings in *opposite* directions, e.g. when brought over the 75° 40′ division the micrometer might read 6′ 10″ and when brought over the 255° 40′ division it might read 6′ 30″. The average reading would then be 75° 46′ 20″. In this case the circle is divided to 10′ and the micrometer head into ten divisions of 1′ each, each subdivided into six divisions of 10″ each. The micrometer drum is shown in the same microscope as the horizontal circle reading; an adjacent microscope shows the vertical circle.

In the Wild or Zeiss Universal Theodolite the two images of the circle are made to *coincide* by moving them equal amounts in opposite directions. This is effected by passing the rays from the opposite sides of the circle through two parallel plate micrometers which rotate equal amounts in opposite directions. The motion of each is read on a micrometer drum in the same microscope, the drum being graduated from 0′ to 10′ in seconds. Fig. 16*d* and Fig. 16*e* show the circle images of Fig. 16*b* and Fig. 16*c* when brought into coincidence, the total motion in each case being y in Fig. 16*b* and Fig. 16*c*. In (*b*) the distance between the last corresponding divisions (75° 40′ and 255° 40′) which have not passed is $20' + y$. The average reading is, therefore, $75° 40' + \frac{20' + y}{2} = 75° 50' + \frac{y}{2}$, where $\frac{y}{2}$ is read on the micrometer drum (say, 6′ 13″), and when brought into coincidence the index mark appears half-way between 75° 40′ and 76° 00′, thus indicating that we must add the micrometer reading to 75° 50′, so that the total reading is 75° 56′ 13″. In (*c*) the distance between the last corresponding divisions (75° 40′ and 255° 40′) which have not passed is y. The average reading is, therefore, $75° 40' + \frac{y}{2}$, where $\frac{y}{2}$ is the micrometer reading (say, 6′ 13″). The total reading is, therefore, 75° 46′ 13″, the index in this case appearing over 75° 40′ when coincidence is obtained. The microscope is placed with its eye-piece alongside that of the telescope. By interposing a prism, the vertical circle readings are read in the same microscope. The horizontal and vertical circles are graduated on glass cylinders, which allows smaller circles, finer graduation marks, and much higher magnification to be employed.

The "Tavistock" Theodolite. In the "Tavistock" Theodolite of Messrs. Cooke, Troughton & Simms, Ltd., a different form of optical micrometer is adopted for double-reading to single seconds. The images of the graduations on the opposite sides of the circle are brought together by reflection through prisms, but the line of separation of the images is now *parallel* to the graduations and the graduations increase in the *same* direction. By the use of travelling prisms, as represented diagrammatically in Fig. 17, the two images can be moved equal amounts (x) in the *same* direction which are proportional to the equal movements (y) of the travelling prisms. Actually, the travelling prisms are mounted side by side on a frame so that they move in the same direction and the emergent rays are each reflected through 90° towards each other before meeting the fixed reflecting surfaces shown. The frame is moved by turning a head outside the instrument and, by means of a rack and pinion, the movement of the frame rotates a glass fine-reading circle, graduated in seconds from 0′ 0″ to 20′ 0″, in the same plane as the images of the circle graduations.

There are two microscopes, one for horizontal, and the

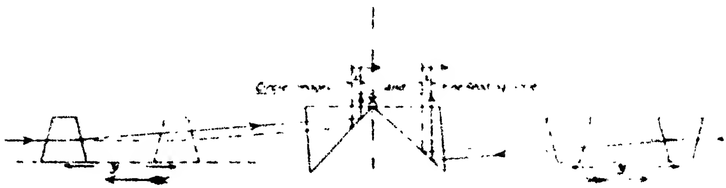


FIG. 17

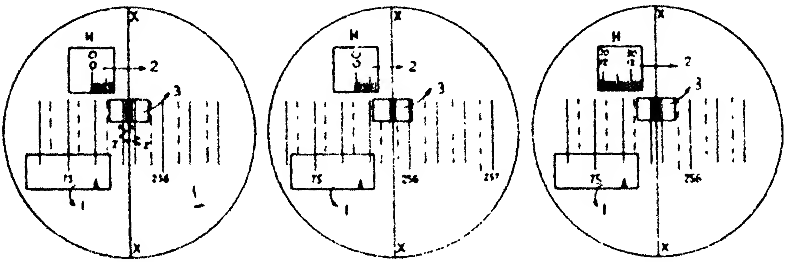
other for vertical, circle readings, one on each side of the telescope eye-piece, and these can be turned over when the telescope is transitted. Each of these shows a screen with three openings (Fig. 18), viz. (1) the coarse reading to the last degree and 20′, (2) the fine reading in minutes and seconds to be added thereto, and (3) the main index mark bisecting the interval between two corresponding opposite graduations.

In Fig. 18*a*, the index in the fine-reading opening (2) reads 0′ 0″, and the main index mark, which is at the junction *X* of the prisms in Fig. 17, appears *midway* between the graduations 75° 40′ and 255° 40′ in the central opening (3), so that the average reading would be

$$\frac{75° \; 40' + z + 75° \; 40' - z}{2} = 75° \; 40'$$

but actually the index in the coarse reading opening (1) reads 75° 20′, so the full reading is 75° 20′ 00″. If the theodolite were now turned through 20′ 0″ to the left, *or* if the fine reading were increased to 20′ 0″ by turning the head controlling it, the next lower pair of graduations would "straddle" the main index, and the index in (1) would read 75° 00′.

In Fig. 18*b*, the theodolite has been turned 12′ 26″ to the right, the opening (2) still reads 0′ 0″, the index in opening (1) is not over a graduation, and in opening (3) only one



(*a*) (*b*) (*c*)

Fig. 18

graduation appears with the main index mark. The micrometer head is then turned, moving both images to the right, until two graduations appear equally spaced on each side of the main index mark in opening (3), Fig. 18*c*, and the index now again appears over 75° 20′ in opening (1). The index in opening (2) now reads 12′ 26″, so that the full reading is 75° 32′ 26″. The main index mark is comparatively broad and, when centered between two graduations, shows narrow spaces on each side, which it is easy to equalize with great accuracy.

The altitude level is read from the telescope eye-piece by a prism bubble reader. The horizontal circle is 3½ in. diameter and the vertical circle 2¾ in. diameter; both are divided on glass to 20′, then silvered, and are read through the glass;

this protects the silver from tarnishing and from dust. The graduations on the horizontal circle are thus only $\frac{1}{100}$ in. apart, the thickness of the etched graduations is about $\frac{1}{8000}$ in., and the figures are $\frac{1}{250}$ in. in height. The displacement of a graduation by $\frac{1}{120000}$ in. would cause an error of 1" at the centre of the circle; these figures give some idea of the perfection of workmanship attainable.* It should be borne in mind that 1" is the angle subtended by $\frac{1}{172}$ in. at 100 ft., or $\frac{1}{17 \cdot 2}$ in. at 1,000 ft.

Eccentricity. If the vernier zeros, or index points, of a theodolite are not diametrically opposite to each other, their readings will not differ exactly by 180° Provided, however, that they rotate round the centre of the graduated circle as axis, the angle measured by each vernier will be unaffected, the two vernier readings always differing by a constant amount. On the other hand, if the axis of rotation does not coincide with the centre of the graduated circle, the angle, as recorded on the two verniers, will be different, and the difference of the readings of the two verniers will not remain constant. However, the *average* angle obtained from the readings of the two verniers, or the angle obtained by averaging each pair of vernier readings (after, of course, adding or subtracting 180° $\left\{\begin{matrix}\text{to}\\ \text{from}\end{matrix}\right.$ the readings of one vernier to reduce its readings to the other) will be free from this error. To prove this (Fig. 19)—

Let O be the centre of the graduated circle, C the axis of rotation, U, V the position of the verniers for one pointing

* "The Tavistock Transit Theodolite" by E. Wilfred Taylor. *Trans. Optical Society*, Vol. 32 (1930–31); also "The Tavistock Theodolite," *Engineering*, Vol. 131 (1931).

of the telescope, $U'V'$ their position for the other pointing. The actual angle turned through is

$$\theta = VCV' = VUV' + CV'U = \frac{VOV'}{2} + \frac{UOU'}{2}$$

$$= \frac{VOV' + UOU'}{2}$$

= average of angles read by each vernier.

Irregularity of Graduation. In spite of the modern perfection of the division of the circles, there must still remain small residual irregularities. To eliminate their effect on the measured angles, these should be measured repeatedly on different arcs of the circle. In the case of vertical angles, only two such measurements are possible, viz. F.R. and F.L. In the case of horizontal angles, however, any number can be taken by "changing zero," i.e. by unlocking the lower clamp and turning the vernier plate through about 90° if the angle is to be measured twice, viz. F.R. and F.L.; if it is to be measured four times, it should be turned through 45° between each measurement; then lock the lower clamp.

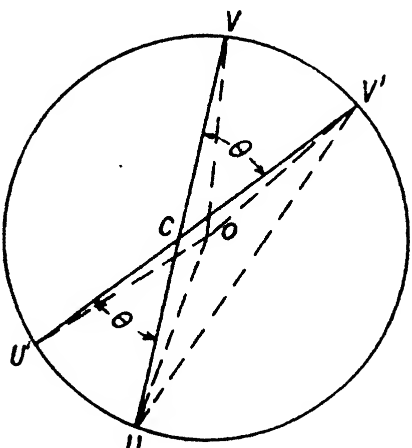


Fig. 19

Swinging Right and Left. To eliminate the turning effect of the sun on the theodolite, and the personal bias of the observer in bisecting a signal, half the observations in measuring horizontal angles should be taken reading the objects in a clockwise order, and the other half taking them in an anti-clockwise order. In the former the cross-hair should be brought on to the signal from the L.H. in the latter from the R.H., by the horizontal tangent screw.

The Reversible Level

Modern levels are constructed of the tilting type, i.e. the telescope has a small motion about a horizontal axis just below it, the motion being controlled by a fine levelling screw at the eye-piece end. At the same time the bubble of the spirit level, which is placed at the *side* of the telescope, can be viewed by a mirror or prism from the eye-piece end of the telescope. Consequently, the plate screws are only employed when setting up to make the vertical axis approximately vertical, using for this purpose a small circular spirit level placed on the base of the instrument. The bubble of the long spirit level on the telescope is brought central *for each reading* by turning the above fine levelling screw. The telescope is made with internal focusing to obtain the advantages already mentioned.

In the best types of tilting level the telescope is made *reversible*, i.e. it can be rotated in a sleeve or collars about its longitudinal axis through 180°, so that the spirit level is now upside down on the opposite side of the telescope. The spirit level tube must, therefore, be convex both top and bottom, i.e. barrel-shaped, and it is essential that the upper and lower "axes" must be parallel. If they are truly parallel the adjustment of the level is very readily effected, and the reversible level is sometimes termed "self-adjusting" on this account. The only adjustment is to place the axis of the bubble tube parallel to the line of collimation. Set up the instrument about 100 ft. from the levelling staff. (Fig. 20.) Bring the bubble central and read the staff to ·001 ft. Turn the telescope 180° about its longitudinal axis and bring the bubble again to its central position. Again read the staff to ·001 ft. If the readings differ there is an error of half the difference of readings. By the fine levelling screw bring the staff reading to the average of the two readings—the line of collimation is now horizontal; then adjust the bubble by the means provided until it is truly central; then check the staff reading again.

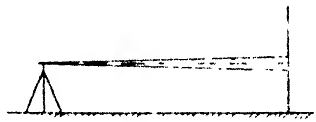
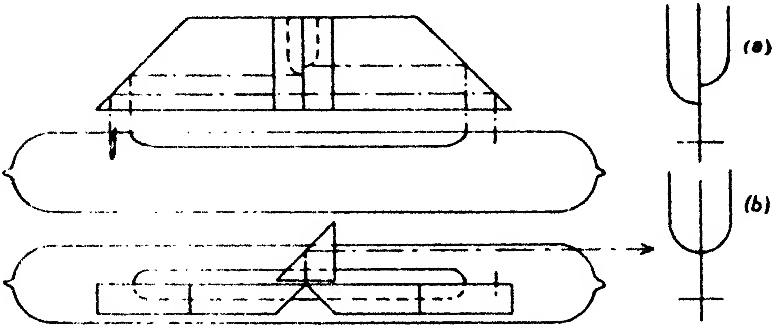


Fig. 20

A tilting level which does not reverse must be tested like a Dumpy Level, on two pegs, whose true difference of level is determined by setting up the level at equal distances from them. Then the level is set up behind one of the pegs, and if the same level-difference of the pegs is not obtained, the level is tilted by the fine screw to give the true level-difference, the bubble being then adjusted to its central position.

Bubble Reading Devices. An inclined mirror is, of course, the simplest, but both ends of the bubble have to be read and their readings equalized, as the bubble shortens with a rise of temperature, and *vice versa*.

Messrs. E. R. Watts & Son have invented a "Constant Bubble" spirit level, in which the tube is of approximately elliptical cross-section and the volumes of air and spirit are so proportioned that the decrease of surface tension with rise of temperature compensates for the expansion of the spirit and renders the length of the bubble constant over a range of at least 100° F. of temperature. With such a "constant" bubble, only one end of the bubble need be observed, and this is easily effected by total internal reflection in a 45° right-angled prism, placed on top of the spirit level case in which the bubble is seen reflected from the eye-piece end of the telescope. In a reversible level there is a second prism on the other side of the case for use in the reversed position. A short bubble is less sensitive than a



From "Surveying Instruments," by R. M. Abraham, by permission of Messrs. C. F. Casella & Co., Ltd.

FIG. 21

long one, so that when a "constant" bubble is not used it is necessary to have an air chamber at one end of the tube, from which air can be drawn to increase the length of the bubble at high temperatures.

When both ends of the bubble are observed, it is best to reflect them together by prisms, as shown in Fig. 21, which shows Connolly's Prism Bubble Reader, as used by Messrs. Zeiss. Both ends of each prism are cut at 45° as shown. The ends of the bubble are, therefore, totally reflected twice and appear together on the inner bevels as shown. In order to view them from the eye-piece end of the telescope, these adjacent images are turned through a right angle by a 45° right-angled prism and then magnified by a lens. If the bubble is uncentred they appear as in (*a*), when centred the ends of the bubble appear as in (*b*).

The only marks on the bubble tube are one at each end, which should appear coincident when the bubble reader is in its correct position. It will be noted that both ends of the bubble are viewed under exactly the same conditions, that the positions of the two ends of the bubble have not to be read on graduations, that as the images of the ends move in opposite directions coincidence, when obtained, is very exact, and that the bubble can be adjusted to give coincidence in another position by moving the whole reader longitudinally. The line joining the two calibration marks on the upper surface of the tube, must be parallel to the axis of the symmetry of the tube, so that, when rotated about this axis of symmetry, the ends of the bubble will still appear in coincidence. In order to set the axis of symmetry of the bubble tube parallel to the longitudinal axis of the telescope, the inclination of the two axes is altered until the bubble remains in coincidence when turned through 180° about the longitudinal axis.*

For the most precise work, however, in addition to this reversibility of rotation about a longitudinal axis through 180°, the eye-piece of a Zeiss level can be removed and inserted into a collar, provided for the purpose, at the object glass end. The diaphragm is marked on the face of a second object glass, and a second diaphragm is marked on the face of the object glass proper. When the eye-piece is thus reversed the object glasses reverse their functions; the one that formerly acted as diaphragm now acts as object glass, and *vice versa*.

* "New Types of Levelling Instruments Using Reversible Bubbles," by T. F. Connolly. *Trans. Optical Society*, Vol. 25 (1923–4).

In adjusting the level, *four* readings are taken, viz. (1) eye-piece direct, level direct; (2) eye-piece direct, level reversed; (3) eye-piece reversed, level reversed; (4) eye-piece reversed, level direct; *in each case the ends of the bubble being brought into coincidence.* In Fig. 22 let β be the angle between the two coincidence positions of the bubble

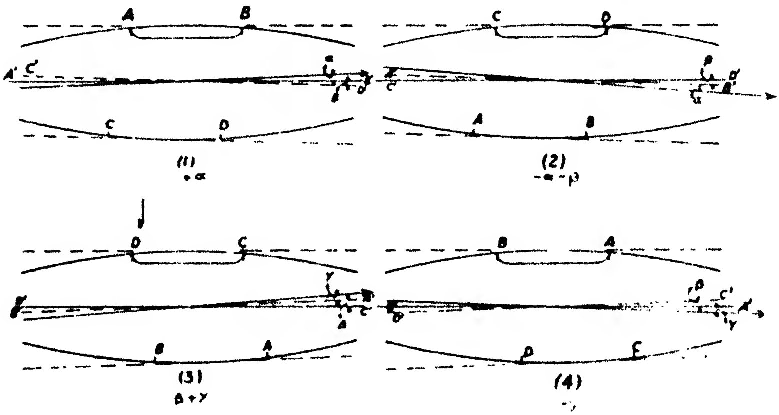


FIG. 22

when reversed, α the inclination of the line of collimation to the coincidence position AB in (1). In (1) the angular error is $+\alpha$, in (2) the angular error is $-\alpha-\beta$. When the eye-piece is reversed the line of collimation may be slightly altered; let it now make an angle γ with the coincidence position AB. In (3) the angular error is $+\beta+\gamma$; in (4) the angular error is $-\gamma$. The average angular error of the line of collimation in the four positions is, therefore, *zero*, so that the average reading on a staff is the reading which is level with the centre of the instrument. The line of collimation is then directed to this mean reading, and the bubble reader prisms are then adjusted longitudinally until the ends of the bubble are in coincidence in the normal position

(1), i.e. with eye-piece in its normal position and spirit level on the left hand; or, if preferred, so that when the bubble is in coincidence in positions (1) and (2) the average of the two readings will be that given by the average of the four readings above described.

The diaphragms are engraved with stadia lines for a constant of 100. This serves two purposes: (*a*) it enables the *lengths* of the sights to be read and thus kept in very approximate equality, and (*b*) by reading the staff on all three marks to avoid misreadings, as, of course, the average of the readings of the two outer marks should equal the middle reading.

The staff consists of an Invar strip, with fine lines engraved at each 0·02 ft., mounted in a groove in a mahogany staff in which it is free to slide, the lower end of the strip being fixed to a hard steel shoe. The staff is provided with a plummet (or circular spirit level on a bracket) to enable it to be kept truly vertical, and is also fitted with handles and two steadying rods. In front of the object glass on the level is a thick sheet of parallel plate glass with a micrometer head, so that the line of sight can be made to intersect an exact division on the staff, the micrometer reading being added to the reading of the next lower graduation on the staff. By this means the staff can be read to ·001 ft. and, by estimation on the micrometer, to ·0001 ft.

CHAPTER IV

DETERMINATION OF LATITUDE, AZIMUTH, TIME, AND LONGITUDE

INSTRUMENTS

A 5-IN. Transit Theodolite with Micrometers, reading to 10″ and by estimation to 1″, is the proper instrument to employ, with a sensitive Altitude Level on the vernier arm of the vertical circle, a Striding Level for application to the Trunnion Axis, a dark glass for the Sun, and, for high altitudes, a Diagonal Eye-piece. An aneroid barometer with thermometer is required for the Refraction correction. In order to see the cross-hairs at night, a simple device is to fix a strip of white paper about ½ in. wide across the object glass by means of a rubber band, making a couple of slits ¼ in. long near top and bottom of the strip, and bending the paper between the slits at 45° so as to serve as a reflector. If the light from an electric torch is directed on this reflector the cross-hairs will be visible. In order to sight on another station at night, or on a "Reference Mark" whose azimuth is to be determined, a lamp should be placed inside a box with a slit in one side, the width of the slit being proportioned to the distance from the instrument, so that it is unnecessary to alter the focus when sighting on a star. The slit, of course, must be accurately centred over the station or reference mark, and should be as far as possible from the instrument.

A Chronometer Watch, keeping mean or sidereal time, is also required. It should be treated with great care, always wound at the same time of the day and kept as horizontal as possible. The important quality of a chronometer is that its "Rate" should be constant, i.e. that it should go fast or slow by a constant amount per day or per hour. If its rate is constant and known, then having determined its "error" or difference from true time at any instant, the true time at any other instant can be determined from the reading of the chronometer at that instant. Obviously, this is best done by a process of "interpolation," i.e. by determining its error before and after the instant of observation, and finding its rate from the increase or decrease in its error; if this is not possible, the instant of observation should be as soon after the second of the two determinations of error as possible, so as to "exterpolate" as little as possible. The "standing rate" of a Chronometer kept at one place differs from its "travelling rate" when taken on a land journey, and these must be separately determined.

If the student has no opportunity of using all the above instruments, he must not suppose that he cannot get useful experience with less perfect ones which may be available. He may, for example, use a 4-in. Theodolite and an ordinary Watch, make his observations, and reduce them as accurately as his instruments permit. It is only by actually making and reducing astronomical observations that this important branch of Surveying will become real to him. If only one observation is possible, the Extrameridian observation for Azimuth and Time on the Star (or on the Sun) should at least be made and reduced. It is the best observation to begin with as it has not to be made at an exact instant, and can be repeated immediately as often as desired.

Latitude

From Chapter II, it will be obvious that this is determined by measuring the Altitude of the Sun, or of a fixed Star, when crossing the meridian; in the case of the Sun at apparent noon, in the case of a Star at upper or at lower transit when the altitude is a maximum or a minimum. The Sun is, of course, the most convenient object for the

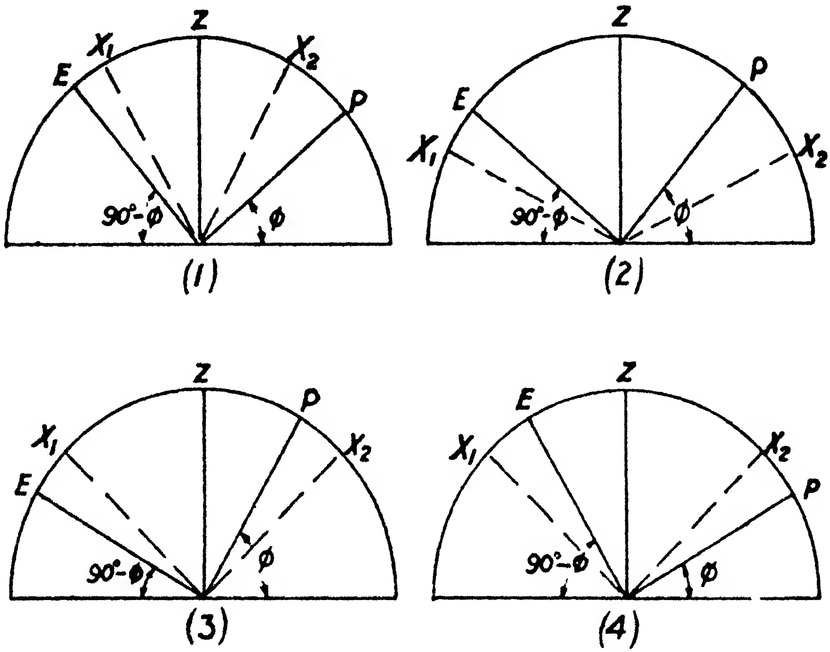


FIG. 1

observation, but the fact that its Declination is constantly varying—in March and September by nearly 1′ per hour or 1″ per minute—requires us to know the Greenwich Mean Time, or the Longitude and Local Mean Time, at least approximately. Owing to this variation in Declination the Sun is not quite at its maximum altitude when crossing the meridian, but the difference is very minute. With the Sun, moreover, the uncertainty as to the correction of Altitude for Atmospheric Refraction cannot be eliminated by "pairing" the observation, i.e. by taking a second observation at nearly the same altitude on the other side of the zenith, and averaging the results obtained from the two observations. Still for some purposes a Sun observation is sufficiently accurate.

(*a*) *Latitude by Zenith-Pair Observations of Two Stars Crossing the Meridian* (Fig. 1). Two stars, X_1, X_2, are selected for observation whose times of transit differ by not more than half an hour, and which transit at nearly equal altitudes but on opposite sides of the zenith. There are four cases. In (1) and (4) the star X_2 nearer the pole is at Upper Transit, and the R.A.'s should differ by $\ngtr$ 30 min. In (2) and (3) the star X_2 is at Lower Transit, and the R.A.'s should differ by from 11 hr. 30 min. to 12 hr. 30 min. Calling the altitudes a_1, a_2, the declinations δ_1, δ_2, and the latitude ϕ, for *equal* altitudes at transit we must have

$$\left.\begin{aligned} &\text{in (1) } 90^\circ - \phi + \delta_1 = \phi + 90^\circ - \delta_2 \\ &\qquad \therefore \delta_2 + \delta_1 = 2\phi. \\ &\text{in (4) } 90^\circ - \phi - \delta_1 = \phi + 90^\circ - \delta_2 \\ &\qquad \therefore \delta_2 - \delta_1 = 2\phi. \end{aligned}\right\} \text{both stars at upper transit.}$$

$$\left.\begin{aligned} &\text{in (2) } 90^\circ - \phi - \delta_1 = \phi - 90^\circ + \delta_2 \\ &\qquad \therefore \delta_2 + \delta_1 = 180^\circ - 2\phi. \\ &\text{in (3) } 90^\circ - \phi + \delta_1 = \phi - 90^\circ + \delta_2 \\ &\qquad \therefore \delta_2 - \delta_1 = 180^\circ - 2\phi. \end{aligned}\right\} \text{one star at lower transit.}$$

These formulae enable us to choose two suitable stars whose apparent declinations are given in the *Nautical Almanac*, but actually the altitudes can never be equal exactly, and we have

in (1) $90° - \phi = a_1 - \delta_1$

$\therefore \quad \phi = 90° - a_1 + \delta_1 \ : \ \phi = a_2 - 90° + \delta_2$

$\therefore$ averaging, $\phi = \dfrac{a_2 - a_1}{2} + \dfrac{\delta_2 + \delta_1}{2}$

in (4) $90° - \phi = a_1 + \delta_1$

$\therefore \quad \phi = 90° - a_1 - \delta_1 \ : \ \phi = a_2 - 90° + \delta_2$

$\therefore$ averaging, $\phi = \dfrac{a_2 - a_1}{2} + \dfrac{\delta_2 - \delta_1}{2}$

in (2) $90° - \phi = a_1 + \delta_1$

$\therefore \quad \phi = 90° - a_1 - \delta_1 \ : \ \phi = a_2 + 90° - \delta_2$

$\therefore$ averaging, $\phi = 90° + \dfrac{a_2 - a_1}{2} - \dfrac{\delta_2 + \delta_1}{2}$

in (3) $90° - \phi = a_1 - \delta_1$

$\therefore \quad \phi = 90° - a_1 + \delta_1 \ : \ \phi = a_2 + 90° - \delta_2$

$\therefore$ averaging, $\phi = 90° + \dfrac{a_2 - a_1}{2} - \dfrac{\delta_2 - \delta_1}{2}$

These four formulae are not given to be memorized, but to show that in all cases the average latitude deduced from the paired observations depends on the *difference* of the altitudes of the two stars. These altitudes have, of course, been corrected for refraction; as the altitudes are nearly equal the corrections will be nearly equal, and any error made in the estimate of the refraction correction will be practically eliminated on the difference, $a_2 - a_1$, of the altitudes. Also, if the face of the theodolite is not changed, the Index Error of the vertical circle is the same for both altitudes and is, therefore, eliminated on their difference. It is necessary, of course, to read the object end (*o*) and the eye-end (*e*) of the altitude level bubble, and to correct each altitude by $\frac{o-e}{2} \times$ angular value of a division of the level, unless the bubble is brought central for each observation.

To make the observation the theodolite is set up carefully in the approximate meridian a few minutes before the estimated time at which the first of the two stars should transmit and directed on the star. The horizontal cross-hair must be made to intersect the star, close to the vertical hair, and kept in contact with the star by turning the vertical tangent screw. The star will move slower and slower in altitude, till at the moment of culmination it will appear to travel along the horizontal hair. When this moment arrives, the two ends of the bubble in the altitude level are read, then the two vertical verniers and the barometer and thermometer. The instrument is then rotated on its vertical axis, and the process is repeated on the second star. If a number of pairs of suitable stars can be observed similarly, the accuracy of the result can be increased by averaging the latitudes obtained from each pair.

EXAMPLE 1. Reduce the following meridian observation for Latitude—

Star	Declination	Right Ascension	Observed Altitude	Altitude Level	
				Object End	Eye End
A .	60° 02′ 45·0″ S.	13h. 59m. 00s.	49° 28′ 15″ S.	5·4	4·6
B .	19° 32′ 09·0″ N.	14h. 12m. 33s.	50° 58′ 10″ N.	5·2	4·8

The value of a level division is ·14″. Take the refraction correction as − 58″ cot altitude. If the longitude is 142° 36′ E.

and the sidereal time of mean noon at Greenwich is 4 hr. 6 min. 17 sec., at what Local Mean Times will the two transits occur?

First make a sketched meridian section thus (Fig. 2). Mark the two ends of the horizon *N* and *S* for North and South, then mark the star of *least* declination (*B*) by its altitude 51° N. Then draw the Equator *E* 20° on the South side of *B*, so as to make the declination of *B* about 20° N. The angle *EON* is then 71° N., and therefore draw *P* the Pole at 19° S. Then mark the other star (*A*) by its altitude 49° S.; *A* will be 30° above the pole which agrees with its declination 60° S. We thus avoid any doubt as to whether *A* is at upper or lower transit. We then find the corrected altitudes of the two stars, thus—

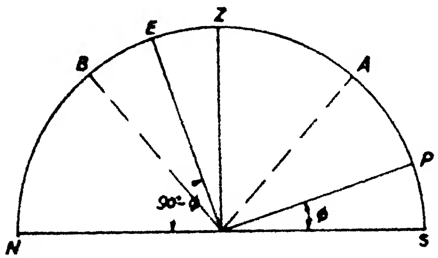


FIG. 2

Star	Observed Altitude	Level Correction	Refraction Correction	Corrected Altitude
A .	49° 28′ 15″ S.	+ 0·4 × 14″ = + 5·6″	− 58″ cot 49° 28′ = − 49·6″	49° 27′ 31·0″ S.
B .	50° 58′ 10″ N.	+ 0·2 × 14″ = + 2·8″	− 58″ cot 50° 58′ = − 47·0″	50° 57′ 25·8″ N.

From *A*	= 49° 27′ 31·0″	From *B*	= 50° 57′ 25·8″
- co-dec.	= 29° 57′ 15·0″	+ dec.	= 19° 32′ 09·0″
Latitude	= 19° 30′ 16·0″	*EON*	= 70° 29′ 34·8″
		∴ Latitude	= 19° 30′ 25·2″

19° 30′ 16·0″
19° 30′ 25·2″

Average = 19° 30′ 20·6″ S. = Latitude

Longitude = 142° 36′ E. = 9 hr. 28 min. + 2 min. 24 sec. = 9 hr. 30 min. 24 sec. = 9·507 hr. before Greenwich.

∴ sidereal time of Mean Noon = 4 hr. 6 min. 17 sec. − 9·507 × 9·857 sec. = 4 hr. 6 min. 17 sec. − 1 min. 34 sec. = 4 hr. 4 min. 43 sec.

	Star *A*	Star *B*
L.S.T.	13h. 59m. 00s.	14h. 12m. 33s.
	4h. 4m. 43s.	4h. 4m. 43s.
Sidereal interval since L.M.N.	9h. 54m. 17s.	10h. 07m. 50s.
Deduct 9·83s. per hour .	1m. 37s.	1m. 40s.
L.M.T.'s of Transit .	9h. 52m. 40s. p.m.	10h. 06m. 10s. p.m.

(*b*) *Latitude by Circum-meridian Altitudes of a Star* (Fig. 3). Instead of observing only the maximum (or minimum)

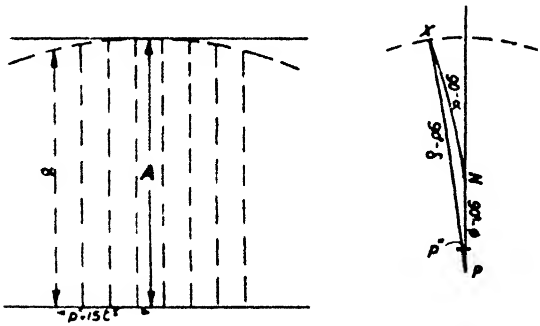


Fig. 3

altitude of each of a pair of North and South stars, increased accuracy can be gained by taking a series of altitudes before and after the transit of the star, noting the times of the observations. There should be an equal number of observations (say 4) before and after transit, and the face of the instrument should be changed, thus *R*, *R*, *L*, *L* (transit) *R*, *R*, *L*, *L*, the observations extending over not more than 20 min. in all. By a formula to be deduced, each of the eight altitudes is corrected to give a value of the maximum (or minimum) altitudes, and the average of the eight altitudes thus corrected is used to determine the latitude. We have

$$\cos P = \frac{\cos ZX - \cos PX \cdot \cos ZP}{\sin PX \cdot \sin ZP} = \frac{\sin a - \sin \delta \cdot \sin \phi}{\cos \delta \cdot \cos \phi}$$

where a, δ, ϕ = altitude, declination. and latitude.

$$\therefore \sin a = \sin \delta \cdot \sin \phi + \cos \delta \cdot \cos \phi \cdot \cos P$$

$$= \sin \delta \cdot \sin \phi + \cos \delta \cdot \cos \phi - 2 \cos \delta \cdot \cos \phi \cdot \sin^2 \frac{P}{2}$$

$$= \cos (\phi - \delta) - 2 \cos \delta \cdot \cos \phi \cdot \sin^2 \frac{P}{2}.$$

Now, if A be the maximum altitude of the star

$$A = 90° - \phi + \delta = 90° - (\phi - \delta)$$

$$\therefore \qquad \sin A = \cos (\phi - \delta)$$

$$\therefore \qquad \sin a = \sin A - 2 \cos \delta \cdot \cos \phi \cdot \sin^2 \frac{P}{2}$$

$$\therefore \qquad \sin A - \sin a = 2 \cos \delta \cdot \cos \phi \cdot \sin^2 \frac{P}{2}$$

But $\sin A - \sin a = 2 \cos \frac{A + a}{2} \cdot \sin \frac{A - a}{2}$

$$\therefore \cos \frac{A + a}{2} \cdot \sin \frac{A - a}{2} = \cos \delta \cdot \cos \phi \cdot \sin^2 \frac{P}{2}$$

But if the observations are taken within a few minutes of transit $\frac{A - a}{2}$ and $\frac{P}{2}$ are small angles, and $\cos \frac{A + a}{2} = \cos A$, practically

$$\therefore A - a = \frac{\cos \delta \cdot \cos \phi}{\cos A} \cdot 2 \sin^2 \frac{P}{2}$$

$$= \frac{\cos \delta \cdot \cos \phi}{\cos A} \cdot \frac{P^2}{2}$$

where $A - a$ and P are in circular measure.

Now if t is the time in sidereal seconds between the observation and the time of transit, $P'' = 15t$, and we have

$$(A - a)'' . \sin 1'' = \frac{\cos \delta . \cos \phi}{\cos A} . \frac{P''^2}{2} (\sin 1'')^2$$

$$\therefore \quad (A - a)'' = \frac{\cos \delta . \cos \phi}{\cos A} . \frac{225t^2}{2 \times 206265}$$

$$= \frac{\cos \delta . \cos \phi}{\cos A} . \frac{t^2}{1833 \cdot 5}$$

$$\text{or } A = a + \frac{\cos \delta . \cos \phi}{\cos A} . \frac{t^2}{1833 \cdot 5}$$

When applied to a number of observations, we have

$$\bar{A} = \bar{a} + \frac{\cos \delta . \cos \phi}{\cos A} \frac{(t^2)}{1833 \cdot 5}$$

which gives the mean value of the maximum altitude. Tables of the factor $\frac{t^2}{1833 \cdot 5}$, usually written as

$$\frac{2 \sin^2 \frac{t}{2}}{\sin 1''}$$

where t stands for our P, are given in Close's *Topographical Surveying* for every second of time up to 20 min. It is easy to show that this formula holds for all cases of Upper Transit, viz. star above or below equator or between the zenith and the pole, also that when the star is at Lower Transit,

$$A = a - \frac{\cos \delta . \cos \phi}{\cos A} . \frac{2 \sin^2 \frac{P}{2}}{\sin 1''} = a - \frac{\cos \delta . \cos \phi}{\cos A} . \frac{t^2}{1833 \cdot 5}$$

in which case P is the angle in arc and t in seconds of time from Lower Transit.

In order to calculate the factor $\frac{\cos \delta . \cos \phi}{\cos A}$, it is sufficient to take the mean of the pair of altitudes nearest the transit and one on each side of it, corrected for altitude level and refraction, for A and to calculate ϕ from it for this purpose. Of course, an accurate knowledge of the local time is required, and the observations should not extend more than 10 min. on each side of the transit. The completed observation should be followed as soon as possible by a similar set of circum-meridian observations on a star at nearly the same altitude on the other side of the zenith in order to eliminate any error due to refraction as already explained.

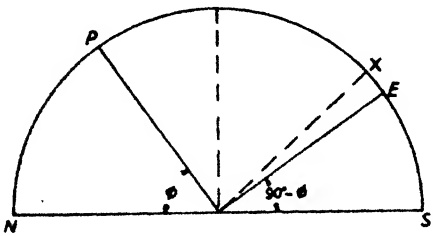


FIG. 4

EXAMPLE 2. Reduce the following observation for Latitude: Star α orionis (R.A. = 5 hr. 51 min. 32·55 sec., Dec. = 7° 23′ 46·0″ N.). Longitude, 1° 17′ 45″ W., Siderial Time of G.M.N. = 21 hr. 40 min. 10·2 sec. Altitudes are South. Watch, 6 sec. slow on G.M.T. Barometer, 29·6 in. Temperature, 40° F. Level, one division = 15″.

Watch Time (M.T.)			Face	Altitude Level		Vertical Circle	
						Vernier C	Vernier D
H.	M.	S.		O	E		
8	5	16 p.m.	R	5·4	5·0	43° 21′ 00″	21′ 10″
8	7	55	R	5·6	4·8	43° 22′ 40″	22′ 50″
8	10	39	L	5·8	4·6	136° 37′ 10″	37′ 30″
8	13	31	} L	5·8	4·6	136° 36′ 50″	37′ 10″
8	16	30	} R	5·6	4·8	43° 24′ 00″	24′ 10″
8	19	16	R	5·0	5·4	43° 23′ 40″	23′ 50″
8	21	47	L	5·2	5·2	136° 38′ 20″	38′ 40″
8	24	40	L	5·4	5·0	136° 39′ 20″	39′ 40″

We first find the watch time of transit:

Longitude = 1° 17′ 45″ W. = 5 min. 11 sec. W.

Acceleration for

5 min. 11 sec. = 0·82 sec. + 0·03 sec. = 0·85 sec.

∴ Sidereal Time at L.M.N. = 21 hr. 40 min. 11·05 sec.

L.S.T. of transit

= 5 hr. 51 min. 32·55 sec. + 24 hr.

= 29 hr. 51 min. 32·55 sec.

Sidereal Int. since L.M.N. = 8 hr. 11 min. 21·5 sec.

Retardation

= 1 min. 18·64 sec. + 1·80 sec. + 0·06 sec.

= 1 min. 20·50 sec.

∴ L.M.T. = 8 hr. 10 min. 01 sec. p.m.

Add 5 min. 11 sec. for Longitude.

G.M.T. = 8 hr. 15 min. 12 sec.

∴ *Watch time of Transit* = 8 hr. 15 min. 06 sec.

We next estimate the meridian altitude A and the latitude (Fig. 4) from the two middle readings thus—

$\frac{1}{2}(43° 23′ 00″ + 43° 24′ 05″) = 43° 23′ 32·5″$

Altitude level $\frac{11·4 - 9·4}{4} \times 15$ $+ \quad 07·5″$

Refraction $\left(61·8″ - \frac{23·7}{60} \times 2·1″\right) 1·017 \times ·999.$ $- \quad 1′ \; 02″$

$43° 22′ 38″ = A$

Declination $= 7° 23′ 46″$

$35° 58′ 52″ = 90° - \phi$

∴ Latitude = $54° 01′ 08″$ N. $= \phi$

$\cos \delta$	$\bar{1}·9964$
$\cos \phi$	$\bar{1}·7690$
	$\bar{1}·7654$
$\cos A$	$\bar{1}·8614$
	$\bar{1}·9040$

$$\therefore \frac{\cos \delta \,.\, \cos \phi}{\cos A} = 0·8017$$

We then complete our table thus—

Altitude Level		Mean Observed Altitude	Mean Interval to Transit		Sidereal Interval to Transit			t^2
O	*E*		M.	S.	M.	S.		
5·4	5·0	43° 21′ 05″	9	50	9	51·6	591·6	349,991
5·6	4·8	43° 22′ 45″	7	11	7	12·2	432·2	186,797
5·8	4·6	43° 22′ 40″	4	27	4	27·7	267·7	71,663
5·8	4·6	43° 23′ 00″	1	35	1	35·3	95·3	9,082
5·6	4·8	43° 24′ 05″	1	24	1	24·2	84·2	7,090
5·0	4·4	43° 23′ 45″	4	10	4	10·7	250·7	62,851
5·2	5·2	43° 21′ 30″	6	41	6	42·1	402·1	161,684
5·4	5·0	43° 20′ 30″	9	34	9	35·6	575·6	331,316
Totals 43·8	39·4	8)346° 59′ 20″						8)1,180,474
Average		43° 22′ 25″						$(\overline{t^2})$ = 147,559

Average level correction $= \frac{43 \cdot 8 - 39 \cdot 4}{16} \times 15''$ = + 4·1″

Add average observed altitude = 43° 22′ 25″

Refraction correction $\left(61 \cdot 8'' - \frac{22 \cdot 5}{60} \times 2 \cdot 1''\right) 1 \cdot 017 \times \cdot 999 =$ − 1′ 02″

Circum-meridian correction $\cdot 8017 \times \frac{147559}{1833 \cdot 5}$ + 1′ 04·5″

∴ Average value of *A* = 43° 22′ 31·6″

∴ 90° − ϕ = 43° 22′ 31·6″ − 7° 23′ 46″ = 35° 58′ 45·6″

∴ Latitude = 54° 01′ 14·4″ N

(*c*) *Latitude by Sun Observation.* The method of circum-meridian altitudes may be applied to the sun if circumstances necessitate a daylight observation. In this case the altitudes should be taken alternately to the upper and lower limbs, letting the vertical hair bisect the limb as nearly as possible, so as to eliminate its semi-diameter. A correction for parallax must be applied when calculating *A*, provisionally and finally; on the other hand the interval, *t*, if recorded on a mean time watch, will not require conversion to sidereal seconds. The declination to be employed will be that at apparent noon, i.e. at the time of transit. It is not, of course, possible to "pair" the observation, so that there will always be some uncertainty as to refraction.

For comparatively rough determinations of latitude a single observation may be made on the sun at its maximum altitude, taking a second observation with changed face on the opposite limb of the sun as soon as the first has been completed at maximum altitude, or a single maximum altitude may be taken by previously determining the index error of the vertical circle, or eliminating it by adjustment. The following example will be found useful in showing the order in which the corrections are made.

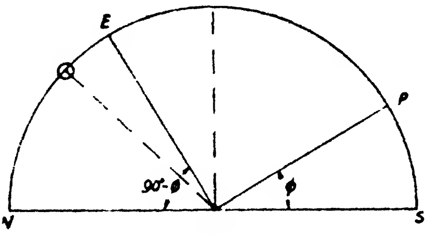


FIG. 5

EXAMPLE 3. At a place in longitude 23° 21′ 50″ E., the observed altitude of the lower limb of the sun crossing the *north* meridian was 43° 17′ 35″. At Greenwich Mean Noon the equation of time was 3 min. 3·0 sec. (Sun before clock), increasing 0·29 sec. per hour, and the declination of the Sun's centre was 15° 19′ 36·0″ N., increasing 45·0″ per hour. The altitude level read 5·6 O., 6·4 E. (one division = 14″), index correction of vertical circle = + 15″. Taking the semi-diameter as 15′ 35″, the parallax as + 8·7″ × cos altitude, refraction as − 58″ × cot altitude, find the latitude of the place and the time of transit on a G.M.T. chronometer. (Fig. 5.)

We first find the G.M.T. of transit in order to find the declination at apparent noon.

Longitude = 23° 21′ 50″ E.

= 1 hr. 32 min. + 1 min. 24 sec. + 3·3 sec.

= 1 hr. 33 min. 27·3 sec. before Greenwich.

But as apparent noon at the place occurs approximately 3 min. 3 sec. before L.M.N., it will occur approximately 1 hr. 36 min. 30·3 sec. before G.M.N., and in this time the *decrease* in the equation of time will be

0·29 sec. × 1·61 hr. = 0·5 sec.

∴ equation of time = 3 min. 2·5 sec. and transit occurs 1 hr. 36 min. 29·8 sec. before G.M.N., that is, at

10 hr. 23 min. 30·2 sec. a.m. G.M.T.

(or 11 hr. 56 min. 57·5 sec. L.M.T.)

∴ Declination of Sun = 15° 19′ 36·0″ − 1·608 × 45″ = 15° 19′ 36·0″ − 1′ 12·4″ = 15° 18′ 23·6″ N.

Observed altitude of lower limb	= 43° 17′ 35″
Index correction	= + 15″
Level correction $-\frac{\cdot 8}{2} \times 14$	= − 5·6″
	43° 17′ 44·4″
Refraction − 58″ cot 43° 18′	= − 1′ 01·6″
	43° 16′ 42·8″
Semi-diameter	= + 15′ 35″
	43° 32′ 17·8″
Parallax + 8·7″ cos 43° 32′	+ 6·3″
Corrected altitude	= 43° 32′ 24·1″
Declination	= 15° 18′ 23·6″
EON	= 58° 50′ 47·7″
∴ Latitude	= 31° 09′ 12·3″ S.

N.B. Strictly we should not have taken the change of declination as 45″ per hour, but as 45″ − (increment of hourly variation of declination in 24 hours) $\times \frac{1{\cdot}608}{2 \times 24}$, i.e. as the average rate over the period between G.M.N. and L.A.N., but no information was given in the question on this point and the interval was short.

AZIMUTH

Usually the theodolite will be set up at one end of a survey line whose azimuth is required, but in other cases the

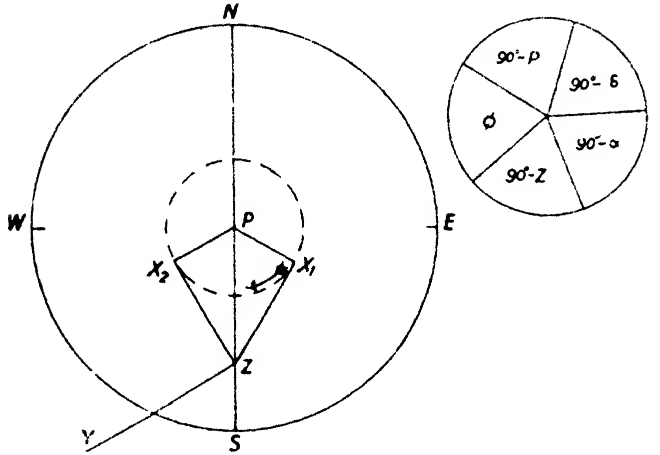


FIG. 6

meridian may have to be set out on the ground, in which case a "Reference Mark" is fixed and its azimuth determined. The observation is also useful to find the Magnetic Variation of the Compass.

(*a*) *By the Elongation of a Circum-polar Star.* A star whose declination is greater than the latitude of the place has its upper transit on the same side of the Zenith as the pole (Fig. 6). The azimuth of such a star varies from a maximum amount East to a maximum amount West, and these positions are called the Eastern and Western Elongations of the star. At elongation the angle $PXZ = 90°$. If such a star is observed through the theodolite at the moment of eastern elongation it will appear to travel up the vertical cross-hair, and at the moment of western elongation it will appear to travel down the vertical cross-hair. Thus an elongation corresponds as regards ***azimuth*** to a meridian transit as regards ***altitude***. If the star is not too far from the pole it is possible to take a second observation with changed face, as the azimuth of the star changes very slowly, and it is important that this should be effected to eliminate the effect of instrumental errors on the horizontal angle YZX between the reference mark Y and the star X, which may be considerable owing to their great difference in altitude. The booking of the observation would be as follows :--

Object	Face	Horizontal Circle*	
		Vernier *A*	Vernier *B*
R.M. . .	*R*		
Star . .	*R*		
Star . .	*L*		
R.M. . .	*L*		

If the exact watch time of elongation is not known, set up a few minutes before the calculated time of elongation and keep the vertical hair on the star near its intersection with the horizontal hair. As soon as the star commences to ascend or descend the vertical hair, read both verniers on the horizontal circle, change face, again bring the vertical hair on to the star near the horizontal hair, and again read the horizontal circle. The reference mark is sighted before and after the observation. If the observer has a chronometer

* For precise work there should also be columns for the readings of the striding level, direct and reversed.

on which he knows the exact instant of elongation he should make his two observations, one before that instant and one after it, and both as close to the instant of elongation as will just allow him to read the verniers, to change face, and to again sight on the star between the two pointings.

The calculations are very simple as the spherical triangle ZPX is right-angled at X. We have

$$\sin(90^\circ - \delta) = \cos\delta = \cos\phi \,.\, \cos(90^\circ - Z)$$
$$= \cos\phi \,.\, \sin Z$$
$$\therefore \sin Z = \frac{\cos\delta}{\cos\phi}$$

From $Z = XZP$ and the observed angle YZX we find YZP.

Also $\sin(90^\circ - P) = \cos P = \tan\phi \,.\, \tan(90^\circ - \delta)$
$$= \tan\phi \,.\, \cot\delta$$
$$\therefore \cos P = \frac{\tan\phi}{\tan\delta}$$

From P = hour angle and the R.A. we find the L.S.T.

Also $\sin\phi = \cos(90^\circ - a) \,.\, \cos(90^\circ - \delta)$
$$= \sin a \,.\, \sin\delta$$
$$\therefore \sin a = \frac{\sin\phi}{\sin\delta}$$

This gives a, the altitude of the star at elongation. This last is useful if the star is a faint one, as it enables the observer to identify the star by setting his instrument to the calculated altitude + the refraction correction for that altitude, and to the approximate azimuth given by Z and the approximate bearing of ZY, the line to the reference mark. If he finds a star elongating close to his pointing he knows this is the star he requires.

Effect of an Error in the Assumed Latitude. If we differentiate the equation $\sin Z = \frac{\cos \delta}{\cos \phi}$, we get

$$\cos Z \,.\, dZ = -\frac{\cos \delta}{\cos^2 \phi}(-\sin \phi)\, d\phi = \frac{\cos \delta \,.\, \sin \phi}{\cos^2 \phi}\, d\phi$$

$$= \sin Z \,.\, \tan \phi \,.\, d\phi$$

$$\therefore\ dZ = \tan Z \,.\, \tan \phi \,.\, d\phi$$

Therefore, an error $d\phi$ in latitude will cause a large error dZ

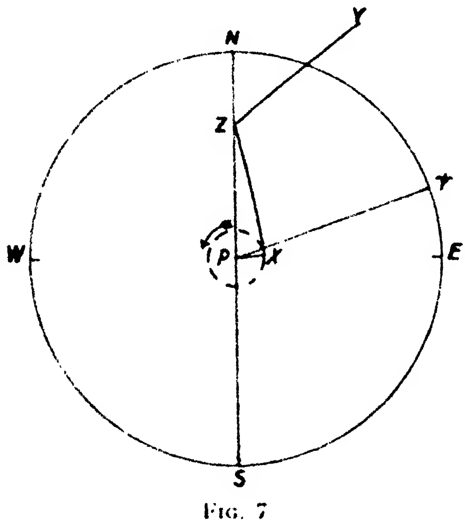


FIG. 7

in Z, (*a*) if Z is large or (*b*) if ϕ is large. We see, therefore, that (*a*) the star should be near the pole, and (*b*) the method is unsuitable in high latitudes.

EXAMPLE 4. At a place in latitude 31° 41′ 40″ S., 121° 32′ 30″ E., a star whose R.A. = 0 hr. 22 min. 15·6 sec. Declination 77° 37′ 54″ S. is observed at eastern elongation, when its clockwise horizontal angle from a survey line ZY is 117° 14′ 50″ Find the azimuth of the survey line and the local mean time of the elongation, if the mean time of transit of ψ at Greenwich is 1 hr. 20 min. 57 sec. from midnight, also the standard time of the elongation if standard time is 8 hr. fast on Greenwich (Fig. 7).

cos 77° 37′ 54″	$\bar{1}$·330,8103	∴ azimuth of line ZY (clockwise from north)
cos 31° 41′ 40″	$\bar{1}$·929,8592	= 180° − 14° 34′ 50″ − 117° 14′ 50″
		= 48° 10′ 20″
sin 14° 34′ 50″	$\bar{1}$·400,9511	
tan 31° 41′ 40″	$\bar{1}$·790,6219	
tan 77° 37′ 54″	0·658,9912	∴ XPZ = 82° 13′ 05″ = 5h. 28m. 52·3s.
cos 82° 13′ 05″	$\bar{1}$·131,6307	∴ ψPZ = 5h. 28m. 52·3s. − 0h. 22m 15·6s = 5h 06m. 36·7s.

∴ L.S.T. = 24h. − 5h. 06m. 36·7s. = 18h. 53m. 23·3s.

Longitude = 121° 32′ 30″ E. = 8h. 06m. 10s. = 8·103h.

∴ Mean time of transit of ψ at 121° 32′ 30″ E.

= 1h. 20m. 57s. + 9·830s. × 8·103 = 1h. 22m. 16·6s.

18h. 53m. 23·3s. sidereal time

= 18h. 53m. 23·3s. − 3m. 05·7s. mean time

= 18h. 50m. 17·6s.

∴ Local mean time of eastern elongation

= 20h. 12m. 34·2s. from midnight

= 8h. 12m. 34·2s. p.m.

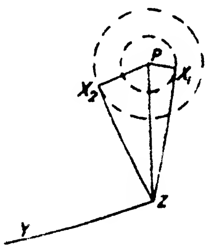


Fig. 8

Standard time is 8 hours fast on Greenwich, i.e. is 6m. 10s. slow on local mean time.

∴ Standard time of eastern elongation is 8h. 6m. 24·2s. p.m.

Azimuths by Elongations of Two Stars. If the latitude is unknown, the azimuth of a line ZY can be found by observing the horizontal angles from it to the elongations of two stars which reach elongation within a short time of each other. Thus (Fig. 8) we measure YZX_1, YZX_2 and, therefore, we know the angle X_1ZX_2 which equals $Z_1 \pm Z_2$, according as the stars are at opposite, or the same, elongation. Let $Z_1 \pm Z_2 = \gamma$ (say).

Then $\sin Z_1 = \dfrac{\cos \delta_1}{\cos \phi} : \sin Z_2 = \dfrac{\cos \delta_2}{\cos \phi}$

$$\therefore \frac{\sin Z_1}{\sin Z_2} = \frac{\cos \delta_1}{\cos \delta_2} = k \text{ (say)}.$$

The problem is, therefore, similar to the "Three Point Problem;" see Chapter VII, or it may be solved thus, $Z_1 = \gamma \mp Z_2$.

$$\therefore \sin Z_1 = \sin \gamma \,.\, \cos Z_2 \mp \cos \gamma \,.\, \sin Z_2$$

$$\therefore \quad k = \frac{\sin Z_1}{\sin Z_2} = \sin \gamma \,.\, \cot Z_2 \mp \cos \gamma$$

$$\therefore \cot Z_2 = \left(\frac{\sin Z_1}{\sin Z_2} \pm \cos \gamma\right) \frac{1}{\sin \gamma}$$

$$= k \operatorname{cosec} \gamma \pm \cot \gamma$$

Azimuth by Circum-Elongation Observations. R. W. Chapman, in *Astronomy for Surveyors*, page 134 (Griffin & Co.), has deduced a formula for the difference of the azimuth of a star from its azimuth at elongation in terms of the difference of its hour angle from its hour angle at elongation, so that a series of observations of azimuth, half of them face right and half face left, may be made on a star for a short time before and after its elongation, just as of altitude in circum-meridian observations.

In our notation his formula becomes

$$(Z_0 - Z)'' = \frac{\sin Z_0 \,.\, \sin \delta}{\sin P} \cdot \frac{2 \sin^2 \dfrac{P_0 - P}{2}}{\sin 1''}$$

$$= \frac{\sin Z_0 \,.\, \sin \delta}{\sin P} \cdot \frac{t^2}{1833{\cdot}5}$$

where Z_0, P_0 = azimuth and hour angle at elongation and t = time in sidereal seconds from elongation.

If the observations are fairly evenly arranged on both sides of elongation, we can write sin P_0 for sin P, and simplify the formula to: $(Z_0 - Z)'' = \tan Z_0 \cdot \sin^2 \delta \cdot \frac{t^2}{1833 \cdot 5}$, as $\cos Z_0 = \sin \delta \ \sin P_0$.

Applying the average value of $Z_0 - Z$ to the average angle YZX, we get the mean angle from Y to the star at elongation. Using this formula for the extreme case of a star whose *polar distance* XP = *half the colatitude* ZP *of the place*, the author finds that for a period of 4 minutes the azimuth of the star at elongation does not vary more than 5″.

(*b*) *By Extra-Meridian Observation of a Star.* This is generally the most convenient observation, as it may be made at any time when the stars are visible and may be repeated as often as desired to obtain a good average result. The observer sights on the reference mark and reads the horizontal verniers, then sets his horizontal cross-hair a little ahead of the star and his vertical cross-hair on the star, following the star with the horizontal tangent screw until it reaches the intersection of the cross-hairs, then he reads both ends of the altitude bubble and both vertical and horizontal verniers, changes face, and repeats the process as quickly as possible, then sights on the reference mark again and reads the horizontal verniers again. This completes a single observation, ready for computation, but it should be repeated at least once for a more accurate result. It is safe to assume that the average azimuth of the star corresponds to its average altitude if the time between the pointings does not exceed 5 minutes. The barometer and thermometer must also be read for the refraction correction.

In the triangle ZPX (Fig. 9), as we know the latitude ϕ and the declination δ, we know the sides $ZP = 90° - \phi$

and $PX = 90° - \delta$, and we have measured the altitude a and, therefore, know the side $ZX = 90° - a$.

From the formula

$$\tan\frac{Z}{2} = \sqrt{\frac{\sin(s - ZX)\,.\,\sin(s - ZP)}{\sin s\,.\,\sin(s - PX)}}$$

we calculate the angle Z, then, by applying the average horizontal angle YZX to Z, we find the azimuth of Y, the reference mark

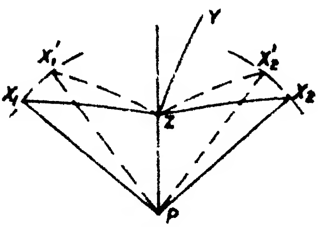


FIG. 9

Pairing Observations. After repeating the observation as many times as required for the desired accuracy on a star X_1, an equal number of similar observations should be made on a star X_2 *as similarly situated as possible on the other side of the meridian*, i.e. the altitude of the second star X_2 and its angle X_2ZP at the zenith should be as nearly as possible the same as the altitude of the first star X_1 and its angle X_1ZP at the zenith or Declinations and Hour Angles X_1PZ, X_2PZ should be equal. This is called *pairing* the observations and its object is to eliminate uncertainty as to refraction. For if we have underestimated the refraction correction, the position of both X_1 and X_2 is too high by the same amount (the altitudes of X_1 and X_2 being assumed equal), and this will make both ZX_1' and ZX_2' too small and the angles $X_1'ZP$, $X_2'ZP$ too large, by the same amount as the two triangles ZPX_1, ZPX_2 were assumed equal in all respects.

From star X_2, the azimuth of $ZY = 360° - YZX_2' - X_2'ZP$. From star X_1 it is $YZX_1' + X_1'ZP$. The average azimuth of ZY from the two sets of observations is, therefore, $180° + \frac{YZX_1' - YZX_2'}{2} + \frac{X_1'ZP - X_2'ZP}{2}$, and as this depends on the *difference* of $X_1'ZP$ and $X_2'ZP$, if both are too large (or too small) by the same amount it will be unaffected by an error in the assumed refraction correction. Of course, in practice, the triangles ZPX_1, ZPX_2 will never be quite the same, so the error will never be entirely eliminated, but the nearer the triangles are to equality the more will the error be minimized.

EXAMPLE 5. The following is an extra-meridian observation for the azimuth of a line ZY,

Object	Face	Altitude Level		Horizontal Circle		Vertical Circle	
		Object	Eye	Vernier *A*	Vernier *B*	Vernier *C*	Vernier *D*
Y	*R*		—	106° 27′ 50″	286° 28′ 00″	—	—
Star	*R*	5·8	5·2	44° 39′ 20″	224° 39′ 30″	28° 15′ 00″	208° 15′ 10″
Star	*L*	5·6	5·4	225° 42′ 20″	45° 42′ 30″	150° 59′ 30″	330° 59′ 40″
Y	*L*			286° 27′ 40″	106° 27′ 50″	—	—

The stars R.A. is 4h. 32m. 04·4s., declination 16° 22′ 34″ N., the latitude 51° 30′ 31″ N. The value of a level division is 20″, the barometer 30·0 in., the temperature 35° F. Find the azimuth of ZY.

The average observed altitude is

$$\frac{28° \ 15' \ 05'' + 29° \ 00' \ 25''}{2} = 28° \ 37' \ 45'':$$

as it is obviously rising, it must be to the east of the meridian (Fig. 10).

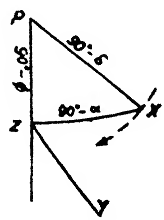


FIG. 10

The altitude level correction is $\frac{11{\cdot}4 - 10{\cdot}6}{4} \times 20'' = +4''$.

On reference to the table of Bessel's Refractions in

Chambers' Seven-Figure Mathematical Tables, we find mean refraction for an altitude of 28° 37′ 49″

$$= 108{\cdot}2'' - \frac{37{\cdot}8}{60} \times 4{\cdot}4'' = 105{\cdot}4''$$

and $B = 1{\cdot}014$ for 30 in., $T = 1{\cdot}028$, $t = 1{\cdot}000$ for 35° F.

∴ Refraction correction = 105·4″ × 1·028 × 1·014 = 109·9″ say, 1′ 50″.

∴ a = corrected altitude = 28° 37′ 49″ − 1′ 50″ = 28° 35′ 59″.

We then prepare a table—

					log sine	
a 28° 35′ 59″	ZX = 61° 24′ 01″	$s - ZX$	= 25° 21′ 27″		$\bar{1}{\cdot}631,7125$	$\bar{1}{\cdot}504,5955$
ϕ = 51° 30′ 31″	ZP = 38° 29′ 29″	$s - ZP$	= 48° 15′ 59″		$\bar{1}{\cdot}872,8830$	
δ = 16° 22′ 34″	PX = 73° 37′ 26″	$s - PX$	= 13° 08′ 02″		$\bar{1}{\cdot}356,4606$	$\bar{1}{\cdot}355,7649$
	2)173° 30′ 56″	s	= 86° 45′ 28″		$\bar{1}{\cdot}999,3043$	
						2)0·148,8306
					log tan $\frac{Z}{2}$	0·074,2153

$$\therefore \frac{Z}{2} = 49^\circ 53' 05{\cdot}6''. \qquad Z = 99^\circ 46' 11''$$

The average horizontal angle from ZY to the star

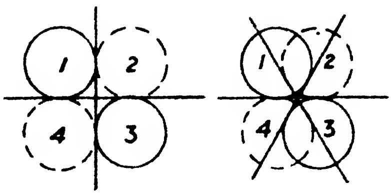
$$= \frac{61^\circ 48' 30'' + 60^\circ 45' 20''}{2} = 61^\circ 16' 55''$$

∴ Azimuth of ZY (clockwise from north)

$$= 99^\circ 46' 11'' + 61^\circ 16' 55'' = 161^\circ 03' 06''.$$

Extra-Meridian Observation for Azimuth on the Sun (Fig. 11). This observation can be made on the sun, but in this case the sun's disc should be made to touch the horizontal and vertical hairs in two opposite quadrants, say (1) and (3) in Fig. 11 (1) in the F.R. and F.L. observations, so as to eliminate its semi-diameter both in altitude and azimuth.

This completes one observation, but it is well to follow it with a second observation, in which the sun's disc is placed successively in the remaining two quadrants, say (2) and (4) in Fig. 11 (1), as it is more difficult than with a star to get both horizontal and vertical contacts simultaneously. If there are three cross-hairs, the contacts can be as shown in Fig. 11 (2). For both contacts the horizontal hair should be set in advance and the side of the disc followed with the vertical (or inclined) hair by means of the horizontal tangent screw until the contact takes place with the horizontal hair. The times must be noted when each contact takes place, so that the sun's declination at the average time may be computed and employed in the calculation. This, of course, demands an approximate knowledge of Greenwich mean time. The altitude must be corrected for parallax. A sun observation can only be "paired" on the sun itself after a considerable time has elapsed, during which the refraction conditions may have altered.



(1) (2)

Fig. 11

Extra-Meridian Observation by Hour Angle. It is obvious that from this observation of altitude we can also find the hour angle P, and hence the L.S.T., from a star or the L.M.T. from the sun. But in the best work the observation is made first as a Time observation simply, altitudes, altitude level, and chronometer times only being recorded, and is paired with a second star on the other side of the meridian to eliminate refraction error. The chronometer error is thus found.

Then the azimuth observation is made, recording chronometer times and horizontal circle readings, but only *approximate* altitude readings for finding the correction of the horizontal angles due to inclination of the horizontal axis, this latter being measured with the striding level, direct and reversed, for each pointing. The hour angle of the star is found by subtracting the star's R.A. from the L.S.T., as found from the corrected chronometer reading at the average time of the two pointings (or the equation of time from the L.M.T. in the case of the sun). Then, in the triangle ZPX, we have the angle P and the two sides $PX = 90° - \delta$, $ZP = 90° - \phi$, and can calculate Z from the formulae for $\tan\frac{Z+X}{2}$ and $\tan\frac{Z-X}{2}$, and hence find the azimuth of ZY from Z and the average horizontal angle YZX.

The advantages are that the vertical hair can be set a little in advance of the star or sun, and the instant noted when contact is made (near the horizontal hair) without touching the instrument; altitudes need only be read roughly, the altitude level need not be read, and refraction does not affect the observation at all—the sun or star can be taken just above the horizon; but, on the other hand, it is obvious that the time must be known very accurately.*

Effect of an Error in Altitude on the Azimuth. We have

$$\cos Z = \frac{\cos PX - \cos ZP \cdot \cos ZX}{\sin ZP \cdot \sin ZX} = \frac{\sin\delta - \sin\phi \cdot \sin a}{\cos\phi \cdot \cos a}$$

Differentiating, we have

$$-\sin Z \cdot dZ = \left(\frac{\sin\delta}{\cos\phi} \cdot \frac{\sin a}{\cos^2 a} - \frac{\sin\phi}{\cos\phi} \cdot \frac{1}{\cos^2 a}\right) da$$

$$= \frac{\sin\delta \cdot \sin a - \sin\phi}{\cos\phi \cdot \cos^2 a} \cdot da$$

* *Astronomy for Surveyors*, by Rice-Oxley and Shearer (Methuen & Co., Ltd., London).

$$\text{but } \cos X = \frac{\cos ZP - \cos PX \,.\, \cos ZX}{\sin PX \,.\, \sin ZX}$$

$$= \frac{\sin \phi - \sin \delta \,.\, \sin a}{\cos \delta \,.\, \cos a}$$

$$\therefore \; dZ = \frac{\sin \phi - \sin \delta \,.\, \sin a}{\sin Z \,.\, \cos \phi \,.\, \cos^2 a} \,.\, da$$

$$= \frac{\cos X \,.\, \cos \delta \,.\, \cos a}{\sin Z \,.\, \cos \phi \,.\, \cos^2 a} \,.\, da$$

$$= \cot X \,.\, \sec a \,.\, da; \text{ as } \frac{\sin Z}{\sin X} = \frac{\cos \delta}{\cos \phi}$$

Therefore, to keep the error in Z as small as possible, X should be as near 90° as possible, i.e. if the star is circumpolar it should be near elongation, if not, it should be as far from the meridian as possible, and the altitude should be as small as possible, but not, of course, less than 20° on account of the uncertainty of refraction at low altitudes. This last restriction does not apply when *Hour Angles* are observed.

Effect of an Error in Latitude on the Azimuth.

$$\cos Z = \frac{\sin \delta - \sin \phi \,.\, \sin a}{\cos \phi \,.\, \cos a}$$

Differentiating, we have

$$-\sin Z \,.\, dZ = \left(\frac{\sin \delta}{\cos a} \cdot \frac{\sin \phi}{\cos^2 \phi} - \frac{\sin a}{\cos a \,.\, \cos^2 \phi}\right) d\phi$$

$$\therefore \; dZ = \frac{\sin a - \sin \delta \,.\, \sin \phi}{\sin Z \,.\, \cos a \,.\, \cos^2 \phi} \,.\, d\phi$$

Now,

$$\cos P = \frac{\cos ZX - \cos PX \,.\, \cos ZP}{\sin PX \,.\, \sin ZP} = \frac{\sin a - \sin \delta \,.\, \sin \phi}{\cos \delta \,.\, \cos \phi}$$

$$\therefore dZ = \frac{\cos P \cdot \cos \delta \cdot \cos \phi}{\sin Z \cdot \cos \alpha \cdot \cos^2 \phi} \cdot d\phi = \cot P \cdot \sec \phi \cdot d\phi,$$

$$\text{as } \frac{\sin Z}{\sin P} = \frac{\sin PX}{\sin ZX} = \frac{\cos \delta}{\cos \alpha}$$

Hence, for the error in azimuth to be as small as possible, the hour angle should be nearly 90° or ± 6 hours. Obviously, too, the method is unsuitable for high latitudes.

Effect of an Error in Declination on the Azimuth. This would occur in a sun observation if there was an error in local time or longitude, producing an error in the assumed Greenwich mean time, and therefore in the assumed declination.

$$\cos Z = \frac{\sin \delta - \sin \phi \cdot \sin \alpha}{\cos \phi \cdot \cos \alpha}$$

$$\therefore \quad -\sin Z \cdot dZ = \frac{\cos \delta}{\cos \phi \cdot \cos \alpha} \cdot d\delta$$

$$\therefore \quad dZ = -\frac{\cos \delta \cdot d\delta}{\sin Z \cdot \cos \phi \cdot \cos \alpha} = -\frac{d\delta}{\sin P \cdot \cos \phi}$$

$$= -\operatorname{cosec} P \cdot \sec \phi \cdot d\delta$$

Therefore, here again the hour angle should be as nearly 90° as possible, and the method is unsuited to high latitudes as sec ϕ would be large.

Daylight Observations. It is sometimes convenient to make an observation for the azimuth of a survey line in the early morning or late afternoon, when the station can be seen without illumination. The meridian must have been previously fixed approximately by a reference mark, and the local sidereal time must be known approximately.

Choosing a bright star, its hour angle is calculated for a convenient instant, and from the hour angle P and the sides $PX = 90° - \delta$, $ZP = 90° - \phi$, the azimuth Z of the

star is calculated and its altitude from $\cos a = \frac{\sin P}{\sin Z} \cdot \cos \delta$.
The refraction correction is *added* to this altitude and just before the instant assumed the theodolite is set to the required azimuth and altitude, when the star should be visible in the field of view and exact observations can be taken on it.

Time

The most obvious way to determine the local time at any instant on the chronometer would be to observe the time of transit of a star or of the sun across the meridian, as in the former case, the R.A. of the star would give the L.S.T. of upper transit, or the R.A. ± 12 hours would give the L.S.T. at lower transit, while in the latter case the equation of time would give the L.M.T. at the average watch time of transit of the sun's western and eastern limbs. But this would involve previous observations to find the true meridian very exactly, and the theodolite would have to be in perfect adjustment so as to sweep out a perfectly vertical great circle, or corrections for inclination of trunnion axis and for collimation must be applied.

Extra-Meridian Observation of a Star for Time. This is conducted similarly to the extra-meridian observation for azimuth, except that the star does not need to be followed by the vertical hair, so long as it crosses the horizontal hair close to the vertical hair, the observer calling "Up" or "Now" to his assistant at the instant of crossing, the latter recording the chronometer time of crossing. The face of the instrument is now changed and the horizontal hair re-set for a second timing of the instant of crossing, the two pointings constituting one observation. The altitude level must, of course, be read each time before the vertical verniers. Having corrected the average vertical angle from the four vernier readings for the altitude level and for refraction, we calculate the hour angle P from

$$\tan\frac{P}{2} = \sqrt{\frac{\sin(s-ZP)\,.\,\sin(s-PX)}{\sin s\,.\,\sin(s-ZX)}}$$

and hence the L.S.T. = R.A. $\pm$ P, according as the hour angle is west or east.

For accuracy, at least another similar observation should be made on the star and then the observation should be "paired" by an equal number of observations on a star similarly situated on the other side of the meridian. This eliminates any uncertainty as to the refraction correction, or at least minimizes it, for in the one case the L.S.T. is $(\text{R.A.})_1 + P_1$ and in the other $(\text{R.A.})_2 - P_2$, so that the *average* L.S.T. is $\frac{(\text{R.A.})_1 + (\text{R.A.})_2 + P_1 - P_2}{2}$, which nearly eliminates any error in the refraction correction, as P_1 and P_2 are almost equally affected. The second pointing in each case should be made as soon as possible after the first, the average value of P from each two pointings should be worked out as a separate observation, and the various chronometer errors thus found should then be averaged, for we are assuming that the star is at the average altitude at the average time, which is only true over a short interval of time.

Just as in the corresponding azimuth observation, the sun may be observed instead of a star, the upper and lower limb being alternately sighted on to eliminate semi-diameter in altitude, the contact being close to the vertical hair, so that the vertical hair apparently bisects the disc. The altitude must be corrected for parallax as well as for altitude level and for refraction. The local mean time = $P \pm$ equation of time, but for both the declination and the equation of time a knowledge of the approximate Greenwich mean time is necessary.

EXAMPLE 6. In Example 5 the times noted on a sidereal time chronometer when the star's altitudes were observed were 0h. 12m. 15·4s. and 0h. 17m. 16·0s. Find the error of the chronometer.

We have already,

log sine $s - ZP$	$\bar{1}$·872,8830	}	$\bar{1}$·229,3436
log sine $s - PX$	$\bar{1}$·356,4606	}	
log sine $S - ZX$	$\bar{1}$·631,7125	}	$\bar{1}$·631,0168
log sine S	$\bar{1}$·999,3043	}	
			2) $\bar{1}$·598,3268
log tan $\frac{P}{2}$			$\bar{1}$·799,1634

$$\therefore \frac{P}{2} = 32° \ 12' \ 01{\cdot}4''.$$

$$\therefore P = 64° \ 24' \ 02{\cdot}8'' = 4\text{h. } 17\text{m. } 36{\cdot}2\text{s.}$$

which we can consider negative as the star is east.

∴ L.S.T. = 4h. 32m. 04·4s. − 4h. 17m. 36·2s.

= 0h. 14m. 28·2s.

Average of the chronometer times = 0h. 14m. 45·7s.

∴ the chronometer error is 17·5s. fast.

Effect of an Error in Altitude on a Time Observation. We have

$$\cos P = \frac{\cos ZX - \cos PX \cdot \cos ZP}{\sin PX \cdot \sin ZP}$$

$$= \frac{\sin a - \sin \delta \cdot \sin \phi}{\cos \delta \cdot \cos \phi}$$

Differentiating, $-\sin P \cdot dP = \frac{\cos a}{\cos \delta \cdot \cos \phi} da$

$$\therefore\ dP = -\frac{\cos a}{\sin P \,.\, \cos \delta \,.\, \cos \phi} \,.\, da$$

$$= -\operatorname{cosec} Z \,.\, \sec \phi \,.\, da\ ;\ \text{as } \frac{\sin P}{\sin Z} = \frac{\cos a}{\cos \delta}$$

Therefore the effect on the hour angle of an error in altitude is least when $Z = 90°$, i.e. when the star is on the Prime Vertical, and will be large in high latitudes.

Effect of an Error in Latitude on a Time Observation.

$$\cos P = \frac{\sin a - \sin \delta \,.\, \sin \phi}{\cos \delta \,.\, \cos \phi}.$$

Differentiating,

$$-\sin P \,.\, dP = \left(\frac{\sin a \,.\, \sin \phi}{\cos \delta \,.\, \cos^2 \phi} - \frac{\sin \delta}{\cos \delta \,.\, \cos^2 \phi}\right) d\phi$$

$$\therefore\ dP = \frac{\sin \delta - \sin a \,.\, \sin \phi}{\sin P \,.\, \cos \delta \,.\, \cos^2 \phi}\, d\phi$$

$$\text{But } \cos Z = \frac{\sin \delta - \sin a \,.\, \sin \phi}{\cos a \,.\, \cos \phi}$$

$$\therefore\ dP = \frac{\cos Z \,.\, \cos a \,.\, \cos \phi}{\sin P \,.\, \cos \delta \,.\, \cos^2 \phi} \,.\, d\phi$$

$$= \cot Z \,.\, \sec \phi \,.\, d\phi\ ;\ \text{as } \frac{\sin P}{\sin Z} = \frac{\cos a}{\cos \delta}$$

Therefore the effect on the hour angle of an error in latitude will be zero, when $Z = 90°$, i.e. when the star is on the Prime Vertical, but will be very large in high latitudes.

Effect of an Error in Declination on a Time Observation (on the Sun).

$$\cos P = \frac{\sin a - \sin \delta \,.\, \sin \phi}{\cos \delta \,.\, \cos \phi}$$

Differentiating,

$$-\sin P \,.\, dP = \left(\frac{\sin a}{\cos \phi} \cdot \frac{\sin \delta}{\cos^2 \delta} - \frac{\sin \phi}{\cos \phi \,.\, \cos^2 \delta}\right) d\delta$$

$$\therefore \quad dP = \frac{\sin \phi - \sin a \,.\, \sin \delta}{\sin P \,.\, \cos \phi \,.\, \cos^2 \delta} \,.\, d\delta$$

$$\text{but} \quad \cos X = \frac{\sin \phi - \sin a \,.\, \sin \delta}{\cos a \,.\, \cos \delta}$$

$$\therefore \quad dP = \frac{\cos X \,.\, \cos a \,.\, \cos \delta}{\sin P \,.\, \cos \phi \,.\, \cos^2 \delta} \,.\, d\delta$$

$$= \cot X \,.\, \sec \delta \,.\, d\delta$$

$$\text{as} \quad \frac{\sin P}{\sin X} = \frac{\sin ZX}{\sin ZP} = \frac{\cos a}{\cos \phi}$$

The effect of an error in the declination is, therefore, less the farther the sun is from the meridian and the nearer it is to the Equator.

Time by Equal Altitudes of a Star. This is a very simple observation, requiring no knowledge of the latitude or declination, but generally demanding a long period of waiting. A star is sighted rising or setting and the time is recorded when it crosses the horizontal hair near the vertical hair, the instrument is then turned in azimuth until the star again appears setting or rising in the field of view. It is then followed with the vertical cross-hair by means of the horizontal tangent screw till it crosses the horizontal hair near to the vertical hair *at the same altitude as before*, when the time is again recorded. Then obviously the average of the two recorded times is the watch time of upper or lower transit. The face of the theodolite must not be changed, and just before each observation the altitude bubble should be made exactly central by means of the clip screws. Better results are obtained if (say) four pairs of observations are taken on the same star, the altitude being read and then altered about $\frac{1}{2}$° for the next observation, the instrument being re-set to the same altitudes in the reverse order as the star returns to them. Then the average of all the times is taken as the watch time of transit.

It is important that the star should be well away from its transit when observed, as its altitude reaches a maximum (or minimum) at transit and consequently changes very slowly there. As shown on page 176, $dP = -\operatorname{cosec} Z \,.\, \sec \phi \,.\, da$. Therefore $\frac{da}{dP} = -\sin Z \,.\, \cos \phi$, and as P changes at a uniform rate of 15" per second, we see that *the rate of increase or decrease of altitude of any star is a maximum when* $Z = 90°$, *i.e. when the star is on the Prime Vertical*, when $\frac{da}{dt} = -15'' \,.\, \cos \phi$ where t is in sidereal seconds. The star should, therefore, be near the Prime Vertical when observed. A possible source of error is a change in the refractive condition of the air between the two observations if the interval is a long one. This interval can be reduced by choosing a star whose declination is not much less than the latitude, so that the interval between its crossings of the Prime Vertical will not be large. If the declination were equal to the latitude it would pass through the zenith, so the altitude of observation will be considerable and a diagonal eye-piece will be needed to turn the line of sight through 90°.

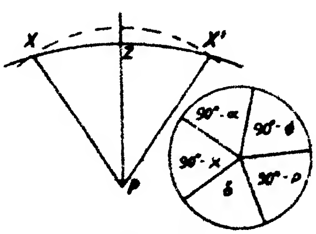


FIG. 12

In Fig. 12, which represents a star crossing the Prime Vertical at X and X', we have a triangle XZP, right-angled at Z, then $\cos P = \tan \delta . \cot \phi = \frac{\tan \delta}{\tan \phi}$, which gives P, the interval being $\frac{2P^\circ}{15}$ hours; also $\sin \delta = \sin a . \sin \phi$, or $\sin a = \frac{\sin \delta}{\sin \phi}$, which gives the altitude of crossing the Prime Vertical.

EXAMPLE 7. a Persei (R.A. 3h. 19m. 31·7s., declination 49° 37′ 28″ N.) is to be observed for time by equal altitudes on the Prime Vertical. Find the altitude when on the Prime Vertical and the Greenwich mean times of crossing it, the latitude being 51° 23′ 00″ N., the longitude 1° 23′ 00″ E., and the sidereal time of previous midnight at Greenwich 5h. 32m. 17·9s.

sin 49° 37′ 28″	$\bar{1}$·881,8504	tan 49° 37′ 28″	0·070,4119
sin 51° 23′ 00″	$\bar{1}$·892,8395	tan 51° 23′ 00″	0·097,5805
sin a	$\bar{1}$·989,0109	cos P	$\bar{1}$·972,8314

∴ a = 77° 09′ 54″
Add refraction = 0′ 13″
∴ Observed altitude = 77° 10′ 07″

∴ P = 20° 03′ 21·2″ = 1h. 20m. 13·4s.
L.S.T. of transit = 3h. 19m. 31·7s.
∴ L.S.T. of prime vertical transits = 1h. 59m. 18·3s. and 4h. 39m. 45·1s.

$$\text{S.T. of 0 hours L.M.T.} = 5\text{h. } 32\text{m. } 17{\cdot}9\text{s.} - \frac{1{\cdot}38}{15} \times 9{\cdot}857\text{s.}$$
$$= 5\text{h. } 32\text{m. } 17{\cdot}0\text{s.}$$

Sidereal intervals from 0h. L.M.T. are

L.S.T. + 24h. − 5h. 32m. 17·0s.
= 20h. 27m. 01·3s. and 23h. 07m. 28·1s.

∴ L.M.T.'s are

20h. 27m. 01·3s. − (3m. 16·59s. + 4·42s. + 0·00s.)
= 20h. 27m. 01·3s. − 3m. 21·0s. = 20h. 23m. 40·3s.
and 23h. 07m. 28·1s. − (3m. 46·08s. + 1·15s. + 0·08s.)
= 23h. 07m. 28·1s. − 3m. 47·3s. = 23h. 03m. 40·8s.

Longitude = 5m. 32s. East

∴ G.M.T.'s are 20h. 18m. 08·3s. and 22h. 58m. 08·8s.

Therefore the transits of prime vertical are at

8h. 18m. 08·3s. p.m. and 10h. 58m. 08·8s. p.m.

The theodolite should therefore, be set to the above altitude and the star observed a little before the first above time (to allow for chronometer error) and the chronometer time noted when the star crosses the horizontal hair. A little before the second above time the star is observed again, the chronometer time being recorded when it crosses the horizontal hair.

Time by Equal Altitudes of the Sun. This demands an approximate knowledge of the latitude and of the Greenwich time, as a correction must be applied for the change of declination of the sun during the observations. In all the observations the same (upper or lower) limb of the sun must be on the horizontal cross-hair, and the vertical hair should bisect the sun's face when the contact is made. On page 177 we had

$$dP = \frac{\sin\phi - \sin\alpha \,.\, \sin\delta}{\sin P \,.\, \cos\phi \,.\, \cos^2\delta} \,.\, d\delta$$

But $\sin\alpha = \sin\delta \,.\, \sin\phi + \cos\delta \,.\, \cos\phi \,.\, \cos P$

$$\therefore\ dP = \frac{\sin\phi - \sin^2\delta \,.\, \sin\phi - \sin\delta \,.\, \cos\delta \,.\, \cos\phi \,.\, \cos P}{\sin P \,.\, \cos\phi \,.\, \cos^2\delta}\, d\delta$$

$$= \left(\frac{\sin\phi \,.\, \cos^2\delta}{\sin P \,.\, \cos\phi \,.\, \cos^2\delta} - \frac{\sin\delta \,.\, \cos\delta \,.\, \cos\phi \,.\, \cos P}{\sin P \,.\, \cos\phi \,.\, \cos^2\delta}\right) d\delta$$

$$= (\operatorname{cosec} P \,.\, \tan\phi - \cot P \tan\delta)\, d\delta$$

Therefore, if the sun's declination has increased by $d\delta''$, the afternoon hour angle is greater than the morning hour angle for the same altitude by $(\operatorname{cosec} P \,.\, \tan\phi - \cot P \,.\, \tan\delta)\, d\delta''$.

The average hour angle is, therefore, increased by half this amount and we must apply a correction of

$$-\frac{d\delta''}{15 \times 2}(\text{cosec } P \,.\, \tan \phi - \cot P \,.\, \tan \delta) \text{ seconds}$$

to the average time of the two equal altitudes. P is half the mean time interval between the two observations expressed in arc, and $\frac{d\delta}{2}$ is the change of declination in this half interval. A number of pairs of observations should be taken and the correction can be calculated for the average interval of the pairs. "Declination increasing" means the sun rising daily higher in the sky. If the sun is falling daily in the sky the correction must be *added* to the average time of the observations.

Azimuth by Equal Altitudes. The method of equal altitudes may be used also for determining azimuth, but in this case horizontal angles must be read instead of chronometer times, and the face of the theodolite should be changed after each observation before transit and re-set to the same altitude and same face successively for the observations after transit. There should be an equal number of F.R. and F.L. observations. The algebraic average of the horizontal angles from the reference mark is then the angle between the reference mark and the south or north, as the case may be. The sun may also be used in this method, but in this case, where the L.H. limb of the sun has touched the vertical cross-hair before noon at any particular altitude, the R.H. limb must be made to touch it at the corresponding altitude in the afternoon and *vice versa,* and there should be an equal number of L.H. and R.H. contacts in both morning and afternoon.

A correction must be applied for the changing declination of the sun. It has been shown on page 172 that an increase of declination $d\delta$ produces a decrease of Z, i.e. XZP, by cosec P sec $\phi \cdot d\delta$, it therefore produces an *increase* of the azimuth of the sun reckoned from the transit T (Fig. 13). From the algebraic average of the two angles between the sun and the reference mark Y we must, therefore, subtract a correction $\frac{d\delta}{2} \cdot$ cosec P . sec ϕ when the declination is increasing and add it when the declination is decreasing, where $P =$ half the mean time interval between the observations expressed in arc and $\frac{d\delta}{2}$ is the change of declination in this half interval. The watch time of each sun observation must, therefore, be taken and the average interval, $2P$, between the morning and afternoon observations used for finding the correction.

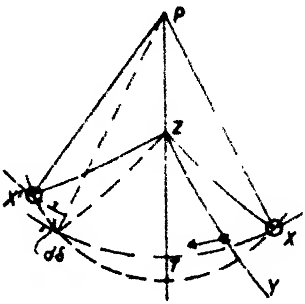


Fig. 13

Longitude

The difference of longitude of two places on the earth's surface is the difference of their local times (sidereal, mean, or apparent) at the same instant. The principal methods of determining longitude differences are by (*a*) *Triangulation* (as explained in Chapter V), (*b*) *Chronometer*, and (*c*) *Signals*.

(*a*) *Triangulation*. This method is the most accurate, but involves a knowledge of the earth's dimensions and is a very expensive method unless the triangulation is required for other purposes.

(*b*) *Chronometer*. Here a number of chronometers keeping local time at place A, with known errors and rates, are transported to place *B* and there compared with chronometers which have been recently checked, as to error and rate, on the local time at *B*. The difficulty is to find the "travelling rate" of the transported chronometers, especially on a land journey, and to be sure that it is uniform.

(*c*) *Signals*. The great developments in wireless telegraphy all over the world render the method of wireless signals the simplest and most accurate method where a triangulation is unavailable. At an increasing number of wireless stations time signals are sent out once or twice a day on the Rhythmic Time system, actuated by a pendulum, at the rate of 61 dots to a mean time minute, lasting for 5 minutes. At the beginning of each minute a dash is sent instead of a dot, the dash commencing at the beginning of the minute. The result is a "time vernier," the interval between two successive dots being $\frac{60}{61} = \cdot 9836$ seconds. The observer notes on his chronometer the hour, minute, and second at the beginning of each minute of the signals and the succeeding second on his chronometer when a dot coincides with a second on his chronometer. By subtraction of his chronometer times he gets the number of seconds, which is the number of beats of the pendulum, since the commencement of the minute of the time signal and this number, multiplied by ·9836 seconds and added to the minute of the time signal, is the signalled time which coincides with the chronometer time at the coincidence.

EXAMPLE 8. Fig. 14 shows that the first minute (18h. 0m. G.M.T.) of the time signal was recorded at 14h. 23m. 54s. on the L.M.T. chronometer, and that the next coincidence occurred at 14h. 24m. 03s., i.e. 9s. later. Therefore, the time signal at 18h. 0m. + 9 × ·9836s., i.e. at 18h. 0m. 8·8524s., coincided with 14h. 24m. 03s. chronometer time. Therefore, the chronometer is 3h. 36m. 5·85s. slow on the time signal. Repeating this observation for the coincidences in each of the remaining 4 minutes of the time signal, the observer averages the five results. We will suppose the average result to be: chronometer 3h. 36m. 5·92s. slow on the time signal.

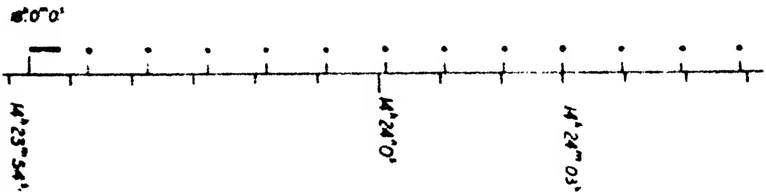


FIG. 14

We will also assume that the chronometer was found to be 1·09s. fast by a paired star observation at 2h. 22m. 30s. and 1·67s. fast by a paired star observation at 22h. 17m. 30s. Then at 14h. 26m. 30s. the chronometer was

$$1{\cdot}09 + \frac{12\text{h. } 04\text{m.}}{19\text{h. } 55\text{m.}} \times 0{\cdot}58\text{s.} = 1{\cdot}44\text{s. fast on L.M.T.}$$

Local time is 3h. 36m. 7·36s. slow on the time signal.

Difference of longitude $= 45° + 9° + 1' \ 50{\cdot}4''$

$= 54° \ 1' \ 50{\cdot}4''$ W

CHAPTER V

THE CURVATURE OF THE EARTH AND ITS EFFECT ON SURVEYS AND LEVELS

Curvature of the Earth

In the chapters on Astronomical Work we have assumed that the earth is a sphere. If it were so, the distance of the surface from its centre would be everywhere the same, viz. R, the radius of the sphere, the vertical at any point of the surface (i.e. the normal to the horizon) would pass through the centre of the sphere and its inclination to the equator would be the latitude, ϕ, of the place (Fig. 1).

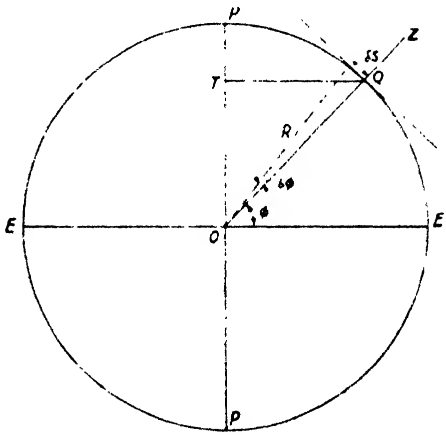


Fig. 1

The equator, meridians, and all (apparently) straight lines would be "great circles" of radius R. The length of 1″ of latitude would be $R \sin 1''$ and would be constant, while that of 1″ of longitude would be $R \cos \phi \,.\, \sin 1''$, as the parallel of latitude ϕ is a small circle of radius $R \cos \phi$. The radius of the sphere could be found by measuring the difference of latitude $\delta\phi$ of any two points on the same meridian and the distance δs between them, as $R = \dfrac{\delta s}{\delta \phi}$.

By the "surface" of the earth we do not, of course, mean the irregular surface of the land but the surface of the "Geoid" or "mean sea level surface," assumed to be continued across the land. This surface, indeed, shows small irregularities, due to variations in gravity and attractions of mountains, so we approximate to it with a "Spheroid of Reference," as the Geoid is found to be nearly an "Oblate Ellipsoid of Revolution," formed by the rotation of an ellipse about its minor axis, and treat the small variations of the Geoid from the spheroid as irregularities. According to the latest measurements the major semi-axis of the ellipse, $a = 20{,}926{,}500$ ft. and the minor semi-axis $b = 20{,}856{,}000$ ft. The compression, $c = \frac{a-b}{a} = \frac{1}{297 \cdot 0}$, and the (eccentricity)^2 $= e^2 = \frac{a^2 - b^2}{a^2} = 0{\cdot}006724$, $e = 0{\cdot}0820$, so that the spheroid is not very different from a sphere.

The vertical, or normal to the surface, at any point P does not pass through the centre (except at the equator and poles) but bisects the angle FPF' between the two focal distances PF, PF', so that we have now two definitions of latitude: the inclination ϕ (Fig. 2) of the normal PG to the equator is called the "geographical latitude," while the inclination L of the radius vector OP to the equator is called the "geocentric latitude." The former is the latitude determined by an astronomical observation and used on maps. The ***radius of curvature of the meridian***, $\rho = \frac{ds}{d\phi}$, increases from the equator to the poles, and if we mark off its length along the normal from P we get the "centre of curvature," H, of the meridian. The length of 1″ of latitude is $\rho \sin 1''$ and increases about 1 per cent from equator to poles. If we produce the normal PG to meet the minor axis at K, K is the centre of curvature perpendicular to the meridian, for as the ellipse rotates about the minor axis the normal always passes through K. The distance $PK = \nu$ is, therefore, the *radius of curvature perpendicular to the meridian*, the radius of the parallel of latitude $PM = \nu \cos \phi$, and the length

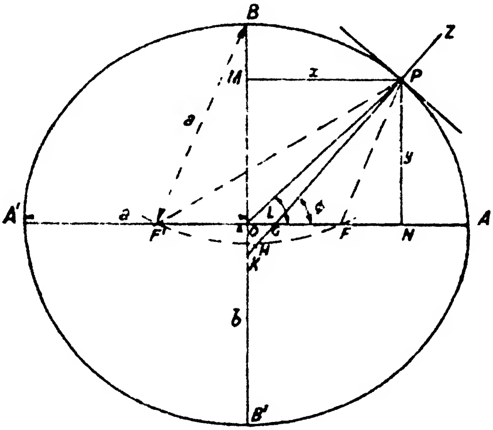


FIG. 2

of 1″ of longitude is $\nu \cos \phi \,.\, \sin 1''$, where ν increases about $\frac{1}{3}$ per cent from the equator to the poles.

To find expressions for these two radii of curvature ρ and ν, the geocentric latitude L, and the radius vector $r = OP$ in terms of the geographical latitude, ϕ.

Taking the usual equation to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and using $e^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2}$, we have

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a^2 - x^2) = (1 - e^2)(a^2 - x^2)$$

$$\therefore \qquad 2y \,.\, \frac{dy}{dx} = -2x(1 - e^2)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y}(1-e^2) = -\frac{x\sqrt{1-e^2}}{\sqrt{a^2-x^2}} = -\cot\phi \quad . \qquad (1)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{\sqrt{a^2-x^2}\,.\,\sqrt{1-e^2} + \dfrac{2x^2\sqrt{1-e^2}}{2\sqrt{a^2-x^2}}}{a^2-x^2}$$

$$= -\frac{(a^2-x^2)\sqrt{1-e^2} + x^2\sqrt{1-e^2}}{(a^2-x^2)^{\frac{3}{2}}} = -\frac{a^2\sqrt{1-e^2}}{(a^2-x^2)^{\frac{3}{2}}}$$

$$\therefore \quad \rho = \frac{\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \frac{\left(1+\dfrac{x^2(1-e^2)}{a^2-x^2}\right)^{\frac{3}{2}}}{\dfrac{a^2\sqrt{1-e^2}}{(a^2-x^2)^{\frac{3}{2}}}}$$

$$= -\frac{(a^2-e^2x^2)^{\frac{3}{2}}}{a^2\sqrt{1-e^2}} \qquad (2)$$

Now, from (1), $\tan\phi = \dfrac{\sqrt{a^2-x^2}}{x\sqrt{1-e^2}}$

$$\therefore \quad x^2(1-e^2)\tan^2\phi = a^2 - x^2$$

$$\therefore \quad x^2 = \frac{a^2}{\sec^2\phi - e^2\tan^2\phi} = \frac{a^2\cos^2\phi}{1-e^2\sin^2\phi}$$

$$\text{or} \quad x = \frac{a\cos\phi}{\sqrt{1-e^2\sin^2\phi}} \quad . \quad . \quad . \qquad (3)$$

$$\therefore \quad \rho = -\frac{\left(a^2 - \dfrac{e^2a^2\cos^2\phi}{1-e^2\sin^2\phi}\right)^{\frac{3}{2}}}{a^2\sqrt{1-e^2}}$$

$$= -\frac{(a^2-e^2a^2)^{\frac{3}{2}}}{a^2\sqrt{1-e^2}\,.\,(1-e^2\sin^2\phi)^{\frac{3}{2}}} = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}} \qquad (4)$$

And $\nu = x\sec\phi = \dfrac{a}{\sqrt{1-e^2\sin^2\phi}}$ from (3) $\qquad (5)$

At the equator, $\phi = 0°$, we have $\rho = a(1 - e^2) = \dfrac{b^2}{a}$, $\nu = a$

At the poles, $\phi = 90°$, we have $\rho = \dfrac{a}{\sqrt{1 - e^2}} = \dfrac{a^2}{b} = \nu$

$$\frac{\tan L}{\tan \phi} = \frac{\dfrac{y}{x}}{\dfrac{\sqrt{a^2 - x^2}}{x\sqrt{1 - e^2}}} = \frac{y\sqrt{1 - e^2}}{\sqrt{a^2 - x^2}}$$

$$= \frac{\sqrt{1 - e^2}\sqrt{a^2 - x^2}\sqrt{1 - e^2}}{\sqrt{a^2 - x^2}}$$

$$= 1 - e^2 = \frac{b^2}{a^2} = 0{\cdot}993276 \quad . \quad . \quad (6)$$

Also $x = r \cos L$, $y = r \sin L$

$$\therefore \quad \frac{r^2 \cos^2 L}{a^2} + \frac{r^2 \sin^2 L}{b^2} = 1$$

$$\therefore \quad r^2 = \frac{a^2b^2}{a^2 \sin^2 L + b^2 \cos^2 L} = \frac{a^2b^2(1 + \tan^2 L)}{a^2 \tan^2 L + b^2}$$

$$= \frac{a^2b^2\left(1 + \dfrac{b^4}{a^4}\tan^2 \phi\right)}{a^2\dfrac{b^4}{a^4}\tan^2 \phi + b^2} = \frac{a^4 + b^4 \tan^2 \phi}{a^2 + b^2 \tan^2 \phi} \quad (7)$$

It can be shown that the radius of curvature of a section by a vertical plane inclined at an angle α to the meridian is $\dfrac{\rho\nu}{\rho \sin^2 \alpha + \nu \cos^2 \alpha}$ which becomes ρ when $\alpha = 0$, and ν when $\alpha = 90°$.

In the determination of the figure of the earth a long triangulation is run for hundreds of miles north and south, and at each station the latitude and the azimuth of the lines are determined astronomically. The stations cannot, of course, be on the same meridian, but they can be projected on to the same meridian; the average radius of curvature between each pair of stations can then be found as $\frac{ds}{d\phi}$. Then if ρ_1 and ρ_2 are two radii of curvature thus obtained for average latitudes ϕ_1 and ϕ_2, we have

$$\rho_1(1 - e^2 \sin^2 \phi_1) = a(1 - e^2) = \rho_2(1 - e^2 \sin^2 \phi_2)^{\frac{3}{2}}$$

an equation which gives e^2; then, by substitution, we find a and hence $b = a\sqrt{1 - e^2}$.

As the variation of the acceleration of gravity, g, with the latitude depends on $c = \frac{a - b}{a}$, valuable confirmation of the shape of the earth is obtained by observations of the value of g at various latitudes, and the value of c thus obtained is in very close agreement with that obtained by the measurement of arcs of the meridian.

For the surveyor the most convenient form in which the two radii of curvature can be given is a table giving at frequent intervals of latitude the length of 1" of latitude and the length of 1" of longitude in feet. Such tables are given in Sir C. F. Close's *Textbook of Topographical and Geographical Surveying*, at intervals of 5′ of latitude from 0° to 60° The following table has been calculated from the most recent determination of the earth's figure—

Latitude, ϕ	1" of Latitude $= \rho \sin 1''$	1" of Longitude $= \nu \cos \phi . \sin 1''$	1" Perp. to Meridian $= \nu \sin 1''$
	Ft.	Ft.	Ft.
0°	100·7724	101·4545	101·4545
30°	101·0270	87·9361	101·5399
51° 30′	101·3981	63·2874	101·6641
60°	101·5394	50·8556	101·7113
90°	101·7973	0·0000	101·7973

EFFECT OF CURVATURE ON SURVEYS

Spherical Excess. Assuming the earth as a sphere, the sum of the three angles of a triangle exceeds 180° by an amount equal to $\frac{\text{Area of triangle}}{(\text{Radius of earth})^2} \times \frac{1}{\sin 1''}$ seconds, but this effect is only appreciable on very large triangles.

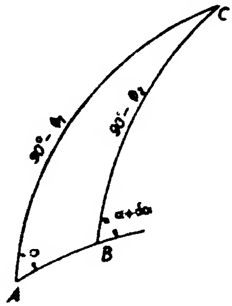


FIG. 3

Taking the average radius of curvature at 30° latitude as

$$\frac{\sqrt{101 \cdot 0270 \times 101 \cdot 5399}}{5280 \sin 1''} = 3956 \cdot 2 \text{ miles}$$

we find the area of triangle necessary for the spherical excess to amount to 1″ is $(\text{radius})^2 \times \sin 1'' = (3956 \cdot 2)^2 \sin 1'' = 75 \cdot 9$ square miles. In large triangulations this must, of course, be allowed for in adjusting the errors of the angles of each triangle, which must be corrected to sum up to 180° + spherical excess. In calculating the sides, one-third of the spherical excess is deducted from each angle and the sides are then calculated by plane trigonometry. The *spheroidal* excess decreases towards the poles as the average radius increases.

Convergence of Meridians. A much more appreciable effect of the curvature of the earth on surveys is that a "straight line" is constantly changing its azimuth. The "Reverse Azimuth," or direction of A from B, is *not* the same as the azimuth of B from $A \pm 180°$. We can deduce a very simple formula for this change of azimuth on a sphere (Fig. 3).

Let AB be the "straight line" (great circle) having an azimuth a at A and $a + \delta a$ at B, and let ϕ_1 and ϕ_2 be the latitudes of A and B. Let C be the pole.

Then in the triangle ABC, $AC = 90° - \phi_1$, $BC = 90° - \phi_2$, $A = \alpha$, $B = 180° - \alpha - \delta\alpha$, C = difference of longitude.

$$\therefore \quad \tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}} . \cot\frac{C}{2}$$

$$\therefore \quad \tan\frac{180° - \delta\alpha}{2} = \frac{\cos\frac{90° - \phi_2 - 90° + \phi_1}{2}}{\cos\frac{90° - \phi_2 + 90° - \phi_1}{2}} \cot\frac{C}{2}$$

$$= \frac{\cos\frac{\phi_1 - \phi_2}{2}}{\sin\frac{\phi_1 + \phi_2}{2}} . \cot\frac{C}{2}$$

$$\therefore \quad \cot\frac{\delta\alpha}{2} = \frac{\cos\frac{\phi_2 - \phi_1}{2}}{\sin\frac{\phi_2 + \phi_1}{2}} \cot\frac{C}{2}$$

$$\therefore \quad \tan\frac{\delta\alpha}{2} = \frac{\sin\frac{\phi_2 + \phi_1}{2}}{\cos\frac{\phi_2 - \phi_1}{2}} \tan\frac{C}{2}$$

Now in all surveying operations the difference of longitude C and the difference of latitude $\phi_2 - \phi_1$ of adjacent stations A and B will be small; we can, therefore, write this last equation

$$\delta\alpha = \sin\frac{\phi_1 + \phi_2}{2} . C = C . \sin\bar{\phi} \qquad (8)$$

i.e. *increase of angle from the meridian = difference of longitude × sine average latitude.* This useful formula, although derived from the sphere, is applicable to a spheroid like the earth, of small eccentricity, provided the line is very short compared to the earth's radius, as in practice it must be.

This change of direction is quite appreciable on surveys of quite moderate size, e.g. if in the latitude of London, 51° 30′ N., a straight line 10 miles long is ranged at 90° to the meridian, the difference of longitude of its ends will be

$$\frac{52800}{63 \cdot 2874} = 834'' = 13' \, 54''$$

Its increase of azimuth will be

$$834'' \sin 51° \, 30' = 653'' = 10' \, 53'' \text{ or } 1' \, 5 \cdot 3'' \text{ per mile}$$

Starting due east, or at azimuth 90°, after 10 miles its azimuth will be 90° 10′ 53″, i.e. S. 89° 49′ 07″ E. We can put this in a general form thus: if d is the length of the line perpendicular to the meridian, the difference of longitude in seconds is $\dfrac{d}{r \cos \phi \, . \sin 1''}$ and the increase of azimuth δa in seconds

$$= \frac{d}{r \, . \cos \phi \, . \sin 1''} \times \sin \phi = \frac{d \tan \phi}{r \sin 1''} \qquad (9)$$

This formula also approximately gives the change of azimuth in a traverse, where d is the total departure between the two points, and ϕ is the average geographical latitude.

Example 1. To set out a portion of a parallel of latitude, say for a distance of 10 miles in latitude 51° 30′ (Fig. 4).

Let A and B be two points on the parallel, P be the pole, and let ABD be the great circle through A and B, and AC a great circle perpendicular to the meridian AP. Then, by symmetry, $PBA = PAB$. But $PBD = PAB + \delta a = 180° - PBA$. $\therefore PAB + \delta a = 180° - PAB$. $\therefore PAB = PBA$

$= 90° - \frac{\delta\alpha}{2}$, and $PBD = 90° + \frac{\delta\alpha}{2}$. $\therefore BAC = 90° - \left(90° - \frac{\delta\alpha}{2}\right)$ $= \frac{\delta\alpha}{2}$, and for short distances we can take $BC = AC \cdot \frac{\delta\alpha}{2} \cdot \sin 1''$.

In this case $AC = 10$ miles and we found above that $\delta\alpha = 653''$; so that the offset at 10 miles will be

$$52800 \times \frac{653}{2} \times \frac{1}{206265} = 83{\cdot}6 \text{ ft.}$$

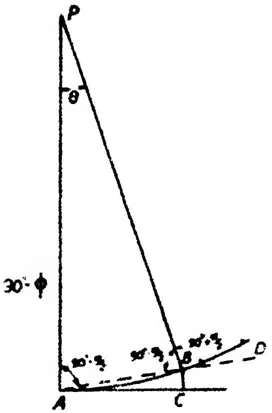


Fig. 4*

As the difference of longitude, and, therefore $\delta\alpha$, is proportional to the distance, the offset will be proportional to the (distance)^2, e.g. at 1 mile it will be 0·836 ft. and at 5 miles it will be 20·9 ft. A parallel of latitude, therefore, appears as a circular curve of very large radius for a short distance. Using formula (9), we can state this in a general form—*offset to great circle perpendicular to meridian at distance d*

$$= \frac{d \tan \phi}{r \cdot \sin 1''} \times \frac{d}{2} \cdot \sin 1'' = \frac{d^2 \tan \phi}{2r} \quad . \quad . \quad (10)$$

Strictly speaking, the offset should not be perpendicular to AC, but along the meridian CP, i.e. at 89° 49′ 07″ to AC.

We can plot a survey over a large region by rectangular co-ordinates with considerable accuracy, but the azimuths of the lines will become more and more incorrect as we depart farther and farther from the origin, as we are in reality using as axes a meridian and a great circle perpendicular to it (Fig. 5). If the meridians are drawn on it they will become more and more inclined to the initial meridian

* *N.B.* In Fig. 4, for $\frac{\alpha}{2}$ read $\frac{\delta\alpha}{2}$ in all cases.

as we depart farther from it the angle between the meridians at the two ends of any line being the "convergence of meridians" for that line. The "bearing" of a line is its angle from a line parallel to the initial meridian, while its "azimuth" is its angle from the meridian at the point in question. This effect is quite noticeable on the British Ordnance Maps, where the longitudes are marked on the upper and lower margins and the meridians are not parallel to the sides of the sheets, except near the central meridian which runs through Cheshire. All methods of map projection, i.e. of representing large regions of the curved surface of the earth on a flat surface, are bound to be defective in some respects, and in this case (which is called Cassini's Projection) distances north and south are exaggerated more and more as we depart from the central (initial) meridian, and such a projection should not be used for more than 150 miles from its central meridian. The exaggeration is approximately as the secant of the angular distance from the central meridian.

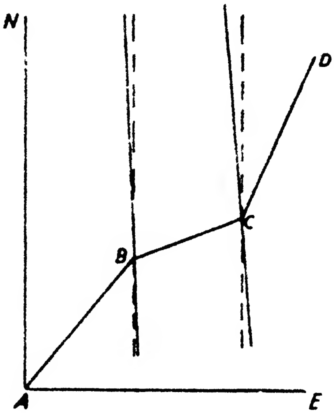


FIG. 5

For larger regions a system of geographical co-ordinates, i.e. latitudes and longitudes, must be used, and some other form of map projection adopted to keep the unavoidable defects as small as possible. It would be very laborious to have to find astronomically the latitudes and longitudes of *all* stations on a large triangulation, so we must be able to calculate the differences of latitude and longitude from the lengths of lines (l) and their azimuths (a), and also to calculate the reverse azimuths of the lines so as to obtain the azimuths of the next line therefrom; by this means only the latitudes and longitudes of a few stations need be found astronomically to serve as "Controls." If the earth were a sphere these differences of latitude and longitude and reverse azimuths for two points A and B could be found by spherical trigonometry (as we should have two sides $90° - \phi_A$, $\frac{l}{R \sin 1''}$ and the included angle α in the spherical triangle PAB, where P is the pole), but as it is a spheroid the formulae become much more complicated and are quite beyond the scope of this book. The following approximate method of "Mean Latitudes" is, however, sufficiently accurate for comparatively short lines (Fig. 6).

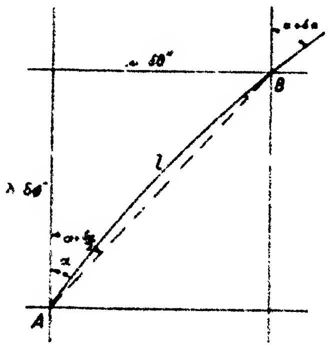


Fig. 6

If A and B are the two stations and $AB = l$, ϕ and θ are the latitude and longitude of A and α is the azimuth of AB at A, we represent the meridians and parallels through A and B by straight lines at right angles to each other at ***mean latitude distances apart***, i.e. we make the distance between the two parallels = difference of latitude ($\delta\phi$) of A and B in seconds × length (λ) of 1″ of latitude at the average latitude of A and B, and the distance between the two meridians = difference of longitude ($\delta\theta$) of A and B in seconds × length (μ) of 1″ of longitude at the average latitude of A and B.

The great circle from A to B will be represented on this diagram by a *curved* line, convex towards the nearer pole, so as to show an increase of azimuth from α at A to $\alpha + \delta\alpha$ at B, while its average azimuth (that of the broken line) can be taken as $\alpha + \frac{\delta\alpha}{2}$. We can then write down the following formulae—

Difference of latitude of A and B

$$\delta\phi = \frac{l \cos\left(\alpha + \frac{\delta\alpha}{2}\right)}{\lambda} \text{ seconds} \quad . \quad . \quad (11)$$

Difference of longitude of A and B

$$\delta\theta = \frac{l \sin\left(\alpha + \frac{\delta\alpha}{2}\right)}{\mu} \text{ seconds} \quad . \quad . \quad (12)$$

Difference of azimuth of AB at A and B

$$\delta\alpha = \delta\theta \times \text{sine of average latitude} \quad . \quad . \quad (13)$$

We will first give an example in which, as $\delta\phi$ and $\delta\theta$ are given in the data, the mean latitude $\bar{\phi}$ and $\delta\alpha$ can be calculated at once.

EXAMPLE 2 (L.U.) Two points, A and B, have the following co-ordinates—

	Latitude	Longitude
A . .	52° 21′ 14″ N.	93° 48′ 50″ E.
B . .	52° 24′ 18″ N.	93° 42′ 30″ E.

Given the following values—

Latitude	1″ of Latitude	1″ of Longitude
	Ft.	Ft.
52° 20′	101·4115	62·1272
52° 25′	101·4129	62·0104

find the azimuths of B from A and of A from B, also the distance AB (Fig. 7).

$$\bar{\phi} = \text{Average latitude} = 52^\circ\, 22'\, 46''$$

$$\text{Difference of latitude} = 3'\, 04''.$$

$$\text{Difference of longitude} = 6'\, 20''$$

$$\delta\alpha = 380'' \sin\bar{\phi} = 301'' = 5'\, 01''$$

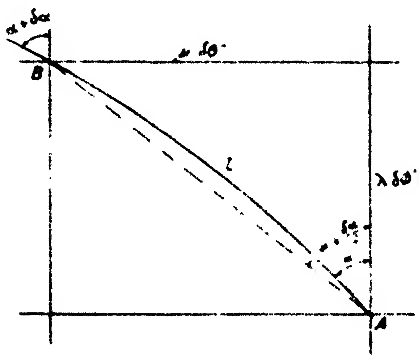


Fig. 7

Length of 1″ of latitude at $52^\circ\, 22'\, 46'' = \lambda$

$$= 101{\cdot}4115 + \frac{166}{300} \times 0{\cdot}0014 = 101{\cdot}4123 \text{ ft.}$$

Length of 1″ of longitude at $52^\circ\, 22'\, 46'' = \mu$

$$= 62{\cdot}1272 - \frac{166}{300} \times {\cdot}1168 = 62{\cdot}0626 \text{ ft.}$$

$$\therefore \tan\left(\alpha + \frac{\delta\alpha}{2}\right) = \frac{380 \times 62{\cdot}0626}{184 \times 101{\cdot}4123} = \tan 51^\circ\, 38'\, 54{\cdot}3''$$

$$\therefore \alpha = 51^\circ\, 38'\, 54{\cdot}3'' - \frac{5'\, 01''}{2} = 51^\circ\, 38'\, 54{\cdot}3'' - 2'\, 30{\cdot}5''$$

$$= 51^\circ\, 36'\, 24''$$

$\therefore\ a + \delta a = 51^\circ\,36'\,24'' + 5'\,01'' = 51^\circ\,41'\,25^\circ$

$\therefore$ Azimuth of AB at A = 308° 23′ 36″ from north

Azimuth of BA at B = 128° 18′ 35″ from north

Length $AB = 184 \times 101{\cdot}4123 \times \sec 51^\circ\,38'\,54{\cdot}3''$

$= 30{,}073$ ft.

More usually, we are given the latitude and longitude of A, the length of the line AB and its azimuth at A, so that we have to find $\delta\phi$, $\delta\theta$, and δa which, when applied to the data, give us the latitude, longitude, and azimuth of the line AB at B. As δa is the last item to be found, we must proceed in the above order by "successive approximations," omitting $\frac{\delta a}{2}$ in the first two formulae and taking the "average latitude" in the first formula as the latitude of A. Then, when δa has been found, we repeat the three calculations in order, taking $\frac{\delta a}{2}$ into account; in fact, when one quantity has been found to a first approximation, we utilize it in the subsequent calculations.

Example 3. A line AB, 52,800 ft. long, has an azimuth of N. 45° 0′ 0″ W. from A in latitude 54° 51′ 30″ N., longitude 101° 13′ 15″ E. Find the latitude and longitude of B and the reverse azimuth of the line at B, given the following data—

Latitude	1″ Latitude	1″ Longitude
	Ft.	Ft.
54° 50′	101·4547	58·5659
54° 55′	101·4561	58·4452

1" of latitude at 54° 51′ 30″

$$= 101{\cdot}4547 + \frac{90}{300}\,{\cdot}0014 = 101{\cdot}4551$$

$$\delta\phi = \frac{52800 \cos 45^\circ}{101{\cdot}4551} = 368{\cdot}00'' = 6'\ 08{\cdot}00''$$

$$\therefore\ \phi = 54^\circ\ 51'\ 30'' + 3'\ 04'' = 54^\circ\ 54'\ 34''$$

$$1''\text{ longitude at }\bar{\phi} = 58{\cdot}5659 - \frac{274}{300} \times 0{\cdot}1207$$

$$= 58{\cdot}4557\text{ ft.}$$

$$\delta\theta = \frac{52800 \sin 45^\circ}{58{\cdot}4557} = 638{\cdot}69'' = 10'\ 38{\cdot}69''$$

$$\delta\alpha = 638{\cdot}69 \sin 54^\circ\ 54'\ 34'' = 522{\cdot}61'' = 8'\ 42{\cdot}61''$$

$$1''\text{ latitude at }\bar{\phi} = 101{\cdot}4561 - \frac{26}{300} \times 0{\cdot}0014$$

$$= 101{\cdot}4560\text{ ft.}$$

$$\delta'\phi = \frac{52800 \cos 45^\circ\ 04'\ 21{\cdot}3''}{101{\cdot}4560} = 367{\cdot}53'' = 6'\ 07{\cdot}53''$$

$$\bar{\phi} = 54^\circ\ 51'\ 30'' + 3'\ 03{\cdot}76'' = 54^\circ\ 54'\ 33{\cdot}76''$$

$$1''\text{ longitude at }\bar{\phi} = 58{\cdot}4452 + \frac{26{\cdot}24}{300} \times 0{\cdot}1207$$

$$= 58{\cdot}4558\text{ ft.}$$

$$\delta'\theta = \frac{52800 \sin 45^\circ\ 04'\ 21{\cdot}3''}{58{\cdot}4558} = 639{\cdot}50'' = 10'\ 39{\cdot}50''$$

$$\delta'\alpha = 639{\cdot}50 \times \sin 54^\circ\ 54'\ 33{\cdot}76'' = 523{\cdot}27'' = 8'\ 43{\cdot}27''$$

∴ Latitude of *B* = 54° 51′ 30″ + 6′ 07·53″
= 54° 57′ 37·53″ N.

Longitude of *A* = 101° 13′ 15″ − 10′ 39·50″
= 101° 02′ 35·50″ E.

Azimuth of *AB* at *B* = N. 45° 08′ 43·27″ W.

$\therefore$ Azimuth of BA at B = S. 45° 08′ 43·27″ E.
= 134° 51′ 16·73″ from north

(This calculation has been checked by Puissant's method as given in Close's *Topographical Surveying*, page 46, and found correct to the nearest 0·01″. (With the azimuth taken as 80°, the greatest discrepancy is 0·02″.)

The repetition of the calculation of $\delta\phi$, $\delta\theta$, and $\delta\alpha$ is rather laborious and may be avoided by applying corrections to their first values thus: we notice that there is hardly any change in the denominators of $\delta\phi$ and $\delta\theta$; disregarding this, we can write λ and μ for these denominators.

$$\delta\phi = \frac{l\cos\left(\alpha + \frac{\delta\alpha}{2}\right)}{\lambda} = \frac{l}{\lambda}\left(\cos\alpha\,.\cos\frac{\delta\alpha}{2} - \sin\alpha\,.\sin\frac{\delta\alpha}{2}\right)$$
$$= \frac{l}{\lambda}\left(\cos\alpha - \sin\alpha\,.\frac{\delta\alpha''}{2}\,.\sin 1''\right)$$
$$= \frac{l}{\lambda}\cos\alpha\left(1 - \tan\alpha\,.\frac{\delta\alpha''}{2}\,.\sin 1''\right)$$

$$\delta\theta = \frac{l\sin\left(\alpha + \frac{\delta\alpha}{2}\right)}{\mu} = \frac{l}{\mu}\left(\sin\alpha\,.\cos\frac{\delta\alpha}{2} + \cos\alpha\,.\sin\frac{\delta\alpha}{2}\right)$$
$$= \frac{l}{\mu}\left(\sin\alpha + \cos\alpha\,.\frac{\delta\alpha''}{2}\,.\sin 1''\right)$$
$$= \frac{l}{\mu}\sin\alpha\left(1 + \cot\alpha\,.\frac{\delta\alpha}{2}\,.\sin 1''\right)^{*}$$

* Or, by Taylor's Theorem—

$$\frac{l}{\lambda}\cos\left(\alpha + \frac{\delta\alpha}{2}\right) = \frac{l}{\lambda}\left(\cos\alpha - \frac{\delta\alpha}{2}\,.\sin\alpha\right)$$
$$\frac{l}{\mu}\sin\left(\alpha + \frac{\delta\alpha}{2}\right) = \frac{l}{\mu}\left(\sin\alpha + \frac{\delta\alpha}{2}\,.\cos\alpha\right)$$

where $\frac{\delta\alpha}{2}$ is in circular measure $= \frac{\delta\alpha''}{2}\sin 1'' = \frac{\delta\alpha''}{412530}$

∴ the correction of—

$$\delta\phi \text{ is } -\text{(first value)} \times \frac{\tan\alpha \, . \, \delta\alpha''}{412530} \quad . \quad . \quad (14)$$

$$\delta\theta \text{ is } +\text{ (first value)} \times \frac{\cot\alpha \, . \, \delta\alpha''}{412530} \quad . \quad . \quad (15)$$

$$\delta\alpha \text{ is } +\text{ (correction of } \delta\theta) \times \sin\bar{\phi} \quad . \quad . \quad (16)$$

Taking the first values,

$$\delta\phi = 368{\cdot}00'' : \text{ correction} = -\frac{368{\cdot}00 \times 1 \times 522{\cdot}61}{412530}$$

$$= -0{\cdot}47'' \quad \therefore \delta\phi = 367{\cdot}53''$$

$$\delta\theta = 638{\cdot}69'' : \text{ correction} = +\frac{638{\cdot}69 \times 1 \times 522{\cdot}61}{412530}$$

$$= +0{\cdot}81'' \quad \therefore \delta\theta = 639{\cdot}50''$$

$$\delta\alpha = 522{\cdot}61'' : \text{ correction} = +0{\cdot}81 \sin 54^\circ\, 54'\, 34''$$

$$= +0{\cdot}66'' \quad \therefore \delta\alpha = 523{\cdot}27''$$

i.e. this much shorter method gives the same results, and the corrections, being relatively small, can be calculated on the slide rule.

Latitudes and Azimuths. In rapid surveys this method is sometimes adopted, viz. to determine astronomically the latitudes of stations visible from each other and measure the azimuths of the lines joining the stations, also astronomically. The average latitude $\bar{\phi}$ is, therefore, known and λ and μ can be interpolated from the tables, while $\delta\phi$ and α are also known.

Then

$$\delta\theta'' = \frac{\lambda\delta\phi'' \cdot \tan\left(\alpha + \frac{\delta\alpha}{2}\right)}{\mu}$$

$$= \frac{\lambda}{\mu}\delta\phi'' \left(\tan\alpha + \frac{\delta\alpha}{2} \cdot \sec^2\alpha\right)$$

$$= \frac{\lambda}{\mu} \cdot \delta\phi'' \tan\alpha\left(1 + \frac{\delta\alpha''}{2}\sin 1'' \frac{2}{\sin 2\alpha}\right)$$

$$l = \lambda\delta\phi'' \cdot \sec\left(\alpha + \frac{\delta\alpha}{2}\right)$$

$$= \lambda \cdot \delta\phi''\left(\sec\alpha + \frac{\delta\alpha}{2} \cdot \frac{\sin\alpha}{\cos^2\alpha}\right)$$

$$= \lambda\,\delta\phi'' \cdot \sec\alpha\left(1 + \frac{\delta\alpha''}{2} \cdot \sin 1'' \tan\alpha\right)$$

$$\delta\alpha'' = \delta\theta'' \cdot \sin\phi$$

Here again, the second term is added as a correction when $\delta\alpha$ has been found.

Another important use of the "convergence of meridians formula" is in checking the angles of a long open traverse by obtaining the azimuths of the first and last lines by an astronomical observation. If the earth's surface were plane the azimuth of the last line would be equal to the azimuth of the first line + the sum of the deflection angles to the right − sum of deflection angles to the left, but owing to the curvature of the earth a correction must be applied which is "difference of longitude of first and last stations × sine of average latitude of first and last stations," provided the length of the traverse is such that the total differences of latitude and longitude are small enough for the formula to apply. If the traverse is too long to be treated as a whole, it can be divided into sections and the convergence of meridians calculated for each section. The first azimuth of each section should be corrected for convergence in such cases before the section is reduced.

EXAMPLE 4. A traverse is run as follows :-

Station	Length	Angle Clockwise from Rear Station	Azimuth
	Ft		
A			
	20,000		45° 0′ 0″ from North
B		210° 0′ 0″	
	20,000		
C		135° 0′ 0″	
	20,000		
D			

The latitude of *A* is 50° 00′ N.

Find the azimuth of line *CD* at *D*, given the following :-

Latitude	1″ Latitude	1″ Longitude
	Ft.	Ft
50° 00′	101·3703	65·3135
50° 05′	101·3717	65·2305

The deflection angle at *B* is 30° *R*

∴ bearing of *BC* is 75° 0′ 0″ from north.

The deflection angle at *C* is 45° *L*.

∴ bearing of *CD* is 30° 0′ 0″ from north.

We have, then, for the three lines —

	Latitude	Departure
AB . .	14142·1	14142·1
BC . .	5176·4	19318·5
CD . .	17320·5	10000·0
Co-ordinates of *D* are .	36639·0	43460·6

$$\therefore \delta\phi = \frac{36639}{101\cdot3703} = 361\cdot44'' = 6'\ 01\cdot44''$$

$$\therefore \phi = 50^\circ\ 03'\ 0\cdot72''$$

$$1'' \text{ longitude at } \phi = 65\cdot3435 - \frac{180\cdot72}{300} \times \cdot1130$$

$$= 65\cdot2754 \text{ ft.}$$

$$\delta\theta = \frac{43461}{65\cdot2754} = 665\cdot81'' = 11'\ 5\cdot81''$$

$$\delta\alpha = 665\cdot81'' \times \sin 50^\circ\ 03'\ 0\cdot72'' = 510\cdot41'' = 8'\ 30\cdot41''$$

∴ Azimuth of *CD* at *D* = 30° ·08′ 30·4″

Azimuth of *DC* at *D* = 210° ·08′ 30·4″

(Latitude of *D* = 50° 06′ 1·44″ N.

Longitude of *D* = 0° 11′ 5·81″ E. of *A*)

When each line of this traverse is treated separately* the azimuth of *BC* is found to be 75° 02′ 46″ at *B*, and the azimuth of *CD* at *C* is found to be 30° 06′ 32·9″ and at *D* 30° 08′ 30·9″, while the latitude of *D* is found to be 50° 06′ 0·8″ N. and the longitude 0° 11′ 06·50″ east of *A*. Therefore, the latitude and longitude of *D*, and the azimuth of *CD* at *D* all agree within 1″, so that in this case it was amply accurate to treat the traverse as a whole.†

* The student will find it a useful exercise to do this.

† If the second corrections are applied agreement is to the nearest 0·1″.

EFFECT OF CURVATURE ON LEVELLING

Trigonometrical Levelling. Another important effect of the earth's curvature is in finding differences of level with the theodolite when the distance is at all considerable. Fig. 8 gives a sketch to illustrate this effect; the sketch is necessarily much distorted.

The angle α of elevation of B from A is measured from the horizontal line AD through A, and the angle of depression of A from B is measured from the horizontal line BE through B. If, from the intersection O of the verticals through A and B, circular arcs AC, BF are described through A and B, these arcs are "level" lines, and as the angles DAC, EBF are each equal to $\frac{c}{2}$, c being the angle subtended by the two verticals FA, BC at the centre of the earth,

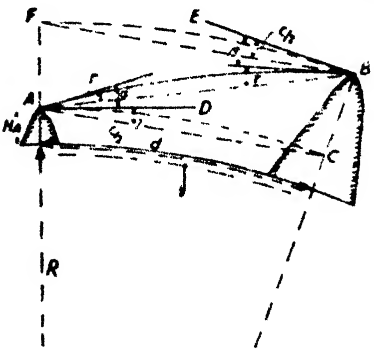


FIG. 8

the elevation α at A must be corrected by $+\frac{c}{2}$ and the depression β at B must be corrected by $-\frac{c}{2}$, in order to determine the difference of level of A and B, which is BC or AF.

The line of sight AB is not a straight line but is curved as shown by atmospheric refraction, the object appearing too high, both from A or B as in astronomical observations. The value of the correction, r, is not the same as in astronomical work, as the ray only passes through a portion of the atmosphere. It is found to be proportional to the distance and to have an *average* value of $r = 0{\cdot}070c = 0{\cdot}14 \times \frac{c}{2}$, or about one-seventh of the curvature correction, though it varies appreciably over sea and land, in different climates and at different times of the day.* The corrected angle of elevation BAC at A is then $\alpha - r + \frac{c}{2}$, and the corrected angle of depression FBA at B is $\beta + r - \frac{c}{2}$. But these are equal, as the chords AC, BF are parallel; we have, therefore,

$$\alpha - r + \frac{c}{2} = \beta + r - \frac{c}{2} = \frac{\beta + \alpha}{2} \quad . \quad . \quad (17)$$

We are assuming that the angles α and β are measured from ground level to ground level, or, what is exactly equivalent, from the instrument to a signal at the same height above ground as the height of the instrument.

From equation (17) we see that $\beta - \alpha = c - 2r$, and, therefore, that the observed angle of depression is greater than the observed angle of elevation, also that each of these corrected angles $= \frac{1}{2}$ their sum $= \frac{\beta + \alpha}{2}$.

Where possible, the levelling should be *reciprocal*, i.e. both angles should be measured, and, best of all, simultaneously (to avoid any risk of change of refraction), and when this is effected the effects of both curvature and refraction are eliminated. By such simultaneous observation we can find the refraction correction as

$$r = \frac{c}{2} - \frac{\beta - \alpha}{2} \quad . \quad . \quad . \quad (18)$$

To find the difference of level BC we solve the triangle ABC, which is nearly a right-angled triangle, as ACB equals

* r averages $\cdot 070c$ over land and $\cdot 080c$ over water.

$90° + \frac{c}{2}$, where c in seconds = geodetic distance d (i.e. at mean sea-level) between A and B ÷ length of 1″ of the earth's surface in the direction AB. AC, the base of this triangle, can be calculated as $2(R + H_A) \sin \frac{c}{2}$, where

$$R = \frac{\text{length of } 1''}{\sin 1''}$$

but for all practical purposes the chord AC = arc AC $= d \cdot \frac{R + H_A}{R}$, and unless H_A is considerable this is inappreciably greater than d. Then the difference of level BC

$$= AC \frac{\sin\left(\alpha - r + \frac{c}{2}\right)}{\sin(90° - \beta - r)} = AC \frac{\sin\left(\alpha - r + \frac{c}{2}\right)}{\cos(\beta + r)}$$

$$= AC \frac{\sin\left(\alpha - r + \frac{c}{2}\right)}{\cos(\alpha - r + c)} = AC \frac{\sin\left(\beta + r - \frac{c}{2}\right)}{\cos(\beta + r)}$$

$$= AC \frac{\sin \frac{\beta + \alpha}{2}}{\cos\left(\frac{\beta + \alpha}{2} + \frac{c}{2}\right)} \quad . \quad . \quad . \quad . \quad (19)$$

Unless the distance AC is very great, the angle $\frac{c}{2}$ is so small and the cosine in the denominator varies so slowly that we may alter the angle in the denominator to agree with that in the numerator, and we can write these

$$BC = AC \tan\left(\alpha - r + \frac{c}{2}\right) = AC \tan\left(\beta + r - \frac{c}{2}\right)$$

$$= AC \tan \frac{\beta + \alpha}{2} \qquad (20)$$

for use when the elevation α, the depression β, or both α and β have been observed, respectively.

Case when Both α and β are Depressions (Fig. 9) From the higher of the two points the angle is always a depression, but if the distance is great compared to the difference of level of the points, the angle at the lower point may be also a depression, but a less depression than the other. In this case α is *negative* and the corrected angle BAC at A is $\frac{c}{2} - \alpha - r$. We now have $\frac{c}{2} - \alpha - r = \beta + r - \frac{c}{2}$ and either corrected angle at A or $B = \frac{\beta - \alpha}{2}$, while $r = \frac{c}{2} - \frac{\beta + \alpha}{2}$, and

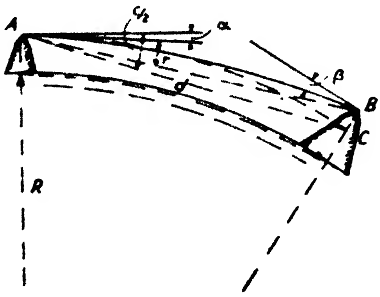


FIG. 9

$$BC = AC \frac{\sin\left(\frac{c}{2} - \alpha - r\right)}{\cos(c - \alpha - r)} = AC \frac{\sin\left(\beta + r - \frac{c}{2}\right)}{\cos(\beta + r)}$$

$$= AC \frac{\sin \frac{\beta - \alpha}{2}}{\cos\left(\frac{\beta - \alpha}{2} + \frac{c}{2}\right)}$$

If, as usual, $\frac{c}{2}$ is very small, we may write these

$$BC = AC \tan\left(\frac{c}{2} - \alpha - r\right) = AC \tan\left(\beta + r - \frac{c}{2}\right)$$

$$= AC \tan \frac{\beta - \alpha}{2}$$

in all of which we have merely written $-\alpha$ for $+\alpha$ in the formulae (19) and (20).

Example 5 (L.U.). From A, a station B, 97,770 ft. distant, is observed with a depression angle of 04′08″ and a reciprocal observation from B to A gave a depression angle of 09′ 41″—in both cases the sights were taken to signals at the same height above ground as the observing theodolite. Calculate the refractive correction and the difference of level.

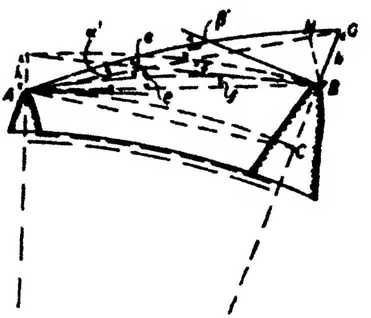


Fig. 10

Take 1″ of arc as 101·4 ft. and log tan 1″ = $\bar{6}$·685,5749.

Here A is the lower station and $\alpha = -04'\,08''$

$$\frac{\beta - \alpha}{2} = \frac{5'\,33''}{2} = 2'\,46{\cdot}5'' = 166{\cdot}5''$$

$\therefore$ rise from A to $B = 97{,}770 \times 166{\cdot}5'' \times \tan 1''$

$\therefore$ log rise $= 4{\cdot}990{,}2056 + 2{\cdot}221{,}4142 + \bar{6}{\cdot}685{,}5749$

$= \bar{1}{\cdot}897{,}1947.$

$$\therefore \text{ rise} = 78{\cdot}93 \text{ ft.} \quad \text{Also } c'' = \frac{97770}{101{\cdot}4} = 964{\cdot}2''$$

$$\therefore \quad r = \frac{c}{2} - \frac{\beta + \alpha}{2} = 482{\cdot}1'' - \frac{13'\,49''}{2}$$

$$= 482{\cdot}1'' - 414{\cdot}5'' = 67{\cdot}6''$$

$$\therefore \quad \frac{r}{c} = \frac{67{\cdot}6}{964{\cdot}2} = 0{\cdot}070$$

Eye and Object Correction. If, as happens frequently, the theodolite at A is sighted on a signal G at B, which is at a greater height above ground than itself, let h = height of signal G at B – height of instrument at A. We must subtract the angle $GAB = e$ (Fig. 10) from a', the observed elevation of the signal at B from A, as an "eye and object" correction, in order to obtain a for the point B.

Draw BH perpendicular to AG. Then BH is inclined at an angle $a' - r$ to the vertical at A and, therefore, is inclined at an angle $a' - r + c$ to the vertical at B.

$$\therefore\ BH = h \cos (a' - r + c)$$

$$\text{Then } e = \text{angle } GAB = \frac{BH}{AB \cdot \sin 1''} = \frac{h \cos (a' - r + c)}{AG \cdot \sin 1''}$$

very nearly, while, as the angle $ACB = 90° + \frac{c}{2}$, it is practically 90°, and therefore,

$$AG = \frac{AC}{\cos GAC} = \frac{AC}{\cos\left(a' - r + \frac{c}{2}\right)}$$

$$\therefore\ e = \frac{h \cos (a' - r + c) \cdot \cos\left(a' - r + \frac{c}{2}\right)}{AC \cdot \sin 1''}$$

$$= \frac{h \cos^2\left(a' - r + \frac{c}{2}\right)}{AC \cdot \sin 1''} \text{ seconds, nearly,}$$

to be subtracted from a' . . (21a)

If a' is negative, this correction must, of course, be added, i.e. a' must be increased by e to give a. Similarly, we can show that if the signal at A is higher above ground than the instrument at B by the amount k, the correction to be added to the depression β' at B is

$$f = \frac{k \cos^2\left(\beta' + r - \frac{c}{2}\right)}{BF \cdot \sin 1''} \quad . \quad . \quad . \quad (21b)$$

where F is on the vertical through A at the level of B. In practice we can usually adopt the much simpler expressions,

$$e = \frac{h}{d \sin 1''}, \text{ and } f = \frac{k}{d \sin 1''} \text{ seconds}$$

because if the angles $\alpha' - r + \frac{c}{2}$ and $\beta' + r - \frac{c}{2}$ are less than $\begin{cases} 5^\circ\ 44' \\ 1^\circ\ 49' \end{cases}$, the value of their (cosine)^2 differs from unity by less than $\begin{cases} 1 \text{ per cent} \\ 0{\cdot}1 \text{ per cent} \end{cases}$, while for a small correction the difference in length of AC and BF from d is inappreciable. These corrections must first be applied to the observed angles α' and β' to obtain α and β for use in formulae (17) to (20) which only hold for equal heights of instrument and signal.

As the length of 1″ of the earth's surface averages about 101·5 ft., the curvature correction in a distance of 1,000 ft. averages $\frac{1000}{2 \times 101{\cdot}5} = 4{\cdot}93''$. The refraction correction averages about 0·14 of this or 0·69″ per 1,000 ft. The combined correction, therefore, averages about 4·24″ per 1,000 ft. or 22·39″ per mile of distance.

EXAMPLE 6 (L.U.). Two stations, A and B, are situated at a distance apart of 11,420 ft. The following observations were recorded: Height of instrument at A = 4·68 ft. Height of signal at A = 14·70 ft. Reduced level of A = 421·20 ft. Elevation of signal at B = 1° 49′ 05″. Height of instrument at B = 4·91 ft. Height of signal at B = 12·80 ft. Depression of signal at A = 1° 45′ 18″. Find the level of B, given that 101·31 ft. subtends 1″ on the earth and find the refraction correction.

Eye and object corrections:

$$h = 12{\cdot}80 - 4{\cdot}68 = 8{\cdot}12 \text{ ft.}$$

$$\therefore\ e = \frac{8{\cdot}12}{11420} \times 206{,}265 = 146{\cdot}65''$$

$$k = 14{\cdot}70 - 4{\cdot}91 = 9{\cdot}79 \text{ ft.}$$

$$\therefore\ f = \frac{9{\cdot}79}{11420} \times 206{,}265 = 176{\cdot}8''$$

$$\therefore\ \alpha = 1^\circ\ 49'\ 05'' - 02'\ 26{\cdot}65'' = 1^\circ\ 46'\ 38{\cdot}35''$$

$$\beta = 1^\circ\ 45'\ 18'' + 02'\ 56{\cdot}8'' = 1^\circ\ 48'\ 14{\cdot}8''$$

$$\therefore\ \frac{\beta + \alpha}{2} = 1^\circ\ 47'\ 26{\cdot}6''$$

$$\therefore\ \text{Rise} = 11{,}420 \tan 1^\circ\ 47'\ 26{\cdot}6'' = 357{\cdot}04 \text{ ft.}$$

$$\therefore\ \text{level of } B = 421{\cdot}20 + 357{\cdot}04 = 778{\cdot}24 \text{ ft.}$$

$$c = \frac{11420}{101{\cdot}31} = 112{\cdot}72''$$

$$\therefore\ r = \frac{c}{2} - \frac{\beta - \alpha}{2} = 56{\cdot}36 - \frac{1'\ 36{\cdot}45''}{2} = 56{\cdot}36 - 48{\cdot}22$$

$$= 8{\cdot}14'' = 0{\cdot}072c$$

In this case if $\cos^2\left(\alpha' - r + \frac{c}{2}\right)$ or $\cos^2\left(\beta' + r - \frac{c}{2}\right)$ were calculated, they would be found to be 0·999, while the lengths *AC* and *BF* would be found inappreciably greater than 11,420 ft.; also cos (1° 47′ 26·6″ + 56·36″) is 0·9995031 as against 0·9995117 for cos 1° 47′ 26·6″, a difference of less than $\frac{1}{100000}$ th part.

Curvature and Refraction in Spirit-Levelling. Taking the average value of 1″ of arc as 101·5 ft., we have $c'' = \frac{d \text{ ft.}}{101 \cdot 5}$ and the correction for curvature in levelling is

$$d \cdot \frac{c''}{2} \sin 1'' = \frac{d^2}{203} \sin 1'' = \frac{D^2 \times 5280^2}{203 \times 206265}$$

$$= \frac{27880000}{41871795} D^2 = 0 \cdot 666 \, D^2$$

where D is the distance in miles.

If $r = 0 \cdot 070c = 0 \cdot 14 \frac{c}{2}$, the average correction for refraction in levelling $= 0 \cdot 14 \times \cdot 666 \, D^2 = 0 \cdot 093 \, D^2$, and the combined correction for curvature and refraction $= 0 \cdot 573 \, D^2$.

In ordinary precise levelling great care is taken to make the length of the foresight equal to the length of the back-sight, in order to eliminate instrumental error, and, incidentally, this should eliminate any error due to curvature and refraction, as the error would be the same on both sights. The length of the sights is kept short in British practice, say not greater than 150 ft., to allow for irregularities in refraction. But when precise levels are carried across a wide river or arm of the sea, a long sight is necessary, and in this case the process of "Reciprocal Levelling" is employed to eliminate curvature and refraction errors. In the best work two levels are used simultaneously, one on each side of the river, each reading on an adjacent bench-mark and on the benchmark on the far side of the river. Then the levels are each taken across the river and the process repeated with the positions of the levels reversed. Here again we must employ a distorted sketch (Fig. 11).

Let the readings taken from A be h_A on benchmark A and H_B on benchmark B, and the readings taken from B be H_A on benchmark A and h_B on benchmark B. Then from

A, B appears lower than A by $H_B - h_A$, and from B, B appears lower than A by $h_B - H_A$.

The average fall from A to B appears, therefore, to be

$$\frac{H_B - h_A + h_B - H_A}{2}$$

Actually, the distant readings H_A and H_B each require a correction of − (curvature − refraction) so that, from A, B is

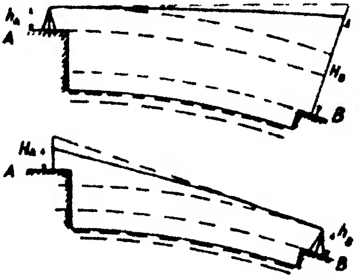


FIG. 11

lower than A by H_B − curvature + refraction − h_A, and from B, B is lower than A by h_B − (H_A − curvature + refraction) = $h_B - H_A$ + curvature − refraction.

The correct average fall from A to B is, therefore,

$$\frac{H_B - h_A + h_B - H_A}{2}$$

as the curvature and refraction cancel out on addition, and this is exactly what it appears to be, showing that the errors due to curvature and refraction have been eliminated by the reciprocal levelling.

The taking of the readings simultaneously from both sides of the river is to eliminate any change of refraction, and the change of station of the levels is to eliminate their instrumental error. If only one level is available it should read from A to B, then the observer should cross the river quickly and read from B to A, then he should re-cross the river and again read from A to B to ensure that there has been no change of refraction in the interval. If only a single reading is available, the corrections must be applied.

EXAMPLE 7. Two B.Ms., A and B, are 2640 ft. apart across an estuary. With the level at A the readings are: A, 4·62; B, 6·88 ft. With the level at B the readings are: A, 2·89; B, 4·71 ft. The level has an error of + 0·002 ft. in 66 ft. Find the level difference and the refraction.

$$\text{Fall } A \text{ to } B = \frac{2{\cdot}26 + 1{\cdot}82}{2} = \underline{2{\cdot}04 \text{ ft.}}$$

$$\text{Curvature correction} = {\cdot}666\, D^2 = {\cdot}1665 \text{ ft.}$$

$$\text{Instrumental error} = \frac{2640}{66} \times {\cdot}002 = +\,0{\cdot}08 \text{ in } 2640 \text{ ft.}$$

$$(6{\cdot}88 - {\cdot}1665 + r - {\cdot}08) - 4{\cdot}62 = 4{\cdot}71 - (2{\cdot}89 - {\cdot}1665 + r - {\cdot}08)$$

$$\therefore\ 2{\cdot}0135 + r = 2{\cdot}0665 - r.$$

$$\therefore\ r = 0{\cdot}0265 \text{ ft.} = {\cdot}159 \times \text{curvature}.$$

CHAPTER VI

TRIANGULATION AND PRECISE LEVELLING

TRIANGULATION: EQUATIONS OF CONDITION—BASE LINES—EXTENSION OF BASE—ANGLE MEASUREMENT—SATELLITE STATION—ANGLE ADJUSTMENT—COMPUTATION OF SIDES AND CO-ORDINATES

PRECISE LEVELLING: FIELDWORK AND ADJUSTMENT OF ERRORS

TRIANGULATION

TRIANGULATION is the determination of position by the use of triangles, in which the length of one side is known and the three angles are measured. If angles could be measured with perfect accuracy only two angles would need to be measured, viz. the two adjacent to the known side, but, in practice, when all three angles are measured their sum always differs slightly from 180° (or from 180° + *spherical excess* where the latter is appreciable in a triangle of large area, see Chapter V) and the discrepancy must be distributed over the angles in the most probable way. If we have a single triangle, or if the triangles forming a system are not interlocked as in Fig. 1*a*, the error in each triangle can only be taken as ⅓ the error on the whole triangle. If, however, we "close" the triangulation as in one of the quadrilaterals with diagonals in Fig. 1*b*, the "polygon with a central point" (Fig. 1*c*), or the "intersecting polygons" of Fig. 1*d* (four polygons are shown, each with its central point on a vertex of another polygon) so as to obtain what, in the "Theory of Structures," would be called "redundant members," we can distribute the angular errors with a much closer probability, owing to the greater number of "Equations of Condition" which must be satisfied.

Equations of Condition. If we take a polygon with a central point (Fig. 2*a*) we have the following conditions:

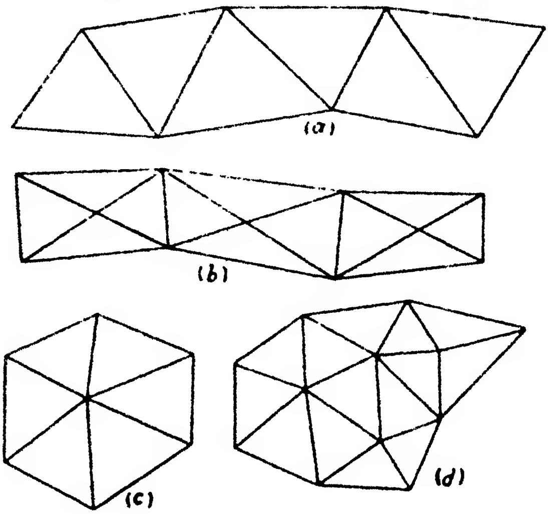


FIG. 1

(i) The sum of the "central angles" taken at the central point O must $= 360°$

$$\text{or } \Sigma(\theta_c) = 360° \quad . \quad . \quad . \quad . \quad (1)$$

If the round of angles taken from O closes, this condition is satisfied by the measured angles, but it must still be satisfied when the angles have been "adjusted" to satisfy other equations of condition.

(ii) The sum of the angles in each of the constituent triangles must $= 180°$ (or, of course, $180°$ + spherical excess if the triangle is large). The angles at A, B, C, D, etc., can be divided into L.H. angles and R.H. angles, according as they appear if we face towards the central point, thus at A, OAB is a L.H. angle, OAD is a R.H. angle, and we call the former A_L and the latter A_R. Each of the constituent triangles OAB, OBC, OCD, ODA thus contains a central

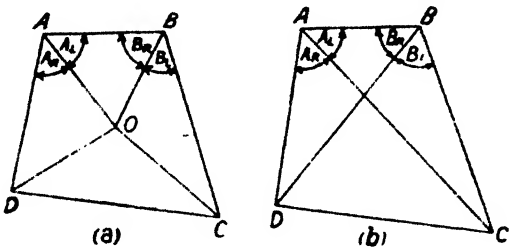


FIG 2

angle, a L.H. angle, and a R.H. angle. This equation of condition can, therefore, be expressed as

$$\theta_C + \theta_L + \theta_R = 180° \text{ for } \textit{each} \text{ of the triangles} \quad . \quad (2)$$

N.B. It would *not* be another independent condition that $A_R + A_L + B_R + B_L + C_R + C_L + D_R + D_L = 360°$, as this is deducible from (1) and (2).

(iii) We have

$$OA = \frac{OB \,.\, \sin B_R}{\sin A_L} = \frac{OC \,.\, \sin C_R}{\sin B_L} \cdot \frac{\sin B_R}{\sin A_L}$$

$$= \frac{OD \sin D_R}{\sin C_L} \cdot \frac{\sin C_R}{\sin B_L} \cdot \frac{\sin B_R}{\sin A_L}$$

$$= \frac{OA \,.\, \sin A_R}{\sin D_L} \cdot \frac{\sin D_R \,.\, \sin C_R \,.\, \sin B_R}{\sin C_L \,.\, \sin B_L \,.\, \sin A_L}$$

$$\therefore \sin A_R \,.\, \sin B_R \,.\, \sin C_R \,.\, \sin D_R = \sin A_L \,.\, \sin B_L \,.\, \sin C_L \,.\, \sin D_L$$

$$\text{or } \log\sin A_R + \log\sin B_R + \log\sin C_R + \log\sin D_R = \log\sin A_L + \log\sin B_L + \log\sin C_L + \log\sin D_L$$

i.e. $$\Sigma(\log\sin\theta_L) = \Sigma(\log\sin\theta_R) \quad . \quad . \quad . \quad (3)$$

In this quadrilateral, composed of four triangles, we have, therefore, six equations of condition to be satisfied.

Now, taking the quadrilateral with diagonals of Fig. 2*b*, which has no central point but angles are measured at each station between the remaining three stations, we can again divide the two angles at each station into L.H. and R.H. angles quite obviously, and the first equation of condition is that

$$A_R + A_L + B_R + B_L + C_R + C_L + D_R + D_L = 360° \qquad (4)$$

A second equation of condition follows from the fact that the two inner angles at A and B must be the supplement of the angle at the crossing of the diagonals and must, therefore, equal the two inner angles at C and D. Therefore,

$$A_L + B_R = C_L + D_R \quad . \quad . \quad . \quad (5)$$

Similarly, a third equation of condition is established, viz.

$$B_L + C_R = D_L + A_R \quad . \quad . \quad . \quad (6)$$

Instead of these three equations (4), (5), and (6) we could have made use of the fact that the sum of the three angles of each of the triangles ABC, BCD, CDA, $DAB = 180°$, but these would still only provide us with three independent equations, e.g. taking triangles ABC, BCD, CDA, we have

$$A_L + B_R + B_L + C_R = 180° \quad . \quad . \quad (7)$$

$$B_L + C_R + C_L + D_R = 180° \quad . \quad . \quad (8)$$

$$C_L + D_R + D_L + A_R = 180° \quad . \quad . \quad (9)$$

Therefore, adding the first and third equations,

$$A_R + A_L + B_R + B_L + C_R + C_L + D_R + D_L = 360°$$

as in (4).

Now, deducting the second equation, we get $A_R + A_L + B_R + D_L = 180°$, i.e. *the sum of the angles of triangle* $DAB = 180°$. As this last has been deduced it is not an independent condition. Again, we have already deduced equation (4) from the three triangles, and we can also deduce equations (5) and (6), thus

From equations (7) and (8),

$$A_L + B_R = 180° - (B_L + C_R) = C_L + D_R$$

which is equation (5); and from equations (8) and (9),

$$B_L + C_R = 180° - (C_L + D_R) = D_L + A_R$$

which is equation (6).

A fourth equation of condition can be established thus—

$$AB = \frac{BC \,.\, \sin C_R}{\sin A_L} = \frac{CD \,.\, \sin D_R}{\sin B_L} \cdot \frac{\sin C_R}{\sin A_L}$$

$$= AD \,.\, \frac{\sin A_R}{\sin C_L} \cdot \frac{\sin D_R}{\sin B_L} \cdot \frac{\sin C_R}{\sin A_L}$$

$$= \frac{AB \sin B_R}{\sin D_L} \cdot \frac{\sin A_R}{\sin C_L} \cdot \frac{\sin D_R}{\sin B_L} \cdot \frac{\sin C_R}{\sin A_L}$$

So that, here again,

$$\sin A_R \,.\, \sin B_R \,.\, \sin C_R \,.\, \sin D_R$$
$$= \sin A_L \,.\, \sin B_L \,.\, \sin C_L \,.\, \sin D_L$$

i.e. the fourth equation of condition is

$$\Sigma(\log \sin \theta_L) = \Sigma(\log \sin \theta_R) \quad . \quad . \quad . \quad (10)$$

Arrangement of Triangles. For the survey of a region of moderate size, e.g. the British Isles, the whole area would be covered with a network of intersecting polygons. This would be too lengthy and costly for a very large country, e.g. India, where chains of triangles at intervals along the meridians of longitude and parallels of latitude are used instead, as in Fig. 3. When the country has been covered with such a grid, chains of triangles can be carried across any mesh from known stations on one chain to known stations on another chain, and from these the details may be filled in as required.

A fundamental principle in Surveying is "to work from the whole to the part," and *not* conversely,* i.e. to cover the country with a "primary triangulation" first, with large triangles or chains of triangles of great accuracy, then to

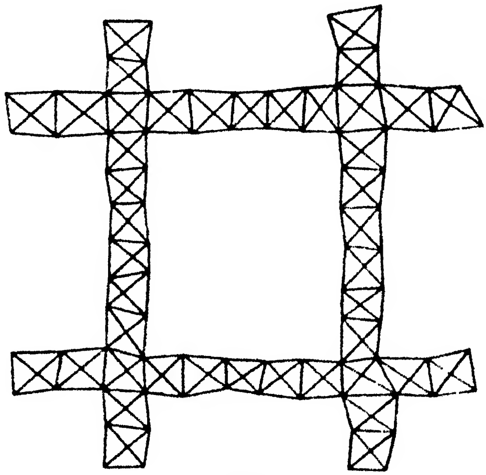


FIG. 3

cover it with smaller triangles of "secondary" accuracy, and lastly with smaller triangles still of "tertiary" accuracy, which will provide stations at such comparatively short distances from each other that the detail may be surveyed by chain surveys, traverses, or even plane table surveys, tied in or "controlled" by the tertiary stations already fixed. No angle of a triangulation should be less than 30°, in order that angular error may not cause too great error in the computation of the sides.

Base-lines. The most expensive part of a triangulation

* *Topographical Surveying*, by Col. C. F. Close. (Wyman & Sons, Ltd.)

is the base-line, the accuracy of which is the upper limit of accuracy of the triangulation. The farther the triangulation proceeds from its base the less the accuracy, owing to accumulated errors, so that, after proceeding a certain distance a "Check base" or "Base of verification" becomes necessary, and this process will require repetition at intervals if the survey covers a very large region. Base-lines are always now measured by suspended wires or tapes hanging in a catenary, a span of about 100 ft. being usual in this country, although in America longer tapes, say, 300 ft. long, supported at a number of intervening points by posts whose tops are set at a uniform gradient, are frequently employed for the sake of greater speed.

For work of first-class importance the tapes or wires are made of "Invar," which is an alloy of 36 per cent nickel and 64 per cent steel, as this alloy possesses the least coefficient of expansion with temperature, viz. about ·000,000,3 per 1° F., while that for steel is ·000,006,25. This is of great importance, as it is always difficult to determine the temperature of the tape or wire itself, especially when the sun is shining. "Invar" increases in length with age, comparatively rapidly at first but more slowly as time goes on, so that the tape or wire requires standardization before and after the base-line is measured. In practice, several of these are used, one being used as a "reference tape," and the others as "working tapes." The reference tape is the one standardized and the working tapes are tested against the reference tape at frequent intervals during the measurement of the base. The tapes are only finely divided for a short distance at each end, the terminal marks (0 ft. and 100 ft., say) being well inside the rings at the ends—wires must have a "reglette," or short divided scale of triangular section, attached at each end, the edge of the scale being along the axis of the wire.

At each end the tape is strained by a cord attached to the ring and passing over a pulley on ball-bearings, carried on a "straining trestle" (P_1, P_2 in Fig. 4), to a suspended weight of not less than twenty times the weight of the tape. Inside

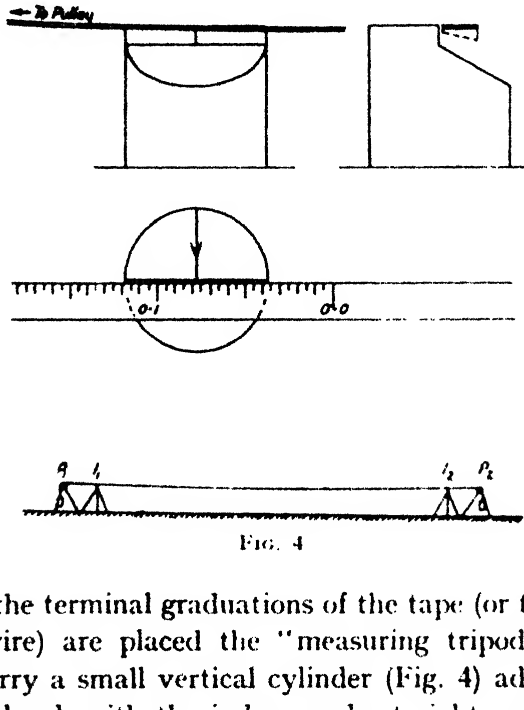


FIG. 4

these at the terminal graduations of the tape (or the reglette of the wire) are placed the "measuring tripods" (I_1, I_2) which carry a small vertical cylinder (Fig. 4) adjustable in line and level, with the index mark at right angles to the tape, and one side cut away so that the tape or reglette may have its upper surface level with the index mark. It is important that the tape or reglette should not disturb the catenary.

There should be a number of such measuring tripods, so that several can be set in advance at approximately correct distances, all being aligned by a theodolite, and their differences of level ascertained by level and staff. The actual measurement is made by a number of simultaneous readings of the index mark at each end with a (hand) magnifying glass, the tape being moved a little alternately in each direction between each pair of readings so as to eliminate pulley friction by the "push and pull" method. The measuring tripods are, of course, always being moved forward from the rear and erected in advance. It is advisable always to leave one *behind* the span actually under measurement in case of accidents. At the end of each day's work the measurement is transferred to a peg in the ground by two theodolites set at right angles to each other. Similarly the permanently marked ends of the base are transferred to and from the index marks on the tripods. The temperature must be read for each span, the bulb of the thermometer touching the tape and shaded from the sun. Accurate work is impossible on windy days.

Corrections of Base. (1) *Standard and Temperature.* The tape or wire must be standardized at a proper establishment such as the National Physical Laboratory at Teddington, near London, where there is a marble slab. 50 metres long, on which tapes can be tested for various lengths, *supported* under known tensions applied by straining pulleys or, preferably, *suspended* in catenary and just touching index marks at the ends of the tape. The base calibrating apparatus is in a gallery kept at a constant temperature, and the distance between the index marks is checked from time to time by a standard H-section "Invar" bar, 12½ ft. long, which is carried on a travelling carriage on rails. Adjustable microscopes fixed to the wall behind the slab read the graduations near the ends of the standard bar, then the bar is moved forward until its near end is opposite the microscope by which the forward end was previously read. The certificate issued will state that the length of the tape or wire between its end marks (say, 0 ft. and 100 ft.) had a certain value at a certain temperature and tension. If desired, the coefficient of expansion with temperature can also be certified, this being ascertained by immersing it at the standard tension in a tank the whole length of the gallery, the water in which can be kept at any required temperature by circulation. It is easy then to calculate at what temperature (t_0) the length of the tape would be correct between its terminal marks. For other temperatures (t) the correction will be $\pm (t - t_0) \cdot \alpha l$, where l is the measured length and α the coefficient of expansion.

(2) *Tension.* If the tape has been standardized at a tension T_0, but is used at a tension T, the correction is $\pm \frac{T - T_0}{aE} l$, where a is the sectional area of the tape and E is Young's Modulus. (E is about 22×10^6 lb. per in.^{2} for Invar.)

(3) *Sag.* As shown in Chapter I, the correction for sag is $-\frac{l}{24} \cdot \frac{W^2}{T^2} = -\frac{l^3}{24} \cdot \frac{w^2}{T^2}$ where W is the weight of the tape *between index marks*, and w = weight per unit length.

(4) *Slope.* As shown in Chapter I, the correction for slope is $-\frac{h^2}{2l}$ where h is the rise (or fall) between index marks, and the ratio $\frac{h}{l}$ is not greater than the limits there stated.

(5) *Height Above Sea.* All survey measurements must be reduced not only to the horizontal but to sea-level. If H is the average height of the base-line above sea level (Fig. 5)

the reduced length (Fig. 5) will be $\frac{R}{R+H}l$. The correction will therefore be $l\left(1-\frac{R}{R+H}\right) = \frac{Hl}{R+H} = \frac{Hl}{R}$ nearly, as $R = 20{,}890{,}000$ ft. nearly, and is large compared to H.

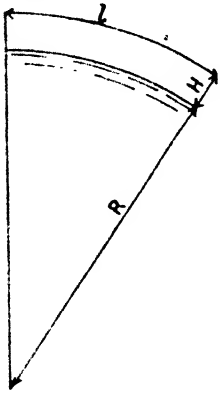


FIG. 5

EXAMPLE 1 (L. U.). A base-line is measured with a suspended steel tape with a tension of 30 lb. The tape was correct when tested at 62° F. supported, and with a tension of 20 lb. Its weight is 0·015 lb. per ft., its sectional area 0·0044 in.^{2}, coefficient of expansion 0·000,006,25 per 1° F., Young's Modulus = 30×10^6 lb. per in.^{2} The measured lengths of the five spans, the rises or falls in the spans, and the temperatures are as follows :--

Span	Rise	Temperature
Feet	Feet	F.
100·191	+ 1·26	45·5°
100·176	+ 2·43	45·7°
100·008	+ 0·20	46·0°
100·142	+ 3·46	46·2°
100·012	− 7·72	46·6°

Find the length reduced to sea level if the average height above sea is 5,000 ft.

Temperature Correction. As the lengths are so nearly equal this can be calculated on the average temperature, viz. 46°.

$$\text{Total correction} = (46° - 62°)\frac{5}{8}\,\frac{1}{100{,}000} \times 500{\cdot}529 = -0{\cdot}0501 \text{ ft.}$$

Tension Correction. Similarly this correction can also be calculated on the total length, viz.

$$\text{Total Correction} = \frac{(30 - 20)\ 500{\cdot}529}{0{\cdot}0044 \times 30 \times 10^6} = +\ 0{\cdot}0379 \text{ ft.}$$

Sag and Slope Corrections.

$l = 100 \times (1 + x)$	$l^3 = 10^6 \times (1 + 3x)$	h	h^2
100·191	1,005,730	+ 1·26	1·588
100·176	1,005,280	+ 2·43	5·905
100·008	1,000,240	+ 0·20	0·040
100·142	1,004,260	+ 3·46	11·972
100·012	1,000,360	− 7·72	59·598
5)500·529	5,015,870		79·103
100·106 av.			

$$\text{Sag Correction} = -\frac{5{,}015{,}870 \times (\cdot015)^2}{24 \times 900} = -0{\cdot}0522 \text{ ft.}$$

$$\text{Slope Correction} = -\frac{79{\cdot}103}{200{\cdot}212} = -0{\cdot}3951 \text{ ft.}$$

$$\textit{Height Above Sea Correction} = -\frac{5{,}000}{20{,}890{,}000} \times 500{\cdot}529 = -0{\cdot}1198 \text{ ft.}$$

$$\text{Reduced Length of Base} = 500{\cdot}529 + 0{\cdot}0379 - 0{\cdot}6172$$
$$= 500{\cdot}529 - 0{\cdot}5793 = 499{\cdot}950 \text{ ft.}$$

Strictly, of course, the value of l used in each of the above corrections should be the value after the corrections above it have been applied. This would, however, make no appreciable difference to the tension correction, as the length has only been reduced $\frac{1}{10{,}000}$ by the temperature correction, and, as it happens, the signs of the tension and temperature corrections are opposite. Even had there been no tension correction, however, each of the lengths for the sag correction would have been reduced by $\frac{1}{10{,}000}$, and the values of l^3 would have been reduced by $\frac{3}{10{,}000}$, say $\frac{1}{3{,}000}$, which would have made no appreciable alteration of the sag correction. Nor would the reduction of the lengths by $\frac{1}{10,000}$ part have affected the slope correction, nor the height above sea correction. The result is, therefore, correct to three decimal places.

A base-line should be measured at least twice—once in each direction. The accuracy of a base so measured would lie between $\frac{1}{100,000}$ and $\frac{1}{2,000,000}$. When less accuracy is required, say $\frac{1}{10,000}$ to $\frac{1}{50,000}$, the base can be measured along the ground, pegs being driven flush with the ground, with zinc strips nailed on their tops. The tape is stretched from peg to peg with a pull of 20 lb. registered on a spring balance, and a fine cut is made simultaneously with a knife, flush with the outside of the handle on the zinc strips at each end at the word of command. The two marks on each peg are distinguished by letters, and the distance between them is afterwards measured with a scale to ·01 in. and allowed for. In this case there is no correction for sag.

Extension of Base. A base-line is usually much shorter than the average length of the side of the triangles which constitute the triangulation. It must, therefore, be extended by well-conditioned triangles, the most usual method being shown in Fig. 6, though the process can be repeated as often as necessary.

AB is the measured base and angles are measured at A, B, C, and D; from these the length of CD (the first extension) is computed, then, by measuring the angles at C, D, E, F, the length of EF (the second extension) is calculated. If no angle is to be less than $30°$, the angles of triangles ABC, ABD, CDE, CDF will have to be about 60° and each extension would enlarge the base about $\sqrt{3}$ times; on the other hand, if it is sufficient that these triangles be "well-conditioned," i.e. no angle less than 30°, each extension will enlarge the base about tan 75°, or 3·732 times.

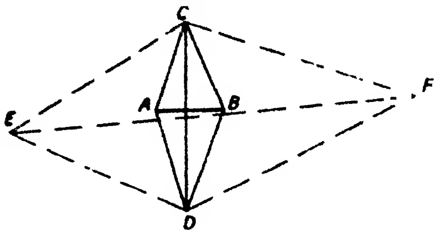


Fig. 6

Marking of Stations. On solid rock, a bolt of some non-rusting metal may be fixed with cement in a drilled hole. Where rock is unavailable, a large stone is buried about 3 ft. deep, the station being marked on the stone by a metal bolt fixed in it. Another stone is then laid flush with ground level, its mark being vertically above that in the buried stone, which latter is only referred to when it is feared that the surface stone has been disturbed.

Angle Measurement. Naturally, the very best theodolites are employed, but these never now exceed 12 in. diameter for primary triangulations. Usually three micrometers are used to read the horizontal circles, reading to single seconds and, by estimation, to 0·1 second. A small vertical telescope is used for centring over the station, and a very sensitive striding level is placed on the horizontal axis. An eye-piece micrometer is frequently fitted in the telescope for the bisection of the signal by a movable vertical hair, the reading of the micrometer being added to that of the circle micrometers. The object of the three micrometers on the horizontal circle is to read the angle equally on all parts of the circle, e.g. if the micrometers (V_1, V_2, V_3, in Fig. 7) are at

0°, 120°, and 240° respectively for the first readings of the first station, and are advanced (i.e. "changing zero") about 70° for each reading till the V_1 reads 350°, then face is

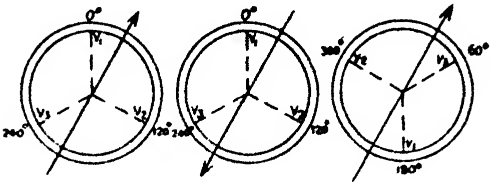


FIG. 7

changed and the micrometers read 180°, 300°, and 60° respectively on the first station, and the micrometers are then advanced about 70° for each reading until the V_1 reads about 170°, it will be seen that the different micrometers have read the circle at 10° intervals all round the circle, and thus errors due to irregularity of graduation are thoroughly eliminated. Actually, the increase of reading is made about 70° 01′ to check the smaller divisions.

The readings on a number of stations are made a large number of times, an equal number face right and face left, and an equal number swinging right and swinging left, and the angles thus obtained between the stations are averaged. In all, each angle would be measured 20 or 30 times. Each horizontal reading is corrected for inclination of the trunnion axis. Atmospheric refraction is the greatest difficulty, and all rays should be kept well away from the ground where possible. Lateral refraction is most dangerous. Observations can be taken at any time in densely cloudy weather—otherwise from 3.30 p.m. to sunset is the best period. For short lines, opaque signals can be used; for long lines, heliotropes (i.e. mirrors to direct the sun's rays from the station observed to the observing station) are best when, of course, the sun is shining. Observations are frequently made at night, lamps being used to sight on, refraction being more constant.

Satellite Station. Occasionally it is convenient to employ as a station a steeple, or other elevated object, on which it would be difficult or impossible to erect the theodolite in order to measure the angles at the station itself. In such cases a "Satellite Station" is employed near the inaccessible station.

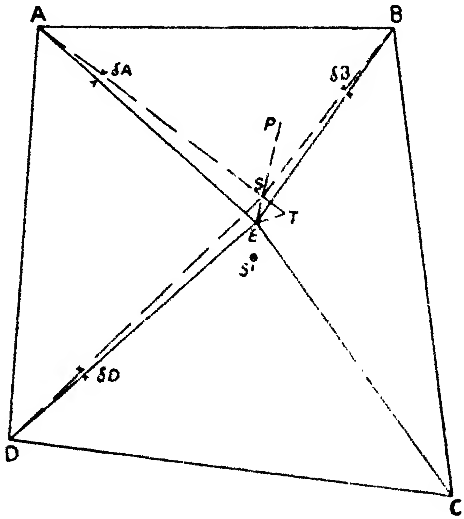


FIG. 8

In Fig. 8 let E be such an inaccessible station to which angles have been read from stations A, B C, and D, but, from which it is impossible to measure the angles AEB, BEC, CED, and DEA. A station S is chosen as near E as possible, and from S as many of the stations are read as can be seen, say D, A, and B; readings must also be made from S to E, and the horizontal distance SE must be measured by laying off a short base-line ST and observing the angles EST, ETS. Then we have

$$\sin \delta A = \frac{SE}{AE} \sin ESA \text{ ; } \sin \delta B = \frac{SE}{BE} \sin ESB,$$

$$\sin \delta D = \frac{SE}{DE} \sin ESD$$

or, as the angles are small,

$$\delta A'' = \frac{SE}{AE} \cdot \frac{\sin ESA}{\sin 1''}; \ \delta B'' = \frac{SE}{BE} \cdot \frac{\sin ESB}{\sin 1''}$$

$$\delta D'' = \frac{SE}{DE} \cdot \frac{\sin ESD}{\sin 1''}$$

so that δA, δB, and δD can be calculated from the measured angles ESA, ESB, ESD, the measured length SE and the values of AE, BE, and DE calculated from the *unadjusted* values of the angles EAB, EBA, EAD, and EDA. Then the angle $AEB = ASB - \delta A - \delta B$ obviously. Producing ES to P, we have

$$\begin{aligned} AED &= PED - PEA = PSD - \delta D - (PSA - \delta A) \\ &= ASD + \delta A - \delta D \end{aligned}$$

N.B. S is *inside* the angles at A and B of triangle AEB, but as regards triangle AED it is *outside* the angle at A and *inside* the angle at D. Another satellite station S' would probably be required in order to observe the stations B, C, and D and thus obtain values for the angles BEC and CED.

A similar procedure would be adopted if it was impossible to sight on a station and angles had to be taken to a signal close to it, or if it was found that a signal was not truly centred over a station.

EXAMPLE 2. A, B, C are three stations, $AB = 39{,}886$ ft., $BC = 24{,}076$ ft., and the angle $CBA = 136° 24' 16''$. The following angles are measured to the top of a spire D, viz.,

$$DCB = 70° 29' 40'', \quad CBD = 63° 21' 40'',$$
$$DBA = 73° 02' 36'', \quad BAD = 44° 25' 54''$$

From E, a point inside the angle BDC and distant 250 ft. from D, the following angles are measured,

$$DEA = 97° 13' 50'', \quad AEB = 62° 42' 00''$$
$$BEC = 46° 30' 50''$$

Find the angles *ADB*, *BDC* and the angular error in the triangles *ADB*, *BDC* (Fig. 9).

The unmeasured angles are

$$ADB = 180° - 117° 28' 30'' = 62° 31' 30''$$

and $$BDC = 180° - 133° 51' 20'' = 46° 08' 40''$$

We first calculate *AD*, *BD*, and *CD*—

39,886	4·600,8205		
sin 62° 31′ 30″	$\bar{1}$·948,0275		
	4·652,7930		4·652,7930
sin 73° 02′ 36″	$\bar{1}$·980,6966	sin 44° 25′ 54″	$\bar{1}$·845,1341
AD	4·633,4896	*BD*	4·497,9271
24,076	4·381,5843		
sin 46° 08′ 40″	$\bar{1}$·857,9887		
	4·523,5956		4·523,5956
sin 70° 29′ 40″	$\bar{1}$·974,3317	sin 63° 21′ 40″	$\bar{1}$·951,2647
BD	4·497,9273	*CD*	4·474,8603
ED	2·397,9400	*ED*	2·397,9400
sin { 97° 13′ 50″ / 82° 46′ 10″	$\bar{1}$·996,5326	sin { 159° 55′ 50″ / 20° 04′ 10″	$\bar{1}$·535,4951
	2·394,4726		1·933,4351
AD	4·633,4896	*BD*	4·497,9271
sin *δA*	$\bar{3}$·760,9830	sin *δB*	$\bar{3}$·435,5080
sin 1″	$\bar{6}$·685,5749	sin 1″	$\bar{6}$·685,5749
δA″	3·075,4081	*δB*″	2·749,9331

∴ *δA* = 1189·6″ = 19′ 49·6″ *δB* = 562·3″ = 9′ 22·3″

ED	2·397,9400	97° 13′ 50″	
sin { 153° 33′ 20″ / 26° 26′ 40″	$\bar{1}$·648,6818	62° 42′ 00″	
	2·046,6218	159° 55′ 50″	*DEB*
CD	4·474,8603	46° 30′ 50″	
sin *δC*	$\bar{3}$·571,7615	206° 26′ 40″	
sin 1″	$\bar{6}$·685,5749	153° 33′ 20″	= *DEC*
δC″	2·886,1866		

∴ *δC* = 769·5″ = 12′ 49·5″

Therefore,

$$ADB = 62° 42' 00'' - 19' 49{\cdot}6 + 9' 22{\cdot}3'' = 62° 31' 32{\cdot}7''$$

and error in triangle *ADB* = + 2·7″

$$BDC = 46° \, 30' \, 50'' - 9' \, 22{\cdot}3'' - 12' \, 49{\cdot}5'' = 46° \, 08' \, 38{\cdot}2''$$

and error in triangle $BDC = -1{\cdot}8''$

Before commencing the question of the adjustment of the angles in a triangulation, we shall work out two examples on the Method of Least Squares applied to angles.

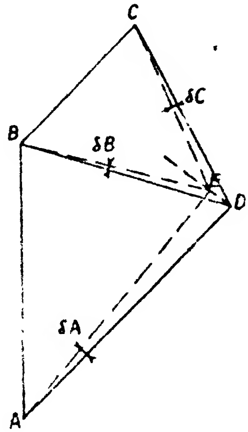


Fig. 9

EXAMPLE 3 (L.U.). A, B, C, D form a round of angles at a station so that $A + B + C + D = 360°$. Their observed values were

$$A = 76° \, 24' \, 40'', \; B = 82° \, 14' \, 25'',$$

$$C = 103° \, 37' \, 50'', \; D = 97° \, 43' \, 15'';$$

the angle $B + C$ was also separately measured *twice* and found to average $185° \, 52' \, 20''$. Find the probable value of each of the four angles if all six measurements were of equal accuracy.

From the equation of condition $A + B + C + D = 360°$, we eliminate D and write

$$A + B + C = 360° - D = 262° \, 16' \, 45''$$

The observational equation $B + C = 185° \, 52' \, 20''$ must have a weight of 2.

w	A	B	C	N
1	1	0	0	76° 24′ 40″
1	0	1	0	82° 14′ 25″
1	0	0	1	103° 37′ 50″
1	1	1	1	262° 16′ 45″
2	0	1	1	185° 52′ 20″

The normal equations are—

(i) $2A + B + C = 338° \, 41' \, 25''$

(ii) $A + 4B + 3C = 716° \, 15' \, 50''$

(iii) $A + 3B + 4C = 737° \, 39' \, 15''$

(i) $2A + B + C = 338° \, 41' \, 25''$

(ii) $2A + 8B + 6C = 1432° \, 31' \, 40''$

$\therefore 7B + 5C = 1093° \, 50' \, 15''$

(ii and iii) $-B + C = 21° \, 23' \, 25''$

$$\left.\begin{aligned} \therefore \; 7B + 5C &= 1093° \, 50' \, 15'' \\ -7B + 7C &= 149° \, 43' \, 55'' \end{aligned}\right\} \therefore 12C = 1243° \, 34' \, 10''$$

$\therefore C = 103° \, 37' \, 50\tfrac{5}{6}'' \quad \therefore B = 82° \, 14' \, 25\tfrac{5}{6}''$

$$A = \frac{338° \, 41' \, 25'' - 185° \, 52' \, 16\tfrac{2}{3}''}{2} = 76° \, 24' \, 34\tfrac{1}{6}''$$

$\therefore D = 360° - 262° \, 16' \, 50\tfrac{5}{6}'' = 97° \, 43' \, 09\tfrac{1}{6}''$ and

$B + C = 185° \, 52' \, 16\tfrac{2}{3}''$

We will now check this by the method of Correlates. We have two equations of condition, viz.

$$A + B + C + D = 360°, \text{ and } B + C = B + C$$

The first entails a correction of $-10''$, and the second of $-5''$, so if e_1, e_2, e_3, e_4, e_5 are the corrections on A, B, C, D, and $B + C$, we must have

$$e_1^2 + e_2^2 + e_3^2 + e_4^2 + 2e_5^2 = \text{minimum}$$

$$e_1 + e_2 + e_3 + e_4 = -10'' \; ; \; e_5 - e_2 - e_3 = -5''$$

$$\therefore \quad \left\{ \begin{aligned} & e_1\delta_1 e + e_2\delta e_2 + e_3\delta e_3 + e_4\delta e_4 + 2e_5\delta e_5 = 0 \\ & \delta e_1 + \delta e_2 + \delta e_3 + \delta e_4 = 0 \\ & \delta e_5 - \delta e_2 - \delta e_3 = 0 \end{aligned} \right.$$

Multiplying the two latter equations by $-\lambda$, $-\mu$ respectively, adding all three equations, and equating the coefficient of each δe to zero, we get

$$e_1 = e_4 = \lambda \,;\, e_2 = e_3 = \lambda - \mu \,;\, e_5 = \frac{\mu}{2}$$

Substituting, we get

$$\begin{cases} 4\lambda - \quad 2\mu = -10'' \\ -2\lambda + 2{\cdot}5\mu = -\ \ 5'' \end{cases} \quad \therefore 3\mu = -20'' \quad \therefore \mu = -6\tfrac{2}{3}''$$

$$\therefore e_5 = -3\tfrac{1}{3}''$$

Also $\lambda = \dfrac{-10 - 13\frac{1}{3}}{4} = -5\tfrac{5}{6}'' = e_1 = e_4$, and $e_2 = e_3 = +\tfrac{5}{6}''$

Therefore, $A = 76^\circ\ 24'\ 34\tfrac{1}{6}''$, $B = 82^\circ\ 14'\ 25\tfrac{5}{6}''$

$C = 103^\circ\ 37'\ 50\tfrac{5}{6}''$ $D = 97^\circ\ 43'\ 09\tfrac{1}{6}''$

$B + C = 185^\circ\ 52'\ 16\tfrac{2}{3}''$, as before.

Example 4 (L.U.). A tunnel is to be run between two points, A and B. The station B is invisible from A, but observations were taken to two other stations, C and D, both to the right of AB, with $ABCD$ clockwise. The following angles (Fig. 10) were recorded -

$CAD = 61^\circ\ 45'\ 00''$, $ADB = 21^\circ\ 14'\ 10''$, $CDB = 44^\circ\ 18'\ 10''$
$ACD = 52^\circ\ 42'\ 50''$, $ACB = 31^\circ\ 20'\ 30''$, $CBD = 51^\circ\ 38'\ 20''$

Calculate the angles BAC and ABD.

The sum of the angles in triangle ACD

$= 180^\circ\ 00'\ 10''$; error $= +10''$

and the sum of the angles in triangle BCD

$= 179^\circ\ 59'\ 50''$; error $= -10''$

Calling the correct values of the 6 given angles θ_1, θ_2, θ_3, θ_4, θ_5, and θ_6, and eliminating θ_1 and θ_6, as there are two equations of condition, we have

$$\theta_2 + \theta_3 + \theta_4 = 180^\circ - 61^\circ\ 45'\ 00'' = 118^\circ\ 15'\ 00''$$
$$\theta_3 + \theta_4 + \theta_5 = 180^\circ - 51^\circ\ 38'\ 20'' = 128^\circ\ 21'\ 40''$$

θ_2	θ_3	θ_4	θ_5	N
1	0	0	0	21° 14′ 10″
0	1	0	0	44° 18′ 10″
0	0	1	0	52° 42′ 50″
0	0	0	1	31° 20′ 30″
1	1	1	0	118° 15′ 00″
0	1	1	1	128° 21′ 40″

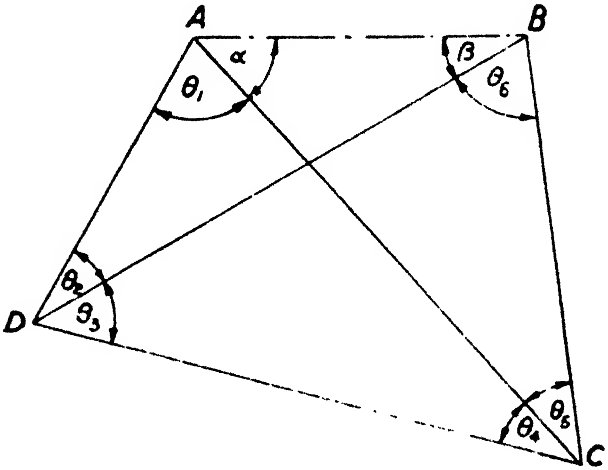


Fig. 10

The normal equations are

$$\text{(i)}\quad 2\theta_2 + \theta_3 + \theta_4 = 139°\,29'\,10''$$
$$\text{(ii)}\quad \theta_2 + 3\theta_3 + 2\theta_4 + \theta_5 = 290°\,54'\,50''$$
$$\text{(iii)}\quad \theta_2 + 2\theta_3 + 3\theta_4 + \theta_5 = 299°\,19'\,30''$$
$$\text{(iv)}\quad \theta_3 + \theta_4 + 2\theta_5 = 159°\,42'\,10''$$

$$\begin{array}{rl}\text{(i) and (iv)} & \left\{\begin{array}{l}\theta_2 + \theta_3 + \theta_4 + \theta_5 = 149°\,35'\,40'' \\ \theta_2 + 3\theta_3 + 2\theta_4 + \theta_5 = 290°\,54'\,50''\end{array}\right. \\ \text{(ii)} & \end{array}$$

$$\therefore\ 2\theta_3 + \theta_4 = 141°\,19'\,10''$$

Also (i) and (iv) $\left\{ \begin{aligned} \theta_2 + \theta_3 + \theta_4 + \theta_5 &= 149^\circ\,35'\,40'' \end{aligned}\right.$
(iii) $\quad \theta_2 + 2\theta_3 + 3\theta_4 + \theta_5 = 299^\circ\,19'\,30''$

$$\therefore \quad \theta_3 + 2\theta_4 = 149^\circ\,43'\,50''$$

$$\therefore \left\{ \begin{aligned} 2\theta_3 + \theta_4 &= 141^\circ\,19'\,10'' \\ 2\theta_3 + 4\theta_4 &= 299^\circ\,27'\,40'' \end{aligned}\right.$$

$$\therefore \theta_4 = \frac{158^\circ\,08'\,30''}{3} = 52^\circ\,42'\,50''$$

$$\therefore \theta_3 = 149^\circ\,43'\,50'' - 105^\circ\,25'\,40'' = 44^\circ\,18'\,10''$$

$$\therefore \theta_2 = \frac{139^\circ\,29'\,10'' - 97^\circ\,01'\,00''}{2} = 21^\circ\,14'\,05''$$

$$\theta_5 = \frac{159^\circ\,42'\,10'' - 97^\circ\,01'\,00''}{2} = 31^\circ\,20'\,35''$$

$$\theta_1 = 180^\circ - (21^\circ\,14'\,05'' + 44^\circ\,18'\,10'' + 52^\circ\,42'\,50'') = 61^\circ\,44'\,55''$$

$$\theta_6 = 180^\circ - (44^\circ\,18'\,10'' + 52^\circ\,42'\,50'' + 31^\circ\,20'\,35'') = 51^\circ\,38'\,25''$$

By the Method of Correlates, let $e_1, e_2, e_3, e_4, e_5, e_6$ be the corrections of angles $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$

Then $\Sigma(e^2) = \text{minimum} \qquad \Sigma(e\,\delta e) = 0$

$e_1 + e_2 + e_3 + e_4 = -10 \qquad \delta e_1 + \delta e_2 + \delta e_3 + \delta e_4 = 0 \qquad -\lambda$

$e_3 + e_4 + e_5 + e_6 = +10 \qquad \delta e_3 + \delta e_4 + \delta e_5 + \delta e_6 = 0 \qquad -\mu$

$\therefore e_1 = \lambda, \quad e_2, e_3 = \lambda + \mu, \quad e_4, e_5 = \mu = e_6$

$\therefore \left\{ \begin{aligned} 4\lambda + 2\mu &= -10 \\ 2\lambda + 4\mu &= 10 \end{aligned}\right. \qquad \therefore \left\{ \begin{aligned} 4\lambda + 2\mu &= -10 \\ 4\lambda + 8\mu &= 20 \end{aligned}\right.$

$\therefore 6\mu = 30. \qquad \therefore \mu = 5''.$

$$\therefore \quad \lambda = \frac{-10 - 20}{2} \ldots = -5'' \text{ and } \lambda + \mu = 0.$$

$\therefore \quad e_1 = e_2 = -5'', \; e_3 = e_4 = 0, \; e_5 = e_6 = +5''$

$\therefore \quad \theta_1 = 61^\circ\,44'\,55''; \; \theta_2 = 21^\circ\,14'\,05''; \; \theta_3 = 44^\circ\,18'\,10''$

$\quad \theta_4 = 52^\circ\,42'\,50''; \; \theta_5 = 31^\circ\,20'\,35''; \; \theta_6 = 51^\circ\,38'\,25''$

as before. The reader will notice the shortening of the work by the Method of Correlates, but the *signs* of the *e*'s require care.]

Calling the angles BAC, ABD, α and β respectively, we have

$$\alpha + \beta = \theta_3 + \theta_4 = 44^\circ\ 18'\ 10'' + 52^\circ\ 42'\ 50'' = 97^\circ\ 01'\ 00''$$

Also

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin 61^\circ\ 44'\ 55''\ .\ \sin 44^\circ\ 18'\ 10''\ .\ \sin 31^\circ\ 20'\ 35''}{\sin 21^\circ\ 14'\ 05''\ .\ \sin 52^\circ\ 42'\ 50''\ .\ \sin 51^\circ\ 38'\ 25''}$$

by condition (iv) for a Quadrilateral [Equation 10, page 221].

sin 61° 44′ 55″	$\bar{1}$·944,9163	sin 21° 14′ 05″	$\bar{1}$·558,9359	$\bar{1}$·505,1894
sin 44° 18′ 10″	$\bar{1}$·844,1353	sin 52° 42′ 50″	$\bar{1}$·900,7059	$\bar{1}$·354,0298
sin 31° 20′ 35″	$\bar{1}$·716,1378	sin 51° 38′ 25″	$\bar{1}$·894,3880	
	$\bar{1}$·505,1894		$\bar{1}$·354,0298	0·151,1596

$$\therefore \frac{\sin\alpha}{\sin\beta} = 1{\cdot}416314 = k$$

And $$\frac{k-1}{k+1} = \frac{0{\cdot}416314}{2{\cdot}416314} = \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta}$$

$$= \frac{2\sin\frac{\alpha-\beta}{2}\ .\ \cos\frac{\alpha+\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\ .\ \cos\frac{\alpha-\beta}{2}} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan 48^\circ\ 30'\ 30''}$$

0·416314	$\bar{1}$·619,4210
2·416314	0·383,1534
	$\bar{1}$·236,2676
tan 48° 30′ 30″	0·053,3188
tan 11° 01′ 23″	$\bar{1}$·289,5864

$$\therefore \frac{\alpha-\beta}{2} = 11^\circ\ 01'\ 23''$$

and $\frac{\alpha+\beta}{2} = 48^\circ\ 30'\ 30''$ $\therefore \alpha = 59^\circ\ 31'\ 53''$; $\beta = 37^\circ\ 29'\ 07''$.

Adjustment of Angles. (i) *Quadrilateral with Diagonals.* This is a useful figure for the survey for a bridge across a wide river, for a chain triangulation and for extension of a

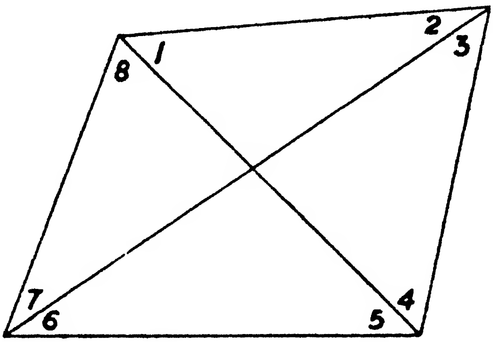


FIG. 11

base-line. In the figure (Fig. 11) the equations of condition are—

(i) $\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7 + \theta_8 = 360^\circ$

(ii) $\theta_1 + \theta_2 = \theta_5 + \theta_6$

(iii) $\theta_3 + \theta_4 = \theta_7 + \theta_8$

(iv) $\Sigma(\log \sin \theta_L) = \Sigma(\log \sin \theta_R)$

Calling the corrections of the angles $e_1, e_2, e_3 \ldots e_8$ respectively, their increases of log sine per 1″ v_1, v_2, v_3, etc., and the total corrections required for each of the above conditions to be fulfilled E_1, E_2, E_3, and E_4, we have—

$e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_7^2 + e_8^2$ = minimum	$\Sigma(e\delta e) = 0$	—
$e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 = E_1$	$\Sigma(\delta e) = 0$	$-\lambda_1$
$e_1 + e_2 - e_5 - e_6 = E_2$	$\delta e_1 + \delta e_2 - \delta e_5 - \delta e_6 = 0$	$-\lambda_2$
$e_3 + e_4 - e_7 - e_8 = E_3$	$\delta e_3 + \delta e_4 - \delta e_7 - \delta e_8 = 0$	$-\lambda_3$
$e_1v_1 - e_2v_2 + e_3v_3 - e_4v_4 + e_5v_5 - e_6v_6 + e_7v_7 - e_8v_8 = E_4$	$\Sigma(v\delta e_L) - \Sigma(v\delta e_R) = 0$	$-\mu$

(N.B. If an angle is greater than 90°, its v will be *negative*.)

$$\therefore\ e_1 = \lambda_1 + \lambda_2 + \mu v_1;\quad e_2 = \lambda_1 + \lambda_2 - \mu v_2;$$
$$e_3 = \lambda_1 + \lambda_3 + \mu v_3;\quad e_4 = \lambda_1 + \lambda_3 - \mu v_4;$$
$$e_5 = \lambda_1 - \lambda_2 + \mu v_5;\quad e_6 = \lambda_1 - \lambda_2 - \mu v_6;$$
$$e_7 = \lambda_1 - \lambda_3 + \mu v_7;\quad e_8 = \lambda_1 - \lambda_3 - \mu v_8$$

Therefore, substituting,

$$8\lambda_1 + \mu(v_1 - v_2 + v_3 - v_4 + v_5 - v_6 + v_7 - v_8) = E_1$$
$$4\lambda_2 + \mu\{(v_1 - v_2) - (v_5 - v_6)\} = E_2$$
$$4\lambda_3 + \mu\{(v_3 - v_4) - (v_7 - v_8)\} = E_3$$
$$\lambda_1\{(v_1 - v_2) + (v_3 - v_4) + (v_5 - v_6) + (v_7 - v_8)\} + \lambda_2\{(v_1 - v_2) - (v_5 - v_6)\} + \lambda_3\{(v_3 - v_4) - (v_7 - v_8)\} + \mu\Sigma(v^2) = E_4$$

These are the normal equations, and the values of λ_1, λ_2, λ_3, and μ obtained therefrom give us the corrections required by the above formulae for e_1, e_2, e_3, etc.

EXAMPLE 5. Find the corrections of the angles given in the table for a Quadrilateral with Diagonals. (See page 243.)

The solution of the normal equations is best executed in tabular form as shown on page 244.

The values of the e's are now inserted in the table and multiplied by the corresponding differences of log sine for 1″, to give the correction of log sines; these, when totalled up, show that the L.H. log sine total is decreased by 936 and the R.H. one increased by 457, so that the difference of ·0001394 in log sine has been decreased to ·000,000,1, which is negligible. The total correction in the angles is found to be − 40·82 + 21·82 = − 19″, as required.

The last column ("sums of angles") now reads—

88° 49′ 40″ − 3·93″ = 88° 49′ 36·07″
91° 10′ 17″ + 6·93″ = 91° 10′ 23·93″
88° 49′ 45″ − 8·93″ = 88° 49′ 36·07″
91° 10′ 37″ − 13·07″ = 91° 10′ 23·93″

We first add up the angles and find E_1, E_2, and E_3, as given below. Then we enter the log sines of the angles and their increases (v) per 1″. Totalling the log sines, we find $E_4 = -\cdot0001394$, or 1,394 shift to R.H.

No.	L.H. Angle	Log sine	v	e	ev	No.	R.H. Angle	Log sine	v	e	ev	Sum of Angles
1	54° 00′ 58″	$\bar{1}$·908,0463	15·30	− 9·87	− 151	2	34° 48′ 42″	$\bar{1}$·756,5454	30·28	+ 5·94	+ 180	88° 49′ 40″
3	42° 16′ 42″	$\bar{1}$·827,8426	23·17	− 3·74	− 87	4	48° 53′ 35″	$\bar{1}$·877,0739	18·37	+ 10·67	+ 196	91° 10′ 17″
5	29° 20′ 36″	$\bar{1}$·690,2332	37·47	− 13·12	− 491	6	59° 29′ 09″	$\bar{1}$·935,2571	12·42	+ 4·19	+ 52	88° 49′ 45″
7	55° 02′ 15″	$\bar{1}$·913,5634	14·72	− 14·09	− 207	8	36° 08′ 22″	$\bar{1}$·770,6697	28·83	+ 1·02	+ 29	91° 10′ 37″
Sum	180° 40′ 31″	$\bar{1}$·339,6855		− 40·82	− 936		179° 19′ 48″	$\bar{1}$·339,5461		+ 21·82	+ 457	
		$\bar{1}$·339,5461		+ 21·82	− 457		180° 40′ 31″					
		1394	→	− 19·00	− 1393		360° 00′ 19″					

$\therefore E_1 = -19''$, $E_2 = 5''$
$E_3 = 20''$, $E_4 = -1394$

$\Sigma(v^2) = 4632$

The normal equations are

$$\therefore \begin{cases} 8\lambda_1 + \mu(-14\cdot98 + 4\cdot80 + 25\cdot05 - 14\cdot11) = -19 \\ 4\lambda_2 + \mu(-14\cdot98 - 25\cdot05) = 5 \\ 4\lambda_3 + \mu(4\cdot80 - (-14\cdot11)) = 20 \\ \lambda_1(29\cdot85 - 29\cdot09) - 40\cdot03\lambda_2 + 18\cdot91\lambda_3 + 4632\mu = -1394 \end{cases}$$

$$\therefore \begin{cases} 8\lambda_1 + 0\cdot76\mu + 19 = 0 & \text{(i)} \\ 4\lambda_2 - 40\cdot03\mu - 5 = 0 & \text{(ii)} \\ 4\lambda_3 + 18\cdot91\mu - 20 = 0 & \text{(iii)} \\ 0\cdot76\lambda_1 - 40\cdot03\lambda_2 + 18\cdot91\lambda_3 + 4632\mu + 1394 = 0 & \text{(iv)} \end{cases}$$

Equation	λ_1	λ_2	λ_3	μ	N	Eliminating Equation
(i)	8	—	—	0·76	19	$\lambda_1 = -0{\cdot}095\mu - 2{\cdot}375$
(ii)	—	4	—	− 40·03	− 5	$\lambda_2 = 10{\cdot}0075\mu + 1{\cdot}25$
(iii)	—	—	4	18·91	− 20	$\lambda_3 = -4{\cdot}7275\mu + 5$
(iv)	0·76	− 40·03	18·91	4632 − ·07220 − 400·600 − 89·397	1394 − 1·8050 − 50·0375 + 94·55	 (Eliminating λ_1) (Eliminating λ_2) (Eliminating λ_3)
				4141·931	1436·71	$\therefore \mu = -\dfrac{1436{\cdot}71}{4142} = -0{\cdot}3469$ $\therefore \lambda_1 = {\cdot}03296 - 2{\cdot}375 = -2{\cdot}3420$ $\lambda_2 = -3{\cdot}4716 + 1{\cdot}25 = -2{\cdot}2216$ $\lambda_3 = 1{\cdot}6400 + 5{\cdot}00 = 6{\cdot}6400$

$$\therefore e_1 = -2{\cdot}3420 - 2{\cdot}2216 - {\cdot}3469 \times 15{\cdot}30 = -\ 9{\cdot}87''$$
$$e_2 = -2{\cdot}3420 - 2{\cdot}2216 + {\cdot}3469 \times 30{\cdot}28 = +\ 5{\cdot}94''$$
$$e_3 = -2{\cdot}3420 + 6{\cdot}6400 - {\cdot}3469 \times 23{\cdot}17 = -\ 3{\cdot}74''$$
$$e_4 = -2{\cdot}3420 + 6{\cdot}6400 + {\cdot}3469 \times 18{\cdot}37 = +\ 10{\cdot}67''$$
$$e_5 = -2{\cdot}3420 + 2{\cdot}2216 - {\cdot}3469 \times 37{\cdot}47 = -\ 13{\cdot}12''$$
$$e_6 = -2{\cdot}3420 + 2{\cdot}2216 + {\cdot}3469 \times 12{\cdot}42 = +\ 4{\cdot}19''$$
$$e_7 = -2{\cdot}3420 - 6{\cdot}6400 - {\cdot}3469 \times 14{\cdot}72 = -\ 14{\cdot}09''$$
$$e_8 = -2{\cdot}3420 - 6{\cdot}6400 + {\cdot}3469 \times 28{\cdot}83 = +\ 1{\cdot}02''$$

so that all the four conditions are complied with by the corrected angles.

(ii) *Polygon with a Central Point.* This is a most adaptable figure, as the polygon may have any number of sides from three upwards. It is the form adopted when the whole surface of a country is to be covered with triangulation, the polygons intersecting each other. We shall take a Quadrilateral for illustration, but the principle is the same for any number of sides.

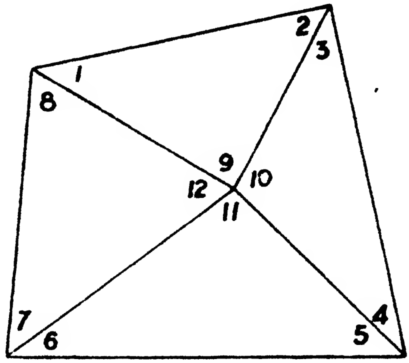


FIG. 12

In the figure (Fig. 12) the equations of condition are—

(i) $\theta_1 + \theta_2 + \theta_9 = 180°$

(ii) $\theta_3 + \theta_4 + \theta_{10} = 180°$

(iii) $\theta_5 + \theta_6 + \theta_{11} = 180°$

(iv) $\theta_7 + \theta_8 + \theta_{12} = 180°$

(v) $\theta_9 + \theta_{10} + \theta_{11} + \theta_{12} = 360°$

(vi) $\Sigma(\log \sin \theta_L) = \Sigma(\log \sin \theta_R)$

We have, therefore,

$\Sigma(e^2)$ = minimum	$\therefore \Sigma(e\delta e) = 0$	—
$e_1 + e_2 + e_9 = E_1$	$\delta e_1 + \delta e_2 + \delta e_9 = 0$	$-\lambda_1$
$e_3 + e_4 + e_{10} = E_2$	$\delta e_3 + \delta e_4 + \delta e_{10} = 0$	$-\lambda_2$
$e_5 + e_6 + e_{11} = E_3$	$\delta e_5 + \delta e_6 + \delta e_{11} = 0$	$-\lambda_3$
$e_7 + e_8 + e_{12} = E_4$	$\delta e_7 + \delta e_8 + \delta e_{12} = 0$	$-\lambda_4$
$e_9 + e_{10} + e_{11} + e_{12} = E_5$	$\delta e_9 + \delta e_{10} + \delta e_{11} + \delta e_{12} = 0$	$-\lambda_5$
$e_1v_1 - e_2v_2 + e_3v_3 - e_4v_4 + e_5v_5 - e_6v_6 + e_7v_7 - e_8v_8 = E_6$	$\Sigma v\delta e_L - \Sigma v\delta e_R = 0$	$-\mu$

$$\therefore e_1 = \lambda_1 + \mu v_1;\quad e_2 = \lambda_1 - \mu v_2;\quad e_3 = \lambda_2 + \mu v_3;$$
$$e_4 = \lambda_2 - \mu v_4;\quad e_5 = \lambda_3 + \mu v_5;\quad e_6 = \lambda_3 - \mu v_6;$$
$$e_7 = \lambda_4 + \mu v_7;\quad e_8 = \lambda_4 - \mu v_8;\quad e_9 = \lambda_1 + \lambda_5;$$
$$e_{10} = \lambda_2 + \lambda_5;\quad e_{11} = \lambda_3 + \lambda_5;\quad e_{12} = \lambda_4 + \lambda_5.$$

Therefore, substituting in the observational equations, we get the normal equations, viz.—

(i) $3\lambda_1 + \lambda_5 + \mu(v_1 - v_2) = E_1$

(ii) $3\lambda_2 + \lambda_5 + \mu(v_3 - v_4) = E_2$

(iii) $3\lambda_3 + \lambda_5 + \mu(v_5 - v_6) = E_3$

(iv) $3\lambda_4 + \lambda_5 + \mu(v_7 - v_8) = E_4$

(v) $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 4\lambda_5 = E_5$

(or, $\lambda_1 + \lambda_2 + \lambda_3 + \ldots \lambda_n + n\lambda_{n+1} = E_{n+1}$, for n sides)

(vi) $\lambda_1(v_1 - v_2) + \lambda_2(v_3 - v_4) + \lambda_3(v_5 - v_6) + \lambda_4(v_7 - v_8)$
$+ \mu\Sigma(v^2) = E_6$

We could solve these equations by the tabular process of elimination used in Example 5, with columns for the coefficients of λ_1, λ_2, λ_3, λ_4, λ_5, μ and N. We should put down equations (i), (ii), (iii) and (iv) in order with eliminating equations for λ_1, λ_2, λ_3 and λ_4 in terms of λ_5 and μ. Then put down equation (v) and substitute for λ_1, λ_2, λ_3, λ_4, giving an eliminating equation for λ_5 in terms of μ. Then put down equation (vi); substitute for λ_1, λ_2, λ_3, λ_4, add up the columns for λ_5, μ and N, and, on substituting for λ_5, we obtain the value of μ. We now retrace our steps and find successively λ_5, λ_1, λ_2, λ_3 and λ_4 and hence the various e's. In Appendix I is given the tabular solution of Example 6 on page 248. It is rather a tedious process without a calculating machine, and the following method of "Successive Approximations" suggested by Professor J. B. Dale will probably be preferred if the errors are small.*

* Or Panton's method may be used, viz.—

Adding (i), (ii), (iii), (iv), we get

$$3\lambda_1 + 3\lambda_2 + 3\lambda_3 + 3\lambda_4 + 4\lambda_5 = E_1 + E_2 + E_3 + E_4,$$

ignoring the μ term.

Multiply (v) by 3—

$$3\lambda_1 + 3\lambda_2 + 3\lambda_3 + 3\lambda_4 + 12\lambda_5 = 3E_5$$

Subtracting, $\lambda_5 = \dfrac{3E_5 - (E_1 + E_2 + E_3 + E_4)^*}{8}$

Using this value of λ_5 and *still ignoring the μ terms*, from (i), (ii), (iii), (iv) we can write

$$\lambda_1 = \frac{E_1 - \lambda_5}{3},\ \lambda_2 = \frac{E_2 - \lambda_5}{3},\ \lambda_3 = \frac{E_3 - \lambda_5}{3},\ \lambda_4 = \frac{E_4 - \lambda_5}{3}$$

Substituting these values of λ_1, λ_2, λ_3, and λ_4 in (vi), we have

$$\mu = \frac{E_6 - \Sigma\lambda(v_L - v_R)}{\Sigma(v^2)}$$

We then proceed to a second approximation, finding $E_1' = E_1 - \mu(v_1 - v_2)$, etc.,

$$\lambda_5 = \frac{3E_5 - (E_1' + E_2' + E_3' + E_4')}{8}$$

and then

$$\lambda_1 = \frac{E_1' - \lambda_5}{3},\ \text{etc.}$$

If the second value of μ differs appreciably from its first value, we must repeat the process until no appreciable change is effected by repeating the process.

EXAMPLE 6 (L.U.). *ABCD* (in clockwise order) is a quadrilateral, with a central point E, which forms a triangulation

Multiply (v) by 3 and subtract it from the sum of (i), (ii), (iii) and (iv). This gives $\mu\Sigma\ (v_L - v_R) - 8\lambda_5 = \Sigma_1{}^4E - 3E_5$. Multiply (vi) by 3, (i) by $v_1 - v_2$, (ii) by $v_3 - v_4$, (iii) by $v_5 - v_6$, (iv) by $v_7 - v_8$ and subtract again. This gives $\mu\{\Sigma(v_L - v_R)^2 - 3\ \Sigma v^2\} + \lambda_5\Sigma\ (v_L - v_R) = \Sigma_1{}^4E\ (v_L - v_R) - 3E_6$. These two equations give a simultaneous equation for λ_5 and μ. Then proceed as above for λ_1, λ_2, λ_3 and λ_4.

* If there were n sides in the polygon the denominator would be $2n$.

for a bridge across a river, E being on a small island. The measured angles are tabulated as shown on page 249.

AB is the base line and its reduced length is 499·857 ft. Adjust the angles to the nearest second by the Method of Least Squares. Calculate the lengths of the sides EB, BC, and EC to the nearest 0·01 ft. P is a point in AB, 247·61 ft. from B, Q is a point in CD such that PEQ is a straight line. Find the angles BPQ and CQP to the nearest second. Calculate the lengths of PE, EQ, and CQ to the nearest 0·01 ft.

We first add up the three angles in each triangle and find $E_1 = -10''$, $E_2 = +10''$, $E_3 = +5''$, $E_4 = +5''$. Then we add up the central angles, total 360° 00′ 00″, $\therefore E_5 = 0$. Then we enter the log sines of each angle (and the increases per 1″) as shown. We add them up and find that they require a shift of 819(·0000819) to the left; $\therefore E_6 = 819$.

First Approximation.

$$\lambda_5 = \frac{0-(-10+10+5+5)}{8} = -1{\cdot}25$$

$$\therefore \lambda_1 = \frac{-10+1{\cdot}25}{3} = -2{\cdot}92; \quad \lambda_2 = \frac{10+1{\cdot}25}{3} = 3{\cdot}75$$

$$\lambda_3 = \frac{5+1{\cdot}25}{3} = 2{\cdot}08; \quad \lambda_4 = \frac{5+1{\cdot}25}{3} = 2{\cdot}08$$

Then we make a table—

v_L	v_L^2	v_R	v_R^2	$v_L - v_R$
11·67	136·2	12·18	148·4	– 0·51
33·63	1131·0	37·15	1380·0	– 3·52
13·93	194·0	10·73	115·1	3·20
38·45	1479·0	37·57	1411·0	0·88
	2940·2		3054·5	

Triangle	Central Angle	Correc-tion	L.H. Angle	Log sine	Increase for 1"	Correc-tion of Angle	Correc-tion of Log sin	R.H. Angle	Log sine	Increase for 1"	Correc-tion of Angle	Correc-tion of Log sine	Sum of Angles	Total Correc-tion
AEB .	59° 03′ 10″	− 4·14″	61° 00′ 54″	$\bar{1}$·941,8823	11·67	− 1·29″	− 15	59° 56′ 06″	$\bar{1}$·937,2458	12·18	− 4·56″	− 56	180° 00′ 10″	− 10″
BEC .	118° 23′ 50″	+ 2·66″	32° 03′ 54″	$\bar{1}$·724,9972	33·63	+ 8·53″	+ 287	29° 32′ 06″	$\bar{1}$·692,8074	37·15	− 1·19″	− 44	179° 59′ 50″	+ 10″
CED .	60° 32′ 05″	+ 0·69″	56° 28′ 01″	$\bar{1}$·920,9407	13·93	+ 3·85″	+ 54	62° 59′ 49″	$\bar{1}$·949,8691	10·73	+ 0·47″	+ 5	179° 59′ 55″	+ 5″
DEA .	122° 00′ 55″	+ 0·79″	28° 42′ 00″	$\bar{1}$·681,4434	38·45	+ 7·32″	+ 281	29° 17′ 00″	$\bar{1}$·689,4232	37·57	− 3·12″	− 117	179° 59′ 55″	+ 5″
	360° 00′ 00″	0·00″		$\bar{1}$·269,2636			+ 607 + 212 + 819		$\bar{1}$·269,3455 $\bar{1}$·269,2636 + 819			− 212		

$\therefore \quad \Sigma(v^2) = 5995$

$$\therefore \mu = \frac{819 - ((-2{\cdot}92)(-0{\cdot}51) + 3{\cdot}75(-3{\cdot}52) + 2{\cdot}08(3{\cdot}20) + 0{\cdot}88(2{\cdot}08))}{5995}$$

$$= \frac{819 - (1{\cdot}49 - 13{\cdot}20 + 6{\cdot}66 + 1{\cdot}83)}{5995} = \frac{822{\cdot}22}{5995} = 0{\cdot}1372$$

Second Approximation.

$$E_1' = -10 - {\cdot}1372(-0{\cdot}51) = -9{\cdot}93$$
$$E_2' = +10 - {\cdot}1372(-3{\cdot}52) = +10{\cdot}48$$
$$E_3' = +5 - {\cdot}1372 \times 3{\cdot}20 = +4{\cdot}56$$
$$E_4' = +5 - {\cdot}1372 \times 0{\cdot}88 = +4{\cdot}88$$

$$\therefore \lambda_5 = \frac{0 - (-9{\cdot}93 + 10{\cdot}48 + 4{\cdot}56 + 4{\cdot}88)}{8} = \frac{-9{\cdot}99}{8}$$

$= -1{\cdot}25$ as before

$$\lambda_1 = \frac{-9{\cdot}93 + 1{\cdot}25}{3} = -2{\cdot}89; \quad \lambda_2 = \frac{10{\cdot}48 + 1{\cdot}25}{3} = 3{\cdot}91$$

$$\lambda_3 = \frac{4{\cdot}56 + 1{\cdot}25}{3} = 1{\cdot}94; \quad \lambda_4 = \frac{4{\cdot}88 + 1{\cdot}25}{3} = 2{\cdot}04$$

$$\therefore \mu = \frac{819 - ((-2{\cdot}89)(-0{\cdot}51) + 3{\cdot}91(-3{\cdot}52) + 1{\cdot}94(3{\cdot}20) + 2{\cdot}04(0{\cdot}88))}{5995}$$

$$= \frac{819 - (1{\cdot}47 - 13{\cdot}76 + 6{\cdot}21 + 1{\cdot}80)}{5995} = \frac{823{\cdot}28}{5995} = 0{\cdot}1373$$

as against 0·1372 before.

Therefore we can take

$\lambda_1 = -2{\cdot}89, \quad \lambda_2 = 3{\cdot}91, \quad \lambda_3 = 1{\cdot}94, \quad \lambda_4 = 2{\cdot}04$

$\lambda_5 = -1{\cdot}25, \quad \mu = 0{\cdot}1373.$

$e_9 = -2{\cdot}89 - 1{\cdot}25 = -4{\cdot}14''; \quad e_{10} = 3{\cdot}91 - 1{\cdot}25 = 2{\cdot}66''$

$e_{11} = 1{\cdot}94 - 1{\cdot}25 = 0{\cdot}69''; \quad e_{12} = 2{\cdot}04 - 1{\cdot}25 = 0{\cdot}79''$

which we enter in the "correction column" for the central angles, where they total 0·00″.

$e_1 = -2{\cdot}89 + {\cdot}1373 \times 11{\cdot}67 = -1{\cdot}29''; \; e_2 = -2{\cdot}89 - {\cdot}1373 \times 12{\cdot}18 = -4{\cdot}56''$

$e_3 = 3{\cdot}91 + {\cdot}1373 \times 33{\cdot}63 = +8{\cdot}53''; \; e_4 = 3{\cdot}91 - {\cdot}1373 \times 37{\cdot}15 = -1{\cdot}19''$

$e_5 = 1{\cdot}94 + {\cdot}1373 \times 13{\cdot}93 = +3{\cdot}85''; \; e_6 = 1{\cdot}94 - {\cdot}1373 \times 10{\cdot}73 = +0{\cdot}47''$

$e_7 = 2{\cdot}04 + {\cdot}1373 \times 38{\cdot}45 = +7{\cdot}32''; \; e_8 = 2{\cdot}04 - {\cdot}1373 \times 37{\cdot}57 = -3{\cdot}12''$

We note

$$e_1 + e_2 + e_9 = -9{\cdot}99'', \quad e_3 + e_4 + e_{10} = +10{\cdot}00''$$
$$e_5 + e_6 + e_{11} = +5{\cdot}01'', \quad e_7 + e_8 + e_{12} = +4{\cdot}99''$$

so that the three angles in each of the four triangles now total 180° 00′ 00″ to 0·01.

Entering these corrections (odd = left hand, even = right hand) in their respective columns, and multiplying by the differences per 1″, we obtain the corrections of log sine, which now total + 607 on L.H. angles, − 212 on R.H. angles, making a shift of 819 = ·0000819 to the left as required.

Now, proceeding with the calculation (Fig. 13)—

499·857	2·698,8458	*EB*	2·707,4261
sin θ_1 corrected	$\bar{1}$·941,8808	sin θ_4 corrected	$\bar{1}$·692,8030
cosec 59° 03′ 5·86″	0·066,6995		3·014,6231
EB	2·707,4261	sin θ_3 corrected	$\bar{1}$·725,0259
		EC	2·739,6490

∴ *EB* = 509·83 ft. ∴ *EC* = 549·10 ft.

sin { 118° 23′ 52·66″ / 61°36′ 07·34″	3·014,6231 as above
	$\bar{1}$·944,3176
BC	2·958,9407

∴ *BC* = 909·79 ft.

$$\tan\frac{EPB - PEB}{2} = \frac{509{\cdot}83 - 247{\cdot}61}{509{\cdot}83 + 247{\cdot}61}\cot\frac{PBE}{2}$$

$$\text{where } \frac{PBE}{2} = \frac{59^\circ\, 56'\, 1{\cdot}44''}{2} = 29^\circ\, 58'\, 0{\cdot}72''$$

262·22	2·418,6658
cot 29° 58′ 0·72″	0·239,1408
	2·657,8066
757·44	2·879,3482
	$\bar{1}$·778,4584

$$\therefore \frac{EPB - PEB}{2} = 30^\circ\, 58'\, 53{\cdot}9''$$

and $\dfrac{EPB + PEB}{2} = 90° - 29° \; 58' \; 0{\cdot}7'' = 60° \; 01' \; 59{\cdot}3''$

$\therefore EPB = BPQ = 91° \; 00' \; 53{\cdot}2''$; $PEB = 29° \; 03' \; 05{\cdot}4''$

$CEQ = 180° - (29° \; 03' \; 05{\cdot}4'' + 118° \; 23' \; 52{\cdot}7'')$

$= 180° - 147° \; 26' \; 58{\cdot}1'' = 32° \; 33' \; 01{\cdot}9''$

$\therefore CQP = 180° - (32° \; 33' \; 01{\cdot}9'' + 56° \; 28' \; 04{\cdot}8'')$

$= 180° - 89° \; 01' \; 06{\cdot}7'' = 90° \; 58' \; 53{\cdot}3''$

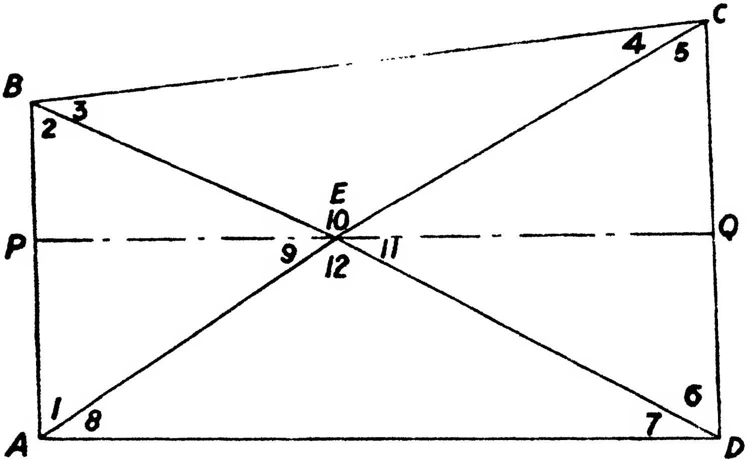


FIG. 13

EB	2·707,4261	*EC*	2·739,6490
sin θ_2 corrected	$\bar{1}$·937,2402	sin *CQP*	$\bar{1}$·999,9362
			2·739,7128
cosec *EPB*	0·000,0682	sin θ_4 corrected	$\bar{1}$·920,9461
PE	2·644,7345	*EQ*	2·660,6589

$\therefore$ *PE* = 441·30 ft. $\therefore$ *EQ* = 457·78 ft.

	2·739,7128 as above
sin *CEQ*	$\bar{1}$·730,8172
CQ	2·470,5300

$\therefore$ *CQ* = 295·48 ft.

(iii) *Two Intersecting Polygons with Central Points.* We will take only a simple case, viz. two Quadrilaterals which intersect, the central point of each being on a vertex of the other. Let the angles be numbered as in Fig. 14. We shall

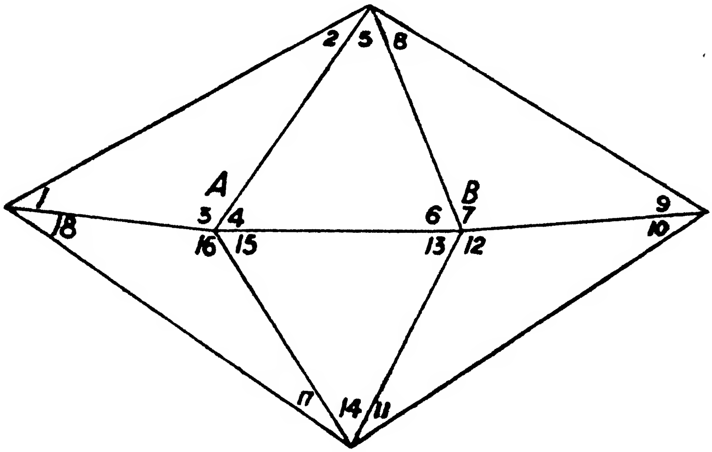


FIG. 14

distinguish the quadrilaterals by their central points *A* and *B*. The normal equations are deduced thus—

$\Sigma(e^2) =$ minimum	$\Sigma(e\delta e) = 0$	—
$e_1 + e_2 + e_3 = E_1$	$\delta e_1 + \delta e_2 + \delta e_3 = 0$	$-\lambda_1$
$e_4 + e_5 + e_6 = E_2$	$\delta e_4 + \delta e_5 + \delta e_6 = 0$	$-\lambda_2$
$e_7 + e_8 + e_9 = E_3$	$\delta e_7 + \delta e_8 + \delta e_9 = 0$	$-\lambda_3$
$e_{10} + e_{11} + e_{12} = E_4$	$\delta e_{10} + \delta e_{11} + \delta e_{12} = 0$	$-\lambda_4$
$e_{13} + e_{14} + e_{15} = E_5$	$\delta e_{13} + \delta e_{14} + \delta e_{15} = 0$	$-\lambda_5$
$e_{16} + e_{17} + e_{18} = E_6$	$\delta e_{16} + \delta e_{17} + \delta e_{18} = 0$	$-\lambda_6$
$e_3 + e_4 + e_{15} + e_{16} = E_7$	$\delta e_3 + \delta e_4 + \delta e_{15} + \delta e_{16} = 0$	$-\lambda_7$
$e_6 + e_7 + e_{12} + e_{13} = E_8$	$\delta e_6 + \delta e_7 + \delta e_{12} + \delta e_{13} = 0$	$-\lambda_8$
$\Sigma_A(e_L v - e_R v) = E_9$	$\delta e_1 v_1 - \delta e_2 v_2 + \delta e_5 v_5 - \delta e_6 v_6 + \delta e_{13} v_{13} - \delta e_{14} v_{14} + \delta e_{17} v_{17} - \delta e_{18} v_{18} = 0$	$-\mu_1$
$\Sigma_B(e_L v - e_R v) = E_{10}$	$\delta e_4 v_4 - \delta e_5 v_5 + \delta e_8 v_8 - \delta e_9 v_9 + \delta e_{10} v_{10} - \delta e_{11} v_{11} + \delta e_{14} v_{14} - \delta e_{15} v_{15} = 0$	$-\mu_2$

$$\begin{aligned}
\therefore\ e_1 &= \lambda_1 + \mu_1 v_1; & e_2 &= \lambda_1 - \mu_1 v_2; \\
e_3 &= \lambda_1 + \lambda_7; & e_4 &= \lambda_2 + \lambda_7 + \mu_2 v_4; \\
e_5 &= \lambda_2 + \mu_1 v_5 - \mu_2 v_5; & e_6 &= \lambda_2 + \lambda_8 - \mu_1 v_6; \\
e_7 &= \lambda_3 + \lambda_8; & e_8 &= \lambda_3 + \mu_2 v_8; \\
e_9 &= \lambda_3 - \mu_2 v_9; & e_{10} &= \lambda_4 + \mu_2 v_{10}; \\
e_{11} &= \lambda_4 - \mu_2 v_{11}; & e_{12} &= \lambda_4 + \lambda_8; \\
e_{13} &= \lambda_5 + \lambda_8 + \mu_1 v_{13}; & e_{14} &= \lambda_5 - \mu_1 v_{14} + \mu_2 v_{14}; \\
e_{15} &= \lambda_5 + \lambda_7 - \mu_2 v_{15}; & e_{16} &= \lambda_6 + \lambda_7; \\
e_{17} &= \lambda_6 + \mu_1 v_{17}; & e_{18} &= \lambda_6 - \mu_1 v_{18}
\end{aligned}$$

Therefore, the normal equations are—

$$\begin{aligned}
&\text{(i)}\ 3\lambda_1 + \lambda_7 + \mu_1(v_1 - v_2) = E_1 \\
&\text{(ii)}\ 3\lambda_2 + \lambda_7 + \lambda_8 + \mu_1(v_5 - v_6) + \mu_2(v_4 - v_5) = E_2 \\
&\text{(iii)}\ 3\lambda_3 + \lambda_8 + \mu_2(v_8 - v_9) = E_3 \\
&\text{(iv)}\ 3\lambda_4 + \lambda_8 + \mu_2(v_{10} - v_{11}) = E_4 \\
&\text{(v)}\ 3\lambda_5 + \lambda_7 + \lambda_8 + \mu_1(v_{13} - v_{14}) + \mu_2(v_{14} - v_{15}) = E_5 \\
&\text{(vi)}\ 3\lambda_6 + \lambda_7 + \mu_1(v_{17} - v_{18}) = E_6 \\
&\text{(vii)}\ \lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 + 4\lambda_7 + \mu_2(v_4 - v_{15}) = E_7 \\
&\text{(viii)}\ \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + 4\lambda_8 + \mu_1(v_{13} - v_6) = E_8 \\
&\text{(ix)}\ \Sigma_{I}\lambda(v_L - v_R) + \mu_1 \Sigma_{I}(v^2) + \lambda_8(v_{13} - v_6)) - \mu_2(v_5^2 + v_{14}^2) \\
&\qquad = E_9 \\
&\text{(x)}\ \Sigma_{II}\lambda(v_L - v_R) + \mu_2 \Sigma_{II}(v^2) + \lambda_7(v_4 - v_{15}) - \mu_1(v_5^2 + v_{14}^2) \\
&\qquad = E_{10}
\end{aligned}$$

For solution by successive approximations, ignore μ terms, add equations (i), (ii), (v), and (vi), and subtract $3 \times$ (vii), thus—

$$3\lambda_1 + 3\lambda_2 + 3\lambda_5 + 3\lambda_6 + 4\lambda_7 + 2\lambda_8 = E_1 + E_2 + E_5 + E_6$$

$$3\lambda_1 + 3\lambda_2 + 3\lambda_5 + 3\lambda_6 + 12\lambda_7 \qquad = 3E_7$$

$$\therefore\quad 8\lambda_7 - 2\lambda_8 = 3E_7 - (E_1 + E_2 + E_5 + E_6) \quad . \quad . \quad (a)$$

Similarly, add (ii), (iii), (iv), and (v), and subtract 3 × (viii)—

$$3\lambda_2 + 3\lambda_3 + 3\lambda_4 + 3\lambda_5 + 2\lambda_7 + 4\lambda_8 = E_2 + E_3 + E_4 + E_5$$
$$3\lambda_2 + 3\lambda_3 + 3\lambda_4 + 3\lambda_5 + 12\lambda_8 \qquad = 3E_8$$
$$\therefore \quad -2\lambda_7 + 8\lambda_8 = 3E_8 - (E_2 + E_3 + E_4 + E_5) \quad . \quad . \quad (b)$$

These simultaneous equations (*a*) and (*b*) give us λ_7 and λ_8, then

$$\lambda_1 = \frac{E_1 - \lambda_7}{3}, \quad \lambda_2 = \frac{E_2 - \lambda_7 - \lambda_8}{3}, \quad \lambda_3 = \frac{E_3 - \lambda_8}{3}$$
$$\lambda_4 = \frac{E_4 - \lambda_8}{3}, \quad \lambda_5 = \frac{E_5 - \lambda_7 - \lambda_8}{3}, \quad \lambda_6 = \frac{E_6 - \lambda_7}{3}$$

When these values are inserted in (ix) and (x), we have a simultaneous equation for μ_1 and μ_2: then proceed as in Case (ii).*

Computation of Sides. This should be done in a methodical manner, for which the following table is suitable, the data being taken from Example 6.

Triangle	Angle	Angle	Log sine	Difference	Log side	Side	Side
EAB	*EBA*	59° 56′ 01·6″	$\bar{1}$·937,2402	0·003,9397	2·702,7855	504·41	*EA*
	AEB	59° 03′ 05·9″	$\bar{1}$·933,3005	0·008,5803	2·698,8458	499·857	*AB*
	EAB	61° 00′ 52·7″	$\bar{1}$·941,8808		2·707,4261	509·83	*EB*
EBC	*BEC*	118° 23′ 52·7″	$\bar{1}$·944,3178	0·251,5146	2·958,9407	909·79	*BC*
	ECB	29° 32′ 04·8″	$\bar{1}$·692,8030	0·032,2229	2·707,4261	509·83	*EB*
	EBC	32° 04′ 02·5″	$\bar{1}$·725,0259		2·739,6490	549·10	*EC*

In this table, the known side is written on the middle line, and the side which is to pass forward into the next triangle is written on the last line. The table is continued up to

* The student should apply this method to *two* intersecting hexagons, with two triangles common; then to *three* intersecting hexagons, one triangle common.

triangle EDA, in which the side EA should check with its value above.

Co-ordinates. By extending the above table with columns for the bearing of the side from some selected meridian, the log cosine and the log sine of this bearing, its latitude and departure, and the total co-ordinates of the stations from some convenient origin, these last can be computed in a systematic manner. Plotting should always be effected from calculated co-ordinates of the stations, (*a*) as a valuable check on the plotting is the measurement of the length of the side when thus plotted, and (*b*) as the survey can be readily plotted on a number of sheets by draughtsmen plotting independently.

Precise Levelling

In good ordinary engineering levelling the error in M miles should not exceed from 0·05 $\sqrt{M}$ to 0·10 $\sqrt{M}$ ft., but in the best precise work the allowable discrepancy between two determinations of the level difference between two B.Ms., M miles apart should not exceed ·012 $\sqrt{M}$ ft.* Precise Levelling is required for the establishment of Bench Marks for general purposes at convenient distances apart all over a country, and for long lines of levels in Water Engineering, where gradients of $\frac{1}{4000}$ or less may be used. Special precautions are, of course, necessary. In this country the Zeiss reversible level, with parallel plate micrometer to read the staff to 0·0001 ft., as described in Chapter III, is employed by the Ordnance Survey; in the United States the levelling on the Coast and Geodetic Survey is executed by a tilting level of Dumpy type. The spirit level is sunk into the telescope so as to be close to the line of collination, and

* *Plane and Geodetic Surveying*, David Clark (Constable & Co.).

"Invar" is used for the telescope to eliminate the effects of temperature. The telescope is long and powerful and the level tube is extremely sensitive. The difference in type of instrument is due to difference in practice, the Americans taking much longer sights where there are long stretches of level ground. As in the Zeiss instrument, the backsight and foresight distances are read by stadia hairs, and the instrument is adjusted by taking the apparent level difference of two pegs from both ends.

Bench Marks. It is, of course, of the highest importance that these should be as invariable as possible. The new primary Ordnance Bench Marks, about 25 miles apart, are fixed in small underground chambers founded from 3 ft. to 13 ft. below ground in solid rock, a secondary surface mark being provided on a granite pillar about 1 ft. above ground.* Secondary Bench Marks, about 1 mile apart, are formed on gun-metal plates securely fixed in the walls of buildings with good foundations. Bench marks in mining districts and on alluvial soil are, of course, liable to vary in level.

Sources of Error. (i) One source of error is *vertical movement* of the level between the reading of the backsight and that of the foresight. If the instrument sinks, this will cause the foresight reading to be less than it should be, and will cause an apparently increased rise or decreased fall. This error can be overcome by having *two* staves and reading the backsight on one and the foresight on the other as quickly as possible; the next time the level is set up the foresight is read first and then the backsight, and so on alternately. The corresponding error due to the vertical movement of the *staff* while the instrument is being moved to a fresh position cannot, however, be entirely eliminated. If the

* "Precise Levelling," Major E. O. Henrici. *Proc. Inst.C.E.*, Vol. 209.

staff sinks, the effect is to increase the backsight reading and, therefore, also to cause an increased rise or decreased fall. The best that can be done is to drive a steel pin with rounded head, from 4 in. to 12 in. long, into the ground on which to hold the staff, and, of course, to choose as good ground as possible for turning points within the limiting distance. Such pins in good ground tend to *rise* with time. A systematic error from this cause can only be avoided by relevelling the line by the *same route* in the *opposite direction*, and taking the mean of the results.

(ii) *Instrumental errors* would be eliminated if the instrument was always exactly midway between the two staves, and the bubble brought always to the same position. The stadia hairs enable the distances to be equalized in most cases or, where inequality cannot be avoided, a correction to be applied for the difference of distance, due to the small residual error in parallelism of the bubble axis and line of collimation which remains after the instrument has been adjusted as carefully as possible. Needless to say, the adjustment of the instrument is frequently checked and its residual error per foot of distance recorded. *Reading errors* are eliminated also by reading all three hairs on the staff and comparing the readings. The staff, of course, must be carefully held vertical, and must be calibrated at every 0·1 ft of its 10 ft. length (all in one piece), so that a correction can be applied for non-uniformity of graduation.

(iii) *Atmospheric Refraction.* The midway position of the level between the two staves would eliminate this error if the line of sight followed a similar curve for both backsights and foresights—actually the curvature is greatest when nearest the ground. On level ground, and with steady atmospheric conditions, the curvature is similar on both sides. The Ordnance Survey minimize this error by restricting the length of sights to 150 ft. in primary work and by avoiding lines of sight close to the ground. The most dangerous condition is when the ground is colder than the air, for a layer of cold dense air then covers the ground. The instrument should be shaded by an umbrella from the sun and wind, and work should only be carried out during the middle of the day--say 10.0 a.m. to 4.0 p.m., when refraction is steadiest.

Adjustment of Observations. If there were no cumulative error, such as movement of the staff, the probable errors would be wholly compensating and would be proportional to the $\sqrt{\text{number of settings of the level}}$. As the distance between staves at each setting of the level is kept as constant as possible (say 300 ft.), the probable errors would be proportional to the $\sqrt{\text{distance}}$. If E is the total discrepancy on a closed circuit of levels, e_1, e_2, e_3 . . . are the errors on the level differences between the various intermediate B.M.'s, and l_1, l_2, l_3 . . . their distances apart on the route levelled, so that $E = e_1 + e_2 + e_3 + e_4 + \ldots$, then, by the Method of Least Squares, we have

$$\Sigma\left(\frac{e^2}{l}\right) = \text{minimum}\,;\ \Sigma\left(\frac{e\delta e}{l}\right) = 0,$$

$$\delta e_1 + \delta e_2 + \delta e_2 + \delta e_3 + \ldots\ldots = 0$$

Multiplying the latter by $-\lambda$, we have

$$\frac{e_1}{l_1} = \frac{e_2}{l_2} = \frac{e_3}{l_3} = \text{etc.} = \lambda$$

$$\therefore E = \lambda(l_1 + l_2 + l_3 + \ldots\ldots)$$

$$\therefore \lambda = \frac{E}{\Sigma l}$$

$$\therefore e_1 = \frac{l_1}{\Sigma l}E,\ e_2 = \frac{l_2}{\Sigma l}E,\ e_3 = \frac{l_3}{\Sigma l}E,\ \text{etc.}$$

We should, therefore, divide the total error found in proportion to the lengths between B.M.'s. In practice, the cumulative error cannot be avoided, but the mean of two levellings taken in opposite directions by the same route should to a large extent be free from this error.* If, therefore, the closed circuit has been levelled in both directions, and the means of the results between each pair of bench marks are taken, these can be corrected for accidental error by the above method.

EXAMPLE 7. In levelling a round of precise levels for four bench marks the following results were obtained: level differences *A* to *B*, + 176·422 ft.; *B* to *C*, − 425·365 ft.; *C* to *D*, + 88·757 ft.; *D* to *A*, + 160·727 ft. In check-levelling in the reverse direction the level differences were: *A* to *D*, − 160·409 ft.: *D* to *C*, − 88·615 ft.; *C* to *B*, + 425·601 ft.; *B* to *A*, − 176·262 ft. The distances *A* − *B* − *C* − *D* were 14·2, 20·4, 10·1, 25·6 miles. Find the probable value of the levels of the bench marks, that of *A* being 325·722 ft.

Bench Mark	Level Difference (1)	Level Difference (2)	Average	Miles	Correction	Level Difference	Reduced Level
A .							325·722
	+ 176·422	− 176·262	+ 176·342	14·2	− 0·023	+ 176·319	
B .							502·041
	− 425·365	+ 425·601	− 425·483	20·4	− 0·033	− 425·516	
C .							76·525
	+ 88·757	− 88·615	+ 88·686	10·1	0·016	+ 88·670	
D .							165·195
	+ 160·727	− 160·409	+ 160·568	25·6	− 0·041	+ 160·527	
A .							325·722
			+ 425·596 − 425·483	Σl = 70·3	0·113	+ 425·516 − 425·516	
	Total Correction . .		− 0·113				

In practice, circuits, each twice levelled in opposite directions, will often intersect one another, and the Method of

* "Precise Levelling," by Major E. O. Henrici, *Proc. Inst. C.E.* Vol. 209 (1919).

Least Squares should be employed, weighting each mean level difference by $\dfrac{1}{\text{distance}}$.

EXAMPLE 8. The same data as Example 7, but add: "A fifth levelling from B to D gives a fall of 336·540 ft., and when checked from D to B gives a rise of 336·760 ft., the length of the route followed being 30·2 miles." Find the probable levels of B, C, and D.

The average value of level difference $BD = -336{\cdot}650$ ft. Fig. 15 shows the data, arranged to form two closed circuits.

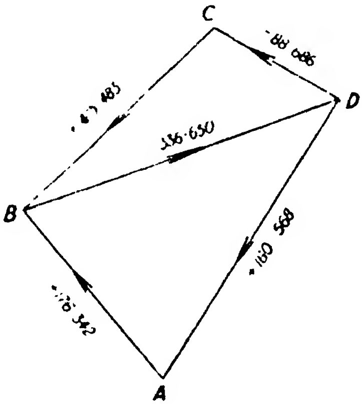


FIG. 15

Calling e_1, e_2, e_3, e_4, e_5, the corrections in AB, CB, DC, DA, BD respectively, we have

$$\frac{e_1^2}{l_1} + \frac{e_2^2}{l_2} + \frac{e_3^2}{l_3} + \frac{e_4^2}{l_4} + \frac{e_5^2}{l_5} = \text{minimum}$$

$$e_1 + e_4 + e_5 = -0{\cdot}260$$

$$e_2 + e_3 + e_5 = -0{\cdot}147$$

$$\therefore \quad \frac{e_1\delta e_1}{l_1} + \frac{e_2\delta e_2}{l_2} + \frac{e_3\delta e_3}{l_3} + \frac{e_4\delta e_4}{l_4} + \frac{e_5\delta e_5}{l_5} = 0 \qquad \Big| \quad - \quad \Big|$$

$$\delta e_1 + \delta e_4 + \delta e_5 = 0 \qquad \Big| \quad -\lambda_1 \quad \Big|$$

$$\delta e_2 + \delta e_3 + \delta e_5 = 0 \qquad \Big| \quad -\lambda_2 \quad \Big|$$

$$\therefore\ e_1 = \lambda_1 l_1\,;\ e_2 = \lambda_2 l_2\,;\ e_3 = \lambda_2 l_3\,;\ e_4 = \lambda_1 l_4\,;\ e_5 = (\lambda_1 + \lambda_2) l_5$$

$$\therefore\ \lambda_1(l_1 + l_4 + l_5) + \lambda_2 l_5 = -0{\cdot}260$$

$$\lambda_1 l_5 + \lambda_2(l_2 + l_3 + l_5) = -0{\cdot}147$$

or $70{\cdot}0\lambda_1 + 30{\cdot}2\lambda_2 = -0{\cdot}260$

$$30{\cdot}2\lambda_1 + 60{\cdot}7\lambda_2 = -0{\cdot}147$$

whence $\lambda_1 = -0{\cdot}00340$, $\lambda_2 = -0{\cdot}00073$, $\lambda_1 + \lambda_2 = -0{\cdot}00413$

$$\therefore\ e_1 = -{\cdot}0034 \times 14{\cdot}2 = -{\cdot}048\,;\quad AB = +176{\cdot}342 - {\cdot}048 = +176{\cdot}294$$

$$e_2 = -{\cdot}00073 \times 20{\cdot}4 = -{\cdot}015\,;\quad CB = +425{\cdot}483 - {\cdot}015 = +425{\cdot}468$$

$$e_3 = -{\cdot}00073 \times 10{\cdot}1 = -{\cdot}007\,;\quad DC = -\ 88{\cdot}686 - {\cdot}007 = -\ 88{\cdot}693$$

$$e_4 = -{\cdot}0034 \times 25{\cdot}6 = -{\cdot}087\,;\quad DA = +160{\cdot}568 - {\cdot}087 = +160{\cdot}481$$

$$e_5 = -{\cdot}00413 \times 30{\cdot}2 = -{\cdot}125\,;\quad BD = -\ 336{\cdot}650 - {\cdot}125 = -\ 336{\cdot}775$$

$\therefore\ A = 325{\cdot}722$ ft.; $B = 502{\cdot}016$ ft.; $C = 76{\cdot}548$ ft.; $D = 165{\cdot}241$ ft.

The assumptions made are: (*a*) that systematic error is eliminated by the check levelling, and (*b*) that the lengths of all sights and other conditions are approximately the same in each levelling.

CHAPTER VII

OTHER METHODS OF PLANE SURVEYING

PLANE TABLING—COMPASS SURVEYING—SEXTANT—RESECTION—PHOTO-SURVEYING—SUBTENSE MEASUREMENTS—BAROMETRIC LEVELLING—ADJUSTMENT OF TRAVERSES

PLANE TABLE SURVEYING

THE Plane Table in its simplest form consists of a drawing-board, mounted on a tripod stand, on which it can be rotated horizontally and clamped in any position, together with an "alidade" or straight-edge fitted with folding sights, so that a line can be drawn on the board in the direction of any visible object. Many convenient elaborations can be made, e.g. a ball and socket joint, or foot-screws, with a circular spirit level sunk in the surface of the table for more accurate levelling of the table, a clamp and tangent screw for preventing rotation and for "orienting" the table more accurately (i.e. for setting any line drawn on the paper mounted on the table in any given direction), and a trough compass for marking magnetic north on the paper or for setting a magnetic north point on the paper to magnetic north. The alidade may be fitted with a telescope, rotating on a horizontal axis carried by a pillar on the straight-edge, with cross hairs, which greatly improves the sighting and enables more steeply inclined sights to be taken, and when such a telescope is provided with stadia hairs, a vertical graduated circle with clamp and tangent screw and a spirit level on its vernier arm, we have a "Tacheometric Plane Table," which is nearly as expensive as an ordinary theodolite.

In plane table surveying, horizontal angles are *drawn* directly on the paper fixed to the board. When at least two known stations, *A* and *B* (Fig. 1), have been plotted on the paper to scale as *a* and *b*, the position of any other points *C*, *D*, and *E*, visible from both stations, can be found by

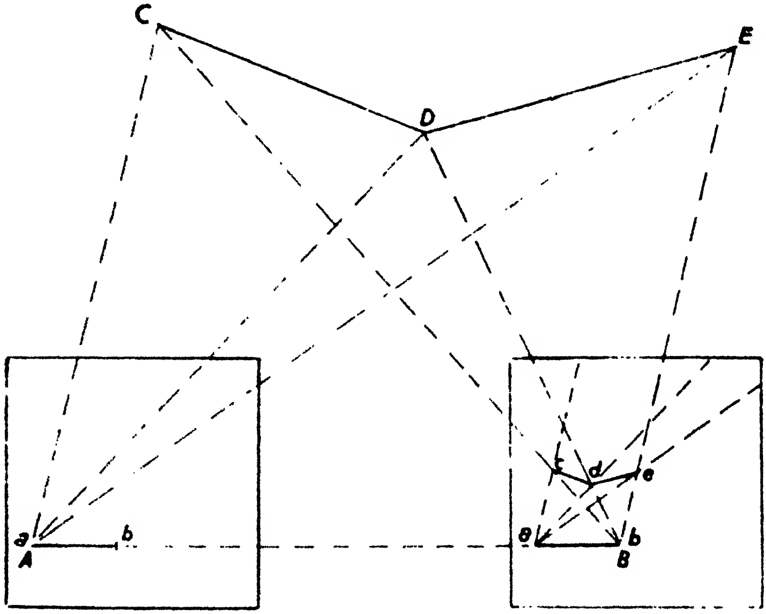


FIG. 1

first setting up the table at *A*, so that *a* is vertically over *A* and the table oriented so that *ab* points to *B*, then setting the alidade in the line *aC* and drawing a "ray" *ac* along the straight-edge, and similarly rays *ad*, *ae* to *D* and *E*. Then the table is moved to *B* and set up so that *b* is vertically over *B*, the alidade is placed along *ba* and the table turned until it points to *A*, the ray *bc* is then drawn through *b* towards *C*. The intersection *c* of the two rays *ac*, *bc* is the position of *C* on the plan, and similarly points *d* and *e* are found.

In any given position, therefore, the "centre" of the instrument is the point on the paper which represents (and is vertically above) the station at which the table is set up,

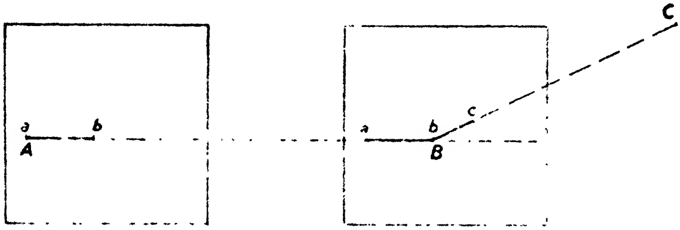


FIG. 2

and the table must always be correctly oriented so that all lines drawn on it are pointing in their correct directions. The position of any point may also be fixed by drawing a ray in its direction and scaling off the distance of the point from the station at which the instrument is set up: this, of course, is only suitable for short distances which can be taped, unless a tacheometric alidade is available. Traversing can also be effected with the plane table by chaining the distance *AB* (Fig. 2), drawing the ray *ab* in the direction *AB*, and scaling *ab* on the paper. Then the table is moved to *B*, set up so that *b* is vertically over *B*, laying the alidade along *ba* and turning the table so that it points to *A*. The ray *bc* can then be drawn to the next station *C*, and when the distance *BC* has been measured it can be scaled off as *bc*. This process is called "orienting by the back ray," and is obviously inferior in accuracy to the method of intersection. Detail adjacent to the traverse lines may be fixed by rays and taped distances from one station, by intersecting rays from two adjacent stations, or by drawing a ray to the point and taping the distance to it from a point already plotted.

The plane table is the instrument *par excellence* for surveying *detail* in topographical or small-scale surveying, as all important details can be fixed by intersections or by a ray and a measured distance, and less important detail can be *sketched* in between such points, thus saving much time; in addition, the survey is plotted on the ground without calculations and its use is readily taught to comparatively

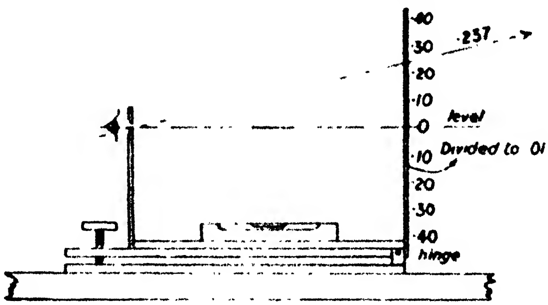


Fig. 3

unskilled assistants. There must, however, be a preliminary triangulation to establish a sufficient number of accurate stations to "control" the plane table survey and prevent errors from accumulating.

Indian Clinometer. This is a simple accessory to the plane table which enables the levels of points, whose position has been found on the plane table, to be determined by simple calculation. It is made of metal (Fig. 3) and consists of a base which rests on the plane table. Hinged to this at one end is an upper base plate which carries vanes at each end, one containing a pin-hole and the other a scale of tangents with its zero at the same level as the pin-hole, when the upper base plate is correctly levelled by means of the screw and spirit level shown. If the distance apart of the vanes is 8 inches, the larger graduations on the vane will be 0·8 inch, and the smaller 0·08 inch, so that the tangents of elevations and depressions can be read to 0·01, and by estimation to 0·001. To find the level of a point plotted on the plane table the observer sights towards the point, levels, the upper base, and estimates the reading on the tangent scale which intersects the point. Then (the scale reading) × (the distance scaled from the survey) gives the difference of level of the table and the point. Conversely, if the level of the plane table is unknown, the observer sights with the clinometer on to a station whose level is known and which is shown on the survey, and thus can deduce the level of the table, which can then be used for the determination of the levels of other points.

Compass Surveying

A compass with sights fixed to its case at opposite ends of a diameter is a useful instrument in rapid or topographical survey, the position of points being fixed by their bearings from (or to) two known stations, or the compass may be used to find the bearings of the lines of a traverse. The smaller sizes of compass, about 3 inches diameter, are usually held in the hand and the reading is taken by a right-angled triangular prism just below the slit, which acts as the nearer sight to the eye; the further sight being a fine vertical wire fixed in a frame. The needle carries a card or light metal ring on which are the graduations, running clockwise from 0° to 360° and commencing at the *south* end of the needle (Fig. 4*a*).

Larger compasses, say 6 inches diameter, are mounted on tripod stands, with a ball and socket joint and clamp, by means of which the compass can be set level by two spirit levels at 90° on the case, and the case, with its sights, can be rotated and fixed in the required direction. The graduated circle is fixed to the case and the graduations run *anti*-clockwise, with 0° at the *farther* sight to the eye (Fig. 4*b*). The north end of the needle is marked in some way—say

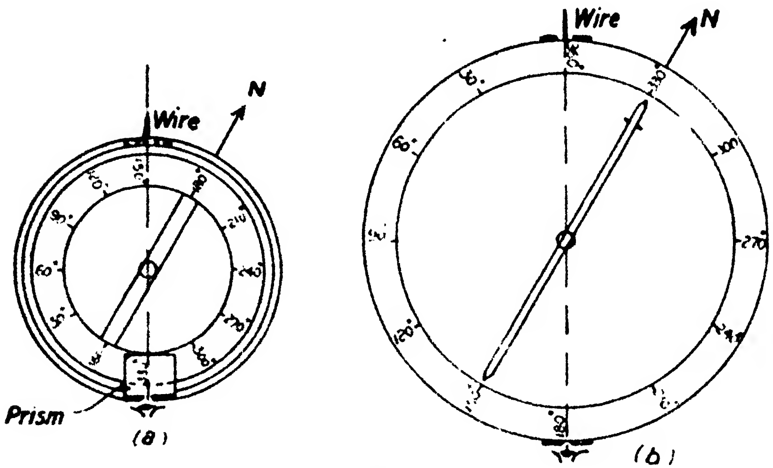


Fig. 4

by a short gold bar and its reading on the graduated circle is the magnetic bearing of the line—the bar needle being sharply pointed at its ends for this purpose.

The needle carries a small adjustable counter-weight to correct its balance for variation in dip in different parts of the world, and has a jewel fixed to its lower side at the centre, which rests on a hard steel point on the bottom of the case. Great care must be taken to avoid blunting of this point by lifting the needle off its bearing by a lever provided, when not in use. The compass is not, of course, a precise instrument, as the direction of Magnetic North does not make a constant angle with True or Geographical North viz.—

(*a*) The magnetic "declination," or deviation of magnetic from true north, varies from place to place on the earth's surface, e.g. in the north-west of Ireland it is more than 6° more westerly than at Dover, i.e. in 515 miles in a W.N.W. direction, an increase of 1′ in less than $1\frac{1}{2}$ miles.

(*b*) The declination at any place varies continuously with the time, e.g. at London it is decreasing (westerly) about 10′ or 11′ per annum.

(*c*) The declination has a diurnal oscillation, of a range at Greenwich of about 7′ in winter and 12′ in summer, about its mean position for the 24 hours.

(*d*) Occasional magnetic storms cause deviations up to 1° or more from the mean values.*

In addition, "*local magnetic attraction*" may cause considerable displacement of the needle. The most common source of this is magnetic material, e.g. steel rails or iron fences near the instrument, but if due to magnetic rock below ground, it may extend over a considerable area. The observer must, of course, see that he has no magnetic material about his person or of a *movable* nature near the instrument, as the movement of this alters its effect on the needle. If the source of attraction is fixed, its effect can usually be eliminated by reading *both* the forward and back bearings of the lines, the readings of which should differ by 180°. If not, the divergence from 180° is due to local attraction at one or both ends of the line, causing, say, the north end of the needle to deflect $\delta°$ to the west and, in this case, causing *all* bearings taken from this station to be increased by the same angle $\delta°$.

* See *Whitaker's Almanack*, p. 156.

Example 1 (L.U.). The following are the forward and back bearings observed in a closed compass traverse *ABCD* –

Line	Bearing	Line	Bearing	Line	Bearing	Line	Bearing
AB	~~74° 20′~~ 75° 0′	*BC*	~~107° 20′~~ 106° 20′	*CD*	~~225° 0′~~ 224° 50′	*DA*	306° 40′
BA	~~256° 0′~~ 255° 0′	*CB*	286° 20′	*DC*	~~44° 40′~~ 44° 50′	*AD*	~~126° 0′~~ 126° 40′

Find the corrected forward bearings.

The discrepancies of forward and back bearings on the four lines are 1° 40′, 1° 0′, 0° 20′, 0° 40′ respectively. The least of these is 0° 20′ on *CD*, and may be due to sluggishness of the needle due to blunting of its bearing. As the compass only appears to be read to 20′, we are justified in correcting *CD* to 224° 50′ and *DC* to 44° 50′ to agree, and we can assume that stations *C* and *D* are both free from attraction, so that the bearings of *CB* and of *DA* are correct. We, therefore, alter *BC* to 106° 20′, i.e. apply a correction of – 1° 0′ at *B*, and, therefore, alter *BA* to 255° 0′, i.e. by the same amount. *AB* is then altered to 75° 0′ to correspond, i.e. we apply a correction of + 0° 40′ at *A*. Consequently, we correct *AD* by the same amount and make *AD* 126° 40′. As this agrees with *DA* 306° 40′, we should feel pretty sure of our corrections. The corrected forward bearings of the four lines are, therefore, 75° 0′, 106° 20′, 224° 50′, 306° 40′, and the local attraction at *A* and *B* deflected the needle 0° 40′ east at *A* and 1° 0′ west at *B*.

The Sextant

This is a precision instrument which is comparable in accuracy with the theodolite. It consists (Fig. 5) of a graduated arc or limb *LL*, which, in the full-size marine sextant, is from 6 inches to 8 inches radius, extends over rather more than a sixth of a circle, and is connected by framework to the centre of the circle. On an axis at this centre is pivoted an arm with a clamp and tangent screw (not shown) for fastening it to the arc, and a vernier *V* and

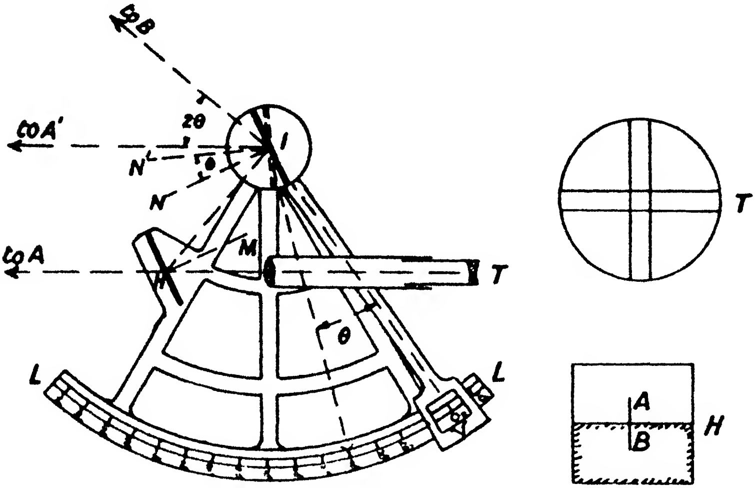


FIG. 5

microscope for reading the graduations. This arm carries a mirror—or "index glass" *I*—with its face perpendicular to the plane of the arc, and the axis about which the arm turns lies in the plane of the mirror. To the framework and also perpendicular to the plane of the limb is fixed a second mirror—the "horizon glass" *H*—so placed that when the vernier on the arm reads 0° 0′ the faces of the two mirrors are parallel. Only the lower half of this mirror is silvered, so that the observer, looking through a small telescope *T*, fixed to the framework, with two pairs of cross-hairs at 90° to each other, sees directly through the upper half of the horizon glass and, by double reflection at both mirrors, through the lower half. When the mirrors are parallel the reflected ray of light $THIA'$ makes, with the normal to each mirror, an angle of reflection = its angle of incidence.

$$\therefore \quad \angle IHT = 2\angle IHM \text{ and } \angle A'IH = 2\angle HIN$$

But as the mirrors are parallel, their normals HM, IN are also parallel.

$$\therefore \quad \angle IHM = \angle HIN$$

$$\therefore \quad \angle IHT = \angle A'IH \text{ and } \therefore \; IA' \text{ is parallel to } HT$$

Consequently, *if the point* A, seen directly through the plain part of the horizon glass, *is at a great distance*, so that the angle IAT is too small to measure, A is also seen by double reflection in coincidence with A, as seen directly. If now the vernier arm, and with it the index glass, is turned through an angle θ, the reflected ray from T follows the same path until it reaches the index mirror, but as the normal IN has been turned through the angle θ to IN' the angle of incidence on the index glass is increased by θ; the angle of reflection on the index glass is, therefore, also increased by θ. Therefore the ray, after reflection from the index glass, makes an angle 2θ with its original direction and now follows the line IB. If, therefore, A is at a great distance, the angle turned through by the vernier arm is one half of the angle AIB, when A and the image of B coincide on the horizon glass.

The graduations on the limb are made in ½ degrees about the centre I, but are numbered as degrees, so that the reading on the arm is automatically doubled and gives the angle AIB directly in a single reading, whereas a theodolite would need two readings. If the point A seen directly through the horizon glass is not so far away that the angle IAT is inappreciable, to see A by reflection as well as directly, the vernier has to be moved behind the 0° 0′ point and reads a small *negative* angle on the arc, which is extended a few degrees for this purpose, which angle must be added as a correction. That is, we must first sight on A direct and by reflection, and read the angle $IAT = A'IA$, then bring B by reflection into coincidence with A seen directly, and read the angle $A'IB$, then add the two together to find the angle AIB. The arc is usually graduated from −5° to 120° or 160°, and read by the vernier to 10″ on a marine sextant. In the smaller "box" sextants, completely enclosed in a metal case of some 3 inches diameter, the vernier reads to 1′. Both are usually held in the hand, but are occasionally mounted on a tripod. Dark glasses are provided to place before the mirrors when observing the sun.

In observations of horizontal and vertical angles taken from a floating vessel, the sextant takes the place of a theodolite. As regards horizontal angles, an angle taken with a theodolite is the difference of the readings on the two points, and the slightest movement of the instrument, even when the vessel is supposed to be at rest, will spoil the accuracy of the observation; with the sextant the observation of both the distant points is made instantaneously, so that even if the vessel is moving rapidly the angle is read accurately. As regards vertical angles to celestial bodies, the slightest motion of the vessel would upset the levels on a theodolite, while with a sextant at sea the instrument is held with its plane vertical, and the object "brought down" by reflection until it appears to touch the visible horizon as viewed directly through the horizon glass: this also can be done instantaneously and accurately, but, of course, a correction for the dip of the horizon,* depending on the height of the observer above the sea, must be subtracted from the measured altitude.

On land, the sextant is not at all so suitable as the theodolite, except in very level country, as it gives the inclined angle θ between the objects, not the horizontal angle ϕ

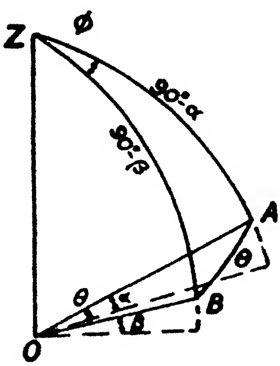


FIG. 6

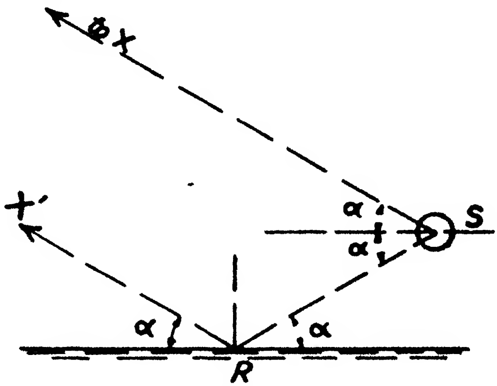


FIG. 7

which is required. It is necessary to measure also, say with a clinometer, the inclinations α and β of the legs of the angle and calculate ϕ by Spherical Trigonometry, thus

$$\text{(Fig. 6)}: \cos\phi = \frac{\cos\theta - \cos(90^\circ - \alpha) \,.\, \cos(90^\circ - \beta)}{\sin(90^\circ - \alpha) \,.\, \sin(90^\circ - \beta)}$$

$$= \frac{\cos\theta - \sin\alpha \,.\, \sin\beta}{\cos\alpha \,.\, \cos\beta}$$

where if α or β is a depression it must be treated as negative, which means that $\sin\alpha$ or $\sin\beta$ will be negative.

When the sextant is used on land for measuring vertical angles to celestial bodies, an "artificial horizon" is required (Fig. 7). This usually consists of the surface of a layer of mercury contained in a tray and protected by a sloping

* *Chambers' Mathematical Tables*, "Apparent Depression of the Horizon."

glass roof. The reflection of the object in the mercury is viewed directly in the horizon glass, and the object itself is brought down by the sextant into apparent coincidence with it. The angle between the object and its reflection is therefore measured, and the figure shows that this is *twice* the altitude of the object. As the object is infinitely distant, the rays XS, $X'R$ are, of course, parallel.

Adjustments of the Sextant

(*a*) *To Make the Index Glass Perpendicular to the Plane of the Arc.* Clamp the vernier arm near the middle of the arc and place the eye near the index glass. If the reflection of the arc in the mirror appears continuous with the arc as seen direct, the adjustment is correct; if not, adjust the index glass by the screw at its back.

(*b*) *To Make the Horizon Glass Perpendicular to the Plane of the Arc.* Sight on a star and move the vernier arm through zero so that the reflected image passes the star as seen direct. If the former does not pass *through* the latter, adjust the horizon glass by the screw behind it until it does so.

(*c*) *To Make the Line of Sight of the Telescope Parallel to the Plane of the Arc.* Make two of the cross-hairs parallel to the plane of the arc by turning the telescope in its support. Choose two stars not less than 90° apart, and make their images coincide on one of the above two cross-hairs. Then, by moving the sextant, try and make them coincide on the other of these cross-hairs. If they do not coincide, alter the inclination of the telescope to the plane of the arc by the screws provided until they can be made to coincide on each cross-hair. When using the sextant, coincidences must always be made in the square formed by the four cross-hairs.

(*d*) *To Ascertain the Index Error of the Sextant.* Sight on a star and make its reflected image coincide with the star as seen direct. Then read the arc. The reading is the index error and may be + or −. It can be corrected by a screw provided, which slightly rotates the horizon glass, but this will necessitate repeating adjustment (*b*). The index error is affected by changes of temperature, so that it should be determined each time the sextant is used. In the day-time a very distant terrestrial object may be used instead of a star.

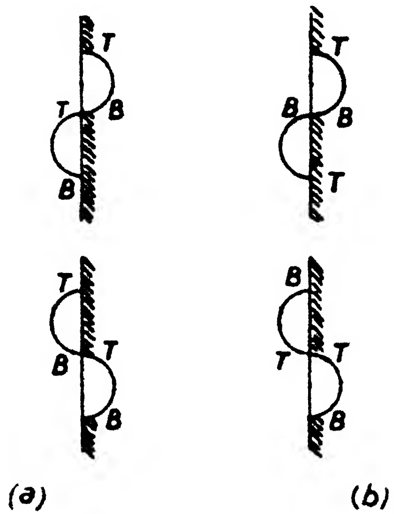


Fig. 8

In the box sextant only adjustments (*b*) and (*d*) are provided for. Square-headed screws are brought through holes in the case and can be turned by a small key provided with the instrument.

Example 2. In order to find the index error of a sextant, the reflected image of the sun is brought into contact with its disc seen directly, first on one side and then on the other. The two readings are 31′ 0″ and −32′ 40″. Find the index error and the sun's semi-diameter. The altitude of the sun's lower limb is then measured at noon, using an artificial horizon, and the observed angle (between the lower limb and its reflection) is found to be 57° 01′ 50″. Find the altitude of the sun's centre corrected for index error, refraction, semi-diameter, and parallax, the barometer reading 28·5 inches, attached thermometer 15° F., air temperature 20° F., horizontal parallax 8·74″.

(Fig. 8*a*): Index error $\dfrac{31'\,0'' - 32'\,40''}{2} = -0'\,50''$.

Diameter $= \dfrac{31'\,0'' + 32'\,40''}{2} = 31'\,50''$.

$\therefore$ Semi-diameter $= 15'\,55''$.

(Fig. 8*b*): Observed altitude of lower limb

$$= \frac{57^\circ\,01'\,50'' + 50''}{2} = \frac{57^\circ\,02'\,40''}{2} = 28^\circ\,31'\,20''$$

On reference to *Chambers' Mathematical Tables*, "Bessel's Refractions," we find

$$\text{Mean refraction for } 28^\circ\,31'\,20'' = 1'\,48{\cdot}2'' - \frac{31{\cdot}3}{60} \times 4{\cdot}4$$
$$= 105{\cdot}9''$$

$B = 0{\cdot}963$ for 28·5 inches, $t = 1{\cdot}002$ for 15° F.
$T = 1{\cdot}060$ for 20° F.

$\therefore$ Refraction correction

$$= -105{\cdot}9 \times 0{\cdot}963 \times 1{\cdot}002 \times 1{\cdot}060 = -108''$$

$\therefore$ Corrected altitude of sun's centre

$$= 28^\circ\,31'\,20'' - 1'\,48'' + 15'\,55'' + 8{\cdot}74'' \cos 28^\circ\,45'\,27''$$
$$= 28^\circ\,45'\,35''$$

N.B. It would have been better to repeat the observation, i.e. immediately to measure the altitude of the upper limb (Fig. 8*b*) and average the results, eliminating the sun's semi-diameter.

Resection

Resection is the fixing of the position of a point by angular observations *from* it to known stations. In general, *three* known stations are required and the method of resection is then called the "Three Point Problem." Any form of angular measuring instrument may be used. With the compass, only two known stations are required, as their bearings from any other point will fix the position of that point, lines being drawn through the known stations at bearings which are the "back bearings" of their given bearings (i.e. given bearing $\pm$ 180°), but as the compass is an instrument of only secondary accuracy, it is well to check the result by taking the bearing of a third station. Resection is most used (*a*) at sea with the sextant, and (*b*) on land with the plane table, and we shall deal with these two cases.

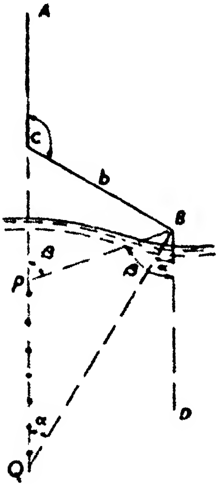


FIG. 9

(*a*) *Resection with the Sextant.* If the point P is in line with two of the known stations A and C (Fig. 9), only one angle β need be measured, viz. BPC. This is the method commonly adopted when soundings are taken in "ranges" or straight lines. A number of pairs of stations such as A, C are fixed so that their lines produced cover uniformly the area to be sounded, and the soundings are taken from a boat rowed in turn along each of these lines. The position of the boat is fixed by the sextant angle β taken to a third known station. If the boat is rowed at a uniform speed and the times noted for each sounding, the sextant observations need only be made occasionally, say at P and Q, the times of the sextant observations being also noted. When the positions fixed by the sextant have been plotted, the positions of the soundings can be found by dividing the distance between the determined positions in proportion to the time intervals. In tidal waters, of course, the water level must be read at, say, half-hourly intervals on a tide gauge, so that its level can be found by interpolation at the time of each sounding, and the soundings can thus be reduced to levels above or below a fixed datum level.

To plot the point P draw a line BD parallel to AC, and draw the line PB so that angle $DBP = \beta$. The intersection of this line with AC produced gives the point P. Or we can calculate the distance PC as

$$b\,\frac{\sin PBC}{\sin \beta} = b\,\frac{\sin (C - \beta)}{\sin \beta}\;;\ b \text{ and } C \text{ being known.}$$

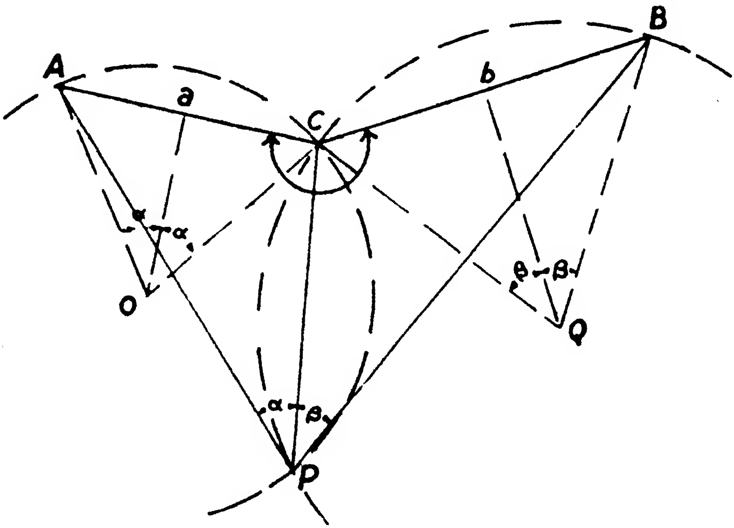


FIG. 10

In the general case (Fig. 10) we measure the angles α and β at P, when we can plot the points geometrically as follows—

From A and C draw AO, CO at angles of $90° - \alpha$ to AC, and from C and B draw CQ and BQ at angles of $90° - \beta$

to CB. From O as centre describe the circle ACP and from Q as centre the circle BCP. The intersection P of these circles is the point required, as the angles AOC, BQC at the centres have, by construction, been made 2α, 2β respectively, which are twice the angles α and β at the circumference, standing on the arcs AC, BC respectively. (More accurately, the radii of these circles can be calculated as

$$OC = \frac{a}{2 \sin \alpha} \text{ and } QC = \frac{b}{2 \sin \beta}$$

then with centres A and C describe arcs of radius OC, and from O as centre an arc ACP of the same radius; with centres B and C describe arcs of radius QC, and from Q as centre an arc BCP of the same radius.)

This construction will, of course, fail if P lies on the circle ACB, as *any* point on such a circle would subtend an angle α to AC and β to BC, and the points O and Q would coincide. The figure $ACBP$ would then be a cyclic quadrilateral and the sum of its opposite angles $ACB + \alpha + \beta = 180°$. It is important to see that this condition is *not* fulfilled, and the nearer the points A, C, B, and P lie to a circle the worse is the "fix" obtained. If the angle ACB in the quadrilateral $ACBP$ is not less than 180°, as in Fig. 10, it cannot occur.

To calculate the position of P, we must first calculate the angles A and B. We have

$$A + C + B + \alpha + \beta = 360°$$

$$\therefore \quad A + B = 360° - C - \alpha - \beta = \gamma \text{ (say)}$$

$$\text{while } PC = \frac{a \sin A}{\sin \alpha} = \frac{b \sin B}{\sin \beta}$$

$$\therefore \quad \frac{\sin A}{\sin B} = \frac{b \sin \alpha}{a \sin \beta} = k \text{ (say)}$$

$$\text{Then } \frac{k-1}{k+1} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{2 \sin \frac{A-B}{2} \cdot \cos \frac{A+B}{2}}{2 \sin \frac{A+B}{2} \cdot \cos \frac{A-B}{2}} = \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}}$$

$$\therefore \tan \frac{A-B}{2} = \frac{k-1}{k+1} \cdot \tan \frac{\gamma}{2}$$

We thus find the value of $\frac{A-B}{2}$ while $\frac{A+B}{2} = \frac{\gamma}{2}$. By addition we get A, then $ACP = 180° - A - \alpha$.

We can then calculate AP and CP by the "rule of sines" and plot P by arcs or, if required, calculate the co-ordinates of P, from the angle A and length AP.

EXAMPLE 3 (L.U.). The co-ordinates of three stations are—

	South	East
A . .	0	0
B . .	0	750
C . .	600	1,200

With a sextant at a point P the angles to A, B, and C are found to be $APB = 50° 10'$, $BPC = 69° 32'$. Find the co-ordinates of P.

$$\text{Obviously, } BC = 750 \text{ ft. and } ABC = 180° - \tan^{-1}\frac{4}{3}$$

$$= 180° - 53° 08' = 126° 52' \text{ (Fig. 11).}$$

$$\therefore \quad A + C = 360° - 126° 52' - 119° 42' = 113° 26' = \gamma$$

$$\text{Also} \quad k = \frac{750 \sin 69° 32'}{750 \sin 50° 10'} = 1{\cdot}2200 = \frac{\sin C}{\sin A}$$

$$\therefore \quad \tan\frac{C-A}{2} = \frac{k-1}{k+1}\cdot\tan\frac{\gamma}{2} = \frac{0{\cdot}2200}{2{\cdot}2200}\tan 56^\circ\,43'$$

$$= \tan 8^\circ\,35' \text{ (using 4-figure logarithms)}$$

$$\therefore \qquad \frac{C-A}{2} = 8^\circ\,35' \text{ and } \frac{C+A}{2} = 56^\circ\,43'$$

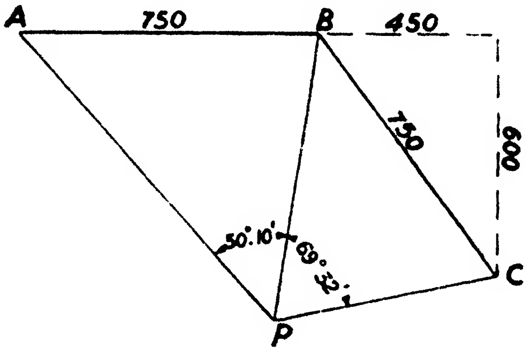


Fig. 11

$$\therefore \qquad A = 48^\circ\,08' \text{ by subtraction.}$$

$$\therefore \qquad ABP = 180^\circ - 48^\circ\,08' - 50^\circ\,10' = 81^\circ\,42'$$

Then $$AP = 750\,\frac{\sin 81^\circ\,42'}{\sin 50^\circ\,10'} = 966{\cdot}5$$

South co-ordinate of $P = 966{\cdot}5 \sin 48^\circ\,08' = 719{\cdot}7$.

East co-ordinate of $P = 966{\cdot}5 \cos 48^\circ\,08' = 645{\cdot}1$.

(*b*) *Resection with the Plane Table.* This process is in constant use in plane tabling as a large number of sub-stations are necessary in order to obtain the detail points by intersections or by "radiation," i.e. by direction and distance. The labour would be very greatly increased if, whenever a new sub-station was found necessary, the surveyor had to proceed to three known stations in order to fix the position of the new sub-station by two rays and check its position by a third ray. By using resection he has merely to find a point, convenient for viewing his detail,

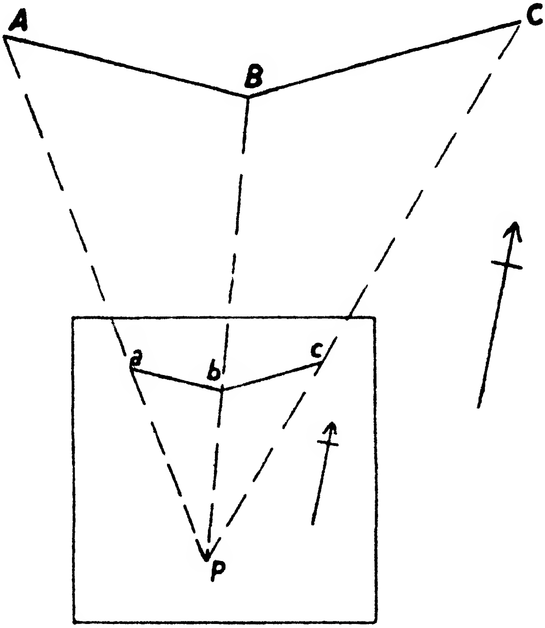


FIG. 12

whence he can see three known stations or sub-stations, and at once to fix his position as will be described.

In Fig. 12, *A*, *B*, *C* are the three known stations, visible from the point where a sub-station *P* is required, and *a*, *b*, *c* are the three stations plotted on the table. If the surveyor orients his table correctly he will have *ab*, *bc*, *ca* parallel to *AB*, *BC*, *CA*. If rays are then drawn through *Aa*, *Bb*, and *Cc* they will obviously intersect in a point *p*, which will be vertically over the point *P* on the ground which it represents. A trough compass is very useful for this purpose if Magnetic

North is already marked on the plan, as it can be placed with its edge along the North Point and the table turned until the north end of the needle reads 0°. The table should then be oriented very approximately and time is saved. If when the three rays are now drawn they are not quite concurrent, they will form a small "triangle of error," indicating that ab is not quite parallel to AB, etc., and the table requires to be rotated a little to make the rays concurrent. The three following rays will be found useful—

1. If the triangle of error is $\left\{\begin{array}{l}\text{inside}\\ \text{outside}\end{array}\right.$ the triangle abc, the correct point p is $\left\{\begin{array}{l}\text{inside}\\ \text{outside}\end{array}\right.$ the triangle of error.

2. If the correct point is outside the triangle of error it will be wholly to the left or right of all three rays.

3. Whether the correct point p is inside or outside the triangle of error, its distance from each ray will be proportional to the length of that ray pa, pb, pc, as measured on the plan.

The reasons for these rules are as follows (Fig. 13): Assuming the correct point p has been found on the plan and fresh rays drawn from it through a, b, and c, the angles which these new rays make with the old ones must all be equal and must all deflect to the left or all to the right of the old rays, as they all represent the angle through which the table must be turned for correct orientation. Rule 3 ensures that all the angles should be equal, and Rules 1 and 2 that they shall all involve turning the table in the same direction. It is assumed in this proof that a, b, and c do not change their positions; this is practically the case, as their movements are small compared to the distances Aa, Bb, and Cc. If a trough compass and Magnetic North Point on the plan are not available, several attempts may be necessary, the diminishing size of the "triangle of error" showing that the "trial and error" process is proceeding correctly. After each attempt the table is turned so that *pa* points to *A*, and the other two rays *Bb*, *Cc* are drawn to see if they are concurrent. When *p* has been thus found on plan, a

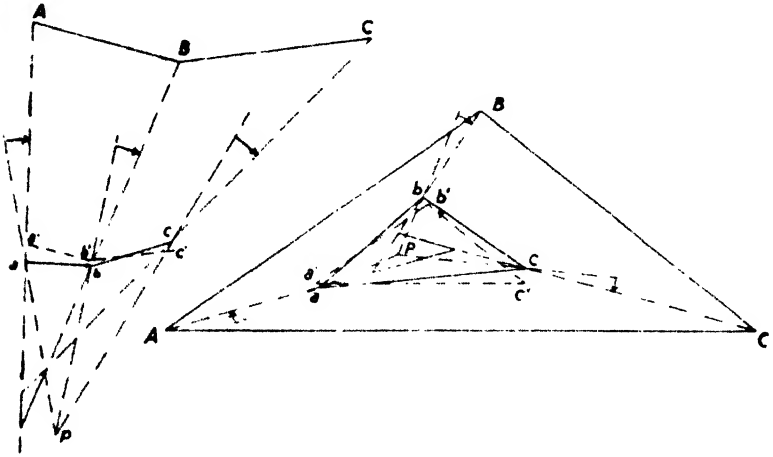


Fig 13

point *P* vertically below it on the ground should be pegged, so that it may serve as one of the three known points for some future resection.

Photographic Surveying

This may be looked on as a development of plane table surveying by intersecting rays from two sub-stations, as two photographs of the same area taken in known directions, one from each sub-station, enable us to draw intersecting rays from the sub-stations on plan to each point, which can be recognized on both photographs and thus to locate the point. The photo-theodolite employed consists essentially of—

1. A photographic camera of fixed known focal length.

2. Horizontal and vertical cross-wires pressed tightly against the sensitive plate, these wires being photographed on the plate. The collimation line of the instrument joins the

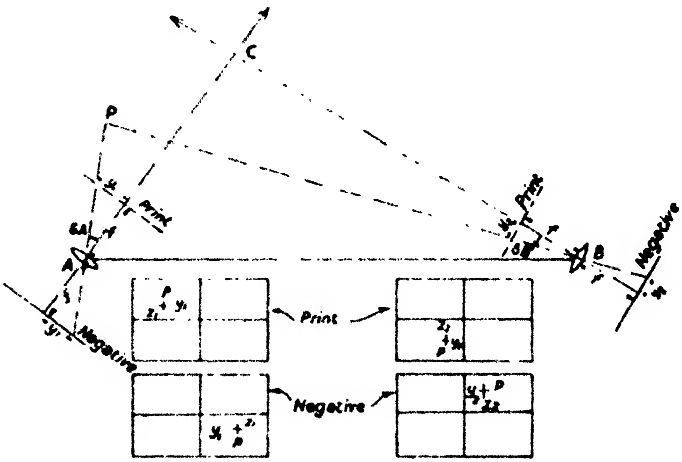


FIG. 14

intersection of the cross-hairs to the optical centre of the object glass, which latter is the centre of the instrument.

3. Three plate screws and a spirit level for levelling the collimation plane.

4. A graduated horizontal circle, with verniers, clamp, and tangent screw below the camera.

5. A telescope with cross-hairs rotating on a horizontal axis above the camera so that the vertical plane of collimation may be sighted on to any station.

In Fig. 14, let A and B be the two sub-stations and AC and BC the two positions of the vertical plane of collimation, the angles CAB and CBA having been measured on the horizontal circle. A point P is shown on both negatives, and both prints, as p: the prints, of course, being complete reversals of the negatives. Let the distances of p from the vertical and horizontal wires at A be y_1, z_1, and from the vertical and horizontal wires at B be y_2, z_2 respectively, then we can plot the point P on plan thus: From A and B on

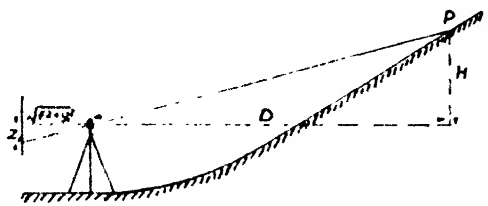


FIG. 15

plan draw AC and BC, making the observed angles with AB. Draw lines perpendicular to AC and BC at a distance f (the focal distance) in *front* of A and B. From AC and BC measure off y_1, y_2 along these lines on the same side as on the prints, and draw lines AP, BP through the points thus obtained, their intersection giving P on plan. In practice, corresponding points on both prints are first marked and numbered, and long threads are used instead of drawing the lines AP and BP.

For the level of the point we measure z_1, the height of p above the horizontal wire, the horizontal distance D of the point having first been scaled, viz. AP on plan (Fig. 15). The ray from p to the centre of the object glass rises a height z_1 in a horizontal distance $\sqrt{f^2 + y_1^2}$, consequently, the height H of P above the horizontal plane of collimation is $D \cdot \frac{z_1}{\sqrt{f^2 + y_1^2}}$, and the level of P can be thus determined. Points intersected by the horizontal cross-wire, of course, lie along a contour.

EXAMPLE 4. The optical axes of two photo-theodolites at A and B are inclined inwards at angles of 57° 26′ and 42° 34′ to a base line AB 346 feet long. The print from A shows a point 1·46 inches to the left of the vertical wire, 0·82 inch above the horizontal wire; that from B shows the same point 2·31 inches to the *right* of the vertical wire. The fixed focal length of both cameras is 6 inches and the level of the collimation at A is 386·70 feet. Calculate the distance and direction of P from A and the level of P.

$$\text{Here } \delta A = \tan^{-1}\frac{1{\cdot}46}{6{\cdot}00} = \tan^{-1} 0{\cdot}2433 = 13^\circ\, 40'$$

$$\therefore \text{ angle } PAB = 57^\circ\, 26' + 13^\circ\, 40' = 71^\circ\, 06'$$

$$\delta B = \tan^{-1}\frac{2{\cdot}31}{6{\cdot}00} = \tan^{-1} 0{\cdot}3850 = 21^\circ\, 03'$$

$$\therefore \text{ angle } PBA = 42^\circ\, 34' + 21^\circ\, 03' = 63^\circ\, 37'$$

$$\therefore \text{ angle } APB = 180^\circ - 71^\circ\, 06' - 63^\circ\, 37' = 45^\circ\, 17'$$

$$\therefore \quad PA = 346\,\frac{\sin 63^\circ\, 37'}{\sin 45^\circ\, 17'} = 436{\cdot}3 \text{ feet at an angle of } 71^\circ\, 06' \text{ to } AB$$

$$\text{and level of } P = 386{\cdot}70 + \frac{{\cdot}820}{\sqrt{6^2 + 1{\cdot}46^2}} \times 436{\cdot}3$$

$$= 386{\cdot}70 + \frac{0{\cdot}82}{6{\cdot}175} \times 436{\cdot}3 = 386{\cdot}7 + 57{\cdot}9$$

$$= 444{\cdot}6 \text{ feet}$$

Stereo-photogrammetry. This is a modern development of photographic surveying where the two photographs are taken with the vertical planes of collimation of the cameras at *right angles* to the base joining the positions of the cameras, so that the two positions of the photographic plate are in the same vertical plane. The trunnion axis of the telescope on the camera is fixed in the vertical plane of collimation of the camera, so that when a photograph has been taken from a known station A (Fig. 16), the second position B of the camera can be fixed in a vertical plane at right angles to the vertical plane of collimation of the first view, and similarly the

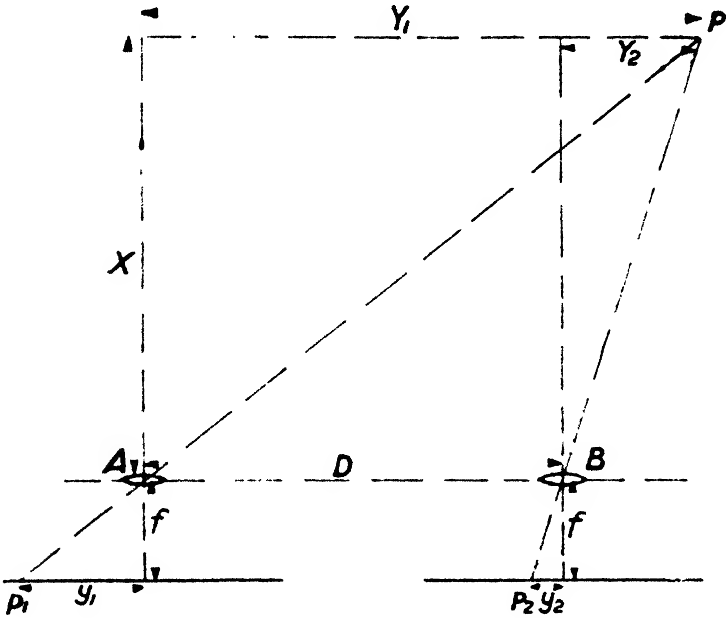


FIG. 16

camera can be set up at B, the telescope pointed at A, and the second view taken with the vertical plane of collimation also at 90° to the base AB. The telescope is fitted with a vertical circle and stadia hairs, so that the horizontal distance D and the difference of level $(Z_1 - Z_2)$ of the collimation planes at A and B can be determined.

Then, if y_1, z_1 ; y_2, z_2 are the distances from the vertical and horizontal wires of the same point p on the plates exposed at A and B respectively, we can calculate the *co-ordinates* of P from the L.H. camera thus

$$Y_1 = \frac{y_1}{f}X, \quad Y_2 = \frac{y_2}{f}X, \text{ and } Y_1 - Y_2 = D$$

$$\therefore \frac{X(y_1 - y_2)}{f} = D. \quad \therefore X = \frac{f}{y_1 - y_2} . D; \; Y_1 = \frac{y_1}{f}X$$

Also if Z_1, Z_2 are the heights of P above the horizontal planes of collimation at A and B, we have

$$\frac{Z_1}{z_1} = \frac{AP}{Ap_1} = \frac{X}{f}. \quad \therefore Z_1 = \frac{z_1}{f}X, \text{ and similarly } Z_2 = \frac{z_2}{f}X$$

Also $Z_1 - Z_2 = \frac{z_1 - z_2}{f}X$ = difference of level of cameras = height of camera B above camera A.

$$\therefore \frac{Y_1}{y_1} = \frac{Z_1}{z_1} = \frac{X}{f} = \frac{D}{y_1 - y_2} = \frac{Y_2}{y_2} = \frac{Z_2}{z_2} = \frac{Z_1 - Z_2}{z_1 - z_2}$$

EXAMPLE 5 (L.U.). Two photographs are taken with a photo-theodolite from positions A and B, 300 feet apart, the lines of collimation being both at 90° to the line AB. A point C appears on the *print* from A as 2·02 inches to the R.H. of the vertical wire and 0·98 inch above the horizontal wire, and on the *print* from B as 3·06 inches to the L.H. of the vertical wire and 1·90 inches above the horizontal wire. B is to the right of A and the focal distance is 6 inches. Find the co-ordinates of C from A as origin and the difference of level of the two collimation planes.

$$Y_1 = \frac{2\cdot02}{6}X, \quad Y_2 = -\frac{3\cdot06}{6}X, \text{ and } 300 = Y_1 - Y_2$$

$$= \frac{X}{6}(2\cdot02 + 3\cdot06) = \frac{5\cdot08}{6}X$$

$$X = \frac{1800}{5 \cdot 08} = 354 \cdot 3 \text{ feet}$$

$$Y_1 = \frac{2 \cdot 02}{6} \times 354 \cdot 3 = 119 \cdot 3 \text{ feet}$$

$$Z_1 = \frac{0 \cdot 98}{6} \times 354 \cdot 3 = 57 \cdot 9 \text{ feet}$$

$$Z_2 = \frac{1 \cdot 90}{6} \times 354 \cdot 3 = 112 \cdot 2 \text{ feet}$$

∴ Collimation at B is 112·2 − 57·9 = 54·3 feet *below* collimation level at A.

In practice, the co-ordinates X, Y, and Z are measured on a "Stereo-Comparator" (Pulfrich's), of which Fig. 17 is a diagrammatic sketch. The two negatives are placed in the instrument and are viewed through a stereoscope—as the negatives are views taken from a distance of, say, 300 feet apart, an intensely stereoscopic effect is produced. At the focus of each eyepiece of the stereoscope is a fixed index, and each eyepiece must first be adjusted separately to see this clearly. Then, using the left eye only, and the screws 1 and 3, the L.H. negative is moved till the index of the L.H. eyepiece coincides with the intersection of the cross-wire lines on the L.H. negative: then, by moving the R.H. negative by the screws 2 and 4 and using the right eye only, the index of the R.H. eyepiece is made to coincide with the intersection of the cross-wire lines on the R.H. negative. Using *both* eyes, the two indexes now appear as a single index at an infinite distance in the combined view. The screw 1 moves both negatives in the direction Y and measures y_1, the "azimuth"; the screw 2 moves the R.H. negative relatively to the L.H. one in the direction Y and measures $y_1 - y_2$, the "parallax." The screw 3 moves both eyepieces in the direction Z and measures z_1, the "altitude," and the screw 4 moves the R.H. negative relatively to the L.H. one in the direction Z and measures $z_2 - z_1$, the "level compensation." Zero readings are now taken on all four screws. Using both eyes, and by turning screw 2, the index is brought to apparently the same *distance* as some point

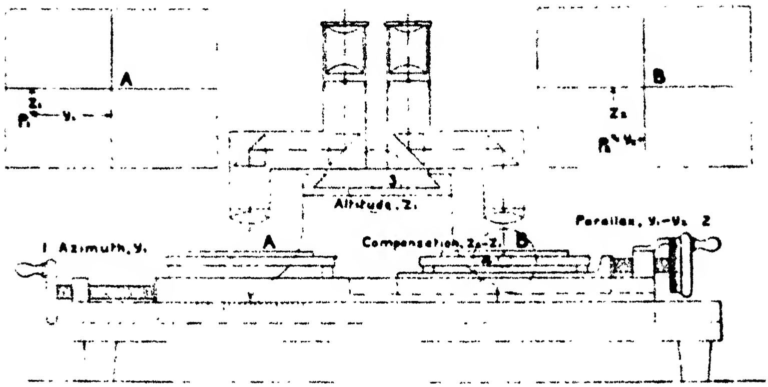


FIG. 17

From *Surveying Instruments* by R. M. Abraham, by permission of Messrs. C. F. Casella & Co., Ltd.

on the ground whose position is required, and the screw 4 is adjusted so that $z_2 - z_1 = \frac{y_1 - y_2}{D}(Z_2 - Z_1)$, where $Z_2 - Z_1$ is the difference of level of the instruments and D is their horizontal distance apart. Then the index is made apparently to touch the point in question by turning the screw 1 for direction and the screw 3 for level.

It is obvious that the "level compensation screw" 4 can be geared to the "parallax screw" 2 in the ratio $\frac{Z_2 - Z_1}{D}$, also that the "altitude screw" 3 can be geared to the "parallax screw" 2 in the ratio $\frac{Z_1}{D}$ so that $z_1 = \frac{Z_1}{D}(y_1 - y_2)$, where

Z_1 is the height of a given contour line above the collimation plane of the L.H. instrument. If these gearings are effected the index can only be made to touch points which are on the required contour level, the co-ordinates X and Y of which can thus be computed from the readings of the micrometer heads of the screws 2 and 1. In a further development, viz. Von-Orel's "Stereo-Autograph," a drawing table is attached to the comparator and a pencil is arranged to move in the co-ordinate directions X and Y to a suitable scale by levers and gearing from screws 2 and 1, and by this means the contour at a given height Z_1 above the collimation plane of the L.H. instrument is drawn automatically on the plan.

Subtense Measurements of Distance and Height

In the usual form of Tacheometry readings are taken on a vertical levelling staff at the upper, middle, and lower stadia hairs in the tacheometer, which readings, together with the readings of the horizontal and vertical circles and the height of the instrument above the ground, enable the direction of the point observed, its horizontal distance, and its difference of level from the observing station, to be found. The reading of the middle hair is a useful check on the readings of the other two hairs. When the instrument is moved forward to a fresh sub-station a set of readings is taken first to the station just left, and the new values of the horizontal and vertical distance thus obtained should agree closely with the values obtained from the forward readings, and should be averaged with them for obtaining the new position and level of the instrument. Only one pointing of the instrument is made for each point observed, and, therefore, only one reading of the vertical and horizontal circles, but the distance at which observations can be taken is limited to that at which the graduations on the levelling staff can be read with sufficient accuracy. For greater distances, "subtense measurements" can be used, i.e. conspicuous targets are fixed at known distances apart horizontally or vertically at the point to be observed, and the theodolite is pointed at each in

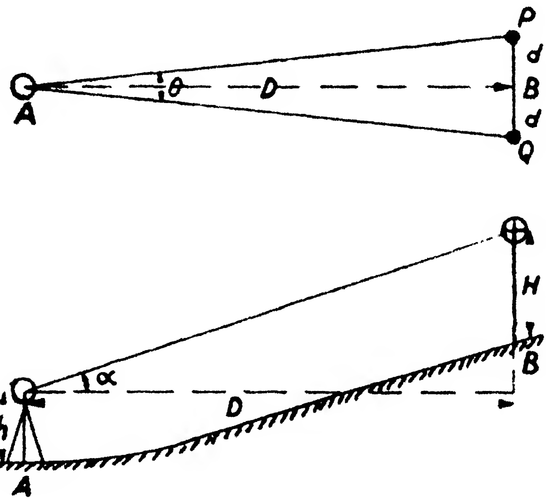


FIG. 18

turn, the horizontal circle or vertical circle being read for both pointings. There are no stadia hairs.

(*a*) *Horizontal Subtense Surveying* (Fig. 18). Here *A* is the theodolite and *B* is the point to be observed. From *B* equal distances $d = BP = BQ$ are marked off at right angles to *AB*. If the distance *AB* is not very great, *P* and *Q* may be targets fixed to a rod from 10 feet to 20 feet in length, held horizontally at *B*, with sights at the centre at right angles to *PQ*, through which the assistant sights on *A*: for great dictances, *P* and *Q* may be targets on vertical poles set out at (say) 50 feet horizontally from B, so that PQ is perpendicular to AB. The horizontal angle $PAQ = \theta$ is read a number of times (say 10) and averaged, and the vertical angle α from A to a target at B is also read. Then the horizontal distance $AB = D = d \cot \frac{\theta}{2}$, while if h, H are the

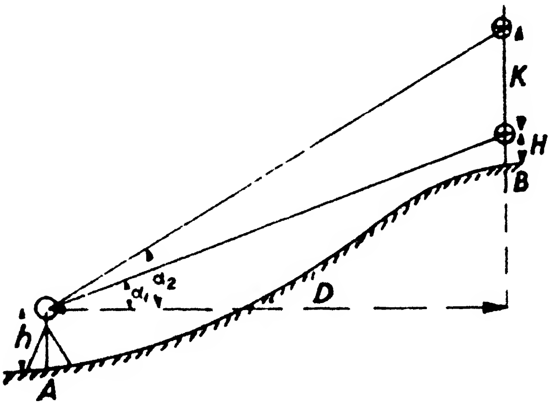


FIG. 19

heights above ground of the theodolite at A and the target at B respectively, the height of B above A is $D \tan \alpha + h - H$. A traverse can be made in this way with considerable accuracy, but the process is a slow one.

(*b*) *Vertical Subtense Survey Surveying* (Fig. 19). Here the tangents are fixed to a vertical staff at a distance, K, apart (say 10 feet). The vertical angles α_1, α_2 are read to the lower and upper targets by the theodolite at A. Then we have

$$K = D \tan \alpha_2 - D \tan \alpha_1 = D(\tan \alpha_2 - \tan \alpha_1)$$

$$\therefore \quad D = \frac{K}{\tan \alpha_2 - \tan \alpha_1} = \text{horizontal distance}$$

and the rise from A to $B = D \tan \alpha_1 + h - H$.

In both methods of subtense surveying the horizontal circle must, of course, be read for the direction of AB, back readings as well as forward readings of the staff should be taken, and the results averaged for a change of station.

Szepessy Tacheometer. This is a new type* in which a graduated vertical levelling staff is read from a theodolite, but the tangent principle is used as in (*b*) above. There are no stadia hairs in the diaphragm, but a scale of tangents of vertical angles is engraved on the fixed cover of the vertical circle, the graduations being at each 0·01 and 0·005. These graduations are reflected by prisms through a window in the size of the telescope so that the scale of tangents is visible in the field of view of the eyepiece alongside the image of the levelling staff. In taking a reading, the horizontal hair is brought on to one of the 0·01 (long and numbered) divisions of the tangent scale and the staff reading taken on the horizontal hair and at the 0·005 tangent divisions immediately above and below it. As in ordinary tacheometry the difference of the lower staff reading and that of the hair should equal the difference of the upper staff reading and that of the hair as a check on the accuracy of reading.

Here, $\tan \alpha_2 - \tan \alpha_1$ is 0·01 and consequently the horizontal distance $D = 100 \times$ difference of the lower and upper readings. The rise from A to $B = D \times$ tangent scale reading of horizontal hair + height of instrument (h) - middle reading on the staff (H).

Depressions are read in the same manner, the "rise" being $-D \times$ middle reading $+ h - H$, i.e. the fall from A to $B = D \times$ middle reading $- h + H$. Calculations are thus much simplified.

* Made by Messrs. E. R. Watts & Son, London

EXAMPLE 6 (L.U.). The following are the readings taken to a point on a Szepessy Tacheometer—

Staff (feet)	Tangent of Elevation
5·19	·265
4·21	·26
3·23	·255

the height of the instrument being 4·50 feet above the station level of 370·61 feet. Find the horizontal distance and level of the point. If the top of the staff had been inclined 2° towards the tacheometer, what would have been the readings and the calculated distance and level? (Fig. 20.)

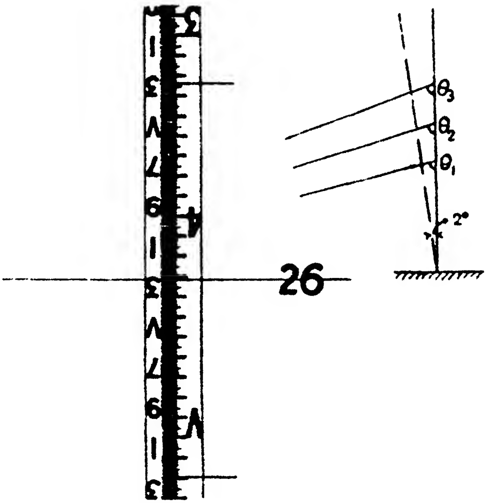


FIG. 20

$$5{\cdot}19 - 4{\cdot}21 = 0{\cdot}98$$
$$= 4{\cdot}21 - 3{\cdot}23$$

Therefore, the readings are consistent.

$$\text{Horizontal distance} = \frac{5{\cdot}19 - 3{\cdot}23}{{\cdot}265 - {\cdot}255} = \underline{196 \text{ feet}}.$$

$$\text{Rise of collimation} = 196 \times 0{\cdot}26 = 50{\cdot}96 \text{ feet}.$$

$$\text{Reduced level of point} = 370{\cdot}61 + 4{\cdot}50 + 50{\cdot}96 - 4{\cdot}21$$
$$= 421{\cdot}57 + 0{\cdot}29 = 421{\cdot}86 \text{ feet}$$

The inclinations of the three rays to the horizontal are 14° 18′, 14° 34½′, 14° 51′ (from the above tangents).

Their inclinations θ to the vertical (Fig. 20) are, therefore, 75° 42′, 75° 25½′, 75° 09′.

Their inclinations to the inclined staff are, therefore,

77° 42′, 77° 25½′, 77° 09′ = $\theta + 2°$

The inclined staff readings are

$$\frac{\text{true reading} \times \sin \theta}{\sin (\theta + 2°)}$$

which are 3·203, 4·175, and 5·145 feet.

Therefore, D would be calculated as
1·942 × 100 = 194·2 ft.

Rise of collimation = 194·2 × 0·26 = 50·492 feet.

Therefore, the apparent level of B would be

370·61 + 4·50 + 50·492 − 4·175
= 421·102 + 0·325
= 421·43 feet.

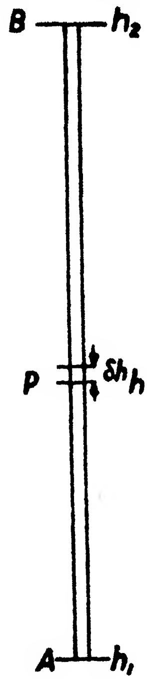


FIG. 21

BAROMETRIC LEVELLING

For the approximate determination of levels (say to an accuracy of 10 feet) in hilly country, the Mercury Barometer may be used, or, more conveniently, the Aneroid Barometer, which is a mechanical copy of it. Fig. 21 represents a column of air at constant temperature between the two levels h_1 and h_2. At an intermediate point P, let p be the pressure, ρ the density, g = acceleration of gravity, then

$$\rho \,.\, g \,.\, \delta h = -\delta p. \quad \therefore \quad \frac{dp}{dh} = -\rho g$$

Also for a perfect gas,

$$pV = \frac{p}{\rho} = R\tau. \qquad \therefore\ \rho = \frac{p}{R\tau}$$

where τ = absolute temperature = temperature Fahrenheit + 460°, and R is a constant for air.

$$\therefore \qquad \frac{dp}{dh} = -\frac{pg}{R\tau}$$

$$\therefore \qquad \frac{dp}{p} = -\frac{g}{R\tau}\,dh. \qquad \therefore\ \log_e \frac{p_1}{p_2} = \frac{g}{R\tau}(h_2 - h_1)$$

$$\therefore \text{ difference of level, } h_2 - h_1 = \frac{R\tau}{g}\log_e \frac{p_1}{p_2} = \frac{R\tau}{g}\log_e \frac{H_1}{H_2}$$

where H_1, H_2 = heights of barometer in any units.

$$= \frac{2{\cdot}3026\ R\tau}{g}(\log H_1 - \log H_2)$$

$$^{*} = 62{,}760\,(\log H_1 - \log H_2)$$

for air at 50° F., 50% saturation and taking g = 32·19 ft./sec.[2] at London.

A table can be prepared of 62,760 log H, thus:

		Height ft	Barometer in
62,760 log 31	93,598	0	31·000
.. .. 30	92,704	1,000	29·883
.. .. 29	91,780	2,000	28·807
.. .. 28	90,824	3,000	27·769
.. .. 27	89,842	4,000	26·769
.. .. 26	88,804	5,000	25·805
.. .. 25	87,735	6,000	24·875
.. .. 24	86,622	7,000	23·979

and with such a table, completed for every 0·1 inch of height of the barometer, and by interpolation, the difference

* "The Aneroid Barometer and Barograph in Engineering," R. H. J. Lee, *Proc. Inst. of Mining and Metallurgy*, Vol. 32, 1922-23.

of level can be found by mere subtraction of the value of 62,760 log H at the upper level from its value at the lower level.

More conveniently, a scale of heights can be graduated round an aneroid barometer used for surveying with its 0 opposite 31 inches. The 1,000 feet mark will be placed at such a reading H_1 of the barometer that

$$\log H_1 = \frac{93598 - 1000}{62760} = 1{\cdot}47543$$

i.e. at 29·883 inches, the 2,000 feet at a reading H_2 such that

$$\log H_2 = \frac{93598 - 2000}{62760}$$

i.e. at 28·807 inches, and so on, as in the above table.

The difference of level of two points *uncorrected for temperature* can then be found by subtraction of the reading of the height scale at the lower level from its reading at the upper level. This scale should *not* be movable, as its graduations are not uniform. In practice, the temperature of the air must be taken at the upper and lower levels and the difference of level must be corrected by multiplying by the factor

$$\frac{\text{average temperature} + 460^\circ}{510^\circ}$$

$$\text{i.e.}\ \frac{\text{average absolute temperature}}{\text{absolute temperature at } 50^\circ\text{ F.}}$$

as indicated by the formula proved above. Also, the variation in the value of g may have to be allowed for if far from the latitude of London, g being given by the formula

$$g = 32{\cdot}09\,(1 + 0{\cdot}0053 \sin^2 \phi)$$

where ϕ = latitude.

Where this latter correction is needed the correcting factor is $\frac{32\cdot19}{g \text{ at place}}$.

As the temperature of the air varies very considerably with great heights, it will be more accurate to measure such heights in stages not exceeding 1,000 feet at a time, correcting each for the average temperature in that stage. To allow for changes in the atmospheric pressure not due to altitude, the surveyor can return to his starting point of known level, or to another point of known level, and adjust any discrepancy in level found over the intermediate readings in proportion to the time, or an assistant may be left to read another instrument at intervals during the day at the starting point, so that by interpolation simultaneous readings at the starting and observed points may be obtained.

The mercury barometer is, of course, the more accurate instrument, but it requires another correction for the unequal expansion of the mercury and the metal scale, depending on the temperature read on the attached thermometer, whereas the aneroid barometer is compensated for changes in its own temperature. Strictly, also, a correction should be made for variation of gravity with altitude, but the errors due to uncertainty as to the actual average *air* temperatures probably render this latter an unnecessary refinement.

The Adjustment of Traverses

Open Traverses. These can only be adjusted as regards their angles and only by finding the azimuths of their first and last lines by astronomical observations. The difference of these azimuths, corrected for the "convergence of meridians," should agree with the difference of azimuth calculated from the observed angles of the traverse, and any discrepancy should be divided equally among the angles. If the traverse is many miles in length the correction for convergence, viz. ***difference of longitude* × *sine of average latitude*,** should be calculated in sections of only a few miles in length as described in Chapter V.

Closed Traverses These are of two kinds, viz. traverses that return to their starting points, and traverses that connect two known points, e.g. two triangulation stations. In the latter case, if the angles between the first and last lines of the traverse and the line joining the triangulation stations have been measured, it is obvious that we have a closed traverse with one side, whose length and bearing must not be adjusted, i.e. its latitude and departure are unalterable. If these angles have *not* been measured, the traverse may be reduced to its first side as meridian, and the length and bearing of the line joining its first and last stations calculated. If this length does not agree with the known distance between the triangulation stations, the sides of the traverse must be altered in the ratio of the known length to the calculated length and the traverse reduced to the true bearing of the triangulated side.

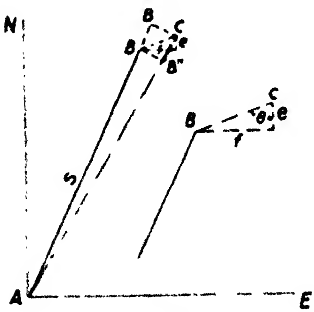


FIG. 22

(*a*) *Bowditch's Method of Adjustment.* If AB (Fig. 22) is a side of a traverse of length s, the probable error in its length will be proportional to $\sqrt{s}$ if of a compensating nature. This would move B to B' (say). Bowditch assumed that the probable error in its bearing due to incorrect angular measurement would produce an *equal* displacement BB'' at right angles to AB, so that B would move to C and the total probable error BC would be proportional to $\sqrt{s}$. Let e and f be the projections of BC on the north and east axes, so that e is the error in latitude and f the error in departure, then $BC = \sqrt{e^2 + f^2}$.

Then if E = total error in latitude and F = total error in departure we have, by the Method of Least Squares, weighting inversely as the (probable error)^2,

$$\Sigma\left(\frac{e^2 + f^2}{s}\right) = \text{minimum}, \ \Sigma(e) = E, \ \Sigma(f) = F$$

Differentiating,

$$\Sigma\left(\frac{e\delta e}{s} + \frac{f \cdot \delta f}{s}\right) = 0, \ \Sigma(\delta e) = 0, \ \Sigma(\delta f) = 0$$

Multiplying the two last equations by $-\lambda$, $-\mu$ respectively, adding all three equations and equating the coefficients of each δe and δf to zero, we obtain

$$\frac{e_1}{s_1} = \lambda; \ \frac{e_2}{s_2} = \lambda; \ \frac{e_3}{s_3} = \lambda, \text{ etc., and } \frac{f_1}{s_1} = \mu; \ \frac{f_2}{s_2} = \mu; \ \frac{f_3}{s_3} = \mu$$

Substituting these values in the original equations we have

$$\lambda \,.\, \Sigma(s) = E, \ \mu\,\Sigma(s) = F. \ \therefore \ \lambda = \frac{E}{\Sigma(s)}, \ \mu = \frac{F}{\Sigma(s)}$$

Therefore the corrections of latitude are

$$e_1 = \frac{s_1}{\Sigma(s)} E; \ e_2 = \frac{s_2 E}{\Sigma(s)}; \ e_3 = \frac{s_3 E}{\Sigma(s)}, \text{ etc.}$$

and the corrections of departure are

$$f_1 = \frac{s_1}{\Sigma(s)} F, \ f_2 = \frac{s_2 F}{\Sigma(s)}, \ f_3 = \frac{s_3 F}{\Sigma(s)}, \text{ etc.}$$

That is: correction of a $\left\{\begin{matrix}\text{latitude}\\ \text{departure}\end{matrix}\right.$

total correction of

$$\left\{\begin{matrix}\text{latitude}\\ \text{departure}\end{matrix}\right. = \frac{\text{length of corresponding side}}{\text{sum of lengths of sides}}$$

Bowditch's Rule, therefore, alters the position of one end B of a line AB (Fig. 22) *relative* to the other by $\frac{s_1}{\Sigma s} \times E$ in latitude and $\frac{s_1}{\Sigma s} \times F$ in departure, i.e. by a resultant amount $BC = \frac{s_1}{\Sigma s}\sqrt{E^2 + F^2} = \frac{s_1}{\Sigma s} \times$ closing error, in a direction θ where $\tan\theta = \frac{F}{E}$, i.e. in a direction parallel to the closing error. The total movement of each station is, therefore, parallel to the closing error and equal

$$\frac{\Sigma(\text{length of sides from start})}{\Sigma(\text{lengths of sides})} \times \text{closing error}$$

This correction can, therefore, be applied graphically as shown in Fig. 23 by drawing a line parallel to the closing error through each station and making the "shifts" $B'B$, $C'C$, $D'D$, $A'A$, proportional to the distance from the starting point A by the diagram shown. Here $AB'C'D'A'$ is the unadjusted traverse, $A'A$ the closing error, $ABCDA$ the adjusted traverse.

The bearings of all sides are altered (unless they lie in the direction of the closing error) and this alteration of bearing will be a maximum if the side is perpendicular to the direction of the closing error, when it will be

$$\frac{s_1}{\Sigma(s)} \times \frac{\text{closing error}}{s_1} = \frac{\text{closing error}}{\Sigma(\text{sides})}$$

in circular measure. Now, even with good chaining, the $\frac{\text{closing error}}{\Sigma(\text{sides})}$ may be as much as $\frac{1}{1000}$, so that the maximum alteration in bearing may be $\frac{3438}{1000} = 3{\cdot}438'$ or $3'\,26''$. Although some closing error is generally found in the angles of a theodolite traverse, the error of the bearing of each side should be much less than 3½′ so that this rule is unsuitable for a *theodolite* traverse, though it is very suitable for a *compass* traverse, where bearings cannot be read to less

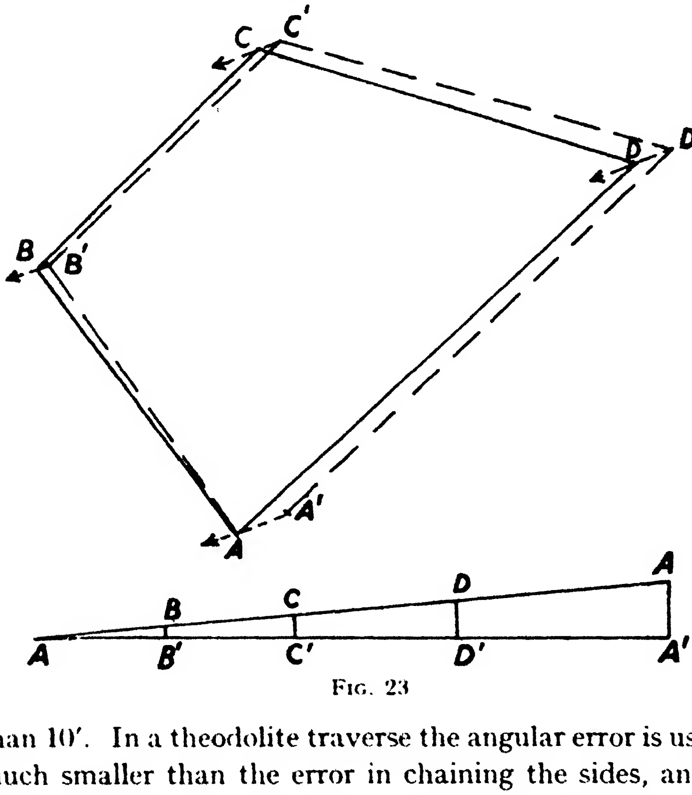


Fig. 23

than 10′. In a theodolite traverse the angular error is usually much smaller than the error in chaining the sides, and the former should first be distributed equally over the angles: the closing error in latitude and departure then found should be adjusted in such a way that the $\frac{\text{correction of departure}}{\text{correction of latitude}}$ for each side should equal $\frac{d}{l}$, the tangent of the bearing of the side, so that the bearing of the side is unaltered (Fig. 24).

(*b*) *Adjustment of a Closed Traverse when the Bearings are to be Unaltered* Let the correction of a side be xs where s is the length of the side, then if l, d are the latitude and departure of the side, the correction of latitude is $l = xl$ and the correction of departure is $f = xd$, where x will vary with different sides. The probable error in the length of the

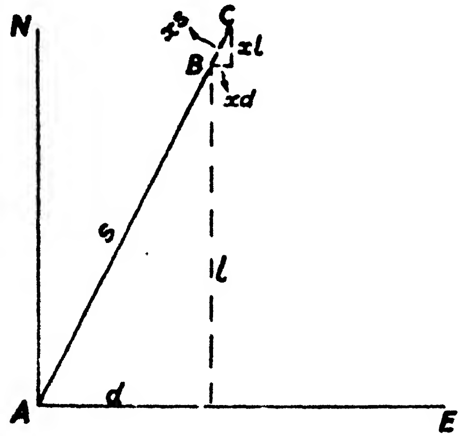


Fig. 24

side will be proportional to $\sqrt{s}$, therefore we must weight the error by $\frac{1}{s}$. Then if E, F are the total corrections in latitude and departure we shall have, by the Method of Least Squares,

$$\Sigma\left(\frac{x^2s^2}{s}\right) = \Sigma(x^2s) = \text{minimum}\,;\ \Sigma(xl) = E\,;\ \Sigma(xd) = F$$

$$\therefore\ \Sigma(xs\delta x) = 0\,;\ \Sigma(l\delta x) = 0\,;\ \Sigma(d\delta x) = 0$$

Multiplying the two latter equations by $-\lambda$, $-\mu$ respectively, adding all three equations together and equating the coefficients of each δx to zero, we have

$$x_1 = \frac{\lambda l_1 + \mu d_1}{s_1}\,;\ x_2 = \frac{\lambda l_2 + \mu d_2}{s_2}\,;\ x_3 = \frac{\lambda l_3 + \mu d_3}{s_3}, \text{ etc.}$$

Substituting in the original equations we get

$$\lambda\Sigma\left(\frac{l^2}{s}\right) + \mu\Sigma\left(\frac{ld}{s}\right) = E;\ \lambda\Sigma\left(\frac{ld}{s}\right) + \mu\Sigma\left(\frac{d^2}{s}\right) = F$$

which equations give λ and μ.

Then the corrections of latitude are

$$e_1 = \lambda\frac{l_1^2}{s_1} + \mu\frac{l_1d_1}{s_1}, \text{ etc.,}$$

and the corrections of departure are

$$f_1 = \lambda\frac{l_1d_1}{s_1} + \frac{\mu d_1^2}{s_1}, \text{ etc.}$$

As a check, we note that

$$\frac{\text{correction of departure}}{\text{correction of latitude}} = \frac{\lambda l_1d_1 + \mu d_1^2}{\lambda l_1^2 + \mu l_1d_1}$$

$$= \frac{d_1(\lambda l_1 + \mu d_1)}{l_1(\lambda l_1 + \mu d_1)} = \frac{d_1}{l_1} \text{ as required above.}$$

Weighting. In Method (*a*) if the measurements of the lengths of the sides are known to vary in accuracy they may be weighted with weights w_1, w_2, w_3, etc. Then, taking $\Sigma\frac{w}{s}(e^2 + f^2)$ = a minimum, it is easy to prove that the corrections will be

$$e_1 = \frac{s_1}{w_1} \cdot \frac{E}{\Sigma\frac{s}{w}}, \text{ etc., in latitude}$$

$$\text{and } f_1 = \frac{s_1}{w_1} \cdot \frac{F}{\Sigma\frac{s}{w}}, \text{ etc., in departure}$$

Similarly in Method (*b*), taking $\Sigma\left(\frac{w}{s}x^2s^2\right)$ = a minimum, we can show that the equations for λ and μ will be

$$\lambda\Sigma\left(\frac{l^2}{ws}\right) + \mu\Sigma\left(\frac{ld}{ws}\right) = E, \quad \lambda\Sigma\left(\frac{ld}{ws}\right) + \mu\Sigma\left(\frac{d^2}{ws}\right) = F$$

and that the corrections will be $e_1 = \lambda \dfrac{l_1^2}{w_1 s_1} + \mu \cdot \dfrac{l_1 d_1}{w_1 s_1}$, etc., in latitude, and $f_1 = \lambda \dfrac{l_1 d_1}{w_1 s_1} + \mu \dfrac{d_1^2}{w_1 s_1}$, etc., in departure.

Method (*b*) is, of course, much more laborious than Method (*a*), but it has a sound theoretical basis, while the method frequently adopted for adjusting a theodolite traverse (viz. by correcting each latitude by

$$\frac{\text{the latitude}}{\text{arith sum of the latitudes}} \times \text{total correction of latitude,}$$

and each departure by $\dfrac{\text{the departure}}{\text{arith. sum of the departures}} \times$ total correction of departure) has no theoretical justification whatever, and it is easily seen that it alters the bearings of the lines.

The adjustment of a traverse means, of course, distributing the *small* unavoidable errors of measurement over a traverse (1) so as to make it close without, in this case, altering the bearings of the sides, and (2) to effect the corrections in the most probable way.

Method (*c*). This is given in Middleton and Chadwick's *Surveying* (Spon, London), and will be found much shorter than Method (*b*); it fulfils condition (1), but not condition (2). Two correction factors x and y are chosen such that about half the sides are corrected by amounts xs and the remainder by amounts ys. The factor x is applied to all sides whose direction lies in the N.E. and S.W. quadrants, and the factor y to all sides whose direction lies in the N.W. and S.E. quadrants.

If E, the total correction in latitude, is $\left\{\begin{matrix}\text{greater}\\ \text{less}\end{matrix}\right.$ than F, the total correction in departure, the signs of ***all*** the $\left\{\begin{matrix}\text{latitude}\\ \text{departure}\end{matrix}\right.$ corrections we made the same as the sign of $\left\{\begin{matrix}E\\ F\end{matrix}\right.$. Then the signs of the $\left\{\begin{matrix}\text{departure}\\ \text{latitude}\end{matrix}\right.$ corrections for each side are made *consistent* with those already chosen for its $\left\{\begin{matrix}\text{latitude}\\ \text{departure}\end{matrix}\right.$ corrections. Thus, if a $\left\{\begin{matrix}+\\ -\end{matrix}\right.$ correction is to be applied to a $\left\{\begin{matrix}+\\ -\end{matrix}\right.$ latitude, the length of the side is assumed to be *increased*, and a $\left\{\begin{matrix}+\\ -\end{matrix}\right.$ correction must be applied to a $\left\{\begin{matrix}+\\ -\end{matrix}\right.$ departure. If, however, a $\left\{\begin{matrix}-\\ +\end{matrix}\right.$ correction is applied to a $\left\{\begin{matrix}+\\ -\end{matrix}\right.$ latitude, the length of the side is assumed to be *decreased* and a $\left\{\begin{matrix}-\\ +\end{matrix}\right.$ correction must be applied to a $\left\{\begin{matrix}+\\ -\end{matrix}\right.$ departure. The latitude corrections are then summed up and equated to E, and the departure corrections are summed up and equated to F, and thus x and y are found from this simultaneous equation. The reader will note that latitudes and departures are spoken of as + or − instead of as "Northings" or "Southings," "Eastings" or "Westings." This lessens the number of columns in the traverse table and enables the corrections to be applied by ordinary algebraical addition.

EXAMPLE 7. The angles of a closed traverse $ABCDEF(A)$ are in order 96° 14′ 00″, 105° 17′ 30″, 124° 22′ 00″, 249° 05′ 40″, 40° 47′ 30″, 104° 14′ 20″ (measured clockwise from the rear station) and the lengths of the sides AB to FA are 701·4, 249·2, 309·6, 1092·8, 1278·5, and 988·5 feet respectively. The bearing of *AB* is 149° 13′ 00″, clockwise from the north. Draw up a traverse table and adjust it to close and determine the co-ordinates of all the stations. Take the co-ordinates of *A* as 1000 N., 00 E.

We draw up a traverse table as shown on page 311.

We first calculate the "deflection angles." If the angle is less than 180° the deflection angle is (180° − angle) *Left*: if more than 180° the deflection angle is (angle − 180°) *Right*. The sum of the deflection angles is 429° 04′ 40″ L. − 69° 05′ 40″ R. = 359° 59′ 00″ L. As this should be 360° L., we *add* 10″ to each L.H. deflection angle and *subtract* 10″ from each R.H. deflection angle. The total is now 360° L. Then, starting from the given bearing of *AB* we successively deduct L.H. deflection angles and add R.H. ones, checking back on to the bearing of *AB*. Then we convert these whole circle bearings into reduced bearings in the usual way. The latitudes and departures are then calculated, using six-figure logarithms. The closing error is, therefore, + 0·98 in latitude, + 4·95 in departure, or 5·05 feet in actual magnitude. The corrections of latitude and departure shown have been worked out by Bowditch's Method, though this is quite unsuitable here, as the bearing of any side nearly perpendicular to the direction of the closing error will be altered by $\frac{5·05}{4620} \times 3438' = 3·76'$, which is obviously wrong when the angles check so closely. The corrections are worked out on the slide rule, viz. $-\frac{0·98}{4620}$ × length of side for latitude, $-\frac{4·95}{4620}$ × length of side for departure. The corrected latitudes and departures being

Stn.	Lengths l (ft.)	Angles	Deflection Angles	Corrected Deflection Angles	Whole Circle Bearings	Reduced Bearings (θ)	Latitude $l \cos \theta$	Departure $l \sin \theta$	Corrected Latitude	Corrected Departure	Co-ordinates North	Co-ordinates East	Stn.
A	701·4	96° 14′ 00″	83° 46′ 00″ L.	83° 46′ 10″ L.	149° 13′ 00″	S. 30° 47′ 00″ E.	− 602·58 − 0·15	+ 358·96 − 0·75	− 602·73	+ 358·21	1000·00	0·00	*A*
B	249·2	105° 17′ 30″	74° 42′ 30″ L.	74° 42′ 40″ L.	74° 30′ 20″	N. 74° 30′ 20″ E.	+ 66·57 − 0·05	+ 240·14 − 0·27	+ 66·52	+ 239·87	397·27	358·21	*B*
C	309·6	124° 22′ 00″	55° 38′ 00″ L.	55° 38′ 10″ L.	18° 52′ 10″	N. 18° 52′ 10″ E.	+ 292·96 − 0·07	+ 100·13 − 0·33	+ 292·89	+ 99·80	463·79	598·08	*C*
D	1092·8	249° 05′ 40″	69° 05′ 40″ R.	69° 05′ 30″ R.	87° 57′ 40″	N. 87° 57′ 40″ E.	+ 38·88 − 0·23	+ 1092·11 − 1·17	+ 38·65	+ 1090·94	756·68	697·88	*D*
E	1278·5	40° 47′ 30″	139° 12′ 30″ L.	139° 12′ 40″ L.	308° 45′ 00″	N. 51° 15′ 00″ W.	+ 800·24 − 0·27	− 997·08 − 1·37	+ 799·97	− 998·45	795·33	1788·82	*E*
F	988·5	104° 14′ 20″	75° 45′ 40″ L.	75° 45′ 50″ L.	232° 59′ 10″	S. 52° 59′ 10″ W.	− 595·09 − 0·21	− 789·31 − 1·06	− 595·30	− 790·37	1595·30	790·37	*F*
A		—									1000·00	0·00	*A*
Total	4620·0	—	359° 59′ 00″ L.	360° 00′ 00″ L.	—		+ 0·98	+ 4·95	0·00	0·00	—	—	—

found, the co-ordinates are calculated and check back to those of A.

The side AB is most nearly perpendicular to the closing error. After adjusting the latitudes and departures, its inclination to the South Meridian is $\tan^{-1}\dfrac{358{\cdot}21}{602{\cdot}73} = 30° \, 43' \, 25''$ so that the bearing of this line has been altered

$$30° \, 47' \, 00'' - 30° \, 43' \, 25'' = 3' \, 35''$$

or nearly the maximum $3{\cdot}76'$ possible.

We shall now work out the corrections, *keeping the bearings unaltered*, as they should be, by Method (*b*). See page 313.

The equations for λ and μ are, therefore,

$$\begin{cases} 1673{\cdot}2\,\lambda - 259{\cdot}6\,\mu = -0{\cdot}98 \\ -259{\cdot}6\,\lambda + 2946{\cdot}4\,\mu = -4{\cdot}95 \end{cases}$$

from which we find $\lambda = -0{\cdot}000858$, $\mu = -0{\cdot}001756$. So far the calculations must *not* be done on the slide rule, as small differences may be involved. The remaining columns, however, can be calculated on the slide rule. It will be noticed how very different these corrections are from those found by Bowditch's Method.

We shall now work out the corrections by Method (*c*), *keeping the bearings unaltered* again—

Factor	Latitude	Departure	Correction of Latitude	Correction of Departure
y	− 602·58	+ 358·96	+ 0·37	− 0·22
x	+ 66·57	+ 240·14	− 0·12	− 0·45
x	+ 292·96	+ 100·13	− 0·54	− 0·19
x	+ 38·88	+ 1,092·11	0·07	− 2·02
y	+ 800·24	997·08	+ 0·49	0·61
x	− 595·09	− 789·31	1·10	1·46
		Totals .	− 0·97	− 4·95

Side	Latitude	Departure	$\frac{\text{Lat.}^2}{\text{Side}}$	$\frac{\text{Lat.} \times \text{Dep}}{\text{Side}}$	$\frac{\text{Dep.}^2}{\text{Side}}$	$\lambda\frac{l^2}{s}$	$\mu\frac{ld}{s}$	Correction of Latitude	$\lambda\frac{ld}{s}$	$\mu\frac{d^2}{s}$	Correction of Departure
701·4	- 602·58	+ 358·96	517·7	- 308·4	183·8	- ·444	+ ·541	+ 0·10	+ ·264	- ·323	- 0·06
249·2	- 66·57	- 240·14	17·8	+ 64·1	231·4	- ·015	- ·113	- 0·13	- ·055	- ·406	- 0·46
309·6	292·96	- 100·13	277·2	- 94·8	32·4	- ·237	- ·166	- 0·40	- ·081	- ·057	- 0·14
1092·8	- 38·88	+ 1092·11	1·4	- 38·9	1091·0	·001	- ·068	- 0·07	- ·033	- 1·916	- 1·95
1278·5	- 800·24	- 997·08	500·9	- 624·1	777·6	- ·430	+ 1·094	+ 0·66	+ ·535	- 1·365	- 0·83
988·5	- 595·09	- 789·31	358·2	+ 475·1	630·2	- ·307	- ·835	- 1·14	- ·407	- 1·106	- 1·51
		Totals	1673·2	- 259·6	2946·4			- 0·98			- 4·95

We first choose the factors, x and y, for each side, then we insert the signs of the corrections as already explained. As the greatest total correction is in departure and is –, we make the signs of all the departure corrections –, and then insert the signs of the latitude corrections to be consistent with them. We then write down the simultaneous equations for x and y, thus—

$$\left.\begin{aligned} 603y - 67x - 293x - 39x + 800y - 595x &= -0{\cdot}98 \\ -359y - 240x - 100x - 1092x - 997y - 789x &= -4{\cdot}95 \end{aligned}\right\}$$

$$\text{or}\quad \left\{\begin{aligned} 994x - 1403y &= 0{\cdot}98 \\ 2221x + 1356y &= 4{\cdot}95 \end{aligned}\right.$$

whence $x = 0{\cdot}001853$, $y = 0{\cdot}000615$. The corrections are then calculated on the slide rule. Comparing the results with those given by Method (*b*) we find no serious disagreement in this case, the greatest discrepancies being on lines *AB* and *EF*, which are most nearly perpendicular to the closing error.

EXAMPLE 8. The co-ordinates of two triangulation stations are as follows—

	North (ft.)	East (ft.)
A . .	2464·2	1242·7
B . .	5463·8	5243·05

A traverse is run from *A* to *B* with sub-stations at *X* and *Y*. The distances *AX*, *XY*, *YB* are 1010·5, 2789·6, and 1849·8 feet respectively, and the angles at *A*, *X*, *Y*, *B* are 28° 22′ 20″, 166° 29′ 00″, 124° 47′ 50″ and 40° 20′ 50″ respectively (clockwise from the rear station). Calculate the co-ordinates of *X* and *Y*.

We first find the azimuth of AB. It is

$$\tan^{-1}\frac{4000\cdot35}{2999\cdot6} = 53^\circ\ 08'\ 10''$$

Therefore, azimuth of $BA = 233^\circ\ 08'\ 10''$. See page 316.

Here again the corrections have been calculated by Bowditch's Rule, which is quite inapplicable here.

To find the corrections exactly by Method (*b*), see page 317.

Here, again, we see how very different the corrections are from those given by Bowditch's Rule.

To find the corrections by the simpler approximate Method (*c*), we note that all the three traverse lines lie in the first quadrant; we must, therefore, select x and y *in quadrants* from *the direction of the given line AB*. The factors for AX, XY, YB are, therefore, x, x, y respectively, and we have

$$\begin{cases} 1195x + 1804y = 1\cdot09 \\ 3586x + 409\cdot5y = 5\cdot14 \end{cases}$$

$\therefore\ x = 0\cdot001476,\ y = -0\cdot0003735$, giving for—

	Latitude Correction	Departure Correction
AX . .	+ 0·22	+ 1·47
XY . .	+ 1·54	+ 3·82
YB . .	− 0·67	− 0·15
	+ 1·09	+ 5·14

Here, again, there is no serious discrepancy from the exact results given by Method (*b*).

Interconnected Traverses. When a survey consists of several closed traverses with some of the sides common to two or more polygons, it is usual to adjust first the polygon with the largest closing error, then that with the next

Stn.	Lengths l (ft.)	Angles	Deflection Angles	Whole Circle Bearings	Reduced Bearings (θ)	Latitude $l \cos\theta$	Departure $l \sin\theta$	Corrected Latitude	Corrected Departure	Co-ordinates North	Co-ordinates East	Stn.
B		—								5463·80	5243·05	*B*
	—			233° 08′ 10″	—	− 2999·6	− 4000·35	− 2999·6	− 4000·35			
A		28° 22′ 20″	151° 37′ 40″ L.							2464·20	1242·70	*A*
	1010·5			81° 30′ 30″	N. 81° 30′ 00″ E.	+ 149·22 + 0·19	+ 999·42 + 0·92	+ 149·41	+ 1000·34			
X		166° 29′ 00″	13° 31′ 00″ L.							2613·61	2243·04	*X*
	2789·6			67° 59′ 30″	N. 67° 59′ 30″ E.	+ 1045·38 + 0·54	+ 2586·32 + 2·54	+ 1045·92	+ 2588·86			
Y		124° 47′ 50″	55° 12′ 10″ L.							3659·53	4831·90	*Y*
	1849·8			12° 47′ 20″	N. 12° 47′ 20″ E.	+ 1803·91 + 0·36	+ 409·47 + 1·68	+ 1804·27	+ 411·15			
B		40° 20′ 50″	139° 39′ 10″ L.							5463·80	5243·05	*B*
Total	5649·9	360° 00′ 00″	360° 00′ 00″ L.			− 1·09	− 5·14					

s	l	d	$\frac{l^2}{s}$	$\frac{ld}{s}$	$\frac{d^2}{s}$	$\lambda\frac{l^2}{s}$	$\mu\frac{ld}{s}$	Correction of Latitude	$\lambda\frac{ld}{s}$	$\mu\frac{d^2}{s}$	Correction of Departure
1010·5	+ 149·22	+ 999·42	+ 22·0	+ 147·6	+ 988·4	− ·017	+ ·267	+ 0·25	− ·112	− 1·789	− 1·68
2739·6	+ 1045·38	+ 2586·32	+ 391·7	+ 969·2	+ 2397·9	− ·298	+ 1·754	− 1·46	− ·737	− 4·340	− 3·60
1849·8	+ 1803·91	+ 409·47	+ 1759·1	+ 399·3	+ 90·6	− 1·338	+ ·722	− 0·62	− ·303	− 0·164	− 0·14
		Totals	2172·8	+ 1516·1	3476·9			+ 1·09			− 5·14

$$\begin{cases} 2173\lambda + 1516\mu = 1{\cdot}09 & \text{whence } \lambda = -0{\cdot}0007608 \\ 1516\lambda + 3477\mu = 5{\cdot}14 & \mu = 0{\cdot}001810 \end{cases}$$

largest closing error, and so on. When the latitude and departure of a common side have once been corrected, they must not be altered but that side must be taken as a "known side" as in the last Example, and the error distributed among the remaining sides of the polygon in question. This is, of course, an empirical method, and the Method of Least Squares can be applied if greater accuracy is required. The student will find it a useful exercise to take the case of two quadrilaterals whose sides are (1), (2), (3), (4): (4), (5), (6), (7) respectively, with closing errors E_1, E_2 in latitude and F_1, F_2 in departure respectively and to deduce the normal equations, (*a*) when the bearings may be altered, and (*b*) when the bearings are correct.

CHAPTER VIII

SETTING OUT

Transition Curves—Compound Curves—Vertical Curves—Tunnels

Transition Curves

When a road or railway is to be suitable for traffic at any but the lowest speeds a "Transition Curve" should be introduced whenever the centre line changes from a straight line to a circular arc. In order to prevent skidding on the road, or undue flange pressure on the outer rail on a curve, the road should be "banked" or given an inward slope, and the outer rail should be "superelevated" or "canted" above the inner one, by an amount depending on the radius of the curve and the speed of traffic.

If m is the mass and v the velocity of the vehicle, ρ the radius of the centre line, and α the inward inclination of the road (or surface of the rails) (Fig. 1), in order to eliminate skidding or flange pressure, the resultant of the weight of the vehicle and of the reaction perpendicular to the road or rail surface must be horizontal, inwards, and of the value $m \dfrac{v^2}{\rho}$.

Therefore $$\tan \alpha = \frac{mv^2}{\rho \, . \, mg} = \frac{v^2}{g\rho} \quad . \quad . \quad . \quad (1)$$

If b is the breadth of the road (or distance between the centre of the rails), the superelevation $e = b \sin \alpha$. If α is small, we have

$$e = \frac{bv^2}{g\rho} \text{ nearly} \quad . \quad . \quad . \quad (2)$$

This superelevation should be applied gradually, so that e is proportional to the distance s from the commencement of the transition curve, or that $s \propto e \propto \frac{1}{\rho}$. We have thus to find such a form of spiral curve that its *radius of curvature*

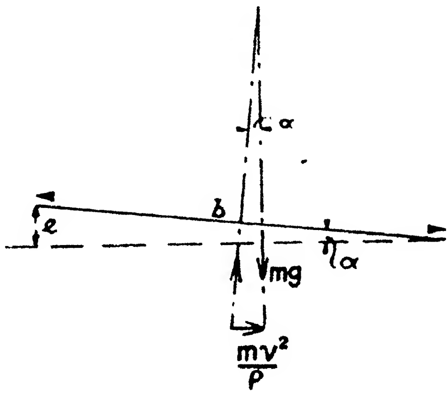


FIG. 1

ρ at any point is inversely proportional to the distance s from the commencement of the curve.

The gauge of a railway line has also to be widened on a curve in order to allow the bogies of the vehicles, with their parallel axles, to travel round it, and this widening, which increases as the radius decreases, can be gradually applied along the transition curve. A transition curve also reduces the shock which would be felt when the path suddenly changed from a straight line to a circle, due to the inward radial acceleration $\frac{v^2}{\rho}$ required on the latter.

Length of Transition Curve. Let a' be the time *rate* at which the radial acceleration $\frac{v^2}{r}$ on the circular curve of radius r may be applied. Then, if l is the length of the transition curve, we have

$$a' = \frac{\frac{v^2}{r}}{\frac{l}{v}} = \frac{v^3}{rl}$$

as the time taken to traverse the curve is $\frac{l}{v}$.

We have, therefore,

$$l = \frac{v^3}{ra'} \quad . \quad . \quad . \quad . \quad (3)$$

i.e. *the length varies inversely as the radius of the circular curve if the velocity is independent of the radius.*

On sharp radius curves, however, the banking angle, a, would be too great for the safety of vehicles at rest on the curve, especially with a strong inward wind, so that on sharp curves it is usual to limit the superelevation e, or banking angle, a, and reduce the speed accordingly. From formula (1) we have $v = \sqrt{gr \tan a}$ as the appropriate speed for a banking angle a, on a curve of radius r. Inserting this in formula (3), we have

$$l = \frac{(gr \tan a)^{\frac{3}{2}}}{ra'} = (g \tan a)^{\frac{3}{2}} \frac{\sqrt{r}}{a'} \quad . \quad . \quad (4)$$

i.e. *when the superelevation or banking is limited, the length of the transition curves varies directly as*

$$\sqrt{\textit{radius of the circular curve}}.$$

If the maximum cant allowed is 6 inches on a standard railway gauge (4′ 8½″), say 4′ 11½″ between rail centres, we have $\sin a = \frac{6''}{59.5} = 0{\cdot}1008$, so that

$$a = 5^\circ\ 47' \text{ and } \tan a = 0{\cdot}1013.$$

The maximum speed for the curve should then be

$\sqrt{32{\cdot}2 \times {\cdot}1013}\,\sqrt{r} = 1{\cdot}806\,\sqrt{r}$ ft. per sec., but we can safely increase this to $2\sqrt{r}$ by allowing a small amount of flange pressure on the outer rail. If a' be taken as 1 ft. per sec.² per sec., we have $l = \dfrac{v^3}{ra'} = 8\sqrt{r}$, where l and r are in feet. Writing the length as L and the radius as R in *Gunter's Chains of* 66 feet, we have $66L = 8\sqrt{66R} = 65\sqrt{R}$, or, approximately,

$$L = \sqrt{R} \quad . \quad . \quad . \quad . \quad (5)$$

a common rule.*

As the radius increases, the allowable velocity, $\sqrt{gr \tan a}$, increases until when $\sqrt{gr \tan a} = v_1$, the maximum velocity allowed on the straight, the velocity again becomes independent of the radius, and the length of the transition curve $= \dfrac{v_1^3}{ra'}$, or varies inversely as the radius for curves of greater radius than $r = \dfrac{v_1^2}{g \tan a}$.

Approximate Treatment. For most purposes it is sufficiently accurate to use the *Cubic Parabola*, $y = Cx^3$, as the transition curve (Fig. 2). Differentiating, we obtain

$$\frac{dy}{dx} = 3Cx^2, \quad \frac{d^2y}{dx^2} = 6Cx$$

Now ρ, the radius of curvature, at any point P

$$= \frac{\left(1 + \left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$$

therefore, if $\dfrac{dy}{dx}$ is small, $6Cx = \dfrac{1}{\rho}$.

* Shortt, "On a Practical Method for the Improvement of Existing Railway Curves," *Proc. Inst. C.E.*, Vol. 176.
If $v = 1{\cdot}806\sqrt{r}$, $l = 5{\cdot}9\sqrt{r}$.

Also, when $\frac{dy}{dx}$ is small, $x = s$, therefore, $s = \frac{1}{6C\rho}$.

So that, *when* $\frac{dy}{dx}$ *is small,* the radius of curvature is inversely proportional to the distance s, from the origin, or

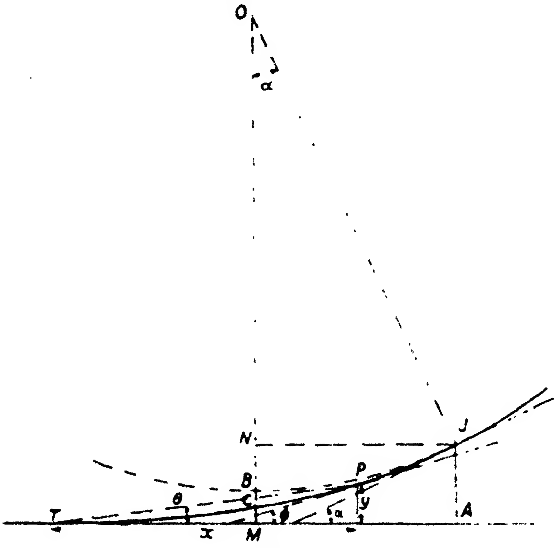


FIG. 2

tangent point, T, which satisfies the condition required for a transition curve.

When $s = l$, at the "junction point" J, where the circular curve begins, we have $\rho = r$.

$$\therefore \quad l = \frac{1}{6Cr}. \qquad \therefore \quad C = \frac{1}{6lr}.$$

so that the equation to the curve is

$$y = \frac{x^3}{6lr} \quad . \quad . \quad . \quad . \quad (6)$$

Let ϕ = the inclination of the tangent at P to the x-axis or straight, α = the "spiral angle," i.e. the value of ϕ at the junction point J, and θ = the "tangential angle" at P = angle PTA, then we have $\frac{dy}{dx} = \tan\phi = 3Cx^2 = \frac{x^2}{2lr}$; $\tan\alpha = \frac{l^2}{2lr} = \frac{l}{2r}$, and as $\frac{dy}{dx}$ is assumed to be small, we may write

$$\phi = \frac{x^2}{2lr};\ \alpha = \frac{l}{2r} \quad . \quad . \quad . \quad (7)$$

Also, $\tan\theta = \frac{y}{x} = \frac{x^2}{6lr} = \frac{\tan\phi}{3}$, or as $\frac{dy}{dx}$ is small,

$$\theta = \frac{\phi}{3} = \frac{x^2}{6lr} \quad . \quad . \quad . \quad (8)$$

It will be noted that the tangential angle θ increases as the *square* of the distance from the tangent point and the offset y as the *cube* of that distance, whereas in a circular curve the tangential angle increases proportionately to the distance and the offset (approximately) as the square of the distance.

The circular curve does not touch the tangent but is moved inwards by an amount $BM = c$, called the "shift." We have

$$BJ = \alpha r = \frac{lr}{2r} = \frac{l}{2}. \quad \text{But } CJ = BJ \text{ (very nearly)} = \frac{l}{2}$$

Therefore, *the shift BM bisects the transition curve*, very approximately.

$$\text{Also, } c = BM = JA - BN = \frac{l^3}{6lr} - r(1 - \cos\alpha)$$

$$= \frac{l^2}{6r} - 2r\sin^2\frac{\alpha}{2} = \frac{l^2}{6r} - 2r\frac{\alpha^2}{4}$$

$$= \frac{l^2}{6r} - \frac{r}{2}\cdot\frac{l^2}{4r^2} = \frac{l^2}{6r} - \frac{l^2}{8r}$$

$$= \frac{l^2}{24r} = \text{shift} \quad . \quad . \quad . \quad . \quad (9)$$

$$\text{and as } CT = \frac{l}{2};\ CM = \frac{\left(\frac{l}{2}\right)^3}{6lr} = \frac{l^2}{48r} = \frac{BM}{2}$$

Therefore, the *shift is bisected by the transition curve.**

To Find the Distance of the Transition Curve from the Circular Curve at any Point. Let y_1 = ordinate to the transition curve, y_2 = ordinate to the circular curve, and x' = distance of the point from A.

Then

$$y_1 = \frac{x^3}{6lr} = \frac{(l - x')^3}{6lr} = \frac{l^3 - 3l^2x' + 3lx'^2 - x'^3}{6lr}$$

$$= \frac{l^2}{6r} - \frac{lx'}{2r} + \frac{x'^2}{2r} - \frac{x'^3}{6lr}$$

$$y_2 = \frac{\left(\frac{l}{2} - x'\right)^2}{2r} + \frac{l^2}{24r} = \frac{l^2}{8r} - \frac{lx'}{2r} + \frac{x'^2}{2r} + \frac{l^2}{24r}$$

$$= \frac{l^2}{6r} - \frac{lx'}{2r} + \frac{x'^2}{2r}$$

Therefore, distance between curves $= y_2 - y_1 = \frac{x'^3}{6lr}$ and, therefore, is proportional to (distance from junction point)³.

If required, therefore, the transition curve could be set out by measuring these distances from the circular curve.

At the quarter points x, or x', $= \frac{l}{4}$

$$\text{and } y, \text{ or } y_2 - y_1 = \frac{l^3}{64 \times 6rl} = \frac{l^2}{16 \times 24r} = \frac{c}{16}$$

so that, in the first quarter of its length, the transition curve

* Glover on "Transition Curves for Railways," *Proc. Inst.C E.*, Vol. 140.

only deviates $\frac{c}{16}$ from the tangent, and in the last quarter of its length $\frac{c}{16}$ from the circular curve.

If the length of the curve (in chains) $= \sqrt{\text{radius (in chains)}}$

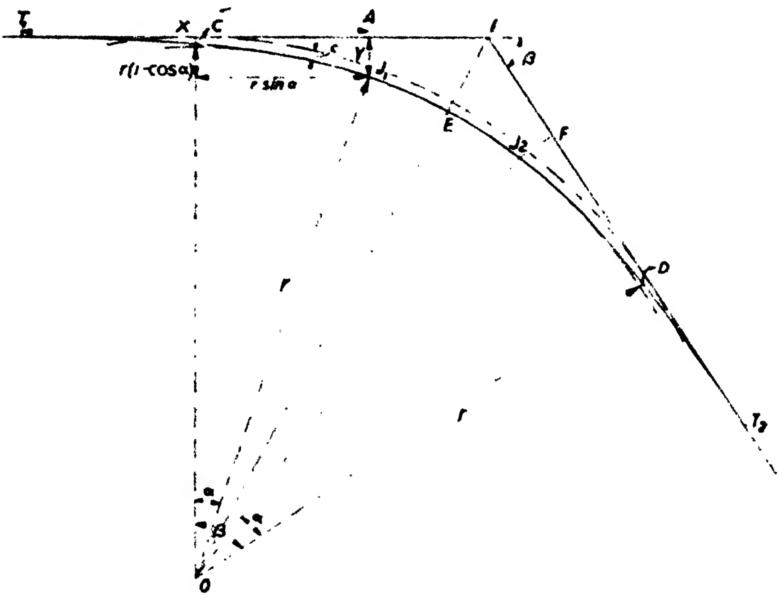


FIG. 3

the shift $c = \frac{l^2}{24r} = \frac{1}{24}$ chain $= 4{\cdot}166$ links. The deviations from the tangent and the circular curve at the quarter and three-quarter points respectively are, therefore, only 0·26 link = 2·06 inches.

The length of the tangent of the combined curve will now be (Fig. 3),

$$T_1I = T_2I = \frac{l}{2} + (r + c) \tan \frac{\beta}{2} \tag{10}$$

i.e. it is increased, by the insertion of the transition curve, by the amount $\frac{l}{2} + c \tan \frac{\beta}{2}$.

The length of the combined curve will be

$$CD + 2T_1C = r \operatorname{arc} \beta + l \quad . \quad . \quad (11)$$

The "external secant,"

$$IE = (r + c) \sec \frac{\beta}{2} - r \quad . \quad . \quad . \quad (12)$$

This last enables the middle point of the curve to be set out from the intersection point of the straights by bisecting the internal angle T_1IT_2.

EXAMPLE 1. (L.U.) A road 30 ft. wide is to turn through an angle of 26° 24′ with a centre line radius of 600 ft., the (forward) chainage of the intersection point being 3640·6 ft. A transition curve is to be used at each end of the circular curve, of such a length that the rate of gain of radial acceleration is 1 ft./sec.³ when the speed is 30 m.p.h. Find the length of the transition curve, the banking of the road for this speed, the chainage of the beginning of the combined curve, and the angle to turn off there for the peg at 3,500 ft.

30 m.p.h. = 44 f.p.s. Rate of gain of radial acceleration

$$= \frac{v^3}{lr} = 1. \quad \therefore\ l = \frac{44^3}{600} = 142 \text{ ft.}$$

$$\tan \alpha = \frac{v^2}{gr} = \frac{44 \times 44}{32{\cdot}2 \times 600} = 0{\cdot}1002. \quad \therefore\ \alpha = 5^\circ\ 43'$$

$$\therefore\ \text{banking} = 30 \sin \alpha = 2{\cdot}99 \text{ ft.}$$

$$\text{Shift, } c = \frac{l^2}{24r} = \frac{142 \times 142}{24 \times 600} = 1{\cdot}40 \text{ ft.}$$

$$\therefore\ \text{tangent} = (600 + 1{\cdot}40) \tan 13^\circ\ 12' + 71{\cdot}0$$

$$= 140{\cdot}7 + 0{\cdot}33 + 71{\cdot}0 = 212{\cdot}0 \text{ ft.}$$

$\therefore$ chainage of $TP_1 = 3640 \cdot 6 - 212 \cdot 0 = 3428 \cdot 6$ ft.

$\therefore$ tangential angle for 3,500 ft. peg $= \dfrac{x^2}{6rl}$

$$= \frac{(71 \cdot 4)^2}{6 \times 600 \times 142} \times 3438' = 0^\circ\ 34 \cdot 3'$$

N.B. It is interesting to calculate the offsets to the transition curve at $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ of its length and at the JP.

These will be $(\frac{1}{4})^3$, $(\frac{1}{2})^3$, $(\frac{3}{4})^3$ of the offset at J, which latter is $\dfrac{l^2}{6r} = 4 \times$ shift $= 5 \cdot 60$ ft.

The offsets are, therefore, 0·09 ft., 0·70 ft., 2·36 ft., and 5·60 ft. at 35·5, 71, 106·5, and 142 ft. from TP_1.

The offset to the circular curve at 106·5 ft. from TP_1

$$1 \cdot 40 \text{ ft.} + \frac{35 \cdot 5^2}{2 \times 600} = 2 \cdot 45 \text{ ft.}$$

Therefore, in the first quarter of its length the transition curve only deviates 0·09 ft. from the tangent, and in the last quarter of its length it only deviates 0·09 ft. from the circular curve.

Example 2 (L.U.). A circular curve, 600 ft. radius, touches a straight line at a chainage of 8576·6 ft., deflects through an angle of 67° 24′ to the right, and touches another straight line. This curve is to be shifted, maintaining the same radius, so as to admit a transition curve 200 ft. long at each end. Calculate the chainages of the beginning (A) and end (B) of the combined curve, the maximum radial displacement of the circular curve, and the angles to be turned off at A from the first straight for the 8,500 ft. peg, and at B from the second straight for the 9,200 ft. peg.

N.B. As 600 ft. and 200 ft. are nearly 9 and 3 Gunter's chains respectively, the length nearly conforms to formula (5).

$$\text{Shift} = \frac{l^2}{24r} = \frac{200 \times 200}{24 \times 600} = 2{\cdot}78 \text{ ft.} = c$$

Increase in length of tangent $= 2{\cdot}78 \tan 33° 42' + 100{\cdot}0$
$= 101{\cdot}85$ ft.

$\therefore$ chainage of $A = 8576{\cdot}6 - 101{\cdot}85 = 8474{\cdot}75$ ft.

Length of combined curve $= 600 \text{ arc } 67° 24' + 200$
$= 905{\cdot}81$ ft.

$\therefore$ chainage of $B = 8474{\cdot}75 + 905{\cdot}81 = 9380{\cdot}56$ ft.

The maximum radial displacement of the circular curve will be at its middle point and will be

$$c \sec \frac{\beta}{2} = 2{\cdot}78 \sec 33° 42' = 3{\cdot}34 \text{ ft.}$$

The tangential angle at A for 8,500 ft. peg

$$= \frac{x^2}{6lr} = \frac{(25{\cdot}25)^2}{6 \times 200 \times 600} \times 3438' = 0° 3{\cdot}04'$$

The tangential angle at B for 9,200 ft. peg

$$= \frac{x^2}{6lr} = \frac{(180{\cdot}56)^2}{6 \times 200 \times 600} \times 3438' = 155{\cdot}7' = 2° 35{\cdot}7'$$

EXAMPLE 3 (L.U.). If the circular curve in Example 2 is not shifted but the radius is sharpened to 540 ft. at its ends, so as to admit transition curves of length $= \sqrt{\text{radius}}$ (in Gunter's chains), find the chainages of the beginning of the combined curve and of the beginning and end of the first curve of 540 ft. radius (Fig. 4).

Length of transition curve in feet

$$= 66\sqrt{\frac{540}{66}} = \sqrt{66 \times 540} = 188{\cdot}8 \text{ ft.}$$

$$\therefore \text{ shift} = \frac{188{\cdot}8 \times 188{\cdot}8}{24 \times 540} = 2{\cdot}75 \text{ ft.}$$

From the figure we see that $OP = 60$ ft.,

$$OQ = 600 - (540 + 2{\cdot}75) = 57{\cdot}25 \text{ ft}$$

$$\therefore \quad \cos POQ = \frac{57{\cdot}25}{60} = 0{\cdot}9542. \quad \therefore \ POQ = 17^\circ\, 24'$$

$$\therefore \quad PQ = 60 \sin 17^\circ\, 24' = 17{\cdot}94 \text{ ft.}$$

$\therefore$ chainage of N or $C = 8576{\cdot}6 + 17{\cdot}94$ ft. $= 8594{\cdot}54$ ft.

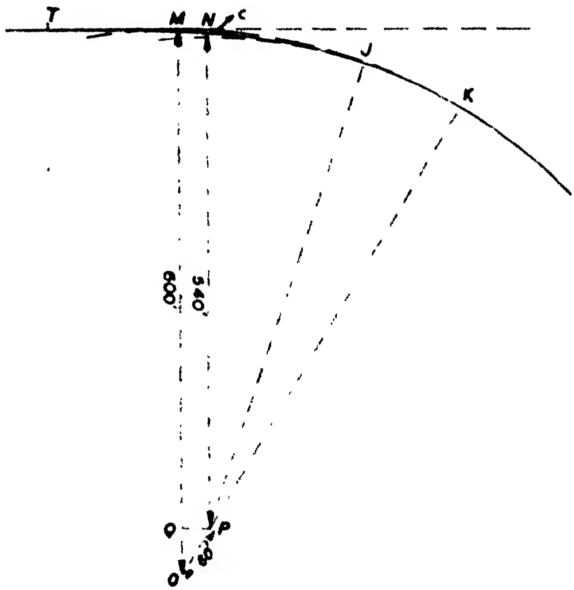


FIG. 4

This is the "shift point" or middle of the transition curve.

$\therefore$ chainage of new tangent point, $T = 8594{\cdot}54 - 94{\cdot}4$
$= 8500{\cdot}14$ ft.

and chainage of the junction point J with the transition curve
$= 8594{\cdot}54 + 94{\cdot}4 = 8688{\cdot}94$ ft.

The length $CK = 540$ arc $17^\circ\, 24' = 164{\cdot}00$ ft.

$\therefore$ chainage of $K = 8594{\cdot}54 + 164{\cdot}00 = 8758{\cdot}54$ ft.

This procedure might be necessary if buildings, etc., prevented inward movement of the centre line on the middle portion of the curve, or outward movement of the adjoining straights, for the transition curve.

Setting Out the Combined Curve. After measuring the deflection angle β at the intersection I of the straights, the tangents would be chained and the tangent points T_1, T_2 fixed.

When chaining these it is well to leave marks at A and F (Fig. 3) on the tangents, opposite to the junction points J_1 J_2, i.e. at distances l from T_1 and T_2. Then the theodolite is set up at T_1, set to zero on I, and the first transition curve is set out by tangential angles and lengths along the curve. When the junction point J_1 is reached the tangential angle will be $\frac{a}{3}$ and the total length of arc l. The offset, $J_1A = \frac{l^2}{6r}$, can then be measured as a check on the position of J_1. The second transition curve is then set out from T_2, and the point J_2 fixed and checked. The theodolite is then set up at J_1 and sighted on T_1, reading $\frac{a}{3}$. The zero line of the horizontal circle is now parallel to IT_1. If the telescope were now transitted and the horizontal circle set to read a, the telescope would be pointing along the tangent at J_1. Actually, the circular curve is set out from J_1 by *adding* a to the tangential angles $\frac{s}{2r}$ for each point, s being the distance from J_1, and the setting out is checked on to the peg at J_2.

Exact Treatment (*Glover's Spiral*).* Starting from the

* Glover on "Transition Curves for Railways," *Proc. Inst. C. E.*, Vol. 140.

fundamental condition $\frac{1}{\rho} = ks$, we have also for all curves $ds = \rho d\phi$. $\therefore$ $d\phi = ksds$. Integrating, $\phi = \frac{ks^2}{2}$, there being no additive constant, as $\phi = 0$, when $s = 0$. At the junction point $\rho = r$, $s = l$,

$$\therefore \quad \frac{1}{r} = kl. \quad \therefore k = \frac{1}{lr} \quad \therefore \phi = \frac{s^2}{2lr} \quad . \quad . \quad (13)$$

Also at the junction point, $\phi = \alpha$,

$$\therefore \quad \alpha = \frac{l^2}{2lr} = \frac{l}{2r} \quad . \quad . \quad . \quad (14)$$

To transform to rectangular co-ordinates, we have

$$s = \sqrt{2lr\phi}\,;\; ds = \sqrt{\frac{lr}{2\phi}}\,.\,d\phi$$

$$dx = ds\,.\cos\phi$$

$$= \sqrt{\frac{lr}{2\phi}}\left(1 - \frac{\phi^2}{2} + \frac{\phi^4}{24}\right)d\phi$$

$$= \sqrt{\frac{lr}{2}}\left(\phi^{-\frac{1}{2}} - \frac{\phi^{\frac{3}{2}}}{2} + \frac{\phi^{\frac{7}{2}}}{24}\right)d\phi$$

$$\therefore x = \sqrt{\frac{lr}{2}}\left(2\phi^{\frac{1}{2}} - \frac{\phi^{\frac{5}{2}}}{5} + \frac{\phi^{\frac{9}{2}}}{108}\right)$$

$$= \sqrt{2rl\phi}\left(1 - \frac{\phi^2}{10} + \frac{\phi^4}{216}\right)$$

$$= s\left(1 - \frac{\phi^2}{10} + \frac{\phi^4}{216}\right)$$

$$dy = ds\,.\sin\phi$$

$$= \sqrt{\frac{lr}{2\phi}}\,.\left(\phi - \frac{\phi^3}{6} + \frac{\phi^5}{120}\right)d\phi$$

$$\sqrt{\frac{lr}{2}}\left(\phi^{\frac{1}{2}} - \frac{\phi^{\frac{5}{2}}}{6} + \frac{\phi^{\frac{9}{2}}}{120} - \right) d\phi$$

$$\therefore y = \sqrt{\frac{lr}{2}}\left(\frac{2}{3}\phi^{\frac{3}{2}} - \frac{\phi^{\frac{7}{2}}}{21} + \frac{\phi^{\frac{11}{2}}}{660} - \right)$$

$$= \sqrt{2lr\phi}\,\frac{\phi}{3}\left(1 - \frac{\phi^2}{14} + \frac{\phi^4}{440} - \right)$$

$$= \frac{s\phi}{3}\left(1 - \frac{\phi^2}{14} + \frac{\phi^4}{440} - \right)$$

$$= \frac{s^3}{6lr}\left(1 - \frac{\phi^2}{14} + \frac{\phi^4}{440} - \right) \quad . \quad . \quad . \quad . \quad (15)$$

If we take the first terms only of these expansions, we have

$$x = s, \quad y = \frac{s^3}{6lr} = \frac{x^3}{6lr}$$

i.e. we have the cubic parabola as a first approximation, as in equation (6). If we take the first two terms we have

$x = s\left(1 - \dfrac{\phi^2}{10}\right)$, $y = \dfrac{s^3}{6lr}\left(1 - \dfrac{\phi^2}{14}\right)$ with ample accuracy.

At the junction point (Fig. 3) let $x = X$, $y = Y$, while $s = l$, $\phi = \alpha$ and we have

$$X = l\left(1 - \frac{\alpha^2}{10}\right), \quad Y = \frac{l^2}{6r}\left(1 - \frac{\alpha^2}{14}\right) \qquad (16)$$

for the co-ordinates of the junction point.

The shift, c, $= Y - r(1 - \cos \alpha)$

The tangent $= (r + c) \tan \dfrac{\beta}{2} + X - r \sin \alpha$

The length of the combined curve $= r \text{ arc } (\beta - 2\alpha) + 2l$

The external secant $= (r + c) \sec \dfrac{\beta}{2} - r$ . . . (17)

$$\text{Also,}\quad \tan\theta = \frac{y}{x} = \frac{s\dfrac{\phi}{3}\left(1-\dfrac{\phi^2}{14}+\dfrac{\phi^4}{440}-\right)}{s\left(1-\dfrac{\phi^2}{10}+\dfrac{\phi^4}{216}-\right)}$$

$$= \frac{\phi}{3}\left(1+\frac{\phi}{35}+\frac{26\phi^4}{51975}+\right)$$

$$= \frac{\phi}{3}+\frac{\phi^3}{105}+\frac{26\phi^5}{155925}+$$

$$\text{but}\quad \tan\frac{\phi}{3} = \frac{\phi}{3}+\frac{1}{3}\left(\frac{\phi}{3}\right)^3+\frac{2}{15}\left(\frac{\phi}{3}\right)^5+\ldots$$

$$= \frac{\phi}{3}+\frac{\phi^3}{81}+\frac{2\phi^5}{3645}+\ldots$$

which is practically the same.

Therefore, as ϕ is generally $\not> 0{\cdot}2$,

$$\theta = \frac{\phi}{3} = \frac{s^2}{6lr} \qquad \ldots \qquad (18)$$

to a very close approximation.

To illustrate the closeness of approximation of a cubic parabola to the true spiral, we will take Example 2 (above) where $l = 200$ ft., $r = 600$ ft., and $\beta = 67°\,24'$.

$$\text{The spiral angle } \alpha = \frac{l}{2r} = \frac{200}{1200} = \frac{1}{6} = \frac{1}{6}\times 3438'$$

$$= 573' = 9°\,33'$$

$$X = 200\left(1-\frac{1}{360}\right) = 199{\cdot}445 \text{ ft.}$$

$$Y = \frac{200\times 200}{6\times 600}\left(1-\frac{1}{36\times 14}\right)$$

$$= \frac{100}{9}\left(1-\frac{1}{504}\right) = 11{\cdot}09 \text{ ft.}$$

The shift, $c = 11{\cdot}09 - 600\,(1 - \cos 9^\circ\,33') = 11{\cdot}09 - 8{\cdot}32$
$= 2{\cdot}77$ ft.

The increase in length of the tangent, above that required for a purely circular curve, $= c \tan \frac{\beta}{2} - r \sin a + X$

$$= 2{\cdot}77 \tan 33^\circ\,42' - 600 \sin 9^\circ\,33' + 199{\cdot}445$$
$$= 1{\cdot}85 - 99{\cdot}54 + 199{\cdot}45 = 101{\cdot}76 \text{ ft.}$$

The length of the combined curve

$$= 600 \text{ arc } (67^\circ\,24' - 19^\circ\,06') + 400$$
$$= 505{\cdot}81 + 400 = 905{\cdot}81 \text{ ft.}$$

Using the cubic parabola, Y would be $\frac{l^2}{6r} = \frac{200 \times 200}{6 \times 600}$ $= 11{\cdot}11$ ft., an error of $0{\cdot}02$ ft., the increase of the tangent was $101{\cdot}85$ ft., an error of $0{\cdot}09$ ft., while the length agreed exactly.

Compound Curves

When the radius of a circular curve changes from one value r_1 to another r_2 we have a compound curve, and there are two cases, (*a*) both curvatures in the same direction, and (*b*) reversed curvature, where the curvature is in opposite directions. In the latter case a length of straight is usually interposed between the curves if no transition curves are employed. We shall assume that the common tangent is known (Fig. 5). Then, if there are no transition curves, in the first case:

$$r_1 \tan \frac{\beta}{2} + r_2 \tan \frac{\gamma}{2} = L, \text{ and } I = \beta + \gamma$$

and in the second case:

$$r_1 \tan \frac{\beta}{2} + l + r_2 \tan \frac{\gamma}{2} = L, \text{ and } I = \beta - \gamma$$

which formulae must be satisfied by the two radii.

When transition curves are employed with reversed curves the length of intervening straight is unnecessary, and we have (Fig. 6),

$$(r_1 + c_1)\tan\frac{\beta}{2} + \frac{l_1}{2} + \frac{l_2}{2} + (r_2 + c_2)\tan\frac{\gamma}{2} = L$$

where l_1, l_2 are the lengths of the transition curves and c_1, c_2 the corresponding shifts.

FIG. 5

EXAMPLE 4 (L.U.). *AB*, *BC*, *CD* are three straights. The length of *BC* is 40 (Gunter's) chains. *BC* deflects 60° right from *AB* and *CD* 45° left from *BC*. Find the radius (r) for two equal circular curves, each with transition curves of length $\sqrt{r}$ at both ends, to connect *AB* and *CD*; *BC* is to be the common tangent without intermediate straight. Find also the total length of curve.

$$\text{Shift} = \frac{l^2}{24r} = \frac{r}{24r} = \frac{1}{24} = 0\cdot042 \text{ chains}$$

$$40\cdot00 = \left(r + \frac{1}{24}\right)\tan 30^\circ + \frac{\sqrt{r}}{2} + \frac{\sqrt{r}}{2} + \left(r + \frac{1}{24}\right)\tan 22^\circ 30'$$

$$= (0\cdot5774 + 0\cdot4142)\left(r + \frac{1}{24}\right) + \sqrt{r}$$

$$\therefore \qquad 0{\cdot}9916r + \sqrt{r} - 39{\cdot}96 = 0$$

Solving as a quadratic in $\sqrt{r}$

$$\sqrt{r} = \frac{-1 \pm \sqrt{1 + 4 \times {\cdot}9916 \times 39{\cdot}96}}{2 \times {\cdot}9916} = \frac{-1 + 12{\cdot}629}{1{\cdot}9832}$$

$$= 5{\cdot}864 \text{ chains.} \quad \therefore\ r = 34{\cdot}39 \text{ chains}$$

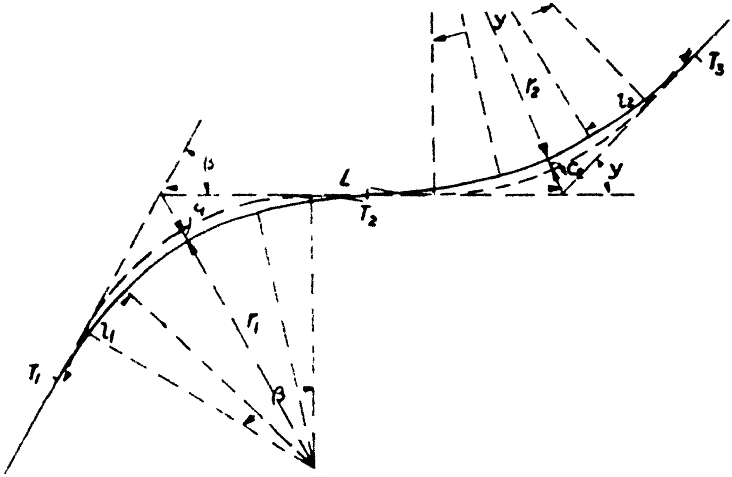


FIG. 6

$$\text{Total length} = 4 \times \frac{\sqrt{r}}{2} + 34{\cdot}39 \text{ arc } (60° + 45°)$$

$$= 11{\cdot}73 + 63{\cdot}02 = 74{\cdot}75 \text{ chains}$$

When a transition curve of length l is required between two circular curves of radii r_1 and r_2, the curvature being in the same direction, Glover* shows that the shift,

c = shortest distance between the two circular curves

$$= \frac{l^2}{24}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \text{ (See Fig. 7)}$$

and the shift bisects the transition curve and is bisected

* "Transition Curves for Railways," *Proc. Inst.C.E.*, Vol. 140.

by it. Curve r_1 is set out up to the shift point and curve r_2 is set out forward from that point. The shift is bisected at C and CJ_1, CJ_2 marked off, each equal to $\frac{l}{2}$. Offsets are

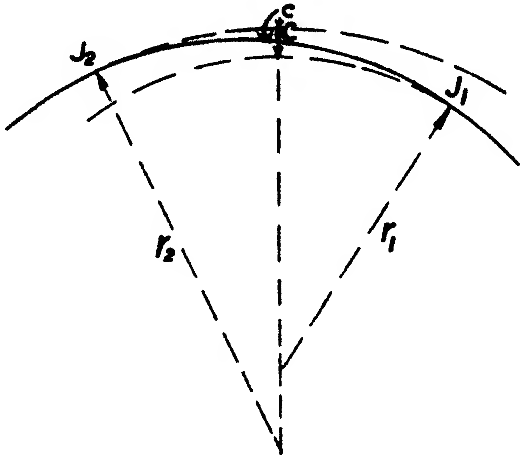


Fig. 7

then set off outside curve r_1 and inside curve r_2 at distances (s) from J_1, J_2 respectively, and of lengths

$$= \frac{s^3}{\left(\frac{l}{2}\right)^3} \times \frac{c}{2}$$

Unless the difference of radii is very considerable, it is unnecessary to employ this refinement; the two circular curves are allowed to touch, and the superelevation is gradually increased from its value on the r_2 curve to its value on the r_1 curve in a length l, of which half is on each curve. In any case, the length l should be decided by the permissible rate of gain of radial acceleration, say 1 ft./sec.^{3}

Vertical Curves

Somewhat analogous to transition curves are the vertical curves used to connect two gradients on a railway or road where the change of gradient is considerable. The curve used is a parabola with vertical axis tangential to the two gradients (Fig. 8).

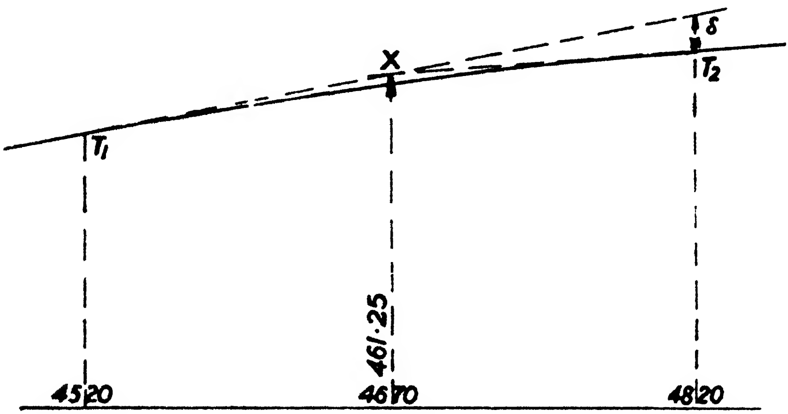


Fig. 8

The vertical through the intersection of the gradients is a diameter of the parabola and bisects the chord joining the two tangent points, so that the tangent points are equidistant (horizontally) from the intersection point. The equation to the parabola, measuring y vertically and x_1 along either tangent is $y = C_1x_1^2$, and as x_1 is proportional to the horizontal distance x from the tangent point, the equation may be written $y = Cx^2$. The constant C is found by calculating the vertical deflection δ of one tangent point from the other tangent produced, thus—

Example 5. A gradient of 1 in 50 meets a gradient of 1 in 400 at chainage 4,670 ft. and level 461·25 ft. A vertical parabola is to be introduced, 300 ft. long, to connect the gradients. Calculate the levels at the tangent points and at each 50 ft. of through chainage.

The chainage of the tangent points will be

$$4670 \pm 150 = 4{,}520 \text{ ft. and } 4{,}820 \text{ ft.}$$

Producing the 2 per cent gradient, its level at 4,820 ft.

$$= 461{\cdot}25 + 3{\cdot}00 = 464{\cdot}25 \text{ ft.}$$

The level of the 0·25 per cent gradient at 4,820 ft.

$$= 461{\cdot}25 + 0{\cdot}375 \text{ ft.} = 461{\cdot}625 \text{ ft.}$$

$$\therefore\ \delta = 2{\cdot}625 \text{ ft. at } x = 300 \text{ ft.}$$

$$\therefore\ C = \frac{2{\cdot}625}{(300)^2} = 0{\cdot}00002917$$

We, therefore, prepare the following table (the level of the 2 per cent gradient at 4,520 ft. = 461·25 − 3·00 = 458·250

Chainage	x	z = Level of 2% gradient	x^2	y	$z - y$ = Level of parabola
4,520	0	458·25	0	0	458·25
4,550	30	458·85	900	0·03	458·82
4,600	80	459·85	6,400	0·19	459·66
4,650	130	460·85	16,900	0·49	460·36
4,700	180	461·85	32,400	0·95	460·90
4,750	230	462·85	52,900	1·54	461·31
4,800	280	463·85	78,400	2·29	461·56
4,820	300	464·25	90,000	2·625	461·625

Setting Out Tunnels

The setting out of tunnels divides itself into three parts: (i) the surface survey or setting out, (ii) the connection of the surface and underground surveys, and (iii) setting out underground. In all three of these, levelling is involved as well as the alignment of the centre line.

(i) *Surface Survey.* In open country the centre line is usually set out on the surface, and the centre line is usually straight for the greater part of its length. An approximate line is chosen on an existing survey, and this is set out with a theodolite from one of the terminal stations and produced until it passes the other terminal station; the first intermediate station is then shifted laterally by an amount equal to the total deviation $\times \frac{\text{distance from starting point}}{\text{total distance}}$. Then the process is repeated, by "trial and error," until a straight line between the terminal points has been obtained. Of course, all observations will be taken, both Face Right and Face Left, and the results averaged. Permanent pillars of

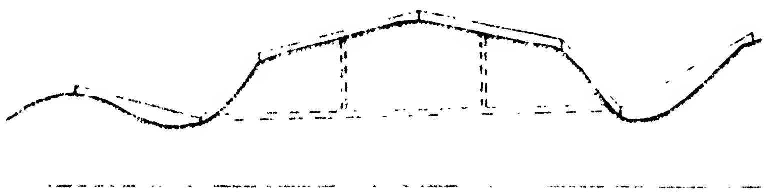


FIG. 9

masonry or concrete should now be built on this line at all salient points, capped with a stone into which a metal plate is fixed, and surrounded with a scaffold, on which the surveyor can stand without shaking the pillar. The centre line should now be set out again, marking it on the metal plates and setting the theodolite (removed from its legs) over these marks. The theodolite should be as powerful as possible, and the process repeated several times, the resultant positions on the pillars being averaged.

It is advisable that two stations should be visible from the terminal stations, and for this purpose it may be necessary to extend the line beyond the terminal stations, as shown in Fig. 9. A line of careful levels should be run over the tunnel and checked in the reverse direction, two bench marks being left near each end, and intermediate bench marks if shafts are to be sunk at intermediate points to expedite construction. The object of the duplicate stations and bench marks is to guard against any disturbance of these marks, in which case such disturbance can be readily detected before it leads to serious error. If there are to be shafts, a peg should be carefully fixed with the theodolite near each shaft, the centre line being marked on the head of a copper nail driven into the top of the peg, or a low pillar may be used.

In towns, it will not usually be possible to set out the centre line on the surface, and a traverse must be run between the terminal points, the stations being marked permanently, say on metal plates let into the curbstones. The work has usually to be executed at night and must be as accurate as possible, the angles being read repeatedly on both faces with a powerful theodolite, and the lengths measured with a steel band along the ground at a constant tension and corrected for temperature and difference in level. If possible, the traverse should be closed; if not, a check on the angles can be obtained by finding the azimuths of the first and last lines by astronomical observation, and comparing the difference of azimuth with the resultant observed deflection angle, allowing, of course, for the effect of the convergence of meridians.

From this traverse, when the co-ordinates of the stations have been reduced, the position, chainage and direction of the centre line can be calculated at the terminal or any intermediate points.

In mountainous country the tunnel will probably be long and deep, and shafts will not be economical, the tunnel being driven wholly from the ends. In such cases a triangulation would be made over the tunnel between the terminal stations, from which the angles between the centre line and an adjacent side of the triangulation at each end can be calculated. In very mountainous country it may not be possible to level over the surface in the usual way, but the difference of level of the ends can be determined by taking vertical angles reciprocally between stations when executing the triangulation.

(ii) *Connection of the Surface and Underground Surveys.* Setting out the centre line of the tunnel from the ends is a straightforward operation. If the centre line has been set out over the surface, the theodolite is set up on the terminal station and sighted on the two visible stations; it is then directed up the tunnel and a permanent mark made in the floor or roof, the average of face right and face left observations being taken.

As the tunnel advances this process is repeated and marks made farther and farther in, until it becomes necessary, for visibility, to move the theodolite to one of these marks in order to produce the centre line still farther; as before, sighting always on two known points before fixing a new one. The most forward mark should be sufficiently near the working face to be of use to the workmen in driving and lining the tunnel, but sufficiently far from it not to be damaged by blasting operations. The levels are carried in by spirit levelling from the two bench marks outside the end, and bench marks are established in the floor or on the sides of the tunnel.

When vertical shafts are employed, as is economical when the tunnel is not too deep, each shaft providing two additional working faces, the method usually adopted is to suspend two long plumb-lines down the shaft, as far as possible apart, to place them exactly in the centre line by a theodolite on the surface, then to transfer the theodolite to the bottom of the shaft and set it up in the tunnel in line with the plumb-lines, when the centre line can be produced both ways and marked in some substantial way on the floor or roof of the tunnel. This is an operation demanding great care, as it amounts to producing a very short base for a long distance, and many precautions are necessary. Long plumb-lines are easily deflected and apt to oscillate slowly, the period of oscillation being proportional to $\sqrt{\text{length}}$; shafts are frequently very wet from ground water, and great care must be taken to see that the plumb-lines do not touch the timbering or rock, and are protected from water drip.

A useful check on the freedom of the plumb-lines is to measure carefully their distance apart at top and bottom of the shaft and see that these agree. Fine piano-wire is used for the plumb-lines with heavy plumb-bobs, up to 50 lb. in weight in some cases, immersed in buckets of water or oil to damp out oscillations as much as possible (Fig. 10). The bobs may have projecting vanes to prevent rotation. At the surface the wires are suspended from cross-beams, frequently passing over a nut on a horizontal screw perpendicular to the centre-line of the tunnel; by rotating the screw the wire can be moved laterally into the required line.

Practice varies as to the position of the theodolite at both top and bottom of the shaft.* Some engineers place the theodolite at a distance from the nearer wire about the same as the distance between the wires, they can then see each of the wires by focusing on it, although one is behind the other, and to the naked eye completely obscured by it, and

* L. H. Cooke, "Underground Orientation by Exact and Approximate Alignments of Plumb-wires in One Shaft," *Proc. Inst. M. and M.*, 1925.

undoubtedly an accurate bisection of the wires is better ensured by this close position. The theodolite is centred over the peg or pillar left near the shaft for the purpose, and sighted on to two of the principal stations, then focused on each wire in turn and the wire moved into line as required. If this operation is repeated with changed face errors due

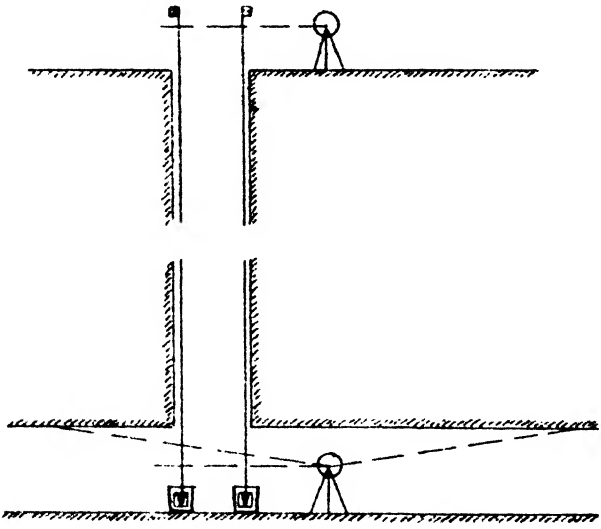


FIG. 10

to non-horizontality of the trunnion axis, non-perpendicularity of the collimation line to the trunnion axis and non-axial draw of the telescope in focusing at very different distances will be eliminated, provided the wires are set in the mean positions obtained by face right and face left observations. Other engineers prefer to set up at 30 ft. to 50 ft. from the nearer wire, so that both wires can be observed at the same focus, although the focus has certainly to be changed when sighting on the permanent stations. In order, however, to be able to see both wires, some special device must be employed, e.g. the supports must be at different levels at the top and the plumb-bobs at different levels at the bottom, or two or three links must be inserted in the nearer wire at top and bottom, through which to see the farther wire, or the farther wire may be made thicker than the nearer one and a white card placed between the two wires, so that, when truly centered, they appear as in Fig. 11.*

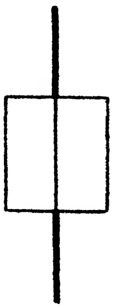


FIG. 11

At the bottom of the shaft there are the additional difficulties of bringing the axis of the theodolite *exactly* into the plane of the wires and the slow oscillations of the plumb-wires, which cannot always be completely damped out with very long wires. The first can be obviated by having the theodolite, above the levelling screws, mounted with a screw motion at right angles to the centre line of the tunnel, so as to move it gently into line.

If the oscillations are very small, they can be averaged by eye on the intersection of the cross-hairs; if too large for this, a fine scale can be fitted in the diaphragm of the theodolite, and the extreme readings on this scale averaged, or a scale may be fixed behind each wire on which similar readings can be read and averaged.

Having set the theodolite exactly in the plane of the wires, a couple of marks should be made in the floor or roof of the tunnel as far from the instrument as possible, and the process should then be repeated with changed face—the permanent marks being the average of those obtained face right and face left.

An alternative method, adopted in mines on the Continent,

* H. A. Bartlett, "Notes on the Construction and Setting-out of Tunnels in the London Clay," *Proc. Inst.C.E.*, Vol. 156.

is that of the "Weisbach Triangle" (Fig. 12). Here the theodolite is set up at C, very slightly out of line with the two wires A and B, so that both wires are visible. The angle ACB is then measured very accurately—it should not exceed a few minutes—and the distances AC, and BC, also AB as a check, also measured very accurately, and from these

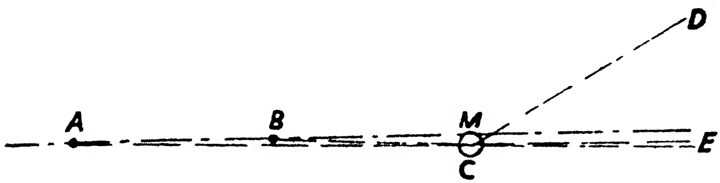


FIG. 12

measurements the azimuth of any other line CD can be deduced if the angle ACD is also read. If desired, the deviation CM of C from the vertical plane of AB can be calculated, and a line CE set out parallel to the centre line at a very short distance CM from it, by sighting on A and deflecting through the angle BAC.

EXAMPLE 6. The vertical plane of two plumb-wires A, B, has an azimuth of 54° 36′ 20″ (clockwise from north). A theodolite is set up at C, south of AB, and the angle ACB is measured as 0° 12′ 30″, the distances AC, BC being 20·065 and 10·122 ft. respectively. An angle ACD is measured clockwise from A as 124° 17′ 50″. Calculate the azimuths of the lines AC and CD and the amount C is distant from the line AB produced.

As the angles BAC, ACB are so small, we can take $AB = AC - BC = 9{\cdot}943$ ft., also the sines of such small angles can be taken as their arcs or circular measures, therefore,

$$\frac{BAC}{ACB} \doteq \frac{\sin BAC}{\sin ACB} = \frac{BC}{AB} = \frac{10{\cdot}122}{9{\cdot}943}$$

$$\therefore \quad \text{angle } BAC = \frac{10{\cdot}122}{9{\cdot}943} \times 750'' = 763{\cdot}5''$$
$$= 0^\circ\ 12'\ 43{\cdot}5''$$
$$\therefore \text{ azimuth of } AC = 54^\circ\ 36'\ 20'' + 0^\circ\ 12'\ 43{\cdot}5''$$
$$= 54^\circ\ 49'\ 03{\cdot}5''$$
$$CD \text{ deflects from } AC \text{ by } 180^\circ - 124^\circ\ 17'\ 50''$$
$$= 55^\circ\ 42'\ 10'' \text{ Left}$$
$$\therefore \text{ azimuth of } CD = 54^\circ\ 49'\ 03{\cdot}5'' - 55^\circ\ 42'\ 10''$$
$$= 359^\circ\ 06'\ 53{\cdot}5''$$

Deviation of C from AB produced $= 20{\cdot}065 \sin 0^\circ\ 12'\ 43{\cdot}5''$
$= 0{\cdot}074$ ft.

20·065	1·302,4392
763·5	2·882,8090
sin 1″	$\bar{6}$·685,5749
0·074272	$\bar{2}$·870,8231

As $\sin x$ agrees with x to $\frac{1}{100{,}000}$ part up to $x = 0^\circ\ 19'$ and $\cos x$ agrees with 1 to $\frac{1}{100{,}000}$ part up to $x = 0^\circ\ 11'$ (see Chapter I) the validity of these approximations is apparent. If desired, a line could be set out from C parallel to the line AB and 0·074 ft. away from it, by sighting on A and turning off an angle of $0^\circ\ 12'\ 43{\cdot}5''$ from the line CA. If AB is the centre line of the tunnel, the marks thus fixed could be altered by 0·074 ft. to give the centre line.

It frequently happens, especially in towns, that the shaft cannot be placed on the centre line of the tunnel, which may be under the road, while the shaft is outside the road and connected with it by an "adit" or short horizontal (temporary) tunnel. In such cases the plumb-wires are suspended in the shaft in some known azimuth so that their line can be produced along the adit into the tunnel proper. The theodolite below ground is then placed in the main tunnel in line with the wires, and at a distance from the wires equal to their distance from the centre line of the tunnel, and the centre line found by turning off the necessary angle.

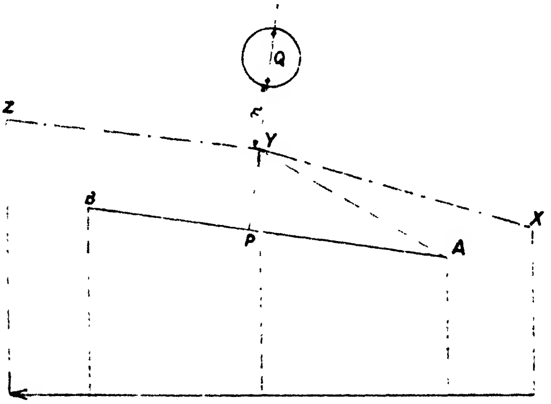


FIG. 13

EXAMPLE 7. The co-ordinates of two stations, A B, on a street traverse and of three intersection points X, Y, Z of straights on a sewer tunnel are as follows (Fig. 13).

	North	East
A . .	5021·56	1032·72
B . .	5872·61	1292·96
X . .	4830·00	996·00
Y . .	5470·00	1192·00
Z . .	6056·00	1320·00

Calculate the distance AP from A of a point P in AB such that PY is perpendicular to AB, the distance PY, and the angles PYX and PYZ.

$$\text{Azimuth of } AB = \tan^{-1}\frac{260{\cdot}24}{851{\cdot}05} = 17°\ 00'\ 10{\cdot}6''$$

$$\text{Azimuth of } AY = \tan^{-1}\frac{159{\cdot}28}{448{\cdot}44} = 19°\ 33'\ 15{\cdot}7''$$

260·24	2·415,3741	159·28	2·202,1612
851·05	2·929,9551	448·44	2·651,7043
tan 17° 0′ 10·6″	$\bar{1}$·485,4190	19° 33′ 15·7″	$\bar{1}$·550,4569

$AY = 448{\cdot}44 \sec 19°\ 33'\ 15{\cdot}7''$

$AP = AY \cos 2°\ 33'\ 05'' = 475{\cdot}42$ ft.

$PY = AY \sin 2°\ 33'\ 05'' = 21{\cdot}18$ ft.

448·44	2·651,7043		
sec 19° 33′ 15·7″	0·025,7995		
AY	2·677,5038	AY	2·677,5038
cos 2° 33′ 05″	$\bar{1}$·999,5692	sin 2° 33′ 05″	$\bar{2}$·648,5098
AP	2·677,0730	PY	1·326,0136

$$\text{Bearing of } XY = \tan^{-1}\frac{196{\cdot}00}{640{\cdot}00} = 17°\ 01'\ 38''$$

$\therefore$ bearing of $YX = 197°\ 01'\ 38''$

$$\text{Bearing of } YZ = \tan^{-1}\frac{128{\cdot}00}{586{\cdot}00} = 12°\ 19'\ 18''$$

196	2·292,2561	128	2·107,2100
640	2·806,1800	586	2·767,8976
tan 17° 01′ 38″	$\bar{1}$·486,0761	tan 12° 19′ 18″	$\bar{1}$·339,3124

Bearing of $YP = 270° + 17°\ 00'\ 11'' = 287°\ 00'\ 11''$

$\therefore$ Angle $PYX = 89°\ 58'\ 33''$ and

Angle $PYZ = 85°\ 19'\ 07''$

In setting out, therefore, we measure off 475·42 ft. from *A* towards *B* and find *P*. Set the theodolite over *P* and turn an angle of 90°, then measure off 21·18 ft. and thus find Y on the ground. If the shaft can be on the tunnel centre-line, we should now set up the theodolite over Y and set out two points on each of the lines YX, YZ by turning off the above angles PYX, PYZ from YP.

After the shaft is sunk and a short length of tunnel driven in the two directions, plumb-wires can be suspended successively in each of the lines YX and YZ, and these centre lines thus carried underground as above described. If, on the other hand, the shaft has to be placed off the line of the tunnel, as indicated in the figure, PY would be extended to Q and the shaft sunk at Q. When the shaft has been sunk, the adit or heading driven back to Y, and a short length of the tunnel driven along YX, YZ, the line PY would be again ranged with the theodolite and the plumb-wires suspended in the shaft in this line. The theodolite would then be ranged into the line of the wires underground at a distance from the nearer wire of n ft., and the angles

$$QYX = 180° - PYX \text{ and } QYZ = 180° - PYZ$$

set off for the centre lines of the tunnel.

To carry the levels down the shaft, the level of a nail in a timber at one side of the top of the shaft is ascertained by levelling from the bench marks left near the top of the shafts for this purpose. A steel tape is then hung down the shaft, care being taken to stretch it vertically and clear of obstructions. If the tape is long enough to measure to the foot of the shaft the depth can be measured in one operation; if not, the lower end of the tape must be marked in some way, the engineer descending the shaft in the skip or cage for this purpose, and the operation is repeated until the bottom of the shaft is reached, where the level of some temporary mark is obtained, this level being transferred to a permanent bench mark on the side of the tunnel by means of the level and staff set up inside the tunnel.

(iii) *Setting Out Underground.* To render plumb-wires and -lines visible a lamp should be held *behind* them, with tracing paper pasted over the glass, or some other form of translucent screen held between the lamp and the line. An electric torch held near the object glass of the level or theodolite will render the cross-hairs visible, and a lamp held near the levelling staff will enable readings of this to be taken. Permanent alignment marks usually consist of file marks on dogs driven into the roof of the tunnel lining (Fig. 14), from which a plumb-line can be suspended. Pencil marks on chalk can be used until the two positions, F.R. and F.L., have been marked, then the file mark can be made, averaging the two marks. When the theodolite is set up below one of these marks for producing the line, a mark must first be plumbed vertically below it, over which the theodolite is set up. Alternatively, the permanent mark can be a centre-punch mark in the head of a nail driven into wooden peg embedded in concrete in the floor and covered over so as to be protected from the traffic. Bench marks are most conveniently formed of spikes driven into the side of the tunnel on which the staff can be held. As the tunnel

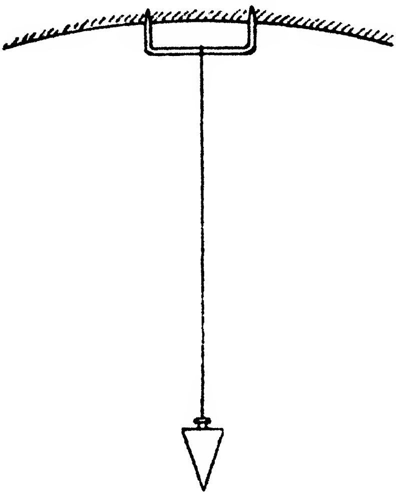


FIG. 14

advances, the centre line marks and bench marks must be advanced and kept as near the face as will be safe from the blasting operations. All productions must be by "double centering," i.e. the mean of observations taken face right and face left, the back sight being as long as possible. The workmen prolong the centre line of the last two marks by eye, using plumb lines and candles held to them. It is advisable at intervals to repeat the process of plumbing down the shaft and checking the alignment marks previously established to ensure that no error is undetected before it has produced too serious results. If possible, the bench marks should be made at a constant height above the invert level of the tunnel, to render the risk of error by the workmen as small as possible, e.g if at intervals a brick in the same course is left projecting a little, or a slight step left in the concrete at a constant height above the actual invert level, the engineer can use these as bench marks and inform the foreman of the amount they are above or below the correct level, so that the levels of future work may be made correct.

The workmen ascertain their level at the working face by sighting along the tops of "boning rods," shaped like T-squares, of constant length, held on the projections thus left, making any necessary corrections by packings under the boning rods. Small deviations from both line and level are certain to occur it is important to ensure that there is no cumulative error. Great care and constant checking of the setting out and completed work are essential if the different lengths of the tunnel are to meet in line and level with that accuracy of which tunnel engineers are justly proud

Where possible, curves are avoided in tunnel work, but they cannot always be avoided, especially near the ends of a tunnel. In such cases, a "heading," i.e. small temporary tunnel, is sometimes driven through in prolongation of the tunnel straight, so that a straight line can be ranged through from end to end, which conduces to greater accuracy. Curves are set out with the theodolite in the usual way, but, of course, the intersection of the two straights is rarely inside the tunnel, nor can the tangent lengths be measured off from the intersection as on the surface. The tangent point must be accurately fixed, and the theodolite moved round the curve as soon as the "long chord" fouls the side of the tunnel. Frequent centre line marks must be left at equal distances apart, and the workmen should be instructed how much to deviate in any given distance from the "short chord" joining the last two marks produced.

EXAMPLE 8 (L.U.) (Fig. 15). The co-ordinates of two points A, B are

	North	East
A . . .	399·60	00
B . . .	201·40	998·40

A straight line AC bears 110° 30′ (clockwise from north) and intersects at C a straight line BC bearing 275° 50′. The chainage of A is 2671·62 ft. Calculate the lengths of AC and CB. The two straights are to be joined by a curve of 500 ft. radius. Calculate the chainage of the tangent points and of B.

The bearing of CB is 95° 50′

$\therefore$ deflection angle at C = 14° 40′ Left

If CB be produced to D,

$$OD = 201{\cdot}40 + 998{\cdot}40 \tan 5^\circ 50'$$
$$= 201{\cdot}40 + 102{\cdot}0007 = 303{\cdot}4007$$
$$AD = 399{\cdot}60 - 303{\cdot}4007 = 96{\cdot}1993 \text{ ft.}$$
$$AC = 96{\cdot}1993 \frac{\sin 84^\circ 10'}{\sin 14^\circ 40'} = 377{\cdot}97 \text{ ft.}$$
$$\text{Easting of } C = 377{\cdot}97 \sin 69^\circ 30' = 354{\cdot}04 \text{ ft.}$$
$$\therefore \quad BC = (998{\cdot}40 - 354{\cdot}04) \sec 5^\circ 50'$$
$$= 644{\cdot}36 \sec 5^\circ 50' = 647{\cdot}71$$

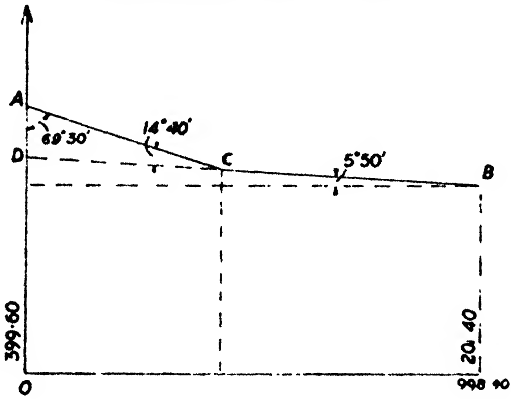


FIG. 15

998·40	2·999,3046
tan 5° 50′	$\bar{1}$·009,2984
102·0007	2·008,6030
96·1993	1·983,1720
sin 84° 10′	$\bar{1}$·997,7453
cosec 14° 40′	0·596,5446
377·97	2·577,4619
sin 69° 30′	$\bar{1}$·971,5876
354·04	2·549,0495
644·36	2·809,1286
sec 5° 50′	0·002,2547
647·71	2·811,3833

$$\text{Length of tangent} = 500 \tan 7^\circ 20' = 500 \times \cdot 1286943 = 64{\cdot}35 \text{ ft.}$$

$$\text{Length of curve} = 500 \text{ arc } 14^\circ 40' = 500 \times \cdot 2559816 = 127{\cdot}99 \text{ ft.}$$

$$\therefore \text{ chainage of } TP_1 = 2671{\cdot}62 + 377{\cdot}97 - 64{\cdot}35 = 2671{\cdot}62 + 313{\cdot}62 = 2985{\cdot}24 \text{ ft.}$$

$$\text{chainage of } TP_2 = 2985{\cdot}24 + 127{\cdot}99 = 3113{\cdot}23 \text{ ft.}$$

$$\text{chainage of } B = 3113{\cdot}23 + 647{\cdot}71 - 64{\cdot}35 = 3113{\cdot}23 + 583{\cdot}36 = 3696{\cdot}59 \text{ ft.}$$

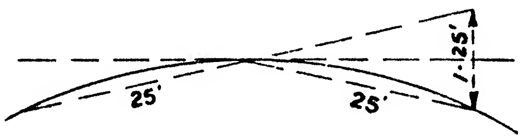


FIG. 16

For setting out the curve the following table is prepared—

Chainage, ft.	Arcs, ft.	Increment of Tangential Angle	Tangential Angle	Tangential Angle	Actual Theodolite Reading
2985·24	—	—	0	0 0′ 0″	360° 00′ 00″
3000	14·76	50·74′	50·74′	0° 50′ 40″	359° 09′ 20″
3025	25	85·94′	136·68′	2° 16′ 40″	357 43′ 20″
3050	25	85·94′	222·62′	3 42′ 40″	356° 17′ 20″
3075	25	85·94′	308·56′	5 08′ 30″	354° 51′ 30″
3100	25	85·94′	394·50′	6 34′ 30″	353 25′ 30″
3113·23	13·23	45·48′	439·98′	7 20′ 00″	352° 40′ 00″

The increment of tangential angle for 25 ft. is

$$\frac{1}{2} \times \frac{25}{500} \times 3437{\cdot}75' = 85{\cdot}94' = 1^\circ 26' \text{ nearly.}$$

The increment for the first length

$$= \frac{14 \cdot 76}{25} \times 85 \cdot 94' = 50 \cdot 74'$$

and that for the last length

$$= \frac{13 \cdot 23}{25} \times 85 \cdot 94' = 45 \cdot 48'$$

The tangential angle for $TP_2 = 7° 20'$, this checks correctly.

In 25 ft. the arc will deviate from the tangent by

$$25 \sin 1° 26' = 0 \cdot 625 \text{ ft.}$$

i.e. from the line of the preceding chord by 1·250 ft., as in Fig. 16.

CHAPTER IX

SURVEYING FROM THE AIR

VERTICAL AND OBLIQUE PHOTOGRAPHS—GRID FOR HIGH OBLIQUES—VERTICAL STEREOSCOPIC PAIRS—STEREOSCOPY—TOPOGRAPHICAL STEREOSCOPE—PLOTTING STEREOSCOPIC PAIRS—DETERMINATION OF GROUND LEVELS—GRID FOR LOW OBLIQUES

SURVEYING by means of photographs taken from aircraft, like Aeronautics itself, has made immense progress during the last 30 years, and for any adequate treatment requires a book to itself, but this volume would be incomplete without some introduction to its principles. Its use enables undeveloped country to be rapidly and economically surveyed to a small scale and existing large-scale surveys (e.g. the 6 in. to 1 mile, Ordnance Survey) to be revised as regards new detail such as roads, buildings, etc. Like all forms of detail surveying it depends on "Ground Control," i.e. on the positions of stations, fixed at intervals by Triangulation (or Traversing), to which corresponding points on the air survey must be adjusted to avoid the accumulation of errors. If the levels of important points of detail have to be shown, or contours of altitude, the levels of the ground control points must also be found. Air photographs are taken from a height of from 5,000 to 10,000 ft., which is measured on an "Altmeter," a form of aneroid barometer. There are two main classes of air survey, viz. (*a*) by Vertical Photographs and (*b*) by Oblique Photographs.

VERTICAL PHOTOGRAPHS

Fig. 1 represents a vertical photograph of *level* ground, y_1y_2 being the negative film (or plate) and C the *perspective* *centre* of the lens through which all rays, e.g. XCx, pass in straight lines: the extent of the photograph is limited to y_1y_2 by the size of the film, say 7 in. square. If a perpendicular Cp is dropped from C to the negative, Cp is called the *principal distance*, and p the *principal point* of the camera.

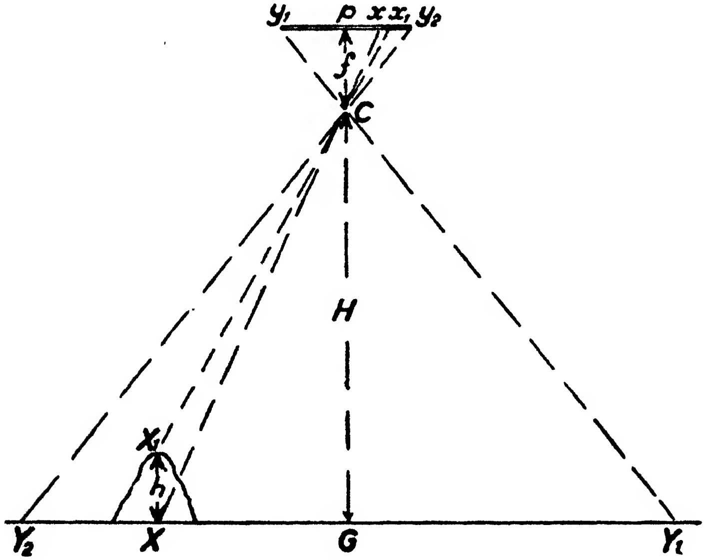


FIG. 1

There are no cross-hairs in the camera but the principal point is marked by a + on a glass plate covering the film (or can be found by joining pairs of marks on the edge of the glass plate), these marks being photographed so as to show on the negative. If H is the height of C above the ground, f the principal distance and X a point on the ground reproduced at x on the plate, the scale of the photograph will be $\frac{Cx}{CX} = \frac{f}{H}$ and it will be constant over the whole of the negative. A plane through C will cut the ground and the negative in parallel straight lines and straight lines at an angle on the ground will be represented by straight lines at the same angle on the negative. If the point observed, however, were X_1, at a height h vertically above X, the point on the negative would be shifted to x_1 and the scale at x_1 would be increased from $\frac{f}{H}$ to $\frac{f}{H-h}$. This is an example

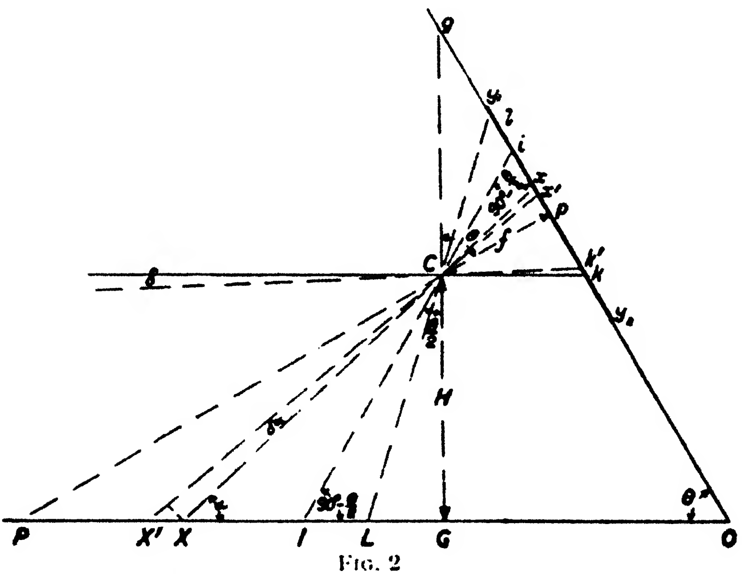


FIG. 2

of *height distortion*. The point G, vertically below C, is called the ***ground plumb-point***—in this case the ***plate plumb-point*** coincides with p the principal point.

HIGH OBLIQUE PHOTOGRAPHS

Fig. 2 represents an oblique photograph of *level* ground, the *tilt* or inclination of the plane of the negative to the horizontal being θ. The plane of the paper (vertical) containing the principal distance Cp ($= f$) is called the ***principal plane***. The plate plumb-point is now at g vertically above the perspective centre C and ground plumb-point G. The principal plane cuts the negative in the ***principal line*** y_1py_2. A horizontal plane through C cuts the negative in a horizontal line through k called the ***horizontal trace***. Straight lines on the ground will still be represented by straight lines on the negative, but parallel straight lines, which may be considered to meet at infinity, will, on the negative, converge to a ***vanishing point*** on the horizontal trace: if they are parallel to the principal plane, this point will be at k, and the lines will be called ***plate meridians***. Parallel straight lines on the ground perpendicular to the principal plane will be represented by parallel straight lines at right angles to the principal line, i.e. horizontal lines or ***plate parallels*** as their vanishing points are at $\pm$ infinity on the horizontal trace.

The scale at any point x along the plate parallel is $\frac{Cx}{CX}$ and decreases from the upper edge of the negative to zero at the horizontal trace at k. The scale along the central meridian, i.e. the ratio of a *short* distance xx' on the negative to the corresponding short distance XX' on the ground,

$$= \frac{f\delta \cot(\alpha+\theta)}{H\delta \cot \alpha} = \frac{f \operatorname{cosec}^2(\alpha+\theta)}{H \operatorname{cosec}^2 \alpha} = \frac{Cx}{CX} \cdot \frac{\operatorname{cosec}(\alpha+\theta)}{\operatorname{cosec} \alpha}$$

where $CXO = \alpha$, and it will also decrease from the upper edge of the negative to zero on the horizontal trace. These scales being different, it is obvious that the *angles* at which lines, other than the ground parallels, cross the central meridian will not be correctly reproduced by the angles between corresponding straight lines on the negative. If we bisect the angles PCG, pCg by the straight line ICi, the points I and i are *isocentres*, and the ground and plate parallels through I and i are called *isometric parallels*. The triangles

CIG and Cip being similar, the scale along this parallel is $\frac{Ci}{CI} = \frac{f}{H}$, the same as for a vertical photograph; the scale along the central meridian there is also $\frac{Ci}{CI}$, as, since

$$\alpha = 90° - \frac{\theta}{2},$$

$$\frac{\operatorname{cosec}(\alpha + \theta)}{\operatorname{cosec}\alpha} = \frac{\operatorname{cosec}\left(90° + \frac{\theta}{2}\right)}{\operatorname{cosec}\left(90° - \frac{\theta}{2}\right)} = 1$$

Consequently a straight line crossing the central meridian at I is represented by a straight line crossing the principal line at i on the negative at the same angle. If this angle is 45°, this line will meet the horizontal trace through k at a distance kq from $k = ik = Ck$ since the angle $iCk = 90° - \frac{\theta}{2} = Cik$. This point q will be the vanishing point for *all* straight lines on the ground at 45° to the central meridian.

Grid for High Obliques

To construct a grid for the positive photograph which will represent a grid of equal squares on the ground with their sides parallel and perpendicular to the central ground meridian OGP, proceed as follows—Draw a straight line kl (Fig. 3) to represent the principal line kpy_1, and a line perpendicular to it, q_1kq_2, to represent the horizontal trace, making $kq_1 = kq_2 = Ck = f \operatorname{cosec} \theta$. If the sides of the squares on the ground are a ft. and we choose the intervals along the parallel at l to be b in., the lateral scale at $l = \frac{b \text{ in.}}{a \text{ ft.}}$

But from Fig. 2 the lateral scale at $l = \frac{Cl}{CL} = \frac{lk}{kO} = \frac{lk}{H \operatorname{cosec} \theta}$

$\therefore lk = \frac{bH \operatorname{cosec} \theta}{a}$, where lk and b are in inches, H and a in feet. Set off l at this distance from k, through l draw a line

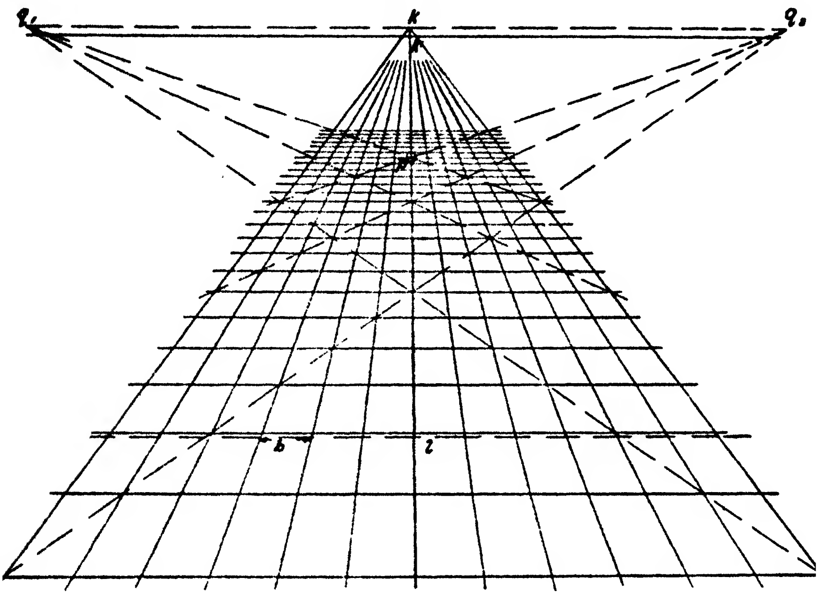


FIG. 3

parallel to q_1q_2, mark off intervals of b in. on each side of l and join the points to k by straight lines. These will be the plate meridians. Then mark off $kp = f \cot \theta$, and draw a parallel through p. Radial straight lines from q_1 and q_2 (commencing with those through p), which represent the diagonals of the squares and are at 45° to the central ground meridian, give the positions of the other parallels. Figs. 2 and 3 represent a *high oblique*, i.e. the *true* horizon shows on the photograph. To draw the *visible* horizon find from tables (see page 101) the *dip* δ of the visible horizon for a height of H ft. Then mark off $kk' = f\{\cot \theta - \cot (\theta + \delta)\}$ and draw a line through k' parallel to q_1kq_2 to represent the visible horizon; the grid is now complete.

The student should draw out such a grid from the following data—Tilt $\theta = 67°$, height $H = 5{,}000$ ft., principal distance = 7 in., sides of grid squares on ground = 660 ft., $b = 1$ in.: plot to $\frac{1}{2}$ full size (i.e. $b = \frac{1}{2}$ in.). Take the dip of the horizon δ, as $59'' \sqrt{H}$ ft. approximately.

To allow for variation in tilts and altitudes a number of grids are made, marked on glass, for various combinations of height H and distances of the visible horizon from the upper edge of the photograph. When a grid has been selected to suit these conditions it is laid on the positive photograph and fitted to the principal point p; the detail and control points can then be transferred from the *perspective* squares on the photograph to the true squares on the ground grid, on which the ground control points are already plotted. To join the photograph to the ones preceding and following it, azimuth lines are drawn on each, near the principal line, through points of detail recognizable on the adjacent photographs.

Surveying with high obliques has been used extensively in Canada for the huge wastes of almost level country covered with forest and innumerable small lakes on the "Laurentian Shield." The aircraft flies in a straight course and photographs are taken at about 2 mile intervals to avoid having to plot too far into the background beyond p, the principal point. Immediately after each exposure the camera is swung 45° to the right and left against stops to take *side obliques* which widen the strip of ground surveyed to about

6 miles. A *low oblique* is a photograph with a smaller tilt, *θ*, so that the horizon does *not* show on the photograph.

Vertical Photographs in Stereoscopic Pairs

This method is suitable for larger scale surveys in more developed or more hilly country. The photographs are

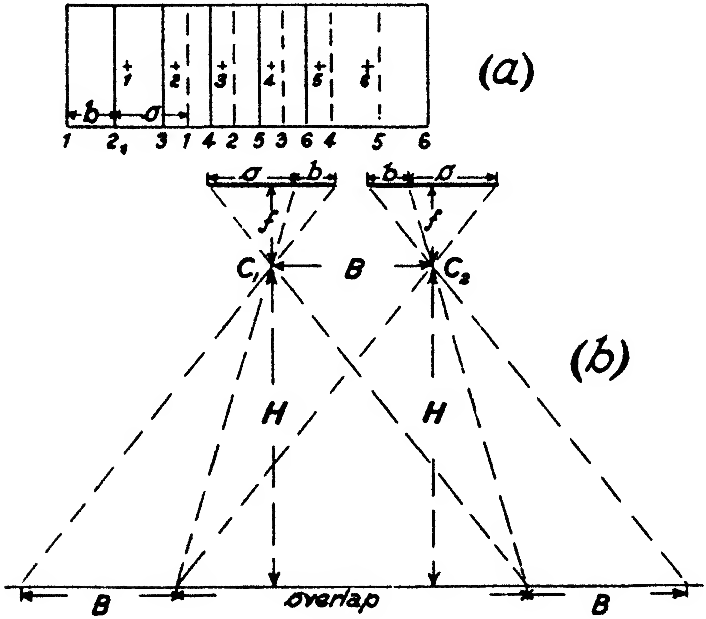


Fig. 4

taken in a series of parallel strips at regular time intervals so arranged in relation to the speed of the aircraft that each photograph overlaps 60 per cent of the adjoining ones in the strip. If a wind is blowing the aircraft will *drift* and the direction of the camera must then be deflected from the forward direction of the aircraft so as to lie along the actual track followed, i.e. the direction of the resultant of the two velocities. The overlap of the strips laterally is usually 25 per cent of their width. It will be seen from Fig. 4 (*a*) that 20 per cent of each photograph appears on three successive photographs in a strip. The distance B between successive positions of the camera is called the ***air base*** and is usually about 3,000 ft. If b is the 40 per cent length of each photograph *not* overlapped by the next one, it is obvious from Fig. 4 (*b*) that $\frac{b}{B} = \frac{f}{H}$, i.e. $B = \frac{Hb}{f}$, so that the time interval for exposures can be found from the actual speed of the aircraft relative to the ground. Also $\frac{fB}{H} = b$, which should be constant.

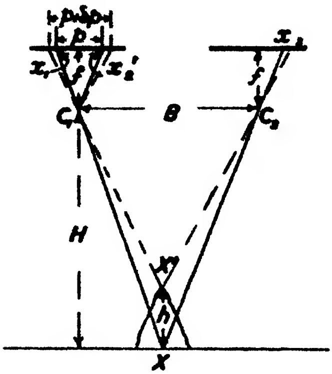


FIG. 5

Again (Fig. 5), if two successive photographs have been taken from an air base B, the point X appears in different positions, x_1, x_2, which show a shift of position or ***absolute parallax, p, in the direction of the air base,*** p being the algebraic difference of position from the principal point, measured parallel to the air base. If we draw C_1x_2' parallel to C_2x_2, we see that $\frac{p}{f} = \frac{B}{H}$, i.e. $p = \frac{Bf}{H}$. If, however, X were at X' on top of a hill of height h, the parallax of X' would be increased to $p + \delta p$. As h is usually small compared to H, we can write $\delta p = -\frac{Bf}{H^2}\delta H = \frac{Bf}{H}\cdot\frac{h}{H} = \frac{bh}{H}$ where b is the length of photograph *not* overlapped by the next one. All objects on the same level have the same absolute parallax, and an increase of ground level (or decrease of depth from the air base) increases the absolute parallax by an amount proportional to that increase (or decrease).

The overlapping portion of each pair of photographs is examined in a *stereoscope*, the principles and use of which will now be explained.

STEREOSCOPY

If two large dots *A* and *B* (Fig. 6) about 1 in. apart on paper are held parallel to the eyes *E*, *E* and viewed by both eyes, with a little practice they can be *fused* into a single dot or virtual image at *D*. If the *separation AB* of the dots is increased to *AC*, the *angle of convergence ADB* is decreased to *AFC* and the *depth* or distance of the image below the dots is increased. This resembles what takes place in ordinary binocular vision and gives the effect of distance and relief to a view; each eye forms its own picture and the two pictures, taken from an *eye base* about 2½ in. long, are fused into one. The object of a stereoscope is to increase by optical means the distance between the eyes and the separation of the corresponding points viewed on the overlap of two successive photographs (taken from the ends of a horizontal air base, say 3,000 ft. long), placed side-by-side without overlapping. Such a stereoscopic view looks solid instead of flat and enables the relative levels of points in the earth's surface to be determined by measuring their difference in parallax or separation. The two photographs must be *in correspondence*, i.e. exactly *oriented* so that the base-line, from the principal point of each photo-

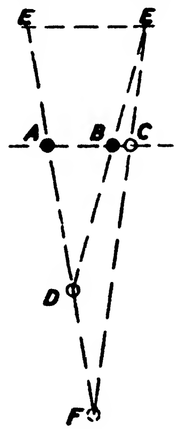


FIG. 6

graph to the position on that photograph of the principal point of the other, is in line with the corresponding line on the other photograph. When this is effected the separation of all pairs of corresponding points at the same level is the same and is parallel to the air base, with *truly vertical* photographs.

If each photograph is covered with an exactly similar *grid* of fine lines engraved on a thin glass plate and the separation of corresponding lines on the grids equals that of the corresponding points at the same level on the photographs, then the fused images of the lines and of the points will be fused at the same depth and the lines will appear to touch the ground at that level, provided, of course, that there are some details showing on the photographs by means of which the lines may be brought into coincidence. By moving one of the grids outwards along the base-line, we can increase the separation of the corresponding lines and form the fused image of the lines and points at a greater depth, so that the lines will appear to touch the ground at a lower level. This increase of separation is the same as the decrease in parallax (δp) due to difference of level, and enables us to measure this difference as $h = \frac{H}{b}.\delta p$. The marking of the grids usually consists of two series of equally spaced lines (Fig. 7) at 45° to the base line, which is marked by short lines. Short lines perpendicular to the base are also marked

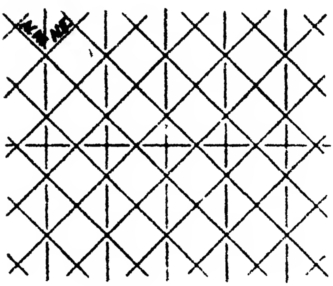


FIG. 7

as shown. The sloping lines are known as "N.W." and "N.E." lines by analogy with directions on a map.

Topographical Stereoscope

Fig. 8 is a diagram of the Barr and Stroud Precision Type of Topographical Stereoscope. The observer's eyes are at

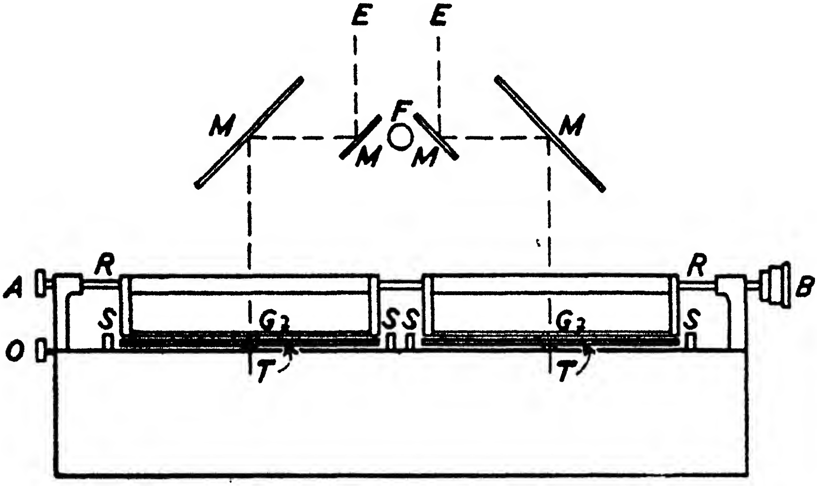


Fig. 8

E, *E* and the two successive photographs are viewed stereoscopically by two pairs of parallel mirrors *M*, *M*; the inclination of the smaller mirrors can be altered by turning a screw *F* at the back of the framework which supports the mirrors from the base of the instrument. The photographs are clamped to two tables *T*, *T*, each of which can be rotated through a small angle by turning the orientation screws *O* at the L.H. end of the base, one at the front and one at the back The centres of the vertical axles of the tables are marked on the tops of the tables. Each table has a pair of stops S, S so arranged that their four faces are all in a line which passes through the centre marks of the tables; thus a straight edge placed against them fixes the position for the base-lines of the grids and photographs on each table. The parallactic grids G, G are fixed in frames supported by arms from a spindle RR at the back so that they can be lowered on to the photographs on the tables (as shown) or raised well

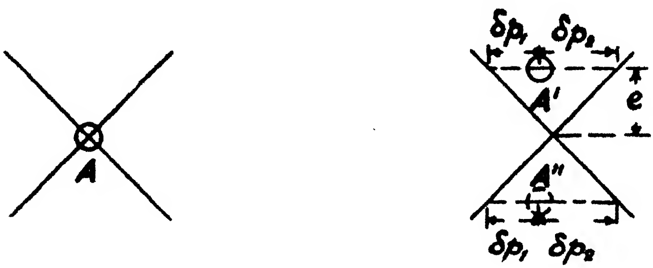


Fig. 9

above them to allow the photographs to be moved or marked. Both frames can be moved parallel to the base-line by equal amounts by turning a milled head A at the L.H. end of the spindle. At the R.H. end is a micrometer head B, which moves the R.H. grid only so as to alter the separation of the grids, and is graduated so that the readings increase as the separation of the grids decreases.

Error in Correspondence. In Fig. 9 an object, which lies at A on the intersection of a cross on the L.H. grid, appears at A' above the intersection of the cross on the R.H. grid. There is clearly an *error in correspondence* e, as AA' is not parallel to the base-line, due to incorrect orientation of the photographs (or to the photographs having been tilted when exposed). When the wires are fused it is still possible with a little eyestrain to fuse the object if e is small, but the effect is that the N.W. line appears to have a smaller separation than the object by δp_1 and, therefore, to be higher than A, while the N.E. line appears to have a larger separation than A and, therefore, to be lower than A. The reverse would be the case if the object were at A'' below the intersection of the cross. If the R.H. grid is moved so that the two lines touch the object in turn, the total movement is $\delta p_1 + \delta p_2$ and the error in correspondence $e = \frac{\delta p_1 + \delta p_2}{2}$.

Plotting Stereoscopic Pairs

Orientation of the Photographs. On both tables the grids are adjusted so that the intersection of a cross is over the centre mark of the table. The photographs are then clamped to the tables with their principal points under the intersections of these crosses, and turned so that corresponding detail near both principal points lies along the base-lines. Then the photographs are viewed through the stereoscope and fused by turning the screw F to the small mirrors, and correspondence observations are made on objects near each of the two principal points. Errors in correspondence are eliminated by turning the orientation screws O, O to rotate the tables slightly and the parallax screw B until both N.E. and N.W. lines of the grid touch the ground near the principal points.

Approximate Measurement of Difference of Level of Two Points, X and Y. By means of screw A bring a short grid line close to each point. By turning B make the mark touch ground successively at X and Y, reading the micrometer head B each time. The difference of the micrometer readings is the difference of parallax, δp, from which the difference of height, $h = \frac{H\delta p}{b}$ may be calculated. For example, if the photographs measure 7 in. × 7 in., the 40 per cent of non-overlap will be 2·8 in. If H is 8,000 ft., a difference of parallax of 0·5 mm. will represent $\frac{8{,}000 \times 0{\cdot}5 \times 0{\cdot}0394}{2{\cdot}8} = 56$ ft. If the principal distance $f = 7$ in., the air base B would be $\frac{Hb}{f} = \frac{8{,}000 \times 2{\cdot}8}{7} =$ 3,200 ft. If the sloping lines are used for observation of either X or Y, both N.W. and N.E. lines must be brought to touch ground one after the other and the average of the micrometer readings taken for that point. This is necessary as X and Y may be at different distances from the base-line.

To Transfer a Point from one Photograph to the Other by the Stereoscope. This may be necessary if there is no well defined detail at some point to which we require to take directions from the principal points of the two photographs. Mark the point required by placing a *correspondence mark*, i.e. a cross like one on the grid marked on a small piece of celluloid film, over the desired position on one of the photographs, then place a similar mark over the approximate position on the other photograph and move it by hand until both lines when fused are at the same depth as the adjacent ground. Then pierce the centres of both marks with a needle on to the photographs.

Plotting from Vertical Photographs by the Stereoscope. Although every care is taken to keep the camera level by the screws provided for this purpose, the photographs are liable to small *tilt distortions* as well as *height distortions*. The former can be shown to be in a radial direction from the isocentre and the latter from the plumb-point on the photograph, neither of which now quite coincides with the principal point as in a truly vertical photograph. However, Hotine* shows that directions taken *from the principal point* of a photograph can be taken as sufficiently accurate for surveys of scales up to $\frac{1}{20000}$, or nearly 3 in. to 1 mile, provided the tilt does not exceed 2° and the range in height of the ground does not exceed a tenth of the height of the aircraft above the average ground level. We cannot assume that the base-lines of the photographs, between each pair of principal points, are in one straight line throughout the whole strip, or that their lengths are as calculated from the *ground speed* of the aircraft and constant interval between the exposures of the photographs; therefore *Minor Control Points* are chosen near the upper and lower edges of the photographs in the 20 per cent portion which appears on three successive photographs and in which lies the principal point of the middle photograph. Lines are drawn to them from the principal points on the different photographs. Then the directions of the base-lines are found by stereoscopic observations of each pair of photographs and the positions of the principal points are fixed by resection from the minor control points. This process gives the *Minor Control Plot* of the strip which can be adjusted to fit the similar plots of adjacent strips until the whole width of the survey is contained on a *Compilation*; this is adjusted again to fit the *ground control points* which should not be more than 7 miles apart longitudinally and the width of 4 or 5 strips laterally.†

Fig. 10 represents three successive photographs, 1, 2 and 3, with their principal points at *A*, *B* and *C*. Photographs 1 and 2 are placed in the stereoscope and carefully oriented. By using a straight edge placed against the stops *S*, the base-

* Capt. M. Hotine; *Surveying from Air Photographs* (1931) (Constable & Co.), p. 152.
† Ibid. p. 153.

line direction Ab is drawn on the overlap as Ab and a short line or *tail* T drawn at the farther edge of the photograph. Similarly in 2 the base-line direction Ba is drawn also with a tail. The process is repeated with photographs 2 and 3 and so on for each pair of photographs, which can then be joined in direction.

The minor control points are then selected in the 20 per cent overlap common to three successive photographs.

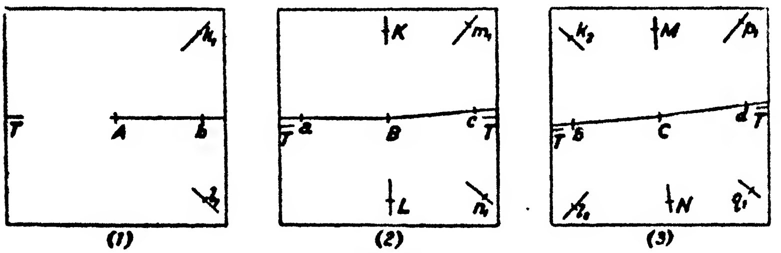


FIG. 10

These should be distinct points of detail lying above and below the principal points at distances about the same as the base-line lengths. If there is no such detail available, they will have to be fixed to correspond by using the stereoscope (see p. 372). These points are shown as K, L in the second photograph and as M, N on the third photograph: similarly, these points are represented by k_1, l_1 and k_2, l_2 on the photographs to the left and right of 2; and by m_1, n_1 and m_2, n_2 on those to the left and right of 3. Short lines are drawn from the principal point of each photograph through these points, the third and subsequent photographs each showing 6 minor control points with short lines through them. A strip of celluloid, with a roughened surface for drawing, is then cut to include the whole length and breadth of the strip surveyed. The first photograph is placed under it and the positions of the principal point *A* and of one of the minor control points k_1 traced on it; also the direction *Ab* and the short line through the other minor control point l_1 (Fig. 11). Then the second photograph is fitted under the celluloid so that *aB* lies along *Ab* (*Ab* being produced so that it lies under the tail of *aB*) and moved until the short line from *B*

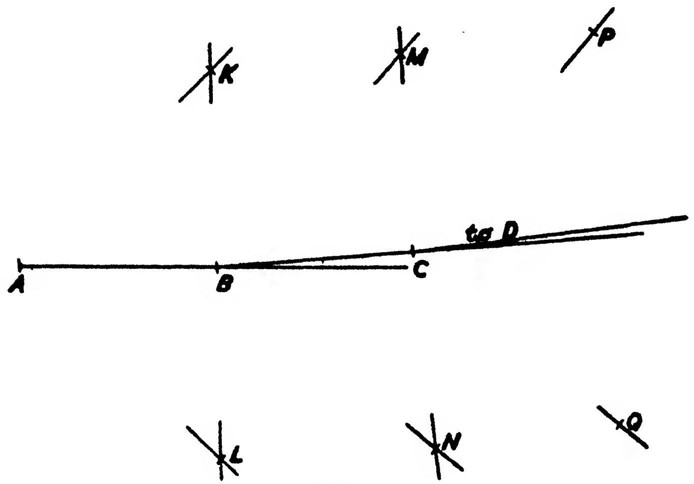


FIG. 11

through *K* passes through k_1. This fixes the position of *B*, which is marked. The short line from *B* through *L* is then traced to intersect at *L* the short line from *A* through l_1. The base direction *Bc* is traced and produced and the short lines through m_1 and n_1 are traced. The third photograph is then inserted with *bC* under *Bc* and moved until the short lines from *C* to k_2 and l_2 pass through the plotted positions of *K* and *L*. This gives the position of *C* and the radial short lines, intersecting those from *B* through m_1 and n_1 fix the points *M* and *N*; the short lines through p_1, q_1 and the base direction *Cd* are then traced and so the process continues.

If, in fixing the position of *C*, the short lines from *C* through k_2 and l_2 cannot *both* be made to pass through K and L, while bC is still kept under Bc, move the photograph so that these latter two lines still coincide, until a short line Ck_2 passes through K and trace this line on the celluloid. Then move the photograph, still keeping bC under Bc, until the short line Cl_2 passes through L and trace this line. These two lines, traced back to meet Bc, form a small ***triangle of error*** with it and the position of C is chosen inside it—at its centre if very small, as it should be. Then the photograph is shifted again so that C on the photograph is under this new position and bC lies under Bc; the points M and N are found by intersection, and the base-line Cd and short lines through p_1 and q_1 are traced.

As angles taken from the principal points of the photographs are correct, any points which can be identified on the overlap of two photographs can be marked and transferred to the minor control plot by placing it over one of the two photographs, so that the principal point (say A) and the base-line Ab on the photograph lie under the corresponding ones on the celluloid, and tracing short radiating lines from A through the marked points. Then substitute the other photograph with its principal point B and base-line Ba under the corresponding ones on the celluloid and intersect the short lines traced on the celluloid with ones traced from B through the corresponding marked points. Thus ground control points, whose positions have been fixed by the triangulation, can be plotted, and ***auxiliary points*** can be fixed on the 25 per cent overlap between adjacent strips, which can thus be plotted on the minor control plots of adjacent strips and which will serve, when adjusted to fit, to connect these plots when the compilation is traced for the whole area of the survey. Other points can be fixed in a similar way to provide triangles for tracing the detail within them from the photographs. The compilation is then adjusted to fit the ground control points. The longest distance between those marked on the compilation compared with their true distance on the ground gives the mean scale of the compilation and the positions of all the ground control points are plotted on paper to this scale. The positions of all ground control points on the compilation are then adjusted to fit those on the paper. When all this adjustment is completed the detail can be traced on the compilation by the triangles traced on it for this purpose and the compilation is ready for reproduction as a survey.

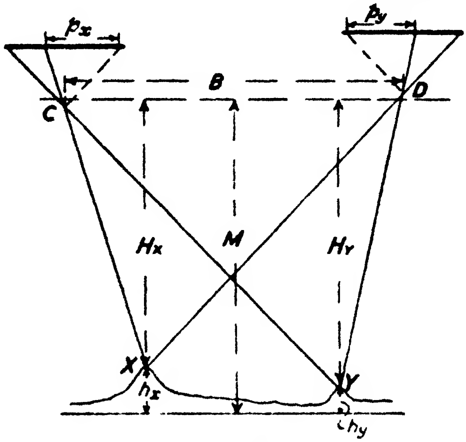


FIG. 12

Determination of Ground Levels

The levels of some salient points will have been found during the ground triangulation by vertical angles, corrected for curvature and refraction, and these will provide ground control for determining spot-levels on the air survey. For this a more accurate method than that already described $\left(\text{viz. } h = \frac{H\delta p}{b}\right)$ is to find the absolute parallax, $p = \frac{fB}{H}$, for both the point X of known level and the point Y whose level is to be found (Fig. 12.) Here B is the average length of the air base CD as measured from the compilation, and H is the average height of the two air stations C and D above the point observed. If h_x, h_y are the heights of X and Y above datum and M the average altitude of C and D above datum as given by altmeter, then $H_x = M - h_x$ and the absolute parallax of $X = p_x = \dfrac{fB}{M - h_x}$. When examined in the stereoscope the difference of parallax of X and Y is found by the micrometer head to be $p_x - p_y = \delta p$. Therefore, the absolute parallax of Y (p_y) is $p_x - \delta p$. Then $H_y = \dfrac{fB}{p_x - \delta p}$ and the height of Y above datum (h_y) is $M - H_y$. This would be true if the camera were absolutely level at both exposures, but owing to small unavoidable tilts there will be defects in correspondence between corresponding points on the two photographs and these must first be determined at four points near the corners of the overlap of each pair of photographs. From these the parallaxes can be corrected for correspondence at other points* and the above method can then be applied. When the levels of suitable points, e.g. summits, ridge and valley lines, tops and bottoms of steep slopes, have thus been found, and, if necessary, adjusted to suit values obtained from neighbouring ground control levels, they can be used to interpolate ground contours on the survey in the usual way.

Grid for Low Obliques

If four points m, n, p, q on a photograph have been identified as corresponding to four ground control points M, N, P, Q on the map, any other point O can be plotted on the

* See Capt. M. Hotine: *Surveying from Air Photographs* (1931) (Constable & Co.), p. 172 *et seq.*

map from its position o on the photograph by the following construction. Join MN, PN, QN, mn, pn, qn, on (Fig. 13). Lay a straight-edged strip of paper across the four rays from n and mark on it where each ray is cut—m', q', o', and p'. Then transfer the strip to the map and move it about until m' falls on MN, p' on PN, q' on QN, and mark o' on the map.

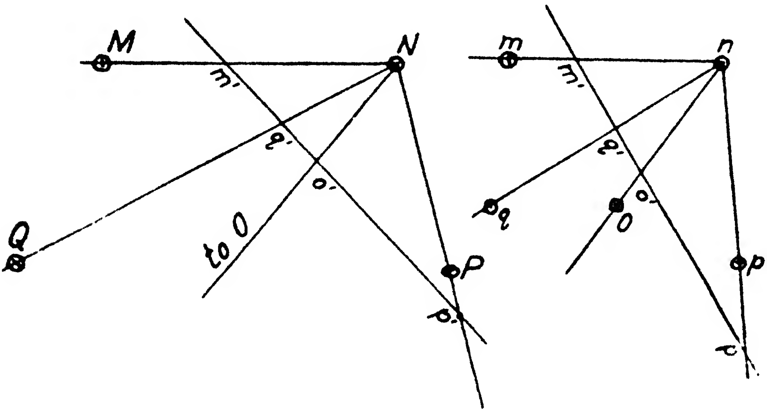


Fig. 13

Join No' and produce. Then draw rays on the map from M (say) and on the photo from m, and repeat the operation, getting a second ray Mo' which fixes O. Reversing the process, this can be done for each corner of a true square on the map, giving a perspective "square" on the photograph. The rays through the perspective centre and m', n', o', and p' form a plane pencil with a constant cross-ratio, so that the cross-ratios of the radiating lines on the photograph are the same as those of the corresponding lines on the map.*

As this process is rather laborious it is usual to use it to project a large "square" first—*abcd* (Fig. 14)—and then to break this up into, say, 16 smaller "squares" for the

* See Appendix II.

grid. Draw the diagonals of the large perspective "square" on the photo and draw lines *ae*, *df* parallel to the diagonals *db*, *ac* respectively. Divide *ae* at *g* in the same ratio as *db* is divided at *i*, and *dc* at *h* in the same ratio as *ac* is divided at *i*, by the construction shown, and draw *hf* parallel to *bc*;

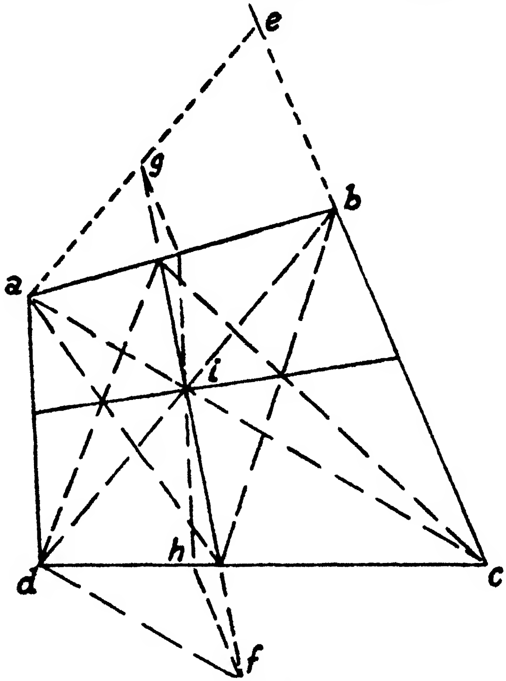


FIG. 14

then *fg** divides the original "square," *abcd*, into two double "squares" which are easily divided into four single "squares" by drawing diagonals as shown. These are each readily divided into four smaller "squares" by drawing their diagonals, so that 16 "squares" are obtained.

* *fg*, *da* and *cb*, as representing parallel lines, converge to the same vanishing point.

APPENDIX I

TABLE FOR ADJUSTMENT

(See p. 246 and Example 6, p. 248.)

Equation	λ_1	λ_2	λ_3	λ_4	λ_5	μ	N	Eliminating Equation
(i)	3	—	—	—	1	− 0·51	10	$\lambda_1 = -\frac{\lambda_5}{3} + 0{\cdot}17\mu - 3{\cdot}3333$
(ii)	—	3	—	—	1	− 3·52	− 10	$\lambda_2 = -\frac{\lambda_5}{3} + 1{\cdot}1733\,\mu + 3{\cdot}3333$
(iii)	—	—	3	—	1	3·20	− 5	$\lambda_3 = -\frac{\lambda_5}{3} - 1{\cdot}0667\,\mu + 1{\cdot}6667$
(iv)	—	—	—	3	1	0·88	− 5	$\lambda_4 = -\frac{\lambda_5}{3} - 0{\cdot}2933\,\mu + 1{\cdot}6667$
(v)	1 — — — —	1 — — — —	1 — — — —	1 — — — —	4 − [illegible] − [illegible] − [illegible] − [illegible]	— + 0·17000 + 1·17333 − 1·06666 − 0·29333	0 − 3·3333 + 3·3333 + 1·6667 + 1·6667	
	—	—	—	—	[illegible]	− 0·01666	3·3333	$\lambda_5 = 0{\cdot}00625\mu - 1{\cdot}2500$
(vi)	[illegible] — — — —	[illegible] — — — —	[illegible] — — — —	[illegible] — — — —	— 0·17000 1·17333 −1·06666 −0·29333	5994·7 − 0·0867 − 4·1301 − 3·4133 − 0·2581	− 819 + 1·7000 − 11·7333 + 5·3333 + 1·4666	
	— —	— —	— —	— —	−0·01666 —	5986·8117 − 0·0001	− 822·2333 + 0·0208	
	—	—	—	—	—	5986·8116	− 822·2125	$\therefore\ \mu = 0{\cdot}1373$

$\therefore\ \lambda_5 = 0{\cdot}00086 - 1{\cdot}25000 = -1{\cdot}24914$

$\lambda_2 = 0{\cdot}4164 + 0{\cdot}1611 + 3{\cdot}3333 = 3{\cdot}9108$

$\lambda_4 = 0{\cdot}4164 - 0{\cdot}0403 + 1{\cdot}6667 = 2{\cdot}0428$

$\lambda_1 = 0{\cdot}4164 + 0{\cdot}0233 - 3{\cdot}3333 = -2{\cdot}8936$

$\lambda_3 = 0{\cdot}4164 - 0{\cdot}1465 + 1{\cdot}6667 = 1{\cdot}9366$

$\therefore\ e_1 = -2{\cdot}894 + 1{\cdot}600 = -1{\cdot}29''$

$e_3 = 3{\cdot}911 + 4{\cdot}611 = +8{\cdot}52''$

$e_5 = 1{\cdot}937 + 1{\cdot}910 = +3{\cdot}85''$

$e_7 = 2{\cdot}043 + 5{\cdot}271 = +7{\cdot}31''$

$e_2 = -2{\cdot}894 - 1{\cdot}670 = -4{\cdot}56''$

$e_4 = 3{\cdot}911 - 5{\cdot}093 = -1{\cdot}18''$

$e_6 = 1{\cdot}937 - 1{\cdot}471 = +0{\cdot}47''$

$e_8 = 2{\cdot}043 - 5{\cdot}151 = -3{\cdot}11''$

APPENDIX II

Any system of four straight lines in a plane radiating from a point, is called a *pencil* of rays. A straight line cutting all four rays is called a *transversal*, e.g. $ABCD$ in Fig. A. Then the ratio $\dfrac{AB \,.\, CD}{BC \,.\, AD}$ is called the *cross-ratio* of the pencil as it is the same as that for any other transversal, *abcd*, of that pencil.

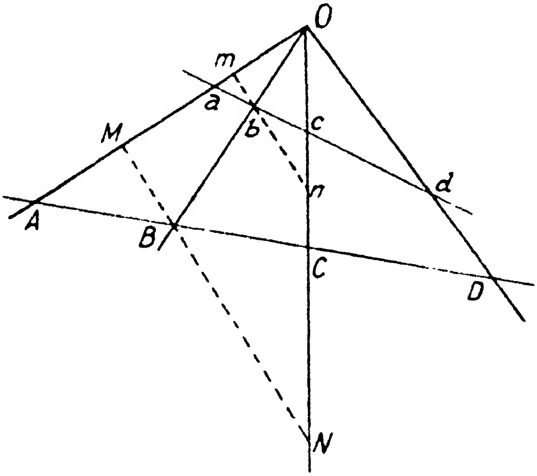


FIG. A

Draw MBN and mbn parallel to OD.

Then $\dfrac{MB}{OD} = \dfrac{AB}{AD}$ and $\dfrac{OD}{BN} = \dfrac{CD}{BC}$.

$$\therefore \quad \frac{MB}{BN} = \frac{AB \,.\, CD}{BC \,.\, AD}; \quad \text{similarly} \quad \frac{mb}{bn} = \frac{ab \,.\, cd}{bc \,.\, ad}.$$

$$\text{But} \quad \frac{mb}{MB} = \frac{Ob}{OB} = \frac{bn}{BN} \quad \therefore \frac{mb}{bn} = \frac{MB}{BN}$$

$$\therefore \frac{ab \,.\, cd}{bc \,.\, ad} = \frac{AB \,.\, CD}{BC \,.\, AD}.$$

(*Q.E.D.*)

INDEX