Chapter 3

Theory of Rectangularly Dualizable Graphs

In this chapter, we derive a necessary and sufficient condition for a plane graph to be an RDG. The main result of this chapter is that in the extended Euclidean plane, a rectangular dual can be realized by a quadrangulation.

3.1 Introduction

The theory of rectangular dualizable graphs plays an important role in floorplanning, particularly at a large scale such as VLSI circuit design. It provides us information at early stage to decide whether a given plane graph can be realized by a rectangular dual.

It is known that every plane graph can be dualized, but not rectangularly dualized [38, 41, 51]. Kozminski and Kinnen [38] were the first who gave the theory of RDGs. We found a critical flaw that invalidates the result given by Kozminski and Kinnen [38] which states as follows:

Theorem 3.1.1. [38, Theorem 5] Let *G* be a separable connected planar graph with all triangular faces except the exterior one. Then *G* is an RDG if and only if

(i.) it has no separating triangle,

(ii.) BNG is a path,

(iii.) each of its maximal blocks corresponding to the endpoints of the BNG contains at most 2 critical CIPs,

(iv.) no other maximal block contains a critical CIP.

As a counter example, consider the separable connected graph *G* shown in Fig. 3.1. Although the given graph in Fig. 3.1 satisfies all the conditions given in Theorem

3.1.1, but it is not an RDG. Using the existing algorithm [8], one can find a rectangular dual for each of its blocks and then a rectangular dual for G can be constructed by gluing them in a rectangular area. But it is not possible because of the occurrence of adjacency of cut vertices v_4 and v_6. Note that corresponding to a cut vertex, there always associate a through rectangle [52] in the rectangular dual of G. But in Fig. 3.1, the cut-vertices are adjacent. Hence, it is not possible to maintain rectangular enclosure while keeping R_4 and R_6 as through rectangles.

Another issue with this theorem is that its proof is not rigorous, but an outline. Except this theorem, there is no result to check the existence of a rectangular dual for separable connected planar graphs.

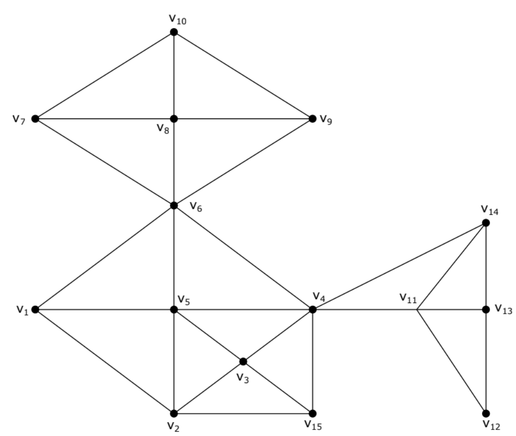


Figure 3.1: A counter example that invalidates Theorem 3.1.1.

Rinsma's work [51] is very restrictive, i.e., it cover the class of rectangular dualizable outer planar graphs only. She didn't develop any theory, but presented a counter example showing that it is not always possible to be realized a rectangular dual for a vertex-weighted outer planar graph having 4 vertices of degree 2. Besides this property, there are infinite outer planar graphs that are not rectangularly dualized. In fact, an outer planar graph having more than four CIPs can not be rectangularly dualized. This can be contradicted by our proposed Theorem 3.4.2 in Section 3.4. Since the graph setting of a VLSI system is obviously non-outer planar due to its large size, this theory can not be preferable for VLSI circuit's design. The theory of rectangular dualizable outer planar graphs plays a limited role in building architecture also.

Lai and Leinward [41] derived the following necessary and sufficient condition for an EPTG to be an RDG:

Theorem 3.1.2. [41, Theorem 3] An EPTG is an RDG if and only if each of its triangular region can be assigned to one of its corner vertices such that each vertex v_i has exactly $d(v_i) - 4$ triangular region assigned to it.

Lai and Leinward [41] showed that solving an rectangular dualization problem of a planar graph is equivalent to a matching problem of a bipartite graph derived from the given graph. This theory relies on the assigned regions to vertices of a graph. The result is not implementable until a method for checking assignments of regions to vertices in an EPTG is known.

The outline of our contribution in this chapter is as follows: Section 3.2 describes the extended RDG construction process. In Section 3.3, we find stereographic projection of dual of an RDG in order to extract some result pertaining to the exterior (unbounded) region of the dual. In Section 3.4, we derive a necessary and sufficient condition for an EPTG to be an RDG. Finally, we conclude our contribution and discuss future scope in Section 3.5.

3.2 Extended RDG

In a graph described by a VLSI system, vertices and edges correspond to component modules and required interconnections respectively. Communication with units outside the given system are modeled by edges having one end incident to a vertex at the infinity (denoted by v_∞, see Fig. 3.2). The vertex v_∞ of an RDG corresponds to the unbounded region of its rectangular dual.

Furthermore, the dual of an RDG needs to be fitted in a rectangular enclosure while connecting to the outside world. Vertices that correspond to regions next to the enclosure are called *enclosure vertices* [40] and those vertices correspond to corner regions are called *corner enclosure vertices*. In Fig. 3.2, vertices $v_7, v_6, v_5, v_4, v_3, v_2, v_1$ are enclosure vertices and v_7, v_5, v_3, v_1 are corner enclosure vertices. Since the enclosure has 4 sides, out of these enclosure vertices, the enclosure corner vertices correspond to corner rectangles or end rectangles of a rectangular dual where a corner rectangle shares its two sides to the unbounded (exterior) region and end rectangle shares three sides to the exterior. Therefore, we need to consider atmost 4 extra edges between the selected enclosure corner vertices and v_∞. These atmost 4 extra edges are known as *construction edges* [41]. A PTG where enclosure vertices are connected to v_∞ together with 4 additional construction edges is called an EPTG . An EPTG of a rectangular dual is depicted in Fig. 3.2 by red edges.

It is interesting to note that all regions including unbounded region are triangulated in EPTG so that every region including unbounded region of the dual of an RDG is quadrangle. This permits the enclosure to be rectangular. A detailed description of unbounded quadrangle region of the dual can be seen in Section 3.3. Since there is one to one corresponding between the edges of a plane graph and its dual, there are multiple edges between an enclosure corner vertex and v_∞. In this thesis, we consider a simple PTG. However, some minor changes (parallel edges between enclosure corner vertices and v_∞ only) in the EPTG is done in order to choose four construction edges.

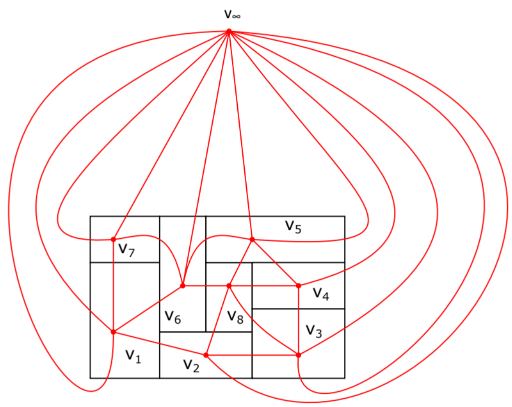


Figure 3.2: Construction of an extended RDG (red edges) of a rectangular dual (dark edges).

3.3 Stereographic Projection

In this section, we describe the process of projection of a rectangular graph to the surface of a sphere.

Let D be the rectangular dual graph of an RDG G. Note that a connected plane graph is a single piece made up of Jordan curves (called edges) joining their ends to pairs of the specified points (called vertices) in the Euclidean plane. Consider a sphere S centered at $(0,0,1/2)$ having radius $1/2$, and a fixed plane embedding D^* of D in the Euclidean plane passing through $z = 0$ (xy-plane). Let $(0,0,1)$ be the north pole N and p be a point of an edge of D^*. Draw a line segment joining the points N and p. Let t be a point where it intersects the surface of S. Thus we see that the point p is mapped to the point t. In this way, the image of each of its points is a curved line on the surface of S and hence each edge of D^* is mapped to a curved line on the surface of S. This results an embedding of D^* on the surface of a sphere.

Now, it is important to identify why the edges of D^* is mapped to the edges on the surface of S? In fact, a connected graph is carried to a connected graph by a continuous map. Thus being the mapping continuous, the image of D^* is again a plane graph on the surface of S with its exterior bounded. Note that the unbounded region is now mapped into a bounded region on S passing through N. This process is known as *stereographic projection* and sphere is known as *Riemann sphere*. But D^* is a rectangular dual graph. Its exterior is a four sided rectangular enclosure. This results the unbounded region of D^* corresponds to a four sided bounded region of the corresponding plane graph embedded on the surface of S. Consequently, when we assign horizontal or vertical orientations to the edges of D^* to transform into a rectangular dual, the unbounded region of D^* corresponds to an unbounded rectangle (region) R_∞ passing through ∞. Thus we see that the exterior of a rectangular dual is a rectangle R_∞ passing through ∞. Note that R_∞ is not a part of a rectangular dual, but is a rectangle that shares its two adjacent sides to each of its enclosure corner rectangles. Recall that a rectangle is a four-sided region with 4 right interior angles formed by its sides. Although in case of R_∞, these interior angles can be realized to be 90° by looking at it from a point at ∞, otherwise we realize every interior angle to be 270°. The role of the point at ∞ is played by N and hence an alternative way is to realize right angle between two sides of the four-sided region passing through N in the stereographic projection of the rectangular dual graph is the angle between the intersection of their tangents to the sides of this region. This discussion realizes us that a rectangular dual is *quadrangulation* of the Euclidean plane.

3.4 RDG Existence Theory

In this section, we derive the necessary and sufficient conditions for PTGs to be RDGs.

Theorem 3.4.1. A necessary and sufficient condition for an EPTG G^* to be an RDG is that it is 4-connected and has atmost 4 critical separating triangles passing through v_∞.

Proof. **Necessary Condition.** Assume that G^* is an RDG. Then it has a rectangular dual D. Let v_i be an arbitrary vertex of G^* dual to some interior region R_i of D. Since every region of D is four-sided, atleast 4 regions are required to fully enclose an interior region of D. This implies that R_i is surrounded by atleast 4 regions of D and hence v_i is adjacent to atleast 4 vertices of G^*, i. e., $d(v_i) \geq 4$. Let v_e be an arbitrary vertex of G^* dual to an enclosure (exterior) region R_e of D. There arise two possibilities:

- R_e surrounds exactly its two sides with R_∞ if it is an enclosure corner region,
- R_e surrounds exactly its one side with R_∞ if it is not an enclosure corner region.

In the first case, R_∞ surrounds the two sides of R_e. There are two edges between v_∞ and v_e where v_∞ corresponds to R_∞. The remaining two sides of R_e are surrounded by atleast two interior regions other than R_∞. This implies that $d(v_e) \geq 4$. In the second case, only one side of R_e is surrounded by R_∞ and the remaining sides are surrounded by atleast three interior regions. This implies that $d(v_e) \geq 4$. Since v_e and v_i are arbitrary vertices of G^*, G^* is 4-connected. This proves the first condition.

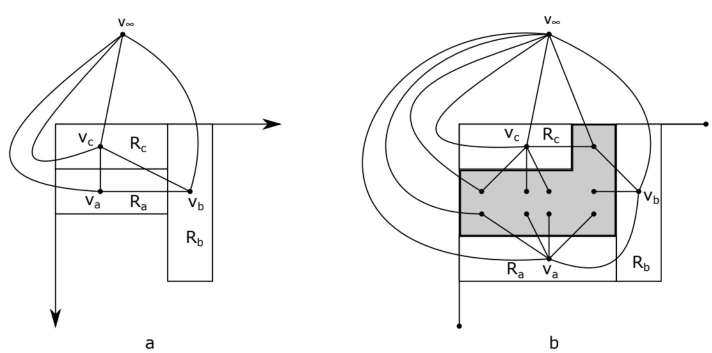


Figure 3.3: Two possibilities of a critical separating triangle enclosing an enclosure corner vertex.

As discussed in Section 3.3, R_∞ surrounds exactly its two adjacent sides to each of the enclosure corner regions of D and exactly one side to the remaining exterior regions of D. Let v_c be a vertex of G^* dual to an enclosure corner region of D. We have already shown that G^* is 4-connected, i.e., $d(v_c) \geq 4$, $\forall v_c \in G^*$. If $d(v_c) = 4$, then two adjacent sides of R_c are surrounded by R_∞ whereas the remaining two sides of R_c are surrounded by two regions R_a and R_b. Clearly, R_a and R_b are the enclosure regions. Since G^* is an EPTG, every region of G^* is triangular. This implies that R_a and R_b are adjacent. Consequently, there is a separating triangle passing through v_∞ and vertices that are dual to R_a and R_b. Clearly, it encloses exactly one vertex v_c. This implies that there is no separating triangle inside this separating triangle and hence it is a critical separating triangle. This situation is depicted in Fig. 3.3a. If $d(v_c) > 4$, there are atleast three interior regions that surround R_c. Vertices that are dual to these interior regions together with v_∞ is a cycle of length atleast 4. Only possibility for the existence of a critical separating triangle passing through v_∞ and enclosing v_c is depicted in Fig. 3.3b. Now it is evident that there is atmost one critical separating triangle passing through v_∞ corresponding to each enclosure corner region. Since a rectangular graph has atmost four enclosure corner regions, there can be atmost 4 critical separating triangles passing through v_∞. This proves the second condition.

Sufficient Condition. Assume that the given conditions hold. We prove the result by applying the induction method on the vertices of G^*. Recall that an EPTG contains atleast two vertices. Let n be the number of vertices of G^*. If $n = 2$, then it is a graph consisting of a single edge and hence it is an RDG. Let us assume that $n > 2$ and the result holds for $n-1$ vertices, i.e., every $(n-1)$-vertex EPTG satisfying the given conditions is an RDG. In order to complete induction, we need to prove that n-vertex EPTG H satisfying the given conditions is an RDG. Since there can be atmost four critical separating triangles in H, there arise two possibilities: (1) there are exactly three edges between v_∞ and atleast one of the enclosure vertices, (2) there are exactly two edges between v_∞ and each enclosure corner vertex. Let v_i be an enclosure corner vertex of H and $A = \{v_1, v_2, \ldots, v_t\}$ be the set of vertices adjacent to v_i.

Consider the first case, i.e., there exist edges (v_i, v_∞), (v_i, v_p), (v_i, v_q) where vertices v_p and v_q are incident to v_∞ as shown in Fig. 3.4a. Construct a new EPTG H_1 by deleting v_i together with the incident edges and introducing new edges (v_∞, v_1), $(v_\infty, v_2) \ldots (v_\infty, v_t)$ (see Fig. 3.4b). We prove that H_1 satisfies the given conditions stated in the theorem.

Consider two vertices v_a and v_b of H such that $i \neq a, b$. As H is 4-connected, by Menger's Theorem 2.1.1, there exist four vertex-disjoint paths between v_a and v_b. Choose each path of the shortest possible length. If none of these paths uses the edges (v_i, v_p) and (v_i, v_q), then the same path would exist in H_1 with the edge (v_∞, v_k), $(1 \leq k \leq t)$ substituted in the place of $(v_k, v_i) \cup (v_i, v_\infty)$ if they occur in the path. Otherwise suppose that one of the four paths passes through (v_i, v_p). Being the shortest possible path, it can not pass through v_∞ or v_k, $(1 \leq k \leq t)$. Consequently, it must use the edge (v_i, v_q). If a path passes through v_∞, it would pass through v_p or v_q, contradicting to the facts that path is the shortest. Thus vertex v_∞ is not used by any of the four paths. Now by substituting the part $(v_i, v_p) \cup (v_i, v_q)$ of the path in H by $(v_p, v_\infty) \cup (v_\infty, v_q)$ in H_1, we can obtain 4 vertex-disjoint paths in H_1 also. Then by Menger's theorem, H_1 is 4-connected.

Next we claim that the number of critical separating triangles in H_1 can not be more than the number of critical separating triangles in H. As discussed in the necessary part that there is atmost one critical separating triangle enclosing an enclosure corner vertex and H has three enclosure corner vertices, there are atmost three critical separating triangles in H. Then the only possibility of occurring a separating triangle in H_1 is as follows.

If an enclosure vertex v_l is incident to both v_p, v_k where $v_k \in A$, then there exists a separating triangle in H_1 passing through v_k, v_p and v_l. Similarly, there can be another separating triangle in H_1 passing through v_t, v_q and $v_s \in A$. Thus there can be atmost two newly separating triangles in H_1. If there exists a critical separating triangle T_c containing v_i in H, then there are three possibilities:

i. there no longer remains T_c in H_1,

ii. T_c is contained in one of the new created separating triangles in H_1,

iii. One of the newly created separating triangle is contained in T_c.

All these possibilities show that there can not be more than four critical separating triangles in H_1. This shows that H_1 has atmost 4 critical separating triangles. Thus, H_1 has $n-1$ vertices satisfying the given conditions. By induction hypothesis, H_1 is an RDG and hence admits a rectangular dual. This rectangular dual can be transformed to another rectangular dual by adjoining a region R_i (corresponding to v_i) as shown in Fig. 3.4c. Then the resultant rectangular dual corresponds to H. Hence H is an RDG.

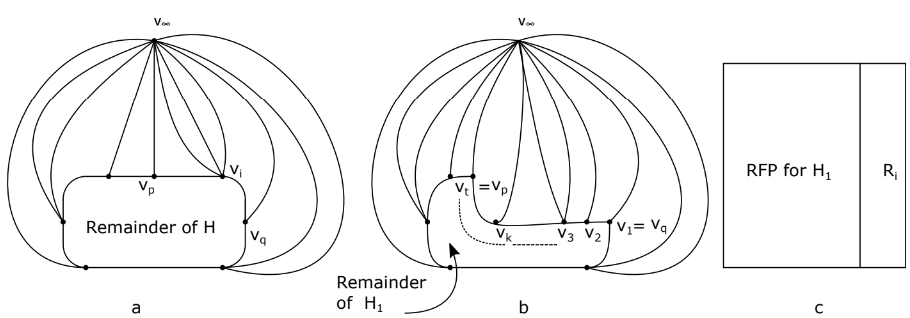


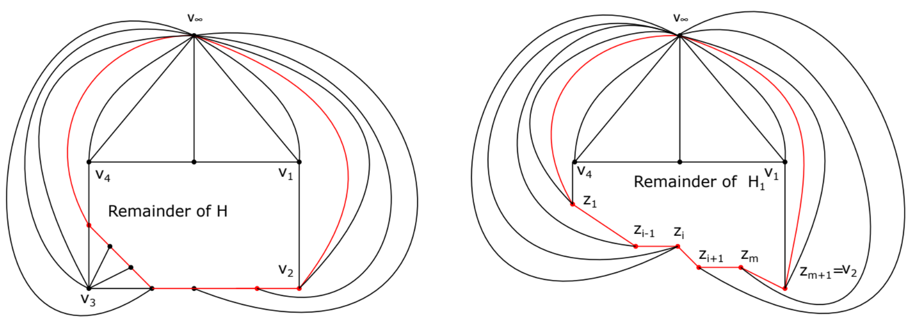
Figure 3.4: (a) Sketch of the graph H when there are three edges between v_∞ and enclosure vertex v_i, (b) sketch of the graph H_1 when there are exactly two edges between each enclosure corner vertex and v_∞, and (c) the construction of an rectangular dual for H.

Consider the second case. In this case, H appears as shown in Fig. 3.5a with atleast four more vertices v_1, v_2, v_3 and v_4. Consider the four enclosure corner vertices v_1, v_2, v_3 and v_4 as shown in Fig. 3.5a. Now we show that there is a separating cycle

C passing through v_i, v_∞ and an enclosure vertex v_d but not passing through v_3 or v_4 such that the removal of vertices of C from H disconnects it into two connected components, each containing atleast one vertex.

If there is an edge (v_1, v_3) in H, there is a separating cycle passing through v_1, v_3 and v_∞. In this case, H is separated into two parts, one of which contains atleast v_2 and another contains atleast v_4.

If there is no edge (v_1, v_3) in H. All vertices adjacent to v_3 lie on a path $y_1 y_2 \dots y_k$ where y_1 and y_k are the enclosure vertices. Let $y_k x_1 x_2 \dots v_2$ be a path of the enclosure vertices starting from y_k and ending with v_2. Then $C = t y_1 y_2 \dots y_k x_1 x_2 \dots v_2$ is a separating cycle which separates H into two parts, one of which contains atleast v_1 and another contains atleast v_3.



a b

Figure 3.5: (a) A separating cycle shown by red edges and (b) the appearance of H_u.

Once a separating cycle exists, there also exists the shortest separating cycle $C_s = v_\infty z_1 z_2 \dots z_m z_{m+1}$. This situation is depicted in 3.5b. Without loss of generality, suppose C_s separates v_1 and v_3. Construct an EPTG H_u from the subgraph contained in the interior of C by adding a vertex v_∞ and edges between v_∞ and enclosure vertices of this subgraph. The new edges in this construction are (v'_∞, z_1), (v'_∞, z_2), $\dots (v'_\infty, z_{m+1})$. Now we show that H_u satisfies the given conditions. Only possibility for creating a separating triangle is a triangle $z_i z_{i+1} v_\infty$ for $1 \leq i \leq n$. If there would exist an edge (z_i, z_{i+1}) in H_u, then it contradicts that C_s is the shortest separating cycle. Therefore, any cycle in H_u is of length atleast 4 and consequently, H_u is 4-connected and can not have more than 4 separating triangles. By induction hypothesis, H_u is an RDG. Similarly, we can show that the EPTG H_b constructed from the remaining part of H is an RDG. Then the corresponding rectangular dual can be placed one above the other and can be merged after applying homeomorphic transformation so as to preserve orthogonal directions of the edges such that the resultant floorplan is a rectangular dual of H as shown in Fig. 3.6. This completes the induction process and hence completes the proof. □

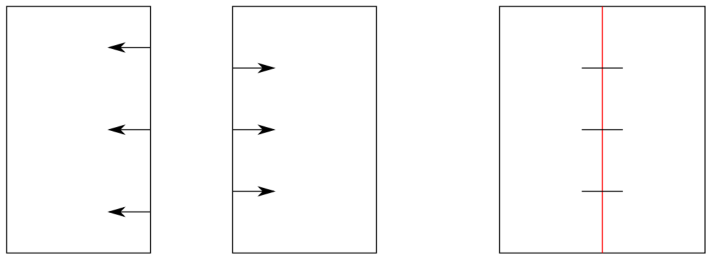


Figure 3.6: (a) Merging two RDGs of H_u and H_b into an RDG for H

Now we turn our attention to derive a necessary and sufficient condition for a PTG to be an RDG. A plane graph can be either nonseparable graph (block) or a separable connected graph. A disconnected graph is also a separable graph. However, we are not considering this case since a rectangular dual or floorplan are concerned with connectivity.

Theorem 3.4.2. A necessary and sufficient condition for a nonseparable PTG G to be an RDG is that it is 4-connected and has atmost 4 critical shortcuts.

Proof. **Necessary Condition.** Assume that G is an RDG. Then it admits a rectangular dual R. Let v_i be an interior vertex of G dual to a rectangle R_i of R. Recall that there require atleast 4 rectangles to surround a rectangle in a rectangular dual. Therefore, there exist atleast 4 rectangles in R enclosing R_i. Then v_i is adjacent to atleast 4 vertices. Since v_i is an arbitrary interior vertex of G, G is 4-connected.

To the contrary, if there exist 5 critical shortcuts in G, the corresponding EPTG G^* would contain 5 critical separating triangles, each passing through exactly one critical shortcut. This is a contradiction to Theorem 3.4.1. This shows that G can not have more than 4 critical shortcuts.

Sufficient Condition. Assume that the given conditions hold. Choose 4 enclosure corner vertices, each on the path joining the endpoints of the critical shortcut lying on its outermost cycle but not as the endpoints of these paths. If the number of critical shortcuts are less than 4, choose the remaining enclosure corner vertices randomly among enclosure vertices. Join each of these 4 vertices to v_∞ by two parallel edges and join each of the remaining $n-4$ enclosure vertices to v_∞ by a single edge. This constructs an EPTG G^* satisfying all the conditions given in Theorem 3.4.1. Hence G is an RDG. This completes the proof. □

Theorem 3.4.3. A necessary and sufficient condition for a separable connected PTG G to be an RDG is that:

i. each of its blocks is 4-connected,

ii. BNG is a path,

iii. both endpoints of an exterior edge of each of its blocks are not cut vertices,

iv. each maximal blocks corresponding to the endpoints of the BNG contains at most 2 critical shortcuts, not passing through cut vertices,

v. other remaining maximal blocks do not contain a critical shortcut, not passing through a cut vertex.

Proof. **Necessary Condition.** Assume that G is an RDG. The proof of the first condition is a direct consequence followed by Theorem 3.4.1. The BNG of G has the following possibilities:

i. it can be a path,

ii. it can be a cycle of length ≥ 3,

iii. it can be a tree.

To the contrary, suppose that the BNG is a cycle of length atleast 3. This implies that atleast three blocks share some cut vertex v_c of G. The construction of an EPTG G^* create more than 4 critical separating triangles, each passing through v_c, v_∞, and a vertex adjacent to v_c that belongs to the outermost cycle of each block. This situation can be depicted in Fig. 3.7a. Then by Theorem 3.4.1, G no longer is an RDG. A similar argument can be applied when it is a tree. This situation can be depicted in Fig. 3.8a. Thus, the BNG is left with one possibility, i.e., the BNG is a path.

To the contrary, suppose that both the endpoints of an exterior edge (v_i, v_j) of a block are cut vertices, then there are more than 4 critical separating triangles passing through v_i, v_j and v_∞ in G^*, which is a contradiction to Theorem 3.4.1. Hence both the endpoints of an exterior edge of a block can not be cut vertices simultaneously.

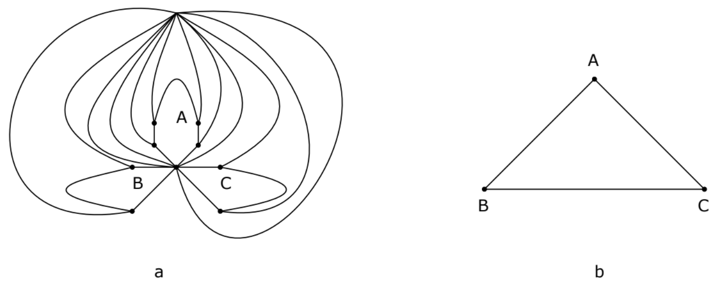


Figure 3.7: (a) A separable connected graph constituted by three blocks A, B and C, and (b) its BNG. Here only the outermost cycles of the blocks are shown.

Let M_i be a maximal block corresponding to the endpoints of the BNG. Since G is an RDG, each of its block is an RDG. Suppose that M_i is an RDG. Then it admits a rectangular dual F_i. It can be easily noted that out of 4 corner rectangular regions of F_i, only two can be the corner rectangular regions of F. Then there can be atmost two critical separating triangles in G^* and hence there can be atmost two critical shortcuts in each M_i. This implies that the fourth condition holds. Also, any other maximal block of the BNG can not share critical separating triangles since any corner rectangular region in R is a rectangular dual. This implies that no other maximal block has a critical separating triangle in G^* and hence there is no critical shortcut in the remaining maximal blocks.

Sufficient Condition. Assume that the given conditions hold. The first condition shows that G^* is 4-connected. The remaining conditions show that there are atmost four critical separating triangles in G^*. By Theorem 3.4.1, G is an RDG. Hence the proof. □

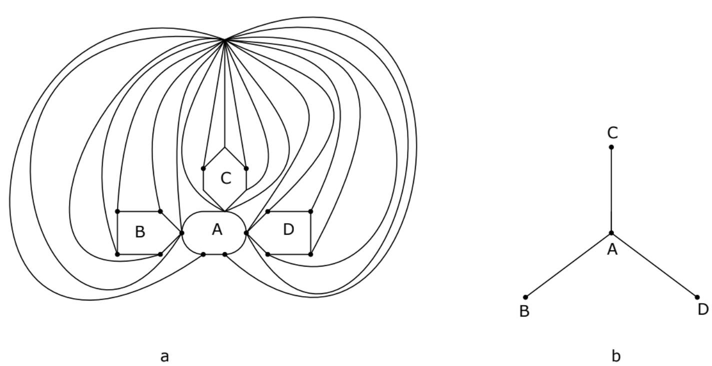


Figure 3.8: A separable connected graph constituted by three blocks A, B, C and D, and (b) its BNG. Here only the outermost cycles of the blocks are shown.

3.5 Concluding Remarks

We showed that a rectangular dual can be realized by a quadrangulation in the extended Euclidean plane. We found a critical flaw that invalidates the result given by Kozminski and Kinnen [38] and renewed existing graph theoretic characterizations of rectangular duals. A new RDG theory we developed, is easily implementable and it simplifies the floorplan construction process of the VLSI circuits as well as architectural buildings.

In future, it would be interesting to transform a non-RDG into an RDG by removing those edges which violates the RDG property and then adding new edges (maintaining RDG property) in such a way that the distances of endpoints of the deleted edges can be minimized. This idea would be useful in reducing the interconnection wire-lengths as well as in complex buildings, it gives the shortest possible paths for those pairs of rooms which are impossible to directly connect.