CONCRETE

PLAIN AND REINFORCED

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BY

The late Frederick W. Taylor; Sanford E. Thompson; and the late Edward Smulski

Volume I. Theory and Design of Concrete and Reinforced Structures

With a chapter by HENRY C. ROBBINS.

Fourth edition. 969 pages, 311 figures, 42 tables. Cloth, 6 x 9.

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VOL. II.

THEORY AND DESIGN OF CONTINUOUS BEAMS, FRAMES, BUILDING FRAMES AND ARCHES

BY

THE LATE FREDERICK W. TAYLOR,
SANFORD E. THOMPSON, S.B.
AND
THE LATE EDWARD SMULSKI, C.E.

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PREFACE TO VOLUME II

This second volume consists of entirely new material not contained in the previous editions of this book. It is adapted as well for use by the practicing engineer as for a textbook for students of Engineering. As far as it is known to the authors, this is the first book in any language in which the subject of statically indeterminate structures has been fully treated both from a theoretical and practical standpoint. Numerous original formulas are given for the easy use in the design of continuous beams, frames and arches.

In all cases the authors attempted to furnish easily understood explanations for the action of the statically indeterminate members. Also the relation between simple structures and statically indeterminate structures is given and the causes for the difference in action explained.

The volume covers the important field of statically indeterminate structures, i.e., structures where the members are either continuous over several supports, form frames of various description, or are arched and restrained at the supports. As is evident from the Table of Contents, the formulas, constants and examples are sufficient to solve practically all the problems likely to occur in the design of concrete structures.

Continuous beams are exhaustively treated. Special treatment is provided for beams consisting of unequal spans. For the student the theory and the derivation of formulas are given. Final formulas are given for practical use. To take care of unusual cases general formulas are given. For more common cases simple final formulas are developed and in many cases constants and diagrams are provided. The use of the formulas is explained by numerous examples.

Special formulas are developed for beams with variable moments of inertia. The use of fixed points for the solution of continuous beams is thoroughly explained.

Rigid frames of several types are also properly treated by giving theoretical treatment as well as simple final formulas and diagrams. The action of rigid frames is explained in a simple manner. The bending moment diagrams show in all cases the type of bending moments to which the various frames are subjected for different types of loading. When used intelligently, these alone may form a basis for approximate solution of problems in cases where accuracy is not of prime importance.

The Chapter on Building Frames contains material specially developed for this volume. Practical formulas ready for use by the practicing engineer on this subject have not ever been published before. The formulas may be used without difficulty for the solution not only of regular arrangement of spans but also for buildings of unusual character. The treatment is also of special value as a guide and explanation as to what happens in a building frame when subjected to bending. The formulas are particularly valuable for the determination of bending moments in wall beams and columns; for the design of buildings two panels wide; and finally for the design of multi-story one-span buildings. The slope-deflection method used for the development of the formulas for building frames is fully discussed and explained. By the general formulas for slope-deflection method here given, and following the plan of the example worked out, cases not specially treated in the book can be solved.

Fixed and two-hinged arches are fully treated. The treatment is prefaced by a description of arch bridges and by an easily understood explanation of the arch action and the action of arches when subjected to loading or to the changes of temperature. Rational approximate method of design of fixed arches is fully treated. The use of the method is explained in an example and simplified by diagrams. Exact method of design is made easy by proper arrangement of tables for computations of the statically indeterminate values. Simple method is given for finding bending moments and thrusts for the several most unfavorable positions of live load for the critical sections without the necessity of computing the statically indeterminate values for every case. The method here given is original with the authors.

In addition to the general treatment of arches, formulas for parabolic arches are given. For these the bending moments and thrusts at the critical sections may be taken directly from tables. The formulas for parabolic arches may be used to advantage in the design of arches with suspended floor and also in all other cases where the dead load is fairly uniform throughout the arch.

In rigid frames and arches the sections are subjected to thrusts and bending moments. To simplify the design of such sections special formulas and diagrams are given. Formulas and diagrams are given also for plain section as well as for reinforced concrete sections. Formulas are given, not only for sections with symmetrically arranged reinforcement but also for sections with unsymmetrical reinforcement. Diagrams are developed by means of which the dimensions of the sections can be obtained directly for given thrust and eccentricity and for specified maximum and minimum stresses. These diagrams are original with the authors. Formulas are given for practically all requirements and their use is shown by numerical examples.

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New York City.

NEW YORK, June, 1928.

IN compiling the book special credit must be given to Mr. Smulski who has drawn liberally from his store of knowledge and experiences and has thoroughly investigated material from all other authorities. The writer and Mr. Miles N. Clair of The Thompson & Lichtner Co., Inc., have carefully reviewed and checked this material.

SANFORD E. THOMPSON.

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CHAPTER I

CONTINUOUS BEAMS

Bending moment coefficients specified for continuous beams by the Joint Committee and by the various building codes apply only where the spans of the beams are equal and the loading uniformly distributed. They obviously do not apply to cases (1) where the spans are unequal, (2) where the loads are concentrated, and (3) where the loading consists of moving loads such as is the case in bridge design. The formulas given in succeeding pages are intended to take care of these cases.

The understanding of the action of continuous beams and of the principles upon which the formulas are based is of prime importance for intelligent design of reinforced concrete structures. Even an ordinary concrete structure often offers problems which cannot be solved by general formulas and require, for safe and economical design, either exact analysis or an approximate treatment based upon sound judgment and full understanding of the principles involved.

Scope of the Chapter.—This chapter gives not only a general treatment of continuous beams but also gives final formulas for bending moments and shears in such shape that they can be readily used in practice. In many instances diagrams and tables are prepared which still farther simplify the use of the formulas. Examples, based on conditions which are constantly arising in practice, show clearly how the material is used in design.

Formulas are given for uniformly distributed and concentrated loadings. Following conditions are treated:

Beams fixed at both supports.

Beams fixed at one support, free at the other.

Continuous beams with free ends.

Two spans with free ends.

Three spans with free ends.
Four spans with free ends.
Continuous beams with fixed ends.
Two spans with fixed ends.
Three spans with fixed ends.
Four spans with fixed ends.
Beams with variable moments of inertia.
Fixed point method of solving continuous beams.

Basis of Formulas.—The formulas given in this chapter are based upon the elastic theory. The three-moment equation, the derivation of which by the slope deflection method is given on p. 644, was used for the development of the working formulas. Separate formulas are given for dead load and for live load since the maximum bending moment coefficients for the live load are different than for the dead load.

Reliability of the Formulas.—The formulas for bending moments and shears are thoroughly reliable and can be applied very simply. The authors recommend their use in preference to the general formulas usually specified which are simply average approximations. Particularly they should be used where the spans are not equal and where the loading is concentrated.

An objection is sometimes raised against formulas based on the elastic theory on the ground that they depend upon the computation of the deflection of the beam. It is being argued that the deflection of the beam depends upon the modulus of elasticity of concrete, which admittedly is variable, and that, since the deflection of the beam cannot be computed with any degree of accuracy, all formulas based upon it must be equally unreliable.

This criticism is based obviously upon insufficient understanding of the theory of elasticity. The formulas are not based upon the magnitude of the deflection at any particular point but upon the relation to each other of the deflections and deformations at the various points. The modulus of elasticity does not enter into any of the formulas for bending moments as it is eliminated in the process of development of the formula. The formulas apply not only to concrete beams with a modulus of elasticity, E, of 2 000 000 in.-lb. but also to steel beams with a modulus of elasticity of 30 000 000 in.-lb. and in fact to beams made of any kind of material.

Criticism of Approximate Formulas for Continuous Beams.—Various specifications and Building Codes specify general bending moment coefficients for continuous beams to be used for dead load as well as for live load. For example, most of the specifications require that for interior spans of a continuous beam a bending moment coefficient $\frac{1}{12}$ should be used at the support and in the center of the span. Similarly in the exterior span a bending moment coefficient $\frac{1}{10}$ is specified. (See Vol. I, p. 279.)

These coefficients were derived originally from the elastic theory for uniformly distributed loading and for equal spans. To make them applicable to dead as well as to live load an assumption had to be made of the ratio between the dead load and live load. It is obvious, therefore, that these general formulas give accurate results only when the actual conditions are sufficiently similar to those assumed in deriving the formulas. When the conditions are different,—namely, when the spans are unequal, when concentrated loads are used for equal or unequal spans, or where the ratio between the dead load and the live load is different,—the general formulas give erroneous results. The error increases with the increase in the difference between the actual and the assumed conditions. As a result many beams designed by these general formulas either do not attain the desired factor of safety or else contain an appreciable excess of material. The examples on p. 178 to 207, giving actual designs of continuous beams with unequal spans, show clearly the impossibility of obtaining proper design by means of the approximate formulas.

In recent years a decided trend has developed towards more logical design methods. More than ever before it is being recognized that accurate design methods give better balanced structures. By elimination of the guess work they permit much more economical design, as no allowance needs to be made for possible errors due to difference in conditions. The ease with which the exact formulas given in this chapter can be used will undoubtedly accelerate the movement towards more logical design methods.

Sometimes continuous beams are being criticised because of distrust in the resistance of the beam at the support to negative bending moments. This criticism is now rarer than it was several years ago. The absurdity of this criticism needs hardly to be demonstrated. The beam at the support is of the same general design as in the center but only reversed. If it were unreliable at the support it would be equally unreliable at the center.

Some engineers prefer the use of the general formulas thinking that they are more conservative than the results obtained by the exact formulas. This is not the case. If the bending moments are computed separately for the dead load and for the most unfavorable positions of the live load they have the same degree of conservatism as was used in the development of the general formulas.

Definition and Assumption.—Continuous beam is a beam which is supported by three or more supports and therefore consists of two or more spans. The general formulas are based on the assumptions that (*a*) there is no connection between the beam and the support, so that the bending moment on one side of any support is transferred in full to the beam on the other side of the support, (*b*) the moment of inertia of the beam is constant and (*c*) the supports are unyielding.

The formulas given in this chapter apply only to cases where the beam is not rigidly connected with the supports but rests upon them freely. In such case the bending moments on both sides of any one support are numerically equal and turn in opposite directions. When a beam is connected with columns the negative bending moment in the beam on one side of the support is transferred partly to the column and partly to the adjoining span of the beam. Then the bending moments in the beam on both sides are not equal. Formulas for such constructions are given in chapters III and IV.

The assumption in the formulas of constant moments of inertia is justified in most cases found in practice. The effect of varying moments of inertia is discussed on page 133.

Formulas in this chapter apply only when there is no uneven vertical displacement of the supports. Uniform settlement of all supports has no effect upon the bending moments. If, however, one support settles more than the adjoining supports, additional bending moments are produced in the beam, the magnitude of which are proportional to the magnitude of the excess settlement. The effect of the movement of supports is discussed on page 151.

Difference between the Behavior of a Simple Beam and a Span of Continuous Beam.—The action of a continuous beam may be better understood by comparing the behavior under load of a simply supported beam, as in Fig. 1, with that of a continuous beam consisting of three equal spans, as in Fig. 2. To make the results comparable, the span length of the simple beam is made equal to that of the continuous beam. The dimensions of the beam also are considered to be the same.

The deflection of the simple beam due to uniformly distributed load extending over its whole span, drawn to appropriate scale, is shown in Fig. 1.

The corresponding bending moments are also shown in Fig. 1. Both the deflections and bending moments are in direct ratio to the magnitude of the load. All bending moments are of the same sign, which for downward loading is plus.

The deflections and bending moments of a continuous beam are shown in Fig. 2 (*a*) and (*b*). Two conditions are illustrated. Figure 2 (*a*) shows the deflection and bending moments, respectively, when

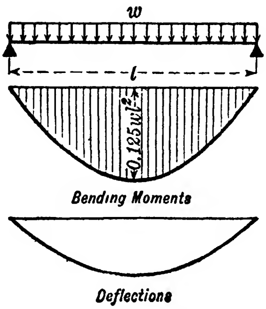


FIG. 1.—Deflection and Bending Moments in Simple Beam. (*See p.* 4.)

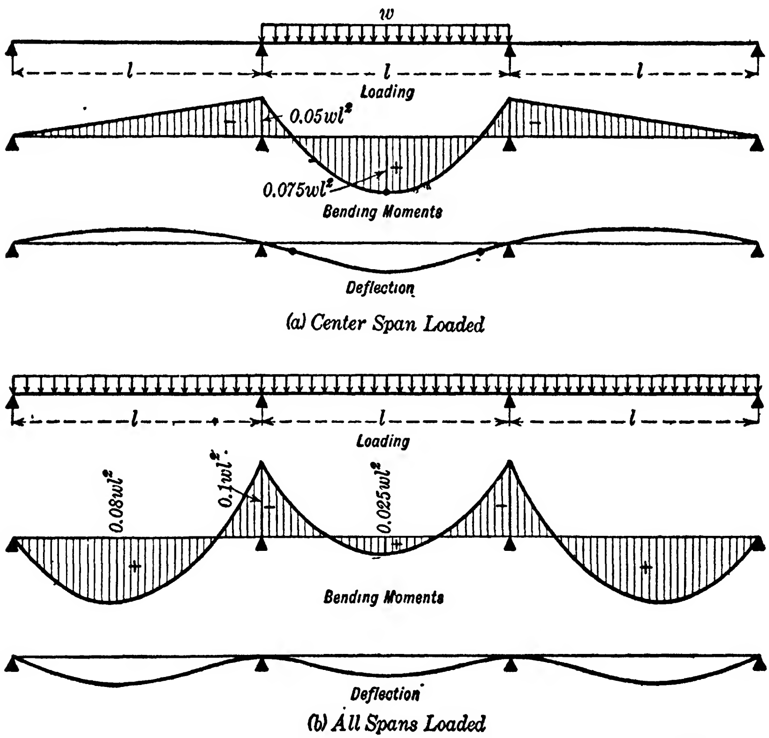


FIG. 2.—Deflection and Bending Moments in Continuous Beam. (*See p.* 4.)

the center span only is loaded, and Fig. 2 (*b*) shows deflection and bending moments when all spans are loaded.

The bending moment and deflection curves for the simple beam and for the continuous beam are drawn to the same scale and therefore may be used directly for comparison.

By comparing the deflection curves it is found, first, that the deflections of the continuous beam in all cases are smaller than those of the simple beam. Second, the shape of the deflection curve for the simple span is different from the shape of the deflection curves for the continuous beam. The deflection curve for the simple span is a simple curve. In continuous beam, the deflection curve of the center span is a reverse curve with two points of contraflexure, and the deflection curve of the end span, when loaded, has one point of contraflexure near the inside support.

In continuous beams the deflection curve extends from the one span to the adjoining spans even if they are not loaded. The deflection curve in one span is always a continuation of the deflection curve in the adjoining span so that both curves have a common tangent at the support.

A comparison of bending moment curves shows that the simple beam is subjected to positive bending moments throughout its length, while the center span of a continuous beam is subjected to positive bending moments in the central portion and to negative bending moments at and near both supports. The location of the points where the bending moments change sign corresponds to the points of contraflexure of the deflection curve.

The maximum positive bending moment in the continuous span is appreciably smaller than in the simple span. Since a continuous span is subjected to negative and positive bending moments it requires a different disposition of reinforcement than a simple beam which is subjected only to positive bending moments.

Effect of the Position of the Span in Continuous Beam.—By comparing the bending moments in the center span of a continuous beam with those of an end span it is evident that the bending moments in the end spans are materially larger than in the center span for the same type of loading.

Effect of Loading of One Span of Continuous Beam upon Adjoining Spans.—The bending moments in a simple beam depend only upon the loading in that beam. No loads placed outside of the beam can have any influence upon it.

In continuous beam, on the other hand, the bending moments in any one span depend not only upon the condition of loading of that span but also upon the condition of loading of the other spans forming the continuous beam. Thus the bending moments in the center span as shown in Fig. 2 (*a*) and (*b*) are different, although in both cases the load on the center span is the same and the only difference is in the loading condition of the end spans.

Compare the bending moments for the condition when the center span, only, is loaded with the condition where all spans are loaded.

By loading of the end spans also the following changes were produced in the bending moments of the center span:

(1) The maximum positive bending moment was reduced from $0.075wl^2$ to $0.025wl^2$.

(2) The maximum negative bending moment was increased from $0.050wl^2$ to $0.1wl^2$.

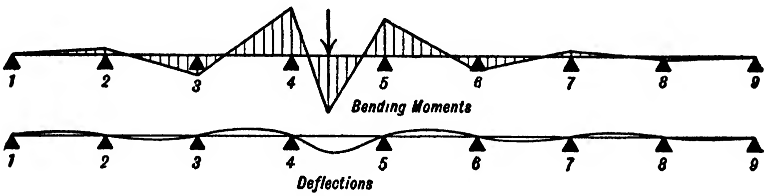


FIG. 3.—Continuous Beam of Eight Span Subjected to Concentrated Load. (*See p.* 7.)

(3) The points of contraflexure in the center span moved towards the center of the span thereby increasing the length of the sections subjected to negative bending moments and reducing the length of the section subjected to positive bending moment.

The effect of the load in one span upon the other spans of a multi-span beam may be studied from Figs. 3 and 4 showing the bending

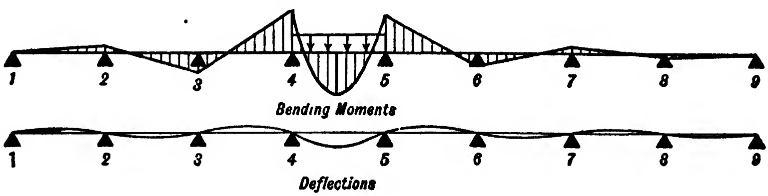


FIG. 4.—Continuous Beam of Eight Span Subjected to Uniform Load. (*See p.* 7.)

moments and deflections throughout a continuous beam produced by a concentrated load and by uniformly distributed load, respectively, placed in one span.

The largest effect of the loading is always upon the loaded spans. The next largest effect is produced on the two adjoining spans at each side of the loaded span. The effect upon succeeding spans becomes smaller and smaller until in the end spans it is negligible.

The bending moment curve in the loaded span depends upon the type of loading and therefore is different for concentrated loads than for the uniformly distributed loads. The bending moment curves in the unloaded spans are straight lines for all types of loading.

It should be noted that the bending moments at both sides of each support are equal, because the bending moment from one span is transferred in full to the adjoining span. At both sides of both supports of the loaded span the bending moments are negative. At the adjoining supports the bending moments are positive, then they change alternately to negative and positive.

In the center of the loaded span the bending moment is positive. In the center of the spans adjacent to the loaded spans the bending moments are negative. In other spans the bending moments in center change alternately to positive and negative.

Position of Load for Absolute Maximum Bending Moments.—It is evident from previous discussion that the loading of other spans has an appreciable influence upon the bending moments in any one span. By proper loading of other spans the bending moments at any one section may be either increased or decreased within certain definite limits. There is always a position of loading for which the bending moment at any one point is an absolute maximum. The position of loading, however, is different for absolute maximum bending moments at the support and in the center of the span.

The following general rules may be formulated for obtaining absolute maximum bending moments:

To get the absolute maximum negative bending moment in a continuous beam at any one support it is necessary to load the two spans on both sides of that support and then every alternate span, as shown in Fig. 5. p. 8.

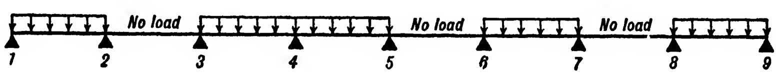


Fig. 5.—Position of Load for Maximum Negative Bending Moment at Support 4. (*See p.* 8.)

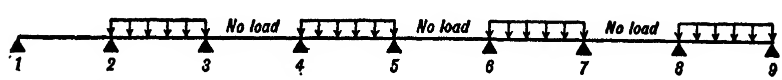


Fig. 6.—Position of Load for Maximum Positive Bending Moment at Center of Span 4–5. (*See p.* 9.)

To get the absolute maximum positive bending moment in the center of any span it is necessary to load the span under consideration and then the alternate spans, as shown in Fig. 6, p. 8.

It will be noted that the positions of the load for the two cases are different. Also it is important to observe that when the negative bending moments at the supports are largest the positive bending moments in the central portion are smallest. For conditions of loading giving largest positive bending moments the negative bending moments are small.

Points of Contraflexure.—Points of contraflexure, also called points of inflection, are the points in a continuous beam where the bending moments change from positive to negative. The bending moment at the points of inflection is zero.

A loaded interior span has two points of contraflexure, one near each support, while a loaded exterior span has one point of contraflexure near the inside support. The locations of the points of contraflexure in the loaded span are not fixed but change according to the condition of loading in adjacent spans. They are nearest the supports for loading producing maximum positive bending moments and farthest from the support for loading producing maximum negative bending moments.

In unloaded interior spans there is only one point of contraflexure. Unloaded exterior spans have no points of contraflexure.

By comparing Fig. 3 for concentrated load and Fig. 4 for uniformly distributed load it is evident that in the unloaded spans the position of points of contraflexure, i.e., the points at which the straight moment lines intersect the horizontals, is the same for both cases. The same is true for any other type of loading, therefore these points are called fixed points (a translation of the German term "Feste Punkte"). Depending only upon the span length, they can be easily located and used to advantage for graphical determination of bending moments in continuous beams. (See p. 153.)

Continuous Beam Replaced by Simple Beam and Cantilevers.—A continuous beam may be replaced by simple beams and cantilevers by cutting the beam at the points of contraflexure (where the bending moments are zero) and connecting the parts by hinged connections sufficiently strong to transfer the shear.

This is illustrated in Fig. 7, p. 10, showing the bending moments in a span of a continuous beam and also simple spans and cantilevers with their corresponding bending moments. To make the matter clear the simple beams are shown as suspended from the cantilevers instead of hinged. This does not affect the bending moments.

The change of the continuous beam into a group of simple beams and cantilevers produces no change in the magnitude of the bending moments. The central portion of the span behaves like a simply supported beam of a span equal to the distance between the points of contraflexure and loaded by the load resting upon it. The portions at the column behave like double cantilevers, pivoted at the support, with arms equal to the distance from the support to the point of inflection. The loading of each cantilever consists of the reaction of the central portion applied at the point of contraflexure and of the load coming directly upon the cantilever.

If the positions of the points of contraflexure are known the bending moments can be easily found by statics. It should be noted that for

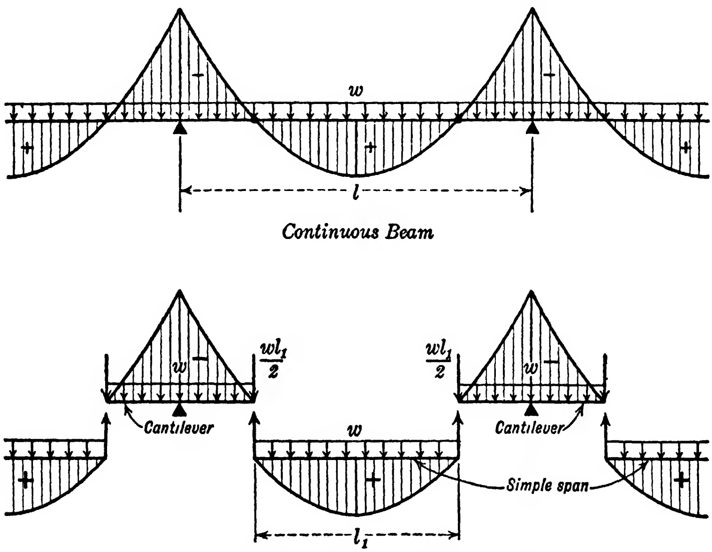


FIG. 7.—Continuous Beam Replaced by Simple Beams. (*See p.* 9.)

each type of loading the location of the points of contraflexure is different. The change of the beam into simple parts would therefore reproduce the conditions in a continuous beam only for the type of loading for which the location of point of contraflexure corresponds to the position of the hinges.

This property is sometimes utilized in bridge construction by using, instead of continuous beams, cantilevered beams combined with simple spans resting on cantilevers.

Signs of Bending Moments.—In this chapter the signs customary in reinforced concrete design are used for designating the character of the bending moments.

The bending moment is negative when it produces in a horizontal beam tension on the top of the beam and compression on the bottom of the beam.

The bending moment is positive when it produces in a horizontal beam tension on the bottom and compression on top.

Thus in a loaded span of a continuous beam, negative bending moments act at the supports and positive bending moments in the center.

This method of designation does not take into account the direction in which the bending moment turns. For instance, the bending moments at both supports of a loaded span are considered as negative although they turn in opposite directions.

Signs of Shear.—The conventional sign method for shears are used. The shear at any section is positive when it tends to move a section upward in relation to its original position. The shear tending to move the section downward is negative.

Thus at left support upward reaction is called positive and the downward loads negative. A shear diagram drawn on this basis will be placed above the base in the left part of the beam and below the base in the right part of the beam.

Notation.—The following notation is used in this chapter.

l_r = length of rth span;
x = distance of any point from left support;
V_s = static reaction due to the loads;
V_r = supplementary reaction due to continuity at the rth support;
V_{r+1} = supplementary reaction due to continuity at the $(r + 1)$th support;
V_l = left end shear in continuous beam;
M_s = static bending moment due to the loads;
M'_x = bending moment at any point caused by V_r and V_{r+1};
M_x = total bending moment in continuous beam.
M_r = bending moment at the rth support;
M_{r+1} = bending moment at the $(r+1)$th support;

Basis for Formulas for Continuous Beams.—Consider a continuous beam consisting of any number of spans and subjected to any type of loading as shown in Fig. 8, p. 12. For the purpose of investigation one span, in this case the rth from the left support, is assumed as detached at the supports from the rest of the beam. It is obvious that, to maintain the same conditions as existed in the span before it was detached, it is necessary to add at the supports to the static reactions and static bending moments produced by the loads additional bending moments and reactions caused by the continuity of the beam. This will be accomplished by adding at each support a bending moment and a supplementary reaction. Thus at the rth support will be added the bending moment M_r and the reaction V_r and at the $(r + 1)$th support the bending M_{r+1} and the reaction V_{r+1}.

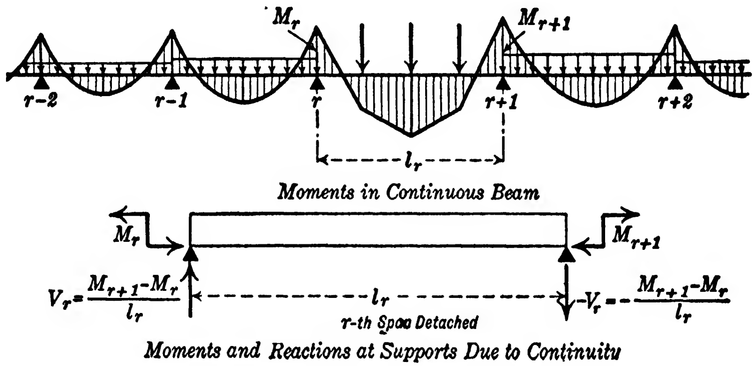


FIG. 8.—Bending Moment and Shears Due to Continuity. (See p. 11.)

Since the signs of these values are not known they are considered as positive. Their actual sign will be determined by the sign of the result of the computations. Thus from computations it may follow that for uniform loading the bending moment is $M_r = -\frac{wl^2}{\alpha}$, which means that the bending moment actually has a negative sign. Also the result may show that $V_r = -\frac{wl}{\beta}$, which means that this reaction acts downward, i.e., in opposite direction to the static reaction.

To keep the span in equilibrium these four values, M_r, M_{r+1}, V_r and V_{r+1}, must be in equilibrium themselves, so that the sum of both reactions must be equal to zero and the sum of all bending moments also must be equal to zero. Since the sum of the reactions is zero the reaction V_r must be equal to the reaction V_{r+1} and they must act in opposite directions. The two equal reactions acting in opposite directions form a couple which produces in a beam a bending moment equal to the reaction V_r multiplied by the span length l_r.

Since the sum of all bending moments must be equal to zero the bending moment M_r plus the bending moment due to the couple must be equal to the bending moment M_{r+1} and must turn in the opposite direction.[1] This may be expressed by

[1] It should be noted that when M_r and M_{r+1} are both negative and therefore of

$$M_r + V_r l_r = M_{r+1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

from which

Supplementary Reaction Due to Continuity,

$$V_r = \frac{M_{r+1} - M_r}{l_r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The three unknown quantities M_r, M_{r+1} and V_r added to replace the effect of continuity in the detached span may, therefore, be expressed in terms of two unknown bending moments at the supports M_r and M_{r+1}. These are called statically indeterminate values since they cannot be determined by the rules of statics alone.

Bending Moments at Any Point Due to Continuity.—The bending moments at the supports M_r and M_{r+1} and the supplementary reactions $+ V_r$ and $- V_r$ produce bending moments at every point of the beam of magnitude given in equation below.

Bending Moment at Any Point Due to Continuity,

$$M'_x = M_r + \frac{M_{r+1} - M_r}{l_r}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Actual Bending Moments in Continuous Beam.—The loads on a span produce at every point a static bending moment M_s. To get the actual bending moment at any point in a continuous beam it is necessary to add to the static bending moment M_s due to the loads the bending moment due to continuity from Formula (3). Thus

Actual Bending Moment in Continuous Beam,

$$M_x = M_r + \frac{M_{r+1} - M_r}{l_r}x + M_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For M_r and M_{r+1} above should be substituted their values with their signs. Usually the first and the second items of Equation (4) are negative so that the actual bending moment M_x is smaller than the static bending moment M_s.

Actual End Shear in Continuous Beam.—The actual end shear at the left support in a continuous beam equals the static end shear V_s plus the supplementary reaction V_r. Substituting for V_r the value from Formula (2)

the same sign as far as the span in question is concerned, they actually turn in opposite directions. The bending moment M_r turns from right to left and the bending moment M_{r+1} turns from left to right.

Left End Shear in Continuous Span,

$$V_l = V_s + \frac{M_{r+1} - M_r}{l_r}. \qquad (5)$$

The shear V_x at any point equals the static shear V_{sx} plus V_r. Thus *Shear at Any point,*

$$V_x = V_{sx} + \frac{M_{r+1} - M_r}{l_r}. \qquad (6)$$

When both M_r and M_{r+1} are negative, the value of $\frac{M_{r+1} - M_r}{l_r}$ is negative if M_{r+1} is larger than M_r. In such case the actual end shear at the left support as obtained from Equation (5) is smaller than the static end shear. The following rule, then, may be formulated:

In a continuous beam the support at which the bending moment is largest has also the largest end shear.

At the right support the end shear is obtained by adding to the static reaction the value $(-V_r)$ or $-\frac{M_{r+1} - M_r}{l_r}$. Also the end shear may be obtained by subtracting from the total load in the span the left end shear.

Bending Moment Diagram for Continuous Beam.—The conditions in a span of continuous beam are clearly shown in Fig. 9 (*a*) to (*c*). In Fig. 9 (*a*) are shown the static bending

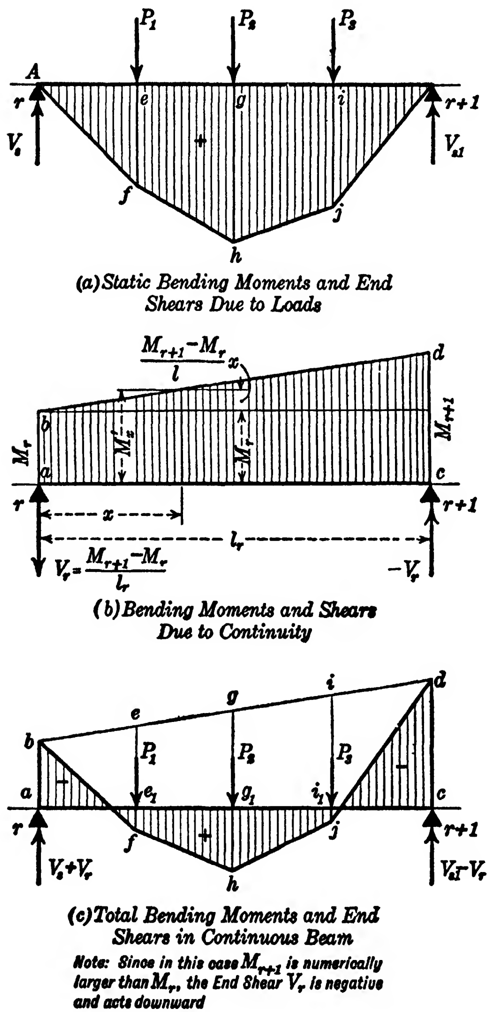


FIG. 9.—Bending Moments in a Span of Continuous Beam. (*See p.* 14.)

moments due to the loads which may be obtained by drawing a funicular polygon. The moments are positive and are plotted below the axis. In Fig. 9 (*b*) are shown the bending moments and reactions produced by the continuity of the beam. The scale for bending moments is same as in (*a*). It is assumed that M_r and M_{r+1} are negative and they are plotted above the axis. In Fig. 9 (*c*) are shown the actual bending moments in a continuous beam obtained by combining bending moments shown in (*a*) and in (*b*). Near the columns the bending moments are negative and in the central portion they are positive. Fig. 9 (*c*) is drawn by plotting the moment area due to continuity *abcd* first. The line *bd* is accepted as closing line for the static bending moments. Starting from this closing line the static bending moments are plotted.

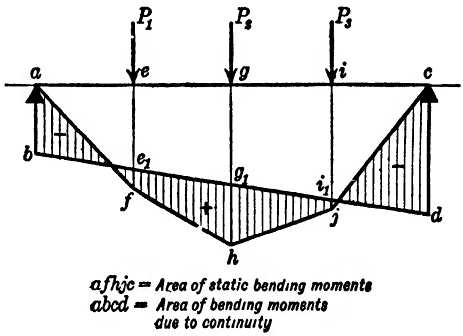


FIG. 10.—Method of Combining Bending Moments in Continuous Beams. (*See p.* 15.)

Another method of combining the static bending moment with the bending moments due to continuity is shown in Fig. 10, p. 15.

After the static bending moment diagram *afhjc* is drawn, the bending moments at the support due to continuity *ab*, and *cd* are plotted on vertical lines below the axis. The line *bd* is accepted as the closing line. All moments below this line are positive and above it negative. The moments are scaled on vertical lines, starting from line *bd*.

BASIC THREE-MOMENT EQUATION FOR CONTINUOUS BEAMS

As evident from Formula 4, p. 13, the bending moment at any point in a continuous beam can be expressed by the static bending moment and the two bending moments acting at the supports. The problem is solved when the two bending moments at the supports are found. They are statically indeterminate values and in their determination it is necessary to use the elastic properties of the beam. A number of methods may be used, the theory of least work and the slope-deflection method given in Chapter IX are the best known.

From elastic properties of the beam following relation is found between the bending moments at any three succeeding supports of a continuous beam. It was first developed by Clapeyron and is justly called the "Clapeyron's equation."

Notation.

Let $r, r + 1$, and $r + 2$ = three succeeding supports of a continuous beam;
l_r = span length of the rth span;
l_{r+1} = span length of the $(r + 1)$th span;
M_r = bending moment at left support, rth span;
M_{r+1} = bending moment at right support, rth span;
M_{r+2} = bending moment at right support, $(r + 1)$th span;
M_{sr} = static bending moment at any point of rth span;
M_{sr+1} = static bending moment at any point of the $(r + 1)$th span.

Then

Basic Three-moment Equation (Clapeyron's Equation). (See. Fig. 11, p. 17.)

$$M_r l_r + 2M_{r+1}(l_r + l_{r+1}) + M_{r+2}l_{r+1} =$$
$$- 6\left[\frac{1}{l_r}\int_0^{l_r} M_{sr}x dx + \frac{1}{l_{r+1}}\int_0^{l_{r+1}} M_{sr+1}(l_{r+1} - x)dx\right]. \quad . \quad . \quad (7)$$

In the above equation it is assumed that the beam has a constant moment of inertia and that the supports are unyielding and on the same level. The equation is developed in Chapter IX.

For solving problems in practice no knowledge of calculus is needed as in all final formulas the integrals are solved and given as simple co-efficients. However, for the benefit of those who desire to go into the mathematical treatment the development of the formulas are also given.

Use of Three-moment Equation.—By the use of the three-moment equation it is possible to find the bending moments at all the supports of a continuous beam. The manner of procedure is shown by the example below.

Let M_1 = bending moment at first support (zero for free ends);
M_2 = bending moment at second support;
M_3 = bending moment at third support;
M_4 = bending moment at fourth support;

M_5 = bending moment at fifth support;
l_1 = span length, first span;
l_2 = span length, second span;
l_3 = span length, third span;
l_4 = span length, fourth span;
M_{s1} = static bending moment at any point, first span;
M_{s2} = static bending moment at any point, second span;
M_{s3} = static bending moment at any point, third span;
M_{s4} = static bending moment at any point, fourth span.

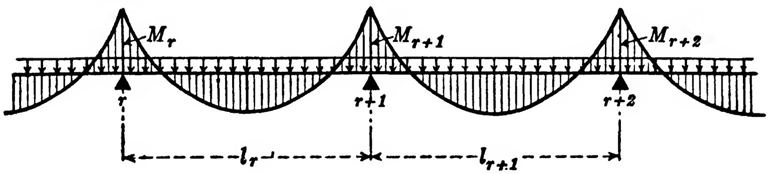


FIG. 11.—Spans Used in Three-moment Equation. (*See p.* 16.)

Assume a continuous beam of, say, four spans. Also assume that the ends of this continuous beam are free. Starting at the left, the following three-moment equation may be written for the first two spans:

$$M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2 = -6\left[\frac{1}{l_1}\int_0^{l_1} M_{s1}x\,dx + \frac{1}{l_2}\int_0^{l_2} M_{s2}(l_2 - x)\,dx\right].$$

Since the left end is free, $M_1 = 0$. Therefore

$$1.\ 2M_2(l_1 + l_2) + M_3l_2 = -6\left[\frac{1}{l_1}\int_0^{l_1} M_{s1}x\,dx + \frac{1}{l_2}\int_0^{l_2} M_{s2}(l_2 - x)\,dx\right].$$

Now consider the second and third spans,

$$2.\ M_2l_2 + 2M_3(l_2 + l_3) + M_4l_3 = -6\left[\frac{1}{l_2}\int_0^{l_2} M_{s2}x\,dx + \frac{1}{l_3}\int_0^{l_3} M_{s3}(l_3 - x)\,dx\right].$$

Finally consider the third and fourth spans,

$$M_3l_3 + 2M_4(l_3 + l_4) + M_5l_4 = -6\left[\frac{1}{l_3}\int_0^{l_3} M_{s3}x\,dx + \frac{1}{l_4}\int_0^{l_4} M_{s4}(l_4 - x)\,dx\right].$$

Since the beam is free at both supports $M_5 = 0$. Therefore

$$3.\ M_3l_3 + 2M_4(l_3 + l_4) = -6\left[\frac{1}{l_3}\int_0^{l_3} M_{s3}xdx + \frac{1}{l_4}\int_0^{l_4} M_{s4}(l_4 - x)dx\right].$$

Thus are obtained three simultaneous equations marked 1, 2, and 3 which are sufficient for finding the three unknown values M_2, M_3, and M_4.

Solving of Integrals.—The integrals in the above equations depend upon the type of loading in the respective spans. In general the integrals are of two types, namely:

$$\int_0^l M_sxdx \quad \text{and} \quad \int_0^l M_s(l - x)dx.$$

The integral $\int_0^l M_sxdx$ represents the static moment about the *left* support of the area obtained by plotting the static bending moments at each point.

The integral $\int_0^l M_s(l - x)dx$ represents the static moment of the same static bending moment area about the *right* support.

The values may be obtained by solving the integral or by drawing the bending moment diagram, computing its area, determining its center of gravity, and finally multiplying the area by the distance of its center of gravity from the respective support. When the bending moment area is complicated it may be divided into triangles and rectangles where areas and centers of gravity can be easily computed.

Analytically the integrals may be solved by substituting a formula for M_s and integrating.

Integrals for Uniformly Distributed Loads.—For uniformly distributed loading w extending over the whole span the values of the integrals [2] are

$$\int_0^l M_sxdx = \int_0^l M_s(l - x)dx = \frac{1}{24}wl^4.$$

Therefore

$$\frac{6}{l}\int_0^l M_sxdx = \frac{6}{l}\int_0^l M_s(l - x)dx = \frac{1}{4}wl^3. \quad . \quad . \quad . \quad . \quad (8)$$

[2] For uniformly distributed loads $M_s = \frac{1}{2}wx(l - x)$. This substituted in the integrals gives

$$\int_0^l M_sxdx = \tfrac{1}{2}w\int_0^l (l - x)x^2dx = \frac{1}{24}wl^4.$$

Integrals for Concentrated Load.—For concentrated load P placed at a distance a from left support the integrals [3] are

$$\frac{6}{l}\int_0^l M_s x dx = \frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]Pl^2 = C_1Pl^2; \quad . \quad . \quad . \quad (9)$$

where

$$C_1 = \frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and

$$\frac{6}{l}\int_0^l M_s(l - x)dx = \frac{a}{l}\left(1 - \frac{a}{l}\right)\left(2 - \frac{a}{l}\right)Pl^2 = C_2Pl^2; \quad . \quad (11)$$

where

$$C_2 = \frac{a}{l}\left(1 - \frac{a}{l}\right)\left(2 - \frac{a}{l}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The values of C_1 and C_2 for different $\frac{a}{l}$ may be taken from Diagram 1, p. 19.

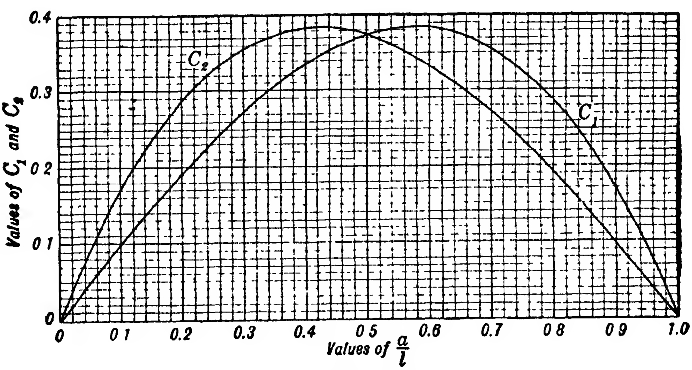


DIAGRAM 1.—Constants C_1 and C_2 for Continuous Beams. (*See p.* 19)

Integrals for Several Loads.—The integrals for several loads may be obtained by computing separately the integrals for each load and adding.

[3] For a concentrated load P, placed at a distance a, the static bending moment is $M_s = P\left(1 - \frac{a}{l}\right)x$ for x smaller than a and $M_s = P\frac{a}{l}(l - x)$ for x larger than a. These substituted in the integrals gives

$$\int^l M_s x dx = \left(1 - \frac{a}{l}\right)P\int_0^a x^2 dx + \frac{a}{l}P\int_a^l (l - x)x dx = P\frac{al^2}{6}\left[1 - \left(\frac{a}{l}\right)^2\right]$$

Similarly can be solved $\int_0^l M_s(l - x)dx$.

Thus for load P_1, P_1' and P_1'' placed at distances a_1, a_1' and a_1'', respectively, from left support the integrals become

$$\frac{6}{l}\int_0^l M_s x dx = l^2\left[\frac{a_1}{l}\left(1 - \left(\frac{a_1}{l}\right)^2\right)P_1 + \frac{a'_1}{l}\left(1 - \left(\frac{a'_1}{l}\right)^2\right)P'_1 + \frac{a''_1}{l}\left(1 - \left(\frac{a''_1}{l}\right)^2\right)P''_1\right].$$

This also may be written

$$\frac{6}{l}\int_0^l M_s x dx = l^2\Sigma\frac{a_1}{l}\left(1 - \left(\frac{a_1}{l}\right)^2\right)P_1 = l^2\Sigma P_1 C_1 \quad . \quad (13)$$

where C_1 can be obtained for $\frac{a_1}{l}$, $\frac{a'_1}{l}$, and $\frac{a''_1}{l}$ from diagram on p. 19.

In the same manner

$$\frac{6}{l}\int_0^l M_s(l - x)dx = l^2\left[\frac{a_1}{l}\left(1 - \frac{a_1}{l}\right)\left(2 - \frac{a_1}{l}\right)P_1 + \frac{a'_1}{l}\left(1 - \frac{a'_1}{l}\right)\left(2 - \frac{a'_1}{l}\right)P'_1 + \frac{a''_1}{l}\left(1 - \frac{a''_1}{l}\right)\left(2 - \frac{a''_1}{l}\right)P''_1\right],$$

also

$$\frac{6}{l}\int_0^l M_s(l - x)dx = l^2\Sigma\frac{a_1}{l}\left(1 - \frac{a_1}{l}\right)\left(2 - \frac{a_1}{l}\right)P_1 = l^2\Sigma P_1 C_2 \quad (14)$$

where C_2 can be obtained for $\frac{a_1}{l}$, $\frac{a'_1}{l}$ and $\frac{a''_1}{l}$ from diagram on p. 19.

When all loads in a span are equal the load P in expression ΣPC_1 and ΣPC_2 may be taken before the summation sign. Thus for equal loads

$$\Sigma PC_1 = P\Sigma C_1 \text{ and } \Sigma PC_2 = P\Sigma C_2 \quad . \quad . \quad . \quad . \quad (15)$$

Integrals for Symmetrical Loading.—For loading symmetrical about the center of the beam the values of both integrals are equal. The area of the statical bending moment diagram is symmetrical about the center so that its center of gravity is in the center of the beam. Therefore the static moment of this area about the left support is equal to the static moment about the right support.

For symmetrical loading

$$\Sigma PC_1 = \Sigma PC_2$$

Use of Three-moment Equations for Beam with Fixed Ends.—In a continuous beam with fixed ends there are unknown bending moments at the fixed ends in addition to the unknown bending moments at the interior supports. The number of unknown values is, therefore, larger than in a continuous beam with simply supported ends by the number of fixed ends. If both ends are fixed there are two more unknown values. With one end fixed there is only one additional unknown value. To get the equations necessary for finding the additional number of unknown values the fixity of each end is expressed by assuming at the fixed end an additional span the length of which is infinitely small. Thus the first two spans considered in making up the three-moment equations will be the imaginary infinitely small span and the first actual span, and the last two spans are the last actual span and the imaginary infinitely small span. In such fashion are obtained just as many more equations as there are fixed ends which is sufficient for finding bending moments at all supports.

RELATION BETWEEN MAXIMUM POSITIVE BENDING MOMENT AND THE NEGATIVE BENDING MOMENTS AT SUPPORTS

Very often the negative bending moments at the supports are known and it is desired to get the corresponding maximum positive bending moments.

The maximum positive bending moments always occur at the point of zero shear. Therefore, to find the maximum positive bending moment compute the left end shear first, then find the point of zero shear and finally compute the bending moment at the point of zero shear. Following formulas can be used.

Notation.—

Let l = span length;
a = location of concentrated load measured from left support;
x_1 = distance of point of maximum positive bending moment from left support;
P = any concentrated load;
w = uniformly distributed loading;
M_l = negative bending moment at left support;
M_r = negative bending moment at right support;
M_{max} = maximum positive bending moment;
V_l = left end shear;
V_{x1} = shear at point x_1;
C_M = constant in $M_{max} = C_M wl^2$;
C_V = constant in $V_l = C_V wl$ and $x_1 = C_V l$.

Concentrated Loads.—

Left End Shear,

$$V_l = \Sigma P\left(1 - \frac{a}{l}\right) + \frac{M_r - M_l}{l}. \quad . \quad . \quad . \quad . \quad (16)$$

If point of zero shear measured from left support is x_1 then

Maximum Positive Bending Moment,

$$M_{\max} = M_l + V_l x_1 - \Sigma P(x_1 - a). \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In the above formulas M_r and M_l must be used with their signs.

Uniformly Distributed Loading.—

Left End Shear,

$$V_l = wl\left(\frac{1}{2} + \frac{M_r - M_l}{wl^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Point of Maximum Positive Bending Moment,[4]

$$x_1 = \left(\frac{1}{2} + \frac{M_r - M_l}{wl^2}\right)l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Maximum Positive Bending Moment,[5]

$$M_{\max} = M_l + \frac{1}{2}\left(\frac{1}{2} + \frac{M_r - M_l}{wl^2}\right)^2 wl^2. \quad . \quad . \quad . \quad (20)$$

Table for Maximum Positive Bending Moment. Uniform Loading.—For known negative bending moments at the supports and for uniformly distributed loading, the maximum positive bending moment may be easily found from table p. 176.

To use the table find the coefficients of the bending moments at the supports, namely, $\frac{M_l}{wl^2}$ and $\frac{M_r}{wl^2}$. Locate these values in the table and obtain a coefficient C_M, which multiplied by wl^2 gives the maximum positive bending moment. Thus

$$M_{\max} = C_M wl^2. \qquad (21)$$

[4] At the point of maximum bending moment the shear is zero. $V_{x1} = V_l - wx_1 = 0$, hence $wx_1 = V_l$ and $x_1 = \frac{V_l}{w}$. By substituting the value for V_l the above formula for x_1 is obtained.

[5] $M_{\max} = M_l + V_l x_1 - \frac{1}{2}wx_1^2$. By substituting the value for V_l and x_1 and combining, the above formula is obtained.

Table for Left End Shear and Point of Maximum Positive Bending Moment. Uniform Loading.—For known negative bending moments at supports and for uniformly distributed loading, the left end shear V_l and the point of maximum positive bending x_1 may be found from table on p. 177.

To use the table find the coefficients of the bending moments at the supports, namely, $\frac{M_l}{wl^2}$ and $\frac{M_r}{wl^2}$. Locate these values in the table and obtain the corresponding value of C_V. Then

Left End Shear,

$$V_l = C_V wl. \tag{22}$$

Point of Maximum Positive Bending Moment,

$$x_1 = C_V l. \tag{23}$$

Example.—The bending moments at the supports are

$$M_l = -0.11wl^2, \quad M_r = -0.08wl^2.$$

Find the maximum positive bending moments, the point of maximum positive bending moment and the left end shear.

Solution.—In table on p. 176, locate $\frac{M_l}{wl^2} = -0.11$ and $\frac{M_r}{wl^2} = -0.08$ and get $C_M = 0.03$.

Therefore

$$M_{\max} = 0.03wl^2.$$

From table on p. 177, for $\frac{M_l}{wl^2} = -0.11$ and $\frac{M_r}{wl^2} = -0.08$ find $C_V = 0.53$.

Therefore

Left End Shear,

$$V_l = 0.53wl^2.$$

Point of Maximum Positive Bending Moment,

$$x_1 = 0.53l.$$

BEAM FIXED AT BOTH SUPPORTS

Beams can be considered as fixed at both supports when their ends are firmly imbedded in the support and the support is sufficiently strong to keep the tangents to the deflection curve horizontal at supports. Such beams have two statically indeterminate values, namely, the two negative bending moments at the supports M_1 and M_2. Using the expedient for fixed ends discussed in the previous paragraphs the following equations may be derived from the three-moment equations.

Basic Equations for Fixed Span,

$$2M_1 + M_2 = -\frac{6}{l^2}\int_0^l M_s(l-x)dx, \quad \ldots . \quad (24)$$

$$M_1 + 2M_2 = -\frac{6}{l^2}\int_0^l M_s x dx, \quad \ldots \ldots \quad (25)$$

from which

Negative Bending Moment at Left Support,

$$M_1 = -\frac{2}{l^2}\left[2\int_0^l M_s(l-x)dx - \int_0^l M_s x dx\right]. \quad . . \quad (26)$$

Negative Bending Moment at Right Support,

$$M_2 = -\frac{2}{l^2}\left[2\int_0^l M_s x dx - \int_0^l M_s(l-x)dx\right] . \quad . . \quad (27)$$

The values of the integrals $\int_0^l M_s x dx$ and $\int_0^l M_s(l-x)dx$ in the above equations are worked out on p. 18 for uniform loading and for concentrated loads.

Several special cases are given below.

Uniform Loading. (See Fig. 12, p. 24.)

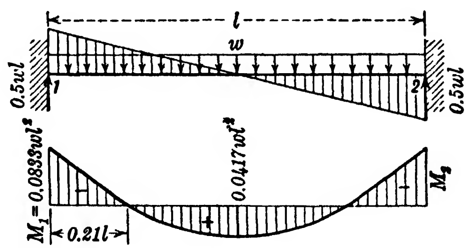


FIG. 12.—Beam Fixed at Both Ends. Uniform Loading. (*See p.* 24.)

End Shears,

$$V_1 = V_2 = \tfrac{1}{2}wl. \quad \ldots\ldots\ldots\ldots \quad (28)$$

Negative Bending Moments,

$$M_1 = M_2 = -\tfrac{1}{12}wl^2. \ldots\ldots\ldots\ldots \quad (29)$$

Maximum Positive Bending Moments,

$$M_{\max} = \tfrac{1}{24}wl^2. \qquad (30)$$

Bending Moment at Any Point x,

$$M_x = \tfrac{1}{12}\left[-1 + 6\frac{x}{l} - 6\left(\frac{x}{l}\right)^2\right]wl^2. \qquad (31)$$

Concentrated Loading. (See Fig. 13, p. 25.)

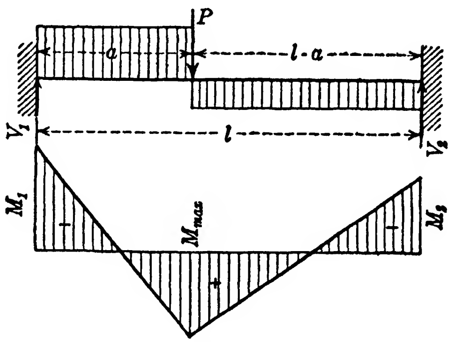


FIG. 13.—Beam Fixed at Both Ends. Concentrated Load. (*See p.* 25.)

End Shears,

$$V_1 = \left[1 - \left(\frac{a}{l}\right)^2\left(3 - 2\frac{a}{l}\right)\right]P = C_3P, \qquad (32)$$

$$V_2 = (1 - C_3)P. \qquad (33)$$

Negative Bending Moments,

$$M_1 = -\frac{a}{l}\left(1 - \frac{a}{l}\right)^2 Pl = -C_4Pl, \qquad (34)$$

$$M_2 = -\left(\frac{a}{l}\right)^2\left(1 - \frac{a}{l}\right)Pl = -C_5Pl. \qquad (35)$$

Maximum Positive Bending Moments,

$$M_{\max} = M_1 + V_1a = 2\left(\frac{a}{l}\right)^2\left(1 - \frac{a}{l}\right)^2 Pl = C_6Pl. \qquad (36)$$

Bending Moment at Any Point x,

$$M_x = \left(\frac{x}{l}C_3 - C_4\right)Pl \text{ for } x \text{ smaller than } a, \quad . \quad . \quad . \quad . \quad (37)$$

$$M_x = \left[\frac{x}{l}(C_3 - 1) + \frac{a}{l} - C_4\right]Pl \text{ for } x \text{ larger than } a. \quad . \quad (38)$$

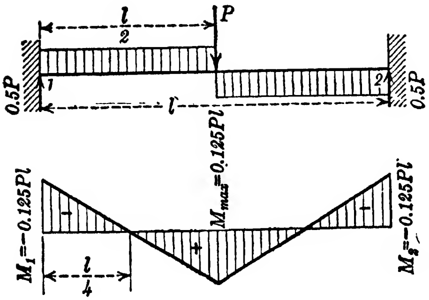


FIG. 14.—Beam Fixed at Both Ends. Load P at Center. (*See p.* 26).

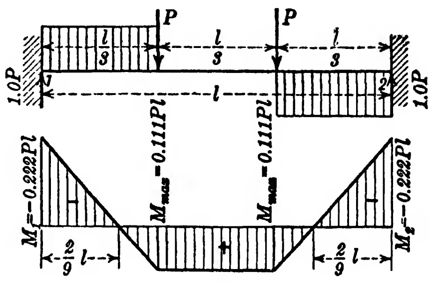


FIG. 15.—Beam Fixed at Both Ends. Loads P at Third Points. (*See p.* 26.)

The values of constants C_3, C_4, C_5 and C_6 for different values of $\frac{a}{l}$ are given in Diagrams 2, p. 27.

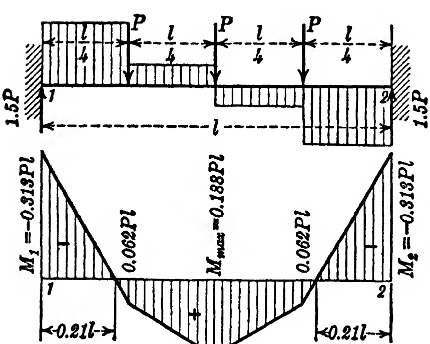


FIG. 16.—Beam Fixed at Both Ends. Loads P at Quarter Points. (*See p.* 26.)

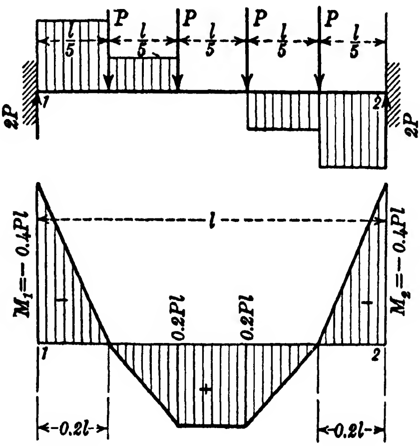


FIG. 17.—Beam Fixed at Both Ends. Loads P at Fifth Points. (*See p.* 26.)

Symmetrical Arrangements of Concentrated Loads. (See Figs. 14 to 17, p. 26.)—Bending moments and end shears for special sym-

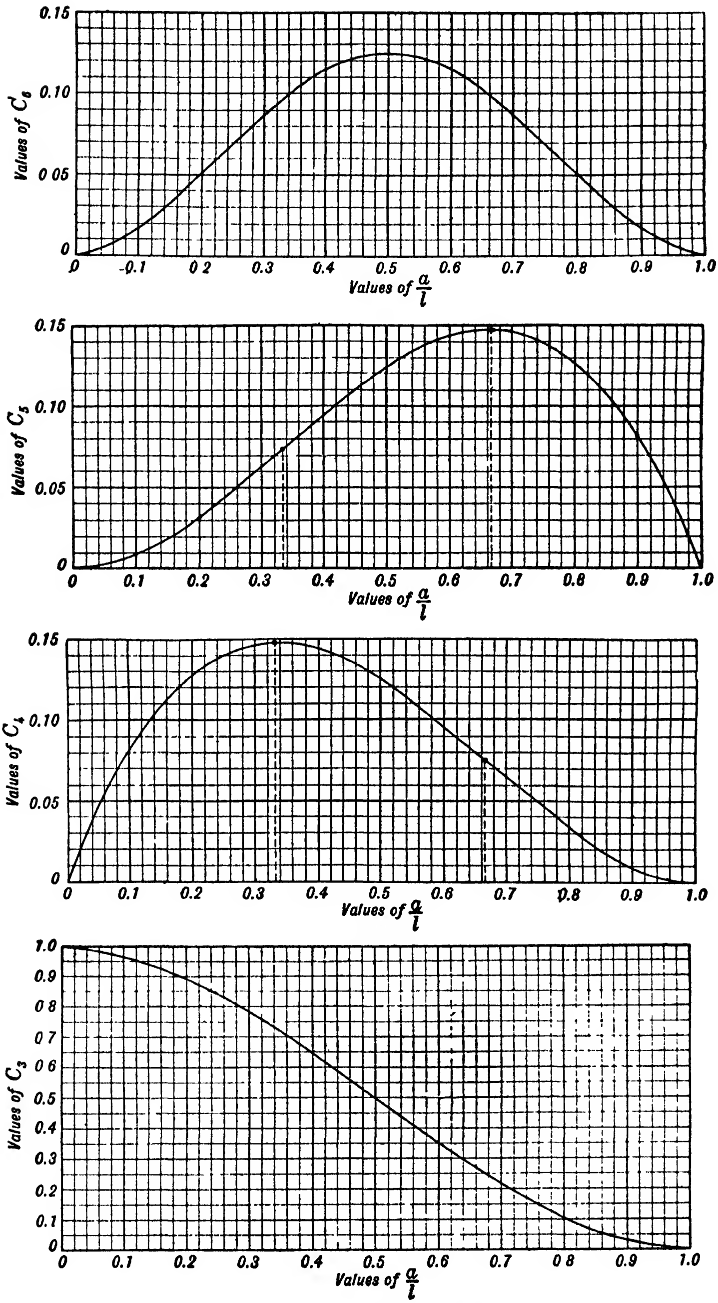


DIAGRAM 2.—Constants C_3, C_4, C_5, and C_6 for Beam Fixed at Both Ends. (*See p.* 26.)

metrical arrangements of equal concentrated loads P are given in table below, and shown in Figs. 14 to 17.

Beam with Both Ends Fixed. Symmetrical Arrangement of Equal Concentrated Loads

Bending Moments and End Shears

Loading	End Shear	Bending Moments		
		Negative at Supports, M_1 and M_2	Maximum Positive, M_{max}	Maximum Static Bending Moment, M_s
$1P$ at center	$0.5P$	$-0.125Pl$	$0.125Pl$	$0.250Pl$
$2P$ at $\frac{1}{3}$ points	$1.0P$	$-0.222Pl$	$0.111Pl$	$0.333Pl$
$3P$ at $\frac{1}{4}$ points..........	$1.5P$	$-0.313Pl$	$0.188Pl$	$0.5Pl$
$4P$ at $\frac{1}{5}$ points....	$2.0P$	$-0.4Pl$	$0.2Pl$	$0.6Pl$

BEAM FIXED AT ONE SUPPORT

A beam fixed at one support and free at the other, has one statically indeterminate value, namely, the bending moment at the fixed support. In the formulas below, the left end is free and the right end is fixed. The general formula is obtained by making in Formula (24) $M_1 = 0$. Therefore

$$2M_2 = -\frac{6}{l^2}\int_0^l M_s x dx, \quad \ldots \ldots \quad (39)$$

from which

Negative Bending Moment at Fixed Support,

$$M_2 = -\frac{3}{l^2}\int_0^l M_s x dx. \quad \ldots \ldots \quad (40)$$

The values of the integral $\int_0^l M_s x dx$ may be found as explained on p. 18.

A number of special cases are developed below.

Uniform Load. (See Fig. 18, p. 29.)

End Shear at Free Support,

$$V_1 = \tfrac{3}{8}wl = 0.375wl. \quad \ldots \ldots \ldots \quad (41)$$

End Shear at Fixed Support,

$$V_2 = \tfrac{5}{8}wl = 0.625wl. \qquad (42)$$

Negative Bending Moment,

$$M_2 = -\tfrac{1}{8}wl^2 = -0.125wl^2. \qquad (43)$$

Maximum Positive Bending Moment,

$$M_{\max} = \tfrac{9}{128}wl^2 = 0.07wl^2. \qquad (44)$$

Concentrated Load General Formulas. (See Fig. 19, p. 29.)

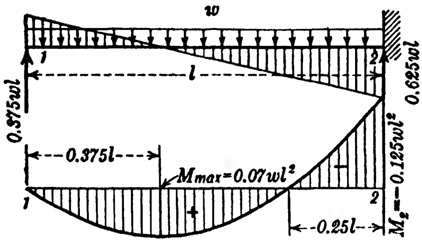


FIG. 18.—Beam Fixed at Right End. Uniform Load. (*See p.* 28.)

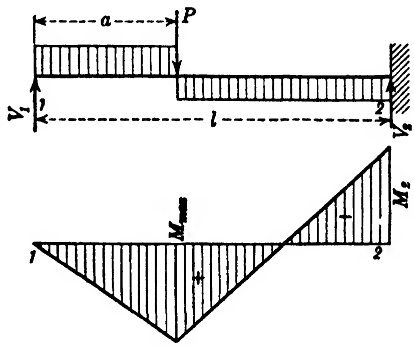


FIG. 19.—Beam Fixed at Right End. Concentrated Load. (*See p.* 29.)

End Shear at Free End,

$$V_1 = \left[1 - \frac{3}{2}\frac{a}{l} + \frac{1}{2}\left(\frac{a}{l}\right)^3\right]P = C_7P. \qquad (45)$$

End Shear at Fixed End,

$$V_2 = P - V_1. \qquad (46)$$

Negative Bending Moment,

$$M_2 = -\frac{1}{2}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]Pl = -C_8Pl. \qquad (47)$$

Positive Bending Moment,

$$M_{\max} = V_1a. \qquad (48)$$

Constants C_7 and C_8 may be taken from Diagram 3, p. 30, for the proper ratio $\frac{a}{l}$.

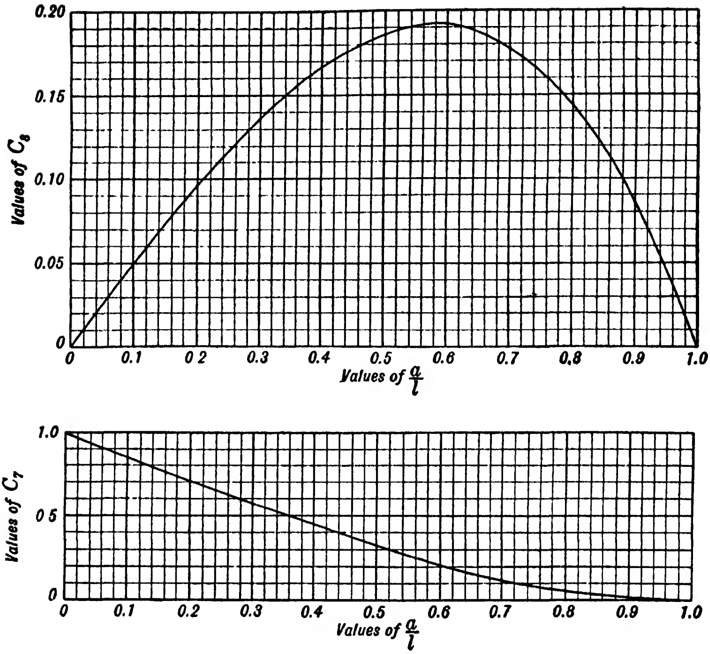


DIAGRAM 3.—Constants C_7 and C_8 for Beam Fixed at One End. (*See p.* 29.)

Special Arrangements of Concentrated Loads. (See Figs. 20 to 23, p. 31.)—Bending moments and end shears for special symmetrical arrangements of equal concentrated loads P are given in table below, and are shown in Figs. 20 to 23.

Beam with One End Fixed. Symmetrical Arrangement of Equal Concentrated Loads

Bending Moment and End Shears

Loading	End Shears		Bending Moments		
	Free End	Fixed End	Negative at Support, M_2	Maximum Positive, M_{max}	Maximum Static Bending, Moment, M_s
$1P$ at center	$0.312P$	$0.688P$	$-0.187Pl$	$0.156Pl$	$0.25Pl$
$2P$ at $\frac{1}{3}$ points	$0.667P$	$1.333P$	$-0.333Pl$	$0.222Pl$	$0.333Pl$
$3P$ at $\frac{1}{4}$ points	$1.031P$	$1.969P$	$-0.469Pl$	$0.266Pl$	$0.5Pl$
$4P$ at $\frac{1}{5}$ points	$1.4P$	$2.6P$	$-0.6Pl$	$0.36Pl$	$0.6Pl$

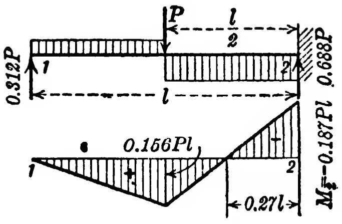


FIG. 20. — Beam Fixed at Right End. Load P in Center. (*See* p. 30.)

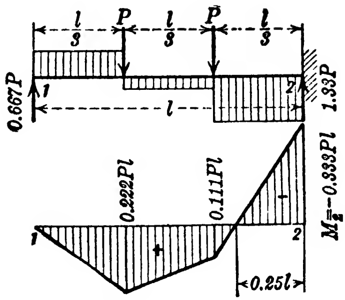


FIG. 21.—Beam Fixed at Right End. Two Loads P, at Third Points. (*See* p. 30.)

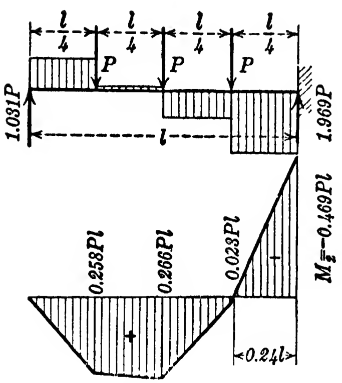


FIG. 22.—Beam Fixed at Right End. Three Loads P, at Quarter Points. (*See* p. 30.)

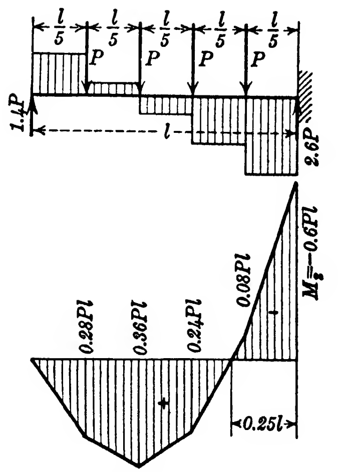


FIG. 23.—Beam Fixed at Right End. Four Loads P, at Fifth Points. (*See* p. 30.)

TWO SPANS WITH FREE ENDS

A continuous beam consisting of two spans with free ends, as shown in Fig. 24, p. 32, has only one statically indeterminate value, namely, the negative bending moment at the center support. Therefore one equation is sufficient for solving the problem. This is obtained from the three-moment equation. Thus

Basic Equation for Two Spans,

$$M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2 = -6\left[\frac{1}{l_1}\int_0^{l_1} M_{s1}xdx + \frac{1}{l_2}\int_0^{l_2} M_{s2}(l_2 - x)dx\right].$$

Since the ends are free $M_1 = M_3 = 0$ and

General Equation for Bending Moment at Support,

$$M_2 = -\frac{\left[\frac{6}{l_1}\int_0^{l_1} M_{s1}xdx + \frac{6}{l_2}\int_0^{l_2} M_{s2}(l_2 - x)dx\right]}{2(l_1 + l_2)} \quad . \quad . \quad (49)$$

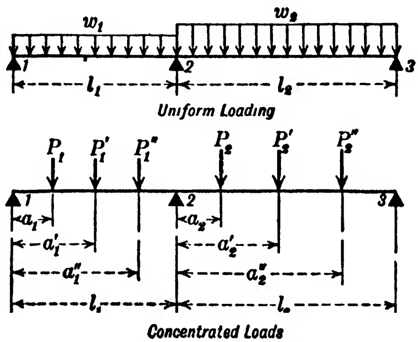


FIG. 24.—Continuous Beam of Two Spans, Free Ends. (*See p.* 31.)

Using the values for integrals $\frac{6}{l_1}\int_0^{l_1} M_{s1}xdx$ and $\frac{6}{l_2}\int_0^{l_2} M_{s2}(l - x)dx$ worked out on p. 18 for uniformly distributed and concentrated loads, respectively, the formulas become:

Negative Bending Moment for Uniform Loading,

$$M_2 = -\frac{w_1l_1^3 + w_2l_2^3}{8(l_1 + l_2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Negative Bending Moment for Concentrated Loading,

$$M_2 = -\frac{l_1^2\Sigma P_1\frac{a_1}{l_1}\left[1-\left(\frac{a_1}{l_1}\right)^2\right] + l_2^2\Sigma P_2\frac{a_2}{l_2}\left(1-\frac{a_2}{l_2}\right)\left(2-\frac{a_2}{l_2}\right)}{2(l_1 + l_2)}. \quad (51)$$

Also using constants in Diagram 1, p. 19,

$$M_2 = -\frac{l_1^2\Sigma P_1C_1 + l_2^2\Sigma P_2C_2}{2(l_1 + l_2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

ΣP_1C_1 means the sum of all loads in first span, each multiplied by the constant C_1 from Diagram 1, p. 19, corresponding to its $\frac{a_1}{l_1}$.

ΣP_2C_2 means the sum of all loads in second span each multiplied by the constant C_2 from Diagram 1, p. 19, corresponding to its $\frac{a_2}{l_2}$.

The maximum shears may be obtained from the following equation.

End Shears for Uniform Load,

$$V_1 = \frac{w_1l_1}{2} + \frac{M_2}{l_1}, \quad \text{(53)}$$

$$V_{2l} = \frac{w_1l_1}{2} - \frac{M_2}{l_1}, \quad \text{(54)}$$

$$V_{2r} = \frac{w_2l_2}{2} - \frac{M_2}{l_2}, \quad \text{(55)}$$

$$V_3 = \frac{w_2l_2}{2} + \frac{M_2}{l_2}. \quad \text{(56)}$$

End Shears for Concentrated Loads,

$$V_1 = \Sigma P_1\left(1 - \frac{a_1}{l_1}\right) + \frac{M_2}{l_1}, \quad \text{(57)}$$

$$V_{2l} = \Sigma P_1\frac{a_1}{l_1} - \frac{M_2}{l_1}, \quad \text{(58)}$$

$$V_{2r} = \Sigma P_2\left(1 - \frac{a_2}{l_2}\right) - \frac{M_2}{l_2}, \quad \text{(59)}$$

$$V_3 = \Sigma P_2\frac{a_2}{l_2} + \frac{M_2}{l_2}. \quad \text{(60)}$$

Reactions on Supports,

The reactions on end supports are equal to the end shears.

The reaction on the central support is equal to the sum of the end shears on both sides. Thus

$$R_2 = V_{2l} + V_{2r}. \quad \text{(61)}$$

Two Equal Spans. Ends Free. Uniform Load

For two equal spans the formulas are obtained by substituting in Formulas (49) to (60), pp. 32 and 33,

$$l_1 = l_2 = l.$$

When the spans are loaded with uniformly distributed load and the intensity of loading in both spans is the same then in Formula (50), p. 32,

$$w_1 = w_2 = w.$$

When one span is loaded and the other not loaded, then for the loaded span $w_1 = w$ and for the unloaded span $w_2 = 0$.

The formulas below are worked out on this basis.

Both Spans Loaded. (See Fig. 25, p. 35.)

Condition for maximum negative bending moment.

End Shears,

$$V_1 = \tfrac{3}{8}wl = 0.375wl. \quad . \quad . \quad (62) \qquad V_{2l} = \tfrac{5}{8}wl = 0.625wl. \quad . \quad . \quad (63)$$

$$V_{2r} = V_{2l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (64) \qquad V_3 = V_1 = 0.375wl. \quad . \quad . \quad (65)$$

Negative Bending Moment,

$$M_2 = -\tfrac{1}{8}wl^2 = -0.125wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

Maximum Positive Bending Moment,

$$M_{\max} = \tfrac{9}{128}wl^2 = 0.0703wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

Bending Moment at Any Point x,

$$M_x = \frac{wx}{8}(3l - 4x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

x measured from free end.

Left Span Loaded. (See Fig. 26, p. 35.)

Condition for maximum positive bending moment in left span.

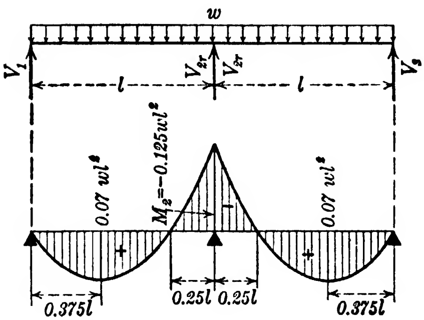
End Shears,

$$V_1 = \tfrac{7}{16}wl = 0.4375wl. \quad . \quad (69) \qquad V_{2l} = 0.5625wl. \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

$$V_{2r} = \tfrac{1}{16}wl = 0.0625wl. \quad . \quad (71) \qquad V_3 = -\tfrac{1}{16}wl = -0.0625wl. \quad . \quad (72)$$

Negative Bending Moment,

$$M_2 = -\tfrac{1}{16}wl^2 = -0.0625wl^2. \quad . \quad . \quad . \quad . \qquad (73)$$



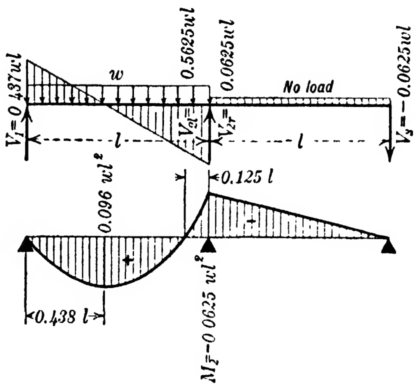


FIG. 25.—Two Equal Spans. Free Ends. Both Spans Loaded. (*See p.* 34.)

FIG. 26.—Two Equal Spans, Free Ends. Left Span Loaded. (*See p.* 34.)

Maximum Positive Bending Moment,

$$M_{\max} = \tfrac{49}{512}wl^2 = 0.096wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (74)$$

Bending Moment at Any Point x in Loaded Span,

$$M_x = \tfrac{1}{16}\frac{x}{l}\left(7 - 8\frac{x}{l}\right)wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (75)$$

TWO EQUAL SPANS. FREE ENDS. CONCENTRATED LOADS

Concentrated Load *P* in Left Span at Distance *a* from Support. (See Fig. 27, p. 35.)

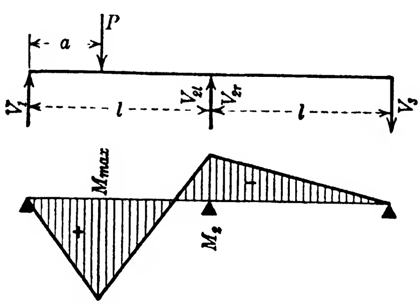


FIG. 27.—Two Equal Spans. Concentrated Load *P* in Left Span.
(*See p.* 35.)

End Shears,

$$V_1 = \left[\left(1 - \frac{a}{l}\right) - \frac{1}{4}\frac{a}{l}\left(1 - \left(\frac{a}{l}\right)^2\right)\right]P = C_9 P. \quad . \quad . \quad . \quad (76)$$

$$V_{2l} = (1 - C_9)P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

$$V_{2r} = \frac{1}{4}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]P = C_{10}P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

$$V_3 = - C_{10}P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

Negative Bending Moment,

$$M_2 = - \frac{1}{4}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]Pl = - C_{10}Pl. \quad . \quad . \quad . \quad (80)$$

Maximum Positive Bending Moment,

$$M_{\max} = V_1 a = C_9 Pa. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

Constants C_9 and C_{10} may be taken from Diagram 4, p. 36 for proper values of $\frac{a}{l}$.

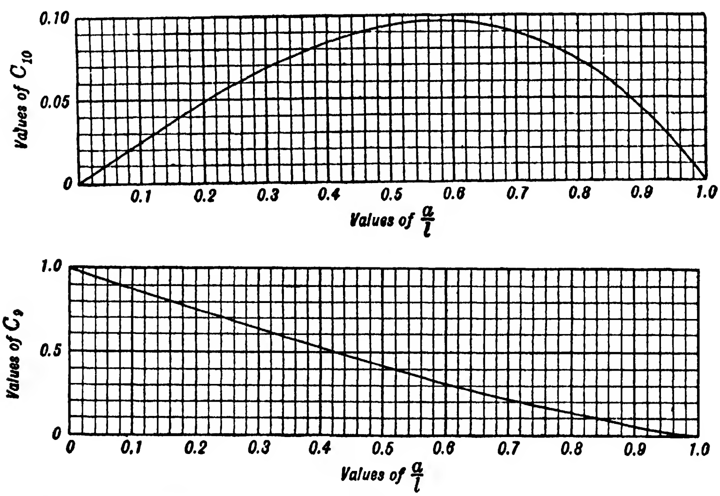


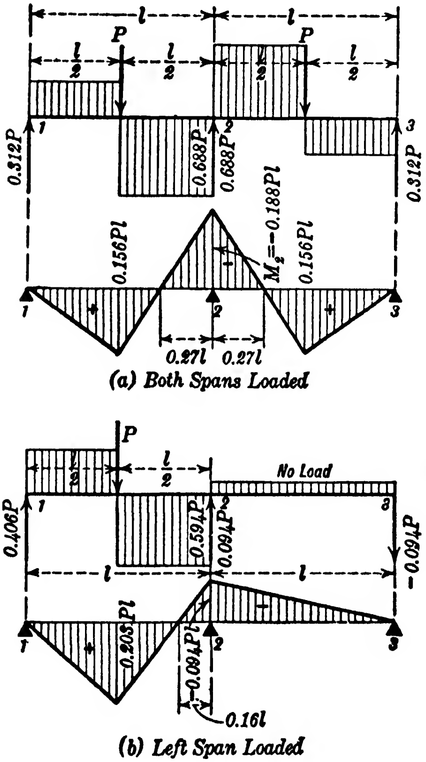
DIAGRAM 4.—Constants C_9 and C_{10} for Two Equal Spans, Concentrated Loads. (*See p.* 36.)

Symmetrical Arrangements of Concentrated Loads.—Following arrangements of concentrated loadings will be considered:

1. One load P at center of span.

2. Two loads P at third points.
3. Three loads P at quarter points.
4. Four loads P at fifth points.

The values in the table on p. 39 may be used for continuous girders which carry cross beams so that the load on the girder is concentrated.



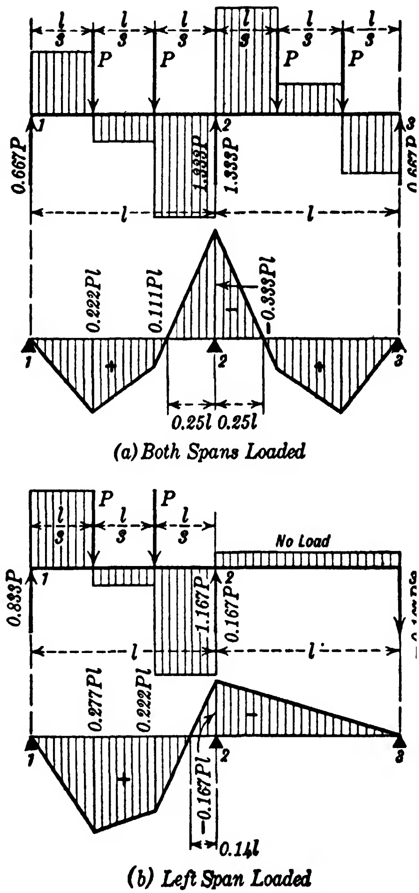


FIG. 28.—Two Equal Spans, Free Ends. Load P at Center. (*See p.* 36.)

FIG. 29.—Two Equal Spans, Free Ends Two Loads P at Third Points. (*See p.* 37).

Two conditions of loading are considered:

(*a*) Both spans loaded.

(*b*) Left span loaded.

The first condition gives maximum negative bending moment at the support and the other condition maximum positive bending moment in the left span.

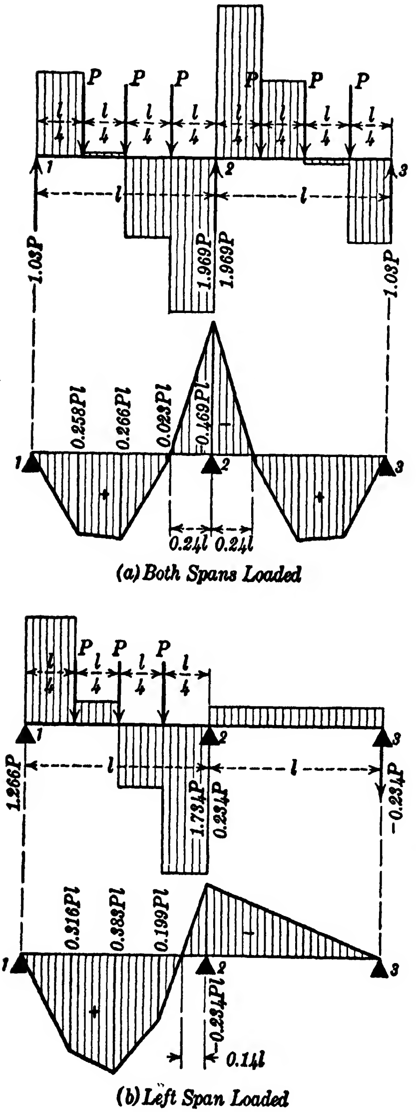


Fig. 30.—Two Equal Spans, Free Ends. Three Loads P at Quarter Points. (*See* p. 37.)

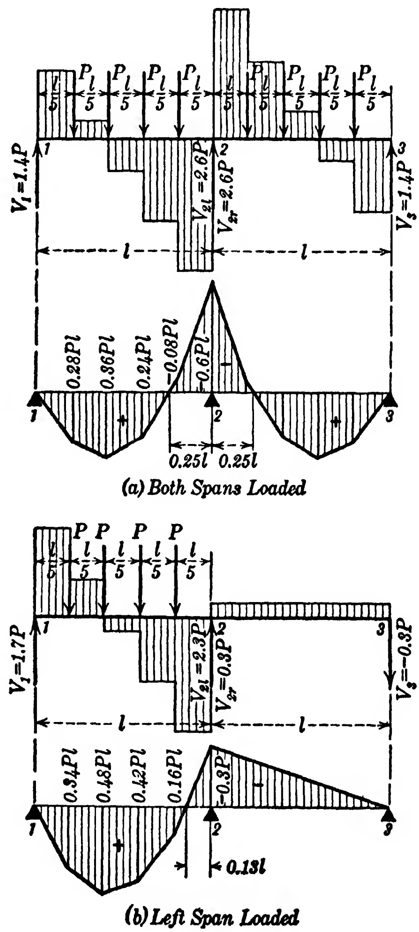


Fig. 31.—Two Equal Spans, Free Ends. Four Loads P at Fifth Points. (*See* p. 37.)

Two Equal Spans, Free Ends. Symmetrical Arrangements of Concentrated Loads P

Bending Moments and End Shears

Spans Loaded	End Shears				Bending Moment		
	V_1	V_{2l}	V_{2r}	V_3	Negative at Support, M_2	Maximum Positive, M_{max}	Static Bending Moment, M_s
				One Load P at Center			
Both Spans....	0 312P	0 688P	0 688P	0 312P	−0 188Pl	0.156Pl	0 25Pl
Left Span.....	0.406P	0.594P	0 094P	−0 094P	−0 094Pl	0 203Pl	
				Two Loads P at Third Points			
Both Spans ...	0 667P	1 333P	1 333P	0.667P	−0 333Pl	0 222Pl	0 333Pl
Left Span.....	0 833P	1.167P	0 167P	−0.167P	−0 167Pl	0.278Pl	
				Three Loads P at Quarter Points			
Both Spans....	1 031P	1.969P	1 969P	1 031P	−0 469Pl	0 266Pl	0.5Pl
Left Span	1.266P	1.734P	0 234P	−0 234P	−0 234Pl	0 383Pl	
				Four Loads P at Fifth Points			
Both Spans....	1 4P	2.6P	2.6P	1 4P	−0 6Pl	0 36Pl	0 6Pl
Left Span.....	1.7P	2.3P	0 3P	−0 3P	−0 3Pl	0 48Pl	

TWO UNEQUAL SPANS. UNIFORM LOAD

The formulas for two unequal spans may be simplified by expressing the length of the shorter span in terms of the longer span. Thus, if $l_1 = 30$ ft. and $l_2 = 20$ ft., then $l_2 = \frac{20}{30}l_1 = \frac{2}{3}l_1$.

Let $l_1 = l =$ length of longer span;

$l_2 = ml =$ length of the shorter span;

$m =$ ratio of span length $\dfrac{l_2}{l_1}$.

Substituting these values in general equations, following simple formulas are obtained.

Uniform Load. Both Spans Loaded. (See Fig. 32, p. 41.)

Condition for maximum negative bending moment.

End Shears,

$$V_1 = \tfrac{1}{8}(3 + m - m^2)wl = D_1wl. \qquad (82)$$

$$V_{2l} = wl - V_1 = (1 - D_1)wl. \qquad (83)$$

$$V_{2r} = wml - V_3 = (1 - D_2)wml. \qquad (84)$$

$$V_3 = \frac{1}{8}\left(3 + \frac{1}{m} - \frac{1}{m^2}\right)wml = D_2wml. \qquad (85)$$

Negative Bending Moment,

$$M_2 = -\tfrac{1}{8}(1 - m + m^2)wl^2 = -D_3wl^2. \quad (86)$$

Points of Inflection (Measured from Center Support):

Long span,

$$x_i = [1 - \tfrac{1}{4}(3 + m - m^2)]l = (1 - 2D_1)l. \quad (87)$$

Short span,

$$x_{im} = \left[1 - \frac{1}{4}\left(3 + \frac{1}{m} - \frac{1}{m^2}\right)\right]ml = (1 - 2D_2)ml. \quad (88)$$

Maximum Positive Bending Moment:

Long span,

$$M_{max} = \tfrac{1}{128}(3 + m - m^2)^2wl^2 = D_4wl^2. \quad (89)$$

Short span,

$$M_{max} = \tfrac{1}{128}\left(3 + \frac{1}{m} - \frac{1}{m^2}\right)^2 w(ml)^2 = D_5w(ml)^2. \quad (90)$$

This formula is good only for m larger than 0.43. For smaller m there is no positive bending moment in the short span.

Points of Maximum Positive Bending Moment:

Long span,

$$x_l = \tfrac{1}{8}(3 + m - m^2)l = D_1l \text{ (measured from support 1).} \quad (91)$$

Short span,

$$x_s = \frac{1}{8}\left(3 + \frac{1}{m} - \frac{1}{m^2}\right)ml = D_2ml \text{ (measured from support 3).} \quad (92)$$

The value of constants D_1 to D_4 are given in Diagram 5, p. 43, and constant D_5 in Diagram 6 p. 44, for different ratios of m.

Uniform Load. Long Span Loaded. Short Span Not Loaded. (See Fig. 33, p. 41.)

Left span assumed to be long span and right span to be short span.

End Shears,

$$V_1 = \frac{3 + 4m}{8(1 + m)}wl = D_6wl. \quad (93)$$

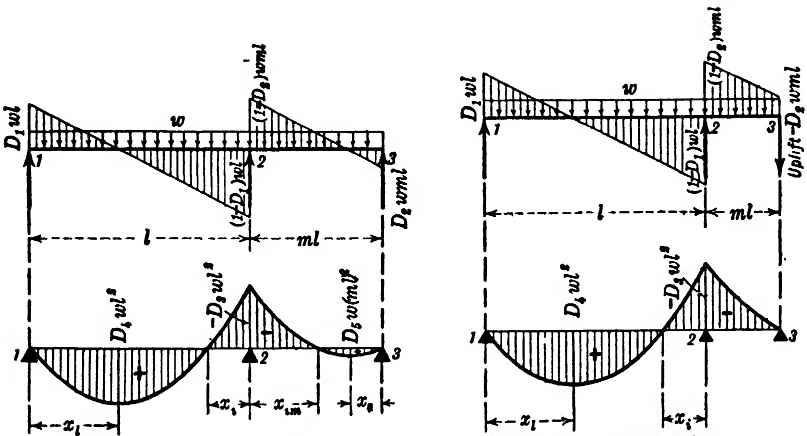
$$V_{2l} = wl - V_1 = (1 - D_6)wl. \quad (94)$$

$$V_{2r} = -V_3 = D_7wl. \quad (95)$$

$$V_3 = -\frac{1}{8m(1 + m)}wl = -D_7wl. \quad (96)$$

Negative Bending Moment,

$$M_2 = -\frac{1}{8(1+m)}wl^2 = -D_8wl^2. \quad . \quad . \quad . \quad (97)$$



(a) Bending Moments when Ratio m Larger than 0.43

(b) Bending Moments when Ratio m Smaller than 0.43

FIG. 32.—Two Unequal Spans, Free Ends. Both Spans Loaded. (*See p.* 39.)

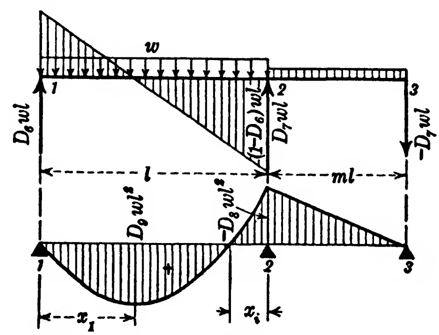


Fig. 33.—Two Unequal Spans, Free Ends. Long Span Loaded. (*See p.* 40.)

Point of Inflection, Long Span (Measured from Support 2),

$$x_i = \left(1 - \frac{3+4m}{4(1+m)}\right)l = (1 - 2D_6)l. \quad . \quad (98)$$

Maximum Positive Bending Moment,

$$M_{max} = \frac{(3+4m)^2}{128(1+m)^2}wl^2 = D_9wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

Point of Maximum Positive Bending Moment,

$$x_1 = \frac{3 + 4m}{8(1 + m)} l = D_6 l \text{ (measured from support 1)} . \quad (100)$$

Uniform Load. Short Span Loaded and Long Span Not Loaded. (See Fig. 34, p. 42.)

Condition for maximum positive bending moment, short span.

End Shears,

$$\left.\begin{aligned} V_1 &= -\frac{m^2}{8(1 + m)} w(ml) = -D_{10}\, w(ml) \\ V_{2l} &= -V_1 = D_{10}\, w(ml) \end{aligned}\right\} \text{long span} \quad (101)$$

$$\left.\begin{aligned} V_{2r} &= w(ml) - V_3 = (1 - D_{11}) w(ml) \\ V_3 &= \frac{3m + 4}{8(1 + m)} w(ml) = D_{11}\, w(ml) \end{aligned}\right\} \text{short span} \quad (102)$$

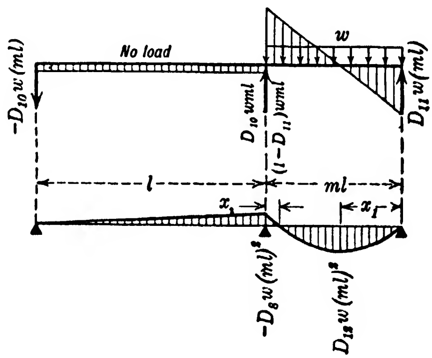


FIG. 34.—Uniform Load Two Unequal Spans, Free Ends. Short Span Loaded. (*See p.* 42.)

Negative Bending Moment,

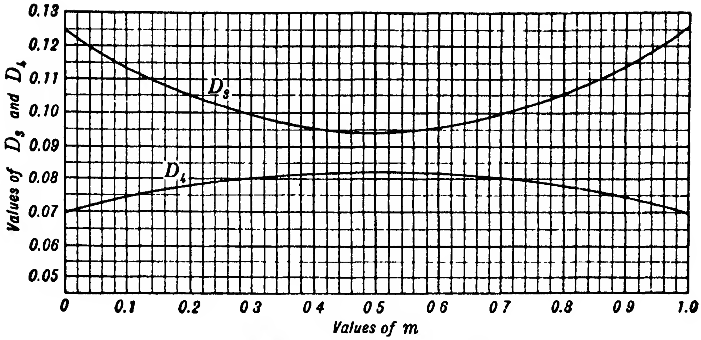
$$M_2 = -\frac{m}{8(1 + m)} w(ml)^2 = -D_{10} wm(l)^2. \quad . \quad . \quad . \quad (103)$$

Point of Inflection, Short Span (Measured from Support 2),

$$x_i = \left(1 - \frac{3m + 4}{4(1 + m)}\right) ml = (1 - 2D_{11}) lm . \quad . \quad . \quad (104)$$

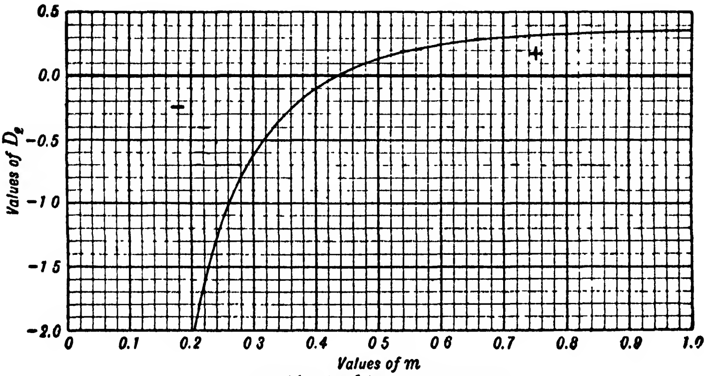
Maximum Positive Bending Moment,

$$M_{max} = \frac{1}{2}\left(\frac{3m + 4}{8(1 + m)}\right)^2 w(ml)^2 = D_{12} w(ml)^2. \quad . \quad . \quad . \quad (105)$$

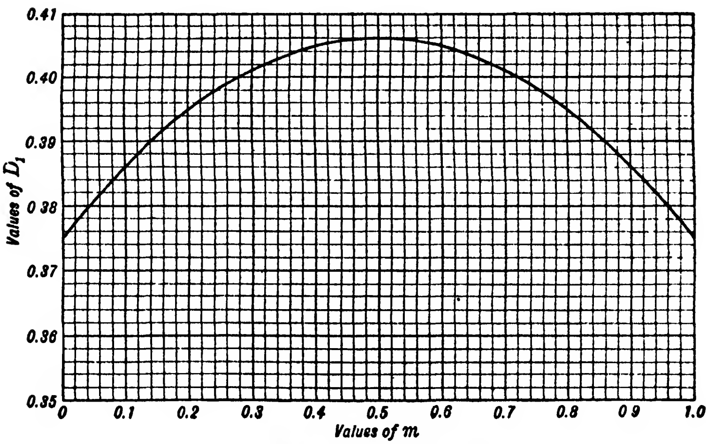


$D_3 = \frac{1}{8}(1 - m + m^2)$ in $M_2 = -D_3 wl^2$

$D_4 = \frac{1}{128}(3 + m - m^2)^2$ in $M_{max} = D_4 wl^2$



$D_2 = \frac{1}{8}(3 + \frac{1}{m} - \frac{1}{m^3})$ in $V_3 = D_2 w(ml)$



$D_1 = \frac{1}{8}(3 + m - m^2)$ in $V_1 = D_1 wl$

DIAGRAM 5.—Constants D_1, D_2, D_3 and D_4 for Two Unequal Spans. *(See p. 40.)*

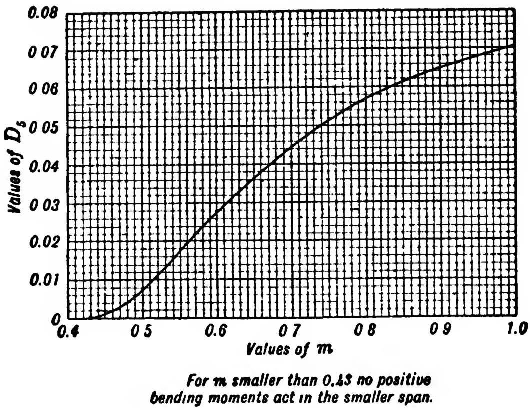
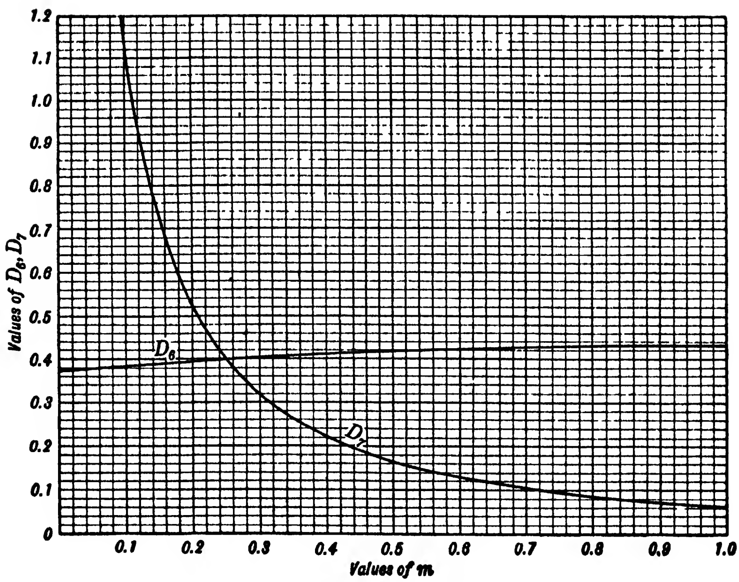


DIAGRAM 6.—Constant D_5 for Two Unequal Spans. (*See p.* 40.)



$$D_6 = \frac{3+4m}{8(1+m)} \text{ in formula } V_1 = D_6 wl$$

$$D_7 = \frac{1}{8m(1+m)} \text{ in formula } V_3 = -D_7 wl$$

DIAGRAM 7.—Constants D_6 and D_7 for Two Unequal Spans. (*See p.* 40.)

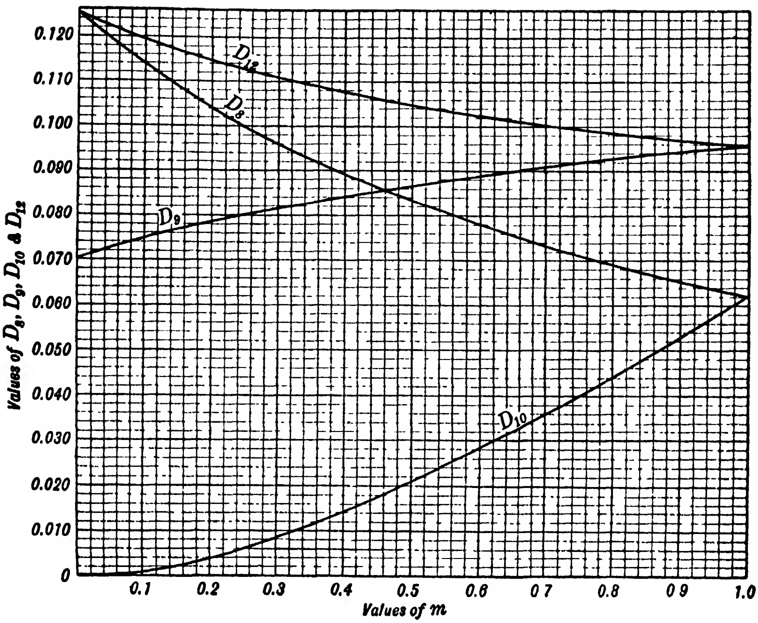


Diagram 8.—Constants D_8, D_9, D_{10} and D_{12} for Two Unequal Spans. (*See p.* 41.)

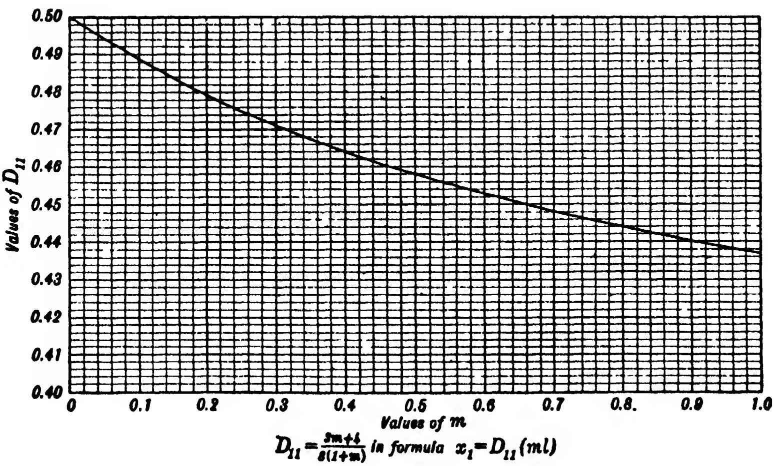


Diagram 9.—Constant D_{11} for Two Unequal Spans. (*See p.* 46.)

Point of Maximum Positive Bending Moment (Measured from Support 3),

$$x_1 = \frac{3m + 4}{8(1 + m)}(ml) = D_{11}(ml). \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

The values of constants are given in Diagrams 6 to 9, pp. 44 and 45 for different ratios of m.

Two Unequal Spans. Concentrated Loads

Concentrated Load P in the Long Span.

End Shears,

$$V_1 = \left[\left(1 - \frac{a}{l}\right) - E_1\right]P. \quad (107) \qquad V_{2l} = P - V_1. \quad . \quad . \quad . \quad (108)$$

$$V_{2r} = \frac{1}{m}E_1P. \quad . \quad . \quad . \quad . \quad (109) \qquad V_3 = - V_{2r}. \quad . \quad . \quad . \quad . \quad (110)$$

Negative Bending Moment,

$$M_2 = -\frac{1}{2(1 + m)}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]Pl = -E_1Pl. \quad . \quad . \quad . \quad (111)$$

Point of Inflection (Measured from Support 2),

$$x_1 = \frac{M_2}{V_{2l}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

Maximum Positive Bending Moment,

$$M_{\max} = \frac{a}{l}\left[\left(1 - \frac{a}{l}\right) - E_1\right]Pl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

Bending Moment at Any Point (x Measured from Support 1),

In loaded span,

$$M_x = V_1x = \frac{x}{l}\left[\left(1 - \frac{a}{l}\right) - E_1\right]Pl \text{ for } x \text{ smaller than } a. \quad . \quad . \quad . \quad (114)$$

$$M_x = V_1x - P(x - a) = \left[\frac{a}{l} - \frac{x}{l}\left(E_1 + \frac{a}{l}\right)\right]Pl \text{ for } x \text{ larger than } a. \quad (115)$$

In unloaded span, x measured from support 2,

$$M_x = (ml - x)\frac{M_2}{ml} = -E_1\left(1 - \frac{x}{ml}\right)Pl. \quad . \quad . \quad . \quad . \quad (116)$$

Concentrated Load P in the Short (Right) Span.

End Shear,

$$V_1 = - E_2P. \quad (117) \qquad V_{2l} = - V_1 = E_2P. \quad (118)$$

$$V_{2r} = P - V_3. \quad (119) \qquad V_3 = \left(\frac{a}{ml} - E_2\right)P. \quad (120)$$

Negative Bending Moment,

$$M_2 = - \frac{m}{2(1+m)}\left(\frac{a}{ml}\right)\left(1 - \frac{a}{ml}\right)\left(2 - \frac{a}{ml}\right)P(ml) = - E_2Pml. \quad (121)$$

Point of Inflection, (x_1 Measured from Support 2),

$$x_1 = \frac{M_2}{P - V_3} = \frac{E_2}{1 - \frac{a}{ml} + E_2}l. \quad (122)$$

Maximum Positive Bending Moment,

$$M_{max} = V_3(ml - a). \quad (123)$$

Bending Moment at Any Point:

In loaded span, x measured from support 2,

$$M_x = V_3(ml - x) - P(a - x)$$
$$= \left[E_5\left(1 - \frac{x}{ml}\right) - \frac{a}{ml} + \frac{x}{ml}\right]P(ml) \text{ for } x \text{ smaller than } a. \quad (124)$$

$$M_x = V_3(ml - x) \text{ for } x \text{ larger than } a. \quad (125)$$

In unloaded span, x measured from support 3,

$$M_x = \frac{M_2}{l}x = - E_2\frac{x}{l}Pl. \quad (126)$$

Anchorage of Short Span.—Whenever the reaction on the outside support of the short span for a proper combination of live and dead load is negative, the beam must be properly anchored to the support and the support must be heavy enough to resist the uplift.

When the ratio, m, of the short span to the long span is smaller than 0.43, there will be uplift at support 3 even for dead load.

The formulas for unequal spans apply only to beams capable of resisting uplift. If no resistance to uplift exists when it is required, the end of the short span moves up and it changes to a cantilever. The bending moment and shear for such case are different than for a continuous beam.

BEAM OF THREE SPANS. FREE ENDS

A continuous beam consisting of three spans with free ends has two statically indeterminate values, namely, the negative bending moment at the interior supports, M_2 and M_3 (see Fig. 35, p. 49). It is necessary to set up two equations using the three-moment theorem given on p. 16.

Following general equations are obtained.

Basic Equations for Three Spans,

$$(1)\quad 2M_2(l_1+l_2)+M_3l_2=-6\left[\frac{1}{l_1}\int_0^{l_1}M_{s1}\,xdx+\frac{1}{l_2}\int_0^{l_2}M_{s2}\,(l_2-x)dx\right]. \quad (127)$$

$$(2)\quad M_2l_2+2M_3(l_2+l_3)=-6\left[\frac{1}{l_2}\int_0^{l_2}M_{s2}\,xdx+\frac{1}{l_3}\int_0^{l_3}M_{s3}\,(l_3-x)dx\right]. \quad (128)$$

The values of the integrals are worked out on p. 18 for uniformly distributed and for concentrated loads. By substituting proper values for the integrals and solving the equations for M_2 and M_3, following formulas for the bending moments at supports are obtained.

Bending Moments at Supports. Uniformly Distributed Load of Different Intensities.

$$M_2=-\frac{2(l_2+l_3)w_1l_1^3+(l_2+2l_3)w_2l_2^3-l_2w_3l_3^3}{16(l_1+l_2)(l_2+l_3)-4l_2^2}, \quad (129)$$

and

$$M_3=-\frac{-l_2w_1l_1^3+(l_2+2l_1)w_2l_2^3+2(l_1+l_2)w_3l_3^3}{16(l_1+l_2)(l_2+l_3)-4l_2^2}. \quad (130)$$

Bending Moment at Supports. Concentrated Loads.

$$M_2=-\frac{2l_1^2(l_2+l_3)\Sigma P_1C_1+2l_2^2(l_2+l_3)\Sigma P_2C_2-l_2^3\Sigma P_2C_1-l_2l_3^2\Sigma P_3C_2}{4(l_1+l_2)(l_2+l_3)-l_2^2}. \quad (131)$$

$$M_3=-\frac{-l_1^2l_2\Sigma P_1C_1-l_2^3\Sigma P_2C_2+2l_2^2(l_1+l_2)\Sigma P_2C_1+2l_3^2(l_1+l_2)\Sigma P_3C_2}{4(l_1+l_2)(l_2+l_3)-l_2^2}. \quad (132)$$

In the above equations the constants are, in general,

$$C_1=\frac{a}{l}\left(1-\left(\frac{a}{l}\right)^2\right), \quad \text{and} \quad C_2=\frac{a}{l}\left(1-\frac{a}{l}\right)\left(2-\frac{a}{l}\right),$$

in which a is the distance of each particular load from left support of the span in which the load is located and l is the length of same spans.

The values of C_1 and C_2 may be taken from Diagram 1, p. 19.

ΣP_1C_1 and ΣP_1C_2 means the sum of all loads in the first span, each load multiplied by a constant C_1 and C_2 respectively corresponding to its ratio $\dfrac{a_1}{l_1}$.

ΣP_2C_1 and ΣP_2C_2 are similar sums for the second span, and finally ΣP_3C_1 and ΣP_3C_2 are similar sums for the third span.

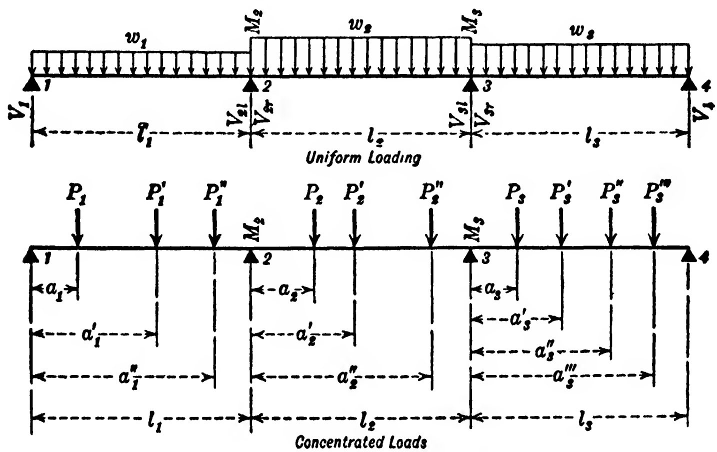


FIG. 35.—Continuous Beam of Three Spans, Free Ends. (*See p.* 48.)

End Shears.—End shears in the various spans may be found from the following equations which are based on Formulas 16 and 18, p. 22.

For Uniform Loads | For Concentrated Loads

End shear at first support

$$V_1 = \frac{w_1l_1}{2} + \frac{M_2}{l_1} \quad . \quad . \quad . \quad (133) \qquad V_1 = \Sigma P_1\left(1 - \frac{a_1}{l_1}\right) + \frac{M_2}{l_1} \quad . \quad . \quad . \quad (134)$$

End shear at second support left

$$V_{2l} = \frac{w_1l_1}{2} - \frac{M_2}{l_1} \quad . \quad . \quad . \quad (135) \qquad V_{2l} = \Sigma P_1\frac{a_1}{l_1} - \frac{M_2}{l_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

End shear at second support, right

$$V_{2r} = \frac{w_2l_2}{2} + \frac{M_3 - M_2}{l_2} \quad . \quad . \quad (137) \qquad V_{2r} = \Sigma P_2\left(1 - \frac{a_2}{l_2}\right) + \frac{M_3 - M_2}{l_2} \quad . \quad . \quad (138)$$

End shear at third support, left

$$V_{3l} = \frac{w_2l_2}{2} - \frac{M_3 - M_2}{l_2} \quad . \quad . \quad (139) \qquad V_{3l} = \Sigma P_2\frac{a_2}{l_2} - \frac{M_3 - M_2}{l_2} \quad . \quad . \quad . \quad (140)$$

End shear at third support, right

$$V_{3r} = \frac{w_3 l_3}{2} - \frac{M_3}{l_3}. \quad . \quad . \quad . \quad (141) \qquad V_{3r} = \Sigma P_3\left(1 - \frac{a_3}{l_3}\right) - \frac{M_3}{l_3}. \quad . \quad . \quad . \quad (142)$$

End shear at fourth support

$$V_4 = \frac{w_3 l_3}{2} + \frac{M_3}{l_3}. \quad . \quad . \quad . \quad (143) \qquad V_4 = \Sigma P_3 \frac{a_3}{l_3} + \frac{M_3}{l_3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

In the above formulas, $\Sigma P_1 \frac{a_1}{l_1}$ and $\Sigma P_1\left(1 - \frac{a_1}{l_1}\right)$ denote sums of static end shears of all loads in the first span, $\Sigma P_2 \frac{a_2}{l_2}$ and $\Sigma P_2\left(1 - \frac{a_2}{l_2}\right)$ are static end shears of all loads in the second span, finally $\Sigma P_3 \frac{a_3}{l_3}$ and $\Sigma P_3\left(1 - \frac{a_3}{l_3}\right)$ are static end shears in the third span.

M_2 and M_3 are bending moments at the supports. It is important to use in the above equations the bending moments with their proper signs. Thus, $V_1 = \frac{w_1 l_1}{2} + \left(- \frac{w_1 l_1}{\alpha}\right)$ when $M_2 = - \frac{w_1 l_1^2}{\alpha}$ is negative and $V_1 = \frac{w_1 l_1}{2} + \left(+ \frac{w_1 l_1}{\alpha}\right)$ when $M_2 = \frac{w_1 l_1^2}{\alpha}$ is positive.

Reactions.—The reactions on end supports 1 and 4 are equal to the end shears V_1 and V_4, respectively, plus any load coming directly on the support.

The reactions at the interior supports 2 and 3 are equal to the sum of end shears to the left and to the right of the support, namely, $(V_{2l} + V_{2r})$ and $(V_{3l} + V_{3r})$, respectively, plus any load coming directly on the support.

THREE EQUAL SPANS. FREE ENDS

For three equal spans the formulas for bending moments become simpler. They are obtained from Formulas (129) to (132) by substituting $l_1 = l_2 = l_3 = l$.

Uniformly Distributed Loads. Different Intensities in Each Span,

$$M_2 = - \frac{(4w_1 + 3w_2 - w_3)l^2}{60}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145)$$

$$M_3 = - \frac{(- w_1 + 3w_2 + 4w_3)l^2}{60}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (146)$$

Concentrated Loads,

$$M_2 = -\frac{(4\Sigma P_1C_1 + 4\Sigma P_2C_2 - \Sigma P_2C_1 - \Sigma P_3C_2)}{15}l. \quad (147)$$

$$M_3 = -\frac{-\Sigma P_1C_1 + 4\Sigma P_2C_1 - \Sigma P_2C_2 + (4\Sigma P_3C_2)}{15}l. \quad (148)$$

For loading symmetrically arranged with regard to the center of span

$$C_1 = C_2$$

so that

$$4\Sigma P_2C_2 - \Sigma P_2C_1 = 3\Sigma P_2C_1,$$

also

$$4\Sigma P_2C_1 + \Sigma P_2C_2 = 3\Sigma P_2C_1.$$

In the above equations ΣP_1C_1 is the sum in first span, ΣP_2C_1 and ΣP_2C_2 are sums in second span and ΣP_3C_1 and ΣP_3C_2 are sums in third span.

C_1 and C_2 are constants from Diagram 1, p. 19.

Three Equal Spans. Free Ends. Uniformly Distributed Load

It is assumed in the formulas below that the intensity of the uniformly distributed loading in all loaded panels is equal. The intensity of loading in unloaded spans is zero.

In practice the beam is loaded by the dead load, which always acts in all spans, and by the live load which cannot be counted upon to be always in all spans. Therefore, to determine the maximum bending moments, it is necessary to compute separately the bending moments for the dead load and for the most unfavorable position of the live load and add the results.

Four conditions of loading are considered below, namely:

(*a*) Condition for dead load all spans loaded. $w_1 = w_2 = w_3 = w$ (Fig. 36 (*a*)).

(*b*) Condition for maximum negative bending moment, two adjoining spans loaded. $w_1 = w_2 = w$ and $w_3 = 0$ (Fig. 36 (*b*)).

(*c*) Condition for maximum positive bending moment in center span, center span loaded, $w_1 = w_3 = 0$ and $w_2 = w$ (Fig. 36 (*c*)).

(*d*) Condition for maximum positive bending moment, in end span, the end spans loaded. $w_1 = w_3 = w$ and $w_2 = 0$ (Fig. 36 (*d*)).

The bending moments and end shears are given in table on p. 52. The bending moment curves as well as shear diagrams are shown in Fig. 36, p. 53. The location of points of maximum positive bending moments as well as of the points of inflection are shown in the figures.

Three Equal Spans. Free Ends

l = Length of Span. w = Uniform Unit Load

End Shear. *(See Fig. 36, p. 53.)*

Condition (Fig. 38)	Spans Loaded	First Span		Second Span		Third Span	
		V_1	V_{2l}	V_{2r}	V_{3l}	V_{3r}	V_4
a	1, 2, 3	0 4wl	0.6wl	0.5wl	0.5wl	0.6wl	0.4wl
b	1, 2, –	0.383wl	**0.617wl**	**0.583wl**	0.417wl	0 033wl	−0 033wl
c	–, 2, –	−0 05wl	0 05wl	0.5wl	0.5wl	0.05wl	−0.05wl
d	1, –, 2	**0 45wl**	0 55wl	0	0	0.55wl	**0 45wl**

Static end shear $V = 0.5wl$

Maximum Bending Moments. *(See Fig. 36, p. 53.)*

Condition (Fig. 38)	Spans Loaded	Negative Bending Moment		Maximum Positive Bending Moments		
		M_2	M_3	First Span	Second Span	Third Span
a	1, 2, 3	−0.1wl^2	−0.1wl^2	0.08wl^2	0.025wl^2	0.08wl^2
b	1, 2, –	**−0.117wl^2**	−0 033wl^2	0 0735wl^2	0.0535wl^2	
c	–, 2, –	−0.05wl^2	−0 05wl^2	....	**0.075wl^2**	
d	1, –, 2	−0.05wl^2	−0.05wl^2	**0 101wl^2**	....	**0.101wl^2**

Static Bending Moment, M_s = 0 125wl^2

Absolute maximum values are shown in black face type.

Maximum Values for Combined Dead and Live Load

Dead Load	Live Load	End Shears			Negative B. M. M_2 and M_3	Maximum Positive B. M.	
		V_1 and V_4	V_{2l} and V_{3r}	V_{2r} and V_{3l}		End Span	Center Span
0.2w	0.8w	0 44wl	0.614wl	0.566wl	−0.114wl^2	0.097wl^2	0.065wl^2
0.3w	0.7w	0.435wl	0.612wl	0.558wl	−0.112wl^2	0.095wl^2	0 060wl^2
0.4w	0.6w	0.43wl	0.610wl	0.550wl	−0.110wl^2	0.093wl^2	0.055wl^2
0.5w	0.5w	0 425wl	0.608wl	0.541wl	−0.108wl^2	0.090wl^2	0 050wl^2
0.6w	0.4w	0.42wl	0.607wl	0 533wl	−0.107wl^2	0.088wl^2	0.045wl^2
0.7w	0.3w	0.415wl	0.605wl	0.525wl	−0.105wl^2	0.086wl^2	0.040wl^2

w = Uniform unit dead plus live load. l = Length of span.

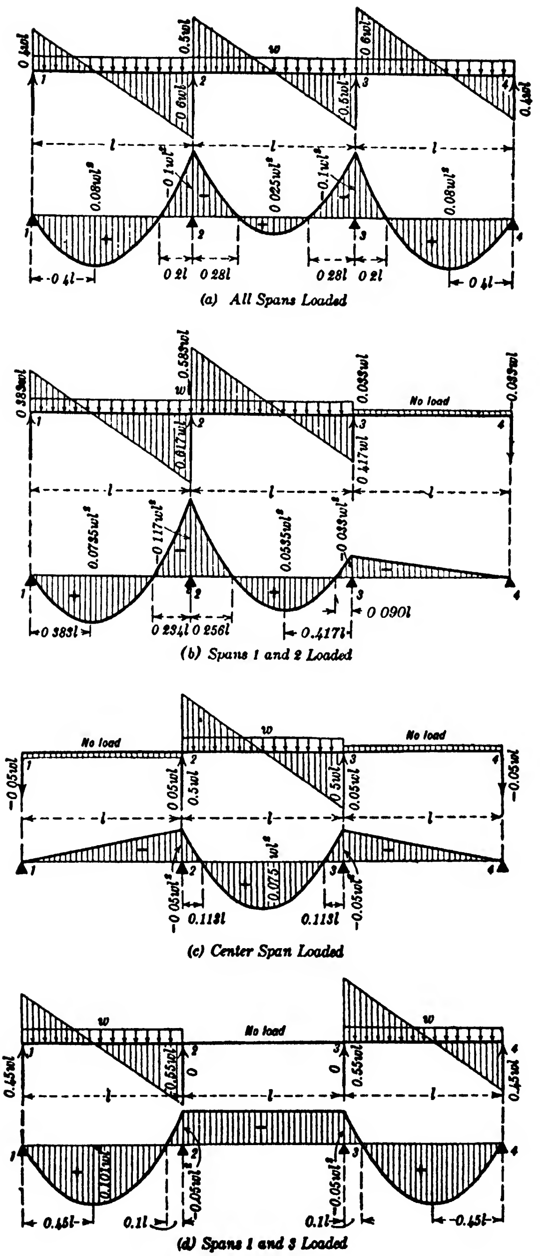


FIG. 36.—Three Equal Spans, Free Ends. Uniformly Distributed Loading. (*See* p. 51.)

Three Equal Spans. Free Ends. Concentrated Loads

Formulas for the bending moments and shears produced by concentrated loads for a beam consisting of three spans are given below.

Load in the First Span at Distance *a* from Left Support. (See Fig. 37, p. 54.)

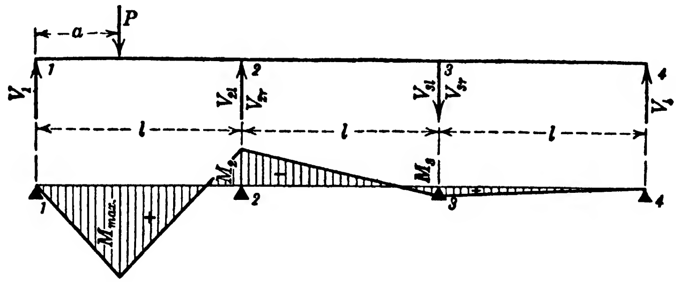


FIG. 37.—Three Equal Spans, Free End. Concentrated Load in End Span. (See p. 54.)

End Shears,

$$V_1 = \left[1 - \frac{a}{l} - \frac{4}{15}\frac{a}{l}\left(1 - \left(\frac{a}{l}\right)^2\right)\right]P = F_1P. \quad . \quad . \quad (149)$$

$$V_{2l} = P - V_1 = (1 - F_1)P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

$$V_{2r} = \frac{1}{3}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]P = F_2P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

$$V_{3l} = -V_{2r} = -F_2P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

$$V_{3r} = -\frac{1}{15}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]P = -\frac{1}{4}F_3P. \quad . \quad . \quad . \quad . \quad (153)$$

$$V_4 = -V_{3r} = \tfrac{1}{4}F_3P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

Bending Moments at Supports,

$$M_2 = -\frac{4}{15}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]Pl = -F_3Pl. \quad . \quad . \quad . \quad . \quad (155)$$

$$M_3 = -\tfrac{1}{4}M_2 = \tfrac{1}{4}F_3Pl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156)$$

Maximum Positive Bending Moment,

$$M_{\max} = V_1a = \frac{a}{l}F_1Pl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

Bending Moment at Any Span (x Measured from Left Support):

Loaded span,

$$M_x = V_1x = F_1\frac{x}{l}Pl \text{ for } x \text{ smaller than } a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (158)$$

$$M_x = V_1x - P(x - a) = \left[F_1\frac{x}{l} - \left(\frac{x}{l} - \frac{a}{l}\right)\right]Pl \text{ for } x \text{ larger than } a. \quad (159)$$

Second span, not loaded,

$$M_x = M_2 + \frac{M_3 - M_2}{l}x = \left(-1 + \frac{5}{4}\frac{x}{l}\right)F_3Pl. \qquad (160)$$

Third span, not loaded,

$$M_x = \frac{M_3}{l}(l - x) = \frac{1}{4}F_3\left(1 - \frac{x}{l}\right)Pl. \qquad (161)$$

Load in Center Span at Distance *a* from Support 2. (See Fig. 38, p. 55.)

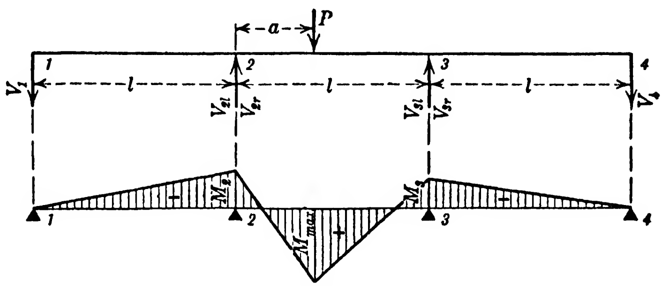


FIG. 38.—Three Equal Spans, Free Ends. Concentrated Load in Center Span. (*See* p. 55.)

End Shears,

$$V_1 = +\frac{M_2}{l} = -F_5P. \qquad (162)$$

$$V_{2l} = F_5P. \qquad (163)$$

$$V_{2r} = \frac{1}{3}\left[3 - 2\frac{a}{l} - 3\left(\frac{a}{l}\right)^2 + 2\left(\frac{a}{l}\right)^3\right]P = F_4P. \qquad (164)$$

$$V_{3l} = (1 - F_4)P. \qquad (165)$$

$$V_{3r} = -\frac{M_3}{l} = F_6P. \qquad (166)$$

$$V_4 = -F_6P. \qquad (167)$$

Negative Bending Moments,

$$M_2 = -\frac{1}{15}\frac{a}{l}\left(1 - \frac{a}{l}\right)\left(7 - 5\frac{a}{l}\right)Pl = -F_5Pl. \qquad (168)$$

$$M_3 = -\frac{1}{15}\frac{a}{l}\left(1 - \frac{a}{l}\right)\left(2 + 5\frac{a}{l}\right)Pl = -F_6Pl. \qquad (169)$$

Maximum Positive Bending Moment,

$$M_{max} = \frac{1}{15}\frac{a}{l}\left[8 + 2\frac{a}{l} - 20\left(\frac{a}{l}\right)^2 + 10\left(\frac{a}{l}\right)^3\right]Pl = F_7Pl. \qquad (170)$$

Bending Moment at Any Point x:

First span, unloaded span,

$$M_x = -V_1 x = -F_5 \frac{x}{l} Pl. \qquad (171)$$

Second span, loaded span,

$$M_x = M_2 - V_{2r} x = \left(F_4 \frac{x}{l} - F_5\right) Pl \text{ for } x \text{ smaller than } a. \qquad (172)$$

$$M_x = \left[(F_4 - 1)\frac{x}{l} - F_5 + \left(\frac{a}{l}\right)\right] Pl \text{ for } x \text{ larger than } a. \qquad (173)$$

Third span, unloaded span,

$$M_x = V_4(l - x) = -F_6\left(1 - \frac{x}{l}\right) Pl. \qquad (174)$$

Symmetrical Arrangement for Symmetrical Loads.—For loadings arranged symmetrically with regard to the center of the span in Formula (147), p. 51, the values ΣPC_1 are equal to ΣPC_2, because the bending moment area is symmetrical and its center is in the center of the span. The final formulas for bending moments for symmetrical loads are:

Negative Bending Moments for Symmetrical Loads,

$$M_2 = -\frac{4\Sigma P_1 C_1 + 3\Sigma P_2 C_1 - \Sigma P_3 C_1}{15} l. \qquad (175)$$

$$M_3 = -\frac{4\Sigma P_3 C_1 + 3\Sigma P_2 C_1 - \Sigma P_1 C_1}{15} l. \qquad (176)$$

If the load groups in all spans are equal, then

$$\Sigma P_1 C_1 = \Sigma P_2 C_1 = \Sigma P_3 C_1.$$

If some spans are loaded and others not loaded substitute proper values for the sum $\Sigma P_1 C_1$ for all loaded spans and in all unloaded spans make the sum equal to zero.

Four conditions of loadings are considered.

(*a*) All spans loaded. Condition for dead load.

$$\Sigma P_1 C_1 = \Sigma P_2 C_1 \qquad \Sigma P_3 C_1 = \Sigma PC_1.$$

(*b*) First and second span loaded. Condition for maximum negative bending moment M_2.

$$\Sigma P_1 C_1 = \Sigma P_2 C_1 = \Sigma PC_1 \quad \text{and} \quad \Sigma P_3 C_1 = 0.$$

(*c*) First and third span loaded. Condition for maximum positive bending moment.

$$\Sigma P_1 C_1 = \Sigma P_3 C_1 = \Sigma PC_1 \quad \text{and} \quad \Sigma P_2 C_1 = 0.$$

(*d*) Center span loaded. Condition for maximum positive bending moment.

$$\Sigma P_2 C_1 = \Sigma PC_1 \qquad \Sigma P_1 C_1 = \Sigma P_3 C_1 = 0.$$

For these loadings the values of negative bending moments are given in the table below.

Negative Bending Moments for Symmetrical Concentrated Loads

Condition	Spans Loaded	M_2	M_3
a	1, 2, 3	$-0.4l\Sigma PC_1$	$-0.4l\Sigma PC_1$
b	1, 2, –	$-0.467l\Sigma PC_1$	$-0.133\Sigma PC_1$
c	–, 2, –	$-0.2l\Sigma PC_1$	$-0.2l\Sigma PC_1$
d	1, –, 3	$-0.2l\Sigma PC_1$	$-0.2l\Sigma PC_1$

where ΣPC_1 is the sum of all loads in a span multiplied by corresponding values of C_1.

Values of $C_1 = \frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]$ for given $\frac{a}{l}$ may be taken from diagram 1, p. 19.

When all loads in a span are equal and symmetrically arranged it is possible to take P before the summation sign so that $\Sigma PC_1 = P\Sigma C_1$. For equal loads, then, the values of C_1 in a span may be added and the sum multiplied by the constant P.

Special Symmetrical Arrangement of Equal Concentrated Loads

Tables are worked out for following special symmetrical arrangements of equal concentrated loads:

1. One load P at center of spans (Fig. 39, p. 59).
2. Two loads P at third points (Fig. 40, p. 61).
3. Three loads P at quarter points (Fig. 41, p. 63).
4. Four loads P at fifth points (Fig. 42, p. 65).

In each case the four types of loading are considered which give maximum values for positive and negative bending moments, respectively.

The bending moments can be used for continuous girders carrying cross beams, which bring concentrated loads on the girders.

For dead load the bending moments for condition a should be used.

For live load use the maximum values of bending moments for proper condition of loadings. Thus for the negative bending moment at the second support use the negative bending moment for condition b.

For positive bending moment in the outside spans use condition d and in the center span use condition c.

The tables also give maximum bending moments and shears for a combination of dead load and live load. For given total load P and ratio of dead load to total load the maximum bending moments and shear can be obtained directly.

One Load P in Center. Three Equal Spans, Free Ends

End Shears

Condition (See Fig. 39)	Spans Loaded	First Span		Second Span		Third Span	
		V_1	V_{2l}	V_{2r}	V_{3l}	V_{3r}	V_4
a	1, 2, 3	0.35P	0 65P	0 5P	0 5P	0 65P	0 35P
b	1, 2, –	0.325P	**0 675*P***	**0 625*P***	0.375P	0.05P	−0 05P
c	–, 2, –	**−0.075*P***	0 075P	0.5P	0.5P	0 075P	**−0 075*P***
d	1, –, 3	**0.425*P***	0 575P	0	0	0.575P	**0.425*P***

Static End Shear, $V_s = 0.5P$

Maximum Bending Moments

Condition (See Fig. 39)	Spans Loaded	Negative at Supports		Max. Positive Bending Moment		
		M_2	M_3	First Span, $M_{\max 1}$	Second Span, $M_{\max 2}$	Third Span, $M_{\max 3}$
a	1, 2, 3	−0.15Pl	−0 15Pl	0 175Pl	0 1Pl	0.175Pl
b	1, 2, –	**−0.175*Pl***	−0 05Pl	0.162Pl	0 137Pl	
c	–, 2, –	−0.075Pl	−0 075Pl	....	**0.175*Pl***	
d	1, –, 3	−0.075Pl	−0 075Pl	**0.212*Pl***		**0.212*Pl***

Max. Static Bending Moment, $M_s = \frac{1}{4} Pl = 0.25\ Pl$

Absolute maximum values are shown in black face type

Maximum Values for Combined Dead and Live Load

Dead Load	Live Load	End Shears			Negative B. M. M_2 and M_3	Maximum Positive B. M.	
		V_1 and V_4	V_{2l} and V_{3r}	V_{2r} and V_{3l}		End Span	Center Span
0.2P	0.8P	0.410P	0.670P	0.60P	−0.170Pl	0.205Pl	0.160Pl
0.3P	0.7P	0.425P	0.667P	0.587P	−0.167Pl	0.201Pl	0.152Pl
0.4P	0.6P	0.395P	0.665P	0.575P	−0.165Pl	0.197Pl	0.145Pl
0.5P	0.5P	0.388P	0.662P	0.562P	− 0.162Pl	0.193Pl	0.137Pl
0.6P	0.4P	0.380P	0.660P	0.550P	−0.160Pl	0.190Pl	0.130Pl
0.7P	0.3P	0.372P	0.657P	0.537P	−0.157Pl	0.186Pl	0.122Pl

P = Concentrated dead plus live load. l = Length of span.

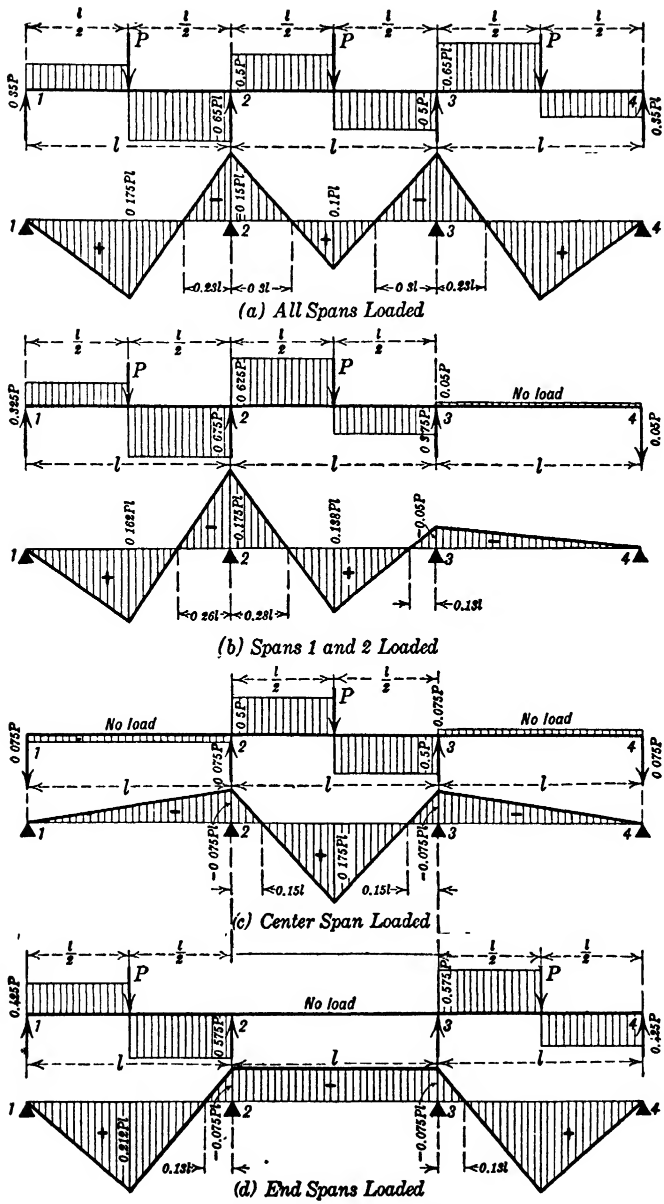


FIG. 39.—Three Equal Spans, Free Ends. One Load P at Center. (See p. 58.)

Three Equal Spans, Free Ends. 2 Loads *P* at Third Points

End Shears

Condition (See Fig. 40)	Spans Loaded	First Span		Second Span		Third Span	
		V_1	V_{2l}	V_{2r}	V_{3l}	V_{3r}	V_4
a	1, 2, 3	0 733*P*	1.267*P*	1 0*P*	1.0*P*	1 267*P*	0 733*P*
b	1, 2, –	0.689*P*	**1 311*P***	**1.222*P***	0 778*P*	0.089*P*	–0 089*P*
c	–, 2, –	**–0.133*P***	0 133*P*	1.0*P*	1.0*P*	0.133*P*	**–0 133*P***
d	1, –, 3	**0.867*P***	1.133*P*	0	0	1.133*P*	**0.867*P***

Static End Shear, $V_s = 1.0P$

Maximum Bending Moments

Condition (See Fig. 40)	Spans Loaded	Negative at Supports		Max. Positive Bending Moment		
		M_2	M_3	First Span, $M_{max\ 1}$	Second Span, $M_{max\ 2}$	Third Span, $M_{max\ 3}$
a	1, 2, 3	–0.267*Pl*	–0.267*Pl*	0.244*Pl*	0.067*Pl*	0.244*Pl*
b	1, 2, –	**–0.311*Pl***	–0 089*Pl*	0.229*Pl*	0.170*Pl*	
c	–, 2, –	–0 133*Pl*	–0.133*Pl*	....	**0.2*Pl***	
d	1, –, 3	–0.133*Pl*	–0 133*Pl*	**0 289*Pl***	....	**0.289*Pl***

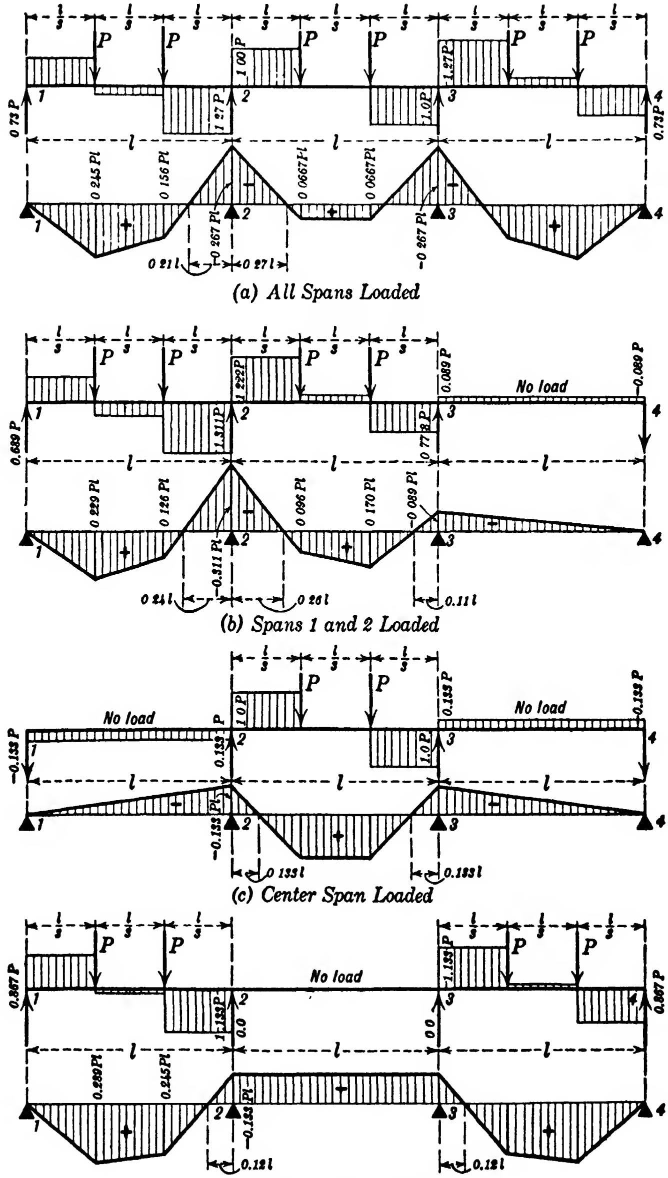
Max. Static Bending Moment, $M_s = \frac{1}{3}Pl$

Absolute maximum values are shown in black face type

Maximum Values for Combined Dead and Live Load

Dead Load	Live Load	End Shears			Negative B. M. M_2 and M_3	Maximum Positive Bending Moment	
		V_1 and V_4	V_{2l} and V_{3r}	V_{2r} and V_{3l}		End Span	Center Span
0.2*P*	0.8*P*	0.840*P*	1.302*P*	1.178*P*	–0.302*Pl*	0.280*Pl*	0.174*Pl*
0.3*P*	0.7*P*	0.827*P*	1.298*P*	1.155*P*	–0.298*Pl*	0.275*Pl*	0.160*Pl*
0.4*P*	0.6*P*	0.813*P*	1.293*P*	1.133*P*	–0.293*Pl*	0.271*Pl*	0.147*Pl*
0.5*P*	0.5*P*	0.800*P*	1.289*P*	1.111*P*	–0.289*Pl*	0.266*Pl*	0.134*Pl*
0.6*P*	0.4*P*	0 787*P*	1.285*P*	1.089*P*	–0.285*Pl*	0.262*Pl*	0.120*Pl*
0.7*P*	0.3*P*	0.773*P*	1.280*P*	1.067*P*	–0.280*Pl*	0.258*Pl*	0.107*Pl*

P = Concentrated dead plus live load. *l* = Span length.



(a) All Spans Loaded

(b) Spans 1 and 2 Loaded

(c) Center Span Loaded

(d) End Spans Loaded

FIG. 40.—Three Equal Spans, Free Ends. Two Loads P at Third Points.

(See p. 60.)

Three Equal Spans, Free Ends. 3 Loads P at Quarter Points

End Shears

Condition (See Fig. 41)	Spans Loaded	First Span		Second Span		Third Span	
		V_1	V_{2l}	V_{2r}	V_{3l}	V_{3r}	V_4
a	1, 2, 3	1 125P	1 875P	1 5P	1.5P	1 875P	1 125P
b	1, 2, –	1 062P	**1 938P**	**1 812P**	1 188P	0 125P	−0 125P
c	–, 2, –	**−0 188P**	0 188P	1 5P	1.5P	0 188P	**−0 188P**
d	1, –, 3	**1.313P**	1 687P	0	0	1 687P	**1 313P**

Static End Shear, $V_s = 1.5P$

Maximum Bending Moments

Condition (See Fig. 41)	Spans Loaded	Negative at Supports		Max. Positive Bending Moment		
		M_2	M_3	First Span, $M_{\max 1}$	Second Span, $M_{\max 2}$	Third Span, $M_{\max 3}$
a	1, 2, 3	−0 375Pl	−0 375Pl	0 313Pl	0 125Pl	0 313Pl
b	1, 2, –	**−0 438Pl**	−0.125Pl	0.281Pl	0.219Pl	....
c	–, 2, –	−0 188Pl	−0 188Pl	....	**0.312Pl**	... ...
d	1, –, 3	−0.188Pl	−0 188Pl	**0.406Pl**	....	**0 406Pl**

Max. Static Bending Moment, $M_s = \frac{1}{2}Pl = 0.5\ Pl$

Absolute maximum values are indicated by black face type

Maximum Values for Combined Dead and Live Load

Dead Load	Live Load	End Shears			Negative B. M. M_2 and M_3	Maximum Positive Bending Moment	
		V_1 and V_4	V_{2l} and V_{3r}	V_{2r} and V_{3l}		End Span	Center Span
0.2P	0.8P	1.275P	1.925P	1.750P	−0.425Pl	0.387Pl	0 275Pl
0.3P	0 7P	1.256P	1.919P	1.718P	−0.419Pl	0.378Pl	0.256Pl
0.4P	0.6P	1.238P	1.913P	1.687P	−0.413Pl	0.369Pl	0 237Pl
0.5P	0.5P	1.219P	1.906P	1.656P	−0.406Pl	0 359Pl	0.218Pl
0.6P	0.4P	1 200P	1.900P	1.625P	−0.400Pl	0.350Pl	0 200Pl
0.7P	0 3P	1.181P	1.893P	1.594P	−0.394Pl	0 341Pl	0 181Pl

P = Concentrated dead plus live load. l = Span length.

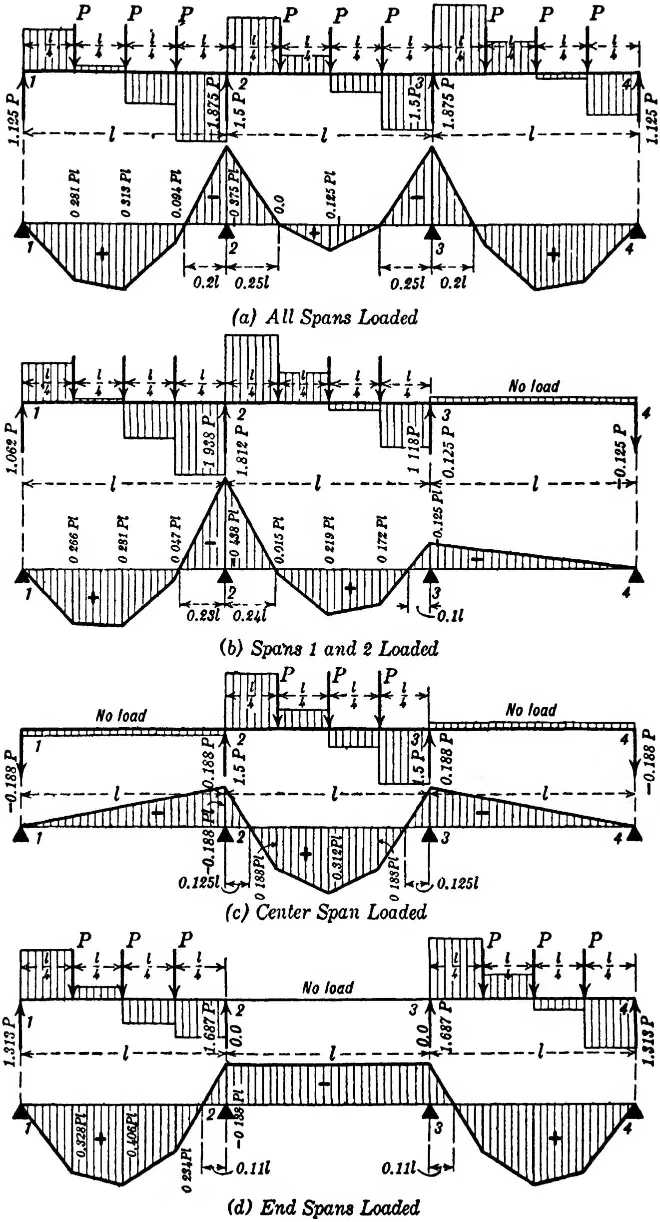


FIG. 41.—Three Equal Spans, Free Ends. Three Loads P at Quarter Points (See p. 62.)

Three Equal Spans, Free Ends. 4 Loads P at Fifth Points

End Shears

Condition (See Fig. 42)	Spans Loaded	First Span		Second Span		Third Span	
		V_1	V_{2l}	V_{2r}	V_{3l}	V_{3r}	V_4
a	1, 2, 3	1.52P	2 48P	2.0P	2.0P	2.48P	1.52P
b	1, 2, –	1.44P	**2.56P**	**2 4P**	1 6P	0 16P	–0 16P
c	–, 2, –	**–0 24P**	0 24P	2 0P	2.0P	0.24P	**–0 24P**
d	1, –, 3	**1 76P**	2 24P	0	0	2.24P	**1.76P**

Static End Shears, $V_s = 2P$

Maximum Bending Moments

Condition (See Fig. 42)	Spans Loaded	Negative at Supports		Max. Positive Bending Moment		
		M_2	M_3	First Span, $M_{\max 1}$	Second Span, $M_{\max 2}$	Third Span, $M_{\max 3}$
a	1, 2, 3	–0.48Pl	–0.48Pl	0 408Pl	0 12Pl	0.424Pl
b	1, 2, –	**–0 56Pl**	–0.16Pl	0.376Pl	0 28Pl	
c	–, 2, –	–0.24Pl	–0.24Pl	......	**0 36Pl**	
d	1, –, 3	–0 24Pl	–0.24Pl	**0 504 Pl**	....	**0 504Pl**

Max. Static Bending Moment, M_s = 0 6Pl

Absolute maximum values are shown in black face type

Maximum Values for Combined Dead and Live Load

Dead Load	Live Load	End Shears			Negative B. M. M_2 and M_3	Maximum Positive Bending Moment	
		V_1 and V_4	V_{2l} and V_{3r}	V_{2r} and V_{3l}		End Span	Center Span
0.2P	0.8P	1.712P	2.544P	2.32P	–0.544Pl	0.485Pl	0.312Pl
0.3P	0 7P	1.688P	2 536P	2.28P	–0.536Pl	0.475Pl	0.288Pl
0.4P	0.6P	1.664P	2.528P	2.24P	–0.528Pl	0.466Pl	0.264Pl
0.5P	0 5P	1.640P	2.520P	2.20P	–0.520Pl	0.456Pl	0.240Pl
0.6P	0.4P	1.616P	2.512P	2.16P	–0.512Pl	0.446Pl	0.216Pl
0.7P	0.3P	1.592P	2.504P	2.12P	–0.504Pl	0.437Pl	0.192Pl

P = Concentrated dead plus live load. l = Span length.

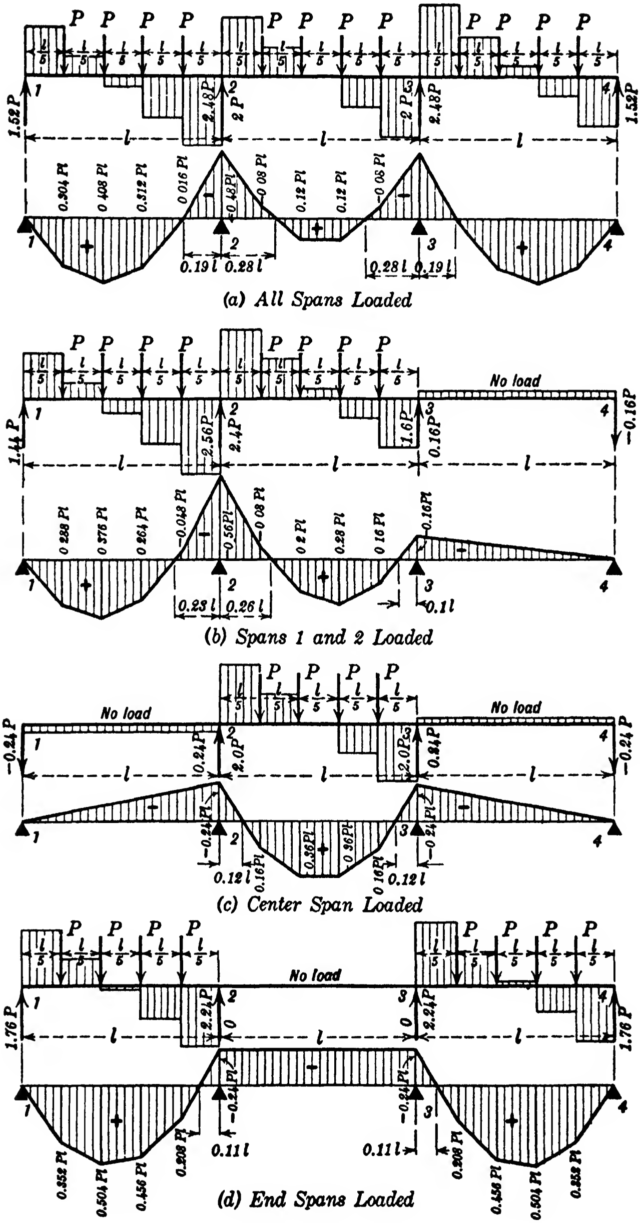


FIG. 42.—Three Equal Spans, Free Ends. Four Loads P at Fifth Points. (*See p.* 64.)

Example of Use of Tables for Concentrated Loads.—A numerical example showing the use of the tables for bending moments and shears for concentrated loads in actual design is given on p. 178.

THREE UNEQUAL SPANS. FREE ENDS

Continuous beams of *equal* spans can be designed, using bending moment coefficients recommended by various regulations. While there is a possibility of error, this ordinarily will not affect seriously the safety of the structure.

In continuous beams with *unequal* spans the possible error due to the use of arbitrary bending moments may be very large and may endanger the safety of the structure. In many cases not only the magnitude, but also the character of the bending moments may be different from those specified for typical spans. Thus in case of large end spans and a small center span the whole center span may be subjected to negative bending moments. Again an end span, the length of which is small in comparison with the center span, may require anchorage to prevent uplift. It also may be subjected to negative bending moment for the whole length. (See Example 2, p. 183 and 3, p. 188.)

For convenience in developing formulas for beams, with unequal spans, the beams with symmetrical arrangement of spans will be treated separately from the beams in which all spans are different.

THREE UNEQUAL SPANS. SYMMETRICAL ARRANGEMENT

Two cases are possible of beams of three unequal spans with symmetrical arrangement, namely:

1. Large center span $l_2 = l$ and small equal end spans $l_1 = l_3 = ml$, m being smaller than unity.
2. Large equal end spans $l_1 = l_3 = l$, and small center span $l_2 = ml$.

1. Large Center Span, Small Equal End Spans, *ml*, *l*, *ml*. Free Ends

Formulas are given below for an arrangement of spans where the center span is largest and the two end spans are equal and smaller than the center span. The ratio m is smaller than unity. This arrangement is shown in Fig. 43, p. 67.

Four conditions of loading are considered.

Uniform Loading. All Spans Loaded.

Conditions of loading for dead load.

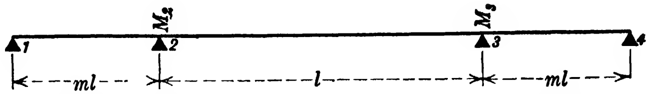
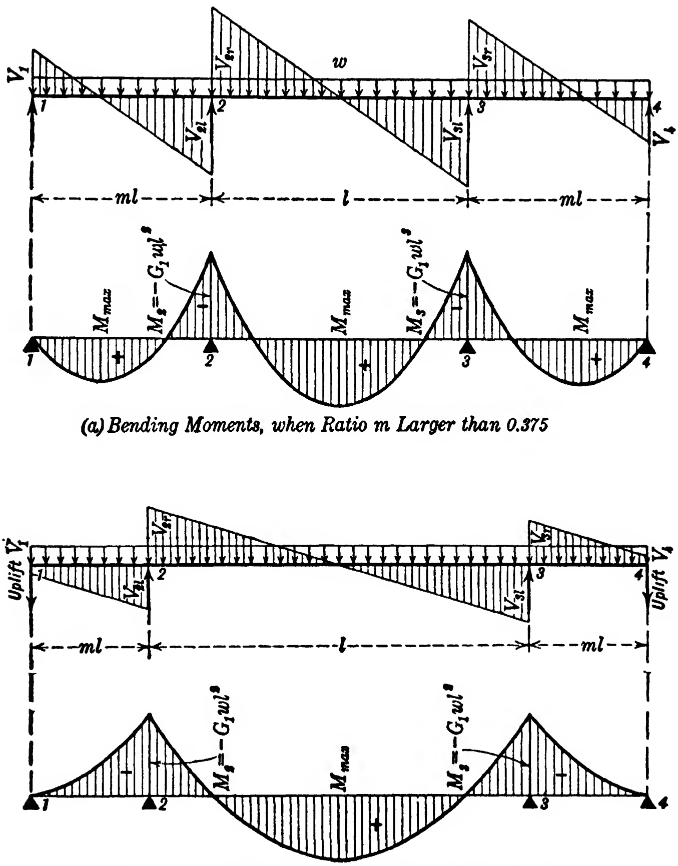


FIG. 43.—Two Small Equal End Spans, Large Center Span, Free Ends. (See p. 66.)



(a) Bending Moments, when Ratio m Larger than 0.375

(b) Bending Moments, when Ratio m Smaller than 0.375

FIG. 44.—Spans *ml*, *l*, *ml*. Bending Moments for Uniform Loading. (See p. 67.)

Maximum Shears,

$$V_1 = \left(\frac{1}{2} - \frac{G_1}{m^2}\right)wml. \quad . \quad (177) \qquad V_{2l} = w(ml) - V_1. \quad . \quad . \quad (178)$$

$$V_{2r} = \frac{wl}{2}. \quad (179) \qquad V_{3l} = \frac{wl}{2}. \quad (180)$$

$$V_{3r} = V_{2l}. \quad (181) \qquad V_4 = V_1. \quad (182)$$

Negative Bending Moments,

$$M_2 = M_3 = -\frac{1 + m^3}{4(2m + 3)}wl^2 = -G_1wl^2. \quad (183)$$

Positive Bending Moments:

First and third span,

$$M_{\max} = \frac{1}{2}\left(\frac{1}{2} - \frac{G_1}{m^2}\right)^2 w(ml)^2. \quad (184)$$

Second span,

$$M_{\max} = \tfrac{1}{8}wl^2 + M_2 = (\tfrac{1}{8} - G_1)wl^2. \quad (185)$$

Point of Maximum Bending Moments:

First and third span, measured from support 1 or 4,

$$x_1 = \frac{V_1}{w} = \left(\frac{1}{2} - \frac{G_1}{m^2}\right)ml. \quad (186)$$

Second span,

$$x_1 = \frac{l}{2}. \quad (187)$$

Values of G_1 may be taken from Diagram 10, p. 70.

Uniform Load. Two Adjoining Spans Loaded.

Condition of loading for maximum negative bending moment.

Maximum Shears,

$$V_1 = \left(\frac{1}{2} - \frac{G_2}{m^2}\right)wml. \quad (188) \qquad V_{2l} = wml - V_1. \quad (189)$$

$$V_{2r} = (\tfrac{1}{2} + G_2 - G_3)wl. \quad (190) \qquad V_{3l} = wl - V_{2r}. \quad (191)$$

$$V_{3r} = -\frac{M_3}{ml} = \frac{G_3}{m^2}wml. \quad (192) \qquad V_4 = -V_{3r}. \quad (193)$$

Negative Bending Moments,

$$M_2 = -\frac{1}{4}\,\frac{2(1 + m)m^3 + 1 + 2m}{4(1 + m)^2 - 1}wl^2 = -G_2wl^2. \quad (194)$$

$$M_3 = -\frac{1}{4}\,\frac{1 + 2m - m^3}{4(1 + m)^2 - 1}wl^2 = -G_3wl^2. \quad (195)$$

Maximum Positive Bending Moments:

First span,

$$M_{\max} = \frac{1}{2}\left(\frac{1}{2} - \frac{G_2}{m^2}\right)^2 w(ml)^2. \quad \ldots \ldots \ldots \quad (196)$$

Second span,

$$M_{\max} = [- G_3 + \tfrac{1}{2}(\tfrac{1}{2} + G_2 - G_3)^2]wl^2. \quad \ldots \ldots \quad (197)$$

Third span,

No positive bending moment.

Points of Maximum Bending Moments:

First span, measured from support 1,

$$x_1 = \left(\frac{1}{2} - \frac{G_2}{m^2}\right)ml. \quad \ldots \ldots \ldots \quad (198)$$

Second span, measured from support 2,

$$x_1 = (\tfrac{1}{2} + G_2 - G_3)l. \quad \ldots \ldots \ldots \quad (199)$$

Values of G_2 and G_3 may be taken from Diagram 10, p. 70.

Uniform Loading. End Spans Loaded.

Conditions of loading for maximum positive bending moments in end spans.

Maximum Shears,

$$V_1 = \left(\frac{1}{2} - \frac{G_4}{m^2}\right)wml. \quad \ldots \quad (200) \qquad V_{2l} = wml - V_1. \quad \ldots \quad (201)$$

$$V_{2r} = V_{3l} = 0. \quad \ldots \ldots \quad (202)$$

$$V_{3r} = V_{2l}. \quad \ldots \ldots \quad (203) \qquad V_4 = V_1. \quad \ldots \ldots \quad (204)$$

Negative Bending Moments,

$$M_2 = M_3 = - \frac{m^3}{4(2m + 3)}wl^2 = - G_4wl^2. \quad \ldots \quad (205)$$

Positive Bending Moments:

First and third span,

$$M_{\max} = \frac{1}{2}\left(\frac{1}{2} - \frac{G_4}{m^2}\right)^2 w(ml)^2. \quad \ldots \ldots \ldots \quad (206)$$

Second span,

No positive bending moment.

Points of Maximum Positive Bending Moment:

First and third span,

$$x_1 = \left(\frac{1}{2} - \frac{G_1}{m^2}\right)ml. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (207)$$

Values of G_4 may be taken from Diagram 10, p. 70.

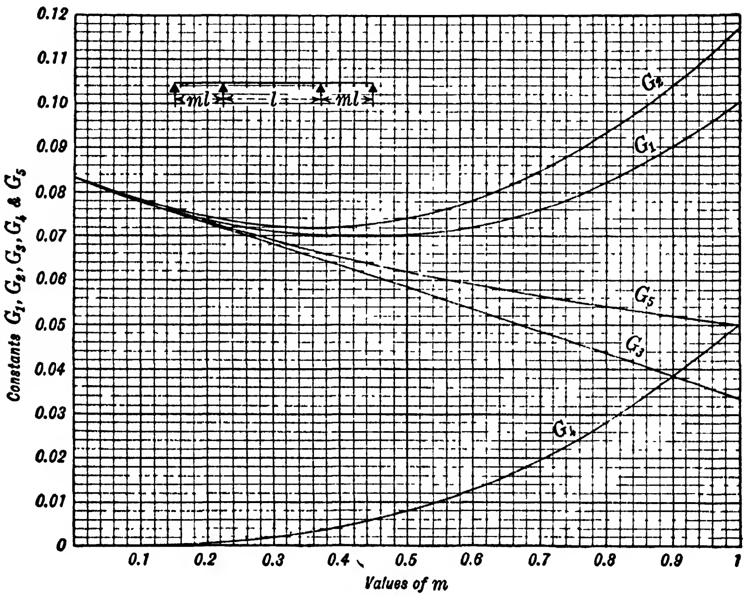


DIAGRAM 10.—Constants G_1 to G_5 for Three Unequal Spans. (*See p.* 68.)

Uniform Load. Center Span Loaded.

Conditions of loading for maximum positive bending moment center span.

End Shears,

$$V_1 = -\frac{M_2}{ml} = -\frac{G_5}{m^2}wml. \quad . \quad (208) \qquad V_{2l} = -V_1. \quad . \quad . \quad . \quad . \quad . \quad (209)$$

$$V_{2r} = \tfrac{1}{2}wl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (210) \qquad V_{3l} = \tfrac{1}{2}wl. \quad . \quad . \quad . \quad . \quad . \quad . \quad (211)$$

$$V_{3r} = -V_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (212) \qquad V_4 = V_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (213)$$

Negative Bending Moments,

$$M_2 = M_3 = -\frac{1}{4(2m+3)}wl^2 = -G_5wl^2. \quad . \quad . \quad . \quad (214)$$

Positive Bending Moments:

First and third span,
No positive bending moment.

Second span,

$$M_{\max} = (\tfrac{1}{8} - G_5)wl^2. \quad \ldots \ldots \quad (215)$$

Points of Maximum Positive Bending Moments:

Second span,

$$x_1 = \frac{l}{2}. \quad \ldots \ldots \quad (216)$$

Values of G_5 may be taken from Diagram 10, p. 70

Example.—A numerical example of the use of the formulas for bending moments and shears in actual design is given on p. 183.

2. Large Equal End Spans, Small Center Span l, ml, l. Free Ends

Bending moments and shears are developed below for a case of two large equal end spans and a small center span. This arrangement is shown in Fig. 45, p. 71.

Four conditions of loading are considered.

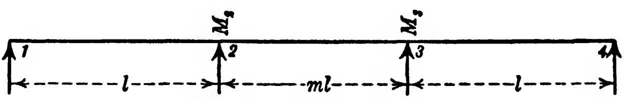


Fig. 45.—Two Equal End Spans, Small Center Span, Free Ends. (*See p.* 71.)

Uniform Load. All Spans Loaded. (See Fig. 46.)
Condition of loading for dead load.

Maximum Shears,

$$V_1 = \tfrac{1}{2} - \frac{1 + m^3}{4(2 + 3m)}wl = (\tfrac{1}{2} - H_1)wl. \quad \ldots \quad (217)$$

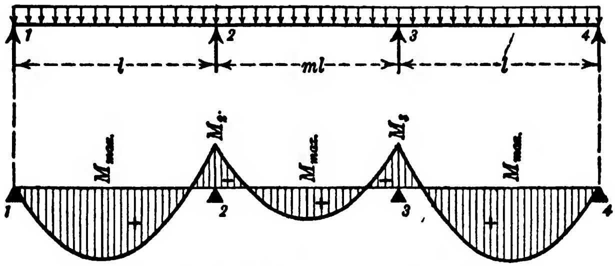
$$V_{2l} = wl - V_1. \quad \ldots \ldots \quad (218)$$

$$V_{2r} = \tfrac{1}{2}wml. \quad \ldots \quad (219) \qquad V_{3l} = \tfrac{1}{2}wml. \quad \ldots \quad (220)$$

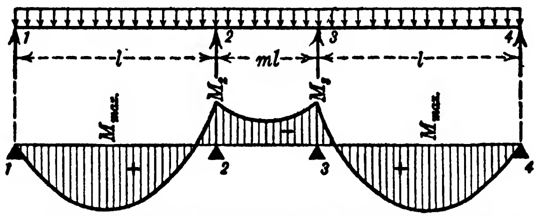
$$V_{3r} = V_{2l}. \quad \ldots \quad (221) \qquad V_4 = V_1. \quad \ldots \quad (222)$$

Negative Bending Moments,

$$M_2 = M_3 = -\frac{1 + m^3}{4(2 + 3m)}wl^2 = -H_1wl^2. \quad . \quad . \quad (223)$$



(a) Bending Moments when Ratio m Larger than 0.84



(a) Bending Moments when Ratio m Smaller than 0.84

FIG. 46.—Spans l, ml, l. Bending Moments for Uniform Loading. (*See* p. 71.)

Maximum Positive Bending Moments:

End spans,

$$M_{max} = \tfrac{1}{2}(\tfrac{1}{2} - H_1)^2wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (224)$$

Center span,

$$M_{2max} = \left(\frac{m^2}{8} - H_1\right)wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (225)$$

Points of Maximum Bending Moments:

End spans,

$$x_1 = \frac{V_1}{w} = (\tfrac{1}{2} - H_1)l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (226)$$

Center span,

$$x_1 = \frac{ml}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (227)$$

Values of H_1 may be taken from Diagram 11, p. 74.

For $m = 0.84$ $M_{2_{max}} = 0$. For smaller values of m there is no positive bending moment in the center span when all spans are loaded.

Two Adjoining Spans Loaded.

Condition for maximum negative bending moment M_2.

End Shears,

$$V_1 = \left[\frac{1}{2} - \frac{2(1+m) + m^3(2+m)}{4(2+m)(2+3m)}\right]wl = (\tfrac{1}{2} - H_2)wl. \quad . \quad (228)$$

$$V_{2l} = wl - V_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (292)$$

$$V_{2r} = \left(\frac{1}{2} + \frac{H_3 - H_2}{m^2}\right)wml. \quad . \quad (230) \qquad V_3l = wml - V_{2r}. \quad . \quad (231)$$

$$V_{3r} = -\frac{M_3}{l} = H_3wl. \quad . \quad . \quad . \quad (232) \qquad V_4 = -V_{3r}. \quad . \quad . \quad . \quad (233)$$

Negative Bending Moments,

$$M_2 = -\frac{2(1+m) + m^3(2+m)}{4(2+m)(2+3m)}wl^2 = -H_2wl^2. \quad . \quad . \quad . \quad (234)$$

$$M_3 = -\frac{(2+m)m^3 - m}{4(2+m)(2+3m)}wl^2 = -H_3wl^2. \quad . \quad . \quad . \quad . \quad (235)$$

Positive Bending Moments:

First span,

$$M_{max} = \tfrac{1}{2}(\tfrac{1}{2} - H_2)^2wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (236)$$

Second span,

$$M_{max} = M_2 + \frac{1}{2}\left(\frac{1}{2} + \frac{H_3 - H_2}{m^2}\right)^2 w(ml)^2 \quad . \quad . \quad . \quad . \quad (237)$$

Third span,

No positive bending moment.

Points of Maximum Bending Moments:

First span,

$$x_1 = \frac{V_1}{w} = (\tfrac{1}{2} - H_2)l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (238)$$

Second span,

$$x_1 = \frac{V_{2r}}{w} = \left(\frac{1}{2} + \frac{H_3 - H_2}{m^2}\right)ml. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (239)$$

Values of H_2 and H_3 may be taken from Diagram 11, p. 74.

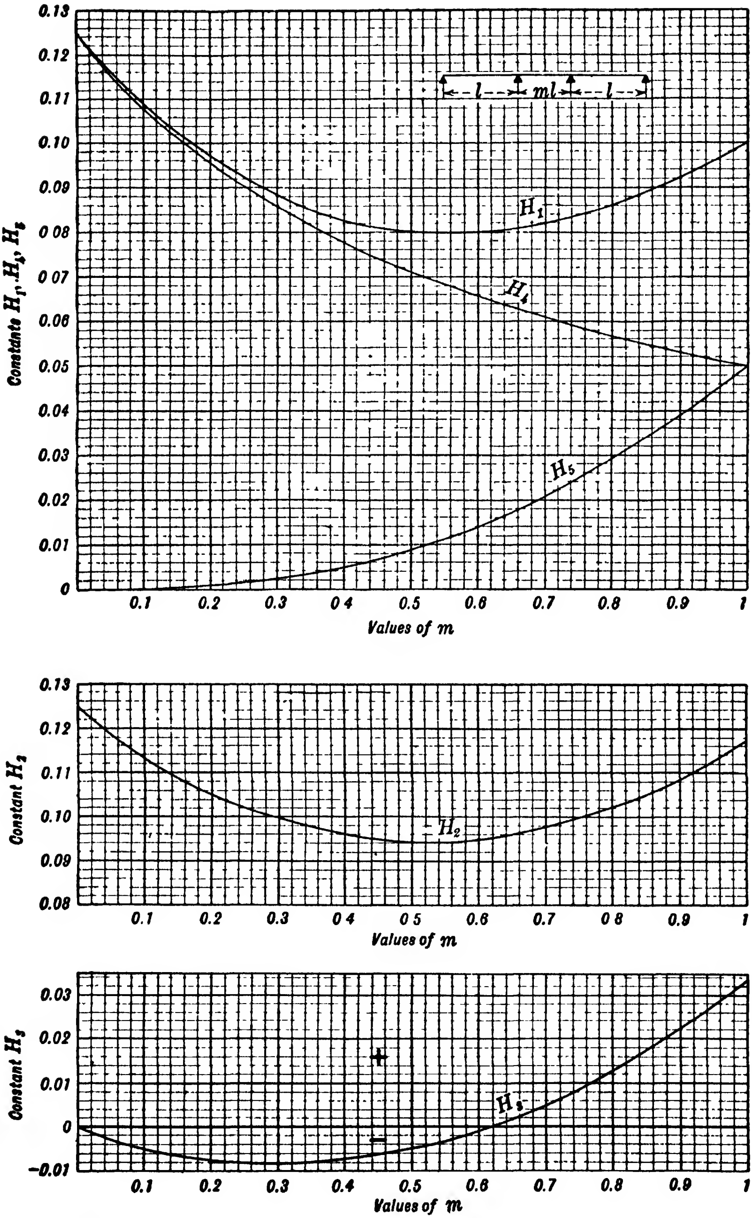


Diagram 11.—Contents H_1 to H_5 for Three Unequal Spans. (See p. 72.)

End Spans Loaded.
Condition for maximum positive bending moment in end spans.

End Shears,

$$V_1 = \left(\frac{1}{2} - \frac{1}{4(2+3m)}\right)wl = (\tfrac{1}{2} - H_4)wl. \quad . \qquad (240)$$

$$V_{2l} = wl - V_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (241)$$

$$V_{2r} = 0. \quad . \quad . \quad . \quad (242) \qquad V_{3l} = 0. \quad . \quad . \quad . \quad . \quad . \qquad (243)$$

$$V_{3r} = V_{2l}. \quad . \quad . \quad (244) \qquad V_4 = V_1. \quad . \quad . \quad . \qquad (245)$$

Negative Bending Moments,

$$M_2 = M_3 = -\frac{1}{4(2+3m)}wl^2 = H_4wl^2. \quad . \quad . \quad . \qquad (246)$$

Maximum Positive Bending Moments:

End spans,

$$M_{max} = \tfrac{1}{2}(\tfrac{1}{2} - H_4)^2wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (247)$$

Center span,

No positive bending moment.

Points of Maximum Bending Moment,

$$x_1 = \frac{V_1}{w} = (\tfrac{1}{2} - H_4)l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (248)$$

Values of H_4 may be taken from Diagram 11, p. 74.

Center Span Loaded.
Condition for maximum positive bending moment in center span.

Maximum Shears,

$$V_1 = -\frac{m^3}{4(2+3m)}wl = -H_5wl. \quad . \qquad (249) \qquad V_{2l} = -V_1. \quad . \qquad (250)$$

$$V_{2r} = \tfrac{1}{2}wml. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (251) \qquad V_{3l} = \tfrac{1}{2}wml. \quad . \qquad (252)$$

$$V_{3r} = V_{2l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (253) \qquad V_4 = V_1. \quad . \quad . \qquad (254)$$

Negative Bending Moments,

$$M_2 = M_3 = -\frac{m^3}{4(2+3m)}wl^2 = -H_5wl^2. \quad . \quad . \qquad (255)$$

Maximum Positive Bending Moment,

$$M_{max} = M_2 + \tfrac{1}{8}w(ml)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (256)$$

Points of Maximum Bending Moments,

$$x_1 = \tfrac{1}{2}ml. \qquad (257)$$

Values of H_5 may be taken from Diagram 11, p. 74.

THREE UNEQUAL SPANS. ALL SPANS DIFFERENT. FREE ENDS

To simplify formulas for continuous beams of three spans when the lengths of all spans are different, the largest span is accepted as a unit and called l and the other spans are expressed in terms of the largest span. Thus the three span lengths are l, m_1l, and m_2l.

Two arrangements of these spans are possible.

1. The center span is largest and is called l, and the end spans are m_1l and m_2l. The arrangement of spans then is m_1l, l, m_2l (see Fig. 47, p. 77.

2. The left end span is largest and is called l, the other two spans are m_1l and m_2l. The arrangement of span then is l, m_1l, m_2l.

The second case applies also when the third span is largest. In such case the numbering begins from the right support instead of the left support as considered in the formulas.

Case 1. Arrangement of Spans m_1l, l, m_2l

General Formulas.—Substituting in Formula 129, p. 48, $l_1 = m_1l$, $l_2 = l$ and $l_3 = m_2l$, following general formulas for negative bending moments are obtained.

General Formulas for Negative Bending Moments,

$$M_2 = -\frac{2(1 + m_2)m_1^3w_1 + (1 + 2m_2)w_2 - m_2^3w_3}{16(1 + m_1)(1 + m_2) - 4}l^2. \qquad (258)$$

$$M_3 = -\frac{-m_1^3w_1 + (1 + 2m_1)w_2 + 2(1 + m_1)m_2^3w_3}{16(1 + m_1)(1 + m_2) - 4}l^2. \qquad (259)$$

This may be simplified by designating certain terms by constants,

$$a = \frac{2(1 + m_2)m_1^3}{16(1 + m_1)(1 + m_2) - 4}. \qquad (260)$$

$$b = \frac{1 + 2m_2}{16(1 + m_1)(1 + m_2) - 4}. \qquad (261)$$

$$c = \frac{m_2^3}{16(1 + m_1)(1 + m_2) - 4}. \qquad (262)$$

$$a_1 = \frac{m_1^3}{16(1 + m_1)(1 + m_2) - 4}. \quad \ldots \ldots \quad (263)$$

$$b_1 = \frac{1 + 2m_1}{16(1 + m_1)(1 + m_2) - 4}. \quad \ldots \ldots \quad (264)$$

$$c_1 = \frac{2(1 + m_1)m_2^3}{16(1 + m_1)(1 + m_2) - 4}. \quad \ldots \ldots \quad (265)$$

All these constants have a common denominator.

Then

$$M_2 = -(aw_1 + bw_2 - cw_3)l^2. \quad \ldots \ldots \quad (266)$$

$$M_3 = -(-a_1w_1 + b_1w_2 + c_1w_3)l^2 \quad \ldots \ldots \quad (267)$$

The constants a, b, c, and a_1, b_1, and c_1 for different values of m_1 and m_2 may be taken from table on p. 79.

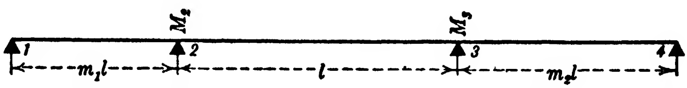


FIG. 47.—Center Span Largest. Arrangement m_1l, l, m_2l. (*See p.* 76.)

Uniform Loading.—Assume that the intensity of uniform loading is the same for all loaded spans and zero in unloaded spans. To get maximum bending moments in different parts of the beam, following arrangements of load are considered.

(*a*) All spans loaded. $w_1 = w_2 = w_3 = w$. Condition for dead load.

(*b*) First and second spans loaded. $w_1 = w_2 = w$ and $w_3 = 0$. Condition for maximum negative bending moment at second support.

(*c*) First and third spans loaded. $w_1 = w_3 = w$, $w_2 = 0$. Condition for maximum positive bending moment at end spans.

(*d*) Second span loaded. $w_1 = w_3 = 0$, $w_2 = w$. Condition for maximum positive bending moments in second span.

To get maximum negative bending moment at the third support, the second and third spans should be loaded. Formulas for this condition can be obtained from case (*b*) by interchanging m_1 and m_2. Then the bending moment M_2 becomes M_3.

(*a*) *All spans loaded.*

$$M_2 = -(a + b - c)wl^2, \quad \ldots \ldots \quad (268)$$

$$M_3 = -(-a_1 + b_1 + c_1)\,wl^2, \quad \ldots \ldots \quad (269)$$

where a, b, c and a_1, b_1, c_1 are constants from table on p. 79 for the known span ratios m_1 and m_2.

(*b*) *First and second span loaded.*

$$M_2 = -(a + b)wl^2, \quad \text{. . . . .} \quad (270)$$

$$M_3 = -(-a_1 + b_1)wl^2, \quad \text{. . . . . . .} \quad (271)$$

where a, b, and a_1 and b_1 are constants from table on p. 79 for the known span ratios m_1 and m_2.

This loading gives M_2 as maximum negative bending moment at second support for uniform loading.

(*c*) *First and third span loaded.*

$$M_2 = -(a - c)wl^2, \quad \text{. . . . .} \quad (272)$$

$$M_3 = -(-a_1 + c_1)wl^2, \quad \text{. . . . . . .} \quad (273)$$

where a, c and a_1, c_1 are constants from table on p. 79 for the known span ratios m_1 and m_2.

The positive bending moments in the two end spans computed on the basis of the above values of M_2 and M_3 are the maximum positive bending moments for uniform loading.

The positive bending moments may be found as explained on p. 22 using table on p. 176.

(*d*) *Second span loaded.*

$$M_2 = -bwl^2, \quad \text{. . . . .} \quad (274)$$

$$M_3 = -b_1wl^2, \quad \text{. . . . .} \quad (275)$$

where b and b_1 are constants from table on p. 79 for the known span ratios of m_1 and m_2.

The positive bending moment in the center span computed on the basis of the above values is the maximum for the center span.

Example.—A numerical example showing the use of the formulas for bending moments and shears in actual design is given on p. 194.

Maximum Positive Bending Moments.—After determining the negative bending moments by the formulas given above, the corresponding positive bending moments may be found from table on p. 176. To use the table find the bending moment coefficients for the negative bending moments at both supports in terms of the span for which the positive bending moment is desired. The coefficient equals the bending moment divided by wl^2 for the center span, and by $w(m_1l)^2$, and $w(m_2l)^2$ for the first and third spans, respectively.

In the table corresponding to the coefficients at both supports the coefficient for positive bending moment is found. The bending

Three Unequal Spans. Arrangement m_1l, l, m_2l

Constants a, b, c and a_1, b_1, c_1

in formulas (258) to (275) pp. 76 to 78

m_1 for a, m_2 for c_1	Constants a and c_1 — m_2 for a, m_1 for c_1									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.0001	0.0001	0.0001	0.0001	0.0001	0 0001	0.0001	0.0001	0.0001	0.0001
0.2	0.0010	0.0010	0.0010	0 0010	0.0010	0.0010	0.0009	0.0009	0.0009	0.0009
0.3	0.0031	0 0031	0.0030	0.0030	0.0030	0.0029	0.0029	0 0029	0.0029	0.0029
0 4	0 0068	0 0067	0.0066	0 0066	0.0065	0.0064	0.0064	0.0064	0.0063	0.0063
0.5	0.0123	0 0121	0.0119	0.0118	0 0117	0.0116	0.0115	0 0115	0.0114	0.0114
0.6	0 0195	0 0194	0.0192	0.0190	0.0189	0.0187	0.0186	0.0185	0.0184	0.0183
0 7	0.0291	0 0287	0.0284	0 0282	0.0280	0.0278	0.0276	0.0275	0.0274	0 0273
0.8	0 0407	0 0402	0.0398	0.0394	0 0392	0 0389	0.0387	0 0385	0.0383	0 0382
0 9	0 0545	0 0540	0.0534	0 0530	0.0526	0.0522	0 0518	0 0517	0.0515	0.0513
1.0	0.0706	0.0698	0.0691	0.0686	0 0682	0.0678	0.0675	0.0673	0.0670	0.0667

m_1 for b, m_2 for b_1	Constants b and b_1 — m_2 for b, m_1 for b_1									
	0.1	0.2	0.3	0.4	0.5	0 6	0 7	0.8	0.9	1 0
0.1	0.078	0.082	0.085	0.087	0.089	0.091	0.093	0.094	0 095	0 096
0.2	0.070	0.073	0.076	0 079	0.081	0.082	0.084	0.085	0.086	0 087
0.3	0 064	0 067	0.069	0 072	0.073	0.075	0 076	0 078	0 079	0 080
0.4	0 058	0 061	0 064	0.066	0.067	0 069	0.070	0.071	0 073	0 073
0.5	0 053	0 056	0.059	0 061	0.062	0.064	0.065	0.066	0.067	0.068
0.6	0 049	0 052	0 055	0 056	0 058	0 059	0.061	0 062	0.064	0.065
0.7	0 046	0 049	0.051	0.052	0.054	0.056	0.057	0.058	0.059	0.060
0 8	0.043	0.046	0 048	0 049	0.051	0 052	0 053	0.054	0 055	0.056
0.9	0 041	0.043	0.045	0 046	0.048	0 049	0 550	0 051	0.052	0.053
1 0	0 038	0.041	0.042	0 044	0.045	0.046	0.048	0 048	0.049	0.050

m_1 for c, m_2 for a_1	Constants c and a_1 — m_2 for c, m_1 for a_1									
	0.1	0 2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	......	0.0005	0.0014	0 0031	0.0056	0.0089	0.0132	0 0185	0.0248	0.0320
0.2	......	0.0004	0.0013	0.0028	0.0051	0.0081	0.0120	0.0167	0.0224	0 0291
0.3	......	0.0004	0 0012	0.0026	0.0046	0.0074	0.0109	0.0153	0.0205	0.0266
0.4	......	0.0003	0.0011	0.0023	0.0042	0.0068	0.0101	0.0141	0.0189	0.0245
0.5	......	0.0003	0.0010	0.0022	0.0039	0.0063	0.0093	0.0131	0.0175	0.0227
0.6	......	0.0003	0.0009	0.0020	0.0036	0.0058	0.0087	0.0122	0.0163	0.0212
0.7	......	0.0003	0.0009	0.0019	0.0034	0.0055	0.0081	0.0114	0.0153	0.0195
0.8	......	0.0003	0 0009	0.0018	0.0032	0.0051	0.0076	0.0107	0.0144	0.0186
0.9	......	0.0002	0.0008	0.0017	0.0030	0.0048	0.0072	0.0101	0.0136	0.0176
1.0	......	0.0002	0.0007	0 0016	0.0028	0.0046	0.0068	0.0095	0.0128	0.0166

moment is obtained by multiplying this coefficient by wl^2, $w(m_1l)^2$ or $w(m_2l)^2$ depending upon the span for which it is found.

For the first and third spans obviously one of the negative bending moments is zero because the ends are free.

The position of the point of maximum bending moment is found from table on p. 177 for the same negative bending moment coefficients. The ratio there found should be multiplied by the length of span for which the positive bending moment is sought.

End Shears.—Having found the bending moments M_2 and M_3 the shears for any span can be found by means of table on p. 177.

For this purpose the coefficient of negative bending moment at both supports should be found by dividing the bending moments by wl^2, $w(m_1l)^2$ or $w(m_2l)^2$, depending upon the span for which the end shears are being determined.

The coefficients of shears are then found from the table. These multiplied by wl, $w(m_1l)$ or $w(m_2l)$ give the end shear at the left support. The end shear at the other support is found by subtracting from the load in the panel the shear at the left support.

Case 2. Arrangement of Spans l, m_1l, m_2l

General Formulas.—General formulas for case 2 shown in Fig. 48, p. 81, are obtained by substituting in Formulas (129) and (130), p. 48, $l_1 = l$, $l_2 = m_1l$, $l_3 = m_2l$. They are

General Formulas for Negative Bending Moments.

$$M_2 = -\frac{2(m_1 + m_2)w_1 + (m_1 + 2m_2)m_1^3w_2 - m_1m_2^3w_3}{16(1 + m_1)(m_1 + m_2) - 4m_1^2}l^2. \quad (276)$$

$$M_3 = -\frac{-m_1w_1 + (m_1 + 2)m_1^3w_2 + 2(1 + m_1)m_2^3w_3}{16(1 + m_1)(m_1 + m_2) - 4m_1^2}l^2. \quad (277)$$

This may be simplified by substituting

$$d = \frac{2(m_1 + m_2)}{16(1 + m_1)(m_1 + m_2) - 4m_1^2}. \quad (278)$$

$$e = \frac{(m_1 + 2m_2)m_1^3}{16(1 + m_1)(m_1 + m_2) - 4m_1^2}. \quad (279)$$

$$f = \frac{m_1m_2^3}{16(1 + m_1)(m_1 + m_2) - 4m_1^2}. \quad (280)$$

$$d_1 = \frac{m_1}{16(1 + m_1)(m_1 + m_2) - 4m_1^2}. \quad . \quad . \quad . \quad . \quad . \quad (281)$$

$$e_1 = \frac{(m_1 + 2)m_1^3}{16(1 + m_1)(m_1 + m_2) - 4m_1^2}. \quad . \quad . \quad . \quad . \quad . \quad (282)$$

$$f_1 = \frac{2(1 + m_1)m_2^3}{16(1 + m_1)(m_1 + m_2) - 4m_1^2}. \quad . \quad . \quad . \quad . \quad . \quad (283)$$

Then

Negative Bending Moments, Simplified,

$$M_2 = -\ (dw_1 + ew_2 - fw_3)l^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (284)$$

$$M_3 = -\ (-\ d_1w_1 + e_1w_2 + f_1w_3)l^2. \quad . \quad . \quad . \quad . \quad . \quad (285)$$

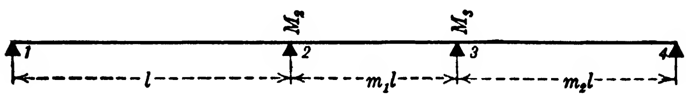


FIG. 48.—Left End Span Largest. Arrangement l, m_1l, m_2l.

The constants d, e, f and d_1, e_1, f_1 for different values of m_1 and m_2 may be taken from tables on pp. 82 and 83.

Uniform Loading of Equal Intensity.—Assuming that $w_1 = w_2 = w_3 = w$, formulas are developed for different arrangement of uniform loads. Four conditions of loading as described on p. 77 are used.

(*a*) *All spans loaded.*

Condition for dead load.

$$M_2 = -\ (d + e - f)wl^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (286)$$

$$M_3 = -\ (-\ d_1 + e_1 + f_1)wl^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (287)$$

where the constants d, e, f and d_1, e_1, f_1 may be taken from tables on pp. 82 and 83 for proper values of m_1 and m_2.

(*b*) *First and second spans loaded.*

Condition for maximum negative bending moment at second support M_2.

$$M_2 = -\ (d + e)wl^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (288)$$

$$M_3 = -\ (-\ d_1 + e_1)wl^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (289)$$

where d, e and d_1, e_1 are constants from tables on pp. 82 and 83 for proper m_1 and m_2.

(*b*$_1$) *Second and third spans loaded.*

Three Unequal Spans. Arrangement l_1, m_1l, m_2l
Constants d, e, f
in formulas (278) to (295), pp. 80 to 84

m_1	Constant d									
	Values of m_2									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.1149	0.1145	0.1143	0.1140	0.1140	0.1140	0.1139	0.1139	0.1139	0.1138
0.2	0.1071	0.1063	0.1059	0.1056	0.1054	0.1052	0.1051	0.1050	0.1049	0.1049
0.3	0.1005	0.0996	0.0990	0.0986	0.0982	0.0980	0.0978	0.0976	0.0975	0.0974
0.4	0.0947	0.0937	0.0931	0.0926	0.0922	0.0919	0.0917	0.0915	0.0913	0.0911
0.5	0.0895	0.0886	0.0879	0.0873	0.0869	0.0866	0.0863	0.0861	0.0859	0.0857
0.6	0.0849	0.0840	0.0833	0.0827	0.0823	0.0819	0.0816	0.0814	0.0812	0.0812
0.7	0.0808	0.0799	0.0792	0.0786	0.0782	0.0778	0.0775	0.0772	0.0770	0.0768
0.8	0.0770	0.0762	0.0755	0.0750	0.0746	0.0742	0.0738	0.0735	0.0732	0.0730
0.9	0.0736	0.0728	0.0722	0.0718	0.0712	0.0708	0.0704	0.0701	0.0699	0.0697
1.0	0.0705	0.0697	0.0691	0.0689	0.0682	0.0678	0.0674	0.0671	0.0669	0.0669

m_1	Constant e									
	Values of m_2									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
0.2	0.0006	0.0006	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007	0.0008	0.0008
0.3	0.0017	0.0019	0.0020	0.0021	0.0021	0.0022	0.0022	0.0023	0.0023	0.0023
0.4	0.0036	0.0040	0.0042	0.0044	0.0046	0.0047	0.0048	0.0049	0.0049	0.0050
0.5	0.0065	0.0071	0.0076	0.0079	0.0082	0.0084	0.0085	0.0087	0.0088	0.0089
0.6	0.0105	0.0113	0.0120	0.0125	0.0129	0.0133	0.0136	0.0138	0.0140	0.0143
0.7	0.0150	0.0167	0.0177	0.0184	0.0190	0.0195	0.0199	0.0203	0.0206	0.0209
0.8	0.0220	0.0234	0.0246	0.0253	0.0264	0.0272	0.0277	0.0282	0.0289	0.0291
0.9	0.0295	0.0314	0.0329	0.0342	0.0352	0.0360	0.0369	0.0376	0.0383	0.0388
1.0	0.0385	0.0407	0.0426	0.0441	0.0454	0.0466	0.0476	0.0485	0.0493	0.0500

m_1	Constant f									
	Values of m_2									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.00003	0.0001	0.0004	0.0007	0.0012	0.0018	0.0024	0.0032	0.0042	0.0052
0.2	0.00003	0.0002	0.0006	0.0011	0.0019	0.0025	0.0040	0.0054	0.0069	0.0087
0.3	0.00004	0.0002	0.0007	0.0013	0.0023	0.0035	0.0050	0.0069	0.0089	0.0112
0.4	0.00004	0.0002	0.0007	0.0015	0.0026	0.0040	0.0057	0.0078	0.0102	0.0130
0.5	0.00004	0.0003	0.0007	0.0015	0.0027	0.0042	0.0058	0.0085	0.0112	0.0143
0.6	0.00004	0.0002	0.0007	0.0016	0.0028	0.0044	0.0065	0.0089	0.0118	0.0154
0.7	0.00003	0.0002	0.0007	0.0016	0.0028	0.0045	0.0066	0.0092	0.0112	0.0158
0.8	0.00003	0.0002	0.0007	0.0016	0.0029	0.0046	0.0068	0.0094	0.0126	0.0162
0.9	0.00003	0.0002	0.0007	0.0016	0.0029	0.0046	0.0068	0.0100	0.0127	0.0165
1.0	0.00003	0.0002	0.0007	0.0016	0.0029	0.0046	0.0068	0.0100	0.0128	0.0167

Three Unequal Spans. Arrangement m_1l, l, m_2l

Constants d_1, e_1, f_1

in formula (278) to (295) p. 80 to 84

m_1	Constant d_1 — Values of m_2									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.0287	0.0191	0.0143	0.0114	0.0095	0.0081	0.0071	0.0063	0.0057	0.0051
0.2	0.0357	0.0265	0.0212	0.0176	0.0150	0.0131	0.0116	0.0105	0.0095	0.0087
0.3	0.0376	0.0298	0.0248	0.0211	0.0184	0.0163	0.0146	0.0133	0.0122	0.0112
0.4	0.0378	0.0312	0.0265	0.0231	0.0205	0.0183	0.0166	0.0152	0.0140	0.0130
0.5	0.0373	0.0316	0.0274	0.0242	0.0217	0.0197	0.0180	0.0165	0.0153	0.0143
0.6	0.0364	0.0315	0.0277	0.0248	0.0224	0.0204	0.0188	0.0174	0.0162	0.0152
0.7	0.0353	0.0311	0.0277	0.0250	0.0228	0.0209	0.0193	0.0180	0.0168	0.0161
0.8	0.0342	0.0304	0.0274	0.0250	0.0229	0.0212	0.0197	0.0183	0.0174	0.0162
0.9	0.0331	0.0298	0.0271	0.0248	0.0243	0.0212	0.0198	0.0186	0.0174	0.0164
1.0	0.0320	0.0296	0.0266	0.0245	0.0227	0.0211	0.0198	0.0186	0.0174	0.0166

m_1	Constant e_1 — Values of m_2									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.0006	0.0004	0.0003	0.0002	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001
0.2	0.0031	0.0023	0.0019	0.0015	0.0013	0.0011	0.0010	0.0009	0.0008	0.0008
0.3	0.0078	0.0062	0.0051	0.0044	0.0038	0.0034	0.0030	0.0028	0.0025	0.0023
0.4	0.0145	0.0120	0.0102	0.0089	0.0079	0.0071	0.0064	0.0058	0.0054	0.0050
0.5	0.0233	0.0198	0.0172	0.0151	0.0136	0.0123	0.0112	0.0103	0.0096	0.0089
0.6	0.0338	0.0293	0.0258	0.0231	0.0209	0.0190	0.0175	0.0162	0.0148	0.0141
0.7	0.0467	0.0411	0.0366	0.0331	0.0301	0.0227	0.0256	0.0238	0.0222	0.0209
0.8	0.0614	0.0546	0.0489	0.0448	0.0411	0.0381	0.0353	0.0329	0.0309	0.0291
0.9	0.0778	0.0700	0.0636	0.0584	0.0538	0.0500	0.0466	0.0436	0.0411	0.0388
1.0	0.0961	0.0872	0.0798	0.0735	0.0682	0.0636	0.0595	0.0560	0.0528	0.0500

m_1	Constants f_1 — Values of m_2									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.0006	0.0033	0.0085	0.0161	0.0261	0.0387	0.0537	0.0713	0.0912	0.1139
0.2	0.0004	0.0025	0.0069	0.0135	0.0226	0.0341	0.0481	0.0645	0.0834	0.1049
0.3	0.0003	0.0021	0.0058	0.0117	0.0200	0.0306	0.0436	0.0582	0.0770	0.0974
0.4	0.0003	0.0017	0.0050	0.0104	0.0179	0.0278	0.0400	0.0533	0.0716	0.0911
0.5	0.0002	0.0015	0.0044	0.0093	0.0163	0.0255	0.0370	0.0492	0.0670	0.0857
0.6	0.0002	0.0013	0.0040	0.0085	0.0150	0.0239	0.0345	0.0457	0.0631	0.0810
0.7	0.0002	0.0012	0.0036	0.0078	0.0138	0.0220	0.0323	0.0426	0.0596	0.0768
0.8	0.0001	0.0011	0.0033	0.0072	0.0129	0.0206	0.0304	0.0400	0.0561	0.0730
0.9	0.0001	0.0010	0.0031	0.0067	0.0121	0.0194	0.0287	0.0376	0.0538	0.0698
1.0	0.0001	0.0009	0.0029	0.0063	0.0114	0.0183	0.0272	0.0355	0.0513	0.0667

Condition for maximum negative bending moment at third support M_3.

$$M_2 = -(e - f)wl^2, \quad \dots \dots \dots \quad (290)$$

$$M_3 = -(e_1 + f_1)wl^2, \quad \dots \dots \dots \quad (291)$$

where e, f and e_1, f_1 are constants from tables on pp. 82 and 83 for proper m_1 and m_2.

(*c*) *First and third spans loaded.*

Condition for maximum positive bending moment in first and third spans.

$$M_2 = -(d - f)wl^2, \quad \dots \dots \dots \quad (292)$$

$$M_3 = -(-d_1 + f_1)wl^2, \quad \dots \dots \dots \quad (293)$$

where d, f and d_1, f_1 are constants from tables on pp. 82 and 83 for proper m_1 and m_2.

(*d*) *Second span loaded.*

Condition for maximum positive bending moment in second span.

$$M_2 = -ewl^2, \quad \dots \dots \dots \quad (294)$$

$$M_3 = -e_1wl^2, \quad \dots \dots \dots \quad (295)$$

where e and e_1 are constants from tables on pp. 82 and 83, for proper m_1 and m_2.

Maximum Positive Bending Moments.—Having determined the negative bending moment, the maximum positive bending moment is found by means of table on p. 176, in the manner described on p. 22.

Maximum Shears.—Knowing the negative bending moments, the maximum end shears are determined by means of table on p. 177, as described on p. 23.

Bending Moment Diagrams for Case 1 and 2

After the bending moments at the support M_2 and M_3 are computed for any particular type of loading, the bending moment diagram may be easily drawn in the following manner.

Draw the bending moment diagrams for each loaded span, considering it as simply supported. The diagrams then are parabolas for which the ordinates at the center are $\frac{1}{8}wl^2$, $\frac{1}{8}w(m_1l)^2$ and $\frac{1}{8}w(m_2l)^2$, respectively. All parabolas to be drawn to same scale.

Plot in a separate figure at the supports 2 and 3 the computed bending moments M_2 and M_3, using same scale as before. The values should be plotted above the axis if negative and below the axis if positive.

Connect the points obtained by plotting M_2 and M_3 with the supports 1 and 3 and with each other. Starting from the lines thus obtained transfer to this figure the ordinates of the parabolas. Part of the diagrams thus obtained will be above the axis and will denote negative bending moments. The balance will be below the axis and will signify positive bending moments.

FOUR SPANS. FREE ENDS.

A beam consisting of four spans with free ends has three statically indeterminate values, namely, the bending moments at the interior supports M_2, M_3 and M_4. There are no bending moments at the free supports, therefore $M_1 = M_5 = 0$.

The static indeterminate values may be found from the following three equations which were derived from the three-moment equation.

$$2M_2(l_1+l_2)+M_3l_2=-6\left[\frac{1}{l_1}\int_0^{l_1}M_{s1}\,xdx+\frac{1}{l_2}\int_0^{l_2}M_{s2}\,(l_2-x)dx\right],$$

$$M_2l_2+2M_3(l_2+l_3)+M_4l_3=-6\left[\frac{1}{l_2}\int_0^{l_2}M_{s2}\,xdx+\frac{1}{l_3}\int_0^{l_3}M_{s3}\,(l_3-x)dx\right],$$

$$M_3l_3+2M_4(l_3+l_4)=-6\left[\frac{1}{l_3}\int_0^{l_3}M_{s3}\,xdx+\frac{1}{l_4}\int_0^{l_4}M_{s4}\,(l_4-x)dx\right].$$

To simplify the work substitute

$$Q_1 = \frac{6}{l_1}\int_0^{l_1}M_{s1}\,xdx,$$

$$Q_2 = \frac{6}{l_2}\int_0^{l_2}M_{s2}\,xdx. \qquad Q'_2 = \frac{6}{l_2}\int_0^{l_2}M_{s2}\,(l_2-x)dx,$$

$$Q_3 = \frac{6}{l_3}\int_0^{l_3}M_{s3}\,xdx, \qquad Q'_3 = \frac{6}{l_3}\int_0^{l_3}M_{s3}\,(l_3-x)dx,$$

$$Q'_4 = \frac{6}{l_4}\int_0^{l_4}M_{s4}\,(l_4-x)dx.$$

These equations solved for M_2, M_3 and M_4 give:

General Equations for Negative Bending Moments,

$$M_2=\frac{[l^2_3-4(l_2+l_3)(l_3+l_4)](Q_1+Q'_2)+2l_2(l_3+l_4)(Q_2+Q'_3)-l_2l_3(Q_3+Q'_4)}{2[4(l_1+l_2)(l_2+l_3)(l_3+l_4)-l_3^2(l_1+l_2)-l_2^2(l_3+l_4)]},$$

$$M_3=\frac{l_2(l_3+l_4)(Q_1+Q'_2)-2(l_1+l_2)(l_3+l_4)(Q_2+Q'_3)+l_3(l_1+l_2)(Q_3+Q'_4)}{[4(l_1+l_2)(l_2+l_3)(l_3+l_4)-l_3^2(l_1+l_2)-l_2^2(l_3+l_4)]}$$

$$M_4=\frac{-l_2l_3(Q_1+Q'_2)+2l_3(l_1+l_2)(Q_2+Q'_3)+[l_2^2-4(l_1+l_2)(l_2+l_3)](Q_3+Q'_4)}{2[4(l_1+l_2)(l_2+l_3)(l_3+l_4)-l_3^2(l_1+l_2)-l_2^2(l_3+l_4)}.$$

Value of Constants for Uniform Loading.—Assume that the loading in the four spans is uniformly distributed but the intensity of the loading in each span is different than in the others, so that

w_1 = uniformly distributed load first span, per lin. ft.;
w_2 = uniformly distributed load second span, per lin. ft.;
w_3 = uniformly distributed load third span, per lin. ft.;
w_4 = uniformly distributed load fourth span, per lin. ft.

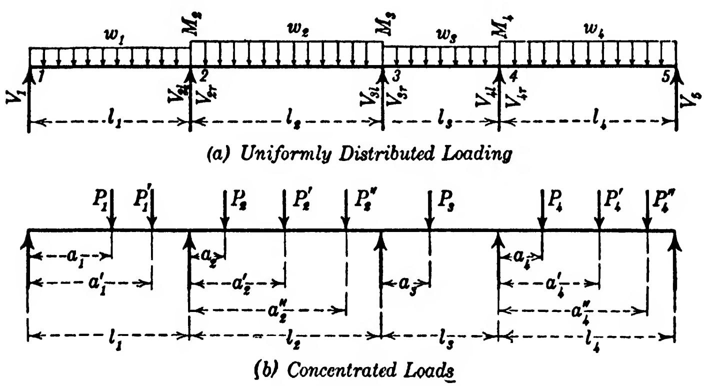


FIG. 49—Four Spans, Free Ends. (*See p.* 86.)

Then the values of constants is

Constants for Uniformly Distributed Loading,

$$Q_1 = \tfrac{1}{4}w_1l_1^3 \text{ for first span,} \quad (296)$$

$$Q_2 = Q'_2 = \tfrac{1}{4}w_2l_2^3 \text{ for second span,} \quad (297)$$

$$Q_3 = Q'_3 = \tfrac{1}{4}w_3l_3^3 \text{ for third span,} \quad (298)$$

$$Q_4 = Q'_4 = \tfrac{1}{4}w_4l_4^3 \text{ for fourth span.} \quad (299)$$

Values of Constants for Concentrated Loads,

Let P'_1, P''_1, P'''_1 = concentrated loads in first span spaced a'_1, a''_1, a'''_1 from left support;

P'_2, P''_2, P'''_2 = concentrated loads in second span spaced a'_2, a''_2, a'''_2 from left support;

P'_3, P''_3, P'''_3 = concentrated loads in third span spaced a'_3, a''_3, a'''_3 from left support;

P'_4, P''_4, P'''_4 = concentrated loads in fourth span spaced a'_4, a''_4, a'''_4 from left support.

Then

$$Q_1 = l_1^2\left[\frac{a'_1}{l_1}\left(1 - \left(\frac{a'_1}{l_1}\right)^2\right)P'_1 + \frac{a''_1}{l_1}\left(1 - \left(\frac{a''_1}{l_1}\right)^2\right)P''_1 + \frac{a'''_1}{l_1}\left(1 - \left(\frac{a'''_1}{l_1}\right)^2\right)P'''_1 + \ldots\right].$$

In simpler form this is written as follows:

Constants for Concentrated Loads,

$$Q_1 = l_1^2\Sigma\frac{a_1}{l_1}\left(1 - \left(\frac{a_1}{l_1}\right)^2\right)P_1 = l_1^2\Sigma P_1C_1 \text{ for first span,} \qquad (300)$$

$$Q_2 = l_2^2\Sigma\frac{a_2}{l_2}\left(1 - \left(\frac{a_2}{l_2}\right)^2\right)P_2 = l_2^2\Sigma P_2C_1 \text{ for second span,} \qquad (301)$$

$$Q_3 = l_3^2\Sigma\frac{a_3}{l_3}\left(1 - \left(\frac{a_3}{l_3}\right)^2\right)P_3 = l_3^2\Sigma P_3C_1 \text{ for third span,} \qquad (302)$$

$$Q_4 = l_4^2\Sigma\frac{a_4}{l_4}\left(1 - \left(\frac{a_4}{l_4}\right)^2\right)P_4 = l_4^2\Sigma P_4C_1 \text{ for fourth span,} \qquad (303)$$

also,

$$Q'_2 = l_2^2\Sigma\frac{a_2}{l_2}\left(1 - \frac{a_2}{l_2}\right)\left(2 - \frac{a_2}{l_2}\right)P_2 = l_2^2\Sigma P_2C_2 \text{ for second span,} \qquad (304)$$

$$Q'_3 = l_3^2\Sigma\frac{a_3}{l_3}\left(1 - \frac{a_3}{l_3}\right)\left(2 - \frac{a_3}{l_3}\right)P_3 = l_3^2\Sigma P_3C_2 \text{ for third span,} \qquad (305)$$

$$Q'_4 = l_4^2\Sigma\frac{a_4}{l_4}\left(1 - \frac{a_4}{l_4}\right)\left(2 - \frac{a_4}{l_4}\right)P_4 = l_4^2\Sigma P_4C_2 \text{ for fourth span.} \qquad (306)$$

The values of C_1 and C_2 for different ratios $\frac{a}{l}$ may be taken from Diagram 1, p. 19.

The process is then as follows. Find for each load in each span the values of $\frac{a}{l}$ by dividing its distance from the left support by the proper span length. For this value find from Diagram 1 on p. 19 the value of C_1 or C_2. Multiply each load by its corresponding constant and add the results in each span.

When the loads are equal, then ΣPC_1 may be written as $P\Sigma C_1$. In such case the values of C_1 or C_2 for all loads in a span are added and the sum multiplied by P.

The value of Q_1, Q_2, Q_3, Q_4, etc., so obtained are substituted in Formulas for M_2 to M_4 and the negative bending moments are obtained.

Maximum Positive Bending Moments.—The Formulas for M_2 to M_4, give bending moments for negative bending moments at the support.

After the negative bending moments are computed the positive bending moments for concentrated loads are obtained as shown in Fig. 9, p. 14, and explained on p. 21.

For uniform loading the maximum positive bending moments for known negative bending moments may be obtained from table on p. 177, in the manner described on p. 22.

Four Equal Spans. Free Ends

For four equal spans $l_1 = l_2 = l_3 = l_4 = l$. This substituted in Formulas for M_2 to M_4 changes the general equations to

General Equations for Equal Spans Any Type of Loading.

Bending Moments at Supports,

$$M_2 = \frac{-15Q_1 - 15Q'_2 + 4Q_2 + 4Q'_3 - Q_3 - Q'_4}{56l}, \quad . \quad (307)$$

$$M_3 = \frac{2Q_1 + 2Q'_2 - 8Q_2 - 8Q'_3 + 2Q_3 + 2Q'_4}{28l}, \quad . \quad . \quad (308)$$

$$M_4 = \frac{-Q_1 - Q'_2 + 4Q_2 + 4Q'_3 - 15Q_3 - 15Q'_4}{56l}. \quad . \quad (309)$$

For values of constants Q_1 to Q_4 and Q'_2 to Q'_4 see Formulas (296) to (306), pp. 86 and 87. They depend upon the loading of the spans.

General Equation for Symmetrical Loading.—When the loading of each span is symmetrical about the center the general equation for bending moments at supports may be simplified because then

$$Q_2 = Q'_2,\ Q_3 = Q'_3,\ Q_4 = Q'_4.$$

Substitute this in Equations (307) to (309).

Bending Moments at Supports for Equal Spans and Symmetrical Loading,

$$M_2 = \frac{-15Q_1 - 11Q_2 + 3Q_3 - Q_4}{56l}, \quad . \quad . \quad . \quad (310)$$

$$M_3 = \frac{2Q_1 - 6Q_2 - 8Q_3 + 2Q_4}{28l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (311)$$

$$M_4 = \frac{-Q_1 + 3Q_2 - 11Q_3 - 15Q_4}{56l}. \quad . \quad . \quad . \quad (312)$$

Foɪ values of constants Q_1 to Q_4 see Formulas (300) to (303).

Uniformly Distributed Loads of Varying Intensities.—If the uniformly distributed loadings of various spans vary as to intensity, so that

w_1 = uniformly distributed loading in first span;
w_2 = uniformly distributed loading in second span;
w_3 = uniformly distributed loading in third span;
w_3 = uniformly distributed loading in fourth span;

the formulas for bending moments become

Bending Moments at Supports for Equal Spans,

$$M_2 = -\frac{15w_1 + 11w_2 - 3w_3 + w_4}{224}l^2. \quad . \quad . \quad . \quad (313)$$

$$M_3 = -\frac{-w_1 + 3w_2 + 3w_3 - w_4}{56}l^2. \quad . \quad . \quad . \quad (314)$$

$$M_4 = -\frac{w_1 - 3w_2 + 11w_3 + 15w_4}{224}l^2. \quad . \quad . \quad . \quad (315)$$

Uniformly Distributed Loading of Uniform Intensities.—In the table below are given bending moments and end shears for cases where the uniformly distributed loading of all loaded spans is of the same intensity.

The bending moments at the supports for each condition of loading were obtained from Equations (313) to (315), p. 89, by substituting w for the loading of all loaded spans and zero for the loading of all unloaded spans. Thus if first and third spans, only, are loaded $w_1 = w_3 = w$ and $w_2 = w_4 = 0$.

Following conditions of loading are considered in the table.

(a) All spans loaded. $w_1 = w_2 = w_3 = w_4 = w$.

Condition for dead load (Fig. 50 (a)).

(b) First, second and fourth spans loaded. $w_1 = w_2 = w_4 = w$, $w_3 = 0$.

Condition for maximum negative bending moment at second support M_2 (Fig. 50 (b)).

(c) Second and third spans loaded. $w_2 = w_3 = w$, $w_1 = w_4 = 0$.

Condition for maximum negative bending moment at third support M_3 (Fig. 50 (c)).

(d) First and third spans loaded. $w_1 = w_3 = w$, $w_2 = w_4 = 0$.

Condition for maximum positive bending moment in first and third span. (Fig. 50 (d)).

Uniform Loading. Four Equal Spans, Free Ends

End Shears

Condition (See Fig. 50)	Spans Loaded	First Span		Second Span		Third Span		Fourth Span	
		V_1	V_{2l}	V_{2r}	V_{3l}	V_{3r}	V_{4l}	V_{4r}	V_5
a	1, 2, 3, 4	0.393wl	0.607wl	0.536wl	0.464wl	0.464wl	0.536wl	0.607wl	0.393wl
b	1, 2, 3, 4	0.380wl	**0.620wl**	**0.603wl**	0.397wl	−0.0402wl	0.0402wl	0.558wl	0.442wl
c	–, 2, 3, –	−0.036wl	0.036wl	0.429wl	**0.572wl**	**0.572wl**	0.429wl	0.036wl	−0.036wl
d	1, –, 3, –	**0.446wl**	0.554wl	0.018wl	−0.018wl	0.482wl	0.518wl	0.054wl	−0.054wl

Maximum values for V_{4l} and V_{4r} are same as for V_{2r} and V_{2l} respectively and act when spans 1, 3 and 4 are loaded.
Maximum value for V_5 is same as for V_1 and acts when spans 2 and 4 are loaded.

Bending Moments

Condition (See Fig. 50)	Spans Loaded	At Supports			Maximum Positive Bending Moments			
		M_2	M_3	M_4	First Span	Second Span	Third Span	Fourth Span
a	1, 2, 3, 4	−0.107wl^2	−0.071wl^2	−0.107wl^2	0.077wl^2	0.036wl^2	0.036wl^2	0.077wl^2
b	1, 2, –, 4	**−0.121wl^2**	−0.018wl^2	−0.058wl^2	0.072wl^2	0.061wl^2	......	0.098wl^2
c	–, 2, 3, –	−0.036wl^2	**−0.107wl^2**	−0.036wl^2	......	0.056wl^2	0.056wl^2	....
d	1, –, 3, –	−0.054wl^2	−0.036wl^2	−0.054wl^2	**0.099wl^2**	....	**0.080wl^2**	....

Absolute maximum values are shown by black-face type.

Maximum values for M_4 is same as for M_2 and acts when spans 1, 3 and 4 are loaded.

Maximum values for positive bending moment in 2nd and 4th spans are same as in 3rd and 1st respectively, and act when spans 2 and 4 are loaded.

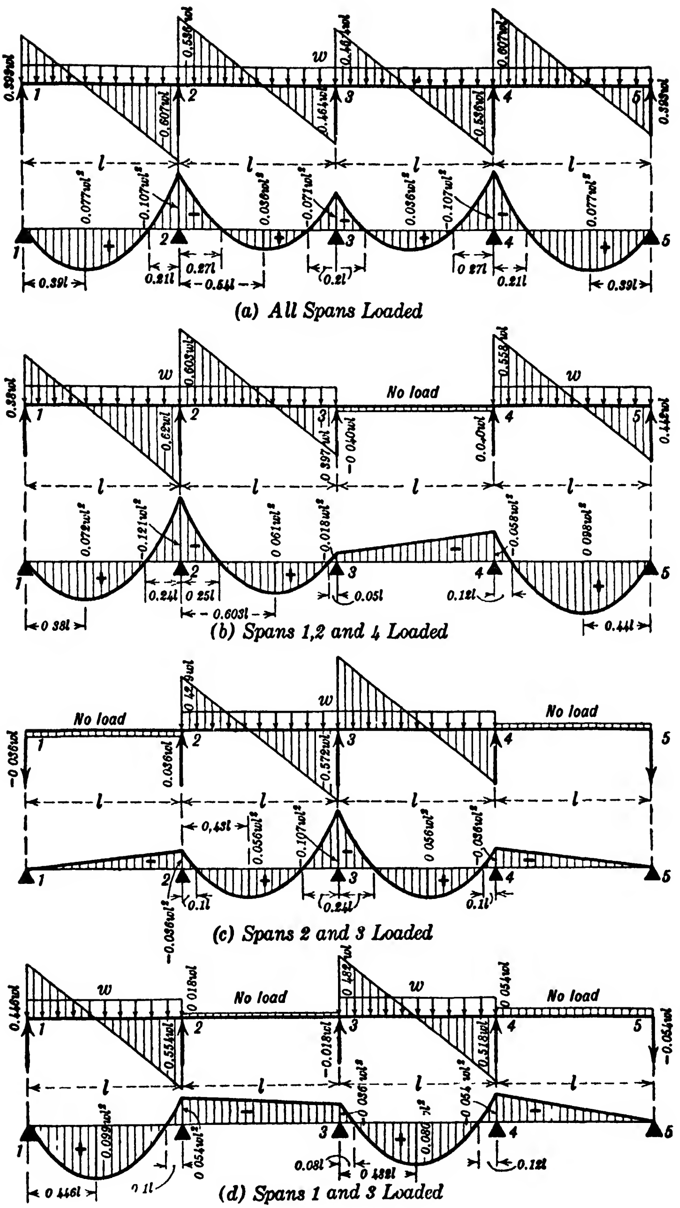


FIG. 50.—Four Equal Spans, Free Ends. Uniformly Distributed Loading. (*See p.* 89.)

Case (*b*) takes care also of the maximum bending moment at the fourth support. This is equal to the maximum bending moment at the second support as determined in case (*b*). It is obtained when the first, third and fourth spans are loaded.

Case (*d*) takes care also of maximum positive bending moments in second and fourth spans. The maximum positive bending moment in the fourth span is the same as the maximum positive bending moment in the first span, while the maximum positive bending moment in the second span is equal to the maximum in the third span. These maximums are obtained by loading the second and fourth spans.

Combining Uniform Dead Load with Uniform Live Load.—Bending moments and shears for a combination of dead load and live load are obtained by computing separately the bending moments and shears for the dead load and for the live load and adding the result. For the dead load the condition (*a*) where all spans are loaded should be used. For the live load the most unfavorable positions of the load for each value should be used.

To facilitate the work, the table below is given where the maximum bending moments and shears are given for various combinations of the dead and live load.

To use this table find the intensity of the dead load w_1 and of the live load w_2. Add these values and get $w = w_1 + w_2$. Find the ratio of the dead load to the total load $\frac{w_1}{w}$. Locate this value in the first column. The bending moments and shears corresponding to this value are maximum values for this combination of dead and live load.

It should be noted that the value w in the table is the sum of the dead plus live load.

Uniform Loading, Four Equal Spans. Free Ends

Maximum Values for Combined Dead and Live Loads. (*See p.* 92.)

Dead Load	Live Load	End Shears				Negative Bending Moment		Maximum Positive Bending Moment	
		V_1 and V_5	V_{2l} and V_{4r}	V_{2r} and V_{4l}	V_{3r} and V_{3l}	M_2 and M_4	M_3	End Spans	Center Spans
0.2w	0.8w	0.435wl	0.617wl	0.590wl	0.550wl	−0.118wl^2	−0.100wl^2	0.095wl^2	0.071wl^2
0.3w	0.7w	0.430wl	0.616wl	0.583wl	0.540wl	−0.117wl^2	−0.096wl^2	0.092wl^2	0.067wl^2
0.4w	0.6w	0.425wl	0.615wl	0.576wl	0.529wl	−0.115wl^2	−0.093wl^2	0.090wl^2	0.062wl^2
0.5w	0.5w	0.419wl	0.613wl	0.569wl	0.518wl	−0.114wl^2	−0.089wl^2	0.088wl^2	0.058wl^2
0.6w	0.4w	0.414wl	0.612wl	0.563wl	0.507wl	−0.113wl^2	−0.085wl^2	0.086wl^2	0.053wl^2
0.7w	0.3w	0.409wl	0.611wl	0.556wl	0.496wl	−0.111wl^2	−0.082wl^2	0.084wl^2	0.049wl^2

w = Uniform unit dead plus live load. l = Length of span.

Four Equal Spans. Concentrated Loads

Formulas are given below for end shears and bending moment produced by single concentrated loads P placed at any distance from left support a.

These formulas can be used for any number of loads on the span adding the results for all loads.

Concentrated Load in First Span at Distance *a* from Left Support. (See Fig. 51, p. 93.)

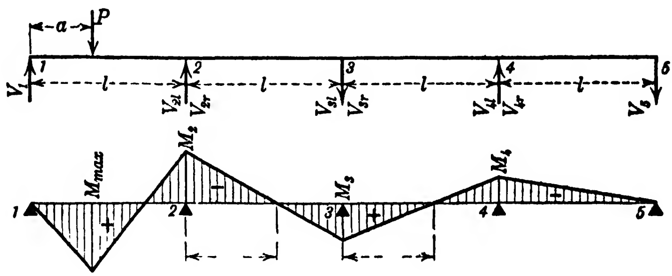


Fig. 51.—Concentrated Load in First Span. (See p. 93.)

End Shears,

$$V_1 = \left(1 - \frac{a}{l}\right)P + \frac{M_2}{l} = \left(1 - \frac{a}{l} - K_1\right)P = K_4P, \quad . \quad (316)$$

$$V_{2l} = P - V_1, \ (317) \quad V_{2r} = \frac{M_3 - M_2}{l} = (K_1 + K_2)P, \ (318)$$

$$V_{3l} = -V_{2r}, \ . \ (319) \quad V_{3r} = \frac{M_4 - M_3}{l} = -(K_2 + K_3)P, \ (320)$$

$$V_{4l} = -V_{3r}, \ . \ (321) \quad V_{4r} = K_3P, \ . \ (322) \quad V_5 = -V_{4r}. \ (323)$$

Negative Bending Moments,

$$M_2 = -\frac{15}{56}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]Pl = -K_1Pl, \ . \ . \ . \ . \ . \ . \ (324)$$

$$M_3 = \frac{1}{14}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]Pl = K_2Pl, \ . \ . \ . \ . \ . \ . \ . \ (325)$$

$$M_4 = -\frac{1}{56}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]Pl = -K_3Pl. \ . \ . \ . \ . \ . \ . \ (326)$$

Bending Moment at Any Point:

First span,

$$M_x = V_1 x = K_4 \frac{x}{l} Pl, \text{ for } x \text{ smaller than } a, \quad (327)$$

$$M_x = V_1 x - P(x - a), \text{ for } x \text{ larger than } a. \quad (328)$$

Second span,

$$M_x = M_2 + \frac{M_3 - M_2}{l} x = \left[- K_1 + (K_1 + K_2) \frac{x}{l} \right] Pl. \quad (329)$$

Third span,

$$M_x = M_3 + \frac{M_4 - M_3}{l} x = \left[K_2 - (K_2 - K_3) \frac{x}{l} \right] Pl. \quad (330)$$

Fourth span,

$$M_x = M_4 - \frac{M_4}{l} x = - K_3 \left(1 - \frac{x}{l} \right) Pl. \quad (331)$$

Maximum Positive Bending Moment,

$$M_{max} = V_1 a. \quad (332)$$

Concentrated Load P in Second Span. (See Fig. 52, p. 94.)

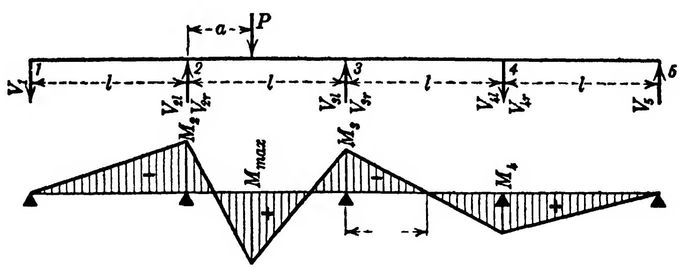


FIG. 52.—Concentrated Load P in Second Span. (*See p.* 94.)

End Shears,

$$V_1 = \frac{M_2}{l} = - K_5 P \quad (333) \qquad V_{2l} = - V_1. \quad (334)$$

$$V_{2r} = \left(1 - \frac{a}{l}\right)P + \frac{M_3 - M_2}{l} = K_7 P. \quad \ldots \ldots \quad (335)$$

$$V_{3l} = P - V_{2r} = (1 - K_7)P. \quad \ldots \ldots \quad (336)$$

$$V_{3r} = \frac{M_4 - M_3}{l} = \frac{5}{4}K_6 P \quad . \quad (337) \qquad V_{4l} = - V_{3r}. \quad . \quad . \quad (338)$$

$$V_{4r} = - \frac{M_4}{l} = -\frac{1}{4}K_6 P. \quad . \quad (339) \qquad V_5 = V_{4r}. \quad . \quad . \quad . \quad (340)$$

Negative Bending Moments,

$$M_2 = -\frac{1}{56}\left[15\frac{a}{l}\left(1 - \frac{a}{l}\right)\left(2 - \frac{a}{l}\right) - 4\frac{a}{l}\left(1 - \left(\frac{a}{l}\right)^2\right)\right]Pl = -K_5 Pl. \quad (341)$$

$$M_3 = -\frac{1}{14}\left[4\frac{a}{l}\left(1 - \left(\frac{a}{l}\right)^2\right) - \frac{a}{l}\left(1 - \frac{a}{l}\right)\left(2 - \frac{a}{l}\right)\right]Pl = - K_6 Pl. \quad (342)$$

$$M_4 = \tfrac{1}{4}K_6 Pl. \quad \ldots \ldots \quad (343)$$

Bending Moment at Any Point:

First span,

$$M_x = V_1 x = - K_5 \frac{x}{l} Pl. \quad \ldots \ldots \quad (344)$$

Second span,

$$M_x = M_2 + V_{2r}x = \left(K_7 \frac{x}{l} - K_5\right)Pl, \text{ for } x \text{ smaller than } a, \quad (345)$$

$$M_x = \left[(1 - K_7)\left(1 - \frac{x}{l}\right) - K_6\right]Pl, \text{ for } x \text{ larger than } a. \quad . \quad (346)$$

Third span,

$$M_x = M_3 + V_{3r}x = \left(-1 + \frac{5x}{4l}\right)K_5 Pl. \quad \ldots \ldots \quad (347)$$

Fourth span,

$$M_x = \frac{M_4}{l}(l - x) = \frac{1}{4}\left(1 - \frac{x}{l}\right)K_6 Pl. \quad \ldots \ldots \quad (348)$$

Beam Loaded by Equal Symmetrical Load Groups

In the table below are given bending moments at supports for beams consisting of four equal spans and loaded by groups of concentrated loads, each group arranged symmetrically in each span about its center. The load groups in each span are equal, but the loading is not fixed so that it may or may not be applied in any span at any particular time.

The values in the table are based on the general formulas (310) to (312), p. 88, for symmetrical loading. The values of constants Q_1, Q_2, Q_3, and Q_4 in all loaded spans were taken equal to Q, and in unloaded spans were made equal to zero.

For example, the loading giving maximum negative bending moment at the second support consists of the first, second and fourth spans loaded. Therefore

$$Q_1 = Q_2 = Q_4 = Q \text{ and } Q_3 = 0.$$

Four conditions of loading are used:

(*a*) All spans loaded, $Q_1 = Q_2 = Q_3 = Q_4 = Q$.
(*b*) First, second and fourth spans loaded, $Q_1 = Q_2 = Q_4 = Q$, $Q_3 = 0$.
(*c*) Second and third spans loaded, $Q_1 = Q_4 = 0$, $Q_2 = Q_3 = Q$.
(*d*) First and third spans loaded, $Q_1 = Q_3 = Q$, $Q_2 = Q_4 = 0$.

For these loadings the values of negative bending moments are given in the table below.

Negative Bending Moments for Symmetrical Concentrated Loads.

Four Equal Spans, Free Ends

Condition	Spans Loaded	M_2	M_3	M_4
a	1, 2, 3, 4	$-0.429\frac{1}{l}Q$	$-0.286\frac{1}{l}Q$	$-0.429\frac{1}{l}Q$
b	1, 2, –, 4	$-0.482\frac{1}{l}Q$	$-0.071\frac{1}{l}Q$	$-0.232\frac{1}{l}Q$
c	–, 2, 3, –	$-0.143\frac{1}{l}Q$	$-0.429\frac{1}{l}Q$	$-0.143\frac{1}{l}Q$
d	1, –, 3, –	$-0.214\frac{1}{l}Q$	$-0.143\frac{1}{l}Q$	$-0.214\frac{1}{l}Q$

where $Q = l^2\Sigma\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]P = l^2\Sigma PC_1$.

Values of C_1 may be taken from Diagram 1, p. 19.

Special Arrangement of Symmetrical Concentrated Loads.—Following arrangement of concentrated loads will be considered:

1. Loads P at center of spans.
2. Two equal loads P at third points.
3. Three equal loads P at quarter points.
4. Four equal loads P at fifth points.

The bending moment can be used for continuous girders which carry cross beams so that the load transferred by the beams to the girders is concentrated.

In each case the four types of loading are considered which give maximum values for negative and positive bending moments, respectively.

For dead load the bending moments for condition (*a*) should be used.

For live load use the maximum values of bending moment for the proper condition of loading. Thus for negative bending moment at the second support use the condition (*b*) and at the third support the condition (*c*). For positive bending moment for all spans use the maximum values obtained from condition (*d*).

The formulas are obtained from general equations in table on p. 96 by substituting the proper values for constant Q.

Combining Concentrated Dead Load with Concentrated Live Load.—The bending moments and shears for concentrated dead and live load are combined in the same manner as explained on p. 92 in connection with the uniform loading.

The table on p. 106 facilitates the work of combining the maximum end shears and bending moments. After the sum of dead load and live load is found, the ratio of dead load to total load is computed for which the maximum end shears and bending moments may be taken directly from the table.

Four Equal Spans, Free Ends. One Load P in Center

End Shears

Condition (See Fig. 53)	Spans Loaded	First Span		Second Span		Third Span		Fourth Span	
		V	V_{2l}	V_{2r}	V_{3l}	V_{3r}	V_{4l}	V_{4r}	V_5
a	1, 2, 3, 4	0 339*P*	0 661*P*	0 554*P*	0 446*P*	0 446*P*	0 554*P*	0 661*P*	0.339*P*
b	1, 2, –, 4	0.320*P*	**0 680*P***	**0.654*P***	0 346*P*	–0 060*P*	0 060*P*	0 587*P*	0 413*P*
c	–, 2, 3, –	–0 054*P*	–0 054*P*	0 393*P*	**0 607*P***	**0 607*P***	0 393*P*	0 054*P*	–0 054*P*
d	1, –, 3, –	**0.420*P***	0 580*P*	0.026*P*	–0.026*P*	0 474*P*	0.526*P*	0 080*P*	–0.080*P*

Maximum values for V_{4l} and V_{4r} are same as for V_{2r} and V_{2l} respectively and act when spans 1, 3 and 4 are loaded.
Maximum values for V_5 is same as for V_1 and acts when spans 2 and 4 are loaded.

Maximum Bending Moments

Condition (See Fig. 53)	Spans Loaded	Negative Bending Moment			Maximum Positive Bending Moment			
		M_2	M_3	M_4	First Span	Second Span	Third Span	Fourth Span
a	1, 2, 3, 4	–0 161*Pl*	–0 107*Pl*	–0 161*Pl*	0 170*Pl*	0 116*Pl*	0.116*Pl*	0.170*Pl*
b	1, 2, –, 4	**–0 181*Pl***	–0 027*Pl*	–0 087*Pl*	0 160*Pl*	0 146*Pl*	....	0 207*Pl*
c	–, 2, 3, –	–0 054*Pl*	**–0 161*Pl***	–0 054*Pl*	. . . .	0 146*Pl*	0 146*Pl*	....
d	1, –, 3, –	–0 080*Pl*	–0 054*Pl*	–0 080*Pl*	**0 210*Pl***	....	**0 183*Pl***	....

Absolute maximum values are shown in black-face type.
Maximum value for M_4 is same as for M_2 and acts when spans 1, 3 and 4 are loaded.
Maximum values for positive bending moments in 2nd and 4th spans are same as in 3rd and 1st spans respectively, and act when spans 2 and 4 are loaded.

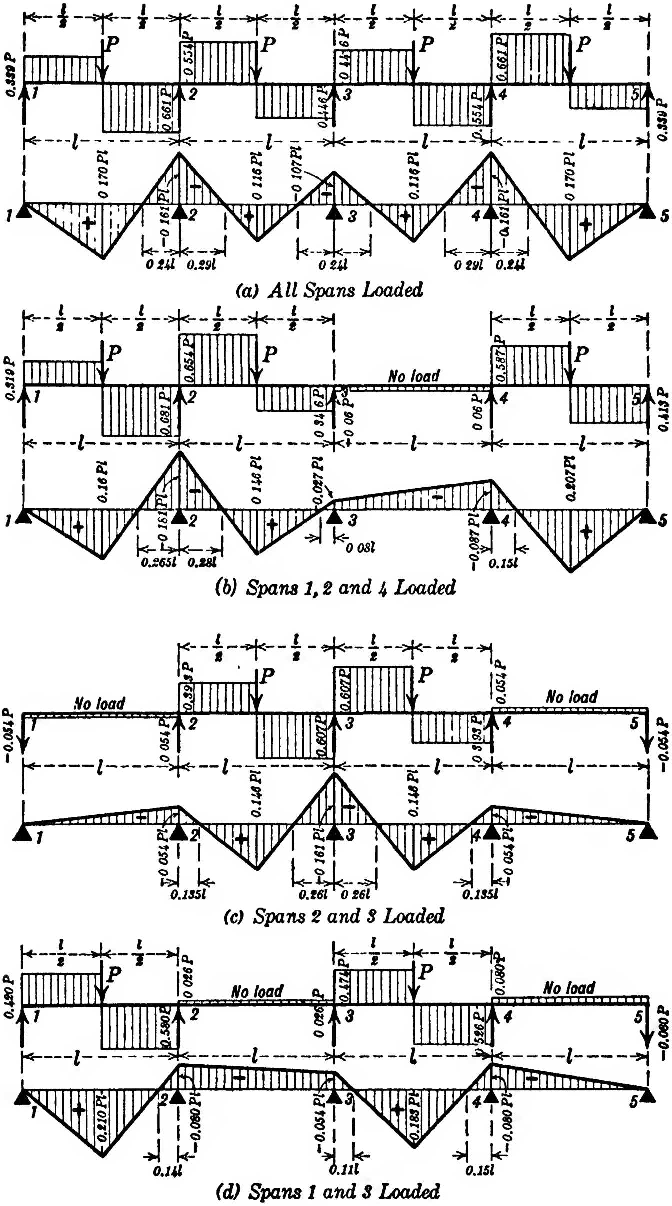


FIG. 53.—Four Equal Spans, Free Ends. One Load P in Center of Span. (*See p.* 98.)

Four Equal Spans, Free Ends. Two Equal Loads P at Third Points

End Shears

Condition (See Fig. 54)	Spans Loaded	First Span		Second Span		Third Span		Fourth Span	
		V_1	V_{2l}	V_{2r}	V_{3l}	V_{3r}	V_{4l}	V_{4r}	V_5
a	1, 2, 3, 4	0.714P	1.286P	1.095P	0.905P	0.905P	1.095P	1.286P	0.714P
b	1, 2, –, 4	0.679P	**1.321P**	**1.274P**	0.726P	−0.107P	0.107P	1.155P	0.845P
c	–, 2, 3, –	−0.095P	0.095P	0.81P	**1.19P**	**1.19P**	0.81P	0.095P	−0.095P
d	1, –, 3, –	**0.858P**	1.143P	0.048P	−0.048P	0.95P	1.05P	0.142P	−0.142P

Maximum values for V_{4l} and V_{4r} are same as for V_{2r} and V_{2l} respectively and when spans 1, 3 and 4 are loaded.
Maximum value for V_5 is same as for V_1 and acts when spans 2 and 4 are loaded.

Maximum Bending Moments

Condition (See Fig. 54)	Spans Loaded	Negative Bending Moment			Maximum Positive Bending Moment			
		M_2	M_3	M_4	First Span	Second Span	Third Span	Fourth Span
a	1, 2, 3, 4	−0.286Pl	−0.192Pl	−0.286Pl	0.238Pl	0.111Pl	0.111Pl	0.238Pl
b	1, 2, –, 4	**−0.321Pl**	−0.048Pl	−0.155Pl	0.226Pl	0.194Pl	....	0.282Pl
c	–, 2, 3, –	−0.095Pl	**−0.286Pl**	−0.095Pl	....	0.175Pl	0.175Pl	....
d	1, –, 3, –	−0.143Pl	−0.095Pl	−0.143Pl	**0.286Pl**	....	**0.222Pl**	....

Absolute maximum values are shown in black face type.
Maximum value for M_4 is same as for M_2 and acts when spans 1, 3 and 4 are loaded.
Maximum values for positive bending moments in 2nd and 4th spans are same as in 3rd and 1st spans respectively, and act when spans 2 and 3 are loaded.

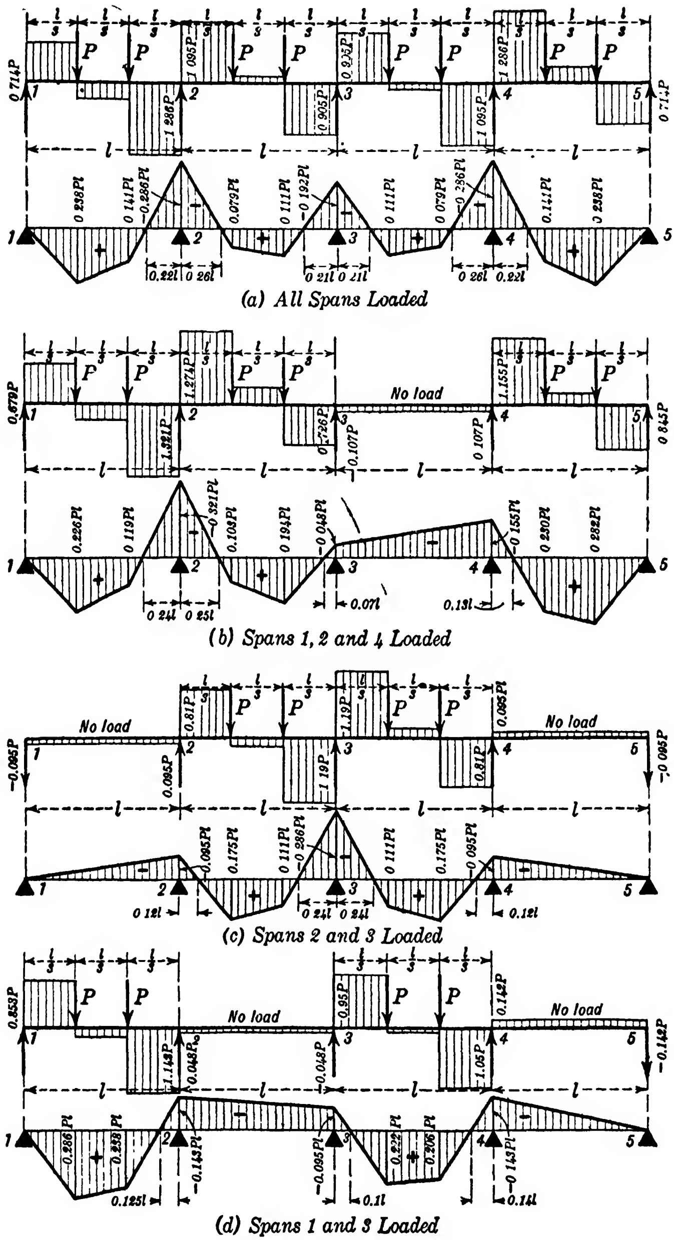


FIG. 54.—Four Equal Spans, Free Ends. Two Loads P at Third Points.
(See p. 100.)

Four Equal Spans, Free Ends. Three Equal Loads *P* at Quarter Points

End Shears

Condition (See Fig. 55)	Spans Loaded	First Span		Second Span		Third Span		Fourth Span	
		V_1	V_{2l}	V_{2r}	V_{3l}	V_{3r}	V_{4l}	V_{4r}	V_5
a	1, 2, 3, 4	1 098*P*	1.902*P*	1.634*P*	1.366*P*	1.366*P*	1.634*P*	1 902*P*	1 098*P*
b	1, 2, –, 4	1.048*P*	**1 952*P***	**1.885*P***	1 115*P*	–0 15*P*	0.15*P*	1.718*P*	1 282*P*
c	–, 2, 3, –	–0 134*P*	0 134*P*	1.232*P*	**1.768*P***	**1 768*P***	1.232*P*	0.134*P*	–0 134*P*
d	1, –, 3, –	**1 299*P***	1 701*P*	0 067*P*	–0.067*P*	1.433*P*	1.567*P*	0.201*P*	–0.201*P*

Maximum values for V_{4l} and V_{4r} are same as for V_{2r} and V_{2l} respectively and act when spans 1, 3 and 4 are loaded.
Maximum value for V_5 is same as for V_1 and acts when spans 2 and 4 are loaded.

Maximum Bending Moments

Condition (See Fig. 55)	Spans Loaded	Negative Bending Moment			Maximum Positive Bending Moment			
		M_2	M_3	M_4	First Span	Second Span	Third Span	Fourth Span
a	1, 2, 3, 4	–0 402*Pl*	–0 268*Pl*	–0.402*Pl*	0 299*Pl*	0.165*Pl*	0.165*Pl*	0 299*Pl*
b	1, 2, –, 4	**–0 452*Pl***	–0 067*Pl*	–0.217*Pl*	0.274*Pl*	0.240*Pl*	......	0.391*Pl*
c	–, 2, 3, –	–0 134*Pl*	**–0 402*Pl***	–0 134*Pl*	....	0 232*Pl*	0 232*Pl*	....
d	1, –, 3, –	–0.201*Pl*	–0 134*Pl*	–0.201*Pl*	**0 400*Pl***	....	**0 333*Pl***	....

Absolute maximum values are shown in black-face type.
Maximum value for M_4 is same as for M_2 and acts when spans 1, 3 and 4 are loaded.
Maximum values for positive bending moments in 2nd and 4th spans are same as in 3rd and 1st spans respectively, and act when spans 2 and 4 are loaded.

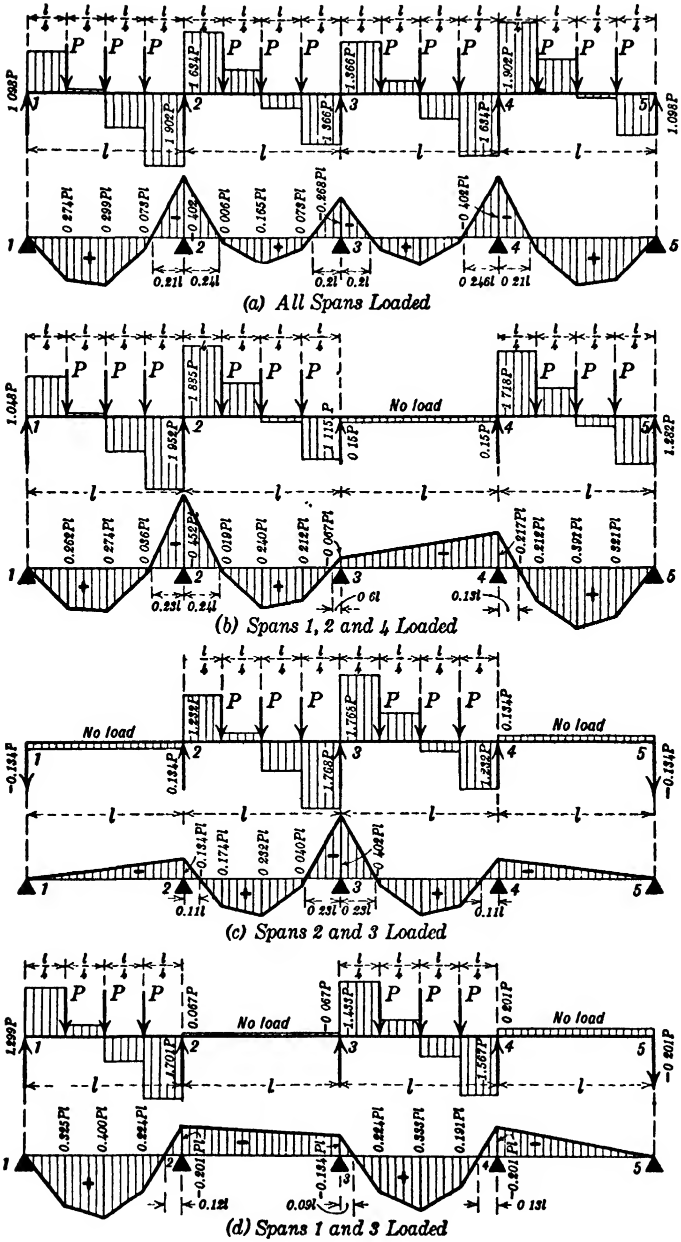


FIG. 55.—Four Equal Spans, Free Ends. Three Loads P at Quarter Points. (See p. 102.)

Four Equal Spans, Free Ends. Four Equal Loads P at Fifth Points

End Shears

Condition (See Fig. 56)	Spans Loaded	First Span		Second Span		Third Span		Fourth Span	
		V_1	V_{2l}	V_{2r}	V_{3l}	V_{3r}	V_{4l}	V_{4r}	V_5
a	1, 2, 3, 4	1.486P	2 514P	2 171P	1 829P	1 829P	2 171P	2 514P	1.486P
b	1, 2, –, 4	1.422P	**2 578P**	**2 493P**	1 507P	–0 193P	0 193P	2 279P	1 721P
c	–, 2, 3, –	–0 171P	0 171P	1 657P	**2 343P**	**2 343P**	1 657P	0 171P	–0 171P
d	1, –, 3, –	**1.743P**	2 257P	0 086P	–0 086P	1 914P	2 086P	0 257P	–0 257P

Maximum values for V_{4l} and V_{4r} are same as for V_{2r} and V_{2l} respectively and act when spans 1, 3 and 4 are loaded.
Maximum value for V_5 is same as for V_1 and acts when spans 2 and 4 are loaded.

Maximum Bending Moments

Condition (See Fig. 56)	Spans Loaded	Negative Bending Moment			Maximum Positive Bending Moment			
		M_2	M_3	M_4	First Span	Second Span	Third Span	Fourth Span
a	1, 2, 3, 4	–0 514Pl	–0 343Pl	–0 514Pl	0 394Pl	0 189Pl	0.189Pl	0 394Pl
b	1, 2, –, 4	**–0 578Pl**	–0 086Pl	–0 279Pl	0.369Pl	0 318Pl	...	0 488Pl
c	–, 2, 3, –	–0 171Pl	**–0 514Pl**	–0 171Pl	...	0 291Pl	0 291Pl	...
d	1, –, 3, –	–0 257Pl	–0 171Pl	–0 257Pl	**0 497Pl**	...	**0 394Pl**	...

Absolute maximum values are shown in black-face type.
Maximum value for M_4 is same as for M_2 and acts when spans 1, 3 and 4 are loaded.
Maximum values for positive bending moments in 2nd and 4th spans are same as in 3rd and 1st span respectively, and act when spans 2 and 4 are loaded.

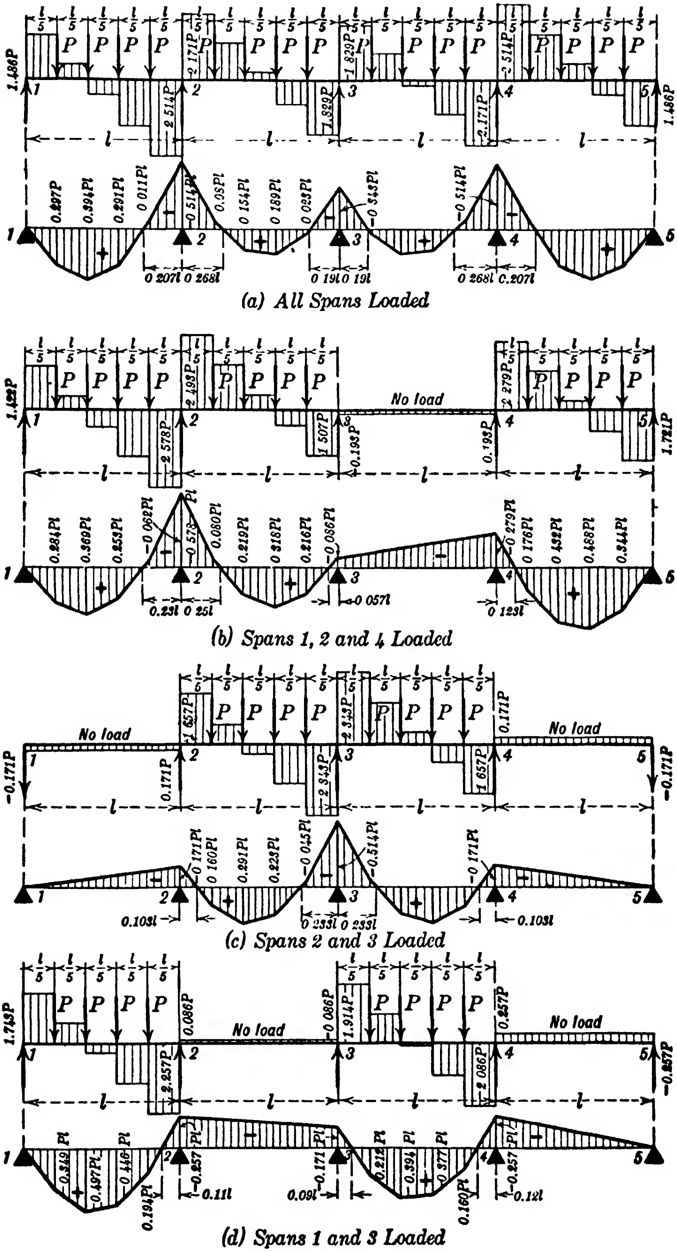


FIG. 56.—Four Loads P at Fifth Points. Four Equal Spans. (*See p.* 103.)

Four Equal Spans, Free Ends. Concentrated Loads

Maximum Values for Combined Dead and Live Loads. *(See p. 97.)*

Dead Load	Live Load	End Shears				Negative B. M.		Maximum Positive B. M.	
		V_1 and V_5	V_{2l} and V_{4r}	V_{2r} and V_{4l}	V_{3r} and V_{3l}	M_2 and M_4	M_3	End Spans	Center Spans
One Load *P* in Center									
0.2*P*	0.8*P*	0.404*P*	0.676*P*	0.634*P*	0.575*P*	−0.177*Pl*	−0.150*Pl*	0.202*Pl*	0.170*Pl*
0.3*P*	0.7*P*	0.396*P*	0.674*P*	0.624*P*	0.559*P*	−0.175*Pl*	−0.145*Pl*	0.198*Pl*	0.163*Pl*
0.4*P*	0.6*P*	0.388*P*	0.672*P*	0.614*P*	0.543*P*	−0.173*Pl*	−0.139*Pl*	0.194*Pl*	0.156*Pl*
0.5*P*	0.5*P*	0.380*P*	0.670*P*	0.604*P*	0.526*P*	−0.171*Pl*	−0.134*Pl*	0.190*Pl*	0.149*Pl*
0.6*P*	0.4*P*	0.371*P*	0.669*P*	0.594*P*	0.510*P*	−0.169*Pl*	−0.129*Pl*	0.186*Pl*	0.143*Pl*
0.7*P*	0.3*P*	0.363*P*	0.667*P*	0.584*P*	0.494*P*	−0.167*Pl*	−0.123*Pl*	0.182*Pl*	0.136*Pl*
Two Equal Loads *P* at Third Points									
0.2*P*	0.8*P*	0.829*P*	1.314*P*	1.238*P*	1.133*P*	−0.314*Pl*	−0.269*Pl*	0.276*Pl*	0.200*Pl*
0.3*P*	0.7*P*	0.815*P*	1.310*P*	1.220*P*	1.104*P*	−0.310*Pl*	−0.258*Pl*	0.272*Pl*	0.189*Pl*
0.4*P*	0.6*P*	0.800*P*	1.307*P*	1.202*P*	1.076*P*	−0.307*Pl*	−0.248*Pl*	0.267*Pl*	0.178*Pl*
0.5*P*	0.5*P*	0.786*P*	1.303*P*	1.184*P*	1.047*P*	−0.303*Pl*	−0.239*Pl*	0.262*Pl*	0.167*Pl*
0.6*P*	0.4*P*	0.772*P*	1.300*P*	1.167*P*	1.019*P*	−0.300*Pl*	−0.230*Pl*	0.256*Pl*	0.155*Pl*
0.7*P*	0.3*P*	0.757*P*	1.296*P*	1.149*P*	0.990*P*	−0.296*Pl*	−0.220*Pl*	0.252*Pl*	0.144*Pl*
Three Equal Loads *P* at Quarter Points									
0.2*P*	0.8*P*	1.259*P*	1.942*P*	1.835*P*	1.688*P*	−0.442*Pl*	−0.375*Pl*	0.380*Pl*	0.299*Pl*
0.3*P*	0.7*P*	1.239*P*	1.937*P*	1.810*P*	1.647*P*	−0.437*Pl*	−0.362*Pl*	0.370*Pl*	0.283*Pl*
0.4*P*	0.6*P*	1.219*P*	1.932*P*	1.785*P*	1.607*P*	−0.432*Pl*	−0.348*Pl*	0.360*Pl*	0.266*Pl*
0.5*P*	0.5*P*	1.198*P*	1.927*P*	1.760*P*	1.567*P*	−0.427*Pl*	−0.335*Pl*	0.350*Pl*	0.249*Pl*
0.6*P*	0.4*P*	1.178*P*	1.922*P*	1.734*P*	1.527*P*	−0.422*Pl*	−0.322*Pl*	0.339*Pl*	0.232*Pl*
0.7*P*	0.3*P*	1.158*P*	1.917*P*	1.709*P*	1.487*P*	−0.417*Pl*	−0.308*Pl*	0.329*Pl*	0.215*Pl*
Four Equal Loads *P* at Fifth Points									
0.2*P*	0.8*P*	1.692*P*	2.565*P*	2.429*P*	2.240*P*	−0.565*Pl*	−0.480*Pl*	0.476*Pl*	0.353*Pl*
0.3*P*	0.7*P*	1.666*P*	2.559*P*	2.496*P*	2.189*P*	−0.559*Pl*	−0.463*Pl*	0.466*Pl*	0.332*Pl*
0.4*P*	0.6*P*	1.640*P*	2.552*P*	2.364*P*	2.137*P*	−0.552*Pl*	−0.446*Pl*	0.456*Pl*	0.312*Pl*
0.5*P*	0.5*P*	1.614*P*	2.546*P*	2.332*P*	2.086*P*	−0.546*Pl*	−0.428*Pl*	0.445*Pl*	0.291*Pl*
0.6*P*	0.4*P*	1.589*P*	2.540*P*	2.300*P*	2.035*P*	−0.540*Pl*	−0.411*Pl*	0.435*Pl*	0.271*Pl*
0.7*P*	0.3*P*	1.563*P*	2.533*P*	2.268*P*	1.983*P*	−0.533*Pl*	−0.394*Pl*	0.425*Pl*	0.250*Pl*

CONTINUOUS BEAMS WITH FIXED ENDS

Continuous beams with fixed ends are seldom found in practice, therefore, only typical cases will be given to demonstrate the effect of fixity at the supports. The same assumptions were made as in connection with beams with free ends (see p. 16).

Ends of continuous beams may be considered as fixed when they are rigidly connected with supports of such rigidity, that after deflection the tangent to the deflection curve at the fixed ends coincides with the original axis of the beam.

Due to fixity of the ends, considerable bending moments develop in the beam at the end supports. These affect the bending moments in all other spans.

Sometimes the ends of the beam are only partially fixed. In such case the bending moments at the ends in the beam will be equal to a fraction of the bending moments for fixed ends. The bending moments in other spans will be somewhere between those for free ends and those for fixed ends.

BEAM OF TWO SPANS WITH FIXED ENDS

Beam consisting of two spans with fixed ends has three statically indeterminate values, namely, the bending moments at the two fixed ends and at the interior support. Following equations may be developed from the three-moment equation. The method of application of the equation to beams with fixed ends is discussed on p. 20.

Fundamental Equations, (see Fig. 57, p. 108),

$$2M_1l_1 + M_2l_1 = -Q'_1. \qquad (349)$$

$$M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2 = -(Q_1 + Q'_2). \qquad (350)$$

$$M_2l_2 + 2M_3l_2 = -Q_2, \qquad (351)$$

where

$$Q_1 = \frac{6}{l_1}\int_0^{l_1} M_s x\,dx, \quad Q'_1 = \frac{6}{l_1}\int_0^{l_1} M_{s1}(l_1 - x)dx.$$

$$Q_2 = \frac{6}{l_2}\int_0^{l_2} M_s x\,dx, \quad Q'_2 = \frac{6}{l_2}\int_0^{l_2} M_{s2}(l_2 - x)dx.$$

These equations solved for M_1, M_2 and M_3 give:

General Equation for Bending Moments at Supports, (see Fig. 57, p. 108),

$$M_1 = -\frac{1}{2}\left(M_2 + \frac{1}{l_1}Q'_1\right). \qquad (352)$$

$$M_2 = -\frac{2Q_1 - Q'_1 + 2Q'_2 - Q_2}{3(l_1 + l_2)}. \quad . \quad . \quad . \quad . \quad . \quad (353)$$

$$M_3 = -\frac{1}{2}\left(M_2 + \frac{1}{l_2}Q_2\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (354)$$

For uniformly distributed loads substitute following values:

$$Q_1 = Q'_1 = \tfrac{1}{4}w_1l_1^3, \quad Q_2 = Q'_2 = \tfrac{1}{4}w_2l_2^3.$$

Therefore the formulas become:

Bending Moments at Supports for Uniform Load of Different Intensities,

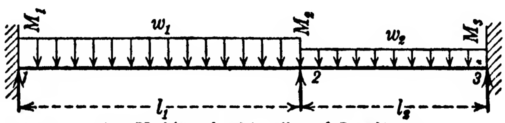
$$M_1 = -\tfrac{1}{2}M_2 - \tfrac{1}{8}w_1l_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (355)$$

$$M_2 = -\frac{w_1l_1^3 + w_2l_2^3}{12(l_1 + l_2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (356)$$

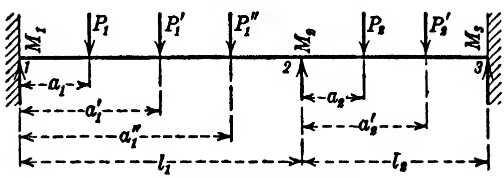
$$M_3 = -\tfrac{1}{2}M_2 - \tfrac{1}{8}w_2l_2^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (357)$$

TWO EQUAL SPANS. FIXED ENDS

For equal spans $l_1 = l_2 = l$ and the formulas change to



(a) *Uniformly Distributed Loading*



(b) *Concentrated Loads*

FIG. 57.—Two Unequal Spans, Fixed Ends. (*See p.* 107.)

Bending Moments at Supports. Two Equal Spans with Fixed Ends,

$$M_1 = -\frac{1}{2}\left(M_2 + \frac{1}{l}Q'\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (358)$$

$$M_2 = -\frac{2Q_1 - Q'_1 + 2Q'_2 - Q_2}{6l_1}. \quad . \quad . \quad . \quad . \quad . \quad (359)$$

$$M_3 = -\frac{1}{2}\left(M_2 + \frac{1}{l}Q_2\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (360)$$

Uniformly Distributed Loading of Different Intensities.

Bending Moments at Supports,

$$M_1 = -\tfrac{1}{2}M_2 - \tfrac{1}{8}w_1l^2. \quad (361)$$

$$M_2 = -\frac{w_1 + w_2}{24}l^2. \quad (362)$$

$$M_3 = -\tfrac{1}{2}M_2 - \tfrac{1}{8}w_2l^2. \quad (363)$$

Uniformly Distributed Loading.—Assume that the intensity of loading in all loaded spans is the same and is equal to w and that in the unloaded spans it is zero.

Two types of loading are considered:

(a) All spans loaded for which the bending moments at center support is maximum.

(b) One span loaded, the other not loaded, for which the positive bending moment is a maximum, also the bending moments at fixed ends are maximum and minimum, respectively.

(a) **All Spans Loaded,** $w_1 = w_2 = w$. (See Fig. 58, p. 109.)

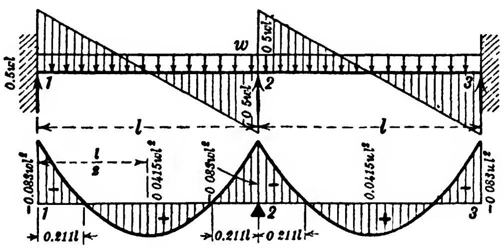


FIG. 58.—Two Equal Spans, Fixed Ends. Both Spans Loaded. (*See p.* 109.)

End Shears,

$$V_1 = V_{2l} = V_{2r} = V_3 = \tfrac{1}{2}wl. \quad (364)$$

Bending Moments at Supports,

$$M_1 = M_2 = M_3 = -\tfrac{1}{12}wl^2 \quad (365)$$

Maximum Positive Bending Moment:

For both spans,

$$M_{\max} = \tfrac{1}{24}wl^2. \quad (366)$$

Point of Maximum Positive Bending Moment,

$$x_1 = \tfrac{1}{2}l \quad (367)$$

(*b*) **One Span Loaded,** $w_1 = w$, $w_2 = 0$. (See Fig. 59, p. 110.)

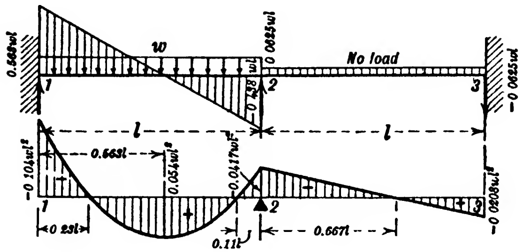


FIG. 59.—Two Spans, Fixed Ends. Left Span Loaded. (*See* p. 110.)

End Shears,

$$V_1 = \tfrac{9}{16}wl. \quad (368) \qquad V_{2l} = \tfrac{7}{16}wl \quad (369)$$

$$V_{2r} = \tfrac{1}{16}wl. \quad (370) \qquad V = -\tfrac{1}{16}wl. \quad (371)$$

Bending Moments at Supports,

$$M_1 = -\tfrac{5}{48}wl^2. \quad (372)$$

$$M_2 = -\tfrac{1}{24}wl^2. \quad (373)$$

$$M_3 = +\tfrac{1}{48}wl^2. \quad (374)$$

Maximum Positive Bending Moment:

Left span,

$$M_{\max} = \tfrac{83}{1536}wl^2. \quad (375)$$

Points of Maximum Positive Bending Moment,

$$x_1 = \tfrac{9}{16}l. \quad (376)$$

THREE SPANS WITH FIXED ENDS

Continuous beam with fixed ends consisting of three spans has four statically indeterminate values, namely, the bending moments at the two fixed ends and at the two intermediate supports. The four equations below, obtained from the three-moment equation, are sufficient to determine the four unknown values. See p. 20 for method of applying the equation to beams with fixed ends.

Fundamental Equations,

$$2M_1l_1 + M_2l_1 = -Q'_1, \quad (377)$$

$$M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2 = -(Q_1 + Q'_2). \quad (378)$$

$$M_2l_2 + 2M_3(l_2 + l_3) + M_4l_3 = -\ (Q_2 + Q'_3). \quad . \quad . \quad (379)$$

$$M_3l_3 + 2M_4l_3 = -\ Q_3, \quad . \quad . \quad . \quad . \quad . \quad (380)$$

where

$$Q_1 = \frac{6}{l_1}\int_0^{l_1} M_{s1}x dx. \qquad Q'_1 = \frac{6}{l_1}\int_0^{l_1} M_{s1}(l_1 - x)dx.$$

$$Q_2 = \frac{6}{l_2}\int_0^{l_2} M_{s2}x dx. \qquad Q'_2 = \frac{6}{l_2}\int_0^{l_2} M_{s2}(l_2 - x)dx.$$

$$Q_3 = \frac{6}{l_3}\int_0^{l_3} M_{s3}x dx. \qquad Q'_3 = \frac{6}{l_3}\int_0^{l_3} M_{s3}(l_3 - x)dx.$$

The values of Q and Q' are developed on pp. 19 and 20. They are in general:

For uniform loading,

$$Q = Q' = \tfrac{1}{4}wl^3.$$

For concentrated load P at a distance a from left support,

$$Q = \frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]Pl^2 = C_1Pl^2$$

and

$$Q' = \frac{a}{l}\left(1 - \frac{a}{l}\right)\left(2 - \frac{a}{l}\right)Pl^2 = C_2Pl^2.$$

For a number of concentrated loads,

$$Q = \Sigma\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]Pl^2 = \Sigma C_1Pl^2$$

and

$$Q' = \Sigma\frac{a}{l}\left(1 - \frac{a}{l}\right)\left(2 - \frac{a}{l}\right)Pl^2 = \Sigma C_2Pl^2.$$

The values of C_1 and C_2 may be taken from Diagram 1, p. 19.

The above equations solved for M_1, M_2, M_3 and M_4 give:

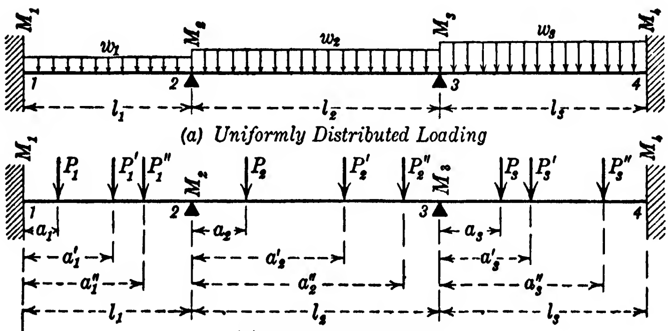
Bending Moments at Supports. General Formulas (see Fig. 60),

$$M_1 = -\ \tfrac{1}{2}M_2 - \frac{1}{2l_1}Q'_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (381)$$

$$M_2 = -\frac{(4l_2 + 3l_3)(2Q_1 - Q'_1 + 2Q'_2) - 2l_2(2Q_2 - Q_3 + 2Q'_3)}{(3l_1 + 4l_2)(4l_2 + 3l_3) - 4l_2^2}. \quad (382)$$

$$M_3 = -\frac{(3l_1 + 4l_2)(2Q_2 - Q_3 + 2Q'_3) - 2l_2(2Q_1 - Q'_1 + 2Q'_2)}{(3l_1 + 4l_2)(4l_2 + 3l_3) - 4l_2^2}. \quad (383)$$

$$M_4 = -\tfrac{1}{2}M_3 - \frac{1}{2l_3}Q_3. \quad (384)$$



(a) *Uniformly Distributed Loading*

(b) *Concentrated Loads*

FIG. 60.—Three Spans, Fixed Ends. (See p. 111.)

Bending Moments at Supports. Uniform Loading of Different Intensities (see Fig. 60),

$$M_1 = -\tfrac{1}{2}M_2 - \tfrac{1}{8}w_1l_1^2. \quad (385)$$

$$M_2 = -\frac{(4l_2 + 3l_3)(w_1l_1^3 + 2w_2l_2^3) - 2l_2(2w_2l_2^3 + w_3l_3^3)}{4[(3l_1 + 4l_2)(4l_2 + 3l_3) - 4l_2^2]}. \quad (386)$$

$$M_3 = -\frac{(3l_1 + 4l_2)(2w_2l_2^3 + w_3l_3^3) - 2l_2(w_1l_1^3 + 2w_2l_2^3)}{4[(3l_1 + 4l_2)(4l_2 + 3l_3) - 4l_2^2]}. \quad (387)$$

$$M_4 = -\tfrac{1}{2}M_3 - \tfrac{1}{8}w_3l_3^2. \quad (388)$$

THREE EQUAL SPANS. FIXED ENDS

For equal spans $l_1 = l_2 = l_3 = l$. Then

General Formulas for Equal Spans.

Bending Moments at Supports. General Formulas,

$$M_1 = -\tfrac{1}{2}M_2 - \frac{1}{2l}Q'_1. \quad (389)$$

$$M_2 = -\frac{7(2Q_1 - Q'_1 + 2Q'_2) - 2(2Q_2 - Q_3 + 2Q'_3)}{45l} \quad . \quad . \quad . \quad . \quad . \qquad (390)$$

$$M_3 = -\frac{7(2Q_2 - Q_3 + 2Q'_3) - 2(2Q_1 - Q'_1 + 2Q'_2)}{45l} \quad . \quad . \quad . \quad . \quad . \qquad (391)$$

$$M_4 = -\tfrac{1}{2}M_3 - \frac{1}{2l}Q_3 \quad . \quad . \quad . \quad . \quad . \qquad (392)$$

Uniformly Distributed Loading. Intensities Different in Each Span.—For uniform loading of different intensities and equal spans the Formulas (385) to (388), p. 112, change to

Bending Moments at Supports. Uniform Loading,

$$M_1 = -\tfrac{1}{2}M_2 - \tfrac{1}{8}w_1l^2 \quad . \quad . \quad . \quad . \quad . \qquad (393)$$

$$M_2 = -\frac{7(w_1 + 2w_2) - 2(2w_2 + w^3)}{180}l^2 = -\frac{7w_1 + 10w_2 - 2w_3}{180}l^2. \qquad (394)$$

$$M_3 = -\frac{7w_3 + 10w_2 - 2w_1}{180}l^2 \quad . \quad . \quad . \quad . \quad . \qquad (395)$$

$$M_4 = -\tfrac{1}{2}M_3 - \tfrac{1}{8}w_3l^2. \quad . \quad . \quad . \quad . \quad . \qquad (396)$$

Uniformly Distributed Loading of Equal Intensities.—Assume that the intensity of the uniformly distributed load is the same in all loaded spans and is zero in unloaded spans.

Following conditions of loading are considered:

(*a*) All spans loaded. $w_1 = w_2 = w_3 = w$. Condition for dead load (Fig. 61 (*a*)).

(*b*) First and second span loaded. $w_1 = w_2 = w$ and $w_3 = 0$. Condition for maximum negative bending moment at second support (Fig. 61 (*b*)).

(*c*) First and third spans loaded. $w_1 = w_3 = w$ and $w_2 = 0$. Condition for maximum positive bending moment in end spans and maximum negative bending moment at fixed end (Fig. 61 (*c*)).

(*d*) Second span loaded. $w_2 = w$ and $w_1 = w_3 = 0$. Condition for maximum positive bending moment in middle span (Fig. 61 (*d*)).

The bending moments and shears for these conditions are given in the table on p. 114 and shown in Fig. 61, p. 115.

Three Equal Spans, Fixed Ends. Uniformly Distributed Loading

End Shears. (*See Fig*. 61, *p*. 115.)

Condition (See Fig. 61)	Spans Loaded	First Span		Second Span		Third Span	
		V_1	V_{2l}	V_{2r}	V_{3l}	V_{3r}	V_4
a	1, 2, 3	$0.5wl$	$0.5wl$	$0.5wl$	$0.5wl$	$0.5wl$	$0.5wl$
b	1, 2, –	$0.483wl$	$\mathbf{0.517wl}$	$\mathbf{0.55wl}$	$0.45wl$	$0.067wl$	$-0.067wl$
c	–, 2, –	$-0.083wl$	$0.083wl$	$0.55wl$	$0.5wl$	$0.083wl$	$-0.083wl$
d	1, –, 3	$\mathbf{0.583wl}$	$0.417wl$	0	0	$0.417wl$	$\mathbf{0.583wl}$

Maximum values for V_{3l} and V_{3r} are same as for V_{2r} and V_{2l} respectively and act when spans 2 and 3 are loaded.

Bending Moments. (*See Fig*. 61, *p*. 115.)

Condition (See Fig. 61)	Spans Loaded	At Supports				Maximum Positive		
		M_1	M_2	M_3	M_4	First Span	Second Span	Third Span
a	1, 2, 3	$-0.083wl^2$	$-0.083wl^2$	$-0.083wl^2$	$-0.083wl^2$	$0.042wl^2$	$0.042wl^2$	$0.042wl^2$
b	1, 2, –	$-0.078wl^2$	$\mathbf{-0.094wl^2}$	$-0.044wl^2$	$0.022wl^2$	$0.039wl^2$	$0.057wl^2$	
c	–, 2, –	$0.028wl^2$	$-0.056wl^2$	$-0.056wl^2$	$0.022wl^2$	....	$\mathbf{0.07wl^2}$	
d	1, –, 3	$\mathbf{-0.111wl^2}$	$-0.0278wl^2$	$-0.0278wl^2$	$\mathbf{-0.111wl^2}$	$\mathbf{0.059wl^2}$	....	$\mathbf{0.059wl^2}$

Absolute maximum values are shown in black-face type.
Maximum value of M_3 is same as for M_2 and acts when spans 2 and 3 are loaded.

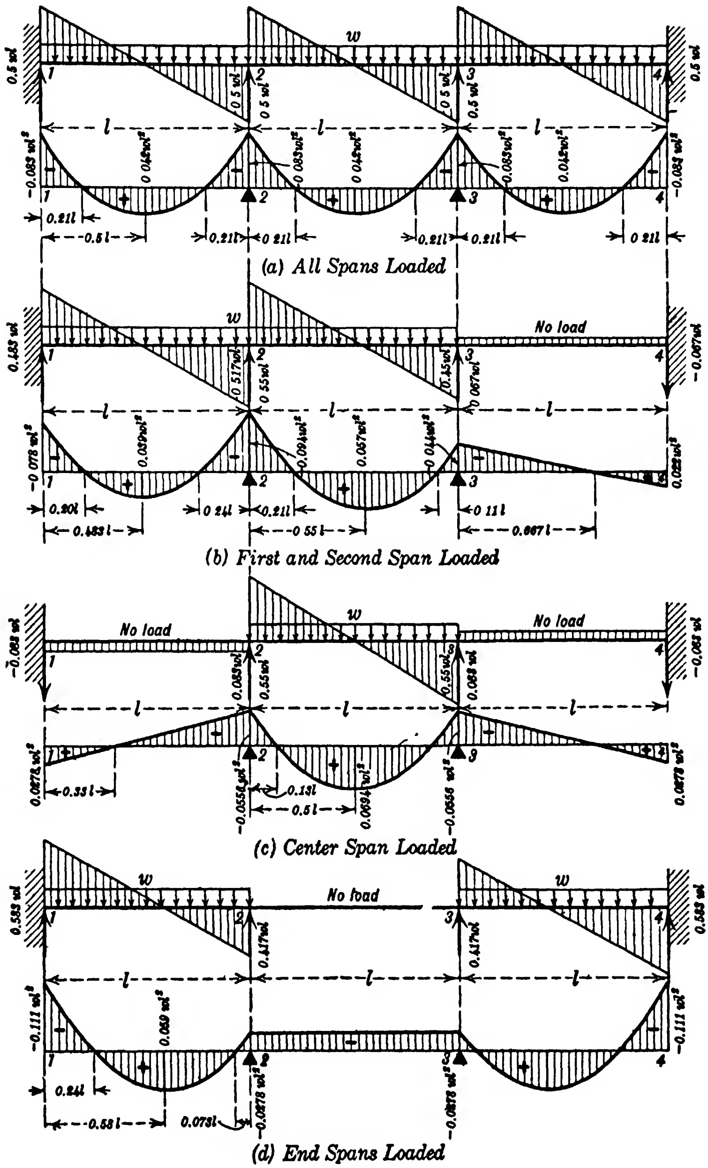


FIG. 61.—Three Equal Spans, Fixed Ends. Uniformly Distributed Loadings.
(See p. 113.)

Combining Dead Load with Live Load.—The bending moments and shears for a combination for dead load and live load may be obtained by computing separately the bending moments and shears for the dead load and adding to them the absolute maximum values for the live load.

In table below are given maximum combined values for different ratios of the dead load to the total load. To use the table find the dead load and the total load. Find the ratio of dead load to the total load. The values from the table corresponding to this ratio give the maximum values for bending moments and shear.

Three Equal Spans, Fixed Ends. Uniform Load

Maximum Values for Combined Dead and Live Loads. (*See p.* 116.)

Dead Load	Live Load	End Shears			Negative Bending Moment		Maximum Positive Bending Moment	
		V_1 and V_4	V_{2l} and V_{3r}	V_{2r} and V_{3l}	M_1 and M_4	M_2 and M_3	End Spans	Center Spans
$0.2w$	$0.8w$	$0.566wl$	$0.514wl$	$0.540wl$	$-0.106wl^2$	$-0.091wl^2$	$0.056wl^2$	$0.072wl^2$
$0.3w$	$0.7w$	$0.558wl$	$0.512wl$	$0.535wl$	$-0.103wl^2$	$-0.091wl^2$	$0.054wl^2$	$0.069wl^2$
$0.4w$	$0.6w$	$0.550wl$	$0.510wl$	$0.530wl$	$-0.100wl^2$	$-0.090wl^2$	$0.052wl^2$	$0.065wl^2$
$0.5w$	$0.5w$	$0.541wl$	$0.508wl$	$0.525wl$	$-0.097wl^2$	$-0.089wl^2$	$0.050wl^2$	$0.061wl^2$
$0.6w$	$0.4w$	$0.533wl$	$0.507wl$	$0.520wl$	$-0.094wl^2$	$-0.088wl^2$	$0.049wl^2$	$0.057wl^2$
$0.7w$	$0.3w$	$0.525wl$	$0.505wl$	$0.515wl$	$-0.092wl^2$	$-0.087wl^2$	$0.047wl^2$	$0.053wl^2$

w = Uniformly distributed unit dead plus live load. l = Length of span.

FOUR SPANS WITH FIXED ENDS

Continuous beam with fixed ends consisting of four spans has five statically indeterminate values, namely, the bending moments at two fixed ends and at the three intermediate supports.

The five equations below, obtained from the three-moment equation, are sufficient to determine the five unknown values.

General Formulas (see Fig. 62, p. 117),

Fundamental Equations,

$$2M_1l_1 + M_2l_1 = -Q'_1, \quad \ldots \quad (397)$$

$$M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2 = -(Q_1 + Q'_2), \quad \ldots \quad (398)$$

$$M_2l_2 + 2M_3(l_2 + l_3) + M_4l_3 = -(Q^2 + Q'_3), \quad \ldots \quad (399)$$

$$M_3l_3 + 2M_4(l_3 + l_4) + M_5l_4 = -(Q_3 + Q'_4), \quad \ldots \quad (400)$$

$$M_4l_4 + 2M_5l_4 = -Q_4, \quad \ldots \quad (401)$$

where

$$Q_1 = \frac{6}{l_1}\int_0^{l_1} M_{s1}x\,dx, \qquad Q'_1 = \frac{6}{l_1}\int_0^{l_1} M_{s1}(l - x)dx,$$

$$Q_2 = \frac{6}{l_2}\int_0^{l_2} M_{s2}x\,dx, \qquad Q'_2 = \frac{6}{l_2}\int_0^{l_2} M_{s2}(l - x)dx,$$

$$Q_3 = \frac{6}{l_3}\int_0^{l_3} M_{s3}x\,dx, \qquad Q'_3 = \frac{6}{l_3}\int_0^{l_3} M_{s3}(l - x)dx,$$

$$Q_4 = \frac{6}{l_4}\int_0^{l_4} M_{s4}x\,dx, \qquad Q'_4 = \frac{6}{l_4}\int_0^{l_4} M_{s4}(l - x)dx.$$

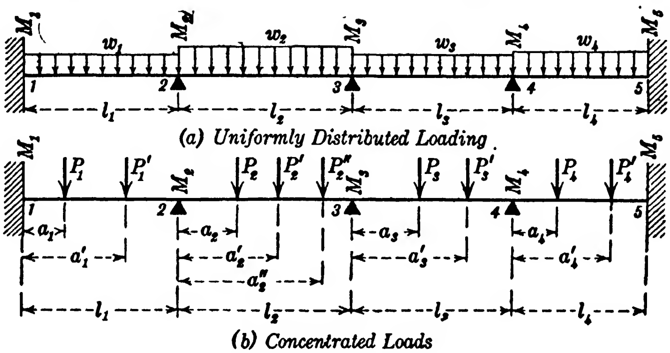


FIG. 62.—Four Spans, Fixed Ends. (*See p.* 116.)

The values of the integral as worked out on p. 19 are in general:
For uniform loads,

$$Q = Q' = \tfrac{1}{4}wl^3.$$

For concentrated loads,

$$Q = \frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]Pl^2 = C_1Pl^2 \quad \text{and} \quad Q' = \frac{a}{l}\left(1 - \frac{a}{l}\right)\left(2 - \frac{a}{l}\right)Pl^2 = C_2Pl^2.$$

The above equations solved for M_1, M_2, M_3, M_4 and M_5 give the following formulas for negative bending moments.

Bending Moments at Supports, General Formulas,

$$M_1 = -\tfrac{1}{2}M_2 - \frac{1}{2l_1}Q'_1, \qquad (402)$$

$$M_2 = -\frac{\left\{\begin{matrix} -l_3^2(2Q_1 - Q'_1 + 2Q'^2) + (4l_3 + 3l_4) \\ [(l_2+l_3)(2Q_1 - Q'_1 + 2Q'_2) - l_2(Q_2 + Q'_3)] + l_2l_3(2Q_3 - Q_4 + 2Q'_4) \end{matrix}\right\}}{-l_2^2(4l_3+3l_4) + (l_2+l_3)(3l_1+4l_2)(4l_3+3l_4) - l_3^2(3l_1+4l_2)}. \qquad (403)$$

$$M_3 = -\frac{\left\{\begin{matrix} -l_2(4l_3+3l_4)(2Q_1 - Q'_1 + 2Q'_2) + (3l_1+4l_2)(4l_3+3l_4) \\ (Q_2 + Q'_3) - l_3(3l_1+4l_2)(2Q_3 - Q_4 + 2Q'_4) \end{matrix}\right\}}{2[-l_2^2(4l_3+3l_4) + (l_2+l_3)(3l_1+4l_2)(4l_3+3l_4) - l_3^2(3l_1+4l_2)]}. \qquad (404)$$

$$M_4 = -\frac{\begin{Bmatrix} l_2l_3(2Q_1-Q'_1+2Q'_2)+(3l_1+4l_2) \\ [(l_2+l_3)(2Q_3-Q_4+2Q'_4)-l_3(Q_2+Q'_3)]-l_2^2(2Q_3-Q_4+2Q'_4) \end{Bmatrix}}{-l_2^2(4l_3+3l_4)+(l_2+l_3)(3l_1+4l_2)(4l_3+3l_4)-l_3^2(3l_1+4l_2)}. \quad (405)$$

$$M_5 = -\tfrac{1}{2}M_4 - \frac{1}{2l_4}Q_4. \quad (406)$$

Since several expressions repeat in all the above equations, the formulas may be simplified by making following substitutions:

$$S_1 = 2Q_1 - Q'_1 + 2Q'_2. \quad (407)$$

$$S_2 = Q_2 + Q'_3. \quad (408)$$

$$S_3 = 2Q_3 - Q_4 + 2Q'_4. \quad (409)$$

Then

$$M_1 = -\tfrac{1}{2}M_2 - \frac{1}{2l_1}Q'_1. \quad (410)$$

$$M_2 = -\frac{-l_3^2S_1+(4l_3+3l_4)[(l_2+l_3)S_1-l_2S_2]+l_2l_3S_3}{-l_2^2(4l_3+3l_4)+(l_2+l_3)(3l_1+4l_2)(4l_3+3l_4)-l_3^2(3l_1+4l_2)}. \quad (411)$$

$$M_3 = -\frac{-l_2(4l_3+3l_4)S_1+(3l_1+4l_2)(4l_3+3l_4)S_2-l_3(3l_1+4l_2)S_3}{2[-l_2^2(4l_3+3l_4)+(l_2+l_3)(3l_1+4l_2)(4l_3+3l_4)-l_3^2(3l_1+4l_2)]}. \quad (412)$$

$$M_4 = -\frac{l_2l_3S_1+(3l_1+4l_2)[(l_2+l_3)S_3-l_3S_2]-l_2^2S_3}{-l_2^2(4l_3+3l_4)+(l_2+l_3)(3l_1+4l_2)(4l_3+3l_4)-l_3^2(3l_1+4l_2)}. \quad (413)$$

$$M_5 = -\tfrac{1}{2}M_4 - \frac{1}{2l_4}Q_4. \quad (414)$$

Values of S_1, S_2 and S_3 for Uniform Loading.—When the loading is uniformly distributed but of different intensity in the different spans the values of Q and S are

$$Q_1 = Q'_1 = \tfrac{1}{4}w_1l_1^3, \qquad Q_2 = Q'_2 = \tfrac{1}{4}w_2l_2^3,$$

$$Q_3 = Q'_3 = \tfrac{1}{4}w_3l_3^3, \qquad Q_4 = Q'_4 = \tfrac{1}{4}w_4l_4^3.$$

Therefore

$$S_1 = \tfrac{1}{4}(w_1l_1^3 + 2w_2l_2^3). \quad (415)$$

$$S_2 = \tfrac{1}{4}(w_2l_2^3 + w_3l_3^3). \quad (416)$$

$$S_3 = \tfrac{1}{4}(2w_3l_3^3 + w_4l_4^3). \quad (417)$$

FOUR EQUAL SPANS. FIXED ENDS

For equal spans $l_1 = l_2 = l_3 = l_4 = l$. Therefore the denominator for M_2 and M_4 is

$$l^3[-(4+3)+2\times 7\times 7-(3+4)] = (98-14)l^3 = 84l^3.$$

The formulas for the bending moment at supports become

Negative Bending Moment, Four Equal Spans, Fixed Ends,

$$M_2 = -\frac{13S_1 - 7S_2 + S_3}{84l} \quad . \quad . \quad . \quad . \quad (418)$$

$$M_3 = -\frac{-7S_1 + 49S_2 - 7S_3}{168l} \quad . \quad . \quad . \quad . \quad (419)$$

$$M_4 = -\frac{S_1 - 7S_2 + 13S_3}{84l} \quad . \quad . \quad . \quad . \quad (420)$$

Values of S_1, S_2 and S_3 are given in Formulas (407) to (409), p. 118.

Uniform Loading.—For uniformly distributed loads and equal spans the values of S_1, S_2 and S_3 are

$$S_1 = \tfrac{1}{4}l^3(w_1 + 2w_2), \quad . \quad . \quad . \quad . \quad (421)$$

$$S_2 = \tfrac{1}{4}l^3(w_2 + w_3), \quad . \quad . \quad . \quad . \quad (422)$$

$$S_3 = \tfrac{1}{4}l^3(2w_3 + w_4), \quad . \quad . \quad . \quad . \quad (423)$$

and the bending moments become

Bending Moments at Supports. Uniform Load Varying Intensities,

$$M_1 = -\frac{M_2}{2} - \frac{1}{8}w_1l_1^2. \quad . \quad . \quad . \quad . \quad (424)$$

$$M_2 = -\frac{\tfrac{1}{4}(13w_1 + 26w_2 - 7w_2 - 7w_3 + 2w_3 + w_4)}{84}l^2$$

$$= -\frac{13w_1 + 19w_2 - 5w_3 + w_4}{336}l^2. \quad . \quad (425)$$

$$M_3 = -\frac{-w_1 + 5w_2 + 5w_3 - w_4}{96}l^2. \quad . \quad . \quad . \quad . \quad (426)$$

$$M_4 = -\frac{w_1 - 5w_2 + 19w_3 + 13w_4}{336}l^2. \quad . \quad . \quad . \quad . \quad (427)$$

$$M_5 = -\frac{M_4}{2} - \frac{1}{8}w_4l_4^2. \quad . \quad . \quad . \quad . \quad (428)$$

Uniformly Distributed Loading.—Assume that the uniform loading is the same in all loaded spans and zero in unloaded spans.

Four types of loading are considered:

(*a*) All spans loaded. $w_1 = w_2 = w_3 = w_4 = w$.

(*b*) First, second and fourth spans loaded. $w_1 = w_2 = w_4 = w$, $w_3 = 0$.

(*c*) Second and third spans loaded. $w_2 = w_3 = w$, $w_1 = w_4 = 0$.

(*d*) First and third spans loaded. $w_1 = w_3 = w$, $w_2 = w_4 = 0$.

The bending moments and shears for the above conditions are given in the table on p. 120 and illustrated in Fig. 63, p. 121.

Four Equal Spans, Fixed Ends. Uniform Loading

End Shears. (*See Fig.* 63, *p.* 121.)

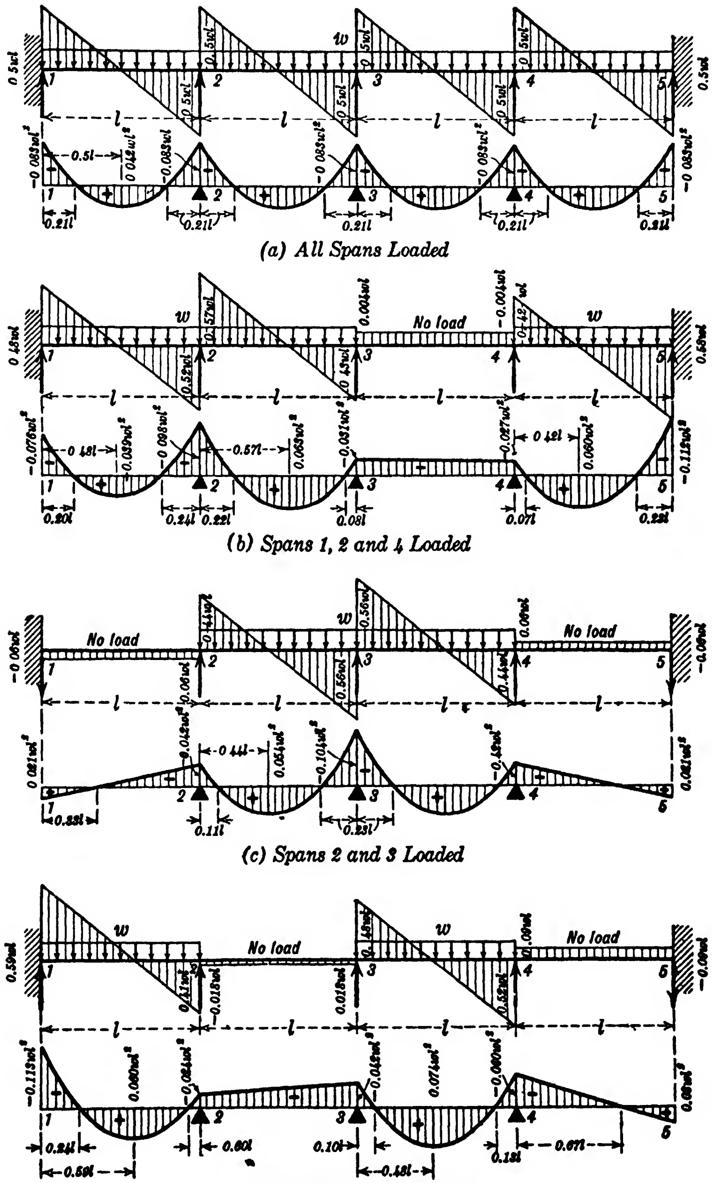
Condition (See Fig. 63)	Spans Loaded	First Span		Second Span		Third Span		Fourth Span	
		V_1	V_{2l}	V_{2r}	V_{3l}	V_{3r}	V_{4l}	V_{4r}	V_5
a	1, 2, 3, 4	0.5*wl*	0 5*wl*	0.5*wl*	0.5*wl*	0.5*wl*	0 5*wl*	0 5*wl*	0.5*wl*
b	1, 2, –, 4	0.48*wl*	**0.52*wl***	**0 57*wl***	0 43*wl*	0 004*wl*	−0.004*wl*	0 42*wl*	0.58*wl*
c	–, 2, 3, –	−0.06*wl*	0 06*wl*	0.44*wl*	**0 56*wl***	**0.56*wl***	0.44*wl*	0 06*wl*	−0.06*wl*
d	1, –, 3, –	**0.59*wl***	0.41*wl*	−0 018*wl*	0 018*wl*	0.48*wl*	0.52*wl*	0.09*wl*	−0.09*wl*

Maximum values for V_{4l} and V_{4r} are same as for V_{2r} and V_{2l} respectively and act when spans 1, 3 and 4 are loaded.
Maximum value for V_5 is same as for V_1 and acts when spans 2 and 4 are loaded.

Maximum Bending Moments

Condition (See Fig. 63)	Spans Loaded	Negative Bending Moment					Positive Bending Moment			
		M_1	M_2	M_3	M_4	M_5	First Span	Second Span	Third Span	Fourth Span
a	1, 2, 3, 4	−0.083*wl*²	−0.083*wl*²	−0 083*wl*²	−0.083*wl*²	−0.083*wl*²	0 042*wl*²	0 042*wl*²	0 042*wl*²	0 042*wl*²
b	1, 2, –, 4	−0 076*wl*²	**−0 098*wl*²**	−0 031*wl*²	−0.027*wl*²	−0.112*wl*²	0.039*wl*²	0 063*wl*²	. . . .	0.060*wl*²
c	–, 2, 3, –	+0.021*wl*²	−0 042*wl*²	**−0 104*wl*²**	−0.042*wl*²	+0 021*wl*²	. . . . .	0.054*wl*²	0 054*wl*²	
d	1, –, 3, –	**−0.113*wl*²**	−0.024*wl*²	−0.042*wl*²	−0.060*wl*²	+0.030*wl*²	**0.060*wl*²**	. . . . .	**0.074*wl*²**	

Absolute maximum values are shown in black-face type.
Maximum values for M_4 is same as for M_2 and acts when spans 1, 3 and 4 are loaded.
Maximum M_5 is same as M_1 and acts when spans 2 and 4 are loaded.
Maximum positive bending moments in 2nd and 4th spans are same as in 3rd and 1st respectively, and act when spans 2 and 4 are loaded.



(a) All Spans Loaded

(b) Spans 1, 2 and 4 Loaded

(c) Spans 2 and 3 Loaded

(d) Spans 1 and 3 Loaded

FIG. 63.—Four Equal Spans. Fixed Ends. Uniformly Distributed Loading. (*See p.* 119.)

Combining Dead Load with Live Load.—The table below is useful for finding the maximum bending moments and shears for a combination of dead load and live load. The method of combining the dead load values with those for live load is explained on p. 92.

To use the table find the dead load and the total load. Find the ratio of dead load to total load. Values for bending moments and shears in the table corresponding to this ratio are maximum for this combination.

Four Equal Spans, Fixed Ends. Uniform Load

Maximum Values for Combined Dead and Live Loads. (*See p.* 122.)

Dead Load	Live Load	End Shears				Negative Bending Moment			Maximum Positive Bending Moment	
		V_1 and V_5	V_{2l} and V_{4r}	V_{2r} and V_{4l}	V_{3l} and V_{3r}	M_1 and M_5	M_2 and M_4	M_3	End Spans	Center Spans
0.2w	0.8w	0.572wl	0.516wl	0.556wl	0.548wl	−0.107wl^2	−0.095wl^2	−0.100wl^2	0.056wl^2	0.068wl^2
0.3w	0.7w	0.563wl	0.514wl	0.549wl	0.542wl	−0.104wl^2	−0.094wl^2	−0.098wl^2	0.055wl^2	0.064wl^2
0.4w	0.6w	0.554wl	0.512wl	0.542wl	0.536wl	−0.101wl^2	−0.092wl^2	−0.096wl^2	0.053wl^2	0.061wl^2
0.5w	0.5w	0.545wl	0.510wl	0.535wl	0.530wl	−0.098wl^2	−0.091wl^2	−0.093wl^2	0.051wl^2	0.058wl^2
0.6w	0.4w	0.536wl	0.508wl	0.528wl	0.524wl	−0.095wl^2	−0.089wl^2	−0.049wl^2	0.049wl^2	0.055wl^2
0.7w	0.3w	0.527wl	0.506wl	0.521wl	0.518wl	−0.092wl^2	−0.088wl^2	−0.089wl^2	0.047wl^2	0.052wl^2

w = Uniformly distributed unit dead plus live load l = Length of span.

BEAMS WITH CANTILEVERS

Cantilevers are the parts of a beam which extend beyond the end supports. One end of the cantilever is connected with beam at the support while the other end is free. Loads placed on a cantilever produce bending moments not only in the cantilever, but also in the beam proper. They also produce shears in the beam and uplift at the opposite end of the beam. The bending moments and shears in a cantilever due to the loads on the cantilever are independent of the conditions of the remainder of the beam. The bending moment and shears in the beam, however, due to the loads on the cantilever depend upon the span of the beam and also upon the conditions at the opposite end of the beam.

Following cases will be considered:

1. Beam freely supported on one end, cantilevered at the other.
2. Beam cantilevered at both ends.
3. Continuous beam of two spans, cantilevered at one end.
4. Beam fixed at one end, cantilevered at the other.

General Information.—The downward loads on the cantilever produce negative bending moments both in the cantilever and in the adjoining span of the beam. The downward loads on the beam, on the other hand, produce positive bending moments in the beam and no bending moments in the cantilever. When the cantilever and the beam are loaded simultaneously, the negative bending moments in the beam due to the cantilever loads reduce the positive bending moments in the beam due to the loads on the beam. As a consequence the beam with a loaded cantilever, when fully loaded, is subjected to smaller bending moments than a similar freely supported beam carrying the same loading.

To get this advantageous condition the cantilever and the beam must be loaded simultaneously. Therefore advantage of this reduction can be taken only for fixed loads, such as the dead load, for which the beam and the cantilever are always sure to be loaded simultaneously. For live load the beam may be loaded when there is no load on the cantilever, also the cantilever may be loaded when there is no load on the beam. Therefore, for live load the most unfavorable condition of loading must be accepted.

In designing beams with cantilevers, the following rules should be observed:

Only fixed loads shall be assumed to act simultaneously on the cantilever and on the beam.

For live load it is necessary to consider following conditions of loading:

(*a*) The cantilever loaded, the main span not loaded. This gives maximum negative bending moments and maximum uplift.

(*b*) The cantilever not loaded, the main span fully loaded. This condition gives maximum positive bending moments in the beam.

(*c*) The cantilever and the main span fully loaded. This gives maximum reactions and maximum end shears at the support adjoining the cantilever.

Each one of these conditions should be combined with the dead load in the manner given below.

The sections must be made strong enough for maximum bending moments and shears thus obtained.

Combining the Dead Load and Live Load Bending Moments and Shears.—In combining the bending moments and shears due to live load and dead load, the following rules should be observed:

Where the bending moment or shear due to dead load and live load is of the same sign, they should be added and the beam designed for the sum of bending moments or shears.

Where the bending moment or shear due to dead load is of opposite sign to that of the live, it is not permissible to deduct the full dead load bending moment from the live load bending moment. Before combining, the dead load bending moments or shears must be divided by proper factor of safety. These reduced dead load moments are then deducted from the bending moments or shears due to the live load. The reason for this is more fully discussed on p. 461, Vol. I.

Particular attention should be given to the uplift. If the reaction due the dead load (divided by the factor of safety) is smaller than the uplift due to the live load, the beam must be anchored to the support. The support naturally must be heavy enough to resist the uplift multiplied by a factor of safety. No live load should be considered as resisting the uplift due to the cantilever loads.

Bending Moments and Shears in Cantilever

The bending moments and shears on the cantilever depend only upon the load on the cantilever and its length. The load on the beam proper or its length have no effect upon the cantilever.

Let P = concentrated load on cantilever;
w = uniformly distributed load on cantilever;
l_1 = length of cantilever, also distance of load P from support.

Then

Bending Moment in Cantilever at Any Point x from Support:

For uniform load,

$$M_x = -\,w(l_1 - x)\frac{l_1 - x}{2} = -\,\tfrac{1}{2}w(l_1 - x)^2. \quad . \quad . \quad . \quad (429)$$

For concentrated load,

$$M_x = -\,P(l_1 - x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (430)$$

Maximum Bending Moment in Cantilever at Support:

For uniform load,

$$M_{max} = -\,\tfrac{1}{2}wl_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (431)$$

For concentrated load,

$$M_{max} = -\,Pl_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (432)$$

End Shears:

For uniform load,

$$V_{1l} = wl_1. \qquad (433)$$

For concentrated load,

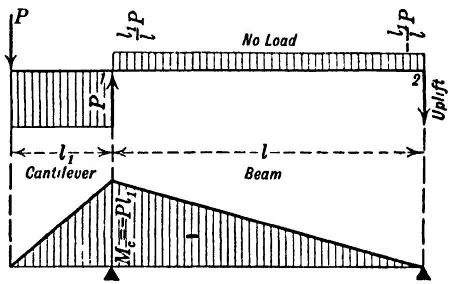
$$V_{1l} = P. \qquad (434)$$

Bending Moments and Shears in Beam

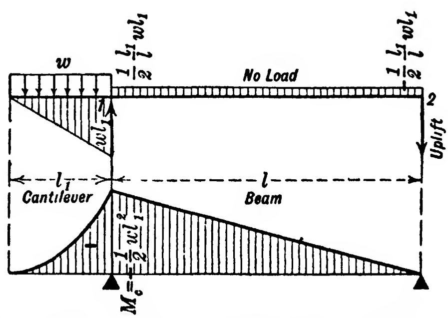
The bending moments in the beam next to the cantilever depend upon the condition at the other support of the beam. Below are given bending moments for several conditions.

Case 1. Beam Free at One End, Cantilevered at Other.—The load on the cantilever produces in the beam negative bending moments with a maximum at the cantilever support where it is equal to the bending moment at the cantilever. At intermediate sections the bending moments vary according to a straight line to zero at the free support. (See Fig. 64, p. 125.)

Fixed Loads.—For fixed loads the bending moments in the beam are a combination of the positive bending moments due to the load on the beam and the negative bending moments produced by load on the cantilever. They are represented by Fig. 65, p. 126.



(a) Bending Moments due to Concentrated Load on Cantilever



(b) Bending Moments due to Uniform Load on Cantilever

Fig. 64.—Beam Free at One End, Cantilevered at Other Concentrated Load. (*See p.* 125.)

For uniformly distributed loading the maximum positive bending moment due to the dead load is

Maximum Positive Bending Moment in Beam, Uniformly Distributed Dead Load,

$$M_{\max} = \frac{1}{8}\left[1 - \left(\frac{l_1}{l}\right)^2\right]^2 wl^2. \quad . \quad . \quad . \quad . \quad . \quad (435)$$

This moment acts at a distance from *free support* equal to

$$x_1 = \left[1 - \left(\frac{l_1}{l}\right)^2\right]\frac{l}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (436)$$

When the fixed loads are concentrated, the maximum bending moment may be found analytically by determining the end shears and the point of zero shear. The maximum positive bending moment acts at the point of zero shear. The same result may be obtained by drawing a moment diagram for the negative bending moment due to the cantilever load and then plotting the positive bending moments due to the loads on the beam starting from the inclined closing line.

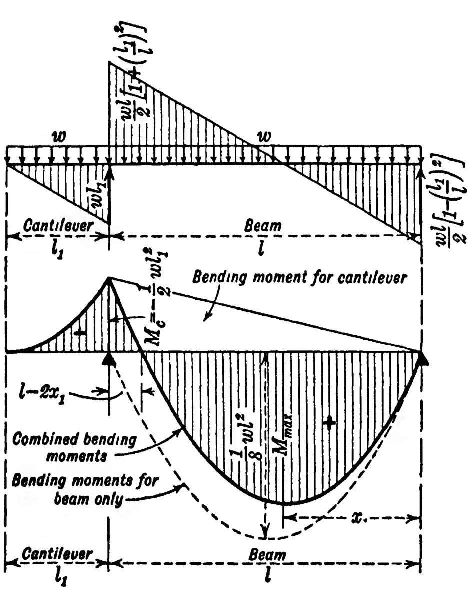


FIG. 65.—Beam Free at One End, Cantilevered at Other. Fixed Loads. (*See p.* 125.)

Live Loads.— Positive bending moment in the beam to be used for live load are the same as for simply supported beam.

Case 2. Beam Provided with Cantilevers at Both Ends.—For fixed loads it may be assumed that both cantilevers are acting at the same time. The resulting bending moments are shown in Fig. 66, p. 127, assuming a symmetrical arrangement of cantilevers. The maximum positive bending moment for fixed loads acts in the center of the beam and is equal to the difference between the maximum positive bending moment for simple span and the negative bending moments at the support.

For live load, maximum negative bending moments are produced in the beam when both cantilevers are loaded and the beam not loaded. This gives a uniform negative bending moment in the beam.

Positive bending moments in the beam to be used for live load are the same as for simply supported beam, which act when the beam is loaded and the cantilever not loaded.

Case 3. Continuous Beams of Two Equal Spans with Cantilever at One End.—The load on the cantilever produces bending moments in both spans of the continuous beam as shown in Fig. 67, p. 128.

Let

M_c = maximum bending moment due to the cantilever load.

Then the bending moments in the beam at the three supports are

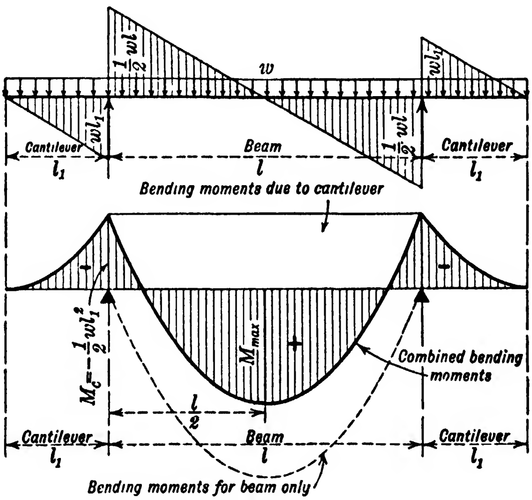


FIG. 66.—Beam with Two Symmetrical Cantilevers. (*See* p. 126.)

Bending Moment at Supports Due to Cantilever Load,

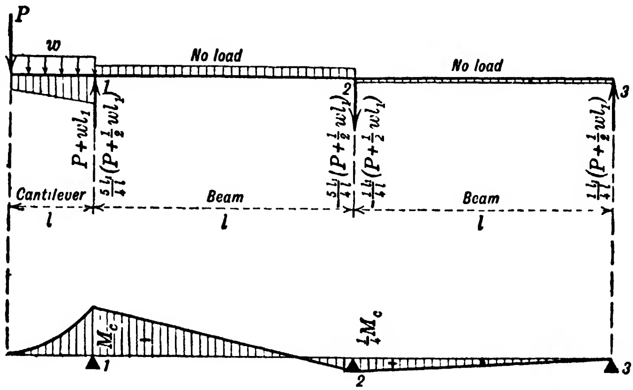
$$M_1 = M_c = Pl_1 + \tfrac{1}{2}wl_1^2. \qquad (437)$$

$$M_2 = -\tfrac{1}{4}M_c. \qquad (438)$$

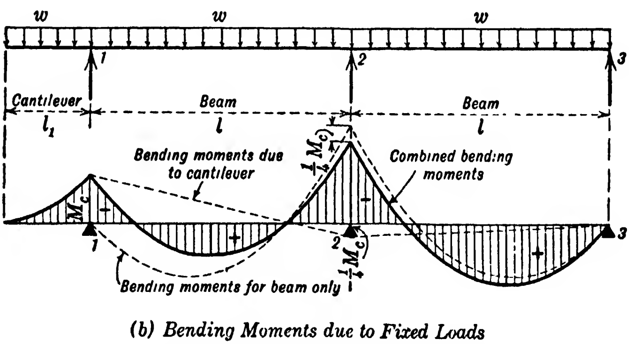
$$M_3 = 0. \qquad (439)$$

The bending moments produced by the load in the cantilever in the span next to the cantilever are mostly negative, while in the second span they are positive. This is illustrated in Fig. 67(*a*), p. 128.

For dead load the bending moments due to the cantilever load may be combined with the bending moments of the continuous beam as shown in Fig. 67(*b*). The bending moments without the cantilever are shown by dotted lines. It should be noted that the load on the cantilever decreases the positive bending moment in the span next to the cantilever but increases the positive bending moment in the other span.



(a) Bending Moments due to Loads on Cantilever



(b) Bending Moments due to Fixed Loads

FIG. 67.—Continuous Beam of Two Spans with Cantilever. (*See p.* 127.)

For live load the negative bending moments in the span next to the cantilever are obtained when the cantilever and the span away from the cantilever are loaded simultaneously.

Maximum negative moment at the second support is obtained when both spans are loaded simultaneously and the cantilever not loaded (see Fig. 25, p. 35).

Maximum positive bending moment for the span next to the cantilever is obtained when that span only is loaded (see Fig. 26, p. 35).

Maximum positive bending moment for the second span is obtained when the second span and the cantilever are loaded simultaneously (see Fig. 67(*b*), p. 128).

No positive bending moment is possible in the cantilever.

Case 4. Beam Fixed at One Support and Cantilevered at the Other.—Bending moment in the cantilever same as for previous cases.

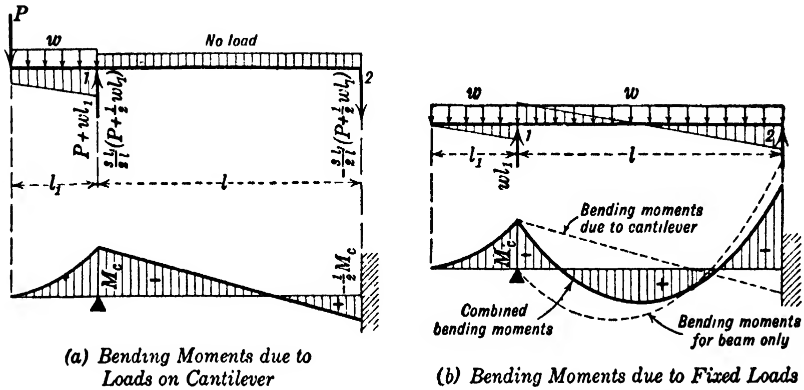


FIG. 68.—Beam Fixed at One End, Cantilevered at the Other. (*See p.* 129.)

Bending moments in the main span due to the cantilever load are shown in Fig. 68(*a*), p. 129. They are

Bending Moment at Supports Due to Cantilever Load,

$$M_1 = M_c. \qquad (440)$$

$$M_2 = -\tfrac{1}{2}M_c. \qquad (441)$$

The bending moments between the supports vary according to a straight line.

Bending Moments Due to Dead Load.—The dead load may be assumed as acting simultaneously on the cantilever and in the main span. The resultant bending moments on the beam are obtained by combining the bending moments due to cantilever load with the bending moments on the main span, computed as given on p. 28.

The resultant bending moments for uniformly distributed load are shown in Fig. 68(*b*), p. 129.

Maximum positive bending moment for dead load may be obtained from following formula:

Maximum Positive Bending Moment, Dead Load. Case 4,

$$M_{\max} = \frac{1}{2}\left[\frac{1}{2} + \frac{3}{4}\left(\frac{l_1}{l}\right)^2\right]^2 wl^2 - \frac{wl_1{}^2}{2} \quad . \; . \; . \quad (442)$$

Distance of Point of Maximum Positive Bending Moment,

$$x_1 = \left[\frac{1}{2} + \frac{3}{4}\left(\frac{l_1}{l}\right)^2\right]l. \quad \ldots \ldots \quad (443)$$

End Shears in Beam Due to Cantilever

The end shears in the beam due to the load on the cantilever are given below. Assume that the cantilever in Case 1, 2 and 4 is located at the left end of the beam.

Also let M_c = bending moment due to left cantilever;
M_{c1} = bending moment due to right cantilever;
l = span of beam;
V_{1r} = end shear to the right of first support;
V_{2l} = end shear to the left of second support.

Case 1. Beam Free at One End, Cantilevered at Other.

End Shears,

$$V_{1r} = \frac{M_c}{l}. \quad \ldots \quad (444) \qquad V_{2l} = -\frac{M_c}{l}. \quad \ldots \quad (445)$$

Case 2. Beam Cantilevered at Both Ends.

End Shears. One Cantilever Loaded:

Same as in previous case.

End Shears. Both Cantilevers Loaded,

$$V_{1r} = \frac{M_c - M_{c1}}{l}. \quad . \quad (446) \qquad V_{2l} = \frac{M_{c1} - M_c}{l}. \quad . \quad . \quad (447)$$

Case 3. Continuous Beam of Two Spans with Cantilever.

End Shears,

$$V_{1r} = 1\frac{1}{4}\frac{M_c}{l} = \frac{5}{4}\frac{l_1}{l}(P + \tfrac{1}{2}wl_1). \quad . \quad (448) \qquad V_2l = -V_{1r}. \quad . \quad . \quad (449)$$

$$V_{2r} = -\frac{1}{4}\frac{l_1}{l}(P + \tfrac{1}{2}wl_1). \quad \ldots \quad (450) \qquad V_3 = -V_{2r}. \quad . \quad . \quad (451)$$

Case 4. Beam Fixed at One End, Cantilevered at Other.

End Shears,

$$V_{1r} = 1\frac{1}{2}\frac{M_c}{l} = \frac{3}{2}\frac{l_1}{l}(P + \tfrac{1}{2}wl_1). \quad . \quad (452) \qquad V_2 = -V_{1r}. \quad . \quad . \quad (453)$$

End Shear in Beam Due to Load in Beam.—The end shears in the beam due to the load on beam are the same as for similar beam without cantilever.

End Shears for Dead Load.—End shears for dead load are obtained by adding the end shears due to the load in the beam to the end shears produced by the cantilever. It should be remembered that at the support away from the cantilever the quantity to be added to the end shear in beam is minus.

Maximum End Shears.—For live load, maximum ends shears at both ends are produced by different conditions of loading.

At the support next to the cantilever, maximum end shears in the beam acts when the beam and the cantilever are fully loaded.

At the other support, maximum end shear acts when the beam is loaded and cantilever not loaded.

Maximum Reactions and Uplift.—Maximum reactions on the support next to the cantilever is obtained when cantilever and the first span are loaded.

Maximum reaction at the other support acts for the beam fully loaded and the cantilever not loaded.

Maximum uplift acts when cantilever only is loaded and rest of span not loaded.

MOMENTS OF INERTIA AND THEIR EFFECT UPON CONTINUOUS BEAM

Method of Determining Moments of Inertia in Reinforced Concrete Construction.—Moments of inertia of a concrete member may be required either for the purpose of computing stresses due to direct stress and bending moments (Chapter II) or for the purpose of determining the variation of moments of inertia to be used in applying the theory of elasticity.

In the first case it is desired to find the actual moment of inertia at a particular section and for a known condition of stresses. The moment of inertia is used directly for determining of stresses. The methods of determining moments of inertia are given in the proper chapters (see Vol. I, p. 174, and Vol. II, p. 245). Where no (or only very small) tensile stresses exist, a substitute homogenous section is used in place of the reinforced concrete section of the member in which the reinforcement is replaced by an area of concrete equal to n times the area of steel, n being the ratio of modulus of elasticity. The moment of inertia of the substitute section is then found as for a homogenous section. On the other hand, where large tensile stresses occur the part of concrete section in tension is neglected and the moment of inertia is found indirectly in the same manner as used in determining stresses in simple bending.

In the second case the moments of inertia are used:

(1) to determine the variation of moments of inertia in a member with variable moments of inertia;
(2) to determine the ratio of moments of inertia for several members as, for instance, in a frame the ratio of moments of inertia of the column to that of the beam. For this purpose it is of more importance to find a representative ratio of moments of inertia for the member than to find the actual moments of inertia at any one section.

Even in a concrete beam of constant dimensions the moment of inertia varies at different sections. The variation is due partly to the variation in the amount of steel and partly to stress conditions. In the sections subjected to large tensile stresses the concrete in the tensile zone is not effective and should be omitted in computing moments of inertia.

In sections nearer the points of zero moment the ineffective concrete area becomes smaller and smaller. The area to be used in computing moments of inertia becomes larger and larger and therefore the moment of inertia increases until it reaches its maximum at and near the point of zero bending moment where the whole concrete section is effective. Therefore the moments of inertia at points of small bending moments are appreciably larger than at points of large stress. This is partly offset by the fact that the deflection of the beam is more affected by the moments of inertia at the points of large stress than at the points of small stress.

Opinions differ as to the proper method of computing the moments of inertia for the purpose of determining the ratio of variation. The preponderance of opinion is in favor of computing the moments of inertia for the concrete section only considering it as a homogenous section and neglecting the reinforcement. This method is recommended by the authors.

Formulas for Moment of Inertia.—The formulas for moments of inertia given below are for concrete section only neglecting the reinforcement. They should be used only for determining the ratio of moments of inertia.

Let b = breadth of rectangular beam; also breadth of flange of T-beam;
b' = breadth of web of T-beam;
h = depth of rectangular beam and T-beam;
t = thickness of flange of T-beam.

Moment of Inertia of Rectangular Section,

$$I = \tfrac{1}{12}bh^3. \qquad (454)$$

Moment of Inertia of T-Beam,

$$I = \tfrac{1}{3}(b - b')t^3 + \tfrac{1}{3}b'h^3 - \frac{[(b - b')t^2 + b'h^2]^2}{4[(b - b')t + b'h]}, \qquad (455)$$

or

$$I = \frac{bh^3}{12}\left[4\left(1 - \frac{b'}{b}\right)\left(\frac{t}{h}\right)^3 + 4\frac{b'}{b} - \frac{3\left[\left(1 - \frac{b'}{b}\right)\left(\frac{t}{h}\right)^2 + \frac{b'}{b}\right]^2}{\left(1 - \frac{b'}{b}\right)\frac{t}{h} + \frac{b'}{b}}\right]. \qquad (456)$$

The value in the square brackets is a constant depending upon the ratios of $\frac{t}{h}$ and $\frac{b'}{b}$. The equation for moment of inertia may be written

$$I = C_I bh^3, \qquad (457)$$

where C_I may be taken from Diagram 12, p. 134, for the proper combination of the ratios $\frac{t}{h}$ and $\frac{b'}{b}$.

Application of Formulas for Constant Moments of Inertia.—In previous pages formulas for continuous beams were developed on the assumption that the moments of inertia of the beam are constant throughout. For beams with constant depth and width this assumption gives accurate enough results. The effect upon bending moments of small haunches, the length of which does not exceed one-tenth of the span, may be considered as negligible.

Effect of Variation in Moments of Inertia.—When the moments of inertia near the supports are appreciably larger than in the central portion of the beam, the bending moments in a continuous beam undergo the following changes.

The negative bending moments at the supports become larger than for beams with constant moments of inertia.

The length of the region subjected to negative bending moments becomes larger.

The positive bending moments in the center becomes smaller than for beams with constant moments of inertia.

The increase of the moment of inertia at the support therefore has the effect of putting more bending moment near the support and reducing the bending moments in the center. The changes will be of course

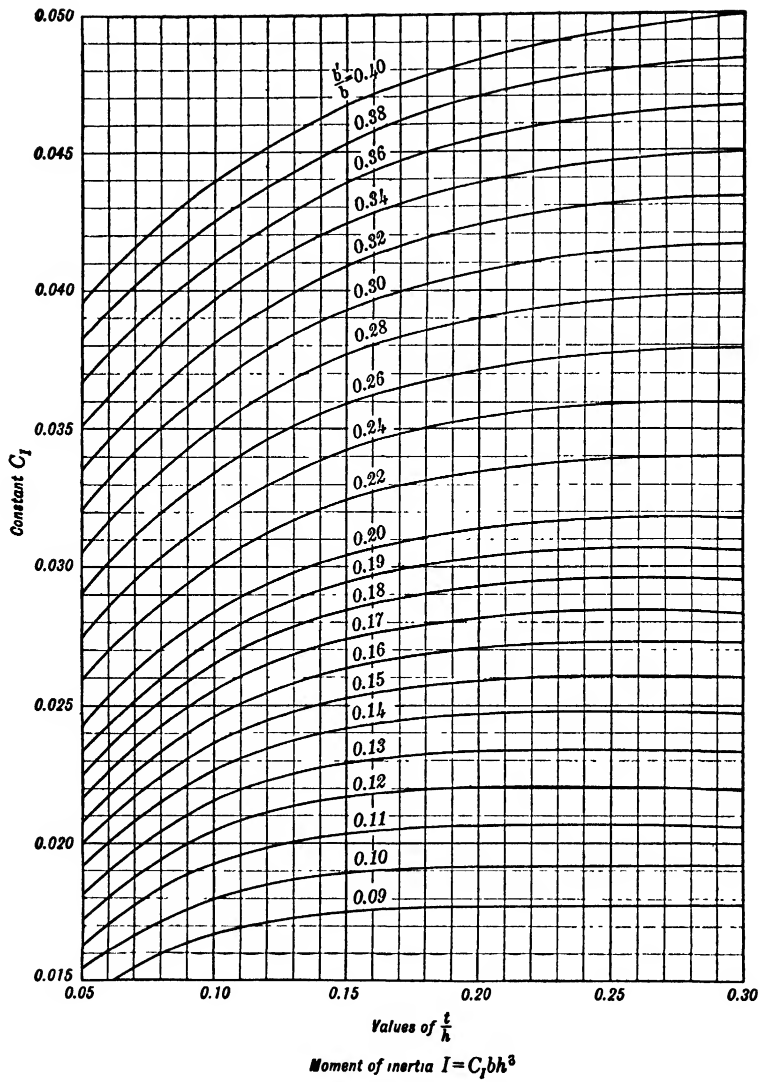


DIAGRAM 12.—Values of C_I for T-Beam for Different $\frac{t}{h}$ and $\frac{b'}{b}$. (See p. 133.)

in some ratio to the increase of the moments of inertia and also to the length of beam affected by the increase.

Variation of Moments of Inertia Found in Practice.—In practice the moment of inertia of the beam is increased by the following means.

1. Beam may be provided with straight haunches at the ends as in Fig. 69. In this case the moment of inertia is constant in the central part of the beam. At the end the depth of the beam varies according

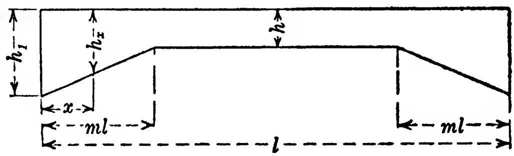


FIG. 69.—Beam Provided with Straight haunch. (*See p.* 135.)

to a straight line. It is assumed that the beam is symmetrical. Referring to Fig. 69 the depth of the section at any point in the haunch may be represented by following formula.

$$h_x=h+\frac{h_1-h}{ml}(ml-x)=h+(h_1-h)\left(1-\frac{x}{ml}\right)=h\left[1+\left(\frac{h_1}{h}-1\right)\left(1-\frac{x}{ml}\right)\right].$$

Hence

$$\frac{h}{h_x}=\frac{1}{1+\left(\frac{h_1}{h}-1\right)\left(1-\frac{x}{ml}\right)}. \quad . \quad . \quad . \quad . \quad (458)$$

Since

$$I=\frac{bh^3}{12}$$

and

$$I_x=\frac{bh_x^3}{12},$$

the ratio of moments of inertia

$$\frac{I}{I_x}=\frac{bh^3}{bh_x^3}=\frac{h^3}{h_x^3}.$$

The variation of moments of inertia, therefore, is

Variation in Moments of Inertia for Straight Haunch,

$$\frac{I}{I_x}=\frac{h^3}{h_x^3}=\frac{1}{\left[1-\left(\frac{h_1}{h}-1\right)\left(1-\frac{1}{m}\frac{x}{l}\right)\right]^3}. \quad . \quad . \quad (459)$$

for $\frac{x}{l} = 0$ to $\frac{x}{l} = m$ and for $\frac{x}{l} = (1 - m)$ to $\frac{x}{l} = 1$,

and
$$\frac{I}{I_x} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (460)$$

for $\frac{x}{l} = m$ to $\frac{x}{l} = (1 - m)$.

2. Beam with parabolic haunches. Sometimes for the sake of appearance the haunches are curved. The curvature may be made according to a parabola as shown in Fig. 70, p. 136.

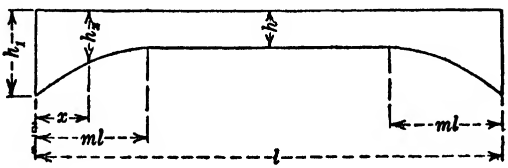


FIG. 70.—Beam with Parabolic Haunch. (*See p.* 136.)

The depth at any point is

$$h_x = h + \frac{h_1 - h}{(ml)^2}(ml - x)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (461)$$

Hence

$$\frac{h}{h_x} = \frac{1}{1 + \left(\frac{h_1}{h} - 1\right)\left(1 - \frac{1}{m}\frac{x}{l}\right)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (462)$$

Assuming that the moments of inertia are proportional to the cubes of the depths of the section, the variation of the moments of inertia may be expressed by the formula below.

Variation of Moments of Inertia for Parabolic Haunch,

$$\frac{I}{I_x} = \frac{h^3}{h_x^3} = \frac{1}{\left[1 + \left(\frac{h_1}{h} - 1\right)\left(1 - \frac{1}{m}\frac{x}{l}\right)^2\right]^3}, \quad . \quad . \quad (463)$$

for $x = 0$ to $x = ml$ and $x = (1 - m)l$ to $x = l$.

$$\frac{I}{I_x} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (464)$$

for $\frac{x}{l} = m$ to $\frac{x}{l} = (1 - m)$.

3. Beam with parabolic bottom. When the bottom is in shape of a parabola the formula above may be used by substituting $m = \frac{1}{2}$. Therefore

$$\frac{I}{I_x} = \frac{1}{\left[1 + \left(\frac{h_1}{h} - 1\right)\left(1 - 2\frac{x}{l}\right)^2\right]^3}. \quad . \quad . \quad . \quad . \quad (465)$$

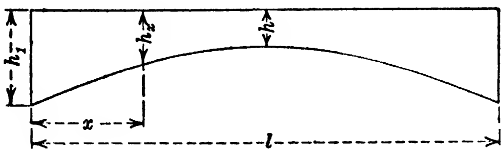


FIG. 71.—Beam with Parabolic Bottom. (*See p.* 137.)

BASIC FORMULAS FOR CONTINUOUS BEAMS WITH VARIABLE MOMENTS OF INERTIA

The basic three-moment equation for continuous beams with variable moments of inertia is given below.

In practice it is not necessary to use the complicated equations nor to solve calculus, as simple equations are given for conditions most likely to occur.

Let l_r and l_{r+1} = length of two adjoining spans;

I_r = least moment of inertia of l_r span;

I_{r+1} = least moment of inertia of l_{r+1} span;

$\frac{I_r}{I_x}$ = variation of moments of inertia in l_r span;

$\frac{I_{r+1}}{I_x}$ = variation of moments of inertia in l_{r+1} span;

M_r, M_{r+1}, M_{r+2} = bending moment at the three supports of span l_r and l_{r+1};

α_r and β_r = constants for l_r span depending upon variation of moments of inertia;

α'_{r+1} and β_{r+1} = constants for l_{r+1} span depending upon variation of moments inertia;

M_{sr}, $M_{s(r+1)}$ = static bending moment due to loads in spans l_r and l_{r+1}, respectively, considering the spans as freely supported.

Basic Three-moment Equation, Variable Moments of Inertia,

$$M_r\beta_r\frac{l_r}{I_r} + 2M_{r+1}\left(\alpha_r\frac{l_r}{I_r} + \alpha'_{r+1}\frac{l_{r+1}}{I_{r+1}}\right) + M_{r+2}\beta_{r+1}\frac{l_{r+1}}{I_{r+1}}$$

$$= -6\left[\frac{1}{l_rI_r}\int_0^{l_r} M_{sr}x\frac{I_r}{I_x}dx + \frac{1}{l_{r+1}I_{r+1}}\int_0^{l_{r+1}} M_{s(r+1)}(l_{r+1} - x)\frac{I_{r+1}}{I_x}dx\right]. \quad (466)$$

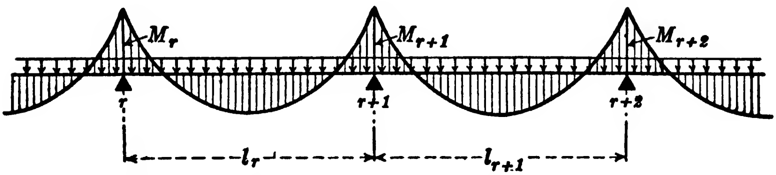


FIG. 72.—Illustration for Three-moment Equation. (*See p.* 138.)

in which the constants α_r, α'_{r+1}, β_r and β_{r+1} depend upon the variation of the moments of inertia. They are:

Constants,

$$\alpha_r = \frac{3}{l_r^3}\int_0^{l_r} x^2\frac{I_r}{I_x}dx, \qquad \alpha'_{r+1} = \frac{3}{l^3_{r+1}}\int_0^{l_{r+1}} (l_{r+1} - x)^2\frac{I_{r+1}}{I_x}dx,$$

$$\beta_r = \frac{6}{l_r^3}\int_0^{l_r} x(l_r - x)\frac{I_r}{I_x}dx, \qquad \beta_{r+1} = \frac{6}{l^3_{r+1}}\int_0^{l_{r+1}} x(l_{r+1} - x)\frac{I_{r+1}}{I_x}dx.$$

The ratios $\frac{I_r}{I_x}$ and $\frac{I_{r+1}}{I_x}$ depend upon the shape of the haunch, and for beams shown in Figs. 69 to 71 they may be represented by Formulas (459), (463) and (465), respectively. Where the haunch forms only part of the beam of a length equal to ml, the integrals $\int_0^{l_r}$ must be solved by dividing it into parts so that

$$\int_0^{l_r} = \int_0^{ml_r} + \int_{ml_r}^{(l_r-ml_r)} + \int_{(l_r-ml_r)}^{l_r}.$$

The values α_r, α'_{r+1}, β_r, and β_{r+1}, are constant for any one design of the beam.

When the minimum moment of inertia is the same in all spans, $I_r = I_{r+1}$ and may be cancelled. This changes the basic equation to

Basic Three-moment Equation when $I_r = I_{r+1}$,

$$M_r\beta_r l_r + 2M_{r+1}(\alpha_r l_r + \alpha'_{r+1}l_{r+1}) + M_{r+2}\beta_{r+1}l_{r+1}$$
$$= -6\left[\frac{1}{l_r}\int_0^{l_r} M_{sr}x\frac{I_r}{I_x}dx + \frac{1}{l_{r+1}}\int_0^{l_{r+1}} M_{s(r+1)}(l_{r+1} - x)\frac{I_r}{I_x}dx\right]. \quad (467)$$

This equation is of the same shape as the three-moment equation with constant moments of inertia.

Constants α and β for Beams with Special Shape of Haunches.—In the three-moment equations just given appear constants α_r, β_r, α'_{r+1} and β_{r+1} which depend upon the length and shape of the haunches and upon the ratio between the maximum and minimum moments of inertia. If the haunches in the various spans are either of different length in proportion to the length of the span, their design is different, or, finally, if the ratios between the maximum and minimum moment of inertia are different, then the constants for each span are different. Thus α_1, α'_1 and β_1 will be constants for the first span, α_2, α'_2 and β_2 will be constants for the second span and so on.

In general there are three constants for each span α, α' and β which must be computed before the bending moments can be determined. For spans for which the haunches are symmetrical the integral $\int_0^l x^2\frac{I}{I_x}dx$ is equal to the integral $\int_0^l (l - x)^2\frac{I}{I_x}dx$. Consequently

For Spans with Symmetrical Haunches,

$$\alpha = \alpha'. \quad \ldots\ldots\ldots \quad (468)$$

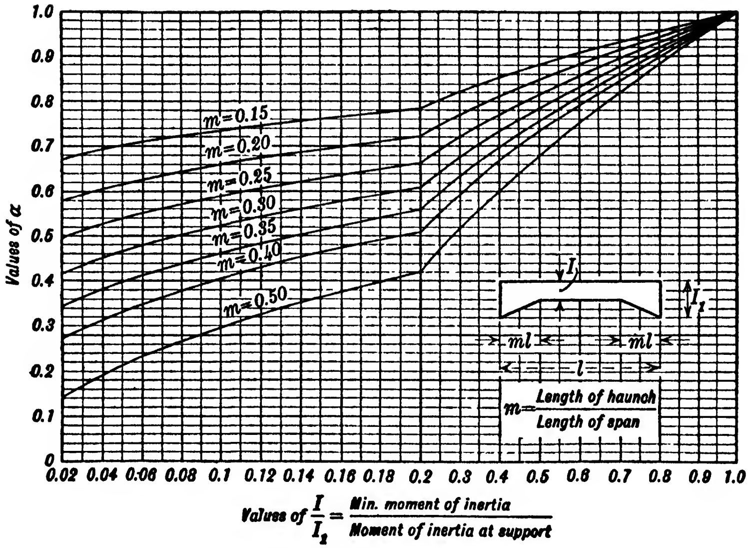
The determination of the constants is complicated. It may be accomplished either by integration for haunches for which the variation of the moments of inertia can be expressed by an equation or by summation for irregular shapes of beam. To facilitate the use of the formulas two shapes of haunches, most common in practice, have been selected and constants for them computed.

The selected shapes of haunches are:

1. Symmetrical beam with straight-line haunches, as described on p. 135 and shown in Fig. 69. Constants for this case are given in Diagram 13, p. 140.

2. Symmetrical parabolic haunches as described on p. 136 and shown in Fig. 70. For this case constants are given in Diagram 14, p. 141.

In each case the constants are given for different ratios of length of haunches to length of spans and for different ratios of the maximum and minimum moments of inertia.



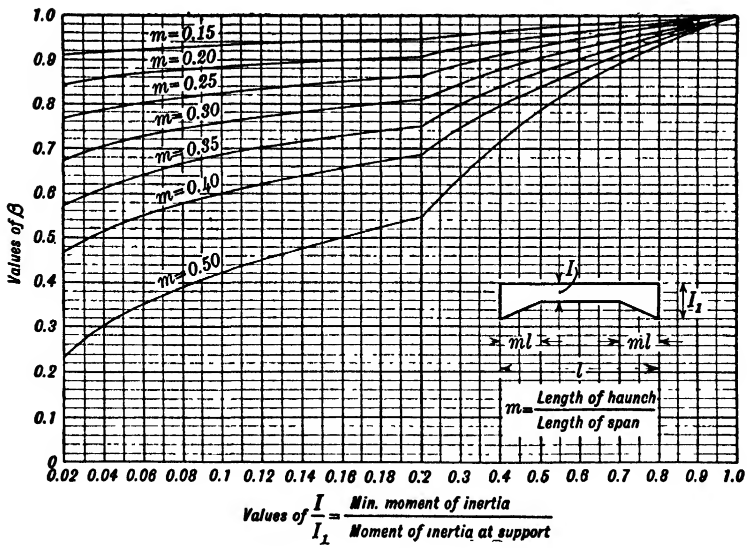
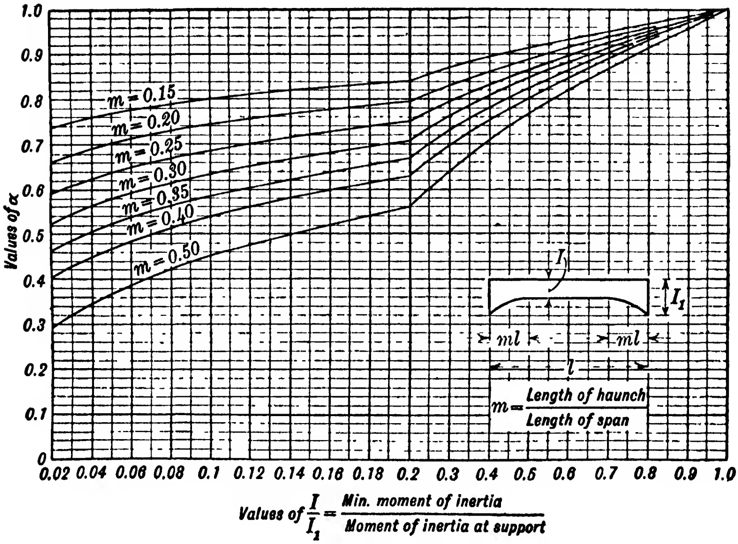


DIAGRAM 13.—Constants α and β for Symmetrical Beam with Straight-line Haunches. (See p. 139.)



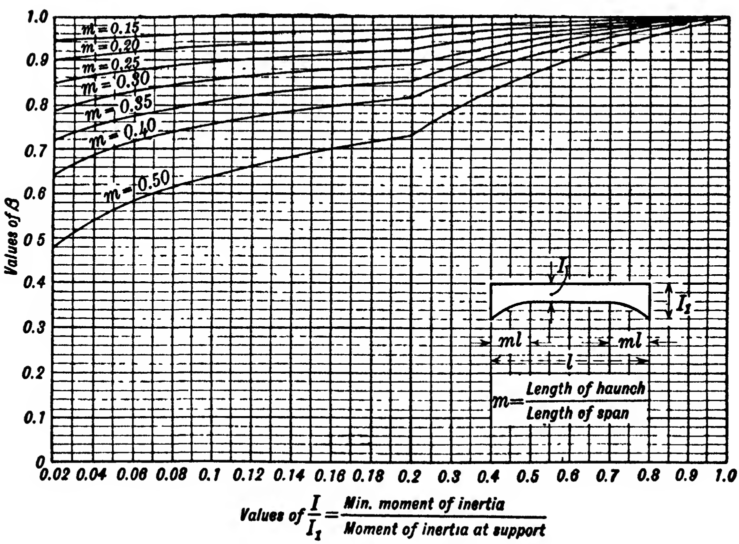


DIAGRAM 14.—Constants α and β for Symmetrical Beam with Parable Haunches.
(*See p* 139.)

For the purpose of preparing diagrams the formulas for constants are changed to the following shape:

$$\alpha = \frac{3l^3}{l^3}\int_0^1 \left(\frac{x}{l}\right)^2 \frac{I}{I_x} d\left(\frac{x}{l}\right) = 3\int_0^1 \left(\frac{x}{l}\right)^2 \frac{I}{I_x} d\left(\frac{x}{l}\right). \quad . \quad . \quad (469)$$

Similarly,

$$\beta = 6\int_0^1 \frac{x}{l}\left(1 - \frac{x}{l}\right)\frac{I}{I_x} d\left(\frac{x}{l}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (470)$$

The values of $\frac{I}{I_x}$ are given by Equations (459) and (463) for the two types of haunches for which diagrams were prepared. To get the constants the integral $\int_0^1$ is divided into three parts, namely, $\int_0^m + \int_m^{1-m} + \int_{1-m}^1$. In each part proper values for $\frac{I}{I_x}$ are substituted and the integrals solved.[6]

Right-hand Parts of Three-moment Equation.—The right-hand parts of the three-moment equation depend upon the loading.

Let w_r = uniformly distributed load in l_r span;

w_{r+1} = uniformly distributed load in l_{r+1} span.

Then, for uniformly distributed loading extending full length of the span the value of the integrals are

Integrals for Uniform Loading,

$$\frac{6}{l_r}\int_0^{l_r} M_{sr} x \frac{I_r}{I_x} dx = \tfrac{1}{4}\beta_r w_r l_r^3. \quad . \quad . \quad . \quad . \quad (471)$$

and

$$\frac{6}{l_{r+1}}\int_0^{l_{r+1}} M_{sr+1}(l_{r+1} - x)\frac{I_r}{I_x} dx = \tfrac{1}{4}\beta_{r+1} w_{r+1} l^3_{r+1}. \quad . \quad . \quad . \quad (472)$$

Values of β_r and β_{r+1} are the same constants as used in the left side of the equation. They are given in Diagrams 13 and 14 for two assumptions as to variation of moment of inertia.

[6] For solution of integrals, see A. Strassner, Neuere Methoden, Band 1, 1925, Wilhelm Ernst and Sohn.

For concentrated loads let

P_r = concentrated load or loads in span l_r;

P_{r+1} = concentrated load or loads in span l_{r+1};

C_{v1} = constant depending upon $\frac{a}{l}$ of the load and the design of the beam (see Diagrams 15 and 16, pp. 144 and 145);

C_{v2} = constant depending upon $\frac{a}{l}$ of the load and the design of the beam (see Diagrams 15 and 16, pp. 144 and 145).

The values C_{v1} and C_{v2} correspond to values C_1 and C_2 (see p. 19) in beam with constant moment of inertia.

For concentrated loads the integrals on the right side of Equation (467), p. 139, become

Single Concentrated Load,

$$\frac{6}{l_r}\int_0^{l_r} M_{sr} x \frac{I_r}{I_x} dx = \beta C_{v1} P_r l_r. \quad . \quad . \quad . \quad . \quad (473)$$

$$\frac{6}{l_{r+1}}\int_0^{l_{r+1}} M_{sr+1}(l - x)\frac{I_r}{I_x} dx = \beta C_{v2} P_{r+1} l_{r+1}. \quad . \quad . \quad (474)$$

Several Concentrated Loads,

$$\frac{6}{l_r}\int_0^{l_r} M_{sr} x \frac{I_r}{I_x} dx = l_r \beta \Sigma C_{v1} P_r, \quad . \quad . \quad . \quad (475)$$

$$\frac{6}{l_{r+1}}\int_0^{l_{r+1}} M_{sr+1}(l - x)\frac{I_r}{I_x} dx = l_{r+1} \beta \Sigma C_{v2} P_{r+1}, \quad . \quad . \quad (476)$$

where $\Sigma C_{v1} P_r$ is the sum of all loads in span l_r each multiplied by its corresponding value of C_{v1} and $\Sigma C_{v2} P_{r+1}$ is the sum of all loads P_{r+1} in l_{r+1} span each multiplied by its corresponding value C_{v2}.

For the selected shapes of beams, described on pp. 135 and 136, C_{v1} and C_{v2} are given in Diagrams 15 and 16, pp. 144 and 145, and constants β are given in Diagrams 13 and 14, pp. 140 and 141.

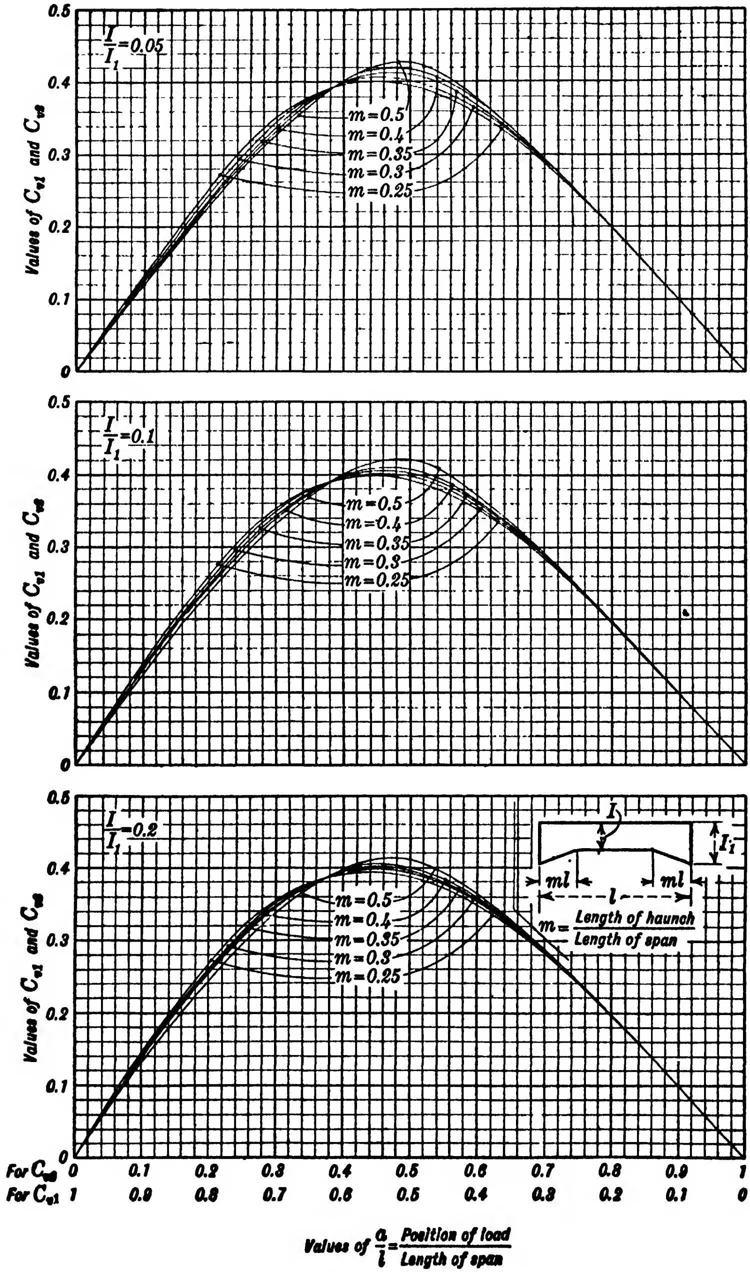


DIAGRAM 15.—Constants C_{v1} and C_{v2} for Symmetrical Beams with Straight-line Haunches. (See p. 143

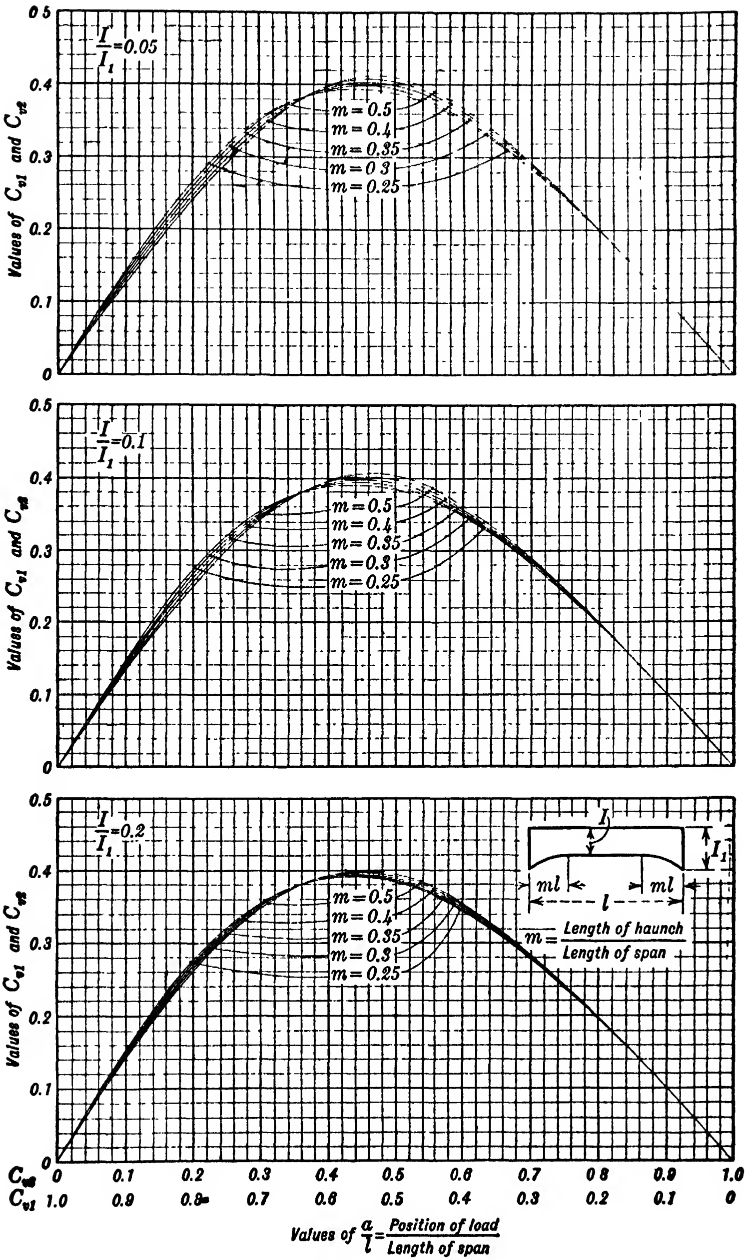


Diagram 16.—Constants C_{v1} and C_{v2} for Symmetrical Beams with Parabolic Haunches. (*See p.* 143.)

TWO SPANS, FREE ENDS. VARIABLE MOMENTS OF INERTIA

Two Unequal Spans.—Using the three-moment Equation (467), p. 139, in the same manner as explained on p. 20 for the beams with constant moments of inertia, the bending moment at support is

Bending Moment at Support. Uniform Loading,

$$M_2 = -\frac{w_1\beta_1 l_1^3 + w_2\beta_2 l_2^3}{8(\alpha_1 l_1 + \alpha_2 l_2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (477)$$

where α_1, β_1 are constants for the first span,

α_2, β_2 are constants for the second span.

Values of α_1, α_2, β_1, β_2 can be taken from Diagrams 13 and 14 for α and β, pp. 140 and 141.

Bending Moment at Support. Concentrated Loading,

$$M_2 = -\frac{l_1^2\beta_1 \Sigma P_1 C_{v1} + l_2^2\beta_2 \Sigma P_2 C_{v2}}{2(\alpha_1 l_1 + \alpha_2 l_2)}. \quad . \quad . \quad . \quad (478)$$

Values of $\alpha_1, \alpha_2, \beta_1, \beta_2$ may be taken from Diagrams 13 and 14, pp. 140 and 141. Values of C_{v1} and C_{v2} may be taken from Diagram 16, p. 145.

Two Equal Spans.—For two equal spans $l_1 = l_2 = l$ and the formulas become

Bending Moment at Support. Two Equal Spans. Uniform Loading,

$$M_2 = -\frac{w_1 + w_2}{16}\frac{\beta}{\alpha} l^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (479)$$

Bending Moment at Support. Two Equal Spans. Concentrated Load,

$$M_2 = -\frac{(\Sigma P_1 C_{v1} + \Sigma P_2 C_{v2})}{4}\frac{\beta}{\alpha} l. \quad . \quad . \quad . \quad . \quad (480)$$

Values of α and β may be taken from Diagrams 13 or 14, pp. 140 and 141. Values of C_{v1} and C_{v2} may be taken from Diagrams 15 or 16, pp. 144 and 145.

End Shears.—After the bending moment at support is found the end shears may be obtained in the same manner as for beams with constant moment of inertia given on p. 14.

THREE SPANS, FREE ENDS. VARIABLE MOMENTS OF INERTIA

Using the three-moment equation in the same manner as for beams with constant moment of inertia, following equations for bending moments at support are obtained:

Let M_2 = bending moment at second support;
M_3 = bending moment at third support;
l_1, l_2, l_3 = span lengths of the three spans;
α_1 and β_1 = constants for first span, depending upon shape and length of haunch from Diagram 13 or 14;
α_2 and β_2 = constants for second span, depending upon shape and length of haunch from Diagram 13 or 14;
α_3 and β_3 = constants for third span, depending upon shape and length of haunch from Diagram 13 or 14;
w_1 = uniform load in first span;
w_2 = uniform load in second span;
w_3 = uniform load in third span.

Uniform Loading, General Equations.

Bending Moment at Support for Uniform Loading, General,

$$M_2 = -\frac{\left\{\begin{matrix}2(\alpha_2 l_2 + \alpha_3 l_3)\beta_1 l_1^3 w_1 + [2(\alpha_2 l_2 + \alpha_3 l_3) - \beta_2 l_2] \\ \beta_2 l_2^3 w_2 - \beta_2\beta_3 l_2 l_3^3 w_3\end{matrix}\right\}}{16(\alpha_1 l_1 + \alpha_2 l_2)(\alpha_2 l_2 + \alpha_3 l_3) - 4\beta_2^2 l_2^2}. \quad (481)$$

$$M_3 = -\frac{\left\{\begin{matrix}-\beta_1\beta_2 l_1^3 l_2 w_1 + [2(\alpha_1 l_1 + \alpha_2 l_2) - \beta_2 l_2] \\ \beta_2 l_2^3 w_2 + 2(\alpha_1 l_1 + \alpha_2 l_2)\beta_3 l_3^3 w_3\end{matrix}\right\}}{16(\alpha_1 l_1 + \alpha_2 l_2)(\alpha_2 l_2 + \alpha_3 l_3) - 4\beta_2^2 l_2^2}. \quad (482)$$

Special Case, Uniform Loading.—If the ratio of maximum and minimum moments of inertia and also the ratio of the length of haunch to total length of span are the same for all spans all constants α are equal and all constants β are equal.

Thus in Formulas (481) and (482) substitute

$$\alpha = \alpha_1 = \alpha_2 = \alpha_3 \quad \text{and} \quad \beta = \beta_1 = \beta_2 = \beta_3.$$

Then

Bending Moment at Support,

$$M_2 = -\frac{2(l_2 + l_3)l_1^3 w_1 + \left[2(l_2 + l_3) - \frac{\beta}{\alpha} l_2\right] l_2^3 w_2 - \frac{\beta}{\alpha} l_2 l_3^3 w_3}{16\frac{\alpha}{\beta}(l_1 + l_2)(l_2 + l_3) - 4\frac{\beta}{\alpha} l_2^2}, \quad (483)$$

$$M_3 = -\frac{-\frac{\beta}{\alpha}l_1{}^3l_2w_1 + \left[2(l_1 + l_2) - \frac{\beta}{\alpha}l_2\right]l_2{}^3w_2 + 2(l_1 + l_2)l_3{}^3w_3}{16\frac{\alpha}{\beta}(l_1 + l_2)(l_2 + l_3) - 4\frac{\beta}{\alpha}l_2{}^2}. \quad (484)$$

Equal Spans, Uniform Loading.—When the spans are of equal length substitute in Equations (483) and (484)

$$l_1 = l_2 = l_3 = l.$$

Then

Bending Moment at Support. Equal Spans,

$$M_2 = -\frac{4w_1 + \left[4 - \frac{\beta}{\alpha}\right]w_2 - \frac{\beta}{\alpha}w_3}{4\left(16\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)}l^2, \quad . \quad . \quad . \quad . \quad (485)$$

$$M_3 = -\frac{-\frac{\beta}{\alpha}w_1 + \left[4 - \frac{\beta}{\alpha}\right]w_2 + 4w_3}{4\left(16\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)}l^2. \quad . \quad . \quad . \quad (486)$$

α and β may be taken from Diagrams 13 and 14 for the proper ratios of $\frac{I}{I_1}$ and m and the proper shape of the haunch.

See p. 51 for method of obtaining most unfavorable loading of spans and absolute maximum moments.

Maximum positive bending moments may be found, using table on p. 177, for known negative bending moments.

Shear can be found as explained on p. 23 for beams with constant moments of inertia.

Concentrated Loads.—General equation for bending moments due to concentrated loads is obtained in the same manner as for beams with constant moment of inertia.

Let in addition to notation on p. 147.

P_1 = load or loads in first span;

P_2 = load or loads in second span;

P_3 = load or loads in third span;

C_{v1} = constant from Diagrams 15 or 16, depending upon $\frac{a}{l}$, the ratio of distance of load from left support to length of span;

C_{v2} = constant from Diagrams 15 or 16, depending upon $\frac{a}{l}$, the ratio of distance of load from left support to length of span.

Constants C_{v1} and C_{v2} correspond to C_1 and C_2 in beam with constant moments of inertia. They depend upon the location of loads and also upon the ratio of $\frac{I}{I_1}$ and the length and shape of haunch.

Bending Moment at Support. Concentrated Load,

$$M_2 = -\frac{\left\{\begin{aligned}&2(\alpha_2 l_2 + \alpha_3 l_3)\beta_1 l_1^2 \Sigma P_1 C_{v1} + 2(\alpha_2 l_2 + \alpha_3 l_3)\\&\quad \beta_2 l_2^2 \Sigma P_2 C_{v2} - \beta_2^2 l_2^3 \Sigma P_2 C_{v1} - \beta_2 \beta_3 l_2 l_3^2 \Sigma P_3 C_{v2}\end{aligned}\right\}}{4(\alpha_1 l_1 + \alpha_2 l_2)(\alpha_2 l_2 + \alpha_3 l_3) - \beta_2^2 l_2^2}. \quad (487)$$

$$M_3 = -\frac{\left\{\begin{aligned}&- \beta_1 \beta_2 l_1^2 l_2 \Sigma P_1 C_{v1} + 2(\alpha_1 l_1 + \alpha_2 l_2)\beta_2 l_2^2 \Sigma P_2 C_{v1}\\&\quad - \beta_2^2 l_2^3 \Sigma P_2 C_{v2} + 2(\alpha_1 l_1 + \alpha_2 l_2)\beta_3 l_3^2 \Sigma P_3 C_{v2}\end{aligned}\right\}}{4(\alpha_1 l_1 + \alpha_2 l_2)(\alpha_2 l_2 + \alpha_3 l_3) - \beta_2^2 l_2^2}. \quad (488)$$

If $\alpha = \alpha_1 = \alpha_2 = \alpha_3$ and $\beta = \beta_1 = \beta_2 = \beta_3$ which occurs when the ratio of maximum and minimum moment of inertia and the ratio m are the same for all spans and the haunch is of same design, the formulas change to

Bending Moment at Support for constant α and β,

$$M_2 = -\frac{\left\{\begin{aligned}&2(l_2 + l_3) l_1^2 \Sigma P_1 C_{v1} + 2(l_2 + l_3)\\&\quad l_2^2 \Sigma P_2 C_{v2} - \frac{\beta}{\alpha} l_2^3 \Sigma P_2 C_{v1} - \frac{\beta}{\alpha} l_2 l_3^2 \Sigma P_3 C_{v2}\end{aligned}\right\}}{4\frac{\alpha}{\beta}(l_1 + l_2)(l_2 + l_3) - \frac{\beta}{\alpha} l_2^2}. \quad (489)$$

$$M_3 = -\frac{\left\{\begin{aligned}&- \frac{\beta}{\alpha} l_1^2 l_2 \Sigma P_1 C_{v1} + 2(l_1 + l_2) l_2^2 \Sigma P_2 C_{v1}\\&\quad - \frac{\beta}{\alpha} l_2^3 \Sigma P_2 C_{v2} + 2(l_1 + l_2) l_3^2 \Sigma P_3 C_{v2}\end{aligned}\right\}}{4\frac{\alpha}{\beta}(l_1 + l_2)(l_2 + l_3) - \frac{\beta}{\alpha} l_2^2}. \quad (490)$$

If first span loaded others not loaded make $P_2 = P_3 = 0$.
If second span loaded others not loaded make $P_1 = P_3 = 0$.

See p. 57 for position of loads for absolute maximum bending moments.

Three Equal Spans, Concentrated Loads.—If all spans are equal, $l_1 = l_2 = l_3 = l$. The formulas become

Bending Moment at Support. Three Equal Spans. All Spans Loaded,

$$M_2 = -\frac{4\Sigma P_1C_{v1} + 4\Sigma P_2C_{v2} - \frac{\beta}{\alpha}\Sigma P_2C_{v1} - \frac{\beta}{\alpha}\Sigma P_3C_{v2}}{16\frac{\alpha}{\beta} - \frac{\beta}{\alpha}}l. \quad . \quad (491)$$

$$M_3 = -\frac{-\frac{\beta}{\alpha}\Sigma P_1C_{v1} + 4\Sigma P_2C_{v1} - \frac{\beta}{\alpha}\Sigma PC_{v2} + 4\Sigma P_3C_{v2}}{16\frac{\alpha}{\beta} - \frac{\beta}{\alpha}}l\,. \quad (492)$$

If end span loaded, $P_2 = P_3 = 0$.

Bending Moment at Supports. Three Equal Spans. Left End Span Loaded,

$$M_2 = -\frac{4}{16\frac{\alpha}{\beta} - \frac{\beta}{\alpha}}l\Sigma P_1C_{v1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (493)$$

$$M_3 = \frac{\frac{\beta}{\alpha}}{16\frac{\alpha}{\beta} - \frac{\beta}{\alpha}}l\Sigma P_1C_{v1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (494)$$

If center span loaded, $P_1 = P_3 = 0$.

Bending Moments at Supports. Three Equal Spans. Center Span Loaded,

$$M_2 = -\frac{4\Sigma P_2C_{v2} - \frac{\beta}{\alpha}\Sigma P_2C_{v1}}{16\frac{\alpha}{\beta} - \frac{\beta}{\alpha}}l. \quad . \quad . \quad . \quad . \quad (495)$$

$$M_3 = -\frac{4\Sigma P_2C_{v1} - \frac{\beta}{\alpha}\Sigma P_2C_{v2}}{16\frac{\alpha}{\beta} - \frac{\beta}{\alpha}}l. \quad . \quad . \quad . \quad . \quad (496)$$

α and β are constants, depending upon design of beam only, to be taken from Diagrams 13 or 14, pp. 140 and 141.

C_{v1} and C_{v2} are constants, depending upon design of beam and location of the various loads, to be taken from Diagrams 15 or 16, pp. 144 and 145.

EFFECT OF DOWNWARD MOVEMENT OF SUPPORT

Formulas previously given are based on the assumption that the beam is straight, that all supports are on the same level and that they remain on the same level after the loads are applied. When the relation of the elevation of the supports becomes disturbed, special bending moments are developed in the beam in addition to those produced by the loading.

Sinking of all foundations by the same amount is not harmful, as it does not affect the bending moments.

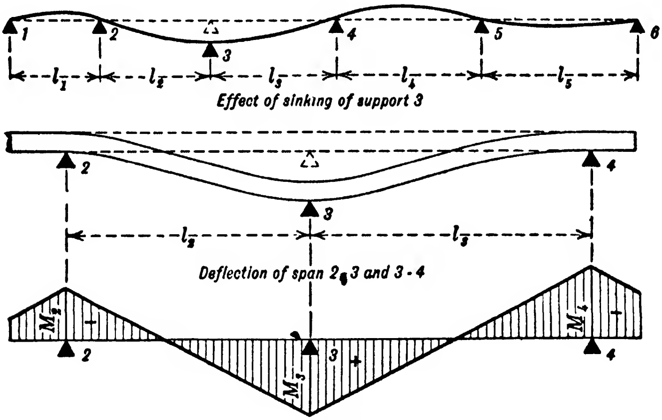


FIG. 73.—Effect of Sinking of Support. (*See p.* 151.)

Sinking of one foundation below the level of other foundations produces bending moments and stresses in the whole beam. The magnitude of the bending moments is proportional to the magnitude of the downward movement.

The effect of vertical downward movement of the support is evident from Fig. 73, p. 151. The maximum bending moments due the sinking of one support are developed in two spans carried by this support.

At both sides of the disturbed support the bending moments are positive with a maximum at the support.

At the ends of the two spans carried by this support the bending moments are negative. In the balance of the beam the bending moments may be found, using the fixed points.

General Formulas for Bending Moments.—Bending moments due to the vertical movement of supports may be found by means of the three-moment equation.

Notation

Let r, $r + 1$, and $r + 2$ = three succeeding supports of a continuous beam;

l_r = span length of rth span, in inches;

l_{r+1} = span length of $(r + 1)$th span, in inches;

I = moment of inertia of the beam, in.^{4};

M_r = bending moment at left support, rth span;

M_{r+1} = bending moment at right support, rth span;

M_{r+2} = bending moment at left support, $r + 1$th span;

Δ_r = vertical movement of rth support, in inches;

Δ_{r+1} = vertical movement of $(r + 1)$th support, in inches;

Δ_{r+2} = vertical movement of $(r + 2)$th support, in inches;

E = modulus of elasticity of concrete, lb. per sq. in.

Values of Δ_r, Δ_{r+1} and Δ_{r+2} are positive when the movement is down and negative when the movement is up.

Then the relation between the bending moments produced at the three supports by these movements of the supports can be expressed by the following equation:

Basic Three-moment Equation for Movement of Supports,

$$M_r l_r + 2M_{r+1}(l_r + l_{r+1}) + M_{r+2}l_{r+1} = 6EI\left(\frac{\Delta_{r+1} - \Delta_r}{l_r} + \frac{\Delta_{r+1} - \Delta_{r+2}}{l_{r+1}}\right). \quad (497)$$

Use of the Three-moment Equation.—The use of the equation just given is evident from the following example in which it is desired to find bending moments in a beam consisting of five spans caused by the sinking of the third support from the left end by an amount equal to Δ.

Using the general equation (497), following four equations may be developed, from which the four unknown bending moments M_2, M_3, M_4 and M_5 may be found.

Let Δ = sinking of third support from left in inches;
I = moment of inertia of beam in in.^{4};
E = modulus of elasticity of concrete, lbs. per sq. in.;
M_2, M_3, M_4, M_5 = bending moment at respective supports in pounds;
l_1, l_2, l_3, l_4, l_5 = span length in inches;

$$2M_2(l_1 + l_2) + M_3 l_2 = - 6EI\Delta\frac{1}{l_2}, \quad . \quad . \quad . \quad . \quad (498)$$

$$M_2 l_2 + 2M_3(l_2 + l_3) + M_4 l_3 = + 6EI\Delta\frac{l_2 + l_3}{l_2\, l_3}, \quad . \quad . \quad (499)$$

$$M_3 l_3 + 2M_4(l_3 + l_4) + M_5 l_4 = - 6EI\Delta\frac{1}{l_3}, \quad . \quad . \quad . \quad . \quad (500)$$

$$M_4 l_4 + 2M_5(l_4 + l_5) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (501)$$

FIXED POINTS

Definition.—Fixed Points [7] (also called conjugate points) are points in which the straight lines representing the bending moment curves in unloaded spans intersect the axis. The position of these points is constant for any one arrangement of spans. It depends upon the properties of the beam such as relative span lengths, conditions at the end supports and relative moments of inertia. They are independent of the loading. Fig. 74, p. 153, shows fixed points of a continuous beam with free ends consisting of n spans.

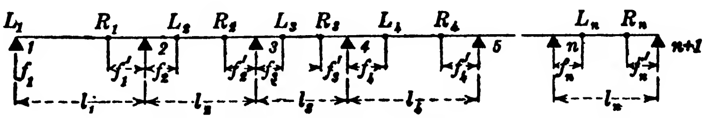


FIG. 74.—Fixed Points in Continuous Beam. (*See p.* 153.)

Each span has two fixed points, one at each end. The left points, marked L in Fig. 74, are used when the loaded span is to the right and the right points marked R are used when the loaded span is to the left of the span under consideration.

[7] In German, "Feste Punkte."

In left end spans with free ends the left fixed point coincides with the end support. The same applies to the right fixed point in a right end span.

In an end span with fixed end the distance of the last fixed point from the end is equal to one-third of the span length.

In an end span with partly restrained ends the last fixed point is located somewhere between the restrained end and the third of the span, depending upon the character of restraint.

Assumptions.—In the discussion and formulas below it is assumed that the moments of inertia of the beam are constant and that the beams rest on unyielding supports. It is assumed also that there is

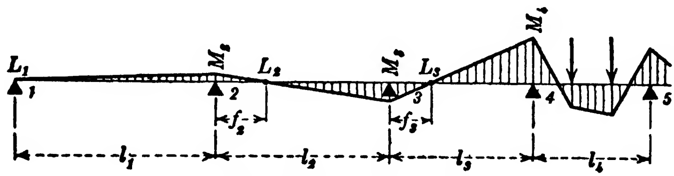


FIG. 75.—Use of Fixed Points. (*See p.* 154.)

no rigid connection between the beam and the support so that the beam can rotate freely at the supports. Formulas for variable moments of inertia are given on p. 164.

Use of Fixed Points.—Assume that, in a continuous beam, consisting of a number of spans, one span is loaded and the other spans are not loaded. When the bending moments at the supports in the loaded span are known, the bending moments in other spans may be found graphically, using the fixed points in the manner shown in Fig. 75, p. 154. In this case the fourth span of a continuous beam consisting of a number of spans is loaded and the other spans are not loaded. The bending moment diagram of the loaded span is drawn first in a manner explained later. The bending moment M_4 is plotted above the axis at the support 4. The apex is connected with the left fixed point L_3 of the third span and the resulting line extended to intersection with a vertical erected at support 3. The point of intersection determines the magnitude of the bending moment M_3 at the support 3. By connecting this point of intersection with the left fixed point L_2 in the second span and extending the line to intersection with a vertical at support 2 the bending moment M_2 is obtained. Finally the bending moment diagram to the left of the loaded span is completed by connecting the apex M_2 with L_1 of the first span, which coincides with the support 1. The bending moment diagram on the right of the loaded span is obtained in the same manner as described for the left half of the diagram by starting at support 5 with bending moment M_5 and using the right fixed points R_6, R_7 . . . . This part of the diagram is not shown in Fig. 75, p. 154. See Figs. 3 and 4, p. 7, for complete diagrams.

Formulas for Location of Fixed Points.—The location of the fixed points may be found by means of the formulas below.

Let $l_1, l_2, l_3 \dots l_n$ = span length of span 1, 2, 3 and n;
$f_1, f_2, f_3 \dots$ = distance of left fixed points $L_1, L_2, L_3, \dots$ from left supports;
$f'_1, f'_2, f'_3 \dots$ = distance of right fixed points $R_1, R_2, R_3 \dots$ from right supports;
f = distance of left fixed point in general;
f' = distance of right fixed point in general.

Formulas for Beams with Free Ends. Constant Moments of Inertia.

Distance of Left Fixed Point L_1 from Support 1, First Span,

$$f_1 = 0. \qquad (502)$$

Distance of Fixed Point L_2 from Support 2, Second Span,

$$f_2 = \frac{1}{2\frac{l_1}{l_2} + 3} l_2 = F_1 l_2. \qquad (503)$$

Distance of Point L_3 from Support 3, Third Span,

$$f_3 = \frac{1}{\frac{l_2}{l_3}\left(2 - \frac{1}{\frac{l_2}{f_2} - 1}\right) + 3} l_3 = F_2 l_3. \qquad (504)$$

Distance of Point L_n from Support n, nth Span,

$$f_n = \frac{1}{\frac{l_{n-1}}{l_n}\left(2 - \frac{1}{\frac{l_{n-1}}{f_{n-1}} - 1}\right) + 3} l_n = F_2 l_n. \qquad (505)$$

Values of F_1 and F_2 may be taken from table, p. 157, for known span ratios and for the computed ratio of the span length to the distance of fixed point for the adjoining span.

Formulas for Beam with Fixed Ends. Constant Moments of Inertia.

Distance of Left Fixed Point L_1 from Support 1, First Span,

$$f_1 = \tfrac{1}{3}l_1. \quad (506)$$

Distance of Left Fixed Point L_2 from Support 2, Second Span,

$$f_2 = \frac{1}{\frac{l_1}{l_2}\left(2 - \frac{1}{\frac{l_1}{f_1} - 1}\right) + 3} l_2 = F_2 l_2. \quad (507)$$

Distance of Point L_n from Support n, nth Span,

$$f_n = \frac{1}{\frac{l_{n-1}}{l_n}\left(2 - \frac{1}{\frac{l_{n-1}}{f_{n-1}} - 1}\right) + 3} l_n = F_2 l_n. \quad (508)$$

Values of F_1 and F_2 may be taken from table on p. 157 for known span ratios and ratios of span to fixed point of the adjoining span to the left.

The distance f_2 for free ends may be found directly. The other distances can be found only in succession by using the value found for one span in the computation of the value for the next span.

Partially Restrained End.—For partially restrained ends the distance f_1 is between 0 and $\tfrac{1}{3}l$, depending upon the degree of restraint. Formulas for other spans are the same as for fixed ends.

Right Fixed Points.—The right fixed points may be found in the same manner as the left fixed points, starting at the right end.

If n is the number of the span counting from the left, then the general equation for the distance of the right fixed point in that span, measured from the right support is

General Formula, Distance of Right Fixed Point R_n in nth Span,

$$f'_n = \frac{1}{\frac{l_{n+1}}{l_n}\left(2 - \frac{1}{\frac{l_{n+1}}{f'_{n+1}} - 1}\right) + 3} l_n. \quad (509)$$

For the fifth span, for instance, $n = 5$, $n + 1 = 6$ and the formula becomes

$$f'_5 = \frac{1}{\frac{l_6}{l_5}\left(2 - \frac{1}{\frac{l_6}{f'_6} - 1}\right) + 3} l_5 \quad (510)$$

The fixed points should be computed first for the last span, counting from the right. If r is the last span, then

Distance of Right Fixed Point for Last Span,

$f'_r = 0$ for free ends, ... (511)

$f'_r = \frac{1}{3}l_r$ for fixed ends, ... (512)

$f'_r =$ between 0 and $\frac{1}{3}l_r$ for partly restrained end. . . (513)

For the next to the last span use the general formula after substituting for n the number of the span. The ratio $\frac{l_{n+1}}{f'_{n+1}}$ refers to the last span and is either 0, 3 or some intermediate value depending upon end conditions.

Values of F_1 in Formula (503), p. 155

Values of $\frac{l_1}{l_2}$														
0.2	0.4	0.6	0 8	1.0	1 2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
0.294	0.263	0.238	0.218	0.200	0.185	0.172	0.161	0.152	0.143	0 135	0 128	0.122	0.116	0 111

These values can be used only when the end of the first span is free.

To find constant for computing right fixed point in the span next to the last use instead of $\frac{l_1}{l_2}$ the ratio of length of the last span to that of the span next to the last.

Values of F_2 in Formulas (504) to (508), pp. 155 and 156

$\frac{l_{n-1}}{l_n}$	Values of $\frac{l_{n-1}}{f_{n-1}}$						
	8	7	6	5	4	3.5	3
0.2	0 297	0.296	0.298	0.299	0 300	0.301	0.303
0.4	0.267	0.268	0.269	0 270	0.272	0.275	0.278
0.6	0.243	0.244	0.245	0 247	0.250	0 252	0 256
0.8	0.223	0 224	0 226	0.227	0.231	0 234	0.238
1.0	0.205	0 207	0.208	0.210	0.214	0 217	0.222
1.2	0.191	0 192	0 194	0.196	0 200	0 204	0.208
1.4	0.180	0.179	0.182	0.184	0.188	0 191	0 196
1.6	0.168	0.169	0.170	0 173	0.177	0.180	0 185
1.8	0 158	0.159	0.160	0 163	0.167	0.170	0 176
2.0	0.149	0.150	0.152	0.154	0.158	0.161	0.167
2.2	0.141	0.142	0.144	0.146	0.150	0.153	0 159
2.4	0.134	0.135	0.137	0.139	0.143	0.146	0.152
2.6	0.128	0.129	0.130	0.132	0.136	0.139	0.145
2.8	0.122	0.123	0.125	0.127	0.130	0.134	0.139
3.0	0.116	0.118	0.119	0.121	0.125	0.128	0.133

This table can be used for computing distance of right fixed points using $\frac{l_n}{l_{n+1}}$ instead of $\frac{l_{n-1}}{l_n}$ and $\frac{l_{n+1}}{f'_{n+1}}$ instead of $\frac{l_{n-1}}{f_{n-1}}$.

Fixed Points for Beams of Equal Spans.—Fig. 76, p. 159, gives the location of fixed points for beams of three, four and five equal spans with free ends.

Derivation of Formulas.—In Fig. 77, p. 160, are represented bending moments produced in the first three spans of a series of spans of a continuous beam, by loads placed in a span to the right of these spans. The spans are not loaded.

From the three-moment equation applied as explained on page 17, following equations are obtained:

$$2M_2(l_1 + l_2) + M_3 l_2 = 0,$$

from which

$$\frac{-M_3}{M_2} = \frac{2(l_1 + l_2)}{l_2} = 2\left(\frac{l_1}{l_2} + 1\right), \quad \text{(514)}$$

$$M_2 l_2 + 2M_3(l_2 + l_3) + M_4 l_3 = 0,$$

from which

$$\frac{M_4}{-M_3} = 2\frac{l_2 + l_3}{l_3} - \frac{l_2}{l_3}\frac{M_2}{(-M_3)}. \quad \text{(515)}$$

From geometry of Fig. 77, p. 160, follows for the second span,

$$\frac{-M_3}{M_2} = \frac{l_2 - f_2}{f_2} = \frac{l_2}{f_2} - 1.$$

Comparing this with equation (514)

$$\frac{l_2}{f_2} - 1 = 2\left(\frac{l_1}{l_2} + 1\right), \quad \text{also} \quad \frac{l_2}{f_2} = 2\frac{l_1}{l_2} + 3,$$

and finally

$$f_2 = \frac{1}{2\frac{l_1}{l_2} + 3} l_2. \quad \text{(516)}$$

For the third span, following geometric relation is apparent

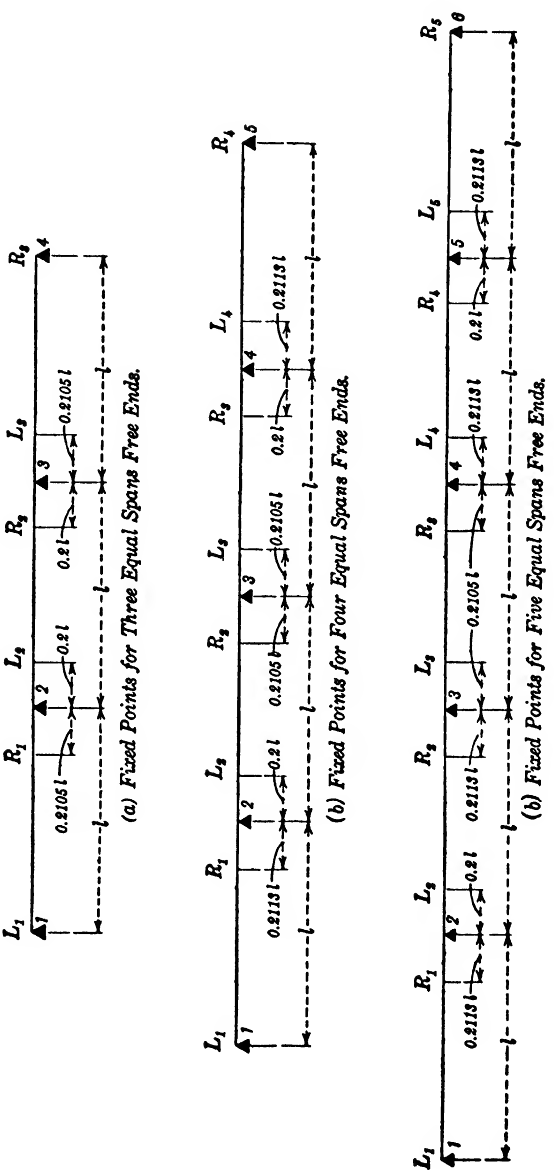
$$\frac{M_4}{(-M_3)} = \frac{l_3 - f_3}{f_3} = \left(\frac{l_3}{f_3} - 1\right). \quad \text{(517)}$$

Compare this with Equation (515) and substitute the value for $\frac{M_2}{-M_3}$,

$$\left(\frac{l_3}{f_3} - 1\right) = 2\left(\frac{l_2 + l_3}{l_3}\right) - \frac{l_2}{l_3}\frac{M_2}{(-M_3)} = 2\left(\frac{l_2}{l_3} + 1\right) - \frac{l_2}{l_3}\frac{1}{\frac{l_2}{f_2} - 1}, \quad \text{(518)}$$

from this

$$\frac{l_3}{f_3} = 2\frac{l_2}{l_3} + 3 - \frac{l_2}{l_3}\frac{1}{\frac{l_2}{f_2} - 1},$$



(a) *Fixed Points for Three Equal Spans Free Ends.*

(b) *Fixed Points for Four Equal Spans Free Ends.*

(b) *Fixed Points for Five Equal Spans Free Ends.*

FIG. 76.—Fixed Points for Equal Spans with Free Ends. (*See p.* 158.)

and finally

$$f_3 = \frac{1}{2\frac{l_2}{l_3} + 3 - \frac{l_2}{l_3}\frac{1}{\frac{l_2}{f_2} - 1}} l_3 = \frac{1}{\frac{l_2}{l_3}\left(2 - \frac{1}{\frac{l_2}{f_2} - 1}\right) + 3} l_3. \quad . \quad . \quad (519)$$

This is a general equation.

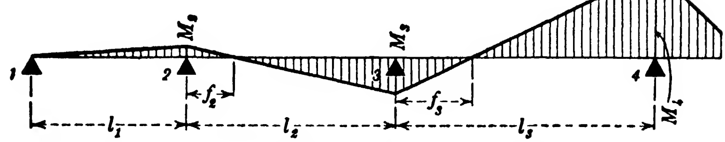


FIG. 77.—Bending Moments in First Three Spans. (*See p.* 158.)

Use of Fixed Points in Loaded Span.—The fixed points may be used to advantage in determining the bending moments at the supports in the loaded spans.

As explained on p. 14 the actual bending moments in a span of a continuous beam may be found by drawing a static bending moment diagram for the loads and plotting at each end of the diagram downward the negative bending moments at the supports. The line connecting the two points at the supports thus obtained is the closing line for the actual bending moments in the span. The values below this line measured on vertical lines represent positive bending moments and above this line negative bending moments.

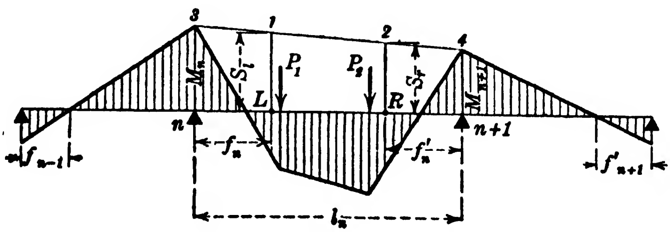


FIG. 78.—Values of S_l and S_r. (*See p.* 161.)

To accomplish this the bending moments at the supports must be found which requires the solution of the three-moment equations.

The work may be simplified by finding, instead of the bending moments at the support, the vertical distances to the closing line at the fixed points.

These distances are given by the following equations.

Distance to Closing Line at Left Fixed Point, Constant Moment of Inertia (see Fig. 78, p. 160),

$$S_l = -fPC_2 \text{ for single load } P. \quad (520)$$

$$S_l = -f\Sigma PC_2 \text{ for number of loads in a span.} \quad (521)$$

$$S_l = -\tfrac{1}{4}fwl \text{ for uniform loading.} \quad (522)$$

Distance to Closing Line at Right Fixed Point, Constant Moment of Inertia (see Fig. 78, p. 160),

$$S_r = -f'PC_1 \text{ for single load } P. \quad (523)$$

$$S_r = -f'\Sigma PC_1 \text{ for number of concentrated loads.} \quad (524)$$

$$S_r = -\tfrac{1}{4}f'wl \text{ for uniform loading,} \quad (525)$$

in which f and f' are distances of the fixed points in the span under consideration and C_1 and C_2 are constants from Diagram 1, p. 19, depending upon the position of the load.

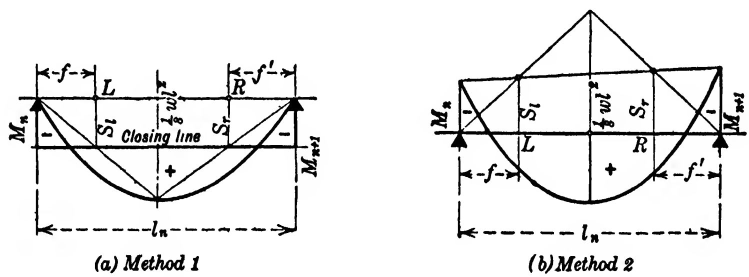


FIG. 79.—Values of S_l and S_r for Uniformly Distributed Loading. (*See p.* 162.)

The solution of the problem is as follows (see Fig. 78).

Find the distances of the fixed points from supports f and f'.

Find for the loaded span the values of S_l and S_r from Formulas (520) to (525), using the distances f and f'.

At the fixed points erect verticals and plot on them the values of S_l and S_r thus obtaining points 1 and 2. In the line $L1$ represents S_l and $R2$ S_r. Connect points 1 and 2 and extend the line on both sides to the supports. Using this line as a closing line draw the static bending moment diagram. All the bending moments in the loaded span are now known.

After the bending moments at the supports of the loaded span are known, the bending moments in the other spans are obtained graphically by using the fixed points.

It should be noted that this method gives the bending moments at the support for the condition of loading where only the span in question is loaded and all other spans are not loaded. If several spans are loaded, the bending moments at the support may be found by considering successively each span as loaded separately and drawing for each case the bending moment curves in all spans. The final bending moments at the supports are then obtained by adding the bending moments produced there by the loads in various spans.

Distance to Closing Line for Uniformly Distributed Load.—When the bending moment diagram for uniformly distributed loading is drawn, the values of S_l and S_r may be easily found graphically by connecting the points on the diagram at each support with the points at the center. The intersections of these two lines with the verticals at the fixed points give the values of S_l and S_r, respectively, in the same scale as used for drawing the bending moments. This is shown in Fig. 79, p. 161, as method 1. When it is desired to get the negative bending moments above the axis, method 2 may be used. This consists of erecting a vertical above the axis in the center of the span, scaling upon it the maximum static bending moment and connecting the apex with the supports. The intersection of these lines with verticals at L and R give S_l and S_r. These points connected form a closing line for the bending moment diagram.

Use of Fixed Points for Drawing Influence Lines.—Fixed points may be used to advantage for preparing influence lines for any desired section by drawing bending moment curves for unit loads placed at different sections throughout the beam. The bending moments at the desired section due to the unit loads are scaled and plotted on verticals passing through the corresponding point of loading.

The method is best illustrated by the example below.

Example.—Determine influence lines for bending moments at the center of the second span for a continuous beam of three spans of different lengths. $l_1 = 30$ ft., $l_2 = 39$ ft. and $l_3 = 24$ ft.

Solution.—First, find the fixed points using Equations (502) and (504), p. 155.

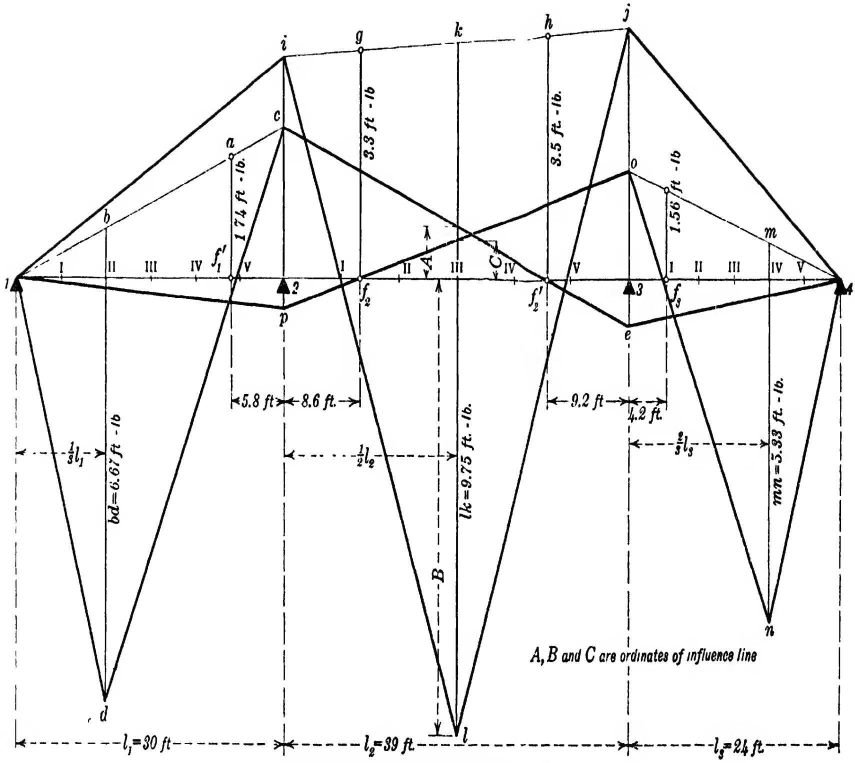
Left fixed points L

$$f_1 = 0.$$

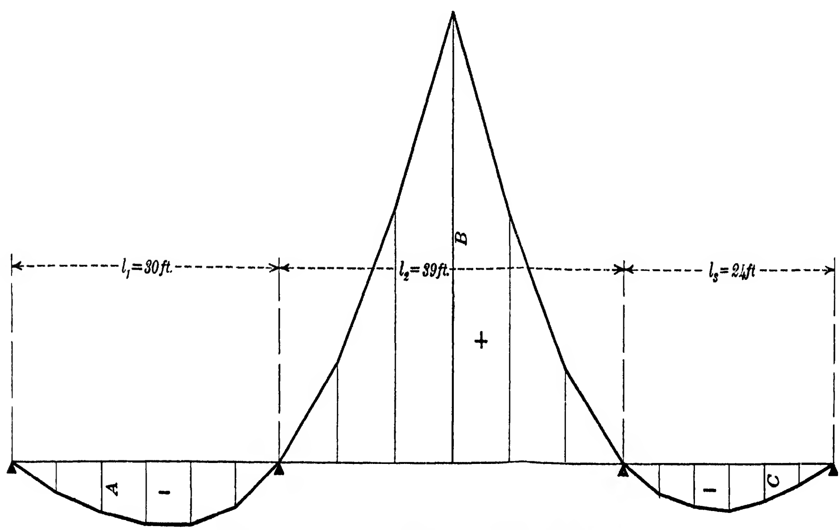
$$f_2 = \frac{1}{2 \times \frac{39}{30} + 3} \times 39 = \frac{1}{1.54 + 3} \times 39 = 8.6 \text{ ft.}$$

$$\frac{l_2}{f_2} - 1 = 4.54 - 1 = 3.54, \quad \frac{1}{\frac{l_2}{f_2} - 1} = 0.28.$$

$$f_3 = \frac{1}{\frac{39}{24}(2 - 0.28) + 3} \times 24 = \frac{1}{\frac{39}{24} \times 1.72 + 3} \times 24 = \frac{1}{5.8} \times 24 = 4.2 \text{ ft.}$$



Method of Constructing Influence Lines



Influence Line for Bending Moments at Center of Second Span

FIG. 80.—Method of Drawing Influence Lines Using Fixed Points. (*See p.* 164.) (*To face p. 162*)

Right fixed points R

$$f'_3 = 0.$$

$$f'_2 = \frac{1}{2 \times \frac{24}{39} + 3} \times 39 = \frac{1}{4.23} \times 39 = 9.2 \text{ ft.}$$

$$\frac{l_2}{f'_2} - 1 = 4.23 - 1 = 3.23, \quad \frac{1}{\frac{l_2}{f'_2} - 1} = \frac{1}{3.23} = 0.31$$

$$f'_1 = \frac{1}{\frac{39}{30}(2 - 0.31) + 3} \times 30 = \frac{1}{5.2} \times 30 = 5.8 \text{ ft.}$$

Divide each span into any desired number of sections. Find for each position of load values of S_l and S_r and plot them at the fixed points above the basis. Connect the points and extend the line till intersection with verticals at support. Draw for each position of load a static bending moment diagram due to a load $P = 1$. This is a triangle with apex under the load and an altitude there equal to $\frac{a}{l}\left(1 - \frac{a}{l}\right)l$, and measured from the line previously obtained.

In this example each span was divided into six parts. However, bending moments at only one point in each span are shown so as not to confuse the drawing. The points selected arbitrarily are in the first span $\frac{a}{l_1} = \frac{1}{3}$, in the second span $\frac{a}{l_2} = \frac{1}{2}$ and in the third span $\frac{a}{l_3} = \frac{2}{3}$.

The static bending moments are,

First span for $\frac{a}{l} = \frac{1}{3}$, $\frac{a}{l}\left(1 - \frac{a}{l}\right)l = \frac{1}{3} \times \frac{2}{3} \times 30 = 6.67$ ft.-lb.

Second span for $\frac{a}{l} = \frac{1}{2}$, $\frac{a}{l}\left(1 - \frac{a}{l}\right)l = \frac{1}{2} \times \frac{1}{2} \times 39 = 9.75$ ft.-lb.

Third span for $\frac{a}{l} = \frac{2}{3}$, $\frac{a}{l}\left(1 - \frac{a}{l}\right)l = \frac{2}{3} \times \frac{1}{3} \times 24 = 5.33$ ft.-lb.

For the selected points the values of S_l and S_r for $P = 1$ are as follows:

First span, $f_1 = 0$, $f'_1 = 5.8$ ft.

$$\frac{a}{l_1} = \frac{1}{3}, C_1 = 0.30, S_l = 0. \quad S_r = -C_1 f'_1 = -0.30 \times 5.8 = -1.74 \text{ ft.-lb.}$$

Second span, $f_2 = 8.6$ ft., $f'_2 = 9.2$ ft.

$$\frac{a}{l_2} = \frac{1}{2}, C_2 = 0.375, S_l = -C_2 f_2 = -0.375 \times 8.6 = -3.3 \text{ ft.-lb.}$$

$$C_1 = 0.375, S_r = -C_1 f'_2 = -0.375 \times 9.2 = -3.5 \text{ ft.-lb.}$$

Third span, $f_3 = 4.2, f_3 = 0.$

$$\frac{a}{l_3} = \frac{2}{3}, C_2 = 0.37. \quad S_l = -C_2 f_3 = 0.37 \times 4.2 = 1.56 \text{ ft.-lb.}$$

$$C_1 = 0.30. \quad S_r = -C_1 \times 0. = 0.$$

For these values bending moment diagrams are drawn in loaded spans as in Fig. 80, opposite p. 162.

Finally, draw the bending moment lines in the unloaded spans. In the end spans only one value, either S_l or S_r, is required.

The bending moment diagram for the point II in the first span is drawn as follows: At point f'_1 plot $S_r = -1.74$ ft.-lb. and get point *a*. Connect point *a* with support *1* and get point *b* above point II, and *c* above support *2*. Plot bd = 6.67 ft.-lb. which is the static bending moment of load $P = 1$ lb. placed at point II. Connect point *c* with f'_2 in the second span and get point *e* below the support *3*. The polygon 1*dce*41 is the bending moment diagram for a load $P = 1$ placed at point II. The bending moments above the axis 14 are positive, and below the axis negative. At the center of the second span (i.e., the point for which the influence line is drawn) the bending moment is equal to *A*. This value should be plotted on the influence diagram.

The bending moment diagram for the point III in the second span is drawn as follows: at points f_2 plot $S_l = -3.3$ ft.-lb. and get point *g*. Similarly get point *h* at f'_2. Connect *gh* and get points *i*, *j* and *k*. Plot the static bending moment lk = 9.75 ft.-lb. To complete the diagram, connect *i* with *1* and *j* with 4. The bending moment diagram is 1*ilj*41. The value for the influence line is *B* to be plotted in the center of the second span.

The bending moment diagram for point IV of the third span was drawn in the same manner as for the first span. The value for the influence line is *C*.

It will be noted that, to get influence line for any point in the center span, it is not necessary to complete the bending moment diagrams in the other spans, as only the line *ce* of the diagram in the first span and the line *op* from the third span are required. Complete diagrams are required when influence lines for different points in all three spans are drawn.

Draw an axis for the influence line parallel to the span and plot the supports and the division points. Erect verticals at each division point. After the bending moment diagrams are drawn scale for each position of the load the bending moment produced at the center of the second span (i.e., the section for which it is desired to draw an influence line) and plot it on the vertical passing through the position of the load. Minus values are plotted above and plus values below the basis.

This is shown in Fig. 80, opposite p. 162.

FIXED POINTS FOR BEAM WITH VARIABLE MOMENTS OF INERTIA

General rules for fixed points for beams with variable moments of inertia are the same as for beams with constant moments of inertia. Fig. 81, p. 165, shows fixed points for a continuous beam consisting of seven spans with free ends. L_2 to L_7 are left fixed points and R_2 to R_6 are right fixed points. The left fixed point in the first span and the right fixed point in the last span coincide the ends of the beam.

Formulas for the distance of the fixed points are derived in the same manner as for beams with constant moments of inertia but using the three-moment equation for variable moments of inertia. They are:

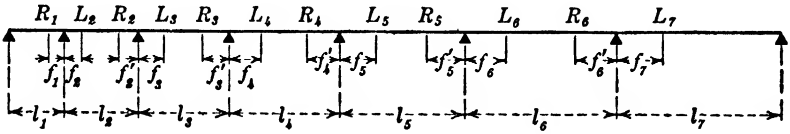


FIG. 81.—Fixed Points for a Continuous Beam with Free Ends. (See p. 165.)

Notation

Let $l_1, l_2, l_3 \ldots l_n$ = span length of continuous beam, numbering from the left;

$I_1, I_2, I_3 \ldots I_n$ = minimum moments of inertia of respective spans;

$\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ = constants from Diagrams 13, p. 140, or 14, p. 141, depending upon shape and length of haunch and ratio of minimum to maximum moments of inertia in each span;

$\beta_1, \beta_2, \beta_3 \ldots \beta_n$ = similar constants from Diagram 13, p. 140, or 14, p. 140;

$f_1, f_2, f_3 \ldots f_n$ = distances of left fixed points $L_1, L_2 \ldots L_n$, from left support;

$f'_1, f'_2, f'_3 \ldots f'_n$ = distances of right fixed points $R_1, R_2 \ldots R_n$ from right support.

General Formulas for Distances of Fixed Points. Haunches Symmetrical.

Left Fixed Points in nth Span, Measured from Left Support,

$$f_n = \frac{\beta_n}{2\alpha_n + \beta_n + \dfrac{l_{n-1}}{l_n}\dfrac{I_n}{I_{n-1}}\left(2\alpha_{n-1} - \dfrac{1}{\dfrac{l_{n-1}}{f_{n-1}} - 1}\beta_{n-1}\right)} l_n. \quad (526)$$

Right Fixed Points in nth Span, Measured from Right Support,

$$f'_n = \frac{\beta_n}{2\alpha_n + \beta_n + \dfrac{l_{n+1}}{l_n}\dfrac{I_n}{I_{n+1}}\left(2\alpha_{n+1} - \dfrac{1}{\dfrac{l_{n+1}}{f'_{n+1}} - 1}\beta_{n+1}\right)} l_n. \quad (527)$$

Special Formulas.—When the minimum moments of inertia of all spans are equal so that $I_1 = I_2 = I_3 = \ldots I_n$, also the shapes of the haunches, the ratios of length of haunch to length of span and finally the ratios of minimum to maximum moments of inertia are the same for all spans, then

$$\alpha_1 = \alpha_2 = \alpha_3 \ldots \alpha_n = \alpha \quad \text{and} \quad \beta_1 = \beta_2 = \beta_3 \ldots \beta_n = \beta$$

and the formulas change to

Special Formula for Distance of Left Fixed Point in nth Span,

$$f_n = \frac{1}{\left(2\frac{\alpha}{\beta} + 1\right) + \frac{l_{n-1}}{l_n}\left(2\frac{\alpha}{\beta} - \frac{1}{\frac{l_{n-1}}{f_{n-1}} - 1}\right)} l_n. \quad . \quad . \quad . \quad (528)$$

Special Formula for Distance of Right Fixed Point in nth Span,

$$f'_n = \frac{1}{\left(2\frac{\alpha}{\beta} + 1\right) + \frac{l_{n+1}}{l_n}\left(2\frac{\alpha}{\beta} - \frac{1}{\frac{l_{n+1}}{f_{n+1}} - 1}\right)} l_n. \quad . \quad . \quad . \quad (529)$$

How to Use Formulas for Fixed Points.—After the dimensions of the beam are decided upon and the shape of the haunches selected, compute the minimum and maximum moments of inertia in each span. Compute the ratios of length of haunches to length of spans. Find for each span values of α and β from the proper diagram. For straight haunch use Diagram 13, p. 140, and for parabolic haunch use Diagram 14, p. 141.

The formulas (526) to (529) are general. If it is desired to get a formula for any particular span substitute for n the number of the span.

Thus for the third span, for example, formula (526) changes to

$$f_3 = \frac{\beta_3}{2\alpha_3 + \beta_3 + \frac{l_2}{l_3}\frac{I_3}{I_2}\left(2\alpha_2 - \frac{\beta_2}{\frac{l_2}{f_2} - 1}\right)} l_3. \quad . \quad . \quad . \quad (530)$$

In this formula all values are known except $\frac{l_2}{f_2}$ which must be found first before f_3 can be computed.

When finding the left fixed points start in the first span from the left. The fixed point there is

$$f_1 = 0 \text{ for free end,}$$

$$f_1 = \tfrac{1}{3}l_1 \text{ for fixed end,}$$

$$f_1 = \text{between } 0 \text{ and } \tfrac{1}{3}l_1 \text{ for partly restrained end.}$$

Next find the fixed point in the second span f_2, making in the proper formula $n = 2$ so that α_n and β_n becomes α_2 and β_2, α_{n-1} and β_{n-1}, α_1 and β_1; l_n becomes l_2 and l_{n-1} changes to l_1. The ratio $\frac{l_{n-1}}{f_{n-1}} = \frac{l_1}{f_1}$ is either 0 for free end or $\frac{1}{3}$ for fixed end.

After f_2 is found compute $\frac{l_2}{f_2}$ and substitute it in the previously given formula for f_3. Proceed thus until the fixed points for all spans are found.

In computing right fixed points start from the right end and proceed in the same manner. In a beam with six spans, for instance, start with the sixth span for which the right fixed points are either zero or $\frac{1}{3}l_6$, depending upon the end condition. Next compute the fixed points consecutively for the fifth span, fourth, span, third span, etc.

Distance at Fixed Points to Closing Line, Loaded Span.—The distance at fixed points to closing line of the bending moment diagram in loaded span is explained on p. 160. The formula for the distance for beams with variable moments of inertia is

Distance to Closing Line at Left Fixed Point. Variable Moment of Inertia,

$S_l = -fP\beta C_{v2}$ for single load P. . . . . . . (531)

$S_l = -f\Sigma P\beta C_{v2}$ for number of loads in a span. . (532)

$S_l = -\frac{1}{4}fwl$ for uniform loading. . . . . (533)

Distance to Closing Line at Right Fixed Point. Variable Moment of Inertia,

$S_r = -f'P\beta C_{v1}$ for single load P. . . . . . (534)

$S_r = -f'\Sigma P\beta C_{v1}$ for number of loads in a span. . (535)

$S_r = -\frac{1}{4}f'wl$ for uniform loading. . . . . (536)

Values of C_{v1} and C_{v2} correspond to C_1 and C_2 in beams with constant moment of inertia. They may be taken from Diagrams 15 and 16, pp. 144 and 145, for proper $\frac{I}{I_1}$, m and $\frac{a}{l}$ and proper design of haunch.

Use of these values is described on p. 161 in connection with beams with constant moments of inertia.

Influence Line for Beams with Variable Moments of Inertia.—Influence lines for beams with variable moments of inertia may be found in the same manner as for beams with constant moments of inertia. The fixed points, however, and the values of S_l and S_r as determined for beams with variable moments of inertia should be used.

INFLUENCE LINES FOR CONTINUOUS BEAMS

When the loading consists of moving loads, as is the case in bridge design, it is necessary to determine the most unfavorable position of loading before the bending moments and shears at the selected section of the beam can be computed. For uniformly distributed loads it is sufficient to know which spans need to be loaded to get maximum results. For concentrated loads it is not sufficient to load some specified spans but, in addition, the largest concentrated loads must be placed in positions producing at the selected section the maximum bending moments or shears.

If it is desired to get exact results the problem can be solved in the easiest manner by means of influence lines.

Definition of Influence Lines.—It has been explained in previous paragraphs that a load placed anywhere on a continuous beam produces bending moments and shears at every cross-section of the beam (see p. 6). To get bending moments and shears at any selected section, it is necessary to find the effect on that section of loads placed in different positions throughout the beam. This can be done by means of the so-called influence lines.

Influence lines for any selected section, therefore, are lines giving the effect upon that section of unit loads placed anywhere on the beam. Since the selected section may be subjected to bending moments and shears, influence lines may be prepared for bending moments and for shears.

In Fig. 83, p. 172 are shown influence lines for bending moments at various points of a beam consisting of three spans. The influence line for any point is partly above and partly below the axis. The loads placed in the part of the beam where the influence line is above the axis produce positive bending moments at the point under consideration. The loads placed where the influence line is below the axis produce negative bending moments. To get maximum positive or negative bending moments the positive or negative sections only respectively should be loaded.

Method of Preparing Influence Lines.—Influence lines for bending moments at a selected section are prepared as follows:

The beam is laid out to any desired scale. Each span is divided into a desired number of divisions, say, ten divisions per span. Vertical lines are erected at each division point. Bending moments produced at the selected section by a unit load placed successively at every division point are then computed. Each computed value is plotted upon the vertical passing through the division at which the load was assumed to act. Negative values are plotted above the axis and positive values below the axis.

The method is illustrated in the following example:

It is desired to get an influence line for bending moments at the center of the center span of a continuous beam of three spans with free ends.

On p. 54 are given formulas for bending moments at any point x in the center span due to loads placed at a distance a from support in the first span and center span, respectively. These are

Bending Moment at Any Point x in the Center Span Due to Load in First Span,

$$M_x = \left(-1 + \frac{5}{4}\frac{x}{l}\right)\frac{4}{15}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\right]Pl. \quad . \quad . \quad . \quad . \quad . \quad (537)$$

Bending Moment at Any Point x in Center Span Due to Load in Center Span,

$$M_x = \left(F_4\frac{x}{l} - F_5\right)Pl \text{ for } x \text{ smaller than } a. \quad . \quad . \quad . \quad . \quad (538)$$

$$M_x = \left[(F_4 - 1)\frac{x}{l} - F_5 + \frac{a}{l}\right]Pl \text{ for } x \text{ larger than } a. \quad . \quad (539)$$

These formulas may be used for drawing influence lines. Since it is desired to get bending moments at the center, substitute in the formulas just given $x = \frac{1}{2}l$. The load is made $P = 1$.

Since in an influence line the position of the load is variable (instead of being fixed as in Formulas (537) to (539)) the constant values of $\frac{a}{l}$ are replaced by variable $\frac{x}{l}$.

Let y = ordinate of the influence line.

Then the formula for bending moments at the center of the center span due to a load $P = 1$ placed anywhere in the beam may be designated by

$$M_{\frac{1}{2}l} = yl.$$

The formula for influence line thus becomes

Formula for Influence Line for Bending Moment at Center of First Span:

First Span,

$$y = \left(-1 + \tfrac{5}{4} \times \tfrac{1}{2}\right)\frac{4}{15}\frac{x}{l}\left[1 - \left(\frac{x}{l}\right)^2\right] = -\frac{1}{10}\frac{x}{l}\left[1 - \left(\frac{x}{l}\right)^2\right].$$

Second span,

$$y = (\tfrac{1}{2}F_4 - F_5) \qquad \text{for } \frac{x}{l} \text{ smaller than } \tfrac{1}{2},$$

$$y = \tfrac{1}{2}(F_4 - 1) - F_5 + \frac{x}{l} \quad \text{for } \frac{x}{l} \text{ larger than } \tfrac{1}{2}.$$

In the same manner influence lines may be prepared for end shears.

Influence Lines for Equal Spans.—Figs. 82 and 83 give influence lines for two and three equal spans. In each case lines are given for bending moments at supports, in the center and at several intermediate points and for end shears.

Use of Influence Lines.—When the influence lines are drawn the bending moments or shear for any type of loading may be found by multiplying the loads by the corresponding ordinate in the influence line and adding the result.

Usually it is sufficient to find by means of influence lines bending moments at the center and at the quarter points. The line of positive bending moments is obtained by plotting the positive values obtained for the three points from influence lines on vertical lines and drawing through them and the supports a curve resembling a parabola. The curve thus obtained represents approximately the positive bending moments due to most unfavorable loading of the span. The positive bending moments near the support are obtained by partial loading of the span. Full loading would have produced negative bending moments at those points.

In the same manner curves are drawn for negative bending moments.

Use of Fixed Points for Influence Lines.—For beams consisting of unequal spans it is easiest to prepare influence lines using the fixed points as explained on p. 162.

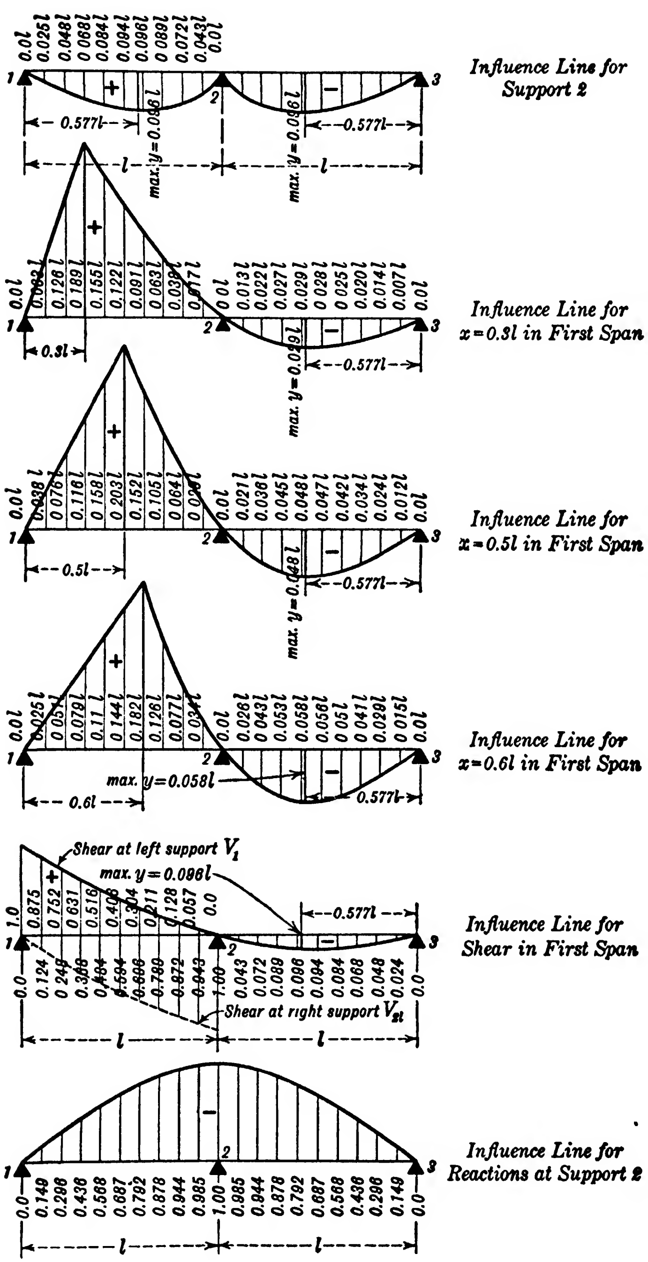


FIG. 82.—Influence Lines for Two Equal Spans. **Free Ends.** (*See p.* 170.)

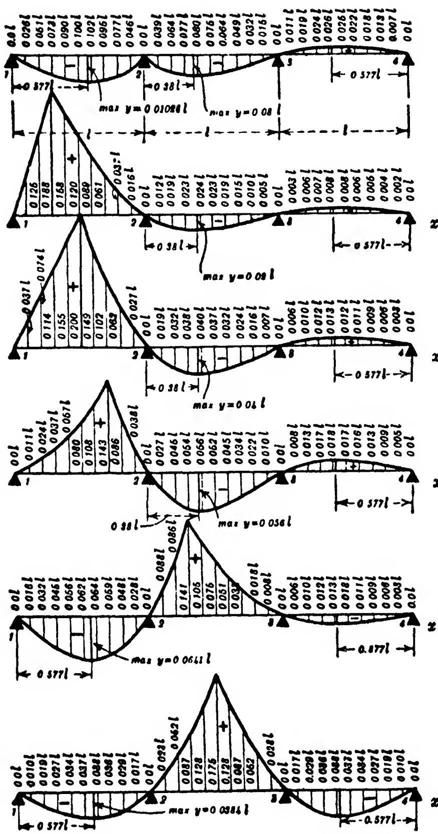


FIG. 83.—Influence Lines for Three Equal Spans. Free Ends. (*See p.* 170.)

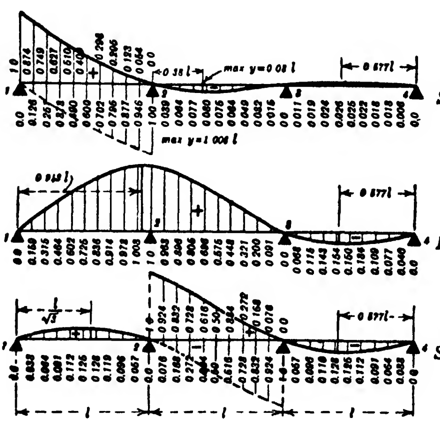


FIG. 84.—Influence Lines for Shears for Three Equal Spans. Free Ends. (*See p.* 170.)

USE OF FORMULAS FOR CONTINUOUS BEAM IN DESIGN

The use of the formulas given in this chapter is illustrated by five examples given on pp. 178 to 207.

In general the object is to determine the absolute maximum bending moments at the critical sections, namely, at the supports and in the middle portion of the spans. These determine the dimensions of the beam unless larger dimensions are required by shear and the required amount of longitudinal reinforcement. Next it is required to determine the absolute maximum bending moments at the intermediate points by drawing the maximum bending moment diagrams. These are useful in determining the points of bending up or down of the longitudinal reinforcement.

Also it is usually necessary to compute the absolute maximum shears at the supports and to draw the absolute maximum shear diagrams. The maximum shear, being a measure of diagonal tension, may determine the size of the section. The shear diagram may be used in determining the diagonal tension reinforcement.

Absolute Maximum Bending Moments or Shears.—Absolute maximum bending moments or shears are the largest values that can be produced at any point by the dead load and the most unfavorable position of the live load.

Bending moments and shears should be computed separately for the dead load and for the most unfavorable positions of the live load.

For the dead load all spans must be considered as loaded simultaneously.

For live load only such spans should be considered as loaded which produce at the section under consideration bending moments or shears of the proper sign. Thus, if it is desired to find the maximum negative bending moment at support 2, the live load should be placed only in positions giving negative bending moments at support 2.

The bending moments and shears to be used in designing the beam are obtained by adding the values due to the dead load to the absolute maximum values due to the live load. The above rule applies in all cases where the bending moments or shears due to the dead load are of the same sign as those for the live load.

If, however, the bending moment due to dead load is of opposite sign to the absolute maximum value due to the live load, so that the dead load balances wholly or partially the effect of the live load, this rule must be modified. In such case the bending moment or shear to be used in design is equal to the maximum value due to the live load plus the value due to the dead load divided by a factor of safety.

Absolute maximum negative bending moments at the supports are always obtained directly from the formulas or tables as just described.

Maximum positive bending moments can be obtained directly by adding the maximum positive bending moments for the dead load to the absolute maximum positive bending moments for the live load only when the location of the points of maximum bending moments in both cases are the same. Where the location of the points of maximum values are different, the maximum positive bending moments are obtained as follows:

Compute in the considered span the negative bending moments at the supports due to dead load.

Compute also in the same span the negative bending moments at the supports due to the position of live load producing absolute maximum positive value.

Add the bending moments at the supports and plot them above a selected axis on verticals erected at the supports.

Connect the points thus obtained and considering this line as a closing line draw a static bending moment diagram for the dead load plus the live load. The part of the diagram below the axis gives the positive bending moments. The maximum value may be found by scaling.

For uniformly distributed loading the maximum positive bending moment for known negative bending moments at the supports may be found by using Table on p. 176. The method is explained on p. 22. The location of the point of maximum positive bending moment also may be found using Table on p. 177.

The absolute maximum positive and negative bending moments are sufficient to determine the required amounts of longitudinal reinforcement at the supports and in the central portion of the beam. The points at which the amount of longitudinal reinforcement may be reduced by bending up some of the bars may be obtained from the absolute maximum bending moment diagram which gives the maximum possible bending moments at all points of the beam. The method of preparing the diagrams is described in the succeeding paragraphs under proper heading. When several beams of similar character are designed it may be sufficient to draw the maximum moment diagram only once and accept the same location of points of bending of bars for all beams. Also the diagrams given in this volume in connection with tables for equal spans may be used as a guide for determining the points at which the reinforcement may be reduced by bending up (or down). In such manner the necessity of drawing the maximum bending moment diagram may be obviated.

In case of unequal spans it is always advisable to draw the maximum bending moment diagrams.

To design the diagonal tension reinforcement it is necessary not only to compute the maximum end shears but also to draw the maximum shear diagrams which give the maximum shears at the intermediate points.

Diagrams of Absolute Maximum Bending Moments.—The diagrams for absolute maximum bending moments are a combination of the bending moment diagram for the dead load with the bending moment diagrams for such positions of live load as give the absolute maximum values at all points in a beam. The curves giving absolute maximum bending moments are therefore composite curves parts of which are due to one type of loading, while other parts may be due to some other type of loading. The curves for maximum negative bending moments always overlap the curves for maximum positive bending moments.

The simplest method of drawing the absolute bending moment diagrams is to draw separate diagrams for the dead load and to the same scale diagrams for each of the critical positions of the live load. The diagrams are drawn as shown in Fig. 85, p. 180, and Fig. 93, p. 198. The ordinates representing the bending moments due to the dead load are added to those giving maximum values for the live load and are plotted. The resulting diagrams are shown at the bottom of Figs. 85 and 93. In this connection it should be remembered that where the dead load bending moment is of opposite sign to the live load bending moment, the dead load ordinates should be divided by a factor of safety before deducting them from the live load ordinates.

The work in connection with drawing the absolute maximum diagrams may be reduced by using the method described and illustrated in the example 2, p. 185.

Absolute Maximum Shear Diagrams.—The method of preparing maximum shear diagrams is clearly shown in examples on pp. 182, 187, 191, 199 and 207. The diagrams are also clearly shown.

Table.—Maximum Positive Bending Moment in Continuous Beam For Known Bending Moments at Support

Constant C_M

in $M_{max} = C_M wl^2$

$\frac{M_r}{wl^2}$	Values of $M_l \div wl^2$								
	0	0 01	0 02	0 03	0.04	0 05	0.06	0 07	0 08
0.00	0.125	0.120	0.115	0 110	0.106	0 101	0.097	0.092	0.088
0.01	0.120	0.115	0 110	0.105	0.100	0.096	0.091	0.087	0.082
0.02	0.115	0.110	0.105	0.100	0.095	0.090	0.086	0.081	0.077
0.03	0.110	0.105	0 100	0.095	0 090	0 085	0.080	0.076	0.071
0.04	0.106	0 100	0.095	0.090	0.085	0.080	0.075	0.070	0.066
0.05	0.101	0.096	0.090	0 085	0.080	0 075	0.070	0.065	0.060
0.06	0.097	0.091	0 086	0.080	0.075	0 070	0.065	0.060	0.055
0.07	0.092	0.087	0 081	0.076	0.070	0.065	0.060	0.055	0.050
0.08	0.088	0.082	0.077	0.071	0.066	0 060	0.055	0.050	0.045
0.0833	0.087	0.081	0 075	0 070	0.064	0.059	0.054	0 048	0.043
0.09	0.084	0.078	0.072	0 067	0.061	0 056	0.050	0.045	0.040
0.10	0.080	0.074	0 068	0 062	0 057	0 051	0.046	0.040	0 035
0.11	0 076	0 070	0.064	0 058	0.052	0.047	0 041	0.036	0.030
0.12	0 072	0.066	0.060	0 054	0 048	0.042	0.037	0.031	0.026
0.125	0.070	0.064	0.058	0 052	0.046	0 040	0 035	0.029	0.023
0.13	0.068	0 062	0.056	0.050	0 044	0.038	0 032	0.027	0.021
0.14	0 065	0 058	0 052	0 046	0.040	0 034	0.028	0.022	0.017
0.15	0 061	0 055	0 048	0 042	0 036	0 030	0 024	0 018	0 012
	0.0833	0.09	0 10	0.11	0.12	0 125	0.13	0.14	0.15
0.00	0.087	0 084	0 080	0 076	0.072	0.070	0.068	0.065	0.061
0 01	0 081	0 078	0.074	0 070	0 066	0 064	0.062	0.058	0.055
0.02	0.075	0 072	0.068	0.064	0 060	0 058	0 056	0.052	0.048
0 03	0.070	0 067	0 062	0 058	0 054	0.052	0.050	0 046	0.042
0.04	0.064	0 061	0.057	0 052	0 049	0 046	0 044	0.040	0 036
0 05	0.059	0 056	0 051	0 047	0 042	0 040	0.038	0.034	0.030
0 06	0.054	0 050	0 046	0 041	0.037	0 035	0.032	0.028	0.024
0.07	0.048	0.045	0.040	0.036	0.031	0.029	0.027	0.022	0.018
0.08	0.043	0 040	0.035	0 030	0 026	0.023	0.021	0.017	0.012
0.0833	0 042	0 038	0.033	0 029	0 024	0 022	0.019	0.015	0.011
0.09	0.038	0.035	0.030	0 025	0 020	0 018	0.016	0.011	0.007
0.10	0.033	0.030	0 025	0.020	0 015	0.017	0 010	0.006	0.001
0.11	0 029	0 025	0.020	0.015	0.010	0.008	0.005	0.000	−0.004
0.12	0.024	0.020	0.015	0.010	0 005	0.002	0.000	−0.005	−0.010
0.125	0 022	0.018	0.013	0 008	0.002	0 000	−0.002	−0.007	−0.013
0.13	0.019	0.016	0.010	0.005	0.000	−0.003	−0.005	−0.010	−0.015
0.14	0.015	0.011	0 006	0.000	−0.005	−0.007	−0.010	−0.015	−0.020
0.15	0.010	0 007	0 001	−0 004	−0 010	−0 012	−0 015	−0.020	−0.025

M_l = Bending moment at left support. M_r = Bending moment at right support. M_{max} = Maximum positive bending moment. w = Unit load. l = Span length.

Table.—Left Reaction in Continuous Beam and Point of Maximum Positive Bending Moment for Known Bending Moments at Support

Constant C_V

in $V_l = C_V wl$ and $x_1 = C_V l$

$\frac{M_r}{wl^2}$	Values of $M_l \div wl^2$								
	0	0 01	0.02	0 03	0.04	0 05	0 06	0.07	0 08
0.00	0 50	0 51	0.52	0 53	0.54	0.55	0.56	0.57	0.58
0.01	0.49	0.50	0.51	0 52	0 53	0 54	0 55	0 56	0.57
0.02	0.48	0.49	0 50	0 51	0.52	0 53	0 54	0.55	0.56
0.03	0 47	0.48	0 49	0 50	0 51	0 52	0 53	0.54	0 55
0.04	0.46	0.47	0 48	0 49	0 50	0 51	0 52	0 53	0.54
0.05	0.45	0.46	0 47	0.48	0 49	0 50	0 51	0.52	0 53
0.06	0.44	0.45	0 46	0 47	0 48	0 49	0 50	0 51	0 52
0 07	0.43	0 44	0.45	0.46	0 47	0.48	0 49	0 50	0.51
0.08	0.42	0.43	0 44	0.45	0 46	0 47	0.48	0 49	0.50
0.0833	0.42	0.43	0.44	0.45	0.46	0 47	0 48	0 49	0 50
0.09	0.41	0.42	0 43	0.44	0 45	0 46	0 47	0.48	0 49
0.10	0.40	0.41	0.42	0 43	0 44	0 45	0.46	0.47	0.48
0 11	0.39	0.40	0.41	0.42	0 43	0 44	0 45	0 46	0.47
0.12	0.38	0.39	0.40	0 41	0.42	0.43	0.44	0 45	0 46
0.125	0.375	0.385	0.395	0.405	0.415	0.425	0 435	0.445	0 455
0.13	0.37	0.38	0.39	0 40	0.41	0.42	0.43	0.44	0 45
0.14	0.36	0.37	0.38	0.39	0 40	0.41	0 42	0 43	0 44
0.15	0 35	0 36	0 37	0 38	0 39	0 40	0 41	0 42	0 43

	0.0833	0 09	0 10	0 11	0 12	0 125	0 13	0 14	0 15
0.00	0.5833	0.59	0.60	0 61	0 62	0.625	0.63	0.64	0 65
0.01	0.5733	0.58	0 59	0.60	0 61	0 615	0.62	0 63	0.64
0.02	0.5633	0.57	0.58	0.59	0 60	0.605	0.61	0.62	0.63
0.03	0.5533	0 56	0.57	0 58	0 59	0.595	0 60	0.61	0.62
0.04	0.5433	0 55	0.56	0.57	0.58	0.585	0.59	0.60	0.61
0.05	0.5333	0.54	0.55	0 56	0 57	0 575	0 58	0.59	0 60
0.06	0.5233	0 53	0.54	0 55	0 56	0.565	0.57	0.58	0 59
0.07	0.5133	0.52	0.53	0.54	0.55	0.555	0.56	0.57	0.58
0.08	0.5033	0 51	0.52	0.53	0 54	0 545	0.55	0.56	0.57
0.0833	0.5000	0.51	0.52	0.53	0 53	0.542	0.55	0.56	0.57
0.09	0.4933	0 50	0 51	0.51	0.53	0.535	0.54	0.55	0.56
0.10	0.4833	0.49	0.50	0 51	0.52	0.525	0.53	0.54	0.55
0.11	0.4733	0.48	0.49	0.50	0.51	0.515	0.52	0.53	0.54
0.12	0.4633	0.47	0.48	0.49	0 50	0.505	0 51	0.52	0 53
0.125	0.4583	0.465	0.475	0.485	0.49	0.500	0.50	0.51	0.52
0.13	0.4533	0.46	0.47	0.48	0.49	0.495	0.50	0.51	0.52
0.14	0.4433	0.45	0 46	0.47	0 48	0.485	0 49	0.50	0.51
0.15	0.4333	0 44	0.45	0 46	0 47	0.475	0.48	0.49	0.50

M_l = Bending moment at left support. M_r = Bending moment at right support. V_l = Left reaction. x_1 = Point of Maximum Positive Bending Moment. w = Unit load. l = Span length.

THREE EQUAL SPANS CONCENTRATED LOADS

Example 1.—Design a continuous girder of three equal spans, supporting longitudinal beams placed at the supports and at the centers of the spans. The span lengths, the general dimensions and the allowable stresses are:

Span of girders, l = 20 ft.
Span of beams, l_1 = 22 ft.
Live load, w = 200 lb. per sq. ft.

Dimensions of slab and beams:

Slab, t = 6 in.
Beam, b = 12 in.
h = 28 in.

Allowable unit stresses:

f_c = 800 lb. per sq. in.
f_c = 900 lb. per sq. in. at supports
f_s = 16 000 lb. per sq. in.
v = 40 lb. per sq. in. in plain concrete
v = 120 lb. per sq. in. with web reinforcement
n = 15

Solution.—The dead and live loads (excepting the weight of the girders) being transmitted to the girders by the beams are, therefore, concentrated. The uniformly distributed weight of the girder which forms only a small proportion of the total load, may be assumed , for the sake of simplicity, to be concentrated at each beam without any appreciable error.

The concentrated loads are:

Concentrated dead loads,

Slab, 75 × 10 × 22,	= 16 500 lb.
Beam (below slab), $\frac{1}{144}$ × 12 × 22 × 150 × 21	= 5 800 lb.
Dead load of girder, considered as concentrated, 300 × 10	= 3 000 lb.
Concentrated dead load at each point,	= 25 300 lb.
Concentrated live load at each point, 10 × 22 × 200	= 44 000 lb.

Bending Moments.—The bending moments may be obtained by multiplying the constants from Table on p. 58, by the concentrated dead load and the concentrated live load, respectively, and by the span. To get the bending moments in inch-pounds use the span length in inches. Thus the constants should be multiplied by

for dead load Pl = 25 300 × 20 × 12 = 6 072 000 in.-lb.
for live load Pl = 44 000 × 20 × 12 = 10 560 000 in -lb.

The results are tabulated in the table below:

	Dead Load		Live Load in					
			1st and 2nd Span		1st and 3rd Span		2nd Span	
	Constant	Actual value	Constant	Actual value	Constant	Actual value	Constant	Actual value
		Inch-kips		Inch-kips		Inch-kips		Inch-kips
M_2	−0.15	− 910.8	−0.175	−1850 0	−0.075	−792 0	−0.075	−792.0
M_3	−0.15	− 910 8	−0 05	− 528 0	−0 075	−792 0	−0.075	−792.0
M_{1max}	0.175	+1062.6	0 163	1710 0	**0.212**	2240.0		
M_{2max}	0 1	+ 607 2	0.137	1150 0	....	......	0 175	1850.0

All bending moments are in inch-kips (1 inch-kip = 1000 in.-lb.).

These bending moments are plotted in Fig. 85, p. 180.

The maximum bending moments are obtained by combining the bending moments due to the dead load with the bending moments due to the most unfavorable condition of the live load.[8] The diagram at the bottom of Fig. 85 gives the combined bending moments for which the beam should be designed. The maximum bending moments are given in the table below:

Absolute Maximum Bending Moments

	Negative Bending Moments at Supports	Positive Bending Moment	
		First span	Second span
	Inch-kips	Inch-kips	Inch-kips
Dead load.........	− 910.8	1062.6	607.2
Live load..........	−1850.0	2240 0	1850.0
Total.........	−2760.8	3302.6	2457 2

Bending moments are in inch-kips (1 inch-kip = 1000 in.-lb.).

External Shears.—The external shears are obtained by multiplying the constants in table on p. 58, by P = 25 300 lb. for the dead load and P = 44 000 lb. for the live load. To get the absolute maximum values use the figures in black-face type.

[8] See p. 92 for method of combining the live load and dead load. Where the bending moments due to dead load are of opposite sign to the bending moments due to live load, divide them by a factor of safety before deducting from the bending moments due to live load.

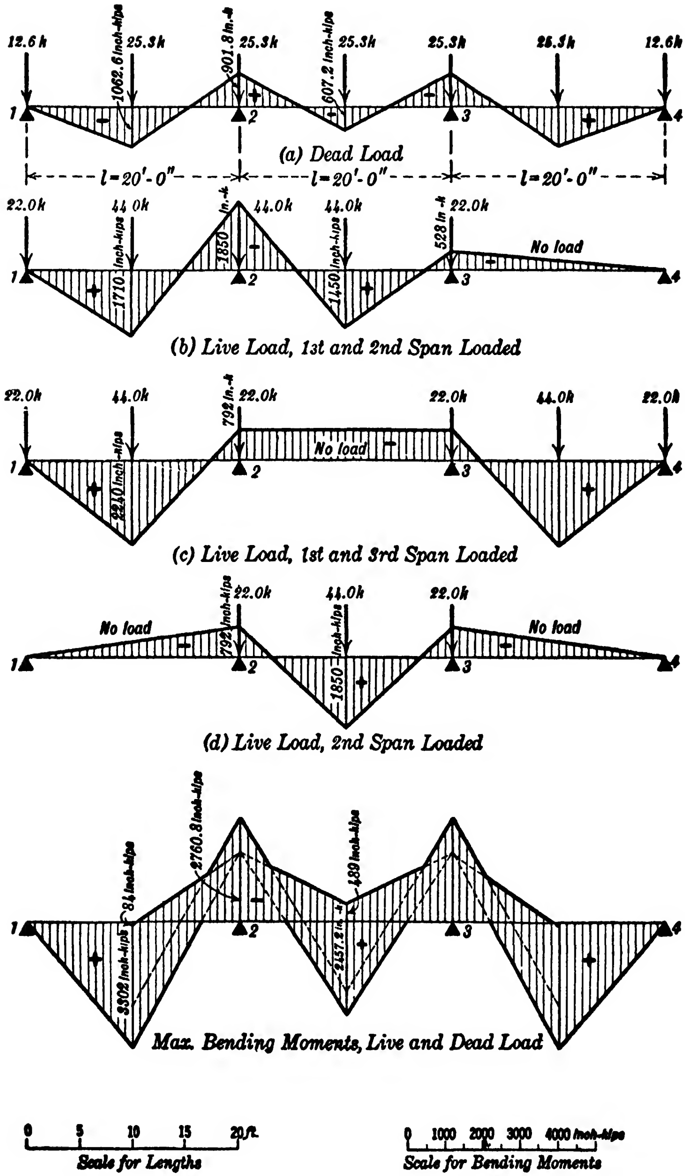


FIG. 85.—Bending Moment Diagrams. (See p. 178.)

The table below gives the maximum end shears at the first and the second supports. Due to symmetry the end shears at supports 1 and 4, and at supports 2 and 3, respectively, are equal.

End Shears Absolute Maximum Values

	V_1	V_{2l}	V_{2r}
	Kips	Kips	Kips
Dead load........	0.35×25 3= 8 9	0 65×25 3=16.4	0 5 ×25.3=12.6
Live load........	0 425×44 0=18 7	0 675×44.0=29 7	0 625×44.0=27.5
Total.........	27 6	46.1	40.1

The shears are given in kips (1 kip = 1000 lb.).

Design of the Girder.—After the maximum bending moments and shears are computed, the girder is designed in the manner described in Vol. I, pp. 215 to 240. The dimensions of the section of the girder are usually controlled either by the maximum shear or by the maximum negative bending moment. The dimensions required at the support are larger than required at the center of the girder because at the support the girder must be considered as a rectangular beam, while in the center it is a T-beam.

Assume width of section $b = 14$ in.

Depth Required by Diagonal Tension.—For $V = 46\,100$ lb., assuming $j = 0.9$ and $v = 120$.

$$d = \frac{46\,100}{0\ 9 \times 14 \times 120} = 30.4 \text{ in.}$$

Depth Required at Support.—Assume that one-half of the required tensile reinforcement will be used as compression reinforcement at the support.

Then $p' = 0.5p_1$. Assume $a = 0.1$.

Using Diagram 7, p. 904, Vol. I, for $f_c = 900$, $f_s = 16\,000$, and $n = 15$ the desired relation $p' = 0.5p_1$ is obtained for $p' = 0.095$ and $p_1 = 0.019$. For $p_1 = 0.019$ the depth is obtained from Formula (26), p. 222, Vol. I,

$$d = 1.05\sqrt{\frac{2\,760\,800}{14 \times 0.019 \times 16\,000}} = 25.4 \text{ in.}$$

Selected Dimensions.—The depth required by shear is larger than the depth required by the bending moment. Therefore the accepted section is (see Fig. 86)

$$b = 14 \text{ in.}$$
$$d = 31 \text{ in.}$$

Required Amount of Steel.

At support,

$$A_s = \frac{2\,760\,800}{0.9 \times 31 \times 16\,000} = 6.2 \text{ sq. in.}$$

Center, first span,

$$A_s = \frac{3\,302\,600}{0.9 \times 31 \times 16\,000} = 7.4 \text{ sq. in.}$$

Use 6—1-in. sq. = 6.0 sq. in.
2—1-in. rd. = 1.6 sq. in.

Total, 7.6 sq. in.

Center, second span,

$$A_s = \frac{2\,457\,200}{0.9 \times 31 \times 16\,000} = 5.5 \text{ sq. in.}$$

Use 4—1-in. sq. = 4.0 sq. in.
2—1-in. rd. = 1.6 sq. in.

Total 5.6 sq. in.

Points of Bending Reinforcement.—The points at which the longitudinal bars may be bent up may be determined from the bending moment diagram showing the maximum bending moments. (See Fig. 86 opposite p. 182.)

The amount of bending moment resisted by each bar is computed and plotted on the bending moment diagram. Where the bar is bent up the corresponding area stops. The steel should be arranged so that the total bending moment diagram is properly covered by the resisting moment due to the bars.

Diagonal Tension Reinforcement.—The amount of diagonal tension reinforcement is determined in the same manner as described in Vol. I, p. 251. The shear is considered as a measure of diagonal tension.

A shear diagram is drawn for the most unfavorable combination of the dead and live loads. (See Fig. 86.)

The amount of shear resisted by the concrete is plotted assuming that concrete alone resists 40 lb. per sq. in.

The bent bars are located and their value for resisting diagonal tension computed.

Diagonal tension resisted by 1 in. sq. bar $= \frac{1.0}{0.7} \times 16\,000 = 22\,900$ lb.

Since the maximum shear in the beam to be resisted by reinforcement is 1 080 lb. per lin. in. and the spacing of bent bars is 15 in., the total amount of shear to be resisted in the distance tributary to one bent bar is $15 \times 1\,080 = 16\,200$ lb. The computed strength of the bar is larger than the amount of diagonal tension to be resisted. Hence the bent bar is sufficient to take care of its tributary distance. The same can be proved about the 1-in. rd. bars.

Stirrups are required only in the sections which are not taken care of by the bent bars. Adopt two pronged stirrups made of $\frac{5}{8}$-in. rd. bars. The value of one stirrup is $2 \times 0.307 \times 16\,000 = 9\,824$ lb. This divided by the shear per lin. in. to be resisted gives the spacing of the stirrups. Thus, at the left end the shear to be resisted by the stirrups is 420 lb. per lin. ft. and the allowable spacing of stirrup is $\frac{9824}{420} = 23$ in. The stirrups will be spaced as shown in Fig. 86, opposite p. 182.

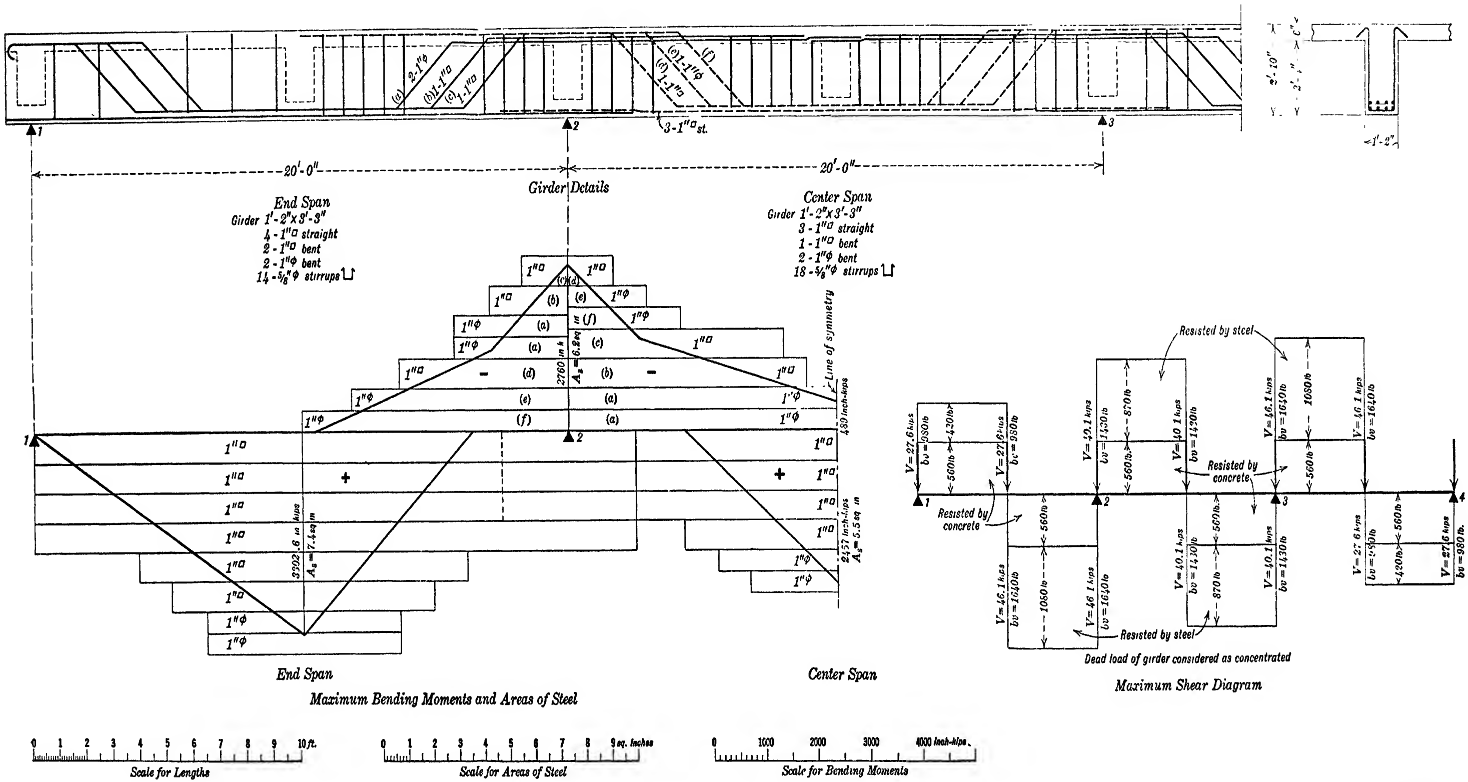


FIG. 86.—Girder Details. (See p. 182.)

(To face page 182)

THREE UNEQUAL SPANS, SYMMETRICAL ARRANGEMENT. UNIFORM LOAD

Example 2.—Design a continuous beam of three unequal spans. The arrangement of spans and their length are

$$l_1 = 14 \text{ ft.} \quad l_2 = 24 \text{ ft.} \quad l_3 = 14 \text{ ft.}$$

Loading

Dead load, $w_1 = 1\,300$ lb. per lin. ft.
Live load, $w_2 = 3\,000$ lb. per lin. ft.

Total, $w = 4\,300$ lb. per lin. ft.

Allowable stresses same as in Example 1.

Solution.—For this condition use formulas given on p. 66 for arrangement of spans

$$ml, \; l, \; ml.$$

Compute ratio m:

$$m = \frac{l_1}{l_2} = \frac{14}{24} = 0.58.$$

Compute:

	Dead Load	Live Load
wl................	31 200 lb.	72 000 lb.
$w(ml)$........... .	18 200 lb.	42 000 lb.
$12wl^2$............	8 908 000 in.-lb.	20 760 000 in.-lb.
$12w(ml)^2$..........	3 060 000 in.-lb.	7 060 000 lb.-lb.
$\frac{1}{8}wl^2 \times 12$...........	1 113 000 in.-lb.	2 593 000 in.-lb.
$\frac{1}{8}(ml)^2 \times 12$.......	383 000 in.-lb.	882 000 in.-lb.

From Diagram 10, p. 70, get the constants G_1, G_2, G_3, G_4, and G_5.

$$G_1 = 0.072 \qquad G_2 = 0.077 \qquad G_3 = 0.055 \qquad G_4 = 0.012 \qquad G_5 = 0.06$$

Using these constants compute all the required coefficients for shears and bending moments for the dead load and for all critical positions of live load using Formulas (178) to (216). The coefficients are given in the following table

	Dead Load		Live Load in					
			1st and 2nd Span		1st and 3rd Span		2nd Span	
	Coefficients	Actual value	Coefficients	Actual value	Coefficients	Actual value	Coefficients	Actual value
		Kips		Kips		Kips		Kips
V_1	0.166	5.2	0.157	11 3	0.27	19.4	−0.108	− 7.8
V_{2l}	0.414	12.9	0.423	30.6	0.31	22.3	0.108	7.8
V_{2r}	0.5	15.6	0 522	38.0	....	....	0.5	36.0
V_{3l}	0.5	15 6	0 478	34.0	....	....	0.5	36 0
V_{3r}	0.414	12.9	0.095	6.8	0.31	22.3	0.108	7.8
V_4	0.166	5 2	−0.095	− 6.8	0.27	19.4	−0.108	− 7.8
		Inch-kips		Inch-kips		Inch-kips		Inch-kips
M_2	0 072	−646 0	0.077	−1590 0	0.012	−249 0	0.06	−1246.0
M_3	0 072	−646 0	0 055	−1140 0	0.012	−249.0	0.06	−1246.0
$M_{1\max}$	0 014	216 0	0.012	249 0	0.037	770.0		
$M_{2\max}$	0.053	476.0	0.059	1220.0	....	....	0.063	1290.0

All shears are in kips (1 kips = 1000 lb.).
All bending moments are in inch kips (1 inch kip = 1000 in.-lb.).

Maximum Negative Bending Moments.—Maximum bending moments at the supports are obtained by adding the bending moments due to dead load to the absolute maximum value for live load.

$$M_2 = M_3 = -\ 646.0 - 1\,590.0 = - \quad 2\,236 \text{ in.-k.}$$
$$= -\ 2\,236\,000 \text{ in.-lb.}$$

Maximum Positive Bending Moments.—To get exact maximum positive bending moment in the first span it is not possible to add the maximum bending moment due to dead load to the absolute maximum positive bending moment due to live load because these two bending moments act at different points. Instead proceed as follows: Add end shear at support 1 for dead load to that of live load when first and third spans are loaded.

$$V_1 = 5.2 + 19.4 = 24.6 \text{ k.} = 24\,600 \text{ lb.}$$

The unit dead and live load is $w = 1\,300 + 3\,000 = 4\,300$ lb. The point of maximum positive bending moment is at the point of zero shear.

$$x_1 = \frac{V_1}{w} = \frac{24\,600}{4\,300} = 5.72 \text{ ft.}$$

Finally,

$$M_{1\max} = \tfrac{1}{2}V_1x_1 = \tfrac{1}{2} \times 24\,600 \times 5.72 = 70\,400 \text{ ft.-lb.} = 845\,000 \text{ in.-lb.}^{9}$$

[9] This bending moment is smaller than the sum of the $M_{1\max}$ for dead load plus the absolute maximum for live load which is 216 + 770 = 986 000 in.-lb.

In the second span the maximum positive bending moment may be obtained directly from the table by adding dead load to absolute maximum live load.

$$M_{2max} = 476.0 + 1\,290.0 = 1\,766.0 \text{ in.-k.} = 1\,766\,000 \text{ in.-lb.}$$

Maximum End Shears.—Maximum end shears are obtained by adding the end shear due to dead load to the absolute maximum values due to live load.

$$V_1 = 5.2 + 19.4 = 24.6 \text{ k.} = 24\,600 \text{ lb.}$$

$$V_{2l} = 12.9 + 30.6 = 43.5 \text{ k.} = 43\,500 \text{ lb.}$$

$$V_{2r} = 15.6 + 38.0 = 53.6 \text{ k.} = 53\,600 \text{ lb.}$$

The end shears at the supports 3 and 4 are the same as at the supports 2 and 1, respectively.

Maximum Bending Moment Diagrams.—To get the points at which the longitudinal bars may be bent up it is necessary to draw the maximum bending moments diagrams. These are combinations of the bending moment diagrams for the dead load with the diagrams for the most unfavorable positions of the live load.

To draw the diagrams proceed as follows: First, the diagrams for maximum negative bending moments are drawn by plotting the bending moment diagrams for the dead load plus the live load extending over the first and second spans. In such case

$$M_2 = -\,646 - 1\,590 = -\,2\,236.0 \text{ in.-k.}$$

$$M_3 = -\,646 - 1\,140 = -\,1\,786.0 \text{ in.-k.}$$

Plot the negative bending moments at the supports 2 and 3, respectively and get points a and b. Connect points a and b, also points 1 and a and get the closing lines $1a$ and ab for the static bending diagrams. Plot in the first and second span the static bending moment diagrams for dead plus live load, for which the maximum static bending moments are

$$\text{First span} \quad M = \tfrac{12}{8} \times (1.3 + 3.0) \times 14^2 = 1\,260.0 \text{ in.-k.}$$

$$\text{Second span} \quad M = \tfrac{12}{8} \times 4.3 \times 24^2 = 3\,720 \text{ in.-k.}$$

To complete the diagram in the first span combine the negative bending moment at the support 2 when the second span, only, is loaded with the bending moments for one-half of the dead load. (See p. 92.)

$$M_2 = -\frac{646\ 0}{2} - 1\,240 = -\,1\,563.0 \text{ in.-k.}$$

Plot this at the support 2 and get point g. Using $1g$ as a closing line draw the bending moment diagram for one-half the dead load for which the maximum static bending moment is $M = \frac{12}{8} \times \frac{1.3}{2} \times 14^2 = 191.5$ in.-k.

The work is simplified by including the values of m and m^2 in the formulas in the coefficients. Then the values of the shears in all spans are obtained by multiplying the coefficients by wl and the values for bending moment are obtained by multiplying them by $12wl^2$. The value of wl and $12wl^2$ are worked out in the preceding table for dead and live loads.

To complete the diagram in the second span combine the negative bending moment produced when first and third spans are loaded with one-half of the dead load

$$M_2 = M_3 = -\frac{646\ 0}{2} - 244 = -567 \text{ in.-k.}$$

Plot this value at both supports and get points h and i. With hi as closing line draw a bending moment diagram for one-half the dead load for which the maximum static bending moment is $M = \frac{12}{8} \times \frac{1\ 3}{2} \times 24^2 = 562$ in.-k.

The diagrams for positive bending moments are now drawn. In the first span combine the bending moment for dead load with bending moments for the live load on the first and third spans.

$$M_2 = -646 - 244 = -890 \text{ in.-k.}$$

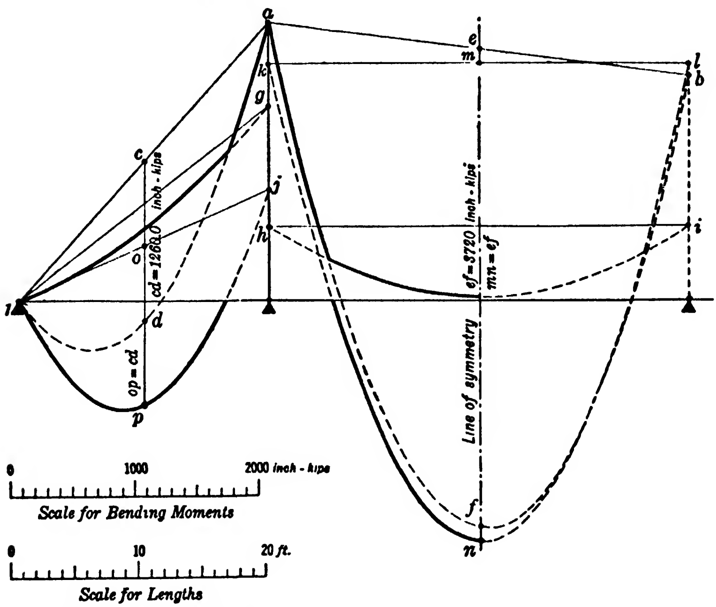


FIG. 87.—Maximum Bending Moment Diagrams. Spans 14 ft., 24 ft., 14 ft (*See p.* 186.)

Plot this at support 2 and get point j. Using $1j$ as a closing line draw the bending moment diagram for dead plus live load.

In the second span combine the bending moments for dead load with those for live load in the second span only for which

$$M_2 = M_3 = -646 - 1\,240 = -1\,886 \text{ in.-k.}$$

Plot this at the supports 2 and 3 and get points k and l. Using kl as a closing line draw a bending moment diagram for dead and live load.

The absolute maximum bending moment diagrams are shown in Fig. 86, p. 186, where they are indicated by heavy lines.

Maximum Shear Diagram.—The maximum shear diagram is shown in Fig. 88, p. 187. Draw the diagram in the following manner:

At support 1 plot the maximum end shears $V_1 = 24.6$ k. and get point *a*. At the support 2 plot the corresponding value of $V_{2l} = wl - V_1$ getting point *b*. Connect points *a* and *b*. The portion of this line above the axis gives the shear at the left side of the end span.

To get the maximum shear at the right side of the end span plot the maximum value for $V_{2l} = 43.5$ k. and the corresponding value of V_1. The line *cd* results and the portion below the axis gives the maximum shear in the right side of the first span.

To get the uplift, combine the value of V_1 for a condition when the second span only is loaded with one-half of the end shear for the dead load $V_1 = 7.2 + \frac{5\ 2}{2} =$ $- 5.2$ k

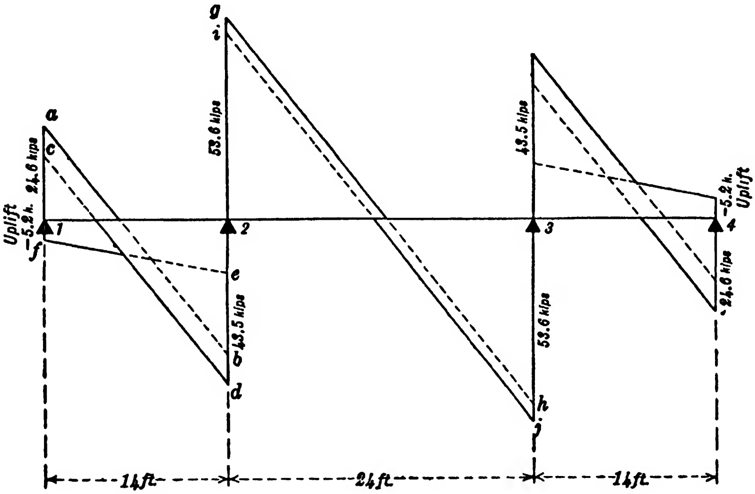


FIG. 88.—Maximum Shear Diagram. (*See p.* 187.)

The shear diagram in the second span is drawn in similar manner as used for the first span.

The resulting diagram shown in Fig. 88 should be used to design the diagonal tension reinforcement.

Dimensions of Beam.—The depth of the section for a selected breadth is determined either by diagonal tension or by requirements at the support.

Absolute maximum end shear, $V_{2l} = 53\,600$ lb.

Depth Required by Diagonal Tension.—Assuming $b = 14$ in., $j = 0.9$,

$$d = \frac{53\,600}{0.9 \times 14 \times 120} = 35.5 \text{ in.}$$

Depth Required by Bending Moment at Support.—The beam at the support is a rectangular beam reinforced for tension and compression. As found on p. 181

for $p_1 = 0.018$ and $p' = 0.0095$ and the specified stresses the ratio between the amount of compression steel and that of tension steel is equal 0.5.

The maximum bending moment at the support is

$$M_2 = -2\,236\,000 \text{ in.-lb.}$$

For $p_1 = 0.019$ the depth is obtained from Formula (22), p. 222, Vol. I.

$$d = 1.05\sqrt{\frac{2\,236\,000}{14 \times 0.019 \times 16\,000}} = 24 \text{ in.}$$

Selected Dimensions.—The depth required by diagonal tension is larger and will be selected.

$$b = 14 \text{ in.}$$
$$d = 36 \text{ in.}$$

Longitudinal Reinforcement.—Use absolute maximum bending moments and the selected depth of section. Use constant $j = 0.9$ in all cases.

First span, center,

$$A_s = \frac{845\,000}{0.9 \times 36 \times 16\,000} = 1.64 \text{ sq. in.}$$

Use 4—$\frac{3}{4}$-in. rd. bars, $A_s = 4 \times 0.44 = 1.76$ sq. in.

Support 2, $$A_s = \frac{2\,236\,000}{0.9 \times 36 \times 16\,000} = 4.3 \text{ sq. in.}$$

Second span, center,

$$A_s = \frac{1\,766\,000}{0.9 \times 36 \times 16\,000} = 3.4 \text{ sq. in.}$$

Use 6—$\frac{7}{8}$-in. rd. bars, $A_s = 6 \times 0.6 = 3.6$ sq. in.

The arrangement of steel is shown in Fig. 89, opposite p. 188.

Example 3.—Design a continuous beam of three unequal spans, the arrangement and lengths of which are

$$l_1 = 24 \text{ ft.} \quad l_2 = 14 \text{ ft.} \quad l_3 = 24 \text{ ft.}$$

Loading:

Dead load. $w_1 = 1\,300$ lb. per lin. ft.
Live load, $w_2 = 3\,000$ lb. per lin. ft.
Total, $w = 4\,300$ lb. per lin. ft.

The allowable stresses are the same as in the previous example. In the central portion the beam is a T-beam for which $t = 6$ in.

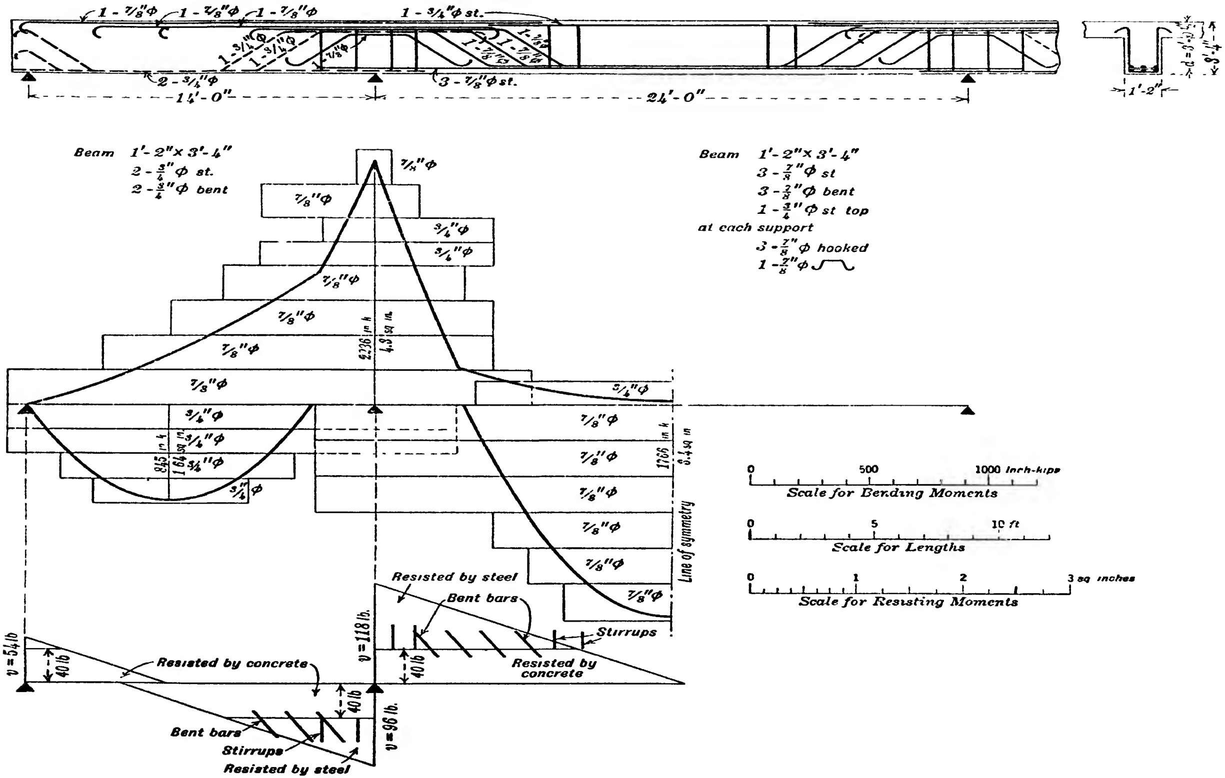


FIG. 89.—Details of Continuous Beam Spans, 14 ft., 24 ft., 14 ft. (See p. 188.) (To face p. 188)

Solution.—For this arrangement of spans use Formulas (217) to (257) for the case $l_1 = l$, $l_2 = ml$, $l_3 = l$.

Find ratio m:

$$m = \frac{l_2}{l_1} = \frac{14}{24} = 0.58.$$

Compute:

	Dead Load	Live Load
wl	31 200 lb.	72 000 lb.
$w(ml)$	18 200 lb.	42 000 lb.
$12 \times wl^2$ . . .	8 908 000 in.-lb.	20 760 000 in.-lb.
$12 \times w(ml)$	3 060 000 in.-lb.	7 060 000 in.-lb.
$\frac{1}{8}wl^2 \times 12$	1 113 000 in.-lb.	2 593 000 in.-lb.
$\frac{1}{8}w(ml)^2 \times 12$. . . .	383 000 in.-lb.	882 000 in.-lb.

The constants may be obtained from Diagram 11, p. 74. For $m = 0.56$ they are

$$H_1 = 0.086, \quad H_2 = 0.094, \quad H_3 = 0.002, \quad H_4 = 0.072, \quad H_5 = 0.014$$

Using these constants, the coefficients for the bending moments and the shears may be found from Formulas (217) to (257). They are tabulated in the table below.

Bending Moment and End Shears

	Dead Load		Live Load in					
			1st and 2nd Span		1st and 3rd Span		2nd Span	
	Constant	Actual value	Constant	Actual value	Constant	Actual value	Constant	Actual value
		Kips		Kips		Kips		Kips
V_1	0 42	13 1	0.406	29.3	0.428	30 8	−0.014	− 1 0
V_{2l}	0 58	18.1	0 594	42.7	0.572	41.2	0.014	1.0
V_{2r}	0 28	9 1	0 46	33 3	....	......	0.28	21.0
V_{3l}	0.28	9.1	0 12	8.7	....	....	0.28	21 0
V_{3r}	0.58	18.1	−0 002	− 1.4	0 428	30.8	0.014	1.0
V_4	0.42	13.1	0.002	1.4	0.572	41.2	−0.014	− 1.0
		Inch-kips		Inch-kips		Inch-kips		Inch-kips
M_2	−0.08	−720.0	−0.094	−1950	−0.072	−1480 0	−0.014	−290.0
M_3	−0.08	−720.0	+0.002	+ 41.0	−0.072	−1480.0	−0.014	−290.0

All shears are given in kips (1 kip = 1000 lb.).
All bending moments are in inch-kips (1 inch-kip = 1 000 in.-lb.)

Maximum Negative Bending Moments.—Maximum negative bending moments at the supports are obtained by adding the bending moments due to the dead load to the absolute maximum value for the live load

$$M_2 = M_3 = -720 - 1\,950 = -2\,670 \text{ in.-k.} = -2\,670\,000 \text{ in.-lb.}$$

Maximum Positive Bending Moments.—In the first span the absolute maximum positive bending moment is obtained as follows:

Find the end shear at the support 1 for dead load plus the live load giving the maximum positive bending moment in the first span, namely, when the first and the third spans are loaded.

$$V_1 = 13.1 + 30.8 = 43.9 \text{ k.} = 43\,900 \text{ lb.}$$

Since the unit load is $w = 1\,300 + 3\,000 = 4\,300$ lb., the point of zero shear is distant from the support 1.

$$x_1 = \frac{V_1}{w} = \frac{43\,900}{4\,300} = 10.2 \text{ ft.}$$

Consequently,

$$M_{1\,\text{max}} = \tfrac{1}{2}V_1x_1 = \tfrac{1}{2} \times 43\,900 \times 10.2 = 223\,000 \text{ ft.-lb.} = 2\,680\,000 \text{ in.-lb.}$$

In the second span the maximum positive bending moment in the center is equal to the positive bending moment for the condition of loading when the center span only is loaded plus one-half of the bending moment due to the dead load. Only one-half of the dead load bending moment is used because it is negative and balances the positive bending moment due to the live load. (See also p. 173.)

Bending moment at supports is

$$M_2 = M_3 = -\tfrac{720}{2} - 290 = -650 \text{ in.-k.}$$

$$M_{2\text{max}} = M_c - M_2 = (882 + \tfrac{383}{2}) - 650 = 424 \text{ in.-k.} = 424\,000 \text{ in.-lb.}$$

Maximum End Shears.—Maximum end shears are obtained by adding the end shears due to the dead load to the absolute maximum values for live load as given in the table on p. 189.

$$V_1 = 13.1 + 30.8 = 43.9 \text{ k.}$$

$$V_{2l} = 18.1 + 42.7 = 60.8 \text{ k.}$$

$$V_{2r} = 9.1 + 33.3 = 42.4 \text{ k.}$$

The end shears at the support 3 are the same as at support 2 and those at the support 4 are the same as at support 1.

Maximum Bending Moment Diagrams.—The absolute maximum bending moments at the various points are obtained by drawing diagrams as shown in Fig. 90, p. 191.

The method of drawing the diagrams is the same as described in Example 2. The bending moments were taken from the table on p. 189. The sums are given on the diagrams. These diagrams are useful in determining the points of bending of the longitudinal reinforcement.

Maximum Shear Diagrams.—The absolute maximum shears at the various points are given in the diagrams, Fig. 91, p. 191. The method of drawing the diagrams is the same as used in Example 2.

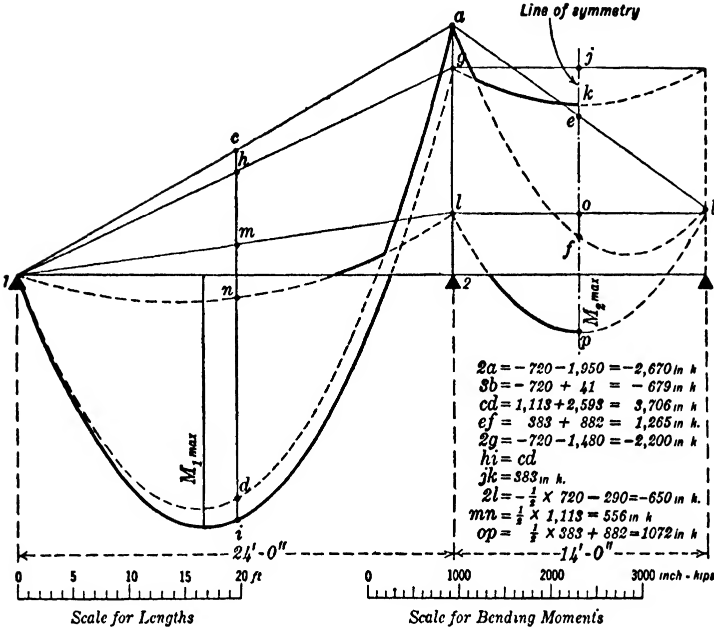


FIG. 90.—Maximum Bending Moment Diagrams. Spans 24 ft., 14 ft., 24 ft. (*See* p. 190.)

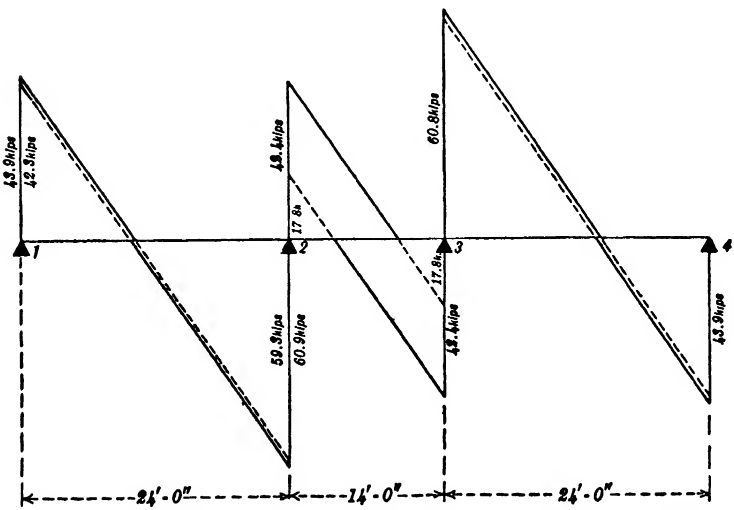


FIG. 91.—Maximum Shear Diagram. Spans 24 ft., 14 ft., 24 ft. (*See* p. 190.)

The maximum shear diagrams may be used in designing the diagonal tension reinforcement.

Dimensions of Beam.—The dimensions are governed by diagonal tension as in Example 2.

Assuming $b = 16$ in. and $j = 0.9$,

$$d = \frac{60\,900}{0.9 \times 16 \times 120} = 35 \text{ in.}$$

Longitudinal Reinforcement.

Negative, at supports 2 and 3,

$$A_s = \frac{2\,670\,000}{0.9 \times 35 \times 16\,000} = 5.3 \text{ sq. in.}$$

Use 7—1-in. rd. bars $A_s = 5.5$ sq. in.

Positive:

Central part of first span,

$$A_s = \frac{2\,680\,000}{0.9 \times 35 \times 16} = 5.4 \text{ sq. in.}$$

Use 7—1-in. rd. bars $A_s = 5.5$ sq. in.

Central part of second span,

$$A_s = \frac{424\,000}{0.9 \times 35 \times 16} = 0.84 \text{ sq. in.}$$

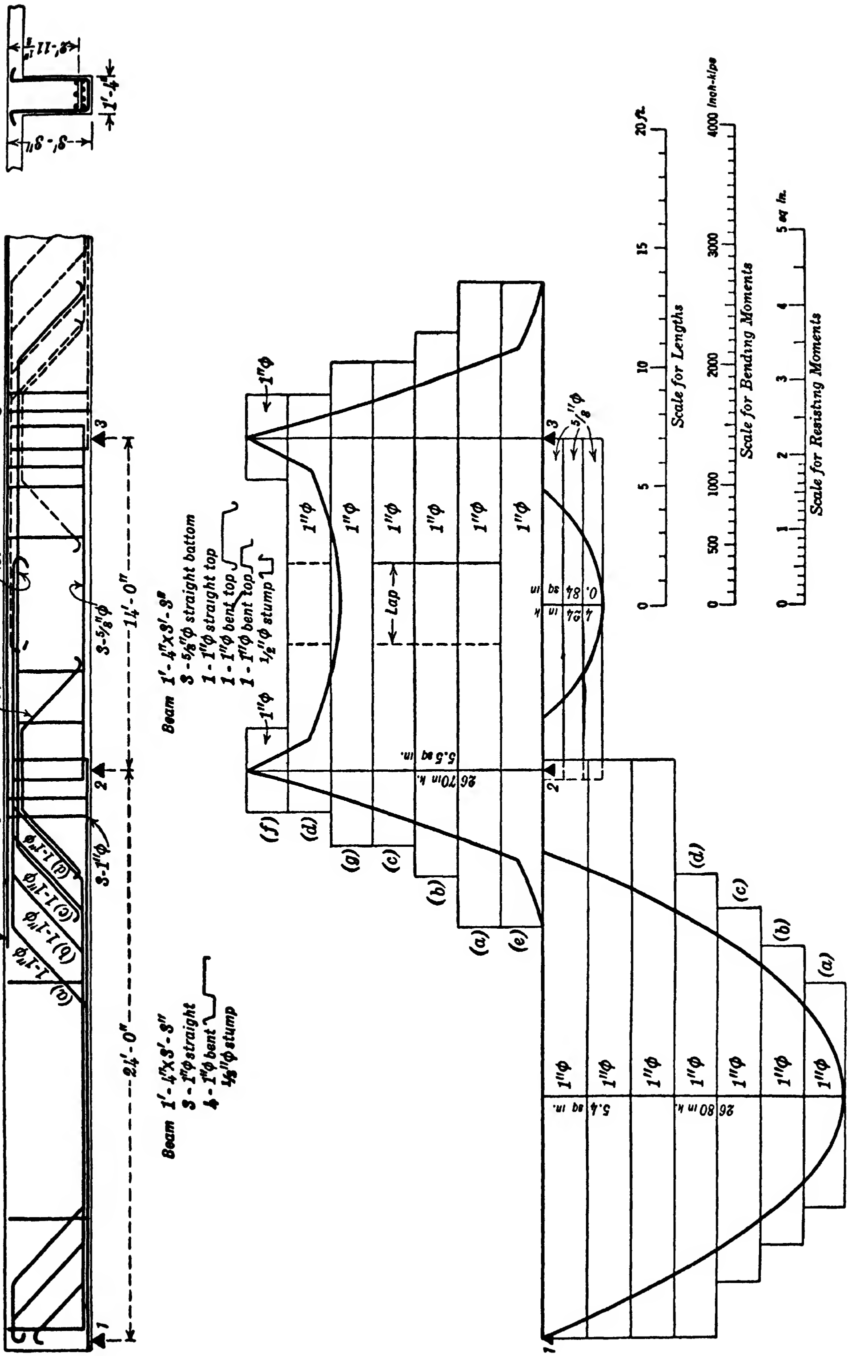
Use 3—⅝-in. rd. bars $A_s = 0.9$ sq. in.

Compression Reinforcement at Supports 2 and 3.

$$p_1 = \frac{5.3}{16 \times 35} = 0.095.$$

Referring to Diagram 7, p. 904, Vol. I, it is evident that for $f_c = 900$, $f_s = 16\,000$, $n = 15$, and $p_1 = 0.0985$ no compression reinforcement is required.

Beam Details.—The details of the beam design are shown in Fig. 92, p. 193. This figure also shows the bending moment diagram and the diagram of the resisting moments. The resisting moments are represented by the areas of effective reinforcement. In this case it was assumed that the bars stop being effective as direct tension reinforcement at the points where they are bent up or down, as the case may be.


(e) 1-1"φ straight
(f) 1-1"φ
(g) 1-1"φ
3-1"φ
3-5/8"φ
24'-0"
14'-0"
Beam 1'-4"×3'-3"
3-1"φ straight
4-1"φ bent
1/2"φ stump
Beam 1'-4"×3'-3"
3-5/8"φ straight bottom
1-1"φ straight top
1-1"φ bent top
1-1"φ bent top
1/2"φ stump
Lap
Scale for Lengths
Scale for Bending Moments
Scale for Resisting Moments

THREE UNEQUAL SPANS, UNSYMMETRICAL ARRANGEMENT

Example 4.—Determine maximum bending moments and shears in a continuous beam with three unequal spans.

Spans, 14 ft., 24 ft., and 18 ft.

Loading,

Dead load, w_1 = 1 300 lb. per lin. ft.
Live load, w_2 = 3 000 lb. per lin. ft.

Total, w = 4 300 lb. per lin. ft.

Solution.—In this case the center span is the largest, therefore Case 1, p. 76, with arrangement m_1l, l, m_2l, applies.

Find first values of m_1 and m_2:

$$m_1 = \tfrac{14}{24} = 0.58 \qquad m_2 = \tfrac{18}{24} = 0.75$$

$$m_1^3 = 0.194 \qquad m_2^3 = 0.422$$

Next find the constants a, b, c and a_1, b_1, c_1. These may be taken from the table on p. 79. If desired they may be worked out as follows, using Formulas (260) to (265), pp. 76 to 77.

Common denominator = 16(1 + 0.58)(1 + 0.75) − 4 = 44.2 − 4 = 40.2.

Numerators	Constants
$2(1+m_2)m_1^3 = 2(1+0.75)\times 0.194 = 0.68$	$a = \frac{0.68}{40.2} = 0.017$
$1+2m_2 = 1+2\times 0.75 = 2.5$	$b = \frac{2.5}{40.2} = 0.062$
$m_2^3 = 0.422$	$c = \frac{0.422}{40.2} = 0.011$
$m_1^3 = 0.194$	$a_1 = \frac{0.194}{40.2} = 0.005$
$1+2m_1 = 1+2\times 0.58 = 2.16$	$b_1 = \frac{2.16}{40.2} = 0.054$
$2(1+m_1)m_2^3 = 2(1+0.58)\times 0.422 = 1.33$	$c_1 = \frac{1.33}{40.2} = 0.033$

Using these constants the bending moments at supports are found for the dead and live loads.

Bending Moments at Supports.

Dead load, $w = 1\,300$ lb per lin. ft.; $wl^2 = 1\,300 \times 24^2 = 749\,000$ ft.-lb.

$$M_2 = -(a + b - c)wl^2 = -(0.017 + 0.062 - 0.01)wl^2 = -0.068wl^2 = -50\,800 \text{ ft.-lb.}$$

$$M_3 = -(-a_1 + b_1 + c_1)wl^2 = -(-0.005 + 0.054 + 0.033)wl^2 = -0.082wl^2 = -61\,400 \text{ ft.-lb.}$$

Live load, $w = 3\,000$ lb. per lin. ft.; $wl^2 = 3\,000 \times 24^2 = 1\,728\,000$ ft.-lb.

First and second span loaded,

$$M_2 = -(a + b)wl^2 = -(0.017 + 0.062)wl^2 = 0.079wl^2 = -136\,500 \text{ ft.-lb.}$$

$$M_3 = -(-a_1 + b_1)wl^2 = -(-0.005 + 0.054)wl^2 = -0.049wl^2 = -84\,600 \text{ ft.-lb.}$$

First and third span loaded,

$$M_2 = -(a - c)wl^2 = -(0.017 - 0.011)wl^2 = -0.006wl^2 = -10\,400 \text{ ft.-lb.}$$

$$M_3 = -(-a_1 + c_1)wl^2 = -(-0.005 + 0.033)wl^2 = -0.028wl^2 = -48\,400 \text{ ft.-lb.}$$

Second and third spans loaded,

$$M_2 = -(b - c)wl^2 = -(0.062 - 0.011)wl^2 = -0.051wl^2 = -88\,000 \text{ ft.-lb.}$$

$$M_3 = -(b_1 + c_1)wl^2 = -(0.054 + 0.033)wl^2 = -0.087wl^2 = -150\,000 \text{ ft.-lb.}$$

Second span loaded,

$$M_2 = -bwl^2 = -0.062wl^2 = -107\,200 \text{ ft.-lb.}$$

$$M_3 = -b_1wl^2 = -0.054wl^2 = -93\,400 \text{ ft.-lb.}$$

Absolute Maximum Negative Bending Moments at Support.—These bending moments are obtained by combining the bending moment due to the dead load with the bending moment for live load when two adjoining spans are loaded.

$$M_2 = -(50\,800 + 136\,500) = -187\,300 \text{ ft.-lb.}$$

$$M_3 = -(61\,400 + 150\,000) = -211\,400 \text{ ft.-lb.}$$

Absolute Maximum Positive Bending Moments.—Absolute maximum positive bending moments are obtained by combining the dead load with the following conditions of loading:

(*a*) For end spans, when the end spans are loaded and the center span not loaded.
(*b*) For center span, when the center span is loaded and the end spans not loaded.
First find bending moments at the supports for these conditions:

(*a*) $M_2 = -(50\,800 + 10\,400) = -61\,200$ ft.-lb.

$M_3 = -(61\,400 + 48\,400) = -109\,800$ ft.-lb.

(*b*) $M_2 = -(50\,800 + 107\,200) = 158\,000$ ft.-lb.

$M_3 = -(61\,400 + 93\,400) = -154\,800$ ft.-lb.

The maximum positive bending moments corresponding to these negative bending moments may be found from table on p. 176.

Left end span,

$$M_2 = -61\,200 \text{ ft.-lb.} \qquad w = 1\,300 + 3\,000 = 4\,300 \text{ lb.} \qquad l = 14 \text{ ft.}$$

$$\text{Coefficient} = \frac{M}{wl^2} = -\frac{61\,200}{4\,300 \times 14^2} = -0.072.$$

The bending moment at support 1 is zero. Referring to table on p. 176, it is found that to the coefficient at supports of 0 and 0.072 corresponds to a maximum positive bending moment coefficient of 0.092, therefore the absolute maximum positive bending moment in first span is

$$M_{max} = 0.092 \times 4\,300 \times 14^2 = 67\,500 \text{ ft.-lb.}$$

This bending moment acts at a distance from support 1 equal to

$$x_1 = 0.43 \times 14 = 6.02 \text{ ft.},$$

where the coefficient 0.43 is taken from table on p. 177.

Similarly for the third span the bending moment coefficients at the supports are 0 and 0.079 and the corresponding positive bending moments coefficient is 089. Therefore

$$M_{max} = 0.089 \times 4\,300 \times 18^2 = 125\,000 \text{ ft.-lb.}$$

and

$$x_1 = 0.421 \times 18 = 7.58 \text{ measured from support 3.}$$

For the center span

$$M_2 = 158\,000 \text{ ft.-lb.}; \quad \text{coefficient} = -\frac{158\,000}{4\,300 \times 24^2} = -0.064.$$

$$M_3 = 154\,000 \text{ ft.-lb.}; \quad \text{coefficient} = -\frac{154\,000}{4\,300 \times 24^2} = -0.062.$$

From table on p. 177, for coefficients 0.064 and 0.062, the maximum positive bending moment coefficient is 0.062. Therefore

$$M_{max} = 0.062 \times 4\,300 \times 24^2 = 154\,000 \text{ ft.-lb.}$$

$$x_1 = 0.502 \times 24 = 12.1 \text{ ft.}$$

Static Bending Moments.—To draw bending moment diagrams it is necessary to find the static bending moments in the center of the various spans for dead load and for live load.

Dead load,

First span $M_s = \frac{1}{8} \times 1\,300 \times 14^2 = 31\,850.0$ ft.-lb.
Second span $M_s = \frac{1}{8} \times 1\,300 \times 24^2 = 93\,600.0$ ft.-lb.
Third span $M_s = \frac{1}{8} \times 1\,300 \times 18^2 = 52\,650.0$ ft.-lb.

Live load,

First span $M_s = \frac{1}{8} \times 3\,000 \times 14^2 = 73\,500.0$ ft.-lb.
Second span $M_s = \frac{1}{8} \times 3\,000 \times 24^2 = 216\,000.0$ ft.-lb.
Third span $M_s = \frac{1}{8} \times 3\,000 \times 18^2 = 121\,500.0$ ft.-lb.

Bending Moment Diagrams.—The points of bending of the longitudinal reinforcement and the length of bars may be obtained from the bending moment diagram. First the bending moment diagrams are drawn for the dead load and the various positions of the live load. Then the bending moments for the dead load are combined with the bending moments for the live load so as to get the maximum results (see p. 173). To get maximum negative bending moments in the center span, combine diagram (*d*) with one-half the ordinates in diagram (*a*). Maximum positive bending moments are obtained by combining diagram (*a*) with diagrams (*d*) and (*e*) respectively. The bending moment diagrams are shown in Fig. 93, p. 199.

The bending moment diagrams are combined by adding and plotting of the ordinates of the diagrams to be combined. Thus to get the maximum negative bending moments at the support 2 the ordinates in diagram (*a*) are added to the ordinates of diagram (*b*). For maximum values at support 3, diagrams (*a*) and (*c*) are combined. To get maximum negative bending moments in the end spans combine diagram (*c*) with one-half of the ordinates of diagram (*a*) (see p. 173).

End Shears.—The maximum end shears at supports 1 and 4 act when the first and the third spans are loaded. For this condition the negative bending moments and the corresponding coefficients were computed previously.

For support 1,

$$M_2 = -61\,200 \text{ ft.-lb.}; \quad \text{coefficient} = -0.072.$$

Therefore the maximum shear at support 1 is

$$V_1 = (0.5 - 0.072)wl = 0.428 \times 4\,300 \times 14 = 25\,800 \text{ lb.},$$

also

$$V_{2l} = 4\,300 \times 14 - 25\,800 = 34\,400 \text{ lb.}$$

For support 4,

$$M_3 = -109\,800 \text{ ft.-lb.}; \quad \text{coefficient} = -0.079.$$

$$V_4 = (0.5 - 0.079)wl = 0.421 \times 4\,300 \times 18 = 32\,700 \text{ lb.},$$

also

$$V_{3r} = 4\,300 \times 18 - 32\,700 = 44\,700 \text{ lb.}$$

At support 2 the maximum end shears act when the first and second spans are loaded.

The bending moments then are

$$M_2 = -187\,300 \text{ ft.-lb.};$$

$$\text{coefficients } \frac{187\,300}{4\,300 \times 14^2} = 0.224 \quad \text{and} \quad \frac{187\,300}{4\,300 \times 24^2} = 0.076.$$

$$M_3 = -(61\,400 + 84\,600) = -146\,000 \text{ ft.-lb.};$$

$$\text{coefficient } \frac{146\,000}{4\,300 \times 24^2} = 0.059.$$

Therefore,

$$V_{2l} = (0.5 + 0.224)wl = 0.724 \times 4\,300 \times 14 = 43\,700 \text{ lb.},$$

also

$$V_1 = (0.5 - 0.224)wl = 0.276 \times 4\,300 \times 14 = 16\,500 \text{ lb.}$$

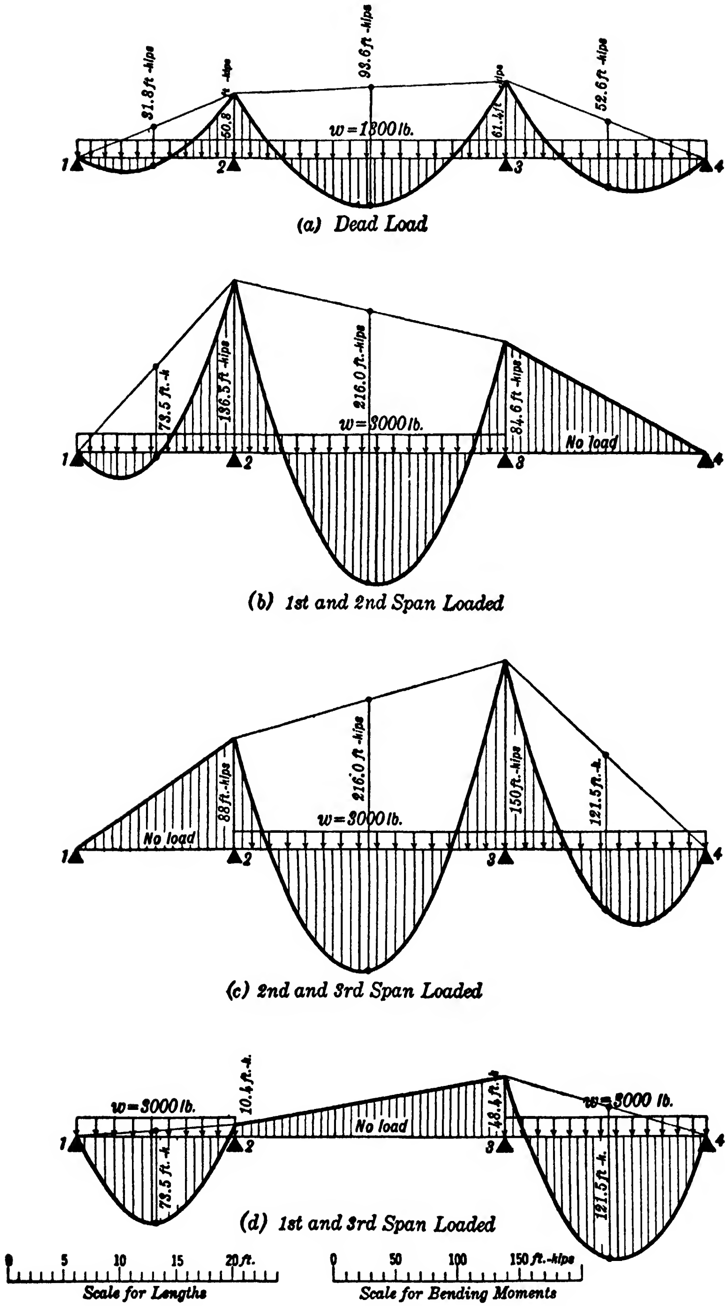
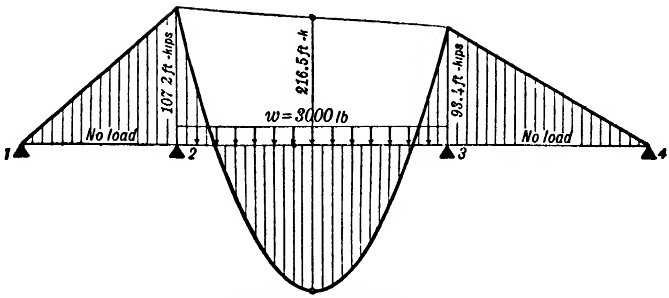
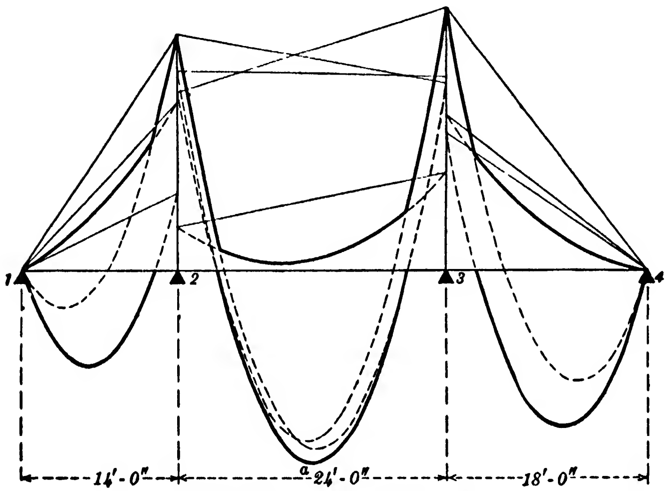


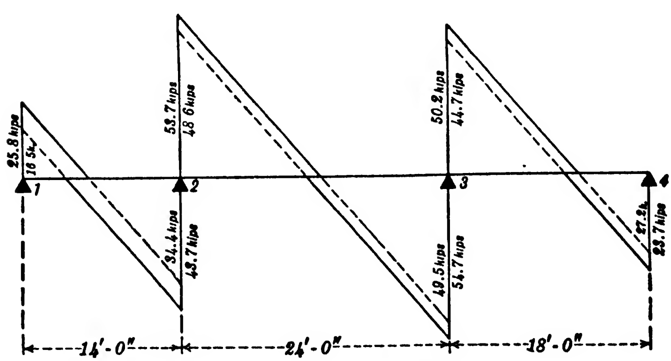
FIG. 93.—Bending Moment and Shear Diagrams for



(e) 2nd Span Loaded



(f) Combined bending moments



Three Unequal Spans, 14 ft., 24 ft., 14 ft. *(See p. 197.)*

The end shear at the right of support 2 may be found by using table on p. 177. Locating the coefficients 0.076 and 0.059 the end shear coefficient is found to be 0.52.

$$V_{2r} = 0.52wl = 0.52 \times 4\,300 \times 24 = 53\,700 \text{ lb.},$$

also

$$V_{3l} = 0.48wl = 49\,500 \text{ lb.}$$

At support 3 the maximum end shears act when second and third spans are loaded. The bending moments are

$$M_2 = -(50\,800 + 88\,000) = -138\,800 \text{ ft.-lb.};$$

$$\text{coefficient} = \frac{138\,800}{4\,300 \times 24^2} = 0.056.$$

$$M_3 = -211\,400 \text{ ft.-lb.};$$

$$\text{coefficients} = \frac{211\,400}{4\,300 \times 24^2} = 0.085 \quad \text{and} \quad \frac{211\,400}{4300 \times 18^2} = 0.15.$$

Therefore,

$$V_{3r} = (0.5 + 0.15)wl = 0.65 \times 4\,300 \times 18 = 50\,200 \text{ lb.},$$

also

$$V_4 = (0.5 - 0.15)wl = 0.35 \times 4\,300 \times 18 = 27\,000 \text{ lb.}$$

The end shear at the left of support 3 may be found by using table on p. 177. Locating the coefficients at support 0.056 and 0.085 the end shear coefficient is found to be 0.53. Hence,

$$V_{3l} = 0.53 \times 4\,300 \times 24 = 54\,700 \text{ lb.}$$

Therefore,

$$V_{2r} = 0.47 \times 4\,300 \times 24 = 48\,600 \text{ lb.}$$

The end shears are plotted in Fig. 93, p. 199. This figure should be used to design the diagonal tension reinforcement.

Determination of Dimensions.—After the maximum bending moments and shears are computed the dimensions of the beam are computed in the same manner as used in Example 2, p. 187. First the depth of the section for an assumed width is found. Then the amount of reinforcement computed at the points of maximum bending moments. Using the bending moment diagram as a guide the points of bending of the bars and their length are determined. Finally the diagonal tension reinforcement is computed.

FOUR UNEQUAL SPANS. SYMMETRICAL ARRANGEMENT. UNIFORM LOAD

Example 5.—Determine maximum bending moments and shears in a continuous beam consisting of four unequal spans. The ends of the beam are considered as simply supported. The span lengths and the loads are

Spans, 36 ft., 46 ft., 46 ft., and 36 ft.

Dead load, $w_1 = 1\,500$ lb. per lin. ft.
Live load, $w_2 = 3\,800$ lb. per lin. ft.

Total, $w = 5\,300$ lb. per lin. ft.

Solution.—The problem will be solved by using the "fixed points" method described on p. 154. The position of the fixed points is determined first using Formulas (502) to (505), p. 155.

Left Fixed Points.

First span,

$$f_1 = 0, \text{ because the end is simply supported.}$$

Second span,

$$f_2 = \frac{1}{3 + \frac{36}{46} \times 2} \times 46 = \frac{1}{4.57} \times 46 = 10.1 \text{ ft.}$$

Third span,

$$f_3 = \frac{1}{3 + \frac{46}{46}\left(2 - \frac{1}{4.57 - 1}\right)} \times 46 = \frac{1}{4.72} \times 46 = 9.77 \text{ ft.}$$

Fourth span,

$$f_4 = \frac{1}{3 + \frac{46}{36}\left(2 - \frac{1}{4.72 - 1}\right)} \times 36 = \frac{1}{5.22} \times 36 = 6.9 \text{ ft.}$$

Right Fixed Points.

Since the beam is symmetrical the positions of the right fixed points is the same as of the left points but arranged in opposite order. Thus,

$$f'_1 = 6.9 \text{ ft.}, f'_2 = 9.77 \text{ ft.}, f'_3 = 10.1 \text{ ft, and } f'_4 = 0.$$

Static Bending Moments.—The static bending moments are:

36-ft. span Dead load, $M_{max} = 243\,000$ ft.-lb.
Live load, $M_{max} = 616\,000$ ft.-lb.

Total, 859 000 ft.-lb.

46-ft. span Dead load, $M_{max} = 397\,000$ ft.-lb.
Live load, $M_{max} = 1\,005\,000$ ft.-lb.

Total, 1 402 000 ft.-lb.

Actual Bending Moments.—The bending moments at the supports are found graphically using the fixed points. First plot the spans to scale and locate the fixed points as shown in Fig. 91, facing p. 203.

Next draw bending moment diagrams when the first and the third span, respectively, is loaded. Since the beam is symmetrical the bending moment diagrams due to loads in the first and the third spans, respectively, are sufficient to get the bending moments for all loadings. The bending moments caused by loading placed in the fourth span are equal to those produced by the loading placed in the first span, only they are placed in reverse order in the beam. The same relation exists between the bending moments due to the loads in the second and those in the third span.

First Span.—In the center of the first span erect a vertical above the base and, by plotting to a convenient scale the maximum positive bending moment due to live load, get point *a*. Connect this point with support *2*. Erect a vertical at f_1', which intersects the line *a2* at *c*. Point *c* connected with support 1 intersects a vertical at support *2* at point *d*. Distance *2d* measured to the same scale as used in plotting *a'a* gives the negative bending moment at support *2* when the first span only is loaded. The bending moments in the loaded span are found by plotting the static bending moment diagram with line *1d* as a closing line. The bending moments above the base are negative and below the base positive.

To get the bending moment in the unloaded spans connect point *d* with the right fixed point in the second span and extend the line to intersection at point *e*. This line gives the bending moments in the second span. The distance *3e* is the bending moment at support *3* due to the load in the first span.

Point *e* connected with the right fixed point in the third span gives the line *ef* indicating bending moments in the third span.

Finally point *f* connected with support 5 gives bending moments in the fourth span.

Third Span.—The bending moments caused by the load placed in the third span are found as follows:

In the center of the third third span plot *g'g* equal to the maximum static bending moment, using the same scale as in the first span. Connect point *g* with supports *3* and *4*. Erect at both fixed points verticals. The points *h* and *i* are intersections of the verticals with lines *g3* and *g4*, respectively. Connect point *h* and *i* and extend the line to intersection with the verticals at the supports *3* and *4*. Distance *3j* and *4k* are negative bending moments at the supports.

To get bending moments in the loaded span plot the static bending moment diagram using *jk* as a closing line.

The bending moments in other spans are represented by *jl* in the second span, l_1 in the first span and *k5* in the fourth span.

The bending moments at all supports due to live load may be found easily by scaling. They are tabulated in the table on p. 203. To get the bending moments due to the dead load, the bending moments due to the live load are multiplied by the ratio of unit loads $\frac{1\,500}{3\,800}$ They are also tabulated in the table.

Since the dead load acts at all spans simultaneously the actual bending moment at each support is obtained by adding the bending moments due to all four conditions of loading.

To get the maximum negative bending moments at the supports add only the negative values.

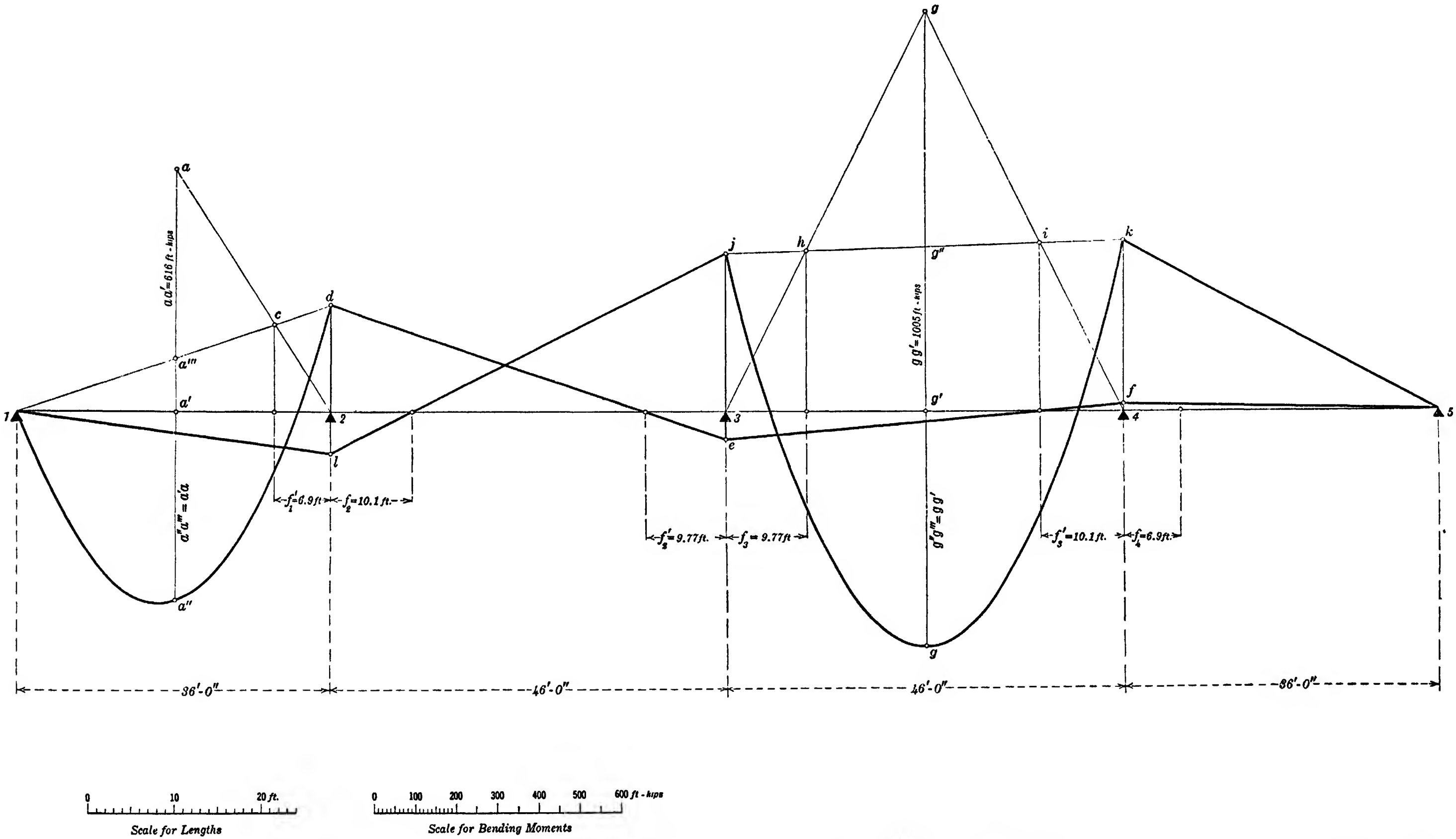


FIG. 94.—Method of Finding Bending Moments by Fixed Points Method. (See p. 202.)

(To face p. 203)

Bending Moment at Supports

	Bending Moments at		
	Support 2	Support 3	Support 4
	Foot-kips	Foot-kips	Foot-kips
Live load:			
In 1st span..................	− 300	+ 82	− 28
In 2nd span.................	− 412	− 425	+ 128
In 3rd span.............	+ 128	− 425	− 442
In 4th Span.............	− 28	+ 82	− 300
Dead Load:			
In 1st span................	− 118	+ 33	− 11
In 2nd span...............	− 175	− 170	+ 50
In 3rd span...........	+ 50	− 170	− 175
In 4th span................	− 11	+ 33	− 118
Dead load in all spans....	− 254	− 274	− 254
Most unfavorable live load.....	− 770	− 850	− 770
Total dead and live load. ..	−1024	−1124	−1024

All bending moments in foot-kips (1 foot-kip = 1000 ft.-lb.).

To get bending moments at supports to be used in designing add the bending moments caused by the dead load to the maximum negative bending moments caused by the live load.

Maximum Positive Bending Moments.—Maximum positive bending moments in the first and the third span occur when these spans are loaded and the other spans are not loaded. For this condition the negative bending moments are given in the table below.

	Bending Moments at		
	Support 2	Support 3	Support 4
	Foot-kips	Foot-kips	Foot-kips
Dead load..................	−254.0	−274.0	−254.0
Live load:			
In 1st span...............	−300.0	+ 82.0	− 28.0
In 3rd span................	+128.0	−425.0	−442 0
Total..................	−426 0	−617.0	−724 0

Due to symmetry of the beam the bending moments in the second span are the same as in the third span only reversed so that the bending moment M_2 is equal to the bending moment M_4.

To get maximum positive bending moments plot the negative bending moments at the support above the axis. Connect the two points thus obtained and, considering this new line as a closing line, plot the static bending moment diagram for the sum of the dead and live load. The portion of this diagram below the axis gives the positive bending moments. The maximum value may be obtained by scaling.

The maximum positive bending moment may also be found by using table on p. 176.

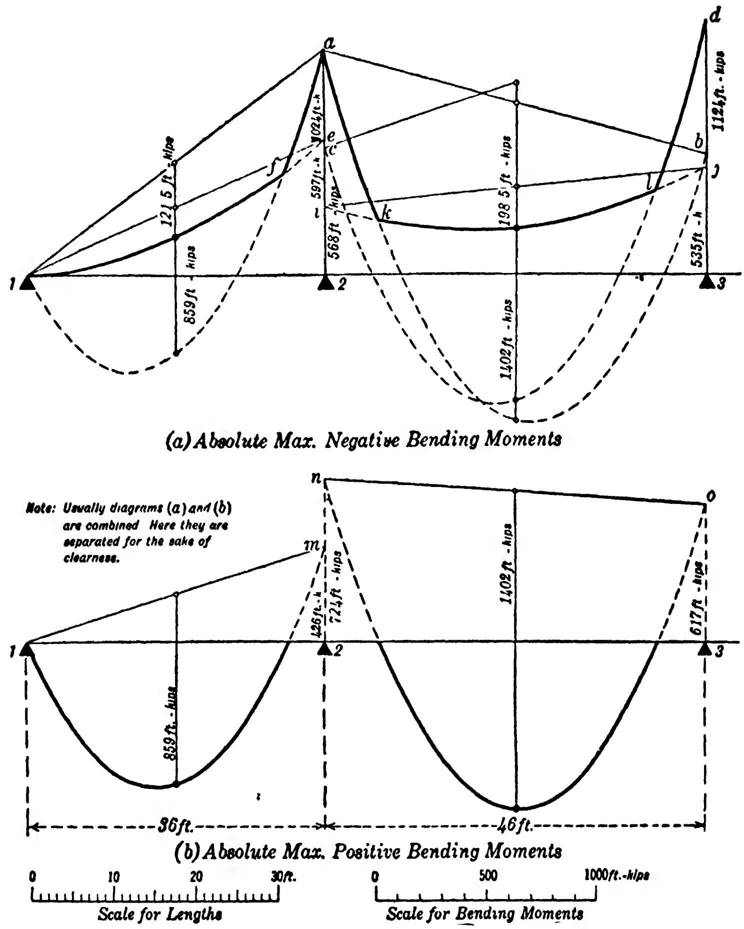


Fig. 95.—Diagrams of Absolute Maximum Bending Moments (*See p.* 204).

Maximum Negative Bending Moment Diagrams.—The exact location of the points at which the reinforcement may be bent up may be obtained from diagrams showing the absolute maximum bending moments at all points in the beam. This diagram is a composite curve drawn for several positions of the live load.

1. To get the maximum bending moments in the region around the support 2

find from the table on p. 203 the bending moments M_2 and M_3 for the dead load on all spans plus the live load on the first, second and fourth spans.

$$M_2 = -\ 1\,024 \text{ ft.-k.}$$

$$M_3 = -\ 274 + 82 - 425 + 82 = -\ 535 \text{ ft.-k.}$$

Plot these bending moments at the supports, obtaining points *a* and *b* in Fig. 95. Considering line *1a* and *ab* as closing lines draw for both spans the static bending moment diagrams for the sum of dead and live load in the manner explained on p. 185. Now consider the region around the support 3. The maximum bending moments there occur when the second and third spans are loaded, for which

$$M_2 = -\ 254 - 442 + 128 = -\ 568 \text{ ft.-k.}$$

$$M_3 = -1\ 124 \text{ ft.-k.} \quad \text{(From Table.)}$$

Plot these values at the supports and get point *c* and *d*. Using *cd* as a closing line draw the static bending moment diagram for dead and live loads in the same manner as for the previous case.

To complete the maximum negative bending moment diagrams draw the negative bending moments that may be produced in the central portions of the first and second spans. Referring to Fig. 50, p. 91, negative bending moments in the first span are produced when the second and fourth spans are loaded and the first span not loaded. For this condition the negative bending moment at the support is

$$M_2 = -\ 442 - 28 = -\ 470 \text{ ft.-k.}$$

As there is no live load in the first span the bending moment varies according to a straight line. In the central span the negative bending moment due to the live load is offset by the positive bending moment due to the dead load. As explained on p. 92, to get proper factor of safety, only one-half of the dead load should be considered as effective in reducing the negative bending moment due to the live load. Therefore add to the above value of M_2 one-half of the bending moment due to the dead load.

$$M_2 = -\ 470 - \frac{254}{2} = -\ 597 \text{ ft.-k.}$$

Plot this at support 2 and get point *e*. Considering *e1* as a closing line draw a bending moment diagram for half of the dead load.

The curve indicating the absolute maximum negative bending moments in the first span is the composite curve *afg*.

In the second span the negative bending moments are developed when the first and the third spans are loaded. The bending moments at the supports when combined with one-half of the dead load are

$$M_2 = -\ \frac{254}{2} - 300 + 128 = 299 \text{ ft.-k.}$$

$$M_3 = -\ \frac{274}{2} + 82 - 425 = -\ 480 \text{ ft.-k.}$$

Plotting these values gives points i and j. Considering ij as a closing line plot static bending moment diagram for one-half of the dead load. The curve indicating the absolute negative bending moments in the span is the composite curve $akld$.

Maximum Positive Bending Moment Diagrams.—The absolute maximum positive bending moment diagrams are easily drawn, using the negative bending moments obtained when computing maximum positive bending moments (see p. 203).

In the first span plot $M_2 = -426$ ft.-k. obtaining point m. Using line $1m$ as a closing line plot a static bending moment diagram for dead and live load. The part of the diagram below the axis 12 is the absolute maximum positive bending moment diagram.

In the second span plot $M_2 = -724$ ft.-k. and $M_3 = 617$ ft.-k.[10] and obtain the closing line no. The part of the bending moment diagram below the axis 23 is the absolute maximum bending moment diagram for the second span.

Maximum Shears.—Maximum shears at the first support occur when live load acts in the first and third spans. This is the same loading as for the maximum positive bending moment. The negative bending moment at support 2 then is

$$M_2 = -426 \text{ ft.-k.},$$

and the corresponding end shear is

$$V_1 = -\tfrac{1}{2}wl_1 + \frac{M_2}{l_1} = 0.5 \times 5\,300 \times 36 - \frac{426\,000}{36} = 95\,400 - 12\,000 = 83\,400 \text{ lb.}$$

The shear at the second support is a maximum when the first, second and fourth spans are loaded. This is the same condition of loading as for maximum negative bending moment at the support 2. The negative bending moments at the supports are

$$M_2 = -1\,024 \text{ ft.-k.}$$

$$M_3 = -535 \text{ ft.-k.}$$

Therefore the maximum end shears are

$$V_{2l} = \tfrac{1}{2}wl_1 - \frac{M_2}{l_1} = \tfrac{1}{2} \times 5\,300 \times 36 + \frac{1\,024\,000}{36} = 95\,400 + 28\,500 = 123\,900 \text{ lb.}$$

and

$$V_{2r} = \tfrac{1}{2}wl_2 + \frac{M_3 - M_2}{l_2} = \tfrac{1}{2} \times 5\,300 \times 46 + \frac{-535\,000 + 1\,024\,000}{46}$$

$$= 122\,000 + 11\,000 = 133\,000 \text{ lb.}$$

At the third support the maximum shear acts when the second and third spans are loaded. This is the same condition of loading as for the maximum negative bending moment at the support 3.

[10] These values are obtained from table on p. 203 which gives values for the condition when the first and third spans are loaded. Since for the present case the second and fourth spans should be loaded the bending moment at support 2 is the same as the bending moment at support 4 in the table.

The negative bending moments at the supports are

$$M_2 = -568 \text{ ft.-k.}$$

$$M_3 = -1144 \text{ ft.-k.}$$

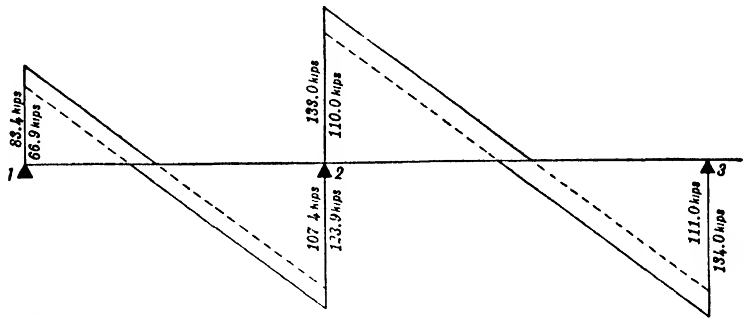


Fig. 96. Maximum External Shears in Continuous Beam of Four Spans
(*See p.* 207.)

Therefore the maximum end shears are

$$V_{3r} = V_{3l} = \tfrac{1}{2}wl_2 - \frac{M_3 = M_2}{l_2} = \tfrac{1}{2} \times 5\,300 \times 46 - \frac{-1\,444 + 568}{46}$$

$$= 122\,000 + 12\,000 = 134\,000 \text{ lb.}$$

The most unfavorable conditions of the external shears are plotted in Fig. 96. This diagram should be used in designing the diagonal tension reinforcement.

CHAPTER II

MEMBERS SUBJECTED TO DIRECT STRESS AND BENDING

In this chapter are given formulas for concrete members such as columns, arch ribs, members of rigid frames, dams and foundations, with and without reinforcement, which are subjected to direct pressures and bending moments acting simultaneously.

The commonly known formulas for direct stress and bending can be used only for computing stresses in the member when its dimensions are known. They cannot be readily used for the determination of dimensions. These have to be assumed first and then the stresses computed.

To simplify the design of members subjected to direct stress and bending, formulas from which dimensions can be obtained directly have been developed, and are here presented, by the authors. The diagrams based on these formulas and given in this chapter make the design of members subjected to direct stress and bending as simple as the design of simple beams and columns.

Examples of Members Subjected to Direct Stress and Bending.—Many of the members of ordinary concrete structures are subjected to direct stress and bending.[1]

The most common examples are the wall columns in a building, which are always subjected to the direct compression caused by the column load and to bending stresses produced by the bending moments due to the floor construction. Interior columns also are often subjected to bending in addition to the column load, especially when either the spans on both sides of the column or the loading are not symmetrical.

Another example of direct stress and bending is found in arch design where all sections are subjected to a concentric thrust and a bending moment.

Dams of gravity type are also subjected to direct pressure and bending when the line of pressure does not coincide with the center line of the dam section.

[1] By direct stress is meant the stress produced by a pressure acting in the centroid of a section. Such pressure produces stresses uniformly distributed over the whole section.

In rigid frame design all members are subjected to thrust and bending moment. It is often possible to neglect the effect of the thrust in the design of the horizontal members forming the frame. The vertical members, however, must be always designed for direct stress and bending.

Pressures on foundations are also determined by the formulas for direct stress and bending when the resultant force acting on the foundation is not concentric with the base. This takes place in foundations for arches, retaining walls and dams.

Stresses Due to Direct Stress and Bending.—When a member is subjected to a concentric force or thrust and a bending moment acting simultaneously, or when it is subjected to an eccentric force, the stresses acting upon it are a combination of compressive stresses produced by the force and bending stresses produced by the bending moment.

The bending stresses consist of compressive stresses acting on one part of the section and tensile stresses acting on the balance of the section. Both the tensile and compressive stresses due to bending moment vary according to a straight line from zero at the neutral axis to a maximum at the extreme fiber.

The stresses produced by the concentric force are uniformly distributed over the whole section.

The stresses produced by the concentric force and the bending moment acting simultaneously are equal to the sum of the stresses produced by each of them separately. The compressive stresses due to the force are increased by the compressive stresses produced by the bending moment. The tensile stresses due to the bending moment, on the other hand, are reduced by the compressive stresses due to the force. The resulting stress is compression over the whole section when the uniformly distributed compressive stress is larger than the maximum tensile stresses. When the maximum tensile stresses are larger than the uniformly distributed compressive stresses part of the section will be in tension. The neutral axis is located where the compressive stresses are equal to the tensile stresses.

Relation between Bending Moment and Eccentrically Applied Thrust.—The stresses produced by a combination of a central thrust and a bending moment are the same as those produced by an eccentrically applied thrust. Thus, a central thrust, N, and a bending moment, M, may be replaced by an eccentric thrust, N, acting at a distance from the axis of the section equal to $e = \frac{M}{N}$. In turn, the eccentric thrust may be replaced by a central load of the same intensity and a bending moment equal to the thrust multiplied by the eccentricity.[2] Therefore, both cases can be solved by the same formulas. The case of an eccentrically applied thrust gives simpler formulas.

Relation between Position of Eccentric Thrust and the Sign of Bending Moment.—A positive bending moment, producing, in a horizontal member,[3] compressive stresses at the top of the section and tensile stresses at the bottom, may be replaced by a positive thrust acting above the axis. (See Fig. 97, p. 210.)

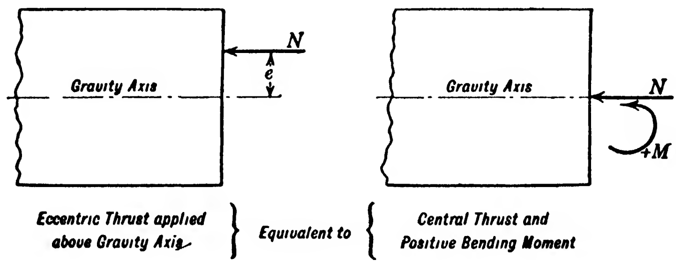


FIG. 97.—Positive Bending Moment and Central Thrust. (*See p.* 210.)

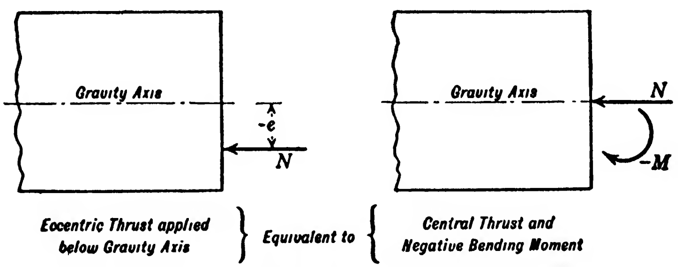


FIG. 98.—Negative Bending Moment and Central Thrust. (*See p.* 210.)

A negative bending moment, producing compressive stresses at the bottom of the section and tensile stresses on the top, may be replaced by a positive thrust acting below the axis. (See Fig. 98, p. 220.)

[2] *Proof of the above statement.*—Assume that a section is submitted to an eccentric thrust N acting at a distance e from the axis. The stress conditions will not be altered if in the center of the section are added two equal forces, but acting in opposite directions, such as $+N$ and $-N$. The section is then exposed to three forces, namely $+N$ at a distance e from center of section and $-N$ and $+N$ in the center of the section. The eccentric force with the negative central force forms a couple, Ne, and may be replaced by a bending moment $M = Ne$. There remains in the center a positive force N. Thus the eccentric thrust is replaced by a central force, N, and a bending moment $M = Ne$. (See also Vol. 1, p. 165.)

[3] Similar action, of course, is true in a vertical or inclined member.

Conversely, a positive thrust acting above the axis produces maximum compression at the top and minimum stresses at the bottom. A positive thrust acting below the axis produces reverse results.

Formulas for Members Subjected to Direct Stress and Bending.—After determining the normal thrust N and the bending moment $M = Ne$, the stresses or the dimensions are found from formulas given below.

For more elementary treatment of the subject reference is made to Vol. I, pp. 164–189, where formulas for stresses in plain concrete sections, symmetrically reinforced section and section with tensile steel only are developed. These formulas are repeated below and new formulas for determining dimensions and amount of reinforcement are added.

To take care of conditions not provided for by formulas in Vol. I additional formulas are given for stresses and dimensions in sections with unsymmetrical reinforcement. Particular attention is called to the formulas for sections with reinforcement near the face subjected to maximum stress only, and also to the formulas for dimensions and amount of reinforcement where it is required that definite stresses f_c and f_s are reached simultaneously.

For the sake of clearness the list of formulas and diagrams are here recapitulated. Unless otherwise designated, the page numbers refer to present Vol. II.

3. Reinforced Concrete Section. One Face in Tension.

a. Symmetrical Reinforcement.

Formulas for maximum compression and tensile stresses, p. 228.

Diagrams for constants C_a, pp. 657 to 661.

Formulas for required steel areas for given stresses, p. 229.

Formulas for dimensions for given stresses and steel ratios, p. 230.

b. Unsymmetrical Reinforcement.

Formulas for areas of tensile and compression reinforcement A_s and A'_s for specified stresses, pp. 233–234.

Formulas for depth of section for specified stresses and accepted ratio of compression steel p_2, p. 235.

Diagrams of constants, pp. 663–665.

c. Section Reinforced for Tension Only.

Formulas for maximum stresses in steel and concrete, p. 236.

Diagrams, pp. 666–667.

Formulas for ratio of tensile steel p for specified stresses, p. 237.

Dimensions of sections for accepted stresses, p. 238.

Derivations of Formulas.

Formulas for plain sections, Vol. I, p. 169.

Formulas for symmetrically reinforced sections under compression, Vol. I, p. 173.

Formulas for section with reinforcement near highly compressed face, p. 244.

Formulas for symmetrical reinforced sections when one surface is in tension, Vol. I, p. 180.

Formulas for unsymmetrically reinforced section, p. 238.

Formulas for section with steel near tension side only, Vol. I, p. 185.

PLAIN SECTIONS UNDER ECCENTRIC THRUST

Plain sections subjected to eccentric thrust (or to a thrust and bending moment) are found very often in practice in structures such as plain concrete or masonry arches, dams, also in foundations for arches, dams and retaining walls.

In such cases, after the magnitude and the position of the thrust are found, the problem resolves itself either into computations of stresses in a given section for possible excessive compression at the highly compressed face or tension at the opposite face, or into computations of dimensions for given maximum and minimum allowable stresses.

Special attention is called to the ease with which the dimensions may be found by using diagrams 1–2, opp. p. 648, and 6–7, pp. 654–655.

Full treatment of the subject with development of formulas is given in Vol. I, pp. 164–194. Final formulas are repeated below.

Notation.

Let N = normal thrust acting on section, lb.;

M = bending moment acting on section, in.-lb.;

e = eccentricity of thrust equals $\frac{M}{N}$, in.;

A = area of cross-section of concrete, sq. in.;

I = moment of inertia of concrete section, in.^{4};

$\frac{e_0}{h}$ = limiting ratio for which $f_t = 0$;

y_1 = distance extreme upper [4] fiber from center of gravity of irregular section, in.;

y_2 = distance extreme lower [4] fiber from center of gravity of irregular section, in.;

b = effective breadth of section, in.;

h = effective depth of section, in.;

f_c = maximum compression stresses, lb. per sq. in.;

f_t = minimum compression stresses, lb. per sq. in.

In vertical and inclined members the depth, h, is assumed to be the dimension, measured at right angles to the axis of the member, between the surfaces of maximum and minimum stress.

If the dimensions are in feet, the bending moment must be in foot-pounds and the resulting stresses will be in lb. per sq. ft.

General Formulas for Stresses. (See Fig. 99, p. 214.)

Maximum Stress,

$$f_c = \frac{N}{A} + \frac{My_1}{I}. \qquad (1)$$

[4] The definition "upper" and "lower" used in defining the distances y_1 and y_2 apply to a condition shown in Fig. 99, p. 214.

Minimum Stress,

$$f_t = \frac{N}{A} - \frac{My_2}{I}. \qquad (1a)$$

See Fig. 99, p. 214, for distribution of stresses.

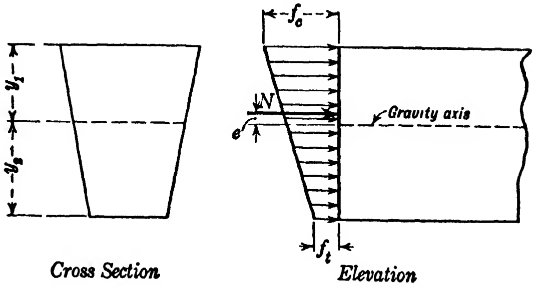


FIG. 99.—Section Subjected to Eccentric Thrust. (*See p.* 214.)

Formulas for Stresses Rectangular Sections. (See Fig. 100, p. 214.)
Maximum Stress,

$$f_c = \frac{N}{bh} + \frac{6M}{bh^2}, \qquad (2)$$

or if $M = Ne$,

$$f_c = \frac{N}{bh}\left(1 + \frac{6e}{h}\right). \qquad (3)$$

Minimum Stress,

$$f_t = \frac{N}{bh} - \frac{6M}{bh^2}, \qquad (4)$$

or if $M = Ne$,

$$f_t = \frac{N}{bh}\left(1 - \frac{6e}{h}\right). \qquad (5)$$

See Fig. 100, p. 214, for distribution of stresses.

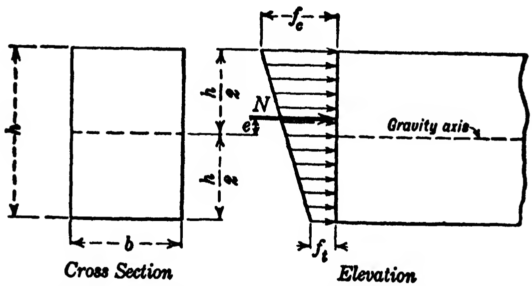


FIG. 100.—Rectangular Section Subjected to Eccentric Thrust. (*See p.* 214.)

Limiting Value $\frac{e_0}{h}$.—Sections without reinforcement can be used only when for the most unfavorable condition of loading no tensile stresses are developed. The limiting value $\frac{e_0}{h}$ may be obtained by making in Formula (5), p. 214 $f_t = 0$.

Limiting value $\frac{e_0}{h}$ for $f_t = 0$,

$$\frac{e_0}{h} = \frac{1}{6}. \qquad (6)$$

When the ratio of $\frac{e}{h}$ is larger than $\frac{1}{6}$, i.e., when the thrust acts outside the middle third, the section must be reinforced.

Formulas for Dimensions of Plain Sections.—The dimensions of a section may be governed either by the maximum stress or by the minimum stress. Usually the width of the section, b, is known and it is desired to find the depth, h, for which the maximum compression stress, f_c, does not exceed the allowable maximum value and for which minimum stress f_t is not smaller than the allowed limit. For plain sections the limit is preferably zero.

The formulas for depth of section h for known width are

Depth of Plain Section Governed by Maximum Stress f_c,

$$h = \frac{1}{2}\left(\frac{N}{bf_c}\right)\left[1 + \sqrt{1 + \frac{24e}{\left(\frac{N}{bf_c}\right)}}\right]. \qquad (7)$$

Use Diagram 1, opp. p. 648, for finding depth h.

Depth of Plain Section Governed by Minimum Stress f_t,

$$h = \frac{1}{2}\left(\frac{N}{bf_t}\right)\left[1 + \sqrt{1 - \frac{24e}{\left(\frac{N}{bf_t}\right)}}\right]. \qquad (8)$$

Use Diagram 2, opp. p. 648, for finding depth h.

Diagrams for Dimensions of Plan Sections.—The solution of the above formulas is complicated. To simplify the design, the Diagrams 1–2, opp. p. 648, are prepared,[5] from which may be obtained the value of h for any value of e and $\frac{N}{bf_c}$ or $\frac{N}{bf_t}$.

[5] The working out of the diagrams was simplified by changing formulas 3 and 5 into the following shape:

(*Continued on next page.*)

These diagrams are of great assistance in selecting proper dimensions of the section. For arches they may be used for the preliminary design of both plain and reinforced ribs. In reinforced concrete arches the dimensions b and h may be fixed first and the required amount of reinforcement found after the final bending moments are determined.

In plain concrete arches the lower limit should be preferably zero or a small value of tension (see p. 453). In reinforced concrete arches the lower limit may be equal to the tensile strength of concrete when it is desired that no cracks should form under working conditions (see p. 453).

Use of Diagrams 1–2, opp. p. 648, for Plain Sections.—The value of N and $M = Ne$ are known. Stresses f_c and f_t are specified or selected as discussed on p. 453.

It is desired to find the smallest section for which the maximum compression is not larger than the maximum allowable compression stress f_c nor the minimum stress smaller than the allowable stress f_t.

Assume a value of b. For barrel arches the value of b is either 12 in. when bending moment and thrust are computed for a strip of arch 1 ft. wide or it is equal to the full width of the arch rib. For rib arches the width of rib must be assumed and several trials may be required before best values of b and h are found.

Compute value of $e = \frac{M}{N}$. Compute the values of $\frac{bf_c}{N}$ and $\frac{bf_t}{N}$ and find depths for each value, using proper diagrams. In each case the depth is found at the intersection of the line for the value of $\frac{bf_c}{N}$ $\left(\text{or } \frac{bf_t}{N}\right)$ and the line for proper e. The larger of the two values should be used.

For use in design computations see examples, p. 247.

Relation between $\frac{bf_c}{N}$, h and e

$$\frac{bf_c}{N} = \frac{1}{h^2}(h + 6e).$$

Relation between $\frac{bf_t}{N}$, h and e

$$\frac{bf_t}{N} = \frac{1}{h^2}(h - 6e).$$

By means of these formulas the work was reversed, i.e., the values of $\frac{bf_c}{N}$ and $\frac{bf_t}{N}$, respectively, were found for successive values of h and e.

SYMMETRICALLY REINFORCED CONCRETE SECTIONS. WHOLE SECTION IN COMPRESSION

The formulas given below apply to symmetrically reinforced sections when the whole section is in compression or only negligible tension stresses occur so that the whole section may be considered as effective in resisting stresses. Such cases occur in arches and rigid frames. When large tension stresses are developed in the section, formulas on pp. 227–231 must be used.

For general treatment of the subject and development of formulas see Vol. I, pp. 173–180. Final formulas are repeated below.

Notation.

Let N = normal thrust acting on section, lb.;

M = bending moment acting on section, in.-lb.;

e = eccentricity of thrust equal to $\frac{M}{N}$ inches;

$\frac{e_0}{h}$ = limiting ratio of $\frac{e}{h}$ for which $f_t = 0$;

A = area of effective [6] cross-section of concrete, sq. in.;

A_s = total area of cross-section of steel in symmetrically reinforced section, sq. in.;

p = ratio of steel $\frac{A_s}{A}$ in symmetrically reinforced section;

p_1 = ratio of steel in tension to area of concrete;

p_2 = ratio of steel in compression to area of concrete;

I = moment of inertia of effective [6] cross-section of concrete, in.^{4};

I_s = moment of inertia of steel, in.^{4};

y_1 = distance of extreme compression fiber from centroid of equivalent [7] section, in.;

y_2 = distance of extreme tension fiber from centroid of equivalent [7] section, in.;

[6] Effective section is the portion of the actual section which is considered as resisting stresses. For instance if it is desired to provide fire proofing, the effective section is the area within the fire proofing.

[7] Equivalent section is a section in which the reinforcement is replaced by a concrete area equal to the area of steel multiplied by the ratio of moduli of elasticity, n.

b = effective [6] breadth of section, in.;

h = effective [6] depth of section, in.;

a = distance of center of steel to centroid of effective [6] concrete section, in.;

d' = distance of center of compression steel to outside face of effective [6] section, in.;

f_c = maximum compressive stresses, lb. per sq. in.;

f_t = minimum compressive stresses, lb. per sq. in.

If the dimensions are in feet, the bending moments must be in foot-pounds and the resulting stresses will be in lb. per sq. ft.

Formulas (9) to (14) may be used to compute stresses when dimensions are given.

Formulas (17) and (18) and diagrams may be used to find dimensions of the section.

General Formulas for Stresses. (No tension or only small tension in section.)

Maximum Compression,

$$f_c = \frac{N}{A + (n-1)A_s} + \frac{My_1}{I + (n-1)I_s}. \quad \text{(9)}$$

Minimum Stress,

$$f_t = \frac{N}{A + (n-1)A_s} - \frac{My_2}{I + (n-1)I_s}. \quad \text{(10)}$$

Formulas for Stresses in Rectangular Sections Symmetrically Reinforced. (See Fig. 101, p. 219.) (No tension or only small tension in section.)

Maximum Compression, if $M = Ne$,

$$f_c = \frac{N}{bh}\left[\frac{1}{1 + (n-1)p} + \frac{6}{1 + 12(n-1)p\left(\frac{a}{h}\right)^2}\frac{e}{h}\right]. \quad \text{(11)}$$

Minimum Stress, if $M = Ne$,

$$f_t = \frac{N}{bh}\left[\frac{1}{1 + (n-1)p} - \frac{6}{1 + 12(n-1)p\left(\frac{a}{h}\right)^2}\frac{e}{h}\right]. \quad \text{(12)}$$

The formula for maximum compression also may be written

Maximum Compression Stresses,

$$f_c = C_e \frac{N}{bh}, \quad \ldots \ldots \ldots \ldots \quad (13)$$

where the constant is

$$C_e = \frac{1}{1 + (n-1)p} + \frac{6}{1 + 12(n-1)p\left(\frac{a}{h}\right)^2} \frac{e}{h}. \quad \ldots \quad (14)$$

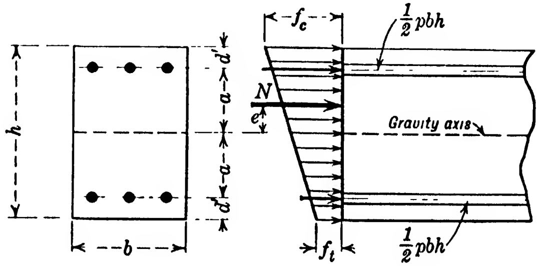


FIG. 101.—Rectangular, Symmetrically Reinforced Section. Whole Section in Compression. (*See p.* 218.)

Diagrams 3–6, pp. 650–653, give values of constants for different ratios of steel p and for ratios of $\frac{2a}{h} = 1.0, 0.9, 0.8$, and 0.7, respectively.

The dash line in each diagram indicates the limits for which $f_t = 0$. For ratios $\frac{e}{h}$ to the right of the dash line, part of the section will be in tension. If the tension is appreciable these formulas are not applicable.

For numerical example of the use of formula for f_c and the diagrams see p. 249.

Limiting value $\frac{e_0}{h}$ for $f_t = 0$.—The formula for the limiting value of $\frac{e}{h}$, for which the minimum stress f_t equals zero, is

Limiting Value $\frac{e_0}{h}$,

$$\frac{e_0}{h} = \frac{1 + 12(n-1)p\left(\frac{a}{h}\right)^2}{6[1 + (n-1)p]}. \quad \ldots \ldots \quad (15)$$

This formula is derived by substituting in formula (12) $f_t = 0$ and solving for $\frac{e}{h}$.

The limiting ratios $\frac{e_0}{h}$ may be obtained from Diagrams 3–6, pp. 650–653, for $2a = h$, $0.9h$, $0.8h$ and $0.7h$ and for different steel ratios. The point at which the straight line for proper p intersects the dash line curve, projected down, gives the limiting value of $\frac{e}{h}$.

Effect of Ratio of Moduli of Elasticity.—The stresses in steel and concrete caused by direct stress and bending are affected not only by the dimensions of the sections and the amount of reinforcement but also by the ratio of the moduli of elasticity of steel to concrete, n.

By inspecting the formulas for reinforced concrete sections it is evident that n and p always appear simultaneously and in some formulas always in the form $(n - 1)p$ and in others in the form of np. The value of a constant in any of the formulas for a definite numerical value of $(n - 1)p$, say, 0.14 will not change if the ratio n is decreased, provided that at the same time the value of the steel ratio p is increased sufficiently to make the result of $(n_1 - 1)p_1 = 0.14$. Therefore for a given bending moment and thrust and given concrete dimensions of the section the stresses will be the same for the following combinations of n and p:

$$n = 15 \quad \text{and} \quad p = 0.01,$$

$$n = 12 \quad \text{and} \quad p = 0.0127,$$

$$n = 10 \quad \text{and} \quad p = 0.0156,$$

because in all three cases the result $(n - 1)p = 0.14$.

The various diagrams can be used for n equal 15, 12, and 10 by properly changing the steel ratios. All diagrams were computed for $n = 15$. The changed steel ratios for $n = 10$ and $n = 12$ which correspond to the steel ratios in the diagrams are given in the table on p. 221.

Changed Steel Ratios for $n = 10$ and 12

Based on Relation $(n - 1)p = (n_1 - 1)p_1$

Ratios p in Diagrams $n = 15$	Changed Ratios for		Ratios p in Diagrams $n = 15$	Changed Ratios for	
	$n = 12$	$n = 10$		$n = 12$	$n = 10$
0.002	0.003	0.003	0.032	0.041	0.050
0.004	0.005	0.006	0.034	0.043	0.053
0.006	0.008	0.009	0.036	0.046	0.056
0.008	0.010	0.012	0.038	0.048	0.059
0.010	0.013	0.016	0.040	0.051	0.062
0.012	0.015	0.019	0.042	0.053	0.065
0.014	0.018	0.022	0.044	0.056	0.068
0.016	0.020	0.025	0.046	0.059	0.072
0.018	0.023	0.028	0.048	0.061	0.075
0.020	0.025	0.031	0.050	0.064	0.078
0.022	0.028	0.034	0.052	0.066	0.081
0.024	0.030	0.037	0.054	0.069	0.084
0.026	0.033	0.040	0.056	0.071	0.087
0.028	0.036	0.044	0.058	0.074	0.090
0.030	0.038	0.047	0.060	0.076	0.093

The use of this table is shown by the following example.

Example.—Find from the proper diagram the value of C_e for $\frac{e}{h} = 0.19$, $p = 0.028$ $n = 10$, when $2a = 0.8h$.

Solution.—To be able to use the diagrams for C_e which are based on $n = 15$, convert the steel ratio for $n = 10$ to a steel ratio for $n = 15$. From the table, p. 221, find that to $p = 0.028$ and $n = 10$ corresponds $p = 0.018$ for $n = 15$. Refer to diagram for C_e on p. 652 marked $2a = 0.8h$ and $n = 15$. Locate $\frac{e}{h} = 0.19$. The vertical line intersects the curve for $p = 0.018$ at $C_e = 1.352$. This is the desired value of C_e. When used in formula (13), p. 219, it gives the maximum stress f_c. Since the point of intersection of $\frac{e}{h} = 0.19$ and $p = 0.018$ is to the left of the heavy dash line, the whole section is in compression.

Method of Determining Required Amount of Reinforcement.—The dimensions of the section b and h are given. The normal thrust N and bending moment $M = Ne$ are known. The problem is to find the required amount of steel for which the maximum compression stresses do not exceed the specified value f_c. The problem can be solved by using Diagrams 3–6, pp. 650–653, for C_e.

First find the ratio $\frac{2a}{h}$ which determines which diagram to use. Also find the ratio $\frac{e}{h}$.

Next from Formula 13, p. 219, find

$$C_e = \frac{bhf_c}{N}. \qquad (16)$$

In the proper diagram locate value of C_e and $\frac{e}{h}$. This determines the ratio p for $n = 15$.

The required amount of reinforcement then is

$$A_s = pbh.$$

For numerical example of determining the amount of reinforcement see p. 249.

For other ratio of n, say $n = 10$, locate the value of p just computed in table on p. 221 under $n = 15$ and find the ratio of steel in column $n = 10$ corresponding to it. This is the required steel ratio.

Formulas for Depth of Section *h*.—As explained in connection with plain sections on p. 215 the dimensions of a section may be governed either by the maximum or the minimum stresses in the section. Therefore formulas are given below in which the depth is governed by the maximum compressive stress and also by the minimum stress. The minimum stress may be small compression, zero or small tension. The maximum stress to be used in the formula is the maximum allowable working stress. Both formulas must be solved and the larger of the two results accepted.

Depth of Section with Symmetrical Reinforcement as Determined by Maximum Compressive Stress, f_c,

$$h = \frac{1}{2}\frac{N}{bf_c}\frac{1}{1+(n-1)p}\left[1+\sqrt{1+\frac{24e}{\left(\frac{N}{bf_c}\right)}\frac{[1+(n-1)p]^2}{1+12(n-1)p\left(\frac{a}{h}\right)^2}}\right]. \qquad (17)$$

Depth of Section with Symmetrical Reinforcement as Determined by Minimum Stress, f_t,

$$h = \frac{1}{2}\frac{N}{bf_t}\frac{1}{1+(n-1)p}\left[1+\sqrt{1-\frac{24e}{\left(\frac{N}{bf_t}\right)}\frac{[1+(n-1)p]^2}{1+12(n-1)p\left(\frac{a}{h}\right)^2}}\right]. \qquad (18)$$

These formulas are rather involved. In practice the depth may be found using the Diagrams 7–8, pp. 654–655.

Diagrams for Finding Depth, h.—To simplify the determination of the depth, diagrams [8] are given on p. 654 from which the depth may be obtained for any value of e, $\frac{N}{bf_c}$ or $\frac{N}{bf_t}$. The diagrams are based on the ratio of steel $p = 0.01$ and $n = 15$. It was assumed that the distance of the center of steel from the nearest surface is 2 in.

Use of Diagrams 7 and 8 for Symmetrically Reinforced Sections. The value of N and $M = Ne$ are known. Stresses f_c and f_t are specified or selected as discussed on p. 453. The problem is to find depth h.

The problem can be solved as follows:

Accept value b in the manner suggested on p. 216 for plain sections.

Compute the value of $e = \frac{M}{N}$ and values of $\frac{bf_c}{N}$ and $\frac{bf_t}{N}$. Find depth for both values, using proper diagrams. The desired depth is found at the intersection of lines for $\frac{bf_c}{N}\left(\text{or } \frac{bf_t}{N}\right)$ with lines for e. The larger of the two values of h should be used.

For numerical example of the use of the diagrams see p. 248.)

SECTION WITH REINFORCEMENT NEAR HIGHLY COMPRESSED FACE ONLY. WHOLE SECTION UNDER COMPRESSION

As evident from Fig. 101, p. 219, in sections subjected to eccentric thrust (or thrust and bending moment) the reinforcement placed near the face with minimum stress resists very small stresses, particularly when the eccentricity is large. If there is no possibility of tension stresses this reinforcement is practically wasted. In such case a symmetrically reinforced section is not economical. Better

[8] Instead of the complicated formulas 17 and 18, following simple formulas were used for working out the diagrams.

Relation between $\frac{bf_c}{N}$, *h and e for symmetrically reinforced sections*

$$\frac{bf_c}{N} = \frac{1}{h[1 + (n - 1)p]} + \frac{6e}{h^2 + 12(n - 1)pa^2},$$

and

Relation between $\frac{bf_t}{N}$, *h and e for symmetrically reinforced sections*

$$\frac{bf_t}{N} = \frac{1}{h[1 + (n - 1)p]} - \frac{6e}{h^2 + 12(n - 1)pa^2}.$$

results are obtained if most of the reinforcement is concentrated near the face under maximum stress and only nominal amount of steel (say, $\frac{1}{4}$ per cent) is placed near the face under light stress.

Position of Thrust N.—Assume that the section is subjected to a normal thrust N, acting in the center of the *concrete* section, and a bending moment $M = Ne$, where e is measured from the center of the concrete section. Due to the unsymmetrical arrangement of the reinforcement the center of gravity of the reinforced concrete section does not coincide with the center of the concrete section but is moved by a distance d_s toward the reinforced face of the section. The thrust

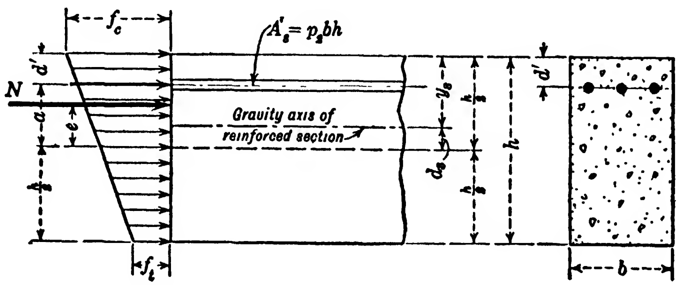


FIG. 102.—Rectangular Section with Steel Near Face under Max. Compression. (*See p.* 224.)

N acting in the center of the concrete section is therefore eccentric as far as the whole section is concerned. The bending moment produced by this eccentricity is of opposite sign to the external bending moment. The resultant bending moment is $M - Nd_s$ and the resultant eccentricity is $(e - d_s)$.

Formulas.—The following formulas may be used for sections with reinforcement near one face only. (See Fig. 102, p. 224.)

Maximum Compression Stresses for Known b, h *and* p_2,

$$f_c = \frac{N}{bh}\left[\frac{1}{1 + (n - 1)p_2} + \frac{6\left[1 + 2(n - 1)p_2\frac{d'}{h}\right]}{1 + (n - 1)p_2\left[1 + 12\left(\frac{a}{h}\right)^2\right]}\,\frac{e - d_s}{h}\right], \quad (19)$$

also

$$f_c = \frac{N}{bh}\left(C_n + C_m\frac{e-d_s}{h}\right), \quad \ldots \ldots \ldots \quad (20)$$

where

$$C_n = \frac{1}{1 + (n - 1)p_2}, \quad \ldots \ldots \ldots \ldots \quad (21)$$

$$C_m = \frac{6\left[1 + 2(n-1)p_2\frac{d'}{h}\right]}{1 + (n-1)p_2\left[1 + 12\left(\frac{a}{h}\right)^2\right]}. \quad . \quad . \quad . \quad . \quad (22)$$

$$d_s = \frac{(n-1)p_2}{1 + (n-1)p_2}a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Constants C_n, C_m and $\frac{d_s}{a}$ are given in table, p. 226, for different values of p_2.

Minimum Stresses for Known b, h and p_2,

$$f_t = \frac{N}{bh}\left[\frac{1}{1 + (n-1)p_2} - \frac{6\left[1 + (n-1)p_2\left(1 + 2\frac{a}{h}\right)\right]}{1 + (n-1)p_2\left[1 + 12\left(\frac{a}{h}\right)^2\right]}\frac{e - d_s}{h}\right]. \quad (24)$$

Limiting Ratio of $\frac{e_0}{h}$ *for which* $f_t = 0$,

$$\frac{e_0}{h} = \frac{1}{6}\frac{1}{1 + (n-1)p_2}\frac{1 + (n-1)p_2\left[1 + 12\left(\frac{a}{h}\right)^2\right]}{1 + (n-1)p_2\left(1 + 2\frac{a}{h}\right)} + \frac{d_s}{h}. \quad . \quad . \quad (25)$$

One-sided reinforcement shall not be used if for any condition of loading the ratio $\frac{e}{h}$ is larger than the value from the above formula.

Required Depth h for Accepted Ratio of Steel p_2,

$$h = \frac{1}{2}C_n\frac{N}{bf_c}\left(1 + \sqrt{1 + \frac{4C_m}{(C_n)^2}\frac{e - d_s}{\frac{N}{bf_c}}}\right), \quad . \quad . \quad . \quad (26)$$

where C_n and C_m are same constants as used in Formula (20), p. 224. They are given in table on p. 226. d_s is given by formula (23), p. 225, in terms of a.

Since a is not known, the value of d_s must be assumed first. After h is computed check the values of d_s and if the difference between the assumed and actual value is large recompute h on the basis of a net value for d_s.

The result can be used only when the value of $\frac{e}{h}$ is smaller than the limiting value from Formula (25), p. 225.

For numerical example of the use of the formula for h see p. 250.

Constants C_n, C_m and $\frac{d_s}{a}$. (*See p.* 225)

Unsymmetrically Reinforced Section, One Face in Tension

p_2	C_n	C_m			$d_s \div a$	
		$\frac{d'}{h} = 0$	$\frac{d'}{h} = 0.05$	$\frac{d'}{h} = 0.1$	$n = 15$	$n = 12$
0.002	0 972	5.39	5 47	5 56	0 027	0 022
0 004	0 946	4 90	5 05	5 21	0 053	0 042
0 006	0 919	4 50	4 70	4 91	0 081	0 062
0.008	0 898	4 15	4 38	4.62	0 101	0 081
0 010	0 877	3 85	4.12	4 38	0 123	0 099
0.012	0 856	3 58	3 88	4 17	0 144	0 117
0 014	0 836	3 36	3 66	3 95	0.163	0 134
0 016	0 817	3 16	3 48	3 79	1 183	0 150
0.018	0 792	2.99	3.32	3 64	0 202	0.165
0.020	0.781	2 83	3.16	3.49	0.219	0.180
0.022	0.764	2 69	3.02	3.36	0 236	0.195
0.024	0 748	2.56	2 89	3 25	0 252	0.209
0.026	0.732	2.44	2 77	3.13	0.267	0 222
0 028	0.718	2 34	2 65	3 01	0.282	0 236
0.030	0 704	2 24	2 50	2 92	0.296	0.248

Required Ratio of Reinforcement p_2 for Given Effective Dimensions b and h and Compression Stress f_c.—The thrust N, the eccentricity $e = \frac{M}{N}$ and dimensions of the section b and h are known. The problem is to find the ratio of reinforcement p_2 required to reduce the maximum stress in concrete to the allowable value of f_c.

Formula for p_2 would be complicated. The problem can best be solved by trial from the following relation.

Relation between f_c, b, h and p_2,

$$f_c\frac{bh}{N} = \frac{1}{1 + (n-1)p_2} + \frac{6\left(1 + 2(n-1)p_2\frac{d'}{h}\right)}{1 + (n-1)p_2\left[1 + 12\left(\frac{a}{h}\right)^2\right]}\,\frac{e - d_s}{h}. \qquad (27)$$

Use of Formula (27), p. 226.—Given dimensions of sections b and h, and the thrust and bending moment N and $M = Ne$.

Specified is the compression stress in concrete f_c.

Stresses in concrete for plain section are too large.

Problem is to find ratio of steel p_2 which, placed near the highly compressed face, would reduce the maximum compression stress to the allowable value of f_c.

To solve the problem compute value of $\frac{f_c bh}{N}$. Find value of $\frac{u}{h}$ (see Fig. 102). Find value of $e = \frac{M}{N}$ and the ratio of $\frac{e}{h}$.

Assume value of p_2 and substitute in Formula (27).

If for the assumed p_2 both sides of the equation are equal, the value p_2 is correct. If the left side is larger than the right side, accept a larger value of p_2. If right side is larger, reduce value of p_2 and refigure.

SYMMETRICALLY REINFORCED SECTION—ONE FACE IN TENSION

Formulas in previous pages are applicable only in cases where no appreciable tension is developed in the section so that the full section may be considered as effective. Where large tensile stresses are developed, the concrete in the portion of the section below the neutral axis, being in tension, should not be considered as resisting stresses. Formulas for symmetrically reinforced rectangular sections with one face in tension are developed in Vol. I, pp. 180–184. Final formulas are repeated below.

Center of Gravity of Symmetrically Reinforced Section when One Face is in Tension.—When the whole concrete section is effective in resisting stresses, the center of gravity of a symmetrically reinforced section coincides with the center of gravity of the concrete section.

When, however, part of the concrete section is in tension and therefore is not considered as effective in resisting stresses, the concrete in this part cannot be considered in computing the center of gravity of the reinforced concrete section. The center of gravity of the effective section does not coincide with the center of the concrete section but moves toward the face under compression by the distance d_s. Then normal thrust, acting in the center of the concrete section, actually is not a central thrust, but acts at an eccentricity equal to d_s. This is taken into account in formulas and it is not necessary to find the actual position of the center of gravity of the effective section.

Stresses in Steel and Concrete for Known Dimensions b, d and p. (See Fig. 103, p. 228.)

Maximum Compression Stresses in Concrete,

$$f_c = \frac{Ne}{C_a bd^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Constant C_a is given by formula (32). It may be taken from Diagrams 10 to 14, pp. 657 to 661.

Maximum Tensile Stresses in Steel,

$$f_s = nf_c\frac{1-k}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

also

$$f_s = C_s\frac{Ne}{C_a bd^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Constant C_s is given by formula (33), p. 228. It may be taken from Diagram 15, pp. 662.

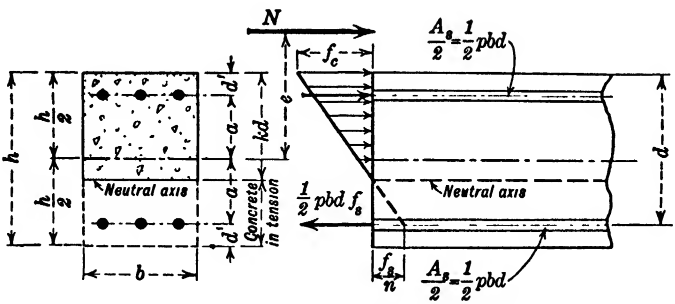


FIG. 103.—Rectangular Section, Symmetrical Reinforcement, One Face in Tension. (*See p. 228.*)

Maximum Compression Stresses in Steel,

$$f'_s = nf_c\left(1 - \frac{d'}{kd}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Constants

$$C_a = \frac{np}{k}\left(\frac{a}{d}\right)^2 + \frac{k}{4}\left(\frac{h}{d}\right) - \frac{k^2}{6} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and

$$C_s = n\frac{1-k}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The value of k is obtained from the following relation

Relation between k, e and p,

$$\frac{e}{d} = \frac{-k^3 + 1.5k^2\frac{h}{d} + 6np\left(\frac{a}{d}\right)^2}{3k^2 + 6npk - 3np\frac{h}{d}} \quad \ldots \ldots \quad (34)$$

The values of C_a and C_s can be found directly if the value of k is known. The value of k cannot be found directly from Equation (34) but must be obtained by trial. Namely, a value of k is assumed and substituted in the equation. If the resulting value of $\frac{e}{d}$ is equal to actual value $\frac{e}{d}$ the value of k is correct, otherwise a new assumption for k, with the previous value as a guide, must be made and the work repeated.

Diagrams for k, C_a and C_s.—From diagrams given on pp. 656–659 the value of k can be obtained for any given ratio of steel p and eccentric ratio $\frac{e}{d}$. The value of C_a and C_s for the value of k so obtained may be taken from diagrams on pp. 657–662.

The diagrams are worked out for the ratios $\frac{h}{d} = 1.0$, 1.1 and 1.2.

An example showing the use of the formulas is given on p. 252.

Required Area of Steel for Known Dimensions b and d and Specified Stresses f_s and f_c.—The dimensions b and d are given and also the thrust N and bending moment $M = Ne$. The stresses f_c, f_s and the ratio n are specified. The problem is to find the required steel ratio p for which neither the tensile stresses in steel f_s nor the compression stresses in concrete f_c exceed the allowable working values.

The problem can be solved by means of the diagrams in the manner described below. Attention is called to the fact that usually the steel ratio p found for the steel stresses will be different than that for the compression stresses in concrete. The larger of the two values should be used. It is obvious that then either the concrete or the steel is understressed. To get the maximum stresses in both materials simultaneously a section with unsymmetrical reinforcement should be used as described on p. 232.

From Formulas (28) and (29) the two values of C_a are obtained.

Constant governed by compression,

$$C_a = \frac{Ne}{bd^2f_c}. \qquad (35)$$

Constant governed by tension,

$$C_a = C_s\frac{Ne}{bd^2f_s}. \qquad (36)$$

These two equations may be used in connection with the diagrams to determine the steel ratio p.

Substitute in the above equations the known values of N, e, b, d, f_c and f_s and find the numerical values of C_a.

Assume value of k and find the corresponding C_s. As a first assumption $k = 0.5$ may be taken. In the diagram for proper $\frac{h}{d}$, find the corresponding values of p for the two computed values of C_a and for the assumed value of k. The larger of the two values should be used.

Check the value of k by means of the diagram for k for the known eccentricity ratio $\frac{e}{d}$ and the value of p just found. In case of large discrepancies between the actual value of k and the assumed values, new values of p should be found for closer value of k.

An example showing the use of the formulas is given on p. 254.

Required Depth of Section d, if it is Desired that the Ratio p Should Not Exceed a Definite Value.—Given is width of section b, also thrust N and bending moment $M = Ne$. Specified are the stresses f_c, f_s and n. The problem is to find depth d for which the ratio p does not exceed a definite chosen value.

Formulas for depth derived from Formulas (28) and (29), p. 228, are

Depth governed by Stresses in Concrete,

$$d = \sqrt{\frac{Ne}{C_abf_c}}. \qquad (37)$$

Depth Governed by Stresses in Steel,

$$d = \sqrt{C_s\frac{Ne}{C_abf_s}}. \qquad (38)$$

To solve the problem assume a ratio of k. As a first approximation k may be assumed as equal to 0.5. For the assumed k and specified value of p find the corresponding values of C_a and C_s. Solve Equations (37) and (38) thereby obtaining two values of depth d. The larger value of d should be used. For this value of d find the ratio of eccentricity $\frac{e}{d}$. Check the value of k for the specified p and computed ratio $\frac{e}{d}$. If the discrepancy between the assumed value of k and the found value is large, make another assumption, taking an intermediate value of k and then recompute the depth d.

An example showing the use of the formulas is given on p. 253.

Use of Diagrams for k and C_a for Different Values of n.—As evident from Formulas (32) to (34), p. 228, the values k and C_a depend not only upon the steel ratios p but also upon the ratio of moduli of elasticity n. It is also evident that the values of p and n always appear in the shape np. Therefore, as explained on p. 220, the diagrams for k and C_a, which were worked out for $n = 15$, can be used for any other n, provided the steel ratio is adapted so that np remains constant. The following table gives the values of p for $n = 10$ and 12, respectively, which correspond to the steel ratios for $n = 15$.

Changed Steel Ratios for $n = 10$ and 12

Based on Relation $np = n_1p_1$

Ratios p in Diagrams $n = 15$	Changed Ratios p for		Ratios p in Diagrams $n = 15$	Changed Ratios p for	
	$n = 12$	$n = 10$		$n = 12$	$n = 10$
0 002	0 0025	0 003	0 032	0 040	0.048
0.004	0.005	0 006	0 034	0.0425	0 051
0.006	0.0075	0 009	0 036	0.045	0 054
0.008	0.010	0 012	0 038	0 0475	0.057
0.010	0.0125	0.015	0 040	0 050	0 060
0.012	0.015	0 018	0.042	0.0525	0 063
0.014	0 0175	0.021	0.044	0 055	0.066
0.016	0.020	0.024	0 046	0 0575	0.069
0.018	0.0225	0.027	0 048	0 060	0.072
0.020	0 025	0.030	0 050	0.0625	0 075
0.022	0.0275	0.033	0 052	0 065	0 078
0.024	0.030	0 036	0.054	0.0675	0.081
0.026	0.0325	0.039	0.056	0 070	0 084
0.028	0.035	0 042	0.058	0 0725	0.087
0.030	0.0375	0.045	0.064	0 075	0.090

UNSYMMETRICALLY REINFORCED SECTION—ONE FACE IN TENSION

Required Reinforcement for Rectangular Sections.—Symmetrically reinforced concrete sections are not the most economical to resist eccentric thrust (or thrust and bending moment) as ordinarily either the amount of compression or tension reinforcement is larger than required by stresses. By using an unsymmetrically reinforced section, i.e., a section where the amount of tension reinforcement is different from that of compression reinforcement, it is often possible to design a section so that both the stresses in the tension reinforcement and the compression stresses in the extreme fiber of concrete reach the maximum allowable values simultaneously. The required amount of compression reinforcement in such cases is nearly always different from the amount of tension reinforcement and therefore the Formulas (28) to (34) for symmetrical sections do not apply.

The condition is illustrated in Fig. 104, p. 232. The formulas are developed on p. 238.

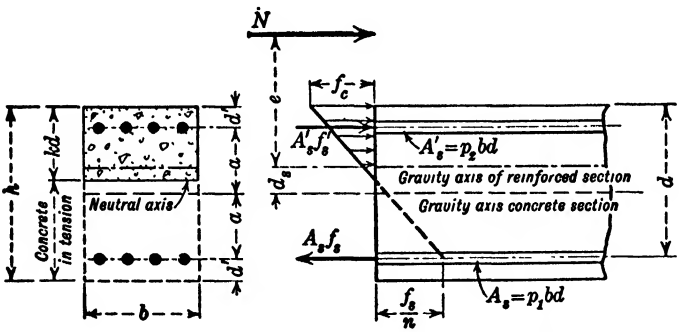


FIG. 104.—Unsymmetrically Reinforced Section Subjected to Eccentric Thrust. One Face in Tension. (*See p* 232.)

Center of Gravity of Section.—When the amount of tension reinforcement is different from the amount of compression reinforcement the center of gravity of the reinforced concrete section does not coincide with the center of gravity of the rectangle composing the concrete section.

The distance of the center of gravity may be found from the following formula:

Distance of Center of Gravity of Reinforced Section from Center of Concrete Section,

$$d_s = \frac{(n-1)(p_2 - p_1)}{\frac{h}{d} + (n-1)(p_1 + p_2)} a. \quad . \quad . \quad . \quad . \quad . \qquad (39)$$

Hence

$$C_e = \frac{d_s}{a} = \frac{(n-1)(p_2 - p_1)}{\frac{h}{d} + (n-1)(p_1 + p_2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

If the value of d_s is positive, the center of gravity is nearer the compression face of beam. For negative value of d_s the center of gravity is nearer the tension face of beam.

Use Diagram 16, p. 663, to determine value of C_c for different values of n, p_1 and p_2 and $\frac{h}{d} = 1.0$. For other values of $\frac{h}{d}$ and when p_2 is smaller than p_1, see explanation on p. 662.

Position of the Central Thrust N.—In the following formulas it is assumed that the section is subjected to a normal thrust N acting in the center of gravity of the reinforced concrete section (and not the center of the rectangle composing the concrete section) and to a bending moment $M = Ne$. The eccentricity e is measured from the center of gravity of the reinforced concrete section.

If the normal thrust acts in the center of the concrete rectangle, it produces, in addition to the moment $M = Ne$, a bending moment $- Nd_s$ which should be added to the main bending moment.

Areas of Reinforcement for Given Stresses.

For known dimensions of the concrete section b and d and specified stresses f_c and f_s the areas of compression and tension reinforcement required by a given N and $M = Ne$ may be found from the following formulas.

Area of Tension Reinforcement Unsymmetrically Reinforced Section.

$$A_s = \frac{1}{2f_s} N\left(\frac{e + d_s}{a} - 1\right) + p\left(1 - \frac{j}{2\frac{a}{d}}\right)bd, \quad . \quad . \quad (41)$$

also

$$A_s = \left(\frac{e}{a} + \frac{d_s}{a} - 1\right)\frac{N}{2f_s} + C_1 bd, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

where

$$C_1 = p\left(1 - \frac{j}{2\frac{a}{d}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Values of C_1 are given in Diagram 17, p. 664.

In the above equations j and p are values corresponding to the stresses f_c and f_s in balanced rectangular beams.

Area of Compression Reinforcement Unsymmetrically Reinforced Section,

$$A'_s = \frac{1}{2f_s}N\left(\frac{e + d_s}{a} + 1\right)\frac{1 - k}{k - \frac{d'}{d}} - p\frac{1 - k}{k - \frac{d'}{d}}\,\frac{j}{2\frac{a}{d}}bd, \quad . \quad (44)$$

also

$$A'_s = C_2\left(\frac{e}{a} + \frac{d_s}{a} + 1\right)\frac{N}{2f_s} - C_3bd, \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

where

$$C_2 = \frac{1 - k}{k - \frac{d'}{d}} \quad \text{and} \quad C_3 = p\cdot\frac{1 - k}{k - \frac{d'}{d}}\cdot\frac{j}{2\frac{a}{d}}. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Values of C_2 and C_3 are given in Diagram 18, p. 665.

In the above equations j, k and p are values corresponding to the stresses f_c and f_s in balanced rectangular beams. See Table 1, p. 649.

Unsymmetrically Reinforced Section. How to Use Equations (40) to (46) and Diagrams 16 to 18.—Equations (40) to (46) and Diagrams 16 to 18 are used for sections subjected to thrust N and bending moment $M = Ne$ when the concrete dimensions b and d are known or assumed and it is desired to find the amount of reinforcement for which the stresses in concrete and tension reinforcement reach the specified values of f_c and f_s.

Proceed as follows. Find the value of k from Table 1 which correspond to the specified stresses f_c, f_s and the ratio of moduli n. Compute the ratio $\frac{2a}{d}$. From proper diagrams select the corresponding values of C_1, C_2 and C_3. (Pages 663 to 665.)

Compute eccentricity e and ratio $\frac{e}{a}$. Since the ratios of tension and compression steel are not known the value of d_s, which is the distance of the center of gravity of concrete section to center of gravity of reinforced section, is not known. It may be assumed as equal 0.05a for small eccentricities and equal 0.15a for large eccentricities. These assumptions give the ratio $\frac{d_s}{a}$ as 0.05 and 0.15, respectively.

The constants found above are substituted in Formulas (41) and (44) and the areas of steel A_s and A'_s computed. The value of d_s should now be found from Diagram 16. If the difference between the assumed d_s and the computed value is considerable, new areas of steel A_s and A'_s should be computed for the new ratio of $C_e = \frac{d_s}{a}$.

Depth of Section for Fixed Ratio of Compression Reinforcement. Unsymmetrically Reinforced Section.—When it is desired that the ratio of compression steel p_2 shall not exceed a certain definite value the depth of the section may be found from the equation given below.

Depth of Section for Fixed Ratio of Compression Steel,

$$d = C_4 \frac{N}{bf_s}\left[1 + \sqrt{1 + \frac{2(e + d_s)}{\frac{a}{d} C_4 \frac{N}{bf_s}}}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where

$$C_4 = \frac{1}{4}\frac{1-k}{k - \frac{d'}{d}} \frac{1}{p_2 + p\frac{1-k}{k - \frac{d'}{d}}\frac{j}{2\frac{a}{d}}} = \frac{C_2}{4(p_2 + C_3)}. \quad . \quad . \quad (48)$$

The values of k, j and p in the Formula (48) are the constants in balanced rectangular beams for the selected unit stresses f_c, f_s and the ratio of moduli n. See Table 1, p. 649. C_2 and C_3 are constant from Diagram 18, p. 665.

The value N is the normal thrust and e is the eccentricity.

The value of d_s is the distance from center of gravity of plain section to center of gravity of reinforced section. It is not known, but it may be assumed as equal to $0.15e$. After A_s and A'_s are found check the assumed value of d_s.

After the depth d is found, the area of compression reinforcement is computed and finally the amount of tension reinforcement is found by Formula (41), p. 233.

SECTION REINFORCED FOR TENSION ONLY

When the concrete section is large enough to resist all compression stresses produced by eccentric thrust, compression reinforcement is not required and it is necessary to provide tension reinforcement only. This stress condition is illustrated in Fig. 105, p. 236. Formulas for such condition are developed in Vol. I, p. 185. Formulas there given are here repeated. Also additional formulas are given to cover all requirements.

Position of Normal Thrust.—The section is assumed to be subjected to a normal thrust acting in the center of gravity of the reinforced concrete section and to a bending moment $M = Ne$.

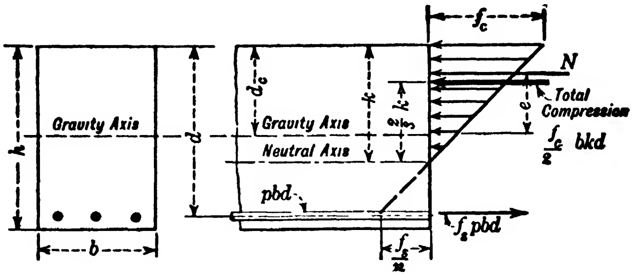


FIG. 105.—Section with Tension Steel Only. (See p. 235.)

Stresses in Section.

Maximum Compression Stress in Concrete,

$$f_c = \frac{M}{C_b bd^2} = \frac{Ne}{C_b bd^2}. \qquad (49)$$

Maximum Tensile Stress in Steel,

$$f_s = n\frac{1-k}{k}f_c, \qquad (50)$$

also

$$f_s = C_s\frac{Ne}{C_b bd^2}, \qquad (51)$$

where

Constants,

$$C_b = \frac{k}{2}\left(\frac{d_c}{d} - \tfrac{1}{3}k\right) + np\frac{1-k}{k}\left(1 - \frac{d_c}{d}\right). \qquad (52)$$

$$C_s = n\frac{1-k}{k}. \qquad (53)$$

$$\frac{d_c}{d} = \frac{1}{2}\frac{h}{d}\,\frac{1 + 2(n-1)p\left(\frac{d}{h}\right)^2}{1 + (n-1)p\frac{d}{h}}. \qquad (54)$$

$$\frac{e}{d} = \frac{k^2\left(\frac{d_c}{d} - \tfrac{1}{3}k\right) + 2np(1-k)\left(1 - \frac{d_c}{d}\right)}{k^2 - 2np(1-k)}. \qquad (55)$$

Values of C_b, k and $\frac{d_c}{d}$ may be taken from diagrams, pp. 666–667. The values of C_s for known k may be taken from Diagram 15, p. 662.

Area of Reinforcement for Given Stresses.—Given dimensions b and d, also the thrust N and bending moment $M = Ne$. Specified are stresses f_c, f_s and ratio n.

Problem is to find the area of steel required to keep the tensile stresses within the allowable limits.

The problem may be solved by means of Diagrams 19 and 20, pp. 666 667.

From Formulas (49) and (51), p. 236, following relations are obtained:

Constant governed by compression,

$$C_b = \frac{Ne}{f_c bd^2}. \qquad (55a)$$

Constant governed by tension,

$$C_b = C_s \frac{Ne}{f_s bd^2}. \qquad (56)$$

These values of the constants may be found from the above formulas because all values in the equations are known.

Assume value of k and find C_s. Find both values of C_b. For the assumed k and the first value of C_b find from Diagram 20 the corresponding value of p. For this value of p and the known ratio of eccentricity check the assumed value of k by means of Diagram 19, p. 666. If the value found from Diagram 19 is appreciably different from the assumed value, new value of p should be found on the basis of a new k.

Similarly find a value of p corresponding to the second value of C_b and k. The larger of the two values should be used.

Finally, knowing the ratio of steel p the area of steel is

$$A_s = pbd. \qquad (57)$$

Dimensions of Section for Accepted Stresses and Ratio of Steel p.—Given width of section b, thrust N and bending moment $M = Ne$ and the stresses f_c and f_s. Find depth of section for which neither the stresses are exceeded nor the ratio of steel is larger than a certain definite value of p.

Problem is solved by using following formulas in conjunction with the diagram.

Depth of Section Governed by Compression Stresses,

$$d = \sqrt{\frac{Ne}{C_b f_c b}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Depth of Section Governed by Tensile Stresses,

$$d = \sqrt{C_s \frac{Ne}{C_b b d f_s}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

In the above equation C_b and C_s are not known. They are found from proper diagrams corresponding to the accepted value of p and an assumed value of k. The values of d are then found, and the value of k checked from Diagram 19, p. 666. If large error was made in assuming value of k the process is repeated.

The depth as governed by f_c will not be equal to the depth governed by f_s. The larger of the two values must be accepted.

DEVELOPMENT OF FORMULAS FOR UNSYMMETRICALLY REINFORCED SECTION

The formulas already given for unsymmetrically reinforced section subjected to thrust and bending moments are developed as follows:

Problem.—It is desired that the section be reinforced so that the stresses in steel and concrete reach simultaneously the specified values f_c and f_s.

Notation.

Let h = depth of rectangular section, in.;
b = breadth of rectangular section, in.;
d = depth of steel in tension, in.;
d_1 = depth of steel in compression, in.;
a = distance from center of concrete section to steel, in.;
d_s = distance of center of gravity of reinforced section from center of gravity of plain section;
f_s = maximum unit tension stress in steel, lb. per sq. in.;
f_c = maximum unit compression in concrete, lb. per sq. in.;
n = ratio of moduli of elasticity;
k = ratio of depth of neutral axis to depth of tension steel d;
$j = 1 - \frac{1}{3}k$ = distance of tension steel from center of gravity of compression stresses in concrete, balanced rectangular section;

p = ratio of tension steel in balanced rectangular section;
p_2 = ratio of actual amount of compression steel;
p_1 = ratio of actual amount of tension steel;
$A_s = p_1bd$ = area of tension steel, sq. in.;
$A'_s = p_2bd$ = area of compression steel, sq. in.

Center of Gravity Axis of Section.—The effective area of the section consists of the concrete area and of the compression and tension areas of steel.

To get the gravity axis of the reinforced section the concrete and steel areas are converted into a homogenous concrete section in which the steel area is replaced by a concrete area equal to n times the steel area and placed in the same relation to the axis of the section as that of the reinforcing bars.

Since the effective section is not symmetrical its center of gravity will not coincide with the center of the rectangle forming the beam but will rest a certain distance d_s above (or below) it. Assume that the compression zone is placed above and the tensile zone below. Then when the amount of compression steel is larger than of tension steel the actual center of gravity will be above the center of the rectangle. In such case d_s is positive. For A_s larger than A'_s the actual center of gravity is below and d_s is negative. The magnitude of d_s is computed as follows.

Referring to Fig. 106, p. 240, the area of the effective section is

$$A = bh + (n - 1)A_s + (n - 1)A'_s,$$

also

$$A = bh + (n - 1)p_1bd + (n - 1)p_2bd.$$

Compute the static moment of all areas about the center of rectangle, considering the moment of the compression areas as positive and of the tensile areas as negative.

Static moment of concrete area is zero.
Static moment of compression steel $(n - 1)p_2bda$.
Static moment of tension steel $- (n - 1)\ p_1bda$.

The total static moment is

$$M = (n - 1)bd(p_2 - p_1)a.$$

The distance d_s is obtained by dividing the total static moment by the total effective area.

$$d_s = \frac{(n - 1)bd(p_2 - p_1)a}{bh + (n - 1)bd(p_1 + p_2)} = \frac{(n - 1)(p_2 - p_1)}{\frac{h}{d} + (n - 1)(p_1 + p_2)}a. \quad (60)$$

Finally

$$C_c = \frac{d_s}{a} = \frac{(n-1)(p_2 - p_1)}{\frac{h}{d} + (n-1)(p_1 + p_2)}. \qquad (61)$$

The values of C_c are given in Diagram 16, p. 663.

Formulas for A_s and A'_s.—A reinforced concrete rectangular section with dimensions b and h is subjected to a thrust N and a bending moment M acting simultaneously. The problem is to find the required areas of compression and tension reinforcement for which the stresses in concrete and in the tension reinforcement will be equal to the selected unit stresses f_c and f_s and the corresponding ratio n.

Making same assumptions as for beams subjected to flexure only (see Vol. I, p. 126), the stresses will be as represented in Fig. 106.

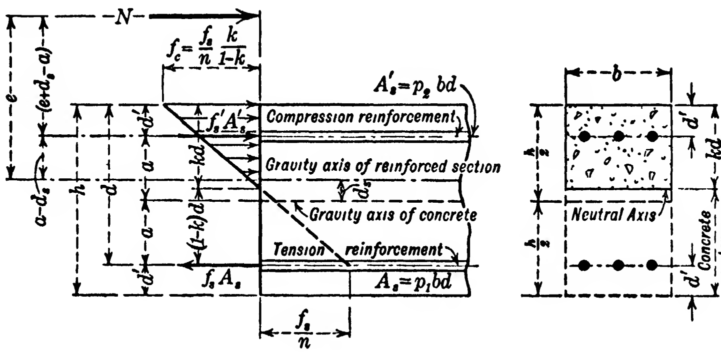


Fig. 106.—Section with Unsymmetrical Reinforcement Subjected to Thrust and Bending, One Face in Tension. (See p. 240.)

The position of neutral axis and the relation between tension and compression stresses are the same as for a beam subjected to flexure only. Thus

$$k = \frac{1}{1 + \frac{f_s}{nf_c}}, \qquad (62)$$

$$f_s = nf_c\frac{1-k}{k}, \qquad (63)$$

also

$$f_c = f_s\frac{k}{n(1-k)}. \qquad (64)$$

The relation of compression stress in steel to the tensile stresses is, as evident from Fig. 106,

$$f'_s = f_s\frac{k - \frac{d'}{d}}{1 - k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

The areas of steel may be obtained from the two requirements of equilibrium.

1. The bending moment about any point of all stresses is equal and of opposite sign to the bending moment about this point of all external forces.

2. The sum of all stresses acting on a section must be equal to the normal thrust N.

Tension reinforcement will be found from the first requirement of equilibrium. To simplify the work the external bending moment M and the central thrust N will be replaced by a thrust acting eccentrically (see Vol. I, p. 165). The eccentricity equals $e = \frac{M}{N}$ and it is measured from the center of gravity of the converted section (and not the center of gravity of plain section).

The section is thus exposed to the external force, N acting eccentrically on the one hand and to the resisting stresses composed of the compression stresses in steel and concrete and tension stresses in steel on the other hand.

Take the moment of all stresses about the center of compression steel. This is equal to total tension times $2a$, minus compression in concrete times $(\frac{1}{3}kd - d')$. The equation is

$$M = 2A_sf_sa - bd\frac{k^2}{2n(1 - k)}f_s(\tfrac{1}{3}kd - d'). \quad . \quad . \quad . \quad . \quad . \quad (66)$$

As evident from the figure the moment arm for the external force, i.e., the distance from the force to the compression steel, is $(e + d_s - a)$ and the moment

$$M = N(e + d_s - a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66a)$$

Equating Formulas (66) and (66a) we have

$$N(e + d_s - a) = 2A_sf_sa - bd\frac{k^2}{2n(1 - k)}fs(\tfrac{1}{3}kd - d'). \quad . \quad . \quad . \quad . \quad . \quad (67)$$

Finally area of tension steel

$$A_s = \frac{1}{2f_s}N\left(\frac{e + d_s}{a} - 1\right) + \frac{k^2}{2n(1 - k)}\frac{(\frac{1}{3}kd - d')}{2a}bd. \quad . \quad (68)$$

Since from Fig. 106 $\frac{1}{3}kd - d' = 2a - jd$ and $\frac{k^2}{2n(1-k)} = p$ for balanced rectangular beams

$$A_s = \frac{1}{2f_s}N\left(\frac{c + d_s}{a} - 1\right) + p\left(1 - \frac{j}{2\frac{a}{d}}\right)bd. \quad . \quad . \quad . \quad . \quad (69)$$

The second requirement of equilibrium will be used to determine the amount of compression reinforcement.

The compression stresses on the section are:

(*a*) in concrete $\frac{f_c bkd}{2}$ also $\frac{k^2}{2n(1-k)}f_s bd$, since $f_c = f_s\frac{k}{n(1-k)}$,

(*b*) in steel $A'_s f'_s = A'_s f_s \frac{k - \frac{d'}{d}}{1-k}$.

Total Compression equals $f_s bd\frac{k^2}{2n(1-k)} + A'_s f_s \frac{k - \frac{d'}{d}}{1-k}$. . . (70)

Total Tension in Steel equals $- f_s A_s$.

Concrete is considered to resist no tension. The sum of tension and compression equals N. Thus

$$N = f_s bd\frac{k^2}{2n(1-k)} + A'_s f_s \frac{k - \frac{d'}{d}}{1-k} - f_s A_s. \quad . \quad . \quad . \quad (71)$$

From this equation we get

$$A_s = bd\frac{k^2}{2n(1-k)} + A'_s \frac{k - \frac{d'}{d}}{1-k} - \frac{1}{f_s}N, \quad . \quad . \quad . \quad . \quad (72)$$

also since $\frac{k^2}{2n(1-k)} = p$

$$A_s = pbd + A'_s \frac{k - \frac{d'}{d}}{1-k} - \frac{1}{f_s}N. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72a)$$

The area of tension steel is also represented by Equation (69), p. 242.

To get the area of compression steel equate Formulas (69) and (72a) and solve for A'_s

$$\frac{1}{2f_s}N\left(\frac{e+d_s}{a}-1\right)+p\left(1-\frac{j}{2\frac{a}{d}}\right)bd = pbd + A'_s\frac{k-\frac{d'}{d}}{1-k}-\frac{1}{f_s}N. \quad (73)$$

From this

$$A'_s\frac{k-\frac{d'}{d}}{1-k} = -\frac{j}{2\frac{a}{d}}pbd+\frac{1}{2f_s}N\left(\frac{e+d_s}{a}+1\right). \quad . \quad . \quad (74)$$

Finally,

Area of Compression Reinforcement,

$$A'_s = \frac{1}{2f_s}N\left(\frac{e+d_s}{a}+1\right)\frac{1-k}{k-\frac{d'}{d}}-\frac{j}{2\frac{a}{d}}\frac{1-k}{k-\frac{d'}{d}}pbd. \quad . \quad (75)$$

In Formulas (69) and (75) k and j are values corresponding to the selected stresses f_c and f_s, and p is the ratio of steel in balanced rectangular beams corresponding to the selected stresses. Expressing the values depending upon the stresses by constants, the formulas become:

Area of Tension Steel,

$$A_s = \frac{N}{2f_s}\left(\frac{e+d_s}{a}-1\right)+C_1bd, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

where

$$C_1 = p\left(1-\frac{j}{2\frac{a}{d}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

Area of Compression Steel,

$$A'_s = C_2\left(\frac{e+d_s}{a}-1\right)\frac{N}{2f_s}-C_3bd, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

where

$$C_2 = \frac{1-k}{k-\frac{d'}{d}}, \quad \text{and} \quad C_3 = p\frac{1-k}{k-\frac{d'}{d}}\frac{j}{2\frac{a}{d}}.$$

The constants are given in Diagram 8, p. 665.

DEVELOPMENT OF FORMULAS FOR SECTION WITH STEEL NEAR SIDE UNDER MAXIMUM STRESS ONLY

As evident from pp. 217 to 220, in sections subjected to eccentric thrust the reinforcement near the side with minimum stress resists very small stresses. If there is no possibility of tension stresses this reinforcement is practically wasted. It may, therefore, be desirable to omit this reinforcement and use steel only in the part with maximum stresses. For such case formulas given below apply.

Let section be provided with reinforcement near one side only, also let

$A'_s = p_2bh$ = steel near side under maximum stress;
p_2 = steel ratio;
b = breadth of section;
h = depth of section;
$a = \frac{h}{2} - d'$ = distance center of concrete section to center of steel;
d_s = distance center of gravity of concrete section to center of gravity of reinforced section.

Due to the assumed arrangement of steel the center of gravity of the reinforced section does not coincide with that of the concrete section. The distance between the two centers is obtained by taking static moments about the center of concrete section and dividing it by the area of the reinforced section.

Moment of concrete section about its center being zero, the moment of the reinforced section is

$$M = (n - 1)p_2bha.$$

The area of reinforced section converted into concrete is

$$A = bh + (n - 1)p_2bh = bh[1 + (n - 1)p_2].$$

Consequently,

$$d_s = \frac{(n - 1)p_2bh}{bh[1 + (n - 1)p_2]}a. \qquad (79)$$

Finally,

Distance between Axis of Gravity of Plain and Reinforced Sections,

$$d_s = \frac{(n - 1)p_2}{1 + (n - 1)p_2}a. \qquad (80)$$

The moment of inertia of the reinforced section about its center of gravity is composed of the moment of inertia of concrete plus moment of inertia of steel.

Moment of inertia of concrete section,

$$I_c = \frac{bh^3}{12} + bhd_s^2 = bh^3\left[\frac{1}{12} + \left(\frac{d_s}{h}\right)^2\right] = bh^3\left[\frac{1}{12} + \left(\frac{(n-1)p_2}{1+(n-1)p_2}\right)^2\left(\frac{a}{h}\right)^2\right].$$

Moment of inertia of steel,

$$I_s = (n-1)p_2bh(a-d_s)^2 = (n-1)p_2bha^2\left(1 - \frac{(n-1)p_2}{1+(n-1)p_2}\right)^2$$

$$= (n-1)p_2bha^2\left(\frac{1+(n-1)p_2-(n-1)p_2}{1+(n-1)p_2}\right)^2$$

$$= \frac{(n-1)p_2}{[1+(n-1)p_2]^2}\left(\frac{a}{h}\right)^2 bh^3.$$

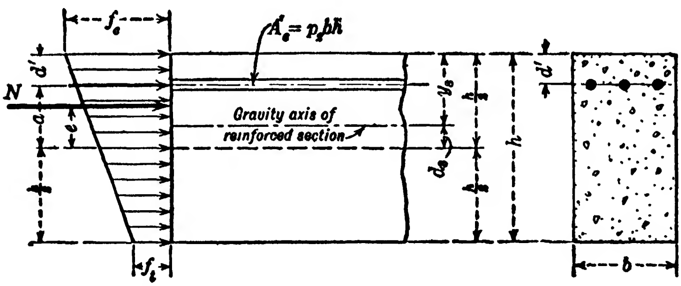


Fig. 107.—Section Reinforced Near One Face, Subjected to Thrust and Bending Moment. (*See p.* 244.)

Total Moment of Inertia,

$$I = I_c + I_s = bh^3\left[\frac{1}{12} + \left(\frac{a}{h}\right)^2\frac{(n-1)p_2[1+(n-1)p_2]}{[1+(n-1)p_2]^2}\right]$$

$$= bh^3\left[\frac{1}{12} + \frac{(n-1)p_2}{1+(n-1)p_2}\left(\frac{a}{h}\right)^2\right]. \qquad (81)$$

The section is subjected to a thrust N acting in the center of the concrete section and the bending moment $M = Ne$. Since the center of gravity of the *concrete* section does not coincide with the center of the *reinforced concrete* section the thrust N produces a bending moment

$M_1 = -Nd_s$ so that the actual bending moment acting on the section is $M = Ne - Nd_s = N(e - d_s)$. The stresses in the section are

$$f_c = \frac{N}{A} + \frac{N(e - d_s)y_s}{I}.$$

The formulas for A and I are given above.

The value if y_s is found as follows:

$$y_s = \frac{h}{2} - d_s = \frac{h}{2} - \frac{(n-1)p_2}{1+(n-1)p_2}a = \left(\frac{1}{2} - \frac{(n-1)p_2}{1+(n-1)p_2}\frac{a}{h}\right)h$$

$$= \frac{1 + (n-1)p_2 - 2(n-1)p_2\frac{a}{h}}{2[1+(n-1)p_2]}h = \frac{1 + (n-1)p_2\left(1 - 2\frac{a}{h}\right)}{2[1+(n-1)p_2]}h.$$

Finally, since $1 - 2\frac{a}{h} = 2\frac{d'}{h}$,

$$y_s = \frac{1 + 2(n-1)p_2\frac{d'}{h}}{2[1+(n-1)p_2]}h. \quad \ldots \ldots \quad (82)$$

Therefore,

$$f_c = \frac{N}{bh[1+(n-1)p_2]} + \frac{\frac{1 + 2(n-1)p_2\frac{d'}{h}}{2[1+(n-1)p_2]}h}{bh^3\left[\frac{1}{12} + \frac{(n-1)p_2}{1+(n-1)p_2}\left(\frac{a}{h}\right)^2\right]}N(e-d_s) \quad (83)$$

This simplified gives

Maximum Compression Stress,

$$f_c = \frac{N}{bh}\left\{\frac{1}{1+(n-1)p_2} + \frac{e-d_s}{h}\,\frac{6\left[1 + 2(n-1)p_2\frac{d'}{h}\right]}{1+(n-1)p_2\left[1 + 12\left(\frac{a}{h}\right)^2\right]}\right\}. \quad (84)$$

The minimum compression stress is obtained from formula

$$f_t = \frac{N}{A} - \frac{h - y_s}{I}N(e-d_s).$$

By substituting values for A, $h - y_s$, and I, and simplifying, following final formula is obtained.

Minimum Compression Stress,

$$f_t = \frac{N}{bh}\left\{\frac{1}{1+(n-1)p_2} - \frac{e-d_s}{h}\,\frac{6\left[1+(n-1)p_2\left(1+2\frac{a}{h}\right)\right]}{1+(n-1)p_2\left[1+12\left(\frac{a}{h}\right)^2\right]}\right\}. \quad (85)$$

PLAIN OR UNREINFORCED SECTION

Example 1. Finding Dimensions and Stresses.—Plain concrete section subjected simultaneously to a normal thrust acting centrally and bending moment.

Given: Normal thrust, $N =$ 300 000 lb.

Bending moment, $M =$ 1 000 000 in.-lb.,

hence,

$$\text{Eccentricity, } e = \frac{1\,000\,000}{300\,000} = 3.33 \text{ in.}$$

Assumed width of section $b =$ 40 in.

Find: depth of section h for which maximum compressive stresses do not exceed 400 lb. per sq. in. No tensile stresses in concrete are permitted.

Such problem may occur in design of plain concrete arches, dams, eccentrically loaded piers, etc.

Solution.—This example is solved by using Diagram 1, opp. p. 648.

$$\text{Compute} \qquad \frac{bf_c}{N} = \frac{40 \times 400}{300\,000} = 0.0533.$$

Locate in Diagram 1, opp. p. 648, $e =$ 3.33 in. and $\frac{bf_c}{N} = 0.0523$ and get $h =$ 31 in.

Since $\frac{h}{6} = \frac{31}{6} =$ 5.17 in. and $l =$ 3.33 in. is smaller than h, the whole section is in compression.

Check the correctness of this solution by using Formulas (3) and (5), p. 214.

$$f_c = \frac{300\,000}{40 \times 31}\left(1 + \frac{6 \times 3.3}{31}\right) = 398 \text{ lb. per sq. in.}$$

$$f_t = \frac{300\,000}{40 \times 31}\left(1 - \frac{6 \times 3\ 3}{31}\right) = 155 \text{ lb. per sq. in.}$$

The depth of section $h =$ 31 in. is satisfactory.

Example 2.—Find dimensions of foundations subjected to eccentric pressure.[a]

Given: Normal thrust, $N =$ 700 000 lb.

Eccentricity, $e =$ 10 in.

[a] Such problem may occur in arch design where the center of the footing was made to coincide with the line of pressure for dead load. The actual dimension of the footing is found for the most unfavorable combination of dead load with live load and temperature stresses.

Allowable maximum pressure on foundation $f_c = 10\,000$ lb. per sq. ft.

No uplift is permitted, therefore $f_t = 0$.

Solution.—Assume $b = 9$ ft. Since the dimensions are in feet change the eccentricity to feet. Therefore $e = \frac{10}{12} = 0.82$ ft.

Compute $$\frac{bf_c}{N} = \frac{9 \times 10\,000}{700\,000} = 0.129.$$

Locate in Diagram 1, opp. p. 648,

$$e = 0.82 \text{ ft.} \quad \text{and} \quad \frac{bf_c}{N} = 0.129$$

which gives $$h = 11.2 \text{ ft.}$$

Check the stresses by Formulas (3) and (5), p. 214.

$$f_c = \frac{700\,000}{9 \times 11.2}\left(1 + \frac{6 \times 0.82}{11.2}\right) = 10\,000 \text{ lb. per sq. ft.}$$

$$f_t = \frac{700\,000}{9 \times 11.2}\left(1 - \frac{6 \times 0.82}{11.2}\right) = 3\,890 \text{ lb. per sq. ft.}$$

To get the most economical results, several assumptions as to the width of foundations should be made and the corresponding depth should be computed as above. The dimensions giving the cheapest foundation should be accepted.

SYMMETRICALLY REINFORCED CONCRETE SECTIONS

Example 3. Finding Dimensions.—Find dimensions of a symmetrically reinforced concrete section subjected to direct stress and bending for the following condition:

Given: Normal thrust, $N = 300\,000$ lb.

Bending moment, $M = 1\,800\,000$ in.-lb.

$$e = \frac{1\,800\,000}{300\,000} = 6 \text{ in.}$$

Allowable unit stresses $f_c = 600$ lb. per sq. in. (compression)

$f_t = -\,40$ lb. per sq. in. (tension)

$n = 15.$

Accepted ratio of reinforcement, $p = 0.01$, and distance center of steel to outside face $d' = 2$ in.

Solution.—Assume width of section $b = 24$ in.

$$f_c = 600 \text{ and } b = 24 \quad \frac{bf_c}{N} = \frac{24 \times 600}{300\,000} = 0.048$$

$$f_t = -\,40 \text{ and } b = 24 \quad \frac{bf_t}{N} = -\frac{24 \times 40}{300\,000} = -\,0.0032.$$

Locating $e = 6$ in. and $\frac{bf_c}{N} = 0.048$ in Diagram 1, opp. p. 648, gives $h = 34$ in. Using Diagram 2, opp. p. 648, and locating $e = 6$ in. and $\frac{bf_t}{N} = -0.0032$ gives $h = 27$ in.

The larger of the two values of h should be used. Therefore the design section fulfilling all requirements is

$$b = 24 \text{ in.}$$

$$h = 34 \text{ in.}$$

$$A_s = 0.01 \times 24 \times 34 = 8.2 \text{ sq. in.}$$

Example 4. Finding Stresses. -Find maximum stresses in concrete for a symmetrically reinforced concrete section subjected to direct stress and bending.

Given: Normal thrust, $N = 180\,000$ lb.

Bending moment, $M = 720\,000$ in.-lb.

$$e = \frac{M}{N} = 4.0 \text{ in.}$$

Dimensions: $b = 30$ in., $h = 20$ in., $d' = 2$ in.

Reinforcement: Ten ⅞-in. round bars arranged symmetrically near both faces.

$$A_s = 10 \times 0.6 = 6.0 \text{ sq. in.}$$

$$p = \frac{6.0}{30 \times 20} = 0.01.$$

Solution.—Since for $h = 20$ in. and $e = 4.0$ in. the ratio of eccentricity

$$\frac{e}{h} = \frac{4.0}{20.0} = 0.2,$$

the whole section is effective and Formulas (13) and (14), p. 219 apply. Diagrams 3 to 6, pp. 650 to 653, will be used to solve the problem.

For $h = 20$ in. and $d' = 2$ in., $2a = 20 - 4 = 16$ in. and $\frac{2a}{h} = \frac{16}{20} = 0.8$. Therefore use Diagram 5 on page 652 marked $2a = 0.8h$

Locate in the diagram $p = 0.01$ and $\frac{e}{h} = 0.2$ and find $C_e = 1.63$.

The maximum compression stress in concrete from Formula (13), p. 219.

$$f_c = 1.63 \times \frac{180\,000}{30 \times 20} = 489 \text{ lb. per sq. in.}$$

Example 5. Finding Areas of Steel.—Find the required amount of reinforcement arrangement symmetrically near two faces for which the maximum compression stresses would not exceed $f_c = 450$ lb. per sq. in.

Given: Normal thrust, $N = 180\,000$ lb.

Bending moment, $M = 720\,000$ in.-lb.

$$e = \frac{M}{N} = 4 \text{ in.}$$

Dimensions of sections: $b = 30$ in. $h = 20$ in., and $d' = 2$ in.

Solution.—For $h = 20$ in., $e = 4$ in., $\frac{e}{h} = \frac{4}{20} = 0.2$.

Since $h = 20$ in. and $d' = 2$ in. Therefore, $2a = 20 - 2 \times 2 = 16$ in. and

$$\frac{2a}{h} = 0.8.$$

Use Diagram 5, p. 652, marked $2a = 0.8h$.

Find $$C_e = \frac{f_c bh}{N} = \frac{450 \times 30 \times 20}{1\,800\,020} = \frac{270\,000}{180\,000} = 1.5.$$

Locate in the diagram

$$C_e = 1.5 \quad \text{and} \quad \frac{e}{h} = 0.2$$

and find

$$p = 0.0141.$$

The required amount of reinforcement, therefore, is

$$A_s = 0.0141 \times 30 \times 20 = 8.46 \text{ sq. in.}$$

SECTION REINFORCED ONLY NEAR HIGHLY COMPRESSED FACE

Example 6. Finding Dimensions.—Find depth of section, reinforced only near highly compressed face, for following conditions:

Given: Normal thrust, $N = 200\,000$ lb.

Bending moment, $M = 700\,000$ in.-lb.

Eccentricity, $e = \frac{M}{N} = 3.5$ in.

Normal thrust acts in the center of the concrete section.

Allowable stress, $f_c = 450$ lb. per sq. in.

Accepted

Width of section, $b = 20$ in.

Steel ratio, $p_2 = 0.02$

$d' = 2$ in.

Solution.—This problem may be solved by using Formula 26, p. 225.

$$h = \tfrac{1}{2}C_n\frac{N}{bf_c}\left(1 + \sqrt{1 + \frac{4C_m}{(C_n)^2}\,\frac{e - d_s}{\frac{N}{bf_c}}}\right)$$

Compute $$\frac{N}{bf_c} = \frac{200\,000}{20 \times 450} = 22.2. \quad \text{Assume} \quad \frac{d'}{h} = 0.1 \text{ and } d_s = 2 \text{ in}$$

For the specified ratio of steel $p_2 = 0.02$ and the assumed $\frac{d'}{h} = 0.1$ the constants C_n and C_m taken from table on p. 226 are

$$C_n = 0.781, \quad C_m = 3.49 \text{ and } \frac{4C_m}{(C_n)^2} = \frac{4 \times 3.49}{0.781^2} = 22.8.$$

$$\text{Finally } h = 0.39 \times 22.2\left(1 + \sqrt{1 + 22.8 \times \frac{3.5 - 2}{22.2}}\right) = 8.65 \times 2.59 = 22.3.$$

For the computed depth $\frac{d'}{h} = \frac{2}{22.4} = 0.082$ and $d_s = 0.219 \times 9.2 = 2.0$. These are sufficiently close to the assumed value. In case of large difference the dimension should be recomputed.

Solution.— $b = 20$ in. $h = 23$ in. (in even inches)

$p_2 = 0.02$ hence $A_s = 20 \times 22.4 \times 0.02 = 8.96$ sq. in.

All the computed reinforcement should be placed near the face under high compression. It is advisable, however, to use some additional steel near the opposite face.

Example 7. Finding Area of Steel.—Find the area of compression reinforcement placed near the highly compressed surface required to reduce the maximum compressive stress to $f_c = 450$ lb. per sq. in. Use ratio of moduli of elasticity $n = 15$.

Given: Normal thrust, $N = 200\,000$ lb.

Bending moment, $M = 900\,000$ in.-lb.

Eccentricity, $e = \frac{M}{N} = 4.5$ in.

Dimensions of section: $h = 28$ in., $b = 30$ in. and $d' = 2$ in.

Allowable compressive stress $f_c = 450$ lb. per sq. in.
Allow 2 in. on each side of concrete section for fireproofing.

Solution.—Since it is necessary to allow 2 in. on each side for fireproofing, the effective section is $h = 24$ in., $b = 26$ in., and $d' = 0$. Therefore $\frac{d'}{h} = 0$.

Compute $f_c\frac{bh}{N} = 450 \times \frac{26 \times 24}{200\,000} = 1.4$ and $\frac{e}{h} = \frac{4.5}{24} = 0.187$.

Substituting these values in Formula (27), p. 226, we find $p_2 = 0.008$. The required area of reinforcement, to be placed near the face under maximum compression, is

$$A_s' = 26 \times 24 \times 0.008 = 5.0 \text{ sq. in.}$$

SYMMETRICALLY REINFORCED SECTION. ONE FACE IN TENSION

Example 8. Finding Stresses.—Find the maximum compressive stresses in concrete f_c and the tensile stresses in steel f_s in a section subjected to direct stress and bending for the following conditions.

Dimensions $b = 24$ in., $h = 30$ in., $d = 28$ in., and $d = 2$ in. Whole section is considered as effective.

Reinforcement symmetrically arranged,

$$A_s = 11.6 \text{ sq. in.},$$

hence

$$p = \frac{11.6}{24 \times 30} = 0.016.$$

$$\text{Normal thrust, } N = 180\,000 \text{ lb.}$$

$$\text{Bending moment, } M = 2\,900\,000 \text{ in.-lb.}$$

$$\text{Eccentricity, } e = \frac{M}{N} = 16.1 \text{ in.}$$

Solution.—Since the eccentricity is large, a part of the section will be in tension. This problem, therefore, must be solved by using Diagrams 9 to 15, pp. 656 to 662.

Since $h = 30$ in., $d = 28$ in., and $\frac{h}{d} = 1.07$ use diagrams marked $h = 1.1d$.

By locating in Diagram 11, $p = 0.016$ and $\frac{e}{d} = \frac{16.1}{28} = 0.57$ find $k = 0.6$.

Locating in Diagrams 12 and 15, respectively, $k = 0.6$ and $p = 0.016$ find values of C_a and C_s:

$$C_a = 0.19$$

$$C_s = 10$$

These substituted in Formulas (28) and (30), p. 228, give

$$f_c = \frac{Ne}{C_a b d^2} = \frac{180\,000 \times 16.1}{0.19 \times 24 \times 28^2} = 810 \text{ lb. per sq. in.}$$

$$f_s = C_s \frac{Ne}{C_a b d^2} = 10 \frac{180\,000 \times 17.7}{0.19 \times 24 \times 28^2} = 8100 \text{ lb. per sq. in.}$$

Answer.—The maximum compression stress in concrete is $f_c = 810$ and the maximum tension stress in steel is $f_s = 8100$.

Example 9. Finding Dimensions.—Find depth of section h, subjected to direct stress and bending and reinforced symmetrically near both faces, for which the ratio of steel will not exceed $p = 0.02$. No fireproofing is required, hence the whole section may be considered as effective.

Given: Normal thrust, $N = 125\,000$ lb.

$$\text{Bending moment, } M = 1\,750\,000 \text{ in.-lb.}$$

$$\text{Eccentricity, } e = \frac{M}{N} = 14.$$

Normal thrust acts in the center of the section.

Specified stresses:

Allowable compressive unit stress in concrete, $f_c = 750$ lb. per sq. in.

Allowable tensile unit stress in steel, $f_s = 16\,000$ lb. per sq. in.

Ratio of elasticity, $n = 15$.

Assume:

Width of section $b = 36$ in. $d' = 2$ in.

Solution.—The depth can be found from Formulas (35) and (36), p. 230.

$$d = \sqrt{\frac{Ne}{C_a b f_c}} \quad \text{and} \quad d = \sqrt{C_s \frac{Ne}{C_a b f_s}}.$$

The larger of the two values must be accepted. Values of C_a and C_s are taken from proper diagrams. First assume $k = 0.5$ and $h = 1.1d$.

Use Diagram 15 for C_s and Diagram 12, p. 659, marked $h = 1.1d$ for C_a.

For the assumed $k = 0.5$ and the accepted $p = 0.02$ find the proper diagrams

$$C_a = 0.218 \quad \text{and} \quad C_s = 15.$$

Compute:

$$d = \sqrt{\frac{Ne}{C_a b f_c}} = \sqrt{\frac{125\,000 \times 14}{0.218 \times 36 \times 750}} = 17.2 \text{ in.}$$

$$d = \sqrt{C_s \frac{Ne}{C_a b f_s}} = \sqrt{15 \times \frac{125\,000 \times 14}{0.218 \times 36 \times 16\,000}} = 14.5 \text{ in.}$$

The depth governed by compression is larger of the two values and will be accepted.

For the depth governed by compression compute

$$\frac{e}{d} = \frac{14}{17.2} = 0.81 \quad \text{and} \quad \frac{h}{d} = \frac{17.2 + 2}{17.2} = 1.11$$

and find from Diagram 11, p. 658, corresponding to $\frac{e}{d} = 0.81$ and $p = 0.02$, $k = 0.55$.

For the new value of k find from Diagram 12, p. 659, $C_a = 0.211$.

Finally,

$$d = \sqrt{\frac{125\,000 \times 14}{0.211 \times 36 \times 750}} = 17.5.$$

Answer.—$b = 36$ in., $h = 18.0 + 2 = 20$ in. (in even inches), and for $p = 0.02$, $A_s = 0.02 \times 36 \times 17.5 = 12.6$ sq. in.

Example 10. Finding Area of Steel.—Find the required area of steel when the dimensions of the section are given and the allowable stresses are specified. Fireproofing of 2 in., measured to center of steel, is required. The section to be symmetrically reinforced.

Given: Normal thrust, $N = 120\,000$ lb.

Bending moment, $M = 2\,160\,000$ in.-lb.

$$\text{Eccentricity, } e = \frac{M}{N} = 18 \text{ in.}$$

Dimensions: $b = 24$ in. $h = 26$ in. $d' = 2$ in.

To get effective area deduct fireproofing.

Specified stresses: $f_c = 800$ lb. per sq. in.

$f_s = 10\,000$ lb. per sq. in.

$n = 15.$

Small value of f_s is used to prevent opening of cracks. This is often desired in structures subjected to effects of the weather.

Solution.—The problem will be solved by means of diagrams for C_a and C_s. Since it is necessary to provide 2-in. fireproofing on each side the effective width is $b = 24 - 4 = 20$ in. and the effective depth $h = 26 - 4 = 22$ in. The value of $d = h$ and $\frac{h}{d} = 1.0$.

Therefore use Diagrams 9 and 10, pp. 656–657, marked $h = 1.0d$.

$$\frac{e}{d} = \frac{18}{22} = 0.82.$$

Assume $k = 0.5$.

From Formulas (35) and (36), p. 230, compute

$$C_a = \frac{Ne}{bd^2 f_c} = \frac{120\,000 \times 18}{20 \times 22^2 \times 800} = 0.279$$

$$C_a = C_s \frac{Ne}{bd^2 f_s} = 15 \times \frac{120\,000 \times 18}{22 \times 22^2 \times 10\,000} = 0.334.$$

Locate in diagram for C_a the value $k = 0.5$ and $C_a = 0.279$ and find $p = 0\,026$. Also locate the value of $k = 0.5$ and $C_a = 0.334$ and find $p = 0.033$. The larger of the two values will be used. Now check the assumed values of k. Value of k is found from Diagram 9, p. 656, for $p = 0.028$ and $\frac{e}{d} = 0.82$. It is

$$k = 0.53.$$

For the revised value of k the required ratio is $p = 0.024$. This will be used.

UNSYMMETRICALLY REINFORCED SECTION. ONE FACE IN TENSION

Example 11. Finding Areas of Steel.—Find the required areas of tension and compression reinforcement for an unsymmetrically reinforced concrete section subjected to direct stress and bending, when it is required that the maximum stresses in steel and concrete should reach the maximum allowable values simultaneously. Concrete dimensions are given. Whole section is assumed as effective.

Given: Normal thrust, $N = 150\,000$ lb.

Bending moment, $M = 2\,850\,000$ in.-lb.

Eccentricity, $e = \frac{M}{N} = 19$ in.

The normal thrust acts in the center of gravity of the reinforced concrete section (and not in the center of the concrete rectangle).

Specified stresses: $f_c = 650$ lb. per sq. in.

$f_s = 16\,000$ lb. per sq. in.

$n = 15.$

Dimensions: $b = 24$ in., $h = 28$ in., $d' = 2$ in.,

$d = 28 - 2 = 26$ in., $2a = 24$ in.

Solution.—The problem can be solved by using Formulas (42) and (45), p. 233.

$$A_s = \left(\frac{e}{a} + \frac{d_s}{a} - 1\right)\frac{N}{2f_s} + C_1bd.$$

$$A'_s = C_2\left(\frac{e}{a} + \frac{d_s}{a} + 1\right)\frac{N}{2fs} - C_3bd.$$

Find ratios:

$$\frac{a}{d} = \frac{12}{26} = 0.46,\quad \frac{e}{a} = \frac{19}{12} = 1.58,\quad \frac{N}{2f_s} = \frac{150\,000}{2 \times 16\,000} = 4.68.$$

Assume $\frac{d_s}{a} = 0.1$. Then $\frac{e}{a} + \frac{d_s}{a} - 1 = 0.68$ and $\frac{e}{a} + \frac{d_s}{a} + 1 = 2.68$.

Find from Table 1, p. 649, value of k for the specified stresses. It is $k = 0.378$.

From Diagrams 17 and 18, pp. 664 and 665, find for $k = 0.378$ and $\frac{a}{d} = 0.46$ constants C_1, C_2 and C_3. They are:

$$C_1 = 0.0004,\quad C_2 = 2.07 \quad\text{and}\quad C_3 = 0.015.$$

Solve equations for A_s and A'_s:

$$A_s = 0.68 \times 4.68 + 0.0004 \times 24 \times 28 = 3.2 + 0.27 = 3.47 \text{ sq. in.}$$

$$A'_s = 2.07 \times 2.68 \times 4.68 - 0.015 \times 24 \times 28 = 25.9 - 10.1 = 15.8 \text{ sq. in.}$$

Find ratios of steel:

$$p_1 = \frac{3.47}{24 \times 28} = 0.0052,\quad p_2 = \frac{15.8}{24 \times 28} = 0.0235.$$

Check values of $\frac{d_s}{a}$ by means of Diagram 16, p. 663. The diagrams are based on $\frac{h}{d} = 1.0$. In this example $\frac{h}{d} = \frac{28}{26} = 1.08$. To be able to use the diagrams divide the steel ratios by 1.08. Thus reduced ratios, to be located in the diagrams, are

$$p_1 = \frac{0.0052}{1.08} = 0.0048 \quad\text{and}\quad p_2 = \frac{0.0235}{1.08} = 0.0218.$$

Locating these in the Diagram 16, p. 663, gives $\frac{d_s}{a} = 0.175$. The assumed ratio was $\frac{d_s}{a} = 0.1$. The difference between the assumed and the computed value is $0.175 - 0.1 = 0.075$. This increases the amount of tension and compression reinforcement. Instead of refiguring the amount of steel from formulas for A_s and A'_s it is easier to compute the additional areas caused by the increase of $\frac{d_s}{a}$. This increase influences only the first term in the formulas for A_s and A'_s.

$$\text{Increase of tension steel} = 0.075 \times 4.68 = 0.351 \text{ sq in.}$$

$$\text{Increase of compression steel} = 1.9 \times 0.075 \times 4.68 = 0.666 \text{ sq. in.}$$

These should be added to the areas previously found.

Example 12. Finding Dimensions.—Find the depth of a section subjected to direct stress and bending, when it is desired that the maximum stresses in concrete and steel should reach simultaneously the maximum allowable values and when it is desired to limit the ratio of the compression reinforcement, to $p_2 = 0.02$.

Given: Normal thrust, $N = 220\,000$ lb.

Bending moment, $M = 3\,500\,000$ in.-lb.

Eccentricity, $e = \frac{M}{N} = 15.9$ in.

Specified stresses: $f_c = 650$ lb. per sq. in.

$f_s = 16\,000$ lb. per sq. in.

$n = 15$

$k = 0.378$ (see table, p. 649).

Assumed: Width of section, $b = 28$ in., $d' = 2$ in.

Whole section may be considered as effective.

Solution.—The depth is found from Formula (47), p. 235,

$$d = C_4 \frac{N}{bf_s}\left[1 + \sqrt{1 + \frac{2(e + d_s)}{\frac{a}{d}C_4\frac{N}{bf_s}}}\right]$$

$$C_4 = \frac{C_2}{4(p_2 + C_3)}.$$

Find constants C_2 and C_3 from Diagram 18, p. 665.

First assume $\frac{a}{d} = 0.46$. Then, for $k = 0.378$, $n = 15$ and $\frac{a}{d} = 0.46$, $C_2 = 2.07$ and $C_3 = 0.015$.

Therefore, $$C_4 = \frac{2.07}{4(0.02 + 0.015)} = \frac{2.07}{0.140} = 14.8.$$

Compute $$\frac{N}{bf_s} = \frac{220\,000}{28 \times 16\,000} = 0.49 \quad \text{and} \quad \frac{N}{2f_s} = 6.9.$$

Assume $d_s = 0.05e$, so that $2(e + d_s) = 2.1e = 33.4$ in.

These values substituted in formula for d give

$$d = 14.8 \times 0.49 \left[1 + \sqrt{1 + \frac{33.4}{0.46 \times 14.5 \times 0.49}} \right]$$

$$= 7.25 [1 + \sqrt{1 + 10.25}] = 31.5 \text{ in.}$$

Since $d' = 2$ in., $2a = 31.5 - 2 = 29.5$ in., and $a = \frac{29.5}{2} = 14.75$ in., the amount of compression reinforcement, therefore, is

$$A'_s = 0.02 \times 28 \times 31.5 = 17.6 \text{ sq. in.}$$

The amount of tension reinforcement is found from formula

$$A_s = \left(\frac{e}{a} + \frac{d_s}{a} - 1 \right) \frac{N}{2f_s} + C_1 bd.$$

C_1 is taken from Diagram 17, p. 664. It is

$$C_1 = 0.0004, \quad \frac{e + d_s}{a} = 1.05 \times \frac{15.9}{14.75} = 1.13 \quad \text{and} \quad \frac{e}{a} + \frac{d_s}{a} - 1 = 0.13.$$

$$A_s = 0.13 \times 6.9 - 0.0004 \times 28 \times 31.5 = 0.896 + 0.353 = 1.25 \text{ sq. in.}$$

and

$$p_1 = \frac{1.25}{28 \times 31.5} = 0.00142.$$

Now check the value of $\frac{d_s}{a}$. The steel ratios are $p_1 = 0.00142$, $p_2 = 0.02$ and $\frac{h}{d} = \frac{33.5}{31.5} = 1.065$. To use the Diagram 16, p. 663, which is based on $\frac{h}{d} = 1$, divide the steel ratios by 1.065. The reduced ratios are $p_1 = \frac{0.00142}{1.065} = 0.00133$ and $p_2 = \frac{0.02}{1.065} = 0.0188$. Located in the diagram they give $\frac{d_s}{a} = 0.19$ and $d_s = 0.19a = 0.19 \times 14.75 = 2.8$ in. The assumed value is $d_s = 0.05e = 0.05 \times 15.9 = 0.795$ in. The difference between the assumed and the computed value of d_s is fairly large, therefore, the depth and the areas of steel will be refigured. $2(e + d_s) = (15.9 + 2.8) = 37.4$ in.

$$d = 14.8 \times 0.49 \left[1 + \sqrt{1 + \frac{37.4}{0.46 \times 14.5 \times 0.49}} \right] = 7.25 [1 + \sqrt{12.45}] = 32.9 \text{ in.}$$

$$A'_s = 0.02 \times 28 \times 32.9 = 18.4 \text{ sq. in.}$$

$$A_s = \left(\frac{15.9}{15.45} + 0.19 - 1 \right) 6.9 + 0.0004 \times 28 \times 32.9 = 1.73 + 0.37 = 2.10 \text{ sq. in.}$$

$$p_1 = \frac{2.10}{28 \times 32.9} = 0.00228$$

and the new $\frac{d_s}{a} = 0.182$, against accepted 0.19. The values of $\frac{d_s}{a}$ are sufficiently close, therefore, the results will be accepted.

CHAPTER III

RIGID FRAMES WITH TWO COLUMNS

Definition.—Rigid frames are structures consisting of a number of vertical and horizontal or inclined members joined in such a manner that at the joints the construction is able to resist all the bending moments and shears that can come upon them. In the types of frames treated in this chapter two members meet at each joint.

As a matter of fact practically all reinforced concrete buildings built by pouring the concrete without positive joints are statically indeterminate structures. This chapter is confined to the treatment of one-story frames supported by two columns. These should always be designed as rigid frames, using the simple formulas given in this chapter.

Structures consisting of a number of spans and a number of stories are often symmetrical in design so that the common general formulas given in Vol. I, p. 279, can be used with safety. More unusual cases should be designed by formulas given in the chapter on Building Frames.

All rigid frames are statically indeterminate structures. The number of statically indeterminate values depends upon the number of spans and the number of vertical members. The character of the end supports has also an influence upon the number of statically indeterminate values.

When the frame is fixed at the ends, statically indeterminate bending moments develop there. When the frame is hinged at the ends the bending moments there are zero. Thus a frame with fixed ends has as many additional statically indeterminate values than a hinged frame as there are fixed supports.

Types of Frames.—Following frames are treated in this chapter:

Frames with two columns and one span:

(*a*) Right-angle frames, hinged ends.
(*b*) Right-angle frames, fixed ends.
(*c*) Ridge frames, hinged ends.
(*d*) Frames with parabolic top, hinged ends.
(*e*) Inclined frames, hinged ends.
(*f*) to (*i*) Sawtooth roofs, several types.
(*j*) Closed rectangular frames.

Basis for Formulas for Rigid Frames.—The statically indeterminate values for rigid frames are found by means of the elastic theory. There are several methods of application of the theory such as method of least work, slope and deflection method, Maxwell's theorem of reciprocity and others. All these are based on the same general principle and the results obtained by one method should agree with the results obtained by any other method. The advantage of one method over the other is only in case of understanding or in case of application.

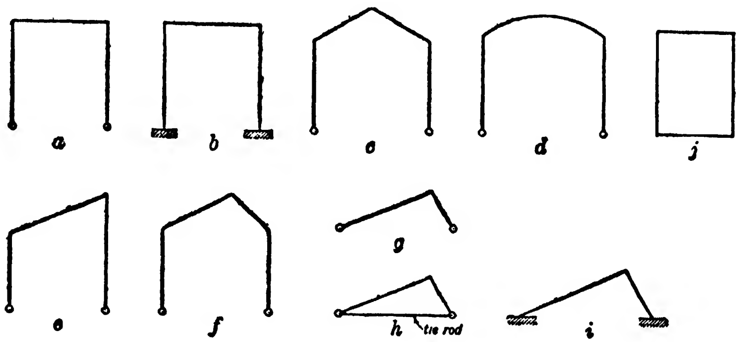


FIG. 108.—Frames of One Span with Two Columns. (*See p.* 258.)

After the statically indeterminate values are found, the bending moments and shears at any point are computed by the ordinary rules of statics.

Reliability of Elastic Theory as Applied to Frames.—The reliability of the elastic theory as applied to reinforced concrete has been definitely established by tests and by the performance of the numerous structures designed by means of formulas based upon it.

The formulas based on this theory give the best available means of consistent designing of the frames. All members in a frame so designed have the required strength, thus combining safety with economy. The use of arbitrary formulas, on the other hand, produces structures which may be too strong in one section and too weak in some other section. The resulting structure, as a whole, is naturally lame.

Objections are sometimes raised to the elastic theory as applied to concrete, that reinforced concrete is not a homogenous material as presumed in the theory, that the modulus of elasticity is not known, that the deflections of the structure cannot be computed with exactness; hence any method based on deflection is faulty.

These objections are not valid. Reinforced concrete construction properly designed acts sufficiently as a homogenous structure for all practical purposes. In applying the elastic theory it is not necessary to compute and use actual deflections of the structure. In determining of the formulas only the relation of deflections of different parts of the structure are used and in that way the value of the modulus of elasticity is finally eliminated so that its magnitude is not material. The formulas are applicable to steel structures as well as concrete structures irrespective of the strength of the concrete and the magnitude of its modulus of elasticity.

Requirements.—To get successful results with a rigid frame, the following requirements must be fulfilled.

1. The frame must be properly designed. At all points the most unfavorable bending moments and shears must be taken care of. Where reversal of bending moments is possible, the most unfavorable negative and positive bending moments must be provided for.

2. Proper foundation must be provided so that no unequal settlement takes place. Where appreciable settlement cannot be avoided, either a rigid frame should not be used or provision should be made to resist stresses produced by unequal settlement. The foundation must be able to resist the horizontal thrust.

3. The frame must be connected to the foundation in the manner contemplated in the design. Obviously a frame designed as fixed at the support and built without provision for fixity will not have the expected factor of safety.

The connection between the frame and the foundation must be strong enough to resist the horizontal thrust. If the frame is considered as fixed at the supports, proper provision should be made to transfer to the foundation the bending moments developed at the ends.

If the frame is considered as hinged, the most effective method would be to provide actual hinges at the bottom. These are seldom used on account of the expense involved. The next best method are the so-called Mesnager hinges illustrated in Fig. 157, p. 366, which consist of inclined bars imbedded in the foundation and in the frame in such a manner that adjacent bars cross each other at the center of the hinge. Such bars resist shear but are not able to resist bending moments. With such construction, to allow free rotation of the frame a clear space should be provided at the bottom by rounding up the top of the foundation and the bottom of the frame. The space may then be filled with asphalt.

Often no special provision is made to permit free rotation of the ends. The frame is joined to the foundation by proper number of dowels arranged so as to be unable to resist bending moments. It is then assumed that rotation of the end will be accomplished by opening of a crack at one edge and compression of concrete at the other. This method is simple, but of course introduces some uncertainties as to the action of the frame.

Another method often used is to provide a rigid connection between the foundation and the frame and to make provision for some bending moments at the ends developed by such connection. No advantage, however, is taken of this in the design of this restraint, and the frame is designed as hinged at the ends. To use this scheme successfully the designer must have sufficient judgment to be able to foresee the effect of the partial restraint.

4. Each frame should be constructed where possible in one continuous operation. Where this is not possible, construction joints should be placed at points of minimum shear. To take care of any possible shear, recesses in concrete should be provided so that old and new concrete should dovetail in the direction of the shear. Proper care should be used in joining old and new concrete. Any laitance should be removed. The surface should be roughened and neat cement paste spread on the top.

Deformation of Rigid Frames under Various Loadings

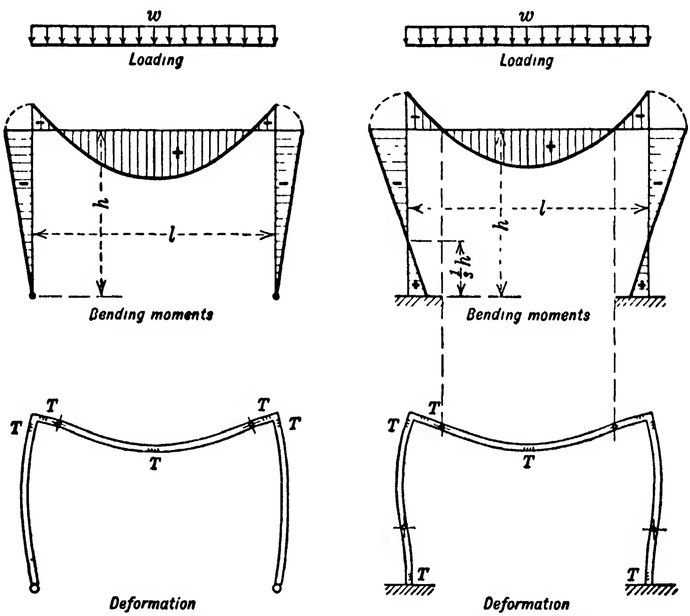
A clear understanding of the action of rigid frames may be obtained by studying their deformation under various loadings. For this purpose the deformations of a rectangular frame are discussed and illustrated below for the following loadings:

(*a*) Vertical loading.
(*b*) Frame with cantilever. Cantilever loaded.
(*c*) One-sided horizontal pressure.
(*d*) Horizontal pressure on both sides.

For purposes of comparison of the character of deformations, frames with hinges are placed side by side with frames with fixed ends. On the deflection diagrams at the points of maximum negative bending moments the tension side is indicated by cracks by the letter T. Points of inflection are shown by small circles. To show the relation between bending moments and deflections, bending moment diagrams are also shown.

Hinged Frame. Vertical Loading. (See Fig. 109 (*a*), p. 262).—Fig. 109 shows deflection curves for uniformly distributed loading. The type of deflection curves is the same, however, for all kinds of vertical loading. The deflection curve of both columns is always the same, differing only as to the magnitude of deflection. The deflection curve of the beam is symmetrical for symmetrical loads and unsymmetrical for unsymmetrical loads.

For vertical loading both columns bulge outward. The outside faces of both columns are under tension in the upper sections where the tensile stresses due to the bending moment are large enough to overcome the compression due to the vertical pressure acting on the column. Cracks are likely to occur at the outside face at the juncture of the beam and the column.



(a) Hinged Ends (b) Fixed Ends

FIG. 109.—Right-angle Frame. Deflection Due to Vertical Loading. (See p. 261.)

The deflection curve of the beam has two points of contraflexure which coincide with the points of zero moments. Tensile stresses occur at the top of the beam between the points of contraflexure and the corners and at the bottom in the central part of the beam. Cracks are likely to occur near the center at the bottom and near the ends at the top.

Fixed Frame. Vertical Loading. (See Fig. 109 (*b*), p. 262).—For vertical loading the deflection curve of the columns is a reverse curve with a point of contraflexure at one third of the height for symmetrical loading. For unsymmetrical loading the points of contraflexure are higher for the lighter loaded column than for the heavier loaded column. The lower portion of the columns below the points of contraflexure are subjected to positive bending moments producing tension at the inside face of the columns. The upper portion of the columns are subjected to negative bending moment producing tension at the outside face. Cracks are likely to occur at the foot of the column at the inside and at the top of the column at the outside.

The deflection of the beam of a frame with fixed ends is of the same character as for a frame with hinged ends except that the total deflection

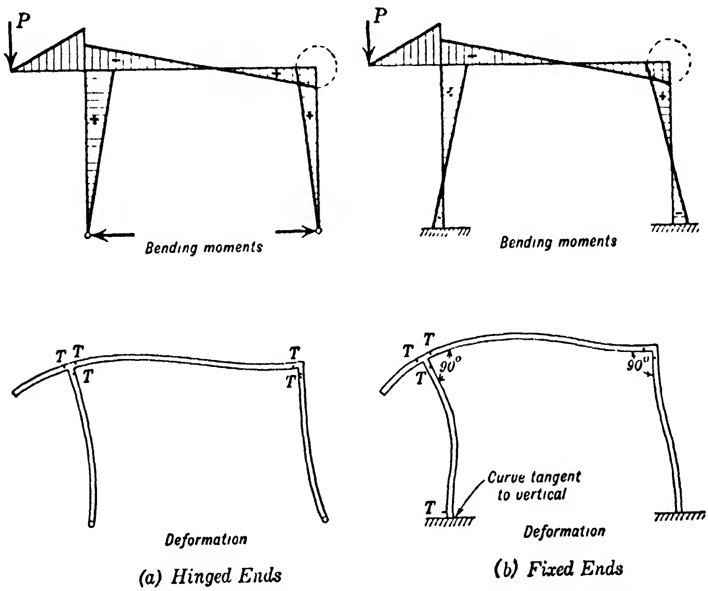


FIG. 110.—Right-angle Frame with Left Cantilever. (*See p.* 263.)

is smaller and the points of contraflexure are farther away from the corners.

Hinged Frame with Left Cantilever. (See Fig. 110 (*a*), p. 263).—The deflection curves of both columns are simple curves. With the left cantilever loaded and the beam not loaded the left column bulges to the right so that its inside face is in tension. The right column bulges inward and its inside face also is in tension. Cracks are most likely to occur at the top of the left columns on the inside, and less likely at the top of the right column also on the inside.

The deflection curve of the beam is a reverse curve with the point of contraflexure near the right corner. The beam bends upward. In the whole left section, including the cantilever, tension acts at the upper part of the beam. In the balance of the beam tension acts at the bottom of the beam. Cracks are most likely to occur at the top on both sides of the left column and less likely at the bottom next to the right column. It should be noted that when the beam is loaded at the same time as the cantilever, the deflection is the sum of the deflections for both types of loadings. The deflection due to one type offsets partly the deflection due to the other type.

Fixed Frame with Left Cantilever. (See Fig. 110 (*b*), p. 263).—The deflection curves of both columns are reverse curves. At the bottom

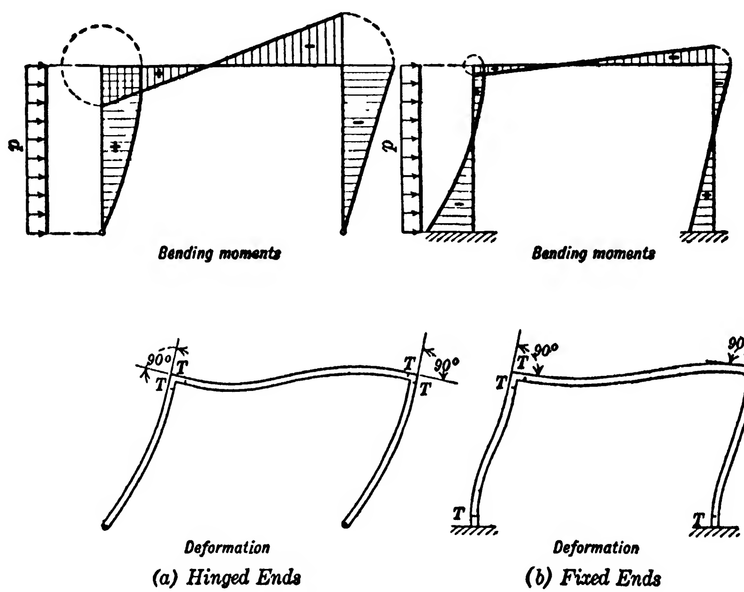


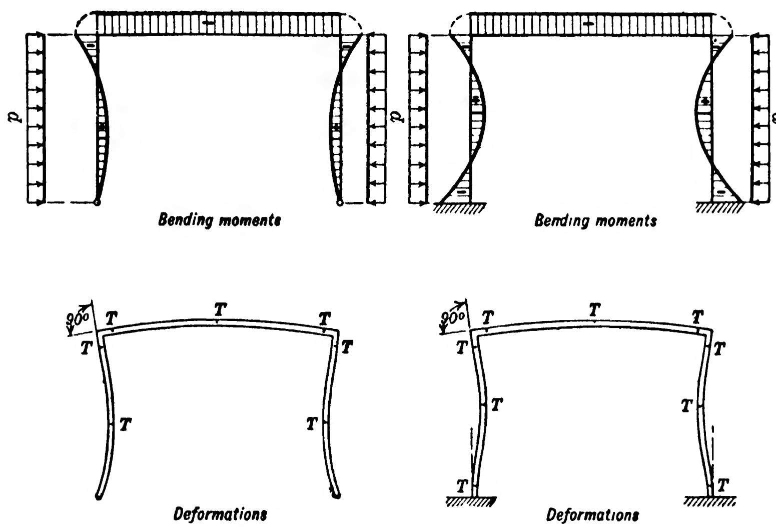
FIG. 111.—Right-angle Frame. Deflection Due to Horizontal Pressure. (*See p.* 265.)

both curves have vertical tangents. The point of contraflexure of the left column is near the bottom while at the right it is in the upper half of the column. The deflection curves above the points of contraflexure is of the same kind as in hinged frame. Below the points of contraflexure the curves bend in opposite directions. The cracks are apt to occur in same places as for hinged frame and in addition at the bottom of columns at the outside.

The deflection curve of the beam is of the same type as for hinged frame. The magnitude of the deflections of beams and columns is appreciably smaller for fixed frames than for hinged frames of same dimensions.

Horizontal Pressure on Left Side. Hinged Frame. (See Fig. 111, p. 264).—The type of deflection curves is the same for all one-sided horizontal pressures, such as uniformly distributed or concentrated wind pressure, earth pressure or traction force. In Fig. 111, p. 264, deflection curves and bending moments are shown for uniformly distributed wind pressure.

With the horizontal pressure acting from the left as in Fig. 111, p. 264, the deflection curves of both columns are simple curves bulging to the right. The tension stresses act on inside face of the left column and at the outside face of the right column. The cracks are likely to



(a) Hinged Ends *(b) Fixed Ends*

Fig. 112.—Right-angle Frame. Horizontal Pressure on Both Sides. (*See p.* 266.)

occur near the top on the inside for left column and on the outside for right column.

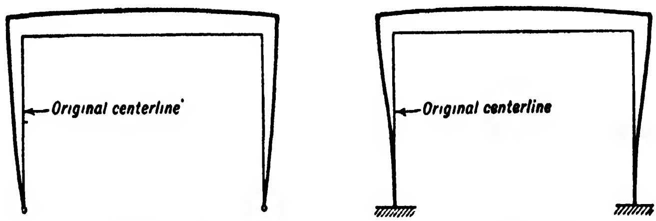
The deflection curve of the beam is a reverse curve, with one point of contraflexure nearer the left corner. The cracks are most likely to occur at the top of the beam near the right corner and less likely at the bottom near the left corner.

Horizontal Pressure on Left Side. Fixed Frame. (See Fig. 112, p. 265.)—With the horizontal pressure acting from left as in Fig. 112, p. 265, the deflection curves of both columns are reverse curves with vertical tangents at the ends. The points of contraflexure are in the lower half of the left column and in the upper half of the right column.

In the left column the cracks are likely to occur in the outside face at the bottom and the inside face at the top. In the right column the cracks are likely to occur in the inside face at the bottom and the outside face at the top.

The deflection curve of the beam is of the same type as for the hinged frame except that for equal dimensions of the frame and equal pressures the deflections are much smaller for the fixed frame.

Horizontal Pressure on Both Sides. Hinged Frame. (See Fig. 112 (*a*), p. 265.)—When horizontal pressure acts on both sides the deflection of both columns is inward. The deflection curve is a simple curve. The cracks are apt to occur in the upper part of the columns in the inside face where the bending moments are large.



(a) Hinged Ends *(b) Fixed Ends*

FIG. 113.—Effect of Rise of Temperature upon Rigid Frame. (*See p.* 266.)

The deflection curve of the beam is also a simple curve bulging upward. The bending moment is uniform throughout the beam so that for this loading alone, the cracks could occur anywhere in the upper face of the beam.

Horizontal Pressure on Both Sides. Fixed Frame. (See Fig. 112 (*b*), p. 265.)—For horizontal force acting on both sides of a fixed frame, the deflection curve of the columns is a reverse curve with a vertical tangent at the bottom and points of inflection in the lower half of the columns. The cracks are likely to occur either in the outside faces at the bottom of the columns or in the inside faces in the upper part of the columns.

The beam deflects upward as in the previous case except that the deflection is appreciably smaller.

Temperature Changes. (See Fig. 113, p. 266.)—The effect of the rise of temperature upon hinged and fixed frames is shown in Fig. 113 (*a*) and (*b*). The type of deflection curves is clearly shown in the figures.

RIGID FRAMES WITH HINGED ENDS

Definition.—The definition of a rigid frame, in general, is given on p. 258. A rigid frame is called "rigid frame with hinged ends" when it is attached to the foundation by means of actual hinges or by such other means which make turning of the frame at the ends possible. When subjected to loading the frame with hinged ends turns at the ends so that no bending moment can be developed there. However, the ends are secured against any horizontal or vertical movement.

Simple frames with hinged ends have one statically indeterminate value, namely, the horizontal thrust at the hinges. This thrust is determined from the requirement that, when subjected to loading, the frame must deform in such a way that there would be no change in the relative position of the hinges. This means that the hinges after deformation must remain on the same levels as before and the horizontal distance between the hinges must not be changed.

Changing a Rigid Frame with Hinged Ends into a Statically Determinate Structure.—For the sake of computation it is necessary to change the statically indeterminate structure into a statically determinate structure.

A rigid frame with hinged ends may be changed into a statically determinate structure by providing one of the hinged ends with rollers. In such case the structure is not capable of resisting any horizontal thrust. At the movable end, the one with rollers, there can act only vertical reactions as any horizontal reaction would cause horizontal movement of the rollers. When loaded, the ends spread sufficiently to allow free deformation of the beam. All reactions are static reactions and the bending moments due to the loads can be determined by simple statics.

For the vertical loads the reactions are vertical and are found in the same mammer as for simple beams.

For the horizontal forces the horizontal reaction can act only at the end not provided with rollers and is equal to the sum of all horizontal forces. In addition the horizontal forces produce vertical reactions at both ends which are equal but act in opposite directions.

This statically determinate construction can be changed back to a statically indeterminate structure by providing horizontal thrusts at the hinge.

Comparison of Rigid Frames with Hinged Ends and Two Hinged Arches.—Rigid frames with hinged ends are, from the standpoint of analysis, structures of the same character as the two-hinged arches. In both cases the horizontal thrust is the only statically indeterminate value. The method of determining the horizontal thrust is the same and the general formula for horizontal thrust is the same in both cases.

A rigid frame differs from an arch in that the bending moments in a rigid frame are very much larger and the thrust much smaller than in an arch. It is possible to design an arch so that no, or only small, tensile stresses are developed at all sections of the arch. It is also possible to adapt the shape of the arch axis so that there are no bending moments in the arch for dead load. This is impossible in rigid frames. All loads, dead and live, produce considerable bending moments in rigid frames.

In many members of the rigid frame the effect of the thrust is negligible. In arches, on the other hand, the thrust at all sections has a considerable effect upon the stresses.

The methods of solving formulas for horizontal thrust for the arch and the rigid frame are also different. A rigid frame usually consists of straight sections for which integrals are easy to solve. Therefore it is possible to solve the integrals for the whole frame by solving separately integrals for each straight section and then adding the results. It is hardly ever necessary to resort to the summation method often required for arches.

General Formula for Rigid Frame with Hinged Ends.—General formula for the horizontal thrust for rigid frame is same as developed for two-hinged arches on p. 552.

Let l = span of rigid frame;
M_s = static bending moment;[1]
I_x = moment of inertia at any point x;
I_1 = moment of inertia of the top section;
y = ordinate for a system with origin at left hinge.

Then

Horizontal Thrust, Rigid Frame of Any Design with Hinged Ends,

$$H = \frac{\int_0^l M_s y \frac{I_1}{I_x} ds}{\int_0^l y^2 \frac{I_1}{I_x} ds} \quad \ldots \ldots \ldots \quad (1)$$

Center of the system of coordinates x, y, is at the left hinge.

The value of ds in the above formula changes to dy for straight

[1] Static bending moment obtained when the frame is changed into a statically determinate structure by providing one end with rollers.

vertical members, to dx for straight horizontal members and to $\frac{dx}{\cos\phi}$ for straight members inclined at an angle ϕ with the horizontal.

Solving the Integrals.—A rigid frame usually consists of a number of straight sections. Each integral in the Equation (1) therefore may be represented by a sum of integrals for the various straight sections. Ordinarily the moments of inertia of each member are constant so that the ratio of moments of inertia may be taken before the integration sign.

In practical design it is not necessary to solve integrals as in the following pages simple formulas are developed for the cases most likely to occur in practice.

Illustration of Solving Integrals in Equation (1).—As an example integrals will be solved for a rigid frame with a ridge roof shown in Fig. 114 for which

h = height of vertical member;

h_1 = height of roof;

s = length of inclined member $= \frac{l}{2\cos\phi}$;

l = span of frame;

ϕ = angle of inclination of inclined members with horizontal;

I_1 = moment of inertia of inclined member;

I = moment of inertia of vertical member.

Denominator.—The integral for the denominator becomes

$$\int_0^l y^2\frac{I_1}{I_x}ds = \int_0^h y^2\frac{I_1}{I}dy + \int_0^{\frac{l}{2}}\left(h + \frac{2h_1}{l}x\right)^2\frac{I_1}{I_1}\frac{dx}{\cos\phi} + \int_{\frac{l}{2}}^{l}\left(h + \frac{2h_1}{l}(l-x)\right)^2\frac{I_1}{I_1}\frac{dx}{\cos\phi} + \int_h^0 (h-y)^2\frac{I_1}{I}dy.^2$$

[2] It is evident from figure that the value of y for any point at a distance x from the left support is $y = h + \frac{2h_1}{l}x$ for left half, and $y = h + \frac{2h_1}{l}(l - x)$ for the right half. Due to symmetry of the structure

$$\int_0^h y^2dy = \int_h^0 (h-y)^2dy$$

and

$$\int_0^{\frac{l}{2}}\left(h + \frac{2h_1}{l}x\right)^2\frac{dx}{\cos\phi} = \int_{\frac{l}{2}}^{l}\left(h + \frac{2h_1}{l}(l-x)\right)^2\frac{dx}{\cos\phi}.$$

$$\int_0^l y^2 \frac{I_1}{I_s} ds = 2\frac{I_1}{I}\int_0^h y^2 dy + \frac{2}{\cos\phi}\int_0^{\frac{l}{2}} \left(h + \frac{2h_1}{l}x\right)^2 dx$$

$$= \frac{2}{3}\frac{I_1}{I}h^3 + \frac{1}{\cos\phi}\left[h^2 l + \frac{h_1}{l}hl^2 + \frac{1}{3}\left(\frac{h_1}{l}\right)^2 l^3.\right]$$

Finally,

$$\int_0^l y^2 \frac{I_1}{I_x} dy = \frac{h^2 l}{3\cos\phi}\left[\frac{I_1}{I}\frac{h}{s} + 3 + \frac{h_1}{h}\left(3 + \frac{h_1}{h}\right)\right]. \quad . \quad . \quad (2)$$

Numerator.—The numerator may also be divided into integrals for the various straight sections.

Let M_{s1} = static bending moment at any point due to loads in left column;

M_{s2} = static bending moment at any point due to loads in inclined member;

M_{s3} = static bending moment at any point due to loads in right column.

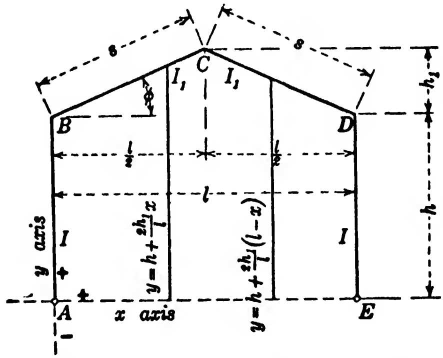


FIG. 114.—Rigid Frame with Ridge Roof. (*See p.* 269).

Center of the system of coordinates at left hinge A.
Then

$$\int_0^l M_s y \frac{I_1}{I_x} ds = \frac{I_1}{I}\int_0^h M_{s1} y dy + \frac{I_1}{I_1}\int_0^{\frac{l}{2}} M_{s2}\left(h + \frac{2h_1}{l}x\right)\frac{dx}{\cos\phi}$$

$$+ \frac{I_1}{I_1}\int_{\frac{l}{2}}^l M_{s2}\left(h + \frac{2h_1}{l}(l - x)\right)\frac{dx}{\cos\phi} + \frac{I_1}{I}\int_h^0 M_{s3}(h - y)dy. \quad . \quad (3)$$

It is necessary to find the equation for M_{s1}, M_{s2} and M_{s3} for any particular type of loading. The working is simplified if the vertical loads and the horizontal loads are treated separately.

For vertical load the static bending moments in the columns are equal to zero. Therefore $M_{s1} = M_{s3} = 0$. The equation changes to

Numerator for Vertical Loads,

$$\int_0^l M_s y \frac{I_1}{I_x} ds = \frac{1}{\cos \phi}\left[\int_0^{\frac{l}{2}} M_{s2}\left(h + \frac{2h_1}{l}x\right)dx + \int_{\frac{l}{2}}^{l} M_{s2}\left(h + \frac{2h_1}{l}(l - x)\right)dx\right]. \quad (4)$$

The numerator must be solved for each type of loading by substituting proper values for M_{s2}.

Thus for uniformly distributed loads

$$M_{s2} = \tfrac{1}{2}x(l - x)w,$$

and the numerator becomes

$$\int_0^l M_s y \frac{I_1}{I_x} ds = \frac{w}{2 \cos \phi}\left[\int_0^{\frac{l}{2}} x(l - x)\left(h + \frac{2h_1}{l}x\right)dx + \int_{\frac{l}{2}}^{l} x(l - x)\left(h + \frac{2h_1}{l}(l - x)\right)dx\right]. \quad (5)$$

The solution of the integrals is simple.

Numerator for Horizontal Pressures.—For horizontal pressures it is usually assumed in computing the static bending moments M_{s1}, M_{s2} and M_{s3} that the windward end of the frame is anchored and the leeward end is supplied with rollers. The anchored end, only, can resist horizontal reaction; therefore, the whole horizontal reaction, which is equal to the total horizontal pressure, acts at the windward end. The static bending moments then can be easily computed. After the horizontal thrust is found from formula, it is then added at both hinges to the static horizontal reaction.

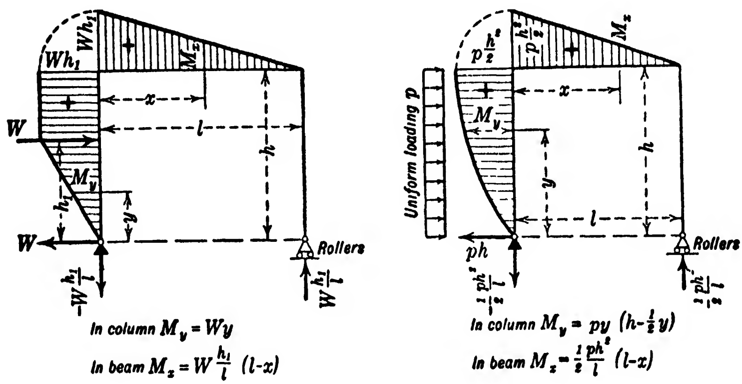
Figure 115, p. 272, shows static bending moments for a concentrated wind pressure and uniformly distributed wind pressure. It is evident that there is no bending moment in the right, i.e., the leeward column, so that the fourth term in Equation (3), p. 270, becomes zero.

Semi-graphical Solution of Integrals.—The integral for the numerator is $\int_0^l M_s y \frac{I_1}{I_x} ds.$

It can be divided into separate parts for each straight section of the frame. Thus for ridge frame the integral for any type of loading becomes

$$\int_0^l M_s y \frac{I_1}{I_x} ds = \frac{I_1}{I}\int_0^h M_{s1} y dy + \int_0^{\frac{l}{2}} M_{s2}\left(h + \frac{2h_1}{l}x\right)\frac{dx}{\cos\phi}$$

$$+ \int_{\frac{l}{2}}^{l} M_{s2}\left(h + \frac{2h_1}{l}x\right)\frac{dx}{\cos\phi} + \frac{I_1}{I}\int_h^0 M_{s3}(h - y)dy.$$



(a) Concentrated Load W (b) Uniform Load p

FIG. 115.—Frame Made Statically Determinate. Bending Moments Due to Horizontal Pressure. (*See p.* 271.)

Examing the integral of each part it is evident that all of them are of the same character. All of them represent the static moment about the origin at A of the area formed by plotting on the member of the static bending moment due to the load.

The semi-graphic method then may be based upon the area of moment principle as follows (see Fig. 116, p. 273):

Lay out the frame to a convenient scale.

Find the static bending moments due to loads and plot them. Area $B_1bdfD_1B_1$ in Fig. 116.

Plot the static bending moments at proper points on lines drawn at right angles to the frame. (In Fig. 116 $ab = a_1b_1$, $cd = Cd_1$, $ef = e_1f_1$.)

Divide the areas thus obtained into simple parts such as rectangles, triangles, etc., for which the areas can be easily found. Compute the areas. (In Fig. 116 triangles Bb_1d_1, Bd_1C, Cd_1D and Dd_1f_1.)

Find the center of gravity of each area (points 1, 2, 3, 4) and draw through each a line at right angles to, and till intersection with, the frame (points 1′, 2′, 3′ and 4′). Measure the vertical distance of these points of intersection from the axis passing through A. In figure, distances 1′1″, 2′2″, 3′3″ and 4′4″.

Multiply each area by proper distance above A of the projected center of gravity (1′1″, 2′2″, 3′3″, 4′4″) and by the ratio $\frac{I_1}{I}$ of each member. The sum of all these values gives the value of the integral.

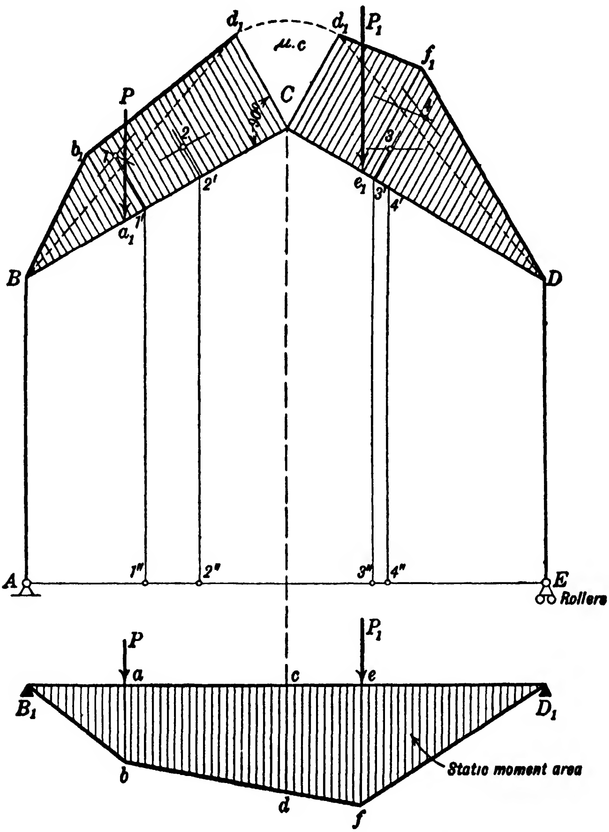


FIG. 116.—Semi-graphical Solution of Integrals. (*See p.* 272.)

RIGHT-ANGLE FRAME. HINGED ENDS

Right-angle frames with hinged ends have one statically indeterminate value, namely, the horizontal thrust H acting at the bottom of the frame. After this thrust is found the bending moments and shears may be computed by ordinary rules of statics.

Notation.

Let l = span of the frame;
h = height of frame;
I_1 = moment of inertia of horizontal member;
I = moment of inertia of vertical member;
H = horizontal thrust;
M_{s1} = static bending moment at any point in left column due to any loading;
M_{s2} = static bending moment at any point in beam due to any loading;
M_{s3} = static bending moment at any point in right column due to any loading;
M_s = static bending moment at any point in beam due to vertical loading;
M_y = actual bending moment in column;
M_x = actual bending moment in beam;
M_B and M_C = corner bending moments.

General Formula for Horizontal Thrust.—General formula for horizontal thrust for any kind of loading is given below.

Horizontal Thrust for Any Kind of Loading,

$$H = \frac{\frac{I_1}{I}\int_0^h M_{s1} y\,dy + h\int_0^l M_{s2}\,dx + \frac{I_1}{I}\int_0^h M_{s3} y\,dy}{lh^2\left(\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + 1\right)} \quad . \quad . \quad . \quad (6)$$

Horizontal Thrust for Vertical Loads.—For vertical loads acting on the beam the static moments in the columns M_{s1} and M_{s3} are zero and $M_{s2} = M_s$. The formula then changes as follows.

Horizontal Thrust for Vertical Loads,

$$H = \frac{\int_0^l M_s\,dx}{lh\left(\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + 1\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This horizontal thrust applies at the hinges and produces in the frame negative bending moments which can be found from formulas below.

Bending Moment in Column at Any Point y above Hinges,

$$M_y = -Hy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Corner Bending Moments,

$$M_B = M_C = - Hh. \quad \quad (9)$$

To get bending moment in the beam it is necessary to subtract from the static bending moment due to the loads the negative bending moment due to the horizontal thrust. This gives following formula.

Bending Moment in Beam at Any Point x,

$$M_x = M_s - Hh. \quad \quad (10)$$

Special Cases of Right-angle Frames

Following special cases will be considered.

Vertical Loading.

Case 1. Beam loaded with uniformly distributed loading.

Case 2. Beam loaded with concentrated loading.

Case 2*a*. Beam loaded with special arrangement of concentrated loading.

Case 3. Beam provided with cantilever, load on cantilever.

Case 3*a*. Beam provided with two cantilevers.

Case 4. Column provided with brackets, load on bracket.

Horizontal Loading.

Case 5. Uniformly distributed horizontal pressure as wind pressure.

Case 6. Concentrated horizontal pressure.

Case 7. Varying horizontal pressure such as earth pressure.

Case 1. Uniformly Distributed Vertical Loading. (See Fig. 117, p. 275.)

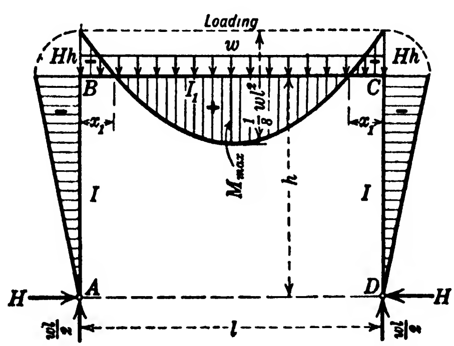


Fig. 117.—Right-angle Frame. Uniformly Distributed Loading. (*See p.* 275.)

End Shear in Beam,

$$V_B = V_C = \tfrac{1}{2}wl. \quad \quad (11)$$

Horizontal Thrust,

$$H = \frac{l}{h} C_1 wl, \quad (12)$$

where

$$C_1 = \frac{1}{12\left(\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + 1\right)}. \quad (13)$$

Maximum Negative Bending Moment in Corners,

$$M_B = M_C = -Hh = -C_1 wl^2. \quad (14)$$

Maximum Positive Bending Moment,

$$M_{max} = \tfrac{1}{8}wl^2 - Hh = (\tfrac{1}{8} - C_1)wl^2. \quad (15)$$

Bending Moment at Any Point x,

$$M_x = \tfrac{1}{2}x(l - x)w - C_1 wl^2. \quad (16)$$

Points of Contraflexure, Distance from Either Corner,

$$x_1 = \tfrac{1}{2}(1 - \sqrt{1 - 8C_1})l = C_2 l, \quad (17)$$

where

$$C_2 = \tfrac{1}{2}(1 - \sqrt{1 - 8C_1}). \quad (18)$$

The constants C_1 and C_2 may be taken from Diagram 17, p. 277.

The bending moments in the frame are shown in Fig. 117, p. 275.

It should be noted that all members are subjected to bending moments and thrusts. In the horizontal member the thrust is equal to the horizontal thrust H. In the vertical member the thrust is equal to the reaction due to the vertical load on the beam plus any additional load coming directly on the vertical member.

The horizontal thrust H produces shearing stresses in the vertical member.

Case 2. Concentrated Vertical Load at Distance *a*. (See Fig. 118.)

Assume that a vertical load P is placed at a distance a from left corner. The formulas are given below.

End Shears,

$$V_B = \left(1 - \frac{a}{l}\right)P. \quad (19) \qquad V_C = \frac{a}{l}P. \quad (20)$$

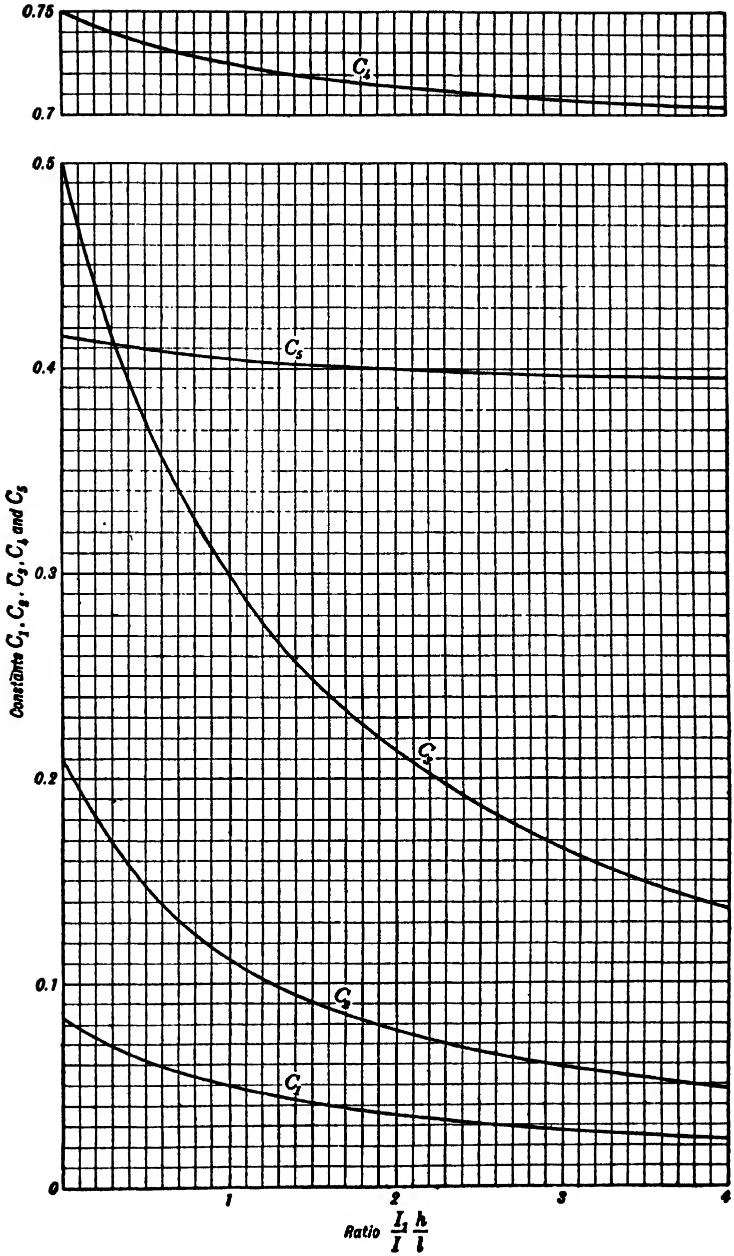


Diagram 17.—Constants C_1 to C_5 in Formulas for Right-angle Frames.

C_1 formula (13), p. 276, C_2 formula (18), p. 276, C_3 formula (22), p. 278, C_4 formula (64), p. 287 and C_5 formula (73), p. 289.

Horizontal Thrust,

$$H = C_3 \frac{a}{l}\left(1 - \frac{a}{l}\right)P\frac{l}{h}, \quad \ldots \ldots \quad (21)$$

where

$$C_3 = \frac{1}{2\left(\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + 1\right)}. \quad \ldots \ldots \quad (22)$$

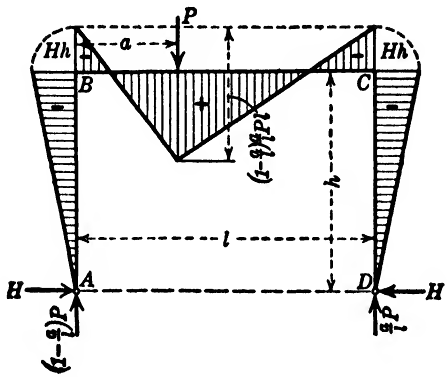


FIG. 118.—Right-angle Frame. Concentrated Vertical Loads. (*See p.* 276.)

The values of C_3 are given in Diagram 17, p. 277.

Bending Moments at Corners,

$$M_B = M_C = -Hh, \quad \ldots \ldots \quad (23)$$

also

$$M_B = M_C = -C_3\frac{a}{l}\left(1 - \frac{a}{l}\right)Pl. \quad \ldots \ldots \quad (24)$$

Bending Moment at Any Point y:

For x smaller than a,

$$M_x = M_s - Hh = \left(1 - \frac{a}{l}\right)\left(\frac{x}{l} - \frac{a}{l}C_3\right)Pl. \quad \ldots \quad (25)$$

For x larger than a,

$$M_x = \frac{a}{l}\left[1 - \frac{x}{l} - \left(1 - \frac{a}{l}\right)C_3\right]Pl. \quad \ldots \ldots \quad (26)$$

Maximum Positive Bending Moment, $x = a$,

$$M_{max} = \frac{a}{l}\left(1 - \frac{a}{l}\right)Pl - Hh = (1 - C_3)\frac{a}{l}\left(1 - \frac{a}{l}\right)Pl. \quad . \quad (27)$$

Values of C_3 are given in Diagram 17, p. 277.

If the loading consists of a number of loads P, P_1, P_2 placed at distances a, a_1, a_2, . . ., the horizontal thrust H is equal to the sum of the horizontal thrusts due to the separate loads. The formula for horizontal thrust for several loads becomes

Horizontal Thrust for Several Vertical Loads,

$$H = \frac{l}{h}C_3\left[P\frac{a}{l}\left(1 - \frac{a}{l}\right) + P_1\frac{a_1}{l}\left(1 - \frac{a}{l}\right) + P_2\frac{a_1}{l}\left(1 - \frac{a_2}{l}\right) + \dots\right]. \qquad (28)$$

Also in shorter form

$$H = \frac{l}{h}C_3\Sigma P\frac{a}{l}\left(1 - \frac{a}{l}\right). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

The values of C_3 depend upon the dimensions of the frame and may be taken from Diagram 17, p. 277.

The negative bending moments at the corners may be found from Formula (23), p. 278.

The maximum positive bending moment in the beam equals the maximum static bending moment due to the loads minus the bending moment at the corner due to the thrust.

Position of Concentrated Moving Loads for Maximum Positive and Negative Bending Moments.—For moving concentrated loads such as used in bridge design, before computing the bending moments, it is necessary to determine the most unfavorable position of the loads on the span by means of influence lines discussed on p. 292. From this it is evident that the exact position of loads for maximum positive bending moment is different than for maximum negative bending moment. It is also found, however, that for the loads there considered it is possible to use for both bending moments the position of loads for maximum static bending moment, without any appreciable error.

Case 2*a*. Special Arrangement of Concentrated Loads.—Following special arrangements of concentrated loads are considered

(*a*) Load P at center.
(*b*) Two loads P at third points.
(*c*) Three loads P at quarter points.
(*d*) Four loads P at fifth points.
(*e*) Five loads P at sixth points.
(*f*) Six loads P at seventh points.

The results are given in table below:

Right-angle Frame Concentrated Symmetrical Loads

All Loads are Equal and Spaced Equal Distances Apart

Condition	Position of Loads	Number of Loads	Horizontal Thrust H	Negative Bending Moment at Corners M_A M_B	Maximum Positive Bending Moment M_{max}
a	At Center	P	$\frac{1}{4}\frac{l}{h}C_3P$	$-\frac{1}{4}C_3Pl$	$\frac{1}{4}(1-C_3)Pl$
b	At Third Points	$2P$	$\frac{4}{9}\frac{l}{h}C_3P$	$-\frac{4}{9}C_3Pl$	$\frac{1}{3}(1-\frac{4}{3}C_3)Pl$
c	At Quarter Points	$3P$	$\frac{5}{8}\frac{l}{h}C_3P$	$-\frac{5}{8}C_3Pl$	$\frac{1}{2}(1-\frac{5}{4}C_3)Pl$
d	At Fifth Points	$4P$	$\frac{4}{5}\frac{l}{h}C_3P$	$-\frac{4}{5}C_3Pl$	$\frac{1}{5}(3-4C_3)Pl$
e	At Sixth Points	$5P$	$\frac{35}{36}\frac{l}{h}C_3P$	$-\frac{35}{36}C_3Pl$	$\frac{3}{4}(1-\frac{15}{9}C_3)Pl$
f	At Seventh Points	$6P$	$\frac{56}{49}\frac{l}{h}C_3P$	$-\frac{56}{49}C_3Pl$	$(\frac{6}{7}-\frac{56}{49}C_3)Pl$

Loads placed directly on top of columns do not produce any bending moments or shears in the frame.

The bending moment diagrams for each case may be drawn in a way similar to that shown in Fig. 119, p. 280, for three equal loads P.

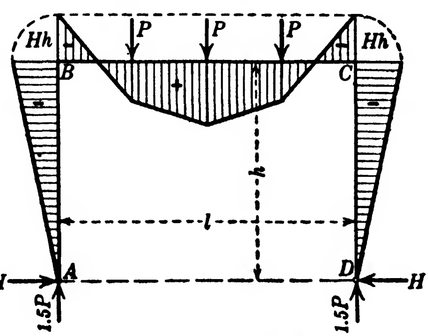


FIG. 119.—Right-angle Frame. Three Equal Loads P. (*See p.* 280.)

The values of

$$C_3 = \frac{1}{2\left(\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + 1\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

may be taken from diagram p. 277.

Case 3. Frame Provided with Cantilever on One Side.—Assume that the frame is provided with a cantilever on left side only as shown in Fig. 120, p. 281.

Also assume that the load on the cantilever produces at the edge of the cantilever an end shear V_1 and a bending moment $-M_1$.

Let, in addition to notation on p. 274,

V_1 = end shear at edge of cantilever, equal to sum of loads on cantilever;

M_1 = bending moment at edge of cantilever.

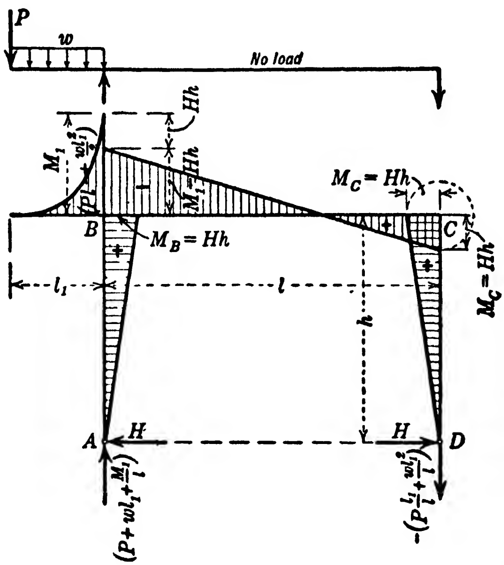


FIG. 120.—Right-angle Frame with Cantilever at One End. (*See p.* 281.)

The bending moments and shears in the frame for this condition are given below and are shown in Fig. 120, p. 281.

Reactions at Hinges,

$$R_A = V_1 + \frac{M_1}{l}, \quad . \quad . \quad . \quad (31) \qquad R_D = -\frac{M_1}{l}. \quad . \quad . \quad (32)$$

End Shears in Beam,

$$V_B = \frac{M_1}{l}, \quad \ldots \ldots \quad (33) \qquad V_C = -\frac{M_1}{l}. \quad \ldots \quad (34)$$

Horizontal Thrust,

$$H = \frac{1}{2\left(\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + 1\right)}\frac{M_1}{h} = C_3\frac{M_1}{h}. \quad \ldots \ldots \quad (35)$$

Bending Moments in Vertical Members at B and C,

$$M_B = M_c = Hh = C_3M_1. \quad \ldots \ldots \quad (36)$$

Bending Moments in Horizontal Member,

$$M_{B1} = -M_1 + M_B = -M_1(1 - C_3), \quad \ldots \quad (37)$$

$$M_{C1} = C_3M_1. \quad \ldots \ldots \quad (38)$$

Point of Contraflexure from Right Corner,

$$x_1 = \frac{1}{2\left(\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + 1\right)}l = C_3l. \quad \ldots \ldots \quad (39)$$

Values of constants C_3 are expressed by formula (30), p. 281, and may be taken from diagram, p. 277.

Bending Moment on Cantilever,

For uniformly distributed load,

$$-M_{1u} = -\tfrac{1}{2}wl_1^2. \quad \ldots \ldots \quad (40)$$

For concentrated load,

$$-M_{1c} = -Pl_1. \quad \ldots \ldots \quad (41)$$

Total Bending Moment,

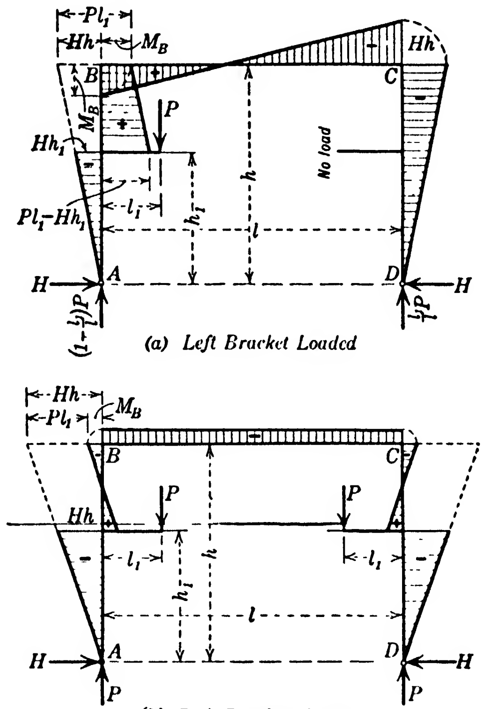
$$-M_1 = -(M_{1u} + M_{1c}). \quad \ldots \ldots \quad (42)$$

Case 3*a*. Frame with Two Cantilevers.—When the horizontal member projects on both sides the bending moments and shear should be computed separately for each cantilever and then the results combined so as to get maximum values.

It should be noted that the negative bending moment in the horizontal member at the corner is a maximum when one cantilever only is loaded. For other members the condition is most unfavorable with both cantilevers fully loaded.

Combining Bending Moment Due to Loads on Cantilever with Bending Moments Due to Loads on Beam.—Bending moments due to loads on cantilever and those due to loads on the beam should be computed separately and then combined so as to get maximum results. The method of combining should be the same as explained on p. 123 for beams with cantilevers.

Case 4. Left Column provided with Brackets. (See Fig. 121(a)).



(a) *Left Bracket Loaded*

(b) *Both Brackets Loaded*

FIG. 121.—Right-angle Frame. Columns Provided with Crane Bracket. (*See p.* 283.)

Let h_1 = depth of bracket above hinges;
l_1 = distance of load to center of column;
P = load on bracket;
M_b = maximum bending moment in bracket.

Reactions at Hinges,

$$R_A = \left(1 - \frac{l_1}{l}\right)P. \quad . \quad (43) \qquad R_D = \frac{l_1}{l}P. \quad . \quad . \quad . \quad . \quad (44)$$

Shear in Beam,

$$V_C = \frac{l_1}{l}P. \quad . \quad . \quad . \quad . \quad (45) \qquad V_D = \frac{l_1}{l}P. \quad . \quad . \quad . \quad . \quad (46)$$

Horizontal Thrust,

$$H = \frac{1}{2\left(\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + 1\right)}\left\{\frac{I_1}{I}\frac{h}{l}\left[1 - \left(\frac{h_1}{h}\right)^2\right] + 1\right\}\frac{l_1}{h}P, \quad . \quad . \quad (47)$$

also

$$H = C_3\left\{\frac{I_1}{I}\frac{h}{l}\left[1 - \left(\frac{h_1}{h}\right)^2\right] + 1\right\}\frac{l_1}{h}P. \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Value of C_3 may be taken from diagram, p. 277.
Bending Moments in Column with Bracket,

Just below bracket,

$$M_{E1} = -Hh_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Just above bracket,

$$M_{E2} = Pl_1 - Hh_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Corner Bending Moments,

$$M_B = Pl_1 - Hh, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$M_C = -Hh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Bending Moment in Bracket,

$$M_b = -Pl_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

The above formulas apply also when the bending moment on the bracket is due to any other cause than a concentrated load. If the bending moment on the bracket is M_b, then substitute in all formulas M_b for Pl_1.

Case 4*a*. Both Columns Provided with Brackets.—In this case it is assumed that both columns are provided with brackets and that the bending moments at both brackets are equal.

This case with Case 4 are sufficient to take care of all conditions possible in practice.

When the bending moments transferred to each of the two columns are different either on account of a difference in length of brackets or a difference in the magnitude of loads, following procedure may be used.

The larger bending moment is divided into two parts one equal to the bending moment in tne opposite bracket and the other equal to the difference between the bending moments at the two brackets.

Bending moments and shears in the frame are then found for a condition of a frame with two brackets with equal bending moment (Case 4*a*) and also for a condition of bending moment at one bracket only (Case 4). The result added give bending moments in frame for two brackets with unequal bending moments.

The bending moments below are given for a frame subjected to two equal bending moments. They are illustrated in Fig. 121 (*b*), p. 283.

Reactions on Hinges. Two Brackets with Equaι Loads,

$$R_A = R_B = P. \quad (54)$$

Shear in Beam,

$$V_B = V_C = 0. \quad (55)$$

Horizontal Thrust. Two Brackets with Equal Loads,

$$H = 2C_3\left\{\frac{I_1}{I}\frac{h}{l}\left[1 - \left(\frac{h_1}{h}\right)^2\right] + 1\right\}\frac{l_1}{h}P. \quad (56)$$

Value of C_3 may be taken from diagram, p. 277.

Bending Moments in Columns,

Just below bracket,

$$M_{E1} = -Hh_1. \quad (57)$$

Just above bracket,

$$M_{E2} = Pl_1 - Hh_1. \quad (58)$$

At corners B and C,

$$M_A = M_B = Pl_1 - Hh. \quad (59)$$

Bending Moment in Bracket,

$$M_b = -Pl_1. \quad (60)$$

The bending moments in the beam at all points due to load in both cantilevers are equal to the corner bending moments. They may be either positive or negative, depending upon the location of the bracket in respect to the total height.

Right-angle Frame Subjected to Horizontal Pressures

Horizontal pressure to which a frame may be subjected is due either to wind pressure as in case of buildings or earth pressure as in case of bridges. It may also be caused by the water pressure.

The wind pressure on the wall may be transmitted directly to the columns by the wall, in which case it is assumed as uniformly distributed. Earth pressure, on the other hand, is represented either by a triangle, when no surcharge is considered, or by a trapezoid when surcharge is considered. The trapezoid loading may be replaced by a uniform loading and a triangular loading.

Wind pressure may also be transferred to the column as a horizontal concentrated load.

The following three cases will therefore cover wind pressures and earth pressure.

Case 5. Uniformly distributed horizontal pressures on the column.
Case 6. Triangular pressures on the column.
Case 7. Concentrated horizontal load on column.

In general the horizontal pressures produce in a frame:

1. Two equal vertical reactions at the hinges acting in opposite directions. The reaction on the windward side is an uplift. The two reactions form a couple, the bending moment of which is equal to the bending moment about the hinge due to the horizontal pressures.

2. Two horizontal reactions acting in opposite direction to the wind pressure. The sum of both horizontal reactions is equal to the sum of the horizontal pressures. The horizontal reaction at the windward hinge, which in the formulas below is called H, is much larger than the horizontal reaction at the leeward hinge.

3. The windward part of the frame, namely, the windward column and the adjacent part of the beam are subjected to positive bending moment. The leeward balance of the frame is subjected to negative bending moment.

Additional notation,

p = uniformly distributed unit wind pressure, lb. per lin. ft.

Case 5. Uniformly Distributed Horizontal Pressure on Column.

Vertical Reactions,

$$R_A = -\tfrac{1}{2}ph\frac{h}{l}. \quad . \quad . \quad (61) \qquad R_D = \tfrac{1}{2}ph\frac{h}{l}. \quad . \quad . \quad (62)$$

Horizontal reactions,

Windward hinge A,

$$H = \frac{1}{8}\frac{11\frac{I_1}{I}\frac{h}{l} + 18}{2\frac{I_1}{I}\frac{h}{l} + 3}ph = C_4 ph, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

where

$$C_4 = \frac{1}{8}\frac{11\frac{I_1}{I}\frac{h}{l} + 18}{2\frac{I_1}{I}\frac{h}{l} + 3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

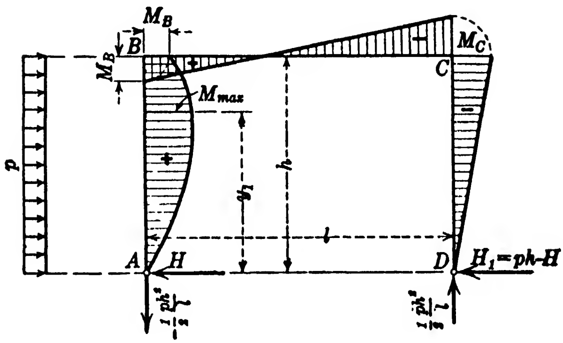


FIG. 122.—Right-angle Frame. Uniformly Distributed Horizontal Pressure. (*See p.* 286.)

Leeward hinge D,

$$H_1 = ph - H = (1 - C_4)ph. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Bending Moment at Any Point y in Column Subjected to Pressure,

$$M_y = Hy - \tfrac{1}{2}py^2.$$

y is measured from the bottom,

Maximum Bending Moment, Windward Column,

$$M_{max} = \frac{1}{2}\left(\frac{1}{8}\frac{11\frac{I_1}{I}\frac{h}{l} + 18}{2\frac{I_1}{I}\frac{h}{l} + 3}\right)^2 ph^2 = \tfrac{1}{2}(C_4)^2 ph^2. \quad . \quad . \quad . \quad . \quad . \quad (66)$$

Point of Maximum Bending Moment above A,

$$y_1 = \frac{1}{8}\frac{11\frac{I_1}{I}\frac{h}{l} + 18}{2\frac{I_1}{I}\frac{h}{l} + 3}h = C_4h. \quad \ldots \ldots \ldots \quad (67)$$

Bending Moments in Corners,

$$M_B = Hh - \tfrac{1}{2}ph^2 = \frac{3}{8}\frac{\frac{I_1}{I}\frac{h}{l} + 2}{2\frac{I_1}{I}\frac{h}{l} + 3}ph^2 = (C_4 - 0.5)ph^2. \quad (68)$$

$$M_C = -H_1h = -(1 - C_4)ph^2. \quad \ldots \ldots \ldots \quad (69)$$

These bending moments act both in the beam and in the columns. Values of C_4 may be taken from Diagram 17, p. 277.

Case 6. Triangular Horizontal Pressure on Column. (See Fig. 123.)

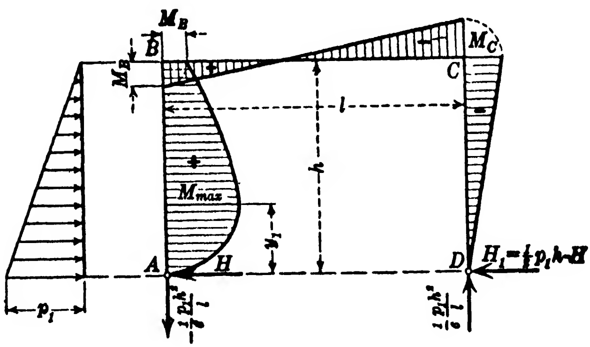


Fig. 123.—Triangular Horizontal Pressure on Column. (*See p.* 288.)

Additional notation,

p_1 = maximum unit pressure at bottom, lb. per lin. ft.

Vertical Reactions,

$$R_A = -\frac{1}{6}p_1h\frac{h}{l}. \quad \ldots \quad (70) \qquad R_D = \frac{1}{6}p_1h\frac{h}{l}. \quad \ldots \quad (71)$$

Horizontal Reactions,

Side next to pressure, hinge A,

$$H = \frac{1.25 + 0.775\frac{I_1}{I}\frac{h}{l}}{\left(3 + 2\frac{I_1}{I}\frac{h}{l}\right)} p_1 h = C_5 p_1 h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (72)$$

where

$$C_5 = \frac{1.25 + 0.775\frac{I_1}{I}\frac{h}{l}}{\left(3 + 2\frac{I_1}{I}\frac{h}{l}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (73)$$

Values of C_5 may be taken from diagram, p. 277.

Side opposite the pressure, hinge D,

$$H_1 = \tfrac{1}{2}p_1 h - H = (\tfrac{1}{2} - C_5)p_1 h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (74)$$

Bending Moment at Any Point y in Column Subjected to Pressure,

$$M_y = Hy - \tfrac{1}{2}\left(1 - \frac{1}{3}\frac{y}{h}\right)p_1 y^2.$$

y is measured from the bottom.

Maximum Bending Moment in Column Subjected to Pressure,

$$M_{\max} = \tfrac{1}{3}C_5(1 - \sqrt{1 - 2C_5})\frac{1 + 2\sqrt{1 - 2C_5}}{1 + \sqrt{1 - 2C_5}} p_1 h^2 = C_6 p_1 h^2, \quad . \qquad (75)$$

where

$$C_6 = \tfrac{1}{3}C_5(1 - \sqrt{1 - 2C_5})\frac{1 + 2\sqrt{1 - 2C_5}}{1 + \sqrt{1 - 2C_5}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (76)$$

Point of Maximum Bending Moment Measured from A,

$$y_1 = h(1 - \sqrt{1 - 2C_5}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (77)$$

Bending Moments in Corners,

$$M_B = Hh - \tfrac{1}{3}p_1 h^2 = (C_5 - \tfrac{1}{3})p_1 h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (78)$$

$$M_C = -H_1 h = -(\tfrac{1}{2} - C_5)p_1 h^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (79)$$

These bending moments act both in the beam and in the columns.

Case 7. Concentrated Horizontal Load W at a Distance $h_1 = mh$.

Additional notation,

W = horizontal concentrated load;

$h_1 = mh$ = vertical distance of point of application of pressure from hinge;

$$m = \frac{h_1}{h}.$$

Vertical Reactions,

$$V_A = -\frac{h_1}{l}W. \quad . \quad . \quad (80) \qquad V_D = \frac{h_1}{l}W. \quad . \quad . \quad . \quad (81)$$

Horizontal Reactions,

Windward hinge A,

$$H = \frac{\frac{I_1}{I}\frac{h}{l}(m^3 - 3m + 4) + 3(2 - m)}{2\left(2\frac{I_1}{I}\frac{h}{l} + 3\right)}W. \quad . \quad . \quad . \quad . \quad (82)$$

Leeward hinge D,

$$H_1 = W - H. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

Maximum Bending Moment in Column Next to Pressure,

$$M_{\max} = Hh_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

Bending Moment in Corners,

$$M_B = Hh - W(h - h_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

$$M_C = H_1h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

Case 7*a*. Concentrated Load W above the Frame.—This case is applicable for tractive force.

If h_1 is larger than h, i.e., the force acts above the frame, the following formulas apply.

Vertical Reactions,

$$V_A = -W\frac{h_1}{l}. \quad . \quad . \quad . \quad (87) \qquad V_D = W\frac{h_1}{l}. \quad . \quad . \quad . \quad . \quad (88)$$

Since the tractive force cannot be exactly determined, it is permissible to use, for the horizontal reactions following approximate formulas.

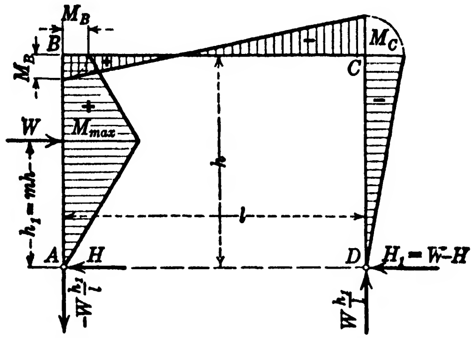


FIG. 124.—Right-angle Frame. Horizontal Concentrated Load. (*See* p. 290.)

Horizontal Thrust,

Left side, hinge A,

$$H = W\left(1 - \frac{h_1}{2h}\right). \quad (89)$$

Right side, hinge D,

$$H = \tfrac{1}{2}W\frac{h_1}{h}. \quad (90)$$

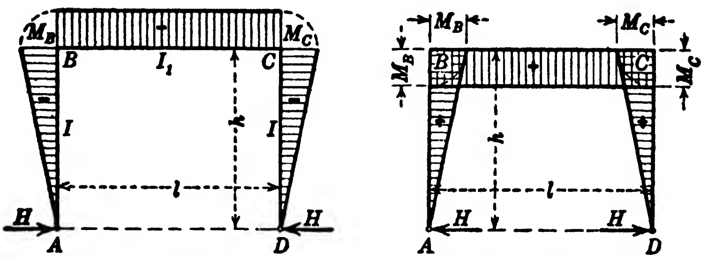
EFFECT OF CHANGES OF TEMPERATURE

Temperature changes cause changes in the length of the columns and the beam composing the frame. Since the frame is firmly anchored at the hinges so that the ends cannot change their relative positions, the change in length of the beam produces bending in the frame with consequent bending stresses.

Temperature changes produce horizontal thrust applied at the hinges. The direction of the thrust is inward for rise of temperature and outward for fall of temperature.

Let α = coefficient of expansion for 1° F.;
E = modulus of elasticity of concrete;
t = change in temperature in degrees.

Case 8. Rise of Temperature. (See Fig. 125 (*a*)).



(a) Rise of Temperature *(b) Fall of Temperature*

Fig. 125.—Right-angle Frame. Effect of Change of Temperature. (*See p.* 292.)

Horizontal Thrust,

$$H = -\frac{\alpha E t}{\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + 1}\cdot\frac{I_1}{h^2}. \qquad (91)$$

Corner Bending Moments,

$$M_B = M_C = Hh = -\frac{\alpha E t}{\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + 1}\cdot\frac{I_1}{h}. \qquad (92)$$

Case 8*a*. Fall of Temperature. (See Fig. 125 (*b*)).
Horizontal Thrust,

$$H = \frac{\alpha E t}{\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + 1}\cdot\frac{I_1}{h^2}. \qquad (93)$$

Corner Bending Moments,

$$M_B = M_C = Hh = \frac{\alpha E t}{\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + 1}\cdot\frac{I_1}{h}. \qquad (94)$$

Influence Lines for Right-angle Frame with Hinged Ends Useful for Concentrated Live Loads

In bridge design it is sometimes required to use concentrated loads for live load. In such case the computations may be simplified by using influence lines.

The significance of influence lines is explained on p. 168.

Influence Lines for Bending Moments for Vertical Loads.—The ordinates of an influence line for bending moments at any selected point is obtained by making in the Formulas (25) and (26), p. 278, $P = 1$, substituting for x the value x_1 for the selected section and making the position of the load variable by changing $\frac{a}{l}$ to $\frac{x}{l}$.

Let x_1 = section for which influence line is drawn;
y_x = ordinate of influence line at any point x.

Then general equation for influence line for bending moments at point x_1 due to vertical load is

Ordinates of Influence Line for Bending Moment at Any Point x_1,

$$y_x = \left(1 - \frac{x}{l}\right)\left(\frac{x_1}{l} - \frac{x}{l}C_3\right)l, \text{ for } x \text{ smaller than } x_1. \quad . \quad . \quad . \quad (95)$$

$$y_x = \frac{x}{l}\left[1 - \frac{x_1}{l} - \left(1 + \frac{x}{l}\right)C_3\right]l, \text{ for } x \text{ larger than } x_1. \quad . \quad . \quad (96)$$

Influence Line for Center of Span.—For center of the span, $x_1 = \frac{1}{2}l$ and $\frac{x_1}{l} = \frac{1}{2}$. The influence line is symmetrical. Thus

Influence Line for Bending Moments at Center of Span,

$$y_{\frac{l}{2}} = \left(1 - \frac{x}{l}\right)\left(\frac{1}{2} - \frac{x}{l}C_3\right)l. \quad . \quad . \quad . \quad . \quad . \quad (97)$$

Influence Line for Quarter Point.—For quarter point, $x_1 = \frac{1}{4}l$ and $\frac{x_1}{l} = \frac{1}{4}$.

Influence Line for Bending Moments at Quarter Point,

$$y_{\frac{l}{4}} = \left(1 - \frac{x}{l}\right)\left(\frac{1}{4} - \frac{x}{l}C_3\right)l, \text{ left side.} \quad . \quad . \quad . \quad . \quad . \quad (98)$$

$$y_{\frac{l}{4}} = \frac{x}{l}\left[\frac{3}{4} - \left(1 + \frac{x}{l}\right)C_3\right]l, \text{ right side.} \quad . \quad . \quad . \quad . \quad (99)$$

Constants C_3 may be obtained from Diagram 17, page 277.

How to Draw Influence Lines.—Lay out the beam to a desired scale. Locate the point for which the influence line is intended. Divide the span into any number of divisions. Usually ten divisions are used.

Erect vertical lines at each division point. Compute the constant C_3 for the frame or take it from diagram, p. 277. Compute for each division point the value $\frac{x}{l}$ and substitute it in proper equation. Compute the ordinate y_x and plot it on a vertical line passing through this division point. Connect the points so obtained by a smooth curve. Note that there is a break in the curve at the point for which the line is drawn.

How to Use Influence Lines.—Place on the beam above the influence line the concentrated loads in the position for which the static bending moment at the selected section is a maximum. Scale the influence line ordinates under the loads and multiply them by the loads. The sum gives the bending moment.

FRAME WITH RIDGE ROOF

General Formula for Vertical Loads.—Frame with ridge roof is shown in Fig. 126, p. 294. The general dimensions are given in the figure.

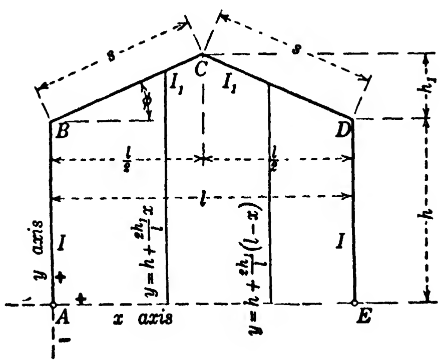


FIG. 126.—General Dimensions of Ridge Frame. (*See p.* 294.)

Let h = height of vertical member;

h_1 = vertical projection of inclined member;

l = horizontal span of the frame;

s = length of inclined portion $= \frac{l}{2 \cos \phi}$;

ϕ = angle of inclination of inclined members with the horizontal;

I_1 = moment of inertia of normal section of inclined member;

I = moment of inertia of vertical member.

The vertical reactions may be determined by statics.

The horizontal thrust is the only statically indeterminate value. The derivation of general formulas is given on p. 268.

General Equation for Horizontal Thrust Due to Vertical Loads,

$$H = \frac{\int_0^{\frac{l}{2}} M_s\left(h + 2\frac{h_1}{l}x\right)dx + \int_{\frac{l}{2}}^{l} M_s\left[h + 2\frac{h_1}{l}(l - x)\right]dx}{\frac{1}{3}h^2 l\left[\frac{I_1}{I}\frac{h}{s} + \frac{h_1}{h}\left(3 + \frac{h_1}{h}\right) + 3\right]}, \quad (100)$$

where M_s is the static moment of the vertical loads considering the beam as freely supported on both ends.

Several special cases are given below.

Case 9. Ridge Frame Uniformly Distributed, Vertical Loading. (See Fig. 127.)

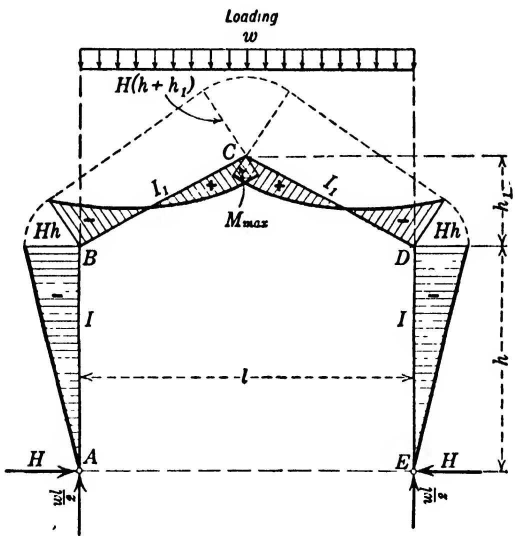


Fig. 127.—Ridge Frame Uniformly Distributed Vertical Loading. (*See p.* 295.)

Vertical Reactions,

$$R_A = \tfrac{1}{2}wl. \quad . \quad . \quad . \quad . \quad (101) \qquad R_E = \tfrac{1}{2}wl. \quad . \quad . \quad . \quad . \quad (102)$$

Horizontal Thrust,

$$H = \frac{1}{32}\,\frac{8 + 5\frac{h_1}{h}}{\frac{I_1}{I}\frac{h}{s} + \frac{h_1}{h}\left(3 + \frac{h_1}{h}\right) + 3}\,\frac{l}{h}wl = C_7\frac{l}{h}wl, \quad . \quad (103)$$

where

$$C_7 = \frac{1}{32} \frac{8 + 5\frac{h_1}{h}}{\frac{I_1}{I}\frac{h}{s} + \frac{h_1}{h}\left(3 + \frac{h_1}{h}\right) + 3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

Corner Bending Moments,

$$M_B = M_D = - Hh = - C_7wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

Bending Moment at Any Point in Inclined Member,

$$M_x = \left[\frac{1}{2}\frac{x}{l}\left(1 - \frac{x}{l}\right) - C_7\left(1 + 2\frac{h_1}{h}\frac{x}{l}\right)\right]wl^2. \quad . \quad . \quad . \quad (106)$$

Bending Moment at Ridge for $x = \frac{l}{2}$,

$$M_C = \left[\frac{1}{8} - C_7\left(1 + \frac{h_1}{h}\right)\right]wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

Case 9*a*. Ridge Frame. Left Half of Roof Loaded Uniformly. (See Fig. 128.)

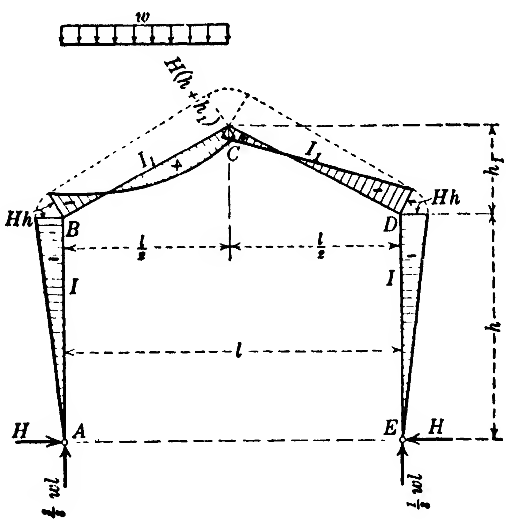


FIG. 128.—Ridge Frame Left Half of Roof Loaded Uniformly. (*See p.* 296.)

Vertical Reactions,

$$R_A = \tfrac{3}{8}wl. \quad . \quad . \quad . \quad . \quad (108) \qquad R_E = \tfrac{1}{8}wl. \quad . \quad . \quad . \quad . \quad (109)$$

Horizontal Thrust,

$$H = \tfrac{1}{2}C_7\frac{l}{h}wl, \qquad (110)$$

where C_7 is same as in previous case.

Corner Bending Moments,

$$M_B = M_D = -\tfrac{1}{2}C_7wl^2. \qquad (111)$$

Bending Moment at Any Point:

Left half of roof, x measured from left end,

$$M_x = \left[\frac{x}{l}\left(\frac{3}{8} - \frac{1}{2}\frac{x}{l}\right) - \frac{1}{2}C_7\left(1 + 2\frac{h_1}{h}\frac{x}{l}\right)\right]wl^2. \qquad (112)$$

Right half of roof, x measured from left end,

$$M_x = \left[\frac{1}{8}\left(1 - \frac{x}{l}\right) - \frac{1}{2}C_7\left(1 + 2\frac{h_1}{h}\frac{x}{l}\right)\right]wl^2. \qquad (113)$$

Bending Moment at Ridge $x = \frac{l}{2}$,

$$M_C = \left[\frac{1}{16} - \frac{1}{2}C_7\left(1 + \frac{h_1}{h}\right)\right]wl^2. \qquad (114)$$

M_C may be positive or negative, depending upon the inclination of the roof.

Case 10. Ridge Frame, Single Concentrated Vertical Load. (See Fig. 129, p. 298.)

Vertical Reaction,

$$R_A = \left(1 - \frac{a}{l}\right)P. \qquad (115) \qquad\qquad R_E = \frac{a}{l}P. \qquad (116)$$

Horizontal Thrust,

$$H = \frac{\frac{l}{h}}{4\left[\frac{I_1}{I}\frac{h}{s}+\frac{h_1}{h}\left(3+\frac{h_1}{h}\right)+3\right]}\frac{a}{l}\left\{6\left(1-\frac{a}{l}\right)+\frac{h_1}{h}\left[3-4\left(\frac{a}{l}\right)^2\right]\right\}P, \quad (117)$$

also

$$H = \frac{l}{h}C_8\frac{a}{l}\left\{6\left(1 - \frac{a}{l}\right) + \frac{h_1}{h}\left[3 - 4\left(\frac{a}{l}\right)^2\right]\right\}P, \qquad (118)$$

where

$$C_8 = \frac{1}{4\left[\frac{I_1}{I}\frac{h}{s} + \frac{h_1}{h}\left(3 + \frac{h_1}{h}\right) + 3\right]}. \quad (119)$$

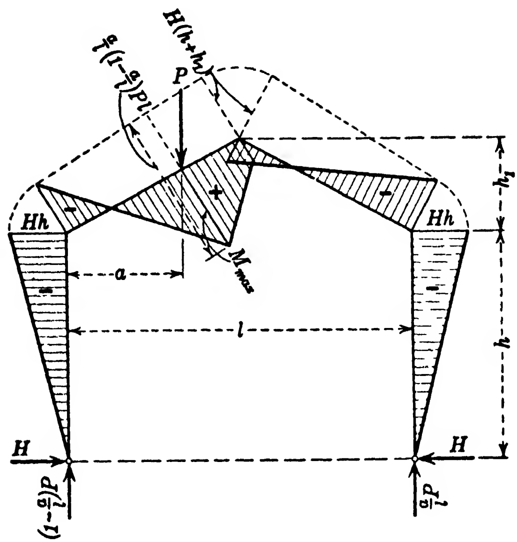


FIG. 129.—Ridge Frame, Single Concentrated Vertical Load. (See p. 297.)

Corner Bending Moments,

$$M_B = M_D = -Hh = C_8Pl. \quad (120)$$

Bending Moment at Ridge,

$$M_C = \tfrac{1}{2}Pa - H(h + h_1). \quad (121)$$

Bending Moment at the Load,

$$M_{\max} = \frac{a}{l}\left(1 - \frac{a}{l}\right)Pl - \left(1 + \frac{2h_1}{h}\frac{a}{l}\right)Hh. \quad (122)$$

Bending Moment at Any Point x,

$$M_x = M_s - \left(1 + \frac{2h_1}{h}\frac{x}{l}\right)Hh. \quad (123)$$

Case 11. Special Arrangement of Concentrated Loads.—Following special arrangements of concentrated loads are considered.

(*a*) Load P at ridge.

(*b*) Three loads P at quarter points.

(*c*) Five loads P at sixth points.

The bending moment diagram for each case may be drawn in the same manner as shown in Fig. 130, p. 299, for five loads.

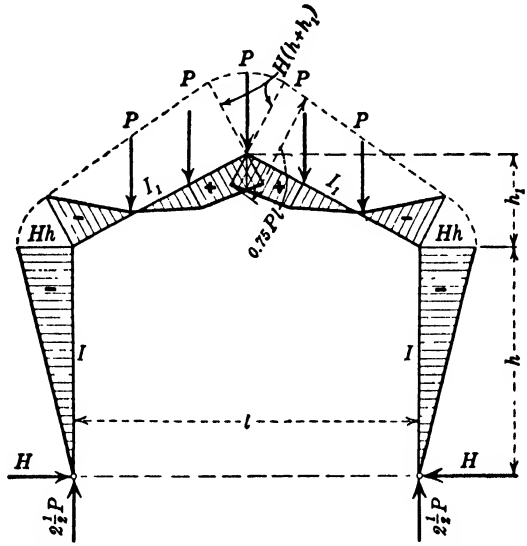


Fig. 130.—Ridge Frame, Hinged Ends. Five Loads P at Sixth Points. (*See p.* 299.)

Ridge Frame Concentrated Symmetrical Loads

All Loads are Equal and Spaced Equal Distances Apart

Case	Number and Position of Loads	Horizontal Thrust H	Corner Bending Moments M_B and M_D	Bending Moment at Loads	Location
a	$1P$ At Center	$\frac{l}{h}\left(1.5+\frac{h_1}{h}\right)C_8P$	$-Hh$	$\frac{1}{4}Pl-\left(1+\frac{h_1}{h}\right)Hh$	Ridge
b	$3P$ At Third Points	$\frac{l}{h}\left(3.75+2.375\frac{h_1}{h}\right)C_8P$	$-Hh$	$\frac{3}{8}Pl-\left(1+\frac{1}{2}\frac{h_1}{h}\right)Hh$	Quarter Point
				$\frac{1}{2}Pl-\left(1+\frac{h_1}{h}\right)Hh$	Ridge
c	$5P$ At Sixth Points	$\frac{l}{h}\left(5.833+3.667\frac{h_1}{h}\right)C_8P$	$-Hh$	$\frac{5}{12}Pl-\left(1+\frac{1}{3}\frac{h_1}{h}\right)Hh$	Sixth Point
				$\frac{2}{3}Pl-\left(1+\frac{2}{3}\frac{h_1}{h}\right)Hh$	Third Point
				$\frac{3}{4}Pl-\left(1+\frac{h_1}{h}\right)Hh$	Ridge

For dimensions see Fig. 130, p. 299.

Constant C_8 was given in Formula (119), p. 298.

Vertical reactions same as for statically determinate structures.

All formulas for bending moments in the inclined members consist of two items, a positive item depending upon the load and a negative item depending upon the value of H, h and h_1. The resulting bending moment may be positive or negative or equal zero depending upon the inclination of the roof.

Loads placed at columns produce vertical reactions in columns, but no bending moments or shear in the frame.

Case 12. Ridge Frame. Left Column Provided with Bracket. (See Fig. 131 (*a*), p. 301.)

In addition to notation, p. 294, let

l_1 = horizontal distance of load on bracket from center of column;

h_2 = vertical distance of center line of bracket from hinge level.

Vertical Reactions,

$$R_A = \left(1 - \frac{l_1}{l}\right)P. \quad . \quad (124) \qquad R_B = \frac{l_1}{l}P. \quad . \quad . \quad . \quad (125)$$

Horizontal Thrust,

$$H = \frac{3\left\{\frac{I_1}{I}\frac{h}{s}\left[1 - \left(\frac{h_2}{h}\right)^2\right] + \frac{h_1}{h} + 2\right\}}{4\left[\frac{I_1}{I}\frac{h}{s} + \frac{h_1}{h}\left(3 + \frac{h_1}{h}\right) + 3\right]}\frac{l_1}{h}P, \quad . \quad . \quad . \quad . \quad . \quad (126)$$

also

$$H = 3\left\{\frac{I_1}{I}\frac{h}{s}\left[1 - \left(\frac{h_2}{h}\right)^2\right] + \frac{h_1}{h} + 2\right\}\frac{l_1}{h}C_8P. \quad . \quad . \quad . \quad . \quad (127)$$

Values of constant C_8 are the same as used in the previous formulas.

Bending Moments in Column with Bracket,

Just below bracket,

$$M_{F1} = -Hh_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

Just above bracket,

$$M_{F2} = Pl_1 - Hh_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

Corner Bending Moments,

$$M_B = Pl_1 - Hh. \quad \ldots \quad (130)$$

$$M_D = - Hh. \quad \ldots \quad (131)$$

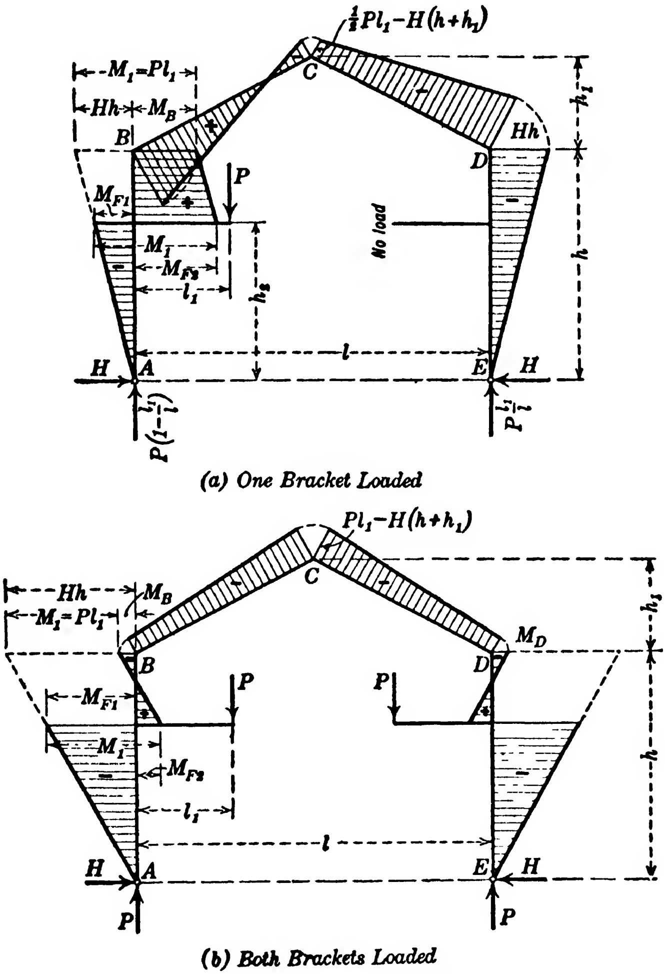


FIG. 131.—Ridge Frame. Columns Provided with Bracket. (*See p.* 300.)

Bending Moment at Ridge,

$$M_C = \tfrac{1}{2}Pl_1 - H(h + h_1) \quad \ldots \quad (132)$$

Bending Moment in Bracket,

$$M_b = - Pl_1. \quad \ldots \quad (133)$$

Case 12*a*. Ridge Frame. Both Columns Provided with Brackets. Loads on Brackets are Equal. (See Fig. 131 (*b*), p. 301.)

Vertical Reactions,

$$R_A = R_E = P. \qquad (134)$$

Horizontal Thrust,

$$H = \frac{6\left\{\frac{I_1}{I}\frac{h}{s}\left[1 - \left(\frac{h_2}{h}\right)^2\right] + \frac{h_1}{h} + 2\right\}}{4\left[\frac{I_1}{I}\frac{h}{s} + \frac{h_1}{h}\left(3 + \frac{h_1}{h}\right) + 3\right]}\frac{l_1}{h}P, \qquad (135)$$

also

$$H = 6\left\{\frac{I_1}{I}\frac{h}{s}\left[1 - \left(\frac{h_2}{h}\right)^2\right] + \frac{h_1}{h} + 2\right\}\frac{l_1}{h}C_8P. \qquad (136)$$

Value of C_8 is the same as in the previous formulas.

Bending Moments in Column,

Just below bracket,

$$M_{F1} = - Hh_2. \qquad (137)$$

Just above bracket,

$$M_{F2} = Pl_1 - Hh_2. \qquad (138)$$

Corner Bending Moments,

$$M_B = M_D = Pl_1 - Hh. \qquad (139)$$

Bending Moment at Ridge,

$$M_C = Pl_1 - H(h + h_1). \qquad (140)$$

Bending Moment in Bracket,

$$M_b = - Pl_1. \qquad (141)$$

Case 12*b*. Bending Moment Applied on Column from Outside.— Formulas given for Cases 12 and 12*a* may also be used when the brackets are outside instead of inside. Also they may be used when one or both columns are subjected to bending moments M acting from the outside and due to any cause. In such cases substitute in all formulas the value of the bending moment $- M$ for Pl_1. It should be noted that the sign of the resulting bending moments for this case are opposite to the signs in Cases 12 and 12*a*.

This case applies also when the frame is provided with a lean-to. The bending moments at the juncture of the lean-to and the main frame may be computed, considering the lean-to as a separate frame fixed at one end. This bending moment then may be considered as applied on the main frame and its effect computed according to Case 12*b*.

Horizontal Forces Acting on Ridge Frame

Since ridge frames are used mainly for buildings, only such horizontal forces will be considered as may be caused by wind pressure. These are:

Case 13*a*. Uniform wind pressure on inclined member.

Case 13*b*. Uniform wind pressure on vertical member.

Case 13*c*. Concentrated horizontal pressure W.

Effect of wind upon the frame is same as described on p. 286 in connection with rectangular frame. The reactions are found in the same manner. As stated there the whole windward portion of the frame is subjected to positive bending moments while the leeward portion is subjected to negative bending moments.

Case 13*a*. Uniform Wind Pressure on Inclined Portion. (See Fig. 132 (*a*), p. 304.)

Additional notation,

p = uniformly distributed unit wind pressure, lb. per lin. ft.

Vertical Reactions,

$$R_A = -\frac{1}{l}(h + \tfrac{1}{2}h_1)ph_1 \quad (142) \qquad R_E = -R_A \quad (143)$$

Horizontal Reaction,

Leeward hinge E,

$$H = \frac{2\frac{I_1}{I}\frac{h}{s} + \frac{5}{4}\frac{h_1}{h}\left(4 + \frac{h_1}{h}\right) + 6}{4\left[\frac{I_1}{I}\frac{h}{s} + \frac{h_1}{h}\left(3 + \frac{h_1}{h}\right) + 3\right]}ph_1$$

$$= \left[2\frac{I_1}{I}\frac{h}{s} + \frac{5}{4}\frac{h_1}{h}\left(4 + \frac{h_1}{h}\right) + 6\right]C_8ph_1 \quad (144)$$

Value of C_8 are the same as in the previous formulas.

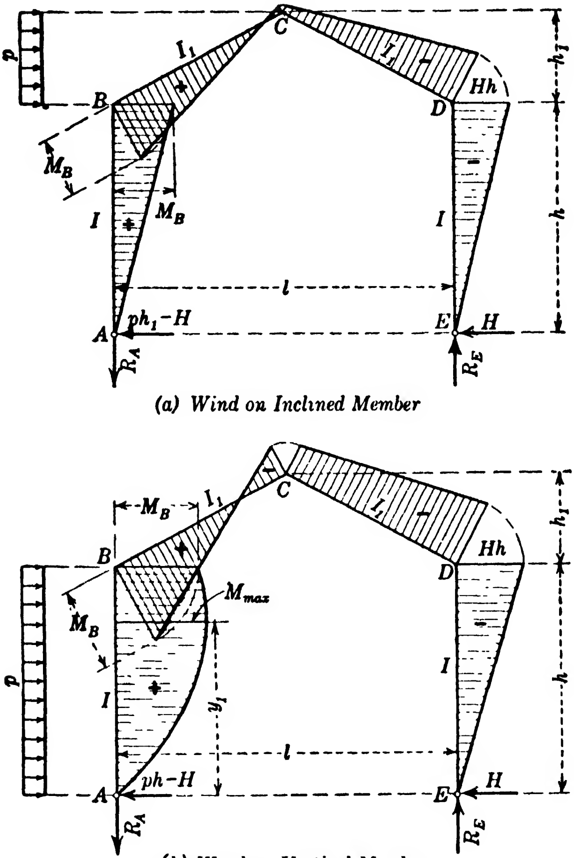
Windward hinge A,

$$H_1 = ph_1 - H \quad (145)$$

Bending Moment at Corners,

$$M_B = H_1 h, \quad \ldots \quad (146)$$

$$M_D = -Hh. \quad \ldots \quad (147)$$



(a) Wind on Inclined Member

(b) Wind on Vertical Member

FIG. 132.—Ridge Frame. Horizontal Wind Pressure Uniformily Distributed. (*See p.* 303.)

Bending Moment at Any Point x,

$$M_x = H_1 h\left(1 + 2\frac{h_1}{h}\frac{x}{l}\right) - V_A x - 2\left(\frac{x}{l}\right)^2 p h_1^2. \quad \ldots \quad (148)$$

Bending Moment at Ridge,

$$M_C = R_E\frac{l}{2} - \left(1 + \frac{h_1}{h}\right)Hh. \quad \ldots \quad (149)$$

Case 13*b*. Uniform Wind Pressure on Vertical Member. (See Fig. 132 (*b*), p. 304.)

Vertical Reactions,

$$R_A = -\frac{1}{2}\frac{h}{l}ph. \quad . \quad . \quad (150) \qquad R_E = -R_A. \quad . \quad . \quad . \quad (151)$$

Horizontal Reaction:

Leeward hinge E,

$$H = \frac{\frac{1}{4}\left[5\frac{I_1}{I}\frac{h}{s} + 6\frac{h_1}{h} + 12\right]}{4\left[\frac{I_1}{I}\frac{h}{s} + \frac{h_1}{h}\left(3 + \frac{h_1}{h}\right) + 3\right]}ph$$

$$= \frac{1}{4}\left[5\frac{I_1}{I}\frac{h}{s} + 6\frac{h_1}{h} + 12\right]C_8 ph. \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

Values of C_8 are the same as in the previous formulas.

Windward hinge A,

$$H_1 = ph - H. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (153)$$

Bending Moments in Corners,

$$M_B = (H_1 - \tfrac{1}{2}ph)h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

$$M_D = -Hh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

Bending Moment at Ridge,

$$M_C = \left[\tfrac{1}{4}ph - H\left(1 + \frac{h_1}{h}\right)\right]h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156)$$

Bending Moment at Any Point in Column,

$$M_y = (H_1 - \tfrac{1}{2}py)y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

Maximum Positive Bending Moment in Column,

$$M_{max} = \tfrac{1}{2}p\left(\frac{H_1}{p}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (158)$$

Point of Maximum Positive Bending Moment,

$$y_1 = \frac{H_1}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

Case 13c. Concentrated Horizontal Pressure at Distance *mh* above Hinge. (See Fig. 133, p. 306.)

Vertical Reaction,

$$R_A = -\,m\frac{h}{l}W. \quad . \quad . \quad (160) \qquad R_E = m\frac{h}{l}W. \quad . \quad . \quad . \quad (161)$$

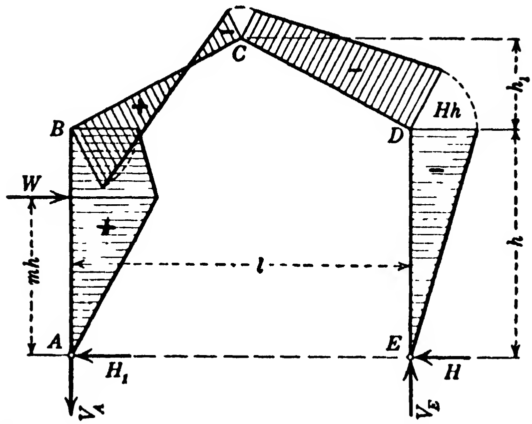


FIG. 133.—Ridge Frame. Concentrated Horizontal Pressure on Vertical Member. (*See p.* 306.)

Horizontal Thrust:

Leeward hinge E,

$$H = \frac{m\left[\frac{I_1}{I}\frac{h}{s}\left(3 - m^2\right) + 3\left(2 + \frac{h_1}{h}\right)\right]}{4\left[\frac{I_1}{I}\frac{h}{s} + \frac{h_1}{h}\left(3 + \frac{h_1}{h}\right) + 3\right]}W. \quad . \quad . \quad (162)$$

Windward hinge A,

$$H_1 = W - H. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (163)$$

Corner Bending Moments,

$$M_B = Wmh - Hh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (164)$$

$$M_D = -\,Hh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (165)$$

Bending Moment at Ridge,

$$M_C = \tfrac{1}{2}R_A l - \left(1 + \frac{h_1}{h}\right)Hh. \quad . \quad . \quad . \quad . \quad . \quad . \quad (166)$$

When the wind applies at A, $m = 1$.

These formulas may be used when the main frame is provided with a lean-to. The wind then is transferred from the lean-to to the main frame at the juncture.

Effect of Temperature Changes

Changes of temperature cause horizontal thrusts at the hinges, which in turn produce bending moment throughout the frame (see also pp. 266 and 291).

Rise of temperature produces thrusts acting inward while fall of temperature produces thrust acting outward.

The resulting bending moments are negative for rise of temperature and positive for fall of temperature.

Let α = coefficient of expansion for 1° F.;
E = modulus of elasticity of concrete;
t = change in temperature in degrees.

Case 14*a*. Rise of Temperature.

Horizontal Thrust,

$$H_t = \frac{12E\alpha t I_1 \frac{l}{2s}}{4\left[\frac{I_1}{I}\frac{h}{s} + \frac{h_1}{h}\left(3 + \frac{h_1}{h}\right) + 3\right]h_2} = 12C_8 E\alpha t I_1 \frac{l}{2s} \quad . \quad . \quad (167)$$

Values of C_8 are the same as in the previous forumlas.

Bending Moments in Corners,

$$M_B = M_D = -H_t h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168)$$

Bending Moment at Ridge,

$$M_C = -H_t(h + h_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169)$$

Case 14*b*. Fall of Temperature.

Horizontal Thrust,

$$H_t = -12C_8 E\alpha t I_1 \frac{l}{2s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (170)$$

Bending Moments in Corners,

$$M_B = M_D = -H_t h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

Bending Moment at Ridges,

$$M_C = -H_t(h + h_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

ROOF FRAME WITH ARCHED ROOF

Roof frames with a curved top are less often used, therefore only typical cases will be considered below. Bending moments due to wind pressure and due to brackets on columns may be computed, without appreciable error, by means of formulas for ridge frames of the same height.

Following assumptions were made regarding the curved roof.

Moments of Inertia.—The moment of inertia of the roof member at the center is I_1. At the other points it varies so that $I_x = \dfrac{I_1}{\cos \phi_x}$, where ϕ_x is the angle of the tangent to the arch with the horizontal.

Shape of Arch.—The arch was assumed to be parabolic in shape as expressed by the following formula.

Formula for Parabolic Roof,

$$y = \frac{4h_1}{l^2}x(l - x), \quad \ldots \ldots \quad (173)$$

where x is measured from left corner B and y above the axis BC.

The formulas can be used without appreciable error when the arch is an arc of a circle, also when the moment of inertia is constant.

Case 15. Parabolic Roof Frame. Whole Span Uniformly Loaded. (See Fig. 134, p. 308.)

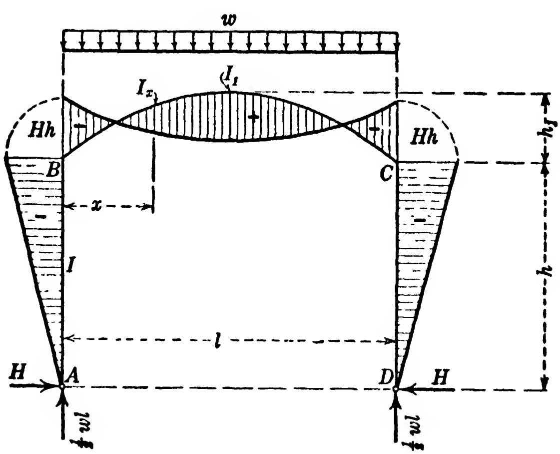


FIG. 134.—Parabolic Roof Frame, Uniformly Distributed Load. (See p. 308.)

Reactions,

$$V_A = V_D = \tfrac{1}{2}wl. \quad \ldots \ldots \quad (174)$$

Horizontal Thrust,

$$H = \frac{1 + \frac{4}{5}\frac{h_1}{h}}{4\left[2\frac{I_1}{I}\frac{h}{l} + 4\frac{h_1}{h}\left(1 + \frac{2}{5}\frac{h_1}{h}\right) + 3\right]}\frac{l}{h}wl. \quad . \quad . \quad (175)$$

Corner Bending Moments,

$$M_B = M_C = -Hh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (176)$$

Bending Moment at Any Point, x Measured from B,

$$M_x = \tfrac{1}{2}x(l - x)w - H(h + y). \quad . \quad . \quad . \quad . \quad . \quad . \quad (177)$$

Maximum Positive Bending Moment,

$$M_{max} = \tfrac{1}{8}wl^2 - H(h + h_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (178)$$

Case 16. Parabolic Roof Frame. Half Span Uniformly Loaded.

Reactions,

$$V_A = \tfrac{3}{8}wl. \quad . \quad . \quad . \quad . \quad (179) \qquad V_B = \tfrac{1}{8}wl. \quad . \quad . \quad . \quad . \quad (180)$$

Horizontal Thrust,

For one-sided loading the horizontal thrust is equal to one-half the thrust for full loading.

Corner Bending Moments,

$$M_B = M_C = -Hh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

Bending Moment at Any Point, x Measured from B,

$$M_x = x(\tfrac{3}{8}l - \tfrac{1}{2}x)w - H(h + y). \quad . \quad . \quad . \quad . \quad (182)$$

Maximum Positive Bending Moment,

This moment is obtained by substituting in formula (182) for x the value of x_1 from the next formula.

Point of Maximum Positive Bending Moment,

$$x_1 = \frac{\frac{3}{8} - \frac{4H}{wl}\frac{h_1}{l}}{1 + \frac{8H}{wl}\frac{h_1}{l}}l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

Case 17. Concentrated Vertical Load. (See Fig. 135, p. 310.)
Reactions,

$$V_A = \left(1 - \frac{a}{l}\right)P. \quad . \quad . \quad (000) \qquad V_D = \frac{a}{l}P. \quad . \quad . \quad . \quad . \quad (184)$$

Horizontal Thrust,

$$H = \frac{3 + 2\frac{h_1}{h} - \frac{a}{l}\left[3 + \frac{h_1}{h}\frac{a}{l}\left(4 - 2\frac{a}{l}\right)\right]}{2\left[2\frac{I_1}{I}\frac{h}{l} + 4\frac{h_1}{l}\left(1 + \frac{2}{5}\frac{h_1}{h}\right) + 3\right]}\frac{a}{h}P. \quad . \quad . \quad (185)$$

Corner Bending Moment,

$$M_B = M_C = -Hh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

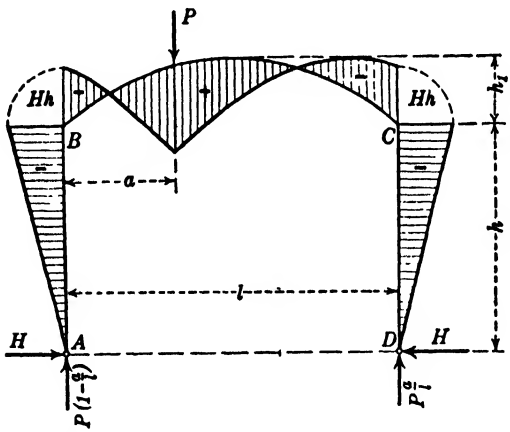


FIG. 135.—Parabolic Roof Frame. Concentrated Load. (*See p.* 310.)

SLANTING ROOF FRAME WITH HINGED ENDS

A slanting roof frame is a frame consisting of three members in which the vertical members are not of the same height as shown in Fig. 136, p. 311. Special formulas must be worked out for this case. These are given below.

Notation.

Let l = length of span;
s = length of inclined member;
h_l = height of left, in this case short, column;
h_r = height of right, in this case long, column;
I = moment of inertia of inclined member;

I_l = moment of inertia of left column;
I_r = moment of inertia of right column;
ϕ = angle of inclination of inclined member.

Case 18. Uniformly Distributed Load. (See Fig. 136.)

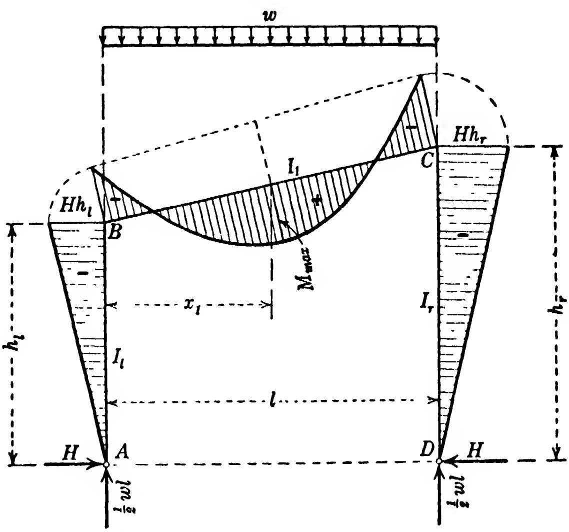


FIG. 136.—Slanting Roof. Uniformly Distributed Load. (See p. 311.)

Vertical Reactions,

$$R_A = R_D = \tfrac{1}{2}wl. \qquad (187)$$

Horizontal Thrust,

$$H = \frac{\dfrac{l}{h_l}\left(1 + \dfrac{h_r}{h_l}\right)}{8\left[\dfrac{h_l}{s}\dfrac{I}{I_l} + \left(\dfrac{h_r}{h_l}\right)^2\left(\dfrac{h_r}{s}\dfrac{I}{I_r} + 1\right) + \dfrac{h_r}{h_l} + 1\right]}wl. \qquad (188)$$

Corner Bending Moments,

$$M_B = -Hh_l. \qquad (189)$$

$$M_C = -Hh_r. \qquad (190)$$

Bending Moment at Any Point x,

$$M_x = \tfrac{1}{2}x(l - x)w - H\left(h_l + \frac{h_r - h_l}{l}x\right). \qquad (191)$$

Point of Maximum Positive Bending Moment,

$$x_1 = \tfrac{1}{2}l - \frac{H}{wl}\,\frac{h_r - h_l}{l}\,l. \qquad (192)$$

Maximum Positive Bending Moment.

This is obtained by substituting the above value of x_1 in equation for bending moment at any point.

Case 19. Concentrated Load at Distance *a*. (See Fig. 137, p. 312.)

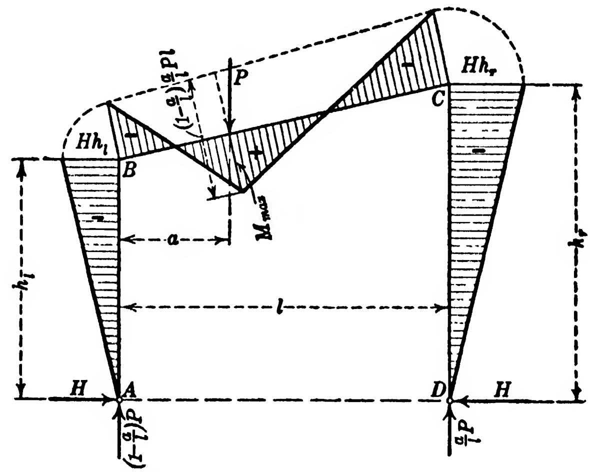


FIG. 137.—Slanting Roof. Concentrated Load. (See p. 312.)

Vertical Reactions,

$$R_A = \left(1 - \frac{a}{l}\right)P, \quad (193) \qquad R_D = \frac{a}{l}P. \quad (194)$$

Horizontal Thrust,

$$H = \frac{\dfrac{a}{l}\left(1 - \dfrac{a}{l}\right)\left[2 + \dfrac{h_r}{h_l} + \dfrac{a}{l}\left(\dfrac{h_r}{h_l} - 1\right)\right]}{2\left[\dfrac{h_l}{s}\dfrac{I}{I_l} + \left(\dfrac{h_r}{h_l}\right)^2\left(\dfrac{h_r}{s}\dfrac{I}{I_r} + 1\right) + \dfrac{h_r}{h_l} + 1\right]}\,\frac{l}{h_l}P. \qquad (195)$$

Corner Bending Moments,

$$M_B = -\,Hh_l, \qquad (196)$$

$$M_C = -\,Hh_r. \qquad (197)$$

Maximum Positive Bending Moment at Load,

$$M = \frac{a}{l}\left(1 - \frac{a}{l}\right)Pl - H\left(h_l + \frac{h_r - h_l}{l}a\right). \quad \ldots \ldots \quad (198)$$

Case 20. Symmetrically Placed Concentrated Loads.

Five positions of concentrated loads are considered, namely:

1. Load P at center.
2. Two loads P at third points.
3. Three loads P at quarter points.
4. Four loads P at fifth points.
5. Five loads P at sixth points.

It should be noted that loads placed directly over columns produce no bending moments and increase only the reactions at the columns. General formula for the horizontal thrust is given below.

Horizontal Thrust for Concentrated Loads.

$$H = \frac{\Sigma\frac{a}{l}\left(1 - \frac{a}{l}\right)\left[2 + \frac{h_r}{h_l} + \frac{a}{l}\left(\frac{h_r}{h_l} - 1\right)\right]}{2\left[\frac{h_l}{s}\frac{I}{I_l} + \left(\frac{h_r}{h_l}\right)^2\left(\frac{h_r}{s}\frac{I}{I_r} + 1\right) + \frac{h_r}{h_l} + 1\right]}\frac{l}{h_l}P. \quad . \quad (199)$$

The values of $\Sigma\frac{a}{l}\left(1 - \frac{a}{l}\right)\left[2 + \frac{h_r}{h_l} + \frac{a}{l}\left(\frac{h_r}{h_l} + 1\right)\right]$ for the five loading conditions are given in table below.

Case	Number and Position of Load	Numerator in Formula for H $\Sigma\frac{a}{l}\left(1-\frac{a}{l}\right)\left[2+\frac{h_r}{h_l}+\frac{a}{l}\left(\frac{h_r}{h_l}-1\right)\right]$	Maximum Positive Bending Moment
1	$1P$ At Center	$\frac{3}{8}\left(\frac{h_r}{h_l}+1\right)$	$\frac{1}{4}Pl-\frac{1}{2}H(h_l+h_r)$
2	$2P$ At Third Points	$\frac{2}{3}\left(\frac{h_r}{h_l}+1\right)$	$\frac{1}{3}Pl-\frac{1}{3}H(2h_l+h_r)$
3	$3P$ At Quarter Points	$\frac{15}{16}\left(\frac{h_r}{h_l}+1\right)$	$\frac{1}{2}Pl-\frac{1}{2}H(h_l+h_r)$
4	$4P$ At Fifth Points	$1\frac{1}{5}\left(\frac{h_r}{h_l}+1\right)$	$\frac{3}{5}Pl-\frac{3}{5}H(h_l+\frac{2}{3}h_r)$
5	$5P$ At Sixth Points	$1\frac{11}{24}\left(\frac{h_r}{h_l}+1\right)$	$\frac{3}{4}Pl-\frac{1}{2}H(h_l+h_r)$

Case 21. Slanting Roof. Wind Pressure on Inclined Member. (See Fig. 138, p. 314.)

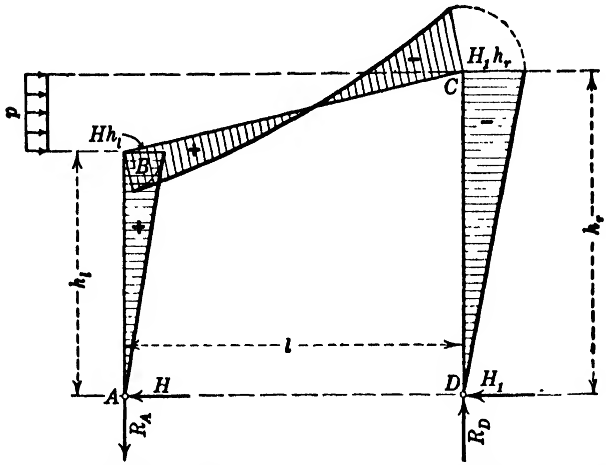


FIG. 138.—Slanting Roof. Wind Pressure on Inclined Member. (*See p.* 314.)

Vertical Reactions,

$$R_A = -\, p(h_r - h_l)\frac{h_r + h_l}{2l}. \quad (200) \qquad R_D = -\, R_A. \quad (201)$$

Horizontal Reactions:

At left hinge A,

$$H = p(h_r - h_l)\frac{\left(\frac{h_r}{h_l}\right)^2\left(8\frac{h_r}{s}\frac{I}{I_r} + 7\right) + 4\frac{h_r}{h_l} + 1}{8\left[\frac{h_l}{s}\frac{I}{I_l} + \left(\frac{h_r}{h_l}\right)^2\left(\frac{h_r}{s}\frac{I}{I_r} + 1\right) + \frac{h_r}{h_l} + 1\right]}. \quad (202)$$

At right hinge D,

$$H_1 = p(h_r - h_l) - H. \quad (203)$$

Corner Bending Moments,

$$M_B = +\, Hh_l. \quad (204)$$

$$M_C = -\, H_1h_r. \quad (205)$$

Case 22. Slanting Roof. Wind Pressure on Vertical Member. (See Fig. 139, p. 315.)

Vertical Reactions,

$$R_A = -\, \tfrac{1}{2}ph_l\frac{h_l}{l}. \quad (206) \qquad R_D = -\, R_A. \quad (207)$$

Horizontal Reaction:

At left hinge A,

$$H = ph_l \frac{8\left(\frac{h_r}{h_l}\right)^2\left(\frac{h_r}{s}\frac{I}{I_r} + 1\right) + 6\frac{h_r}{h_l} + 3\frac{h_l}{l}\frac{I}{I_l} + 4}{8\left[\frac{h_l}{s}\frac{I}{I_l} + \left(\frac{h_r}{h_l}\right)^2\left(\frac{h_r}{s}\frac{I}{I_r} + 1\right) + \frac{h_r}{h_l} + 1\right]}. \quad . \quad . \quad (208)$$

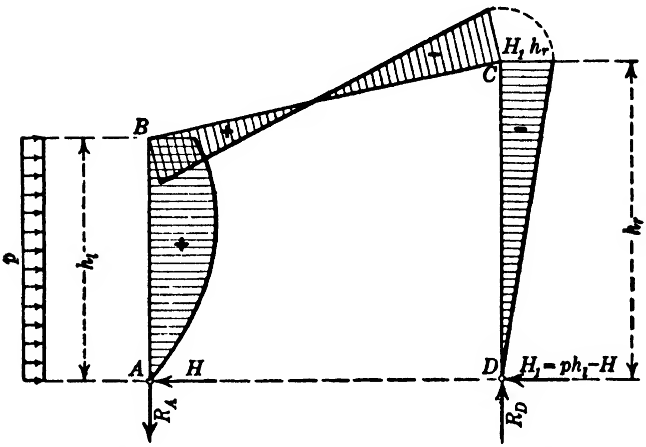


FIG. 139.—Slanting Roof. Wind Pressure on Vertical Member. (*See p.* 314.)

At right hinge D,

$$H_1 = ph_l - H. \quad . \quad . \quad . \quad (209)$$

Corner Bending Moments,

$$M_A = Hh_l - \tfrac{1}{2}ph_l^2. \quad . \quad . \quad . \quad (210)$$

$$M_B = - H_1 h_l. \quad . \quad . \quad . \quad (211)$$

Bending Moment at Any Point in Left Column,

$$M_y = H_y - \tfrac{1}{2}py^2. \quad . \quad . \quad . \quad (212)$$

Point of Maximum Bending Moment,

$$y_1 = \frac{H}{p}. \quad . \quad . \quad . \quad (213)$$

Maximum Bending Moment,

$$M_{\max} = \frac{1}{2}\frac{H^2}{p}. \quad . \quad . \quad . \quad (214)$$

SAW-TOOTH ROOF FRAME

Saw-tooth roof frame as shown in Fig. 140, p. 316, is similar to a ridge roof frame, the difference between the two being in the location of the ridge.

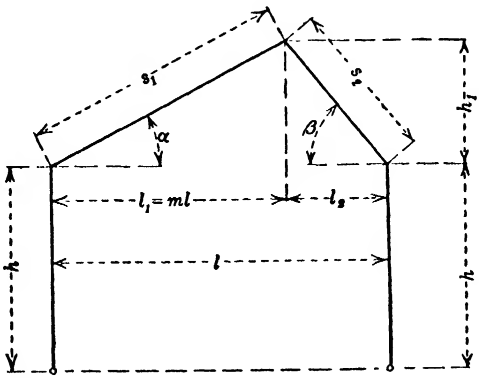


FIG. 140.—Saw-tooth Roof Frame. (*See p.* 316.)

Let l = horizontal span of frame;
h = height of vertical member of frame;
h_1 = vertical projection of the inclined member;
$l_1 = ml$ = horizontal projection of the left inclined member;
$l_2 = (1 - m)l$ = horizontal projection of the right inclined member;
s_1 = length of left inclined member;
s_2 = length of right inclined member;
I_1 = moment of inertia of normal section of inclined members;
I = member of inertia of vertical members;
α and β = angles of the inclined members with horizontal.

The derivation of formulas is similar to that for ridge frames.

Vertical Reactions.—Vertical reactions are same as for statically determinate structures.

Horizontal Thrust.—General formula for horizontal thrust for vertical loads is given below.

Horizontal Thrust. General Formula,

$$H = \frac{\int_0^{l_1} M_s\left(h + \frac{h_1}{l_1}x\right)ds + \int_{l_1} M_s\left[h + \frac{h_1}{l_2}(l - x)\right]ds}{lh^2\left\{\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + \left(\frac{s_1}{l} + \frac{s_2}{l}\right)\left[1 + \frac{h_1}{h} + \frac{1}{3}\left(\frac{h_1}{h}\right)^2\right]\right\}}. \quad (215)$$

Types of Loading Considered.—Usually a saw-tooth roof is loaded by uniformly distributed loading and by a concentrated load on the ridge. Formulas for these loadings are given below.

Uniformly Distributed Loading.—The unit dead load on the flatter part, which is provided with a concrete slab, is always larger than of the glass-covered steeper part. The live load on the flatter part of the roof is also heavier. Therefore it will be assumed that the intensity of the uniformly distributed loading is different for the two sections. The condition is illustrated in Fig. 141, p. 317.

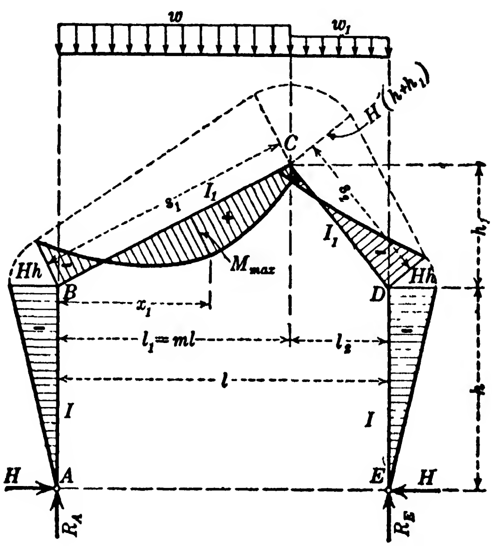


FIG. 141.—Saw-tooth Roof. Uniformly Distributed Loading. (*See p.* 317.)

Let, in addition to notation on p. 316,

w = uniformly distributed vertical unit load, left part of the frame;

w_1 = uniformly distributed vertical unit load, right part of the frame.

Then

Vertical Reaction for Loading on Both Sides,

$$R_A = \left(1 - \frac{m}{2}\right)wl_1 + \tfrac{1}{2}(1 - m)w_1l_2 \quad . \quad . \quad . \quad . \quad . \quad (216)$$

$$R_E = wl_1 + w_1l_2 - R_A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (217)$$

To get reactions when one side only is loaded make, in the above equations, the unit load of the unloaded side equal zero.

Horizontal Thrust:

For uniform loading on the flatter side,

$$H_1 = \frac{ml}{12h} \frac{3(1-m)\left(\frac{s_1}{l} + \frac{s_2}{l}\right)\left(1 + \frac{2}{3}\frac{h_1}{h}\right) + \frac{s_1}{l}\left(1 + \frac{1}{2}\frac{h_1}{h}\right)}{\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + \left(\frac{s_1}{l} + \frac{s_2}{l}\right)\left[1 + \frac{h_1}{h} + \frac{1}{3}\left(\frac{h_1}{h}\right)^2\right]} wl_1. \quad (218)$$

For uniform loading on the steep side,

$$H_2 = \frac{(1-m)l}{12h} \frac{3m\left(\frac{s_1}{l} + \frac{s_2}{l}\right)\left(1 + \frac{2}{3}\frac{h_1}{h}\right) + \frac{s_2}{l}\left(1 + \frac{1}{2}\frac{h_1}{h}\right)}{\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + \left(\frac{s_1}{l} + \frac{s_2}{l}\right)\left[1 + \frac{h_1}{h} + \frac{1}{3}\left(\frac{h_1}{h}\right)^2\right]} w_1 l_2. \quad (219)$$

Total Thrust for Loads on Both Sides,

$$H = H_1 + H_2. \quad \ldots\ldots\ldots \quad (220)$$

The denominator of both equations is the same.

If the uniformly distributed loading is the same on both sides the formula for horizontal thrust becomes

Horizontal Thrust, Both Sides Loaded with Loading w,

$$H = \frac{l}{12h} \frac{\left\{ 3m(1-m)\left(\frac{s_1}{l} + \frac{s_2}{l}\right)\left(1 + \frac{2}{3}\frac{h_1}{h}\right) + \left(m^2\frac{s_1}{l} + (1-m)^2\frac{s_2}{l}\right)\left(1 + \frac{1}{2}\frac{h_1}{h}\right) \right\}}{\frac{2}{3}\frac{I_1}{I}\frac{h}{l} + \left(\frac{s_1}{l} + \frac{s_2}{l}\right)\left[1 + \frac{h_1}{h} + \frac{1}{3}\left(\frac{h_1}{h}\right)^2\right]} wl. \quad (221)$$

Bending Moments at Corners,

$$M_B = M_D = -Hh. \quad \ldots\ldots\ldots \quad (222)$$

Bending Moment at Ridge,

$$M_C = l_1(R_A - \tfrac{1}{2}wl_1) - H(h + h_1). \quad \ldots \quad (223)$$

This bending moment may be positive or negative, depending upon the inclination of the members and the ratio m.

Bending Moment at Any Point:

Left side, x measured from left support,

$$M_x = R_A x - \tfrac{1}{2}wx^2 - H\left(h + \frac{h_1}{l_1}x\right). \quad \ldots \quad (224)$$

Right side, x measured from left support,

$$M_x = R_B(l - x) - \tfrac{1}{2}w_1(l - x)^2 - H\left[h + \frac{h_1}{l_2}(l - x)\right]. \quad . \quad (225)$$

The point of maximum bending moment depends not only upon the load, but also upon the inclination of the roof. It is given by the following equation.

Point of Maximum Bending Moment,

$$x_1 = \frac{1}{w}\left(R_A - H\frac{h_1}{l_1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (226)$$

Maximum Positive Bending Moment,

Maximum positive bending moment is obtained by substituting the value of x_1 in equation for M_x for the left side.

Concentrated Load P at Ridge. (See Fig. 142, p. 319.)

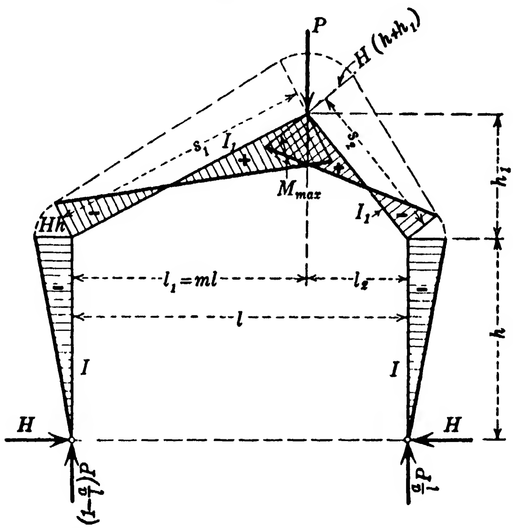


FIG. 142.—Saw-tooth Roof. Load P at Ridge. (*See p.* 319.)

Vertical Reactions,

$$R_A = (1 - m)P. \quad . \quad (226a) \qquad R_B = mP. \quad . \quad . \quad . \quad . \quad . \quad (227)$$

Horizontal Thrust,

$$H = \frac{1}{2}\,\frac{m(1 - m)\left(\dfrac{s_1}{l} + \dfrac{s_2}{l}\right)\left(1 + \dfrac{2}{3}\dfrac{h_1}{h}\right)}{\dfrac{2}{3}\dfrac{I_1}{I}\dfrac{h}{l} + \left(\dfrac{s_1}{l} + \dfrac{s_2}{l}\right)\left[1 + \dfrac{h_1}{h} + \dfrac{1}{3}\left(\dfrac{h_1}{h}\right)^2\right]}\,\frac{l}{h}P. \quad . \quad . \quad (228)$$

Corner Bending Moment,

$$M_A = M_D = -Hh. \qquad (229)$$

Bending Moment at Ridge,

$$M_C = (1 - m)Pl_1 - H(h + h_1). \qquad (230)$$

SPECIAL SHAPE OF SAW-TOOTH ROOF

The special shape of saw-tooth roof shown in Fig. 143 is often used. The equations for this case are worked out below.

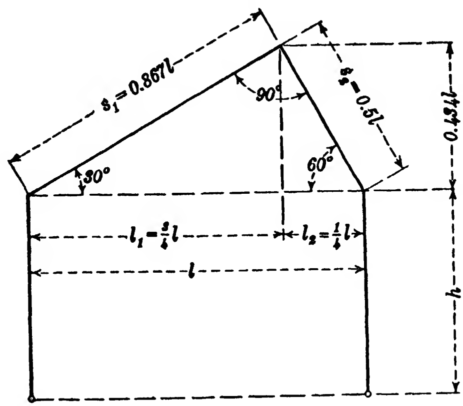


FIG. 143.—Special Shape of Saw-Tooth roof. (*See p.* 320.)

For this special case

$$l_1 = \tfrac{3}{4}l, \; m = \tfrac{3}{4},$$

$$s_1 = \frac{\sqrt{3}}{2}l, \; \frac{s_1}{l} = \frac{\sqrt{3}}{2} = 0.867; \; s_2 = \tfrac{1}{2}l, \; \frac{s_2}{l} = \frac{1}{2},$$

$$\frac{s_1}{l} + \frac{s_2}{l} = 1.367,$$

$$h_1 = \frac{\sqrt{3}}{4}l = 0.434l.$$

These values are substituted in general formulas.

Uniform Loading. Special Shape of Saw-tooth Frame.

Vertical Reaction,

$$R_A = (\tfrac{15}{32}w + \tfrac{1}{32}w_1)l. \qquad (231)$$

$$R_B = (\tfrac{9}{32}w + \tfrac{7}{32}w_1)l. \qquad (232)$$

Horizontal Thrust:

For load on flat side,

$$H_1 = 0.047 \frac{1.03\left(1+0.289\frac{l}{h}\right) + \left(0.867 + 0.188\frac{l}{h}\right)}{0.667\frac{I_1}{I}\frac{h}{l} + 1.367\left[1 + 0.433\frac{l}{h} + 0.0625\left(\frac{l}{h}\right)^2\right]} wl. \quad (233)$$

For load on steep side,

$$H_2 = 0.005 \frac{3.075\left(1 + 0.289\frac{l}{h}\right) + \left(0.5 + 0.108\frac{l}{h}\right)}{0.667\frac{I_1}{I}\frac{h}{l} + 1.367\left[1 + 0.433\frac{l}{h} + 0.0625\left(\frac{l}{h}\right)^2\right]} w_1 l. \quad (234)$$

Total thrust, both sides loaded,

$$H = H_1 + H_2. \quad \ldots \quad (235)$$

Bending moment formulas are same as for general case.

Concentrated Load at Ridge. Special Shape of Saw-tooth Roof.

Vertical Reaction,

$$R_A = \tfrac{1}{4}P. \quad \ldots \quad (236) \qquad R_B = \tfrac{3}{4}P. \quad \ldots \quad (237)$$

Horizontal Thrust,

$$H = \frac{1}{2} \frac{0.256\left(1 + 0.289\frac{l}{h}\right)}{0.667\frac{I_1}{I}\frac{h}{l} + 1.367\left[1 + 0.433\frac{l}{h} + 0.0625\left(\frac{l}{h}\right)^2\right]}. \quad (238)$$

Formulas for bending moments same as for general case.

Saw-tooth Roof Frame Not Connected with Columns

If the connection between the column and the frame is not rigid, but the column is held against side movement, the saw-tooth roof may be considered as hinged at the juncture of the roof and the column (see Fig. 144, p. 322). In such case the value of h in Formulas (218) to (226) is zero and following formulas result.

Uniformly Distributed Loading.

Reactions:

Both sides loaded,

$$R_A = \left(1 - \frac{m}{2}\right)wl_1 + \tfrac{1}{2}(1 - m)w_1 l_2. \quad \ldots \quad (239)$$

$$R_C = (wl_1 + w_1 l_2) - R_A. \quad \ldots \quad (240)$$

Horizontal Thrust:

Left side loaded,

$$H_1 = \frac{1}{2}\frac{ml}{h_1}\frac{(1\frac{1}{4} - m)\frac{s_1}{l} + (1 - m)\frac{s_2}{l}}{\frac{s_1}{l} + \frac{s_2}{l}}wl_1. \quad . \quad . \quad . \qquad (241)$$

Right side loaded,

$$H_2 = \frac{1}{2}\frac{(1 - m)l}{h_1}\frac{m\frac{s_1}{l} + \left(m + \frac{1}{4}\right)\frac{s_2}{l}}{\frac{s_1}{l} + \frac{s_2}{l}}w_1l_2. \quad . \quad . \quad . \qquad (242)$$

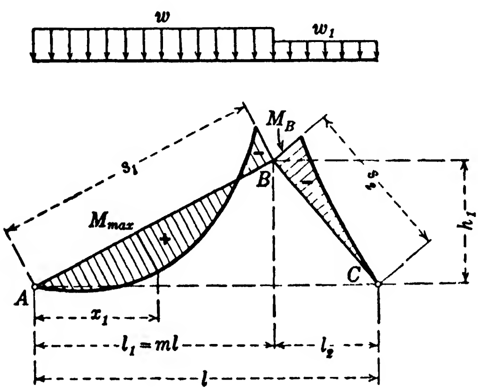


FIG. 144.—Saw-tooth Roof Hinged on Top of Columns. Uniform Loading. (*See p.* 321.)

Both sides loaded,

$$H = H_1 + H_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (243)$$

Bending Moment at Any Point,

Left side, x measured from left support,

$$M_x = R_Ax - \tfrac{1}{2}wx^2 - H\frac{h_1}{l_1}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (244)$$

Right side, x measured from right support,

$$M_x = R_C(l - x) - \tfrac{1}{2}w_1(l - x)^2 - H\frac{h_1}{l_2}(l - x). \quad . \quad . \qquad (245)$$

Points of Maximum Bending Moment,

$$x_1 = \frac{1}{w}\left(R_A - H\frac{h_1}{l_1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (246)$$

Bending Moment at the Ridge,

$$M_B = l_1(R_A - \tfrac{1}{2}wl_1) - Hh_1. \quad \ldots \ldots \quad (247)$$

Concentrated Load P at Ridge.

Reactions,

$$R_A = (1 - m)P. \quad . \quad (247a) \qquad R_C = mP. \quad . \quad . \quad (248)$$

Horizontal Thrust:

$$H = m(1 - m)\frac{l}{h_1}P. \quad \ldots \ldots \quad (249)$$

Bending Moment at Ridge,

$$M_B = Hh_1. \quad \ldots \ldots \quad (250)$$

Saw-tooth Roof with Tension Member

When the columns upon which the saw-tooth rests are not held firmly on the top and, therefore, are not capable of resisting the horizontal thrust, a frame shown in Fig. 144 cannot be used because under the action of the load the ends of the frame would spread and it would act as a simple supported beam. In such case the frame should be provided with a horizontal tension member which resists the horizontal thrust. Such frame is shown in Fig. 145, p. 323.

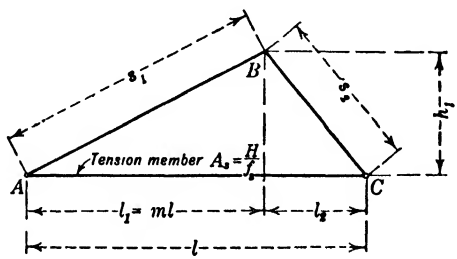


FIG. 145.—Saw-tooth Roof with Horizontal Tension Member. (*See p.* 323.)

Vertical Reactions.—The vertical reactions of such frame are same as in the previous case.

Horizontal Thrust.—The horizontal thrust is resisted by the tension member. Under the action of the thrust the tension member lengthens thus allowing a small increase of the span. To allow for this increase the Formulas (241) to (243) for horizontal thrust is changed to

Let E_s = modulus of elasticity of steel.

Horizontal Thrust, Uniform Load, Frame with Tension Member:

Left side, loaded,

$$H_1 = \frac{1}{2}\frac{ml}{h_1}\frac{(1\frac{1}{4} - m)\frac{s_1}{l} + (1 - m)\frac{s_2}{l}}{\frac{s_1}{l} + \frac{s_2}{l} + \frac{f_s}{E_s}\frac{l}{h_1}}wl_1. \quad . \quad . \quad (251)$$

Right side, loaded,

$$H_2 = \frac{1}{2}\frac{(1 - m)l}{h_1}\frac{m\frac{s_1}{l} + (m + \frac{1}{4})\frac{s_2}{l}}{\frac{s_1}{l} + \frac{s_2}{l} + \frac{f_s}{E_s}\frac{l}{h_1}}wl_2. \quad . \quad . \quad (252)$$

Horizontal Thrust, Load Concentrated at Ridge,

$$H = \frac{m(1 - m)\left(\frac{s_1}{l} + \frac{s_2}{l}\right)}{\frac{s_1}{l} + \frac{s_2}{l} + 3\frac{f_s}{E_s}\frac{l}{h_1}}\frac{l}{h_1}P. \quad . \quad . \quad . \quad . \quad . \quad (253)$$

In the expression $\frac{f_s}{E_s}\frac{l}{h_1}$ the values f_s and E_s and l and h_1, respectively, must be in the same units.

The bending moments are found in the same manner as in previous case.

Tension Member.—The area of the tension member is found by dividing the horizontal thrust by the allowable unit stress f_s.

$$A_s = \frac{H}{f_s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (254)$$

The tension member may consist of one or more heavy bars. These may be exposed or covered with a shell of concrete. The bars must be firmly anchored to the concrete member at the ends. In such construction no bending moment is developed at the ends.

Sometimes the tension member consists of a number of small bars imbedded in a concrete beam. The juncture of the beam with the inclined members in rigid and must be reinforced to take care of the bending moment developed there.

EFFECT OF FIXING OR RESTRAINING IN RIGID FRAMES OF COLUMNS AT BOTTOM

The formulas for rigid frames given on the previous pages are based on the assumption that the columns are hinged at the bottom, i.e., that they are free to rotate at the ends so that no bending moment can develop there.

This condition may not always exist in practice. The columns either may be actually fixed at the bottom, as, for instance, when they are connected with heavy foundations, or they may be partly restrained. In both cases proper changes must be made in the design to take care of the bending moments produced by the restraint at the ends.

Ends of Columns Fixed.—The effect of fixing the ends of the columns is as follows:

1. The maximum bending moments at the top of the columns are increased.

2. The negative bending moments in the loaded spans are increased and the positive bending moments decreased.

3. Bending moments are developed at the bottom of the columns the magnitude of which is equal to one-half of the bending moments at the top of the columns. The sign of the bottom bending moments is opposite to the sign at the top.

For vertical loading the bending moments in a right-angle frame with fixed angles are equal to the bending moments of a frame with hinged ends of the same span, but with a height of columns equal to three-fourths of the height of the fixed frame. The bending moments obtained from the hinged frame may be used for the beam and for the columns at the top. At the bottom of the columns will act a bending moment of opposite sign and equal to one-half the bending moment at the top. Between the top and bottom of the columns the bending moments vary according to a straight line, which fixes the point of contraflexure in the columns at one-third of the height of the column, measured from the bottom.

Partial Restraint of Column Ends.—When the column ends are partially restrained the bending moments in the frame will be between the bending moments for the fixed and hinged end condition. If the restraint at the end can be expected but cannot be positively counted upon, it is best to design the frame as hinged at the bottom and in addition to provide reinforcement at the bottom to take care of the bending moment developed there by the restraint.

RIGID FRAMES WITH FIXED ENDS

Definition.—Rigid frames are considered as having fixed ends when the columns are rigidly attached to heavy foundations, or placed on top of other heavy construction, in such a way that when loaded the tangent to the deflection curve at the bottom coincides with the original axis of the member.

Rigid frames are structures of same nature as fixed arches. As is the case with fixed arches, rigid frames with fixed ends have three statically indeterminate values.

Since rigid frames usually consist of straight members it is possible to solve the integrals so that it is not necessary to resort to the summation method often required in arch design.

Notation

Let l = span of frame;
I_1 = minimum moment of inertia of the horizontal or inclined member;
I_x = moment of inertia at any point of frame;
X, Y = coordinates referred to left support A as center;
X_c, Y_c = coordinates of elastic center;
x, y = coordinates referred to elastic center as center of coordinates;
V_A = vertical reaction at left support;
H_A = horizontal reaction at left support;
M = auxiliary bending moment;
M_s = static bending moment at any point due to loading, considering frame as a cantilever fixed at right support. It is negative (see Formulas (264) to (266);
M_x = actual bending moment in frame at any point.

General Formulas.—General formulas for rigid frames with fixed ends are the same, and are developed in the same manner, as for arches. The formulas are developed in Chapter VIII.

Elastic Center.—Usually it is necessary to find the elastic center first, which is then accepted as the center of coordinates. The formulas for the location of the elastic center for a symmetrical frame are given below.

Position of Center of Coordinates with Reference to Left Support,

$$X_c = \tfrac{1}{2}l. \qquad (255)$$

$$Y_c = \frac{\int_0^{\frac{l}{2}} Y \frac{I_1}{I_x} ds}{\int_0^{\frac{l}{2}} \frac{I_1}{I_x} ds} \qquad (256)$$

Statically Indeterminate Values.—The three statically indeterminate values are:

1. Vertical reaction at left support V_A.
2. Horizontal thrust at left support H_A.
3. Auxiliary bending moment M.

The formulas for the statically indeterminate values are:

Vertical Reaction at Left Support,

$$V_A = -\frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s x \frac{I_1}{I_x} ds}{\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \frac{I_1}{I_x} ds} \qquad (257)$$

Since M_s is negative, V_A is positive.

Horizontal Thrust at Left Support,

$$H_A = -\frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s y \frac{I_1}{I_x} ds}{\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{I_1}{I_x} ds} \qquad (258)$$

This value is negative.

Auxiliary Bending Moment,

$$M = -\frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s \frac{I_1}{I_x} ds}{\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{I_1}{I_x} ds} \qquad (259)$$

Since M_s is negative M is positive.

The values of x and y in the Formulas (257) to (259) refer to the center of coordinates at the elastic center.

After the above indeterminate values are found the bending moments at the supports M_A and at any point M_x are computed from the following formulas.

Bending Moment at Left Support,

$$M_A = M - V_A\frac{l}{2} - H_AY_e. \quad . \quad . \quad . \quad (260)$$

Bending Moment at Any Point x and y,

$$M_x = M + V_Ax + H_Ay + M_e, \quad . \quad . \quad (261)$$

where x and y are coordinates referred to the elastic center as origin. They must be used with their proper signs. Thus for all points to the left of the center use $(-x)$ for x and for points below the axis $(-y)$ for y.

Example of Application of General Formulas to Frames

Frames usually consist of straight sections for which the integrals may be easily solved. The integrals for the frame are evaluated by solving the integrals for each straight section and adding the results.

To illustrate the application of general formulas, special formulas will be developed for a ridge frame as shown in Fig. 146, p. 328. Moments of inertia are constant throughout each member.

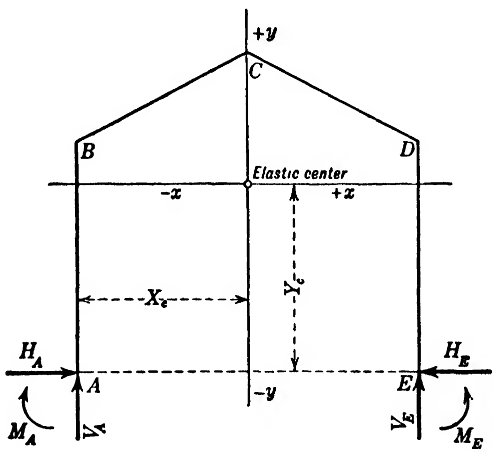


FIG. 146.—Ridge Frame with Fixed Ends. (*See p.* 328.)

In addition to notation on p. 326,

Let I_1 = constant moment of inertia of inclined member;
I = constant moment of inertia of vertical member;
h = height of vertical member;

h_1 = vertical projection of inclined member;
ϕ = angle of inclination, inclined member.

Elastic Center for Ridge Frame.—The integrals forming Formula (256) are solved as follows: Referring to Fig. 146 the numerator of Formula (256) is

$$\int_0^{\frac{l}{2}} Y\frac{I_1}{I_x}ds = \frac{I_1}{I}\int_0^h YdY + \frac{I_1}{I_1}\int_0^{\frac{l}{2}}\left(h + \frac{2h_1}{l}x\right)\frac{dx}{\cos\phi}$$
$$= \frac{I_1}{I}\frac{h^2}{2} + \frac{1}{\cos\phi}\left(\frac{hl}{2} + \frac{h_1l^2}{4l}\right)$$

$$\int_0^{\frac{l}{2}} Y\frac{I_1}{I_x}ds = \frac{1}{2}\left[\frac{I_1}{I_x}h^2 + \frac{l}{\cos\phi}\left(h + \frac{h_1}{2}\right)\right].$$

The denominator of Formula (256) is

$$\int_0^{\frac{l}{2}}\frac{I_1}{I_x}ds = \frac{I_1}{I}\int_0^h dy + \frac{I_1}{I_1}\int_0^{\frac{l}{2}}\frac{dx}{\cos\phi} = \frac{I_1}{I}h + \frac{l}{2\cos\phi}.$$

Therefore

Ordinates for Elastic Center for Ridge Frame,

$$X_c = \frac{l}{2}. \qquad (262)$$

$$Y_c = \frac{\frac{I_1}{I}h^2 + \frac{l}{\cos\phi}\left(h + \frac{h_1}{2}\right)}{2\frac{I_1}{I}h + \frac{l}{\cos\phi}}. \qquad (263)$$

This is the new center of coordinates.

Denominator for H_A for Ridge Frame.—The denominator for Formula (258), p. 327, is solved as follows:

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{I_1}{I_x}ds = 2\frac{I_1}{I}\int_{-Y_c}^{h-Y_c} y^2dy + 2\int_0^{\frac{l}{2}}\left(h + \frac{2h_1}{l}x - Y_c\right)^2\frac{dx}{\cos\phi}.$$

This solved gives

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{I_1}{I_x}ds = h\frac{I_1}{I}\left(\frac{2}{3}h^2 - 2Y_ch + 2Y_c^2\right)$$
$$+ \frac{l}{\cos\phi}\left(h^2 + hh_1 + \frac{h_1^2}{3} - 2Y_ch - Y_ch_1 + Y_c^2\right)$$

Denominator for V_A for Ridge Frame.—The denominator for Formula (257), p. 327, is solved as follows:

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \frac{I_1}{I_x} ds = 2\left(\frac{l}{2}\right)^2 \frac{I_1}{I} h + 2\int_{-\frac{l}{2}}^{0} x^2 \frac{dx}{\cos\phi} = \frac{l^2}{2}\left(\frac{I_1}{I} h + \frac{1}{6}\frac{l}{\cos\phi}\right).$$

Denominator for M for Ridge Frame.

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{I_1}{I_x} ds = 2\frac{I_1}{I} h + \frac{l}{\cos\phi}.$$

Numerators for Ridge Frame.—Numerators for all formulas must be determined separately for each type of loading as they depend upon the value of the static bending moments M_s. For vertical loading M_s acts only on the inclined member so that the formulas for denominators become

Numerator for H for Vertical Loading,

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s y \frac{I_1}{I_x} ds = \frac{1}{\cos\phi}\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s\left(h + \frac{2h_1}{l}x - Y_c\right) dx.$$

Numerator for V_A for Vertical Loading,

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s x \frac{I_1}{I_x} ds = \frac{1}{\cos\phi}\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s x dx.$$

Numerator for M,

$$\int_{0}^{l} M_s \frac{I_1}{I_x} ds = \frac{1}{\cos\phi}\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s dx.$$

After the value for M_s is substituted the integrals can be easily solved. The values of M_s are

For uniformly distributed load,

$$M_s = -\frac{1}{2}w\left(\frac{l}{2} + x\right)^2. \quad \ldots\ldots\ldots\ldots \quad (264)$$

For concentrated load P at a distance a from center,

$$M_s = -(x - a)P \text{ for points to the right of the load.} \quad (265)$$

$$M_s = 0 \text{ for points to the left of the load.} \quad \ldots\ldots \quad (266)$$

See also p. 600 in arch chapter.

Semi-graphical Solution.—The numerators also may be solved by means of the semi-graphical method explained on p. 272.

The numerator for H represented by $\frac{I_1}{I}\int M_s\, yds$ is the static moment of the moment area about the x-axis passing through the elastic center multiplied by $\frac{I_1}{I}$.

The numerator for V_A represented by $\frac{I_1}{I}\int M_s\, yds$ is the static moment of the moment area about the y-axis passing through the elastic center multiplied by $\frac{I_1}{I}$.

The numerator for M represented by $\frac{I_1}{I}\int M_s ds$ is the moment area multiplied by $\frac{I_1}{I}$.

The bending moment diagram should be drawn first. Then the bending moments should be plotted on the frame on lines at right angle to the frame. The ends should be connected to form a curve. The total area becomes divided into areas belonging to each member. The area enclosed by this curve should be computed for each member separately and its center of gravity determined. The center of gravity should be projected on the member and the ordinate x_1 and y_1 of this point should be found. The moment area multiplied by x_1 and y_1, respectively, gives the desired respective static moment of the moment area. This now should be multiplied by proper $\frac{I_1}{I}$. By adding the partial static moments the total numerator is obtained.

Solution for Variable Moments of Inertia.—When the moments of inertia of the member are not constant, it is impossible to take the value $\frac{I_1}{I_x}$ before the integration sign. In such case draw the bending moment diagram as before. Each ordinate of the diagram multiply by the corresponding value of $\frac{I_1}{I_x}$. Plot this new diagram. Determine the area thus obtained and find its center of gravity. Finally proceed as explained in connection with the solution with constant moments of inertia.

RIGHT-ANGLE FRAME. FIXED ENDS

Formulas for following conditions of loading are given below.

1. Uniformly distributed vertical loading.
2. Concentrated vertical loading.
3. Horizontal pressure.

Let l = length of span;
h = height of frame;
I_1 = moment of inertia of horizontal member;
I = moment of inertia of vertical member.

Position of Elastic Center.—The position of elastic center is common for all loadings as it depends upon the dimensions of the frame.

Position of Elastic Center in Reference to Left Support,

$$X_c = \frac{l}{2}. \qquad (267)$$

$$Y_c = \frac{\frac{I_1}{I}\frac{h}{l} + 1}{2\frac{I_1}{I}\frac{h}{l} + 1}h. \qquad (268)$$

Denominator for H, V_A and M.—The values of numerator for H, V_A and M depend only upon the shape of the frame. They are:

Denominator for H,

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{I_1}{I_x} ds = \tfrac{1}{3}(h - Y_c)\left(2 + \frac{I_1}{I}\frac{h}{l}\right)hl. \qquad (269)$$

Denominator for V_A,

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \frac{I_1}{I_x} ds = \tfrac{1}{12}l^3\left(1 + 6\frac{I_1}{I}\frac{h}{l}\right) \qquad (270)$$

Denominator for M,

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{I_1}{I_x} ds = l\left(1 + 2\frac{I_1}{I}\frac{h}{l}\right). \qquad (271)$$

These values may be used for solving problems not covered below.

Case 1. Uniformly Distributed Vertical Loading. (See Fig. 147.)

Vertical Reaction,

$$V_A = V_B = \tfrac{1}{2}wl. \qquad (272)$$

Horizontal Thrust,

$$H = -\frac{\frac{l}{h}}{4\left(\frac{I_1}{I}\frac{h}{l} + 2\right)}wl. \quad \ldots \ldots \ldots \quad (273)$$

Auxiliary Moment,

$$M = \frac{1}{12\left(2\frac{I_1}{I}\frac{h}{l} + 1\right)}wl^2. \quad \ldots \ldots \ldots \quad (274)$$

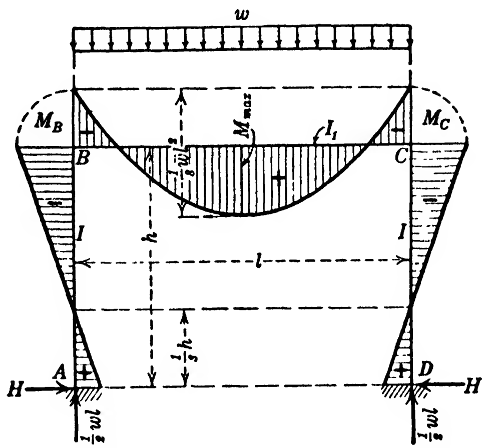


FIG. 147.—Rectangular Frame, Fixed End. Uniform Loading. (*See p.* 332.)

This does not need to be computed.

Corner Bending Moments,

$$M_B = M_C = -\frac{1}{6\left(\frac{I_1}{I}\frac{h}{l} + 2\right)}wl^2. \quad \ldots \ldots \quad (275)$$

Bending Moment at Supports,

$$M_A = M_D = \frac{1}{12\left(\frac{I_1}{I}\frac{h}{l} + 2\right)}wl^2. \quad \ldots \ldots \ldots \quad (276)$$

Bending Moment in Beam at Any Point x,

$$M_x = \tfrac{1}{2}wx(l - x) + M_B. \quad \ldots \ldots \ldots \ldots \quad (277)$$

Maximum Positive Bending Moment,

$$M_{max} = \frac{1}{8}wl^2 + M_B = \frac{1}{24} \frac{3\frac{I_1}{I}\frac{h}{l} + 2}{\frac{I_1}{I}\frac{h}{l} + 2} wl^2. \quad . \quad . \quad . \quad (278)$$

Case 2. Concentrated Vertical Load *P* at Distance *a*. (See Fig. 148, p. 334.)

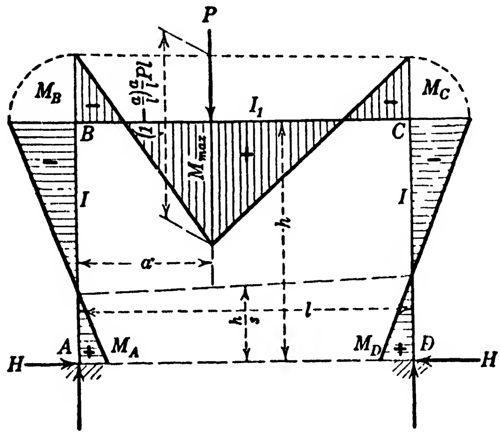


FIG. 148.—Rectangular Frame, Fixed Ends. Concentrated Load. (*See p.* 334.)

Reactions,

$$V_A = \left(1 - \frac{a}{l}\right)\left[1 + \frac{a}{l}\frac{1 - 2\frac{a}{l}}{1 + 6\frac{I_1}{I}\frac{h}{l}}\right]P \quad . \quad . \quad . \quad . \quad . \quad . \quad (279)$$

$$V_D = P - V_A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (280)$$

Horizontal Thrust,

$$H = \frac{a}{l}\left(1 - \frac{a}{l}\right)\frac{\frac{l}{h}}{\frac{2}{3}\left(\frac{I_1}{I}\frac{h}{l} + 2\right)}P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (281)$$

Bending Moments at Corners,

$$M_B = -\frac{a}{l}\left(1 - \frac{a}{l}\right)\left[\frac{1}{2 + \frac{I_1}{I}\frac{h}{l}} + \frac{1 - 2\frac{a}{l}}{2\left(1 + 6\frac{I_1}{I}\frac{h}{l}\right)}\right]Pl. \quad . \quad (282)$$

$$M_C = -\frac{a}{l}\left(1 - \frac{a}{l}\right)\left[\frac{1}{2 + \frac{I_1}{I}\frac{h}{l}} - \frac{1 - 2\frac{a}{l}}{2\left(1 + 6\frac{I_1}{I}\frac{h}{l}\right)}\right]Pl. \quad . \quad (283)$$

Bending Moments at Bottom,

$$M_A = +\frac{a}{l}\left(1 - \frac{a}{l}\right)\left[\frac{1}{2\left(2 + \frac{I_1}{I}\frac{h}{l}\right)} - \frac{1 - 2\frac{a}{l}}{2\left(1 + 6\frac{I_1}{I}\frac{h}{l}\right)}\right]Pl. \quad (284)$$

$$M_D = \frac{a}{l}\left(1 - \frac{a}{l}\right)\left[\frac{1}{2\left(2 + \frac{I_1}{I}\frac{h}{l}\right)} + \frac{1 - 2\frac{a}{l}}{2\left(1 + 6\frac{I_1}{I}\frac{h}{l}\right)}\right]Pl. \quad (285)$$

Bending Moment at Load P,

$$M_{max} = M_A - Hh + V_A a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (286)$$

Case 2a. Symmetrically Placed Concentrated Vertical Loads.— For symmetrically placed loads the points of inflection of the columns are distant $\frac{h}{3}$ above the bottom (see p. 325). Also the vertical reaction is equal to one half of the loads applied on the beam. This property may be utilized for determining the bending moments. The only statically indeterminate value is the horizontal thrust which acts at the points of inflections of the columns.

Reactions,

$$V_A = V_B = \tfrac{1}{2}\Sigma P = \tfrac{1}{2}(P_1 + P_2 + P_3 + \ldots). \quad . \quad . \quad . \quad (287)$$

Horizontal Thrust,

$$H = \frac{\frac{l}{h}}{\frac{2}{3}\left(2 + \frac{I_1}{I}\frac{h}{l}\right)}\left[P_1\frac{a_1}{l}\left(1 - \frac{a_1}{l}\right) + P_2\frac{a_2}{l}\left(1 - \frac{a_2}{l}\right) + P_3\frac{a_3}{l}\left(1 - \frac{a_3}{l} + \ldots\right)\right], \quad (288)$$

also

$$H = \frac{\frac{l}{h}}{\frac{2}{3}\left(2 + \frac{I_1}{I}\frac{h}{l}\right)} \Sigma P \frac{a}{l}\left(1 - \frac{a}{l}\right). \quad . \quad . \quad . \quad . \quad (289)$$

Corner Bending Moments,

$$M_B = M_C = -\tfrac{2}{3} Hh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (290)$$

Bending Moment at the Bottom,

$$M_A = M_D = +\tfrac{1}{3} Hh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (291)$$

Bending Moment at Any Point,

$$M_x = M_s - \tfrac{2}{3} Hh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (292)$$

Case 3. Left Columns with Load on Crane Bracket. (See Fig. 149, p. 336.)

Notation.

In addition to notation on p. 332.

Let l_1 = distance of load P on bracket from center line of vertical member;

h_1 = height of bracket above the level of the hinges.

Reactions,

$$V_A = \left(1 - \frac{l_1}{l}\right)P + \frac{M_D - M_A}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (293)$$

$$V_B = \frac{l_1}{l}P - \frac{M_D - M_A}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (294)$$

Horizontal Thrust,

$$H = \frac{1}{2}\frac{l_1}{h} \frac{\frac{I_1}{I}\frac{h}{l}\left[1 - 6\frac{h_1}{h}\left(1 - \frac{h_1}{h}\right)\right] + \left[2 - 3\frac{h_1}{h}\left(2 - \frac{h_1}{h}\right)\right]}{\frac{I_1}{I}\frac{h}{l} + 2} P. \quad (295)$$

Corner Bending Moments,

$$M_B = \frac{1}{2}\frac{I_1}{I}\frac{h}{l}\left[\frac{3\left(1 - 2\frac{h_1}{h}\right)}{6\frac{I_1}{I}\frac{h}{l} + 1} - \frac{\frac{h_1}{h}\left(3\frac{h_1}{h} - 2\right)}{\frac{I_1}{I}\frac{h}{l} + 2}\right] Pl_1. \quad . \quad . \quad (296)$$

$$M_C = -\frac{1}{2}\frac{I_1}{I}\frac{h}{l}\left[\frac{3\left(1 - 2\dfrac{h_1}{h}\right)}{6\dfrac{I_1}{I}\dfrac{h}{l} + 1} + \frac{\dfrac{h_1}{h}\left(3\dfrac{h_1}{h} - 2\right)}{\dfrac{I_1}{I}\dfrac{h}{l} + 2}\right]Pl_1 \quad . \quad . \quad (297)$$

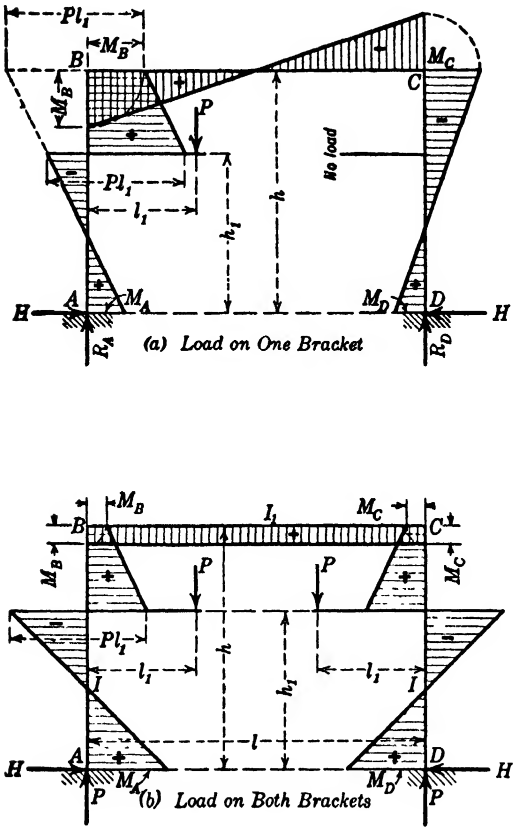


FIG. 149.—Rectangular Frame, Fixed Ends, with Crane Brackets. (*See p.* 336.)

Bending Moments at the Bottom,

$$M_A = M_B + Hh - Pl_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (298)$$

$$M_D = M_C + Hh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (299)$$

Bending Moment at the Bracket:

Just below bracket,

$$M_E = M_A - Hh_1.$$

Just above bracket,

$$M_{E1} = Pl_1 + M_E. \quad (300)$$

Case 3a. Equal Concentrated Loads on Both Brackets. (See Fig. 149 (b), p. 337.)

Vertical Reactions,

$$V_A = V_B = P.$$

Horizontal Thrust,

Equal double the thrust for single load.

Corner Bending Moments,

$$M_B = M_C = \frac{\frac{h_1}{h}\left(2 - 3\frac{h_1}{h}\right)}{\frac{I_1}{I}\frac{h}{l} + 2} Pl_1. \quad (301)$$

Bending Moments at Bottom,

$$M_A = M_B + Hh - Pl_1. \quad (302)$$

$$M = M_C + Hh. \quad (303)$$

Bending Moments at Bracket:

Just below bracket,

$$M_E = M_A - Hh_1. \quad (304)$$

Just above bracket,

$$M_{E1} = Pl_1 + M_E. \quad (305)$$

Case 4. Uniformly Distributed Horizontal Pressure. (See Fig. 150, p. 339.)

Let p = uniformly distributed horizontal pressure.

Vertical Reaction,

$$V_A = -\frac{\frac{I_1}{I}\frac{h}{l}}{\frac{l}{h}\left(6\frac{I_1}{I}\frac{h}{l} + 1\right)} ph. \quad (306) \qquad V_D = -V_A. \quad (306a)$$

Horizontal Thrust:

At left support,

$$H = \frac{1}{8}\,\frac{6\frac{I_1}{I}\frac{l}{h} + 13}{\frac{I_1}{I}\frac{l}{h} + 2}\,ph. \qquad (307)$$

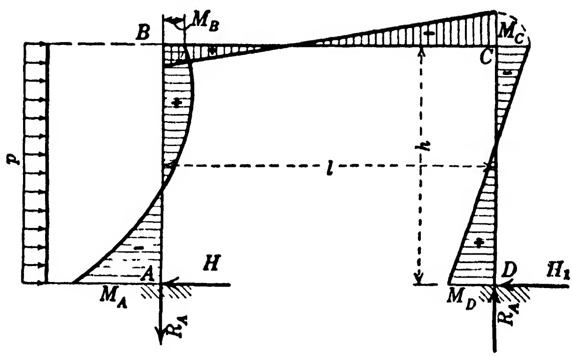


FIG. 150.—Uniformly Distributed Horizontal Pressure. (*See p.* 338.)

At right support,

$$H_1 = ph - H. \qquad (308)$$

Bending Moments at Bottom,

$$M_A = -\frac{1}{2}\left(1 - \frac{5\frac{I_1}{I}\frac{h}{l} + 9}{12\left(\frac{I_1}{I}\frac{h}{l} + 2\right)} - \frac{\frac{I_1}{I}\frac{h}{l}}{6\frac{I_1}{I}\frac{h}{l} + 1}\right)ph^2. \qquad (309)$$

$$M_D = \frac{1}{2}\left(\frac{5\frac{I_1}{I}\frac{h}{l} + 9}{12\left(\frac{I_1}{I}\frac{h}{l} + 2\right)} - \frac{\frac{I_1}{I}\frac{h}{l}}{6\frac{I_1}{I}\frac{h}{l} + 1}\right)ph^2. \qquad (310)$$

Bending Moments at Corners,

$$M_B = M_A - Hh + \tfrac{1}{2}ph^2. \qquad (311)$$

$$M_C = M_D - H_1h. \qquad (312)$$

RIGHT-ANGLE HINGED FRAME WITH VARYING MOMENTS OF INERTIA

In bridge construction it is often desirable to make the beam composing a frame deeper at the ends than in the center, either for the sake of appearance or to increase the head room in the center. In such cases the top of the beam is made straight, but the bottom is either curved or provided with haunches.

If the bottom is curved, the curve either extends throughout the whole length of the span or a part of the beam at each end is curved and the central part is straight.

The vertical members may also have variable dimensions.

If the depth of any member of the frame is not constant throughout its length, its moments of inertia are not constant but vary with the variation in depth. This affects the formulas for bending moments and thrusts.

The formulas below are for frames with variable moments of inertia.

In succeeding pages special cases are worked out so that in practical design the use of calculus is avoided and the calculations can be made with comparative ease.

General Formula.—The hinged frame has one statically indeterminate value, namely, the horizontal thrust at the hinges. The general formula for this thrust is

Let M_{s1} = static bending moment in left column;
M_{s2} = static bending moment in beam;
M_{s3} = static bending moment in right column;
I_1 = smallest moment of inertia of the beam;
I_x = variable moment of inertia in beam at point x;
I = smallest moment of inertia of the columns;
I_y = variable moment of inertia in column at point y;
H = horizontal thrust.

General Formula for Horizontal Thrust,

$$H = \frac{\frac{1}{I}\int_0^h M_{s1} y \frac{I}{I_y} dy + \frac{h}{I_1}\int_0^l M_{s2} \frac{I_1}{I_x} dx + \frac{1}{I}\int_0^h M_{s3} y \frac{I}{I_y} dy}{\frac{2}{I}\int_0^h y^2 \frac{I}{I_y} dy + \frac{h^2}{I_1}\int_0^l \frac{I_1}{I_x} dx}. \quad (313)$$

In the above formula M_{s1} and M_{s3} are zero for vertical loads as these produce static bending moments in the beam only.

For horizontal loading such as earth pressure there are static bending moments in the columns as well as in the beam (see p. 272).

Assumptions as to Variation of Moments of Inertia.—To make the solution of the integrals possible it is necessary to establish a formula for the variation of the moments of inertia.

Vertical Member.—The vertical members are either of constant depth or the depth is made smallest at the bottom and is increased according to a straight line to a maximum at the juncture of the horizontal and vertical members. In such case it is accurate enough to assume that the vertical member is of constant moment of inertia and make this assumed constant moment of inertia equal to the actual moment of inertia at a height equal to 0.65 the theoretical height of the vertical member.

Horizontal Member.—The horizontal member is assumed to have a straight top and to be provided at the bottom with haunches symmetrically arranged about the center of the beam.

Two types of haunches will be considered:

1. Straight haunches.
2. Parabolic haunches.

When the length of the parabolic haunches is equal to one-half of the span the bottom of the horizontal member assumes the shape of a continuous parabola. The two cases are illustrated in Fig. 151, p. 344.

Denominator for H.—With the above assumptions the integrals forming the denominator for H assume following values.

Let I_{av} = moment of inertia of cross-section of vertical member at a height above hinge equal to $0.65h$;

h = theoretical height of vertical member;

l = span length;

ml = length of haunch;

d_1 = minimum depth of beam at center;

d_2 = maximum depth of beam at support;

d_x = depth of beam at any point x;

I_1 = minimum moment of inertia of beam;

I_2 = maximum moment of inertia of beam;

I_x = moment of inertia at any point x.

Then

Vertical Member,

$$\frac{2}{I}\int_0^h y^2\frac{I}{I_y}dy = \frac{2}{I_{av}}\int_0^h y^2dy = \frac{2}{3}\frac{h^3}{I_{av}}. \quad . \quad . \quad . \quad . \quad (314)$$

Horizontal Member:

Straight haunch,[2] as shown in Fig. 151 (*a*),

$$\frac{h^2}{I_1}\int_0^l \frac{I_1}{I_x}dx = \frac{h^2}{I_1}\left(1 - \frac{c(3+2c)}{(1+c)^2}m\right)l,$$

where

$$c = \frac{d_2}{d_1} - 1.$$

Finally

$$\frac{h^2}{I_1}\int_0^l \frac{I_1}{I_x}dx = \frac{h^2}{I_1}(1 - \gamma m)l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (315)$$

where

$$\gamma = \frac{c(3+2c)}{(1+c)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (316)$$

Values of γ may be taken from table on p. 346 for proper values of c.

[2] For a beam with symmetrically arranged straight haunches of a length ml at each side and a portion of constant depth in the middle, the integral $\int_0^l \frac{I_1}{I_x}dx$ should be divided into three parts namely,

$$\int_0^l \frac{I_1}{I_x}dx = \int_0^{ml} \frac{I_1}{I_x}dx + \int_{ml}^{(l-ml)} dx + \int_{(l-lm)}^{l} \frac{I_1}{I_x}dx.$$

Due to symmetry the first and the third parts are equal. To solve the integral it is necessary to express $\frac{I_1}{I_x}$ by an equation.

For straight haunch from geometric relation

$$d_x = d_1 + \frac{d_2 - d_1}{ml}(ml - x) \quad \text{and} \quad \frac{d_x}{d_1} = \left[1 + \left(\frac{d_2}{d_1} - 1\right)\left(1 - \frac{x}{ml}\right)\right].$$

Consequently,

$$\frac{I_x}{I_1} = \left[1 + \left(\frac{d_2}{d_1} - 1\right)\left(1 - \frac{x}{ml}\right)\right]^3.$$

Substituting this in equation above and making $c = \frac{d_2}{d_1} - 1$

$$\int_0^l \frac{I_1}{I_x}dx = 2\int_0^{ml} \frac{dx}{\left[1 + c\left(1 - \frac{x}{ml}\right)\right]^3} + \int_{ml}^{l-ml} dx$$

$$= \left[\frac{2+c}{(1+c)^2}ml + l - 2ml\right] = \left(1 - \frac{c(3+2c)}{(1+c)^2}m\right)l.$$

Parabolic haunch,[3] as shown in Fig. 151 (*b*),

$$\frac{h^2}{I_1}\int_0^l \frac{I_1}{I_x}dx = \frac{h^2}{I_1}\left\{1 - 2m + \frac{m}{4}\left[\frac{5 + 3c}{(c + 1)^2} + \frac{3}{\sqrt{c}} \text{ arc tan } \sqrt{c}\right]\right\}l,$$

where ml = length of haunch,

$$c = \frac{d_2}{d_1} - 1.$$

Finally

$$\frac{h^2}{I_1}\int_0^l \frac{I_1}{I_x}dx = \frac{h^2}{I_1}(1 - \delta m)l, \quad \text{(317)}$$

where

$$\delta = 2 - \frac{1}{4}\left[\frac{5 + 3c}{(c + 1)^2} + \frac{3}{\sqrt{c}} \text{ arc tan } \sqrt{c}\right]. \quad \text{(318)}$$

Values of δ may be taken from table on p. 346 for proper values of c.

Total Denominator.—To get the total denominator add to the value of the integral for vertical member, Equation (314), the value for horizontal member either Equation (316) or (318).

[3] For a beam with symmetrical parabolic haunches the integral $\int_0^l \frac{I_1}{I_x}dx$ also equals $2\int_0^{ml} \frac{I_1}{I_x}dx + \int_{ml}^{l-ml} dx.$

The variation in the depth of beam in the haunches may be expressed by

$$d_x = d_1 + \frac{d_2 - d_1}{ml}(ml - x)^2.$$

Consequently,

$$\frac{d_1}{d_x} = \frac{1}{1 + c\left(1 - \frac{x}{ml}\right)^2} \quad \text{and} \quad \frac{I_1}{I_x} = \frac{1}{\left[1 + c\left(1 - \frac{x}{ml}\right)^2\right]^3}.$$

Hence,

$$2\int_0^{ml} \frac{I_1}{I_x}dx = 2\int_0^{ml} \frac{1}{\left[1 + c\left(1 - \frac{x}{ml}\right)^2\right]^3}dx = \frac{1}{4}\left[\frac{5 + 3c}{(c + 1)^2} + \frac{3}{\sqrt{c}} \text{ arc tan } \sqrt{c}\right]ml.$$

The whole integral therefore is

$$\int_0^l \frac{I_1}{I_x}dx = \frac{1}{4}\left[\frac{5 + 3c}{(c + 1)^2} + \frac{3}{\sqrt{c}} \text{ arc tan } \sqrt{c}\right]ml + l - 2ml$$

$$= \left\{1 - 2m + \frac{m}{4}\left[\frac{5 + 3c}{(c + 1)^2} + \frac{3}{\sqrt{c}} \text{ arc tan } \sqrt{c}\right]\right\}l.$$

Total Denominator for H,

$$h^2\frac{l}{I_1}\left[\frac{2}{3}\frac{I_1h}{I_{av}l}+1-\gamma m\right] \quad \text{Straight Haunch.} \quad . \quad . \quad . \quad (319)$$

$$h^2\frac{l}{I_1}\left[\frac{2}{3}\frac{I_1h}{I_{av}l}+1-\delta m\right] \quad \text{Parabolic Haunch.} \quad . \quad . \quad . \quad (320)$$

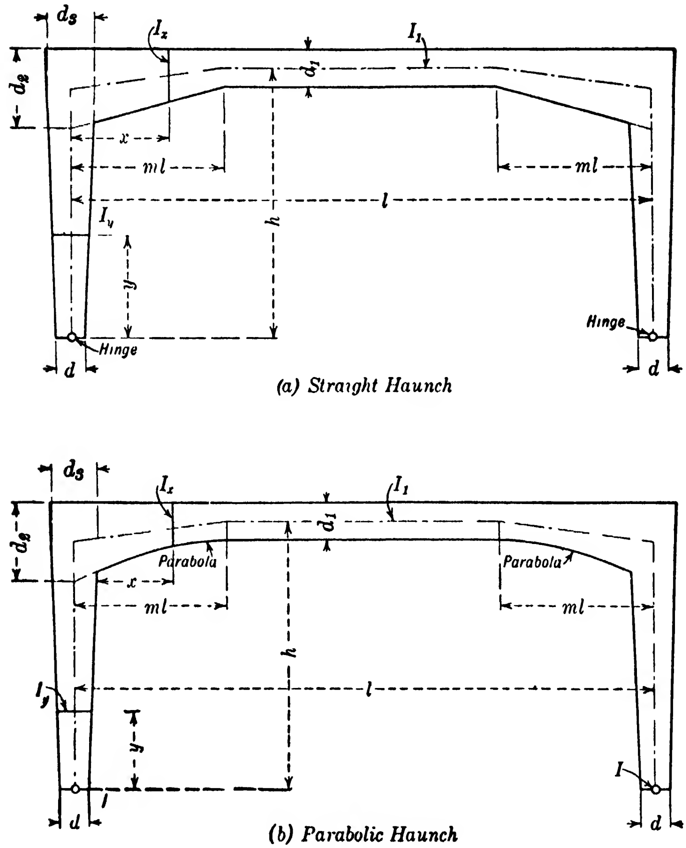


FIG. 151.—Right-angle Frame. Members with Varying Moments of Inertia. (See p. 341.)

Where γ and δ are given in table on p. 346 for different values of $c=\frac{d_2}{d_1}-1.$

Numerator of H.—The numerator for H for vertical loads is $\frac{h}{I_1}\int_0^l M_s \frac{I_1}{I_x} dx$. The integral represents the area of the reduced static bending moment diagram obtained by multiplying the static bending moment at all points by the corresponding ratios of $\frac{I_1}{Ix}$ and plotting the so-obtained values. This problem can be solved graphically for any shape of beam. The numerator may also be obtained from formulas below for the two types of haunches described on p. 341. Values are given for uniformly distributed and concentrated loads.

Numerator for H. Uniformly Distributed Loading,[4]

$$\frac{h}{I_1}\int_0^l M_s \frac{I_1}{I_x} dx = \frac{1}{12}\frac{h}{I_1}\beta w l,^3 \quad . \quad . \quad . \quad . \quad . \quad (321)$$

where β is a constant from Diagram p. 140 for straight haunches and Diagram p. 141 for parabolic haunches. It depends upon the length of haunch ml and the ratio of minimum to maximum moment of inertia.

Horizontal Thrust, Variable Moments of Inertia.—By substituting the denominators and numerators previously developed in Formula (313), p. 340, final formulas for horizontal thrust are obtained.

Horizontal Thrust, Uniform Loading,

$$H = \frac{\beta \frac{l}{h}}{12\left[\frac{2}{3}\frac{I_1 h}{I_{av} l} + 1 - \gamma m\right]} wl. \quad \text{Straight Haunch.} \quad . \quad . \quad . \quad (322)$$

$$H = \frac{\beta \frac{l}{h}}{12\left[\frac{2}{3}\frac{I_1 h}{I_{av} l} + 1 - \delta m\right]} wl. \quad \text{Parabolic Haunch.} \quad . \quad . \quad (323)$$

[4] For uniformly distributed loading $M_s = \frac{w}{2}x(l - x)$. The denominator for H then is $\frac{h}{I_1}\int_0^l M_s \frac{I}{I_x} dx = \frac{h}{I_1}\frac{w}{2}\int_0^l x(l - x)\frac{I_1}{I_x} dx$. Since, as explained on page 138, $\int_0^l x(l - x)\frac{I_1}{I_x} dx = \frac{1}{6}\beta l^3$, the value of the integral becomes $\frac{1}{12}\frac{h}{I_1}\beta w l^3$, where β is a constant from Diagrams, pp. 140 or 141.

β is a constant from Diagrams pp. 140 or 141. δ and γ are constants from the Table that follows.

Constants γ and δ

	Values of $c = \frac{d_2}{d_1} - 1$									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
γ	0.26	0.47	0.64	0.78	0.89	0.98	1.07	1.13	1.19	1.25
δ	0.18	0.32	0.44	0.54	0.62	0.69	0.76	0.82	0.87	0.91

CLOSED RECTANGULAR FRAME

Following loading will be considered for a closed rectangular frame shown in Fig. 152, p. 346.

(*a*) Uniformly distributed vertical loading on BC.
(*b*) Uniformly distributed vertical loading on AD.
(*c*) Concentrated horizontal load W at top corner B.

Let l = span of frame;
h = height of frame;
I_1 = moment of inertia of the top beam BC;
I_2 = moment of inertia of the bottom beam AD;
I = moment of inertia of the columns;
w = uniformly distributed vertical unit load;
W = concentrated horizontal load.

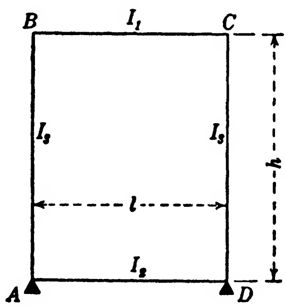


FIG. 152.—Closed Rectangular Frame. (*See p.* 346.)

(*a*) Uniformly Distributed Vertical Loading on Top Beam *BC*. (See Fig. 153 (*a*), p. 347.)

Reactions,

$$V_1 = \frac{wl}{2}. \quad . \quad . \quad . \quad . \quad . \quad (324) \qquad V_2 = \frac{wl}{2}. \quad . \quad . \quad . \quad . \quad (325)$$

Corner Bending Moments,

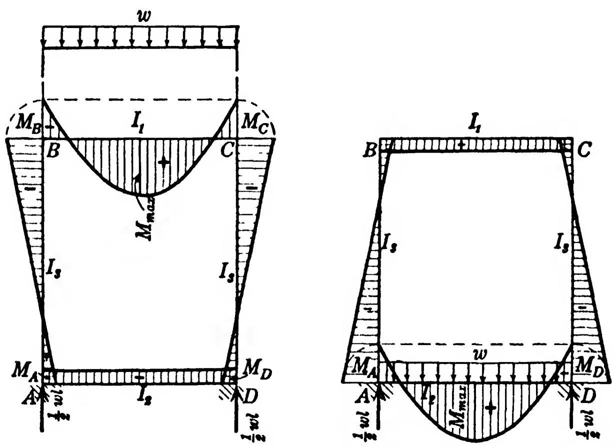
$$M_A = M_D = \frac{1}{12} \frac{1}{\frac{I_1}{I}\frac{h}{l} + 2\left(1 + \frac{I_1}{I_2}\right) + 3\frac{I}{I_2}\frac{l}{h}} wl^2. \quad . \quad . \quad (326)$$

$$M_B = M_C = -\frac{1}{12} \frac{2 + 3\frac{I}{I_2}\frac{l}{h}}{\frac{I_1}{I}\frac{h}{l} + 2\left(1 + \frac{I_1}{I_2}\right) + 3\frac{I}{I_2}\frac{l}{h}} wl^2. \quad . \quad . \quad (327)$$

Maximum Positive Bending Moment in BC,

$$M_{\max} = \tfrac{1}{8}wl^2 + M_B. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (328)$$

(b) Uniformly Distributed Vertical Loading at Bottom Beam *AD*. (See Fig. 153 (*b*), p. 347.)



(a) Top loaded *(b) Bottom loaded*

FIG. 153.—Closed Rectangular Frame. Uniformly Distributed Vertical Loading. (*See p.* 346.)

Reactions,

$$V_1 = \frac{wl}{2}. \quad . \quad . \quad . \quad . \quad . \quad (329) \qquad V_2 = \frac{wl}{2}. \quad . \quad . \quad . \quad (330)$$

Corner Bending Moments,

$$M_A = M_D = -\frac{1}{12}\frac{\left(2\frac{I_1}{I_2} + 3\frac{I}{I_2}\frac{l}{h}\right)}{\frac{I_1}{I}\frac{h}{l} + 2\left(1 + \frac{I_1}{I_2}\right) + 3\frac{I}{I_2}\frac{l}{h}}wl^2. \quad . \quad (331)$$

$$M_B = M_C = \frac{1}{12}\frac{\frac{I_1}{I_2}}{\frac{I_1}{I}\frac{h}{l} + 2\left(1 + \frac{I_1}{I_2}\right) + 3\frac{I}{I_2}\frac{l}{h}}wl^2. \quad . \quad . \quad (332)$$

Maximum Positive Bending Moment in AD,

$$M_{\max} = \tfrac{1}{8}wl^2 + M_A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (333)$$

(*c*) **Concentrated Horizontal Force *W* in Corner *B*.** (See Fig. 154, p. 349.)

Reactions,

$$V_1 = -W\frac{h}{l}. \quad . \quad . \quad . \quad (334) \qquad V_2 = W\frac{h}{l}. \quad . \quad . \quad . \quad . \quad (335)$$

Corner Bending Moments,

$$M_A = -\frac{3\frac{I_1}{I}\frac{h}{l} + 1}{2\left(6\frac{I_1}{I}\frac{h}{l} + \frac{I_1}{I_2} + 1\right)}Wh. \quad . \quad . \quad . \quad . \quad (336)$$

$$M_B = \frac{3\frac{I_1}{I}\frac{h}{l} + \frac{I_1}{I_2}}{\left(6\frac{I_1}{I}\frac{h}{l} + \frac{I_1}{I_2} + 1\right)}Wh. \quad . \quad . \quad . \quad . \quad . \quad (337)$$

$$M_C = -M_D. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (338)$$

$$M_D = -M_A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (339)$$

Closed Frame Resting on the Ground.—The formulas given in the preceding paragraphs apply when the frame is supported at the corners. If a frame rests directly on the ground the reaction of the soil will be distributed over the entire bottom member of the frame. The distribution will depend upon the type of loading.

When the frame is loaded by uniformly distributed loading acting downward on the top member the reaction of the soil acting upward

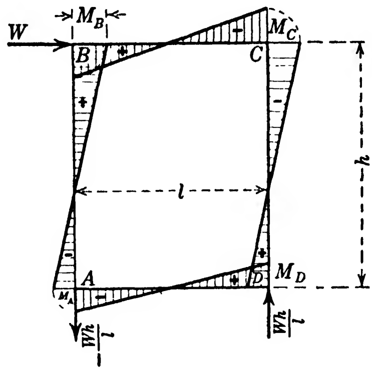
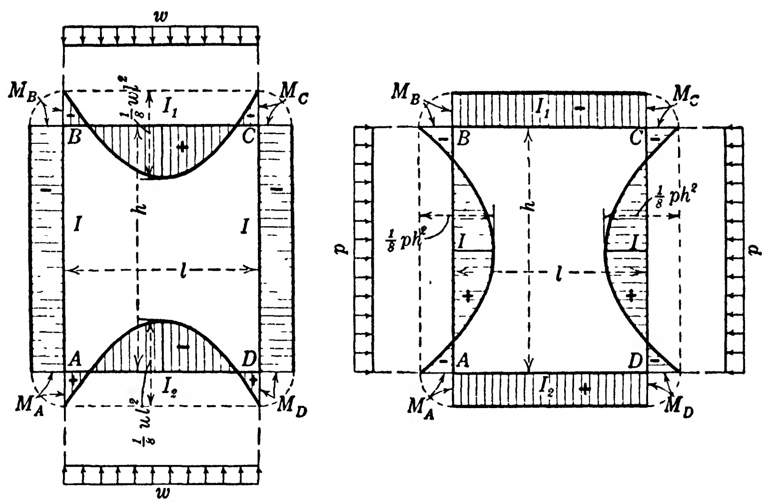


FIG. 154.—Closed Rectangular Frame. Horizontal Force W in Corner B. (*See p.* 348.)



(a) Uniformly Distributed Vertical Loading Top and Bottom

(b) Uniformly Distributed Horizontal Loading on Both Sides

FIG. 155.—Closed Frame. Loading Applied at Opposite Sides. (*See p.* 348.)

on the bottom member will also be uniformly distributed. This condition of loading is shown in Fig. 155(*a*), p. 349. The bending moments at the corners for such condition become

Bending Moments at Corners, Top and Bottom Uniformly Loaded:

Upper corners,

$$M_B = M_C = -\frac{1}{12}\frac{2 + 3\frac{I}{I_2}\frac{l}{h} - \frac{I_1}{I_2}}{\frac{I_1}{I}\frac{h}{l} + 2\left(1 + \frac{I_1}{I_2}\right) + 3\frac{I}{I_2}\frac{l}{h}}wl^2. \quad . \quad (340)$$

Lower corners,

$$M_A = M_D = -\frac{1}{12}\frac{2\frac{I_1}{I_2} + 3\frac{I}{I_2}\frac{l}{h} - 1}{\frac{I_1}{I}\frac{h}{l} + 2\left(1 + \frac{I_1}{I_2}\right) + 3\frac{I}{I_2}\frac{l}{h}}wl^2. \quad . \quad (341)$$

Bending Moment in the Beam at Any Point,

$$M_x = M_A + \tfrac{1}{2}wx(l - x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (342)$$

Maximum Bending Moment in Beam,

$$M_{\max} = M_A + \tfrac{1}{8}wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (343)$$

This bending moment is positive in the top beam and negative in the bottom beam.

Bending Moments in Columns,

$$M_y = M_A + \frac{M_B - M_A}{h}y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (344)$$

Closed Frame Loaded by Uniformly Distributed Inside Pressures.—The formulas given in the preceding paragraphs may also be used when the frame is subjected to inside pressures. In such case the signs of the bending moments will be reversed.

Horizontal Pressures Acting on Both Sides of the Frame.—When both vertical sides are loaded by horizontal pressures, and when the pressures on both sides are equal and act in opposite directions the bending moments in the corners are given by formulas below. This condition of loading is shown in Fig. 155, p. 349.

Let M_{fA} = bending moment at A due to loads on vertical member AB, considering it as fixed at both ends;

M_{fB} = bending moments at B on vertical member AB, considering it as fixed at both ends.

Then

Bending Moments at the Corners,

$$M_A = M_D = -\frac{M_{fA} + M_{fB}\left(\frac{I_1 h}{Il} + 2\right)}{\frac{I_1}{I}\frac{h}{l} + 2\left(1 + \frac{I_1}{I_2}\right) + 3\frac{Il}{I_2 h}}. \quad . \quad . \quad . \quad (345)$$

$$M_B = M_C = -\frac{M_{fA}\left(\frac{I_1 h}{Il} + 2\frac{I_1}{I_2}\right) + M_{fB}\frac{I_1}{I_2}}{\frac{I_1 h}{Il} + 2\left(1 + \frac{I_1}{I_2}\right) + 3\frac{Il}{I_2 h}}. \quad . \quad . \quad . \quad (346)$$

End Shears,

$$V_A = V_s - \frac{M_B - M_A}{h}, \quad . \quad (346a) \qquad V_B = V_s + \frac{M_B - M_A}{h}. \quad (347)$$

Bending Moment in Columns at Any Point y Above Point A,

$$M_y = M_s + M_A - \frac{M_B - M_A}{h}y. \quad . \quad . \quad . \quad . \quad (348)$$

Bending Moments in Beams,

In beam AD,

$$M_x = M_A \text{ (constant)}.$$

In beam BC,

$$M_x = M_B \text{ (constant)}.$$

Values of M_{fA} and M_{fB} for Different Types of Loading,

For uniformly distributed loading,

$$M_{fA} = M_{fB} = \tfrac{1}{12}ph^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (349)$$

For triangular loading with zero at top of frame and maximum pressure at bottom p_1,

$$M_{fA} = \tfrac{1}{20}p_1h^2. \quad . \quad . \quad (350) \qquad M_{fB} = \tfrac{1}{30}p_1h^2. \quad . \quad . \quad . \quad (351)$$

GENERAL INSTRUCTIONS FOR DESIGNING RIGID FRAMES

Before designing a frame, all dimensions independent of the frame design must be selected. Thus the span and the height of the construction must be definitely established and the spacing of frames selected.

Next step is to design the construction which transmits the load to the frame. In roof construction the beams supporting the slabs are usually placed longitudinally with the building. Their spacing should be made just close enough so as to permit the use of the minimum thickness of slab. Such arrangement gives minimum dead load on the frame.

Sometimes the construction supported by the concrete frame is made of wood or some other material. This reduces to some extent the cost of construction.

In bridge design the spacing of the frames is much closer than in building construction. No floor beams are needed and the slab is supported directly on the frames.

Preliminary Dimensions of Frame.—It is necessary to make assumptions as to the preliminary dimensions of the frame to get the moments of inertia as well as the dead load of the frame.

Preliminary dimensions of the frame members may be accepted by judgment. After the bending moments are computed, the dimensions may be changed sufficiently to take care of the stresses. Any appreciable change in dimensions, however, may affect the ratio of stiffness of the various members thereby affecting the bending moments sufficiently to require refiguring. This may be avoided by computing the dimensions, using arbitrary coefficients for moments for dead and live load, varying from $\frac{1}{16}$ to $\frac{1}{12}$, depending upon conditions. This bending moment may be assumed to act at the joint of vertical and horizontal members. There the cross-section is usually rectangular, the slab being in the tensile zone, reinforced in tension and compression. The ratio of tension and compression steel is selected and the depth of the frame member computed by Formula (26), p. 222, Vol. I.

In selecting the dimensions of the vertical member for roof frames, it must be borne in mind that the magnitude of the horizontal thrust and, therefore, the amount of reduction of the positive bending moment depends upon the stiffness of the vertical members. The stiffer the vertical member in relation to the horizontal members the smaller are the bending moments in the center of the frame. It is often economical to make the vertical members especially heavy and thereby reduce the dimensions of the horizontal or inclined members. This method reduces the dead load of the member producing bending moments, and is especially advantageous in long spans.

The vertical members may be made rectangular in shape. Their width is usually same as the width of the flange of the other members. Where the wall between the vertical members is of reinforced concrete it may be considered as a part of the vertical member under the same conditions as used for T-beams. When any part of the wall is considered as a part of the vertical member it must be built monolithic with the member. Sufficient reinforcement must be used to prevent the separation of the wall and vertical member.

Improved Design of Vertical Members.—The vertical members of hinged frames may be made of constant depth for their full height. Such design, however, is not economical.

Referring to the bending moment diagrams for hinged rigid frame it is evident that the bending moments at the bottom of the vertical members is zero and increases gradually to its maximum at the joint between the vertical member and the beam. The reaction is practically constant. It is obvious that much smaller depth is required at the bottom where there is no bending moment than at the top where the bending moment is a maximum.

An economical design of the vertical member may be obtained by designing it at the bottom to resist the vertical reaction in the same manner as a centrally loaded column. The smallest percentage of steel permitted for reinforced columns should be used there.

At the top the vertical member should be designed for the direct stress and bending moment. The section should be considered as reinforced for tension and compression. Obviously, much larger depth is required there. Usually, both faces of the column are made straight so that no additional computations of depth for intermediate sections are required unless it is desired to reduce amount of reinforcement in the lower parts of the columns.

A good design is obtained by making the outside face of the vertical member vertical and slanting the inside face. In such design the actual axis of the vertical member is inclined somewhat inward, which affects advantageously the bending moments.

Moment of Inertia of Vertical Member with Slanting Face.—The moments of inertia of the vertical member designed as recommended above are not constant as assumed in the formulas. This may be taken into account as explained on p. 341. Ordinarily it is exact enough to use the formulas with constant moments of inertia and accepting for the moment of inertia of the columns the value at a height above the hinge equal to $0.65h$. The depth of the column there is $d\left(0.65\frac{d_1}{d} + 0.35\right)$ where d is the minimum depth and d_1 is maximum depth.

Horizontal and Inclined Members.—The depth of the horizontal and inclined members may be uniform. Since at the quarter points the bending moments are very small, the depth of the member there may be reduced.

Such design may not give pleasing results when the member is horizontal. With inclined members, on the other hand, the reduction of the depth at the quarter points may be easily accomplished in the manner shown in Fig. 157, opposite p. 366.

Rounding Up of Corners.—Sharp corners in rigid frames are objectionable. At the joint the construction should be provided either with a fillet or, preferably, the corners should be rounded up. The rounding up of corners not only reduces the compression stresses but also obviates the necessity of making sharp bends in the reinforcement.

To reduce formwork instead of rounding up the corners, two or three straight fillets are often used as shown in Fig. 215, opp. p. 665, Vol. I.

Reinforcement

The amount of reinforcement at the critical sections should be computed according to the formulas accepted for reinforced concrete construction.

All members composing a frame are subjected to direct stress and bending moments. However, the normal thrust in the horizontal and inclined members is usually small in comparison with the bending moment. Therefore, the effect of the thrust may be disregarded and members may be designed for bending moments only in the same manner as ordinary beams and girders.

The moment diagram should be used as a guide where to bend the reinforcement. The main reinforcement of the horizontal or inclined member should be made independent of the vertical member reinforcement. No attempt should be made to run bars from the beam into the column as such bars would be awkward to handle in the field. The bending of the bars should be made as simple as possible. The problem of erection should be always kept in mind when designing the reinforcement. Sometimes an excess of steel is more economical than over-elaborateness in bending of bars.

The normal thrust in the vertical members is much larger; therefore, they should be designed for direct stress and bending. The required tension and compression reinforcement may be obtained by formulas given in Chapter II. It is not necessary to use the same amount of reinforcement near the compression face as used near the tension face.

The joint between the horizontal and vertical members should be particularly well reinforced. All the reinforcement of the joint should extend from the column into the beam (and not from the beam into the column). A sufficient amount of ties or hoops should be used at the joint. Where compression reinforcement is curved it should be held in position by ties, else the bars might buckle when stressed. All reinforcement must be extended beyond the point where it is needed a sufficient distance to develop the bar by bond. When the end of a bar is located within a region subjected to tensile stresses it should be provided with a hook.

When a member is bent, as is the case with the inclined members of a ridge frame, the bars at the bend must be curved gently and not bent. Where the curvature is small, continuous bars may be used, extending from one half of the frame to the other. To prevent straightening of the bars a sufficient number of stirrups must be used at the curved section. When the curvature is large, a part or all the bars must be made up of two parts, one for each half of the member. The two parts of the bar then intersect at the bend and are lapped there by extending each bar in a straight line a sufficient distance beyond the point of intersection to develop its strength by bond. Hooks at the ends are also advisable as additional precaution.

Diagonal Tension Reinforcement

As in other types of reinforced concrete construction, the shearing stresses in a frame are accepted as a measure of diagonal tension.

The shearing stresses in any member are produced by external shears acting at right angles to the members. In a rigid frame the external forces are vertical and horizontal. After the horizontal thrust and the vertical reactions are computed, the external shears at any point can be found by statics.

In a right-angle frame the external horizontal shears produce shearing stresses and diagonal tension in vertical members and direct compression or tension in horizontal members. The vertical forces and reactions, on the other hand, produce shearing stresses and diagonal tension in horizontal members and compression in vertical members.

In a frame with inclined members, the inclined members are at an angle to both the horizontal and vertical external shears. In such case each external shear must be resolved into a component acting at right angles to the inclined member and a component parallel to the inclined member. The components at right angle to the member produce shearing stresses which may be used as a measure of diagonal tension.

The components parallel to the member produce direct compression or tension in the member and should be combined with the bending moments in determining flexual stresses.

To get maximum shearing stresses, the dead load should be combined with such positions of live load and wind as would give maximum values for shearing stresses.

The maximum shearing stresses in the left column are produced by the dead and live load and by wind pressure acting from the right. The opposite is true of the right column.

The maximum shearing stresses in an inclined member of a frame are produced also by dead and live load and by wind acting on the opposite side of the member.

EFFECT OF CRANE LOADS ON A RIGID FRAME

If a rigid frame is provided with brackets for cranes, the effect of the crane loads should be found in the following manner.

1. Find the capacity of the crane as well as the weight of the machinery. The capacity of the crane should be multiplied by a proper factor to allow for impact.

2. Find the most unfavorable reactions of the crane on the left bracket. This is obtained when the crane load is as near the left column as possible. Also find the reaction on the right bracket for the same loading.

3. The reaction at the left bracket separate into two parts, one of which is equal to the reaction at the right bracket.

4. Using Formulas (56) to (60), p. 285, for right-angle frames (or Formulas (136) to (141), p. 302, for ridge frames), find bending moments in the frame due to symmetrically loaded brackets. The symmetrical load on each bracket is equal to the reaction on the right support.

5. Using Formulas (44) to (53), p. 302, for right-angle frames (or Formulas (125) to (133) for ridge frames), find bending moments in the frame due to one-sided load on the left bracket. This load equals the left reaction minus the right reaction.

After the bending moments and shears are computed combine them with the bending moments due to dead load, live load and wind load so as to get the largest possible bending moment, both positive and negative, at the various sections. It is obvious that the maximum values at various points will be obtained for different combinations. The dead load must be included in all combinations. Any or all other loadings may be omitted when necessary to get maximum values.

EFFECT OF WIND PRESSURE ON FRAMES

The intensity of wind pressure should depend upon the possibility of high winds in the locality. In most cities the intensity of wind load is specified. In New York City, for instance, it is required to assume a horizontal wind pressure of 30 lb. per sq. ft. Larger values should be used in locations subject to tornadoes and hurricanes.

The bending moments due to the wind should be computed by approximate formulas. This should be combined with the bending moment due to dead load or dead and live load to get the most unfavorable results.

Usually for a combination including the wind load, unit stresses 30 per cent larger are allowed than for live and dead load alone. In many cases this increase is sufficient to take care of the wind pressure except in such places where the wind load produces bending moments of different sign to that due to the dead load, of sufficient magnitude to change the sign of the resultant bending moment. Thus it may happen that the bending moment due to wind load is negative where the positive bending moment due to dead load is small. Then negative bending moment must be provided for.

EFFECT OF EARTH PRESSURE

In bridge design earth pressure must be considered in design. Following conditions should be considered.

1. Bridge without live load subjected to earth pressure with surcharge on one or both sides.

2. Bridge with live load subjected to earth pressure with surcharge on one side.

The bending moments due to earth pressure are found by Formulas (61) to (69), p. 286, and (70) to (79), p. 288. They should be combined with the dead and live load in such a manner as to give the most unfavorable results.

Some designers consider earth pressure with surcharge as a fixed load in the same sense as the dead load of the frame and include the earth pressure in computing the bending moments due to fixed loads. Since on both sides of the frame pressure produces negative bending moments in the beam, the effect of such assumption is that for the fixed loads the negative bending moments due to the dead load are increased and the positive bending moment due to dead load are decreased.

The authors do not approve of this method.

It is possible for the actual intensity of the earth pressure to be materially smaller than that assumed in computations so that the actual reduction in positive bending moments would be materially smaller than obtained form computations. The positive bending moment for fixed load may be larger than obtained by considering the earth pressure as fixed.

The earth pressure should best be considered as a live load and included or omitted in combinations same as any other live load. If it is desired to get largest economy, part of the earth pressure may be considered as fixed load and part as live load.

DESIGN OF RIGID FRAME WITH RIDGE-HINGED ENDS

Example.—Design a Ridge Frame having the following general dimensions:

Span l = 50 ft.
Height of columns h = 30 ft.
Height of roof h_1 = 6 ft.
Spacing of frames = 20 ft.

Assumptions as to live load, wind and dead load:

Live Load: 40 lb. per sq. ft.
Weight of roofing: 20 lb. per sq. ft.
Wind Load: 30 lb. per sq. ft., acting horizontally.

Unit Stresses:

In flexure:
f_c = 800 lb. per sq. in.
f_c = 900 lb. per sq. in. at supports
n = 15
f_s = 16 000 lb. per sq. in.

In direct compression:
f_c = 500 lb. per sq. in.

The unit stresses may be 30 per cent higher when used for a combination of dead load, live load and wind.

Solution.—To get the dead load supported by the frame and the points of application of the concentrated loads, it is necessary to decide upon the arrangement of the longitudinal beams and slabs forming the roof. In this case the beams will be arranged as shown in Fig. 156, p. 359. The thickness of the slab is 3½ in. The dimensions of the beams are 10 in. by 18 in. These dimensions are sufficient to determine the dead load.

Concrete Dimensions of Frame.[1]

Inclined Member.—The inclined member is a T-Beam as shown in Fig. 156, p. 359. The assumed dimensions are b = 68 in., b' = 12 in., h = 40 in. and t = 3½ in.

The moment of inertia of the inclined member is found by Formula (457), p. 133, using diagram on page 134.

$$\frac{t}{h} = 0.0875 \quad \text{and} \quad \frac{b'}{b} = \frac{12}{68} = 0.18.$$

The constant from diagram corresponding to these values is C_I = 0.0258.
Hence

$$I_1 = 0.0258 \times 68 \times 40^3 = 112\,400 \text{ in.}^4$$

[1] To get the moments of inertia and the dead load of the frame, it is necessary to assume the concrete dimensions of the frame. These may be assumed by judgment or computed on the basis of approximate formulas for bending moments.

Vertical Member.—Vertical Member is a rectangle for which $b = 12$ in. and $h = 38$ in.

Moment of Inertia:

$$I = \frac{1}{12} \times 12 \times 38^3 = 54\,900 \text{ in.}^4$$

Ratio of Rigidity.—Since $l = 50$ ft. and $h_1 = 6$ ft., $s = \sqrt{25^2 + 6^2} = 25.7$ ft., therefore,

$$\frac{I_1}{I}\frac{h}{s} = \frac{112\,400 \times 30}{54\,900 \times 25.7} = 2.4.$$

Loading of Frame.—Both dead and live loads due to the roof are concentrated at the intersections of the beams and the frame. Hence the loads are placed at sixth points. The weight of the inclined member of the frame is uniformly distributed.[2]

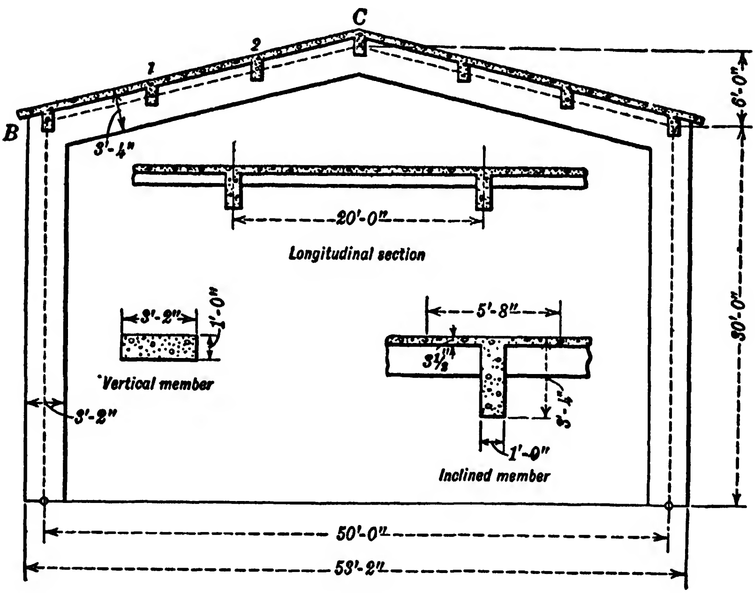


FIG. 156.—Assumed Dimensions of Frame. (*See p.* 358.)

Dead Load:

Weight of inclined member equals $\frac{12 \times 40}{144} \times 150 \times \frac{25.7}{25} = 515$ lb. per lin. ft. Call this 520 lb.

[2] In this example the uniformly distributed weight of the inclined member is considered separately so as to show the use of formulas. In practice it would be accurate enough to concentrate this weight at points of application of the other loads and add it to the dead load of the roof.

Concentrated loads at intersections of beams and inclined member:

Slab and Roofing $8.33 \times 1.03 \times (44 + 20) = 550$

Beam below Slab $\frac{10 \times 14.5}{144} \times 150 = 150$

700 lb. per lin. ft. of beam

Total concentrated load:

$$P_d = 700 \times (20 - 1) = 13\,300 \text{ lb. placed at sixth points.}$$

Vertical Live Load:

Unit Load: 40 lb. per sq. ft.

Since live load is transferred to the frame by beams, it is concentrated and the concentrated load is

$$P_l = 8.33 \times 40 \times 20 = 6664 \text{ lb., call } 6700 \text{ lb., placed at sixth points.}$$

Wind Load:

Unit Load: 30 lb. per sq. ft.

$$p = 30 \times 20 = 600 \text{ lb. per lin. ft. of frame.}$$

DEAD LOAD

Uniformly Distributed Weight of Inclined Member:

$$w = 520 \text{ lb. per lin. ft.}$$

$$wl = 520 \times 50 = 26\,000 \text{ lb.}$$

$$wl^2 = 26\,000 \times 50 = 1\,300\,000 \text{ ft.-lb.}$$

$$\frac{I_1}{I}\frac{h}{s} = 2.4, \quad \frac{h_1}{l} = \frac{6}{30} = 0.2.$$

Use Formula (103), p. 295 for horizontal thrust

$$H = C_7 \frac{l}{h} wl.$$

Find constant C_7, from Formula (101), p. 296.

$$C_7 = \frac{1}{32} \frac{8 + 5 \times 0.2}{2.4 + 0.2(3 + 0.2) + 3} = \frac{9}{193.3} = 0.046.$$

Hence

Horizontal Thrust:

$$H = 0.046 \times \tfrac{50}{30} \times 26\,000 = 2000 \text{ lb.}$$

Vertical End Shear:

$$V_B = 520 \times 25 = 13\,000 \text{ lb.}$$

Corner Bending Moments:

$$M_B = M_D = -2000 \times 30 = -60\,000 \text{ ft.-lb.}$$

Bending moment at loads (Formulas 106 and 107, p. 296),

at $x = \frac{1}{6}l$

$$M_1 = [\tfrac{1}{2}\times\tfrac{1}{6}(1-\tfrac{1}{6})-0.046(1+0.067)]wl^2=(0.0694-0.0491)wl^2=26\,000\text{ ft.-lb.}$$

at $x = \frac{1}{3}l$

$$M_2 = [\tfrac{1}{2}\times\tfrac{1}{3}(1-\tfrac{1}{3})-0.046(1+0.133)]wl^2=(0.111-0.052)wl^2=76\,700\text{ ft.-lb.}$$

at ridge

$$M_C = [\tfrac{1}{8} - 0.046 \times 1.2]wl^2 = (0.125 - 0.055)wl^2 = 91\,000\text{ ft.-lb.}$$

Concentrated Dead Load:

$$P = 13\,300\text{ lb., placed at sixth points,}$$

$$Pl = 665\,000\text{ ft.-lb.}$$

Horizontal thrust is found from table, p. 299, Case *c*.

$$H = \frac{l}{h}\left(5.833 + 3.667\frac{h_1}{h}\right)C_8 P,$$

in which C_8 is obtained from Formula 119,

$$C_8 = \frac{1}{4}\,\frac{1}{2.4 + 0.2(3 + 0.2) + 3} = 0.041.$$

Horizontal Thrust: (See table, p. 299, Case *c*.)

$$H = \tfrac{50}{30}(5.833 + 3.667 \times \tfrac{6}{30}) \times 0.041P = 0.45 \times 13300 = 6000\text{ lb.}$$

Vertical End Shear:

$$V_B = 2.5 \times 13\,300 = 33\,250\text{ lb.}$$

Corner Bending Moments:

$$M_B = M_D = -Hh = -6000 \times 30 = -180\,000\text{ ft.-lb.}$$

Bending Moments at Loads: [From formula in table, p. 299.]

$$M_1 = \tfrac{5}{12}Pl - (1 + 0.067)Hh = 277\,100 - 192\,000 = 85\,100\text{ ft.-lb.}$$

$$M_2 = \tfrac{2}{3}Pl - (1 + 0.133)Hh = 443\,400 - 203\,900 = 239\,500\text{ ft.-lb.}$$

$$M_C = \tfrac{3}{4}Pl - (1 + 0.2)Hh = 498\,800 - 216\,000 = 282\,800\text{ ft.-lb.}$$

Summary of Dead Load Bending Moments:

	M_B	M_1	M_2	M_C
Uniform D. L.	60.0	26 0	76 7	91 0
Concentrated D. L.	180.0	85 1	239.5	282 8
Total.......	240.0	111.1	316.2	373.8

The bending moments in thousand ft.-lb. or ft. kips.

These bending moments are plotted in Fig. 157, opposite p. 366. The columns in this figure are shown as horizontal lines to make the plotting clearer.

LIVE LOAD

The Live Load is also considered as concentrated at sixth points.

Both Sides Loaded:

$$P_l = 6700 \text{ lb. at sixth points.}$$

The bending moments are proportionate to the bending moments for concentrated dead load and may be obtained by multiplying the values for dead load by $\frac{6700}{13\,300}$ = 0.504

$$V_A = 16\,750 \text{ lb.}$$

$$H_1 = 6\,000 \times 0.504 = 3\,020 \text{ lb.}$$

$$M_B = 180\,000 \times 0.504 = -\,91\,000 \text{ ft.-lb.}$$

$$M_1 = 85\,100 \times 0.504 = 42\,900 \text{ ft.-lb.}$$

$$M_2 = 239\,500 \times 0.504 = 121\,000 \text{ ft.-lb.}$$

$$M_C = 282\,800 \times 0.504 = 143\,000 \text{ ft.-lb.}$$

Left Side Loaded:

$$P_l = 6700 \text{ lb. at sixth and third points.}$$

$$P_l = 3350 \text{ lb. at ridge.}$$

Horizontal thrust for one-sided load is equal to one-half the thrust for full load,

$$H_{\frac{1}{2}l} = \tfrac{1}{2} \times 3020 = 1510 \text{ lb.}$$

Reactions

$$V_A = (\tfrac{5}{6} + \tfrac{4}{6} + \tfrac{1}{4})P = \tfrac{21}{12} \times 6\,700 = 11\,700 \text{ lb.}$$

Corner Bending Moments:

$$M_B = M_D = -\,H_{\frac{1}{2}l}h = -\,1\,510 \times 30 = -\,45\,300 \text{ ft.-lb.}$$

Bending Moments at Loads:

$$M_1 = 11\,700 \times \tfrac{50}{6} - 1.067 \times 45\,300 = 97\,500 - 48\,300 = 49\,200 \text{ ft.-lb.}$$

$$M_2 = (11\,700 \times \tfrac{1}{3} - 6\,700 \times \tfrac{1}{6})50 - 1.133 \times 45\,300 = 140\,000 - 51\,300 = 88\,700 \text{ ft.-lb.}$$

$$M_C = 5\,050 \times 25 - 1\,510 \times 36 = 126\,250 - 54\,360 = 72\,000 \text{ ft.-lb.}$$

WIND PRESSURE

Wind on Left Inclined Member:

$$p = 30 \times 20 = 600 \text{ lb. per lin. ft.}$$

$$ph_1 = 600 \times 6 = 3\,600 \text{ lb.}$$

$$ph_1^2 = 21\,600 \text{ ft.-lb.}$$

Vertical Reactions:

$$V_A = -3\,600 \times \tfrac{33}{50} = -2\,376 \text{ lb.} \qquad V_D = +2\,376 \text{ lb.}$$

Horizontal Thrust. (Formulas (144) and (145) p. 303.)

Right Column

$$H = [2 \times 2.4 + \tfrac{5}{4} \times \tfrac{6}{30}(4 + \tfrac{6}{30}) + 6]0.041 \times 3\,600 = 1\,750 \text{ lb.}$$

Left Column

$$H_1 = 3\,600 - 1\,750 = 1\,850 \text{ lb.}$$

Bending Moments: (Formulas (146) to (149), p. 304.)

$$M_B = H_1 h = 1\,850 \times 30 = 55\,500 \text{ ft.-lb.}$$

$$M_C = -1\,750 \times 36 + 2\,376 \times 25 = -63\,000 + 59\,400 = -3\,600 \text{ ft.-lb.}$$

$$M_D = -1\,750 \times 30 = -52\,500 \text{ ft.-lb.}$$

Bending Moments left half of beam, intermediate points

$$M_x = H_1 h\left(1 + 2\frac{h_1}{h}\frac{x}{l}\right) - V_A\,x - 2\left(\frac{x}{l}\right)^2 p h_1^2.$$

Bending Moments in Inclined Member, Wind on Inclined Member

x	$\frac{x}{l}$	$2\frac{h_1}{h}\frac{x}{l}$	$1+2\frac{h_1}{h}\frac{x}{l}$	$H_1 h\left(1+2\frac{h_1}{h}\frac{x}{l}\right)$	$-V_A x$	$\left(\frac{x}{l}\right)^2$	$-2\left(\frac{x}{l}\right)^2 p h_1^2$	M_x
8 33	$\frac{1}{6}$	0 067	1 067	59 2	−19 8	0 028	−1.22	38 2
16 66	$\frac{1}{3}$	0 133	1.133	62 9	−39.4	0 111	−4.80	18 7
.25	$\frac{1}{2}$	0 2	1 2	66.5	−59 4	0 25	−10.8	−3 6

These bending moments are plotted in Fig. 157, opposite p. 366.

Wind from Left on Vertical Member:

$$p = 600 \text{ lb. per lin. ft.}$$

$$ph = 18\,000 \text{ lb.}$$

Reactions: (Formula 150, p. 305.)

$$R_A = -\tfrac{1}{2}\,\tfrac{30}{50} \times 18\,000 = -5\,400 \text{ lb.} \qquad R_E = 5\,400 \text{ lb.}$$

Horizontal Thrust: (Formulas (152) and (153), p. 305.)

Right hinge B

$$H = \tfrac{1}{4}(5 \times 2.4 + 6 \times 0.2 + 12)0.041ph = 0.26 \times 18\,000 = 4\,700 \text{ lb.}$$

Left hinge A

$$H_1 = ph - H = 18\,000 - 4\,700 = 13\,300 \text{ lb.}$$

Bending Moments: (Formulas 151 to 156, p. 305.)

$$M_B = \left(13\,300 - \frac{18\,000}{2}\right)30 = 129\,000 \text{ ft.-lb.}$$

$$M_C = \left(\frac{18\,000}{4} - 4\,700 \times 1.2\right)30 = -\,34\,200 \text{ ft.-lb.}$$

$$M_D = -\,4\,700 \times 30 = -\,141\,000 \text{ ft.-lb.}$$

Bending moment at intermediate points in Beam:

$$M_1 = 129\,000 - \tfrac{1}{3}(129\,000 + 34\,200) = 74\,600 \text{ ft.-lb.}$$

$$M_2 = 129\,000 - \tfrac{2}{3}(129\,000 + 34\,200) = 20\,200 \text{ ft.-lb.}$$

Moments at intermediate points in Column:

$$M_y = (H_1 - \tfrac{1}{2}py)y.$$

Bending Moments in Column, Wind on Vertical Member

y	$\frac{1}{2}py$	$H_1 - \frac{1}{2}py$	M_y	Remarks
6.0	1 8	11 5	69 0	
12.0	3 6	9 7	116 4	
18.0	5 4	7.9	142.2	
22.2	6 65	6 65	147.6	Maximum
24.0	7.2	6 1	146.4	
30.0	9.0	4.3	129 0	Moment M_B

Sum of Moments Due to Wind in Inclined Members

Wind on	M_B	M_1	M_2	M_C	M_D
Inclined member...........	55.5	38.2	18.7	−3.6	−52 5
Vertical member...........	129 0	74 6	20.2	−34 2	−141 0
Total.................	184.5	112.8	38.9	−37.8	−193 5

Sum of Bending Moments Due to Wind in Column

Wind on	$y=6$ ft.	$y=12$ ft.	$y=18$ ft.	$y=22.2$	$y=24$ ft.
Inclined member...........	11.1	22.2	33.3	41.1	44 4
Vertical member...........	69.0	116.4	142.2	147.6	146.4
Total.................	80.1	138.6	175.5	188.7	190.8

SUMMARIES OF BENDING MOMENTS

Bending Moments for Dead and Live Load

	Point B	Load 1	Load 2	Point C
Dead load..............	−246.3	102 6	303.8	355.9
Live load*	−91 0	49 2	121.0	143.0
Total..............	−337.3	151.8	424 8	498.9

Bending moments are in ft.-kip (1 kip = 1000 lb.)

Maximum Negative Bending Moments

	Point B	Load 1	Load 2
D. L.	−246.3	102 6	303.3
L. L.†.................	−45 5		
Wind..................	−193 5	−141 6	− 89.7
Total....	−485 3	−39 0	213 6

Bending moments are in ft.-kip (1 kip = 1 000 lb.).

Maximum Positive Bending Moments

	Point B	Load 1	Load 2	Point C
D. L..................	−246.3	102 6	303 8	355 9
L. L.†..................	....	49.2	88 7	143 0
Wind..................	184.5	112 8	38 9	
Total...............	−62.8	264 6	431.4	498.9

Bending moments are in kip ft. (1 kip = 1 000 lb.).

The bending moments are shown in Fig. 157, opposite p. 366. This figure should be used to determine the points of bending of the reinforcement.

* Values of bending moments for load on whole span were used, except at load 1, where load at half span gives larger value.

† Where Bending Moments due to wind are combined with bending moments due to live load, it is assumed that the live load acts only upon the half of the frame not subjected to wind. It is not conceivable that, with high wind acting on the roof, snow would remain on the part of the roof subjected directly to the wind.

DIAGONAL TENSION REINFORCEMENT

Inclined Members.—As explained on page 355, to find the diagonal tension in the inclined members, it is necessary to find components at right angles to the inclined member, due to the external vertical shears and to the horizontal forces.

The forces producing maximum shearing stresses are:

Dead load
Live load acting on the member
Wind acting on opposite side of the member under consideration.

The vertical end shears for dead load and live load are:

	V_B	V_D
Dead load uniform.........	13 000	13 000
concentrated.....	33 200	33 200
Live load on one side.......	11 700	5 050
Total..................	57 900 lb.	51 250 lb.

To get the reaction at the hinge, it is necessary to add to the end shear the load coming directly on the column.

The external vertical shears at the different sections of the frame are shown in Fig. 157, opposite p. 366.

The wind on the right produces following reactions and thrusts:

Horizontal Thrusts	At A	At E
Wind on inclined portion....	1 750 lb.	1 850 lb.
Wind on vertical portion....	4 700 lb.	13 300 lb.
Total.................	6 550 lb.	15 150 lb.

Vertical Reactions:

Wind on inclined portion....................	± 2 376 lb.
Wind on vertical portion....................	± 5 400 lb.
	± 7 776 lb.

The external shears acting at right angles to the member are obtained by multiplying the vertical shears by cos ϕ and the horizontal shears by sin ϕ, where ϕ is the angle of inclination of the inclined members.

$$\text{In this case } \cos\phi = \frac{l}{25} = \frac{50}{2 \times 25.7} = 0.972$$

$$\sin\phi = \frac{2h_1}{l} = \frac{12}{50} = 0.24.$$

Normal shear due to wind:

$$V_n = 7.8 \times 0.972 = 6.5 \times 0.24 = 7.6 - 1.6 = 6.0 \text{ kips.}$$

The external shear normal to the inclined members at various sections are shown in Fig. 157.

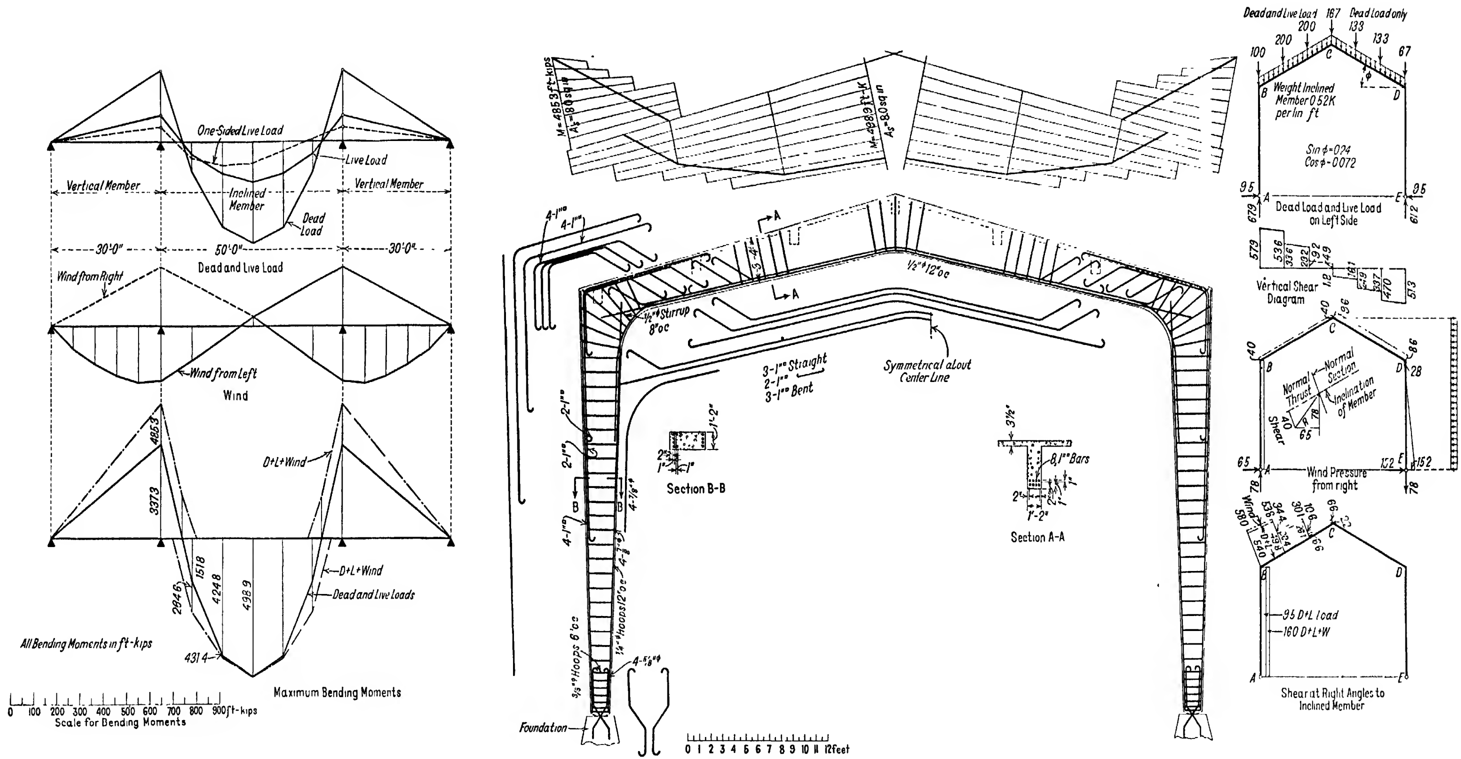


Fig. 157.—Example of Rigid Frame Design (See p. 361)

To face page 366

After the external shears are found, the diagonal tension reinforcement is computed in the same manner as for other reinforced concrete structures. This is explained on page 251, Vol. 1.

Vertical Members.—Maximum external shear in vertical members is obtained by adding the horizontal thrusts due to dead load, the one-sided live load and the wind. The sum of horizontal thrusts is

$$9.5 + 6.5 = 16 \text{ kips}$$

It produces shearing stresses and diagonal tension at all sections of the vertical member.

FINAL DESIGN OF FRAME

After the bending moments and shears are determined, the dimensions of the frame are computed by formulas and methods given in Vol. I. The general instructions regarding the design of rigid frames given on p. 351 will be found of assistance. The final design of the frame is shown in Fig. 157, opposite p. 366.

Inclined Member.—The inclined member is subjected to bending moments and to a thrust the magnitude of which is equal to the sum of the components acting at right angles to the normal section of the member of the external shear and the horizontal thrust. The magnitude of this thrust is small and it may be neglected in computing the dimensions or stresses.

At the center of the span the maximum positive bending moment is $M_{max} =$ 498.9 ft.-kip, as evident from the summary on p. 365. It is produced by the live and dead loads. Steel areas are computed for this bending moment.

At the support the bending moment is $M_B = 337.3$ ft.-kip for the live and dead loads and $M_B = 485.3$ ft.-kip for a combination of the dead and live loads with the wind pressure. Since the second amount is more than 30 per cent larger than the first, it should be used in determining the dimensions and the amount of reinforcement, using, however, unit stresses 30 per cent larger than for the dead and live load, alone.

To accommodate the reinforcement and also to reduce the compression stresses, the breadth of the beam has been increased from 12 to 14 inches. Compression reinforcement was found necessary.

Diagonal tension is resisted by the bent bars, and by stirrups in the portion in which the bent bars are not effective. At the center of the span stirrups are introduced to prevent the curved bars from straightening under the effect of the tensile stresses.

Vertical Members.—The dimension of the vertical member is smallest at the hinge and increases towards the top where it is determined by the maximum bending moment and thrust. The vertical member is subjected to a thrust equal to the reaction and to bending moments. It should be noted that the effect of the thrust is very much smaller in comparison with that of the bending moments, than found in arch design. Considerable tensile stresses are developed in the member requiring a large amount of tensile reinforcement.

The member is designed using the formulas and diagrams in Chapter II. The maximum bending moment acts at the joint of the vertical and inclined members and is of the same magnitude as was used for the inclined member.

Shearing stresses are produced at all sections of the vertical member by the horizontal thrust. Their magnitude, however, does not exceed the allowable stresses for plain concrete.

No actual hinges are used. To permit rotation of the member the bottom of the column was rounded up as shown in Fig. 157. The horizontal thrust is resisted by eight $\frac{5}{8}$-in. rd. bent bars extending from the foundation into the column, four on each side, and arranged so that they can resist shear and direct tension but are incapable of resisting bending moments. (See Fig. 157, opposite p. 366.)

Points of Bending of Reinforcement.—The maximum bending moment diagram was used to determine the points of bending of the reinforcement. For this purpose the bending moment diagram was plotted and upon it were superimposed the areas of reinforcement resisting this bending moment. Since the depth of the member decreases toward the end, the effect of the reinforcement is similarly decreased. Each bar is therefore represented by two slanting lines.

CHAPTER IV

BUILDINGS CONSIDERED AS FRAMES

WHEN a continuous beam is rigidly connected with columns, as, for example, in ordinary reinforced concrete skeleton buildings, the bending moments in the beam depend not only upon the number and the length of spans composing the beam itself, but also upon the rigidity of the columns with which it is connected. Since the formulas for bending moments in continuous beams given in Chapter I are not applicable to such construction, special formulas are developed in this chapter.

Difference between Frames and Continuous Beams.— When continuous beams are independent of the supports, the bending moment in one span at the support is transmitted in full to the beam in the next span. Conditions of this nature exist in continuous bridges, where the beam rests on supports but is not connected with them, and approximately in constructions where the amount of column reinforcement running into the beams is not sufficient to tie the members rigidly together. The intermediate beams in a floor construction running into the girders (and not into the columns) furnish another example of continuous beam action.

The condition is different when beams run into concrete columns of considerable rigidity as ordinarily found in building construction. The bending moment at the end of one span of the beam cannot be transferred to the beam in the next span without subjecting the columns to bending. Instead of transmitting the bending moment in full to the beam in the next span, part of the moment is transferred to the column above the beam, part to the column below the beam and the balance to the beam. Consequently the effect of loading of one span upon the other spans is much smaller than in continuous beams not connected with columns. The range of bending moments to which a beam may be subjected is considerably smaller in a frame than in a continuous beam.

As far as columns are concerned, the difference between continuous beams and frames is that the columns of continuous beams are subjected to vertical loads only, while the columns of frame structures are also subjected to vertical loads and in addition to considerable bending, particularly due to unbalanced loadings.

General Principles of Frame Action.—When in a structure several members meet and are rigidly connected, thus forming a rigid joint, the bending moment in one member composing the joint must be balanced or resisted by the bending moments in the remaining members. For equilibrium the algebraic sum of the bending moments acting on all members composing the joint must be zero. This means that the sum of the bending moments turning in one direction must be equal to the sum of bending moments turning in the opposite direction.

Members composing a joint may be considered as rigidly connected when each member at the plane of juncture with the other members is capable to resist the bending and shearing stresses to which it may be subjected.

Signs of Bending Moments.—The signs used for making summations must be based on a different principle than commonly used in concrete design. A bending moment in a member at any section is positive when the resisting forces at that section turn the member clockwise, i.e., from left to right. A bending moment turning counter-clockwise is negative. This method differs from that commonly used in concrete construction where a bending moment in a beam is negative when it produces tension on the top of the beam or outside face of column, and positive when it produces tension at the bottom of the beam or inside face of column, irrespective of the direction in which it turns. Thus in a beam subjected to downward vertical loading, according to common designation, the bending moments at both supports are negative. If designated according to the direction of turning explained above, the bending moment at the left support is negative while at the right support it is positive (see also p. 631).

Illustration of Frame Action.—The principle of frame action will be illustrated by following examples. Clear understanding of the principle is essential for intelligent designing of concrete structures.

First Case.—Assume a beam continuous over three spans but simply resting on the top of the four supports (see Fig. 158, p. 371). The ends of the beam are held against uplift but they are free to turn.

To simplify the explanation the middle span only is considered as loaded. The bending moments are shown in the figure. Consider the condition at the second support marked 2. There are only two members at joint 2, i.e., the beam in the center span and the beam in the end span. The bending moment produced in the center span by the loads, the resisting forces of which in this case turn counter-clockwise, may be called minus. It must be resisted by a bending moment in the only other member in the joint. The bending moment is therefore transferred in full from the center span to the end span. In the end span, the resisting forces at the left end turn clockwise and may be called plus. The algebraic sum of the moments at the joint 2 is therefore zero.

Second Case.—Assume a beam of the same length and the same number of spans as before but rigidly connected with the columns by sufficient reinforcement (see Fig. 159, p. 372). Let the center span be loaded in the same manner as in the previous case. The bending moments and deflection for this case are shown in the figure.

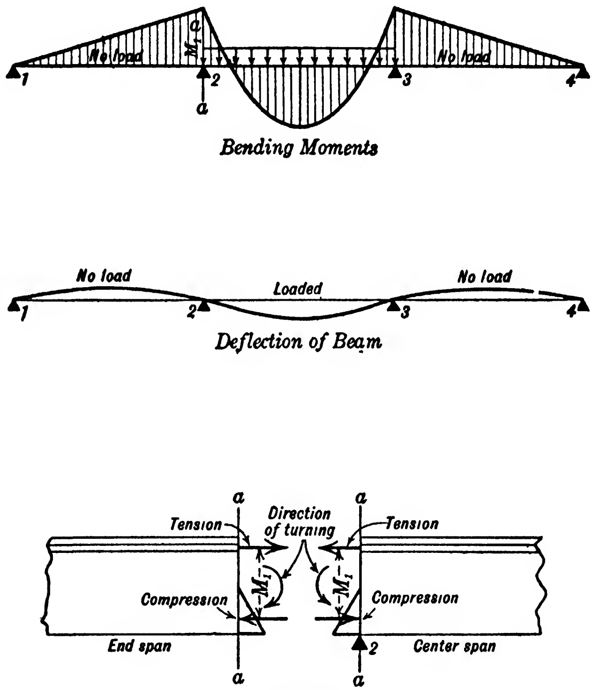


FIG. 158.—Continuous Beam. Condition at Support. (*See p.* 370.)

Consider the joint at the interior column marked 2. It consists of three members, namely, one column and two beams. At the joint 2 the loads produce in the center span a bending moment M_2 which turns clockwise and is negative. Since at this joint three members are rigidly connected no member can turn without turning the other two members. All three members, therefore, will be subjected to bending moments.

The bending moment M_2 is not transferred in full to the beam on the other side of the support as in a continuous beam but only in part. Therefore M_1 is smaller than M_2. The ratio between M_1 and M_2 is called the ratio of transference.

For equilibrium the algebraic sum of all the moments at the joint must equal zero; therefore, the sum of the resisting bending moments in the column M_3 and in the end span M_1 must be equal to the bending

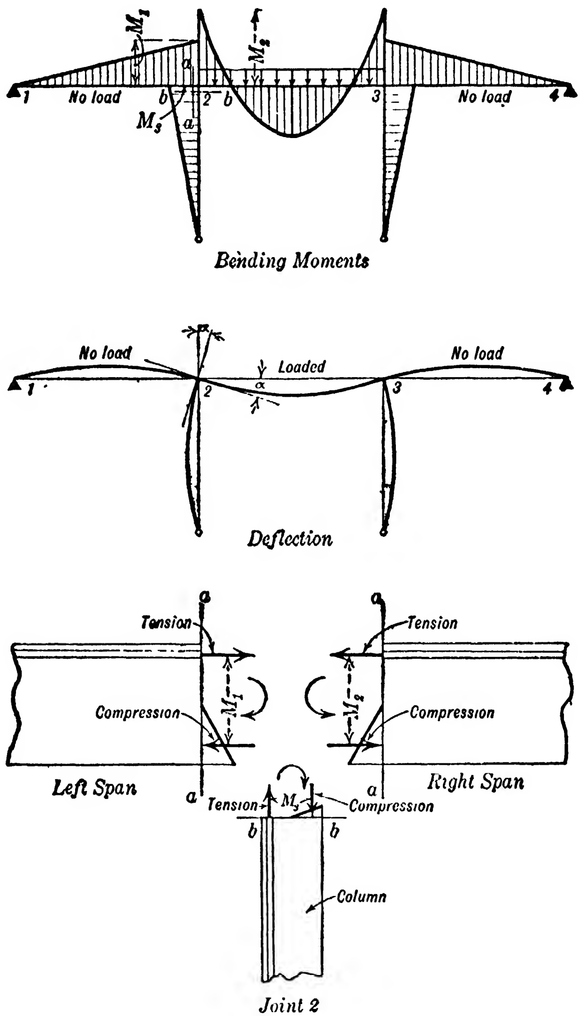


FIG. 159.—One Story Frame. (See p. 371.)

moment M_3 produced by the loads at the support in the middle span. Or $M_1 + M_2 + M_3 = 0$ and $M_2 = - M_1 - M_3$. The bending moments in the column and the end beam are of opposite signs to the bending moment in the center span; therefore they turn clockwise and are positive.

Since the joint 2 is rigid, the beam in both spans has the same tangent to the deflection curve and the angle with the vertical of the tangent to the deflection curve for the column is equal to the angle with the horizontal of the tangent for the beam.

Third Case.—Assume a beam of the same length and number of spans as in the previous two cases but rigidly connected with the columns above and below the beam (see Fig. 160, p. 373). Let the center span be loaded for which case the bending moments are as shown in the figure.

Consider the joint 2 at the interior column. It consists of four members, namely, two columns and two beams. The bending moment M_3 in the middle span produced by the loads turns counter-clockwise as in the previous cases. This moment will be resisted by bending moments in the other three members. The algebraic sum of the bending moments again is zero or $M_1 + M_2 + M_3 + M_4 = 0$, therefore, $M_3 = - M_1 - M_2 - M_4$.

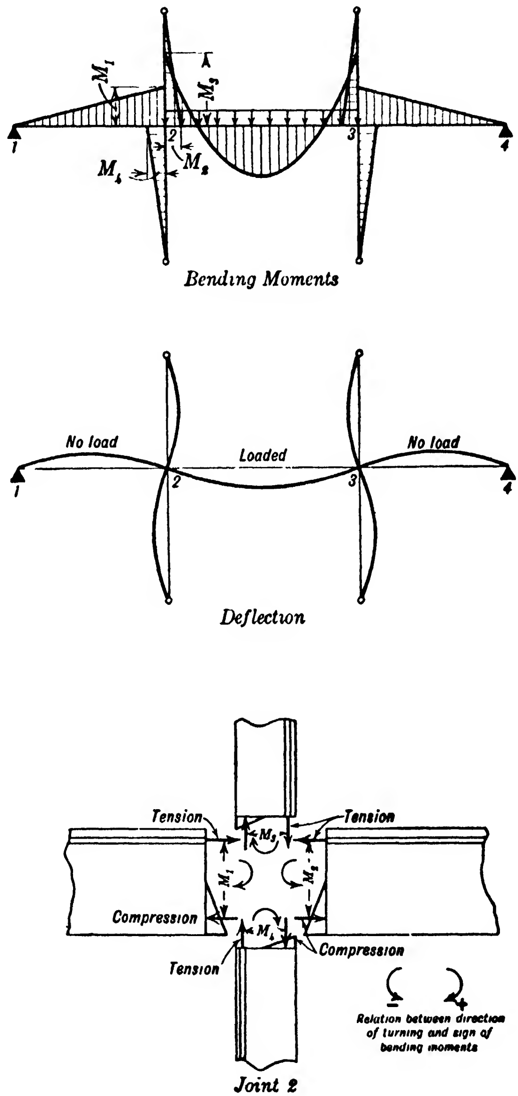


Fig. 160.—Two-story Frame. (*See* p. 373.)

The bending moments in the three resisting members turn in opposite direction to the moment in the center span.

Effect of Rigidity of Restraining Members on Bending Moments.—The restraint exerted by one member upon another member composing a joint depends upon the relative rigidity of the two members. The rigidity of a member is expressed by the ratio $\frac{I}{l}$ where I is the moment of inertia of the member and l its length (span in case of a beam and height in case of column). In the above ratio both items are in first power. The rigidity of a member is therefore directly proportional to its moment of inertia and inversely proportional to its length. This means that the rigidity of a member increases with the increase of its moment of inertia and decreases with the increase of its length.

The magnitude of the bending moment produced by the load at the support of the loaded member increases with the increase of the ratio of rigidity of the restraining member $\frac{I_1}{l_1}$ in proportion to the rigidity of the loaded member, $\frac{I}{l}$. One extreme condition is $\frac{I_1}{l_1} \div \frac{I}{l} = 0$ when no restraint exists. Another extreme when the ratio $\frac{I_1}{l_1} \div \frac{I}{l} = \frac{I_1}{I}\frac{l_1}{l}$ is very large in which case the member is rigidly fixed.

Effect of End Condition of Restraining Member.—The restraining effect of one member upon the other members forming a joint depends also upon the condition existing on the other end of the restraining member. The end may be either free to turn, partially restrained or rigidly fixed. The restraining effect is largest for the rigidly fixed condition of the end and smallest for free end. For partially restrained end the effect is intermediate between the two extreme conditions. As a general rule the restraining effect of a fixed member is one-third larger than the restraining effect of a member of the same rigidity $\frac{I}{l}$, but with the end free to turn.

A member may be considered as free to turn when it is provided with a hinge (conditions very seldom found in practice) or when the end simply rests upon a support. Also the end of a column provided with a small footing may be considered as free to turn.

A member may be considered as fixed when it is built monolithic with a member of comparatively large rigidity. This condition is approached when the ratio of $\frac{I_1}{l_1} \div \frac{I}{l} = 10$. The characteristic of this condition is that irrespective of the loading the angle between the tangent to the elastic curve and the original axis of the member is zero.

The end of a member may be considered as partly restrained when it runs into another joint composed of several members. This condition is most often found in concrete skeleton construction. A column with a footing of considerable size may also be considered as partly restrained at its end.

Effect of the Number of Restraining Members upon Bending Moments.—The restraint upon a loaded member by other members in a joint increases with the number of restraining members. This can be readily understood if it is considered that, since each member, acting separately, restrains the loaded member in proportion of their ratios of rigidity, the effect of all members acting together must be equal to the sum of all individual effects.

METHOD OF DETERMINING MOMENTS IN BUILDING FRAME

The determination of exact bending moments in a structure consisting of a number of spans and several stories high is a complicated problem. A simplification, however, accurate enough for all practical purposes may be obtained by using, instead of the whole frame, simpler substitute frames.

Substitute Frames.—Consider a frame shown in Fig. 161 representing a vertical section of a building six stories high and five panels wide. Suppose that it is desired to compute bending moments in any beam or column, for instance *ab* and columns *ac* and *ad*. The bending moments in these members are influenced to a larger or smaller degree by the condition of loading of all spans. Actual computations, however, show that the members in question are affected mainly by the condition of loading of the portion of the frame shown by heavy lines and to a smaller extent by the portions of the frame indicated by dash lines. The effect of the balance of the frame is negligible. Therefore, instead of making computations for the complete structure it is permissible to accept the simpler frames as a substitute frame, for which it is possible to develop workable formulas.

Types of Substitute Structures.—It will be found that for ordinary conditions the four types of substitute structures treated in this chapter will be sufficient.

Fig. 163 shows a three-span structure with two-story columns.
Fig. 167 shows a substitute frame for wall columns.
Fig. 168 shows a substitute frame for a building two panels wide.
Fig. 169 shows a substitute frame for a building one panel wide.

BASIS OF FORMULAS AND DEGREE OF ACCURACY

The formulas for the building frames are based on the elastic theory and were worked out by means of the slope deflection method given on p. 631. To simplify the formulas it was assumed that in no case does any horizontal or vertical movement of any of the joints take place. This assumption is not correct in all cases. It introduces some inaccuracies into the formulas, the degree of which varies with conditions.

For symmetrical and symmetrically loaded frames the formulas give exact results. For unsymmetrical frames the error increases with the

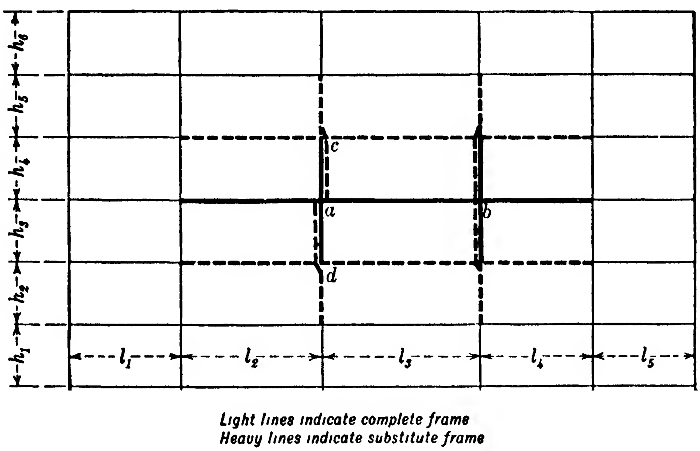


FIG. 161.—Complete Building Frame and Substitute Frames. (*See p. 375.*)

degree of variation from the symmetrical condition. In no case, however, is the error appreciable. The formulas may be used with confidence for the purposes for which they are intended.

ABSOLUTE MAXIMUM BENDING MOMENTS

The bending moments in the beams are usually affected mainly by the condition of loading of the members on the same level with the beam under consideration. The maximum bending moments, therefore, found for the beam, considering it as a center span of a substitute frame (or end span in case of a wall beam), may be considered as the maximum bending moment to be used in design. Thus the substitute frame shown by heavy lines in Fig. 161, p. 376 gives maximum bending moments for span *ab*.

The bending moments in a column obtained from the formulas for substitute frames are not the absolute maximum values possible in a building as a whole. Each tier of columns may be a lower column in one substitute frame, when the frame is selected in one story, and also an upper column in another substitute frame, when the frame is placed in the story below. The bending moments produced in the columns in the two cases are of the same sign. When the floor just below a tier of columns and the floor just above the tier of columns are loaded simultaneously the bending moments in the columns are equal to the sum of bending moments produced by the two substitute frame condi-

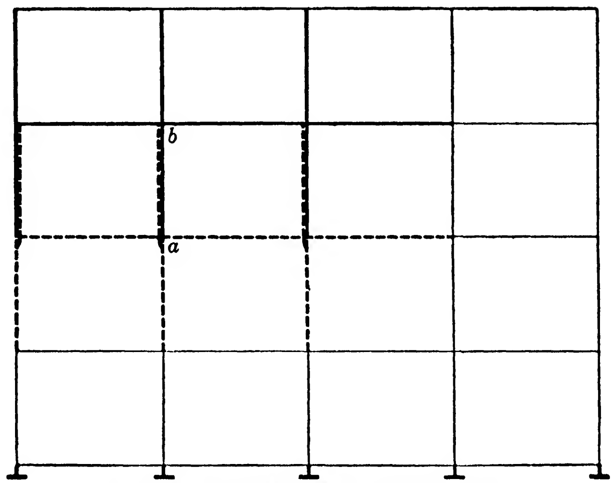


FIG. 162.—Substitute Frames for Absolute Maximum Bending Moment in Column. (*See p. 377.*)

tions. Other floors also affect the bending moments in the columns under consideration, but the effect never exceeds more than 5 per cent of the total bending moment.

From the above explanation follows a rule that for the absolute maximum bending moment in any column may be taken the sum of bending moments produced in that column by considering it as a part of two substitute frames placed so that the column forms first the upper column of one frame and then the lower column of the other frame. The two frames are shown in Fig. 162, p. 377. The two substitute frames, one of which is shown by heavy lines and the other by heavy dash lines, give maximum bending moment in the column *ab*. It should be noted that it is necessary to compute bending moments in the columns not only at the joints but also at the ends. This may be accomplished by using Formulas 13 to 15, p. 383.

If it is not desirable to compute bending moments in two frames, sufficiently accurate results may be obtained by multiplying the maximum bending moment in the column as obtained from one frame by 1.43.

USE OF THE BUILDING FRAME FORMULAS

The formulas given in this chapter may be used for determining bending moments in beams and columns in any multistory structure with equal and unequal span lengths.

Following procedure may be followed in design:

Determine the spacing of the beams and columns. Design the slab. Determine the dimensions of the beams on the basis of approximate formulas for bending moments and shears, such as recommended in Vol. I, p. 279. Compute the column loads and using them as a basis determine the column sizes. Where the difference in spans is great the accepted dimensions should be larger than required by vertical loads to take care of the bending moments.

Compute the moments of inertia of the beams and columns, using formulas given on p. 133. The diagram on p. 134 may also be used to advantage.

Select the substitute frame in which the span of the beam for which bending moments are desired forms the center span. This substitute frame may be moved from story to story and in this manner bending moments determined in the various floors. Usually similar beams in all floors are made of the same dimensions and provided with the same amount of steel. Therefore one substitute frame may be sufficient when placed in a position in the structure for which the bending moments are the largest. To get bending moments in the wall columns and wall beams, substitute frames described on p. 390 should be used. When the spans of beams are not equal, substitute frames may have to be used in which the largest span forms the center span and also frames in which the smallest span forms the center span. Several trial computations may have to be made to get the frame for which the bending moments are maximum.

Compute the rigidity ratios $\frac{I_1 l_2}{I_2 l_1}$, $\frac{I_3 l_2}{I_2 l_3}$, $\frac{I_4 l_2}{I_2 h_2}$, $\frac{I_5 l_2}{I_2 h_1}$, $\frac{I_6 l_2}{I_2 h_2}$, $\frac{I_7 l_2}{I_2 h_1}$.

Determine the end ratios c_1, c_3, c_4, c_5, c_6, and c_7 according to the existing end conditions in the substitute frame. Where the ends of the substitute frame are restrained by members of the structure not forming a part of the frame, the values of c may be computed by Formula (30), p. 640.

Compute Frame Constants A and B, or A, B and C.

Find the bending moments in the beams due to dead load and the maximum bending moments due to the most unfavorable position of the live load. Add the bending moments for dead load and live load. For the sum of the bending moments compute the amount of reinforcement required by the positive and negative bending moments. Also check the compression stresses.

In columns the absolute maximum bending moments should be obtained as explained on p. 377. To get the stresses the maximum bending moments should be combined with the column loads, using formulas given in Chapter II.

To get maximum compression stresses in columns combine maximum vertical loads with maximum bending moments.

To get maximum tensile stresses in columns combine the maximum bending moments with the minimum vertical loads. Minimum vertical loads are obtained when the structure is considered as unloaded except in the panels producing the bending moments.

The magnitude of the bending moments in beams and columns respectively, depends upon the relative rigidity of the beams and columns. Usually the beams are made of the same dimensions in all floors (except when the intensity of the loading changes materially). The dimensions of columns on the other hand are smallest at the top of the building and increase with the increase of the load upon them. In the upper floors, therefore, the ratio of the rigidity of the beam to that of the column is larger than in the lower floors. By inspecting the formulas it is found that for equal rigidity of the beams the positive bending moments in the beams increase with the decrease of the rigidity of the columns. The negative bending moments in beams, on the other hand, increase with the increase of the rigidity of the columns. Therefore in a building the positive bending moments are largest in the upper stories where the columns are least rigid, and the negative bending moments are maximum in the lower stories where the columns are most rigid.

As far as columns are concerned the bending moments in the columns increase with their rigidity; therefore they are largest in the lower stories. The effect of the bending moments in columns, however, is much larger in the upper floors, because there the dimensions of the columns are the smallest and also the vertical load much smaller than in the lower stories. The possibility of tensile stresses in columns is also much larger in upper stories than in lower stories.

FRAME LOADED BY SYMMETRICALLY ARRANGED CONCENTRATED LOADS

If the frame is loaded by symmetrically arranged concentrated loads the formulas in this chapter may be used by substituting in them for $\frac{1}{12}wl^2$ the bending moment M_f produced at the support by the concentrated loading when the beam is considered as fixed at both supports.

For special cases the bending moments M_f are given in the table on page 28.

TWO-STORY FRAME OF THREE SPANS. IRREGULAR SPACING OF COLUMNS

Three-span, two-story substitute structure may be used for determining bending moments in beams and columns for irregular spacing of columns. The formulas are based on a substitute structure shown in

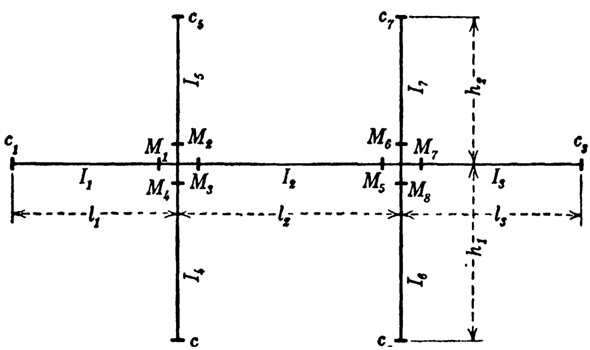


FIG. 163.—Two-Story Frame of Three Spans. (*See p.* 380.)

Fig. 163. The condition of restraint of the ends of the members is expressed by the ratio c.

In investigating maximum bending moments in a building such substitute structure should be selected for which the largest spans is in the center. Maximum bending moments in columns are found as explained on p. 377.

Formulas are given first for a two-story frame consisting of three spans of unequal lengths, as shown in Fig. 163, p. 380. For equal spans or equal end spans the formulas are much simpler as given in subsequent pages.

Four conditions of loading are considered:

Case *a.*—All spans loaded. Condition for dead load.

Case *b.*—Center span loaded. Condition for maximum positive bending moment in center span.

Case *c.*—Two adjoining spans loaded. Condition for maximum negative bending moment at support.

Case *d.*—End spans loaded. Condition for maximum positive bending moment in end spans and maximum negative bending moment in center spans.

To determine the bending moments for dead loads the first condition of loading should be accepted. To determine the maximum negative bending moment at the support in the beams the third condition of loading, for which two adjoining spans are loaded, should be adopted. The maximum bending moment in the columns should be determined from the second condition of loading with middle span loaded. The bending moments due to live load and dead load should be combined.

Let I_1, I_2, I_3 = moments of inertia of the three spans;
c_1, c_3 = constants depending upon end conditions of first and third span;
I_4 and I_6 = moments of inertia of lower columns;
c_4, c_6 = constants depending upon end conditions of lower columns;
I_5 and I_7 = moments of inertia of upper columns;
c_5, c_7 = constants depending upon end conditions of upper columns;
l_1, l_2, l_3 = span lengths of the beam;
h_1 = height of lower columns;
h_2 = height of upper columns;
A and B = constants, Formulas (1) and (2).

Constants for All Cases.

Members at left joint,

$$A = (6 - c_1)\frac{I_1}{I_2}\frac{l_2}{l_1} + (6 - c_4)\frac{I_4}{I_2}\frac{l_2}{h_1} + (6 - c_5)\frac{I_5}{I_2}\frac{l_2}{h_2} + 4. \qquad (1)$$

Members at right joint,

$$B = (6 - c_3)\frac{I_3}{I_2}\frac{l_2}{l_3} + (6 - c_6)\frac{I_6}{I_2}\frac{l_2}{h_1} + (6 - c_7)\frac{I_7}{I_2}\frac{l_2}{h_2} + 4, \qquad (2)$$

where $c = 2$ for members fixed at ends,
$c = 3$ for members hinged at ends.

For restrained ends use for c intermediate values between fixed and hinged or compute by Formula (30), p. 640.

Signs of Bending Moments.—The bending moments are negative for downward loads: in a beam, when tension is produced at its upper face; in end columns, when tension is produced at their outside faces; in a center column (for frames with three columns), when tension is produced at the left face.

Case *a*. All Spans Loaded.—Using constants for A and B given on p. 381 the formulas for bending moments are

Negative Bending Moments in Beams:

Left end span,

$$M_1 = -\frac{c_1}{24}\left[1 - (6 - c_1)\frac{I_1}{I_2}\frac{l_2}{l_1}\frac{B}{AB - 4}\right]wl_1^2 - \frac{1}{12}(6 - c_1)\frac{I_1}{I_2}\frac{l_2}{l_1}\frac{B + 2}{AB - 4}wl_2^2 + \frac{c_3}{12}(6 - c_1)\frac{I_1}{I_2}\frac{l_2}{l_1}\frac{1}{AB - 4}wl_3^2. \qquad (3)$$

Center span,

$$M_3 = -\frac{c_1}{6}\frac{(B - 1)}{AB - 4}wl_1^2 - \frac{1}{12}\frac{(B + 2)(A - 4)}{AB - 4}wl_2^2 + \frac{c_3}{12}\frac{A - 4}{AB - 4}wl_3^2. \qquad (4)$$

$$M_5 = +\frac{c_1}{12}\frac{B - 4}{AB - 4}wl_1^2 - \frac{1}{12}\frac{(A + 2)(B - 4)}{AB - 4}wl_2^2 - \frac{c_3}{6}\frac{A - 1}{AB - 4}wl_3^2. \qquad (5)$$

Right end span,

$$M_7 = \frac{c_1}{12}(6 - c_3)\frac{I_3}{I_2}\frac{l_2}{l_3}\frac{1}{AB - 4}wl_1^2 - \frac{1}{12}(6 - c_3)\frac{I_3}{I_2}\frac{l_2}{l_3}\frac{A + 2}{AB - 4}wl_2^2 - \frac{c_3}{24}\left(1 - (6 - c_3)\frac{I_3}{I_2}\frac{l_2}{l_3}\frac{A}{AB - 4}\right)wl_3^2. \qquad (6)$$

Bending Moments in Column:

Left column,

$$M_2 = (6-c_5)\frac{I_5}{I_2}\frac{l_2}{h_2}\left(-\frac{c_1}{24}\frac{B}{AB-4}wl_1^2 + \frac{1}{12}\frac{B+2}{AB-4}wl_2^2 - \frac{c_3}{12}\frac{1}{AB-4}wl_3^2\right). \quad . \quad . \quad . \quad (7)$$

$$M_4 = -\frac{I_4}{I_5}\frac{h_2}{h_1}\frac{6-c_4}{6-c_5}M_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Right column,

$$M_6 = (6-c_7)\frac{I_7}{I_2}\frac{l_2}{h_2}\left(-\frac{c_1}{12}\frac{1}{AB-4}wl_1^2 + \frac{1}{12}\frac{A+2}{AB-4}wl_2^2 - \frac{c_3}{24}\frac{A}{AB-4}wl_3^2\right). \quad . \quad . \quad . \quad (9)$$

$$M_8 = -\frac{I_5}{I_7}\frac{h_2}{h_1}\frac{6-c_5}{6-c_7}M_6.$$

Bending Moments at Ends of Beams and Columns.—The bending moments at ends of beams and columns depend upon the degree of restraint. They may be found from the following formulas:

Negative Bending Moment at End of Beams,

$$M_0 = 0 \text{ for hinged end} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$M_0 = -\tfrac{1}{2}M_1 - \tfrac{1}{8}wl_1^2 \text{ for fixed end.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$M_0 = -\frac{2(3-c_1)}{6-c_1}M_1 - \frac{3-c_1}{2(6-c_1)}wl_1^2 \text{ for restrained end,} \quad (12)$$

where c is between 2 and 3 or may be computed by Formula (30), p. 640.

If the span is not loaded, wl_1^2 in the formulas is zero.

To get bending moment at other end, substitute in above formulas M_7 for M_1 and l_3 for l_1 and c_3 for c_1.

The bending moments at end of columns vary from zero for hinged ends to one-third of the bending moments at the joints for fixed ends, and are of opposite signs.

Negative Bending Moment at Ends of Columns:

For hinged ends,

$$M_0 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

For fixed ends,

$$M_0 = -\tfrac{1}{2}M_2. \qquad (14)$$

For restrained ends,

$$M_0 = -\frac{2(3-c)}{6-c}M_2, \qquad (15)$$

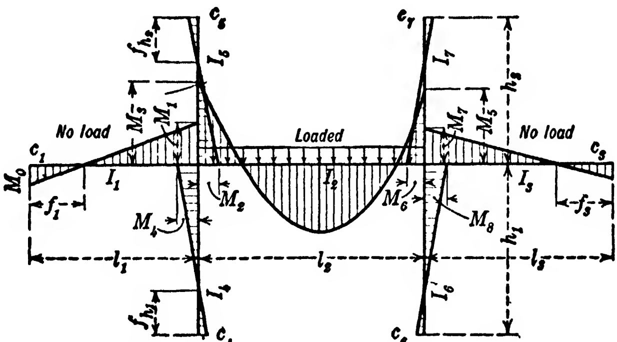
where c is a constant of a value between 2 and 3.

The formulas just given are for the end of the left top columns. By substituting for M_2 the bending moments M_4, M_6 and M_8, respectively, the bending moments at the ends of the other columns may be found.

Positive Bending Moments in Beams.—For known negative bending moments at supports, the maximum positive bending moment for this loading may be found graphically by plotting the negative bending moments above the axis and then drawing a parabola (see p. 14). It may also be found using table on p. 176.

Case *b*. Center Span Loaded.—(See Fig. 164, p. 384.)

This condition of loading gives maximum positive bending moment in the center span.



$\frac{f_1}{l_1}, \frac{f_3}{l_3}, \frac{f_{h_1}}{h_1}$ and $\frac{f_{h_2}}{h_2}$ vary from 0 for hinged ends to $\frac{1}{3}$ for fixed ends

$\frac{M_0}{M_1}$ varies from 0 for hinged ends to $\frac{1}{3}$ for fixed ends

FIG. 164.—Center Span Loaded. (*See p.* 384.)

The bending moments in the columns are also maximum for this loading.

Using the constants for A and B as given by Formulas (1) and (2), p. 381, the bending moments are

Bending Moments in Beams at Supports:

Left end span,

$$M_1 = -\frac{1}{12}(6 - c_1)\frac{I_1}{I_2}\frac{l_2}{l_1}\frac{B+2}{AB-4}wl_2^2. \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

Center span,

$$M_3 = -\frac{1}{12}\frac{(B+2)(A-4)}{AB-4}wl_2^2, \text{ left.} \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

$$M_5 = -\frac{1}{12}\frac{(A+2)(B-4)}{AB-4}wl_2^2, \text{ right.} \quad . \quad . \quad . \quad . \qquad (18)$$

Right end span,

$$M_7 = -\frac{1}{12}(6 - c_3)\frac{I_3}{I_2}\frac{l_2}{l_3}\frac{A+2}{AB-4}wl_2^2.$$

Maximum Bending Moments in Columns:

Left column,

$$M_2 = \frac{1}{12}(6 - c_5)\frac{I_5}{I_2}\frac{l_2}{h_2}\frac{B+2}{AB-4}wl_2^2, \text{ upper column.} \qquad (19)$$

$$M_4 = -\frac{I_4}{I_5}\frac{h_2}{h_1}\frac{6-c_4}{6-c_5}M_4, \qquad \text{lower column.} \qquad (20)$$

Right column,

$$M_6 = \frac{1}{12}(6 - c_7)\frac{I_7}{I_2}\frac{l_2}{h_2}\frac{A+2}{AB-4}wl_2^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

$$M_8 = -\frac{I_5}{I_7}\frac{h_2}{h_1}\frac{6-c_5}{6-c_7}M_6. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

Positive Bending Moment.—After finding M_3 and M_5 the maximum positive bending moment may be obtained from table on p. 176.

Case *c*. Two Adjoining Spans Loaded. (See Fig. 165, p. 386.)—This condition of loading gives maximum negative bending moments in the center span at the left column.

Negative Bending Moments at Supports in Beams:

End span,

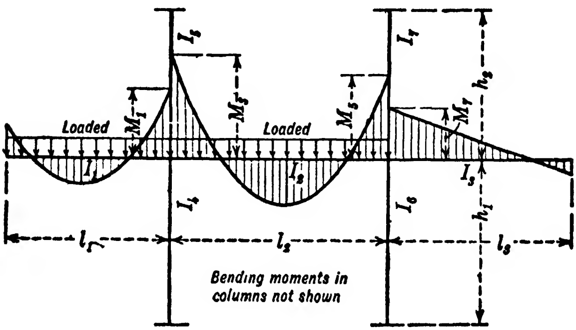
$$M_1 = -\frac{c_1}{24}\left(1 - (6 - c_1)\frac{I_1 l_2}{I_2 l_1}\frac{B}{AB-4}\right)wl_1^2$$

$$-\frac{1}{12}(6 - c_1)\frac{I_1 l_2}{I_2 l_1}\frac{B+2}{AB-4}wl_2^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

Center span,

$$M_3 = -\frac{c_1}{6}\frac{B-1}{AB-4}wl_1^2 - \frac{1}{12}\frac{(B+2)(A-4)}{AB-4}wl_2^2. \quad . \quad . \qquad (24)$$

$$M_5 = \frac{c_1}{12}\frac{B-4}{AB-4}wl_1^2 - \frac{1}{12}\frac{(A+2)(B-4)}{AB-4}wl_2^2. \quad . \quad . \qquad (25)$$



Loading for Max. M_1 and M_5

FIG. 165.—Two Adjoining Spans Loaded. (*See p.* 385.)

Case *d*. End Spands Loaded. (See Fig. 166, p. 386.)

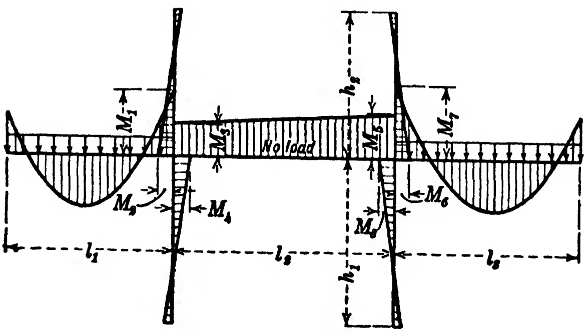


FIG. 166.—End Spans Loaded. (*See p.* 386.)

This condition of loading gives maximum positive bending moments in the end spans. also maximum bending moments in columns.

Negative Bending Moments at Supports in Beams:
End span,

$$M_1 = -\frac{c_1}{24}\left[1 - (6 - c_1)\frac{I_1 l_2}{I_2 l_1}\frac{B}{AB - 4}\right]wl_1^2$$

$$+ \frac{c_3}{12}(6 - c_1)\frac{I_1 l_2}{I_2 l_1}\frac{1}{AB - 4}wl_3^2. \quad . \quad . \quad . \quad . \quad (24)$$

Center span

$$M_3 = -\frac{c_1}{6}\frac{B - 1}{AB - 4}wl_1^2 + \frac{c_3}{12}\frac{A - 4}{AB - 4}wl_3^2. \quad . \quad . \quad . \quad (25)$$

$$M_5 = \frac{c_1}{12}\frac{B - 4}{AB - 4}wl_1^2 - \frac{c_3}{6}\frac{A - 1}{AB - 4}wl_3^2. \quad . \quad . \quad . \quad . \quad (26)$$

Right End Span

$$M_7 = \frac{c_1}{12}(6 - c_3)\frac{I_3 l_2}{I_2 l_3}\frac{1}{AB - 4}wl_1^2$$

$$-\frac{c_3}{24}\left[1 - (6 - c_3)\frac{I_3 l_2}{I_2 l_3}\frac{A}{AB - 4}\right]wl_3^2. \quad . \quad . \quad . \quad (27)$$

Bending Moments in Columns:
Left Column

$$M_2 = (6 - c_5)\frac{I_5}{I_2}\frac{l_2}{h_2}\left(-\frac{c_1}{24}\frac{B}{AB - 4}wl_1^2\right.$$

$$\left.-\frac{c_3}{12}\frac{1}{AB - 4}wl_3^2\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$M_4 = -\frac{I_2}{I_3}\frac{h_1}{h_2}\frac{6 - c_2}{6 - c_3}M_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Right Column

$$M_6 = (6 - c_7)\frac{I_7 l_2}{I_2 h_2}\left(-\frac{c_1}{12}\frac{1}{AB - 4}wl_1^2\right.$$

$$\left.-\frac{c_3}{24}\frac{A}{AB - 4}wl_3^2\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$M_8 = -\frac{I_5 h_2}{I_7 h_1}\frac{6 - c_5}{6 - c_7}M_6. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Use results either from Formulas (19) and (20) or (28) and (29), which ever give larger results.

Bending moments at the ends of beams and columns may be found by Formulas (10) to (15) p. 383.

TWO-STORY FRAME. ALL SPANS EQUAL

Use notation on p. 381 except that c_1 applies to both end spans, c_2 to both lower columns and c_3 to both upper columns.

For equal spans $l_1 = l_2 = l_3 = l$ also $I_1 = I_2 = I_3 = I$.

The column sizes in each story are also equal so that $I_4 = I_6 = I_l$, $I_5 = I_7 = I_u$ The ratios of rigidity become

$$\frac{I_1}{I_2}\frac{l_1}{l_2} = 1, \quad \frac{I_3}{I_2}\frac{l_2}{l_1} = 1, \quad \frac{I_4}{I_2}\frac{l_2}{h_1} = \frac{I_6}{I_2}\frac{l_2}{h_1} = \frac{I_l}{I}\frac{l}{h_1} \quad \text{and} \quad \frac{I_5}{I_2}\frac{l_2}{h_2} = \frac{I_7}{I_2}\frac{l_2}{h_2} = \frac{I_u}{I}\frac{l}{h_2}.$$

The frame constants for both joints are equal so that $A = B$.

Value of Frame Constant,

$$A = (6 - c_1) + (6 - c_2)\frac{I_l}{I}\frac{l}{h_1} + (6 - c_3)\frac{I_u}{I}\frac{l}{h_1} + 4. \quad . \quad (32)$$

Case *a*. Center Span Loaded.

Bending Moments in Beams at Supports:

End span,

$$M_1 = -\frac{1}{12}\frac{6 - c_1}{A - 2}wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Center span,

$$M_3 = -\frac{1}{12}\frac{A - 4}{A - 2}wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Bending Moments in Columns:

Upper columns,

$$M_2 = \frac{1}{12}\frac{I_u}{I}\frac{l}{h_2}\frac{6 - c_3}{A - 2}wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Lower columns,

$$M_4 = -\frac{I_l}{I_u}\frac{h_2}{h_1}\frac{6 - c_2}{6 - c_3}M_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Case *b*. End Span Loaded.

Bending Moments in Beams at Supports:

Loaded span,

$$M_1 = -\frac{c_1}{24}\left(1 - \frac{(6 - c_1)A}{A^2 - 4}\right)wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Center span,

$$M_3 = -\frac{c_1}{6}\frac{A-1}{A^2-4}wl^2. \quad \ldots \ldots \quad (38)$$

$$M_5 = \frac{c_1}{12}\frac{A-4}{A^2-4}wl^2. \quad \ldots \ldots \quad (39)$$

Unloaded span,

$$M_7 = \frac{c_1}{12}\frac{(6-c_1)}{A^2-4}wl^2. \quad \ldots \ldots \quad (40)$$

Bending Moments in Columns:

Left column,

$$M_2 = -(6-c_3)\frac{I_u l}{I h_2}\frac{c_1}{24}\frac{A}{A^2-4}wl^2. \quad \ldots \quad (41)$$

$$M_4 = -\frac{I_l}{I_u}\frac{h_2}{h_1}\frac{6-c_2}{6-c_3}M_2. \quad \ldots \ldots \quad (42)$$

Right column,

$$M_6 = -(6-c_3)\frac{I_u}{I}\frac{l}{h_2}\frac{c_1}{12}\frac{1}{A^2-4}wl^2. \quad \ldots \quad (43)$$

$$M_8 = -\frac{I_l h_2}{I_u h_1}\frac{6-c_2}{6-c_3}M_6. \quad \ldots \ldots \quad (44)$$

Maximum Bending Moments.—Maximum bending moments are produced by the following loading conditions:

Maximum negative bending moments in beams when two adjoining spans are loaded.

Maximum positive bending moments in the end beams when both end spans are loaded.

Maximum positive bending moment in the center beam when the center span is loaded.

Maximum bending moments in columns when both end spans are loaded.

The formulas below are obtained by proper combination of the formulas given in the previous paragraphs for end span loaded and center span loaded.

Maximum Negative Bending Moments in Beams for Live Load,

$$M_1 = -\frac{1}{12}\left\{\frac{6-c_1}{A-2} + \frac{c_1}{2}\left[1 - \frac{(6-c_1)A}{A_2-4}\right]\right\}wl^2.$$

$$M_3 = -\frac{1}{12}\left(2c_1\frac{A-1}{A^2-4} + \frac{A-4}{A-2}\right)wl^2. \quad \ldots \quad (45)$$

Maximum Positive Bending Moment in Center Beam for Live Load,

$$M_{max} = \frac{1}{12}\left(1.5 - \frac{A-4}{A-2}\right)wl^2. \quad \ldots \ldots \quad (46)$$

Maximum Bending Moment in Column for Live Load,

$$M_2 = -\frac{1}{24}(6 - c_3)\frac{I_u}{I}\frac{l}{h_2}\frac{c_1}{A-2}wl^2. \quad \ldots \ldots \quad (47)$$

$$M_4 = -\frac{I_l}{I_u}\frac{h_2}{h_1}\frac{6-c_2}{6-c_3}M_2. \quad \ldots \ldots \quad (48)$$

The above formulas give the maximum bending moments in the column for the substitute frame. To get the actual maximum bending moments in the column for the structure the effect of other substitute frames must be considered as explained on p. 377. The actual maximum bending moment in the columns of the structure is about 43 per cent larger than obtained from the last two formulas.

BENDING MOMENTS IN WALL COLUMNS AND END BEAMS

Wall columns are always subjected to bending not only due to the live load but also due to the dead load. It is of great importance for the safety of the structure that these bending moments be properly provided for. In flat slab construction this necessity has been generally recognized and most modern flat slab specifications include a requirement for bending moments in wall columns. In beam and girder designs the bending moments in columns are often neglected with great detriment to the structure.

A comparatively simple method of finding the bending moments in the wall columns of a structure is by the use of the substitute frames of a type shown in Fig. 167, p. 391. This consists of three spans and three two-story columns, one of which is the wall column.

Notation.

Let l_1, l_2, l_3 = length of the three spans;
h_1, h_2 = length of lower and upper columns;
I_1, I_2, I_3 = moments of inertia of beams in the three spans;
c_3 = constant at the end of third span;
I_4, I_6, I_8 = moments of inertia of lower columns;
c_4, c_6, c_8 = constants at ends of lower columns;
I_5, I_7, I_9 = moments of inertia of upper columns;
c_5, c_7, c_9 = constants at ends of upper columns;
A, B, C, D = frame constants from Formula (49).

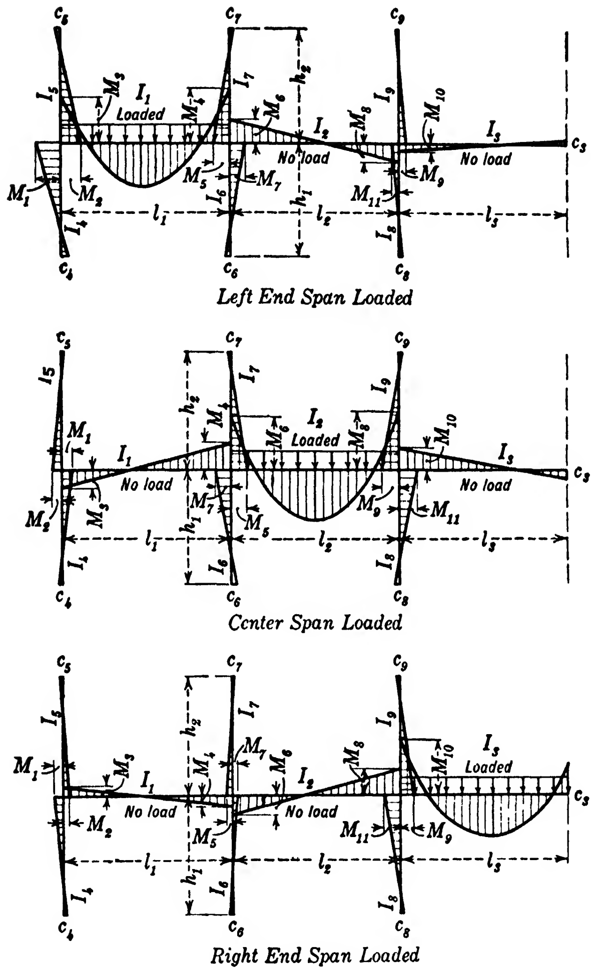


FIG. 167.—Substitute Frame for Wall Columns. (See p. 390.)

Frame Constants,

$$\left.\begin{aligned}
A &= (6 - c_4)\frac{I_4 l_1}{I_1 h_1} + (6 - c_5)\frac{I_5 l_1}{I_1 h_2} + 4 \\
B &= (6 - c_6)\frac{I_6 l_1}{I_1 h_1} + (6 - c_7)\frac{I_7 l_1}{I_1 h_2} + 4\left(1 + \frac{I_2 l_1}{I_1 l_2}\,\frac{C - \dfrac{I_2 l_1}{I_1 l_2}}{C}\right) \\
C &= (6 - c_8)\frac{I_8 l_1}{I_1 h_1} + (6 - c_9)\frac{I_9 l_1}{I_1 h_2} + (6 - c_3)\frac{I_3 l_1}{I_1 l_3} + 4\frac{I_2 l_1}{I_1 l_2} \\
D &= (6 - c_6)\frac{I_6 l_1}{I_1 h_1} + (6 - c_7)\frac{I_7 l_1}{I_1 h_2} + 4\left(\frac{I_2 l_1}{I_1 l_2} + \frac{A - 1}{A}\right)
\end{aligned}\right\}, \qquad (49)$$

where $c_3, c_4, c_5, c_6, c_7, c_8 = 2$ for members with fixed ends,

and $= 3$ for members with hinged ends.

For restrained ends use intermediate values for c or compute it from Formula (30), p. 640.

Formulas are given separately for dead load and for live load. In the formulas for dead load all spans are assumed to be loaded. The formulas for live load are based upon the loading giving the maximum values at the section under consideration.

Signs of Bending Moments.—The bending moments are negative for downward loads; in beams, when they produce tension at the top; in left end and center columns, when the tension is produced at their left faces.

Bending Moments for Loads Placed in Separate Spans.—Figure 167, p. 391, illustrates bending moments in the frame produced when each one of the three spans is loaded separately. Formulas for bending moments for each of the three conditions were developed separately and then combined so as to get the most unfavorable results. The formulas for separate loadings are not given. If desired they may be obtained from the formulas for dead load by retaining only the values applying to the loaded span and omitting the values for the unloaded spans. Thus for a condition when the left end span only is loaded formula (50) changes to

$M_3 = -\frac{1}{12}\frac{(A - 4)(B + 2)}{AB - 4}wl_1^2$. The other two values in the formula containing l_2 and l_3 are omitted.

Maximum Bending Moments for Dead Load.

Bending Moment in Beams at Supports Due to Dead Load:

Wall span, at supports,

$$M_3 = -\frac{1}{12}\,\frac{(A-4)(B+2)}{AB-4}wl_1^2 + \frac{1}{6}\,\frac{(A-4)\left(C+2\frac{I_2l_1}{I_1l_2}\right)}{A\left[CD-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]}wl_2^2$$

$$-\frac{c_3}{6}\,\frac{I_2l_1}{I_1l_2}\,\frac{A-4}{A\left[CD-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]}wl_3^2. \qquad (50)$$

$$M_4 = -\frac{1}{12}\,\frac{(A+2)(B-4)}{AB-4}wl_1^2 - \frac{1}{3}\,\frac{(A-1)\left(C+2\frac{I_2l_1}{I_1l_2}\right)}{A\left[CD-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]}wl_2^2$$

$$+\frac{c_3}{4}\,\frac{I_2l_1}{I_1l_2}\,\frac{A-1}{A\left[CD-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]}wl_3^2. \qquad (51)$$

Center span, at supports,

$$M_6 = -\frac{1}{3}\,\frac{I_2l_1}{I_1l_2}\,\frac{(A+2)\left(C-\frac{I_2l_1}{I_1l_2}\right)}{C(AB-4)}wl_1^2 - \frac{1}{12}\,\frac{\left(C+2\frac{I_2l_1}{I_1l_2}\right)\left(D-4\frac{I_2l_1}{I_1l_2}\right)}{CD-4\left(\frac{I_2l_1}{I_1l_2}\right)^2}wl_2^2$$

$$+\frac{c_3}{12}\,\frac{I_2l_1}{I_1l_2}\,\frac{D-4\frac{I_2l_1}{I_1l_2}}{CD-4\left(\frac{I_2l_1}{I_1l_2}\right)^2}wl_3^2. \qquad (52)$$

$$M_8 = \frac{1}{6}\,\frac{(A+2)\left(C-4\frac{I_2}{I_1}\,\frac{l_1}{l_2}\right)}{C(AB-4)}wl_1^2 - \frac{1}{12}\,\frac{\left(C-4\frac{I_2l_1}{I_1l_2}\right)\left(D+2\frac{I_2l_1}{I_1l_2}\right)}{CD-4\left(\frac{I_2l_1}{I_1l_2}\right)^2}wl_2^2$$

$$-\frac{c_3}{6}\,\frac{I_2l_1}{I_1l_2}\,\frac{D-\frac{I_2l_1}{I_1l_2}}{CD-4\left(\frac{I_2l_1}{I_1l_2}\right)^2}wl_3^2. \qquad (53)$$

Maximum Positive Bending Moments Due to Dead Load.—Maximum positive bending moments, due to the dead load, are obtained from table on p. 176, using the negative bending moments obtained from the above formulas.

Bending Moments in Columns Due to Dead Load,

Wall columns,[1]

$$M_2 = -(6 - c_5)\frac{I_5 l_1}{I_1 h_2}\frac{1}{A - 4}M_3. \quad . \quad . \quad . \quad . \qquad (54)$$

$$M_1 = -\frac{6 - c_4}{6 - c_5}\frac{I_4 h_2}{I_5 h_1}M_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (55)$$

Second column,[9]

$$M_5 = \frac{(6-c_7)}{12}\frac{I_7 h_2}{I_2 l_1}\left[-\frac{A+2}{AB-4}wl_1^2 + \frac{C+2\dfrac{I_2 l_1}{I_1 l_2}}{CD-4\left(\dfrac{I_2 l_1}{I_1 l_2}\right)^2}wl_2^2 - \frac{c_3\dfrac{I_2 l_1}{I_1 l_2}}{CD-4\left(\dfrac{I_2 l_1}{I_1 l_2}\right)^2}wl_3^2\right] \qquad (56)$$

$$M_7 = -\frac{6 - c_6}{6 - c_7}\frac{I_6 h_2}{I_7 h_1}M_5. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (57)$$

Maximum Bending Moments for Live Loads.—To get maximum bending moments at any section due to live load, only the spans producing bending moments of the desired sign are considered as loaded. The following formulas are for maximum bending moments. It should be noted that all the fractions in these formulas are the same as used in formulas for dead load for the corresponding spans.

Maximum Negative Bending Moments in Beams for Live Loads,

Wall span,

$$M_3 = -\frac{1}{12}\frac{(A-4)(B+2)}{AB-4}wl_1^2 - \frac{c_3}{6}\frac{I_2 l_1}{I_1 l_2}\frac{A-4}{A\left[CD-4\left(\dfrac{I_2 l_1}{I_1 l_2}\right)^2\right]}wl_3^2. \qquad (58)$$

$$M_4 = -\frac{1}{12}\frac{(A+2)(B-4)}{AB-4}wl_1^2 - \frac{1}{3}\frac{(A-1)\left(C+2\dfrac{I_2 l_1}{I_1 l_2}\right)}{A\left[CD-4\left(\dfrac{I_2}{I_1}\dfrac{l_1}{l_2}\right)^2\right]}wl_2^2. \quad . \qquad (59)$$

Center span,

[1] Positive bending moment produces tension at the right face of the column and negative bending moment produces tension at the left face of the column.

$$M_6 = -\frac{1}{3}\frac{I_2 l_1}{I_1 l_2}\frac{(A+2)\left(C-\frac{I_2 l_1}{I_1 l_2}\right)}{C(AB-4)}wl_1^2 - \frac{1}{12}\frac{\left(C+2\frac{I_2 l_1}{I_1 l_2}\right)\left(D-4\frac{I_2 l_1}{I_1 l_2}\right)}{CD-4\left(\frac{I_2 l_1}{I_1 l_2}\right)}wl_2^2. \quad (60)$$

$$M_8 = -\frac{1}{12}\frac{\left(D+2\frac{I_2 l_1}{I_1 l_2}\right)\left(C-4\frac{I_2 l_1}{I_1 l_2}\right)}{CD-4\left(\frac{I_2 l_1}{I_1 l_2}\right)^2}wl_2^2 - \frac{c_3}{6}\frac{I_2 l_1}{I_1 l_2}\frac{D-\frac{I_2 l_1}{I_1 l_2}}{CD-4\left(\frac{I_2 l_1}{I_1 l_2}\right)^2}wl_3^2. \quad (61)$$

Maximum Positive Bending Moments for Live Loads.—First find negative bending moments at the supports from formulas below. The maximum positive bending moments corresponding to these bending moments at the supports may then be taken from table on p. 176.

Negative Bending Moments Used to Determine Maximum Positive Bending Moment:

Wall span,

$$M_3 = -\frac{1}{12}\frac{(A-4)(B+2)}{AB-4}wl_1^2 - \frac{c_3}{6}\frac{I_2 l_1}{I_1 l_2}\frac{A-4}{A\left[CD-4\left(\frac{I_2 l_1}{I_1 l_2}\right)^2\right]}wl_3^2. \quad (62)$$

$$M_4 = -\frac{1}{12}\frac{(A+2)(B-4)}{AB-4}wl_1^2 + \frac{c_3}{4}\frac{I_2 l_1}{I_1 l_2}\frac{A-1}{A\left[CD-4\left(\frac{I_2 l_1}{I_1 l_2}\right)^2\right]}wl_3^2. \quad (63)$$

Center span,

$$M_6 = -\frac{1}{12}\frac{\left(C+2\frac{I_2 l_1}{I_1 l_2}\right)\left(D-4\frac{I_2 l_1}{I_1 l_2}\right)}{CD-4\left(\frac{I_2 l_1}{I_1 l_2}\right)^2}wl_2^2. \quad (64)$$

$$M_8 = -\frac{1}{12}\frac{\left(D+2\frac{I_2 l_1}{I_1 l_2}\right)\left(C-4\frac{I_2 l_1}{I_1 l_2}\right)}{CD-4\left(\frac{I_2 l_1}{I_1 l_2}\right)^2}wl_2^2. \quad (65)$$

Maximum Bending Moments in Columns for Live Loads:

Wall columns,

$$M_2 = (6-c_5)\frac{I_5l_1}{I_1h_2}\left\{\frac{1}{12}\frac{B+2}{AB-4}wl_1^2+\frac{c_3}{6}\frac{\frac{I_2l_1}{I_1l_2}}{A\left[CD-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]}wl_3^2\right\}. \quad (66)$$

$$M_1 = -\frac{6-c_4}{6-c_5}\frac{I_4h_2}{I_5h_1}M_2. \quad \ldots \quad (67)$$

Second column (first and third spans loaded),[9]

$$M_5 = -(6-c_7)\frac{I_7l_1}{I_1h_2}\left[\frac{1}{12}\frac{A+2}{AB-4}wl_1^2+\frac{c_3}{12}\frac{\frac{I_2l_1}{I_1l_2}}{CD-4\left(\frac{I_2l_1}{I_1l_2}\right)^2}wl_3^2\right]. \quad (68)$$

$$M_7 = -\frac{6-c_6}{6-c_7}\frac{I_6h_2}{I_7h_1}M_5. \quad \ldots \quad (69)$$

Second column (second span loaded),[2]

$$M_5 = \frac{1}{12}(6-c_7)\frac{I_7l_1}{I_1h_2}\frac{C+2\frac{I_2l_1}{I_1h_2}}{CD-4\frac{I_2l_1}{I_1h_2}}wl_2^2. \quad \ldots \quad (70)$$

$$M_7 = -\frac{6-c_4}{6-c_7}\frac{I_6h_2}{I_7h_1}M_5. \quad \ldots \quad (71)$$

The absolute maximum bending moment in the columns are obtained as explained on p. 377.

Bending Moments at End of Beams and Columns.—Bending moments at end of beams and columns may be found by Formulas (13) to (15) p. 383.

Numerical Example.—For numerical example of the application of the preceding formulas, see p. 410.

WALL COLUMNS AND END BEAMS. ALL SPANS EQUAL

Assume that all spans composing the frame are equal and that the moments of inertia of all beams are equal. Also assume that the moments of inertia of the second and third columns are equal. Then

$$l_1 = l_2 = l_3 = l, \quad I_1 = I_2 = I_3 = I, \quad I_6 = I_8 \text{ and } I_7 = I_9.$$

[2] See note, p. 394.

The frame constants become

$$\left.\begin{aligned} A &= (6 - c_4)\frac{I_4 l}{I h_1} + (6 - c_5)\frac{I_5 l}{I h_2} + 4. \\ B &= C - \left[(2 - c_3) + \frac{4}{C}\right]. \\ C &= (6 - c_6)\frac{I_6 l}{I h_1} + (6 - c_7)\frac{I_7 l}{I h_2} + (6 - c_3) + 4. \\ D &= C - \left[(2 - c_3) + \frac{4}{A}\right]. \end{aligned}\right\} \quad . \quad (72)$$

These values substituted in the general formulas given in the preceding pages give:

Maximum Bending Moments for Dead Load.

Bending Moments in Beam for Dead Load:

Wall panel, negative bending moments,

$$M_3 = -\frac{1}{12}\left[\frac{(A-4)(B+2)}{AB-4} - \frac{2(A-4)(C+2)}{A(CD-4)} + \frac{2c_3(A-4)}{A(CD-4)}\right]wl^2. \quad (73)$$

$$M_4 = -\frac{1}{12}\left[\frac{(A+2)(B-4)}{AB-4} + \frac{4(A-1)(C+2)}{A(CD-4)} - \frac{3c_3(A-1)}{A(CD-4)}\right]wl^2. \quad (74)$$

Center panel, negative bending moments,

$$M_6 = -\frac{1}{12}\left[\frac{4(A+2)(C-1)}{C(AB-4)} + \frac{(C+2)(D-4)}{CD-4} - \frac{c_3(D-4)}{CD-4}\right]wl^2 \quad . \quad (75)$$

$$M_8 = -\frac{1}{12}\left[-\frac{2(A+2)(C-4)}{C(AB-4)} + \frac{(C-4)(D+2)}{CD-4} + \frac{2c_3(D-1)}{CD-4}\right]wl^2. \quad (76)$$

Maximum Positive Bending Moments for Dead Load.—Maximum positive bending moments are obtained from the table on p. 176, using the negative bending moments at the supports found from the above formulas.

Bending Moments in Columns for Dead Load:

Wall column,[3]

$$M_2 = \frac{1}{12}(6-c_5)\frac{I_5 l}{I h_2}\left[\frac{B+2}{AB-4} - \frac{2(C+2)}{A(CD-A)} + \frac{2c_3}{A(CD-4)}\right]wl^2. \quad (77)$$

$$M_1 = -\frac{6-c_4}{6-c_5}\frac{I_4 h_2}{I_5 h_1}M_2. \quad (78)$$

Second column,[3]

$$M_5 = -\frac{1}{12}(6-c_7)\frac{I_7 l}{I h_2}\left[\frac{A+2}{AB-4} - \frac{C+2}{CD-4} + \frac{c_3}{CD-4}\right]wl^2, \quad (79)$$

$$M_7 = -\frac{6-c_6}{6-c_7}\frac{I_6 h_2}{I_7 h_1}M_5. \quad (80)$$

Maximum Bending Moments for Live Load.

Maximum Negative Bending Moments in Beams: ;

Wall span,

$$M_3 = -\frac{1}{12}\left[\frac{(A-4)(B+2)}{AB-4} + \frac{2c_3(A-4)}{A(CD-4)}\right]wl^2. \quad (81)$$

$$M_4 = -\frac{1}{12}\left[\frac{(A+2)(B-4)}{AB-4} + \frac{4(A-1)(C+2)}{A(CD-4)}\right]wl^2. \quad (82)$$

Center span,

$$M_6 = -\frac{1}{12}\left[\frac{4(A+2)(C-1)}{C(AB-4)} + \frac{(C+2)(D-4)}{CD-4}\right]wl^2. \quad (83)$$

$$M_8 = -\frac{1}{12}\left[\frac{(C-4)(D+2)}{CD-4} + \frac{2c_3(D-1)}{CD-4}\right]wl^2. \quad (84)$$

Maximum Positive Bending Moments in Beams,

To get maximum positive bending moments in the beams compute the negative bending moments at the support for the proper loading and then find from the table on p. 176 the corresponding maximum positive bending moment.

The negative bending moments to be used are:

For wall span,

$$M_3 = -\frac{1}{12}\left[\frac{(A-4)(B+2)}{AB-4} + \frac{2c_3(A-4)}{A(CD-4)}\right]wl^2, \quad (85)$$

[3] Positive bending moment produces tension at the right side of the column and the negative bending moment produces tension at the left side of the column.

$$M_4 = -\frac{1}{12}\left[\frac{(A+2)(B-4)}{AB-4} - \frac{3c_3(A-1)}{A(CD-4)}\right]wl^2. \quad . \quad . \quad (86)$$

For center span,

$$M_6 = -\frac{1}{12}\frac{(C+2)(D-4)}{CD-4}wl^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

$$M_8 = -\frac{1}{12}\frac{(C-4)(D+2)}{CD-4}wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

Maximum Bending Moments for Live Load in Columns:

Wall columns,[4]

$$M_2 = \frac{1}{12}(6-c_5)\frac{I_5 l}{I h_2}\left[\frac{B+2}{AB-4} + \frac{2c_3}{A(CD-4)}\right]wl^2. \quad . \quad (89)$$

$$M_3 = -\frac{6-c_4}{6-c_5}\frac{I_4 h_3}{I_5 h_1}M_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

Second column,[4]

$$M_5 = -\frac{1}{12}(6-c_7)\frac{I_7 l}{I h_2}\left[\frac{A+2}{AB-4} + \frac{c_3}{CD-4}\right]wl^2. \quad . \quad (91)$$

$$M_7 = -\frac{6-c_6}{6-c_7}\frac{I_6 h_2}{I_7 h_1}M_5. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Wall Columns and End Beams. Special Case of Equal Spans

To simplify the use of the formulas for this type of substitute frames, formulas are given in which the spans of beam and their moments of inertia are assumed to be equal. Also it is assumed that the rigidity ratios for the columns are as follows:

$$\frac{I_4 l}{I h_1} = 1, \quad \frac{I_5 l}{I h_2} = 0.8, \quad \frac{I_6 l}{I h_1} = 0.8, \quad \frac{I_7 l}{I h_2} = 0.7.$$

The constants at the ends are assumed to be

$$c_3 = c_4 = c_5 = c_6 = c_7 = 2.5.$$

Substitute these values in Formulas (81) to (92) for live load. For dead load use Formulas (73) to (80) in which the items previously found for live load may also be substituted.

[4] See note, p. 398.

For these assumptions the frame constants become

Frame Constants,

$$A = 3.5 + 3.5 \times 0.8 + 4 = 10.3.$$

$$B = 12.75 + (-0.5 + 0.31) = 12.56.$$

$$C = 3.5 \times 0.8 + 3.5 \times 0.7 + 3.5 + 4 = 12.75.$$

$$D = 12.75 + (-0.5 + 0.39) = 12.64.$$

Also

$$AB = 129, \quad AB - 4 = 125, \quad C(AB - 4) = 1600.$$

$$CD = 161, \quad CD - 4 = 157, \quad A(CD - 4) = 1620.$$

Bending Moments for Dead Loads.

Bending Moments in Beams for Dead Load:

Wall panel, negative bending moments,

$$M_3 = -\frac{1}{12}\left[0.754 - \frac{2 \times 6.3 \times 14.75}{1620}\right]wl^2 = -0.639\frac{wl^2}{12}. \quad (93)$$

$$M_4 = -\frac{1}{12}\left[1.18 - \frac{4 \times 9.3 \times 14.75}{1620}\right]wl^2 = -0.84\frac{wl^2}{12}. \quad (94)$$

Center panel, negative bending moments,

$$M_6 = -\frac{1}{12}\left[1.17 - \frac{2.5 \times 8.64}{157}\right]wl^2 = -1.03\frac{wl^2}{12}. \quad (95)$$

$$M_8 = -\frac{1}{12}\left[1.19 - \frac{2 \times 12.3 \times 8.75}{1600}\right]wl^2 = -1.06\frac{wl^2}{12}. \quad (96)$$

Maximum Positive Bending Moments for Dead Load,

Wall span,

$$M_{max} = 0.061wl^2. \quad (97)$$

Center span,

$$M_{max} = 0.052wl^2. \quad (98)$$

Bending Moments in Columns Due to Dead Load:

Wall column,[5]

$$M_2 = 2.8\left[0.12 - \frac{2 \times 14.75}{1600}\right]\frac{wl^2}{12} = 0.28\frac{wl^2}{12}. \quad (99)$$

$$M_1 = -\frac{1}{0.8} \times 0.28\frac{wl^2}{12} = -0.35\frac{wl^2}{12}. \quad (100)$$

[5] See note, p. 398.

Second column[6]

$$M_5 = 2.5\left[0.099 - \frac{14.75}{157}\right]\frac{wl^2}{12} = 0.018\frac{wl^2}{12}. \quad . \quad . \quad . \quad . \quad . \qquad (101)$$

$$M_7 = -\frac{0.8}{0.7} \times 0.018\frac{wl^2}{12} = -0.02\frac{wl^2}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (102)$$

Maximum Bending Moments for Live Load.

Maximum Bending Moments in Beams,

Wall span,

$$M_3 = -\frac{1}{12}\left[\frac{6.3 \times 14.56}{125} + \frac{5.0 \times 6.3}{1620}\right]wl^2 = -0.754\frac{wl^2}{12}. \quad . \quad . \qquad (103)$$

$$M_4 = -\frac{1}{12}\left[\frac{12.3 \times 8.56}{125} + \frac{4 \times 9.3 \times 14.75}{1620}\right]wl^2 = -1.18\frac{wl^2}{12}. \qquad (104)$$

Center span,

$$M_6 = -\frac{1}{12}\left[\frac{4 \times 12.3 \times 11.75}{12.75 \times 125} + \frac{14.75 \times 8.64}{157}\right]wl^2 = -1.17\frac{wl^2}{12}. \qquad (105)$$

$$M_8 = -\frac{1}{12}\left[\frac{8.75 \times 14.64}{157} + \frac{2 \times 2.5 \times 11.64}{157}\right]wl^2 = -1.19\frac{wl^2}{12}. \qquad (106)$$

Maximum Positive Bending Moments,

Wall span,

$$M_{\max} = 0.066wl.^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (107)$$

Center span,

$$M_{\max} = 0.057wl.^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (108)$$

Maximum Bending Moments in Columns. Live Load:

Wall column,

$$M_2 = 3.5 \times 0.8\left[\frac{14.8}{124} + \frac{2 \times 2.5}{1620}\right]\frac{wl^2}{12} = 0.34\frac{wl^2}{12}. \quad . \quad . \qquad (109)$$

$$M_1 = -\frac{1}{0.8} \times M_2 = -0.43\frac{wl^2}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (110)$$

Second column,

$$M_5 = -3.5 \times 0.7\left[\frac{12.3}{125} + \frac{2.5}{157}\right]\frac{wl^2}{12} = -0.25\frac{wl^2}{12}. \quad . \quad . \qquad (111)$$

$$M_7 = -\frac{0.8}{0.7} \times 0.25\frac{wl^2}{12} = 0.29\frac{wl^2}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (112)$$

[6] See note. p. 398.

STRUCTURES TWO PANELS WIDE

For structures two panels wide the substitute frame shown in Fig. 168, p. 403, may be used

Let l = length of the left span;
l_1 = length of the right span;
h = height of the lower column;
h_1 = height of the upper column;
I_1, I_2 = moments of inertia of the left and right spans;
I_3, I_5, I_7 = moments of inertia of the lower columns;
c_3, c_5, c_7 = constants depending upon end conditions of lower columns;
I_4, I_6, I_8 = moments of inertia of the upper columns;
c_4, c_6, c_8 = constants depending upon end conditions of upper columns;
A, B, C = frame constants given by Formula (113) below.

Frame Constants.—The frame constants to be used in the formulas for bending moments are

Frame Constants,

$$\left.\begin{aligned} A &= (6 - c_3)\frac{I_3 l_1}{I_1 h} + (6 - c_4)\frac{I_4 l_1}{I_1 h_1} + 4. \quad . \quad . \quad . \\ B &= (6 - c_5)\frac{I_5 l_1}{I_1 h} + (6 - c_6)\frac{I_6 l_1}{I_1 h_1} + 4\left(1 + \frac{I_2 l_1}{I_1 l_2}\right) \\ C &= (6 - c_7)\frac{I_7 l_1}{I_1 h} + (6 - c_8)\frac{I_8 l_1}{I_1 h_1} + 4\frac{I_2 l_1}{I_1 l_2} \end{aligned}\right\}, \quad . \quad . \quad (113)$$

where

$c_3, c_4, c_5, c_6, c_7, c_8$ = 2 for members with fixed ends,

and

= 3 for members with hinged ends.

For partial restraint use an intermediate value for c or compute it by means of Formula (30), p. 640.

Case *a*. Left Span Loaded. (See Fig. 168, p. 403.)—This condition of loading gives (1) maximum positive bending moment in the loaded span; (2) maximum negative bending moment at wall support of loaded span; (3) maximum bending moment in the center column and (4) maximum bending moment in wall column at loaded span.

The bending moments may be found from formulas given below.

Bending Moments in Beam at Supports:

Left span,

$$M_3 = -\frac{1}{12}\frac{(A-4)\left[C(B+2)-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]}{A\left[BC-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]-4C}wl_1^2. \quad . \quad (114)$$

$$M_4 = -\frac{1}{12}\frac{(A+2)\left[C(B-4)-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]}{A\left[BC-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]-4C}wl_1^2. \quad . \quad (115)$$

Right span,

$$M_6 = -\frac{1}{3}\frac{(A+2)\frac{I_2l_1}{I_1l_2}\left(C-\frac{I_2l_1}{I_1l_2}\right)}{A\left[BC-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]-4C}wl_1^2. \quad . \quad . \quad (116)$$

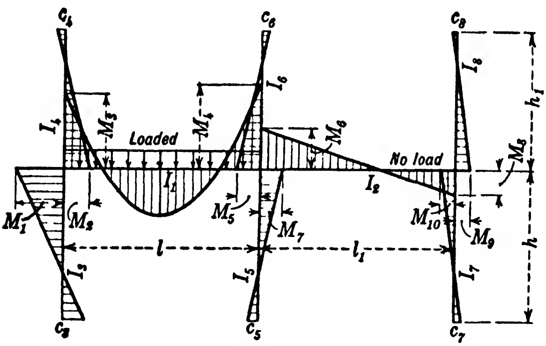


FIG. 168.—Substitute Frame Two Panels Wide. (*See p.* 402.)

$$M_8 = \frac{1}{6}\frac{(A+2)\frac{I_2l_1}{I_1l_2}\left(C-4\frac{I_2l_2}{I_1l_1}\right)}{A\left[BC-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]-4C}wl_1^2. \quad . \quad . \quad . \quad . \quad (117)$$

Bending Moments in Columns:

Left end column,

$$M_1 = -\frac{1}{12}\frac{(6-c_3)\frac{I_3l_1}{I_1h}\left[C(B+2)-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]}{A\left[BC-4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right]-4C}wl_1^2. \; (118)$$

$$M_2 = -\frac{I_4h}{I_3h_1}\frac{6 - c_4}{6 - c_3}M_1. \quad \quad (119)$$

Center column,

$$M_5 = \frac{1}{12}\frac{(6 - c_6)\frac{I_6l_1}{I_1h}C(A + 2)}{A\left[BC - 4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right] - 4C}wl_1^2. \quad \quad (120)$$

$$M_7 = -\frac{I_5h}{I_6h_1}\frac{6 - c_5}{6 - c_6}M_5. \quad \quad (121)$$

Right end column,

$$M_9 = -\frac{1}{6}\frac{(6 - c_8)\frac{I_8l_1}{I_1h_1}(A + 2)\frac{I_2l_1}{I_1l_2}}{A\left[BC - 4\left(\frac{I_2l_1}{I_1l_2}\right)^2\right] - 4C}wl_1^2. \quad \quad (122)$$

$$M_{10} = -\frac{I_7h_1}{I_8h}\frac{6 - c_7}{6 - c_8}M_9. \quad \quad (123)$$

Maximum Positive Bending Moment in Left Span.—For known bending moments at the supports M_3 and M_4 the corresponding maximum positive bending moment may be taken from table on p. 176.

Case *b*. Right Span Loaded.—This condition of loading gives (1) maximum positive bending moment in the right span; (2) maximum negative bending moment in the right span at the wall column; (3) maximum bending moment in the center column; (4) maximum bending moment in the right wall column.

The bending moments are given in the formulas below.

Bending Moments in Beam at Supports:

Left span,

$$M_3 = \frac{1}{6}\frac{\left(C + 2\frac{I_2l_1}{I_1l_2}\right)(A - 4)}{C(AB - 4) - 4A\left(\frac{I_2l_1}{I_1l_2}\right)^2}wl_2^2. \quad \quad (124)$$

$$M_4 = -\frac{1}{3}\frac{\left(C + 2\frac{I_2l_1}{I_1l_2}\right)(A - 1)}{C(AB - 4) - 4A\left(\frac{I_2l_1}{I_1l_2}\right)^2}wl_2^2. \quad \quad (125)$$

Right span,

$$M_6 = -\frac{1}{12}\frac{\left(C + 2\frac{I_2l_1}{I_1l_2}\right)\left[A\left(B - 4\frac{I_2l_1}{I_1l_2}\right) - 4\right]}{C(AB - 4) - 4A\left(\frac{I_2l_1}{I_1l_2}\right)^2}wl_2^2. \quad (126)$$

$$M_8 = -\frac{1}{12}\frac{(C - 4)\left[A\left(B + 2\frac{I_2l_1}{I_1l_2}\right) - 4\right]}{C(AB - 4) - 4A\left(\frac{I_2l_1}{I_1l_2}\right)^2}wl_2^2 \quad . \quad . \quad (127)$$

Bending Moments in Columns:

Left end column,

$$M_1 = \frac{1}{6}\frac{(6 - c_3)\frac{I_3l_1}{I_1h}\left(C + 2\frac{I_2l_1}{I_1l_2}\right)}{C(AB - 4) - 4A\left(\frac{I_2l_1}{I_1l_2}\right)^2}wl_2^2. \quad . \quad . \quad . \quad . \quad (128)$$

$$M_2 = -\frac{I_4h}{I_1h_1}\frac{6 - c_4}{6 - c_3}M_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

Center column,

$$M_7 = \frac{1}{12}\frac{(6 - c_6)\frac{I_6l_1}{I_1h_1}A\left(C + 2\frac{I_2}{I_1}\frac{l_1}{l_2}\right)}{C(AB - 4) - 4A\left(\frac{I_2l_1}{I_1l_2}\right)^2}wl_2^2. \quad . \quad . \quad . \quad (130)$$

$$M_5 = -\frac{I_5h_1}{I_6h}\frac{6 - c_5}{6 - c_6}M_5. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

Right end column,

$$M_9 = \frac{1}{12}\frac{(6 - c_8)\frac{I_8l_1}{I_1h}\left[A\left(B + 2\frac{I_2l_1}{I_1l_2}\right) - 4\right]}{C(A - B) + 4A\left(\frac{I_2l_1}{I_1l_2}\right)^2}wl_2^2. \quad . \quad (132)$$

$$M_{10} = -\frac{I_7h}{I_8h_1}\frac{6 - c_7}{6 - c_8}M_9. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

In end columns the bending moment is negative when it produces tension on outside faces of the columns. In center column negative bending moment produces tension at the left face of the column.

Case *c*. Both Spans Loaded.—This condition of loading should be used for dead load and also for maximum bending moments at the center support. The bending moments are obtained by computing the bending moments due to case *a* and *b* and adding the results.

Maximum Bending Moments in Columns. See p. 377 for method of determining maximum bending moments in columns.

Numerical Example. For numerical example of application of the formulas for a building two panels wide, see p. 416.

ONE-SPAN MULTI-STORY FRAME

Following formulas may be used for structures one span wide and a number of stories high. They are based upon the substitute frame shown in Fig. 169, p. 406.

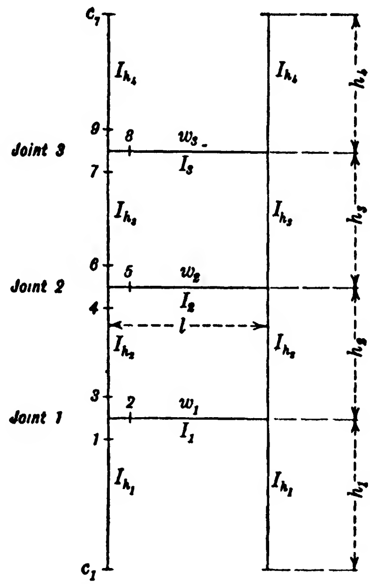


FIG. 169.—One-span, Multi-story Frame. (*See p.* 406.)

Notation.

Let I_1, I_2, I_3 = moments of inertia of beams;

I_{h_1}, I_{h_2}, I_{h_3}, I_{h_4} = moments of inertia of columns;

l = span of beams;

h_1, h_2, h_3, h_4 = height of columns;

c_1, c_4 = constants depending upon end conditions of columns;

w_1, w_2, w_3 = uniformly distributed loading in the three floors;
M_2, M_5, M_8 = bending moments in beams at supports at points 2, 5 and 8 (see Fig. 169);
$M_1, M_3, M_4, M_6, M_7, M_9$ = bending moments in columns at supports (see (Fig. 169);
X, Y, Z = angles of tangents to deflection curves at joints 1, 2 and 3 multiplied by $E\frac{I_1}{l}$
A, B, C = frame constants as given by Formulas (134) to (136).

The frame is symmetrical and the loading is assumed to be symmetrical also. Therefore the bending moments at both sides at symmetrical points are equal.

The constants are

Frame Constants,

$$A = (6 - c_1)\frac{I_{h_1}l}{I_1h_1} + 2 + 4\frac{I_{h_2}l}{I_1h_2}. \quad (134)$$

$$B = 4\frac{I_{h_2}l}{I_1h_2} + 2\frac{I_2}{I_1} + 4\frac{I_{h_3}l}{I_1h_3}. \quad (135)$$

$$C = (6 - c_4)\frac{I_{h_4}l}{I_1h_4} + 2\frac{I_3}{I_1} + 4\frac{I_{h_3}l}{I_1h_3}. \quad (136)$$

Values of X, Y, and Z,

$$Y = \frac{-2C\frac{I_{h_2}l}{I_1h_2}w_1 + ACw_2 - 2A\frac{I_{h_3}l}{I_1h_3}w_3}{ABC - 4C\left(\frac{I_{h_2}l}{I_1h_2}\right)^2 - 4A\left(\frac{I_{h_3}l}{I_1h_3}\right)^2}\frac{l^2}{12}. \quad (137)$$

$$X = \frac{1}{A}\frac{1}{12}w_1l^2 - \frac{2}{A}\frac{I_{h_2}l}{I_1h_2}Y. \quad (138)$$

$$Z = \frac{1}{C}\frac{1}{12}w_3l^2 - \frac{2}{C}\frac{I_{h_3}l}{I_1h_3}Y. \quad (139)$$

To get the bending moments compute the values of X, Y, and Z and substitute in the following formulas

Formulas for Bending Moments,[7]

First joint,

$$M_1 = -(6 - c_1)\frac{I_{h_1}l}{I_1h_1}X \qquad \text{Column 1st Tier} \quad (140)$$

[7] The formulas are developed by the slope-deflection method.

$$M_2 = 2X - \frac{1}{12}w_1l^2 \qquad \text{Beam} \quad \ldots\ldots \quad (141)$$

$$M_3 = 2\frac{I_{h_2}l}{I_1h_2}(2X + Y) \qquad \text{Column 2nd Tier} \quad \ldots \quad (142)$$

Second joint,

$$M_4 = -\ 2\frac{I_{h_2}l}{I_1h_2}(X + 2Y) \qquad \text{Column 2nd Tier} \quad \ldots \quad (143)$$

$$M_5 = 2\frac{I_2}{I_1}Y - \frac{1}{12}w_2l^2 \qquad \text{Beam} \quad \ldots\ldots \quad (144)$$

$$M_6 = 2\frac{I_{h_3}l}{I_1h_3}(2Y + Z) \qquad \text{Column 3rd Tier} \quad \ldots \quad (145)$$

Third joint,

$$M_7 = -\ 2\frac{I_{h_3}l}{I_1h_3}(Y + 2Z) \qquad \text{Column 3rd Tier.} \quad \ldots \quad (146)$$

$$M_8 = 2\frac{I_3}{I_1}Z - \frac{1}{12}w_3l^2 \qquad \text{Beam} \quad \ldots\ldots \quad (147)$$

$$M_9 = (6 - c_4)\frac{I_{h_4}l}{I_1h_4}Z \qquad \text{Column 4th Tier} \quad \ldots \quad (148)$$

The signs of the bending moments in the formulas just given are the conventional signs used in concrete design. These are based upon the position of tension and not upon the direction of rotation.

The formulas for M_1 to M_9 are obtained directly from Formulas (140) to (148). Since the frame is symmetrical and symmetrically loaded, the angles of the tangents to the deflection curves on both sides of each beam are equal and of opposite sign. The values of X, Y, and Z were obtained by making the sum of bending moments at each joint equal to zero. This gave three equations which were sufficient to determine the unknown values of X, Y, and Z. To simplify the formulas the sums of elastic ratios at each joint were represented by A, B, and C, respectively.

Formulas for Maximum Bending Moments.—The formulas given above may be used for dead and live loads by substituting proper values for unit loads w_1, w_2 and w_3.

For dead load all three floors must be co idered as loaded simultaneously.

For live load only such floors should be loaded which give maximum bending moments at the points considered.

The bending moments due to loads placed in different floors are evident from Fig. 170 (*a*) to (*c*).

For numerical example of application of the formulas, see p. 424.

Use of Formulas.—The formulas given above may be used for structures of any number of stories.

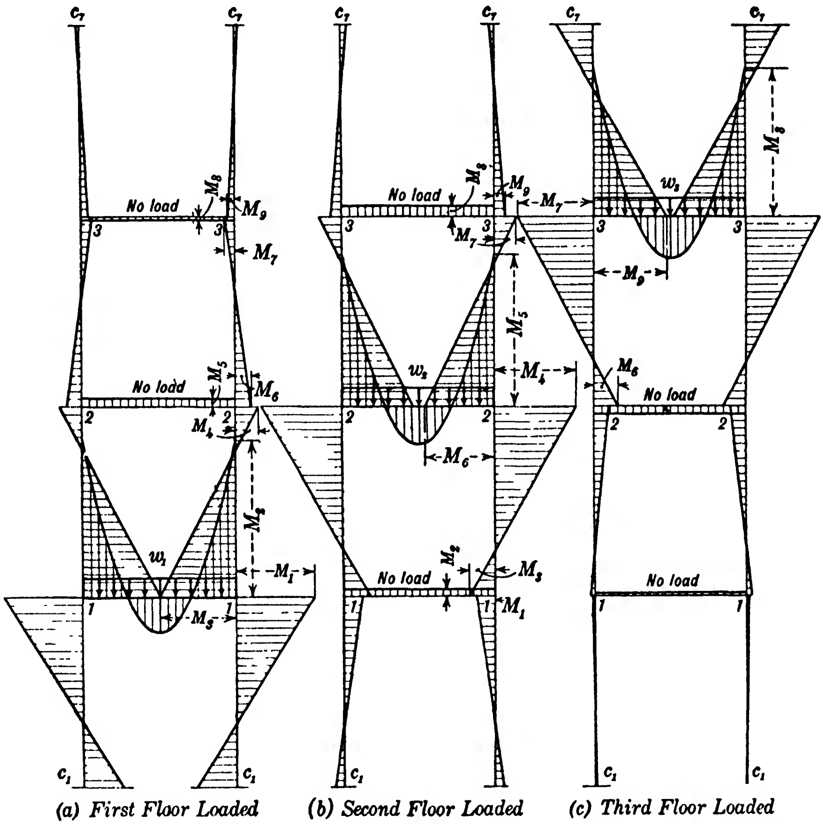


FIG. 170.—Bending Moments in One-span, Multi-floor Frame. (*See p.* 409.)

When used for a frame consisting of a roof and two floors, the top floor of the substitute frame should be considered as the roof. The columns above the top floor should be omitted by making $Ih_4 = 0$. M_9 is then equal to zero and M_7 and M_8 are equal.

When the number of floors in a structure is larger than three the bending moments are found by using the frame as a substitute frame. It may be placed in two or three positions in the structure and in such manner all bending moments in the structure may be computed with sufficient accuracy.

BENDING MOMENTS IN FRAME WITH UNEQUAL SPANS

Example 1.—Find bending moments in beams and columns in a building frame of three unequal spans shown in Fig. 171, p. 410.

The main dimensions of the frame are:

	Span of beams,	26 ft., 12 ft., 26 ft.
	Story heights,	11 ft. 6 in.
Loadings of floors,		
	Dead load,	1 100 lb. per lin. ft.
	Live load,	1 600 lb. per lin. ft.
	Total,	2 700 lb. per lin. ft.

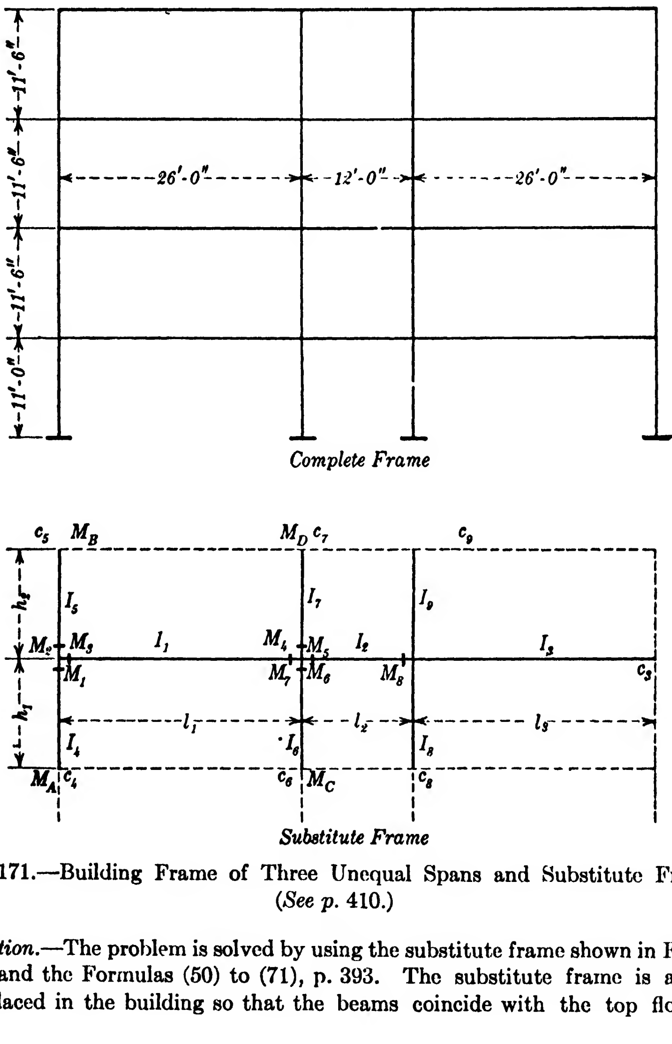


FIG. 171.—Building Frame of Three Unequal Spans and Substitute Frame. (*See p.* 410.)

Solution.—The problem is solved by using the substitute frame shown in Fig. 171, p. 410, and the Formulas (50) to (71), p. 393. The substitute frame is assumed to be placed in the building so that the beams coincide with the top floor. In this example only one position of the substitute frame is considered. For more accurate work, the substitute frame should be placed successively in each story.

The beam is of uniform cross-section throughout, therefore $I_1 = I_2 = I_3$. Since the frame is symmetrical, $I_7 = I_9$ and $I_6 = I_8$.

Ratios of Rigidity.—From preliminary design of the sections based on approximate formulas, the ratios of rigidity are:

$$\frac{I_2 l_1}{I_1 l_2} = \frac{29}{12} = 2.41, \quad \frac{I_3 l_1}{I_1 l_3} = 1, \quad \frac{I_4 l_1}{I_1 h_1} = 1.2, \quad \frac{I_5 l_1}{I_1 h_2} = 1.1, \quad \frac{I_6 l_1}{I_1 h_1} = 0.9, \quad \frac{I_7 l_1}{I_1 h_2} = 0.8.$$

These ratios are found by computing for the beam and for each column the moments of inertia. All moments of inertia must be in the same units. Also the span and the column height must be in the same units. The results are not affected if the moments of inertia are in in.[4] and the values of h and l in feet.

The beam is usually a T-beam and its moment of inertia is found by Formula (457), p. 133. Diagram on p. 134 may also be used. No reinforcement needs to be considered in computing the moments of inertia of the beams or columns.

End Constants c.—The constants c_3 to c_9 are computed by Formula (30), p. 610. The ends of each member is assumed to be restrained by the members of the building frame, outside of the substitute frame, meeting at the respective ends. These are shown in Fig. 171 by dash lines. The ratios of rigidity of the outside members are either computed in the same manner as for the members of the substitute frame, or estimated, using the computed ratios as a guide.

$$c_3 = 2 + \frac{4}{4 + 3.5 \times 1.1 + 3.5 \times 1.2} = 2.33, \qquad 6 - c_3 = 3.67.$$

$$c_4 = 2 + \frac{4}{4 + 3.5 \times 1.1 + 3.5 \times 0.83} = 2.37, \qquad 6 - c_4 = 3.63.$$

$$c_5 = 2 + \frac{4}{4 + 3.5 \times 0.8} = 2.59, \qquad 6 - c_5 = 3.41.$$

$$c_6 = c_8 = 2 + \frac{4}{4 + 3.5 \times 1.1 + 3.5 \times 1.1 + 3.5 \times 2.6} = 2.19, \qquad 6 - c_6 = 3.81.$$

$$c_7 = c_9 = 2 + \frac{4}{4 + 3.5 \times 1.2 + 3.5 \times 2.9} = 2.22, \qquad 6 - c_7 = 3.78.$$

Frame Constants.

$$A = 3.63 \times 1.2 + 3.41 \times 1.1 + 4 = 12.11,$$

$$B = 3.81 \times 0.9 + 3.78 \times 0.8 + 4\left(1 + 2.41\,\frac{19.76 - 2.41}{19.76}\right) = 18.93,$$

$$C = 3.81 \times 0.9 + 3.78 \times 0.8 + 3.67 \times 1 + 4 \times 2.41 = 19.76,$$

$$D = 3.81 \times 0.9 + 3.78 \times 0.8 + 4\left(2.41 + \frac{12.11 - 1}{12.11}\right) = 19.76.$$

Values of wl^2 and $\frac{1}{8}wl^2$.

Dead load, $wl_1^2 = wl_3^2 = 1.1 \times 26^2 = 743.6$ ft.-k., $\quad \frac{1}{8}wl_1^2 = 92.95$ ft.-k.

$wl_2^2 = 1.1 \times 12^2 = 158.4$ ft.-k., $\quad \frac{1}{8}wl_2^2 = 19.8$ ft.-k.

Live load, $wl_1^2 = wl_3^2 = 1.6 \times 26^2 = 1\,081.6$ ft.-k., $\frac{1}{8}wl_1^2 = 135.2$ ft.-k.
$wl_2^2 = 1.6 \times 12^2 = 230.4$ ft.-k., $\frac{1}{8}wl_2^2 = 28.8$ ft.-k.

The bending moments are in foot-kips. 1 ft.-k. = 1000 ft.-lb.

Maximum Negative Bending Moments in Beams for Dead Load (use Formulas (50) to (53), p. 393).

Wall span,

$$M_3 = -\frac{1}{12}\,\frac{8.11\times 20.93}{12.11\times 18.93-4}\times 1.1\times 26^2+\frac{1}{6}\,\frac{8.11\times 24.58}{12.11(19.76\times 19.76-4\times 2.41^2)}\,1.1\times 12^2$$

$$-\frac{2.33}{6}\times 2.41\,\frac{8.11}{4\,450}\times 1.1\times 26^2 = -\frac{1}{12}\times\frac{169.7}{225.2}\times 1.1\times 26^2+\frac{33.2}{4\,450}\times 1.1\times 12^2$$

$$-\frac{7.6}{4\,450}\times 1.1\times 26^2 = -(.063+0.002)\times 743.6+0.0074\times 158.4 = -47.1 \text{ ft.-k.}$$

$$M_4 = -\frac{1}{12}\,\frac{14\ 11\times 14\ 93}{225.2}\times 1.1\times 26^2 - \frac{1}{3}\,\frac{11\ 11\times 24\ 58}{4\,450}\times 1.1\times 12^2$$

$$+\frac{2\ 33}{3}\times 2.41\,\frac{11.11}{4\,450}\times 1.1\times 26^2 = (-0.078+0.0047)\times 743.6$$

$$-0.02\times 158.4 = -57.6 \text{ ft.-k.}$$

Center span,

$$M_6 = -\tfrac{1}{8}\times 2.41\,\frac{14.11\times 17.35}{19.76\times 225.2}\times 1.1\times 26^2-\frac{1}{12}\,\frac{24.58\times 10.12}{19.76\times 19.76-4\times 2.41^2}\times 1.1\times 12^2$$

$$+\frac{2.33}{12}\times 2.41\,\frac{10.12}{367}\times 1.1\times 26^2 = (-0.044+0.013)\times 743.6-0.057\times 158.4$$

$$= -32.1 \text{ ft.-k.}$$

Due to symmetry of the building frame, the same value will be accepted for bending moment M_8 as for M_6.

Bending Moments in Columns for Dead Load.—(Use Formulas (54) to (57), p. 394.)

Wall columns,

$$M_2 = 3.41\times 1.1\times\frac{1}{8.11}\times 47.1 = 21.8 \text{ ft.-k.,}$$

$$M_1 = -\frac{3\ 63}{3\ 41}\times\frac{1\ 2}{1\ 1}\times 21.8 = -25.4 \text{ ft.-k.}$$

Second column,

$$M_5 = \frac{3.78}{12}\times 0.8\left[-\frac{14.11}{225.2}\times 1.1\times 26^2+\frac{24.58}{367}\times 1.1\times 12^2-\frac{2.33\times 2.41}{367}\times 1.1\times 26^2\right]$$

$$= 0.252[-(0.063+0.015)\times 743.6+0.067\times 158.4] = -11.8 \text{ ft.-k.}$$

$$M_7 = \frac{3.81}{3.78}\times\frac{0\ 9}{0.8}\times 11.9 = 13.7 \text{ ft.-k.}$$

Maximum Negative Bending Moments in Beams for Live Load.—Using the proper coefficients found for the dead load, the maximum negative bending moments for live load are:

Wall span,

$$M_3 = -(0.063 + 0.002) \times 1.6 \times 26^2 = -70.0 \text{ ft.-k.},$$

$$M_4 = -0.078 \times 1.6 \times 26^2 - 0.02 \times 1.6 \times 12^2 = -89.0 \text{ ft.-k.}$$

Center span,

$$M_6 = -0.044 \times 1.6 \times 26^2 - 0.057 \times 1.6 \times 12^2 = -60.7 \text{ ft.-k.}$$

Due to symmetry of the building, the same value for M_8 will be accepted as found for M_6.

Maximum Bending Moments in Columns for Live Load.

Wall columns,

$$M_2 = 3.41 \times 1.1 \times \frac{1}{8.11} \times 70.0 = 32.4 \text{ ft.-k.},$$

$$M_1 = -\frac{3.63}{3.41} \times \frac{1.2}{1.1} \times 32.4 = -37.6 \text{ ft.-k.}$$

Second column,

$$M_5 = -0.252(0.063 + 0.015) \times 1.6 \times 26^2 = -21.3 \text{ ft.-k.}$$

$$M_7 = \frac{3.81}{3.78} \times \frac{0.9}{0.8} \times 21.3 = 24.2 \text{ ft.-k.}$$

Total Bending Moments.—The bending moments for dead load should be added to the maximum bending moments for live load.

$$M_1 = -(25.4 + 37.6) = -63.0 \text{ ft.-k.},$$

$$M_2 = 21.8 + 32.4 = 54.2 \text{ ft.-k.}$$

$$M_3 = -(47.1 + 70.0) = -117.1 \text{ ft.-k.},$$

$$M_4 = -(57.6 + 89.0) = -146.6 \text{ ft.-k.},$$

$$M_5 = -(11.8 + 21.3) = -33.1 \text{ ft.-k.},$$

$$M_6 = -(32.1 + 60.7) = -92.8 \text{ ft.-k.},$$

$$M_7 = 13.7 + 24.2 = 37.9 \text{ ft.-k.},$$

$$M_A = -\frac{2(3 - c_4)}{6 - c_4} M_1 = \frac{1.26}{3.63} \times 63.0 = 21.8 \text{ ft.-k.},$$

$$M_B = -\frac{2(3 - c_5)}{6 - c_5} M_2 = -\frac{0.82}{3.41} \times 54.2 = -13.1 \text{ ft.-k.},$$

$$M_C = -\frac{2(3 - c_6)}{6 - c_6} M_7 = -\frac{1.62}{3.81} \times 37.9 = -16.1 \text{ ft.-k.},$$

$$M_D = -\frac{2(3 - c_7)}{6 - c_7} M_5 = \frac{1.56}{3.78} \times 33.1 = 13.7 \text{ ft.-k.}$$

The bending moments in the beams just given may be considered as maximum. To get maximum bending moments in the columns, add to the maximum bending moments for this frame, the bending moments produced by substitute frames placed just above and below the frame under consideration (see p. 377).

Maximum Positive Bending Moments in Beams.—Maximum positive bending moment in the end span is produced when the two end spans are loaded and the center span not loaded. The negative bending moments for such condition are:

$$M_3 = -70 \text{ ft.-k.},$$

$$M_4 = (-0.078 + 0.004) \times 1.6 \times 26^2 = -80.2 \text{ ft.-k.}$$

Combine these negative bending moments with the corresponding negative bending moments for the dead load and draw a bending moment diagram as shown in Fig. 172, p. 415. The maximum positive bending moment, then, may be scaled. Also the maximum positive bending moment may be found, using the table on p. 176.

$$M_3 = -70 - 47.1 = -117.1 \text{ ft.-k.} = -0.0643 \times (1\,081.6 + 743.6),$$

$$M_4 = -80.2 - 57.6 = -137.8 \text{ ft.-k.} = -0.0745 \times (1\,081.6 + 743.6).$$

From the table the coefficient for maximum positive bending moment corresponding to the coefficients 0.0643 and 0.0745 is 0.056. Hence

$$M_{max} = 0.056(1\,081.6 + 743.6) = 102.2 \text{ ft.-k.}$$

For the center span, the maximum positive bending moment is produced when the center span only is loaded. Then $M_6 = M_8 = -0.057 \times 1.6 \times 12^2 = -13.1$ ft.-k. for live load.

For dead load alone the bending moment in the center span at the points of maximum positive bending moment is negative, and it balances partly the positive bending moment due to the live load. Therefore, to get the most unfavorable condition, the live load bending moments should be combined with one-half of the bending moments for the dead load as explained on p. 92,

$$M_6 = -\frac{32.1}{2} - 13.1 = -29.2 \text{ ft.-k.}$$

The static bending moment is $M_s = \dfrac{19.8}{2} + 28.8 = 38.7$ ft.-k. and the maximum positive bending moment in the center is

$$M_{max} = 38.7 - 29.2 = 9.5 \text{ ft.-k.}$$

Maximum Bending Moment Diagrams.—If desired, the maximum bending moment diagrams are drawn as follows:

The diagram drawn to find the maximum positive bending moment also gives the maximum negative bending moments near the wall support.

To get the maximum bending moments at the center support find the negative bending moments for a condition when the first and the second span are loaded and the third not loaded.

$$M_3 = -\ 0.063 \times 1.6 \times 26^2 + 0.0074 \times 1.1 \times 12^2 = -\ 66.6 \text{ ft.-k.},$$

$$M_4 = -\ 89.0 \text{ ft.-k.},$$

$$M \ = -\ 60.7 \text{ ft.-k.},$$

$$M_8 = \frac{1}{6}\,\frac{14.11 \times 10\ 12}{4\ 450} \times 1.6 \times 26^2 - \frac{1}{12}\,\frac{10.12 \times 24.58}{367} \times 1.6 \times 12^2$$

$$= 0.005 \times 1\ 081.6 - 0.056 \times 230.4 = -\ 7.5 \text{ ft.-k.}$$

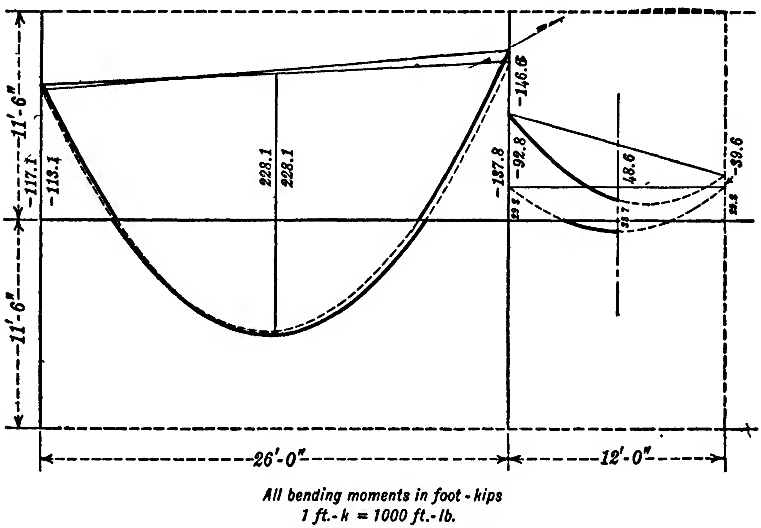


FIG. 172.—Bending Moment Diagrams for Beams. (*See p.* 415.)

Add to these the corresponding bending moments for dead load and plot the diagrams as shown in Fig. 172, p. 415.

$$M_3 = -\ (47.1 + 66.6) = -\ 113.7 \text{ ft.-k.},$$

$$M_4 = -\ (57.6 + 89.0) = -\ 146.6 \text{ ft.-k.},$$

$$M_6 = -\ (32.1 + 60.7) = -\ \ 92.8 \text{ ft.-k.},$$

$$M_8 = -\ (32.1 + 7.5) = -\ 39.6 \text{ ft.-k.}$$

The static bending moments, to be used in drawing the diagrams, are:
End span,

$$M_s = 92.95 + 135.2 = 228.1 \text{ ft.-k.}$$

Center span,

$$M_s = 19.8 + 28.8 = 48.6 \text{ ft.-k.}$$

The diagrams may be used to determine the points of bending of the reinforcement.

BENDING MOMENTS FOR BUILDING FRAME TWO SPANS WIDE.

Example 2.—Find bending moments in beams and columns of a two-span, four-story building frame shown in Fig. 173.

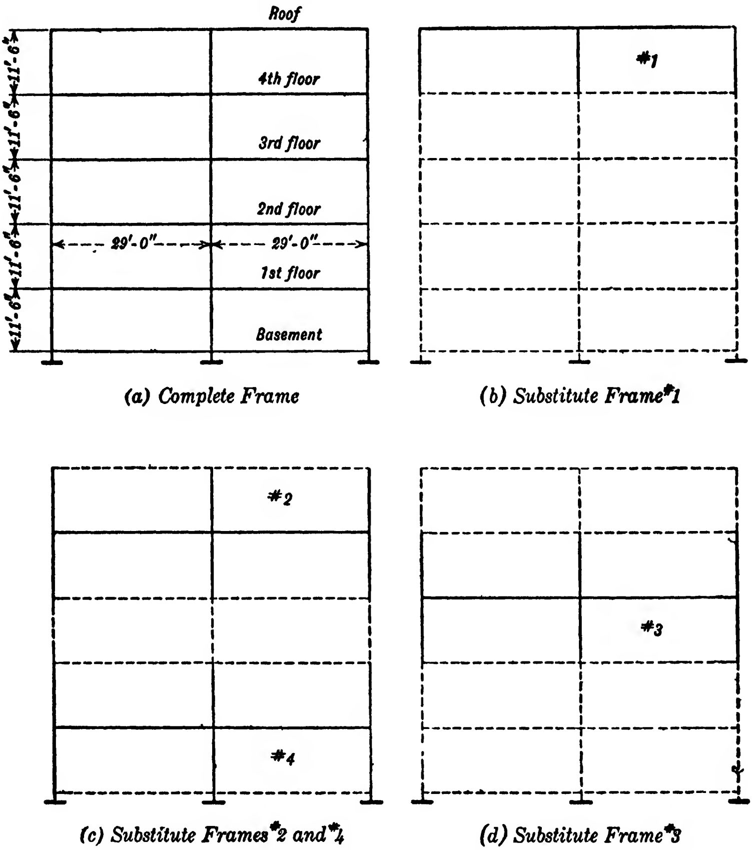


FIG. 173.—Two-span, Four-story Building Frame. (*See p.* 416.)

The main dimensions of the frame are:

Span of a beam, 29 ft.

Story heights, 11 ft. 6 in.

Loadings,

All floors	Dead load,	1 800 lb. per lin. ft.
	Live load,	3 200 lb. per lin. ft.
	Total,	4 000 lb. per lin. ft.
Roof	Dead load,	1 600 lb. per lin. ft.
	Live load,	700 lb. per lin. ft.
	Total,	2 300 lb. per lin. ft.

Solution.—The problem is solved by using two-span substitute frames shown on p. 403, formulas for which are given on p. 402. The substitute frame is placed consecutively in four locations in the building, and bending moments are computed for each position. The location of the substitute frames in the building are shown in Fig. 173, p. 416.

Substitute Frame No. 1.—In this frame the beam is placed at the roof level (see Fig. 174, p. 418). Since both spans are of equal length and the loads are equal, their moments of inertia are equal. The moments of inertia of the wall columns on both sides are equal for the same reason. Since there are no upper columns in the frame, in the formulas the moments of inertia of the upper columns are made equal to zero.

Therefore $\qquad I_1 = I_2, \quad I_3 = I_7, \quad I_4 = I_6 = I_8 = 0.$

The ends of the columns are restrained by the members of the building frame which are not a part of the substitute frame. The values of c_3, c_5 and c_7 are assumed to be equal to 2.4 (see p. 402), and $(6 - c_3) = (6 - c_5) = (6 - c_7) = 3.6$.

For more accurate work compute the values of c as in example 1.

Constants A, B and C (see p. 402).

From a preliminary design it was found that

$$\frac{I_3 l_1}{I_1 h} = \frac{I_7 l_1}{I_1 h} = 0.9 \quad \text{and} \quad \frac{I_5 l_1}{I\ h} = 0.5.$$

Also
$$\frac{I_2 l_1}{I_1 l_2} = 1 \quad \text{and} \quad \frac{I_4 l_1}{I_1 h_1} = \frac{I_6 l_1}{I_1 h_1} = \frac{I_8 l_2}{I_1 h_1} = 0.$$

The constants are, therefore,

$$A = 3.6 \times 0.9 + 4 = 7.24,$$
$$B = 3.6 \times 0.5 + 8 = 9.8,$$
$$C = A = 7.24.$$

Bending Moments, Left Span Loaded. (Use Formulas (114) to (123), p. 403.)

Beam, left span,

$$M_3 = -\frac{1}{12}\,\frac{(7\ 24 - 4)[7\ 24(9\ 8 + 2) - 4]}{7.24(9\ 8 \times 7\ 24 - 4) - 4 \times 7.24}\,wl_1^2 = -\,0.049wl_1^2,$$

$$M_4 = -\frac{1}{12}\,\frac{(7\ 24 + 2)[7\ 24(9\ 8 - 4) - 4]}{454\ 6}\,wl_1^2 = -\,0.065wl_1^2.$$

Beam, right span,

$$M_6 = -\frac{1}{3}\,\frac{(7\ 24 + 2)(7.24 - 1)}{454\ 6}\,wl_1^2 = -\,0.042wl_1^2,$$

$$M_8 = \frac{1}{6}\,\frac{(7\ 24 + 2)(7\ 24 - 4)}{454\ 6}\,wl_1^2 = 0.011wl_1^2.$$

Left column,

Top, $\qquad M_1 = M_3 = -\,0.049wl_1^2.$

Bottom, $\qquad M_B = -\frac{2(3 - c_3)}{6 - c_3}\,M_1 = \frac{1.2}{3.6} \times 0.049wl_1^2 = 0.016wl_1^2.$

Center column,

Top, $M_7 = -(M_4 - M_6) = 0.023wl_1^2.$

Bottom, $M_C = -\frac{1.2}{3.6} \times 0.022wl_1^2 = -0.007wl_1^2.$

Right column,

Top, $M_{10} = M_8 = 0.011wl_1^2.$

Bottom, $M_A' = -\frac{1.2}{3.6}M_{10} = -0.004wl_1^2.$

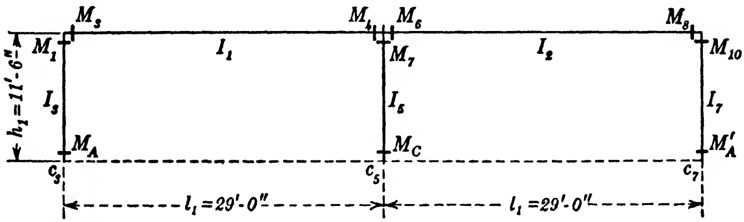


FIG. 174.—Substitute Frame No. 1. (*See p.* 417.)

Due to the symmetry of the frame, the bending moments produced by the uniformly distributed loading on the right span are the same as produced by the loading on the left span, but acting in reverse order. Thus M_1, when right span is loaded, equals M_{10} when left span is loaded.

The bending moment coefficients just computed are tabulated in the following table. In computing final bending moments, both spans are considered as loaded

Substitute Frame No. 1. Bending Moments at Supports

	Bending Moment Coefficients				Bending Moments		
	Span Loaded		Dead Load	Max. Live Load	Dead Load	Live Load	Total
	Left	Right					
					Inch-kips	Inch-kips	Inch-kips
M_1	−0.049	0.011	−0 038	−0 049	− 613.8	−346.2	− 960 0
M_A	0 016	−0 004	0 012	0 016	193 8	113.0	306.8
M_3	−0 049	0 011	−0 038	−0 049	− 613.8	−346.2	− 960 0
M_4	−0 065	−0 042	−0.107	−0 107	−1 728.0	−756 0	−2 484.0
M_7	0.022	−0 022	....	±0 022	....	±155.4	± 155 4
M_C	−0.007	0.007	....	±0 007	....	∓ 49.4	∓ 49.4

for dead load. For live load, the loads are placed so as to produce maximum values at the points considered. Multiply the coefficients by wl_1^2, which is for dead load, $1\,600 \times 29^2 \times 12 = 16\,150\,000$ in.-lb. $= 16150$ in.-k.

$$M_S = \tfrac{1}{8}wl^2 = 2\,020\,000 \text{ in.-lb.} = 2020 \text{ in.-k.}$$

for live load, $700 \times 29^2 \times 12 = 7\,064\,000$ in.-lb. $= 7064$ in.-k.

$$M_S = \tfrac{1}{8}wl^2 = 883\,000 \text{ in.-lb.} = 883 \text{ in.-k.}$$

All bending moments are in inch-kips (1 in.-k. = 1 000 in.-lb.). The bending moments in the right side are same as for the symmetrically placed points on the left side.

Maximum Positive Bending Moment in Beam.—The maximum positive bending moment in the beam occurs when only one span is loaded. To find this bending moment, combine the negative bending moments at the supports for the dead load and for the live load acting on one span.

$$M_3 = -(0.038 \times 16\,150\,000 + 0.049 \times 7\,064\,000) = -960\,200 \text{ in.-lb.}$$

$$M_4 = -(0.107 \times 16\,150\,000 + 0.042 \times 7\,064\,000) = -1\,728\,000 \text{ in.-lb.}$$

For these bending moments draw a bending moment diagram as explained on p. 175 and shown in Fig 175, p. 420. The static bending moment is

$$M_S = \tfrac{1}{8}(16\,150\,000 + 7\,064\,000) = 2\,903\,000 \text{ in.-lb.}$$

This diagram gives not only the maximum positive bending moment, which can be obtained by scaling, but also the points where the bars may be bent up.

For the known negative bending moments, the corresponding maximum positive bending moment may be found also by using table on p. 176.

Diagrams of Maximum Negative Bending Moments.—The points where the bars at the supports may be bent down are obtained by drawing bending moment diagrams for the conditions of loading producing maximum values at the supports.

At the wall column the diagrams previously drawn may be used.

At the center column draw a diagram when both spans are loaded.

The bending moments at supports are

$$M_3 = -(613.800 + 0.038 \times 7\,064\,000) = -882\,300 \text{ in.-lb.},$$

$M_4 = -2\,484\,000$ in.-lb. (*See Table*, p. 418.)

The static bending moment is

$$M_S = 2\,903\,000 \text{ in-lb.}$$

Maximum Bending Moments in Columns.—As explained on p. 377, the maximum bending moments in columns are obtained by combining bending moments produced by two successive frames. In this case the bending moments in the upper columns are obtained by combining bending moments for Substitute Frames, Nos. 1 and 2.

Substitute Frame No. 2.—This frame is shown in Fig. 176, p. 420. Use Formulas (117) to (123), p. 403.

Due to symmetry $I_1 = I_2$, $I_3 = I_7$ and $I_4 = I_8$. The ends of the frame are restrained by one, two and three members. Due to symmetry, $c_3 = c_7$ and $c_4 = c_8$. Using Formula (30), p. 640, the values of c are found as follows:

The upper wall column is restrained by one member. For this member the relative rigidity is the reciprocal for $\frac{I_3}{I_1}\frac{l_1}{h}$ in Frame No. 1. It is $\frac{1}{0.9} = 1.1$. Therefore

$$c_4 = 2 + \frac{4}{4 + 3.5 \times 1.1} = 2.51, \quad 6 - c_4 = 3.49.$$

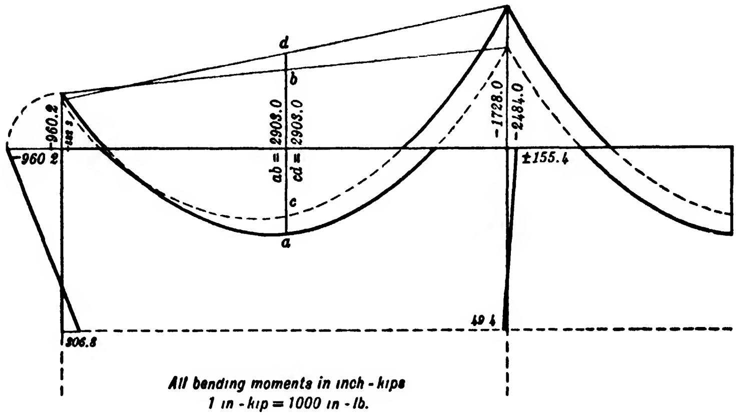


FIG. 175.—Maximum Bending Moments in Roof Beam. (*See p.* 419.)

The upper center column is restrained by two members. The relative rigidity of each of these members and of the column is the reciprocal of $\frac{I_5}{I_1}\frac{l_1}{h}$ in Frame No. 1. It is $\frac{1}{0.5} = 2$. Therefore

$$c_6 = 2 + \frac{4}{4 + 3.5 \times 2 + 3.5 \times 2} = 2.22, \quad 6 - c_6 = 3.78.$$

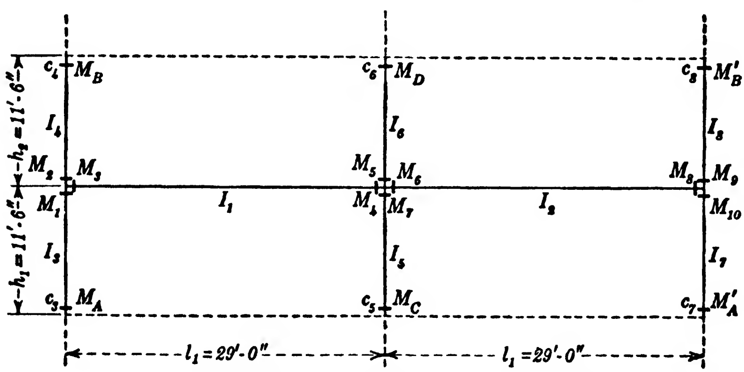


FIG. 176.—Substitute Frame No. 2. (*See p.* 419.)

Similarly are found the ratios for lower columns

$$c_3 = 2.28, \quad 6 - c_3 = 3.62; \quad c_5 = 2.18, \quad 6 - c_5 = 3.82.$$

Constants A, B and C for Frame No. 2. (See p. 402.).

From preliminary figures (here not reproduced) the ratios of rigidity are

$$\frac{I_3}{I_1}\frac{l_1}{h} = 1.0,\quad \frac{I_4}{I_1}\frac{l_1}{h_1} = 0.9,\quad \frac{I_4}{I_3}\frac{h}{h_1} = 0.9,\quad \frac{I_5}{I_1}\frac{l_1}{h} = 0.6,\quad \frac{I_6}{I_1}\frac{l_1}{h_1} = 0.5,\quad \frac{I_6}{I_5}\frac{h}{h_1} = \frac{0.5}{0.6} = 0.83.$$

Due to symmetry, $\frac{I_2}{I_1}\frac{l_1}{l_2} = 1$, and the rigidity ratios of the right wall columns are the same as for the left wall columns. Also $c_8 = c_4 = 2.51$, and $c_9 = c_3 = 2.28$.

The frame constants are:

$$A = 3.72 \times 1.0 + 3.49 \times 0.9 + 4 = 10.86,$$

$$B = 3.82 \times 0.6 + 3.78 \times 0.5 + 8 = 12.18,$$

$$C = A = 10.86.$$

Bending Moments. Left Span Loaded. (Use Formulas (114) to (123)).

In beam, left span,

$$M_3 = -\frac{1}{12}\,\frac{(10\ 86 - 4)[10\ 86(12\ 18 + 2) - 4]}{10.86[10.86 \times 12.18 - 4] - 4 \times 10\ 86}wl_1^2$$

$$= -\frac{1}{12}\,\frac{1\ 029}{1\ 350}wl_1^2 = -0.063wl_1,$$

$$M_4 = -\frac{1}{12}\,\frac{(10\ 86 + 2)[10\ 86(12\ 18 - 4) - 4]}{1\ 350}wl_1^2 = -\ 0.068wl_1^2.$$

In beam, right span,

$$M_6 = -\frac{1}{3}\,\frac{(10\ 86 + 2)(10\ 86 - 1)}{1\ 350}wl_1^2 = -\ 0.032wl_1^2,$$

$$M_8 = \frac{1}{6}\,\frac{(10\ 86 + 2)(10.86 - 4)}{1\ 350}wl_1^2 = 0.011wl_1^2.$$

In left wall column,

$$M_1 = -\frac{1}{12}\,\frac{3\ 72 \times 1\ 0[10\ 86(12\ 18 + 2) - 4]}{1\ 350}wl_1^2 = -\ 0.034wl_1^2,$$

$$M_A = -\frac{2(3 - c_3)}{6 - c_3}M_1 = \frac{1\ 44}{3.62} \times 0.034wl_1^2 = 0.014wl_1^2,$$

$$M_2 = -\ 0.9\,\frac{3\ 49}{3.62}M_1 = 0.029wl_1^2,$$

$$M_B = -\frac{2(3 - c_4)}{6 - c_4}M_2 = -\frac{0.98}{3.49} \times 0.029wl_1^2 = -\ 0.008wl_1^2.$$

In center column,

$$M_7 = \frac{1}{12}\,\frac{3.82 \times 0.6 \times 10.86(10.86 + 2)}{1\,350}wl_1^2 = 0.02wl_1^2,$$

$$M_C = -\frac{2(3 - c_5)}{6 - c_5}M_7 = -\frac{1.64}{3.82} \times 0.02 = -0.009wl_1^2,$$

$$M_5 = -0.83\frac{3.78}{3.82}M_7 = -0.016wl_1^2,$$

$$M_D = -\frac{2(3 - c_6)}{6 - c_6}M_5 = -\frac{1.56}{3.78}M_5 = 0.007wl_1^2.$$

In right wall column,

$$M_9 = -\frac{1}{6}\,\frac{3.49 \times 0.9(10.86 + 2)}{1\,350}wl_1^2 = -0.005wl_1^2,$$

$$M'_B = -\frac{2(3 - c_8)}{6 - c_8}M_9 = \frac{0.98}{3.49} \times 0.005wl_1^2 = 0.0014wl_1^2,$$

$$M_{10} = -\frac{1.0}{0.9} \times \frac{3.62}{3.49}M_9 = 0.0058wl_1^2,$$

$$M'_A = -\frac{2(3 - c_7)}{6 - c_7}M_{10} = -\frac{1.44}{3.62} \times 0.0058wl_1^2 = -0.0023wl_1^2.$$

Due to the symmetry of the frame, the bending moments produced when the right span is loaded are the same as at the symmetrically placed points of the frame when the left span is loaded. Thus M_3, when right span is loaded, equals M_8 when left span is loaded. Similarly $M_4 = M_6$, $M_2 = M_9$ and $M_1 = M_{10}$.

The bending moment coefficients and the final negative bending moments are tabulated in the following table. For the Frame No. 2 the dead load is 1 800 lb. per sq. ft. and the live load 3 200 lb. per sq. ft. Hence for dead load

$$12wl_1^2 = 12 \times 1\,800 \times 29^2 = 18\,200\,000 \text{ in.-lb.},$$

$$M_S = 2\,280\,000 \text{ in-lb.},$$

and for live load

$$12wl_1^2 = 12 \times 3\,200 \times 29^2 = 32\,400\,000 \text{ in.-lb.},$$

$$M_S = 4\,050\,000 \text{ in.-lb.}$$

Static bending moment for dead plus live load is 2 280 000 + 4 050 000 = 6 330 000 in.-lb.

Due to symmetry, bending moments are given only for the left half of the frame. For the right half the maximum bending moments are same as for the symmetrically placed points in the left half.

Maximum Positive Bending Moments in Beam.—The maximum positive bending moments in the beam are found in the same manner as explained on p. 419 in connection with Frame No. 1.

Substitute Frame No. 2. Bending Moments at Supports

	Bending Moment Coefficient				Bending Moments		
	Span Loaded		Dead Load	Max. Live Load	Dead Load	Live Load	Total
	Left	Right					
					Inch-kips	Inch-kips	Inch-kips
M_1	−0 034	0 0058	−0 0282	−0.034	− 513 1	−1 102 0	−1 615 1
M_A	0 014	−0.0023	0 0117	0.014	212 9	453.6	666.5
M_2	0 029	−0 005	0 024	0 029	436 8	939 6	1 376 4
M_B	−0 008	0 0014	−0 0066	−0 008	− 120 1	− 259 2	− 379.3
M_3	−0.063	0 011	−0 052	−0 063	− 946 4	−2 041 0	−2 987 4
M_4	−0 068	−0 032	−0 10	−0 10	−1 820.0	−3 240 0	−5 060 0
M_6	−0 016	0 016	....	∓0 016	....	∓ 518 4	∓ 518 4
M_7	0 02	−0 02	....	±0 02	....	± 648 0	± 648 0
M_C	−0 008	0 008	....	∓0 008	....	∓ 259 2	∓ 259 2
M_D	0 007	−0 007	......	±0.007	....	± 226 8	± 226 8

The negative bending moments are

$$M_3 = -\ 2\,987\,400 \text{ in.-lb.},$$

$$M_4 = -\ (1\,820\,000 + 0.068 \times 32\,400\,000) = 4\,024\,000 \text{ in.-lb.},$$

and the corresponding maximum positive bending moment is

$$M_{\max} = 2\,280\,000 + 4\,050\,000 = 6\,330\,000 \text{ in.-lb.}$$

Maximum Bending Moments in Top Columns.—The top columns of Frame No. 2 are also parts of Frame No. 1 where they form the bottom columns. The bending moments produced by the two conditions are of the same sign. To get maximum bending moments in the columns, the bending moments from Frame No. 1 should be added to the bending moments from Frame No. 2.

The maximum values for the columns in the top story are

Wall column	Top,	$M_B = -\ (379\,300 + 960\,000) = -\ 1\,339\,300$ in.-lb.
	Bottom,	$M_2 = 1\,376\,400 + 306\,800 = 1\,682\,800$ in.-lb.
Center column	Top,	$M_D = \mp\ (259\,200 + 155\,400) = \mp\ 414\,600$ in.-lb.
	Bottom,	$M_7 = \pm\ (648\,000 + 49\,400) = \pm\ 697\,400$ in.-lb.

The bending moments at intermediate points in the column vary according to a straight line. The variation may be obtained by plotting the top and bottom moments at the ends of the columns and connecting them by a straight line. It should be noticed that the bending moments at the ends are of opposite sign and should be plotted on opposite sides of the axis.

The maximum bending moments in the lower columns of Frame No. 2 are obtained by combining the bending moments for Frame No. 2 with bending moments in upper columns of Frame No. 3.

Bending Moment Diagrams.—The bending moment diagrams for Frame No. 2 are shown in Fig. 177, p. 424. They were drawn as explained in connection with Frame No. 1.

Frames No. 3 and 4.—The bending moments for Frames No. 3 and No. 4 may be found in the same manner as illustrated for Frame No. 2.

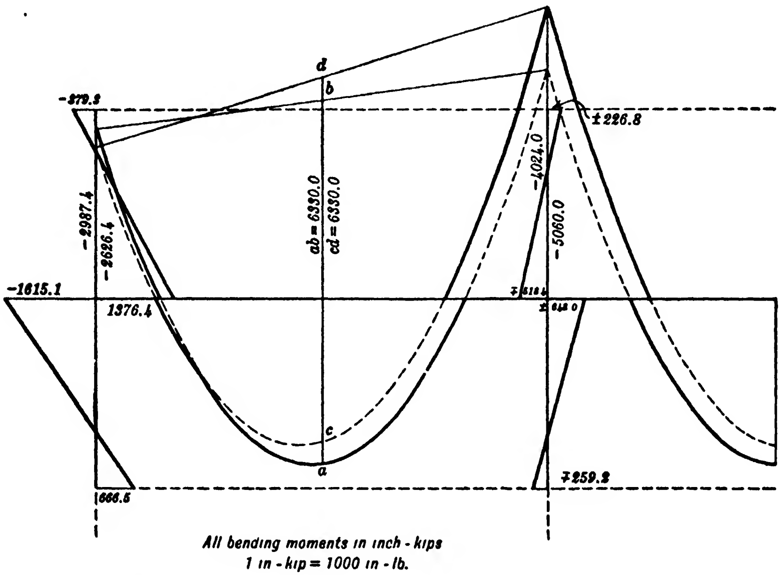


FIG. 177.—Maximum Bending Moments in Beam, Frame No. 2. (*See p.* 424.)

Design of the Building.—After the bending moments are found for each member, the beam and columns are designed as explained in Vol. I. The beams are subjected to thrusts in addition to the bending moments, but the effect of the thrusts are insignificant. For this reason no formulas are given for the computation of the thrusts. After the bending moments in the beams are found, the shears may be computed in the same manner as for continuous beams (see p. 14) and rectangle frame (see p. 355). The beams should be made strong enough to take care of the shears and bending moments. The points of bending of reinforcement may be taken from the bending moment diagram. In this connection reference is made to the example on p. 181 where the complete design of a continuous beam is worked out.

The columns are subjected to the direct pressure due to the vertical load, and also to bending moments. For designing, formulas given in Chapter II should be used.

BENDING MOMENTS IN BUILDING FRAME ONE-SPAN WIDE

Example 3.—Find bending moments in beams and columns of a four-story building frame one-span wide, as shown in Fig. 178, p. 425.

The main dimensions of the frame are:

Span of beams, 32 ft.

Story heights, 11 ft. and 12 ft. (see Fig. 178).

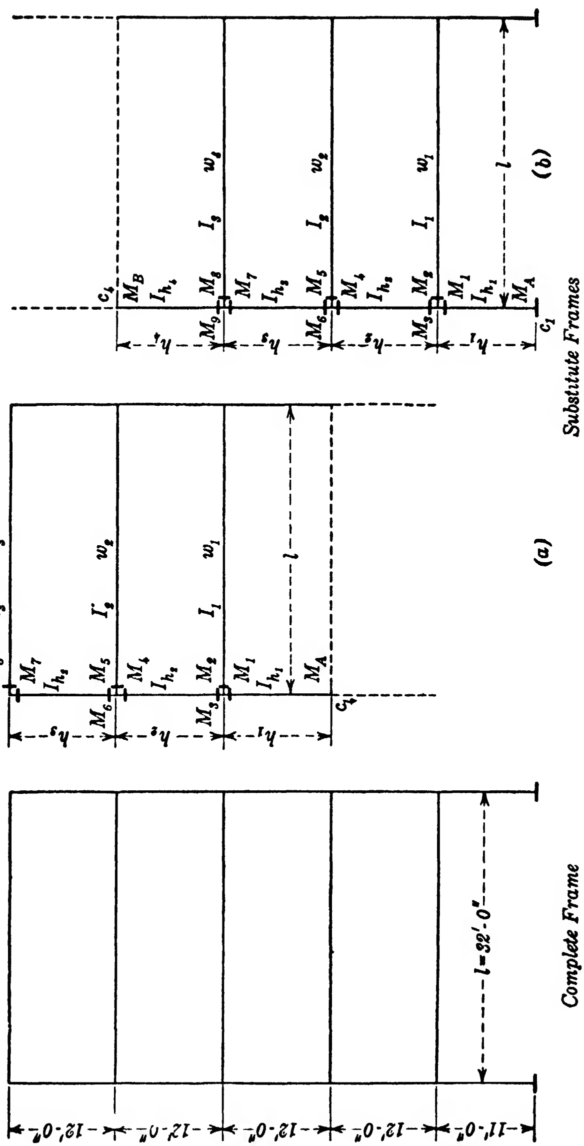


FIG. 178.—Four-story, One-span Building Frame. (See p. 424.)

Loadings,

Roof,	Dead load,	1 800 lb. per lin. ft.
	Live load,	900 lb. per lin. ft.
	Total,	2 700 lb. per lin. ft.
Floors,	Dead load,	2 200 lb. per lin. ft.
	Live load,	4 000 lb. per lin. ft.
	Total,	6 200 lb. per lin. ft.

Solution.—The problem is solved by using the two substitute frames marked (*a*) and (*b*) in Fig. 178, p. 425.

Substitute Frame No. 1.—The bending moments in this frame are found by using Formulas (134) to (148), p. 408.

Detail computations are given for Frame No. 1 only, because the work for Frame No. 2 is identical with that for Frame No. 1.

Since in this frame no columns extend above the floor, $I_{h_4} = 0$.

From preliminary design of sections, based on approximate bending moments, the moments of inertia of the section are

$$I_1 = I_2, \ \frac{I_2}{I_1} = 1, \ I_3 = 0.83I_1, \ \frac{I_3}{I_1} = 0.83, \ \frac{I_{h_1}l}{I_1h_1} = 1.25, \ \frac{I_{h_2}l}{I_1h_2} = 1.2, \ \frac{I_{h_3}l}{I_1h_3} = 1.1, \ \frac{I_{h_4}l}{I_1h_4} = 0.$$

Also from Formula (30), p. 640, the effect of restraint by two members is

$$c_1 = 2 + \frac{4}{4 + 3.5 \times 1.5 + 3.5 \times 1.1} = 2.31, \ 6 = c_1 = 3.69.$$

Frame Constants for Frame No. 1.

$$A = (6 - 2.31) \times 1.25 + 2 + 4 \times 1.2 = 11.4,$$

$$B = 4 \times 1.2 + 2 + 4 \times 1.1 = 11.2,$$

$$C = 0+2\times 0.8+4\times 1.1=6.0.$$

Values of X, Y and Z for Frame No. 1.

(1) First floor loaded, other floors not loaded, $w_2 = w_3 = 0$,

$$Y = \frac{-2 \times 6.0 \times 1.2}{11.4 \times 11.2 \times 6.0 - 4 \times 6.0 \times 1.2^2 - 4 \times 11.4 \times 1.1^2}\frac{w_1l^2}{12} = -0.0214\frac{w_1l^2}{12},$$

$$X = \left(\frac{1}{11.4} + \frac{2}{11.4} \times 1.2 \times 0.0214\right)\frac{w_1l^2}{12} = 0.092\frac{w_1l^2}{12},$$

$$Z = 0 + \frac{2}{6.0} \times 1.1 \times 0.214\frac{w_1l^2}{12} = 0.0079\frac{w_1l^2}{12}.$$

(2) Top floor loaded, $w_1 = 0$, $w_3 = 0$,

$$Y = \frac{11.4 \times 6.0}{676.4}\frac{w_2l^2}{12} = 0.101\frac{w_2l^2}{12},$$

$$X = 0 - \frac{2}{11.4} \times 1.2 \times 0.101\frac{w_2l^2}{12} = -0.0213\frac{w_2l^2}{12},$$

$$Z = 0 - \frac{2}{6.0} \times 1.1 \times 0.101\frac{w_2l^2}{12} = -0.037\frac{w_2l^2}{12}.$$

(3) Roof loaded, $w_1 = 0$, $w_2 = 0$,

$$Y = -\frac{2 \times 11.4 \times 1.1}{676.4}\frac{w_3l^2}{12} = -0.037\frac{w_3l^2}{12},$$

$$X = 0 + \frac{2}{11.4} \times 1.2 \times 0.037\frac{w_3l^2}{12} = 0.0078\frac{w_3l^2}{12},$$

$$Z = \left(\frac{1}{6.0} + \frac{2}{6.0} \times 1.1 \times 0.037\right)\frac{w_3l^2}{12} = 0.181\frac{w_3l^2}{12}.$$

For known values of X, Y and Z all the bending moment coefficients are found using Formulas (140) to (148) as given in the following table.

Bending Moment Coefficients

Bending Moment	Roof Loaded, $w_1=w_2=0$	Top Floor Loaded, $w_1=w_3=0$	Bottom Floor Loaded, $w_2=w_3=0$
$M_8=2\frac{I_3}{I_1}Z-\frac{1}{12}w_3l^2$	$-0.70\frac{w_3l^2}{12}$	$-0.061\frac{w_2l^2}{12}$	$0.013\frac{w_1l^2}{12}$
$M_7=M_8$	$-0.70\frac{w_3l^2}{12}$	$-0.061\frac{w_2l^2}{12}$	$0.013\frac{w_1l^2}{12}$
$M_6=2\frac{I_{h_3}}{I}\frac{l}{h_3}(2Y+Z)$	$0.234\frac{w_3l^2}{12}$	$0.363\frac{w_2l^2}{12}$	$-0.076\frac{w_1l^2}{12}$
$M_5=2\frac{I_2}{I_1}Y-\frac{1}{12}w_2l^2$	$-0.074\frac{w_3l^2}{12}$	$-0.798\frac{w_2l^2}{12}$	$-0.043\frac{w_1l^2}{12}$
$M_4=-2\frac{I_{h_2}l}{I_1h_2}(X+2Y)$	$0.159\frac{w_3l^2}{12}$	$-0.435\frac{w_2l^2}{12}$	$-0.119\frac{w_1l^2}{12}$
$M_3=M_1-M_2$	$-0.052\frac{w_3l^2}{12}$	$0.140\frac{w_2l^2}{12}$	$0.392\frac{w_1l^2}{12}$
$M_2=2X-\frac{1}{12}w_1l^2$	$0.016\frac{w_3l^2}{12}$	$-0.043\frac{w_2l^2}{12}$	$-0.816\frac{w_1l^2}{12}$
$M_1=-(6-c_1)\frac{I_{h_1}}{I_1}\frac{l}{h_1}X$	$-0.036\frac{w_3l^2}{12}$	$0.097\frac{w_2l^2}{12}$	$-0.424\frac{w_1l^2}{12}$

Using the bending moment coefficients given in the table, the bending moments for dead load and live load are computed. For dead load consider all spans as loaded. For live load consider that the load is placed only in the floors which produce at the considered section the bending moments of the desired sign.

Bending Moments for Dead Load

$w_1 = 1.8$ kips, $\frac{1}{12}w_1l^2 = 154.0$ foot-kips

$w_2 = w_3 = 2.2$ kips, $\frac{1}{12}w_2l^2 = \frac{1}{12}w_3l^2 = 188.0$ foot-kips

(1 kip = 1 000 lb.)

Bending Moment	Roof Loaded	Top and Bottom Floor Loaded	Total
			Foot-kips
M_8	$-\ 0.7\frac{w_1l^2}{12} = -108.0$	$(-0.061+0.013)\frac{w_2l^2}{12} = -\ 9.1$	-117.1
M_7	$-\ 0.7\frac{w_1l^2}{12} = -108.0$	$(-0.061+0.013)\frac{w_2l^2}{12} = -\ 9.1$	-117.1
M_6	$0.234\frac{w_1l_2}{12} = \ 36.0$	$(0.363-0.077)\frac{w_2l^2}{12} = \ 53.7$	89.7
M_5	$-0.074\frac{w_1l^2}{12} = -\ 11.4$	$-(0.798+0.043)\frac{w_2l^2}{12} = -158.0$	-169.4
M_4	$0.159\frac{w_1l^2}{12} = \ 24.5$	$(-0.435+0.118)\frac{w_2l_2}{12} = -\ 59.6$	$-\ 35.1$
M_3	$0.052\frac{w_1l^2}{12} = \ 8.0$	$(0.140+0.392)\frac{w_2l^2}{12} = \ 100.0$	108.0
M_2	$0.016\frac{w_1l^2}{12} = \ 2\ 5$	$-(0.043+0.816)\frac{w_2l^2}{12} = -161.0$	-158.5
M_1	$-0\ 036\frac{w_1l^2}{12} = -\ 5\ 6$	$(0\ 097-0.424)\frac{w_2l^2}{12} = -\ 61\ 5$	$-\ 55\ 9$

The bending moment at the end of the bottom column is obtained from formula

$$M_A = -\frac{2(3 - c_1)}{6 - c_1}M_1 = \frac{1\ 38}{3\ 69} \times 202.7 = 76.0 \text{ foot-kips.}$$

Maximum Positive Bending Moments for Beams.—To get maximum positive bending moments in beams, combine the bending moments at the supports due to dead load with bending moments due to live load when the load is placed on the span under consideration and on any other span producing in the span under consideration positive bending moments.

Thus for maximum positive bending moment in the roof, the roof and the bottom floor should be loaded.

$$M_8 = -\ 117.1 - 53.6 + 0.013 \times 342.0 = -\ 166.2 \text{ ft-k.}$$

Since the static bending moment at the roof is $M_S = \frac{1}{8}(1.8 + 0.9)32^2 = 344.0$ ft.-k. the maximum positive bending moment in the roof is

$$M_{\max} = 344.0 - 166.2 = 177.8 \text{ ft.-k.}$$

Maximum Bending Moment for Live Load and Total Bending Moments

$w_1 = 0.9$ kips, $\frac{1}{12}w_1l^2 = 76\ 9$ foot-kips

$w_2 = w_3 = 4.0$ kips, $\frac{1}{12}w_2l^2 = \frac{1}{12}w_3l^2 = 342.0$ foot-kips

(1 kip = 1000 lbs.)

Bending Moment	Roof	Floors	Total Live Load	Total Dead and Live Load
			Foot-kips	Foot-kips
M_8	$-\ 0.7\frac{w_1l^2}{12} = -53.6$	$-0.061\frac{w_2l^2}{12} = -\ 20.8$	− 74.4	−191.5
M_7	$-\ 0.7\frac{w_1l^2}{12} = -53.6$	$-0.061\frac{w_2l^2}{12} = -\ 20.8$	− 74.4	−191.5
M_6	$0.234\frac{w_1l^2}{12} = 18.0$	$0\ 363\frac{w_2l^2}{12} = 124.0$	142.0	231.7
M_5	$-0.074\frac{w_1l^2}{12} = -\ 5.7$	$-0.841\frac{w_2l^2}{12} = -287.0$	−293.7	−463.1
M_4	....	$-0.554\frac{w_2l^2}{12} = -189.0$	−189.0	−224.1
M_3	....	$0.532\frac{w_2l^2}{12} = 181.0$	181 0	289.0
M_2	....	$-0.869\frac{w_2l^2}{12} = -296.0$	−296.0	−454.5
M_1	$-0.036\frac{w_1l^2}{12} = -\ 2\ 8$	$-0.424\frac{w_2l^2}{12} = -144\ 0$	−146 8	−202 7

For the top floor of the frame, place the load in the top floor only and consider the roof and the bottom floor as not loaded.

$$M_5 = -\ 169.4 - 0.798 \times 342.0 = -\ 441.4 \text{ ft.-k.}$$

Since $M_S = 780.0$ ft.-k.

$$M_{\max} = 780.0 - 441.4 = 338.6 \text{ ft.-k}$$

For the bottom floor, load the bottom floor and the roof

$$M_2 = -\ 158.5 + 0.016 \times 154.0 - 0.816 \times 342.0 = -\ 435.0 \text{ ft.k.},$$

consequently

$$M_{\max} = 780.0 - 435.0 = 345.0 \text{ ft.-k.}$$

Maximum Bending Moment Diagrams.—For the computed maximum positive and negative bending moments, bending moment diagrams may be drawn as shown in

Fig. 179, p. 430. These are useful for determining the points of bending of the reinforcement.

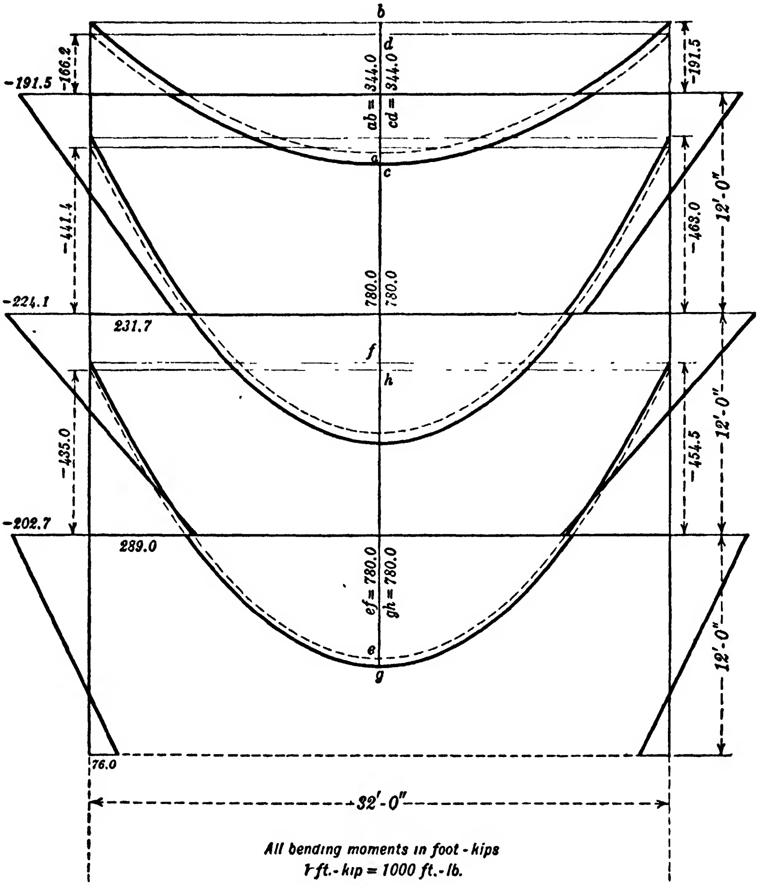


FIG. 179.—Maximum Bending Moment Diagrams. (*See p.* 430.)

CHAPTER V

CONCRETE AND REINFORCED CONCRETE ARCH BRIDGES

THE development of concrete and reinforced concrete increased considerably the use of arches for bridges. An arch bridge is subjected mostly to compressive stresses and since concrete is particularly adapted to resisting compression it is the logical material for this type of construction. With reinforcement to resist any possible tensile stresses a reinforced concrete arch bridge is much superior to stone or brick arch, where reliance must be placed entirely upon resistance in compression and tension must be avoided.

The first concrete arches were of moderate spans such as used in ordinary masonry construction. With better understanding of this type of construction the length of spans increased materially. The most marked increase took place within the last decade. Up to 1915 the Walnut Lane Bridge in Philadelphia with a span of 233 feet was one of the longest concrete arch spans in existence. Since then spans of that length have become comparatively common. In 1927 the longest span built in America was the Cappellen Memorial Bridge in Minneapolis with a clear span of 400 ft. This has been surpassed by an arch erected in France with a span of 433 ft. A bridge is now (1928) in process of construction in France with an arch span of 558 ft.

Advantages of Arch Construction.—Concrete arch construction has following advantages:

1. Permanency. A properly designed and built arch is permanent. Instead of deteriorating it gains in strength with age.

2. Small cost of upkeep. A properly designed and built arch bridge entails practically no expense for upkeep.

3. Aesthetic appearance. An arch bridge lends itself admirably to artistic treatment. It may be fitted into a landscape without destroying any of its natural beauty.

4. Less vibration and less noise. Due to the large mass there is no appreciable vibration in concrete arch bridges. The noise, so common in steel bridges, is entirely eliminated.

Comparative Costs.—The cost of a bridge depends upon a number of factors and local conditions which may affect an arch bridge in a different manner than a steel truss or a concrete girder. Therefore it is not possible to establish any fixed rule showing where any particular type of bridge design is most economical.

In comparing the cost of arch bridges with other types of construction, it is necessary to consider not only the cost of superstructure but also the cost of foundations.

The cost of superstructure for arches with spans longer than 50 ft. is cheaper than that of concrete girder bridges and also cheaper than steel trusses.

The relative economy of foundation depends upon conditions of the ground. Where rock or other hard foundation is not far from the surface of the ground, the cost of foundations for arches may be even lower than for other types of construction. The total cost of the arch bridge therefore will be lower than for a well-designed steel bridge or concrete girder bridge. Where the foundation work is more difficult, the cost of foundations for arches will be larger than for steel bridges, not only because the arch bridge is heavier, but also because the arch foundation must be made unyielding and also provision must be made to resist the horizontal thrust. The extra cost of foundation, therefore, reduces the advantage of the arch bridge and in bad ground may make the cost of an arch bridge higher.

Where difficult foundations are encountered, it may be more economical to consider the use of bow string arches with horizontal ties in which the thrust is resisted by ties. The foundations for such structures are subject only to vertical forces the same as for simple girders and trusses.

Where an unyielding foundation is hard to obtain and fixed arches are not advisable, the use of hinged arches may be considered, the stresses in which are not affected by yielding of foundations.

CONCRETE ARCHES *vs.* STEEL TRUSSES

In comparing the relative advantages of concrete arches and steel trusses, not only the initial cost but also the cost of upkeep must be taken into consideration. A steel bridge must be regularly painted, otherwise it corrodes and in a comparatively short time becomes dangerous to traffic. Where proper maintenance can be counted upon, the yearly cost of this should be estimated and an amount added to the estimated cost of the steel bridge sufficient to yield the required yearly expenditure. In outlying districts or in small municipalities having no efficient maintenance forces, a bridge is likely to receive no attention. In such cases concrete arch bridges should be used even if the initial cost is appreciably larger than for steel bridges because an unattended steel bridge may become useless within a very short time.

GENERAL REQUIREMENTS FOR SUCCESSFUL ARCH BRIDGE

To get a successful structure the following requirements must be fulfilled:

1. The arch bridge must be designed by a competent engineer experienced in designing of arches.

2. Not only the general design but also details, such as expansion joints in spandrel walls in filled spandrel arches and in floor construction of open spandrel arches, must be carefully worked out.

3. Proper water proofing and drainage must be provided. Water must not be allowed to accumulate in pockets or cracks where in freezing it might damage the structure.

4. The structure should be built under the supervision of an experienced engineer and by a reliable contractor who has had experience in building concrete arches.

5. Proper foundation must be provided as required by the plans. If unyielding foundation was anticipated but is not obtainable, the design of the arch might have to be changed to take care of a possible yielding of foundations. Foundation in running water must be properly protected against underscoring.

6. Concrete of first class quality and of required strength must be used. Particular pains should be taken to produce dense concrete, as porous concrete, irrespective of strength, is in danger of deterioration under the influence of weather and frost.

7. Method of placing concrete as well as the location of construction joints must be carefully worked out by the engineer in charge and not allowed to be determined in the field.

CLASSIFICATIONS OF ARCH BRIDGES

The arch bridges may be classified according to their method of design and also according to the type of construction. The method of design depends upon the number of hinges. The classification then may be:

Three-hinged Arches,
Two-hinged Arches,
One-hinged Arches (hinge at the crown) and
Fixed (or hingeless) Arches.

The difference between these types is not only in construction as required by the presence or absence of hinges but also in methods of design. Separate formulas are given for two-hinged and fixed arches.

According to their construction the arch bridges may be divided into

Filled Spandrel Arches,
Open Spandrel Arches.

Filled Spandrel Arches.—The oldest type of concrete arch bridges is the filled spandrel type illustrated in Fig. 180, p. 435. In this type of construction the space between the extrados of the arch and the roadway is filled with earth. This fill, after it is properly tamped and rolled, supports the roadway. The fill is retained on both sides by spandrel walls. The spandrel walls perform the same duty as an ordinary retaining wall. They may be built of any kind of masonry.

Spandrel Walls.—Spandrel walls are the walls placed at the sides of the arch bridge and extending from the top of the arches up to the level of the roadway. Usually they are built separately from the arch and are not considered as a part of the load-carrying structure. A coping and balustrade or railing may be placed on the top of the spandrel wall.

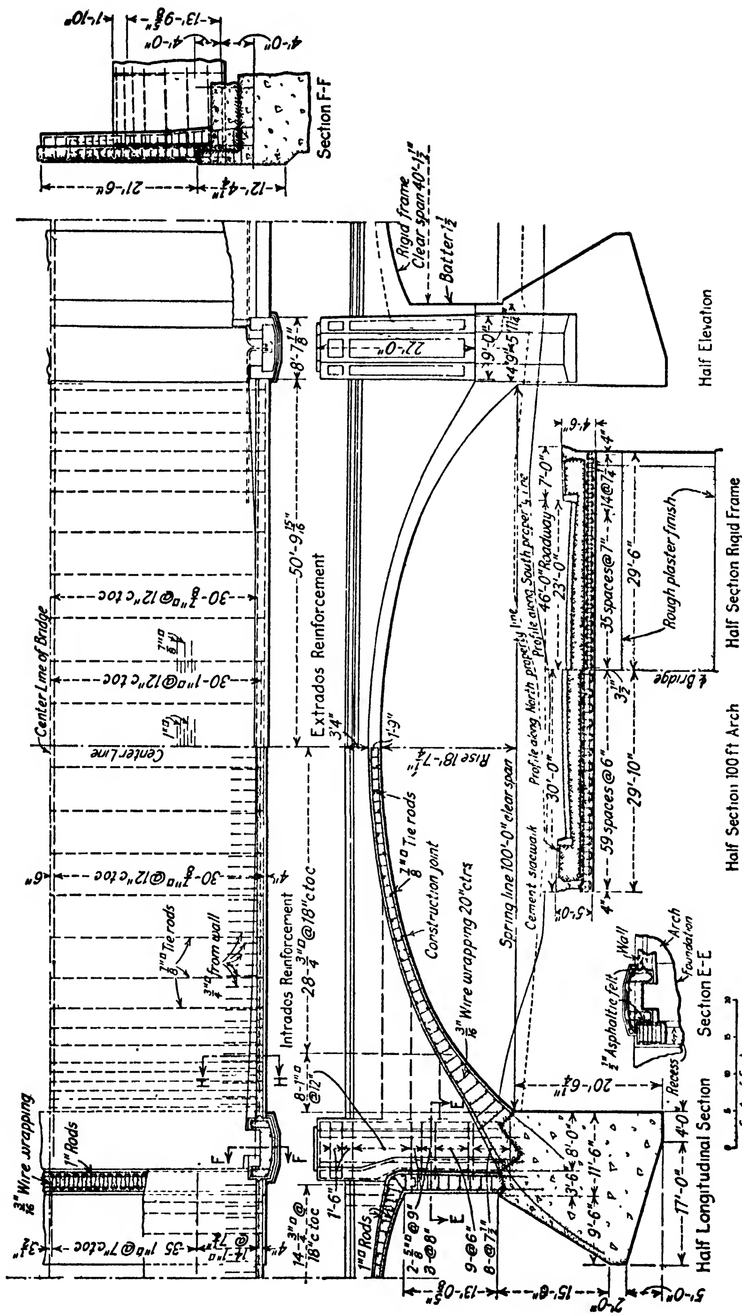
The main purpose of the spandrel wall is to retain the filling material, and its structural action is similar to that of a retaining wall. The force to be resisted is the earth pressure of the fill increased by the side pressure caused by the live load. The effect of live load may be replaced by a proper surcharge in the manner described in connection with retaining walls.[1]

Ordinarily in concrete construction the spandrel walls are made of concrete, plain or reinforced. When desirable, as for instance to give the arch the appearance of a masonry arch, the walls may be made of stone, brick or any other type of masonry.

Masonry or plain concrete spandrel wall must be proportioned in the same manner as a gravity type of retaining wall. Proper provision must be made against sliding and overturning. As the thickness of the wall at the bottom is much larger than required at the top, a reduction of the thickness may be obtained by stepping or sloping of the back surface.

A reinforced concrete wall may consist of a simple upright slab as shown in Fig. 181, p. 436, designed as the upright slab of a T-shape retaining wall, the arch itself serving as the base. The slab is then a cantilever fixed at the top of the arch and loaded by the horizontal earth pressures with surcharge. The maximum bending moment acts at the bottom of the spandrel, i.e., where the spandrel joins the arch. The slab may be made of uniform thickness. In arches with large rise,

[1] For design of retaining walls, see Vol. I, p. 841.

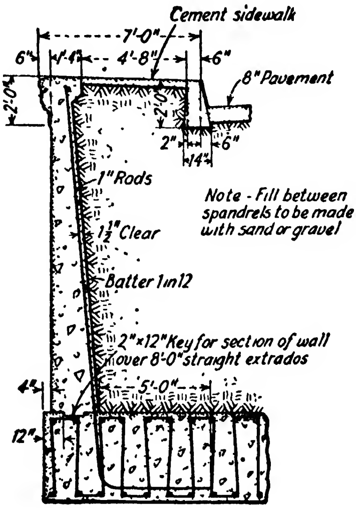


For section *H-H* see Fig. 181, p. 436.

Avenue Sixty Bridge, Los Angeles, California. Merrill Butler, Engineer of Bridges.

FIG. 180.—Filled Spandrel Arch. (*See p.* 434.)

however, the inside face of the walls is made slanting. The main reinforcement is upright and is placed near the inside face of the wall and parallel to it. It receives its maximum stress at the bottom of the wall, therefore the wall reinforcement must be anchored in the arch a sufficient distance to develop the strength of the bar by bond (see p. 267, Vol. I). It is apt to be difficult to keep the long slender wall bars in position during the concreting of the arch. Therefore, instead of anchoring the bars in the arch, separate dowels are used of a length equal to double the distance required for anchorage. Half of each bar is imbedded in the arch and the other half serves as a lap for the reinforcement in the wall.



For position of section *H-H*, see Fig. 180, p. 435.

FIG. 181.—Reinforced Concrete Spandrel Wall. (*See p.* 434.)

To prevent sliding of the wall, a recess should be provided in the arch sufficient to take the shear.

Horizontal temperature reinforcement should be used to prevent cracks.

When the wall is of considerable depth, its cost may be decreased by using buttresses as shown in Fig. 182, p. 437. The wall then acts as a slab supported by the buttresses and loaded by the earth pressure. Usually it should be treated as a continuous slab. The main reinforcement in the wall is horizontal and is placed near the outside face between buttresses and near the inside face at the buttresses. The amount of the steel should be proportioned according to the earth pressure. The largest amount of steel is required at the bottom of the arch and may be decreased with the decrease of the intensity of pressure.

The buttresses must resist the pressure transferred to them by the wall. They act as cantilevers supported on the top of arch and resist the pressures equal to the unit pressures multiplied by the spacing of the buttresses. The main reinforcement is placed near their inside faces and must be anchored in the arch either by extending it into the arch a sufficient distance or by means of dowels in the same manner as explained in connection with upright walls. Recesses capable of resisting the shear must be provided in the arch.

Shearing stresses must be computed in the buttress to determine whether diagonal tension reinforcement is required. The wall must be anchored to the buttresses by means of horizontal stirrups extending from the buttress into the wall a sufficient length to develop the bar by bond.

Buttressed walls are particularly advantageous when it is desirable to cantilever out the sidewalk beyond the face of the wall as shown in Fig. 182, p. 437. The sidewalk then is supported on brackets built from every buttress. The brackets act as cantilevers having a maximum bending moment at the face of the wall. The main reinforcement is placed horizontally near the top and is anchored in the buttress. Also diagonal tension reinforcement may be required. The sidewalk slab is considered as a slab carried by the brackets.

The buttress for this case is subjected to horizontal forces in the same manner as explained before. In addition it must resist the bending moment transferred to it by the sidewalk bracket. This bending moment should be computed about the center of the buttress section on the top. It is therefore equal to the bending moment for which the bracket is designed plus the total load on the bracket multiplied by one-half the depth of the buttress section on the top. This bending moment acts at all sections of the buttress. Particular care should be used in designing the joint between the buttress and the bracket so as to make it capable of resisting the shear and bending moment.

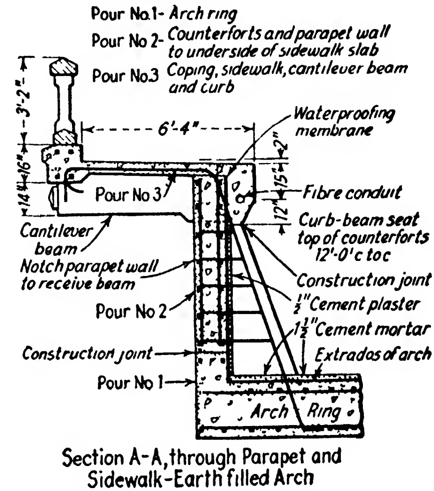


FIG. 182.—Spandrel Wall with Buttresses. (See p. 436.)

Cantilever Sidewalk for Plain Concrete Spandrel Walls.—Sometimes it is desirable to extend the sidewalk beyond the outside edge of the spandrel wall when spandrel of plain section is used. The construction then will depend upon the length of the projection. Small projection may be made without any special provision relying on concrete to resist shear and bending moment. Of course the bending stresses produced by the projection must not exceed the allowable working stresses for plain concrete in tension.

Larger projections require brackets properly proportioned and reinforced. The tension reinforcement from the brackets must extend into the wall. It is not sufficient to extend the bars sufficiently to develop them by bond. Such construction would take care of the bracket but would not provide for the bending moment transferred by the bracket to the wall. Usually it is necessary to extend the bars the whole height of the wall and anchor it in the arch because the bending moment from the bracket acts with equal force on all sections. The reinforcement could be stopped only in case of a very heavy wall when its weight is sufficient to keep the eccentricity caused by the bending moments well within the middle third of the wall.

Contraction Joints in Spandrel Walls.—Ordinarily the spandrel wall is not considered a part of the carrying construction, therefore it should be built so as to enable the arch to deflect independently of the spandrel walls. For this purpose a number of vertical contraction joints are provided in the spandrel wall, which not only enable the arch to deform but also provide for any movement of the wall due to the temperature changes.

Filling Material.—Any available material may be used for filling between the spandrel walls. It is important, however, to have the fill properly tamped, otherwise it may settle, and cause damage to the roadway.

In flat arches when it is desirable to increase the thrust due to the dead load in order to reduce the tensile stresses due to the live load and changes of temperature, material with high specific gravity such as very lean concrete may be used for fill. It is placed on top of the arch proper and is not considered as a part of the arch ring. This method is used in Europe but seldom in the United States where tensile stresses are provided for by proper reinforcement.

Fill for Arches on Grade.—When the roadway of the arch is on a grade but the springing lines are placed on a level, the depth of fill on one side of the crown is much larger than on the other side. When filled with same kind of material on both sides the dead load is unsymmetrical and the line of pressure for the dead load is an unsymmetrical curve. It is obvious, therefore, that in such case it is impossible to make the arch axis coincide with the line of pressure for dead load and consequently some bending moments will be developed by the dead load.

To remedy this, attempts are often made to equalize the load on both sides. This can be done by using on the side with smaller depths filling material of larger specific gravity than on the other side. Where this is not possible the load on the side with larger depths may be reduced by making part of the spandrel space hollow and support the fill above the extrados by small arches or concrete slabs.

Often the expense of the equalizing of the loads would be much larger than the extra cost of providing for the bending moments due to the dead load. This is particularly true in America.

Waterproofing and Drainage.—The extrados of the arch and the sides of the spandrel walls must be properly waterproofed. The waterproofing membrane must be applied on the top of the concrete and properly protected before the spandrel is filled. All rain water must be promptly carried off by proper drainage.

OPEN SPANDREL ARCHES

In recent years the open spandrel arch construction became most favored for arch bridges, especially arches with large ratio of rise to span and for spans over 100 ft. In open spandrel arch bridges the fill above the arch ribs is omitted and the construction consists of (*a*) arch ribs, (*b*) a system of vertical supports above the arch ribs, (*c*) a horizontal floor construction carrying the roadway and supported by the vertical supports.

The economy of open spandrel arch construction is due to following reasons:

1. The dead load is reduced by omitting the fill so that the arches and the foundation may be made lighter.

2. The arches do not need to be made the full width of the bridge. The barrel type arch rib may be replaced by two or more independent narrow ribs.

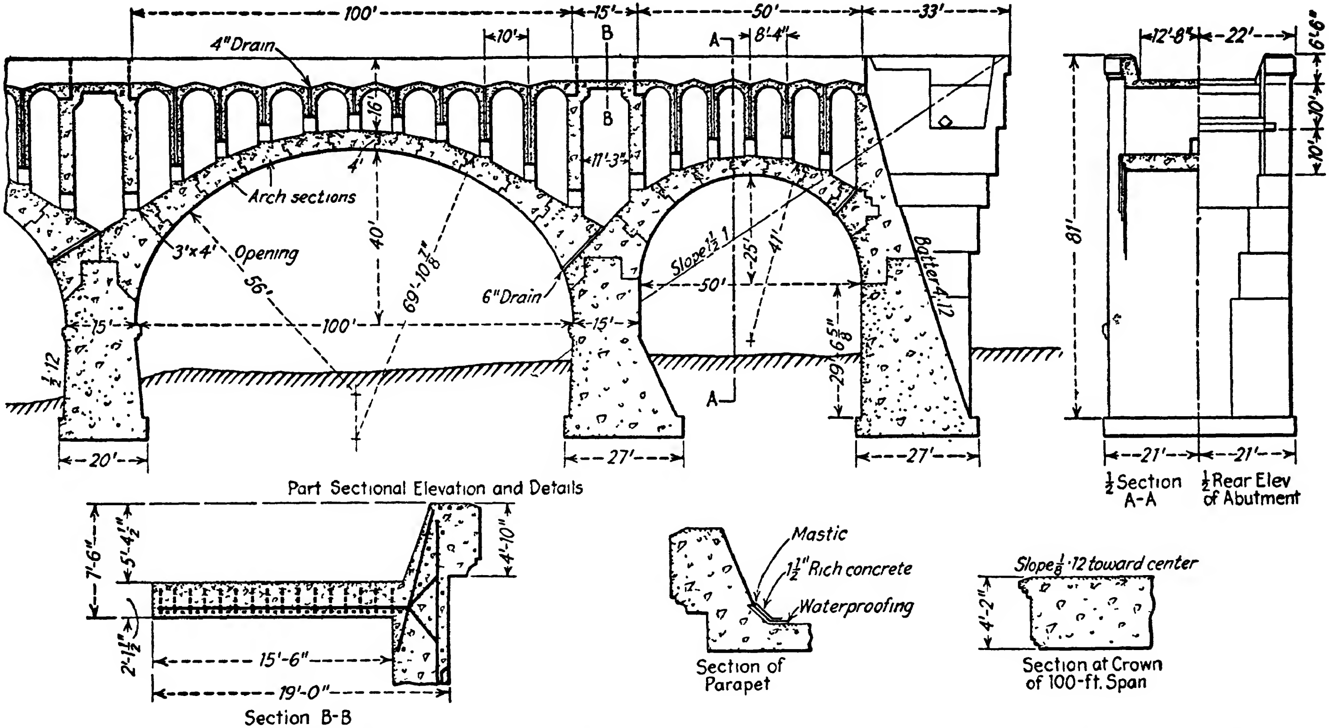
3. The independent ribs may be made deeper than is possible with barrel arches, thus reducing the effect of the bending moments and reducing the tensile stresses.

4. The ribs may be made of rich concrete properly reinforced, with consequent reduction in cost.

By the introduction of open spandrel arches the usefulness of this arch construction has been considerably increased.

Arch Ribs.—The arch rib may be a barrel rib extending the full width of the bridge same as in filled spandrel arches. This is shown in Fig. 183, p. 440. The width of the rib, however, may be reduced if desired by cantilevering the sidewalks beyond the faces of the ribs as shown in Fig. 184, p. 441.

The cost of construction may be reduced by using instead of one barrel rib two separate ribs as shown in Fig. 185, p. 442. The combined width of the two ribs is smaller than the width that would have been

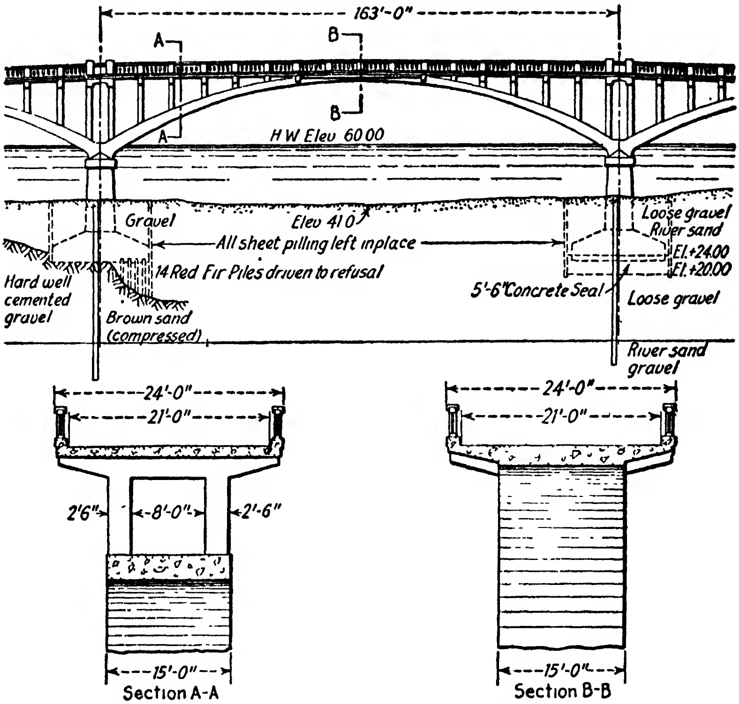


Note: Longitudinal reinforcement 1 in. sq. bars 12 in on centers top and bottom in large span and 18 in. on centers in small span. Cross bars 3/4 in. sq. bars 4 ft. on centers top and bottom. For description see *Engineering News-Record*, Feb. 21, 1928, p. 342.

FIG. 183.—Railroad Arch Bridge, Chicago and Eastern Illinois R. R. (*See p.* 429.)

required for full barrel arches. In very wide bridges three separate ribs may be used, a wide one in the center and two narrower ones at the sides.

In the above cases the width of the arch rib is large in comparison with its depth. Each rib is stable by itself so that no lateral bracing is required.

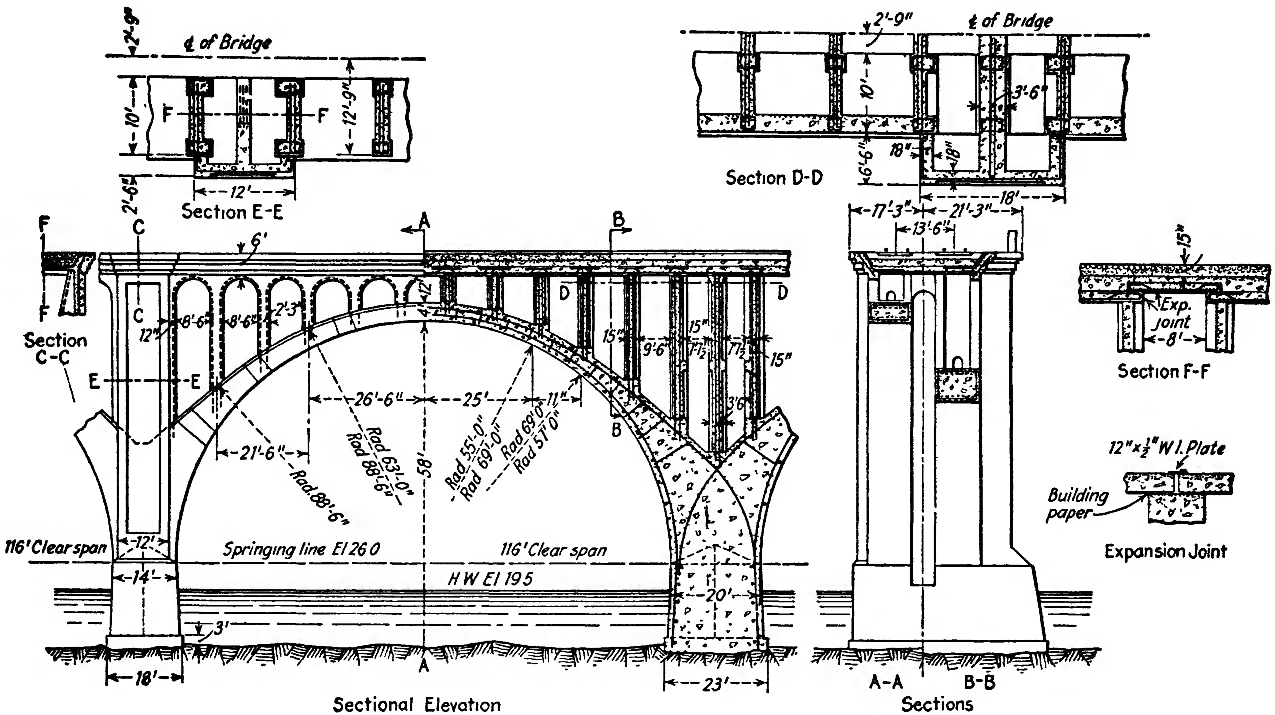


R. M. Morton, State Highway Engineer. Harlan D. Miller, Bridge Engineer.

FIG. 184.—Bridge across Van Duren River, near Alton, California. (*See p.* 439.)

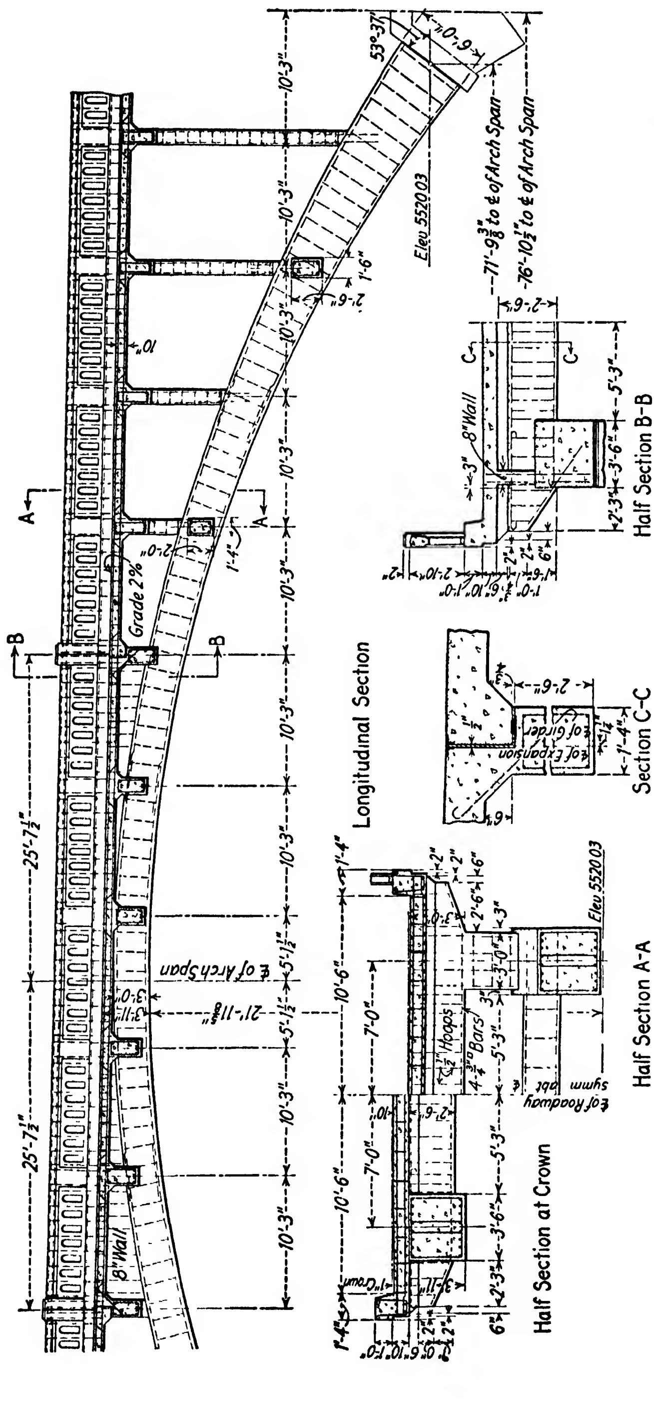
The cost of construction may be farther reduced by using two or more narrow ribs as illustrated in Figs. 186 and 188. This type lends itself very well to cases where the bridge is designed for combined traffic such as vehicular traffic and railroad traffic. Separate arch ribs are used under the railroad tracks for vehicular traffic and sidewalks respectively. Each set of ribs is proportioned for the loads coming upon them. No question is then involved as to the distribution of the loads over the ribs.

Since the ribs are narrow, they require lateral bracing not only to increase the unsupported length of the ribs but also to resist wind stresses.



For description see *Engineering News-Record*, July 12, 1917, p. 73. J. E. Greiner & Co., Engineers.

FIG. 185.—James River Bridge, Richmond, Va. (*See p.* 439.)



Harlan D. Miller, Bridge Engineer.

FIG. 186.—Bridge across Truckee River, California. (See p. 441.)

A modification of this type of construction is the construction with suspended roadway, treated on p. 446; also the construction in which the ribs extend to the roadway as discussed on p. 445.

Vertical Supports.—The load from the floor construction is transmitted to the arch ribs by means of vertical supports. It is well to remember that these vertical supports not only must carry the load with a proper factor of safety but also must transmit properly and uniformly the load to the arch ribs. The type and arrangement of the vertical supports will depend upon the arrangement and type of arch ribs.

For barrel arches or for wide arch ribs, the most effective type of vertical supports from the standpoint of distributing the load on arch

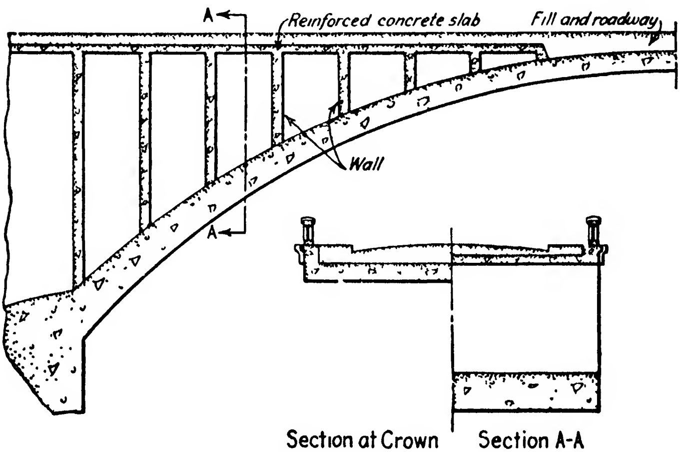


Fig. 187.—Typical Arch with Open Spandrel. (*See p.* 445.)

ribs consists of cross walls, as shown in Figs. 183 and 187. In Fig. 183 the cross wall is provided with an opening in the center which reduces the amount of material in the wall and also its weight. This opening gives convenient access to the top of the arch rib for inspection. The material in the cross walls is not properly utilized because its thickness is governed not by strength but by lateral stiffness and also by construction reasons. The compression stresses are therefore small.

To reduce the cost of vertical supports, the cross walls may be replaced by independent columns of proper strength to carry the vertical load. They must be placed so as to distribute properly the load over the arch ribs. A distributing block of concrete should be placed under the column. When one row of columns is used per rib, it should be placed in the center of the rib. When two rows of columns are used per rib, they should be placed so that the center of gravity of the loads on the two columns will coincide with the center of the arch rib.

To distribute the load over the arch rib, reinforcement should be used on the top and bottom so as to take any tensile stresses in case of cross bending of the ribs. Also stiffening cross ribs are often used over the rib between the columns.

The columns are designed according to formulas given in the chapter on columns in Vol. I. Proper proportions must be maintained between the height and the least dimension of the columns. Slender columns must be properly braced. Any horizontal traction force must be provided for by making the columns able to resist bending.

As the columns and walls are poured separately from the arch rib, dowels should be provided in the arches of same number and size as used for column or wall reinforcement. A proper seat also should be provided in the arch rib with a horizontal bed to receive the column or wall.

Longitudinal Walls.—Vertical supports may also consist of a series of longitudinal walls extending full length of the arch. This construction is but rarely used in America, as it has no advantages over the cross-wall type of construction. It has merits only when it is desirable to give the arch bridge an outside appearance of a filled spandrel arch or where the longitudinal walls form a part of the arch.

Floor Construction.—The floor construction may rest upon vertical supports above the arch rib the whole length of the bridge as shown in Fig. 183 or the roadway in the central part of the span may rest directly upon the arch rib with vertical supports on both ends of each span (see Fig. 187, p. 444).

The type of floor construction depends upon the type of vertical supports. Where vertical supports consist of cross walls the floor construction usually consists of:

1. Arches spanning between cross walls.
2. Slabs spanning between cross walls.

Since sliding joints are required in the floor to take care of temperature changes, the floor arches can seldom be considered as actual arches and for this reason they are treated as curved beams and provided with reinforcement. When slab construction is used, the spacing of the walls should be made small enough to permit the use of a slab thickness not larger than 8 in.

The main reinforcement consists of bars running longitudinally with the bridge. When the slabs are continuous, proper negative reinforcement must be provided over the walls. To prevent longitudinal cracks cross bars are also used.

When the vertical support consists of independent columns, the floor construction may consist of (1) cross beams running between columns and slab spanned between them, (2) stringers running longitudinally, supported by cross beams, and a slab running between stringers, the cross beams often being cantilevered out to support the sidewalk, and (3) flat slab construction where beams and girder are omitted and, instead, a massive slab is supported on columns with enlarged heads.

ARCH RIBS EXTENDING ABOVE ROADWAY

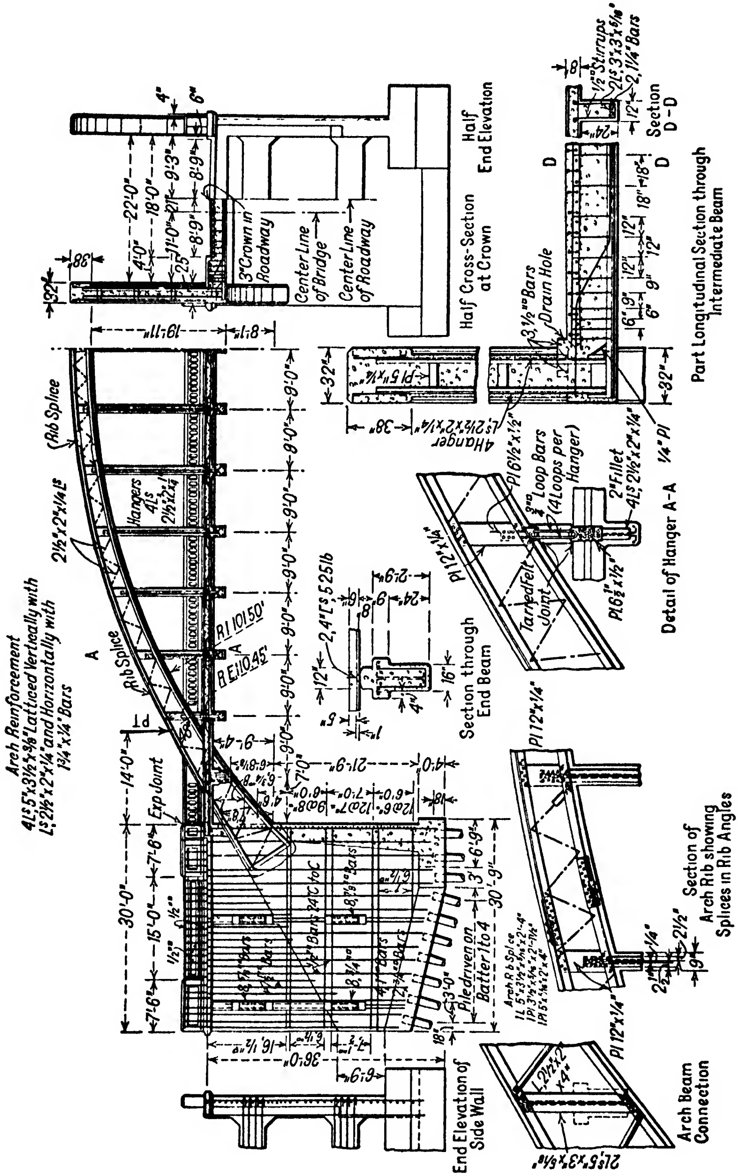
When the clear headroom under the bridge is not sufficient for an economical arch, it is possible to place the arch ribs partially or wholly above the roadway. Such construction is shown in Figs. 188 and 201.

Fixed Arch Ribs Extending above Roadway.—In Fig. 188 the arch consists of two ribs placed on both sides of the roadway with the roadway suspended from the arch ribs. In this construction the ribs are rigidly attached to the abutment so that the arches may be considered as fixed arches. The horizontal thrust is resisted by the abutments.

The arch is located below the roadway at the supports and above the roadway at the crown. The floor construction consists of floor beams running from arch rib to arch rib, longitudinal beams extending between the floor beams, and slab carried by the longitudinal beams. The floor beams in parts of the construction are suspended from the arches while next to the abutment they are supported on posts.

Bow-String Bridges.—When it is not desirable to transfer the horizontal thrust to the abutment the arch rib may be connected by horizontal ties as shown in Fig. 201. The ties are made strong enough to resist full horizontal thrust. To make sure that no horizontal pressure will be exerted against the abutment, it is advisable to rest one end of the bridge upon a sliding bearing. When the arch is of considerable size it is advisable to support it in the same manner as is customary in steel trusses—namely, one end is supported on a steel shoe with a pin bearing while the other end is placed upon roller bearings. Such a bridge is able to move longitudinally with the movement due to temperature, without exerting any pressure on the abutment except that required to overcome friction.

In Europe the expansion is often taken care of by means of concrete rocker bearings. These are usually cast of rich concrete and are heavily reinforced with spirals. The rockers are rectangular in shape and their upper and lower faces are segmental. For a description of a



For description see *Engineering News-Record*, Dec. 11, 1918, p. 994.

FIG. 188.—Fixed Arch with Suspended Roadway. (*See p.* 446.)

rocker used for a $286\frac{1}{2}$-ft. bow-string arch see Engineering News-Record, Aug. 18, 1927, p. 273.

From the standpoint of design, bow-string arches are considered as two-hinged arches with a horizontal tie.

The ties may consist of bars of proper number and size properly anchored at the supports. These may be spliced by laps of proper length. More positive results are obtained by using tie rods of large diameter with upset ends connected with turnbuckles. Proper care should be used to transfer the stress in the ties at the ends to the concrete of the arch. Anchor plates are often required, specially when the ties consist of heavy bars. In each case the tie must be computed for the maximum horizontal thrust.

Vertical Suspenders.—The floor beams are suspended from the arches by means of vertical suspenders. These may consist of exposed steel rods or of steel construction imbedded in concrete. They are subjected to the following stresses: (1) direct tension caused by vertical load; (2) bending due to the wind; (3) bending due to rigidity of connection between the floor beam and the suspender; (4) distortion caused by shortening or lengthening of the arch ribs. This is specially marked in hangers of short lengths.

The stresses due to wind pressure are caused by the action of the wind on the suspenders alone. This pressure may act from both sides, so that reinforcement on both sides is required. Also bending stresses in the suspenders may be produced by wind pressure acting upon the ribs.

Lateral Stability of Ribs.—When the ribs extend far enough above the roadway it may be necessary to connect them by lateral bracing so as to give them lateral stability and also prevent excessive stresses due to wind pressure.

USE OF REINFORCEMENT IN CONCRETE ARCHES

Concrete arches may be built either of plain concrete or of reinforced concrete.

Plain concrete arches may be used when the compression stresses in concrete due to the dead load are sufficient to balance the tensile stresses caused by bending moments produced by the live load and the changes of temperature.

This is often the case in arches of spans more than 180 ft. where the arches and the superstructure are of massive construction. For instance, the Walnut Lane Bridge, in Philadelphia, with a clear span of 233 ft. was built without any reinforcement in the arches.

The authors, however, do not recommend the use of plain concrete for arches, because in most cases reinforced concrete arches may be built cheaper. The reinforced concrete arch section may be made more slender and, therefore, be less subject to temperature stresses. Finally reinforced concrete arches are more able to resist unexpected stresses due to any disarrangement of foundation or any tensile stresses due to any causes whatever.

Plain concrete arches should never be used unless they rest directly on rock foundation.

Reinforcement in concrete arches is used when the compression due dead load stresses is not sufficient to balance the tensile stresses caused by bending moments. The reinforcement then serves to resist tensile stresses. Reinforced concrete arches have the additional advantage that the allowable unit compression stresses in reinforced concrete arches are larger than for plain concrete. In addition to this increase in unit stresses the reinforcement may be assumed as resisting compression directly in the same manner as in columns.

When the reinforcement consists of rigid structural shapes, it may be used to support either partially or wholly the framework.

Reinforcement is particularly necessary for flat arches because there the effects of rib shortening and changes of temperature is especially large.

It is obvious that isolated ribs (unless of very large width in proportion to depth) should be fully reinforced in the same manner as recommended for columns.

Reinforcement Consisting of Bars.—The most common type of reinforcement for arches consists of bars of diameters usually employed in reinforced concrete. The main reinforcement runs longitudinally with the arch. Since for different conditions tension can occur near the intrados just as well as near the extrados, reinforcement is usually placed near both faces of the arch. In addition to the longitudinal bars, cross bars are used which tie the main bars and also prevent any longitudinal cracks. To prevent buckling of longitudinal bars hoops are used running around the top and the bottom bars. (Typical arrangement of reinforcement is seen in Fig. 180, p. 434.)

When the reinforcement consists of two layers of bars (one near the intrados and the other near the extrados) the bottom bars may be placed first before construction has started and supported on blocks or spacers. The concrete may then be poured and the top reinforcement placed after concreting has reached the level of top reinforcement. While with proper care and proper supervision this method may give satisfactory results it is not recommended by the authors. More satisfactory results are obtained by placing and inspecting all the reinforcement before beginning the concreting.

It is important to keep the reinforcement in place during concreting. It is particularly harmful for the reinforcement to come too near the surface or to be exposed as then its rusting would materially impair the safety of the structure. Also it is harmful for the steel to be too far in from the surface, as then its effect in resisting bending stresses is lessened and the concrete must crack on the outside before the steel can come into action. Sagging or bent bars are also of comparatively little use. Therefore some positive means must be employed to keep the bars in the position shown on the plans.

The lower layer of bars can be kept in place easily and the difficulty is encountered only with the top reinforcement. This layer must be supported rigidly enough so that during concreting it will not become misplaced. It is well to remember that the steel is subjected to considerable ill usage while concrete is being poured.

Sometimes a sufficient number of stirrups with sufficient number of cross bars in the top layer may keep the top bars in place. When it is desired to use more positive means, it is possible to use horizontal steel cross frames consisting of small angles.

Such frame was used in the construction of Park Avenue Bridge at Cincinnati, Ohio.[2] It consists of $3\frac{1}{2} \times 2\frac{1}{2} \times \frac{3}{8}$-in. horizontal angles and $2 \times 2 \times \frac{5}{8}$-in. vertical inclined angles. Notches are provided in the upstanding legs of the horizontal angles to receive the bars. Between frames the bars were held in position laterally by two $1\frac{1}{2} \times \frac{3}{8}$ flats placed across the bars one on top and one on bottom. The flats were bolted tightly.

Bars must be spaced far enough apart to permit pouring of the concrete through the top layer of steel.

The amount of longitudinal reinforcement usually ranges between one-half and 1 per cent of the cross section of the arch. Usually the reinforcement is placed symmetrically about the arch axis, one-half of the total area being used near each face. The bars are spliced to develop their full tensile strength. Sometimes at points of maximum bending stresses a larger amount of steel is required than the amount used throughout the arch. In such cases extra bars are added in these heavily stressed sections which extend only as far as needed.

In some designs bars near one face, only, are used. They are placed near the lower face of the arch at the crown and then are bent up and carried near the upper face of the arch at the springing. The authors do not recommend this method because it does not take care

[2] See *Engineering News-Record,* Year 1917, p. 193.

of the condition where tensile stresses develop on the top at the crown and at the bottom at the springing.

The reinforcement of the arch is considered as increasing the cross section of the arch as indicated in formulas pp. 217 to 238. Where no considerable tensile stresses occur Formula (100), p. 219, may be used. Where considerable tensile stresses occur near one face Formulas (28) p. 228 should be used.

Unsymmetrical reinforcement may be computed by means of Formulas on p. 233.

Spiral Reinforcement of Arches.—The longitudinal bars described in previous paragraph are used mainly to take care of possible tension and only incidentally as compression reinforcement. Sometimes in arches consisting of separate ribs it is desirable to increase the compression strength to reduce the section of the rib. In such cases the heavily stressed sections of the ribs may be reinforced by means of spiral reinforcement in addition to the longitudinal reinforcement. As explained in Vol. I, spiral reinforcement increases considerably the allowable compression strength of the concrete.

When the cross section of the rib is round or octagonal, one spiral is used as in round columns. In this manner the compression strength of the whole section is increased.

Usually it is desirable to strengthen by spirals only the section of the arch which is apt to be subjected to largest compression stresses. This leads to a design where separate spirals of small diameters are placed near the upper and lower face of the arch section, respectively. To reinforce fully the compression zone for its whole width it is necessary to use several intersecting spirals of small diameters.

Since the spiral increases the compressive strength of the highly stressed zone of the arch section, its width may be made much smaller than would be otherwise required. The unit stresses in spiraled concrete may be accepted as recommended in connection with spiral columns, Vol. I, p. 421.

To get largest benefit from spiral reinforcement the cross-section of the rib is made I-shaped. Such design gives a much larger moment of inertia than obtained for a rectangular section of the same area. Furthermore, a much larger proportion of the cross-section can be strengthened by the spiral as both flanges are spiraled and only the web has no spirals.

Spiral reinforcement in arches is used to some extent in England and France, but is used very rarely in the United States. It has the advantage that the required section of the rib is reduced, thereby reducing the dead load. The disadvantage is the increased cost of labor for formwork and for fabrication and placing of the spiral reinforcement. Finally the placing of concrete in the portions of the rib reinforced by spirals is much more difficult. The condition is much worse than in spiral columns because there the spiral is vertical and concrete is deposited from the top, and only the small amount of concrete outside of the core needs to pass through the spiral. In arch section, on the other hand, the core concrete must be poured through the spiral.

Melan System

This system of arch reinforcement invented by Joseph Melan of Brünn, Austria, in 1892, consists of structural ribs built of angles connected by lattice work. The depth of the ribs is such that the steel is protected by 2 in. of concrete. In small arches the ribs may consist of I-beams or channels curved to the curvature of the arch. The ribs are usually spaced laterally from 2 to 4 ft., at an average 3 ft., apart. Transversely they are connected by cross frames spaced 10 to 15 ft. apart. Very often in addition to the steel ribs longitudinal reinforcement consisting of bars is used.

The main advantage of the Melan System is that the rigid ribs may be used to carry partially or wholly the forms for the construction of the arches. In such manner the dead load stresses are resisted by the ribs. Another advantage is that the steel ribs can be easily kept in position during erection.

Another modification of the Melan arch is the construction used in the Springfield arch bridge.[3] The arch ribs in that construction were provided with three hinges. After the arch was constructed the top hinge was removed.

To get full benefit of this type of reinforcement the arch ribs must be carried into the abutment for sufficient distance to offer full anchorage.

The importance of this is brought home by the failure of an arch bridge in Dayton, Ohio, described in Engineering News-Record, May 24, 1921, p. 511. The failure was caused by yielding of the pier. The steel ribs in this case extended only a short distance into the pier and pulled out. The failure probably would have been prevented had the reinforcement been properly anchored.

Usually the abutment is constructed separately from the arch rib. To make the pouring of concrete in the abutment possible before the steel ribs for the arch are placed, short dowel ribs are used which extend into the abutment and project beyond the springing line sufficiently for proper connection with the steel rib.

[3] See *Engineering News-Record*, March 31, 1922, p. 514.

The reinforcement of two adjoining arches should be connected at the pier.

Wünsh System

This system has comparatively a very limited application. The reinforcement consists of rigid frames spaced from $1\frac{1}{2}$ to $2\frac{1}{2}$ ft. apart. A horizontal upper member is placed near the extrados and a curved lower member near the intrados. The two members are connected at each abutment to vertical members imbedded in concrete. In this manner a steel frame is obtained. The amount of reinforcement at the crown usually amounts to 1 to 2 per cent.

Arches of that type are applicable only for very shallow arches in which the ratio of the rise to the span is from $\frac{1}{10}$ to $\frac{1}{15}$.

The bridge at Sarajevo, Bosnia, with an 83-ft. span is one of the largest spans of this system.

Emperger System

The Emperger System of arch bridges consists of cast iron encased in spiraled concrete. Considerable economy is claimed by the inventor for this system. Arches of considerable size have been built or projected which distinguish themselves by lightness and low cost. Arches of that type while common in Europe have not as yet been used in the United States.

ALLOWABLE UNIT STRESSES IN AN ARCH

Allowable Compression Stresses.—The allowable unit stresses in an arch should not exceed the values given in the table below.

Allowable Compressive Unit Stresses f_c in Concrete Arches

Description	Concentric or Nearly Concentric Load	Thrust and Bending Moment	
		e Smaller than $\frac{1}{6}h$	e Larger than $\frac{1}{6}h$
(1)	(2)	(3)	(4)
Plain concrete	$0.18f'_c$	$0.21f'_c$	
Reinforced concrete, min. $p = 0.01$	$0.225f'_c$	$0.265f'_c$	$0.315f'_c$

f'_c = ultimate compression strength of concrete at 28 days tested in cylinders.

The compressive stresses in an arch should be computed by formulas in Chapter II on Direct Stress and Bending.

The reasons for the above recommendations are as follows:

If an arch is subjected to a central thrust, the stresses will be uniformly distributed over the whole section. The stress conditions, therefore, as far as compressive stresses are concerned, are the same as in columns. The allowable unit stresses for this condition must not exceed the allowable unit stress for columns. We have, therefore, the requirement in column 2 of the above table which may be expressed as follows:

The allowable unit stresses in concrete for central thrust must not exceed the allowable unit stresses for columns. For plain concrete arches the allowable stresses are 0.8 of those for reinforced concrete arches.

If an arch is subjected to bending moments and normal thrusts the section is subjected to a stress of varying intensity. The maximum stress acts only on a small area. The character of the stresses changes gradually from column stresses to flexural stresses. As explained in Vol. I, p. 30, the allowable unit stress for flexural stresses are larger than for the column stresses. Therefore, the allowable maximum compression stresses in arches should vary between those allowed for columns and the allowable unit stresses in flexure depending upon the variation in stresses.

Allowable Tension Stresses and Required Amount of Tension Reinforcement.—The tensile stresses must be investigated first for working conditions. The stresses due to the dead load, including rib shortening, are combined with stresses due to the most unfavorable position of the working live load, the change of temperature producing at the section bending moments of the same sign as those for live load and the shrinkage.

Second, the tensile stresses must be investigated to determine whether the design has a sufficient factor of safety. This is accomplished by combining the stresses due to the dead load, including rib shortening, with the live load and temperature stresses used for previous case but multiplied by a desired factor of safety. The factor of safety may be taken as two.

For the first, i.e., the working conditions, the tensile stresses in concrete should not exceed

(*a*) for plain concrete sections preferably no tensile stresses and never more than $0.02f'_c$ or 40 lb. per sq. in. for 2000 lb. concrete, using formula $f_t = \frac{N}{bh}\left(1 - \frac{6e}{h}\right)$;

(b) for reinforced concrete sections $0.08f'c$ or 160 lb. per sq. in. for 2 000 lb. concrete, using formula

$$f_t = \frac{N}{bh}\left[\frac{1}{I - (n-1)p} - \frac{6he}{h^2 + 12(n-1)pa^2}\right].$$

For the second condition, i.e., when the live load and temperature stresses are multiplied by factor of safety, the tensile stresses must not exceed

(a) for plain concrete sections $0.10f'_c$ or 200 lb. per sq. in. for 2 000 lb. concrete, using formula $f_t = \frac{N}{6h}\left(1 - \frac{6e}{h}\right)$;

(b) for reinforced concrete sections the stresses in steel must not exceed 14 000 lb. per sq. in. multiplied by the same factor of safety. Formulas 29, p. 228, or 51, p. 236, should be used.

Required Amount of Tensile Reinforcement.—The required amount of tensile reinforcement must not be less than 0.25 per cent of the largest gross cross-section of the arch. The exact amount of reinforcement to be used should be computed for the combination used for the second condition. The allowable tensile stress in steel must not exceed the values specified above.

The object of the requirement for the first condition is plain. It is to prevent under working conditions open cracks in concrete. These are particularly harmful in structures subjected to atmospheric conditions as they may lead to gradual disintegration of the concrete at the cracks. The condition is particularly unfavorable in arches where the sections are subjected to reversal of stresses, i.e., a section under tension for one position of the load becomes subjected to compression for another position of the load. An open crack usually is irregular. After the load producing it is removed the crack may not close completely due to small relative displacement between the projections and the opposing indentations. When the section is subjected to compression, the crack may be closed forcibly with the consequent crushing and grinding of the projecting particles of concrete. Again subjected to tension, the resulting crack becomes wider than it was originally under the same type of loading.

Cracking is particularly undesirable in plain concrete sections as then it automatically reduces the depth of the section. This is not the case in reinforced concrete sections where the reinforcement supplements the lost tension due to cracked concrete. When the arch section fulfills the first requirement no cracks in concrete need to be expected.

The object of the requirement for the second condition is not as self-evident, especially as no such requirement is made in beam design. The reason for it is given below.

A beam, the sections for which are determined for allowable unit stresses, has automatically the factor of safety upon which the unit stresses are based. If for working loads the steel is stressed theoretically to 16 000 lb. per sq. in., the stress will be 32 000 lb. per sq. in. when the load is doubled.

In arch design the stresses are not a simple result of the loading as in beam design. Instead, they are resultants of several items, each independent of the others. Moreover the effect of each item is of different nature from that of the other items. When the live load and temperature, for instance, are doubled the stresses in the section are not doubled, as was the case with the beam. On the contrary, the increase in the stresses, particularly the tensile stresses, will be much out of proportion to the ratio of increase in the live load and temperature.

In properly designed arches the dead load produces a central thrust, causing compression stresses uniformly distributed over the arch sections. To these should be added the bending stresses caused by rib shortening due to dead load thrust. The amount of the stresses due to dead load, including rib shortening, is fixed and is not affected by the magnitude of the live loading or by the temperature changes.

Live load and changes of temperature produce mainly bending moments. The thrusts produced by unfavorable positions of live load and temperature are comparatively small. Fall of temperature and shrinkage, even, produce a pull, i.e., a thrust acting outward.

When the dead load stresses, which are compression, are combined with stresses due to live load and changes of temperature, the tensile stresses produced by the bending moments are balanced in part or in full by the compression stresses due to the dead load. It often happens that for working conditions the tensile stresses are either entirely balanced by the dead load compression stresses or only very small tension remains. In such cases, according to prevailing practice and the recommendations of textbooks treating on the subject of arch design, this would signify that the arch is safe as far as tensile stresses are concerned and that no tensile reinforcement is required.

This conclusion usually is erroneous as no factor of safety is provided. For working conditions the tensile stresses may be small. However, for unusual conditions, as, for instance, when the arch is subjected to a larger live load than that accepted in design, when larger temperature changes occur than assumed or when any other emergency arises for which the factor of safety should be provided, the tensile stresses increase out of proportion to the increase in these items.

Assume that the accidental live load is twice the working live load. The tensile stresses produced by this increased load doubles, but since the balancing compression stresses produced by fixed dead load remains the same as for working load, the resulting tension is very largely increased. For further explanation and example see Vol. I, p. 462. To provide sufficient factor of safety the amount of tensile reinforcement should be designed according to following rule.

Combine thrust and bending moment due to the dead load and rib shortening with the bending moment and thrust due to working live load and temperature changes multiplied by a desired factor of safety. If there is a possibility of yielding of the foundations or bending of the pier proper, bending moments should be added.

If for this condition the tensile stresses in concrete do not exceed the values specified on p. 454 no reinforcement is required. If tensile stresses are larger, provide sufficient reinforcement so that the stresses in concrete and steel do not exceed double the allowable unit stresses in flexure. Formulas on p. 228 or those on p. 236 may be used.

This method provides sufficient factor of safety as far as tensile stresses are concerned.

CHAPTER VI

FORMULAS FOR DESIGN OF ARCHES FIXED AT SUPPORTS

Arch action, contrary to widespread belief, is governed by the same rules of mechanics as ordinary beam action. Once the reactions in an arch are computed, the bending moments and shears at any point are found by common rules of statics. The formulas for computing stresses in arches are based on the same assumptions as in other reinforced concrete structures. The characteristic difference between a beam and an arch is that while the cross-sections of a beam are subjected only to bending moments and shears, the cross-sections of an arch are subjected to bending moment, shears and normal thrusts.

An arch is always a curved beam, but a curved beam is not necessarily an arch.

A curved beam supported in such a manner that its ends are free to slide and thereby increase the span of the beam is not an arch but a simply supported beam. When subjected to vertical loading, the beam will have a tendency to straighten. Its ends will slide outward and tensile stresses will develop on the under side of the beam throughout its length in the same manner as in simply supported beams. At the support there are only vertical reactions and their magnitude is the same as for ordinary beams carrying the same loading. The cross-sections are subjected to bending moments and shears only, which are found in the same manner as for simply supported beams. These are resisted by the cross-section of the beam in substantially the same manner as with straight beams. No benefit is derived from the curvature of the beam.

A curved beam, provided at the ends with hinges firmly attached to solid unyielding supports, is a hinged arch. When loaded with vertical loads it can rotate at the ends, but the ends cannot move outward because the tendency of the curved beam to straighten is overcome by the resistance of the unyielding supports. When loaded with vertical loads, the beam exerts vertical pressure on the supports. In an attempt to straighten, the beam exerts upon the support in addition a horizontal pressure which is resisted by a horizontal reaction called the horizontal thrust. The direction of the horizontal thrust is inward so that it produces compression in the arch. It also produces a negative bending moment in the beam equal to the thrust multiplied by the vertical distance of the section from the support. The curved beam under such conditions is a two-hinged arch. The bending moments at any section are appreciably smaller than for simple beams because the static bending moments are reduced by the negative bending moments produced by the horizontal thrust. The actual bending moments are not positive throughout the span, as in the previous case, but are positive in some sections and negative in others. In this case two benefits are derived from the curvature of the beam. First, the bending moments are greatly reduced. Second, the whole section is subjected to compression stresses due to the thrust which reduce the tensile stresses produced in the section by the bending moment.

A curved beam built into solid unyielding supports, so that the ends can neither rotate nor spread, is a fixed arch. Not only the tendency of the curved beam to straighten under load but also the tendency to rotate at the ends is prevented. At the support not only horizontal thrusts are developed, as in hinged arches, but also bending moments. The bending moments at any section of the arch are, therefore, a combination of static bending moments produced by the loads, bending moments produced by the horizontal thrust, and bending moments developed at the supports. The benefits derived from the curvature of the beam are similar to those in the previous case. In addition, the resulting bending moments are smaller.

Characteristics of Arch Action.—As follows from the preceding paragraphs, the main characteristic of arches common to all is the presence at the support of a horizontal thrust induced there, because the unyielding supports prevent the curved beam from straightening under vertical loads. The horizontal thrust acts towards the center of the arch. It produces compression stresses at all sections of the arch.

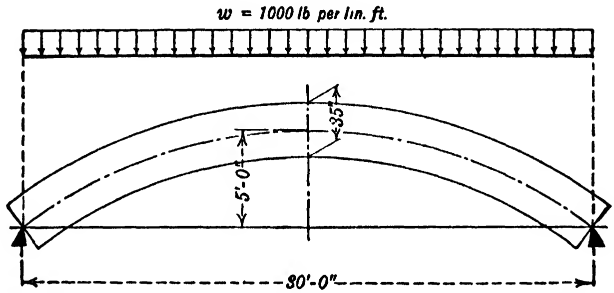
The horizontal thrust also produces negative bending moments at all sections of the arch which counteract the positive bending moments due to the loads. Thus the second characteristic of arch action is that in addition to the bending moment each section is subjected to a direct thrust and also that at all sections the static bending moment due to the load is considerably reduced by the bending moment due to horizontal thrust.

The above characteristics are common to fixed as well as hinged arches.

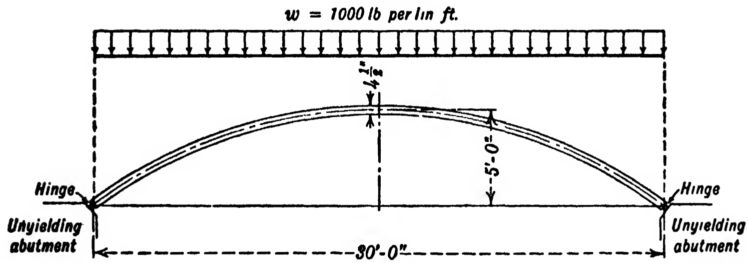
The difference between hinged arches and fixed arches is that in hinged arches at the hinges there is no bending moment and the thrust is applied at the center of the hinge. In fixed arches all sections are subjected to bending moment. The location of the point of application of the thrust at the springing is different for different positions of loading.

Illustration of Arch Action.—Following example illustrates the difference between beam action and arch action.

Assume a curved beam as shown in Fig. 189 (a) and (b), p. 460, parabolic in shape with following dimensions: span $l = 30$ ft., rise $r = 5$ ft., width of beam 1 ft. The uniformly distributed loading equals $w = 1\,000$ lb. per lin. ft.



(a) Curved Beam Free to Move Horizontally at Supports



(b) Curved Beam Hinged to Unyielding Abutments

FIG. 189.—Illustration of Arch Action. (*See p.* 460.)

In Fig. 189 (a) the ends of the beam are hinged and one bearing is movable. In Fig. 189 (b) the ends are also hinged, but both bearings are securely fastened to the unyielding abutments.

Curved Beam as Simply Supported Beam.—Consider the beam in Fig. 189 (a) with movable support. When subjected to loading the free end of the beam will move outward, so that no horizontal thrust can develop. Because of the hinges no bending moment can be developed at the support. The reaction at the supports and the bending moments at any point, therefore, are the same as for a straight simply supported beam of the same span and loading.

Maximum static bending moment in the center

$$M = \tfrac{1}{8}wl^2 \times 12 = \tfrac{1}{8} \times 1\,000 \times 30^2 \times 12 = 1\,350\,000 \text{ in. lb.}$$

For allowable unit stresses $f_s = 16\,000$, $f_c = 800$ using the corresponding constants from Table 2, p. 880, Vol. I, and the formulas on p. 204.

$$h = 0.079\sqrt{\frac{1\,350\,000}{12}} + 1.5 = 33.5 + 1.5 = 35 \text{ in.}$$

$$A_s = pbd = 33.5 \times 12 \times 0.0118 = 4.75 \text{ sq. in.}$$

This means that it is necessary to use a section 35 in. deep, 12 in. wide with 4.75 sq. in. of tension reinforcement to resist the bending moments produced by the load.

Curved Beam as an Arch.—Now consider a curved beam in Fig. 189 (*b*) identical in curvature with the previously described beam and subjected to the same loading, but placed between unyielding abutments. Since the ends are hinged, no bending moment is developed there. The vertical reactions and the static bending moment due to the loads are the same as in previous case. However, in addition to vertical reaction we have at each support a horizontal thrust H acting inward. As is shown on p. 574 the magnitude of this thrust is

$$H = -\frac{1}{8}\frac{wl^2}{r} = -\frac{1\,000 \times 30^2}{8 \times 5} = -\,22\,500 \text{ lb.}$$

This thrust produces at each section of the arch a negative bending moment, the magnitude of which is $M_x = Hy$, where y is the vertical distance of the center of the section from the support. In the center of the span y equals the rise, or $y = r$ so that the bending moment produced by the thrust is

$$M = Hr = -\frac{1}{8}\frac{wl^2}{r}r = -\frac{1}{8}wl^2,$$

This bending moment is equal and of opposite sign to the static bending moment produced by the load. Therefore, the resultant bending moment in the center of the span is zero and the arch section is subjected only to the horizontal thrust, $H = -\,22\,500$ lb.

If the allowable unit stress for direct compression is $f_c = 450$ lb. per sq. in., the required area of concrete section is

$$A = \frac{22\,500}{450} = 50 \text{ sq. in.}$$

or a section $4\frac{1}{2}$ in. deep and 12 in. wide is sufficient to resist stresses due to this particular loading.

Compare the two constructions. In the first case, where the curved beam acts as a simple beam, it is necessary to use in the center a section 35 in. deep and 12 in. wide with 4.75 sq. in. of steel, while in the second case, where the beam is an arch, a section $4\frac{1}{2}$ in. deep and 12 in. wide is sufficient to resist the stresses due to this special type of loading.

In the above illustration arch action was produced because the ends of beams were attached to unyielding supports. Similar results may be obtained by tying the ends of the beam by a tension member or tie which resists the horizontal thrust. The area required by the tie equals the horizontal thrust divided by the allowable stress in steel.

$$A_s = \frac{H}{f_s} = \frac{22\,500}{16\,000} = 1.4 \text{ sq. in.}$$

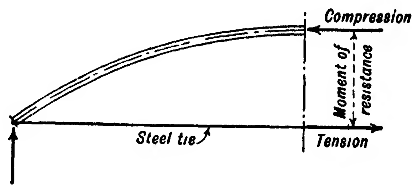


FIG. 190.—Arch with Tension Tie Member. (*See p.* 462.)

In Fig. 190 is shown a section through the center of an arch provided with a tie. The concrete section is subjected to compression, the steel tie to tension. These forces form a resisting couple which opposes the bending moments due to the loads. The moment arm of the resisting couple equals the rise of the arch.

In the above illustration for the sake of clearness the simplest form of arch action was assumed. In practice the problem is complicated by one-sided loading, temperature stresses, etc., so that larger dimensions would have to be used for the arch than result from the computations just given.

Deflection of Arch under Different Types of Loading.—There is a considerable difference between the manner of deflection of an arch under vertical loading and that of a beam.

A beam subjected to vertical loading always deflects downward. After deflection, all points on the axis of the beam (except the supports) are below the original position of the axis of the unloaded beam.

An arch subjected to vertical loading deflects downward throughout only for loads extending over the whole span. For partial loading part of the axis deflects downward and the balance deflects upwards. Figure 191 shows in exaggerated form the shape assumed by the arch for different types of partial loading.

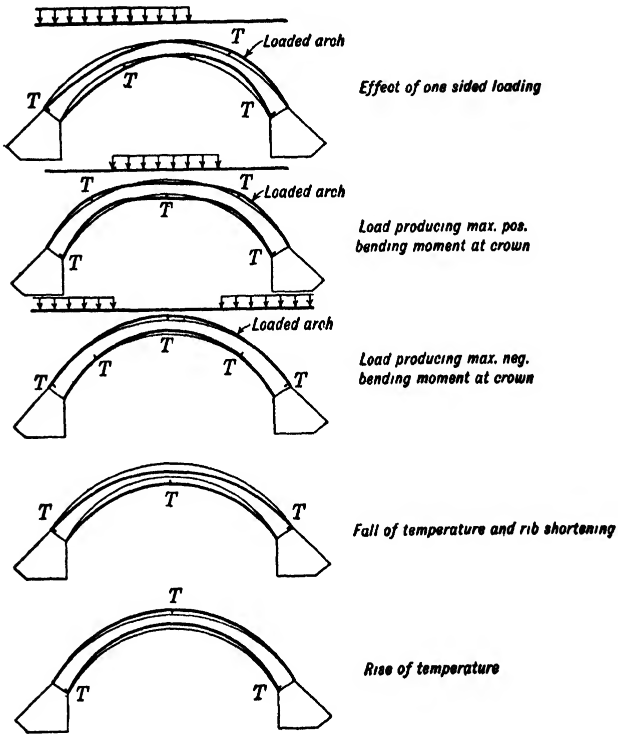


FIG. 191.—Deflection of Arch under Different Types of Loading. (*See p.* 463.)

The most unfavorable loadings for an arch are one-sided loadings shown in Fig. 191. The loaded part of the arch moves downward while the unloaded part moves up. The points of maximum tension, i.e., where cracks may be expected, are marked by T.

The partial loading shown in Fig. 191, p. 463, producing maximum tension at the crown pushes the crown downward and the haunches outward. The exaggerated deflection is shown in the figure.

The loading producing maximum negative bending moment at crown shown in Fig. 191 has the opposite effect to that in the previous case. The arch is pushed downward at the haunches and forced up at the crown.

Effect of Temperature.—As will be explained later, temperature changes have a very marked effect upon arches. Due to the fall of temperature the arch shortens and assumes a shape shown in Fig. 191. The crown lowers. The bending moments are positive at the crown and negative at the springings.

The rise of temperature has the opposite effect to the fall of temperature. The deflections caused thereby are shown in Fig. 191, p. 463.

Effect of Yielding of the Abutments.—Yielding of the abutments has a very unfavorable effect upon the arch as it nullifies the arch action, partly or fully, depending upon the extent of the yielding. In a general way yielding of the abutments has an effect similar to that of the fall of temperature. The deflection caused by it is shown in Fig. 191.

METHOD OF DESIGNING FIXED ARCH BRIDGES

In designing arch bridges following problems must be solved.

First, proper type of construction must be selected, i.e., either the filled spandrel type or the open spandrel type. The various types are discussed on p. 434.

Second, the curvature of the arch axis must be determined as explained on p. 465 under "Curvature of Arch Axis."

Third, preliminary dimensions must be determined as discussed on p. 476 under "Proportioning of Arch Sections."

Fourth, complete analysis of the arch should be made, using formulas for the indeterminate values given on pp. 492 to 496.

Fifth, after the bending moments and thrusts are determined maximum stresses at various sections should be computed. When the stresses are either too small or too large, proper change in the dimensions of the arch section or the amount of reinforcement should be made.

CRITICAL CROSS-SECTIONS AND CRITICAL POSITIONS OF LIVE LOAD

Critical Cross-sections.—In the design of arches the depths of the cross-sections are measured upon lines drawn at right angles to the tangent to the arch axis. The dimensions of the cross-sections are not constant throughout the length of the arch but are smallest at the crown and increase gradually toward the springing, i.e., the support of the arch.

Experience teaches that, where the variation of the magnitude of the cross-sections at the intermediate points is properly selected, as suggested on page 477, only three critical sections need to be examined. If the stresses at these are satisfactory, the other intermediate sections are also safe. The critical cross-sections are at: 1. The Springing; 2. The Quarter Point; and 3. The Crown.

In arches of special magnitude or with unusual design of the arch ribs it may be advisable to consider several additional sections.

Critical Positions of Live Load.—As evident from the influence lines for bending moments shown in Fig. 199, p. 535, and the discussion on pp. 543 and 544, the maximum bendings at the critical sections are produced not by a live load extending the whole length of the arch span but by different positions of partial loadings. The positions of the live load producing maximum bending moments at the critical sections are given on p. 505.

CURVATURE OF THE ARCH AXIS

The axis of a properly designed arch is a continuous consistent curve the radius of curvature of which is largest at the crown and decreases consistently towards the support. As will be shown later the ratio of change of the radius of curvature depends upon the ratio of the unit dead load at the crown to that at the springing. The larger this ratio, the larger is the decrease in the radius of curvature and also in the consistent increase in curvature.

Dead Load for Arches with Filled Spandrel.—The determination of the dead load in arches with filled spandrel is simple. The dead load consists of the weight of paving, the weight of fill and the weight of the arch.

The weight of the spandrel and the balustrade may be considered as distributed over the whole width of the arch.

The weight of paving is constant for the whole arch and is usually known before the design of the arch is started. After the arch is laid out the weight of the fill and the arch proper may be obtained by scaling the vertical distances and multiplying these by the unit weight of fill and concrete, respectively. When a graphical method of determining the arch axis is used, the work may be simplified by reducing the ordinates of the fill to the basis of the unit weight of the concrete by multiplying them by the ratio of unit weight of fill to unit weight of masonry. The reduction can also be done by graphics. The scaled dimensions at various sections can then be used directly for the purpose of drawing the line of pressure without the necessity of changing them into weights by computations. The actual horizontal thrust then may be obtained by proper adjustment of the scale.

The dead load may be computed either for a horizontal strip of arch 1 foot wide or for the whole width of the arch. The last method should be used when the width of the arch rib is not constant but increases towards the support.

The unit dead load varies with depth of fill and thickness of arch and therefore is a minimum at the crown and increases towards the springing consistently. The ratio of increase depends upon the ratio of rise to span of the arch. Obviously it is smaller for shallow arches than for deep arches. The unit dead load plotted along the span of the arch on vertical ordinates gives a consistent curve. Often for approximate work this curve is assumed to be a parabola. More accurate results are obtained by assuming the increase to be proportional to the increase in the vertical ordinate of the arch axis measured from the crown.

q_c = dead load at crown per unit of length of arch span;
q_s = dead load at springing per unit of length of arch span;
$m = \frac{q_s}{q_c}$ = ratio of dead loads at springing and crown;
l = span length of arch axis;
r = rise of arch axis.

Dead Load for Arches with Open Spandrel.—In arches with open spandrel the dead load consists of the weight of paving and the weight of floor construction supporting the roadway, the weight of the vertical supporting members for the floor construction and the weight of the arch rib. Usually the floor construction and the supporting members are designed before the work on the design of the arch rib is started so that the largest part of the dead load is accurately known. The only unknown value is the weight of the arch rib. The weight of the roadway and floor constructions in many cases is constant throughout the whole length of the arch, and the variable items are the weights of the vertical supports, which increase towards the support because of the increasing height, and the weight of the arch rib. While there is some variation between the unit dead load at the crown and at other points, the ratio of variation is much smaller than in arches with filled spandrel. In some cases, particularly in arches with suspended roadway, the total unit dead load is almost uniform throughout the whole length of the arch.

The dead load due to paving, the floor construction and the vertical supports are concentrated at the points of application of these vertical supports. These concentrations should be used when determining the line of pressure. Obviously, for divisions of the arch between the vertical supports, the dead load consists only of the dead load of the arch ribs.

When the arch consists of separate parallel ribs, proper proportion of the floor load for each rib must be determined by computing reactions of the floor construction. Where the construction consists of two arch ribs only, and the floor is symmetrical, one-half of the weight of the floor comes upon each rib. When there are three ribs, the interior rib usually carries more load than the outside ribs unless the load is adjusted by cantilevering the sidewalk beyond the outside ribs.

Often it is desired to determine preliminary dimensions of arches with open spandrel by means of the approximate method given on p. 480 using the diagrams on pp. 669 to 677. These are based upon an assumption that the dead load is distributed over the arch (and not concentrated as is actually the case), that the variation of the dead load is proportional to the variation in the ordinates of the arch axis and that the curve representing the variation is fixed by the ratio $m = \frac{q_s}{q_c}$, which is the ratio of the unit load at the crown and at the springing. To make possible the use of these diagrams it is necessary to replace the concentrated dead load at the crown and springing by distributed load of an intensity equal to the concentrated loads divided by the spacing of concentrated loads.

It must be kept in mind that the object of finding these unit loads is to find such a ratio $m = \frac{q_s}{q_c}$ for which the corresponding curve would properly represent the distribution of the dead load throughout the arch in question. Where the floor construction is the same throughout the whole span of the arch as in Fig. 183, p. 440, the actual unit loads at the crown and springing may be used. However, when the roadway construction in the central portion of the arch rests on shallow fill and at the ends on concrete floor construction such as in Fig. 187, p. 444, the use of actual unit loads at the crown and at the springing would not represent properly the variation of the dead load throughout the arch. The dead load in such case does not increase consistently but there is a break at the point where the fill ends and concrete construction begins. In such case closer results can be obtained for the purpose of computing the ratio of $m = \frac{q_s}{q_c}$ if, instead of the actual dead load at the crown, the load is used that would be produced by extending the concrete floor construction over the whole length of the arch.

Method of Determining Arch Axis.—The shape of the arch axis has a decided effect upon the economy of an arch construction. It cannot be accepted arbitrarily. Proper arch shape may be obtained by one of the two well-recognized methods.

1. The arch axis is made to coincide with the line of pressure for dead load so that the arch, when not loaded, is subjected only to the normal thrust due to the dead load. This method is recommended by the authors.

2. The arch axis is made to coincide with the line of pressure for the dead load plus one-half of the live load uniformly distributed over the whole arch, a method originally recommended for masonry arches by Tolkmitt. For dead load alone, such an arch is subjected to negative bending moments at the crown and positive bending moments at the springing. The purpose of these initial bending moments is to counteract partially the bending moments due to the live load and rib shortening.

In both cases the effect of rib shortening due to the dead load thrust should be considered. The thrust and bending moments due to this cause should be computed separately as given on p. 494 and the stresses produced by it combined with the axial stresses due to the dead load thrust.

Improved Shape of Arch Axis.—The stresses in the arch can be decreased somewhat by using the so-called improved shape of arch. To eliminate the effect of rib shortening the arch axis is made to deviate from the line of pressure at points where bending moments are produced by rib shortening. In this way bending moments are produced in the arch by the dead load thrust. If proper shape is used these bending moments may be equal and of opposite sign to the bending moments produced by rib shortening, thereby neutralizing each other. Formulas for the shape of the improved axis were developed by Osterfeld.[1] The method is complicated and is not recommended to any but experts in arch design.

Shape of Arch Axis as Affected by Variation in Intensity of Dead Load.—The shape of the arch axis in methods 1 and 2 coincides with the line of pressure for the dead load (or the dead load plus live load). It is obvious, therefore, that the shape will depend upon the distribution of the dead load along the span of the arch.

Dead Load Uniform.—For constant intensity of dead load for the whole length of the arch span, the line of pressure is a parabola of

[1] Beton und Eisen, year 1923, p. 70.

second power and may be expressed by the equation $y = \frac{4r}{l^2}x^2$, where the x and y are ordinates referred to a system of coordinates passing through the crown. The ordinate at the quarter point for which $x = \frac{l}{4}$ is $y_{\frac{1}{4}} = \frac{1}{4}r$. It should be noted that in this case both methods of determining the arch axis give the same result.

Variable Intensity of Dead Load.—When the intensity of the dead load is not uniform and the unit dead load increases towards the support, the arch axis becomes flatter near the crown and steeper at the springing. The shape of the arch axis approaches parabolas of third or fourth power, depending upon the magnitude of the variation between the unit dead load at the crown and at the springing. The depth of the axis below the crown for $\frac{l}{4}$ becomes smaller than the $0.25r$ for the parabolic axis. The arch axis for this case is usually a composite curve and usually cannot be expressed by any closed mathematical function.

Approximate Formulas for Arch Axis.—The arch axis may be expressed by approximate formulas by assuming that the dead load between the crown and the support varies according to some mathematical formula. Such approximate shapes sometimes give satisfactory results. One of the approximate methods for the shape of the arch axis and also for the magnitude of the horizontal thrust due to dead load is given on p. 486. It should be used only for the purpose of determining the preliminary shape. Since the amount of work connected with the determination of the line of pressure for the exact loads is comparatively small, it is preferable to use the actual line of pressure for the final arch axis.

Determining Shape of Arch Axis when it Coincides with Line of Pressure for Fixed Loads.—Below are outlined methods of determining the shape of arch axis for any given fixed loads. The fixed loads may be dead loads only or they may be dead loads plus any portion of the live load. Horizontal thrust for these loads is obtained incidentally.

The rise and the span of the arch must be fixed before the work on the arch axis is started. This fixes three points on the arch axis, namely, the crown and the springings. The arch axis must pass through these three points.

As fully explained on p. 626 a line of pressure is a funicular polygon for given loads drawn with a pole distance equal to the horizontal thrust produced by these loads. When the arch axis corresponds with the line of pressure for any type of loading then this loading produces no bending moment in the arch but only concentric normal thrusts.

To become an arch axis the line of pressure must pass through the selected crown and springings. With this additional requirement the line of pressure may be drawn without the necessity of a prior determination of the horizontal thrust. **The magnitude of the horizontal thrust is obtained incidentally while the shape is being determined.** The shape of the arch axis does not depend upon the statically indeterminate values. This means that the arch axis is fixed only by the magnitude and location of the fixed loads and is entirely independent of the elastic properties of the arch. For equal loads the shape of the arch will be the same for fixed arches, arches with one, two, or three hinges.

The line of pressure may be determined either analytically or graphically. In both cases the following preliminary work must be performed.

1. The rise and span of the arch must be selected, thus fixing the position of the crown and the springings.

2. The preliminary dead load must be determined. For this purpose the type of roadway must be decided upon to get the weight of paving. The relation of the top of the roadway to the arch axis must be fixed. For arches with filled spandrel the depth of fill at the crown must be fixed. In open spandrel arches the floor construction must be designed and its weight computed. This can be done prior to the design of the arch because the design of the floor is independent of the actual shape of the arch axis.

3. An approximate arch axis is selected for the purpose of determining the dead loads in arches with filled spandrel. Often it is close enough to accept for first approximation an arc of a circle passing through the selected crown and springings. More accurate results are obtained when two or more points on the arch axis are computed by means of approximate Formula (10), p. 482. Usually it is sufficient to determine points at quarter points of the arch, i.e., for $x = \frac{l}{4}$. The preliminary axis is then fitted to pass through these selected points.

4. The depth of the arch rings at various points is assumed. This fixes the dead load of the arch. Having drawn the arch axis and the thickness of the arch the dead load of the fill can be determined.

5. The arch axis is divided into a desired number of divisions. The larger the number of divisions the more accurate is the result.

The arch axis may be divided in a number of ways.

(*a*) The arch axis may be divided into a number of divisions of equal length. Then ds for all divisions are equal. Since the inclination of the arch axis at various divisions is variable, the horizontal length dx will be variable, being larger near the crown than near the springings.

(*b*) The span of the arch may be divided into a number of equal divisions so that the values of dx are constant. In such case the length of the sections of the ribs ds are variable, being larger near the springing than near the crown.

3. The arch axis may be divided so that the ratio of $\frac{ds}{I_x}$ for each division is the same. This reduces the work of computations as it eliminates the $\frac{ds}{I_x}$ values. In such case the divisions near the crown are smallest and increase towards the support. Where the difference between the moment of inertia of the arch ring at the crown and at the springings is large, the length of divisions at the springing may become too large for accuracy.

For arches with open spandrel the division should be made so that their center will coincide with the center of vertical supports. If the spacing between the supports is too large an intermediate section between these supports may be taken.

The dead load for each division may be assumed to act in the center of the division, unless the variation in load within that division is large, in which case the point of application should coincide with the center of gravity of the load.

Analytic Method of Determining the Line of Pressure.—Knowing the magnitude and the positions of dead loads at the various divisions, the line of pressure may be determined analytically by computing the static bending moments of the loads for each division point, considering the arch as a simply supported beam, and by dividing these bending moments by the horizontal thrust. The result gives at the respective points the vertical distance of the line of pressure from a base line passing through the springings. The horizontal thrust is found by dividing the static bending moment in the center by the rise.

This method follows directly from the requirement that for the fixed loads there should be no bending moment at any point in the arch. To make this possible the positive static bending moments due to the loads must be balanced by the negative bending moment due to the horizontal thrust. The horizontal thrust applied at the springing and its bending moment equals Hy, that is the thrust times the vertical distance of the point at the arch axis from the springing. By equating the static bending to Hy the value of y may be found.

The amount of work may be reduced by finding the increments of the static bending moments starting from the crown. These divided by the horizontal thrust give the depth of the line of pressure below the crown. The procedure is shown in the table on p. 473. The loads and dimensions used in the table are taken from Fig. 192, p. 472. The values a_1, a_2, a_3 . . . a_6 are distances of the centers of gravity of loads in each division from their left end. The loads P_1 to P_6 are dead loads. The arch is shown as divided into six divisions. Usually a larger number of divisions is advisable. The ordinates of the line of pressure referred to an axis passing through the crown are given in column (6) of the table.

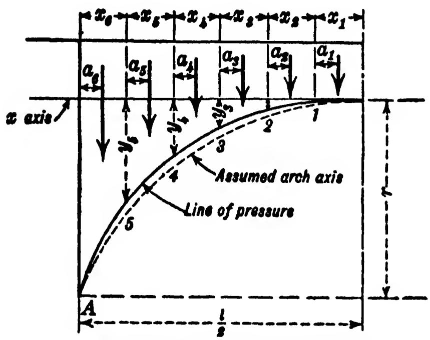


FIG. 192.—Analytical Determination of Line of Pressure. (*See p.* 472.)

Graphical Determination of Line of Pressure for Arch Axis.—When it is required to make the arch axis coincide with the line of pressure, the axis may be determined graphically in the following manner.

Preliminary shape of arch axis and thickness of arch are accepted first and laid out to scale. The top of roadway is also drawn. This for arches with filled spandrels determines the dead load.

The arch is divided into a desired number of divisions. For filled spandrel arches divide the span into divisions of equal length. The points of application of dead load are in center of gravity of the divisions. For open spandrel arches the points of application of load coincides with the location of vertical supports.

Dead load for each division is computed (see p. 465). In filled spandrel arches the work may be simplified by reducing the ordinates of the fill to the basis of unit weight of concrete by multiplying them by a ratio of unit weight of fill to unit weight of concrete. To this should be added a length corresponding to weight of paving. These reduced values of ordinates are plotted above the top of the arch ring. Then, instead of actual dead loads for each division, it is possible to use the ordinates at the points of application of the loads measured from the bottom surface of arch rib to the top of the reduced surface. These are laid out as a force polygon to a convenient scale. The use of the ordinates instead of the actual loads is permissible because the actual loads are proportional to the ordinates, and for graphical

Line of Pressure for Symmetrical Dead Load

Points	Loads	Distance of of Load	$P_n a_m$	Length of Division	$(P_1+P_2+\ldots P_n)x_n$	M_{n-1}	Bending Moment M_n (4)+(6)+(7)	$y'_n=\frac{M_n}{H}$ (8)÷H
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	P_1	a_1	P_1a_1	x_1	0	O	$M_1 = P_1a_1$	$y'_1=M_1\div H$
2	P_2	a_2	P_2a_2	x_2	P_1x_2	M_1	$M_2 = P_2a_2+P_1x_1+M_1$	$y'_2=M_2\div H$
3	P_3	a_3	P_3a_3	x_3	$(P_1+P_2)x_3$	M_2	$M_3 = P_3a_3+(P_1+P_2)x_3+M_2$	$y'_3=M_3\div H$
4	P_4	a_4	P_4a_4	x_4	$(P_1+P_2\ P_3)x_4$	M_3	M_4	$y'_4=M_4\div H$
5	P_5	a_5	P_5a_5	x_5	$(P_1+P_2+P_3+P_4)x_5$	M_4	M_5	$y'_5=M_5\div H$
A	P_6	a_6	P_6a_6	x_6	$(P_1+P_2+P_3+P_4+P_5)x_6$	M_5	M_A	$y'_A=M_A\div H$

A is the springing.
The value of H in the last column equals bending moment at springing divided by the rise or $H = \frac{M_A}{r}$.
Values in Col. (8) are obtained by adding items of Col. (4), (6), and (7).
Values in Col. (7) are equal to the values in Col. (8) for the previous point.

work the proportions of the various loads and not their magnitude are required.

Draw a force polygon for one-half of the arch starting at the top with the section next to the springing. Select a convenient pole distance and draw a funicular polygon. When the arch axis for the left side is drawn, place the pole distance to the right of the force polygon. The resulting funicular polygon is concave as shown in Fig. 193. Extend the end rays of the polygon till intersection. Draw a vertical line through this point of intersection. This line indicates the position of the resultant of the forces on the left half of the arch. Draw a horizontal at the crown. This line is the outside line of the line of pressure at the crown. Extend this line to intersection with the resultant just found. Connect this new point of intersection with the springing. The line thus obtained is the end line of the line of pressure at the springing. Draw parallel lines to these two lines at the ends of the force polygon, the horizontal line at the bottom and the inclined line at the top. The point of intersection of these two lines gives the pole for the line of pressure.

The horizontal thrust is equal to the horizontal distance of the new pole from the force polygon, measured to the same scale as used in drawing the force polygon.

When the force polygon represents not actual forces but ordinates scaled in the manner described above for spandrel filled arches, then the horizontal thrust is obtained by multiplying the pole distance by the length of the division in feet, by the weight of the concrete per cubic foot and by the number by which the ordinate was divided before using in the force polygon.

To check the work, continue the outside rays of the funicular polygon to intersection with the verticals passing through springing and the crown, respectively. The points of intersection connected give the closing line of this polygon.

Draw from the original pole a line parallel to this closing line. This intersects the force polygon at a certain point *a*.

Connect the springing with the crown. This is the closing line for the line of pressure. From the new pole draw a line parallel to this closing line. If the work is correct this line will intersect the force polygon at the point *a*.

After checking the position of the new pole draw a new funicular polygon which is the desired line of pressure. The outside rays of this should pass through the springing and the crown, respectively.

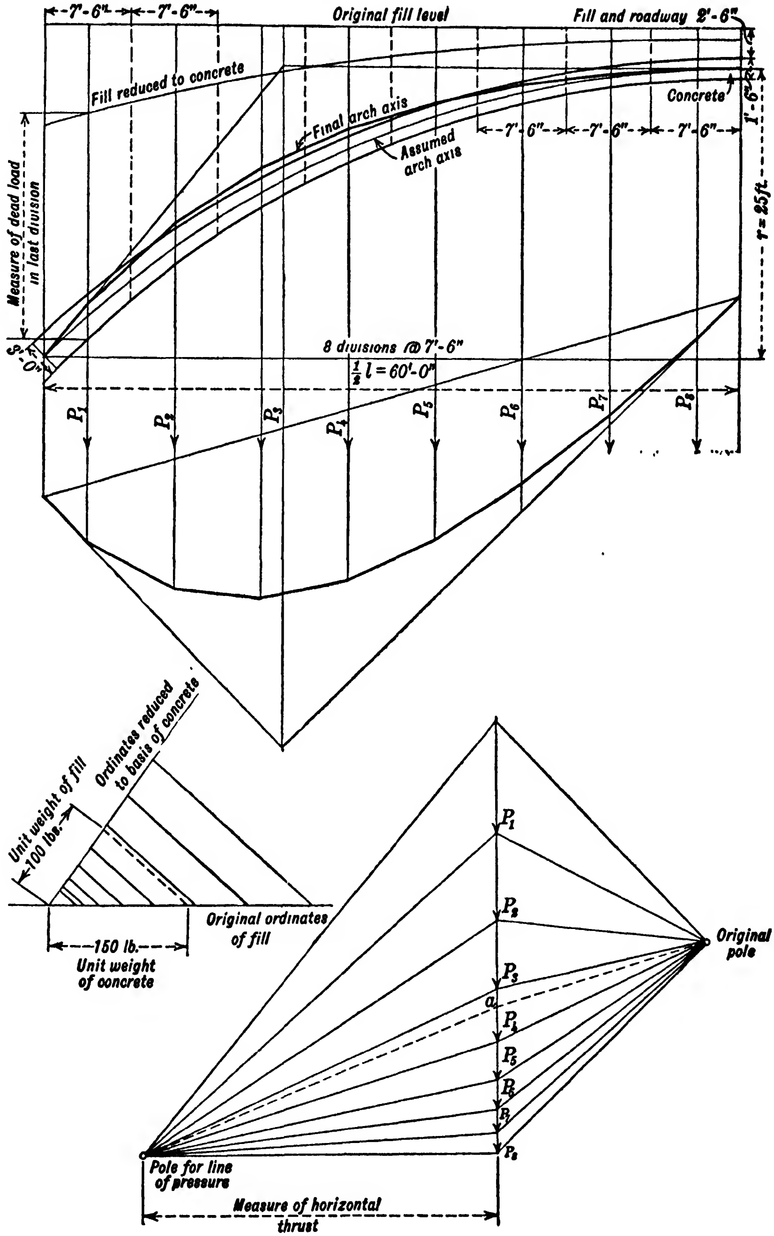


FIG. 193.—Graphical Determination of Line of Pressure. (*See p.* 472.)

PROPORTIONING OF ARCH RIB SECTIONS

Dimensions of Arch Rib Sections.—The cross-sections of the fixed arch rib are not constant throughout the length of the arch but are smallest at the crown and increase gradually until they reach a maximum at the springing. In arch design normal cross-sections are used, i.e., sections on planes perpendicular to the tangents to the arch axis.

Let

I = moment of inertia of section at the crown;
I_x = moment of inertia of section at an intermediate point x;
I_s = moment of inertia of section at the springing;
ϕ_s = angle between the section and the vertical at the springing;
ϕ_x = angle of inclination of intermediate section,
n = ratio $\dfrac{I}{I_s \cos \phi_s}$,
r = rise of arch;
l = span of arch;
h = depth of section at crown;
h_x = depth of section at intermediate point;

The relation between the moment of inertia at the crown and at the springing is usually expressed by the ratio n, which is the ratio of the moment of inertia at the crown, I, to the vertical projection of the moment of inertia at the springing, $I_s \cos \phi_s$. This ratio is not constant for all arches but depends mostly upon the ratio of rise to span of the arch. The ratio n decreases with the decrease of the ratio $\frac{r}{l}$. For shallow arches with $\frac{r}{l} = 0.1$ the ratio may be as small as $n = 0.1$. For larger values of $\frac{r}{l}$, n increases rapidly. For average conditions it is:

Ratio of n for average conditions:

$$n = \frac{I}{I_s \cos \phi_s} = 0.3.$$

Variation of Moments of Inertia at Intermediate Points.—At intermediate points the variation of the moments of inertia is usually represented by the ratio $\dfrac{I}{I_x \cos \phi_x}$. The moments of inertia at the intermediate points must be large enough to take care of the bending moments and thrusts there developed. It has been found from computations that both the parabolic variation of the ratios as expressed by Formula

(1) and the straight line variation as expressed by Formula (4) give large enough sections at the intermediate points when proper n is selected and when the section at the crown is large enough. The parabolic variation is more economical then the straight line variation and is recommended by the authors.

Parabolic Variation of Moments of Inertia:

$$\frac{I}{I_x \cos \phi_x} = 1 - 4(1 - n)\left(\frac{x}{l}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

where x is measured from the crown.

For a known ratio of $n = \dfrac{I}{I_s \cos \phi_s}$ and a known moment of inertia at the crown the moment of inertia at any intermediate point is

Moment of Inertia at Intermediate Point, Parabolic Variation

$$I_x = \frac{1}{\cos \phi_x \left[1 - 4(1 - n)\left(\frac{x}{l}\right)^2\right]} I \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

For rectangular sections the depth of the section at any intermediate point is

Depth of Rectangular Section at Intermediate Point, Parabolic Variation

$$h_x = \sqrt[3]{\frac{1}{\left[1 - 4(1 - n)\left(\frac{x}{l}\right)^2\right] \cos \phi_x}}\, h. \quad . \quad . \quad . \qquad (3)$$

The variation of moments of inertia given in Formula (1) was used in developing the formulas for parabolic arches given on p. 545. In the approximate method given on p. 480 a straight line variation was used to simplify the integration.
This is

Straight Line Variation of Moments of Inertia

$$\frac{I}{I_x \cos \phi_x} = 1 - (1 - n)\frac{2x}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Moment of Inertia at Intermediate Point, Straight Line Variation

$$I_x = \frac{1}{\cos \phi_x \left[1 - (1 - n)\frac{2x}{l}\right]} I, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

where x is measured from the crown.

Arch sections proportioned by these formulas require somewhat more material than by the parabolic variation given by previous formulas.

After the sections at the crown and the springing are selected, the ratio n is computed and the type of variation selected, the sections at intermediate points can be easily computed from the appropriate formula.

How to Determine Dimensions of Cross-sections.—The required dimensions of cross-sections of the arch cannot be computed directly as in case of a beam design, because the magnitude of the bending moments and thrusts required to determine the dimensions of the sections are dependent upon the same dimensions of cross-sections.

The work of determining the required dimensions of the arch rib, therefore, is somewhat involved. It consists of (*a*) determining preliminary dimensions of the arch rib, (*b*) determining statically indeterminate values and bending moments and thrusts at various points of the arch, based upon these preliminary dimensions, (*c*) checking of the preliminary dimensions to determine whether the stresses are satisfactory and determining the exact amount of reinforcement.

If the stresses are either too large or too small the dimensions of the sections are reduced or increased. Small changes in the dimensions of the arch do not affect to any extent the statically indeterminate values and therefore do not change the bending moments and thrusts. Appreciable changes, however, particularly if they are different for different parts of the arch, may require new computations for the statically indeterminate values. For this reason it is important to select the preliminary dimensions of the arch with care so as to require only as small changes in the dimensions as possible.

The preliminary dimensions are often selected by judgment or by "rule-of-thumb" formulas. More satisfactory results are obtained by actually computing the preliminary dimensions, using the approximate method described on p. 480. The results obtained by this method are sufficiently accurate so that only small final adjusting of dimensions will be necessary.

Determining of Preliminary Dimensions.—It is sufficient to compute preliminary dimensions at the crown and at the springing only. Preliminary dimensions of arch ribs are computed for approximate bending moments and thrusts, which are found as explained on p. 480. The computations may be made as follows:

1. The span and rise of the arch should be fixed. The type of roadway should be selected. Position of the top of the roadway in respect of the top of the arch should be accepted. For open spandrel arches

the floor construction should be designed as it is not dependent upon the dimensions of the arch ring.

2. The dead load on the arch is determined next in the manner described on p. 466. To get the total dead load it is necessary to assume the thickness of the arch. This first approximation of the arch dimensions may be selected by judgment or computed from rule-of-thumb Formula (6), p. 480. These dimensions are to be used only for computation of the dead load.

3. Find the values of q_s and q_c which are the intensities of the dead load, at the springing and at the crown, respectively (see p. 466). These determine the ratio of $m = \frac{q_s}{q_c}$. The approximate dead load thrust H_d is then found from Formula (35), p. 486, also the position of the arch axis $y_{\frac{1}{4}}$ at the quarter point, $x = \frac{l}{4}$, from Formula (10), p. 482.

4. The type of construction of the arch rib is selected, i.e., whether of plain concrete or reinforced concrete. For reinforced concrete section the minimum steel ratio to be used is selected (see p. 448).

5. The live load for which the bridge is to be designed is decided upon and also the extent of the temperature changes to be provided for.

6. Approximate bending moments and thrusts for live load, temperature changes and rib shortening are found by the approximate method from diagrams in Chapter X. It is sufficient to find values for the crown and the springing. The bending moments and thrust for various conditions are combined so as to get the most unfavorable combination (see p. 487).

7. For the most unfavorable combination of the bending moments and thrusts found as above, preliminary dimensions of the arch rib are determined, using diagrams opp. p. 648 for plain concrete sections and diagrams pp. 654–655 for reinforced concrete sections. The allowable unit stresses in compression are given on p. 453. As far as tensile stresses are concerned, only first condition, p. 454, needs to be considered for preliminary dimensions. The amount of steel actually required is found after final computations are made.

Rule-of-Thumb Formula for Arch Thickness.—The following rule-of-thumb formula, originally developed by F. F. Weld,[2] and revised by the authors, may be used as a guide for the first approximation of the arch thickness to be used for computing the dead load.

[2] *Engineering Record*, Nov. 4, 1905, p. 529.

Let h = thickness of arch at crown, ins.;
l = length of span of arch, ft.;
w = uniformly distributed live load, lb. per sq. ft.;
w' = weight of fill at crown, lb. per sq. ft.;
c = constant;
f_c = allowable compression stress at crown.

Then

Thickness of Arch at Crown,

$$h = c\left(\sqrt{l} + \frac{l}{10} + \frac{w}{200} + \frac{w'}{490}\right), \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where

$$c = \frac{450}{f_c} \text{ for plain concrete,}$$

$$c = \frac{450}{1.14 f_c} \text{ for reinforced concrete.}$$

The rule-of-thumb formula should never be used for the final design.

Relation between the thickness at the crown and that at the springing is discussed on p. 476. The thicknesses at intermediate points should be varied as given on p. 477.

APPROXIMATE METHOD OF DESIGNING FIXED ARCHES

The approximate method [3] given below is based upon the same elastic theory as used in the exact method. It differs from the exact method in that instead of the actual dead loads, assumed dead loads are used which vary according to a fixed rule. For this assumed dead load the line of pressure can be represented by a closed mathematical function and the formulas for the statically indetermined values may be solved by integration.

Following notation is used in the formulas for the approximate method.

[3] This method has been evolved partly by Dr. Ing. Färber and partly by A. Strassner. (See Deutsche Bauzeitung No. 3, year 1915, "Neues Verfahren zur raschen Ermittelung der Formen und Normalkräfte in Gewölben," by Dr. Ing. R. Färber. Also A. Strassnei, "Neure Methoden," Vol. 2. Wilhelm Ernst & Son, publishers.) Subsequently this method with modifications was embodied in a paper presented by Mr. Charles S. Whitney before the American Society of Civil Engineers. See Transactions of Am. S. of C. E., Vol. 88, year 1925, p. 931.

Notation.

Let Y = ordinate for any point on arch axis, center of coordinates at crown;

Y_c = vertical distance of elastic center from crown,

$n = \dfrac{I}{I_s \cos \phi_s}$ = ratio of moments of inertia at crown and of vertical projection at springing.

x = abscissa for any point on arch axis, center of coordinates at crown;

l = span length of arch axis, ft.;

r = rise of arch axis, ft.;

q_c = unit dead load at crown, lb. per lin. ft. of arch;

q_s = unit dead load at springing, lb. per lin. ft. of arch;

q_x = unit dead load at any point lb. per lin. ft. of arch;

$m = \dfrac{q_s}{q_c}$ = ratio of dead loads;

I = moment of inertia at crown;

I_s = moment of inertia at springing;

I_x = moment of inertia at any point x;

ϕ_x = angle of inclination of tangent to arch axis at point x;

ϕ_s = angle of inclination of tangent to arch axis at springing.

Use of Approximate Method.—The approximate method is very useful:

1. For proper selection of preliminary dimensions;
2. For determining the preliminary shape of arch axis.

As the amount of work involved in the determination of the bending moments and thrusts is comparatively small, computations may be made for several ratios of $\dfrac{I}{I_s}$ and finally the most economical arch sections selected.

After the most suitable dimensions of the arch are selected an actual line of pressure for the dead load should be drawn and this should be accepted as the actual arch axis.

The criterion of the correctness of the results obtained by this approximate method is the degree to which the arch axis as obtained by this method coincides with the arch axis obtained by drawing the actual line of pressure for the dead load. If the two curves coincide, the results by the approximate method are exact.

For small arches the dimensions obtained by the approximate method may be used for final design without the exact analysis, unless the line of pressure for dead load differs considerably from the arch axis obtained by the approximate formula.

For more important structures complete analysis should be made after satisfactory dimensions are obtained by the approximate method.

Assumption.—The dead load at the various points in the arch is assumed to be represented by the equation:

Dead Load at Any Point,

$$q_x = q_c\left[1 - \frac{Y}{r}(m - 1)\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $m = \frac{q_s}{q_c}$ and Y is the ordinate of arch axis when center of coordinates is at the crown.

Arch Axis as Line of Pressure.—A formula for line of pressure is found for the dead load given by the above equation and this line of pressure is accepted as the arch axis.[4] The formula for arch axis is

Ordinates of Arch Axis. Center of Coordinates at Crown,

$$Y = \frac{7}{m - 1}(\text{hyper. cos } 2k - 1), \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where

$$k = \log\,(m + \sqrt{m^2 - 1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$m = \frac{q_s}{q_c}.$$

Finally

Ordinates of arch axis.

$$Y = C_o r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where the constant C_o is given in Diagram 22, p. 670, for various ratios of m and for $\frac{x}{l} = \frac{1}{8}, \frac{1}{4}$, and $\frac{3}{8}$.

Where tables of hyperbolic functions are not available and the constants in Diagram 22, p. 670, do not apply, the following approximate formula may be used.

[4] For note (see p. 480).

Approximate Formula for Arch Axis. Center of Coordinates at crown.

$$Y=\frac{4}{3}\left(\frac{x}{l}\right)^2\left\{\frac{16}{2+\sqrt{2(m+1)}}-1-4\left(\frac{4}{2+\sqrt{2(m+1)}}-1\right)\left(\frac{x}{l}\right)^2\right\}r. \quad (11)$$

Variation in Moments of Inertia.—To solve the integrals in the formulas for static indeterminate values, following equation was assumed for the variation of the moments of inertia.

Variation of Moments of Inertia,

$$\frac{I}{I_x \cos \phi_x} = 1 - (1 - n)\frac{2x}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where

$$n = \frac{I}{I_s \cos \phi_s}.$$

Elastic Center.—The formulas for Y and $\frac{I}{I_x \cos \phi_x}$ substituted in the general equation for the elastic center gives [3]

$$Y_c=\frac{2}{(m-1)(n+1)}\left\{\frac{\sqrt{m^2+1}}{k}-1-(1-n)\left[\frac{\sqrt{m^2-1}}{k}-\frac{m-1}{k}-\frac{1}{2}\right]\right\}r. \quad (13)$$

Finally, where k is given by Formula (9), p. 482.

Distance of Elastic Center from Crown.

$$Y_c = C_e r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

C_e is given in Diagram 23, p. 670, for different ratios of m and n.

Denominators of Horizontal Thrust.—The denomination for the horizontal thrust is

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{I}{I_x}ds + \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{Ax}.$$

The second item may be omitted. Then

Denominator for Horizontal Thrust,[5]

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{I}{I_x}ds = C_h l r^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where C_h is given in Diagram 24, p. 671, for different ratios of m and n.

This formula is used in determining the horizontal thrust for temperature changes, rib shortening for dead load and shrinkage.

[5] For development of these formulas see A. Strassner, Neuere Methoden.

Denominator for Vertical Reaction V_A.—Substituting values for $\frac{I}{I_x}ds$ from Formula (12), the denominator for V_A becomes

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \frac{I}{I_x} ds = \frac{1 + 3n}{48} l. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Denominator for Auxiliary Moment M.—The denominator for M for the variation of moments of inertia expressed by Formula (12) is

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{I}{I_x} ds = \frac{1 + n}{2} l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Maximum Bending Moments for Live Load.—Using the above formulas for the statically indeterminate values and the assumed formulas for the arch axis and the variation of moments of inertia, formulas were developed by A. Strassner for maximum bending moments at the crown, the quarter point and springing due to uniformly distributed loading. On the basis of these formulas constants in Diagrams 26 to 31 were developed by means of which the bending moments and the corresponding thrusts can be obtained.

Crown.

Maximum Positive Bending Moment,

$$M_c = C_{(+c)} wl^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Corresponding Horizontal Thrust,

$$H_c = - C_{(+hc)} wl \frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Maximum Negative Bending Moment,

$$M_c = - C_{(-c)} wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Corresponding Horizontal Thrust,

$$H_c = - C_{(-hc)} wl \frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Values of $C_{(+c)}$, $C_{(-c)}$, $C_{(+hc)}$, $C_{(-hc)}$ may be taken from Diagrams 26 and 27, pp. 672 and 673, for proper ratios of moments of inertia n and dead load ratio m.

Quarter Point $x = \frac{l}{4}$.

Maximum Positive Bending Moment,

$$M_{\frac{1}{4}} = C_{(+\frac{1}{4})} wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Corresponding Horizontal Thrust,

$$H_{\frac{1}{4}} = - C_{(+\frac{1}{4}h)} wl\frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Maximum Negative Bending Moment,

$$M_{\frac{1}{4}} = - C_{(-\frac{1}{4})} wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Corresponding Horizontal Thrust,

$$H_{\frac{1}{4}} = - C_{(-\frac{1}{4}h)} wl\frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Values of $C_{(+\frac{1}{4})}$, $C_{(-\frac{1}{4})}$, $C_{(+\frac{1}{4}h)}$, $C_{(-\frac{1}{4}h)}$ may be taken from Diagrams 28 and 29, pp. 674 and 675, for the proper ratios of moments of inertia n and the dead load ratio m.

Springing.

Maximum Positive Bending Moment,

$$M_s = C_{(+s)} wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Corresponding Horizontal Thrust,

$$H_s = - C_{(hs)} wl\frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Maximum Negative Bending Moment,

$$M_s = - C_{(-s)} wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Corresponding Horizontal Thrust,

$$H_s = -C_{(-hs)} wl\frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Values of $C_{(+s)}$, $C_{(-s)}$, $C_{(hs)}$, $C_{(-hs)}$, may be taken from Diagrams 30 and 31, pp. 676 and 677, for proper ratios of moments of inertia n and the dead load ratio, m.

Normal Thrusts.—Normal thrusts can be obtained from the following formula.

Normal Thrust for Dead or Live Load,

$$N_x = \frac{H}{\cos \phi_x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

This formula gives exact results for dead load, when the line of pressure coincides with the arch axis. For live load the error is not appreciable.

Effect of Temperature Changes.—Using the denominator for horizontal thrust given by Formula (15) and the distance for elastic center from Formula (14) and letting

$\pm t$ = total expected change of temperature in degrees Fahrenheit from temperature at closing of arch (+ for rise of temperature, − for fall of temperature);
α = coefficient of expansion due to temperature changes;
l = span of arch, ft.;
r = rise of arch, ft.;
I = moment of inertia of arch section at the crown;
C_h = constant from Diagram 24, p. 671.

Then
Horizontal Thrust Due to Changes of Temperature,

$$H_t = -\frac{\alpha EI(\pm t)}{r^2}\frac{1}{C_h}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where C_h is a constant from Diagram 24, p. 671.
Bending Moments Due to Changes of Temperature:

Crown,

$$M_{ct} = \pm H_t Y_c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Springing,

$$M_{st} = \mp H_t(r - Y_c). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

For Fall of Temperature H is positive, the bending moment at the crown is positive and the bending moment at the springing is negative.
For Rise of Temperature H is negative, the bending moment at the crown is negative and the bending moment at the springing is positive.

Horizontal Thrust for Dead Load.—For known unit dead load at crown q_c and at the springing q_s and the assumed variation of dead load given on p. 482 by Formula (7), and assuming that the arch axis corresponds with the line of pressure for the dead load, the formula for the horizontal thrust becomes

Horizontal Thrust for Dead Load,

$$H_d = (0.1080 + 0.0190m - 0.0005m^2)q_c\frac{l^2}{r}, \quad . \quad . \quad . \quad (34)$$

also

$$H_d = C_d q_c \frac{l^2}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where C_d is a constant from Diagram 25, p. 671.

Rib Shortening Due to Dead Load.—The effect of rib shortening for known H_d may be obtained from the general formula

$$H_s = -\frac{I\dfrac{l}{A_{av}}}{\displaystyle\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x}} H_d$$

by substituting for the denominator the value given by Formula (15), p. 483.

Horizontal Thrust for Rib Shortening,

$$H_s = -\frac{I\dfrac{l}{A_{av}}}{C_h l r^2} H_d = -\frac{I}{A_{av} r^2}\frac{1}{C_h} H_d. \qquad (36)$$

Values of C_h may be taken from Diagram 24, p. 671.

Bending Moments Due to Rib Shortening:

At crown,

$$M_c = H_s Y_c. \qquad (37)$$

At springing,

$$M_s = -H_s(r - Y_c). \qquad (38)$$

Effect of Shrinkage.—The effect of shrinkage may be taken care of by assuming it to be equivalent to a drop of temperature of 15 degrees Fahrenheit, because the shortening of the arch rib caused by shrinkage is about the same as the shortening produced by the specified drop of temperature.

Angles ϕ_s and $\phi_{\frac{1}{4}}$ at Springing and Quarter Point.—To get the normal thrusts it is necessary to know the magnitude of the angles of inclination of the tangent to the arch axis at the springing and at the quarter points. These can be taken from Diagram 21, p. 669, for different assumptions as to the shape of the arch.

Computation of Dimensions.—After the bending moments and thrusts are found for the various conditions they should be tabulated as outlined in table on p. 488. To simplify the summation the positive bending moments and the negative bending moments with the corresponding thrusts are tabulated in separate columns. Separate tables should be made for the crown, the quarter point and the springing.

The thrusts due to the dead load and the rib shortening and the bending moment due to these causes should be entered with their signs both in the negative and the positive columns. The items in the positive and the negative columns should then be added. Attention is called to the fact that not all the thrusts in each column are of the same sign. This should be taken into account when adding the items.

After the resulting bending moments are found the required sections may be obtained by using Diagrams 1–2, opp. p. 648, for plain concrete sections and Diagrams 7–8, pp. 654 and 655, for reinforced concrete sections. The accepted sections should be such that neither the maximum allowable compressive unit stresses nor the allowable tensile unit stresses in the section should be exceeded (see p. 453).

Section at Springing

Allowable compressive unit stress f_c =
Allowable tensile unit stress f_t =
$\cos \phi_s$ =

Type of Loading	Positive Bending Moment		Negative Bending Moment	
	Thrust	Bending Moment	Thrust	Bending Moment
Dead Load............				
Rib Shortening..........				
Live Load..............				
Temperature Changes....				
Shrinkage..............				
Yielding of Piers.......				
Sum...................				

Enter the Thrusts and Bending Moments for Dead Load and Rib Shortening, with their signs, both in the positive and the negative columns.

Use the sum of bending moments and the corresponding thrusts in each column to determine the dimensions of the section.

Find the normal thrust from formula $N = \dfrac{H}{\cos \phi_s}$.

Make similar tables for the Crown and Quarter Point.

EXACT METHOD OF ANALYSIS FOR FIXED ARCHES

As explained in the previous pages a fixed arch is a statically indeterminate structure with three statically indeterminate values. These are

1. Horizontal thrust, H.
2. Vertical reaction, V_A.
3. Auxiliary bending moment, M.

To solve the problem of computing the stresses in an arch it is necessary to compute these three statically indeterminate values. The bending moments and shears at any section of the arch may then be computed by statics. The formulas for the statically indeterminate values, full derivation of which is given in Chapter VIII on Theory of Arches, are repeated here for convenient use.

The design of an arch usually consists of the determination of the arch axis, the selection of the dimensions either by judgment or by using approximate formulas, the computation of bending moments and thrusts using the statically indeterminate values, finally the computation of stresses.

The final analysis should include:

Dead load, including the effect of rib shortening;
Live load;
Effect of changes of temperature;
Effect of shrinkage;
Effect of yielding of abutments.

Dead Load.—When the arch axis coincides with the line of pressure for the dead load, no bending moments exist in the arch except those produced by the rib shortening. The horizontal thrust for dead load is easily determined from line of pressure as explained on p. 471. The effect of rib shortening is then found from Formula 51, p. 494.

Live Load.—The live load may be considered either as uniformly distributed or as consisting of concentrated wheel loads.

For uniformly distributed loading compute first the denominators for H, V_A, and M, using Tables 1 and 2. These are constant for all positions of the load. The numerators for the statically indeterminate values depend upon the position of the live loads on the arch. As explained on p. 505, it is sufficient to determine the numerators for the live load extending over the whole span, using Table 3, p. 502, and for the live load extending over $\frac{5}{8}$ of the span length, using Table 4, p. 504. By proper combination of these values in the manner described on p. 506 it is possible to get statically indeterminate values for the most unfavorable positions of the live load producing maximum stresses at the crown, the quarter point and the springing.

The actual steps to be taken may be seen more clearly from the example on p. 511.

For concentrated wheel loads the problem is solved most conveniently by the use of the influence lines.

After the statically indeterminate values are determined, bending moments and thrusts are found at the critical sections, namely, the crown, the quarter point and the springing.

Computation of Stresses.—After the bending moments and thrusts are found for the various conditions they should be tabulated as outlined in table, p. 491. Separate tables should be made for the crown, the quarter point and the springing. The negative bending moments and the positive bending moments with the corresponding thrusts should be tabulated in separate columns.

The thrusts due to the dead load and the rib shortening and the bending moment due to the rib shortening should be combined with the positive bending moments and the corresponding thrusts and then with the negative bending moments and the corresponding thrusts.

Finally stresses should be computed for the resulting bending moments and thrusts. If the stresses are too large or too small, either the dimensions of the section or the amount of reinforcement should be changed. The formulas for the computation of the stresses are given in Chapter II on Direct Stress and Bending. The formula to be used depends upon the design of the section, i.e., whether plain or reinforced, and upon the character of the stresses, i.e., whether the whole section is in compression or whether part of it is in tension.

If additional reinforcement is required it may be added at both faces so that the section is symmetrically reinforced or only at the face at which the stresses are excessive. In the last case an unsymmetrically reinforced section will result for which formulas given on p. 232 may be used.

If change in section is required, the new section may be found by using Diagrams 1–2, opp. p. 648, for plain sections and Diagrams 7, 8, p. 654, for reinforced concrete sections. The final dimensions should be such that neither the maximum allowable compressive unit stress nor the allowable tensile unit stresses in the section will be exceeded (see p. 453).

Section at Springing

Concrete area A = Steel area A_s = Moment of inertia I_s =
Depth of section h = $\cos \phi_s$ =

Type of Loading	Positive Bending Moment		Negative Bending Moment	
	Thrust	Bending Moment	Thrust	Bending Moment
Dead Load.				
Rib Shortening				
Live Load				
Temperature Changes. . .				
Shrinkage.				
Yielding of Piers. . . .				
Sum.				

Combine Thrust and Bending Moment for Dead Load and Rib Shortening with both the sum of all positive and the sum of all negative bending moments and their corresponding thrusts.

Get the normal thrust from formula $N = \dfrac{H}{\cos \phi_s}$.

Use the sum of bending moments and thrust in each column to compute stresses or to determine dimensions of the section.

Make similar tables for the Quarter Point and the Crown.

FINAL EXACT FORMULAS FOR FIXED ARCHES

Notation.

Let Y = abscissa of the arch axis referred to left support as center of coordinates;

X_s and Y_s = location of elastic center with reference to left support;

ds = length of a division of the arch;

l = span of arch;

A_x = area of average section in each division of the arch;

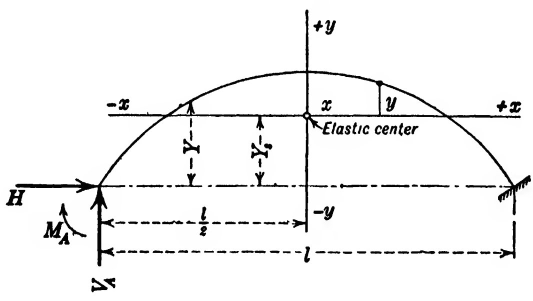
I_x = moment of inertia of average section in each division of arch;

I = moment of inertia of section at the crown;
x, y = coordinates of the center of the division of the arch referred to axes through elastic center;
ϕ_x = angle with vertical of any normal section of the arch;
M_s = static bending moment of loads considering arch as cantilevered at right support (see p. 600);
H = horizontal thrust at both supports for vertical loads;
M_A = bending moment at left support;
V_A = vertical reaction at left support;
M = auxiliary bending moment;
M_x = bending moment at any section of the arch;
N_x = normal thrust at any section of the arch.

Assumptions.

Symmetrical arch;
Center of coordinates in elastic center of arch;
Arch fixed, supports unyielding.

Final Formulas.—The final formulas for exact analysis of an arch are given in the succeeding pages. Formulas are given for vertical loads, for the effect of change of span length, and for the effect of rib shortening and temperature changes. (See Fig. 194, p. 492.)



Arch considered as fixed at right support and free at left support. V_A, H and M_A replace the effect of fixity at left support.

FIG. 194.—Bending Moment and Reactions at Left Support of Fixed Arch. (*See p.* 492.)

Position of Center of Coordinates with Reference to Left Support,

$$X_s = \tfrac{1}{2}l, \quad \text{(39)}$$

$$Y_s = \frac{\sum_0^{\frac{l}{2}} Y \frac{I\,ds}{I_x}}{\sum_0^{\frac{l}{2}} \frac{I\,ds}{I_x}}, \quad \text{(40)}$$

where Y are the ordinates of the arch axis referred to the left support as the center.

Horizontal Thrust Due to Vertical Loads,

$$H = -\frac{\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_s y \frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}}. \quad . \quad . \quad . \quad (41)$$

This value is negative. $\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_s}$ may be omitted or made $\frac{Il}{A_{av}}$. (See p. 608.)

Vertical Reaction Due to Vertical Loads,

$$V_A = -\frac{\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_s x \frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \frac{Ids}{I_x}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Since M_s is negative (see p. 600), V_A is positive.

Auxiliary Bending Moment,

$$M = -\frac{\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_s \frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Since M_s is negative (see p. 600), M is positive.

Bending Moment at Left Support,

$$M_A = M - V_A \frac{l}{2} - H_A Y_s. \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Bending Moment at Any Section of the Arch with Ordinates x and y,

$$M_x = M + V_A x + H_A y + M_s. \quad . \quad . \quad . \quad (45)$$

Normal Thrust and Shear at Any Point,

$$N_x = V_x \sin \phi_x + H_A \cos \phi_x, \quad . \quad . \quad . \quad . \quad (46)$$

$$V_x = V_A - \Sigma P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

In the above equations M_s is the static bending moment of the vertical loads at the various points, obtained for each point by multiplying the loads to the left of it by their distance from the point under consideration as explained on p. 600.

Effect of Change of Span Length

Let Δl = change in span length, in.;
E = modulus of elasticity, lb. per sq. in.

Δl is positive for shortening of span and negative for lengthening of span. Formulas are developed on p. 602.

Horizontal Thrust Due to Change of Span Length,

$$H = -\frac{EI\Delta l}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}} \quad . \quad . \quad . \quad . \quad (48)$$

Bending Moments at Support,

$$M_A = M_B = -HY_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Bending Moment at Any Point x,

$$M_x = Hy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Use H and y with their signs. Y_s is positive.

Effect of Rib Shortening for Dead Load

The formulas given below are developed on p. 606.

Let H_d = horizontal thrust for dead load;
H_s = horizontal thrust for rib shortening.

Horizontal Thrust Due to Rib Shortening,

$$H_s = \frac{\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}} H_d. \quad . \quad . \quad . \quad . \quad (51)$$

This thrust is positive. $\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_s}$ may be replaced by $\frac{Il}{A_{av}}$. (See p. 608.)

Bending Moment at Any Point x,

$$M_x = H_s y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Maximum Negative Bending Moment at Support, $y = -Y_s$,

$$M_s = -H_s Y_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Maximum Positive Bending Moment at Crown, $y = (r - Y_s)$,

$$M_c = H_s(r - Y_s). \quad (54)$$

Effect of Temperature Changes

Formulas are developed and fully discussed on p. 608.

Let t = change of temperature in degrees;
α = coefficient of expansion per 1 degree Fahrenheit;
E = modulus of elasticity;

Average values: E = 2 000 000 lb. per sq. in.; α = 0.0000055 and αE = 11. Value of t for average conditions equals $\pm$ 30 degrees.

Fall of Temperature.—Formulas below give effect of a fall of temperature by t degrees below the temperature at closing of the arch.

Horizontal Thrust,

$$H_t = \frac{\alpha EtI}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}}. \quad (55)$$

Bending Moment at Any Point,

$$M_{tx} = H_t y. \quad (56)$$

Maximum Bending Moment at Springing, $y = -Y_s$,

$$M_{ts} = -H_t Y_s. \quad (57)$$

Maximum Bending Moment at Crown, $y = (r - Y_s)$,

$$M_{tc} = H_t(r - Y_s). \quad (58)$$

Rise of Temperature.—Formulas below give effect of rise of temperature by t degrees above the temperature at closing of arch.

Horizontal Thrust,

$$H_t = -\frac{\alpha EtI}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}}. \quad (59)$$

Maximum Bending Moment at Springing, $y = -Y_s$,

$$M_{ts} = -H_t Y_s. \quad (60)$$

Maximum Bending Moment at Crown, $y = (r - Y_s)$,

$$M_{tc} = H_t(r - Y_s). \quad (61)$$

Effect of Shrinkage

The effect of shrinkage may be obtained by computing the expected shrinkage and considering this as equal to the lengthening of the span Δl. This value is negative. Substituted in Formulas (48) to (50), p. 494, it gives the horizontal thrust and bending moments due to shrinkage. (See also p. 487.)

METHOD OF SOLVING FORMULAS FOR H, V_A AND M

Formula (41) to (43) must be solved in most cases by the summation method. This means that the arch must be divided into small divisions, then computations must be made for each division and finally the results must be added. The work can be simplified by arranging the computations in table form in the manner shown in Tables 1 to 4. The tables are:

Table 1.—Method of Finding Elastic Center Y_s and Denominator for M;

Table 2.—Method of Finding Denominators for H and V_A;

Table 3.—Method of Finding Numerator for H and M for full uniform load;

Table 4.—Method of Finding Numerators for H, V and M for uniform load on $\frac{5}{8}$ of the span;

After the values of H, V and M are computed for the full uniform load (Table 3) and the partial load (Table 4), the values for the most unfavorable positions of loads may be found as explained on p. 505.

Determining of Elastic Center of Arch.—After the shape of the arch is determined (see p. 469) and the thickness of arch sections decided upon the elastic center is found as follows:

1. Accept first a system of coordinates with a center at the left support. If the springings are on the same level, then the X-axis is horizontal. If the road is on a grade the springings may be placed on a line parallel to the road. In such case the line connecting the springings and not the horizontal line should be accepted as the X-axis. The Y-axis is always vertical.

2. Divide the arch into a number of divisions. While the divisions may be of any length, usually the work is simplified if either their projection on the X-axis is constant or if the ratio $\frac{Ids}{I_x}$ is constant. The first method is recommended.[6] The span is divided into a number

[6] While in some cases the method with constant $\frac{Ids}{I_x}$ requires somewhat less work, it has the disadvantage that the end divisions are too large. Also if after first com-

of equal parts. Vertical lines through the division points intersect the arch axis at the division points. The number of divisions depends upon the span. For ordinary spans 8–10 divisions for each half of the arch are sufficient.

3. Find the values of X and Y relating to the original axes for the center of each division.

4. Find the length of each division of the arch ds, measured along the arch axis. This is best done by scaling.

5. Find thickness of arch ring at the center of each division.

The width of the section for barrel arches may be assumed equal to 1 ft. In rib arches the actual width must be taken.

6. The value of Y_s is found by tabulating values as shown in the table below. Since the arch is symmetrical only one-half of the arch needs to be considered.

Table 1.—Method of Finding Elastic Center Y_s, New Values of y and Denominator for M

(Arch has eight divisions for each half. I = moment of inertia at crown)

No. of Section	X	Y	ds	Properties of Sections				$\frac{Ids}{I_x}$	$Y\frac{Ids}{I_x}$	y	$y\frac{Ids}{I_x}$	x
				b	d	$A_x = bd$	$I_x = \frac{bd^3}{12}$	$I \times \frac{(4)}{(8)}$	(3) × (9)	(3) − Y_s	(11) × (9)	(2) − $\frac{l}{2}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
1												
2												
3												
4												
5												
6												
7												
8												

Add all values in column (10) to get $\sum Y\frac{Ids}{I_x}$ for one-half of the arch.

Add all values in column (9) to get $\sum\frac{Ids}{I_x}$ for one-half of the arch.

$$Y_s = \frac{\sum Y\frac{Ids}{I_x}}{\sum\frac{Ids}{I_x}} = \frac{\text{Sum Col. (10)}}{\text{Sum Col. (9)}}.$$

As a check, add values in column (11). If the sum is zero the work is correct.

Denominator for the auxiliary bending moment M equals sum of Col. (9) multiplied by **2**.

Col. (11) and (13) give new ordinates referred to elastic center.

putation it is necessary to change the thickness of some parts of the arch, the constant $\frac{Ids}{I_x}$ may be upset and a new division required before the arch can be refigured.

Table 2.—Method of Finding Denominators for H and V_A

(In this table arch assumed to have eight divisions per each half)

No. of Section	y	$\frac{Ids}{I_x}$	$y\frac{Ids}{I_x}$ (2)×(3)	$y^2\frac{Ids}{I_x}$ (2)×(4)	x	x^2	$x^2\frac{Ids}{I_x}$ (7)×(3)	Ids	A_x	$\frac{Ids}{A_x}$	$x\frac{Ids}{I_x}$ (6)×(3)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1 2 3 4 5 6 7 8		From Table 1	From Table 1								
Sum			$\frac{1}{2}\sum y^2\frac{Ids}{I_x}=$			$\frac{1}{2}\sum x^2\frac{Ids}{I_x}=$			$\frac{1}{2}\Sigma=$		

$\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{Ids}{I_x}$ equals twice the sum of column (5).

$\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{A_x}$ equals twice the sum of column (11). Columns (9), (10) and (11) may be omitted when the second member of the denominator for H is made equal to $\frac{Il}{A_{av}}$.

The sum of two above values is denominator in Formula (41), p. 493, for H.

$\sum_{-\frac{l}{2}}^{\frac{l}{2}} x^2\frac{Ids}{I_x}$ is twice the sum of column (8). It is the denominator in Formula (42), p. 493, for V_A.

Column (12) is required only for influence lines for H.

Denominator for Auxiliary Bending Moment.—The denominator for the auxiliary moment M in Formula (43), p. 493, is $\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}$.

It does not depend upon the character of the loading but only upon the dimensions of the arch. The denominator for Y_s previously computated is $\sum_{0}^{\frac{l}{2}} \frac{Ids}{I_x}$. By comparing the two, it is evident that the denominator for M consists of sums for the whole arch while for Y_s it consists of sums of the same values but only for one-half of the arch. Since the arch is symmetrical the desired denominator is equal to

$$\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x} = 2\sum_{0}^{\frac{l}{2}} \frac{Ids}{I_x},$$

where $\sum_{0}^{\frac{l}{2}} \frac{Ids}{I_x}$ is the sum of column (9) in the previously computed table 1 on p. 497.

Denominators for H and V_A.—The denominator for H is

$$\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}$$

or

$$\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \frac{Il}{A_{av}}$$

where A_{av} is the average cross-sections area, and for V_A

$$\sum_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \frac{Ids}{I_x}.$$

The computation is shown in Table 2 p. 498. In symmetrical arches it is sufficient to compute the values for one-half of the arch. The sum for the whole arch is then double the sum for one-half the arch.

NUMERATOR FOR H, M AND V_A FOR DEAD LOAD

The numerators for H, M and V_A are dependent upon the character of the loading as well as upon the shape and dimensions of the arch.

When the arch axis coincides with the line of pressure for dead load, no computation for H, M and V_A are required because $M = 0$, V_A is static reaction and H is obtained from line of pressure. (See p. 471.)

When the arch axis does not coincide with the line of pressure for dead load then the statically indeterminate values H and M must be computed just as for the live load. For symmetrical arches V_A for dead load equals one-half of the total load.

The denominators for the statically indeterminate values are the same as used for live load. The first step in determining the numerators is to compute the dead load for each division of the arch and consider it as applied in the center of the division.[7]

The values of M_s which appear on the numerators can be best found graphically by drawing a funicular polygon as shown in Fig. 195. The values are then scaled at each section.

For symmetrical arches the work of computing the numerator for H and M may be simplified by adding the values of M_s for symmetrical

[7] Actually the load should be applied in the center of gravity for the dead load in that division. Usually such refinement is not warranted.

points and multiplying the sum by the common values of $y\frac{Ids}{I_x}$ and $\frac{Ids}{I_x}$. The procedure of computing the numerators is outlined in Table 3.

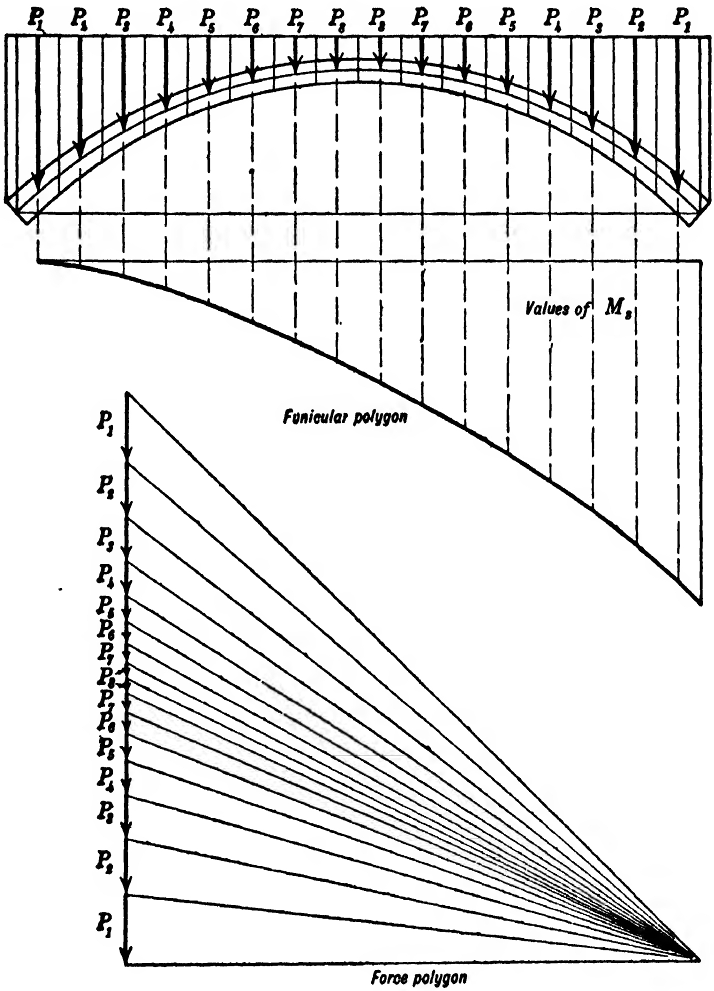


Fig. 195.—Static Bending Moments for Dead Load. (*See p.* 499.)

For symmetrical loads the numerator for V_A does not need to be computed because the vertical reaction equals one-half of the dead load upon the arch.

NUMERATOR FOR H, M AND V_A UNIFORMLY DISTRIBUTED LIVE LOAD

Full Span Loaded.—Assume that the arch is symmetrical.

Numerator for V_A for symmetrical arches loaded by symmetrically disposed loads does not need to be computed because the value of V_A is equal to one-half of the load upon the arch.

The numerator for H is $\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_s y \frac{Ids}{I_x}$ and for M it is $\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_s \frac{Ids}{I_x}$. The values could be found by computing M_s for each section of the arch, multiplying them by the corresponding values of $y\frac{Ids}{I_x}$ and $\frac{Ids}{I_x}$, respectively, and adding the results.

For symmetrical arches and loads the work may be simplified. The values of $y\frac{Ids}{I_x}$ and $\frac{Ids}{I_x}$ are the same for points located symmetrically about the y-axis on both sides of the arch. The ordinates of the two points are $+ x$ and $- x$. Instead of multiplying separately the value of M_s for $+ x$ by $y\frac{Ids}{I_x}$ or $\frac{Ids}{I_x}$ and the value of M_s for $- x$ by the same $y\frac{Ids}{I_x}$ or $\frac{Ids}{I_x}$, the values of M_s for $+ x$ and $- x$ may be added and the sum multiplied.

The bending moment M_s at any point x for full uniform load may be found from Formula (47), p. 602. It is for the point at the right half of the arch for which $x = + x$.[8]

$$M_{s(x)} = -\frac{w}{8}(l + 2x)^2.$$

For the symmetrical point at the left half of the arch for which $x = - x$ the bending moment becomes

$$M_{s(-x)} = -\frac{w}{8}(l - 2x)^2.$$

The sum of the moments for $+ x$ and $- x$ is

$$M_{s(x)} + M_{s(-x)} = -\frac{w}{8}(l + 2x)^2 - \frac{w}{8}(l - 2x)^2 = -w\left[\left(\frac{l}{2}\right)^2 + x^2\right].$$

These values may be substituted in the formulas for the numerator

[8] $M_{s(x)}$ denotes bending moment M_s at section x. $M_{s(-x)}$ denotes bending moment M_s at section $-(x)$.

for H_A and M. The summations then need to be made only for one-half of the arch; the limits of the sums will be $\sum_0^{\frac{l}{2}}$.

Numerator for H_A for Full Load.—Substitute these values in the formula for the numerator for H_A.

$$\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_s y \frac{Ids}{I_x} = -w \sum_0^{\frac{l}{2}} \left[\left(\frac{l}{2}\right)^2 + x^2\right] y \frac{Ids}{I_x}$$

$$= -w\left[\sum_0^{\frac{l}{2}} x^2 y \frac{Ids}{I_x} + \left(\frac{l}{2}\right)^2 \sum_0^{\frac{l}{2}} y \frac{ds}{I_x}\right].$$

Table 3.—Numerators for H and M for Full Uniform Load

(Arch axis assumed to consist of eight divisions per half arch)

Number of Section	x	y	$x^2 \frac{Ids}{I_x}$	$x^2 y \frac{Ids}{I_x}$ (3)×(4)	$l+2x$	$(l+2x)^2$ (6)^2	$\frac{M_s}{w} = \frac{(l+2x)^2}{8}$ (7)÷8
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1 and 1′							
2 and 2′							
3 and 3′							
4 and 4′							
5 and 5′			From Table 2				
6 and 6′							
7 and 7′							
8 and 8′							
Sum			Σ=				

The numerator for H, $\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_s y \frac{Ids}{I_x} = -w \sum_0^{\frac{l}{2}} x^2 y \frac{Ids}{I_x}$ is equal to the sum of column (5) multiplied by unit load w.

The numerator for M, $\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_s \frac{Ids}{I_x} = w\left[\sum_0^{\frac{l}{2}} x^2 \frac{Ids}{I_x} + \left(\frac{l^2}{2}\right) \sum_0^{\frac{l}{2}} \frac{Ids}{I_x}\right]$ equals the sum of column (4) plus sum of column (3) from Table 2 times $\left(\frac{l}{2}\right)^2$, the total multiplied by unit load w.

To get values of H and M divide the above values by the denominators found in Table 2.

Values in column (8) will be used in Table 5, p. 510, for determining bending moments at any point. Usually neither Table 5 nor columns (6) to (8) in this table need to be computed.

Since $\sum y\frac{Ids}{I_x} = 0$ the formula for the numerator for H_A becomes

$$\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_s y\frac{Ids}{I_x} = -w\sum_0^{\frac{l}{2}} x^2 y\frac{Ids}{I_x}.$$

Numerator for M for Full Load.—Substituting the values for M_s the numerator for M becomes

$$\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_s\frac{Ids}{I_x} = -w\sum_0^{\frac{l}{2}}\left[\left(\frac{l}{2}\right)^2 + x^2\right]\frac{Ids}{I_x}$$

$$= -w\left[\sum_0^{\frac{l}{2}} x^2\frac{Ids}{I_x} + \left(\frac{l}{2}\right)^2\sum_0^{\frac{l}{2}}\frac{Ids}{I_x}\right].$$

Values of H_1, V_1, and M_1 for Loading Scheme 1.—In this scheme live load extends over $\frac{5}{8}$ of the span and is placed on the right side of the arch. The values of H_1, V_{A1} and M_1 are determined by Formulas (41) to (43), p. 493. The denominators in these equations are the same as found on p. 498 because they are functions of the arch and independent of the loading. See p. 506 for explanation of Scheme 1.

The numerators are:

$$\text{for } H_1 \qquad \sum_{-\frac{1}{8}l}^{\frac{1}{2}l} M_s y\frac{Ids}{I_x}; \quad \ldots\ldots \quad (62)$$

$$\text{for } V_{A1} \qquad \sum_{-\frac{1}{8}l}^{\frac{1}{2}l} M_s x\frac{Ids}{I_x}; \quad \ldots\ldots \quad (63)$$

$$\text{and for } M_1 \qquad \sum_{-\frac{1}{8}l}^{\frac{1}{2}l} M_s\frac{Ids}{I_x}. \quad \ldots\ldots \quad (64)$$

From inspection of Fig. 196 (*b*), p. 507, it is evident that the bending moment M_s at any point x is

$$M_s = -\tfrac{1}{2}w(\tfrac{1}{8}l + x)^2.$$

For points on the left half of the arch and the values of x to be subtracted in the above equation are negative. For points on the arch to the left of the loaded area, i.e., where $-x$ is larger than $-\frac{1}{8}l$, M_s is zero.

The formula for numerators for H_1, V_{A1} and M_1 are obtained by substituting the above value for M_s in Equations (62) to (64).

The numerators are:

$$\text{for } H_1 \qquad -w\sum_{-\frac{1}{8}l}^{\frac{l}{2}}\tfrac{1}{2}(\tfrac{1}{8}l + x)^2 y\frac{Ids}{I_x};$$

$$\text{for } V_{A1} \qquad -w\sum_{-\frac{1}{8}l}^{\frac{l}{2}} \tfrac{1}{2}(\tfrac{1}{8}l + x)^2 x \frac{Ids}{I_x};$$

$$\text{for } M_1 \qquad -w\sum_{-\frac{1}{8}l}^{\frac{l}{2}} \tfrac{1}{2}(\tfrac{1}{8}l + x)^2 \frac{Ids}{I_x}.$$

An easy method of obtaining the values is by using the tabulation below.

Table 4.—Numerators for H_1, M_1 and V_{A1} for Loading Scheme 1.

(Load extends from right springing for $\frac{5}{8}l$)

No. of Section	x	$\frac{1}{8}l+x$	$\frac{1}{2}(\frac{1}{8}l+x)^2$	$\frac{Ids}{I_x}$	$\frac{Ids}{I_x}\frac{1}{2}(\frac{1}{8}l+x)^2$	$\frac{Ids}{I_x}\frac{1}{2}(\frac{1}{8}l+x)^2x$	$\frac{Ids}{I_x}\frac{1}{2}(\frac{1}{8}l+x)^2y$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
3′							
2′							
1′							
1							
2							
3							
4							
5							
6							
7							
8							
9							
10							
Sum							

The numerator for H_1 is $w\sum_{-\frac{1}{8}l}^{\frac{1}{2}l} \frac{1}{2}(\frac{1}{8}l + x)^2 y \frac{Ids}{I_x}$ is equal to the sum of column (8) multiplied by w.

The numerator for V_{A1} is $w\sum_{-\frac{1}{8}l}^{\frac{1}{2}l} \frac{1}{2}(\frac{1}{8}l + x)^2 x \frac{Ids}{I_x}$ is equal to the sum of column (7) multiplied by w.

The numerator for M_1 is $\frac{w}{2}\sum_{-\frac{1}{8}l}^{\frac{1}{2}l} \frac{1}{2}(\frac{1}{8}l + x)^2 \frac{Ids}{I_x}$ is equal to the sum of column (6) multiplied by w.

To get values of H_1, V_1 and M_1 divide the numerators by the appropriate denominators on p. 498. (See p. 522 for numerical example.)

Simplification of Work in Tables 3 and 4.—The work may be somewhat simplified if the horizontal projections of the divisions of the arch are equal. If the number of divisions per half arch is n, and the horizontal length of each division a, the span length may be expressed by $l = 2na$. The ordinate of the center of each division then is $x = 0.5a$, $1.5a$, $2.5a \ldots (n - 0.5)a$. In general the ordinate of any point is $x = m_x a$. This may be introduced in the various equations.

For instance, $\sum_0^{\frac{l}{2}} x^2 y \frac{Ids}{I_x}$ becomes $a \sum_0^n m_x^2 y \frac{Ids}{I_x}$

and $\sum_0^{\frac{l}{2}} x^2 \frac{ds}{I_x}$ becomes $a \sum_0^n m_x^2 \frac{Ids}{I_x}$.

In column (2) of Table 3 and (2) of Table 4 instead of values of x introduce the corresponding values of m_x, namely, 0.5, 1.5, 2.5, . . . n. The balance of the work is performed in the same manner as given in the table. The sums must then be multiplied by the common length of the divisions a.

PARTIAL LOADINGS PRODUCING MAXIMUM BENDING MOMENTS

As explained on pp. 543 and 544 the following loadings produce maximum stress at the critical three cross-sections of the arch ribs.

At Left Springing.

Maximum positive bending moment is produced by loading scheme 1 shown in Fig. 196 (b), p. 507. (Load on $\frac{5}{8}$ of span length from B.)

Maximum negative bending moment is produced by loading scheme 2 shown in Fig. 196 (c), p. 507. (Load on $\frac{3}{8}$ of span length from A.)

At Left Quarter Point.

Maximum positive bending moment is produced by loading scheme 2 shown in Fig. 196 (c), p. 507. (Load on $\frac{3}{8}$ of span length from A.)

Maximum negative bending moment is produced by loading scheme 1 shown in Fig. 196 (b), p. 507. (Load on $\frac{5}{8}$ of span length of B.)

At Crown.

Maximum positive bending moment is produced by loading scheme 3 shown in Fig. 196 (d), p. 507. (Load on $\frac{1}{8}$ of span length each side of crown.)

Maximum negative bending moment is produced by loading scheme 4 shown in Fig. 196 (e), p. 507. (Load on $\frac{3}{8}$ of span on each end.)

To find the statically indeterminate values H, V_A and M, for each of the above conditions, it is sufficient to find the values for a load extending over the whole span of the arch and for a load extending over $\frac{5}{8}$ of the span of the arch. The values for other cases may then be found by combining the values for these two loading schemes in the manner given below.

Notation.

Let M = auxiliary moment full load;
M_1 = auxiliary moment for left springing, loading scheme 1;
M_2 = auxiliary moment for left springing, loading scheme 2;
M_{2R} = auxiliary moment full right springing, loading scheme 2;
M_3 = auxiliary moment for left springing, loading scheme 3;
M_4 = auxiliary moment for left springing, loading scheme 4;
V_A = vertical reaction at left springing, full load;
V_{A1} = vertical reaction at left springing, loading scheme 1;
V_{A2} = vertical reaction at left springing, loading scheme 2;
V_{B2} = vertical reaction at right springing, loading scheme 2;
V_{A3} = vertical reaction at left springing, loading scheme 3;
V_{A4} = vertical reaction at left springing, loading scheme 4;
H = horizontal thrust, full load;
H_1 = horizontal thrust, scheme 1;
H_2 = horizontal thrust, scheme 2;
H_3 = horizontal thrust, scheme 3;
H_4 = horizontal thrust, scheme 4;
w = uniformly distributed load per unit of length;
l = span of arch.

Full Span Loaded.—The statically indeterminate values for load extending over the whole span may be found as outlined in Table 2, p. 498, and 3, p. 502. The values for this condition are called V_A, H and M. Due to symmetry of loading $V_A = \frac{1}{2}wl$.

Scheme 1.—Five-eighths of span on right side loaded. This loading scheme shown in Fig. 196 (*b*), p. 507. The statically indeterminate values for this condition are called V_{A1}, H_1 and M_1 and are found as outlined in Table 2, p. 498, and 4, p. 504. This loading produces maximum positive bending moment at left springing and maximum negative bending moment at left quarter point.

Scheme 2.—Three-eighths of span on the left side loaded. By comparing scheme 1 and scheme 2 it is evident that simultaneous loading of both schemes would be equivalent to loading extending over the whole span. Therefore, if the indeterminate values for full load and for scheme 1 are known, the indeterminate values for scheme 2 may be obtained by simple subtraction. This scheme of loading produces

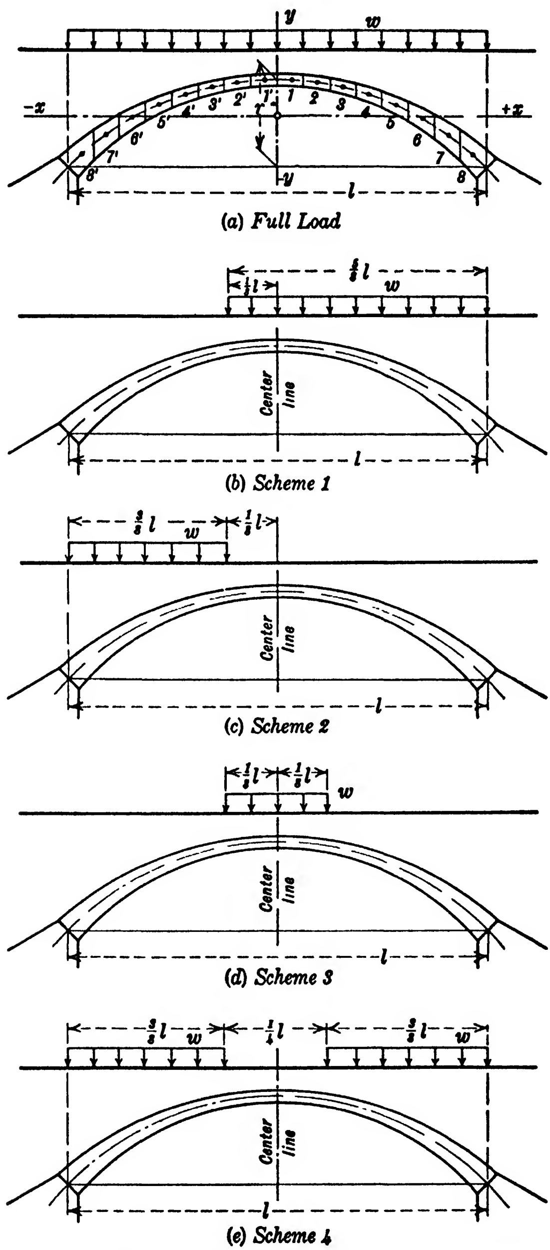


FIG. 196.—Position of Loading for Maximum Bending Moments. (See p. 505.)

maximum negative bending moment at left springing and also maximum positive bending moment at left quarter point.

Thus

$$M_2 = M - M_1;$$

$$H_2 = H - H_1;$$

$$V_{A2} = \frac{wl}{2} - V_{A1}.$$

For determining the indeterminate values for scheme 3 it is necessary to have values of scheme 2 at the right support. These can be readily expressed in terms of values at the left support. Thus from statics the vertical reaction at right support is

$$V_{B2} = \tfrac{3}{8}wl - V_{A2}.$$

The auxiliary moment M_{2R} at right support may be found from equation,[9]

$$M_{2R} = M_2 - \tfrac{15}{128}wl^2 = M - M_1 - \tfrac{15}{128}wl^2.$$

Scheme 3.—Load extends on each side of crown for $\frac{1}{8}$ of span. This loading scheme is shown in Fig. 196 (*d*), p. 507. From inspection of schemes 1 and 2, it is evident that scheme 3 may be obtained by subtracting from scheme 1 the reversed loading of scheme 2. For this purpose it is necessary to find the vertical reaction at right springing V_{B2} for scheme 2 and the auxiliary moment for right springing M_{2R}. For

[9] From Formula (44), p. 493, after substituting M_{2R} for M, etc., and solving for M_{2R}

$$M_{2R} = M_{B2} + V_{B2}\frac{l}{2} + HY_s.$$

From ordinary statics

$$M_{B2} = M_{A2} + V_{A2}l - (\tfrac{3}{8}wl)\tfrac{13}{16}l = M_{A2} + V_{A2}l - \tfrac{3}{8} \times \tfrac{13}{16}wl^2$$

and

$$V_{B2} = \tfrac{3}{8}wl - V_{A2}.$$

Substitute this in above equation

$$M_{2R} = M_{A2} + V_{A2}l - \tfrac{3}{8} \times \tfrac{13}{16}wl^2 + \tfrac{3}{16}wl^2 - V_{A2}\frac{l}{2} + HY_s$$

$$M_{2R} = M_{A2} + V_A\frac{l}{2} + H_AY_s - \tfrac{15}{128}wl^2.$$

Since

$$M_{A2} + V_A\frac{l}{2} + H_AY_s = M_2$$

the above form changes to

$$M_{2R} = M_2 - \tfrac{15}{128}wl^2.$$

reversed loading of scheme 2 shown in Fig. 196 (*d*), p, 507, V_{B2} and M_{2R} become values at the left springing.

The vertical reaction from scheme 3 due to symmetry of loading equals one-half of the load in the arch. Therefore

$$V_{A3} = \tfrac{1}{8}wl.$$

Values of H_3 and M_3 are obtained by subtracting from scheme 1 the reversed scheme 2. Thus, taking M_{2R} as found in footnote, p. 508,

$$M_3 = M_1 - M_{2R} = M_1 - M + M_1 + \tfrac{15}{128}wl^2$$
$$= 2M_1 - M + \tfrac{15}{128}wl^2.$$
$$H_3 = H_1 - H_2 = 2H_1 - H.$$

Scheme 4.—Three-eighths of each half of the span loaded. This loading scheme is shown in Fig. 196 (*e*), p. 507. It produces maximum negative bending moments at the crown. The statically indeterminate values may be obtained by subtracting from full load the load in scheme 3.

The vertical reaction due to symmetry of loading equals one-half of the load on the arch.

$$V_{A4} = \tfrac{3}{8}wl.$$

The other values are found by subtracting from the values for full load corresponding values for scheme 3.

Thus

$$M_4 = M - M_3 = M - 2M_1 + M - \tfrac{15}{128}wl^2$$
$$= 2(M - M_1) - \tfrac{15}{128}wl^2,$$

and

$$H_4 = H - H_3 = H - 2H_1 + H = 2(H - H_1).$$

Bending Moment at Any Point of Arch

Usually in arches in which proper variation of moments of inertia at intermediate points has been accepted it is sufficient to find bending moments at the crown, the springing and the quarter points. In important structures it may be advisable to investigate bending moments at all intermediate points in the manner explained in the following paragraphs.

After the values of V_A, H_A and M are found the bending moments at all points in the arch may be either computed analytically as explained below or found graphically by drawing a line of pressure as described on p. 626.

Analytically the values are computed from Equation (45), p. 493, which is

$$M_x = M + V_A x + H_A y + M_s.$$

To reduce the amount of work the tables may be computed for $w = 1$ and the final result multiplied by the unit load w.

The equation then changes to

$$\frac{M_x}{w} = \frac{M}{w} + \frac{V_A}{w}x + \frac{H}{w}y + \frac{M_s}{w}.$$

Uniformly Distributed Load. Whole Span Loaded.—In symmetrical arches loaded uniformly throughout, the bending moments, at sections of the left half of the arch are equal to the bending moments at symmetrically placed sections of the right half of the arch. Therefore, computations need be made for one-half of the arch only.

The values of $\frac{H}{w}$ and $\frac{M}{w}$ are taken from p. 502. Since $V_A = \frac{wl}{2}$, $\frac{V_A}{w} = \frac{l}{2}$. The values of $\frac{M_s}{w}$ are taken from column (8) of Table 3, p. 502.

The work may be performed according to the scheme given in Table 5.

Table 5.—Bending Moments at any Point for Full Load

(Bending moments in this table are seldom required, see p. 509.

$(A) = \frac{V_A}{w} = \frac{l}{2}$. $(B) = \frac{H}{w}$ = see p. 502. $(C) = \frac{M}{w}$ = see p. 502

Point	x	$\frac{V_A}{w}x$ (2) × (A)	y	$\frac{H}{w}y$ (4) × (B)	$\frac{M_s}{w}$	$\frac{M}{w}+\frac{M_s}{w}$ (C) + (6)	$\frac{M_x}{w}$ (3) − (5) + (7)	M_x (8) × w
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1								
2								
3								
4								
5								
6								
7								
8								

Uniform Load. Partial Loading.—For one-sided loading it is necessary to compute bending moments for all sections of the arch, as the bending moments at symmetrically placed sections are not equal. It may be noted, however, that for symmetrical sections, the values of $\frac{V_A}{w}x$ are numerically the same but of opposite signs, while values $\frac{H}{w}y$ are the same numerically and of the same sign. Considering the right half loaded, the values of M_s for the unloaded section are zero. Values of M_s for right half of the arch may be taken from Table 4, p. 504.

The bending moments may be worked out by the scheme suggested in Table 6.

Table 6.—Bending Moments M_x for Partial Loading

(Bending moments in this table are seldom required, see p. 509

$(A) = \frac{V_A}{w}$ = see p. 504. $(B) = \frac{H}{w}$ = see p. 504. $(C) = \frac{M}{w}$ = see p. 504.

Point	x	$\frac{V_A}{w}x$ $(A)\times(2)$	y	$\frac{H}{w}y$ $(B)\times(4)$	$\frac{M_s}{w}$ See p. 504	$\frac{M}{w}+\frac{M_s}{w}$ $(C)+(6)$	$\frac{M_x}{w}$ $(3)-(5)+(7)$	M_x $(8)\times w$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1								
2								
3								
4								
5								
6								
7								
8								
8′								
7′								
6′								
5′								
4′								
3′								
2′								
1′								

DESIGN OF A FIXED ARCH BRIDGE WITH FILLED SPANDRELS

Example.—Design a fixed arch with filled spandrels to serve as a highway bridge, for which

theoretical span, $l = 120$ ft.
theoretical rise, $r = 22$ ft.

Live load, 100 lb. per sq. ft. Impact, 20 per cent.
Changes of temperature, $\pm 25°$ F.
Shrinkage equivalent to a drop of temperature of 15° F.

Reinforcement: Minimum 1 per cent of concrete area.

Stresses in concrete: Concentric compression, $f_c = 450$ lb. per sq. in.
Direct stress and bending, small eccentricity, $f_c = 530$ lb. per sq. in.
large eccentricity, $f_c = 630$ lb. per sq. in.
Maximum tension, $f_t = 120$ lb. per sq. in.

$E = 2\,000\,000$ lb. per sq. in., $\alpha = 0.0000055$, hence, $\alpha E = 11$.

Depth of fill at crown equals 1 ft. Unit weight of fill = 100 lb. per cu. ft. Roadway on top of fill, 100 lb. per sq. ft. Roadway is assumed to be level longitudinally.

Solution.—In solving this problem following steps are taken: (1) the dimensions of the arch are found by the approximate method given on p. 480; (2) the exact line of pressure for dead load is determined and adopted for the arch axis; (3) bending moments and thrusts are computed by the exact method; and (4) the stresses in the arch sections are computed.

Use of Approximate Method.—To be able to use the approximate method, it is necessary to determine the unit dead loads at the crown and at the springing, respectively, and then compute the ratio $\frac{q_c}{q_s}$. For the purpose of computing the dead load, the thickness of the arch at the crown is obtained by the rule-of-thumb Formula (6), p. 480

$$h_c = \frac{450}{530 \times 1.14}\left(\sqrt{120} + \tfrac{120}{10} + \tfrac{100}{200} + \tfrac{200}{400}\right) = 18 \text{ in.}$$

Hence for a strip of arch 12 in. wide

$$I_c = \frac{12 \times 18^3}{12} = 5\,832 \text{ in}^4.$$

To get the depth at the springing, assume $n = \frac{I_c}{I_s \cos \phi_s} = 0.3$ (see p. 476). From diagram, p. 669 for $\frac{l}{r} = \frac{120}{22} = 5.5$ the value of $\frac{1}{\cos \phi_s} = 1.45$ at an average. $I_s = \frac{12 \times h_s^3}{12} = h_s^3$. Therefore

$$0.3 = \frac{18^3}{h_s^3} \times 1.45 \quad \text{and} \quad h_s = 18\sqrt[3]{\frac{1.45}{0.3}} = 30.4 \text{ in.}$$

Accept $h_s = 30$ in. (measured at right angle to the arch axis).

Plot the rise and span of the arch for one-half of the arch. Plot at the springing and at the crown the computed thicknesses of the arch. Also plot the fill and the roadway. The unit dead loads at the crown and at the springing are now obtained by scaling the fill and the arch thicknesses and multiplying by their respective unit weights. For the purpose of determining the dead load the arch thickness at the springing is scaled on a vertical line. The unit dead loads are

$$q_c = 1.5 \times 150 + 100 + 100 = 425 \text{ lb. per sq. ft.}$$

$$q_s = 3.2 \times 150 + 21 \times 100 + 100 = 2\,680 \text{ lb. per sq. ft.}$$

and the ratio $\frac{q_s}{q_c} = \frac{2680}{425} = 6.3.$

The constants to be used in the approximate method are

$$m = \frac{q_s}{q_c} = 6.3 \quad \text{and} \quad n = \frac{I_c}{I_s \cos \phi_s} = 0.3.$$

$$\text{Live load, } w = 100 + 20 \text{ per cent} = 120 \text{ lb. per sq. ft.}$$

$$wl = 120 \times 120 = 14\,400 \text{ lb.}; \; wl^2 = 120 \times 120^2 = 1\,728\,000 \text{ ft.-lb.}$$

Live Load at Springing.—Find constants from Diagrams 30 and 31, pp. 676 and 677, corresponding to $m = 6.3$ and $n = 0.3$. Use Formulas (26) to (29), p. 485.

Positive bending moment,

$$M_s = 0.032wl^2 = 55\,300 \text{ ft.-lb}; \qquad H = -0.104wl\frac{l}{r} = -8260 \text{ lb.}$$

Negative bending moment,

$$M_s = -0.0163wl^2 = -28\,200 \text{ ft.-lb.}; \quad H = -0.032wl\frac{l}{r} = -2530 \text{ lb.}$$

Live Load at Crown.—Find constants from Diagrams 26 and 27, pp. 672 and 673, corresponding to $m = 6.3$ and $n = 0.3$. Use Formulas (18) to (21), p. 484.

Positive bending moment,

$$M_c = 0.0068wl^2 = 11\,700 \text{ ft.-lb.}; \qquad H = -0.08wl\frac{l}{r} = -6340 \text{ lb.}$$

Negative bending moment,

$$M_c = -0.0029wl^2 = -5\,000 \text{ ft.-lb.}; \quad H = -0.057wl\frac{l}{r} = -4520 \text{ lb.}$$

Effect of Changes of Temperature and Shrinkage.—The horizontal thrust is obtained from Formula (31), p. 486, in which the constant C_h is obtained from Diagram 24, p. 67. The formula is $H_t = -\frac{\alpha EI(\pm t)}{r^2}\cdot\frac{1}{C_h}$.

As given on p. 512 $\alpha E = 11$, $I = 18^3 = 5\,832$. From Diagram 24, $C_h = 0.037$.

$$H_t = -\frac{\alpha EI}{r^2C_h}(\pm t^\circ) = -\frac{11 \times 5\,832}{(12 \times 22)^2 \times 0.037}(\pm t) = -24.75(\pm t^\circ)\text{lb.}$$

For rise of temperature, $t = 25^\circ$ and $H_t = -24.75 \times 25 = -620$ lb.

For fall of temperature plus shrinkage, $t = -40^\circ$ and $H_t = 24.75 \times 40 = 990$ lb.

Bending Moments Due to Temperature.—Find the location of the elastic center from Formula (14), p. 483. $Y_c = C_0r$, in which C_0 taken from Diagram 22, p. 670, is 0.194.

$$Y_c = 0.194 \times 22 = 4.268 \text{ ft.}$$

Therefore

$$r - Y_c = 22 - 4.268 = 17.732 \text{ ft.}$$

Springing	Rise,	$M_s = 620 \times 17.732 = 11\,000$ ft.-lb.;	$H_t = -\ 620$ lb.
	Fall,	$M_s = -\ 990 \times 17.732 = -\ 17\,600$ ft.-lb.;	$H_t = \ 990$ lb.
Crown	Rise,	$M_c = -\ 620 \times 4.268 = -\ 2\,650$ ft.-lb.;	$H_t = -\ 620$ lb.
	Fall,	$M_c = 990 \times 4.268 = 4\,230$ ft.-lb.;	$H_t = \ 990$ lb.

Effect of Dead Load.—The arch axis is assumed to coincide with the line of pressure for dead load. Therefore, dead load produces no bending moments in the arch. The dead load thrust is found from Formula (34), p. 486, in which the constant C_d, from Diagram 25, p. 671, is 0.213

$$H_d = -\ 0.213 \times 425 \times \frac{120^2}{22} = -\ 59\,500 \text{ lb.}$$

Effect of Rib Shortening.—The horizontal thrust for rib shortening is found from Formula (36), p. 487, in which C_h is the same as used previously for the effect of temperature. Assume that the average thickness of the arch is 22 in. and $A_{av} = 12 \times 22 = 264$ sq. in. The rise in inches is $22 \times 12 = 264$ in. Then the horizontal thrust is

$$H_s = -\frac{5\,832}{264 \times 264^2}\,\frac{1}{0\ 037}H_d = 0.0086 \times 59\,500 = 510 \text{ lb.}$$

The bending moments due to rib shortening are (see Formulas (37) and (38), p. 487.

Springing, $M_s = -\ 510 \times 17.732 = -\ 9\,040$ ft.-lb.

Crown, $M_c = 510 \times 4.268 = 2\,220$ ft.-lb.

Summary of Bending Moments and Thrust.—The bending moments and thrusts previously computed are tabulated in the following table.

Approximate Method. Summary of Bending Moments and Thrusts

Type of Loading	Positive Bending Moments		Negative Bending Moments	
	Horizontal Thrusts	Bending Moments	Horizontal Thrusts	Bending Moments
	Springing			
	Pounds	Foot-lb.	Pounds	Foot-lb.
Dead load	−59 500	....	−59 500	
Rib shortening	....	....	+ 510	− 9 040
Live load	− 8 260	55 300	− 2 530	−28 200
Effect of temperature and shrinkage	− 620	11 000	+ 990	− 17600
Total	−67 380	66 300	−62 030	−54 840
			+ 1 500	
			−60 530	

Crown

	Pounds	Foot-lb.	Pounds	Foot-lb.
Dead load	−59 500	....	−59 500	
Rib shortening	+ 510	2 200		
Live load	− 6 340	11 700	− 4 520	− 5 000
Effect of temperature and shrinkage	+ 990	4 230	− 620	− 2 650
Total	−65 840 + 1 500 −64 340	18 130	64 640	− 7 650

Dimensions of Arch Section, Approximate Method.—The dimensions are found by using the maximum bending moment given in the previous table.

$$\text{Springing,} \quad M_s = 66\,300 \times 12 = 796\,000 \text{ in.-lb.}$$

$$H = -67\,380 \text{ lb.}$$

$$N = \frac{H}{\cos \phi_s} = 67\,380 \times 1.47 = 99\,000 \text{ lb.}$$

$$e = \frac{796\,000}{99\,000} = 8.05 \text{ in.}$$

For f_c = 630 lb. per sq. in. and f_t = 120 lb. per sq. in., the depth of the section reinforced with 1 per cent of steel, is obtained from Diagrams 7, p. 654, and 8 p. 655.

$$\frac{bf_c}{N} = \frac{12 \times 630}{99\,000} = 0.0764 \text{ and corresponding } h_s = 29 \text{ in.}$$

$$\frac{bf_t}{N} = -\frac{12 \times 120}{99\,000} = -0.01455 \text{ and corresponding } h_s = 28 \text{ in.}$$

$$\text{Crown,} \quad M_c = 18\,130 \times 12 = 218\,000 \text{ in.-lb.}$$

$$H = -64\,340 \text{ lb.}$$

$$N = H$$

$$e = \frac{218\,000}{64\,340} = 3.39 \text{ in.}$$

For small eccentricity use f_c = 530 lb. per sq. in.

$$\frac{bf_c}{N} = \frac{12 \times 530}{64\,340} = 0.0988 \text{ and corresponding } h_c = 18 \text{ in.}$$

$$\frac{bf_t}{N} = -\frac{12 \times 120}{64\,340} = -0.0224 \text{ and corresponding } h_s = 14 \text{ in.}$$

Adopted Preliminary Dimensions.—From the previous figures following dimensions of the arch sections are adopted

Springing, $h_s = 29$ in.

Crown, $h_c = 18$ in.

EXACT ANALYSIS OF ARCH

The adopted preliminary dimensions will be checked by the exact method.

First determine the line of pressure for dead load, which will be accepted as the axis of the arch. For this purpose lay out the arch to scale using an approximate arch axis as determined by constants in Diagram 22, p. 670. Take constants corresponding to $m = 6.3$.

$\frac{x}{l}$	Constants	Constants × 22 ft.
0.125	0.039	0 86 ft.
0.25	0.172	3 78 ft.
0.375	0.454	9.99 ft.

These figures give the vertical distances from the crown of the arch axis at the one-eighths, quarter and three-eighths points. They are sufficient for plotting the preliminary arch axis.

Plot the fill and the roadway. (See Fig. 197, p. 518.)

Divide the span into 16 equal horizontal divisions, each division 7 ft. 6 in. long. Find center of gravity of the dead load in each division. For the two end divisions find the actual center of gravity of the trapezoid forming the load. For other divisions assume that the loads act in the center of each division.

Find dimensions at the intermediate points using the parabolic variation of the moments of inertia given on p. 477. For $n = 0.3$

$$\frac{I}{I_x \cos \phi_x} = 1 - 4(1 - n)\left(\frac{x}{l}\right)^2 = 1 - 2.68\left(\frac{x}{l}\right)^2,$$

where x is measured from the crown.

$$h_x = \frac{1}{\sqrt[3]{\cos \phi_x\left(1 - 2.68\left(\frac{x}{l}\right)\right)^2}} h_c.$$

In this formula all values are known except $\cos \phi_x$. This is obtained graphically as shown in Fig. 197, p. 518. Computations for h_x are given in the following table.

After the thicknesses of the arch are found at the intermediate points they are plotted and the outline of the intrados and extrados is drawn. The dead loads acting at the various arch sections may now be computed. At each center of gravity scale the thickness of the arch (measured on a vertical line) and the depth of the fill. Multiply each value by the corresponding specific gravity. Add the weight

Dimensions of Arch Axis at Intermediate Points

Division	$\frac{x}{l}$	$\left(\frac{x}{l}\right)^2$	$2.68\left(\frac{x}{l}\right)^2$	$1-2.68\left(\frac{x}{l}\right)^2$	$\frac{1}{\cos\phi_x}$	$\frac{1}{\cos\phi_x\left[1-2.68\left(\frac{x}{l}\right)^2\right]}$	$\frac{1}{\sqrt[3]{\cos\phi_x\left[1-2.68\left(\frac{x}{l}\right)^2\right]}}$	h_x
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	$\frac{1}{32}$	0.001	0.003	0.997	1.0	1.01	1.0	17
2	$\frac{3}{32}$	0.009	0.024	0.976	1.01	1.04	1.01	17.2
3	$\frac{5}{32}$	0.024	0.064	0.936	1.02	1.09	1.03	17.5
4	$\frac{7}{32}$	0 048	0 128	0.872	1.03	1.18	1.06	18.1
5	$\frac{9}{32}$	0.080	0.214	0.786	1.06	1.35	1.10	18.8
6	$\frac{11}{32}$	0.118	0.316	0.684	1.11	1.63	1.18	20.2
7	$\frac{13}{32}$	0.166	0.445	0.555	1.22	2.20	1.30	22.2
8	$\frac{15}{32}$	0.220	0 590	0 410	1.35	3.30	1.49	25.4

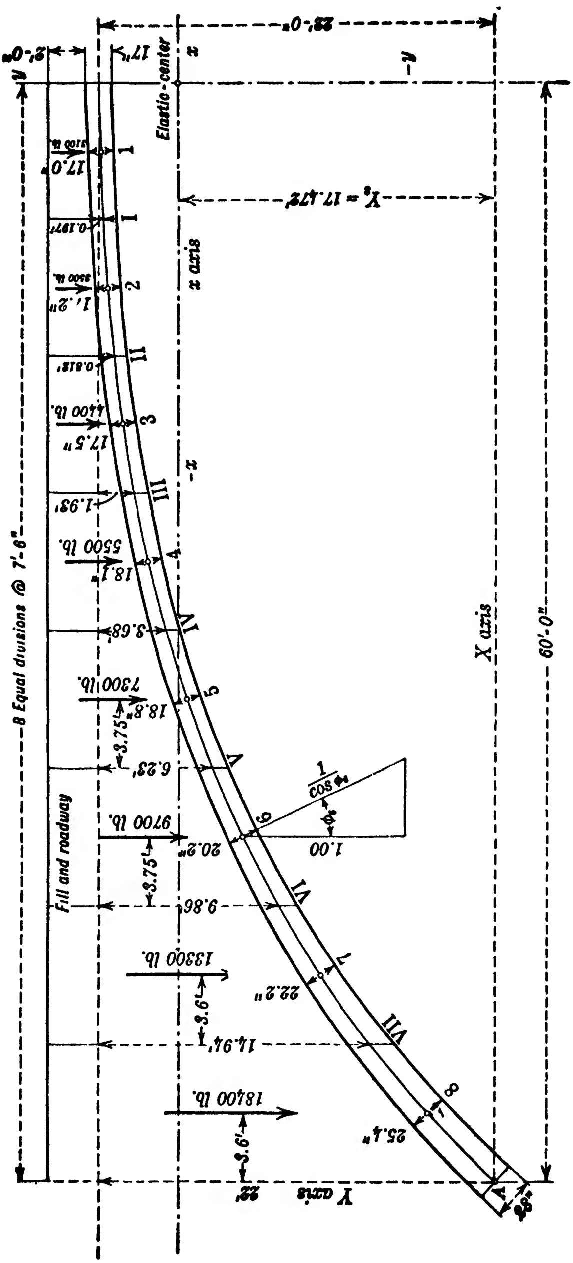


FIG. 197.—Example of Arch Design. (See p. 516.)

of fill and concrete and multiply by the length of the division. The work is shown in the following table.

Dead Loads Acting on Arch

Division	Fill + Roadway	Arch	Arch+Fill	Dead Load per Division
	Pounds per Sq. Ft.	Pounds per Sq. Ft.	Pounds per Sq. Ft.	Pounds
1	200	1.4×150=210	410	3 100
2	235	1.5×150=230	465	3 500
3	350	1.6×150=240	590	4 400
4	475	1.7×150=255	730	5 500
5	700	1.8×150=270	970	7 300
6	1 000	2.0×150=300	1 300	9 700
7	1 425	2.3×150=345	1 770	13 300
8	2 000	3.0×150=450	2 450	18 400

Knowing the dead loads and their points of application, the line of pressure is found in the manner shown in the following table.

Line of Pressure for Dead Load

Point	Loads P_n	Distance of Loads a_n	$P_n a_n$	Length of Division		M_{n-1}	M_n	$y'_n = \frac{M_n}{H}$
I	3 100	3 75	11 630	7.5	....	....	11 630	0.197
II	3 500	3 75	13 130	7.5	3 100×7 5= 23 250	11 630	47 980	0 812
III	4 400	3 75	16 500	7.5	6 600×7.5= 49 500	47 980	113 980	1.930
IV	5 500	3 75	20 630	7 5	11 000×7.5= 82 500	113 980	217 110	3.68
V	7 300	3 75	27 380	7.5	16 500×7 5=123 750	217 110	368 240	6 23
VI	9 700	3.75	36 400	7.5	23 800×7 5=178 500	368 240	583 140	9 86
VI	13 300	3.6	47 900	7.5	33 500×7.5=251 300	583 140	882 340	14 94
A	18 400	3 6	66 200	7.5	46 800×7.5=351 000	882 340	1 299 540	22 00

$$H_d = \frac{1\ 299\ 540}{22} = 59\ 070 \text{ lb.}$$

The table gives the values of y' measured from the crown for the arch axis at the end of each division. These values are plotted and the arch axis drawn.

Now find the elastic center for the new arch axis and for the cross-sections of the arch worked out in the table of dimensions on p. 517. The work is done in the manner outlined in Table 1, p. 497. It should be noted, however, that, since the values of $\frac{I}{I_x \cos \phi_x}$ are already worked out, the values of $\frac{Ids}{I_x}$ are obtained by multiplying $\frac{I}{I_x \cos \phi_x}$ by the horizontal length of the division dx, because $\frac{dx}{\cos \phi_x} = ds$. The values of Y are measured from the X-axis passing through the springing, as origin, to the centers of the divisions. The values x and y in Table 1 refer to the system of coordinates with an origin at the elastic center.

Table 1.—Finding Elastic Center Y_s

Point	Y	X	$\frac{I}{I_x \cos \phi_x}$	$\frac{Ids}{I_x}$	$Y\frac{Ids}{I_x}$	y	$y\frac{Ids}{I_x}$	x
0	22	60	....	....	....	4.528		
1	21.88	56.25	0.997	7.49	163 9	4.408	33.02	± 3.75
2	21.50	48.75	0.976	7.32	157.4	4.028	29.48	±11 25
3	20.58	41.25	0.936	7 02	144.5	3.108	21.82	±18.75
4	19.21	33.75	0 872	6 54	125.6	1.738	11.36	±26.25
5	17.08	26.25	0.786	5.90	100.8	− 0.392	− 2.31	±33.75
6	14.00	18.75	0.684	5.13	71.8	− 3.472	−17.81	±41.25
7	9 58	11.25	0.555	4.16	39 8	−7 892	−32.83	±48.75
8	3.58	3.75	0.410	3 07	11.0	−13.892	−42.65	±56.25
A	......	......	....	....	....	−17.472	....	±60.00
				46.63	814.8		+95.68 −95.60	

$$Y_s = \frac{\Sigma Y \frac{Ids}{I_x}}{\sum \frac{Ids}{I_x}} = \frac{814.8}{46.63} = 17.472; \quad y = Y - 17.472; \quad x = X - 60.$$

Since in Table 1 the sum of $y\frac{Ids}{I_x}$ column is almost zero, the work is sufficiently accurate.

The denominator for H, V_A and M are now found by means of Table 2. In computing $\frac{Il}{A_{av}}$ assume $A_{av} = 1 \times 1.66 = 1.66$ sq. ft.

Table 2.—Denominators for H, V_A and M

Section	y	$\frac{Ids}{I_x}$	$y\frac{Ids}{I_x}$	$y^2\frac{Ids}{I_x}$	x	x^2	$x^2\frac{Ids}{I_x}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	4.408	7.49	33.02	145.53	± 3.75	14.06	105.33
2	4.028	7.32	29.48	118.75	±11.25	126.56	926.41
3	3.108	7.02	21.82	67.82	±18.75	351.56	2 468.00
4	1.738	6.54	11.36	19.74	±26.25	689.06	4 507.00
5	− 0.392	5.90	− 2.31	0.91	±33.75	1 139.06	6 721.00
6	− 3.472	5.13	−17.81	61.84	±41.25	1 701.56	8 729.00
7	− 7.892	4.16	−32.83	259.10	±48.75	2 376.56	9 885.00
8	−13.892	3.07	−42.65	592.50	±56.25	3 063.96	9 715.00
		46.63		1 266.19			43 056.74

$$\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} = 2 \times 1\,266.19 = 2\,532.38; \quad \frac{Il}{A_{av}} = \frac{1.42^3 \times 120}{12 \times 1.66} = 17.3.$$

Denominator for $H = 2\,532.38 + 17.3 = 2\,549.68$ ft.^{3}

Denominator for $V_A = 2 \times 43\,056.74 = 86\,113.48$ ft.^{3}

Denominator for $M = 46.63 \times 2 = 93.26$ ft.

For live load the four schemes of loading, which give maximum bending moments at the springing, the quarter point and the crown respectively, will be considered. For this purpose find the numerators for H and M for full load by means of Table 3 and the numerators for H_1, M_1 and V_{A1} for Scheme 1 by means of Table 4. The value of H, M and V_A are then found for these two loadings. Finally, by combining these in the proper manner, the values are found for all the other schemes of loading.

Table 3.—Numerator for Full Loading

Section	x	y	$x^2\frac{Ids}{I_x}$	$x^2y\frac{Ids}{I_x}$
(1)	(2)	(3)	(4)	(5)
1	3 75	4 408	105 33	464.35
2	11.25	4.028	926.41	3 734 00
3	18 75	3 108	2 468 00	7 671 00
4	26 25	1.738	4 507 00	7 383.00
5	33 75	− 0 392	6 721.00	− 2 634.50
6	41 25	− 3 472	8 729 00	− 30 308.00
7	48 75	− 7 892	9 885 00	− 78 020.00
8	56.25	−13.892	9 715 00	−134 930.00
			43 056 74	−245 892.50
				19 252.35
				−226 640.15

Numerator for $H = -\ 226\,640.15w$ lb.-ft.^{3}

Numerator for $M = 43\,056.74 + 60^3 \times 46.63 = 210\,924.7w$ lb.-ft.^{2}

***H*, *M* and V_A for Full Load.**—Using the numerator from Table 3 and the denominators from Table 2 the statically indeterminate values for uniformly distributed loading extending over the whole span are

$$H = -\frac{226\,640.15}{2\,549.68} = 88.9w \text{ lb.},$$

$$M = \frac{210\,924\ 7}{93.26} = 2\,260w \text{ ft.-lb.}$$

$$V_A = \tfrac{1}{2}wl = 60w \text{ lb.}$$

Table 4.—Numerator for Loading Scheme 1

Load extends over $\frac{5}{8}$ of the span.

Section	x	$\frac{1}{8}l+x$	$(\frac{1}{8}l+x)^2$	$\frac{1}{2}(\frac{1}{8}l+x)^2$	$\frac{Ids}{I_x}$	$\frac{Ids}{I_x}\frac{1}{2}\left(\frac{1}{8}l+x\right)^2$	$x\frac{Ids}{I_x}\frac{1}{2}\left(\frac{1}{8}l+x\right)^2$	y	$y\frac{Ids}{I_x}\frac{1}{2}\left(\frac{1}{8}l+x\right)^2$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
2′	−11.25	3.75	14.06	7.03	7.32	51 5	− 578.9	4.028	207.3
1′	− 3.75	11.25	126.57	63 28	7.49	474 0	− 1 777.5	4.408	2 089.2
1	3.75	18.75	351.55	175.77	7.49	1 316 3	4 924 0	4.408	5 803.5
2	11.25	26.25	689.10	344 55	7.32	2 522 0	28 372.0	4.028	10 159.0
3	18.75	33.75	1 139.10	569 55	7.02	3 998 0	74 960 0	3 108	12 425.0
4	26.25	41.25	1 701.60	850.80	6.54	5 564 0	146 050.0	1.738	9 670.0
5	33 75	48.75	2 376.60	1 188 3	5 90	7 010.0	236 600.0	− 0.392	− 2 748.0
6	41.25	56.25	3 164.20	1 582.1	5 13	8 116.0	334 800 0	− 3.472	− 28 180.0
7	48 75	63.75	4 064.10	2 032.1	4.16	8 454 0	416 350.0	− 7.892	− 67 400.0
8	56.25	71.25	5 076.20	2 538.1	3.07	7 762 0	436 600 0	−13.892	−107 850.0
						45 267 8	1 678 656 0		−206 178.0
							− 2 356 4		40 353 0
							1 676 299.6		−165 825.0

Numerator for H_1 for Scheme 1 = − 165 825.0w lb.-ft.^{3}
Numerator for M_1 for Scheme 1 = 45 267.8w lb.-ft.^{2}
Numerator for V_{A1} for Scheme 1 = 1 676 299.6 lb.-ft.^{3}

H_1, M_1 and V_{A1} for Loading Scheme 1.—Using the numerator found in Table 4 and the denominator from Table 2, the static indeterminate values for Scheme 1 are

$$H_1 = -\frac{165\,825.0}{2\,549.68}w = 65.05w \text{ lb.,}$$

$$M_1 = \frac{45\,267.8}{93\;26}w = 485.4w \text{ ft.-lb.}$$

$$V_{A_1} = \frac{1\,676\,299.6}{86\,113\;48} = 19.4w \text{ lb.}$$

Loading Scheme 2.—The statically indeterminate values for Scheme 2, in which the loading extends from left support for a distance equal to $\frac{3}{8}l$, are obtained by subtracting from the values for full load the value for Scheme 1.

$$H_2 = (88.9 - 65.05)w = 23.85w \text{ lb.,}$$

$$M_2 = (2\,260 - 485.4)w = 1\,774.6w \text{ ft.-lb.,}$$

$$V_{A_2} = (60 - 19.46)w = 40.54w \text{ lb.}$$

Loading Scheme 3.—The statically indeterminate values for Scheme 3, in which the loading extends on both sides of the crown for a distance equal $\frac{1}{8}l$, are obtained as explained on p. 509.

$$H_3 = H_1 - H_2 = (65.05 - 23.85)w = 41.2w \text{ lb.,}$$

$$M_3 = 2M_1 - M + \tfrac{15}{128}wl^2 = (970.8 - 2\,260)w + \tfrac{15}{128} \times (120)^2w = 399.8w \text{ ft.-lb.}$$

$$V_{A_3} = \tfrac{1}{8}wl = 15w \text{ lb.}$$

Loading Scheme 4.—The loading in this scheme extends on each side of the arch from the springing for a distance equal to $\frac{3}{8}l$. The statically indeterminate values are obtained as explained on p. 509.

$$H_4 = H - H_3 = (88.9 - 41.2)w = 47.7w \text{ lb.,}$$

$$M_4 = M - M_3 = (2\,260 - 399.8)w = 1\,860.2w \text{ ft.-lb.,}$$

$$V_{A_4} = \tfrac{3}{8}wl = 45w \text{ lb.}$$

The statically indeterminate values for all schemes of loading are tabulated in the following table.

Statically Indeterminate Values for All Schemes of Loading

Type of Loading	H	V_A	M
	Pounds	Pounds	Foot-lb.
Full load........	$88.9w = 10\,668$	$60w = 7\,200$	$2\,260w = 271\,200$
Scheme 1........	$65\ 05w = 7\,806$	$19\ 46w = 2\,335$	$485.4w = 58\,260$
Scheme 2........	$23\ 85w = 2\,862$	$40.54w = 4\,865$	$1\,774\ 6w = 212\,980$
Scheme 3........	$41\ 2w = 4\,945$	$15\ 0w = 1\,800$	$399\ 8w = 47\,980$
Scheme 4........	$47.7w = 5\,725$	$45w = 5\,400$	$1\,860.2w = 223\,250$

w = 120 lb. per sq. ft.

All values are for a width of arch equal to 1 foot.

Bending Moments for Live Load.—Using the values given in the previous table, the bending moments at the springing, crown and quarter point are found as follows:

Scheme 1:

Bending moment at springing,

$$M_A = M_1 - V_{A1}\frac{l}{2} - H_1Y_s = 58\,260 - 130\,100 + 7\,806 \times 17.472 = 64\,546 \text{ ft.-lb.}$$

Bending moment at quarter point,

$$M_{1/4} = M_1 - V_{A1}\frac{l}{4} - H_1y_{1/4} = 58\,260 - 70\,050 - 7\,806 \times 0.848 = -18\,409 \text{ ft.-lb.}$$

Scheme 2:

Bending moment at springing,

$$M_A = M_2 - V_{A2}\frac{l}{2} - H_2Y_s = 212\,980 - 291\,900 + 2\,862 \times 17.472 = -28\,915 \text{ ft.-lb.}$$

Bending moment at quarter point,

$$M_{1/4} = M_2 - V_{A2}\frac{l}{2} - H_2y_{1/4} - \tfrac{1}{2}w\left(\frac{l}{4}\right)^2 = 212\,980 - 145\,950 - 2\,862 \times 0.848$$

$$- \tfrac{1}{2} \times 120 \times 30^2 = 10\,603 \text{ ft.-lb.}$$

Scheme 3:

Bending moment at crown,

$$M_c = M_3 - H_3(r - y_s) - \tfrac{1}{2}w\left(\frac{l}{8}\right)^2 = 47\,980 - 4\,945 \times 4.528 - \tfrac{1}{128} \times 120 \times 15^2$$

$$= 11\,410 \text{ ft.-lb.}$$

Scheme 4:

Bending moment at crown,

$$M_c = M_4 - H_4(r - y_s) - \tfrac{3}{8}wl \times \tfrac{5}{16}l = 223\,250 - 5\,725 \times 4.528 - \tfrac{3}{8} \times 120 \times 120 \times \tfrac{5}{16} \times 120 = -\,5\,172 \text{ ft.-lb.}$$

Effect of Rib Shortening.—The horizontal thrust due to rib shortening is found from Formula (51), p. 494, in which the numerator is taken from Table 2. The value of $\frac{Il}{A_{av}}$ is the same as worked out under Table 2.

$$H_s = \frac{17.3}{2\,549.68} \times 59\,070 = 0.0068 \times 59\,070 = 402 \text{ lb.}$$

Effect of Temperature Changes and Shrinkage.—The horizontal thrust due to temperature changes are found from Formula (59), p. 495, in which $\alpha E = 11 \times 144 = 1\,584$ lb. per sq. ft., l is in feet and I in ft.^{4} The numerator is the same as above.

$$H_t = -\frac{1\,584 \times 120 \times 0.238}{2\,549.68} \times (\pm t°) = 17.7 \times (\pm t°)$$

Rise of Temperature $t° = 25°$,

$$H_t = -\,17.7 \times 25 = -\,443 \text{ lb.}$$

Fall of Temperature Plus Shrinkage $t° = -\,40°$,

$$H_t = 17.7 \times 40 = 708 \text{ lb.}$$

Bending Moments Due to Rib Shortening and Temperature Changes.—The bending moments at the springing, the quarter point and the crown are obtained by multiplying the horizontal thrusts by the proper values of y. They are tabulated in the following table.

Bending Moments Due to Temperature Changes and Rib Shortening

	H	Springing $y = -17.472$ ft.	Quarter Point $y = 0.848$ ft.	Crown $y = 4.528$ ft.
	Pounds	Foot-lb.	Foot-lb.	Foot-lb.
Fall.............	708	−12 370	600	3 188
Rise............	−443	7 740	−376	−2 007
Rib shortening...	402	−7 020	341	1 820

Combined Bending Moments.—The following table gives the final bending moments and thrusts for a combination of live load, dead load, rib shortening and temperature changes. These values are used to compute the stresses in the arch sections.

Final Bending Moments and Thrusts

Type of Loading	Positive Bending Moments		Negative Bending Moments	
	Horizontal Thrusts	Bending Moments	Horizontal Thrusts	Bending Moments
Springing				
	Pounds	Foot-lb.	Pounds	Foot-lb.
Dead load	−59 070	....	−59 070	
Rib shortening	....	....	+ 402	− 7 020
Live load	− 7 806	64 546	− 2 862	−28 915
Temperature and shrinkage	− 443	7 740	+ 708	−12 370
Total	−67 319	72 286	−60 822	−48 305
Quarter Point				
	Pounds	Foot-lb.	Pounds	Foot-lb.
Dead load	−59 070	....	−59 070	
Rib shortening	+ 402	341		
Live load	− 2 872	10 603	− 7 806	−18 409
Temperature and shrinkage	+ 708	600	− 443	− 376
Total	—60 832	11 544	−67 319	−18 785
Crown				
	Pounds	Foot-lb.	Pounds	Foot-lb.
Dead load	−59 070	....	−59 070	
Rib shortening	+ 402	1 820		
Live load	− 4 945	11 410	− 5 725	− 5 172
Temperature and shrinkage	+ 708	3 188	− 443	− 2 007
Total	−63 015	16 418	−65 238	− 7 179

Stresses Due to Final Bending Moments and Thrusts.

Springing,

$$H = -67\,319 \text{ lb.},$$

$$M_A = 72\,286 \text{ ft.-lb.} = 865\,000 \text{ in.-lb.}$$

$$N_A = \frac{H}{\cos \phi_x} = 67\,319 \times 1.47 = 99\,000 \text{ lb.},$$

$$e = \frac{865\,000}{99\,000} = 8.7 \text{ in.}$$

Section: $b = 12$ in., $h = 28$ in., $d' = 2$ in., $d = 26$ in.,

$$A_s = 0.01 \times 12 \times 28 = 3.36 \text{ sq. in.}$$

Since part of the section is in tension, use Formula (28), p. 228.

$$\frac{e}{d} = \frac{8\ 7}{26} = 0.334, \quad \frac{h}{d} = \frac{28}{26} = 1.08.$$

Use diagrams for $h = 1.1d$, pp. 658 and 659.

From diagram for k, for $\frac{e}{d} = 0.334$ and $p = 0.01$, $k = 0.79$.

From diagram for C_a, for $k = 79$ and $p = 0.01$, $C_a = 0.152$.

From diagram on p. 662 for C_s, for $k = 0.79$ and $n = 15$, $C_s = 4.0$.

The final stresses are

$$f_c = \frac{M}{C_a bd^2} = \frac{865\,000}{0\ 152 \times 12 \times 28^2} = 610 \text{ lb. compression,}$$

$$f_s = C_s f_c = 4 \times 610 = 2\,440 \text{ lb. tension.}$$

The stresses are satisfactory.

Crown,

$$H = -\ 63\,015 \text{ lb.,}$$

$$M_c = 16\,418 \text{ ft.-lb.} = 197\,000 \text{ in.-lb.,}$$

$$e = \frac{197\,000}{63\,015} = 3.12.$$

Section: $b = 12$ in., $h = 17$ in., $d' = 2$ in., $2a = 13$ in., $\frac{2a}{h} = 0.77$,

$$A_s = 0.01 \times 12 \times 17 = 2.04 \text{ sq. in.}$$

Since the eccentricity is small, the Formula (13), p. 219 may be used. Use diagram marked $2a = 0.8h$.

Find ratio $\frac{e}{h} = \frac{3.12}{17} = 0.184$.

From Diagram 5, p. 652, for $p = 0.01$ and $\frac{e}{h} = 0.184$, $C_e = 1.76$ and

$$f_c = 1.76 \times \frac{63\,015}{12 \times 17} = 546 \text{ lb. per sq. in.}$$

The stress is slightly larger than 530 lb. allowed for small eccentricties. The section should be enlarged to 18 in. or the steel area should be increased.

Quarter Points,

$$H = -\ 67\,319 \text{ lb.,}$$

$$M_{1/4} = -\ 18\,785 \text{ ft.-lb.} = -\ 226\,000 \text{ in.-lb.,}$$

$$N_{1/4} = \frac{H}{\cos \phi_x} = 67\,319 \times 1.09 = 73\,200 \text{ lb.,}$$

$$e = \frac{226\,000}{73\,200} = 3.1 \text{ in.}$$

Section: $b = 12$ in., $h = 18.5$ in., $d' = 2$ in., $2a = 14.5$ in., $\frac{2a}{h} = 0.78$,

$$A_s = 0.01 \times 12 \times 18.5 = 2.2 \text{ sq. in.}$$

Use Diagram 5, p. 652, marked $2a = 0.8h$.

$\frac{e}{h} = \frac{3.1}{18.5} = 0.168$. From diagram, for $p = 0.01$ and $\frac{e}{h} = 0.168$, $C_e = 1.68$ and

$f_c = 1.68 \times \frac{73\,200}{12 \times 18\ 5} = 550$ lb. per sq. in.

Results.—From the computations of stresses it is evident that the stresses at the crown and at the quarter point are too large. This can be remedied by increasing the section at the crown to 18 in. The remaining section will be increased in proportion by maintaining the ratio $\frac{I}{I_x \cos \phi_x}$ as in the original design.

INFLUENCE LINES

Definition of Influence Lines.—Influence lines are lines representing the effect of unit loads $P = 1$ placed at different points on the arch, upon the magnitude of the statically indeterminate values in an arch or upon the bending moment at any selected section of the arch.

Usually influence lines are drawn for the three statically indeterminate values H, V_A and M. These are common for the whole arch. Typical influence lines for H, V_A and M are shown in Fig. 198, p. 529.

In addition, there are drawn influence lines for bending moments at selected sections. A separate influence line is required for each section.

Sections for which Influence Lines are Drawn.—The sections for which bending moment influence lines are usually drawn are:

1. Springing line of arch;
2. Quarter-point of arch;
3. Crown of arch.

For important arches other intermediate sections may have to be investigated. In such case additional influence lines for bending moments are drawn.

Figure 199, p. 530, shows typical influence lines for bending moments at the springing line, quarter point and crown.

How Influence Lines are Prepared.—An influence line for bending moments at any section (say, the crown) is prepared as follows.

Divide the arch into a number of divisions. Place the load $P = 1$ successively at the end of each division. For each position of the load

$P = 1$ compute the bending moment produced by that load at the selected section (say, the crown) by the appropriate formulas given in previous paragraphs. The determined values are plotted to any convenient scale on vertical lines passing through the corresponding positions of the load. Starting from a common horizontal axis, the positive values are plotted above the axis and the negative values below the axis. The curve passing through the points thus obtained is the influence line for the bending moments at the selected section.

In similar fashion can be drawn influence lines for the horizontal thrust H, vertical reaction V_A and the auxiliary bending moment M.

Purpose of Influence Lines for H, V_A and M.—The influence lines for the three statically indeterminate values of H, V_A and M are necessary in order to determine the influence lines for bending moments at the selected section.

The influence lines for H and V_A are also used to determine the normal thrust N_x to be used in conjunction with the bending moments to determine stresses in the arch. It is obvious that the same position of loading must be used in determining H and V_A as was used for determining the bending moment.

If it is desired to get maximum values for H and V_A the whole arch must be loaded because a load placed in any position produces values of the same sign.

When the loading consists of concentrated loads the maximum horizontal thrust is obtained when the heaviest loads are placed in the center of the span. Maximum vertical reaction, on the other hand, is obtained when the heaviest loads are placed near the support at which the reaction is desired.

Purpose of Influence Lines for Bending Moments.—Influence lines for bending moments are used to determine maximum positive and negative bending moments at the selected sections. The position of the loads for maximum bending moments can be obtained from the study of the influence lines.

As evident from Figs. 199 (a) to (c), p. 535, the influence line for bending moments lies partly below and partly above the axis. The parts above the axis signify positive bending moments, while the parts below signify negative bending moments.

When it is desired to get maximum positive bending moment, those parts of the arch should be loaded for which the influence line is above the axis. For maximum negative bending moments the remaining parts of the arch should be loaded.

For concentrated loads the heaviest loads should be placed where the ordinates of the influence line are largest. Two or three trials may

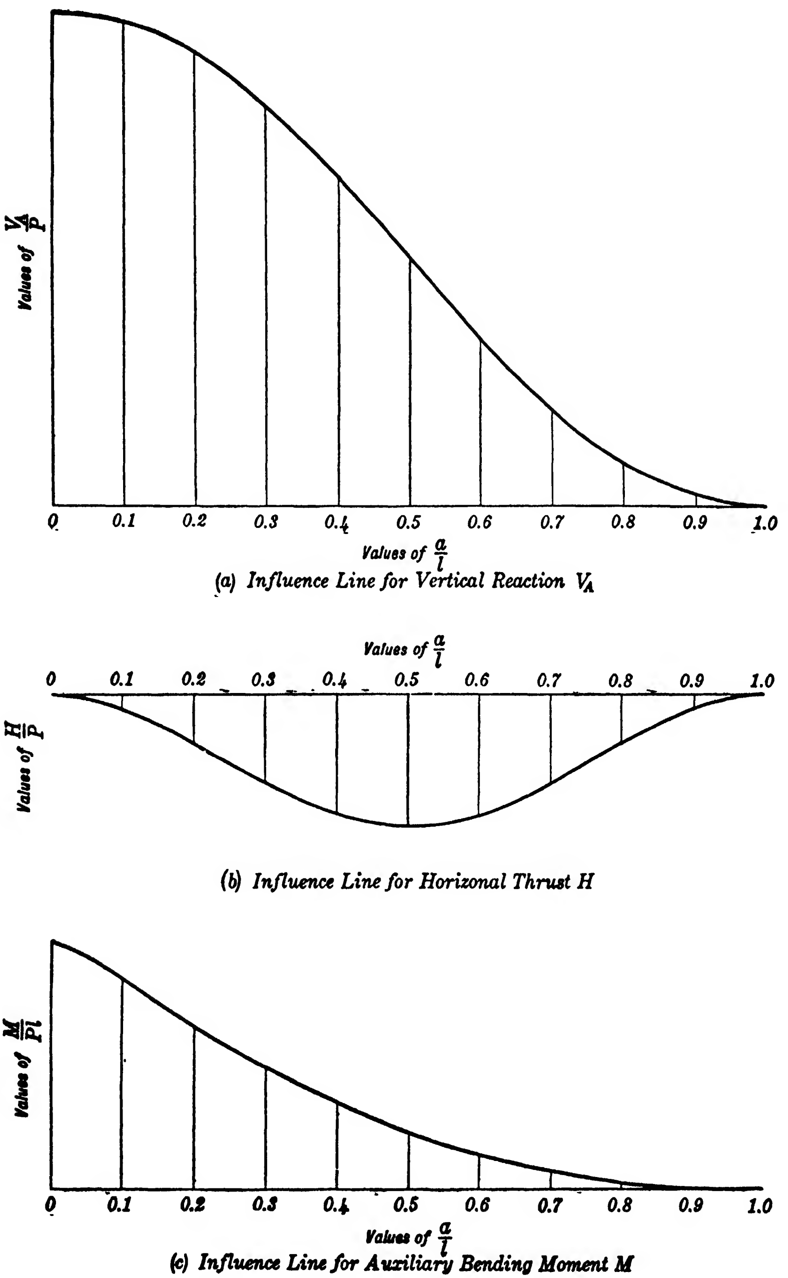


FIG. 198.—Typical Influence Lines for H, V_A and M. (*See p.* 528.)

be necessary before absolute maximum value of the bending moment is obtained.

To find stresses at any section due to any particular loading, not only the bending moment but also the thrust is necessary. The thrust is a function of H and V_A which are obtained from the respective influence lines. It is important to remember that the same loading must be used for determining the H and V_A as was used for determining of the bending moment.

Influence Line for *H*.—This line gives the value of the horizontal thrust, H, caused by a vertical unit load $P = 1$ placed at any point on the arch. It is obtained by computing H for a load $P = 1$ placed successively at each point of the arch and plotting the result as an ordinate under the load. The line connecting the points is the influence line.

A typical influence line for H is shown in Fig. 198. Since all downward loads produce horizontal thrusts of the same sign, the whole line is below the horizontal base.

The values of H for unit loads may be obtained either analytically or graphically.

The analytic method is given below.

General formula for horizontal thrust is (see p. 493)

$$H = -\frac{\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_s y \frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}}.$$

Since for symmetrical arches the influence line is symmetrical, the values need to be found only for one-half of the arch. To simplify the work the loads will be placed on the right half of the arch.

For $P = 1$ placed at a distance x_1 from the crown the static bending moment M_s is (see p. 330)

$$M_s = -(x - x_1) \text{ when } x \text{ is larger than } x_1$$

and

$$M_s = 0 \text{ when } x \text{ is smaller than } x_1.$$

Substituting this value for M_s in formula for H the equation for the influence line becomes

Horizontal Thrust for $P = 1$ Placed at Distance x_1 from the Crown,

$$H = \frac{\sum_{x_1}^{\frac{l}{2}} (x - x_1) y \frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}} \quad \ldots \ldots \quad (62)$$

The denominator of this formula, constant for all positions of the load, is found in Table 2, p. 498. The numerator may be found by the scheme suggested in Table 7, p. 533.

Influence Line for V_A.—Typical influence line for V_A is shown in Fig. 198. It should be noticed that the values of V_A for all positions of downward load are positive, therefore the whole influence line is above the horizontal base.

The general formula for V_A is (see p. 493)

$$V_A = -\frac{\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_s x \frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \frac{Ids}{I_x}}.$$

Values of M_s are same as used for H.

For a load $P = 1$ at x_1 from center the bending moment is $M_s = -(x - x_1)$. Substitute this in the equation for V_A.

Vertical Reaction for $P = 1$ Placed at a Distance x_1 from Crown,

$$V_A = \frac{\sum_{x_1}^{\frac{l}{2}} (x - x_1) x \frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \frac{Ids}{I_x}} \quad \ldots \ldots \ldots \quad (63)$$

The denominator, constant for all positions of the load, is worked out in Table 2, p. 498. The numerator may be found by the scheme suggested in Table 8, p. 534.

In determining the influence line for V_A the following facts may be useful.

For load $P = 1$ placed at the center the reactions at both supports are equal. Therefore, the ordinate of the influence line there is 0.5.

For load $P = 1$ placed at the right support, the reaction V_A on the left support is equal zero.

For load $P = 1$ placed at the left support, the reaction there is equal unity.

For two equal loads each equal to $P = 1$ placed symmetrically on the

Table 7.—Method of Determining Influence Line for *H*

(The Values of $y\frac{Ids}{I_x}$ are worked out in Table 1, p. 497.)

Assumption: span $l = 96$ ft.; divided into 16 divisions each 6 ft. long.

Section	$\frac{Ids}{I_x}$	y	$y\frac{Ids}{I_x}$	x	Load at End of First Division $x_1 = 42$		Load at End of Second Division $x_1 = 36$		Load at End of Third Division $x_1 = 30$		Load at End of Fourth Division $x_1 = 24$		Load at Center $x_1 = 0$	
					$x - x_1$	$(x - x_1)y\frac{Ids}{I_x}$	$x - x_1$	$(x - x_1)y\frac{Ids}{I_x}$	$x - x_1$	$(x - x_1)y\frac{Ids}{I_x}$	$x - x_1$	$(x - x_1)y\frac{Ids}{I_x}$	$x - x_1$	$(x - x_1)y\frac{Ids}{I_x}$
(1)	(2)	(3)	(4)	(5)	(6_1)	(7_1)	(6_2)	(7_2)	(6_3)	(7_3)	(6_4)	(7_4)	(6_8)	(7_8)
1′				45	3		9		15		22		45	
2′				39	....		3		9		15		39	
3′				33			....		3		9		33	
4′				27					....		3		27	
5′				21							....		21	
6′				15									15	
7′				9									9	
8′				3									3	
Sum of $(x - x_1)y\frac{Ids}{I_x}$					$\Sigma =$		$\Sigma =$		$\Sigma =$		$\Sigma =$		$\Sigma =$	

To find H for $P = 1$ placed successively at different points, add in Table 7 each column (7) and divide the sum of each column by $\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{Ids}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}$ obtained from Table 2, p. 498.

To draw the influence line for H, plot to a convenient scale each of the values found above on vertical lines passing through the point at which the load was assumed to be placed, starting from a common horizontal base.

After the influence line is drawn, the horizontal thrust due to any load P placed at any point is equal to the load P multiplied by the ordinate of the influence line directly under the load.

arch and acting simultaneously, the reaction V_A equals 1. Therefore, if the reaction for one load acting separately is known, the reaction for the other load acting separately may be found by subtracting from unity the known reaction of the fourth load. This simplifies the preparation of the influence line, because it is only necessary to compute the reactions for the loads placed on the half of the arch next to the right support. The ordinates for the other half are obtained by subtracting from unity the computed ordinate of the symmetrically placed load in the other half of the arch.

Table 8.—Method of Determining Influence Line for V_A

(Values of $x\frac{Ids}{I_x}$ are taken from Table 2.)

Assumption: Span, $l = 98$ ft., divided into 16 divisions, each 6 ft. long.

Section	x	$x\frac{Ids}{I_x}$ (See p. 498)	Load at End of First Division $x_1 = 42$		Load at End of Second Division $x_1 = 36$		Load at Center $x_1 = 0$	
			$x - x_1$	$(x-x_1)x\frac{Ids}{I_x}$ $(4_1)\times(3)$	$x - x_1$	$(x-x_1)x\frac{Ids}{I_x}$ $(4_2)\times(3)$	$x - x_1$	$(x-x_1)x\frac{Ids}{I_x}$ $(4_8)\times(3)$
(1)	(2)	(3)	(4_1)	(5_1)	(4_2)	(5_2)	(4_8)	(5_8)
1	45		3		9		45	
2	39		.....		3		39	
3	33				.....		33	
4	27						27	
5	21						21	
6	15						15	
7	9						9	
8	3						3	
Sum of Col. (5_1) to (5_8)			$\Sigma =$		$\Sigma =$		$\Sigma =$	

$$V_A(\text{for } P = 1) = \frac{\text{Sum of Col. (5)}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} x^2\frac{Ids}{I_x}}.$$

To find V_A for $P = 1$ at each successive section add in table above each column (5), and divide each sum by $\sum_{-\frac{l}{2}}^{\frac{l}{2}} x^2\frac{Ids}{I_x}$ obtained from Table 2, p. 498.

Plot the values obtained to same scale as used for H on verticals passing through the positions of the load, starting from common horizontal base.

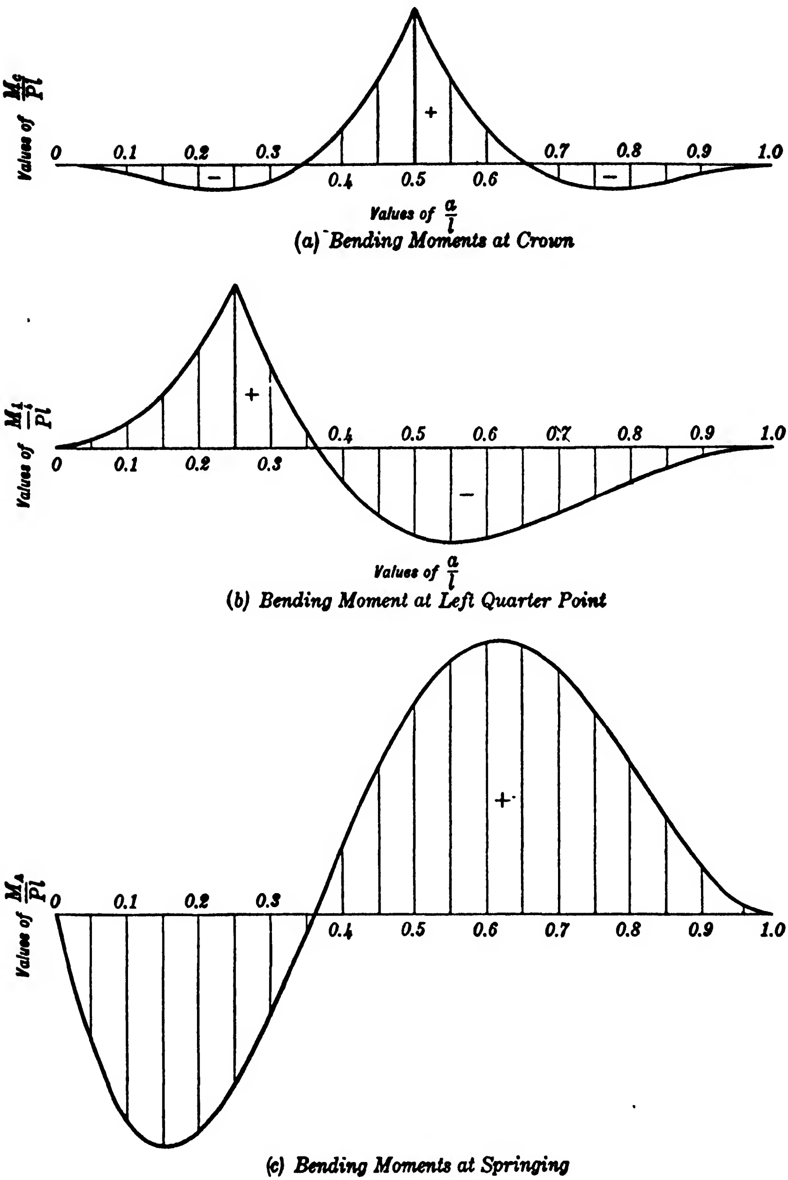


FIG. 199.—Typical Influence Lines for Bending Moments. (*See p.* 528.)

The vertical reaction V_A due to any load placed at any point is equal to the load P multiplied by the ordinate of the influence line below that point.

Influence Line for Auxiliary Moment M.—Influence lines for M has the same significance as the two previously described influence lines. It is obtained in the same manner as described for H.

The general formula for M is (see p. 493)

$$M = -\frac{\sum_{x_1}^{\frac{l}{2}} M_s \frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}}.$$

Substituting the values of M_s for $P = 1$ in same manner as for H on p. 531.

Auxiliary Moment M for $P = 1$ Placed at Distance x_1,

$$M = \frac{\sum_{x_1}^{\frac{l}{2}} (x - x_1)\frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

The denominator, constant for all loadings, is worked out in Table 2, p. 498. The values $\frac{Ids}{I_x}$ used in the numerator are also given in Table 1, p. 497. The numerator may be found as suggested in Table 9, p. 538.

The work will be simplified as follows. As is demonstrated in the foot-note,[10] the ordinate of the influence line for a load $P = 1$ on the

[10] The numerator for a load placed on the left side of the arch for which

$$x = -x_1$$

is

$$\sum_{-x_1}^{\frac{l}{2}} (x - (-x_1))\frac{Ids}{I_x} = \sum_{-x_1}^{\frac{l}{2}} (x + x_1)\frac{Ids}{I_x}.$$

This sum can be replaced by a sum

$$\sum_{-x_1}^{x_1} (x + x_1)\frac{Ids}{I_x} + \sum_{x_1}^{\frac{l}{2}} (x + x_1)\frac{Ids}{I_x}$$

$$= \sum_{-x_1}^{x_1} x\frac{Ids}{I_x} + x_1\sum_{-x_1}^{x_1} \frac{Ids}{I_x} + \sum_{x_1}^{\frac{l}{2}} (x - x_1)\frac{Ids}{I_x} + 2x_1\sum_{x_1}^{\frac{l}{2}} \frac{Ids}{I_x}.$$

The first two items are obtained by multiplying out the item $\sum_{-x_1}^{x_1} (x + x_1)\frac{Ids}{I_x}$ and left side of the arch is equal to the ordinate or symmetrical load on the right side of the arch plus x_1, which is the distance of the load from the center. Therefore, only the ordinates for the right half of the arch need to be computed from the formula.

Influence Line for M_A.—Having determined the influence lines for V_A, H and M the influence line for the bending moment at the left support, M_A may be obtained from Formula (44), p. 493.

$$M_A = M - \frac{l}{2}V_A - Y_s H_A.$$

This formula may be used for loads placed on the right half of the arch. For loads placed on the left half the bending moment at left support may be expressed in terms of the bending moment due to the symmetrically placed load on the right half. The relation is expressed

taking the constant x_1 before summation sign. The last two items were obtained by adding and subtracting $2\,\Sigma x_1 \frac{Ids}{I_x}$. The first term $\sum_{-x_1}^{x_1} x \frac{Ids}{I_x} = 0$ because the terms for between $-x_1$ and 0 are equal and of opposite sign to those between 0 and $+x_1$. The term $x_1 \sum_{-x_1}^{x_1} \frac{Ids}{I_x}$ equals $2x_1 \sum_{0}^{x_1} \frac{Ids}{I_x}$ because the values of $\frac{Ids}{I_x}$ are the same for both sides. The expression then is reduced to

$$\sum_{x_1}^{\frac{l}{2}} (x - x_1)\frac{Ids}{I_x} + 2x_1\left[\sum_{0}^{x_1}\frac{Ids}{I_x} + \sum_{x_1}^{\frac{l}{2}}\frac{Ids}{I_x}\right] = \sum_{x_1}^{\frac{l}{2}} (x - x_1)\frac{Ids}{I_x} + 2x_1 \sum_{0}^{\frac{l}{2}}\frac{Ids}{I_x}.$$

Substituting this numerator in the formula for M, the formula becomes

$$M\ (\text{for } -x_1) = \frac{\sum_{x_1}^{\frac{l}{2}} (x - x_1)\frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}} + \frac{2x_1 \sum_{0}^{\frac{l}{2}} \frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}} = \frac{\sum_{x_0}^{\frac{l}{2}} (x - x_1)\frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}} + x_1.$$

because $2\sum_{0}^{\frac{l}{2}} \frac{Ids}{I_x} = \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}$. The term $\dfrac{\sum_{x_1}^{\frac{l}{2}} (x - x_1)\frac{Ids}{I_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}}$

is equal to M for a point placed at a distance $x = x_1$ on the right side of the arch. Thus the above statement is demonstrated.

Table 9.—Influence Line for Auxiliary Moment *M*

(Values of $\frac{Ids}{I_x}$ are given in Table 2, p. 498.)

Assumption: Span, $l=96$ ft., divided into 16 divisions, each 6 ft. long.

No. of Section	(See Table 2)		Load at End of First Division $x_1=42$		Load at End of Second Division $x_1=30$		Load at Center $x_1=0$	
	x	$\frac{Ids}{I_x}$	$x-x_1$	$(x-x_1)\frac{Ids}{I_x}$ $(4_1)\times(3)$	$x-x_1$	$(x-x_1)\frac{Ids}{I_x}$ $(4_2)\times(3)$	$x-x_1$	$(x-x_1)\frac{Ids}{I_x}$
(1)	(2)	(3)	(4_1)	(5_1)	(4_2)	(5_2)	(4_3)	(5_3)
1	45		3		9		45	
2	39		......		3		39	
3	33				......		33	
4	27						27	
5	21						21	
6	15						15	
7	9						9	
8	3						3	
Sum of Column (5)			$\Sigma=$		$\Sigma=$		$\Sigma=$	

$$M \text{ for } P=1 \text{ right side} = \frac{\text{Sum of Col. (5)}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}}.$$

M for left side $=$ M for right side plus x_1.

To find M for $P=1$ at each successive point on right side of arch, add in table above each column (5) and divide each sum by $\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}$ obtained from Table 2, p. 498.

To find M for $P=1$ at each successive point on left side of arch, add to the values of M for corresponding points on right side the value of x_1.

Plot the values obtained to any convenient scale on verticals passing through the positions of the load, starting from common horizontal base.

Auxiliary bending moment M due to any load P placed at any point is equal to the load multiplied by the ordinate of the influence line below that point.

by the formula[11] where $M_A(x_1)$ and $M_A(-x_1)$ signify bending moments at springing A due to a load $P = 1$ placed at x_1 and $(-x_1)$, respectively.

$$M_{A(-x_1)} = M_{A(x_1)} + lV_{A(x_1)} - \left(\frac{l}{2} - x_1\right). \quad . \quad . \quad . \quad (66)$$

The ordinates of the influence line for M_A may be found as suggested in Table 10. The values for the right side of the arch are computed first and then the values for the left side from Formula (66).

[11] The bending moment M_A for a load $P = 1$ located $x = x_1$ is (right side of arch).

$$M_{A(x_1)} = M_{(x_1)} - \frac{l}{2}V_{A(x_1)} - Y_sH_{(x_1)}.$$

For load $P = 1$ at $x = -x_1$ (left side of arch) the formula is

$$M_{A(-x_1)} = M_{(-x_1)} - \frac{l}{2}V_{A(-x_1)} - Y_sH_{(-x_1)}.$$

For loads at points on the arch located symmetrically about y-axis of the values of Y_sH are equal.

Therefore

$$Y_sH_{(x_1)} = Y_sH_{(-x_1)}.$$

The relation of values of V_A for loads $P = 1$ at symmetrically located points as given on page 534 is $V_{A(-x_1)} = 1 - V_{A(x_1)}$.

Therefore

$$\frac{l}{2}V_{A(-x_1)} = \frac{l}{2} - \frac{l}{2}V_{A(x_1)}.$$

The relation between values M for loads $P = 1$ at x_1 and at $-x_1$, as given on page 536, is

$$M_{(-x_1)} = M_{(x_1)} + x_1.$$

Using the above values following relation may be found between the bending moments M_A for $P = 1$ placed separately at symmetrically located point.

Substitute for $M_{A(-x_1)}$, $V_{A(-x_1)}$ and $H_{(-x_1)}$ values in terms of $M_{(x_1)}$, $V_{A(x_1)}$ and $H_{(x_1)}$. Then

The moment M_A for left side of arch namely for $x = -x_1$ is

$$M_{A(-x_1)} = M_{(x_1)} + x_1 - \left(\frac{l}{2} - \frac{l}{2}V_{A(x_1)}\right) - Y_sH_{(x_1)}.$$

Also

$$M_{A(-x_1)} = M_{(x_1)} + \frac{l}{2}V_{A(x_1)} - Y_sH_{(x_1)} + x_1 - \frac{l}{2}.$$

Add and subtract at the right side $V_{A(x_1)}$,

$$M_{A(-x_1)} = M_{(x_1)} - \frac{l}{2}V_{A(x_1)} - Y_sH_{(x_1)} + \frac{l}{2}V_{A(x_1)} + \frac{l}{2}V_{A(x_1)} - \left(\frac{l}{2} - x_1\right).$$

Finally

$$M_{A(-x_1)} = M_{A(x)} + lV_{A(x_1)} - \left(\frac{l}{2} - x_1\right).$$

Table 10.—Influence Line for Bending Moment at Left Support M_A

(Values of H, V_A and M are taken from proper influence lines)

Section	$-H$ (Table 7)	$-HY_s$	V_A (Table 8)	$\frac{l}{2}V_A$	M (Table 9)	M_A (3) − (5) + (6)	x_1	$\frac{l}{2}-x_1$	lV_A (5) × 2	Section	M_A (7) + (9) + (10)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1										1′	
2										2′	
3										3′	
4										4′	
5										5′	
6										6′	
7										7′	
8										8′	

INFLUENCE LINES FOR BENDING MOMENTS AT ANY SECTION

Influence lines for bending moments at any selected section of the arch for which the ordinates are x_n and y_n may be found from Formula (45), p. 493, which is

$$M_{xn} = M + V_A x_n + H y_n + M_s.$$

The load $P = 1$ is assumed to be placed successively at each division of the arch. For each location of the load values of M_s are computed. This is the static bending moment of the load $P = 1$ about the selected section, considering the arch as a cantilever fixed at right support. The values of M, V_A and H_A for each location of the load are taken from the influence lines previously prepared.

After the bending moment M_{xn} for $P = 1$ placed at any one point is found, it is plotted to a convenient scale on a vertical line passing through that point starting from a horizontal base. The positive values are placed above and the negative values below the horizontal base. In symmetrical arches the influence line for symmetrically placed sections are the same, but placed in reverse positions.

The work is simplified if the section for which influence line is desired is assumed to be on the left half of the arch, as then the values M_s need to be computed only for the loads to the left of the section. For loads placed at points to the right of the section $M_s = 0$.

Table 11 outlines a convenient method of determining the ordinates of the influence lines.

PROPERTIES OF INFLUENCE LINES FOR BENDING MOMENTS

Influence lines are regular consistent curves. Any irregularity in their shape is a sure sign of error in computations. For all fixed arches the shape of influence lines for bending moments for points similarly located is similar in character irrespective of the shape of the arch. Below will be shown and discussed influence lines for bending moments at the crown, springing line and haunch. These points are the critical points in an arch.

It will be noted that each influence line consists of positive areas, i.e., areas above the base line, and negative areas below the base line. This means that loads placed where the influence line is positive produce at the selected section positive bending moments and loads placed where the influence line is negative produce negative bending moments. For maximum bending moment of any one sign the load must cover the whole length of the influence line of that sign.

Table 11.—Influence Line for Bending Moment M_{x_n}

Ordinates of Section: $x_n = (-a)$; $y_n = b$

(Values of H, V_A and M are taken from proper influence lines, H is negative.)

Right Half of Arch							Left Half of Arch								
Load at Section	M	V_A	$V_A x_n$	H	$H y_n$	M_{x_n} (2)+(4)+(6)	Load at Section	M	V_A	$V_A x_n$	H	$H y_n$	x	$M_s = x_n - x$	M_{x_n} (9)+(11)+(13)+(15)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
1							1′								
2							2′								
3							3′								
4							4′								
5							5′								
6							6′								
7							7′								
8							8′								

Plot the values from column (7) and (13) on verticals passing through appropriate points starting from a common horizontal base. Place positive values above the base and negative values below the base.

To find bending moment due to any loading multiply each load by the ordinate vertically below it.

The bending moment due to any load is equal to the load, times the corresponding ordinate of the influence line. For uniformly distributed load extending over any part of the arch the bending moment equals the area between the influence line and the base in the section below the loaded part of the arch multiplied by the unit load. The area of influence line therefore is the measure of the magnitude of the bending moments. The larger the area of the influence line the larger is the moment.

Influence Line at the Crown.—Fig. 199 shows a typical influence line for bending moment at the crown. It consists of two symmetrical arms. The positive section is in the central part of the arch and the negative sections on each side.

The positive section extends on both sides of the crown a distance equal to about $\frac{1}{8}l$. Each negative section extends from the support for a distance equal to about $\frac{3}{8}l$.

To get maximum positive bending moment at the crown, the central part of the arch only must be loaded. For maximum negative bending moment this central part must be left without load and instead the remaining parts of the arch must be loaded.

The area of the positive section of the influence line always predominates, which means that when live loads are placed in the most unfavorable position, the maximum positive bending moment at the crown due to live load is larger than the maximum negative bending moment. The difference between the positive and negative areas is smallest for parabolic arch axis and is due only to the effect of rib shortening. Were the effect of rib shortening neglected, the sum of negative areas would be equal to the positive area, consequently for full span loaded uniformly the bending moment at the crown would be zero.

For arch axis such as used for arches with filled spandrels (where the arch axis approaches a parabola of third or fourth power) the negative areas of the influence lines decrease with the increase of the radius of curvature at the crown.

The maximum positive and negative bending moments for different shapes of arch axis, based on influence lines, are given in Diagrams 26–27, pp. 672–673.

Influence Line at the Quarter Point of the Span.—Fig. 199, p. 535, show a typical influence line for bending moments at the quarter point. It consists of two areas intersecting at the vertical passing through the quarter point. The shorter arm is wholly above the axis, while the longer arm is partly above and partly below the axis. The section of positive bending moments starts from the support nearest to the load and extends for a distance equal to from 0.35 to 0.40 of the span. The lower value is for parabolic arch axis while the higher value is for arch axes resembling parabolas of fourth power such as would be used for arches with filled spandrels. Negative bending moments are caused by loads placed on the remaining 0.65 to 0.60 of the span.

Influence Lines for Bending Moments at Springing.—Typical influence lines for bending moments M_A at the left springing is given in Fig. 199. It consists of a negative section next to the left support and a positive section on the rest of the arch. The loads producing the maximum negative and maximum positive bending moments, respectively, are shown in Fig. 196, p. 507.

The shape of the influence line and, therefore, the length of the negative and the positive sections depends mainly upon the shape of the arch axis and also, but to a smaller degree, upon the ratio of the moments of inertia at the crown to the moment of inertia at the springing. The negative section extends from the left support for a distance varying from 0.35 to 0.40 of the span. The positive section extends over the rest of the span. At an average the section of negative bending moments may be taken as $\frac{3}{8}$ of the span while that of the positive bending moments as $\frac{5}{8}$ of the span.

Although the negative section is shorter than the positive section, the negative ordinates are much larger than the positive ordinates. Consequently the difference between the areas is not as large as would seem from the difference of their length. For parabolic arch axis and $\dfrac{I}{I_x \cos \phi_x}$ constant the two areas are almost equal, the difference being only due to the effect of rib shortening. If the effect of the rib shortening is neglected the two areas would be equal. When the arch axis becomes flatter in the center and steeper at the ends and, therefore, resembles the shape of parabolas of higher power, the negative areas become smaller and the positive areas larger.

In diagrams, pp. 676–677, are given maximum negative and positive bending moments for different shaped arch axes.

PARABOLIC FIXED ARCHES

Use of Parabolic Arches and Parabolic Arch Formulas.—When the dead load is practically of uniform intensity over the whole length of the arch, the line of pressure for the dead load is a parabola. In such case the arch axis should be made parabolic in shape. The formulas for parabolic arches, then, give exact results.

The dead load is nearly uniform in open spandrel arches, particularly in arches consisting of separate ribs and in arches with suspended roadway. In such cases the weight of the floor system, which forms the largest part of the dead load, is uniform. The difference between the load at the various points is due only to the difference in the weight of the arch rib. This small difference in dead load produces a negative bending moment at the crown which partly offsets the bending moment due to rib shortening. The actual bending moments due to this excess of the dead load may be readily found from the influence lines. When the ratio of the dead load at the springing and at the crown exceeds 1.2 but is not more than 1.5 the arch axis should not be parabolic, but should be made to coincide with the line of pressure for dead load. However, the parabolic formulas for bending moments and thrusts and the influence lines give reliable enough results and their use is permissible.

For ratios of dead load at springing to that at the crown larger than 1.5 the error in using the formulas for parabolic arches is at least 8 per cent and increases with the increase of the dead load ratio. In such cases the use of parabolic formulas is not advisable. Preliminary bending moments may be taken from diagrams on pp. 671 to 677 and the statically indeterminate values should be computed in the manner given on p. 481.

Properties of the Arch.

Equation of Arch Axis, Center of Coordinates at Crown,

$$Y = \frac{4r}{l^2}x^2. \qquad (67)$$

Elastic Center, Distance from Crown,

$$Y_c = \frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} Y\frac{Ids}{d_x}}{\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}} = \frac{3n+2}{5(n+2)}r, \qquad (68)$$

where

$$n = \frac{I}{I_s \cos \phi_s}.$$

Equation of Arch Axis, Center of Coordinates at Elastic Center,

$$y = Y_c - \frac{4r}{l^2}x^2. \qquad (69)$$

Variation in Moment of Inertia,

$$\frac{I}{I_x \cos \phi_x} = 1 - 4(1 - n)\left(\frac{x}{l}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

where

$$n = \frac{I}{I_s \cos \phi_s}.$$

Moment of Inertia at Any Point,

$$I_x = \frac{\sqrt{1 + 64\left(\frac{r}{l}\right)^2\left(\frac{x}{l}\right)^2}}{1 - 4(1 - n)\left(\frac{x}{l}\right)^2} I. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Depth of Rectangular Section at Any Point,

$$h_x = \sqrt[3]{\frac{\sqrt{1 + 64\left(\frac{r}{l}\right)^2\left(\frac{x}{l}\right)^2}}{1 - 4(1 - n)\left(\frac{x}{l}\right)^2}}\, h. \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Formulas for Statically Indeterminate Values H, V_A and M.—Formulas for statically indeterminate values H, V_A and M for parabolic fixed arches are given on pp. 618, 620, and 622 to 624.

Uniformly Distributed Loading.—The tables below give reactions, thrusts and bending moments for most unfavorable positions of uniformly distributed live load at springing, quarter point and crown.

Springing

	Values of n						
	1.0	0.8	0.6	0.4	0.3	0.2	0.1
	Maximum Positive Bending Moment (Formulas (92) to (95) p. 623)						
M_A	$0.017wl^2$	$0.018wl^2$	$0.019wl^2$	$0.021wl^2$	$0.022wl^2$	$0.023wl^2$	$0.025wl^2$
$(r/l)H$	$0.085wl$	$0.086wl$	$0.086wl$	$0.087wl$	$0.087wl$	$0.087wl$	$0.088wl$
V_A	$0.151wl$	$0.150wl$	$0.148wl$	$0.146wl$	$0.145wl$	$0.143wl$	$0.141wl$
	Maximum Negative Bending Moment (Formulas (96) to (99) p. 623)						
M_A	$-0.017wl^2$	$-0.018wl^2$	$-0.019wl^2$	$-0.021wl^2$	$-0.022wl^2$	$-0.023wl^2$	$-0.025wl^2$
$(r/l)H$	$0.040wl$	$0.039wl$	$0.039wl$	$0.039wl$	$0.038wl$	$0.038wl$	$0.037wl$
V_A	$0.349wl$	$0.35wl$	$0.35wl$	$0.35wl$	$0.36wl$	$0.36wl$	$0.36wl$

Quarter Point

	Values of n						
	1.0	0.8	0.6	0.4	0.3	0.2	0.1
	Maximum Positive Bending Moment (Formulas (100) to (103) p. 624)						
$M_{\frac{1}{4}}$	$0.0090wl^2$	$0.0087wl^2$	$0.0081wl^2$	$0.0077wl^2$	$0.0075wl^2$	$0.0071wl^2$	$0.0065wl^2$
$(r/l)H$	$0.0345wl$	$0.0342wl$	$0.0338wl$	$0.0333wl$	$0.0329wl$	$0.0324wl$	$0.0318wl$
V_A	$0.333wl$	$0.333wl$	$0.335wl$	$0.337wl$	$0.338wl$	$0.340wl$	0 $342wl$
	Maximum Negative Bending Moment (Formulas (104) to (107) p 624)						
$M_{\frac{1}{4}}$	$0.0090wl^2$	$0.0087wl^2$	$0.0081wl^2$	$0.0077wl^2$	$0.0075wl^2$	$0.0071wl^2$	$0.0065wl^2$
$(r/l)H$	$0.090wl$	$0.0091wl$	0 $0091wl$	$0.0092wl$	$0.0092wl$	$0.0093wl$	$0.0093wl$
V_A	$0.167wl$	$0.167wl$	$0.165wl$	$0.163wl$	$0.162wl$	$0.160wl$	0 $158wl$

Crown

	Value of n						
	1.0	0.8	0.6	0.4	0.3	0.2	0.1
	Maximum Positive Bending Moment (Formulas (84) to (87) p. 622)						
M_C	0 $0051wl^2$	$0.0049wl^2$	$0.0048wl^2$	$0.0045wl^2$	$0.0044wl^2$	0 $0042wl^2$	$0.0040wl^2$
$(r/l)H$	$0.056wl$	$0.057wl$	$0.057wl$	$0.058wl$	0 $059wl$	$0.060wl$	0 $061wl$
V_A	← $\frac{1}{2}wl$ →						
	Maximum Negative Bending Moment (Formulas (88) to (91), p. 622)						
M_C	$0.0051wl^2$	0 $0049wl^2$	0 $0048wl^2$	0 $0045wl^2$	0 $0044wl^2$	0 $0042wl^2$	$0.0040wl^2$
$(r/l)H$	$0.069wl$	$0.068wl$	$0.068wl$	0 $067wl$	0 $066wl$	$0.065wl$	0 $064wl$
	← $\frac{1}{2}wl$ →						

Effect of Rib Shortening, Parabolic Fixed Arch.—In Formula (51), p. 494, make in the numerator

$$\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x} = \frac{I}{A_{av}} l;$$

in the denominator omit the second term and for the first term substitute a value from Formula (76), p. 619. This gives

Horizontal Thrust for Rib Shortening, Parabolic Arch,

$$H_s = -\frac{175(n+2)}{4(n^2+8n+2.667)r^2}\frac{I}{A_{av}}H_d = -C_p\frac{I}{A_{av}}H_d. \quad (73)$$

This thrust should be used as explained on p. 494.

Effect of Temperature, Parabolic Fixed Arch.—Making in Formula (55), p. 495, similar substitutions as in the previous case, the following formula is obtained.

Horizontal Thrust Due to Temperature Changes,

$$H_t = \pm \frac{175(n+2)}{4(n^2+8n+2.667)r^2}\alpha EtI = C_p\alpha EtI. \quad . \quad . \quad (74)$$

This thrust should be used as explained on p. 495.

Values of C_p

	Value of n						
	1 0	0.8	0.6	0.4	0.3	0.2	0.1
C_p	11.3	12.6	14.5	17.4	19.6	22.4	26.4

Influence Lines for Parabolic Fixed Arches.

Influence Line for Horizontal Thrust,

$$H_{\text{for } P=1} = -\frac{\left[\frac{1}{4}-\left(\frac{x}{l}\right)^2\right]\left\{3n(n+4)+8(1-n)(n+2)\left[\frac{1}{4}-\left(\frac{x}{l}\right)^2\right]\right\}}{\frac{12}{35}\left(n^2+8n+\frac{8}{3}\right)+15(n+2)\frac{I}{r^2A_{av}}}\frac{l}{r}. \quad (75)$$

Influence Line for Vertical Reaction,

$$V_{A\,\text{for}P=1} = \frac{1}{2}\left(1-2\frac{x}{l}\right)^2\left[\left(1+\frac{x}{l}\right)-\frac{3(1-n)}{2(2+3n)}\frac{x}{l}\left(1+2\frac{x}{l}\right)^2\right]. \quad (76)$$

Influence Line for Auxiliary Bending Moment,

$$M_{\text{for } P=1} = \frac{1}{8}\left(1-2\frac{x}{l}\right)^2\left\{1+\frac{1-n}{2(2+n)}\left(1+2\frac{x}{l}\right)^2\right\}l^2. \quad . \quad . \quad . \quad (77)$$

Influence Line for Bending Moment at Springing, M_A,

$$M_A = M - V_A\frac{l}{2} - H(r - Y_c). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

Take values of M, V_A and H from proper influence lines.

Influence Line for Bending Moment at Crown, M_c,

$$M_c = M + HY_c. \quad \text{Right side of arch.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

Take values of M and H from proper influence lines.

CHAPTER VII

TWO-HINGED ARCHES

Description of Two-hinged Arches.—Two-hinged arches are arches provided with hinges at both springings. Ordinarily the hinges are placed at the abutments or pier. More rarely the hinges are placed above the springing of the arch, thus dividing the construction into a two-hinged arch and two cantilevers, one at each abutment. The cantilevers are built monolithic with the abutments.

Behavior of Two-hinged Arches under Load.—The figure 200 illustrates the behavior of the arch under different positions of the live

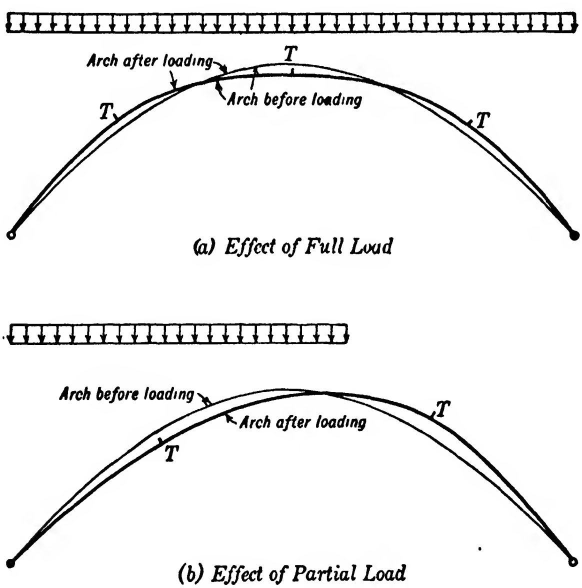
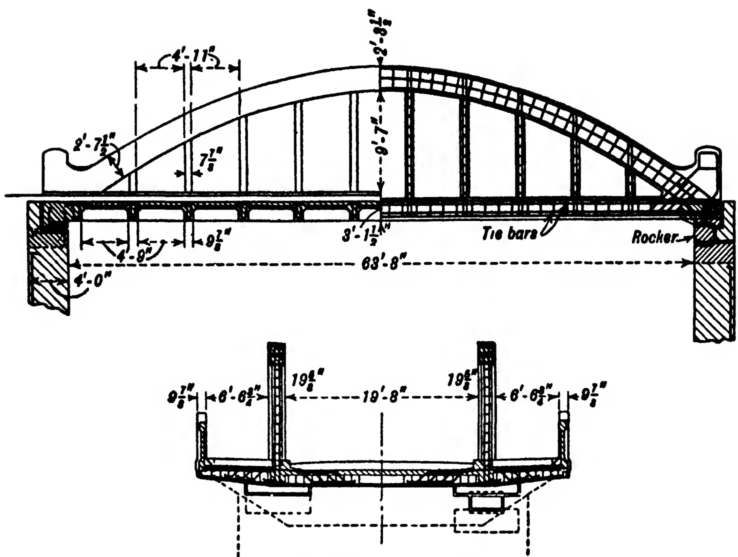
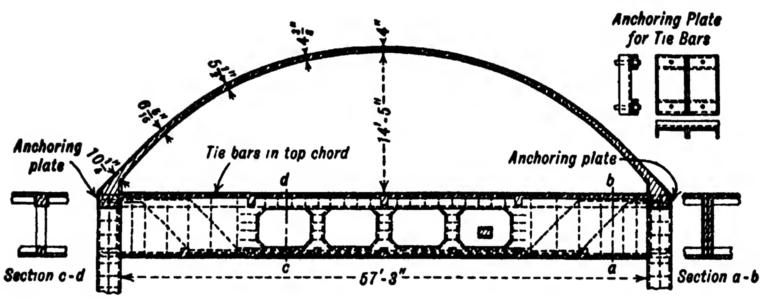


FIG. 200.—Two-hinged Arch Subjected to Various Loads. (*See p.* 549.)

load. As is the case with fixed arches, partial load produces larger bending moments at the critical sections than a load extending the full length of the arch span.



Bow-String Arch Bridge



Roof Arch

Fig. 201.—Types of Two-hinged Arches.[1] (*See p.* 551.)

[1] The Bow-string Arch Bridge represents the bridge at Kristianstad in Sweden, described in Beton und Eisen, year 1916, p. 6.

The Roof Arch represents the Arch in a theatre at Kopenhagen, described in Beton und Eisen, year 1909, p. 146.

Use of Two-hinged Arches.—Two-hinged arches are used, particularly in Europe, for long span roof construction, where the horizontal thrust is usually resisted by horizontal steel ties connecting the ends of the arch. Bow-string arch bridges may also be designed as two-hinged arches with or without ties. For ordinary arch bridges two-hinged arches are seldom used, three-hinged arches being preferable.

Notation.

Let l = length of span;

Δl = change in span length;

A_x = area of cross-section of arch normal to arch axis at any point x;

A_{av} = area of average cross-section of the arch, normal to arch axis;

I_c = moment of inertia of cross-section at the crown;

I_x = moment of inertia of cross-section of arch A_x;

E = modulus of elasticity of concrete;

M_x = actual bending moment in the arch at any point x due to loads;

M_S = static bending moment due to loads at any point x considering arch as a simply supported beam;

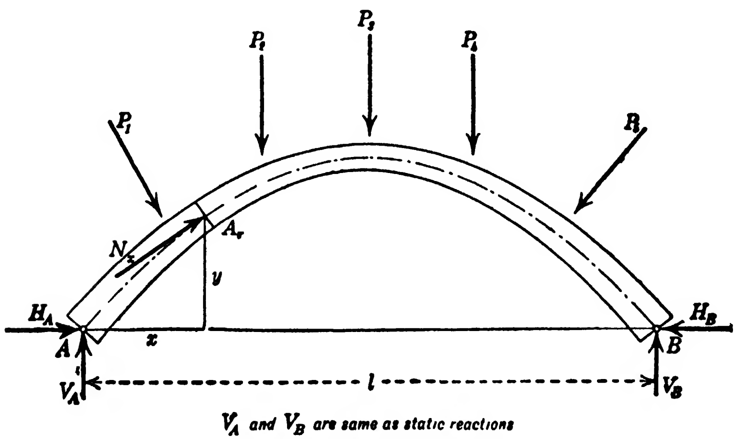


FIG. 202.—Forces and Reactions in Two-hinged Arch. (*See p.* 552.)

ϕ_x = angle of inclination of section with the vertical at point x;

V_A = vertical reaction at left support;

V_B = vertical reaction at right support;

H_A = horizontal thrust at left support;

H_B = horizontal thrust at right support;

H = horizontal thrust for vertical loads;

N_x = thrust normal to cross-section A_x of the arch;
x and y = ordinates relating to left support as center of coordinates;
ds = infinitely small division of the arch axis used in integration;
Δs = finite division of the arch axis used in summation.

Reactions and Bending Moment in Two-hinged Arches at the Hinges.—In two-hinged arches no bending moment can exist at the hinges. At each hinge A and B there exists only a vertical reaction V_A and V_B, respectively, and a horizontal thrust H_A and H_B, respectively. Where the loading is vertical the horizontal thrusts at both hinges are equal so that $H_A = H_B = H$. (See Fig. 202, p. 551.)

Vertical Reactions.—Vertical reactions are statically determinate. Since there are no bending moments at the hinges, one vertical reaction can be computed as for freely supported beams by taking bending moments of loads and reactions about one of the hinges and equating it to zero. The other reaction is then found from the requirement that the sum of both reactions must be equal to the sum of the loads.

Horizontal Thrust.—The horizontal thrust H in two-hinged arches is the only statically indeterminate value. It is formed from the following formulas.[2]

[2] The horizontal thrust is found from the requirement that the horizontal movement of the support due to the bending moments and thrusts be equal zero.

The equation for the horizontal movement of the support is the same as for fixed arches. (See p. 592.)

$$\Delta l = -\int_0^l \frac{M_x}{EI_x} yds + \int_0^l \frac{N_x}{EA_x} ds \cos \phi_x.$$

The ordinates x and y refer to a system of coordinates with an origin at the left support A.

Substitute in the first term of the above equation for the bending moment at any point of arch $M_x = M_S + Hy$. (See Formula (10), p. 275.) The second term of the above equation is small in comparison with the first so that it is permissible to simplify it by substituting $N_x = -H$ and also using instead of the variable A_x an average value A_{av}. This makes the solution of the integral possible. The integral is:

$$\int_0^l \frac{N_x}{A_x} ds \cos \phi_x = -\frac{H}{A_{av}} \int_0^l dx = -H \frac{l}{A_{av}}.$$

The equation then becomes

$$\Delta l = -\int_0^l \frac{M_S}{EI_x} yds - H \int_0^l y^2 \frac{ds}{EI_x} - \frac{H}{E} \frac{l}{A_{av}}. \quad (1)$$

Make $\Delta l = 0$. Eliminating the constant value of E and solving for H.

Horizontal Thrust Two-hinged Arch, by Integration

$$H = -\frac{\int_0^l M_S y \frac{I_c}{I_x} ds}{\int_0^l y^2 \frac{I_c}{I_x} ds + \frac{lI_c}{A_{av}}}.$$

Horizontal Thrust Two-hinged Arch, by Integration,

$$H = -\frac{\int_0^l M_S y \frac{I_c}{I_x} ds}{\int_0^l y^2 \frac{I_c}{I_x} ds + \frac{lI_c}{A_{av}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By substituting for the integral sign the summation sign and for the infinitely small ds the finite value of Δs,

Horizontal Thrust Two-hinged Arch, by Summation,

$$H = -\frac{\sum_0^l M_S y \frac{I_c}{I_x} \Delta s}{\sum_0^l y^2 \frac{I_c}{I_x} \Delta s + \frac{lI_c}{A_{av}}} \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

The last term in the denominator $\frac{lI_c}{A_{av}}$ represents the effect of rib shortening due to live load thrust. It is small and may be omitted except in very flat arches.

Bending Moment at Any Point.—After the horizontal reaction is computed, the bending moments are found from the formula.

Bending Moment at Any Point, Two-hinged Arch,

$$M_x = M_S + Hy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where M_S is the static bending moment for simply supported beams. H should be used with its sign. For downward loads Hy is negative.

Normal Thrust.

Formula for normal thrust is

Normal Thrust, Two-hinged Arch,

$$N_x = V_x \sin \phi_x - H \cos \phi_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Determining of Horizontal Thrust by Summation

Where the arch axis cannot be expressed by an equation the horizontal thrust H cannot be obtained by integration and the summation method must be used.

Denominator.—As evident from Formula (2), p. 553, the denominator for H is $\sum_0^l y^2 \frac{I_c}{I_x} \Delta s + \frac{lI_c}{A_{av}}$. The denominator can be computed as follows:

Lay out the arch axis to a convenient scale and also the assumed thickness of the arch. Divide the arch into a desired number of divisions. Find the values of y for the center of each division. Measure the length of each division Δs. Scale the thickness h_x of the arch at the center of each division, also scale h_c at the crown. Prepare a table as shown on p. 554 and make the computations there indicated. For symmetrical arches only one-half of the arch needs to be used and the sum multiplied by 2, because the value of $\sum_0^l y^2 \frac{I_c}{I_x} \Delta s$ for the whole arch equals twice the value for the half arch. For rectangular cross-sections $\frac{I_c}{I_x} = \left(\frac{h_c}{h_x}\right)^3$.

To get the denominator add $\frac{lI_c}{A_{av}}$ unless this value is omitted.

Table 1.—Denominator for H, Two-hinged Arch

Value of $\sum_0^l y^2 \frac{I_c}{I_x} \Delta s$

Arch Divided into $2n$ Divisions

y = ordinate of the center of each division

Number of Division	y	y^2	Δs	h_x	$\frac{h_c}{h_x}$	$\left(\frac{h_c}{h_x}\right)^3$	$\left(\frac{h_c}{h_x}\right)^3 \Delta s$	$y^2\left(\frac{h_c}{h_x}\right)^3 \Delta s$	$y\left(\frac{h_c}{h_x}\right)^3 \Delta s$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1									
2									
3									
4									
5									
n									
Sum of Col. (9) = $\sum_0^{\frac{l}{2}} y^2 \frac{I_c}{I_x} \Delta s$ =									

To get the denominator for H multiply the sum of Col. (9) by 2 and add $\frac{lI_c}{A_{av}}$.

Numerator.—The numerator $\sum_0^l M_S y \frac{I_c}{I_x} \Delta s$ depends upon the loads and must be determined separately for each arrangement of the loading for which the arch is investigated.

Uniform Load Extending Over Whole Span.—In symmetrical arches for uniform loading extending over the whole span the values of M_S

for two symmetrically located points are equal. Since the values of y, Δs and I_x are also equal, the sum $\sum_0^l$ for the whole arch is equal to twice the sum $\sum_0^{\frac{l}{2}}$ for one-half of the arch. Therefore, the sums for one-half of the arch, only, need to be computed.

The general formula for static bending moment is $M_S = \frac{1}{2}wx(l - x)$. If the arch is divided into $2n$ equal divisions so that $l = 2nl_1$ the equation may be simplified by using the following formula for static bending moment.

Let l_1 = length of division of the span $= \frac{l}{2n}$.

$$M_S = \frac{wl_1{}^2}{2}\frac{x}{l_1}\left(2n - \frac{x}{l_1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Table 2.—Numerator for Horizontal Thrust, Two-hinged Arch

Uniform Load, Whole Span Loaded

$$\sum_0^{\frac{l}{2}} M_S y\frac{I_c}{I_x}\Delta s = \frac{wl_1{}^2}{2}\sum_0^{\frac{l}{2}}\frac{x}{l_1}\left(2n - \frac{x}{l_1}\right)y\frac{I_c}{I_x}\Delta s$$

x and y are ordinates of the centers of each division. l_1 is horizontal length of each division.

Number of Division	$y\frac{I_c}{I_x}\Delta s$	$\frac{x}{l_1}$	$2n-\frac{x}{l_1}$ $2n-(3)$	$\frac{x}{l_1}\left(2n-\frac{x}{l_1}\right)$ $(3)\times(4)$	$\frac{x}{l_1}\left(2n-\frac{x}{l_1}\right)y\frac{I_c}{I_x}\Delta s$ $(5)\times(2)$
(1)	(2)	(3)	(4)	(5)	(6)
1	From Table 1	0.5			
2		1.5			
3		2.5			
n		$n-0.5$			
Sum of Col. (6) = $\sum_0^{\frac{l}{2}}\frac{x}{l_1}\left(2n - \frac{x}{l_1}\right)y\frac{I_c}{I_x}\Delta s =$					

To get the numerator multiply the sum $\sum_0^{\frac{l}{2}}\frac{x}{l_1}\left(2n - \frac{x}{l_1}\right)y\frac{I_c}{I_x}\Delta s$ obtained from the table by $2\frac{wl_1{}^2}{2} = wl_1{}^2$.

Uniform Load Extending over One-half of the Span.—For symmetrical arches the horizontal thrust for uniform load w, extending over one-half of the span is equal to one-half of the horizontal thrust for full loading.

Uniform Loading Extending over Five-eighths of Span Length. (See Fig. 203 (*a*).)—This loading gives approximately the maximum negative bending moment at the unloaded quarter point of the arch. If the load covers the left side of the arch the static bending moments are

$$M_S = \frac{1}{2}\frac{x}{l}\left(\frac{55}{64} - \frac{x}{l}\right)wl^2 \quad \text{for } x \text{ smaller than } \tfrac{5}{8}l. \qquad (5)$$

$$M_S = \frac{25}{128}\left(1 - \frac{x}{l}\right)wl^2 \quad \text{for } x \text{ larger than } \tfrac{5}{8}l. \qquad (6)$$

To simplify the work, divide the arch into $2n$ equal parts and substitute as before $l = 2nl_1$ and $\dfrac{x}{l_1} = n_x$. Then the formulas change to

$$M_S = \tfrac{1}{2}n_x(3.44n - n_x)wl_1{}^2 \quad \text{for } x = n_xl_1 \text{ smaller than } \tfrac{5}{8}l$$

and

$$M_S = 0.39n(2n - n_x)wl_1{}^2 \quad \text{for } x = n_xl_1 \text{ larger than } \tfrac{5}{8}l.$$

The numerator for H becomes

$$\sum_0^l M_S y\frac{I_c}{I_x}\Delta s = wl_1{}^2\left\{\sum_0^{\frac{5}{8}\frac{l}{l_1}} \tfrac{1}{2}n_x(3.44n - n_x)y\frac{I_c}{I_x}\Delta s + \sum_{\frac{5}{8}\frac{l}{l_1}}^{2n} 0.39n(2n - n_x)y\frac{I_c}{I_x}\Delta s\right\}. \qquad (7)$$

It is best to divide the arch into sixteen equal divisions. Then ten divisions are loaded and six are not loaded. Also $2n = 16$ and $\dfrac{5}{8}\dfrac{l}{l_1} = 10$. The computations should be worked out as suggested in table below. The loading of the arch and the divisions are shown in Fig. 203(*a*), p. 552.

Uniform Load Extending over Three-eighths of Span Length. (See Fig. 203(*b*).)—This loading gives approximately the maximum positive bending moment at the loaded quarter point of the arch. The horizontal thrust is obtained by subtracting the horizontal thrust for load extending over five-eighths span found above from the horizontal thrust for full load.

Uniform Load Extending Over One-eighth Span on Both Sides of Crown. (See Fig. 204(*a*).)—This loading gives approximately the maximum positive bending moment at the crown. The horizontal thrust for this loading may be obtained by subtracting, from the hori-

Table 3.—Numerator for Horizontal Thrust, Two-hinged Arch

Five-eighths of Span Uniformly Loaded

$$\Sigma_0^l M_s y\frac{I_c}{I_x}\Delta s = wl_1^2\left\{\Sigma_0^{10}\tfrac{1}{2}n_x(3.44n - n_x)y\frac{I_c}{I_x}\Delta s + \Sigma_{10}^{16}0.39n(2n - n_x)y\frac{I_c}{I_x}\Delta s\right\}$$

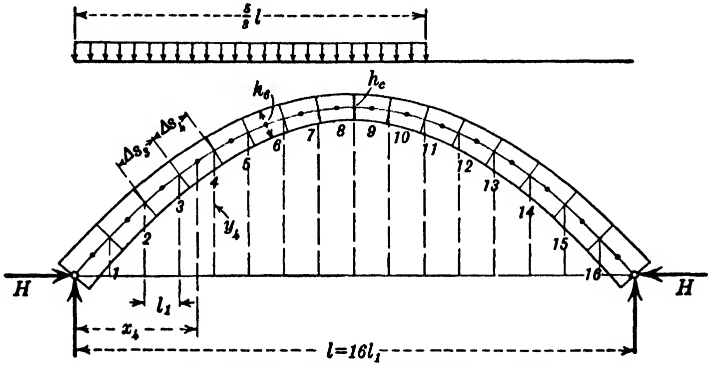
$$l = 16l_1, \quad 2n = 16, \quad \frac{5}{8}\frac{l}{l_1} = 10, \quad 1.72n = 13.75, \quad 0.39n = 3.12$$

Number of Division	$y\frac{I_c}{I_x}\Delta s$	n_x	$1.72n - n_x$ $1.72n - (3)$	$\frac{1}{2}n_x(3.44n - n_x)$ $(3)\times(4)\times\frac{1}{2}$	$\frac{1}{2}n_x(3.44n - n_x)y\frac{I_c}{I_x}\Delta s$ $(2)\times(5)$	$2n - n_x$ $2n - (3)$	$0.39n(2n - n_x)$ $0.39n\times(7)$	$0.39n(2n - n_x)y\frac{I_c}{I_x}\Delta s$ $(2)\times(8)$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	Take from Table 1	0.5	27.02			…	…	…
2		1.5	26.02				…	…
3		2.5	25.02			…	…	…
4		3.5	24.02			…	…	…
5		4.5	23.02			…	…	…
6		5.5	22.02			…	…	…
7		6.5	21.02			…	…	…
8		7.5	20.02			…	…	…
9		8.5	19.02			…	…	…
10		9.5	18.02				…	…
11		10.5	…	…	…	5.5		
12		11.5	…	…	…	4.5		
13		12.5	…	…	…	3.5		
14		13.5	…	…	…	2.5		
15		14.5	…	…	…	1.5		
16		15.5	…	…	…	0.5		
				Sum of Col. (6)			Sum of Col. (9)	

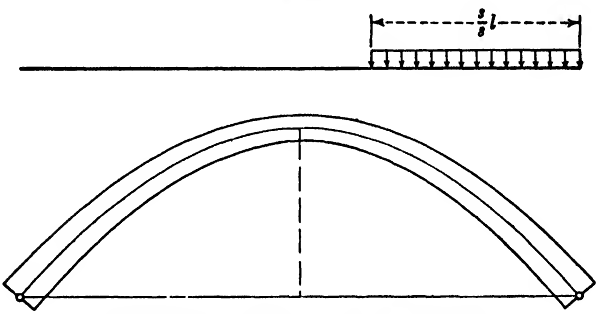
To get numerator add sums of Col. (6) and Col. (9) and multiply by wl_1^2.
The figures given in the table apply to all arches irrespective of span or rise.

zontal thrust for load over five-eighths of the span, the horizontal thrust for load extending over three-eighths of the span.

Uniform Load Extending at Each End for Three-eighths of Span. (See Fig. 204(*b*).)—This loading gives approximately the maximum negative bending moment at the crown. The horizontal thrust for this



(a) Loading for Maximum Negative Bending Moment at Right Quarter Point

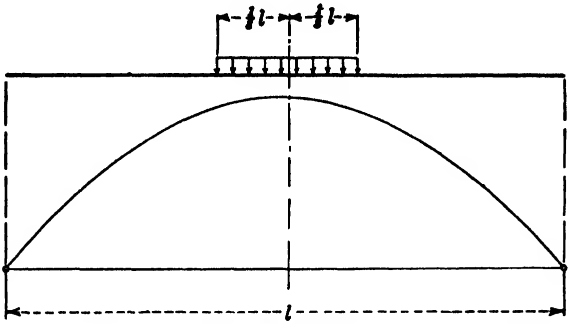


(b) Loading for Maximum Positive Bending Moment at Right Quarter Point

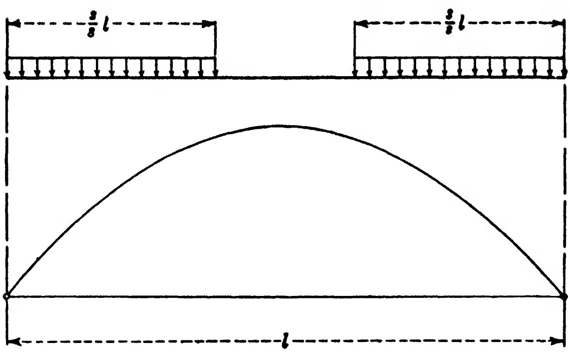
FIG. 203.—Loading for Maximum Bending Moments at Quarter Point. (*See p.* 556.)

loading equals twice the horizontal thrust for load extending on one side only for three-eighths of the span.

Horizontal Thrust for Concentrated Loads.—The work of determining the numerator for horizontal thrust for a single concentrated load may be simplified by applying the characteristics that the horizontal thrust produced by one load is equal to one-half of the horizontal thrust produced by two equal loads placed symmetrically on the arch. Therefore the numerator for two symmetrical loads is computed and then divided by two. This method is advantageous because it is necessary to make summations for one-half of the arch only, the summation for the both halves being equal, and because the formulas for static bending moments M_S are simpler. Thus for two loads P placed at a distance a from each support, for which $x = a$ and $x = l - a$,



(a) Loading for Maximum Positive Bending Moment at Crown



(b) Loading for Maximum Negative Bending Moment at Crown

FIG. 204.—Loading for Maximum Bending Moments at Crown. (*See p.* 556.)

respectively, the static bending moment is $M_S = Px$ for all points between the support and the load and $M_S = Pa$ for all points between the two loads. Then the formula for the numerator becomes

Numerator for Two Symmetrically Placed Loads,

$$\sum_0^l M_S y \frac{I_c}{I_x} \Delta s = 2P\left(\sum_0^a xy \frac{I_c}{I_x} \Delta s + a \sum_a^{\frac{l}{2}} y \frac{I_c}{I_x} \Delta s\right). \quad . \quad . \quad (8)$$

The table below gives an easy method of finding the numerator.

The work may be still more simplified by dividing the arch span into an even number of equal divisions. If the horizontal length of each division is Δs and their number $2n$, then the span length is $l = 2nl_1$, the location of the load from left support $a = ml_1$ and any point $x = n_x l_1$.

The formula for the numerator for two symmetrical loads becomes

$$\sum_0^l Msy\frac{I_c}{I_x}\Delta s = 2l_1P\left[\sum_0^m n_x y\frac{I_c}{I_x}\Delta s + m\sum_m^n y\frac{I_c}{I_x}\Delta s\right]. \quad . \quad (9)$$

Table 4.—Numerator for H for Concentrated Load placed at $a = ml_1$, Two-hinged Arch

$$\sum_0^l Msy\frac{I_c}{I_x}\Delta s = l_1P\left[\sum_0^m n_x y\frac{I_c}{I_x}\Delta s + m\sum_m^n y\frac{I_c}{I_x}\Delta s\right]$$

Arch Divided into $2n$ Divisions.

Make summations for one-half of the arch, only.

Number of Divisions	$\frac{x}{l_1} = n_x$	$y\frac{I_c}{I_x}\Delta s$	$n_x y\frac{I_c}{I_x}\Delta s$	$y\frac{I_c}{I_x}\Delta s$
(1)	(2)	(3)	(4)	(5)
1	0.5	From Table 1	For n_x smaller than m	
2	1.5			
3	2.5			
4	3.5			For n_x larger than m
...	...			
n	$n - 0.5$			
	Sum of column (4) and (5)	=		=

To get numerator for two symmetrical loads P add the sums of Col. (4) and Col. (5) and multiply by $2Pl_1$.

For one concentrated load P multiply the sum of Col. (4) and Col. (5) by Pl_1.

LINE OF PRESSURE, TWO-HINGED ARCH

Having determined the horizontal thrust, H, the bending moments at all points may be obtained by drawing a line of pressure. As explained on p. 626 a line of pressure is a funicular polygon for the forces acting upon the arch drawn with a pole distance equal to the horizontal thrust, H.

When the vertical reactions are known, the vertical forces are laid out to a convenient scale on a vertical line starting with the force nearest the right support. The right reaction is then scaled off from the top. A horizontal line is drawn to the right upon which the horizontal thrust, H, is laid out to the same scale as used for the forces. A funicular polygon is now drawn, starting at the left hinge. If the work is done correctly, the polygon will pass through the right hinge.

If the line of pressure is correctly drawn the distance between the line of pressure and the arch axis represents the eccentricity of application of the thrust. The magnitude of the resultant thrust at any point may be obtained from the force polygon by scaling the length of the ray parallel to the line of pressure. The normal thrust may be easily found by resolving the resultant thrust into normal thrust and shear. The line of pressure, therefore, in connection with the force polygon gives all the required information for computing stresses.

Line of Pressure for One Concentrated Load.—For one concentrated load the line of pressure consists of two straight lines passing through the hinges, and intersecting at a point located on the vertical line indicating the force. The location of the two lines is fixed when the location of the point of intersection is known. The distance of this point of intersection above the arch axis is the eccentricity and, for known horizontal thrust, H, it may be obtained by dividing the bending moment M_x at that point by the thrust, H.

Since according to Formula (3), p. 553, $M_x = M_S + Hy$ and for a load placed at $x = a$, $M_S = P\frac{l-a}{l}a$,

$$M_x = P\frac{l-a}{l}a + Hy.$$

Therefore

$$e = \frac{M_x}{H} = \frac{P}{H}\frac{l-a}{l}a - y.$$

INFLUENCE LINES, TWO-HINGED ARCH

Influence Line for Horizontal Thrust.—Influence line for horizontal thrust is a curve by means of which can be found the horizontal thrust produced by any force P placed at any point on the arch. The influence line is drawn as follows:

The horizontal thrusts are computed for a unit load $P = 1$ placed successively at all division points in the arch. The numerators of H may be found as outlined in Table 3, p. 557. For denominator of H

use scheme in Table 2, p. 555. The computed values of H are plotted to a convenient scale on verticals passing through the respective points starting from a horizontal base. The resulting points connected give a a curve called the influence line for H. The ordinate of this curve multiplied by the load P gives the horizontal thrust due to that load.

A typical influence line for H is shown in Fig. 207, p. 575.

Influence Line for Bending Moments at Any Selected Section.—Influence lines for bending moments at any selected section are curves by means of which it is possible to get, by scaling, the bending moments due to any kind of loading. An influence line may be prepared for any section. Usually, it is sufficient to get influence lines for the bending moments at the crown and at the quarter point.

The influence line for any selected section is obtained by computing for that section the bending moment M_x due to a load $P = 1$ placed successively at each division of the arch. The values are plotted to a convenient scale above or below a horizontal base on vertical lines passing through the locations of the load. They are plotted above the base line when the bending moment is positive and below when it is negative. The curve connecting the points forms the influence line. The ordinate of this curve at any point multiplied by the load P gives the bending moment of this load placed at that point about the selected section.

Let l = span length;
x_1, y_1 = coordinates of the selected section for which influence line is drawn;
x = coordinate of any point at which load $P = 1$ is placed;
M_x = bending moment at selected section.

Then

Bending Moment at Selected Section, Two-hinged Arch

When load $P = 1$ is placed to the right of the selected section

$$M_x = Hy_1 + \frac{l - x}{l}x_1. \qquad (10)$$

When load $P = 1$ is placed to the left of the selected section

$$M_x = Hy_1 + \frac{l - x_1}{l}x. \qquad (11)$$

In both formulas H is the horizontal thrust due to the load $P = 1$ when placed at point x. Its magnitude changes with the change in the position of the load. It may be taken from the influence line for the horizontal thrust. H is negative and should be used in formulas and tables with its sign.

When the arch is divided into $2n$ sections of equal length l_1, then the span length is $l = 2nl_1$. The ordinate of the section for which influence is drawn, is $x_1 = ml_1$ and any point it is $x = n_x l_1$.

$$\text{Let } 2n = \frac{l}{l_1} = \text{ratio for span length;}$$

$$m = \frac{x_1}{l_1} = \text{ratio for selected section;}$$

$$n_x = \frac{x}{l_1} = \text{ratio for any point at which load } P = 1 \text{ is placed.}$$

The Equations (10) and (11) change to:
When load $P = 1$ is placed to the right of the section,

$$M_x = Hy_1 + \frac{2nl_1 - n_x l_1}{2nl_1} ml_1 = Hy_1 + \frac{m(2n - n_x)}{2n} l_1. \quad . \quad (12)$$

When load $P = 1$ is placed to the left of the section,

$$M_x = Hy_1 + \frac{n_x(2n - m)}{2n} l_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The values for the influence line may be found as outlined in Table 5, p. 564.

Use of Influence Lines for Bending Moment.—Similarly as in fixed arches, the influence lines are partly above and partly below the axis. This means that the loads in one position produce positive bending moment and in the other position negative bending moment. To get maximum positive bending moment, place the loads only at points producing positive bending moment. For uniformly distributed load multiply the area of the influence line by unit load. For concentrated loads place the heaviest loads at points producing maximum moment. Multiply the loads by the ordinates below them.

To get maximum negative bending moments, place the loads only at points producing negative bending moments.

In both cases the horizontal thrust and vertical reactions to be used with these bending moments should be taken for the same type of loading.

Table 5.—Influence Line for Bending Moment at Quarter Point $x_1 = 0.25l$, $y_1 = \dots$, Two-hinged Arch

Length of span $l = 20l_1$, $2n = 20$, $x_1 = 0.25l = 5l_1$, $m = 5$, $l_1 = \dfrac{l}{20}$

$$\frac{n_x(2n - m)}{2n}l_1 = \frac{3}{4}n_x l_1, \quad \frac{m(2n - n_x)}{2n}l_1 = \frac{1}{4}(20 - n_x)l_1$$

Section	H	Hy_1	Loads to the Left of Quarter Point			Loads to the Right of Quarter Point			
		$(2)\times y_1$	n_x	$\frac{3}{4}n_x l_1$ $\frac{3}{4}l_1\times(4)$	M $(5)+(3)$	n_x	$20-n_x$ $20-(7)$	$\frac{1}{4}l_1(20-n_x)$ $(8)\times\frac{1}{4}l_1$	M $(9)+(3)$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	Take from influence line for H	H and Hy_1 are negative	0.5						
2			1.5						
3			2.5						
4			3.5						
5			4.5						
6			.....	....	....	5.5	14.5		
7			.....	....	....	6.5	13.5		
8			.....	....	....	7.5	12.5		
9			.....	....	....	8.5	11.5		
10			.....	....	....	9.5	10.5		
10′			.....	....	....	10.5	9.5		
9′			.....	....	....	11.5	8.5		
8′			.....	....	....	12.5	7.5		
7′			.....	....	....	13.5	6.5		
6′			.....	....	....	14.5	5.5		
5′			.....	....	....	15.5	4.5		
4′			.....	....	....	16.5	3.5		
3′			.....	....	....	17.5	2.5		
2′			.....	....	....	18.5	1.5		
1′			.....	....	....	19.5	0.5		

EFFECT OF CHANGE OF SPAN LENGTH, TWO-HINGED ARCH

It has been proved on p. 602 in connection with fixed arches that change of span length produces a horizontal thrust and bending moments at all points of the arch, but no vertical reaction. At the springing of a fixed arch there is a bending moment and horizontal thrust. In two-hinged arches there can be no bending moment at the hinges, therefore there the arch is subjected to a central horizontal thrust only.

Let the center of coordinates to which the arch is referred coincide with the left hinge. Also

Let H = horizontal thrust due to change of length of span;
M_x = bending moment due to change of length of span;
y = ordinate of any point on arch axis;
Δl = change in length of span.
For signs of Δl see p. 584.

Then
Bending Moment at Any Point,

$$M_x = Hy. \qquad (14)$$

Introducing this value in formula for change of span length and making $N_x = -H$

$$\Delta l = -\int_0^l \frac{Hy}{EI_x} y ds - \int_0^l \frac{H}{EA_x} \cos \phi_x ds.$$

From this the horizontal thrust is

$$H = -\frac{\Delta l}{\int_0^l \frac{ds}{EI_x} y^2 + \int_0^l \frac{\cos \phi_x}{EA_x} ds}.$$

Taking the constant E before the integration sign, multiplying top and bottom by I_c and replacing $\int_0^l \frac{\cos \phi_x}{A_s} ds$ by $\frac{l}{A_{av}}$, where A_{av} is average area of section, the equation changes to

Horizontal Thrust Due to Change in Span Length Δl.

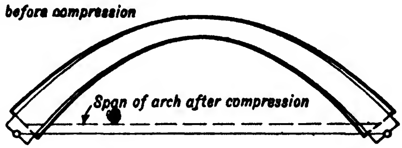
$$H = -\frac{I_c E \Delta l}{\int_0^l y^2 \frac{I_c}{I_x} ds + \frac{l I_c}{A_{av}}}. \qquad (15)$$

This formula can be used when the span length actually changes due to a movement of the support. Also it can be used in case when the span remains fixed, but the length of arch changes, thereby producing the same effect as if the arch axis remained constant but the span changed.

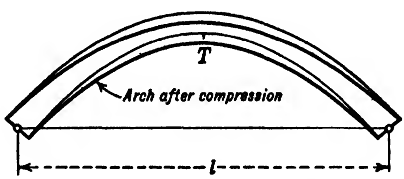
EFFECT OF RIB SHORTENING, TWO-HINGED ARCH

When subjected to a normal thrust the arch rib compresses. The heavy lines in Fig. 205(a) show the position an arch would assume after being compressed if it was free to move at the ends. In such case not only the length of rib but also the length of span would shorten. Since the hinges are firmly attached to unyielding supports this change in span cannot take place. Instead, the shortened arch must change as shown in Fig. 205(*b*), where light lines show the arch before rib shortening and heavy lines after rib shortening. The crown of the arch is lowered and tensile stresses are produced at the bottom of the arch. Their intensity is maximum at the crown and reduces towards the support.

The effect of rib shortening is the same as if the arch rib remained constant, but, instead, the span length increased. Therefore Formula (15), p. 565, may be used to solve the problem.



(a) Compression of Arch Due to Thrust, if Ends Free to Move



(b) Actual Effect of Rib Shortening

FIG. 205.—Effect of Rib Shortening. (*See p.* 565.)

The unit stress produced by a normal thrust N_x on any section A_x is $f_c = \frac{N_x}{A_x}$. The shortening due to this stress of a division of arch axis equal in length to ds is $\frac{N_x}{EA_x}ds$. The horizontal component of this shortening is $\frac{N_x \cos \phi_x}{EA_x}ds$. The shortening of the whole span is $\Delta l = \int_0^l \frac{N_x \cos \phi_x}{EA_x}ds$.

It is accurate enough to make $N_x = -H$ and replace the varying A_x by average A_{av}. Then

$$\Delta l = \int_0^l \frac{N_x \cos \phi_x}{EA_x} ds = -\frac{H}{A_{av}} \int_0^l \cos \phi_x ds = -\frac{H}{EA_{av}} \int_0^l dx = -\frac{Hl}{EA_{av}}.$$

Substituting this value in Equation (15) the thrust due to rib shortening becomes

Horizontal Thrust Due to Rib Shortening,

$$H_r = -\frac{\dfrac{lI_c}{A_{av}}}{\displaystyle\int_0^l y^2 \frac{I_c}{I_x} ds + \frac{lI_c}{A_{av}}} H. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This thrust is positive because H in the formula is negative. This formula needs to be used for dead load only. For live load the effect of rib shortening is taken care of by the item $\frac{lI_c}{A_{av}}$ in the formula for H.

EFFECT OF CHANGES OF TEMPERATURE, TWO-HINGED ARCH

Fall of temperature produces contraction, and rise of temperature produces expansion of the arch rib.

Let t° = rise or fall of temperature in degrees;
c = coefficient of expansion per degree of Fahrenheit;
l = length of span in inches;
s = length of arch rib in inches.

The change in length of rib due to temperature changes, obtained by multiplying the length of rib by the number of degrees and by the coefficient of expansion, is $st^\circ c$. The corresponding horizontal change in length is obtained by multiplying the horizontal component of each section by $t^\circ c$. Since the sum of the horizontal components is the span l, the change of the length of span due to the change in length of rib is

$$\Delta l = \pm\, lt^\circ c.$$

This change is positive for rise of temperature and negative for fall of temperature.

The effect of the changes of temperature and consequent changes in length of arch rib is the same as if the arch rib remained constant but the span shortened or lengthened by $lt°c$. The thrust may be found by substituting this value for Δl in Formula (15), p. 565.

Thrust Due to Fall of Temperature, Two-hinged Arch

$$H = \frac{I_c E l t° c}{\int_0^l y^2 \frac{I_c}{I_x} ds + \frac{l I_c}{A_{av}}} \quad \text{. . . . . (17)}$$

Thrust Due to Rise of Temperature, Two-hinged Arch

$$H = -\frac{I_c E l t° c}{\int_0^l y^2 \frac{I_c}{I_x} ds + \frac{l I_c}{A_{av}}} \quad \text{. . . . . . . (18)}$$

If the lengths are in inches, areas in square inches, moments of inertia in inches^4, E must be in pounds per square inch. The resulting force will be in pounds.

Bending Moment Due to Changes of Temperature.—The thrust found above acts at the level of the hinges. The bending moment produced by it at any point of the arch a distance y above the hinges is

Bending Moment Due to Changes of Temperature,

$$M = Hy. \quad \text{. . . . . (19)}$$

Since the value y is always positive the moment is positive for positive H, and negative by negative H.

Fall of temperature produces positive bending moments throughout the arch varying from zero at the support to a maximum Hr at the crown. These bending moments produce tension at the bottom of the arch.

Rise of temperature produces negative bending moments throughout the arch varying from zero at the support to a maximum $-Hr$ at the crown. These bending moments cause tension at the top of the arch.

This should be contrasted with the condition in fixed arches where the thrust acts at the elastic center of the arch and produces bending moment of one sign above and of another sign below the elastic center.

ECONOMICAL SHAPE OF ARCH AXIS

The best results are obtained when the arch axis coincides with the line of pressure for the dead load. The line of pressure is drawn in the same manner as explained for fixed arches on p. 468. Arches so designed, when not loaded, are subjected to normal thrust only and also to the effects of rib shortening due to dead load computed as explained on p. 565.

When the dead load is practically uniformly distributed as is the case in bow-string arches the arch axis will be a parabola. In such case the simplified formulas for parabolic arch axis may be used.

PRELIMINARY DIMENSIONS FOR TWO-HINGED ARCHES

Preliminary dimensions of the arch can be found as follows:

Assume depth of section and compute the dead load. The horizontal thrust for dead load is then found by dividing the static bending moment by the rise in the center.

The live load thrusts and bending moments are found by using parabolic arch formulas (see p. 573).

Temperature thrusts and bending moments are also found as for parabolic arches (see p. 576).

The thrust and bending moments are combined and the dimensions found by using Diagrams 1 and 2, opp. p. 648 or 7 and 8, pp. 654–655.

The dimensions may be found either at the crown only or at the crown and at the quarter point.

Having the preliminary dimensions the dead load is checked. The arch axis is then determined to coincide with the line of pressure for the new dead load. The arch is then laid out to scale and the final computations made.

Critical Sections for Two-hinged Arches.—The following sections need to be investigated:

1. Crown;
2. Quarter point;
3. Hinge.

The bending moments at the crown are smaller than at the quarter point. Therefore the depth of the section at the crown is made smallest and then increases slowly at first and then more rapidly, reaching its maximum at the quarter point. From the quarter point toward the hinge the bending moments decrease and the sections may also be decreased. At the hinge the bending moment is zero and the section is determined only by the required bearing of concrete on the hinge.

As the appearance of the arch with such variations in sections would not be pleasing, it is often made either of constant depth throughout or with the depth increasing from the crown toward the hinge with the section at the hinge a maximum. With such arrangement, naturally, some material is wasted in the portion of the arch between the quarter point and the hinge.

Most Unfavorable Position of Loading.—In arches, partial loading, when properly placed, produces larger bending moments than a loading extending the full length of the span. The position of the most unfavorable loading is different for different sections. Thus a different loading must be used in the section at the crown from that used at, say, the quarter point.

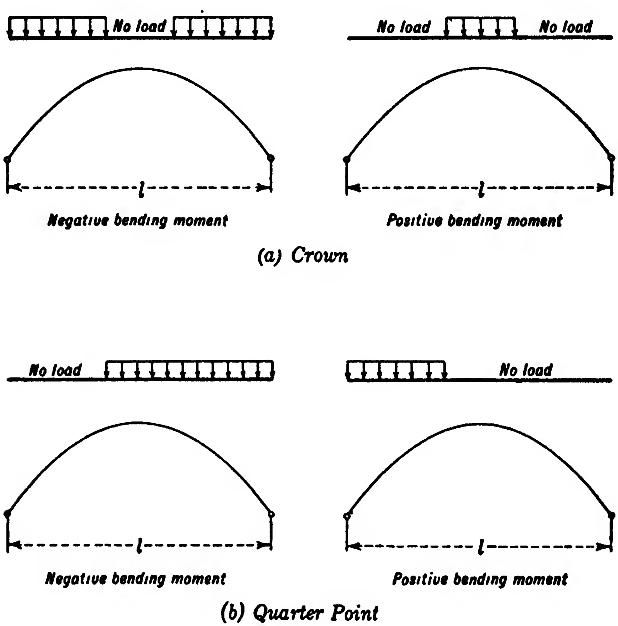


Fig. 206.—Most Unfavorable Position of Live Load. (*See p.* 570.)

Influence lines for bending moments furnish the best method of determining the most unfavorable position. Their use for the purpose is explained on p. 563. They are particularly useful when the loading consists of concentrated loads as then the loads must be placed not only within a proper region but also the largest loads must be placed at points producing largest bending moments. For uniformly distributed loads it is sufficient to know the extent of the length of arch to be loaded.

The position of points where bending moments change sign is not constant but depends upon the curvature of arch and upon the $\frac{\Delta s}{I_x}$

ratios. However, it is accurate enough for practical purposes to accept the following average positions of the most unfavorable loading for critical sections.

For Crown.

For maximum positive bending moments. Load extends from $x = 0.35l$ to $x = 0.65l$. (See Fig. 206(a), p. 570.

For maximum negative bending moments. Load extends on each side from support for a distance equal $0.35l$. (See Fig. 206(a), p. 570.

At the Left Quarter Point.

For positive bending moment. Load extends from the left hinge for a distance equal to 0.41 of the span. (See Fig. 206(b), p. 570.

For negative bending moment. Load extends from right hinge for a distance equal to 0.59 of the span. (See Fig. 206(b), p. 570.

At the Hinges.

Full load produces maximum compression at the hinges.

TWO-HINGED ARCHES WITH HORIZONTAL TIE

Two-hinged arches with horizontal tie are often used in bridge construction for the so-called bow-string arches and in roof construction.

Horizontal ties are used when it is impossible or undesirable to make the abutments strong enough to resist the horizontal thrust. In such case the horizontal thrust is resisted by the tie, and the supports are designed for vertical reactions only.

The horizontal tie under the action of the horizontal thrust lengthens; therefore the span length increases by the same amount. This must be taken into account in formulas for the arches with ties.

In Formula (1), p. 552, instead of making $\Delta l = 0$ as was done for arches with fixed hinges, it is made equal to the elongation of the ties under the action of the thrust.

Let H = horizontal thrust, lb.;
E_s = modulus of elasticity of steel, lb. per sq. in.;
A_s = area of tie, sq. in.;
l = length of tie, in.;
Δl = elongation of the tie, in.

Then

$$\Delta l = \frac{H}{A_s E_s} l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Substituting this value for Δl in Equation (1), p. 552.

$$\int_0^l \frac{M_s}{E} \frac{y}{I_x} ds + H \int_0^l \frac{y^2}{E} \frac{ds}{I_x} + H \int \frac{\cos \phi_x}{EA_x} ds = - \frac{Hl}{A_s E_s}.$$

This solved for H, and making $\int \frac{\cos \phi_x}{EA_x} ds = \frac{l}{EA_{av}}$ also multiplying top and bottom by I_c

$$H = -\frac{\frac{1}{E}\int_0^l M_s y \frac{I_c}{I_x} ds}{\frac{1}{E}\int_0^l y^2 \frac{I_c}{I_x} ds + \frac{lI_c}{EA_{av}} + \frac{lI_c}{A_s}\frac{1}{E_s}}.$$

Multiplying by E and since $\frac{E_s}{E} = n$.

Horizontal Thrust, Two-hinged Arch with Ties,

$$H = -\frac{\int_0^l M_s y \frac{I_c}{I_v} ds}{\int_0^l y^2 \frac{I_c}{I_x} ds + \frac{lI_c}{A_{av}} + \frac{lI_c}{nA_s}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Comparing this formula with Formula (2), p. 553, for fixed position of hinges, the two formulas differ in that the term $\frac{lI_c}{nA_s}$ denoting the lengthening of the tie is added in the denominator for arches with ties. The denominator is found in the same manner as explained on p. 553 and after the summation is made, the fixed value $\frac{lI_c}{nA_s}$ is added. There is no difference in the numerator.

Bending Moments and Influence Lines.—The bending moments and influence lines are found in the same manner as for arches without ties.

Since in arches with ties the support is not capable of resisting the horizontal thrust the arch should be designed so that it is free to move at one end. The span length then is governed only by the ties. When no means are provided to enable free movement of the arch the thrust may be transferred to the abutment instead of bringing the ties into action. This may be harmful to the abutment.

Influence of Temperature Changes.—When an arch with a tie is subjected to changes of temperatures, not only the arch but also the tie undergoes a change in length. If the change in temperature is the same in the arch as in the tie and when the coefficient of expansion is the same the whole structure will change its length without producing any stresses. The free end of the arch will move to adjust itself to the new length of the arch and the new length of the span.

The difference between an arch with fixed hinges and unyielding supports and an arch with a tie (and one movable end) is that with fixed hinges, under influence of temperature, the length of the arch changes but the span remains the same, while with a tie not only the arch but also the span, which is governed by the tie, changes.

PARABOLIC ARCH WITH TWO HINGES

The formula for the only statically indeterminate value in a two-hinged arch is

Horizontal Thrust, General Formula, Two-hinged Arch

$$H = -\frac{\int_0^l M_S y \frac{I_c}{I_x} ds}{\int_0^l y^2 \frac{I_c}{I_x} ds + \frac{lI_c}{A_{av}}}. \qquad (22)$$

When the curve of the arch axis can be expressed by a mathematical equation, the integrals can be solved.

On the following pages information is given so that the value of H for parabolic arches may be determined without the necessity of solving the integrals.

The formula for a parabolic arch axis, considering the left hinge as the center of coordinates, is

$$y = \frac{4r}{l^2}(lx - x^2). \qquad (23)$$

It is also necessary to express I_x by a formula. The simplest assumption is that $I_x \cos \phi_x = I_c$ so that $\frac{I_c}{I_x} = \cos \phi_x$. This substituted in the above formulas gives

$$H = -\frac{\int_0^l M_S y \cos\phi_x ds}{\int_0^l y^2 \cos \phi_x ds + \frac{lI_c}{A_{av}}}.$$

The value of I_c is constant and therefore may be taken before the integration sign and canceled. Also $ds \cos \phi_x = dx$. This changes the formula to

$$H = -\frac{\int_0 M_S y dx}{\int^l y^2 dx + \frac{lI_c}{A_{av}}}. \qquad (24)$$

Denominator for H Formula, Parabolic Two-hinged Arch.—Substituting in the denominator for Formula (24) $y = \frac{4r}{l^2}(lx - x^2)$ and solving the integral[3] the denominator for H becomes

$$\int_0^l y^2dx + \frac{lI_c}{A_{av}} = \frac{8}{15}r^2l + \frac{lI_c}{A_{av}}.$$

The formula then becomes
Horizontal Thrust for Parabolic Two-hinged Arches,

$$H = -\frac{\int_0^l M_S y dx}{\frac{8}{15}r^2l + \frac{lI_c}{A_{av}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Horizontal Thrust for Uniformly Distributed Load in Parabolic Arch.—The horizontal thrust for uniformly distributed load is found by substituting in Formula (25), p. 574, the proper value for M_S and solving the integral.

For loading extending the whole length of the span the thrust is
Horizontal Thrust, Uniform Load, Whole Span Loaded,

$$H = \frac{1}{8}wl\left(\frac{l}{r}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

For loading extending for a distance ml from support the thrust is
Horizontal Thrust, Uniform Load over Length ml,

$$H = \tfrac{5}{16}m^2(1 - m^2 + \tfrac{2}{5}m^3)wl\frac{l}{r}. \quad . \quad . \quad . \quad . \quad (27)$$

For loading extending for a distance of ml from both supports, the horizontal thrust is equal to twice the value in Formula (27).

For loading extending for a distance m_1l on each side of the crown, the horizontal thrust is found by computing from Formula (27) the horizontal thrust for loads extending from support a distance $ml = (\frac{1}{2} - m_1)l$, multiplying it by two and subtracting it from the thrust for full load from Formula (26).

[3] $\int_0^l y^2dx = \frac{16r^2}{l^4}\int_0^l (l^2x^2 - 2lx^3 + x^4)dx = \frac{16r^2}{l^4}l^5(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}).$

Finally

$$\int_0^l y^2dx = \tfrac{8}{15}r^2l.$$

Horizontal Thrust for Concentrated Load in Parabolic Arch.—For a load P located at point a from left support the horizontal thrust [4] is represented by the following formula

Horizontal Thrust for Concentrated Load,

$$H = -\frac{5}{8}\left[1 - \left(\frac{a}{l}\right)^2\left(2 - \frac{a}{l}\right)\right]\frac{l}{r}\frac{a}{l}P. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Ordinates for Influence Line for H.—In the Equation (26) for H make $P = 1$ and replace a by x to get the influence line for the horizontal thrust.

Ordinate for Influence Line for H, Parabolic Two-hinged Arch

$$\eta = \frac{5}{8}\frac{l}{r}\frac{x}{l}\left[1 - \left(\frac{x}{l}\right)^2\left(2 - \frac{x}{l}\right)\right]. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Figure 207, p. 575, gives influence lines for different ratios of $\frac{l}{r}$.

Bending Moments for Concentrated Loads in Parabolic Arch.—The bending moment about a section located at a distance x from the left support produced by a load placed at a distance a from left support may be found from following formulas.

For load P placed between the section and the left support (a is smaller than x),

$$M_x = \left(1 - \frac{x}{l}\right)\left\{\frac{a}{l} - \frac{5}{2}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\left(2 - \frac{a}{l}\right)\right]\frac{x}{l}\right\}Pl. \quad . \quad . \quad (30)$$

[4] This formula is obtained by using two symmetrically placed loads for which the static bending moments are $M_s = Px$ for x smaller than a and $M_s = Pa$ for x larger than a. Substitute these values in the equation for the numerator and solve the integrals for one half of the span only then the numerator can be used for single loads (see p. 558).

$$\int_0^{\frac{l}{2}} M_s y dx = \int_0^a Px\frac{4r}{l^2}(lx - x^2)dx + \int_a^{\frac{l}{2}} Pa\frac{4r}{l^2}(lx - x^2)dx$$
$$= \frac{4r}{l^2}P\left(\int_0^a (lx^2 - x^3)dx + a\int_a^{\frac{l}{2}} (lx - x^2)dx\right).]$$

This solved gives for the numerator

$$\int_0^{\frac{l}{2}} M_s y dx = \frac{1}{3}\frac{r}{l}\frac{x_1}{l}\left[1 - \left(\frac{a}{l}\right)^2\left(2 - \frac{a}{l}\right)\right]Pl^3.$$

The denominator is, neglecting the second term, $\frac{8}{15}r^2l$.

The horizontal thrust, then, is

$$H = -\frac{\frac{1}{3}\frac{r}{l}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\left(2 - \frac{a}{l}\right)\right]}{\frac{8}{15}r^2l}Pl^3 = -\frac{5}{8}\left[1 - \left(\frac{a}{l}\right)^2\left(2 - \frac{a}{l}\right)\right]\frac{l}{r}\frac{a}{l}P.$$

For loads placed between the section and the right support (a_1 is larger than x),

$$M_x = \frac{x}{l}\left\{\left(1 - \frac{a}{l}\right) - \frac{5}{2}\frac{a}{l}\left[1 - \left(\frac{a}{l}\right)^2\left(2 - \frac{a}{l}\right)\right]\left(1 - \frac{x}{l}\right)\right\}Pl. \quad (31)$$

It should be noted that the bending moments due to loads are independent of the rise of the arch. They are the same for all arches having the same span and the same loading even if the rise is different.

Influence Line for Bending Moments.—The influence line for bending moments at any definite point consists of two branches. The

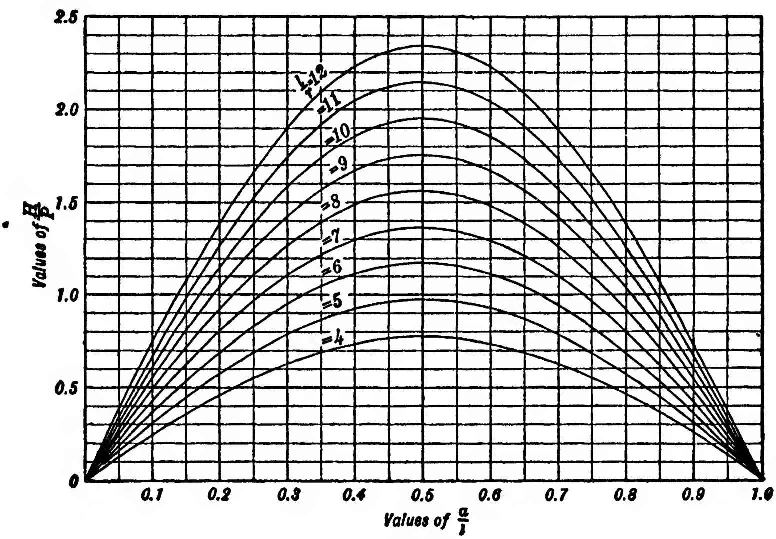


FIG. 207.—Influence Line for Horizontal Thrust. Parabolic Arch. (*See p.* 575.)

equations for these two curves may be obtained by substituting in Equations (28) and (29) $P = 1$, also for x the ordinate x_1 of the point for which the influence line is desired and making a a variable value. Thus

Left Branch of Influence Line, Parabolic Two-hinged Arch

$$\eta_m = \left(1 - \frac{x_1}{l}\right)\left\{\frac{x}{l} - \frac{5}{2}\frac{x}{l}\left[1 - \left(\frac{x}{l}\right)^2\left(2 - \frac{x}{l}\right)\right]\frac{x_1}{l}\right\}l. \quad . \quad . \quad (32)$$

Right Branch of Influence Line, Parabolic Two-hinged Arch

$$\eta_m = \frac{x_1}{l}\left\{1 - \frac{x}{l} - \frac{5}{2}\frac{x}{l}\left[1 - \left(\frac{x}{l}\right)^2\left(2 - \frac{x}{l}\right)\right]\left(1 - \frac{x_1}{l}\right)\right\}l. \quad . \quad (33)$$

The influence lines for the bending moment are independent of the rise.

Rib Shortening Due to Dead Load, Parabolic Two-hinged Arch.—The thrust due to rib shortening is obtained by substituting in Formula (16), p. 567, the value of the denominator for parabolic arch.

Horizontal Thrust Due to Rib Shortening Produced by Thrust H_d,

$$H_r = \frac{15}{8}\frac{I_c}{r^2 A_{av}} H_d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Moment Due to Rib Shortening,

$$M_x = H_r y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Effect of Temperature, Parabolic Two-hinged Arch.—Substituting in Formula (17), p. 568, the value for the denominator for parabolic arch, the formula for horizontal thrust due to changes of temperature becomes

Thrust Due to Fall of Temperature by $t°$ Degrees,

$$H_t = \frac{15}{8}\frac{I_c}{r^2} Et^0 c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Thrust Due to Rise of Temperature by t^0 Degrees,

$$H_t = -\frac{15}{8}\frac{I_c}{r^2} Et^0 c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

When r is in inches and E in pounds per square inch the horizontal thrust H is in pounds.

Bending Due to Temperature Changes,

$$M = H_t y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

In the above formula the value for H_t must be substituted with the proper sign.

CHAPTER VIII

THEORY OF ARCHES

In this chapter are developed formulas which are used in Chapter VI where final formulas necessary for design of arches are given. The formulas there given are in complete form, so that no reference is necessary to the chapter on Theory of Arches. However, for proper understanding of the formulas it is advisable for the designer to acquaint himself with the manner in which the formulas were derived.

Assumption as to Fixed Arches.—An arch is called fixed when:

(1) the supports are unyielding so that the arch can neither increase nor decrease its span, and
(2) the arch is built into the supports so that it cannot turn at the springings. The last requirement means also that there cannot be any changes in the central angle, i.e., the angle formed by lines drawn at the springings at right angles to the tangents to the arch axis remains constant.

Formulas will be first developed based upon these assumptions. Later, on p. 602, will be considered the effect upon the arch if these assumptions are not entirely fulfilled.

Reactions and Moments at Support.—As a general rule the arch action produces at each support an inclined reaction R, which can be resolved into a vertical reaction V, and horizontal reaction H. Thus at the left support marked A, the inclined reaction is R_A and its components V_A and H_A. At the right support marked B, the inclined reaction is R_B with components V_B and H_B. For downward loading of an arch the vertical reactions act upward and the horizontal reactions inward. (See Fig. 208, p. 583.)

In fixed arches, since the ends of the arch are not free to turn, bending moments will be developed there in addition to the reactions. Thus at support A a bending moment M_A and at support B a bending moment M_B will be developed. They may be either positive or negative. In computations the bending moments at the support M_A and M_B are assumed as positive. Their actual sign depends upon the plus or minus sign of the result.

Thus at the support there are six unknown quantities, V_A, H_A, M_A at left and V_B, H_B, M_B at right. Three of these can be determined from static rules of equilibrium.

Notation.

Let l = span length;

x, y = distance of any point from left support;

P_1, P_2, P_3, P_4 = vertical downward forces;

a_1, a_2, a_3, a_4 = distances of downward forces from left support;

$P_{H_1}, P_{H_2}, P_{H_3}$ = horizontal forces;

y_1, y_2, y_3 = vertical distances of horizontal forces from support;

V_A and V_B = vertical reactions;

H_A and H_B = horizontal reactions or thrusts;

M_A = bending moment in arch at left support;

M_B = bending moment in arch at right support;

M_S = static bending moment considering arch as freely supported beam;

M_s = static bending moment at any point due to vertical forces, considering arch as a cantilever fixed at the right support B.

Three Static Equations of Equilibrium.—The three static conditions of equilibrium are:

1. The algebraic sum of all vertical forces and reactions (or vertical components of inclined forces and reactions) must be equal to zero. From which follows that the sum of vertical forces must be equal to the sum of vertical reactions, and act in opposite direction.

2. The algebraic sum of all horizontal forces and reactions (or horizontal components of inclined forces and reactions) must be equal to zero. From which follows that the sum of horizontal forces must be equal to the sum of horizontal reactions and act in opposite direction.

3. The moment of all forces and reactions about any point of the structure must be equal zero.

These conditions are sufficient for finding all reactions in simply supported beams.

When the loading acts at right angles to the beam, only conditions 1 and 3 are necessary because horizontal forces are absent.

Sign of Forces.—The sign of downward vertical forces, such as loads, is accepted as minus. The sign of upward vertical forces, such as reactions, is taken as plus.

The sign of horizontal forces acting inwards, such as horizontal thrust due to downward loads, is accepted as minus. The sign of horizontal forces acting outward, such as thrust due to rib shortening and fall of temperature, is taken as plus.

Sign of Bending Moments.—The sign of bending moments producing tension at the bottom of the arch member is accepted as plus, while bending moments producing tension at the top of the member is taken as minus.

Relation between Reactions.—From the first rule of equilibrium following relation is obtained:

$$V_A + V_B - (P_1 + P_2 + P_3 + P_4) = 0$$

also

$$V_A = (P_1 + P_2 + P_3 + P_4) - V_B.$$

Thus if one reaction is known, the other can be easily found.

From the second rule of equilibrium

$$H_A + H_B - (P_{H_1} + P_{H_2} + P_{H_3}) = 0$$

also

$$H_A = (P_{H_1} + P_{H_2} + P_{H_3}) - H_B.$$

If all external forces are vertical there are no horizontal forces, therefore both horizontal reactions are equal and act in opposite directions. Thus

$$H_A = - H_B.$$

From the above it is evident that by the use of the two equations of equilibrium the four unknown reactions at the support R_A, R_B, H_A, and H_B can be reduced to two unknown values, namely, R_A and H_A. These two values must be found by formulas based on the elastic properties of the arch to be determined in succeeding pages. The bending moments at the support M_A and M_B are additional unknown quantities.

Relation between Bending Moments at Support.—According to the third equation of equilibrium, by taking bending moments about the right support, the following relation may be formed between the bending moments at the support M_A and M_B.

$$M_A + V_A l - [P_1(l - a_1) + P_2(l - a_2) + P_3(l - a_3)] + [P_{H_1}y_1 + P_{H_2}y_2 + P_{H_3}y_3] + M_B = 0.$$

The horizontal thrusts H_A and H_B do not appear in the above equation as they act at the level of the supports and therefore do not produce any bending moments at the supports.

There being two unknown quantities M_A and M_B the above equation is not sufficient for determining of both of them. However, if the bending moment at one support is known, the bending moment at the other support can be found from the above equation. This makes one value determinable by statics while the other value, say, M_A, is statically indeterminate.

Statically Indeterminate Values.—As evident from above discussion, in fixed arches, the following values cannot be determined by the three rules of equilibrium,

$$V_A,\ H_A, \text{ and } M_A.$$

These are the reactions and the bending moment at one (in this case the left) support. The statically indeterminate values will be found from rules which take into consideration the elastic properties of the arch. Having found these values, the reactions and the bending moment at the other support can be found by statics.

Bending Moment in the Arch at Any Intermediate Point.—When all the loads are vertical the bending moment at any point may be expressed in terms of reactions and loads as follows:

$$M_x = M_A + V_A x - P_1(x - a_1) - P_2(x - a_2) - P_3(x - a_3) + H_A y. \qquad (1)$$

Since $-P_1(x - a_1) - P_2(x - a_2) - P_3(x - a_3)$ is the static bending moment of the vertical loads considering the arch as a cantilever fixed at the right support B, it may be replaced by M_s.

Therefore

Bending Moment at Any Point x, in Terms of M_A, V_A and H_A,

$$M_x = M_A + V_A x + H_A y + M_s, \quad . \ . \ . \ . \ . \ . \quad (2)$$

where $M_s = -P_1(x - a_1) - P_2(x - a_2) - P_3(x - a_3)$.

It should be noted that for vertical loading H_A is negative so that $H_A y$ is actually negative. If instead of the vertical reaction V_A the bending moment at the other support M_B is known the formula becomes

Bending Moment at Any Point x, in Terms of M_A, M_B and H_A,

$$M_x = M_A + \frac{M_B - M_A}{l} x + M_S + H_A y. \quad . \ . \ . \quad (3)$$

Where M_S is the static bending moment of the load considering the arch as a beam simply supported on both supports. The bending moment M_x may be positive or negative, depending upon the sign of the result.

Notation.

Let l = span of arch;
r = rise of arch;
Δl = horizontal movement of one support in respect to the other due to thrust and bending moment;
$\Delta_1 l$ = horizontal movement of support due to thrust;
$\Delta_2 l$ = horizontal movement of support due to bending moment;
Δr = vertical movement of one support in respect to the other due to thrust and bending moment;
$\Delta_1 r$ = vertical movement of support due to thrust;
$\Delta_2 r$ = vertical movement of support due to bending moment;
$\Delta\phi$ = change in central angle;
H_A and H_B = horizontal thrusts at supports A and B, respectively;
V_A and V_B = vertical reaction at supports A and B, respectively;
M_A and M_B = bending moments at supports A and B, respectively;
M_s = static bending moment considering arch as cantilever fixed at right support B;
N_x = normal thrust at any point x;
S_x = shear at any point x;
A_x = area of normal cross-section at any point x;
I_x = moment of inertia of normal cross-section A_x;
I = moment of inertia of normal cross-section at the crown;
ϕ_x = inclination of normal cross-section A_x;
ds = length of a division of arch.

DETERMINATION OF STATICALLY INDETERMINATE VALUES FOR FIXED ARCHES

The forces and the consequent reactions at the supports in a fixed arch are shown in Fig. 208, p. 583.

As evident from the figure, there are three unknown quantities at each support. The static rules of equilibrium (see p. 579) furnish three equations sufficient to find three unknown quantities. The remaining three unknown values are statically indeterminate and must be found from formulas developed on the basis of the elastic properties of the arch. The method of attack is to develop expressions for deformation of the arch under various conditions and from these to obtain formulas for three of the unknown reactions and moments.

In the succeeding pages formulas will be developed for the three values at the left support, namely, H_A, M_A and V_A. After these are found, the values at the right support as well as bending moments and thrusts at any point may be found from statical equations.

Assumption for Determining the Statically Indeterminate Values.—All sections of the arch are subjected to a bending moment and a thrust. Each factor causes deformation of the arch sections. The deformation of the arch due to each one of these two factors will be found separately and then they will be added. After the general formulas for deformation due to the thrust and bending moment are found, the final formulas

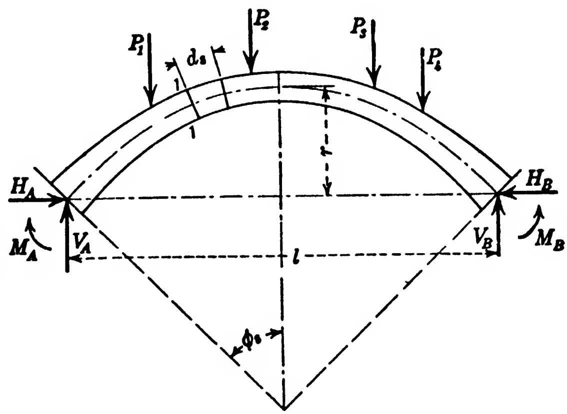


FIG. 208.—Forces and Reactions in a Fixed Arch. (*See p.* 582.)

for the indeterminate values are obtained from the following requirement:

The magnitude and the disposition of the bending moments and thrusts in the arch must be such that the aggregate of all the deformations at the various sections of the arch will not increase its span, raise or lower it at its supports, nor change its central angle.

Mathematically this requirement may be expressed by the following formulas:

Requirement for Fixed Arches with Unyielding Supports,

$$\Delta l = 0. \qquad (4)$$

$$\Delta r = 0. \qquad (5)$$

$$\Delta \phi = 0. \qquad (6)$$

The procedure in developing formulas then is as follows:

1. Find general formula for horizontal and vertical movements, respectively, of the support due to the thrust.

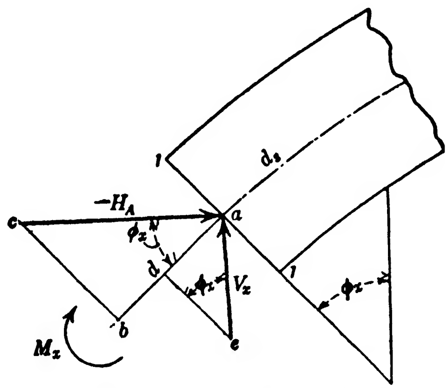
2. Find general formula for horizontal and vertical movements, respectively, of the support due to the bending moments.

3. Add the horizontal and vertical movements separately, due to the two above causes, and equate them to zero.

4. Find general equation for change of central angle due to the bending moments and equate it to zero.

Sign of Movements of Support.—Following signs are adopted for the movement of the support:

A horizontal movement which shortens the arch axis is positive.
A horizontal movement which lengthens the arch axis is negative.
A vertical movement which raises the left support is positive.
A vertical movement which lowers the left support is negative.



$$ab = -H_A \cos\phi_x, \quad ad = V_A \sin\phi_x, \quad N_x = ab + ad = -H_A \cos\phi_x + V_A \sin\phi_x$$

$$bc = -H_A \sin\phi_x, \quad de = V_A \cos\phi_x, \quad S_x = -bc + de = H_A \sin\phi_x + V_A \cos\phi_x$$

FIG. 209.—Forces and Reactions Acting on an Arch Section. *(See p. 585.)*

Normal Arch Section and Forces Acting upon it.—The work is simplified by using in formulas normal arch sections and the bending moments and thrusts acting upon these normal arch sections. The normal sections are sections at right angles to the tangent to the arch axis at the considered points. In circular arches the normal sections coincide with the radial lines.

In Fig. 208, *I–I* is a section normal to the arch axis at point x. This section is subjected to a horizontal thrust H_A, a vertical shear $V_x = V_A - P_1$, both acting centrally, and a bending moment M_x composed of the bending moment at the support M_A, the bending moment due to the reaction V_A and the bending moment due to the load P_1.

The forces acting upon this section are more clearly shown in Fig. 209, p. 584. H_A and V_x combined form the resultant thrust.

Effect of the Thrust upon the Arch

To get the deformation due to the forces H_A and V_x they are resolved into two components, one normal and the other parallel to the section. The two normal components added, form the normal thrust N_x while the two components parallel to the section form the shear S_x. The angle of inclination of the section I–I with the vertical being ϕ_x, the components of the two forces H_A and V_x normal to the section are $H_A \cos \phi_x$ and $V_x \sin \phi_x$ and the components parallel to the section are $H_A \sin \phi_x$ and $V_x \cos \phi_x$. The normal thrust N_x is equal to the sum of the normal components and the shear S_x to the sum of the parallel components. Their equations are

$$N_x = V_x \sin \phi_x - H_A \cos \phi_x. \qquad (7)$$

$$S_x = V_x \cos \phi_x + H_A \sin \phi_x. \qquad (8)$$

Effect of Shear S_x.—In an arch of the dimensions used in practice the effect of the shear upon deformation is very small and will be neglected.

Effect of Normal Thrust N_x.—As will be proved below the normal thrust acting upon an arch tends to reduce its span and its rise. If free to move, the left end of the arch would perform a horizontal and vertical movement of the magnitude given by the formulas below.

Horizontal Movement of the Left Support Due to Thrust N_x,

$$\Delta_1 l = \int_0^l \frac{N_x}{EA_x} ds \cos \phi_x. \qquad (9)$$

Vertical Movement of Left Support Due to Thrust N_x,

$$\Delta_1 r = \int_0^l \frac{N_x}{EA_x} ds \sin \phi_x. \qquad (10)$$

The above formulas are developed in the following manner: Divide the arch axis into n small divisions, the lengths of which are ds_1, ds_2, . . . ds_n. Consider any division of arch of a length equal ds, with area of cross section A_x, subjected to normal thrust, N_x acting in the center of the cross section. The unit stress on the section A_x due to the thrust is $f_c = \frac{N_x}{A_x}$. The unit shortening due to this stress is equal to $\frac{f_c}{E} = \frac{N_x}{EA_x}$. The shortening of the whole division, the length of which is ds, is obtained by multiplying the unit shortening by ds.

Therefore, the shortening of the whole division or $\Delta_1 ds = \frac{N_x}{EA_x}ds$. This is shown in Fig. 210, p. 586, in which the division of the arch before deformation is shown by solid lines, and after deformation by dash lines.

As evident from the figure the shortening $\Delta_1 ds$ takes place along the arch axis. The point 1 on the arch axis moved to point 1′. This movement may be resolved into a horizontal movement 1–1″ and vertical movement 1′–1″. The horizontal movement is called $\Delta_1 dx$ and the vertical movement $\Delta_1 dy$. Their magnitude may

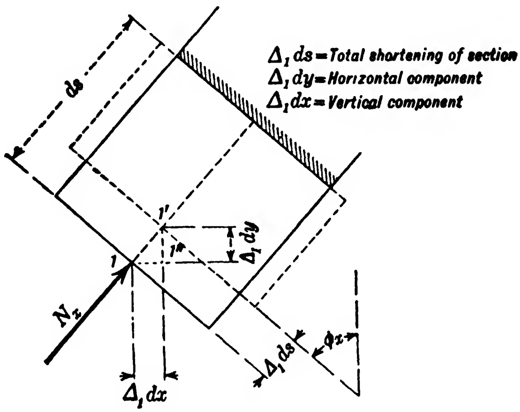


FIG. 210.—Deformation Due to Normal Thrust. *(See p. 586.)*

be obtained by multiplying the movement along the axis $\Delta_1 ds$ by cos ϕ_x and sni ϕ_x, respectively. Since $\Delta_1 ds = \frac{N_x}{EA_x}ds$, the formulas for partial movements are:

Partial Horizontal Movement Due to Thrust N_x,

$$\Delta_1 dx = \Delta_1 ds \cos \phi_x = \frac{N_x}{EA_x} ds \cos \phi_x. \quad \ldots \ldots \quad (11)$$

Partial Movement Due to Thrust N_x,

$$\Delta_1 dy = \Delta_1 ds \sin \phi_x = \frac{N_x}{EA_x} ds \sin \phi_x. \quad \ldots \ldots \quad (12)$$

These formulas apply to all divisions of the arch. Now consider the division next to the right support. Let its partial movements be $\Delta_1 dx_1$ and $\Delta_1 dy_1$. This means, due to the thrust acting upon it, the end section of this division moves by this amount towards the right support. Since the rest of the arch is connected with this division, all the points on the arch to the left of this division must also perform the same movement. Consider a division adjoining the first division. Due to the thrust acting upon it, this division will perform a movement $\Delta_1 dx_2$ and $\Delta_1 dy_2$. All the points on the arch to the left of the second division must also move by the same amount. The whole arch has already performed the movement $\Delta_1 dx_1$ and $\Delta_1 dy_1$, due to the compression of the first section, therefore the total movement will be equal to the sum of the two partial movements. The same reasoning may be applied to the movements of the succeeding sections. Consequently the movement of the point A at the left support caused by the compression of all sections by the thrust, N_x equals the sum of the partial movements of all the divisions in the arch.

Let $\Delta_1 l$ = total horizontal movement of point A at the left support due to thrusts acting on all sections.

$\Delta_1 r$ = total vertical movement of point A at the left support due to the thrusts acting on all sections.

$\Delta_1 dx_1, \Delta_1 dx_2, \ldots \Delta_1 dx_n$ = horizontal movements of divisions 1, 2, . . . n, due to normal thrusts action upon them.

$\Delta_1 dy_1, \Delta_1 dy_2, \ldots \Delta_1 dy_n$ = individual vertical movements of the division due to normal thrusts acting upon them.

$$\Delta_1 l = \Delta_1 dx_1 + \Delta_1 dx_2 + \Delta_1 dx_3 + \ldots \Delta_1 dx_m = \Sigma \Delta_1 dx,$$

$$\Delta_1 r = \Delta_1 dy_1 + \Delta_1 dy_2 + \Delta_1 dy_3 + \ldots \Delta_1 dy_n = \Sigma \Delta d_1 y.$$

Substituting the proper values for Δdx and Δdy from Formulas (11) and (12), the above formulas may be written

$$\Delta_1 l = \sum \frac{N_x}{EA_x} ds \cos \phi_x. \qquad (13)$$

$$\Delta_1 r = \sum \frac{N_x}{EA_x} ds \sin \phi_x. \qquad (14)$$

If the divisions are infinitely small, the total movements may be expressed by integrals

$$\Delta_1 l = \int_0^l \frac{N_x}{EA_x} ds \cos \phi_x. \qquad (13a)$$

$$\Delta_1 r = \int_0^l \frac{N_x}{EA_x} ds \sin \phi_x. \qquad (14a)$$

Effect of Bending Moment M_x upon the Arch

Horizontal and Vertical Movement of Supports Due to Bending Moments.—As will be proved below the bending moments acting upon all the sections of an arch tend to produce a horizontal and a vertical movement of point A of the left support, the magnitude of which is given by the following formulas.

Horizontal Movement of the Left Support Due to the Bending Moments,

$$\Delta_2 l = -\int_0^l \frac{M_x y}{EI_x} ds. \qquad (15)$$

Vertical Movement of the Left Support Due to Bending Moments,

$$\Delta_2 r = \int_0^l \frac{M_x x}{EI_x} ds. \qquad (16)$$

To develop these formulas consider a division of the arch of a length equal to ds, subjected to a bending moment, M_x. Since the length of the division is very small the bending moment at all normal sections within this division may be considered as constant. Assume that the normal cross sections of the arch are homogeneous and symmetrical and that their depth equals h and the moment of inertia equals I_x. The bending moment M_x produces tensile stresses in one-half of the cross section and compression stresses in the other half. The stresses are zero at the central axis of the cross section and increase according to a straight line to a maximum at the extreme fibers where it reaches the value $f_c = \frac{\pm M_x h}{2I_x}$. The sign plus designates the compression stresses and minus the tension stresses.

These stresses cause deformation of the concrete, namely, shortening of the fibers where the section is under compression and lengthening of the fibers in the part of the section under tension. The deformation at the center is zero. Letting E equal the modulus of elasticity and assuming that the modulus of elasticity in tension is the same as in compression, the unit deformation at any point is equal the

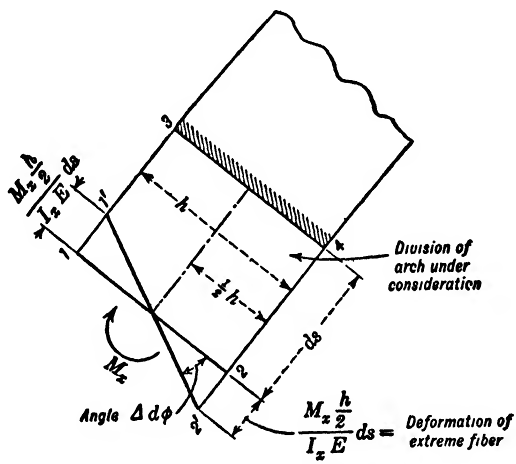


FIG. 211.—Deformation Due to Bending Moment. (See p. 588.)

unit stress divided by the modulus of elasticity, E. At the extreme fibers the unit deformation is expressed by $\frac{f_c}{E} = \frac{M_x h}{2EI_x}$. The lengthening and shortening of the extreme fibers for the whole division, the length of which is ds, is obtained by multiplying the unit deformation by ds. The total deformation of the extreme fibers of the section due to the bending moment is therefore $ds\frac{f_c}{E} = \frac{M_x h}{2EI_x}ds$. As evident in Fig. 211, p. 588, due to the deformation caused by the bending moment the cross section rotates about its gravity axis so that the section 1–2 assumes the position 1′–2′. The angle between the two positions of the section measured in radians is obtained by dividing the total movement of the extreme fibers by the radius of the movement which, for rectangular sections, is equal to one-half of the depth of the section.

Thus
Angular Movement of Section Due to Bending Moment, in Radians,

$$\Delta d\phi = \frac{M_x h}{2EI_x}ds \div \frac{h}{2} = \frac{M_x}{EI_x}ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The deformation of the section affects not only the division of the arch under consideration but also the remainder of the arch. In Fig. 212 is shown the division of the arch 1 2 3 4 and also the balance of the arch 1 2 5 6 to the left of the section 1–2, before and after deformation. For the present the section 1 2 5 6 is considered as unstressed and as free to move. The section 1–2 moves to the position 1′–2′. The balance of the arch 1 2 5 6 being connected with the section 1–2 must follow and therefore must assume the position 1′ 2′ 5′ 6′. The left support A, which also is considered as free to move, moves to the position A'. The object now is to determine the horizontal and the vertical component of the movement AA', caused by the bending moment acting upon the division 1 2 3 4.

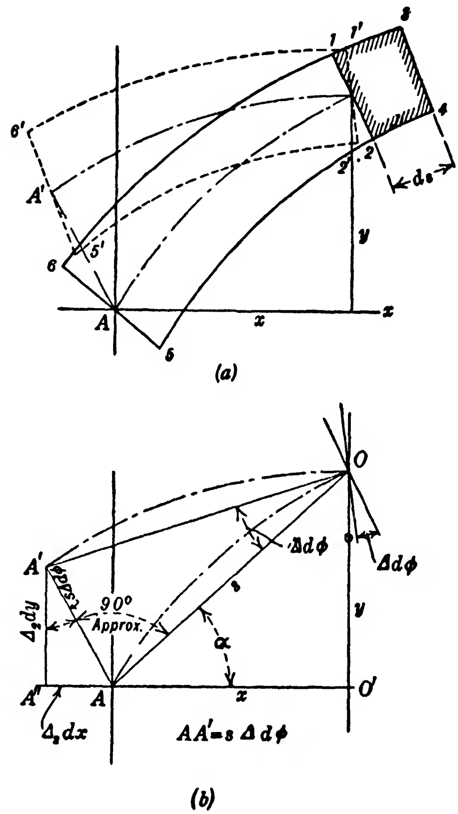


FIG. 212.—Movement of Support Due to Bending Moment M_x. (*See p.* 589.)

To avoid confusion, in Fig. 212 (*b*) the center lines of the arch shown in Fig. 212 (*a*) are redrawn, while the outlines of the arch are omitted. In Fig. 212 (*b*) connect point O with point A and A'. Also erect a perpendicular at A'. Then AA'' is the horizontal movement of point A and may be called $\Delta_2 dx$, while $A'A''$ is the vertical movement of point A caused by the bending moment M_x and may be called $\Delta_2 dy$. From the figure it is evident that the angle between OA and OA' is equal to the angular movement of the section 1–2. AA' is the arc of a circle, the radius of which is equal to $OA = s$ and the center angle equal to $d\phi$. The magnitude of this arc equals $sd\phi$. Since the angle $d\phi$ is very small AA' may be considered not only as the arc of the circle but also as the tangent to the circle at point A. The angle $A'AO$ may be, therefore, considered as a right angle. With this assumption the angle at A' in a triangle $AA'A''$ is equal to the angle at A in the triangle OAO'. Therefore, the triangle $AA'A''$ is similiar to the triangle OAO'

From the similarity of triangles, following proportion is obtained. $s\Delta_2 dy = xs\Delta d\phi$ from which $\Delta_2 dy = \frac{xs\Delta d\phi}{s} = x\Delta d\phi$. Since from Equation (17), $\Delta d\phi = \frac{M_x}{EI_x}ds$ the vertical movement of the left support due to the bending moment M_x is

$$\Delta_2 dy = \frac{M_x x}{EI_x}ds.$$

Similarly the horizontal movement of the left support due to the bending moment M_x is $\Delta_2 dx = -\frac{M_x y}{EI_x}ds$. Following signs are adopted for the movement of the support.

A horizontal movement which tends to shorten the arch axis is considered positive.

A horizontal movement which lengthens the arch is negative.

A vertical movement which raises the left support is called positive.

A vertical movement which lowers the left support is called negative.

From Fig. 212 (*a*) it is evident that the positive bending moment lengthens the arch and raises the support upward. Therefore, it produces a negative horizontal and positive vertical movement. The opposite is true of the negative bending moment. In other words, the horizontal movement is of opposite sign to the sign of the bending moment while the vertical movement is of the same sign as the bending moment.

In the above discussion only one division of the arch was considered as subjected to bending moment and the balance of the arch as not stressed. Actually, however, in a loaded arch all divisions are subjected to bending moments. The angular deformation of the section of each division will produce partial movements of the left support. Thus, for the first division next to the right support, for which the values are M_1, I_1, x_1, y_1, and ds_1, the movement of the left support according to the above formulas is

$$\Delta_2 dx_1 = y_1 \Delta d\phi_1 = -\frac{M_1 y_1}{EI_1}ds_1.$$

$$\Delta_2 dy_1 = x_1 \Delta d\phi_1 = \frac{M_1 x_1}{EI_1}ds_1.$$

For the adjoining section with values M_2, I_2, x_2, y_2, and ds_2, the movement of the left support is

$$\Delta_2 dx_2 = -\frac{M_2 y_2}{EI_2}ds_2.$$

$$\Delta_2 dy_2 = \frac{M_2 x_2}{EI_2}ds_2.$$

The total horizontal movement of the left support due to the bending moments acting at all divisions of the arch is, therefore, equal to

$$\Delta_2 l = \Delta_2 dx_1 + \Delta_2 dx_2 + \ldots \Delta_n dx_n,$$

and substituting the values for $\Delta_2 dx_1$, $\Delta_2 dx_2$

$$\Delta_2 l = -\left[\frac{M_1 y_1}{EI_1}ds_1 + \frac{M_2 y_2}{EI_2}ds_2 + \ldots \frac{M_n y_n}{EI_n}ds_n\right].$$

Finally
Total Horizontal Movement of Left Support Due to Bending Moments,

$$\Delta_2 l = -\sum \frac{M_x y}{EI_x} ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Similarly the total vertical movement of the left support may be obtained. It is
Total Vertical Movement of Left Support Due to Bending Moments,

$$\Delta_2 r = \sum \frac{M_x x}{EI_x} ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Change in Central Angle Due to Bending Moment.—The angular change of the section 1–2 in Fig. 212 due to the bending moment not only produces a movement $\Delta_2 dx$ and $\Delta_2 dy$ at the support but also changes the central angle.

From inspection of Fig. 212 it is evident that the increase or decrease in the central angle due to a bending moment acting upon a section of arch is the same as the change of the angle of inclination of the section, namely, $\Delta d\phi$. Decrease in angle is called positive and increase negative. Positive bending moment decreases the angle while negative bending moment increases it.

If an arch is divided into a number of sections and each section is submitted to a bending moment M_x, the total change in the central angle is equal to the sum of the changes of the angle of the individual sections or

$$\Delta\phi = \Delta d\phi_1 + \Delta d\phi_2 + \Delta d\phi_3 + \; . \; . \; . \; \Delta d\phi_n.$$

This can be expressed by

$$\Delta\phi = \frac{M_1}{EI_1} ds_1 + \frac{M_2}{EI_2} ds_2 + \frac{M_3}{EI_3} ds_3 + \; . \; . \; . \; \frac{M_n}{EI_n} ds_n.$$

This may also be expressed as

$$\Delta\phi = \sum \frac{M_x}{EI_x} ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

When the arch axis and the moments can be represented by mathematical functions, the above equation may be expressed by an integral.

Change of Central Angle Due to Bending Moment,

$$\Delta\phi = \int_0^l \frac{M_x}{EI_x} ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20a)$$

FINAL EQUATIONS

As explained in the previous discussion, an arch, for the present considered as fixed at the right support and free at the left support, when subjected at the various sections to bending moments and thrusts, deforms and as a result the left end of the arch tends to

(1) move horizontally;
(2) move vertically;
(3) turn at the springing line, i.e., change the central angle.

The horizontal and vertical movements are caused by the bending moments and the thrusts while the turning effect is caused only by bending moments. Formulas for movements due to each individual cause separately are developed under proper headings on pp. 585 and 587. The total movement is the sum of movements caused separately by the bending moment and by the thrust.

Consider the arch as referred to rectangular coordinates, with the center at the left support. Designate the coordinates by capital letters X and Y to distinguish them from another set of coordinates to be introduced later. (See Fig. 213, p. 593.)

Then adding the movement from Formulas (9) and (10), p. 585, due to bending moment and from Formulas (15) and (16), p. 587, due to thrust.

Total Horizontal Movement of Left Springing Due to Bending Moment and Thrust,

$$\Delta l = -\int_0^l \frac{M_x Y}{EI_x} ds + \int_0^l \frac{N_x}{EA_x} ds \cos \phi_x. \quad . \quad . \quad . \quad (21)$$

Total Vertical Movement at Left Springing Due to Bending Moment and Thrust,

$$\Delta_r = \int_0^l \frac{M_x X}{EI_x} ds + \int_0^l \frac{N_x}{EA_x} ds \sin \phi_x. \quad . \quad . \quad . \quad . \quad (22)$$

Total Angular Movement at Left Springing Due to Bending Moment,

$$\Delta\phi = \int_0^l \frac{M_x}{EI_x} ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

In the above equations the bending moments M_x and thrusts N_x are functions of the three unknown values at the left support H_A, V_A and M_A and of the loads. If the values Δl, Δr and $\Delta\phi$ are known, the above three equations may be used for finding these three statically indeterminate values.

Requirements for Fixed Arches.—In fixed arches the supports are unyielding and the connection between the arch and the support rigid. From this it follows directly that in a fixed arch no movements of the arch at the supports can take place.

The above requirement may be represented by following equations.

$$\Delta l = 0, \quad \Delta r = 0 \quad \text{and} \quad \Delta\phi = 0.$$

The bending moments and thrusts at the various sections, therefore, must be such that the movement of the support due to one set of bend-

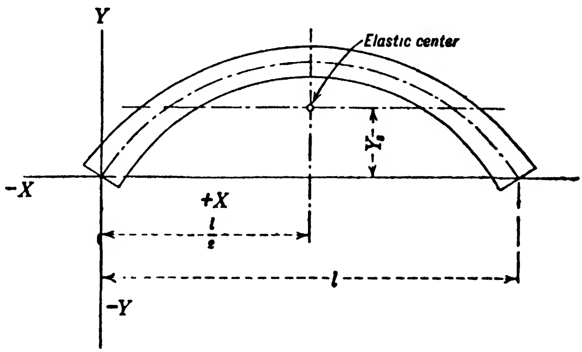


FIG. 213.—Systems of Coordinates. (*See p.* 592.)

ing moments acting on one part of the arch is neutralized by a movement in opposite direction due to another set of bending moments acting on another part of the arch. Thus the arch must be subjected to negative and positive bending moments of such magnitudes, and so placed that the effect of positive bending moments would neutralize the effect of the negative bending moments and the thrust.

Substituting in the above equation the values for Δl, Δ_r and $\Delta\phi$ from Formulas (21) to (23) and dividing by the constant value of E, following three formulas are obtained:

$$-\int_0^l \frac{M_x Y}{I_x} ds + \int_0^l \frac{N_x}{A_x} ds \cos\phi_x = 0. \quad . \quad . \quad . \quad (24)$$

$$\int_0^l \frac{M_x}{I_x} X ds + \int_0^l \frac{N_x}{A_x} ds \sin\phi_x = 0. \quad . \quad . \quad . \quad (25)$$

$$\int_0^l \frac{M_x}{I_x} ds = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The term $N_x \sin\phi_x$ in Equation (25) may be neglected because it is zero in the center of the arch where $\sin\phi_x = 0$. It increases slowly with the increase of ϕ_x, but even at its maximum it assumes only a small value.

The bending moment M_x and thrust N_x will now be expressed in terms of H_A, V_A and M_A.

Referring to p. 581 the bending moment and thrust at any point at a distance X and Y from the left support are

$$M_x = M_A + V_A X + H_A Y + M_s, \quad . \quad . \quad . \quad . \quad (27)$$

and

$$N_x = V_x \sin \phi_x - H_A \cos \phi_x, \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where

$$V_x = V_A - [P_1 + P_2 + P_3]. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The effect of the thrust N_x upon the total deformation of the arch is small in comparison with that of the bending moment; therefore, the total result will be affected very little by substitutions for the complicated expression for N_x in Formula (7) the expression

$$N_x = - H.$$

This is exact in the central portion of the arch where the depth of the section is small and therefore the effect of the thrust the largest. Near the springing there is a difference between the actual thrust and the assumed thrust, but the effect is small due to the increased section.

Substituting in Equations (24) and (26) the values of M_x and N_x (with the simplifications mentioned above) the following basic equations for arches are obtained:

$$1. \quad -\int_0^l \left[\frac{M_A}{I_x} Y ds + \frac{V_A}{I_x} XY ds + \frac{H_A}{I_x} Y^2 ds + \frac{M_s}{I_x} Y ds\right] - \int_0^l \frac{H_A \cos \phi_x}{A_x} ds = 0. \quad (30)$$

$$2. \quad \int_0^l \left[\frac{M_A}{I_x} X ds + \frac{V_A}{I_x} X^2 ds + \frac{H_A YX}{I_x} ds + \frac{M_s X ds}{I_x}\right] = 0. \quad (31)$$

$$3. \quad \int_0^l \left[\frac{M_A ds}{I_x} + \frac{V_A X}{I_x} ds + \frac{H_A Y ds}{I_x} + \frac{M_s}{I_x} ds\right] = 0. \quad . \quad . \quad (32)$$

Finally, taking the constant values M_A, H_A and V_A before the integration and since $ds \cos \phi_x = dx$, the equations become [1]

[1] M_s cannot be taken before the integration sign because it is different for every point. The moment of inertia I_x also varies for different points because the cross sections vary.

General Equations,

$$1.\ M_A\int_0^l Y\frac{Ids}{I_x}+V_A\int_0^l XY\frac{Ids}{I_x}+H_A\int_0^l Y^2\frac{Ids}{I_x}+\int_0^l M_sY\frac{Ids}{I_x}$$
$$+H_A\int_0^l \frac{Idx}{A_x}=0. \quad (33)$$

$$2.\ M_A\int_0^l X\frac{Ids}{I_x}+V_A\int_0^l X^2\frac{Ids}{I_x}+H_A\int_0^l YX\frac{Ids}{I_x}+\int_0^l M_sX\frac{Ids}{I_x}=0. \quad (34)$$

$$3.\ M_A\int_0^l \frac{Ids}{I_x}+V_A\int_0^l X\frac{Ids}{I_x}+H_A\int_0^l Y\frac{Ids}{I_x}+\int_0^l M_s\frac{Ids}{I_x}=0. \quad . \quad (35)$$

X and Y are coordinates with the center at the left support.

The above equations were multiplied by the moment of inertia at the crown I to simplify the mathematical work.

From these three equations the unknown values M_A, V_A and H_A could be determined. The resulting formulas can be simplified by moving the center of coordinates from the left support to the elastic center of the arch, i.e., a point for which the $\int y\frac{Ids}{d_x}$, $\int x\frac{Ids}{I_x}$ and $\int xy\frac{Ids}{I_x}$ terms are zero.

Location of the New Center of Coordinates.—The new center of coordinates, i.e., the elastic center of the arch, is found by considering the values $\frac{Ids}{I_x}$ for each division of the arch as loads applied at their centers and locating the center of gravity for these loads.

Let X_s = horizontal distance of the elastic center from Y-axis;
Y_s = vertical distance of the elastic center from X-axis.

For symmetrical arches the elastic center is located on the axis of symmetry of the arch, i.e., the vertical line through the center of the span. Therefore

Horizontal Distance of Elastic Center from Y-axis,

$$X_s=\frac{l}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The vertical distance of the center of gravity from the original X-axis is found from the requirement that the static moment about the X-axis of the values $\frac{Ids}{I_x}$ for all sections be equal to the sum of all values multiplied by Y_e. The static moment of any value $\frac{Ids}{I_x}$ about the X-axis is $Y\frac{Ids}{I_x}$. The sum of the static moments is $\sum_0^l Y\frac{Ids}{I_x}$ and the sum of all the values is $\sum_0^l \frac{Ids}{I_x}$. Therefore

$$Y_e\sum_0^l \frac{Ids}{I_x} = \sum_0^l Y\frac{Ids}{I_x},$$

from which

Vertical Distance of Elastic Center from X-axis,

$$Y_e = \frac{\sum_0^l Y\frac{Ids}{I_x}}{\sum_0^l \frac{Ids}{I_x}}. \qquad \qquad (37)$$

Transferring the Center of Coordinates to the Elastic Center.—The old and the new systems of coordinates are shown in Fig. 213. The old axes passing through the left support A are shown by solid lines, while the new axes passing through the elastic center are shown by dash lines.

Formulas (33) to (35) are based on the system of coordinates passing through the left support. To change the center of coordinates from the left support to the elastic center it is necessary to substitute for the old coordinates X and Y values based on new coordinates.

Let X and Y = old coordinates with origin at left support;
x and y = new coordinates with origin at elastic center.

Then for any point the relation between the old and new coordinates may be expressed by

$$X = \frac{l}{2} + x$$
$$Y = Y_e + y.$$

The transfer is accomplished by substituting these values for X and Y in the Equations (33) to (35).

The values M_A, V_A and M_e and $\frac{Ids}{I_x}$ are not affected by the change of the axes of coordinates.

Thus Equation (1), p. 595, will change to

$$(1a)\quad M_A\left(\int_{-\frac{l}{2}}^{\frac{l}{2}} Y_s\frac{Ids}{I_x} + \int_{-\frac{l}{2}}^{\frac{l}{2}} y\frac{Ids}{I_x}\right) + V_A\int_{-\frac{l}{2}}^{\frac{l}{2}}\left(\frac{l}{2}+x\right)(Y_s+y)\frac{Ids}{I_x}$$

$$+ H_A\int_{-\frac{l}{2}}^{\frac{l}{2}}(Y_s+y)^2\frac{Ids}{I_x} + \int_{-\frac{l}{2}}^{\frac{l}{2}} M_s(Y_s+y)\frac{Ids}{I_x} + H_A\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Idx}{A_x} = 0.$$

Since for coordinates passing through the elastic center $\int x\frac{Ids}{I_x} = 0$, $\int y\frac{Ids}{I_x} = 0$ and $\int xy\frac{Ids}{I_x} = 0$, the integrals in the above equation may be simplified as follows:

$$\int_{-\frac{l}{2}}^{\frac{l}{2}}\left(\frac{l}{2}+x\right)(Y_s+y)\frac{Ids}{I_x} = \frac{l}{2}Y_s\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Ids}{I_x} + Y_s\int_{-\frac{l}{2}}^{\frac{l}{2}} x\frac{Ids}{I_x}$$

$$+\frac{l}{2}\int_{-\frac{l}{2}}^{\frac{l}{2}} y\frac{Ids}{I_x} + \int_{-\frac{l}{2}}^{\frac{l}{2}} xy\frac{Ids}{I_x} = \frac{l}{2}Y_s\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Ids}{I_x};$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}}(Y_s+y)^2\frac{Ids}{I_x} = Y_s^2\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Ids}{I_x} + 2Y_s\int_{-\frac{l}{2}}^{\frac{l}{2}} y\frac{Ids}{I_x} + \int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{Ids}{I_x}$$

$$= Y_s^2\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Ids}{I_x} + \int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{Ids}{I_x}.$$

Substituting these values in (1a)

$$M_AY_s\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Ids}{I_x} + V_A\frac{l}{2}Y_s\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Ids}{I_x} + H_AY_s^2\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Ids}{I_x} + H_A\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{Ids}{I_x}$$

$$+ Y_s\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s\frac{Ids}{I_x} + \int_{-\frac{l}{2}}^{\frac{l}{2}} M_s y\frac{Ids}{I_x} + H_A\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Idx}{A_x} = 0.$$

Finally

$$(\mathrm{I})\quad \left(M_A+V_A\frac{l}{2}+H_AY_s\right)Y_s\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Ids}{I_x}$$

$$= -H_A\left[\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{Ids}{I_x} + \int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Idx}{A_x}\right] - Y_s\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s\frac{Ids}{I_x} - \int_{-\frac{l}{2}}^{\frac{l}{2}} M_s y\frac{Ids}{I_x}.$$

Making similar substitutions for X and Y the Equations (2) and (3) change to:

$$\text{(II)}\quad \frac{l}{2}\left(M_A + V_A\frac{l}{2} + H_A Y_s\right)\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Ids}{I_x}$$

$$= - V_A\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\frac{Ids}{I_x} - \frac{l}{2}\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s\frac{Ids}{I_x} - \int_{-\frac{l}{2}}^{\frac{l}{2}} M_s x\frac{Ids}{I_x}.$$

$$\text{(III)}\quad \left(M_A + V_A\frac{l}{2} + H_A Y_s\right)\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Ids}{I_x} = - \int_{-\frac{l}{2}}^{\frac{l}{2}} M_s\frac{Ids}{I_x}.$$

If the value for $\left(M_A + V_A\frac{l}{2} + H_A Y_s\right)\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Ids}{I_x}$ from Equation (III)

is substituted in Equation (I) we get

$$- Y_s\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s\frac{Ids}{I_x} = - H_A\left(\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{Ids}{I_x} + \int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Idx}{A_x}\right)$$

$$- Y_s\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s\frac{Ids}{I_x} - \int_{-\frac{l}{2}}^{\frac{l}{2}} M_s y\frac{Ids}{I_x}.$$

Final Formulas.—After canceling, following formula is obtained.
Horizontal Thrust at Left Support,

$$H_A = - \frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s y\frac{Ids}{I_x}}{\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{Ids}{I_x} + \int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{Idx}{A_x}}. \quad . \quad . \quad . \quad . \quad (38)$$

The horizontal thrust is negative for vertical loads.

Similarly from Equation (II)
Vertical Reaction at Left Support,

$$V_A = - \frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s x\frac{Ids}{I_x}}{\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\frac{Ids}{I_x}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Since for vertical loads, M_s is negative and $\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s x \frac{Ids}{I_x}$ is negative, the vertical reaction will be positive.

The value for the bending moment at the support M_A may be found from Formula (III) by substituting in it first the determined values for H_A and V_A and then solving for M_A. The work is simplified if, in place of the expression $\left(M_A + V_A \frac{l}{2} + H_A Y_s\right)$ an auxiliary bending moment M is introduced, then the Equation (III) becomes

$$M \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x} = - \int_{-\frac{l}{2}}^{\frac{l}{2}} M_s \frac{Ids}{I_x},$$

from which

Auxiliary Bending Moment,

$$M = - \frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s \frac{Ids}{I_x}}{\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}}. \quad \ldots \ldots \ldots \quad (40)$$

Since for vertical loads, M_s is negative the numerator is negative, consequently the value of M will be positive.

Bending Moments.—Having computed the values for H_A, V_A and M, the bending moment at the left support is found from the relation

$$M = M_A + V_A \frac{l}{2} + H_A Y_s.$$

It is

Bending Moment at Left Support A,

$$M_A = M - V_A \frac{l}{2} - H_A Y_s. \quad \ldots \ldots \ldots \quad (41)$$

To get bending moment at any point using new coordinates, substitute for X and Y values from equations on p. 596 in Formula (2), p. 581.

$$M_x = M_A + V_A\left(\frac{l}{2} + x\right) + H_A(Y_s + y) + M_s.$$

This also may be written

$$M_x = \left(M_A + V_A \frac{l}{2} + H_A Y_s\right) + V_A x + H_A y + M_s.$$

Since the expression $\left(M_A + V_A\frac{l}{2} + H_A Y_s\right)$ is equal to the auxiliary bending moment M, the formula may be written in following simple form.

Bending Moment at Any Point,

$$M_x = M + V_A x + H_A y + M_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

The values M, V_A and H_A are fully explained. Since H_A is negative the value $+ H_A y$ is also negative. The value of M_s is explained in next paragraphs. It is negative. Values of x and y are ordinates of any point referred to the system passing through the elastic center.

Static Bending Moments M_s at Any Point.—Bending moment M_s is the static bending moment at any point of the vertical loads, when the arch is considered as a cantilever free at the left support and fixed at the right support B. The static moment M_s is negative.

From the nature of the bending moment M_s follow these two rules:

A concentrated load, to produce a bending moment M_s at any point, must be placed to the left of the point. Loads placed between the point under consideration and the right support produce no bending moment M_s at that point.

The bending moment M_s of any concentrated load about any section is equal to the load multiplied by its distance from the section.

When, instead of the distance between the load and the section, the ordinates of the load and the section are given, following general formula may be used.

Let x = distance from center of coordinates to section at which bending moment is required;
a = distance from center of coordinates to loads;
P = concentrated load.

Then

Bending Moment M_s for Concentrated Load,

$$M_s = - P(x - a) \text{ for all points to the right of the load.} \quad (43)$$

$$M_s = 0 \qquad \text{for all points to the left of the load.} \quad . \quad (44)$$

The above equations are general. The values for x and a must be substituted with their signs.

Thus for load P_1 at a distance $(- a)$ the formula becomes

$$M_s = - P_1[x - (- a)] = - P_1(x + a).$$

The method of computing the bending moment M_s from Formula (43) is illustrated below. Bending moments will be found at section 1, 2 and 3, due to the loads P_1 and P_2 shown in Fig. 214, p. 601.

Consider section 1 at a distance x from the center of coordinates. The bending moment produced at that section by the load P_1 placed at a distance a_1 from the center is $-P_1(x - a_1)$.

For load P_2 placed at a distance $-a_2$ from the center the value to be substituted for a in Formula (43) is $-a_2$. The bending moment due to P_2 about section 1 is $-P_2[x-(-a_2)] = -P_2(x + a_2)$.

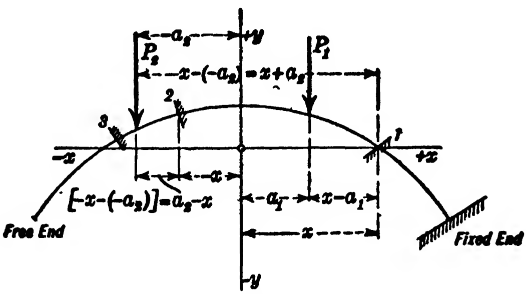


FIG. 214.—Static Bending Moments Due to Concentrated Loads. (*See p.* 601.)

This is correct because the distance between the load and the section as evident from Fig. 214 is $x + a_2$.

The total bending moment about section 1 of loads P_1 and P_2

$$M_s = -[P_1(x - a_1) + P_2(x + a_2)].$$

Consider now section 2 at a distance $-x$ from the center. The only load producing bending moment there is P_2 for which the distance is $-a_2$. Substituting these values in the general equation

$$M_s = -P_2[(-x) - (-a_2)] = -P_2(a_2 - x).$$

At section 3 there is no bending moment M_s because all the loads are at the right of the section and produce moment in this part of the cantilever.

Bending Moments M_s for Unit Loads.—A general formula for M_s of a load $P = 1$ placed at a distance a from the center about a section at a distance x from the center is

Bending Moment for Unit Load $P = 1$,

For x larger than a,

$$M_s = -(x - a). \quad (45)$$

For x smaller than a,

$$M_s = 0. \qquad (46)$$

x and a must be used with their signs.

Bending Moment M_s for Uniformly Distributed Load.—For uniformly distributed load the bending moment M_s equals the load to the left of the section multiplied by the distance of the center of gravity of this load from the section.

Let w = uniform load per unit of length;
x = distance from center to the section;
l = span of arch.

Then

Bending Moment M_s at Section x when Whole Span Loaded,

$$M_s = -w\left(\frac{l}{2}+x\right)\frac{1}{2}\left(\frac{l}{2}+x\right) = -\tfrac{1}{8}w(l+2x)^2. \qquad (47)$$

If the section is located on the left half of the arch substitute for x the value $(-x)$. Then $M_s = -\frac{1}{8}w(l-2x)^2$.

Bending Moment M_s at Section x when Only Right Half of Span Loaded,

$$M_s = -\frac{wx^2}{2} \text{ for point at the right.} \qquad (48)$$

EFFECT OF SHORTENING OR LENGTHENING OF ARCH SPAN

In the previous discussion it was assumed that the span of the arch is fixed so that there is no lengthening nor shortening of the span of the arch. In the discussion below will be found the effect upon the arch of any change of its span length.

Occasions When Change of Span Takes Place.—Actual change of the span takes place when the supports yield either due to give of the ground or due to bending of the piers.

An effect upon the arch similar to the effect of a change in span length is also produced, (*a*) when the arch shortens or lengthens due to temperature changes; (*b*) when the arch rib becomes compressed due to the normal thrust; (*c*) when the arch rib shortens due to shrinkage of concrete. In these three cases the length of the span, i.e., the distance between supports, remains constant and the change takes place in the length of the arch rib. Thus under the effect of normal thrust or fall of temperature the arch rib itself shortens by a distance Δs. If the ends of the arch could move, the distance between springings of the shortened arch would be equal to $l - \Delta l$. Actually, however, the distance between supports remains equal to l. The effect upon the shortened arch, therefore, is the same as if the length of the arch remained constant, but, instead, the span increased by the distance equal to Δl.

The effect of the fall of temperature, the rib shortening and the shrinkage are of the same nature as the effect of a horizontal yielding of the supports, i.e., a lengthening of the span.

The effect of the rise of temperature is similar to that of a horizontal movement of the support inward, i.e., a shortening of the span.

Effect of Change of Span Length Equal to Δl.—A change of span length equal to Δl produces bending moments and reactions at the supports and also bending moments and thrusts at all points of the arch. The relations between the bending moments and reactions and statically indeterminate values must be the same as between bending moments and reactions for the loads.

Assume, as before, that the axis of coordinates passes through the elastic center of the arch.

Let V_A, M_A and H = reactions and the bending moments at left support;
V_B, M_B and H = reactions and bending moment at right support;
M = auxiliary bending moment.

Then according to Formula (42) and since the static bending moment due to loads $M_s = 0$, the bending moment at any point is

$$M_x = M + V_A x + Hy$$

and the bending moments at the supports for which

$$x = \pm \tfrac{l}{2} \quad \text{and} \quad y = -Y_s$$

$$M_A = M - \tfrac{1}{2}V_A l - HY_s$$

$$M_B = M + \tfrac{1}{2}V_A l - HY_s.$$

Subtracting the first equation from the second we get

$$M_B - M_A = V_A l.$$

Due to the symmetry of the arch the bending moments at both supports must be equal. Therefore,

$$M_B - M_A = 0.$$

From this it follows that $V_A l$ is zero, also $V_A = 0$. For the same reason the other vertical reaction is equal zero.

The formula for bending moment at any point is reduced to

$$M_x = M + Hy.$$

The relation between a horizontal movement of the support and the bending moments causing it is given on p. 592. By transferring the arch to the new system of coordinates and making $N_x = -H$ this relation becomes

$$\Delta l = \frac{1}{E}\left[-\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_x y \frac{ds}{I_x} - \sum_{-\frac{l}{2}}^{\frac{l}{2}} H \frac{dx}{A_x}\right].$$

According to Maxwell's law of reciprocity the relation between the movement of the support and the bending moments in the arch causing this movement is the same as the relation between the movement of support and the bending moments in the arch which this movement produces. We can, therefore, use the above equation for solving our problem by substituting for Δl the movement of the support and for M_x the bending moment from formula above. Thus

$$\Delta l = \frac{1}{E}\left[-\sum_{-\frac{l}{2}}^{\frac{l}{2}} (M + Hy) y \frac{ds}{I_x} - \sum_{-\frac{l}{2}}^{\frac{l}{2}} H \frac{dx}{A_x}\right]$$
$$= -M \sum_{-\frac{l}{2}}^{\frac{l}{2}} y \frac{ds}{I_x} - H \sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{ds}{I_x} - H \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dx}{A_x}.$$

Since for the accepted coordinates $\sum_{-\frac{l}{2}}^{\frac{l}{2}} y \frac{ds}{I_x} = 0.$

$$\Delta l = -\frac{H}{E}\left(\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{ds}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dx}{A_x}\right)$$

and finally

$$H = -\frac{E\Delta l}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{ds}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dx}{A_x}}.$$

It will be noticed that the denominator is the same as for horizontal thrust due to the loads. For signs of Δl see p. 584.

The bending moment M will be found from the Equation (23), p. 592. This transferred to new system of coordinates becomes

$$\Delta \phi = \frac{1}{E}\left(\sum_{-\frac{l}{2}}^{\frac{l}{2}} M_x \frac{ds}{I_x}\right).$$

Since

$$\Delta \phi = 0$$

we have after substituting value for M_x

$$0 = \sum_{-\frac{l}{2}}^{\frac{l}{2}} (M + Hy)\frac{ds}{I_x} = M\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{ds}{I_x} + H\sum_{-\frac{l}{2}}^{\frac{l}{2}} y\frac{ds}{I_x}.$$

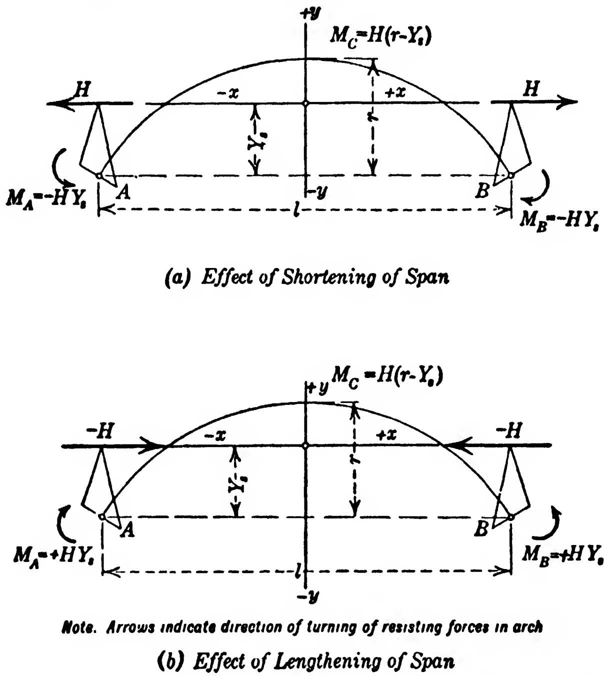
Since

$\sum_{-\frac{l}{2}}^{\frac{l}{2}} y\frac{ds}{I_x} = 0$ the second item cancels and we have

$$M\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{ds}{I_x} = 0.$$

Therefore

$$M = 0.$$



(a) Effect of Shortening of Span

(b) Effect of Lengthening of Span

FIG. 215.—Effect of Change in Span Length of Arch. (*See p.* 606.)

This demonstrates that when the span length changes by Δl, V_A and M are equal zero and H equals

Final Equations for Change of Length:

Horizontal thrust (after multiplying the numerator and the denominator by I)

$$H = -\frac{E\Delta l I}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{Ids}{I_x} + \sum \frac{Idx}{A_s}}. \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Bending moment at the support,

$$M_A = M_B = -HY_s. \qquad (50)$$

Bending moment at any point,

$$M_x = Hy. \qquad (51)$$

From the above it is evident that change of length of the span produces a horizontal thrust applied at the arch at the level of x-axis as shown in Fig. 215 (a) and (b), p. 605.

Lengthening of the span produces a positive, H, as shown in Fig. 215 (a), while shortening produces a negative, H, acting as shown in Fig. 215 (b).

RIB SHORTENING

The positive thrust acting on the arch compresses the arch ring. If free to move, the compressed arch would assume the shape of an arch with a shorter span. If l is the original length of span, while $l_1 = l - \Delta l$ is the span a free arch would assume when compressed.

Since the arch is not free to move, the span of the compressed arch remains the same as before compression and the shortened arch rib must adapt itself to the larger span by spreading. The crown is lowered and the arch bends. The maximum negative bending moment acts at the springing line, where it produces tension at the top and maximum positive bending moment is at the crown where it produces tension at the bottom. The effect of rib shortening is the same as that in the fall of temperature shown in Fig. 215 (a), p. 605.

For live load the effect of rib shortening is small because the thrust is small in comparison with the bending moment. It could be neglected without any appreciable error. It is represented by the term $\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}$ in the denominator of the formulas for the horizontal thrust. All formulas based on these formulas for horizontal thrust automatically include the effect of rib shortening.

For dead load the effect of rib shortening is appreciable particularly for shallow arches and in most cases needs to be computed.

Usually the axis of the ring is selected so that it coincides with the line of pressure for the dead load (or the dead load plus one-half of the live load) (see p. 468). In such case the thrust acts centrally and there is no bending moment due to the thrust. However, this does not take into account the effect of the rib shortening due to the thrust which needs to be computed separately.

The effect of rib shortening is small when the rise of the span is more than one-quarter of the span. For flatter arches the effect increases rapidly with the decrease of the ratio of rise to span. Prof. Mörsh [2] has computed the stresses due to rib shortening due to dead load for arches of different ratios of rise to span. The axis of all arches were madè to coincide with the line of pressure so that the thrust acted centrally.

From his computations it is evident that stresses due to rib shortening are fairly small for high arches with ratio of rise to span $\frac{r}{l} = \frac{1}{4}$ but increase rapidly with the decrease of the ratio of rise to span.

Formula for Rib Shortening.

Let H_d = horizontal thrust due to dead load;
H_s = horizontal thrust due to rib shortening;
N_{dx} = normal thrust due to dead load at any point x;
A_x = area of cross section of the arch at any point x;
ϕ_x = angle of inclination of any section with vertical;
E = modulus of elasticity;
ds = length of division of arch.

The shortening of the arch caused by the dead load thrust may be found in the following manner: The normal thrust at any section may be expressed with sufficient exactness by the formula $N_{dx} = - H_d$ (see p. 594.) The unit stress due to the thrust is $\frac{N_{dx}}{A_x} = - \frac{H_d}{A_x}$ and the unit shortening of the arch along arch axis is $\frac{H_d}{EA_x}$. The shortening of the division ds along its axis is $\frac{H_d}{EA_x}ds$. The horizontal projection of this shortening, i.e., the shortening along the span is $- \frac{H_d}{EA_x} ds \cos \phi_x = - \frac{H_d}{EA_x} dx$.

When the rib shortens and the span remains the same the effect is the same as if the span lengthened and the rib length remained constant.

The total lengthening of the span, therefore, is

$$\Delta l = - \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{H_d \cos \phi_x}{EA_x} ds = - \frac{H_d}{E} \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dx}{A_x}.$$

[2] Berechnung von eingespannten Gewölben, by Prof. E. Morsh. *Schweizerishe Bauzeitung*, Band XLVII, No. 7 und 8.

This value of Δl introduced in Formula (49) gives the thrust due to rib shortening. It is (after multiplying numerator and denominator by I)

Horizontal Thrust Due to Rib Shortening,

$$H_s = - \frac{\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Id_x}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}} H_d. \quad (52)$$

The expression $\sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}$ may be replaced by $\frac{Il}{A_{av}}$ where A_{av} is area of average cross-section.

This thrust is positive and applies at the level of the center of gravity of the elastic weights of the arch, the position of which is already determined. All values in the equation are already computed for the arch.

Bending Moment Due to Rib Shortening.—The horizontal thrust produces a bending moment in every point of the arch, the general equation for which

Bending Moment at Any Point,

$$M_x = H_s y. \quad (53)$$

Maximum Negative Moment at Support, $y = - Y_s$,

$$M_s = - H_s Y_s. \quad (54)$$

Maximum Positive Moment at Crown, $y = (r - Y_s)$,

$$M_c = H_s(r - Y_s). \quad (55)$$

The bending moment is positive at the crown and negative at the springing.

TEMPERATURE STRESSES

Changes of temperature from that at which the arch ring was closed cause either lengthening or shortening of the arch rib. If the arch were free to move horizontally at either end, the arch ring could expand and contract freely. The change in length of the rib would cause either shortening or lengthening of the span and no stresses would be developed in the arch.

Since, however, a fixed arch is held at the supports, the arch rib after expanding or contracting cannot change its span length but must accommodate itself to the fixed span which is either too short or too long for the changed length of rib. Bending moments and a horizontal thrust are thereby developed in the arch.

The effect of shortening of the arch rib caused by fall of temperature and the bending moments and thrust produced by it are of the same character as for rib shortening. As a consequence the arch lowers at the crown and tensile stresses are developed at the springing at the bottom and at the crown at the top, as shown in Fig. 216 (*a*).

The lengthening of the arch, caused by rise of temperature, produces bending moments and a horizontal thrust of opposite sign. The crown rises and tensile stresses are developed at the springing at the bottom and at the crown at the top, as shown in Fig. 216 (*b*).

The effect of temperature changes is shown in Fig. 216 (*a*) and (*b*).

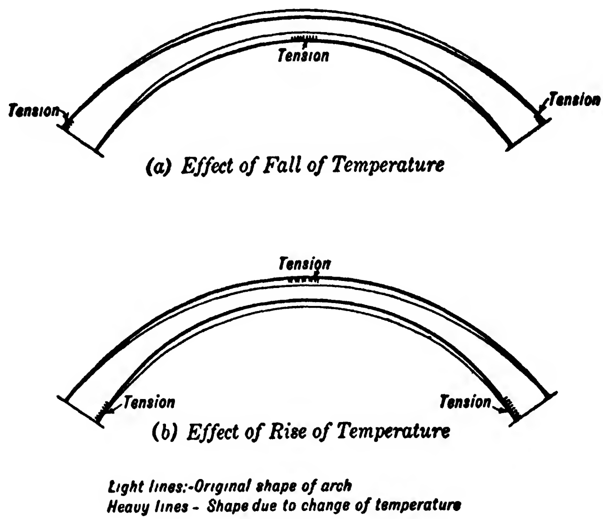


FIG. 216.—Effect of Temperature Changes on Arches. (*See p.* 609.)

Thrust and Bending Moment Due to Changes of Temperature.

Let t = change of temperature in degrees;
α = coefficient of expansion per 1 degree Fahrenheit;
E = modulus of elasticity, lb. per sq. in.;
I = moment of inertia at crown;
l = span of arch.

The change in span which would take place in a free arch due to change in temperature

$$\Delta l = \pm\, l\alpha t.$$

The sign + is for rise of temperature and sign − for fall of temperature. Substitute this in Equation (49), p. 605.

The horizontal thrust produced by the change of temperature is

$$H_t = -\frac{\pm \alpha t E l I}{\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}} = -\frac{\pm \alpha t E l I}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}}.$$

Horizontal Thrust for Rise of Temperature,

$$H_t = -\frac{\alpha t E l I}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}}. \quad \ldots \ldots \ldots \quad (56)$$

The sign, −, signifies that the thrust acts inward. It is of the same sign as the thrust due to loading. Stresses produced by this thrust on the section are compression.

Horizontal Thrust for Fall of Temperature,

$$H_t = +\frac{\alpha t E l I}{\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \sum_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}}. \quad \ldots \ldots \ldots \quad (57)$$

The sign, +, signifies that the thrust acts outward. It is of opposite sign to the thrust due to loading and of the same sign as the thrust due to rib shortening. Stresses produced by it on the section are tension.

The numerator of the above formulas is the same as for the thrust due to loading.

This thrust applies at the center of gravity of the elastic ratios $\frac{Ids}{I_x}$. In Formulas (56) and (57) all values must be in the same units.

Bending Moments.—The bending moments produced by the thrust may be found from general formula

$$M_t = H_t y, \quad \ldots \ldots \ldots \quad (58)$$

where y is the distance of the point from x-axis. It is zero at the level of x-axis and reaches maximum values at the springing line and the crown. The signs of bending moments are as follows:

	At Crown	At Springing
Rise of temperature..........	minus	plus
Fall of temperature...........	plus	minus

Line of pressure for temperature is a straight line coinciding with the x-axis.

Peculiarities of Temperature Stresses.—One of the peculiarities of the temperature stresses is that the magnitude of the thrust caused by temperature changes is comparatively small but the bending moments are large so that the stresses are mainly bending stresses. The tensile stresses are particularly harmful to the arch.

Another peculiarity of the effect of temperature changes which gives a good deal of trouble in design is that the stresses caused by them not only cannot be reduced but actually are increased, by increasing the dimensions of the sections of the arch. This is apparent from the study of the Formula (56) for temperature thrust. Divide the numerator and denominator by the common I.

The numerator of this formula, αtEl, is not affected by the dimensions of the arch sections.

In the denominator, on the other hand, the magnitude of the arch sections affects both terms. The second term is small and will not be discussed. In the first term $\sum_{-\frac{l}{2}}^{\frac{l}{2}} y^2\frac{ds}{I_x}$ the values of y and ds depend upon the shape of the arch axis only. The moment of inertia I_x depends upon the size of the section. For a rectangular section the value I_x is equal $\frac{bh^3}{12}$. If h increases, the value of I_x increases with the third power of the increase of h.

Let h = original depth of section;

h_1 = increased depth of section;

$\frac{h_1}{h} = m$ = ratio of increase of the section;

I_{x_1} = moment of inertia of increased section;

I_x = moment of inertia of original section;

f_1 = temperature stress increased section;

f = temperature stress original section.

The moment of inertia of the increased section is equal to $I_{x1} = \frac{bh_1^3}{12}$. The ratio of moments of inertia $\frac{I_{x_1}}{I_x} = \left(\frac{h_1}{h}\right)^3 = m^3$. Therefore $I_{x_1} = m^3 I_x$. Since I_x is in the denominator, the value of the term $\sum y^2\frac{ds}{I_x}$ decreases according to m^3, i.e., according to the third power of the ratio of increase of the depth of section h. Consequently the denominator for the thrust H_t decreases according to third power of the ratio of increase of the depth. If the ratio of increase of the depth of section $\frac{h_1}{h}$ is constant for all section of the arch, then

$$\Sigma y^2 \frac{ds}{I_{x_1}} = \Sigma y^2 \frac{ds}{m^3 I_x} = \frac{1}{m^3} \Sigma y^2 \frac{ds}{I_x}.$$

This substituted in formula for H_t gives

$$H_{t_1} = \pm \frac{\alpha t E l}{\frac{1}{m^3} \Sigma y^2 \frac{ds}{I_x}} = \pm\, m^3 \frac{\alpha t E l}{\Sigma y^2 \frac{ds}{I_x}}$$

or

$$H_{t_1} = m^3 H_t.$$

The value of H_{t_1} for the arch with increased dimensions increases according to the third power of the ratio of increase of the depths of sections. The same is true of the bending moments due to changes of temperature,

$$M_{t_1} = m^3 M_t.$$

The stresses due to the bending moment for the increased section are

$$f_1 = \frac{M_{t_1} \frac{h_1}{2}}{I_{x_1}} = m^3 M_t \frac{m \frac{h}{2}}{m^3 I_x} = m M_t \frac{h}{2 I_x}$$

since $M_t \frac{h}{2I_x}$ is f the stress for the smaller section. Therefore

$$f_1 = mf.$$

From the above it is evident that for the increased sections the temperature stresses are larger than for the original section.

At first glance this result seems unreasonable. However, the correctness of it is evident from the following reasoning: Suppose that the arch has very small thickness such as would be the case for an arch made of sheet metal. Such an arch can adapt itself very easily to the changes of temperature and practically no stresses will be developed due to the changes of temperature. With the increase of the stiffness of the arch the stresses increase rapidly because the stiffness of the arch is denoted by the moment of inertia which increases according to the third power of the thickness of the arch. It is important, therefore, to remember that a slender arch is much more desirable than a heavy arch.

If the stresses due to the temperature changes are excessive the only way to decrease them is by reducing the thickness of the arch at the crown and increasing the thickness at the springing. In this manner the thrust may remain of the same magnitude as before, but the center of gravity of the elastic ratios $\frac{Ids}{I_x}$ for the new arch will be higher than for the original arch. The moment arm at the crown is reduced, thus reducing the bending moment at the crown. The stresses due to the temperature changes are either reduced or remain of the same magnitude as before. Due to the decrease of the section at the crown, however, the dead load stresses, which are compression, become larger and may reduce sufficiently the tensile stresses due to the temperature.

Influence of Shrinkage.—Shrinkage has the same effect upon the arch as the fall of temperature or rib shortening. If the coefficient of shrinkage is known the shortening of the span Δl may be easily found. The thrust and moments, then, may be computed by Formulas (57) and (58) in which the shortening due to shrinkage is substituted for the shortening due to temperature. Shrinkage is often taken care of by adding 15 degrees Fahrenheit to the assumed fall of temperature.

It should be noticed that when the arches are built in transverse strips, the arch is not closed until most of the strips have set and thereby undergone the largest part of the shrinkage. The arch as a whole is then affected only by the additional shrinkage of the cured concrete.

Methods of Eliminating the Effect of Rib Shortening and Shrinkage.—In shallow arches the stresses caused by the rib shortening and the shrinkage are considerable and it is often economical to eliminate their effect by the introduction of temporary hinges or by the method used by Freyssinet in France. Freyssinet's method consists of compressing the arch ribs artificially, before closing them, by means of hydraulic jacks inserted between two planes which cut the crown of the arch. After the ribs are properly compressed and the centering removed the crown section is closed and the jacks are removed.[3]

FORMULAS FOR FIXED PARABOLIC ARCHES

If the arch axis is a parabola, it can be represented by a mathematical equation and the formulas for H, V_A and M may be solved by integration instead of the summation method.

The general formulas for the statically indeterminate values are (see p. 598)

$$H = -\frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s y \frac{Ids}{I_x}}{\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{Ids}{I_x} + \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x}} \quad . \quad . \quad . \quad . \quad . \quad (59)$$

[3] See *Engineering News-Record*, Sept. 18, 1924, p. 463.

$$V_A = -\frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s x \frac{Ids}{I_x}}{\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \frac{Ids}{I_x}} \quad \text{. . . . . (60)}$$

$$M = -\frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s \frac{Ids}{I_x}}{\int_{-\frac{l}{2}}^{\frac{l}{2}} x \frac{Ids}{I_x}} \quad \text{. . . . . (61)}$$

To make integration of the above formulas possible it is necessary to express y and $\frac{Ids}{I_x}$ in terms of x. A formula for the curvature of the arch axis gives the values of y and ds. To get the value of I_x it is necessary to express the variation of moments of inertia by a formula. It should be noted that the formula for y must be referred to a system of coordinates passing through the elastic center of the arch.

Let r = rise of arch axis, ft.;
l = span of arch axis, ft.;
Y = vertical ordinate, when the x-axis passes through the crown;
y = vertical ordinate, when the x-axis passes through elastic center;
x = horizontal ordinate (same for both systems of coordinates);
ds = small element of arch axis;
ϕ_x = angle of inclination at any point x of the tangent to the arch axis horizontal. Also angle of inclination of normal section with vertical;
I_x = moment of inertia of a normal section at any point;
I = moment of inertia of arch section at the crown;
I_s = moment of inertia of normal arch section at the springing;
ϕ_s = angle of inclination of tangent at the springing;
Y_c = vertical distance of elastic center from crown;
Y_s = vertical distance of elastic center from springing.

Variation of Moment of Inertia.—The ratio of the moment of inertia of the arch at the crown and springing is known or assumed. It is usually expressed as

Ratio of Moments of Inertia,

$$\frac{I}{I_s \cos \phi_s} = n. \quad \text{. . . . . (62)}$$

To make integration possible, it is necessary to express the variation of the moments of inertia at the intermediate points of the arch by means of a formula. As explained on p. 477, the design of an arch is most economical when the moments of inertia at the various sections vary according to the following formula:

$$\frac{I}{I_x \cos \phi_x} = 1 - \left(1 - \frac{I}{I_s \cos \phi_s}\right)\left(\frac{2x}{l}\right)^2$$

or since $\dfrac{I}{I_s \cos \phi_s} = n$ and is a constant

$$\frac{I}{I_x \cos \phi_x} = 1 - 4(1 - n)\left(\frac{x}{l}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad (63)$$

Knowing the value of I at the crown and n, the moment of inertia for intermediate section may be found from

$$I_x = I\frac{1}{1 - 4(1 - n)\left(\dfrac{x}{l}\right)^2}\,\frac{1}{\cos \phi_x}. \quad . \quad . \quad . \quad . \quad (64)$$

In the above formula only $\cos \phi_x$ depends upon the shape of the arch. From general rules $\cos \phi_x = \dfrac{1}{\sqrt{1 + \tan^2 \phi_x}}$. For a parabola $dy = 2\dfrac{4r}{l^2}x\,dx$, therefore $\tan \phi_x = \dfrac{dy}{dx} = 2\dfrac{4r}{l^2}x = 8\left(\dfrac{r}{l}\right)\left(\dfrac{x}{l}\right)$. The formula for $\cos \phi_x$ becomes

$$\cos \phi_x = \frac{1}{\sqrt{1 + 64\left(\dfrac{r}{l}\right)^2\left(\dfrac{x}{l}\right)^2}}$$

and

$$\frac{1}{\cos \phi_x} = \sqrt{1 + 64\left(\frac{r}{l}\right)^2\left(\frac{x}{l}\right)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Therefore for a parabola the moment of inertia at an intermediate point is

$$I_x = I\frac{\sqrt{1 + 64\left(\dfrac{r}{l}\right)^2\left(\dfrac{x}{l}\right)^2}}{1 - 4(1 - n)\left(\dfrac{x}{l}\right)^2} = \frac{C_1}{C_2}I, \quad . \quad . \quad . \quad (66)$$

where $C_1 = \sqrt{1 + 64\left(\dfrac{r}{l}\right)^2\left(\dfrac{x}{l}\right)^2}$ and $C_2 = \left[1 - 4(1 - n)\left(\dfrac{x}{l}\right)^2\right]$.

For rectangular section with constant width b and variable depth h_x the moment of inertia at any point is $I_x = \frac{bh^3_x}{12}$ and $I = \frac{bh^3}{12}$. Substituting this in Equation (66) following relation between the depths is found

$$h_x = h \sqrt[3]{\frac{\sqrt{1 + 64\left(\frac{r}{l}\right)^2\left(\frac{x}{l}\right)^2}}{1 - 4(1 - n)\left(\frac{x}{l}\right)^2}} = \sqrt[3]{\frac{C_1}{C_2}}h. \quad . \quad (67)$$

Equation of Parabolic Axis Referred to Coordinates through Crown.—The equation of the parabolic axis referred to a system of coordinates passing through the crown is

$$Y = \frac{4r}{l^2}x^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Elastic Center.—Elastic center is the center of gravity of the ratios of $\frac{Ids}{I_x}$. As explained on p. 595 to simplify computations the system of coordinates is transferred to the elastic center, as then for the new coordinates the values $\int y\frac{Ids}{I_x}$, $\int x\frac{Ids}{I_x}$ and $\int xy\frac{Ids}{I_x}$ are equal to zero. For symmetrical arches the elastic center is on a vertical line passing through the crown. The vertical distance of the elastic center from the crown, i.e., from the original center of coordinates is given by the general equation

$$Y_c = \frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} Y\frac{Ids}{I_x}}{\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Ids}{I_x}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

To solve this, substitute proper values for Y and for $\frac{Ids}{I_x}$. The value for Y is taken from Formula (68), p. 616. In the expression $\frac{Ids}{I_x}$, $ds = \frac{dx}{\cos \phi_x}$.[4] Therefore $\frac{Ids}{I_x}$ becomes $\frac{Idx}{I_x \cos \phi_x}$.

[4] ds is the hypothenuse in a right-angle triangle in which the two other sides are dx and dy. Consequently $dx = ds \cos \phi_x$ and $ds = \frac{dx}{\cos \phi_x}$.

Substitute the value of I_x from Formula (64), p. 615,

$$I\frac{ds}{I_x} = \frac{Idx}{I_x \cos \phi_x} = \left[1 - 4(1 - n)\left(\frac{x}{l}\right)^2\right]dx.$$

Substitute this value and $Y = \frac{4r}{l^2}x^2$ in Equation (69),

$$Y_c = \frac{4r\int_{-\frac{l}{2}}^{\frac{l}{2}} \left(\frac{x}{l}\right)^2\left[1 - 4(1 - n)\left(\frac{x}{l}\right)^2\right]dx}{\int_{-\frac{l}{2}}^{\frac{l}{2}} \left[1 - 4(1 - n)\left(\frac{x}{l}\right)^2\right]dx} \quad . \quad . \quad . \quad . \quad (69a)$$

After the integrations are solved the equation changes to

Distance of Elastic Center from Crown.

$$Y_c = \frac{3n + 2}{5(n + 2)}r = C_3 r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

where

$$C_3 = \frac{3n + 2}{5(n + 2)}.$$

For $n = 1$ the above equation changes to

$$Y_c = \tfrac{1}{3}r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Formula of the Arch Axis when Center of Coordinates is at the Elastic Center.—Following relation is apparent between the vertical coordinates of the two systems for fixed parabolic arch.

$$y = Y_c - Y = \frac{3n + 2}{5(n + 2)}r - Y.$$

Substituting for Y the value from Formula (68), namely, $\frac{4r}{l^2}x^2$, we get

Equation of Arch Axis, Referred to Coordinates Passing through Elastic Center,

$$y = \frac{3n + 2}{5(n + 2)}r - \frac{4r}{l^2}x^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Final Formulas for H, V_A and M, Fixed Parabolic Arch.—By substituting in Formulas (59) to (61), p. 613, $\frac{Ids}{I_x} = \left[1 - 4(1 - n)\left(\frac{x}{l}\right)^2\right]dx$, and $\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{Idx}{A_x} = \frac{Il}{A_{av}}$, the formulas become

$$H = \frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s y\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx}{\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx + \frac{lI}{A_{av}}} \quad . \quad . \quad . \quad . \quad (73)$$

$$V_A = -\frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s x\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx}{\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx} \quad . \quad . \quad . \quad . \quad (74)$$

$$M = -\frac{\int_{-\frac{l}{2}}^{\frac{l}{2}} M_s\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx}{\int_{-\frac{l}{2}}^{\frac{l}{2}} \left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx} \quad . \quad . \quad . \quad (75)$$

The problem now resolves itself into solving for the parabolic axis of the integrals $\int y^2\left[1-4(1-n)\left(\frac{x}{l}\right)^2\right]dx$, $\int x^2\left[1-4(1-n)\left(\frac{x}{l}\right)^2\right]dx$ and $\int\left[1-4(1-n)\left(\frac{x}{l}\right)^2\right]dx$, which depend only upon the arch axis, and of the integrals $\int M_s y\left[1-4(1-n)\left(\frac{x}{l}\right)^2\right]dx$, $\int M_s x\left[1-4(1-n)\left(\frac{x}{l}\right)^2\right]dx$ and $\int M_s\left[1-4(1-n)\left(\frac{x}{l}\right)^2\right]dx$ which depend upon the arch axis and also upon the character of the loading as expressed by M_s.

Denominator for H, Fixed Parabolic Arch.

Substituting in $\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx$, $y = Y_c - \frac{4r}{l^2}x^2$ the integral changes to

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} \left(Y_c - \frac{4r}{l^2}x^2\right)^2\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx$$
$$= l\left\{\left(Y_c^2 - \frac{2}{3}rY_c + \frac{1}{5}r^2\right) - (1-n)\left[\frac{1}{3}Y_c^2 - \frac{2}{5}rY_c + \frac{1}{7}r^2\right]\right\}.$$

Substitute for $Y_c = \frac{3n+2}{5(n+2)}r$. Finally the denominator for H becomes

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx = \frac{4}{5 \times 35}\,\frac{lr^2}{(n+2)}\left(n^2 + 8n + \frac{8}{3}\right). \quad (76)$$

Denominator for V_A, Fixed Parabolic Arch.—After integrating, the denominator becomes

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx = \frac{1}{60}(2+3n)l^3. \quad \ldots\ldots \quad (77)$$

Denominator for M, Fixed Parabolic Arch.

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} \left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx = \frac{1}{3}(2+n)l. \quad \ldots\ldots \quad (78)$$

Numerators for H, V_A and M.—The numerators for Formulas (73) to (75) depend not only upon the shape of the arch but also upon the static bending moment of the load M_s. Therefore they must be found for each particular type of loading.

Using for M_s in the denominators the value for a unit load $P = 1$ placed at any point x, the resulting formulas would give equations for influence lines for H, V_A and M.

Influence Lines for H.—The value of M_s at any point for a unit load $P = 1$ placed at a distance x_1 from the crown is $M_s = -(x - x_1)$. The numerator for H then becomes

$$\int_{x_1}^{\frac{l}{2}} M_s y\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx$$

$$= -\int_{x_1}^{\frac{l}{2}} (x - x_1)\left(Y_c - \frac{4r}{l^2}x^2\right)\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx.$$

This solved, gives

$$\int_{x_1}^{\frac{l}{2}} M_s y\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right]dx = -\frac{rl^2}{15(n+2)}\left[\frac{1}{4} - \left(\frac{x_1}{l}\right)^2\right]^2$$

$$\left\{3n(n+4) + 8(1-n)(n+2)\left[\frac{1}{4} - \left(\frac{x_1}{l}\right)^2\right]\right\}.$$

Substituting in Formula (73) the above numerator as well as the value for the denominator from Formula (76), p. 619, and replacing the specific value x_1 by a general value x, the formula for influence line for H becomes

$$H_{\text{for } P=1} = -\frac{\dfrac{rl^2}{15(n+2)}\left[\dfrac{1}{4}-\left(\dfrac{x}{l}\right)^2\right]^2\left\{3n(n+4)+8(1-n)(n+2)\left[\dfrac{1}{4}-\left(\dfrac{x}{l}\right)^2\right]\right\}}{\dfrac{4}{5\times 35}\dfrac{lr^2}{n+2}\left(n^2+8n+\dfrac{8}{3}\right)+\dfrac{lI}{A_{av}}}.$$

Finally

Influence Line for H,

$$H_{\text{for } P=1} = -\frac{\left[\dfrac{1}{4}-\left(\dfrac{x}{l}\right)^2\right]^2\left\{3n(n+4)+8(1-n)(n+2)\left[\dfrac{1}{4}-\left(\dfrac{x}{l}\right)^2\right]\right\}}{\dfrac{12}{35}\left(n^2+8n+\dfrac{8}{3}\right)+15(n+2)\dfrac{I}{r^2A_{av}}}\frac{l}{r}. \tag{79}$$

The last term in the denominator, which signifies the effect of rib shortening, may be omitted as for live loads it is insignificant.

Influence Line for V_A.—Substituting $M_s = x - x_1$ the numerator in Formula (74) for a unit load $P = 1$ placed at x_1 becomes

$$\int_{x_1}^{\frac{l}{2}} M_s x\left[1-4(1-n)\left(\frac{x}{l}\right)^2\right]dx = -\int_{x_1}^{\frac{l}{2}}(x-x_1)x\left[1-4(1-n)\left(\frac{x}{l}\right)^2\right]dx.$$

This integrated and simplified becomes

$$\int_{x_1}^{\frac{l}{2}} M_s x\left[1-4(1-n)\left(\frac{x}{l}\right)^2\right]dx$$
$$= -\frac{2+3n}{40}\left(1-2\frac{x_1}{l}\right)^2\left[\frac{1}{3}\left(1+\frac{x_1}{l}\right)-\frac{1-n}{2(2+3n)}\frac{x_1}{l}\left(1+2\frac{x_1}{l}\right)^2\right]l^3.$$

Substituting in Formula (74) the above value for the numerator also the value from Formula (77) for the denominator, and replacing x_1 by x the formula for influence line becomes

Influence Line for V_A,

$$V_{A\ \text{for } P=1} = \frac{1}{2}\left(1-2\frac{x}{l}\right)^2\left[\left(1+\frac{x}{l}\right)-\frac{3(1-n)}{2(2+3n)}\frac{x}{l}\left(1+2\frac{x}{l}\right)^2\right]. \tag{80}$$

Substituting in the above formula consecutively the values for $\frac{x}{l}$ for different points on the axis and plotting the result under the corresponding points, we obtain an influence line for V_A.

Influence Line for M. — Substitute in Formula (75)

$$M_s = -(x - x_1).$$

The resulting equation is

$$\int_{x_1}^{\frac{l}{2}} M_s \left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right] dx = -\int_{x_1}^{\frac{l}{2}} (x - x_1)\left[1 - 4(1-n)\left(\frac{x}{l}\right)^2\right] dx.$$

This solved and simplified gives

$$\int_{x_1}^{\frac{l}{2}} M_s \left[1 - 4(1 - n)\left(\frac{x}{l}\right)^2\right] dx$$

$$= -\frac{l^2(2 + n)}{24}\left(1 - 2\frac{x_1}{l}\right)^2\left\{1 + \frac{1 - n}{2(2 + n)}\left(1 + 2\frac{x_1}{l}\right)^2\right\}.$$

Substituting in Formula (75) the above numerator and the denominator from Formula (78), p. 619, also replacing the specific value x_1 by a general value x we get a formula for influence line for M.

Influence Line for M,

$$M_{\text{for } P=1} = \frac{3}{l(2 + n)} \frac{l^2(2 + n)}{24}\left(1 - 2\frac{x}{l}\right)^2\left\{1 + \frac{1 - n}{2(2 + n)}\left(1 + 2\frac{x}{l}\right)^2\right\}$$

$$= \frac{1}{8}\left(1 - 2\frac{x}{l}\right)^2\left\{1 + \frac{1 - n}{2(2 + n)}\left(1 + 2\frac{x}{l}\right)^2\right\}l. \quad . \quad (81)$$

Influence Line for M_A.—The formula for bending moment at the left support is as explained on p. 599,

$$M_A = M - V_A\frac{l}{2} - H(r - Y_c). \quad . \quad . \quad . \quad . \quad . \quad (82)$$

The values of M, V_A and H for successive points taken from the influence lines and substituted in the above formula give the points of the influence line for M_A.

Influence Line for Bending Moment at the Crown.—The bending moment at the crown is obtained from the following general formula.

$$M_c = M + HY_c + M_s.$$

To get an influence line for the bending moment at the crown substitute in the above formula values for M and H from the influence lines for different positions of the load $P = 1$ and also the corresponding value of M_s. For the loads placed at the right side of the arch the moment M_s at the crown is zero. Therefore

Bending Moment at Crown for Loads at Right side of Arch,

$$M_c = M + HY_c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

Since the influence line is symmetrical, the values for the right side only need to be computed. This simplifies the work somewhat.

MAXIMUM BENDING MOMENTS FOR UNIFORMLY DISTRIBUTED LOADING PARABOLIC ARCH

Formulas below give maximum bending moments at the crown, the springing, and at the quarter points for parabolic arch. These are found by integration and are based on the most unfavorable loading for each particular case.

Bending Moments and Thrusts at the Crown.—For maximum positive bending moment the loading extends from $x = -\frac{l}{8}$ to $x = \frac{l}{8}$.

For maximum negative bending moment the load is placed at each end and extends from springing for a distance equal to $\frac{3}{8}l$.

Formulas for H, V_A and M as well as for the bending moment at the crown M_c are given below.

Maximum Positive Bending Moment at Crown,

$$M_c = M + HY_c - \tfrac{1}{128}wl^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

$$H = -2\left(\frac{0.2515 + 0.719n + 0.085n^2}{2.677 + 8n + n^2} - \frac{1}{16}\right)\frac{l}{r}wl, \quad . \quad . \quad (85)$$

$$V_A = \tfrac{1}{8}wl, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

$$M = \frac{1}{2+n}(0.484 + 0.0466n)wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

In table on p. 547 are given values for different ratios of moment of inertia n.

Maximum Negative Bending Moment at Crown,

$$M_c = M + Hy_c - \tfrac{15}{128}wl^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

$$H = -2\left(\frac{1}{8} - \frac{0.2515 + 0.719n + 0.085n^2}{2.677 + 8n + n^2}\right)\frac{l}{r}wl, \quad . \quad . \quad (89)$$

$$V_A = \tfrac{3}{8}wl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

$$M = \frac{1}{2+n}(0.252 + 0.153n)wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

In table on p. 547 are given values for different ratios of moments of inertia n.

Bending Moments and Thrusts at the Springing.—For maximum positive bending moment at left springing the load extends from the right springing for a distance equal to $0.6l$.

For maximum negative bending moment at left springing the load extends from left springing for a distance equal to $0.4l$.

Formulas for H, V_A and M as well as for the bending moment at the springing M_A are given below. They were obtained by substituting in the denominators for H, V_A and M the value for M_s, which for uniformly distributed load is $M_s = -\frac{1}{2}w(x_1 - x)^2$, and solving the integrals within the limits of the loaded section of the arch.

Maximum Positive Bending Moment at Springing,

$$M_A = M - V_A\frac{l}{2} - H(r - y_c), \qquad (92)$$

where

$$H = -\frac{0.2368 + 0.678n + 0.0806n^2}{2.667 + 8n + n^2}\frac{l}{r}wl, \qquad (93)$$

$$V_A = \frac{1}{2 + 3n}(0.275 + 0.481n)wl, \qquad (94)$$

$$M = \frac{1}{2 + n}(0.0493 + 0.0587n)wl^2. \qquad (95)$$

In table on p. 546 are given values for different ratios of moments of inertia n.

Maximum Negative Bending Moment at Springing,

$$M_A = M - V_A\frac{l}{2} - H(r - y_c), \qquad (96)$$

$$H = -\left(\frac{1}{8} - \frac{0.02368 + 0.678n + 0.0806n^2}{2.677 + 8n + n^2}\right)\frac{l}{r}wl, \qquad (97)$$

$$V_A = \left[\frac{1}{2} - \frac{1}{1 + 3n}(0.275 + 0.481n)\right]wl, \qquad (98)$$

$$M = \frac{1}{2 + n}(0.251 + 0.141n)wl^2. \qquad (99)$$

In table on p. 546 are given values for different ratios of moments of inertia n.

Maximum Bending Moments at Quarter Points.—For maximum positive bending moment at left quarter point the load extends from the left support for a distance equal to $\frac{3}{8}l$.

For maximum negative bending moment at right quarter point the load extends from the right support for a distance equal to $\frac{3}{8}l$.

Formulas for H, V_A and M were found in the same manner as explained in connection with the bending moments at the springing.

Maximum Positive Bending Moment at Quarter Point,

$$M_{\frac{1}{4}l} = M - \frac{l}{4}V_A + H(Y_c - \tfrac{1}{4}r) - \tfrac{1}{32}wl^2, \quad . \quad . \quad . \quad . \quad (100)$$

$$H = -\left[\frac{1}{8} - \frac{0.2515 + 0.719n + 0.085n^2}{2.667 + 8n + n^2}\right]\left(\frac{l}{r}\right)wl, \quad . \quad (101)$$

$$V_A = \left[\frac{1}{2} - \frac{1}{2 + 3n}(0.3096 + 0.5292n)\right]wl, \quad . \quad . \quad . \quad . \quad (102)$$

$$M = \frac{1}{2 + n}(0.243 + 0.135n)wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

In table on p. 547 are given values for different ratios of moments of inertia n.

Maximum Negative Bending Moment at Quarter Point,

$$M_{(-\frac{1}{4}l)} = M - \frac{l}{4}V_A + H(Y_c - \tfrac{1}{4}r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

$$H = -\frac{0.0215 + 0.719n + 0.085n^2}{2.667 + 8n + n^2}\left(\frac{l}{r}\right)wl, \quad . \quad . \quad . \quad . \quad (105)$$

$$V_A = \frac{1}{2 + 3n}(0.3096 + 0.5292n)wl, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

$$M = \frac{1}{2 + n}(0.0572 + 0.0648n)wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

In table on p. 547 are given values for different ratios of moments of inertia n.

LINE OF PRESSURE, FOR FIXED ARCHES

Thrust at Any Point.—The thrust R is the resultant of the vertical external shear and horizontal external shear acting at any point.

The vertical external shear at any point, V_x equals the vertical reaction minus all the vertical downward loads between the point and the reaction.

The horizontal external shear at any point equals the horizontal thrust at the support minus all horizontal forces between the support and the point under consideration. If there are no horizontal forces, the external horizontal shear at any point is constant and equal to the thrust at the support, H.

If V_x and H_x are the external shears, the resultant thrust is

$$R = \sqrt{V_x^2 + H_x^2}. \qquad (108)$$

The resultant thrust, R, is usually inclined at an angle to the normal section. All stresses in an arch are computed on a section normal to the arch axis. To get stresses on the section the inclined thrust, R, must be resolved into a thrust at right angles to the section, i.e., the normal thrust, N, and force parallel to the section, i.e., the shear. The normal thrust, N, causes uniform compression of the section. In arches the effect of the shear S is negligible.

Knowing the direction and magnitude of the resultant thrust, R, and the position of the section, the normal thrust may be found graphically by resolving it into two components, one perpendicular and the other parallel to the section.

Analytically the normal thrust may be found as follows:

Let ϕ = angle of the section with the vertical.

Then the normal thrust consists of the normal thrust due to the vertical shear, V_x, and the horizontal shear, H_x.

$$N_x = V_x \sin \phi - H_x \cos \phi. \qquad (109)$$

In the above equation H_x for vertical loads is a negative value, consequently $-H_x$ is positive.

Eccentric Normal Thrust Replaces Bending Moment and Central Thrust.—Each section of an arch is subjected to a central thrust N_x and a bending moment M_x. As explained on p. 210, a bending moment and thrust may be replaced by a thrust applied eccentrically on the section. The eccentricity measured on normal section equals

$$e_x = \frac{M_x}{N_x}. \qquad (110)$$

If the bending moment M_x is positive and the thrust positive, the location of the eccentric thrust is above the axis.

If the bending moment M_x is negative and the thrust positive the location of the eccentric thrust is below the axis.

Characteristics of Line of Pressure.—Line of pressure in an arch is a curve obtained by connecting the points of application of the eccentric thrust at various sections of the arch. The eccentricities are found as explained in the previous paragraph.

Line of pressure is also a funicular polygon for the forces acting upon the arch drawn with a pole distance equal to the horizontal thrust at the support, H. To draw a line of pressure it is necessary to know the three statically indeterminate values, namely, V_A, H_A and M_A. Having determined these values, a line of pressure is drawn in the following fashion.

First, draw to any convenient scale a force polygon for the loads acting on the arch starting with the force at the extreme left side of the arch.

For vertical loads the force polygon is a vertical line. For inclined loads the sides of the polygon are parallel to the corresponding loads.

Second, on a vertical passing through the starting point of the force polygon, starting from the top, lay out the vertical reaction V_A at the end of which erect a horizontal extending to the left and upon this scale off the value of H_A to the same scale as used for other forces.

Third, determine the eccentricity for a vertical section at the left support by dividing M_A by H_A and lay off this eccentricity on a vertical through the left support. The point thus obtained will be the starting point of the line of pressure. Where the bending moment is positive this point will be above the axis; while for negative bending moment it will be below the axis.

Fourth, from this starting point draw the funicular polygon in the ordinary fashion.

It is obvious that a separate line of pressure corresponds to each type of loading.

The method of drawing the line of pressure is shown in Fig. 217, p. 628 for forces P_1, P_2 and P_3. As evident from the figure the force polygon is drawn first starting with P_1. The left reaction is then scaled at the top of the polygon. The balance is the right reaction. The distance AC at the left support is equal M_A divided by H, and the distance BD is equal to M_B divided by H.

Properties and Use of Line of Pressure.—The line of pressure in conjunction with the force polygon gives all the information necessary for computing stresses at all sections of the arch for the particular loading for which the line of pressure is also drawn. The location and the direction of the eccentric resultant thrust may be taken from the line of pressure and the magnitude of resultant thrust may be found from the force polygon by measuring the ray parallel to the line of pressure in the same scale as was used for drawing the force polygon.

The bending moment may be found as follows: For any vertical section of the arch the bending moment is equal to the horizontal thrust at that point H_x multiplied by the eccentricity e_x, i.e., the vertical distance from the arch axis to the line of pressure. The value of H_x may be taken from the force polygon. If the arch is subjected to vertical forces only, the horizontal thrust is constant throughout the arch. When, in addition, there are horizontal forces (or horizontal components of inclined forces) the value H_x equals the horizontal thrust at the support plus the horizontal forces between the section in question and the support.

For any section normal to the arch axis the bending moment is found by multiplying the normal thrust N_x by the eccentricity of the line of pressure measured along the normal section. The normal thrust N_x, i.e., the component of the resultant thrust acting at right angles to a section normal to the arch axis, may be obtained from the force polygon by drawing at the top of the ray representing the resultant thrust, R, a line at right angles to the section of the arch and at the bottom a line parallel to the section. The length of the line at right angles to the section gives the normal thrust, N_x, and the other line gives the shear, S_x.

In both cases the eccentricity is measured to the same scale as was used in drawing the arch. When the line of pressure is above the axis the bending moment is positive while when below the axis it is negative.

For symmetrical arches and symmetrical loading the line of pressure for the two halves of the arch is symmetrical. In such case it is necessary to draw only the line of pressure for one-half of the arch.

The correctness of the line of pressure for fixed arches may be checked from the following two properties:

First, the line of pressure must intersect the arch axis in at least three points.

Second, when the value of $I_x \cos \phi_x$ is constant the areas between the line of pressure and the arch axis placed above the arch axis must be equal to the areas placed below the arch axis. This follows directly from the requirement that $\int_0^l M_x \frac{ds}{I_x} = 0$ (see p. 593). For constant $I_x \cos \phi_x$ this may be written $\int_0^l M_x dx = 0$. If e_x is the eccentricity at any point, then the bending moment M_x equals He_x. Substituting this in the equation, $\int M_x dx = \int He_x dx = H \int e_x dx = 0$. Finally $\int e_x d_x = 0$.

The differential $e_x dx$ is the area of an infinitely small section of a width equal to dx. The integral of this value is the total area between the arch axis and the line of pressure. Since the sum of all areas must be equal zero, the positive areas above the axis must equal the negative areas below the axis.

Line of Pressure for Single Loads.—The line of pressure for a single load P consists of two straight lines intersecting at P. To draw this line of pressure the same information is required as for multiple loads. To each position of load corresponds a different line of pressure and a different force polygon. The magnitude of the horizontal thrust is different for different location of the load. In comparing the bending moments produced by two equal loads but placed at different points on an arch, not only the eccentricities from the line of pressure but also the horizontal thrusts should be compared because the bending moment depends upon the eccentricities and also upon the horizontal thrust.

Proof that Funicular Polygon Drawn as Described is Line of Pressure.—That a funicular polygon drawn as described above is a line of

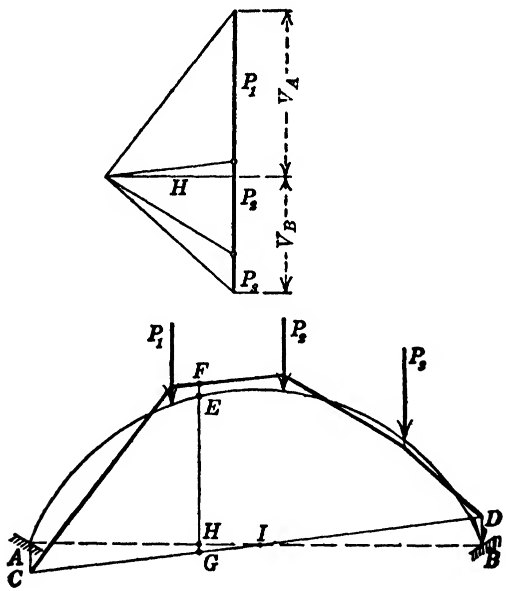


Fig. 217.—Line of Pressure. (*See p.* 628.)

pressure will be proven, if it is shown, that at any point the bending moment M_x on a vertical section of the arch equals the horizontal thrust multiplied by the vertical eccentricity. The equation for bending moment at any point in an arch from Formula (3), p. 581, is

$$M_x = M_A + M_S + \frac{M_B - M_A}{l}x - Hy. \quad . \quad . \quad . \quad (111)$$

In Formula (111) H is considered as a positive value.

In Fig. 217 in which the force polygon and the line of pressure were drawn according to the rules laid out on p. 626, connect the points C and D by a straight line.

From the general property of funicular polygon it follows that at any point the vertical distance from the closing line CD to the polygon multiplied by the pole distance (which in this case is equal to the horizontal thrust, H) is equal to the static bending moment at that section. Thus at point E the distance FG multiplied by the pole distance H is equal to the static bending moment M_S considering the arch as a simply supported beam.

Thus

$$M_S = H \times \overline{FG},$$

consequently,

$$\overline{FG} = \frac{M_S}{H}.$$

From the figure it is evident that the distance $\overline{FG}$ consists of three sections, $\overline{FE}$, $\overline{EH}$ and $\overline{HG}$, so that

$$\overline{FG} = \overline{FE} + \overline{EH} + \overline{HG}.$$

In the above, $\overline{FG} = \frac{M_S}{H}$, $\overline{FE}$ is the eccentricity e_x and $\overline{EH}$ is equal to y. Therefore, $\frac{M_S}{H} = e_x + y + \overline{HG}$ and

$$e_x = \frac{M_S}{H} - \overline{HG} - y. \quad \ldots \ldots \ldots \quad (112)$$

The value of $\overline{HG}$ is found as follows:

From proportions $\overline{HG} = \frac{\overline{AC} + \overline{BD}}{l}x - \overline{AC}$. Since as explained on p. 626 $\overline{AC} = \frac{M_A}{H}$ and $\overline{BD} = -\frac{M_B}{H}$, the value of $\overline{HG} = \frac{M_A - M_B}{H}\frac{x}{l} - \frac{M_A}{H}$. By substituting this in Equation (111) and multiplying by H, we get

$$e_xH = M_A + M_S + \frac{M_B - M_A}{l}x - Hy = M_x.$$

In the above H is considered as a positive value.

Graphical Determination of Line of Pressure.—As explained in previous paragraphs, the location of the line of pressure for a given load is fixed when the three statically indeterminate values H_A, V_A and M_A are known.

The line of pressure also may be drawn without computing the statically indeterminate values when three points of the line of pressure are known or assumed. The effect of fixing of the three points is the same as if the arch were provided at these three points with hinges. Graphical method is used to determine the line pressure due to the dead load (or dead load plus one-half live load) if it is desired to make the arch axis to coincide with this line of pressure for this loading. In such case the selected springing points and the crown point are the fixed points of the line of pressure. Also this method may be used for approximate design of arches when the eccentricities of the line of pressure at the supports and at the crown are found from approximate empirical formulas.

For symmetrical arches and symmetrical loading the line of pressure needs to be drawn for one-half of the arch only, because the other half is symmetrical.

For unsymmetrical loading the whole line of pressure must be determined.

CHAPTER IX

SLOPE DEFLECTION METHOD OF SOLVING STATICALLY INDETERMINATE STRUCTURES

The method of determining statically indeterminate values used for arches and explained on p. 582 can be used for all statically indeterminate structures. Its application to frames consisting of a large number of straight members, however, is too involved. The slope deflection method explained below gives a simpler solution of such problems. It should be understood that the basic principle of this method is identically the same as used for solving arches. The two methods differ only in the manner of application of a common principle. If a problem is solved correctly using the two methods the results must be identical.

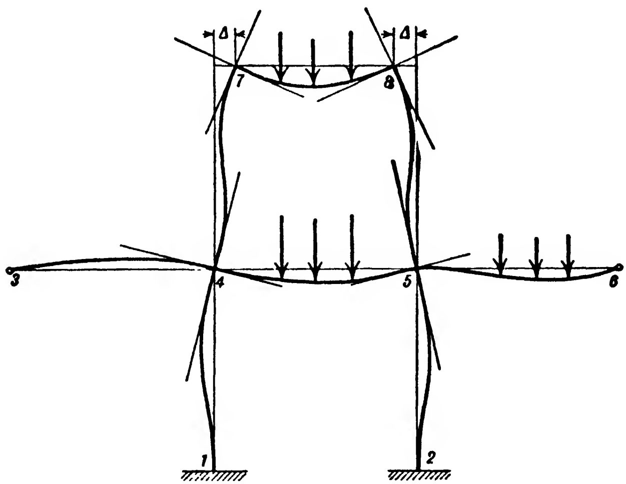


FIG. 218.—Deflection of Frame Consisting of Number of Straight Members. (*See p.* 631.)

Consider a rigid frame consisting of a number of straight members joined rigidly at each intersection. Such frame is shown in Fig. 218, p. 631. When any member of the frame is loaded, all members of the frame, whether loaded or not, deflect. For vertical loading the deflection is largest in members near the loaded spans and diminishes for the members away from the loaded spans.

After deflection the axis of each straight member becomes a curve, the shape of which depends upon the end conditions of the member and also upon whether the member is loaded and, if so, how. Although the slope-deflection method is based upon the investigation of the deflection of the members, the actual deflection is not computed as it becomes eliminated in the final formulas for bending moments.

Types of Deflection Curves.—The type of deflection curve assumed by a member depends upon the end conditions of the member and also upon the loading acting directly upon the member. The various types of deflection curves are shown in Fig. 218, p. 631, and are described below.

1. Member 3–4, hinged at one end and restrained by the structure at the other end, not loaded directly. Its deflection curve is a simple curve. The angles of rotation at the two ends are of opposite sign to each other.

2. Member 5–6, hinged at one end and restrained at the other, and loaded. Deflection curve has one point of contra-flexure. Angles of rotation at the ends may be of same or opposite signs.

3. Member 1–4, fixed at one end and restrained by the structure at the other, not loaded directly. Deflection curve is a reverse curve. At the fixed end the deflection curve is tangent to the original position of the axis of the member so that the angle of rotation there is zero.

4. Member 4–7, restrained at both ends by the structure and not loaded directly. The deflection curve is a reverse curve. Tangents to the deflection curve at both ends of the member are inclined to the original position of the member. The angles of rotation at the ends are of the same sign.

5. Member 4–5, restrained at both ends as before and loaded. Its deflection curve has two points of contra-flexure. Similar curve would be obtained if one of the ends were fixed. However, the angle of rotation at the fixed end would be zero.

Angle of Rotation.—Angle of rotation at any point of the member is the angle between the original position of the axis of the member and the tangent to the deflection curve at that point. It is measured in radians, i.e., arcs of a circle the radius of which is unity. Thus the measure of one degree is $\frac{\pi}{180}$.

The sign of the angle of rotation is positive when the rotation of the tangent, starting from the original position, is from left to right, i.e., clockwise (see Fig. 219, p. 633).

The sign of the angle of rotation is negative when the rotation is from right to left, i.e., counter-clockwise (see Fig. 219, p. 633).

When a joint in which several members meet is rigid, the angles between the axes of the members meeting there must remain the same after deflection as they were before deflection. **Therefore the angles of rotation of all members in a joint must be the same.** This rule will be used in determining the statically indeterminate values.

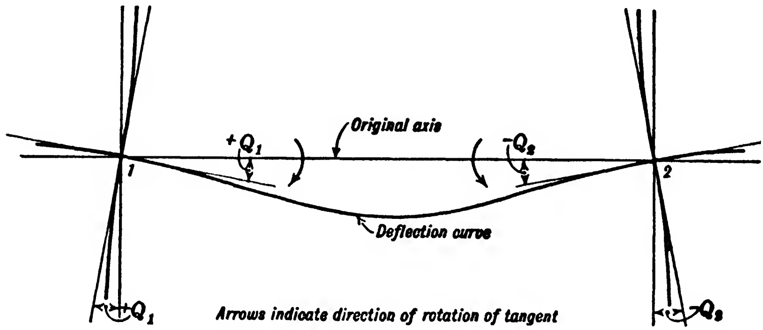


FIG. 219.—Sign of Angles of Rotation. (*See p.* 632.)

Movement of the Joint.—The center of a rigid joint after deflection may remain in the same position as before deflection, or it may move. Thus the joints 3, 4, 5, 6, in Fig. 218, p. 631, remained in the same position as before deflection, while the joint 7 and 8 moved by a dis-distance Δ measured on a line drawn at right angles to the original position of the member.

Notation.—Following notation is used in formula below.

Let l = length of the member;

Q_1 = angle of rotation at left end of a member, i.e., angle between tangent to deflection curve and original position of member, in radians;

Q_2 = angle of rotation at right end, in radians;

M_x = actual bending moment at any point x of the member;

M_s = static bending moment due to the loads at point x;

M_{f_1} = bending moment at left support, considering member as fixed at both ends (see pp. 24 to 28 for constants);

M_{f_2} = bending moment at right support, considering member as fixed at both ends (see pp. 24 to 28 for constants);

I_x = moment of inertia of the member at any point x;

Δ = movement of the joint at right angles to the member;

E = modulus of elasticity.

Basis for Slope Deflection Method.—The slope deflection method is based upon the following two propositions.

1. In a member subjected to bending the difference between the angles of rotation at ends of the member Q_1 and Q_2 is equal to the area of the $\frac{M_x}{EI_x}$ diagram for this member. Mathematically this is expressed by the formula below, using Q_1 and Q_2 with proper signs. (See Fig. 220, p. 634.)

$$Q_1 - Q_2 = \int_0^l \frac{M_x}{EI_x} dx, \qquad (1)$$

$$Q_2 - Q_1 = \int_l^0 \frac{M_x}{EI_x} dx. \qquad (2)$$

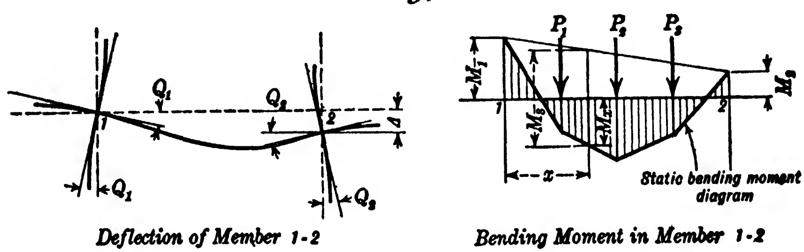


FIG. 220.—Basis for Slope Deflection Method. (See p. 634.)

2. If, in a member subjected to bending, one end of the member moves in relation to the other, then the distance of end 1 from the tangent to the deflection curve at end 2 measured on a line at right angles to the initial position of the member is equal to the static moment about the end 1 of the area of the $\frac{M_x}{EI_x}$ diagram. This is expressed by

$$\Delta - Q_2 l = \int_0^l \frac{M_x}{EI_x} x dx. \qquad (3)$$

(See Fig. 220, p. 634.)

Proposition 1 follows directly from the reasoning on p. 591 relating to the change of the center angle caused by bending moments. Using the signs for angles of rotation on p. 632 the difference $Q_1 - Q_2$ for the condition shown in Fig. 220, p. 634, actually is equal to the numerical sum of the two angles. As evident from the figure, the center angle is also equal to the numerical sum of $Q_1 - Q_2$ and represents the change in center angle for this member. Taking this into account Formula 20, p. 591, is identical with Formula (3), p. 634.

Proposition 2 follows from reasoning relating to the movement of the end of the member caused by bending moment, explained on p. 587.

Basic Formulas for Slope-deflection Method.—Using propositions 1 and 2 it is possible to express the bending moments at each end of a member in terms of the angles of rotation Q_1, Q_2, the movement of the end Δ and the bending moments due to the load M_{f_1} and M_{f_2}. Assume that the moment of inertia for the span under consideration is constant. Then I_x becomes I and may be taken before the integration sign.

Basic Formulas for Slope-deflection Method,[1]

$$M_1 = 2E\frac{I}{l}\left(2Q_1 + Q_2 - \frac{3\Delta}{l}\right) + M_{f_1}. \qquad (4)$$

$$M_2 = 2E\frac{I}{l}\left(2Q_2 + Q_1 - \frac{3\Delta}{l}\right) + M_{f_2}, \qquad (5)$$

where M_{f_1} and M_{f_2} are bending moments at supports for beams fixed at both ends and are given by the following formulas:

$$M_{f1} = -\frac{2}{l^2}\left(2l\int_0^l M_s dx - 3\int_0^l M_s x dx\right). \qquad (6)$$

$$M_{f2} = \frac{2}{l^2}\left(3\int_0^l M_s x dx - l\int_0^l M_s dx\right). \qquad (7)$$

(See pp. 24 to 28 for values of M_{f_1} and M_{f_2} for special loadings.) These are general formulas. For special end conditions of the members special formulas may be developed as given below.

[1] These basic equations are derived as follows:

The bending moment at any point may be expressed in terms of the bending moments at the ends and the static bending moment. Referring to Fig. 220 and using signs as explained on page 370 the bending moment at any point M_x is

$$M_x = M_s - (-M_1) + \frac{-M_1 - M_2}{l}x.$$

Substituting this in integrals of Equations (1) and (3) and solving gives

$$\int_0^l M_x dx = \int_0^l \left(M_s + M_1 + \frac{-M_1 - M_2}{l}x\right)dx = \tfrac{1}{2}l(M_1 - M_2) + \int_0^l M_s dx.$$

$$\int_0^l M_x x dx = \int_0^l \left(M_s + M_1 + \frac{-M_1 - M_2}{l}x\right)x dx = \tfrac{1}{3}l^2(\tfrac{1}{2}M_1 - M_2) + \int_0^l M_s x dx.$$

Substitute these values in Equation (1) and (3),

$$Q_1 - Q_2 = \frac{1}{EI}\left[\frac{l}{2}(M_1 - M_2) + \int_0^l M_s dx\right].$$

$$\Delta - Q_2 l = \frac{1}{EI}\left[\frac{l^2}{3}(\tfrac{1}{2}M_1 - M_2) + \int_0^l M_s x dx\right].$$

(*Note continued on next page*)

Possible Conditions at the Ends of Members.—There are four conditions possible at the ends of the members. They are illustrated in Fig. 221, p. 636.

Condition 1. Both ends are restrained by the other members of the structure. Such condition exists in the member 2–3. The angles at both ends are determined by considering other members of the structure. Formulas (8) to (11) apply.

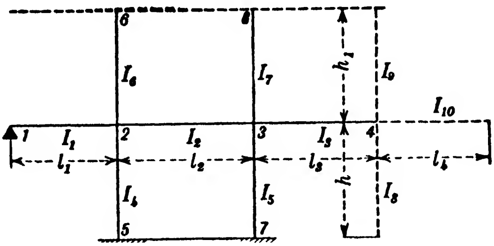


FIG. 221.—Conditions of Members at Support. (*See p.* 636.)

Condition 2. One end fixed, other end restrained by other members of the structure as in members 2–5 and 3–7. The angle at the fixed end is zero. Formulas (12) to (19) apply.

Condition 3. One end free or hinged, other end restrained by other members of the structure as in member 1–2. The bending moment at the free end is zero. Formulas (20) to (25) apply.

Solving these equations for M_1 and M_2,

$$M_1 = 2E\frac{I}{l}\left[2Q_1 + Q_2 - 3\frac{\Delta}{l}\right] - \frac{2}{l}\left(-\frac{3}{l}\int_0^l M_s x\,dx + 2\int_0^l M_s\,dx\right).$$

$$M_2 = 2E\frac{I}{l}\left[Q_1 + 2Q_2 - 3\frac{\Delta}{l}\right] + \frac{2}{l}\left(\frac{3}{l}\int_0^l M_s x\,dx - \int_0^l M_s\,dx\right).$$

When the member is fixed at both ends the deflection curve is tangent to the original position of the member. Therefore, $Q_1 = Q_2 = 0$, and the bending moment for fixed beams fixed at both ends become:

$$M_1 = M_{f_1} = -\frac{2}{l}\left(-\frac{3}{l}\int_0^l M_s x\,dx + 2\int_0^l M_s\,dx\right).$$

$$M_2 = M_{f_2} = \frac{2}{l}\left(\frac{3}{l}\int_0^l M_s x\,dx - \int_0^l M_s\,dx\right).$$

These values may be substituted in the general equations for the integrals, which results in the final formulas.

Condition 4. One end restrained by other members of the structure, other end restrained by members outside of the structure under consideration. Thus in member 3–4 end 3 is restrained by the structure while end 4 is restrained by three members shown by dash lines not considered a part of the structure. Similarly ends 6 and 8 are restrained by two members not considered a part of the structure.

Condition 4 occurs when a substitute structure is used instead of the complete structure. The omitted members exert a restraint upon the end of the substitute structure which is intermediate between a free condition and fixed condition. The effect of these restraining members may be taken into account without incorporating them into the substitute structure by using Formula (30), p. 640.

Formulas for Special End Conditions.—The basic Formulas (4) and (5) assume following shape for special end conditions.

Notation.

Let E = modulus of elasticity;

I = moment of inertia of the member;

M_1 = bending moment at left end;

M_2 = bending moment at right end;

M_{f_1} = bending moment at left end, considering member fixed at both supports (see pp. 24 to 28 for constants);

M_{f_2} = bending moment at right end, considering member fixed at both supports (see pp. 24 to 28 for constants);

M_f = bending moment at supports for symmetrical load, considering member fixed at both supports (see pp. 24 to 28 for constants);

l = length of member;

Q_1 = angle of rotation at left support, in radians;

Q_2 = angle of rotation at right support, in radians;

Δ = movement of one end of the member at right angles to original position.

Condition 1. Both Ends Restrained.

(*a*) Both ends remain on same level,

$$M_1 = 2E\frac{I}{l}(2Q_1 + Q_2) + M_{f_1}, \quad \text{(8)}$$

$$M_2 = 2E\frac{I}{l}(Q_1 + 2Q_2) + M_{f_2}. \quad \text{(9)}$$

(*b*) Right end moved by distance Δ,

$$M_1 = 2E\frac{I}{l}\left(2Q_1 + Q_2 - 3\frac{\Delta}{l}\right) + M_{f_1}, \quad \ldots \ldots \quad (10)$$

$$M_2 = 2E\frac{I}{l}\left(Q_1 + 2Q_2 - 3\frac{\Delta}{l}\right) + M_{f_2}. \quad \ldots \ldots \quad (11)$$

Condition 2. One End Restrained, Other End Fixed.

(*a*) Both ends remain on same level:

Right end fixed,

$$M_1 = 4E\frac{I}{l}Q_1 + M_{f_1} \quad \text{at restrained end,} \quad \ldots \quad (12)$$

$$M_2 = \tfrac{1}{2}(M_1 - M_{f_1}) + M_{f_2} \quad \text{at fixed end.} \quad \ldots \quad (13)$$

Left end fixed,

$$M_1 = \tfrac{1}{2}(M_2 - M_{f_2}) + M_{f_1} \quad \text{at fixed end,} \quad \ldots \quad (14)$$

$$M_2 = 4E\frac{I}{l}Q_2 + M_{f_2} \quad \text{at restrained end.} \quad \ldots \quad (15)$$

(*b*) One end moved by distance Δ,

Right end fixed

$$M_1 = E\frac{I}{l}\left(Q_1 - 3\frac{\Delta}{l}\right) + M_{f_1} \quad \text{at restrained end,} \quad \ldots \quad (16)$$

$$M_2 = \tfrac{1}{2}(M_1 - M_{f_1}) + M_{f_2} \quad \text{at fixed end.} \quad \ldots \quad (17)$$

Left end fixed,

$$M_1 = \tfrac{1}{2}(M_2 - M_{f_2}) + M_{f_1} \quad \text{at fixed end,} \quad \ldots \quad (18)$$

$$M_2 = 4E\frac{I}{l}Q_2 + M_{f_2} \quad \text{at restrained end.} \quad \ldots \quad (19)$$

Condition 3. One End Restrained, Other End Hinged.

(*a*) Both ends remain on same level,

Right end hinged. $M_2 = 0$,

$$M_1 = 3E\frac{I}{l}Q_1 + (M_{f_1} - \tfrac{1}{2}M_{f_2}). \quad \ldots \ldots \quad (20)$$

Left end hinged. $M_1 = 0$,

$$M_2 = 3E\frac{I}{l}Q_2 + (M_{f_2} - \tfrac{1}{2}M_{f_1}). \quad \ldots \ldots \quad (21)$$

(b) One end moved relative to the other end by distance Δ,

Right end hinged. $M_2 = 0$,

$$M_1 = 3E\frac{I}{l}\left(Q_1 - \frac{\Delta}{l}\right) + (M_{f_1} - \tfrac{1}{2}M_{f_2}). \quad \quad (22)$$

Left end hinged. $M_1 = 0$,

$$M_2 = 3E\frac{I}{l}\left(Q_2 - \frac{\Delta}{l}\right) + (M_{f_2} - \tfrac{1}{2}M_{f_1}). \quad \quad (23)$$

For symmetrical loads,

$$M_{f_1} = -M_{f_2} = M_f. \quad \quad (24)$$

and

$$M_{f_1} - \tfrac{1}{2}M_{f_2} = -(M_{f_2} - \tfrac{1}{2}M_{f_1}) = 1\tfrac{1}{2}M_f. \quad \quad (25)$$

Condition 4. One End Restrained by Members Outside the Frame, Other End Restrained by the Members of the Frame. (See 222, p. 641.)

Assume that a member of a frame at one end is restrained by the frame itself and at the other end by three members not forming a part of the frame, with length l_1, h_1 and h_2 and moments of inertia I_1, I_2 and I_3, respectively. The length of the member is l and its moment of inertia I.

This case occurs when the frame under consideration is a substitute frame for a more complicated structure. The ends of the substitute frame are then restrained by the members of the structure which are outside of the substitute frame. In Fig. 222, p. 641, the member AB, which is a part of the frame shown by heavy lines is restrained at B by the frame and at A by the three members outside of the frame shown by dash lines.

Right end restrained,

$$M_1 = (6 - c)\,E\frac{I}{l}Q_1 + \frac{c}{2}M_{f_1}, \quad \quad (26)$$

$$M_2 = \frac{2(3-c)}{6-c}M_1 + \frac{6(3-c)}{6-c}M_{f_2}. \quad \quad (27)$$

Left end restrained,

$$M_1 = \frac{2(3-c)}{6-c}M_2 + \frac{6(3-c)}{6-c}M_{f_1}, \quad \quad (28)$$

$$M_2 = (6 - c)E\frac{I}{l}Q_2 + \frac{c}{2}M_{f_2}, \quad \quad (29)$$

where

$$c = 2 + \frac{4}{4 + a_1\frac{I_1 l}{I l_1} + a_2\frac{I_2 l}{I h_1} + a_3\frac{I_3 l}{I h_2}}. \quad . \quad . \quad . \quad . \quad (30)$$

For members with hinged ends, a_1, a_2 and a_3 each equals 3; for members with partly fixed ends, a_1, a_2 and a_3 equals 3.5; for members with fixed ends a_1, a_2 and a_3 equals 4.

Signs of Bending Moments, Angles and Movements of Ends.—In the above formulas following signs are used:

Bending Moments.—In applying the slope deflection method it is necessary to use different signs for bending moments than ordinarily used in reinforced concrete design. The customary signs used in concrete design depend upon the location of the tensile zone, whether near the upper or lower surface of the member. In the slope-deflection method, on the other hand, the sign depends upon the direction in which resisting forces produced by the bending moment tend to turn the member.

The bending moments are positive (+) when the resisting stresses tend to turn the member, or a portion of the member under consideration, clockwise, i.e., from left to right.

According to this sign method in a span of a continuous beam subjected to downward loads the bending moment at the left support is negative and the bending moment at the right support is positive. (In ordinary method both bending moments would be negative.)

In a rigid right-angle frame subjected to downward loads at the left corner the sign of the bending moment in the beam is negative while the bending moment in the column is positive. At the right corner the signs are reverse. (In ordinary method all bending moments would be minus.)

Sign of Angles of Rotation.—The angle Q is called positive when the tangent to the deflection curve, starting from its initial position, rotates clockwise.

The angle Q is negative when the tangent to the elastic curve, starting from its initial position, rotates counter-clockwise.

For example, in Fig. 218 the angles at points 4 and 7 are positive while the angles at points 5 and 8 are negative.

Sign of Movement of the Ends Δ.—When the movement of the end results from a clockwise rotation of the member from its original position, the value of Δ is positive. When it is due to a counter-clockwise rotation, it is negative.

For example, the movement of point 7 is positive while point 8 is negative.

Sign of Bending Moments M_{f_1} and M_{f_2}.—When the loads act downward the bending moments M_{f_1} and M_{f_2} have the same sign as used in Equations (8) to (29). For loads acting upward the sign of M_{f_1} and M_{f_2} in Formulas (8) to (29) would have to be reversed.

For horizontal pressure acting from the left upon a vertical or inclined member the upper end of this member corresponds to the right end of the beam and the lower end to the left end.

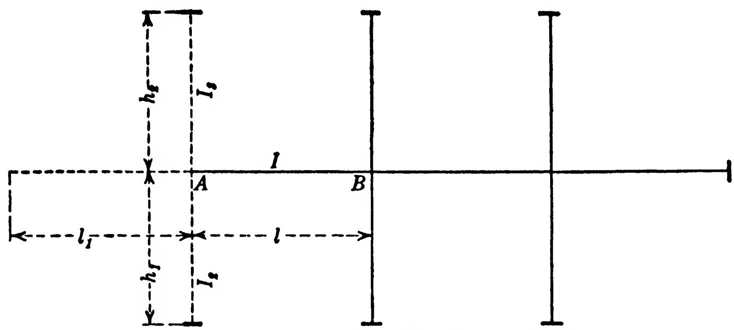


FIG. 222.—End of Member of a Frame Restrained by Members Outside of the Frame. (*See p.* 640.)

How to Apply Slope Deflection Method.—The method is applied in the following manner:

1. Equations for bending moments at each end are set down, using one of the Formulas (8) to (29). For each member there are two unknown angles of rotation, one at each end. The actual number of unknown angles in the whole structure is decreased, however, by the fact that all members meeting at a joint have a common angle of rotation. The total number of unknown angles, therefore, is equal to the number of joints.

2. Equations are then formed for each joint based on the requirement that for equlibrium the sum of bending moments for all members meeting a joint is equal zero. This means also that of the members meeting at a joint some are subjected to bending moments turning to the left while the others are subjected to bending moments turning to right. The sum of bending moments turning to the right must be equal to the sum of the bending moments turning to the left.

3. When no movement Δ occurs at any joint due to the deflection of

the frame, the joint equations are sufficient to find all the unknown angles of rotation Q_1 because the number of unknown angles is equal to the number of joints and the number of joint equations is also equal to the number of the joints.

When any joints move, due to the deflection of the frame, the values of Δ so created are additional unknown quantities. To find these values, additional equations are written from the requirement that the frame as a whole must be in equilibrium.

4. After all equations are prepared the values of the unknown angles are found by solving the simultaneous equations in the well-known

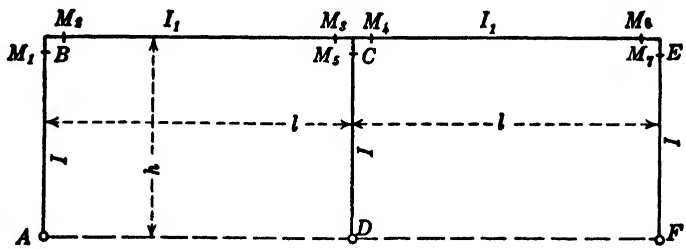


FIG. 223.—Three-legged, Two-span Frame. (See p. 642.)

manner. When solved, the values of Q are expressed in terms of the known M_{f_1} and M_{f_2}.

5. Finally, bending moments at all joints are obtained by substituting in proper equations the proper values of Q.

Example of Application of the Slope Deflection Method.—To illustrate the use of the slope deflection method, formulas are determined for a three-legged, two-span frame shown in Fig. 223, p. 642.

The three unknown angles at the three joints are Q_B at joint B, Q_C at joint C and Q_E at joint E.

An additional unknown quantity is the horizontal movement of the joints B, C and E called Δ. This movement is included only in formulas for columns. The number of unknown values therefore is four. Four equations are required to solve the problem.

Bending Moments at the Ends of Members:

Bending moment in beams,

$$M_2 = 2E\frac{I_1}{l}(2Q_B + Q_C) + M_{f_1}. \quad \dots \quad (31)$$

$$M_3 = 2E\frac{I_1}{l}(2Q_C + Q_B) + M_{f_2}. \quad \dots \quad (32)$$

$$M_4 = 2E\frac{I_1}{l}(2Q_C + Q_E). \quad \ldots \quad (33)$$

$$M_6 = 2E\frac{I_1}{l}(2Q_E + Q_C). \quad \ldots \quad (34)$$

Bending moment in columns,

$$M_1 = 3E\frac{I}{h}\left(Q_B - \frac{\Delta}{h}\right). \quad \ldots \quad (35)$$

$$M_5 = 3E\frac{I}{h}\left(Q_C - \frac{\Delta}{h}\right). \quad \ldots \quad (36)$$

$$M_7 = 3E\frac{I}{h}\left(Q_E - \frac{\Delta}{h}\right). \quad \ldots \quad (37)$$

Equations for Joints,

(1) $M_1 + M_2 = 0$ for joint B. . . . . (38)

(2) $M_3 + M_4 + M_5 = 0$ for joint C. . . . . (39)

(3) $M_6 + M_7 = 0$ for joint E. . . . . (40)

Equation for structure as a Whole,[2]

(4) $\frac{M_1}{h} + \frac{M_5}{h} + \frac{M_7}{h} = 0.$ . . . . (41)

By substituting values for the bending moments, following equations are obtained.

(1) $3E\frac{I}{h}\left(Q_B - \frac{\Delta}{h}\right) + 2E\frac{I_1}{l}(2Q_B + Q_C) + M_{f_1} = 0.$ . . . . (42)

(2) $2E\frac{I_1}{l}(2Q_C + Q_B) + M_{f_2} + 2E\frac{I_1}{l}(2Q_C + Q_E) + 3E\frac{I}{h}\left(Q_C - \frac{\Delta}{h}\right) = 0.$ (43)

(3) $2E\frac{I_1}{l}(2Q_E + Q_C) + 3E\frac{I}{h}\left(Q_E - \frac{\Delta}{h}\right) = 0.$ . . . . (44)

(4) $\frac{I}{h}\left[3E\frac{I}{h}\left(Q_B - \frac{\Delta}{h}\right) + 3E\frac{I}{h}\left(Q_C - \frac{\Delta}{h}\right) + 3E\frac{I}{h}\left(Q_E - \frac{\Delta}{h}\right)\right] = 0.$ . . (45)

[2] $\frac{M_1}{h}$, $\frac{M_5}{h}$ and $\frac{M_7}{h}$ are equal to the horizontal thrust at the bottom of the legs For equilibrium the sum of horizontal thrust must be equal zero.

To simplify, call $Q_B = X$, $Q_C = Y$, $Q_E = Z$ and $E\frac{I}{h} = k$, $E\frac{I_1}{l} = k_1$. The equations then become

Final Equations for Determining X, Y, Z and Δ,

$$(1) \qquad (4k_1 + 3k)X + 2k_1Y - 3\frac{k}{h}\Delta = -M_{f_1}. \quad . \quad . \quad (46)$$

$$(2) \qquad 2k_1X + (8k_1 + 3k)Y + 2k_1Z - 3\frac{k}{h}\Delta = -M_{f_2}. \quad . \quad . \quad (47)$$

$$(3) \qquad 2k_1Y + (4k_1 + 3k)Z - 3\frac{k}{h}\Delta = 0. \quad . \quad . \quad . \quad . \quad (48)$$

$$(4) \qquad X + Y + Z - 3\frac{1}{h}\Delta = 0. \quad . \quad . \quad . \quad . \quad (49)$$

The above four equations must be solved for X, Y, Z and Δ. After the values of the unknown are found the bending moments M_1 to M_7 are obtained by means of the Equations (31) to (37).

Derivation of Three-Moment Equation for Continuous Beams.— The three-moment equation for solving continuous beams may be developed by means of the slope-deflection method.

Consider two spans of a continuous beam consisting of any number of spans as shown in Fig. 224, p. 645.

Let l_r = span length of the rth span;
l_{r+1} = span length of the $r + 1$th span;
M_r = bending moment at left support, rth span;
M_{r+1} = bending moment at left support, $r + 1$th span;[3]
M_{r+2} = bending moment at left support, $r + 2$th span;[3]
M_{sr} = static bending moment at any point of rth span;
M_{sr+1} = static bending moment at any point of $r + 1$th span;
M_{f_1r} = bending moment at left support, rth span
M_{f_2r} = bending moment at right support, rth span
$M_{f_1(r+1)}$ = bending moment at left support, $r + 1$th span;
$M_{f_2(r+1)}$ = bending moment at right support, $r + 1$th span;
} Considering the spans as fixed at both ends.

[3] At support $r + 1$ the bending moments at both sides are equal but of opposite sign. It is positive at the right.

At support $r + 2$ the bending moment is positive at the right of the support and negative at the left.

Q_r = angle of rotation at rth support;
Q_{r+1} = angle of rotation at $r + 1$th support;
Q_{r+2} = angle of rotation at $r + 2$th support;
E = modulus of elasticity of concrete.

As explained on p. 637, it is possible to establish following relations between the angles of rotation and the bending moment.

First span [3] (for signs see footnote),

$$M_r = 2E\frac{I_r}{l_r}(2Q_r + Q_{r+1}) + M_{f_1 r}. \qquad (50)$$

$$-M_{r+1} = 2E\frac{I_r}{l_r}(2Q_{r+1} + Q_r) + M_{f_2 r}. \qquad (51)$$

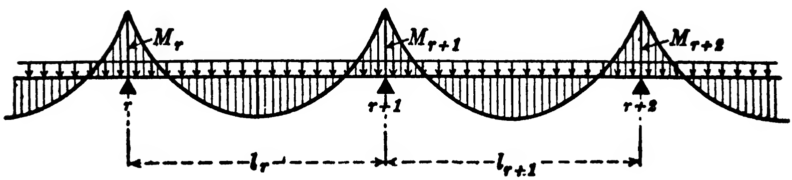


FIG. 224.—Two Spans of a Continuous Beam. (*See p.* 644.)

Second span,[3]

$$M_{r+1} = 2E\frac{I_{r+1}}{l_{r+1}}(2Q_{r+1} + Q_{r+2}) + M_{f1(r+1)}. \qquad (52)$$

$$-M_{r+2} = 2E\frac{I_{r+1}}{l_{r+1}}(2Q_{r+2} + Q_{r+1}) + M_{f2(r+1)}. \qquad (53)$$

Multiply Equation (51) by 2 and subtract from the result to Equation (50). In this way Q_r is eliminated.

$$-2M_{r+1} - M_r = 6E\frac{I_r}{l_r}Q_{r+1} + (M_{f_1 r} - 2M_{f_2 r}). \qquad (54)$$

In the same manner eliminate Q_{r+2} from Equations (52) and (53).

$$M_{r+2} + 2M_{r+1} = 6E\frac{I_{r+1}}{l_{r+1}}Q_{r+1} + (2M_{f_1(r+1)} - M_{f_2(r+1)}). \qquad (55)$$

[3] See note, p. 644.

Multiply Equation (55) by $\frac{I_r l_{r+1}}{I_{r+1} l_r}$ and subtract from the result Equation (54).

$$M_r + 2M_{r+1}\left(1 + \frac{I_r l_{r+1}}{I_{r+1} l_r}\right) + \frac{I_r l_{r+1}}{I_{r+1} l_r} M_{r+2}$$
$$= + \left[(M_{f_1r} - 2M_{f_2r}) + \frac{I_r l_{r+1}}{I_{r+1} l_r}(2M_{f_1(r+1)} - M_{f_2(r+1)})\right]. \quad (56)$$

Finally, since[4] $M_{f_1r} - 2M_{f_2r} = -\frac{6}{l_r^2}\int_0^{l_r} M_s x dx$ and $2M_{f_1(r+1)} - M_{f_2(r+1)} = -\frac{6}{l^2_{r+1}}\int_0^{l_{r+1}} M_s(l_{r+1} - x)dx$,

Three-moment Equation,

$$M_r + 2M_{r+1}\left(1 + \frac{I_r l_{r+1}}{I_{r+1} l_r}\right) + \frac{I_r l_{r+1}}{I_{r+1} l_r} M_{r+2}$$
$$= -6\left[\frac{1}{l_r^2}\int_0^{l_r} M_s x dx + \frac{1}{l^2_{r+1}} \frac{I_r l_{r+1}}{I_{r+1} l_r}\int_0^{l_{r+1}} M_s(l_{r+1} - x)dx\right]. \quad (57)$$

When the moments of inertia of both spans are equal the equation changes to the form most commonly known

Three-moment Equation, Constant Moment of Inertia,

$$M_r l_r + 2M_{r+1}(l_r + l_{r+1}) + M_{r+2} l_{r+1}$$
$$= -6\left[\frac{1}{l_r}\int_0^{l_r} M_{sr} x dx + \frac{1}{l_{r+1}}\int_0^{l_{r+1}} M_{s(r+1)}(l_{r+1} - x)dx\right]. \quad (58)$$

[4] Substitute for M_{f1r}, M_{f2r}, $M_{f1(r+1)}$ and $M_{f2(r+1)}$, values from Equations (6) and (7) and combine.

CHAPTER X

DIAGRAMS

This chapter gives diagrams required for the design of sections subjected to direct stress and bending moments and also diagrams required for approximate designs of fixed arches.

DIAGRAMS FOR DIRECT STRESS AND BENDING

The diagrams required for the design of sections subjected to direct stress and bending moments are described in Chapter II. They are:

PLAIN CONCRETE SECTIONS

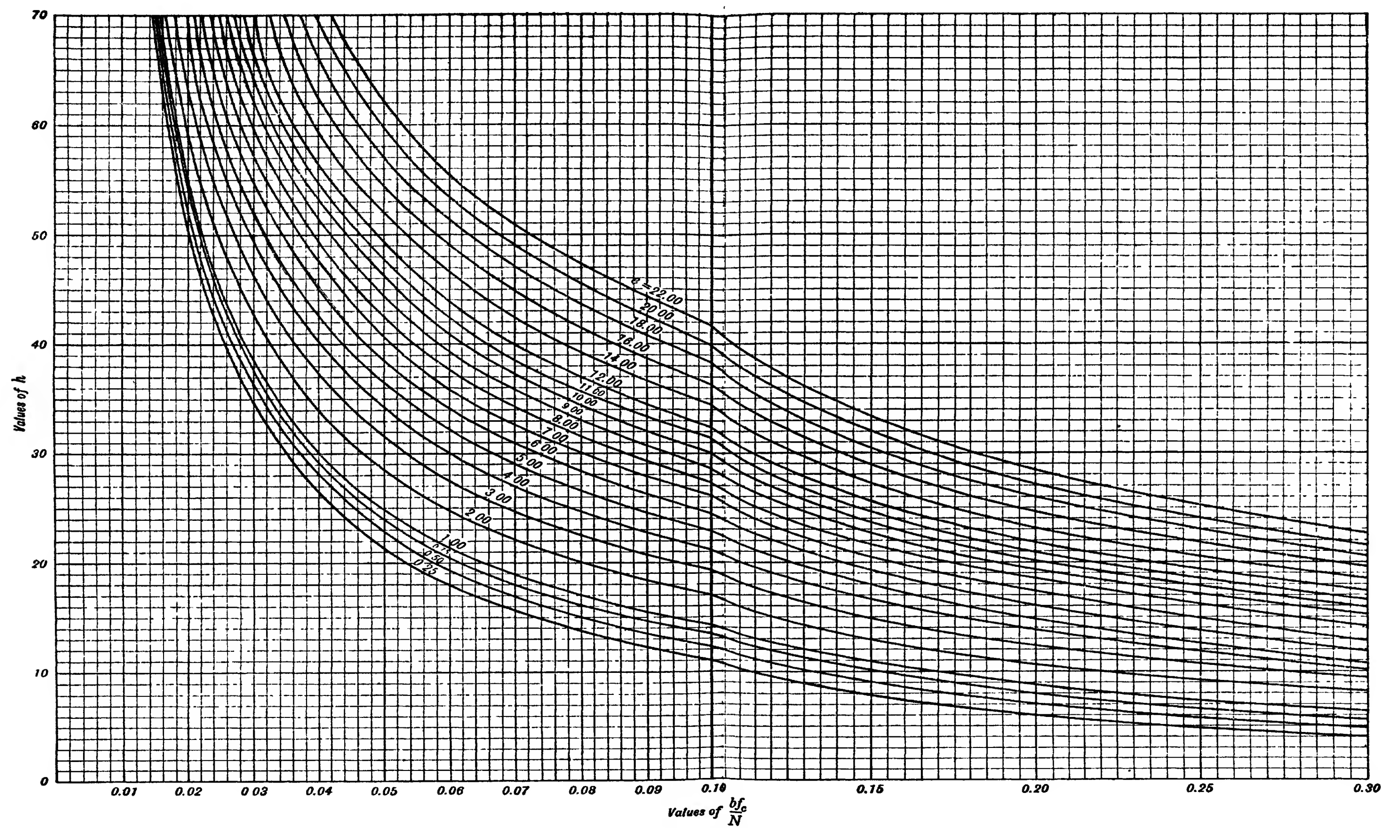
REINFORCED CONCRETE SECTION SUBJECTED TO BENDING ONLY

REINFORCED CONCRETE SECTION. WHOLE SECTION IN COMPRESSION

SYMMETRICALLY REINFORCED CONCRETE SECTION
ONE FACE IN TENSION

UNSYMMETRICALLY REINFORCED CONCRETE SECTION
ONE FACE IN TENSION

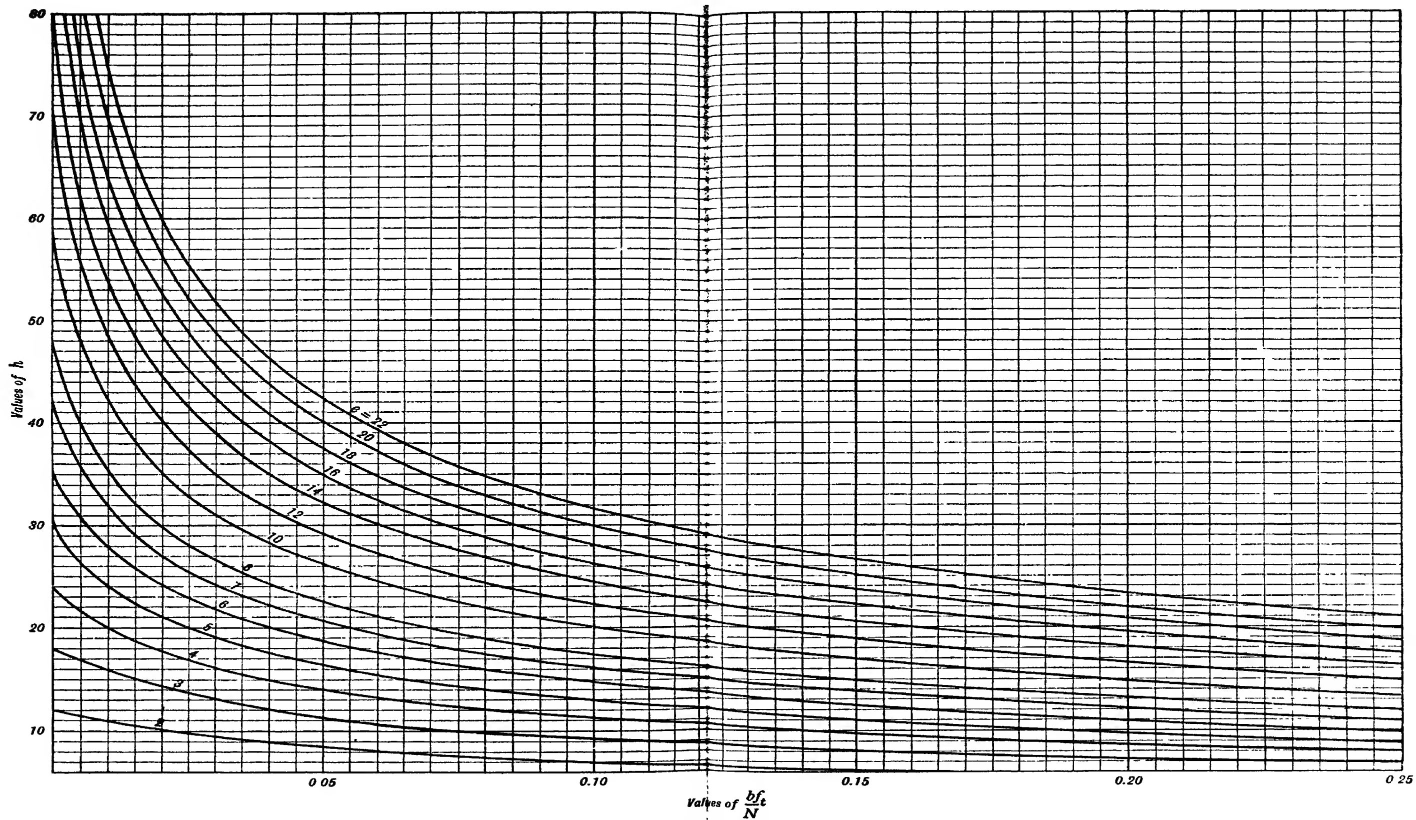
SECTION WITH TENSION STEEL ONLY
ONE FACE IN TENSION



For example, see p. 247.

DIAGRAM 1.—Depth of Unreinforced Sections for Known Eccentricity e and Ratio $\frac{bf_c}{N}$. Upper Limit. (*See p.* 215.)

(*To face page* 648)



For example, see p. 247. All values of $\frac{bf_t}{N}$ in this diagram are negative.

DIAGRAM 2.—Depth of Unreinforced Sections for Known Eccentricity e and Ratio $-\frac{bf_t}{N}$. Lower Limit. (*See p.* 215.)

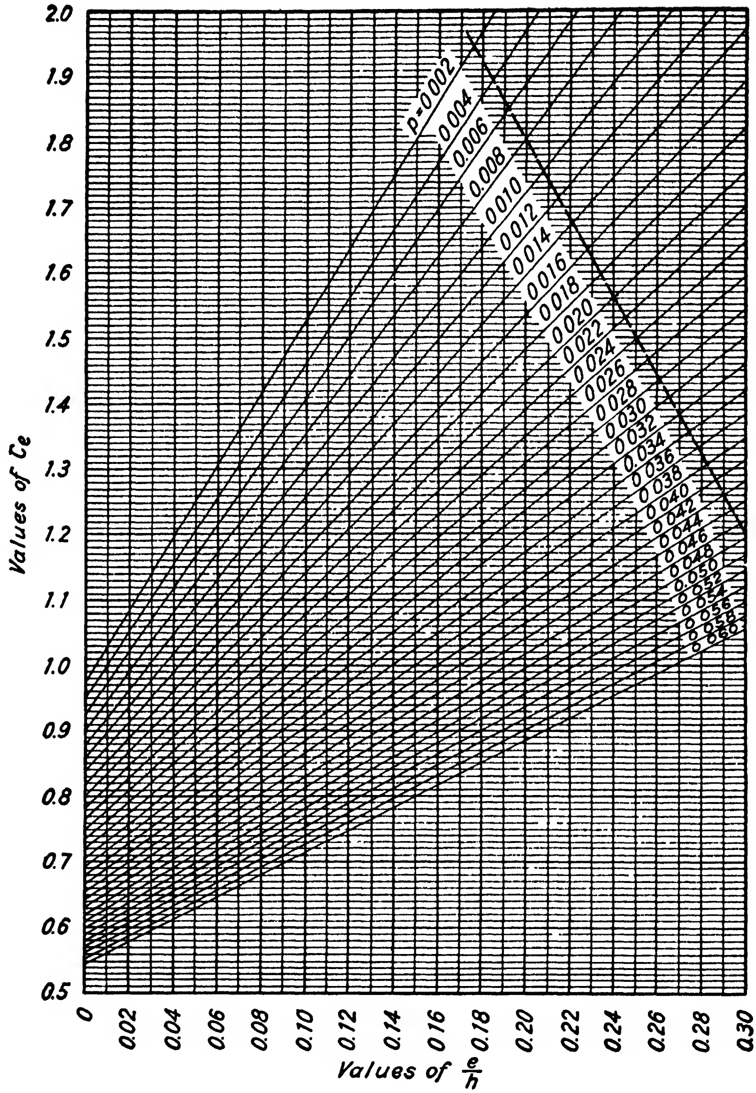
(*To face page* 648)

Table 1.—Constants for Rectangular Beams and Slabs

To be used in formulas for Depth of Beam, $d = C\sqrt{\frac{M}{b}}$ or $d = \sqrt{\frac{M}{Rb}}$ and Depth of Slab, $d = C_1\sqrt{M}$; in formulas for Moment of Resistance, $M = \frac{bd^2}{C^2}$, or $M = Rbd^2$; in formula for Area of Reinforcement, $A_s = pbd$, or $A_s = \frac{M}{jdf_s}$.

Ratio of Moduli of Steel to Concrete, $n = 15$.

Working Strength of Steel, Lb. per Sq. In.	Working Strength of Concrete, Lb. per Sq. In.	Ratio Depth of Neutral Axis to Depth of Steel.	Ratio of Moment Arm to Depth of Steel $\left(1-\frac{k}{3}\right)$	Ratio Area of Steel to Beam Above Steel.	Constants. For Beams.		For Slabs.
f_s	f_c	k	j	p	C	R	C_1
14 000	500	0.349	0.884	0.0062	0.114	77.1	0 0329
	550	0.372	0.876	0 0073	0.106	89 4	0.0306
	600	0.392	0.869	0 0084	0.099	101 9	0 0286
	650	0.411	0.863	0 0095	0.093	115.2	0.0268
	700	0.428	0.857	0 0107	0 088	128 6	0 0254
	750	0 446	0.851	0 0119	0 084	142 2	0.0243
	800	0.462	0.846	0 0132	0.080	156.4	0.0231
	850	0.477	0.841	0 0145	0 077	170.4	0.0222
16 000	500	0.319	0.894	0 0050	0.119	71.3	0 0341
	550	0.339	0.887	0 0058	0 110	83.0	0.0318
	600	0 360	0.881	0 0067	0.103	95 0	0 0297
	650	0.378	0.874	0.0077	0.096	107 5	0 0277
	700	0.397	0.868	0 0087	0.091	120 4	0 0263
	750	0 414	0.862	0 0097	0 087	133 5	0 0251
	800	0 429	0.857	0 0107	0 083	146 7	0 0240
	850	0 444	0.852	0 0118	0.079	160.4	0 0228
	900	0.458	0.847	0 0129	0.076	174.5	0.0219
18 000	500	0 294	0.902	0 0041	0 123	66.3	0.0355
	550	0 314	0.895	0 0048	0.114	77 1	0 0329
	600	0 333	0.889	0 0056	0.106	88.9	0 0306
	650	0 351	0.883	0.0063	0.100	100.8	0.0289
	700	0 369	0.877	0 0072	0 094	112.9	0 0271
	750	0 385	0.872	0 0080	0.089	125.9	0 0257
	800	0.400	0.867	0 0089	0.085	138 7	0 0245
	850	0 415	0 862	0 0098	0.081	151 9	0 0234
	900	0.429	0.857	0 0107	0.078	165 4	0.0225
20 000	500	0 272	0 909	0 0034	0 127	62 0	0 0367
	550	0 292	0 903	0 0040	0 118	72 5	0 0341
	600	0 311	0 896	0 0047	0 110	83 5	0 0318
	650	0 328	0 891	0 0053	0 103	94.9	0 0297
	700	0 344	0 885	0 0060	0 097	106 5	0 0280
	750	0.359	0 880	0 0068	0 092	118.8	0 0266
	800	0 374	0 875	0 0075	0 087	131 3	0 0251
	850	0.389	0 870	0 0083	0 083	143 8	0 0240
	900	0 403	0 866	0 0091	0 080	157.0	0 0231

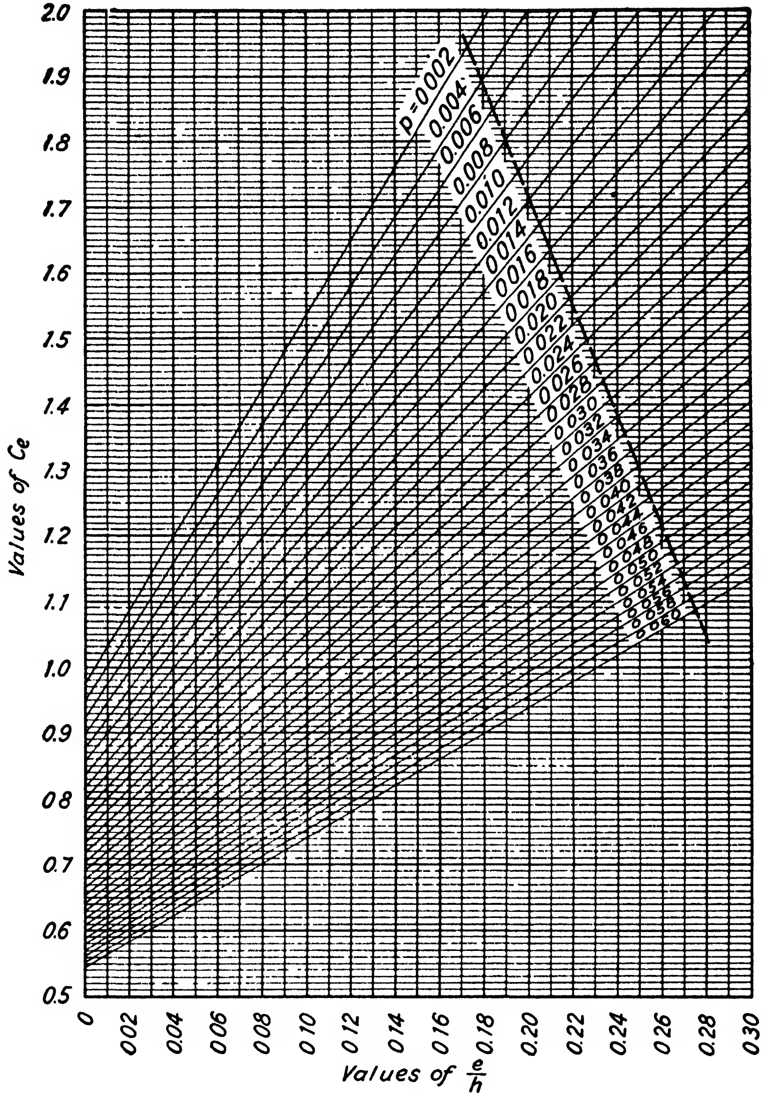


$2a = h$, $n = 15$. See p. 221 for different n.

Use in Formula for Max. Compression Stress, $f_c = C_e \dfrac{N}{bh}$. (*See p.* 219.)

For example, see p. 249.

DIAGRAM 3.—Constants C_e for Symmetrically Reinforced Sections. Whole Section in Compression. (*See p.* 219.)

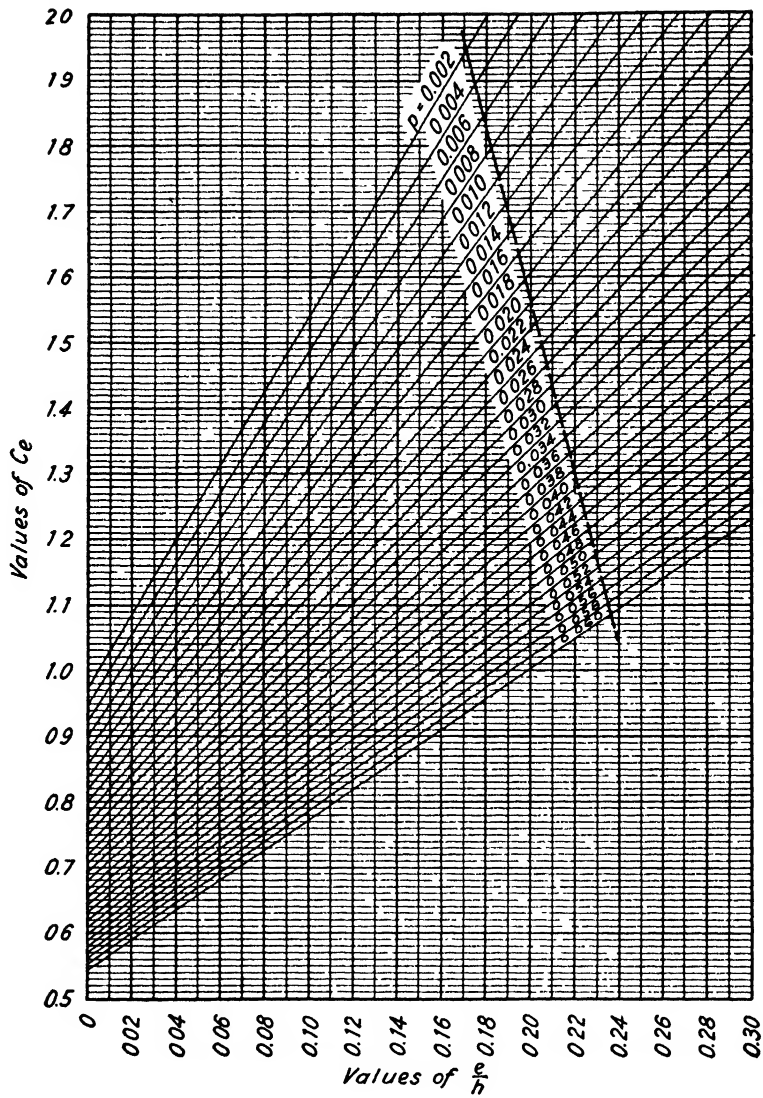


$2a = 0.9h$, $n = 15$. See p. 221 for different n.

Use in Formula for Max. Compression Stress, $f_c = C_e \frac{N}{bh}$. (*See p.* 219.)

For example, see p. 249.

DIAGRAM 4.—Constants C_e for Symmetrically Reinforced Sections. Whole Section in Compression. (*See p.* 219.)

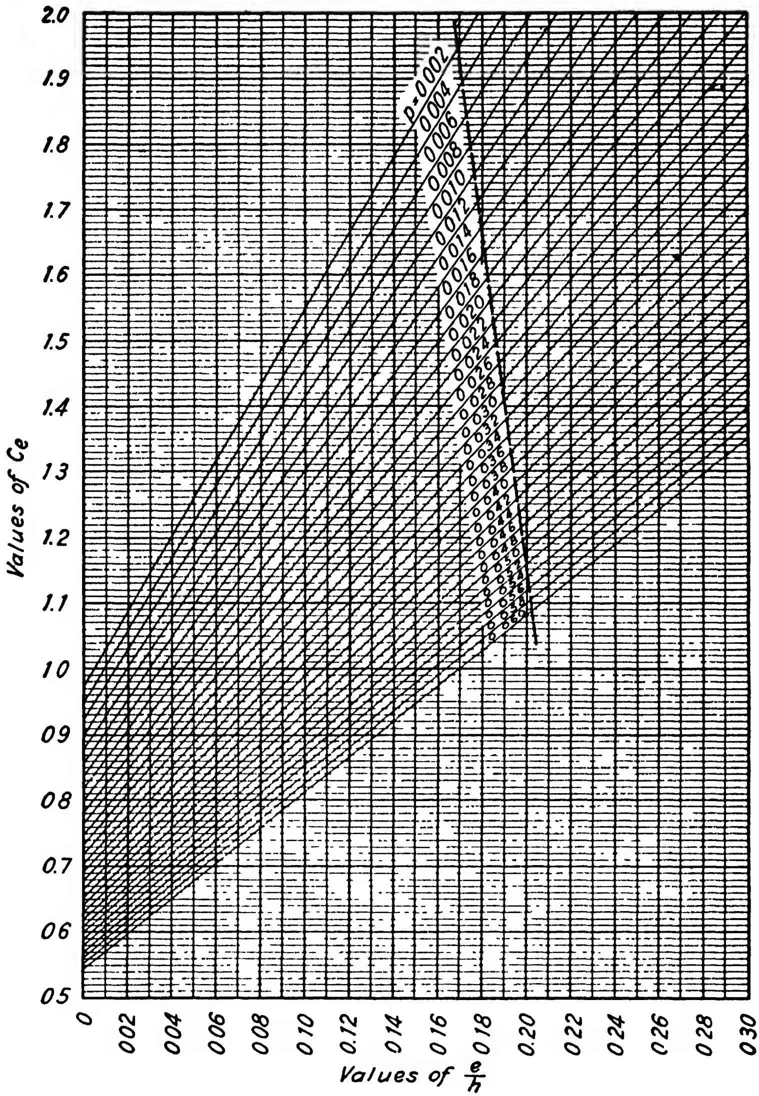


$2a = 0.8h$, $n = 15$. See p. 221 for different n.

Use in Formula for Max. Compression Stress, $f_c = C_e \frac{N}{bh}$. (*See p.* 219.)

For example, see p. 249.

Diagram 5.—Constants C_e for Symmetrically Reinforced Sections. Whole Section in Compression. (*See p.* 219.)



2a = 0.7h, n = 15. See p. 221 for different *n*.

Use in Formula for Max. Compression Stress, $f_c = C_e \frac{N}{bh}$. (*See p.* 219.)

For example, see p. 249.

Diagram 6.—Constants C_e for Symmetrically Reinforced Sections. Whole Section in Compression. (*See p.* 219.)

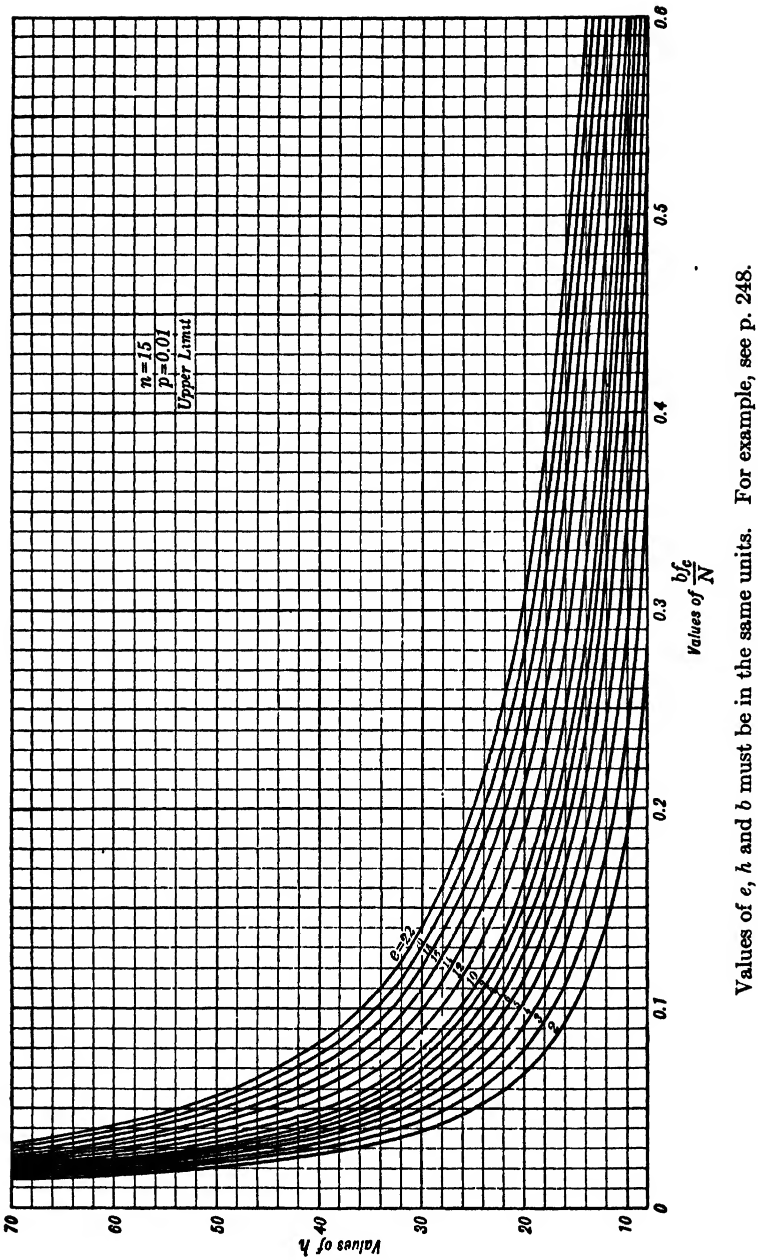


Diagram 7.—Depth of Symmetrically Reinforced Sections for Known Eccentricity e and Ratio $\frac{bf_c}{N}$. Upper limit. (*See p.* 223.)

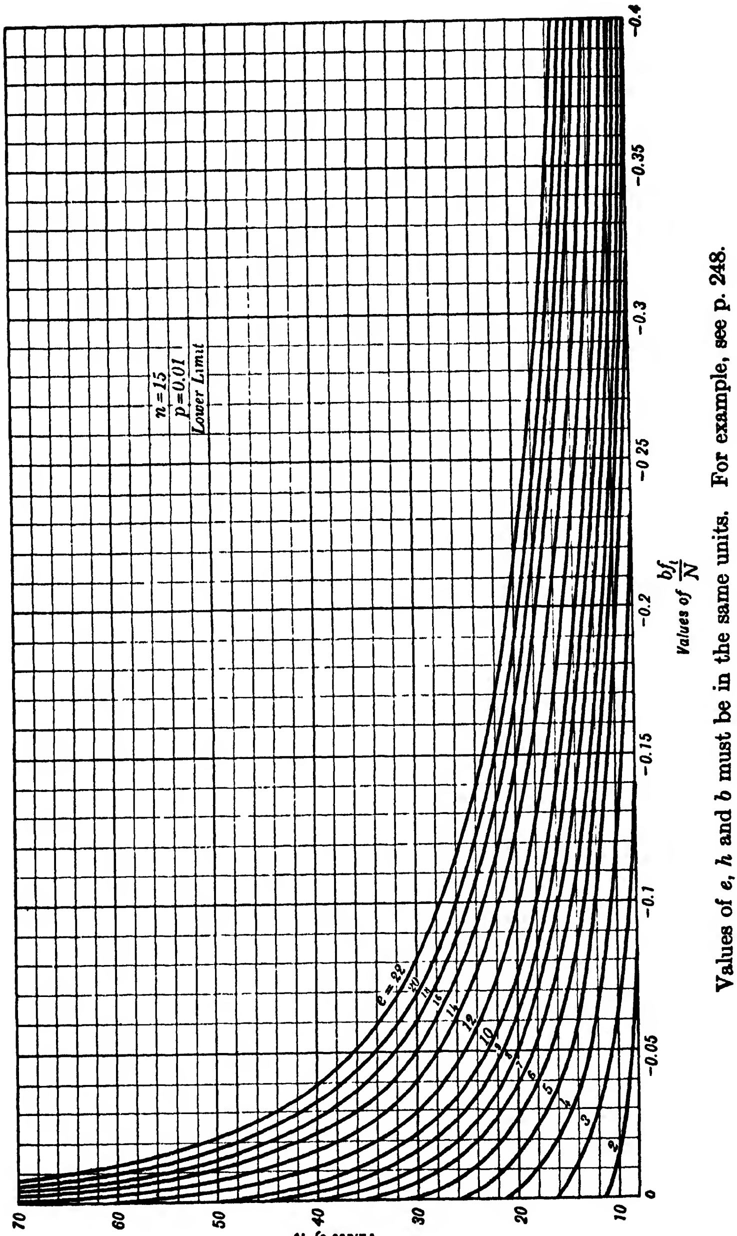
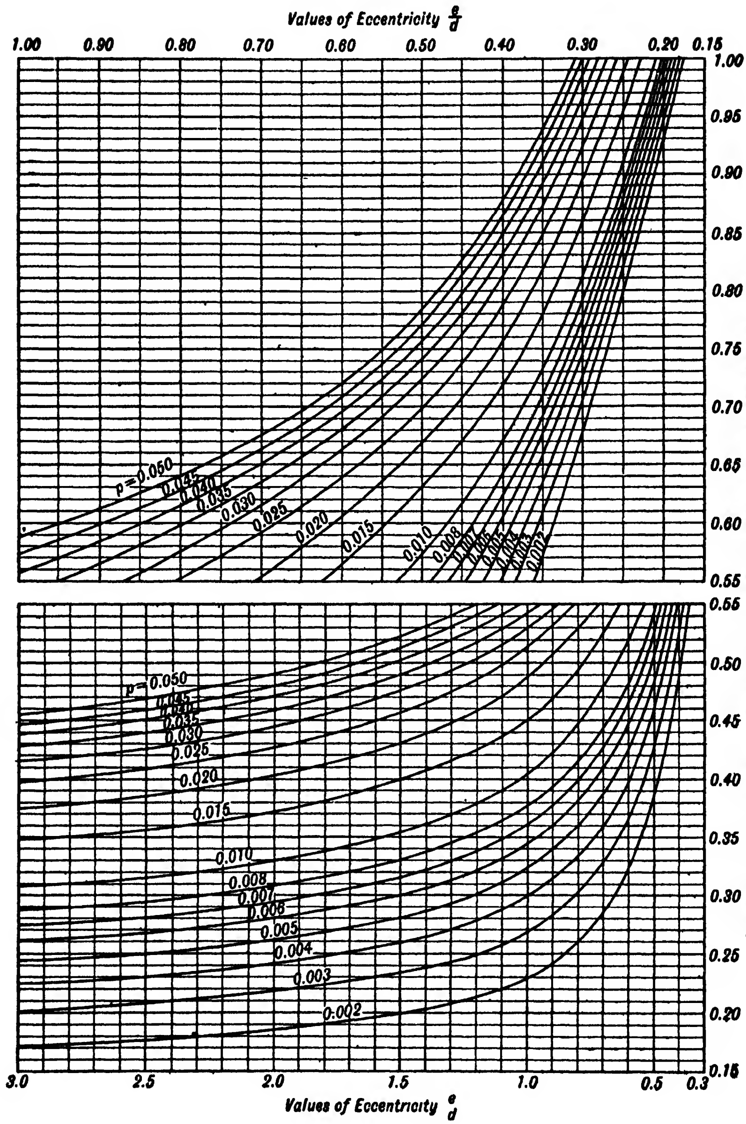
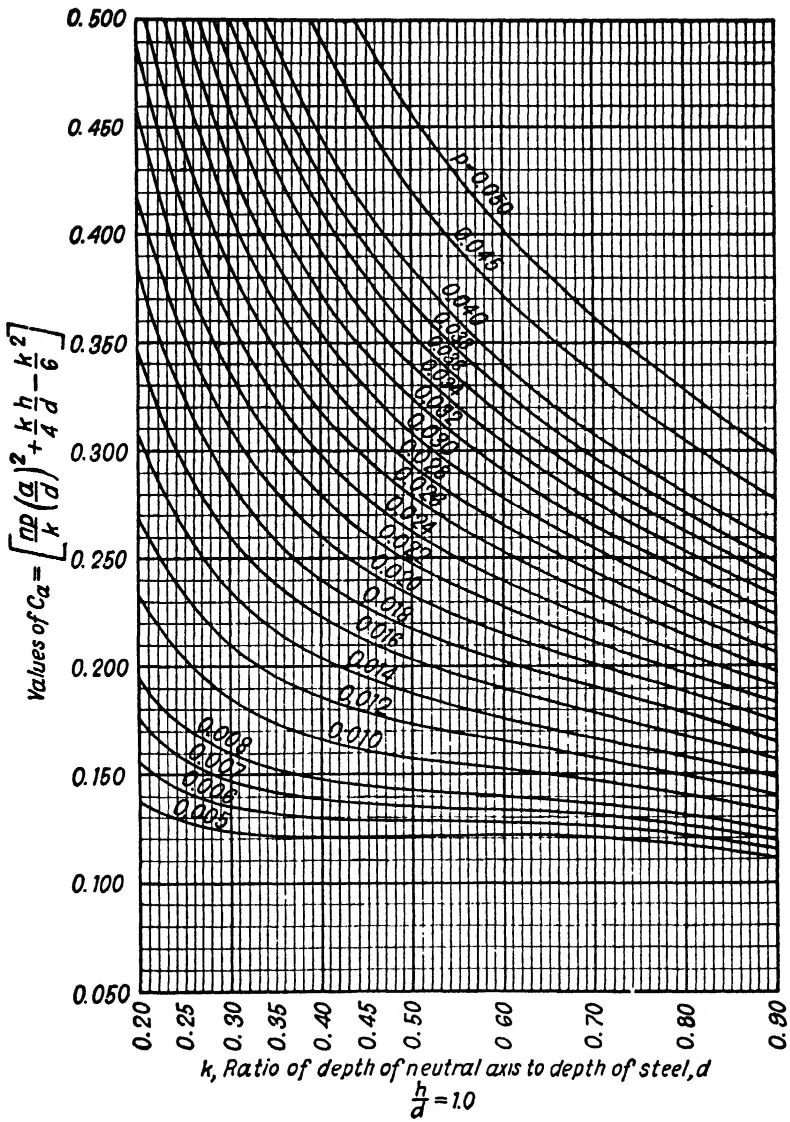


DIAGRAM 8.—Depth of Symmetrically Reinforced Sections for Known Eccentricity e and Ratio $\frac{bf_t}{N}$. Lower limit. (*See p.* 223.)



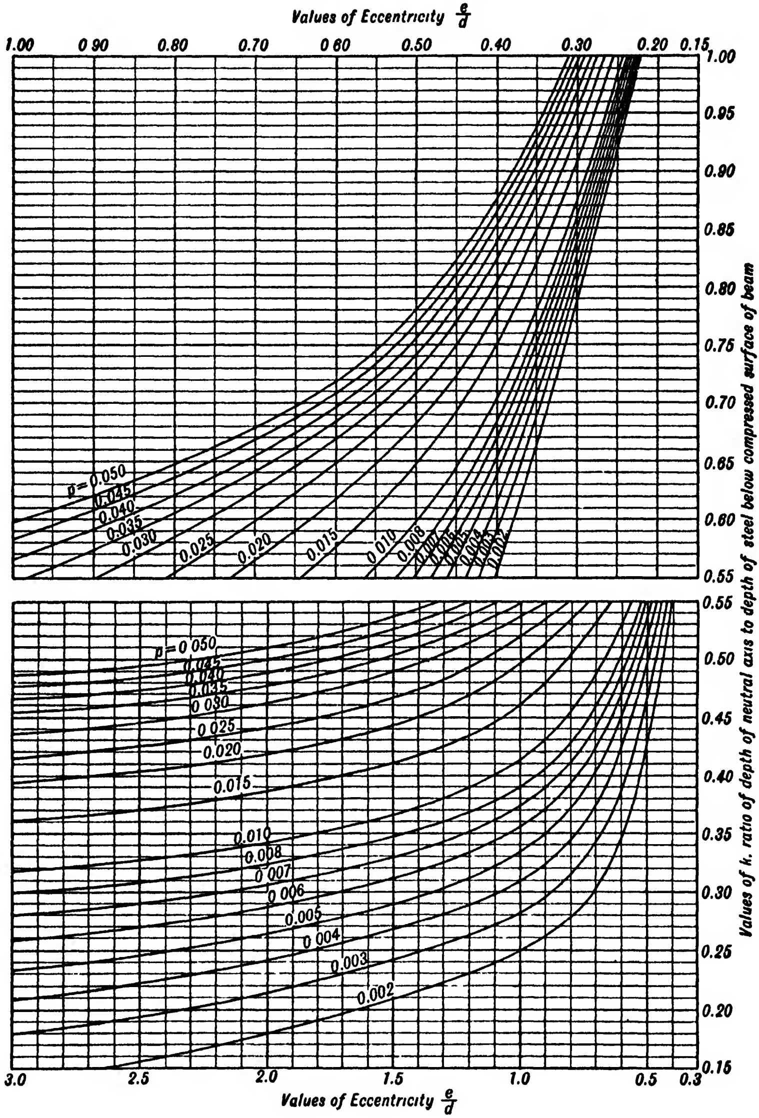
$h = 1.0d$, $n = 15$. See p. 231 for different n. $p = \frac{A_s}{bd}$ where A_s is Steel at Both Sides. For Example, see p. 252.

DIAGRAM 9.—Ratio of Depth of Neutral Axis, k, for Different Eccentricities Part of Section in Tension. (*See p.* 229.)



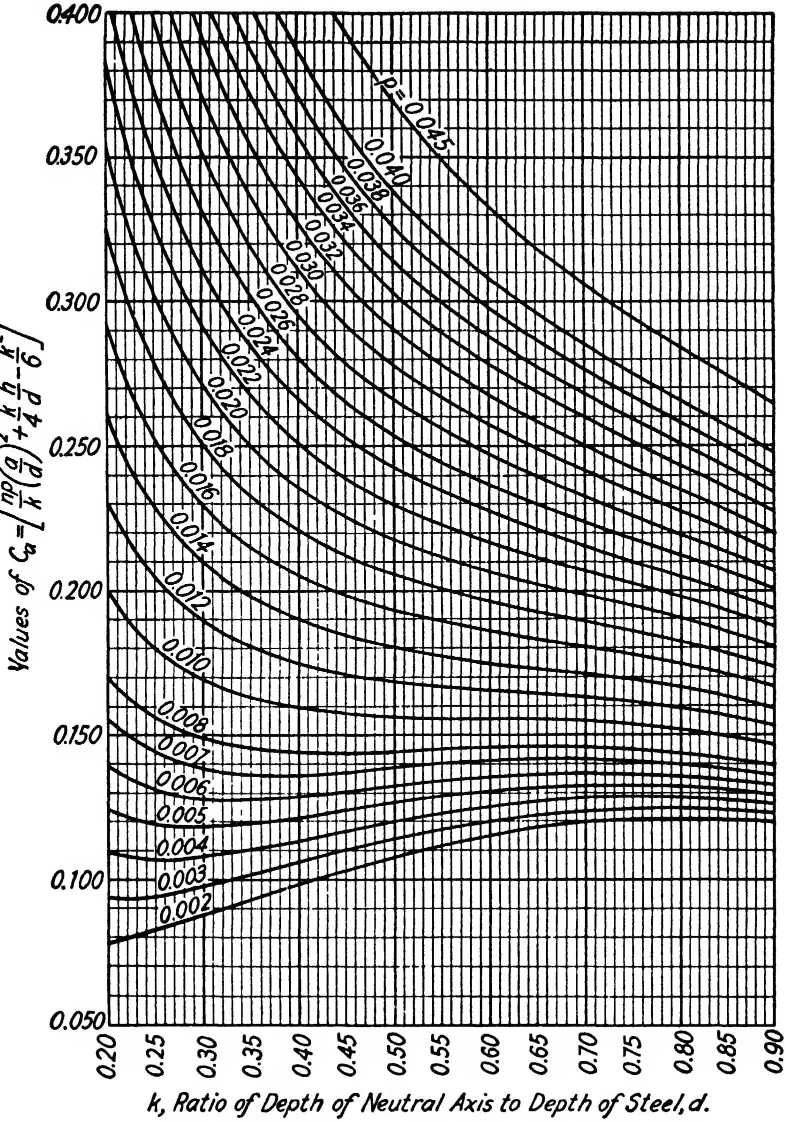
$h = 1.0d$, $n = 15$. See p. 231 for different n. $p = \frac{A_s}{bd}$, where A_s is Steel at Both Sides. For Example, see p. 252.

DIAGRAM 10.—Constants C_a in Formula (29), p. 228. Part of Section in Tension. (*See p.* 229.)



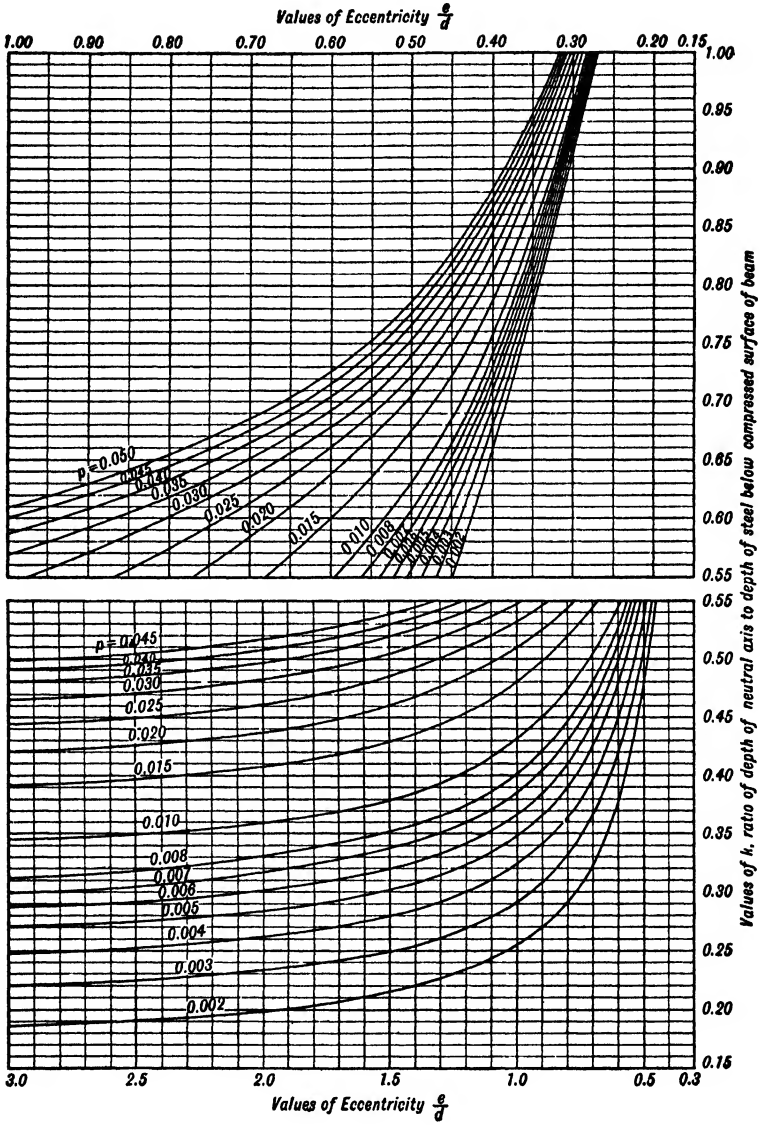
$h = 1.1d$, $n = 15$. See p. 231 for different n. $p = \frac{A_s}{bd}$ where A_s is Steel at Both Sides. For Example see p. 252.

DIAGRAM 11.—Ratio of Depth of Neutral Axis, k, for Different Eccentricities Part of Section in Tension. (*See p.* 229.)



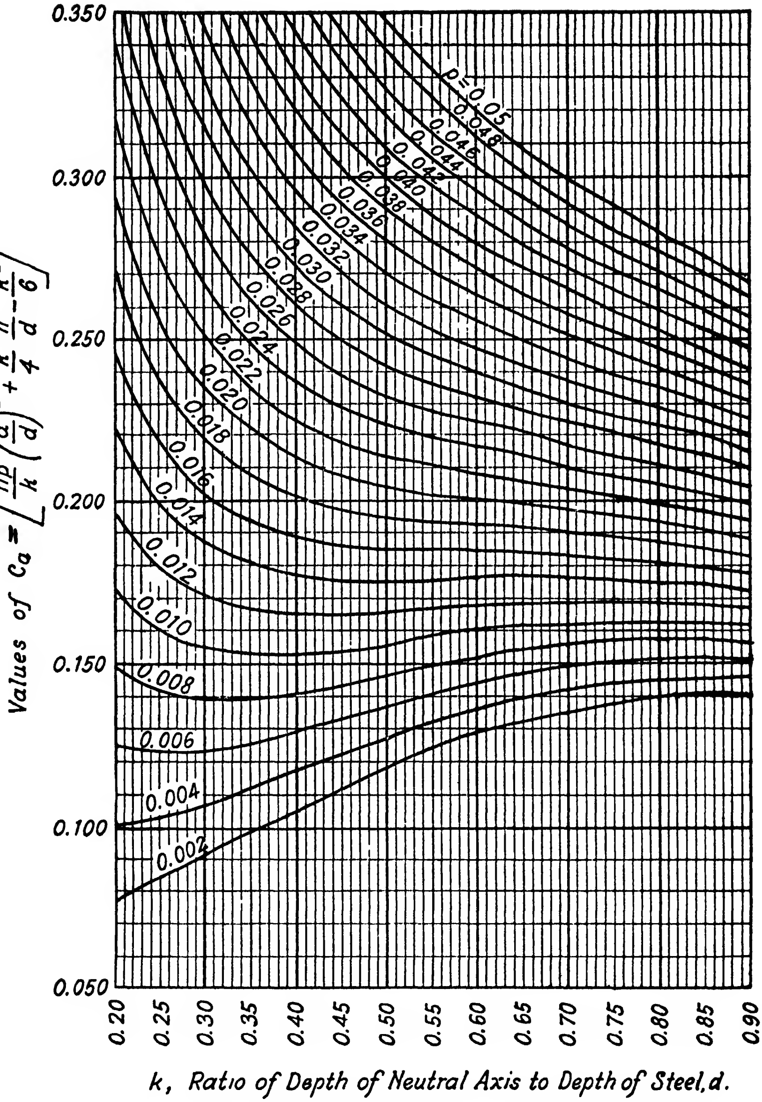
h = **1.1**d, n = **15.** See p. 231 for different n. $p = \frac{A_s}{bd}$, where A_s is Steel at Both Sides. For Example, see p. 252.

DIAGRAM 12.—Constants C_a in Formula (28), p. 228. Part of Section in Tension. (*See p.* 229.)



$h = 1.2d$, $n = 15$. See p. 231 for different n. $p = \frac{A_s}{bd}$, where A_s is Steel at Both Sides. For Example, see p. 252.

DIAGRAM 13.—Ratio of Depth of Neutral Axis, k, for Different Eccentricities Part of Section in Tension. (*See p.* 229.)



$h = 1.2d$, $n = 15$. See p. 231 for different n. $p = \frac{A_s}{bd}$, where A_s is Steel at Both Sides. For Example, see p. 252.

DIAGRAM 14.—Constants C_a in Formula (29), p. 228. Part of Section in Tension. (*See p.* 229.)

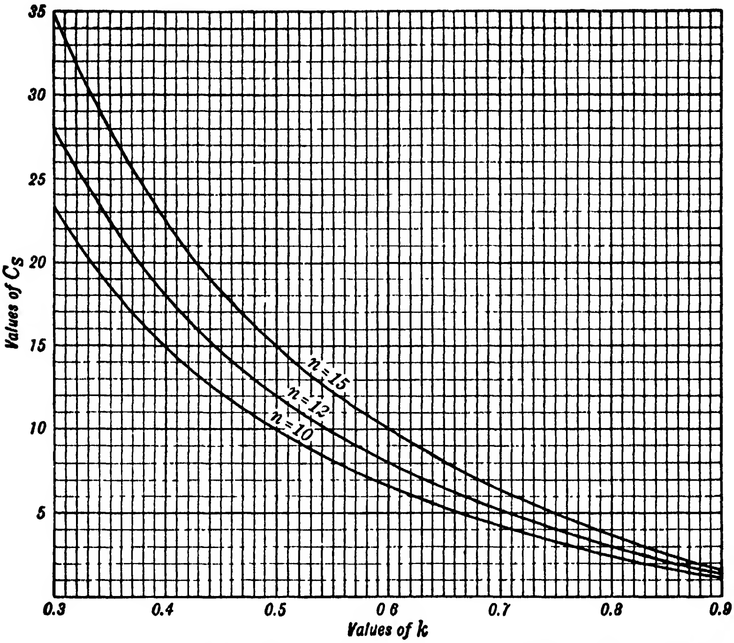


DIAGRAM 15.—Constant C_s in Formulas (33), p. 228 and (53), p. 236. (*See p.* 229.)

BASIS FOR DIAGRAM 16

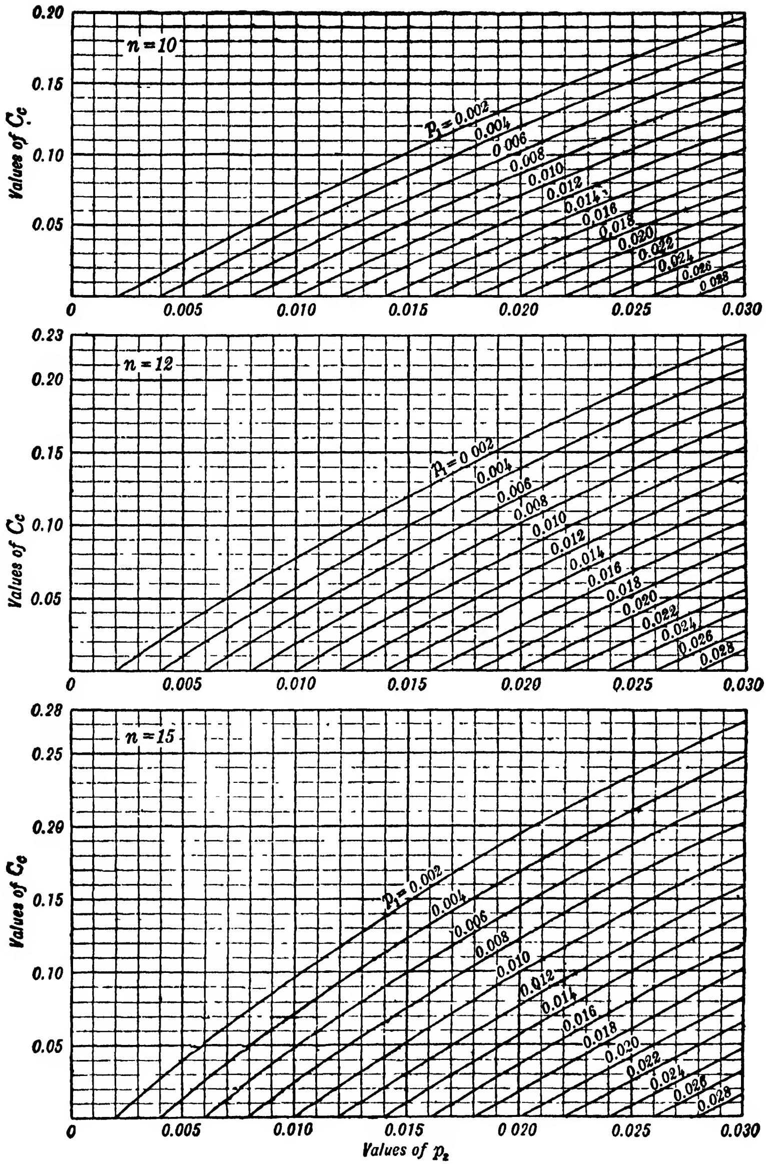
Constants in Diagram 16 are based on Formula (40), p. 233. Values are given for $\frac{h}{d} = 1.0$ and $n = 10$, 12 and 15, respectively.

Use of Diagrams for Different $\frac{h}{d}$.—The diagrams may be used for different values of $\frac{h}{d}$ by locating in the diagrams not the actual values of p_1 and p_2 but the values of $p_1 \div \frac{h}{d}$ and $p_2 \div \frac{h}{d}$. Thus for $\frac{h}{d} = 1.1$, $p_1 = 0.01$ and $p_2 = 0.02$, locate in the diagram the values $p_1 = \frac{0.01}{1.1} = 0.091$ and $p_2 = \frac{0.02}{1.1} = 0.0182$.

Use of Diagrams for p_2 Smaller than p_1.—In diagram values of p_2 are always larger than p_1 and the resulting constant is positive denoting that center of gravity is nearer the compression face. When p_2 is smaller than p_1, interchange p_1 and p_2 before using the diagram and consider the result obtained from diagram as negative.

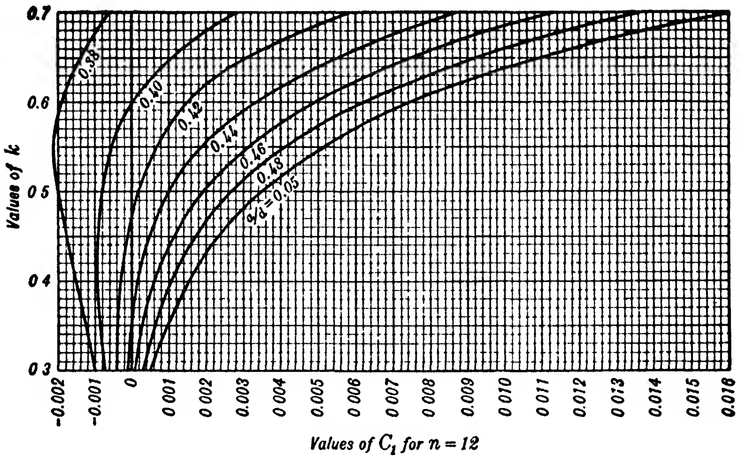
Thus for $p_1 = 0.017$, $p_2 = 0.022$ and $n = 15$, $C_c = 0.045$.

For $p_1 = 0.022$ and $p_2 = 0.017$ the value of C_c is the same numerically as in previous case but is of opposite sign. $C_c = -0.045$.



$\frac{h}{d} = 1$. For different $\frac{h}{d}$, see note on opposite page.

DIAGRAM 16.—Constants C_c in Formula (40), p. 233 for Unsymmetrically Reinforced Sections. One Face in Tension. (*See p.* 233.)



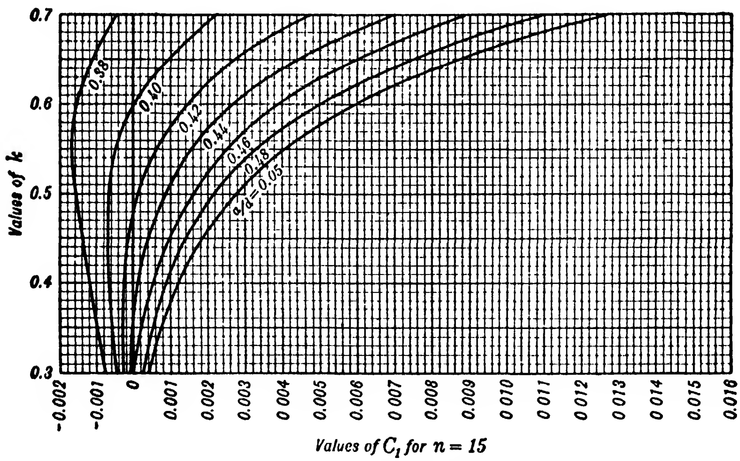


DIAGRAM 17.—Constants C_1 in Formula (43), p. 233. Unsymmetrically Reinforced Section. One Face in Tension. (*See p.* 233.)

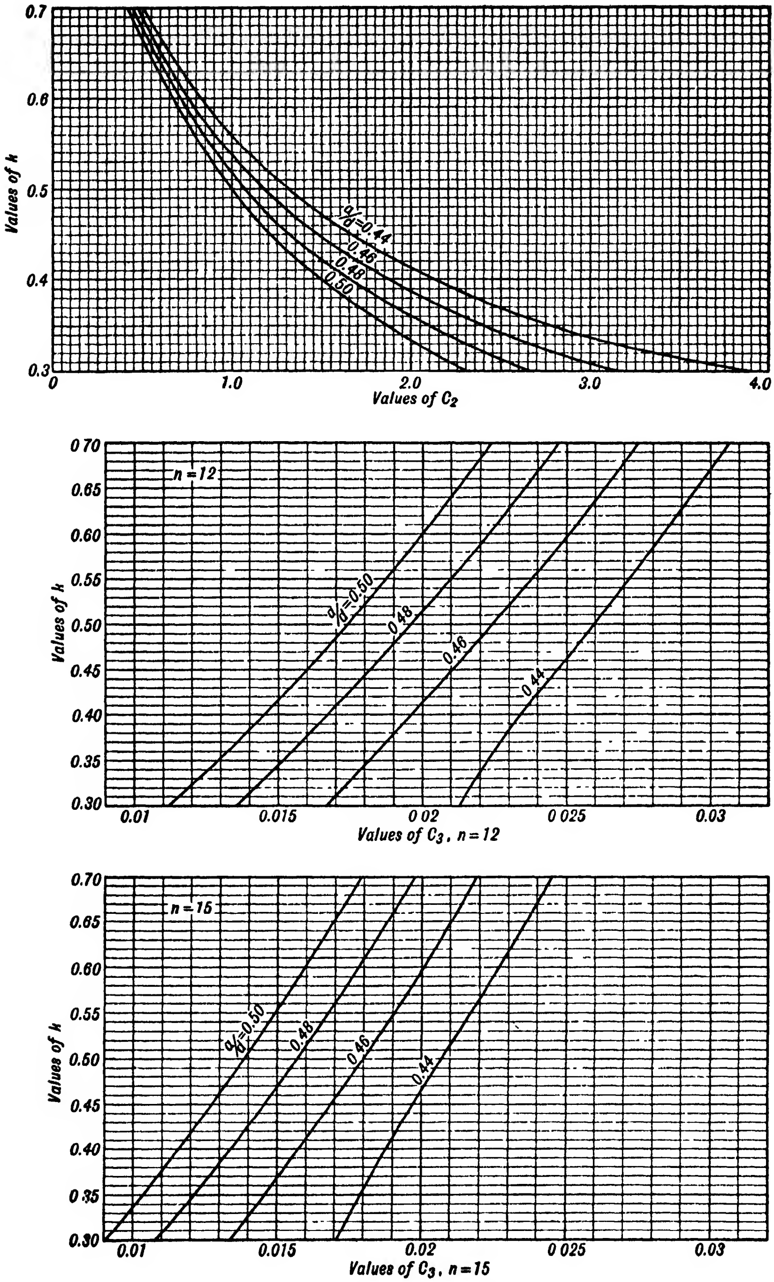
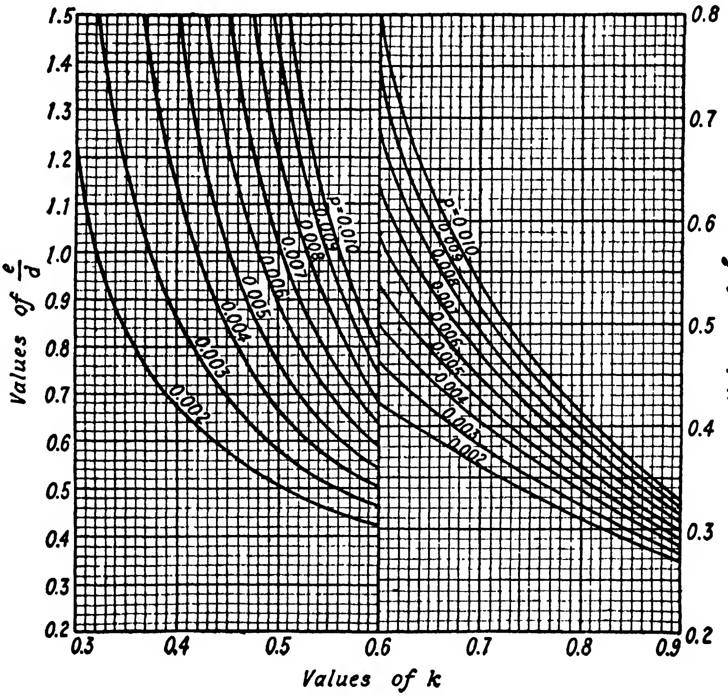


DIAGRAM 18.—Constants C_2 and C_3 in Formula (45), p. 234. Unsymmetrically Reinforced Section. One Face in Tension. (*See p.* 234).

Ratio of $\frac{d_c}{d}$ for Different Steel Ratios p. (*See p.* 236.)

d_c is measured from compression face of section.

	Steel Ratios p										
	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.010	0.011	0.012
$\frac{d_c}{d}$	0.565	0.570	0.576	0.581	0.586	0.590	0.595	0.600	0.605	0.609	0.613



$d = 0.9h$, $n = 15$

Tension Side Only Reinforced. $p = \frac{A_s}{bd}$.

DIAGRAM 19.—Ratio of Depth of Neutral Axis, k, for Different Eccentricities. Part Section in Tension. (*See p.* 236.)

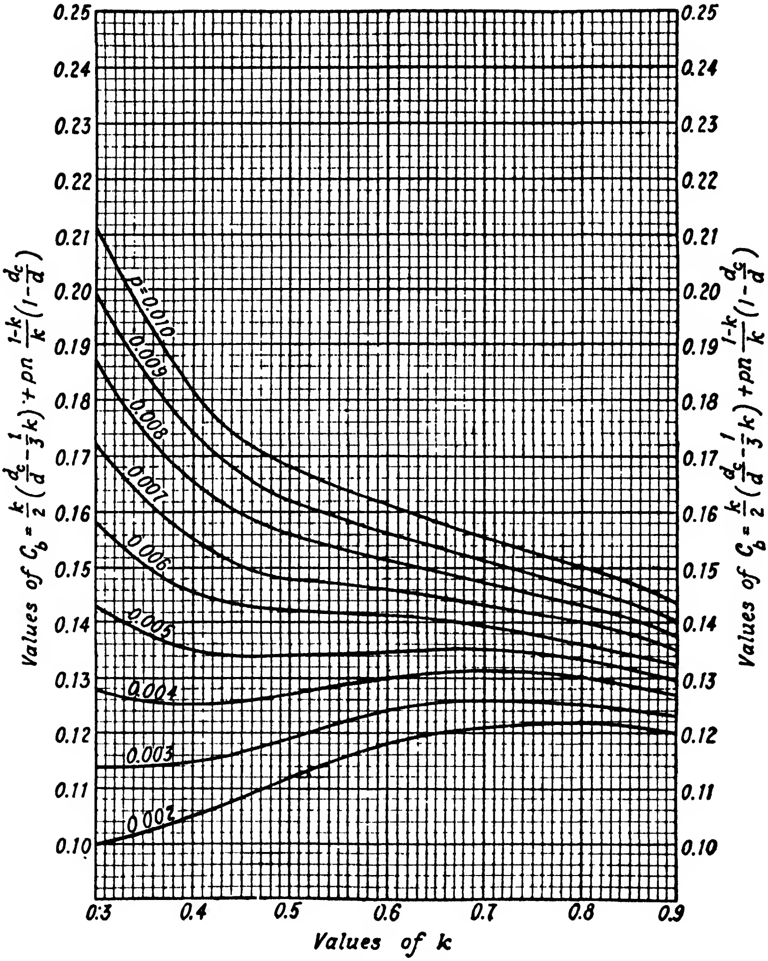
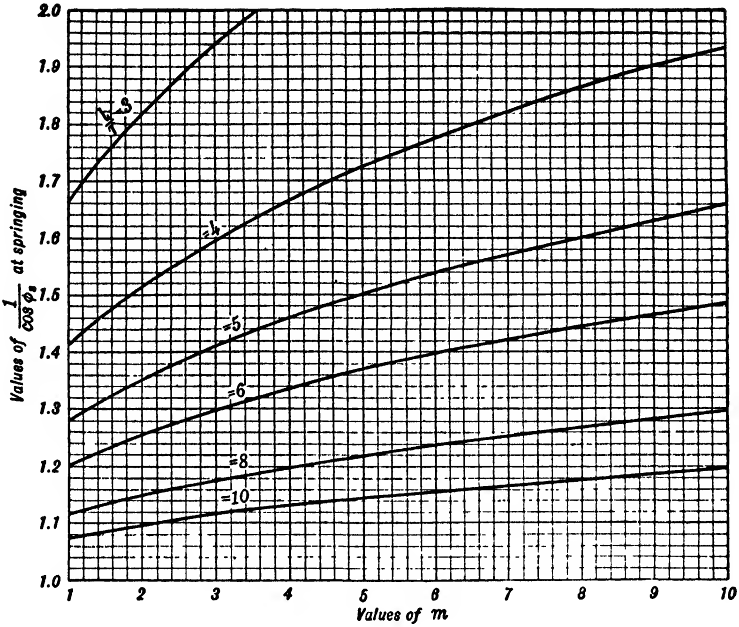


DIAGRAM 20.—Constants C_b in Formula (49), p. 236. *(See p. 236.)*

DIAGRAMS FOR APPROXIMATE DESIGN OF FIXED ARCHES

The diagrams required for the design of fixed arches according to the approximate method described on page 480 are as follows:



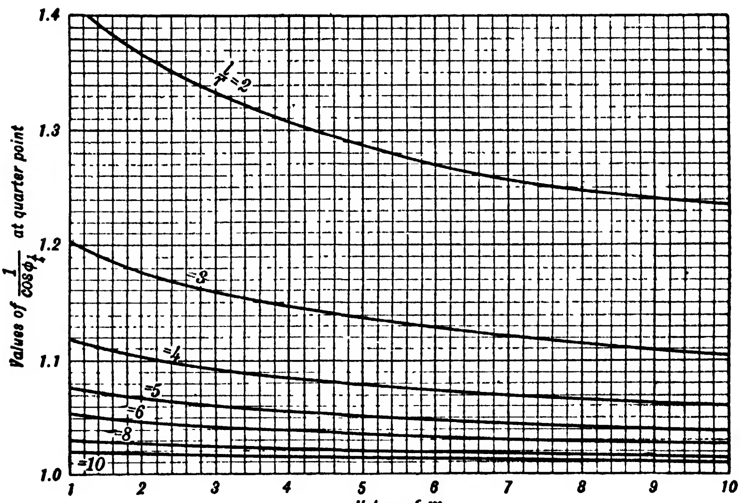


DIAGRAM 21.—Angles $\phi_{\frac{1}{4}}$ and ϕ_s at Quarter Point and Springing, Respectively.
(*See p.* 487.)

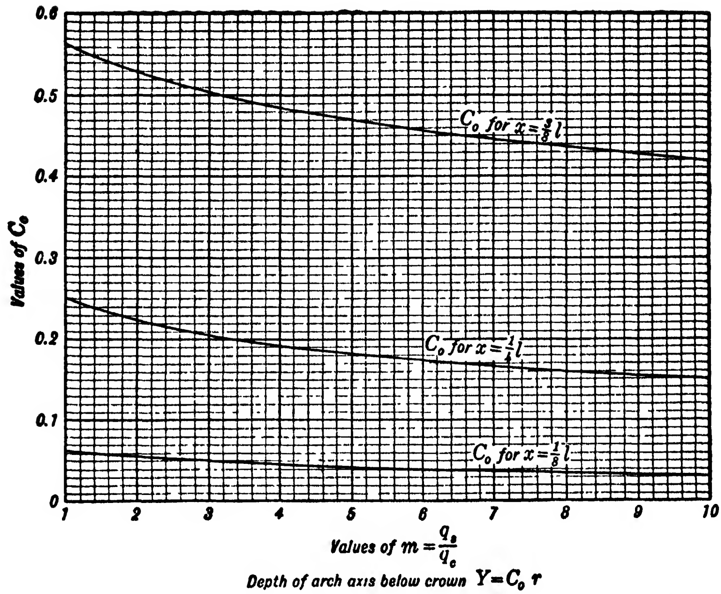
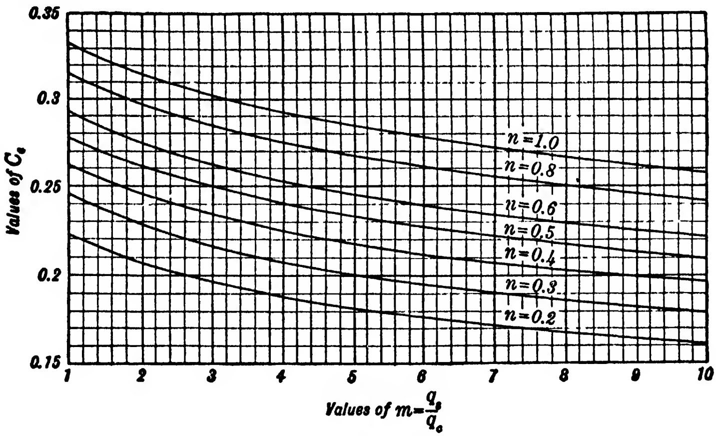
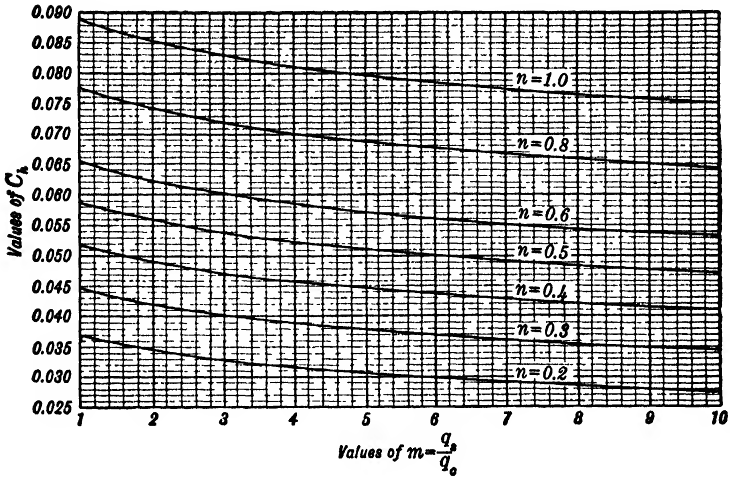


DIAGRAM 22.—Ratio of Depth of Arch Axis below Crown C_0, for $x = \frac{1}{8}l$, $\frac{1}{4}l$ and $\frac{3}{8}l$ from Crown. (*See p.* 482.)



Distance Elastic Center from Crown $Y_e = C_e r$

DIAGRAM 23.—Constant C_e in Formula (14), p. 483 for Location of Elastic Center. (*See p.* 483.)



$$C_h = \frac{1}{lr^2}\int_{-\frac{l}{2}}^{\frac{l}{2}} y^2 \frac{I}{I_x} ds$$

DIAGRAM 24.—Constant C_h in Formula (15), p. 483. (*See p.* 483.)

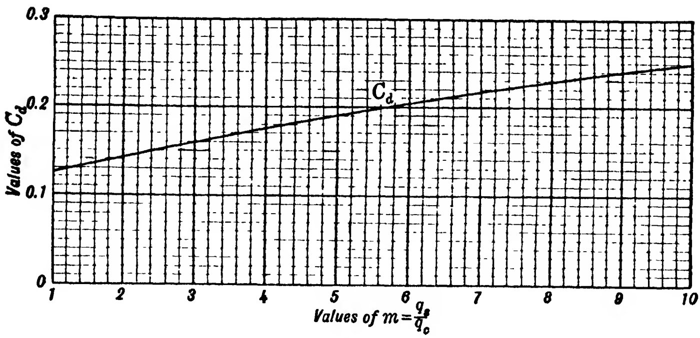
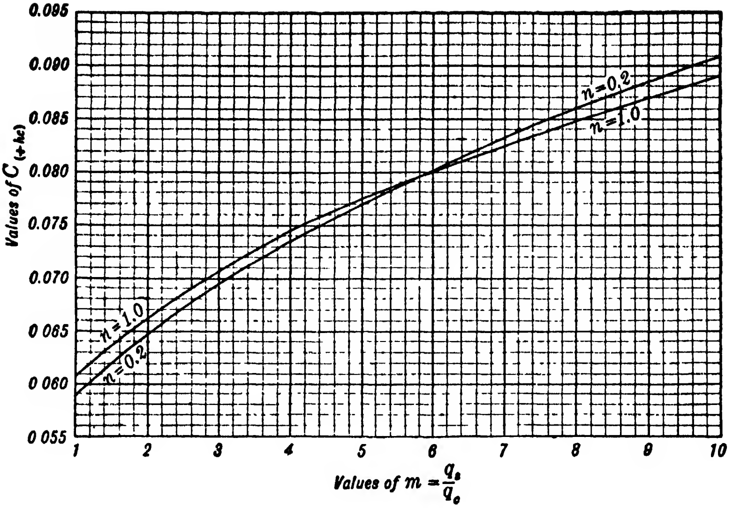
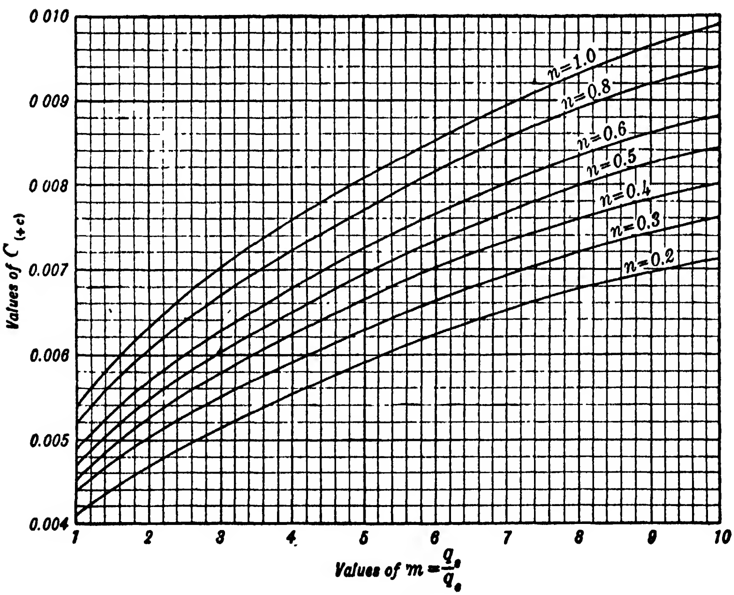


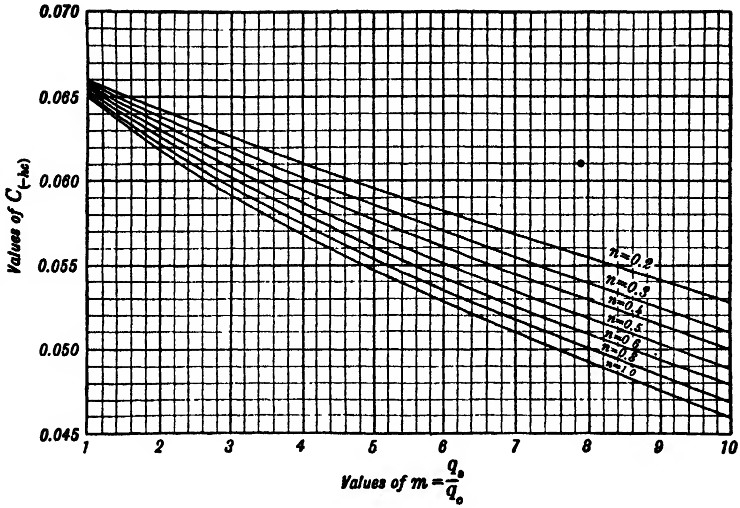
DIAGRAM 25.—Constant C_d in Formula (35), p. 486 for Horizontal Thrust for Dead Load. (*See p.* 486.)

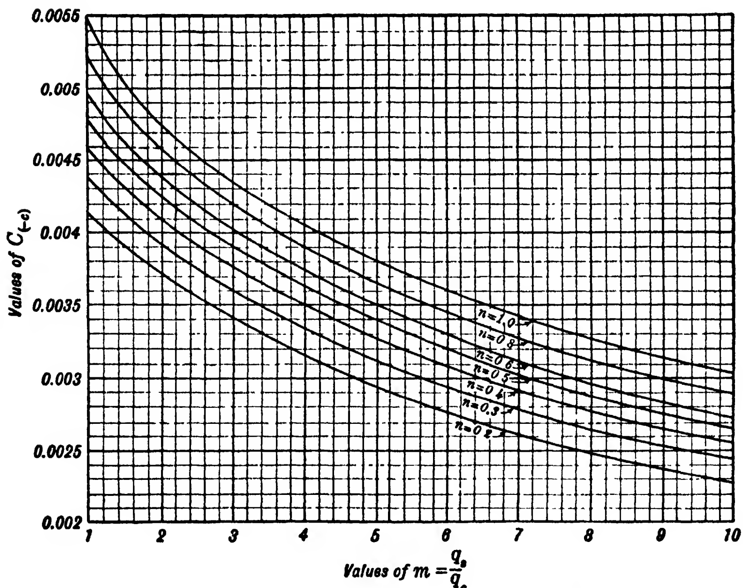




$$M_c = C_{(+c)}wl^2, \qquad H_c = C_{(+hc)}wl\frac{l}{r}$$

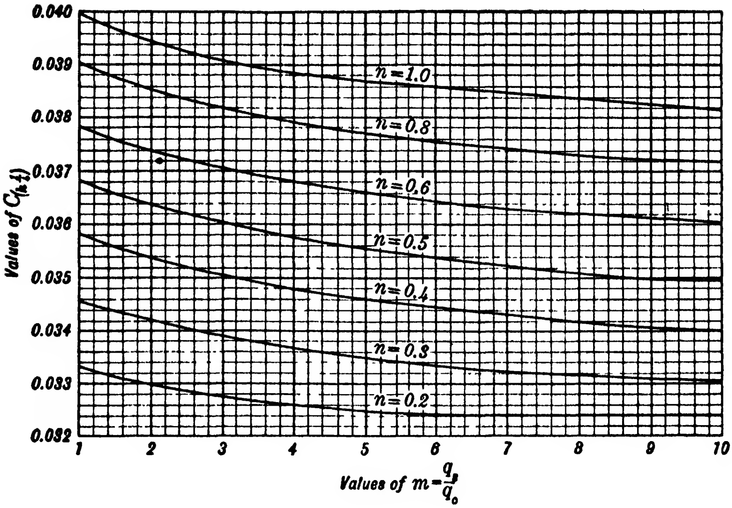
Diagram 26.—Coefficients for Maximum Positive Bending Moment and Corresponding Horizontal Thrust at Crown. (*See p.* 484.)

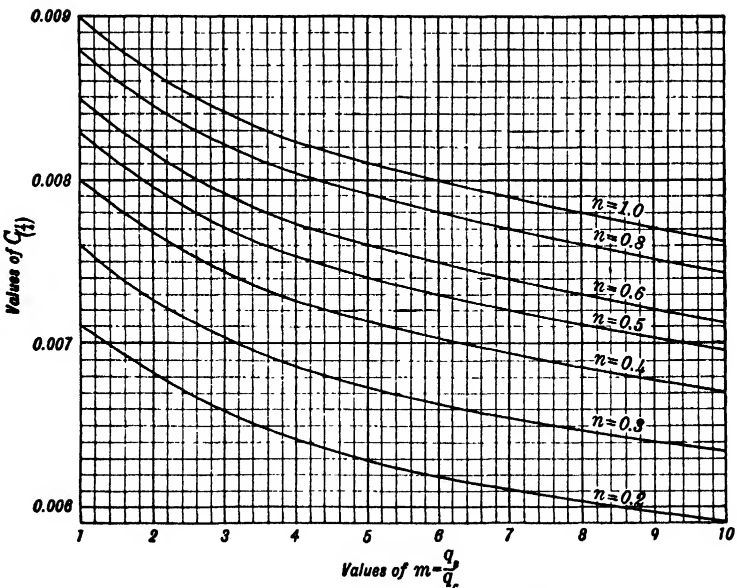




$$M_c = -C_{(-c)}wl^2, \qquad H_c = -C_{(-hc)}wl\frac{l}{r}$$

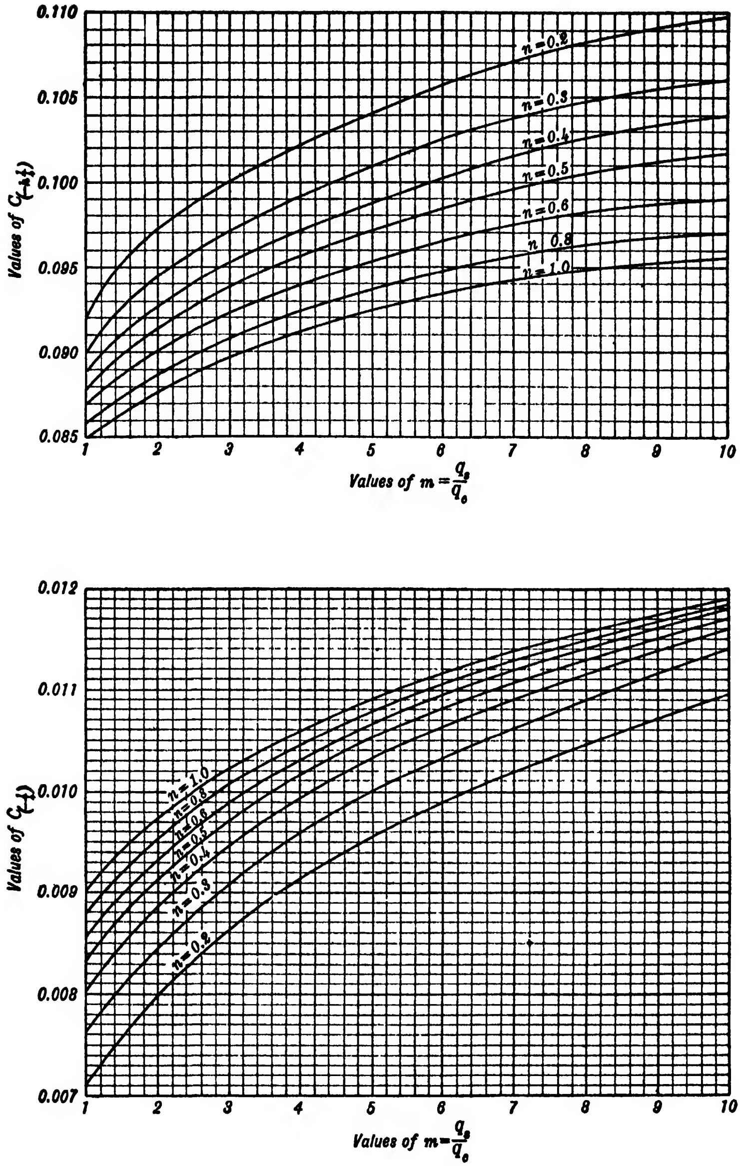
DIAGRAM 27.—Coefficients for Maximum Negative Bending Moment and Corresponding Horizontal Thrust at Crown. (*See p.* 484.)





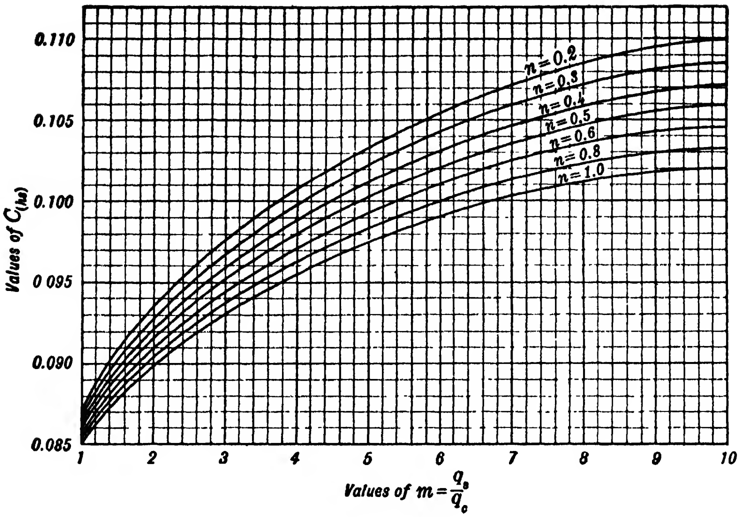
$$M_{\frac{1}{4}} = C_{(\frac{1}{4})}wl^2, \qquad H_{\frac{1}{4}} = C_{(h\frac{1}{4})}wl\frac{l}{r}$$

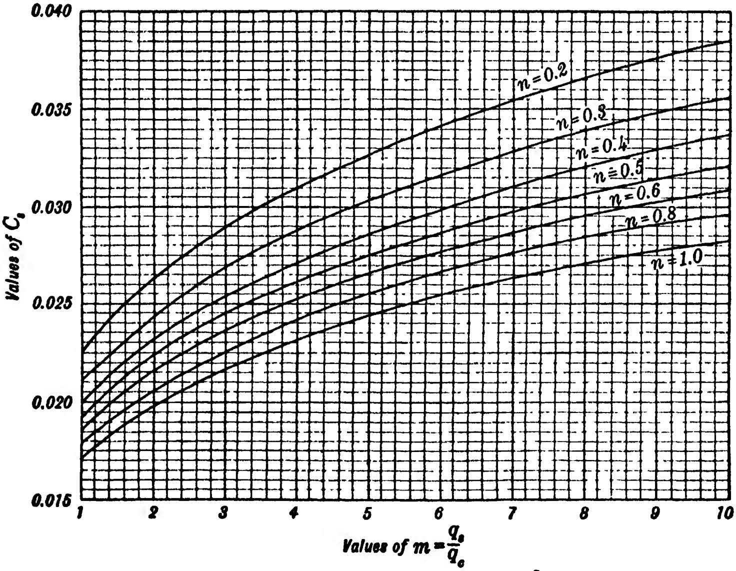
DIAGRAM 28.—Coefficients for Maximum Positive Bending Moment and Corresponding Horizontal Thrust at Quarter Point. (*See p.* 485.)



$$M_{\frac{1}{4}} = -C_{(-\frac{1}{4})}wl^2, \qquad H_{\frac{1}{4}} = -C_{(-h\frac{1}{4})}wl\frac{l}{r}$$

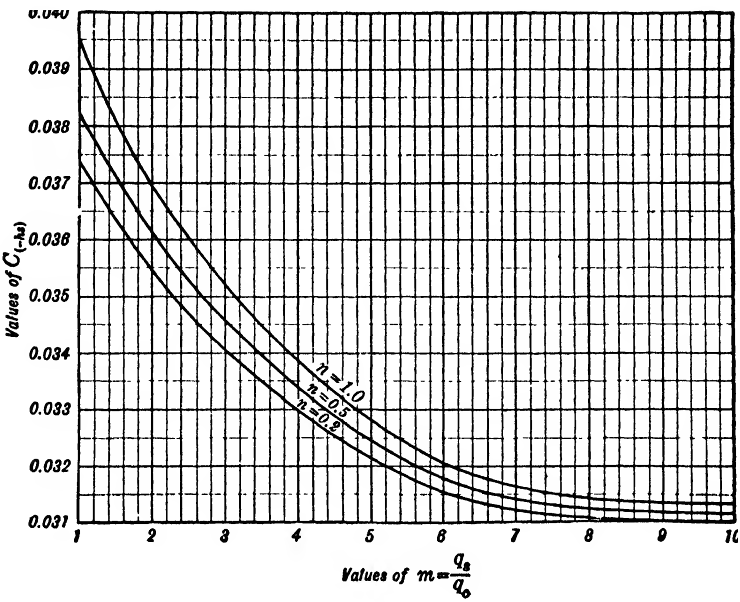
Diagram 29.—Coefficients for Maximum Negative Bending Moment and Corresponding Horizontal Thrust at Quarter Point. (*See p.* 485.)

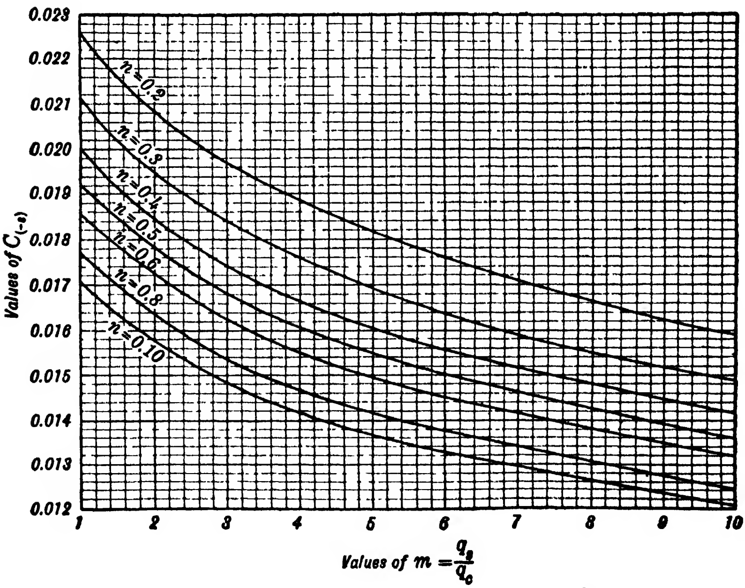




$$M_s = C_s wl^2; \qquad H_s = -C_{(hs)} wl\frac{l}{r}$$

DIAGRAM 30.—Coefficients for Maximum Positive Bending Moment and Corresponding Horizontal Thrust at Springing. (*See p.* 485.)





$$M_s = -C_{(-s)}wl^2, \qquad H_s = -C_{(-hs)}wl\frac{l}{r}$$

Diagram 31.—Coefficients for Maximum Negative Bending Moment and Corresponding Horizontal Thrust at Springing. (*See p.* 485.)

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