Chapter 5

Edge-Reducible Rectangularly Dualizable Graphs

In Chapter 4, we derived a new RDG from a given RDG by introducing new adjacencies among the vertices of the given RDG while preserving all the existing adjacencies among the vertices of the RDG until no more adjacency can be added. In this chapter, we are doing reverse of the previous Chapter 4 , i. e., the goal of this chapter is: we are removing adjacencies of the vertices of an RDG to construct a new RDG and looking for those minimal RDGs from which no adjacency among the vertices can be removed without violating RDG property. We also present an algorithm that constructs the new RDG and an algorithm that directly transform a rectangular dual to another rectangular dual by removing adjacencies of its rectangles.

5.1 Introduction

In Modern VLSI technology, one of constraints for circuit's floorplan known as boundary constraint is essential for better establishing input-output connections between VLSI circuit and outside world. In fact, adding this constraint in floorplanning increases the quality of circuit's floorplan [77, 78]. A boundary constraint refers to rectangles which need to be packed on the boundary of a floorplan. There lacks graph theoretic characterization for this constraint in floorplanning. In this chapter, we deal with this constraint with the help of graph notion.

There is some work on generation of rectangular duals from a given rectangular dual [37, 40, 64, 67]. In these methods except [67], a topologically distinct rectangular dual was obtained from an existing one for a given graph preserving the adjacencies of the existing one. Wang *et al.* [67] developed a method to generate a new rectangular dual by adding or deleting rectangles from an existing one.

A series of papers [2, 25, 27, 50, 56, 70, 72] studies enumerations of rectangular duals. Though it is trending recently to generate all rectangular partitions, each with n-rectangles of a given rectangle, but it is not preferable because computationally it is very expensive to pick a rectangular partition with the desired number of rectangles on its boundary from the large solution space.

In Chapter 4, we have seen that the generation of a new RDG by adding edges to a given RDG reduces the size of the new RDG. For a better understanding, consider an RDG G and its rectangular dual R shown in Fig. 5.1a and Fig. 5.1b respectively. If we want v_4 to be an exterior vertex in G, its corresponding rectangle R_4 needs to be shifted to the boundary of R as shown in Fig. 5.1c or Fig. 5.1d. Clearly, it either creates dead space (the shaded area as shown in Fig. 5.1d) which is not desirable [55] or it generates a rectilinear dual with an $L-$shaped region, but we are interested in rectangular duals only. From this example, it is clear that it is not always possible to shift a rectangle to the exterior while maintaining the rectangularity of a floorplan. Hence in this chapter, we present a graph theoretical characterization of rectangular duals for addressing their boundary constraints.

Mathematically, it is interesting to identify whether a given rectangular dual is transformable to another given rectangular dual by reducing adjacencies of its rectangles which may not always be true. Therefore we study the methods of transformations for rectangular dual from a graph theoretic perspective by introducing the concept of edge-reduction in an RDG. In this chapter, we introduce edge-reducible as well as edge-irreducible RDGs. Then we derive a necessary and sufficient condition for an RDG to be edge-reducible to another RDG. Further, we derive a necessary and sufficient condition for an RDG to be edge-irreducible. Then we show that such RDGs have no proper subgraph (except Hamiltonian path) which is an RDG. We also present a polynomial time algorithm to transform an edge-reducible RDG to an edge-irreducible RDG.

The chapter is structured as follows: In Section 5.2, we introduce the concept of edge-reduction of an RDG. Section 5.3 describes a necessary and sufficient condition for an RDG to be edge-reducible to another RDG. Section 5.4 describes a necessary and sufficient condition for an RDG to be edge-irreducible (a minimal one). In Section 5.5, we describe mainly two algorithms: the first one transforms an RDG to another RDG and the second directly transforms a rectangular dual to another rectangular dual. Finally, we conclude the derived results in Section 5.6.

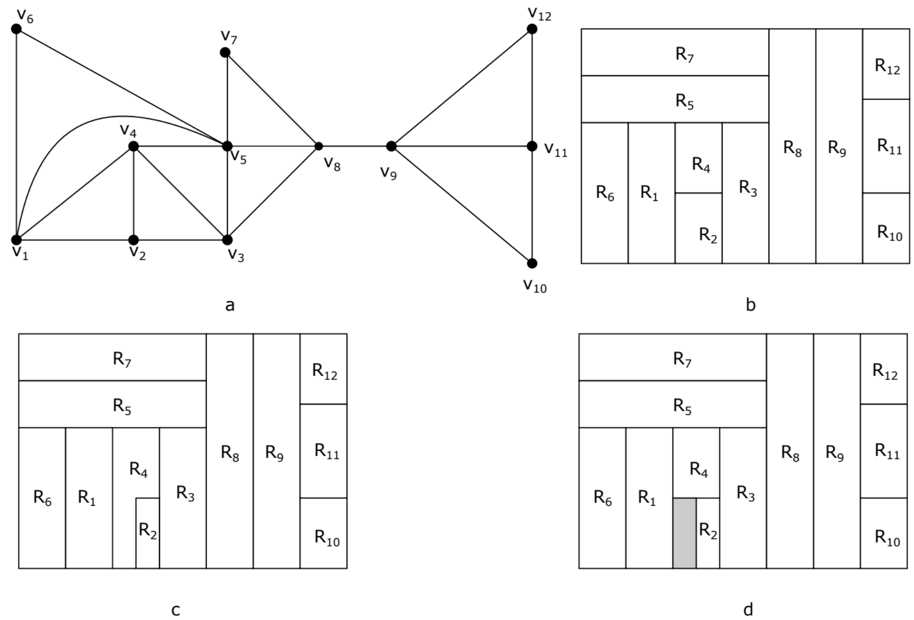

Figure 5.1: Example showing that it is not always possible to shift a rectangle to the boundary while preserving the rectangularity of a floorplan.

5.2 Concept of Edge-Reduction

In this section, we introduce two type of RDGs: an edge-reducible RDG and an edge-irreducible RDG.

Definition 5.2.1. Any two adjacent vertices v_i and v_j in an RDG $G = (V,E)$ are said to be *separable* if $G' = (V,E')$ is an RDG where $E' = E - (v_i, v_j)$.

Definition 5.2.2. An RDG is called *edge-reducible* if it has separable vertices, otherwise it is called *edge-irreducible* RDG.

Consider the two RDGs G_1 and G_2 shown in Fig. 5.2a and 5.2b respectively. Here G_1 is edge-reducible because the pairs of adjacent vertices v_5 and v_7, and v_3 and v_5 in G_1 are separable. After deleting the edges (v_5, v_7) and (v_5, v_3), G_1 reduces to G_2. Now we claim that G_2 is an edge-irreducible RDG.

Since G_2 is nonseparable, the removal of any interior edge (v_i, v_j) of G_2 makes one of its interior faces quadrangle. Then the corresponding four rectangles in its a rectangular dual meet at a point. This contradicts the assumption that no four rectangles meet at a point in a rectangular dual .

Now to the contrary suppose that any two exterior adjacent vertices v_i and v_j are separable in G_2. For instance, assume $v_i = v_2$ and $v_j = v_3$. Then BNG of $G_2 - (v_2, v_3)$ is a path of two vertices such that one of the maximal blocks corresponding to the end vertex of BNG has three critical CIPs $(v_6v_7v_8, v_2v_1v_{10}, v_4v_5v_6)$, which is a contradiction to the third condition of Theorem 2.2.2. Similarly we arrive at contradiction if we choose any other exterior edge. This implies that none of the exterior adjacent vertices are separable, i.e., G_2 is an edge-irreducible RDG.

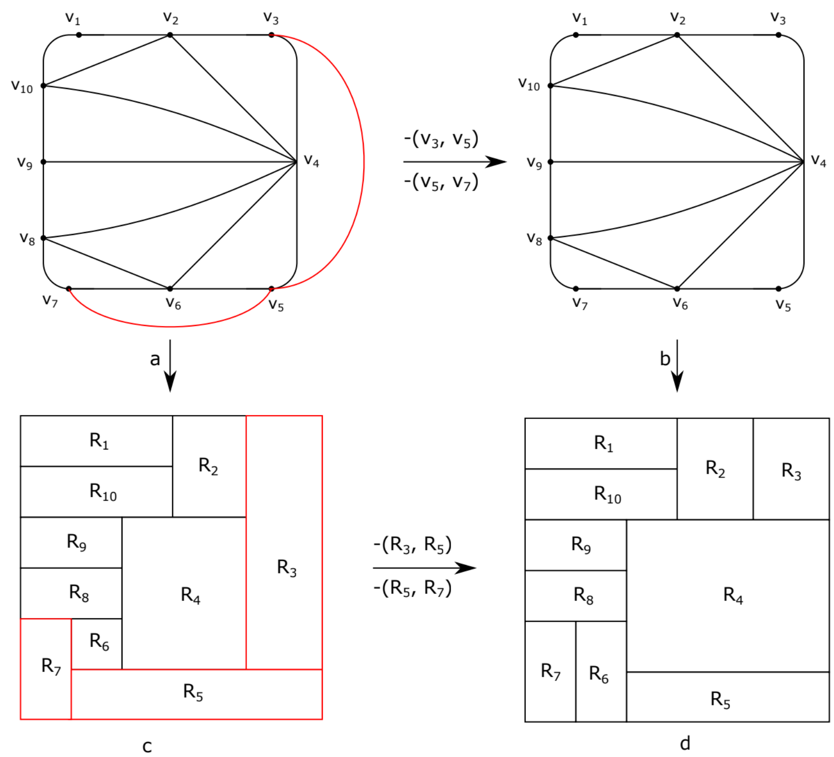

Figure 5.2: (a) An edge-reducible RDG G_1, (b) an edge-irreducible RDG G_2, (c) a rectangular dual for G_1 and (d) a rectangular dual for G_2

5.3 Theory of Edge-Reducible RDG

In this section, we derive a necessary and sufficient condition for an RDG to be edge-reducible to another RDG.

Lemma 5.3.1. If (v_i, v_j) is an exterior edge of a nonseparable RDG G, then $N(v_i) \cap N(v_j)$ is singleton.

Proof. To the contrary, suppose that $N(v_i) \cap N(v_j)$ is not singleton. Then there exist two triangles $v_i v_j v_s$ and $v_i v_j v_t$ in the plane embedding of G such that one of them is contained in other where v_s and v_t belongs to $(N(v_i) \cap N(v_j))$. Clearly, the one which encloses the other is a separating triangle of G. This is a contradiction to Theorem 2.2.1 since G is an RDG. Hence the result. □

Lemma 5.3.2. If (v_i, v_j) is an exterior edge of a nonseparable plane graph G with $d(v_i) > 2$ and $d(v_j) > 2$, then the number of CIPs of $G' = G - \{(v_i, v_j)\}$ is atleast the number of CIPs of G.

Proof. Suppose that P is a CIP in G. Then the following two cases arise:

i. (v_i, v_j) lies on P

   Since (v_i, v_j) is an exterior edge of G, by Lemma 5.3.1, there is exactly one common vertex v_k belonging to $N(v_i) \cap N(v_j)$. If v_k is an exterior vertex of G, then v_i, v_j and v_k form a triangle in G such that either $d(v_i) = 2$ or $d(v_j) = 2$, which is a contradiction. This implies that v_k is an interior vertex of G. Also, P and the edge joining its endpoints forms a cycle because of which v_k can not be a cut vertex of G. Hence, the removal of (v_i, v_j) from P increases its length by one in G' since on removing (v_i, v_j) from P, two sides of the triangle passing through (v_i, v_j) become a part of P. Thus we see that P is a CIP in G'.

ii. (v_i, v_j) does not lie on P

   Here P is of course a CIP in G'. But in this case, if $N(v_i) \cap N(v_j)$ is adjacent to an exterior vertex v_t in G, then $(N(v_i) \cap N(v_j), v_t)$ is a shortcut in G' and hence there is a new CIP in G' joining $N(v_i) \cap N(v_j)$ and v_t.

Thus, we see that the number of CIPs of $G - \{(v_i, v_j)\}$ always exceeds the number of CIPs of G. □

Lemma 5.3.3. If G is an RDG, then each of its face (region) is triangular.

Proof. Since G is an RDG, it admits a rectangular dual R. By the definition of a rectangular dual , no four rectangles of R meet at a point, i.e., there can only be 3-joints in R. Let R_1, R_2, R_3 be three rectangles of R meeting at a point or forming a 3-joint. Then three vertices of G which are duals to these rectangles of R form a cycle of length 3 in the interior of G. Hence each interior face (region) of G is triangular. □

Theorem 5.3.1. A necessary condition for two adjacent vertices v_i and v_j of an RDG to be separable is that (v_i, v_j) is an exterior edge of the RDG.

Proof. Let C and C' be the exterior faces of G and G' respectively. Suppose that there exist separable vertices v_i and v_j in an RDG G. Then $G' = (V, E')$ is an RDG where $E' = E - (v_i, v_j)$. By Lemma 5.3.3, all interior faces of both G and G' are of equal length (i.e., of length 3). But $E' \subsetneq E$ and, G and G' have the same number of vertices. This implies that C and C' have different length, i.e., $|C| < |C'|$. Also, when (v_i, v_j) is removed from C, the two other edges of the triangle passing through (v_i, v_j) becomes a part of C', i.e., removing an edge from C increases the size of C' by one. Hence, $|C'| - |C| = 1$ and (v_i, v_j) belongs to C, i.e., (v_i, v_j) is an exterior edge of G. □

It is interesting to note that for an exterior edge (v_i, v_j), v_i and v_j may not be separable. For example, refer to the RDG G_2 in Fig. 5.2b where none of the exterior vertices are separable. Hence, the converse of Theorem 5.3.1 is not true. Also, we can conclude that any two interior vertices v_i and v_j of an RDG G can never be separable.

Theorem 5.3.2. Suppose that C and C' are the exterior regions (faces) of RDGs $G = (V, E)$ and $H = (V, E')$ respectively. Denote $E - E'$ by Q. If G is edge-reducible to H, then $|Q| = |C'| - |C|$ and $E' \subsetneq E$.

Proof. Suppose that G is edge-reducible to H. By the definition of an edge-reducible RDG, we have $E' \subsetneq E$.

Consider $G_1 = G$. Since G is edge-reducible, by Theorem 5.3.1, there exists a nonempty set Q_1 of exterior edges incident to separable vertices of G_1. If $G_1 - Q_1 \neq H$, then there exists a nonempty set Q_2 of exterior edges incident to separable vertices of $G_1 - Q_1$. Denote $G_1 - Q_1$ by G_2. Similarly, if $G_2 - Q_2 \neq H$, then there exists a nonempty set Q_3 of exterior edges incident to separable vertices of $G_2 - Q_2$. Continuing in this way until $G_k - Q_k = H$. Consequently there exists a partition[1] $\{Q_1, Q_2, \ldots, Q_k\}$ such that each Q_i contains the exterior edges incident to separable vertices of G_i where $G_1 = G$, $G_{k+1} = H$ and $G_{i+1} = G_i - Q_i$, $(2 \leq i \leq k)$.

[1]A partition of a nonempty set A is a collection of its nonempty subsets A_1, A_2, $\ldots$, A_n such that $A_i \cap A_j$ $(i \neq j)$ and $A = A_1 \cup A_2 \cup \cdots \cup A_n$.

Suppose that C_i is the exterior face of G_i. Then

$$|Q_1| = |C_2| - |C_1| \tag{5.1}$$
$$|Q_2| = |C_3| - |C_2| \tag{5.2}$$
$$\dots \tag{5.3}$$
$$|Q_k| = |C_{k+1}| - |C_k| \tag{5.4}$$

adding (5.1)-(5.4), we get

$$|Q_1| + |Q_2| + \cdots + |Q_k| = |C_{k+1}| - |C_1|$$
$$\implies |Q| = |C'| - |C|$$

Hence the proof. □

Theorem 5.3.3. Suppose that $G = (V,E)$ and $H = (V,E')$ are two nonseparable RDGs where $E' \subsetneq E$. Denote $E - E'$ by Q. A necessary and sufficient condition for G to be edge-reducible to H is that there exists a partition $\{Q_1, Q_2, \dots, Q_k\}$ of Q such that each Q_i contains the edges of G_i and each G_i has atmost four CIPs where $G_1 = G$, $G_{k+1} = H$ and $G_{i+1} = G_i - Q_i,\ (1 \leq i \leq k)$.

Proof. **Necessary Condition:** Suppose that G is edge-reducible to H. By Theorem 5.3.2, there exists a partition $\{Q_1, Q_2, \dots, Q_k\}$ of Q such that each Q_i contains the edges of G_i where $G_1 = G$, $G_{k+1} = H$ and $G_{i+1} = G_i - Q_i,\ (1 \leq i \leq k)$.

It is given that G and H are nonseparable graphs. This implies that each G_i is also a nonseparable graph. Also, G_1 is an RDG. By Theorem 2.2.1, G_1 has atmost four CIPs. Now Q_1 contains the edges (edge) incident to separable vertices of G_1. Therefore $G_2 = G_1 - Q_1$ is an RDG. By Theorem 2.2.1, G_2 has atmost four CIPs. Again Q_2 contains the edges (edge) of G_2 incident to separable vertices of G_2. This implies that $G_3 = G_2 - Q_2$ is an RDG. By Theorem 2.2.1, G_3 has atmost four CIPs. Continue in this way, we can show that each G_i, $(i \geq 4)$ has atmost four CIPs.

Sufficient Condition: Suppose that there exists a partition $\{Q_1, Q_2, \dots, Q_k\}$ of Q such that each Q_i contains the edges of G_i and each G_i has atmost four CIPs where

$G_1 = G$, $G_{k+1} = H$ and $G_{i+1} = G_i - Q_i$, $(1 \leq i \leq k)$. Note that

$$G_2 = G_1 - Q_1 \tag{5.5}$$
$$G_3 = G_2 - Q_2 = G_1 - (Q_1 \cup Q_2) \tag{5.6}$$
$$\dots \tag{5.7}$$
$$G_{k+1} = G_1 - (Q_1 \cup Q_2 \cup Q_3 \cup \cdots \cup Q_k) \tag{5.8}$$
$$H = G - Q. \tag{5.9}$$

Consequently, we get

$$H = G_{k+1} \subsetneq G_k \subsetneq G_k \subsetneq \cdots \subsetneq G_3 \subsetneq G_2 \subsetneq G_1 = G. \tag{5.10}$$

In order to prove that G is edge-reducible to H, it is sufficient to prove that each RDG G_i is edge-reducible to RDG G_{i+1} since edge-reduction of RDGs is an equivalence relation.

Now G_2 is a proper subgraph of G_1. As G_1 is an RDG, each of its interior faces is triangular and Q_1 contains the exterior edges of G_1. Hence, each interior face of G_2 is triangular.

Now we claim that G_1 is edge-reducible to G_2, i.e., we need to show that Q_1 is a set of those edges which are incident to separable vertices of G_1, i.e., $G_1 - Q_1$ (G_2) is an RDG. Since G_1 is an RDG, it has no separating triangle and it is obvious to see that the removal of a set Q_1 of exterior edges from G_1 does not create any separating triangle. Also, G_2 is a nonseparable graph as it is a supergraph of H and it is given that G_2 has atmost four CIPs. Thus, by Theorem 2.2.1, G_2 is an RDG. In the similar fashion, we can show that each G_i, $(i \geq 3)$ is an RDG. □

Corollary 5.3.1. Suppose that $G = (V, E)$ is a nonseparable RDG and $G' = (V, E')$ is a nonseparable graph where $E' = E - (v_i, v_j)$ such that (v_i, v_j) is an exterior edge of G. Then the vertices v_i and v_j of G are separable if and only if G' has atmost four CIPs.

Theorem 5.3.4. Suppose that $G = (V, E)$ and $H = (V, E')$ are separable RDGs where $E' \subsetneq E$. Let $Q = E - E'$, $G_1 = G$, $G_{k+1} = H$ and $G_{i+1} = G_i - Q_i$, $1 \leq i \leq k$. A necessary and sufficient condition for G to be edge-reducible to H is that there exists a partition $\{Q_1, Q_2, \dots, Q_k\}$ of Q such that

i. each Q_i contains the edges of G_i,

ii. each maximal block corresponding to the endpoints of the BNG of G_i has atmost two critical CIPs,

iii. other maximal blocks of the BNG of G_i has no critical CIP.

Proof. It directly follows by applying Theorem 5.3.3 on each of its blocks and complying with Theorem 2.2.2. □

Corollary 5.3.2. Suppose that $G = (V,E)$ is a nonseparable RDG and $G' = (V,E')$ is a separable graph where $E' = E - (v_i, v_j)$. Then vertices v_i and v_j of G are separable if and only if each maximal block corresponding to endpoints of BNG of G' has atmost 2 critical CIPs and other maximal blocks of the BNG has no critical CIP [2.2.4].

Consider two RDGs $G = (V,E)$ and $H = (V,E')$ as shown in Fig. 5.3a and 5.3b respectively. Their respective rectangular duals are shown in Fig. 5.3c and 5.3d where v_i is dualized to R_i. It is clear that $E' \subsetneq E$.

$Q = E - E' = \{(v_1, v_3), (v_1, v_9), (v_7, v_9)\}$.

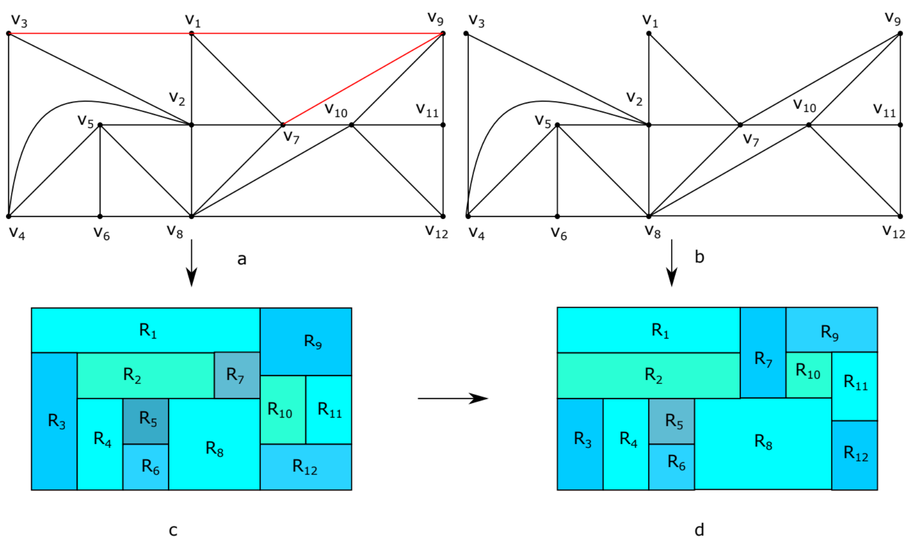

Figure 5.3: (a) A nonseparable RDG G is reducible to (b) another nonseparable RDG H. Their respective rectangular duals are shown in (c) and (d).

$$Q_1 = \{(v_1, v_3), (v_1, v_9)\}$$
$$Q_2 = \{(v_7, v_9)\}$$
$$G_1 = G$$
$$G_2 = G_1 - \{(v_1, v_3), (v_1, v_9)\}$$
$$G_3 = G_2 - \{(v_7, v_9)\}.$$

We can see that G_1 has no critical shortcut. This implies that there is no CIP in G_1. (v_2, v_4) and (v_2, v_7) are the critical shortcuts in G_2. Then there are two CIPs in G_2. By Theorem 5.3.3, G is edge-reducible to H.

5.4 Theory of Edge-Irreducible RDGs

In Section 5.3, we presented a necessary and sufficient condition for an RDG to be edge-reducible to another RDG. Now a natural question arises: Are there RDGs which are not reducible to any of RDGs? In this section, we derive a necessary and sufficient condition for a given RDG to be an edge-irreducible RDG.

Theorem 5.4.1. An RDG G has a vertex of degree 2 if and only if there exists a CIP of length 2 in G.

Proof. First assume that v_t is a vertex of degree 2 in G. Let G admit a rectangular dual R. We can see that any interior rectangle R_i of R is surrounded by atleast four rectangles of R and hence the corresponding vertex has degree atleast 4 in G. This shows that v_t is an external vertex. Let v_a and v_b be the two external vertices that are adjacent to v_t. By Lemma 5.3.3, every interior face of G is triangular. This implies that the vertices v_a, v_t and v_b form a triangle such that (v_a, v_b) is a shortcut and hence there exists a CIP joining vertices v_a and v_b in G.

Conversely, assume that there exists a CIP P of length 2 in G. Since P has length 2, there are three vertices v_i, v_j and v_k such that v_i and v_k are the endpoints of P and v_j is an interior vertex in P. As each interior face of G is triangular, vertices v_i, v_j and v_k forms a triangle and hence $d(v_j) = 2$. This proves our result. □

Lemma 5.4.1. If an RDG G has a vertex of degree 2, then it corresponds to a corner rectangle or an end rectangle in its rectangular dual. In particular, if there are 4 vertices of degree 2 in G, each corresponds to a corner rectangle in its rectangular dual .

Proof. Assume that G admits a rectangular dual R. Then by Lemma 5.3.3, each interior face of G is triangular. Also, it is given that G has a vertex v_a of degree 2. This implies there exists two vertices v_1 and v_2 that are adjacent to v_a only. Also, v_1 and v_2 must be adjacent since each interior face of G is triangular. Then there is a shortcut (chord) joining vertices v_1 and v_2 and hence there exists a CIP P containing vertices v_1, v_a and v_2 where v_1 and v_2 are the endpoints of P in G.

From [38], we know that a vertex on a CIP (except initial or end vertex of path) corresponds to either a corner rectangle or an end rectangle in the corresponding rectangular dual . It implies that there exists a vertex v_a in P that corresponds to either a corner rectangle or an end rectangle in R.

If there are four vertices of degree 2 in G, by Theorem 5.4.1, there are 4 CIPs each containing a vertex of degree 2 and hence, a vertex of degree 2 lying on each CIP correspond to a corner rectangle in R.

□

Theorem 5.4.2. A necessary condition for a nonseparable RDG G to be an edge-irreducible RDG is that it has exactly 4 vertices of degree 2.

Proof. In order to prove that G has exactly 4 vertices of degree 2, it is enough to show that there exist four CIPs, each of length 2, in G. To the contrary, first suppose that there does not exist a CIP in G. Consider a graph G_1 defined by $G_1 = G - \{(v_i, v_j)\}$ where (v_i, v_j) is an exterior edge of G. To arrive at a contradiction, we prove that G_1 is an RDG. By Lemma 5.3.1, $N(v_i) \cap N(v_j)$ is singleton. Suppose that $N(v_i) \cap N(v_j) = \{v_t\}$. For all possible cases for v_t, G_1 can have atmost one CIP as follows:

- If v_t is an interior vertex of G and is incident to some exterior vertex v_s of G, then (v_s, v_t) is a shortcut in G_1. Hence G_1 has a CIP.
- If v_t is an interior vertex of G and is not incident to any exterior vertex of G, then there is no shortcut in G_1 and hence G_1 has no CIP.
- If v_t is an exterior vertex but not a cut vertex of G, then either (v_i, v_t) or (v_j, v_t) is a shortcut in any plane embedding of G_1 and hence there exists a CIP joining vertices either v_i and v_t or v_j and v_t in G_1.
- If v_t is a cut vertex of G, then clearly there is no CIP in G_1.

Thus we see that G_1 can have atmost one CIP. Now if G_1 is a nonseparable graph, by Corollary 5.3.1, v_i and v_j are separable and hence G_1 is an RDG. If G_1 is a separable connected graph, by Corollary 5.3.2, v_i and v_j are separable and hence G_1 is an RDG. This contradicts our assumption that G is an edge-irreducible RDG and therefore G has atleast one CIP.

Again consider a graph $G_2 = G_1 - \{(v_k, v_l)\}$ where (v_k, v_l) is an exterior edge of G_1. Using the same analogy used for G_1, we can show that G_2 has a CIP and hence G has atleast two CIPs. Continuing in this way until G has exactly 4 CIPs. In fact, if G has more than four CIPs, then by Theorem 2.2.1 it no longer remains an RDG, which is a contradiction.

Now we prove that each of the four CIPs has length 2 in G. To the contrary, suppose that P is a CIP of length greater than 2 in G. Consider a graph $G_3 = G - \{(v_p, v_q)\}$ where (v_p, v_q) belongs to P. To contradict our assumption, we prove that G_3 is an RDG as follows:

- Every interior face of G_3 is triangular

  Since G is an RDG, by Lemma 5.3.3 each of its interior faces is triangular. Since (v_p, v_q) is an exterior edge of G, every interior face of G_3 is triangular.

- G_3 is a nonseparable graph

  Since G is a nonseparable graph, $d(v_p) \geq 2$ and $d(v_q) \geq 2$. If either $d(v_p) = 2$ or $d(v_q) = 2$, by Theorem 5.4.1 there is a CIP of length 2 in G which contradicts our assumption. This implies that $d(v_p) > 2$ and $d(v_q) > 2$ in G. Also by Lemma 5.3.1, $N(v_p) \cap N(v_q)$ is singleton. Suppose that $N(v_p) \cap N(v_q) = \{v_s\}$. Then v_s lies on the cycle C_P formed by CIP P and the edge joining to its endpoints or lies inside C_P in any plane embedding of G_3. Now if v_s lies on C_P, then either (v_p, v_s) or (v_q, v_s) is a shortcut in any plane embedding of G and hence there exists a CIP joining vertices either v_p and v_s or v_q and v_s in G having length 2 which contradicts our assumption. Therefore it lies inside C_P. Also, we have shown that $d(v_p) > 2$ and $d(v_q) > 2$ in G. Thus G_3 is a nonseparable graph.

- G_3 has atmost 4 CIPs

  Since (v_p, v_q) is an exterior edge of G and every face of G is triangular, by Lemma 5.3.1, $N(v_p) \cap N(v_q)$ has exactly one vertex and hence the edge $(N(v_p) \cap N(v_q), v_t)$ is well defined for some vertex v_t of G_3 where $v_t \notin (N(v_p) \cap N(v_q))$. If v_t is an interior vertex of G, then $(N(v_p) \cap N(v_q), v_t)$ is not a shortcut in G and hence G_3 has no CIP joining the vertices $N(v_p) \cap N(v_q)$ and v_t. If v_t is an exterior vertex of G, $(N(v_p) \cap N(v_q), v_t)$ is a shortcut in G_3 and hence there exists a CIP joining vertices $N(v_p) \cap N(v_q)$ and v_t in G_3. Note that P is no longer a CIP in G_3 since P contains endpoints of the shortcut joining vertices $N(v_p) \cap N(v_q)$ and v_t. This implies that the number of CIPs in G_3 can not exceed the number of CIPs in G. But G has exactly 4 CIPs. Thus G_3 has atmost 4 CIPs.

- G_3 has no separating triangles

  Since G is an RDG, by Lemma 5.3.3 each of its interior faces of G is triangular. Also by Theorem 2.2.1, it is independent of separating triangles. Note that (v_p, v_q) is an exterior edge of G. Clearly, G_3 is independent of separating triangles and every interior face of G_3 is triangular.

Thus by Theorem 2.2.1, G_3 is an RDG which contradicts our assumption that G is an edge-irreducible RDG. This implies that P is of length 2. Since P is arbitrary, every CIP has length 2. Thus G has exactly 4 CIPs, each of length 2. □

Consider a nonseparable RDG G shown in Fig. 5.4a whose rectangular dual is shown in Fig. 5.4b. It is interesting to note that G has 4 vertices of degree 2 and it is an edge-reducible RDG. Here vertices v_8 and v_{10} are separable and $G-(v_8, v_{10})$ is an RDG admitting rectangular dual shown in Fig. 5.4c. Thus, the condition in Theorem 5.4.2 is not sufficient.

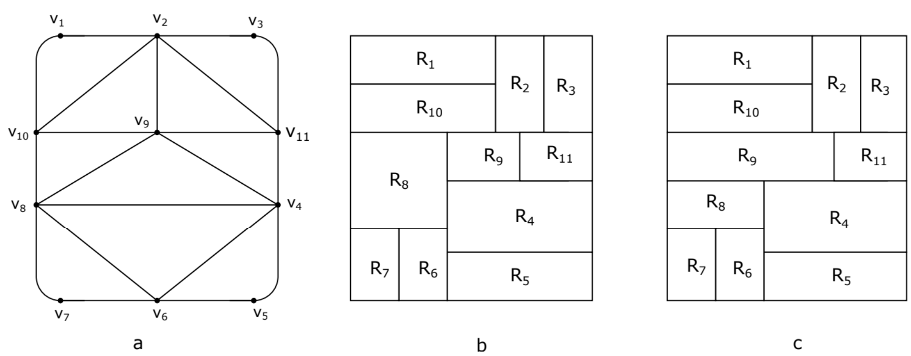

Figure 5.4: Showing that the converse of Theorem 5.4.2 is not true.

Theorem 5.4.3. A necessary and sufficient condition for a nonseparable RDG G to be an edge-irreducible RDG is that:

i. it has exactly 4 vertices of degree 2 on its outermost cycle C,

ii. for any edge (v_i, v_j) of C with $d(v_i) > 2$, $d(v_j) > 2$ and for an exterior vertex v_t on C, $(N(v_i) \cap N(v_j), v_t)$ $(t \neq i, j)$ is an interior edge such that there exists no vertex of degree 2 on a path lying on C joining the vertices $N(v_i) \cap N(v_j)$ and v_t.

Proof. If G has atmost two vertices, then necessary and sufficient conditions are trivial. Let us consider that G has atleast 3 vertices.

Necessary Condition: First suppose that G is an edge-irreducible RDG and C is the outermost cycle of G. The first condition follows from Theorem 5.4.2.

Consider an edge (v_i, v_j) on C such that $d(v_i) > 2$ and $d(v_i) > 2$. By Lemma 5.3.1, $N(v_i) \cap N(v_j)$ is singleton and hence the edge $(N(v_i) \cap N(v_j), v_t)$ is well defined.

Now consider a graph G_1 defined as $G_1 = G - \{(v_i, v_j)\}$. By Theorem 5.4.2, G has exactly 4 vertices of degree 2 and therefore by Theorem 5.4.1, G has exactly four CIPs, each of length 2. Then by Lemma 5.3.2, G_1 has at least 4 CIPs. Specifically, all four CIPs of G are the CIPs of G_1 and considering the proof of Theorem 5.4.2, G_1 may have one more CIP other than these 4 CIPs.

Since G is an edge-irreducible RDG, no two adjacent vertices in G are separable. This implies that G_1 is not an RDG. Also, it is given that G is an RDG. By Theorem 2.2.1, it is independent of separating triangles and by Lemma 5.3.3, each of its interior faces is triangular. Now (v_i, v_j) is an exterior edge of G. Therefore, G_1 is independent of separating triangles and each interior face of G_1 is triangular. At the same time, G_1 is not an RDG. It implies that G_1 has more than 4 CIPs. Consequently, G_1 has a CIP P, joining vertices $N(v_i) \cap N(v_j)$ and v_t, other than those of 4 CIPs which are common to both G and G_1. Then there exists an exterior vertex v_t such that $(N(v_i) \cap N(v_j), v_t)$ is an interior edge (critical shortcut) in G. Now $(N(v_i) \cap N(v_j), v_t)$ is a critical shortcut in G. It has no endpoints of any of the four CIPs. But each of these four CIPs has length 2. By Theorem 5.4.1, there exists no vertex of degree 2 on a path lying on C joining vertices $N(v_i) \cap N(v_j)$ and v_t.

Sufficient Condition: Assume that the given conditions holds. Consider a graph G_2 defined by $G_2 = G - \{(v_p, v_q)\}$, where (v_p, v_q) is an exterior edge such that $d(v_p) > 2$ and $d(v_q) > 2$. By the second condition of the assumption, there exists some exterior vertex v_t in G such that $(N(v_p) \cap N(v_q), v_t)$ is an interior edge in G. Then $N(v_p) \cap N(v_q)$ and v_t are the exterior vertices in G_2 which is in turn implies that $(N(v_p) \cap N(v_q), v_t)$ is a shortcut in G_2. Now we claim that $(N(v_p) \cap N(v_q), v_t)$ is a critical shortcut in G_2. By the first condition of the assumption, G has 4 vertices of degree 2. Then by Theorem 5.4.1, G has exactly 4 CIPs, each of length 2. By Lemma 5.3.2, G_2 has atleast four CIPs. Also, it is given that there exists no vertex of degree 2 on a path P_1 lying on C joining vertices $N(v_p) \cap N(v_q)$ and v_t. This means that the endpoints of the four CIPs (each having length 2) does not lie on the path joining vertices $N(v_p) \cap N(v_q)$ and v_t. Then $(N(v_p) \cap N(v_q), v_t)$ is a critical shortcut in G_2 and hence there exists a CIP P_1 joining vertices $N(v_p) \cap N(v_q)$ and v_t in G_2. Consequently, G_2 has exactly 5 CIPs. Now if G_2 is a nonseparable graph, then by Theorem 2.2.1, G_2 is not an RDG. If G_2 is a separable connected graph, then we claim that all the CIPs of G_2 are critical, i.e., none of the CIPs contains a cut vertex in its interior. Since G is a nonseparable graph, the vertex $N(v_p) \cap N(v_q)$ is the only cut vertex of G_2. Also, it is not the interior vertex of P_1 since $N(v_p) \cap N(v_q)$ is one of the endpoints of P_1. This implies that P_1 is a critical CIP of G_2. Now each of the remaining four CIPs of G_2 is of length 2. By Theorem 5.4.1, each of them has exactly one vertex of degree 2. We first show that $d(N(v_p) \cap N(v_q)) > 2$. To the contrary, suppose that $d(N(v_p) \cap N(v_q)) = 2$. Then $N(v_p) \cap N(v_q)$ is an exterior vertex of G_2. This implies that either $d(v_p) = 2$ or $d(v_q) = 2$ which is a contradiction since $d(v_p) > 2$ and $d(v_q) > 2$. This shows that $d(N(v_p) \cap N(v_q)) > 2$ and hence it cannot be in the interior of any of the four CIPs having length 2. Therefore all the four CIPs each of length 2 are also critical. Thus, we have shown that all CIPs of G_2 are critical. Recall that G_2 has only one cut vertex. This implies that it has only two blocks and clearly one of the two blocks has atleast three critical CIPs, since G_2 has 5 critical CIPs. Then by Theorem 2.2.2, G_2 is not an RDG.

Thus we see that the pair of vertices v_p and v_q is not separable in G. Since (v_p, v_q) is arbitrary, no pair of adjacent vertices in G is separable. Hence G is an edge-irreducible RDG. □

The RDG shown in Fig. 5.2b fulfills all the conditions of Theorem 5.4.3 and hence it is an edge-irreducible RDG admitting a rectangular dual shown in Fig. 5.2d.

Theorem 5.4.4. A necessary and sufficient condition for a separable connected RDG G to be an edge-irreducible RDG is that:

i. each maximal block corresponding to the endpoints of BNG of G is either a complete graph with two vertices or has exactly two vertices of degree 2 on its outermost cycle C and for any edge (v_i, v_j) on C with $d(v_i) \neq 2$, $d(v_i) \neq 2$ and for an exterior vertex v_t on C, $(N(v_i) \cap N(v_j), v_t)$ $(t \neq i, j)$ is an interior edge such that there exists no vertex of degree 2 on a path lying on C joining vertices $N(v_i) \cap N(v_j)$ and v_t.

ii. any other maximal block is a complete graph with two vertices or for any edge (v_i, v_j) on C and an exterior vertex v_t on C, $(N(v_i) \cap N(v_j), v_t)$ $(t \neq i, j)$ is an interior edge.

Proof. It directly follows by applying Theorem 5.4.3 on each of its blocks and complying with Theorem 2.2.2. □

Fig. 5.5a fulfills all the conditions of Theorem 5.4.4 and hence it is an edge-irreducible RDG whose rectangular dual is shown in Fig. 5.5b.

Theorem 5.4.5. A necessary and sufficient condition for RDG $G = (V, E_1)$ to be edge-irreducible RDG is that no proper subgraph $H = (V, E_2)$ (except Hamiltonian path) of G is an RDG.

Proof. **Necessary Condition:** Assume that G is an edge-irreducible RDG. This implies that there do not exist two adjacent vertices v_i and v_j in G which are separable and hence there exists no RDG such that $G' = (V, E')$ where $E' \subsetneq E$. Consequently, there does not exist no proper subgraph H that is an RDG.

Sufficient Condition: Assume that H is not an RDG. Since H is an arbitrary proper graph such that $|E_2| < |E_1|$, consider $E_2 = E_1 - (v_i, v_j)$. As H is not an RDG, vertices v_i and v_j are not separable in G and hence G is an edge-irreducible RDG. □

As discussed in Section 5.2, no pair of the vertices of G_2 shown in Fig. 5.2b is separable. This implies that no subgraph of G_2 with the same number of vertices is an RDG except Hamiltonian path of G_2. A rectangular dual for Hamiltonian path of G_2 can be constructed by arranging unit rectangles corresponding to its vertices in a row.

5.5 Constructive Algorithms for Rectangular Duals

In this section, we present algorithms for constructing new rectangular duals from the existing one by reducing its adjacencies: with graph notion and without graph notion.

We first mould proposed results in the form of algorithms. Algorithm 3 transforms an edge-reducible biconnected RDG to an edge-irreducible biconnected RDG and Algorithm 2 determines the number of CIPs in an RDG, which is a input requirement for Algorithm 2. We also analyze the complexity of these algorithms. In spite of developing the method of transformations of a rectangular dual to another rectangular dual from the context of graph theoretic notion, Algorithm 5 directly generates a new rectangular dual from a given rectangular dual. Algorithm 4 is a input requirement of Algorithm 3.

5.5.1 RDG Transformation Algorithm

In the most design problems, the underlying graphs of floorplans are biconnected. Therefore abiding by common design practice, we have described Algorithm 3 for transforming a biconnected RDG to another biconnected RDG. In fact, Algorithm 3 gives an edge-irreducible biconnected RDG as an output for an input biconnected RDGs only. Also, one can obtain output as the edge-reducible RDG by imposing some restrictions to Z or W (in the lines 10 and 13 of Algorithm 3 respectively). Suppose that one desires that a particular set X of adjacency relations must not be removed from the given RDG. Then Z or W needs to be replaced by $Z - X$ or $W - X$. Thus we see that Algorithm 3 can be made easily applicable to design problems. Algorithm 2 determines the number of CIPs in a nonseparable graph and it is used as a call function in Algorithm 3.

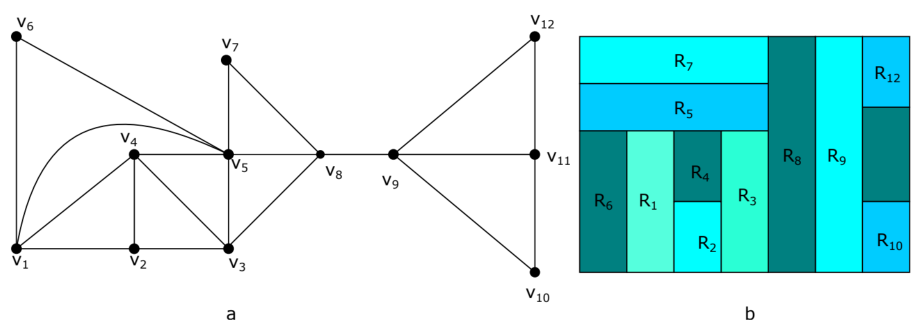

Figure 5.5: (a) A separable connected edge-irreducible graph G and (b) its rectangular dual .

Algorithm 2 ListOfCIPs(G, W)

Input: A biconnected RDG $G = (V, E)$

Output: The list of CIPs in G

```
W ← ϕ, L ← ϕ, U ← ϕ, X ← ϕ
for all (v_i, v_j) ∈ E do
    s ← |N(v_i) ∩ N(v_j)|
    if s == 1 then
        L ← L ∪ {(v_i, v_j)}
        U ← U ∪ {v_i, v_j}
    else
        continue
    end if
end for
for all (v_i, v_j) ∈ (E − L) do
    if v_i, v_j ∈ U then
        W ← W ∪ {(v_i, v_j)}
        X ← X ∪ {v_i, v_j}
    else
        continue
    end if
end for
for all (v_i, v_j) ∈ W do
    if (v_i, v_{i+1}), (v_{i+1}, v_{i+2}), ..., (v_{j−1}, v_j) ∈ L then
        if v_k ∈ X, i+1 ≤ k ≤ j−1 then
            W ← W − {(v_i, v_j)}
        else if (v_i, v_{i−1}), (v_{i−1}, v_{i−2}), ..., (v_{j+1}, v_j) ∈ L then
            if v_k ∈ X, i+1 ≤ k ≤ j−1 then
                W ← W − {(v_i, v_j)}
            end if
        end if
    else
        continue
    end if
end for
return W
```

Algorithm 3 **Reducing a biconnected RDG to an edge-irreducible biconnected RDG**

Input: A biconnected RDG $G = (V,E)$
Output: An edge-irreducible biconneted RDG $G' = (V,E')$

1: $Z \leftarrow \phi$
2: **for all** $(v_i, v_j) \in E$ **do**
3: $s \leftarrow |N(v_i) \cap N(v_j)|$
4: **if** $s == 1$ **then**
5: $Z \leftarrow Z \cup \{(v_i, v_j)\}$
6: **else**
7: **continue**
8: **end if**
9: **end for**
10: **for all** $(v_i, v_j) \in Z$ **do**
11: **if** $|N(v_i)| > 2 \wedge |N(v_j)| > 2 \wedge (N(v_i) \cap N(v_j)) == \{v_t\}$ **then**
12: ListOfCIPs($G = (V, E - \{(v_i, v_j)\}), W$)
13: **if** $|W| \leq 4$ **then**
14: $E \leftarrow E - \{(v_i, v_j)\}$
15: $Z \leftarrow Z \cup \{(v_i, v_t), (v_t, v_j)\} - \{(v_i, v_j)\}$
16: **else**
17: **print** G is an edge-irreducible biconnected RDG.
18: **end if**
19: **end if**
20: **end for**
21: **return** G'

Analysis of computational complexity

- The computational complexity of Algorithm 2 is linear

  The complexity of the lines $2-10$ is $|N(v_s)||N(v_t)||E| = K_1K_2|E| \cong \mathrm{O}(n)$, the complexity of the lines $11-18$ is $|U||E-L| \cong \mathrm{O}(n)$ and the complexity of the lines $19-31$ is $|W||L||X|^2 \cong \mathrm{O}(n)$. Hence complexity of Algorithm 2 is linear.

- The computational complexity of Algorithm 3 is $\mathrm{O}(n^2)$.

  The complexity of the lines $3-9$ is $|N(v_s)||N(v_t)||E| = K_1K_2|E| \cong \mathrm{O}(n)$,. The computational complexity of the lines $10-20$ is the product of $|N(v_i)||N(v_j)||Z|.|P_c||A|$ and the computational complexity of Algorithm 2. But $|N(v_i)|.|N(v_j)|.|Z||P_c||A| \cong \mathrm{O}(n^2)$. Hence complexity of Algorithm 3 is quadratic.

Remark 5.5.1. If for some graphs, $|N(v_s)||N(v_t)|$ or $|N(v_s)|$ is near to $|V|$, then complexity of Algorithm 2 and Algorithm 3 is cubic. However, in design problem such graphs do not appear quite often. It can be noted that both $|N(v_s)|$ and $|N(v_s)|$ can not be near to $|V|$ in a plane graph simultaneously.

Remark 5.5.2. The proof of correctness follows from the above sequence of theorems.

5.5.2 Transformation algorithm for rectangular duals

In this section, we present a transformation algorithm (Algorithm 5) to derive a new rectangular dual from a given rectangular dual. In Algorithm 5, a list of transformations is to be applied on a rectangular dual to reduce it to another rectangular dual. If a rectangular dual is rotated by 90°, its length and height get interchanged. Two rectangles m_1 and m_2 in a rectangular dual are said to be adjacent vertically (horizontally) if they share a wall or a section of a wall which is aligned vertically (horizontally).

Denote a rectangular dual with n-rectangles by RD(n). Let $m_i((x_i,y_i),w_i,h_i)$ denote the i^{th}-rectangle with lower left coordinate (x_i,y_i), length w_i and height h_i corresponding to a vertex v_i. Let m_i and m_j be two external rectangles that need to be separated and m_c be a rectangle adjacent to both m_i and m_j. Identify i^{th} and j^{th} rectangles accordingly as $h_j > h_i$ and if $h_i = h_j$, then choose j^{th} rectangle satisfying $x_i > x_j$. Let (x_i,y_i) be the lower left co-ordinate of m_i and $c_{i,j}$ represents the part of a wall that is common to both m_i and m_j. Further, denote i^{th} condition and i^{th} transformation by C_i and T_i respectively.

Conditions and Transformations

Here, we provide all conditions and transformations that arise due to all possible cases of rectangles m_i and m_j that need to be separated with the help of rectangles m_c adjacent to both m_i and m_j. The remaining cases are covered by rotations, flips of a rectangular dual as defined in Algorithm 5.

1. (a) C_1 : If $h_j > h_i$, $x_c > x_j$ and m_i and m_j are adjacent horizontally (Fig. 5.6a).

   (b) $T_1 : h_j \to h_j + c_{ij}, w_i \to w_i - c_{ij}, (x_j,y_j) \to (x_j, y_j - c_{ij})$ and $(x_i,y_i) \to (x_i + c_{ij}, y_i)$ (Fig. 5.6b).

2. (a) $C_2 : w_c < w_i$, $h_i < h_j$ (Fig. 5.6c).

   (b) $T_2 : w_i \to w_i - w_c, h_c \to h_c + h_i, (x_i,y_i) \to (x_i + w_c, y_i)$ and $(x_c,y_c) \to (x_c, y_c - h_i)$ (Fig. 5.6d).

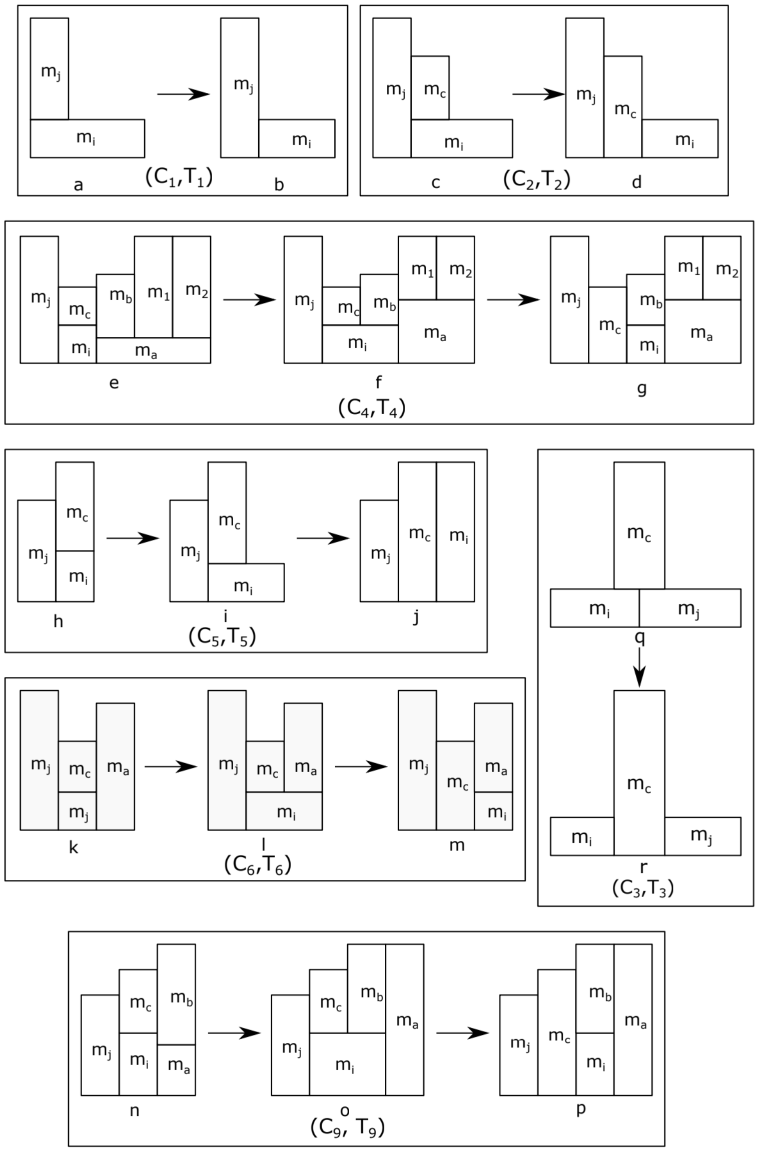

Figure 5.6: Geometric demonstrations of conditions and transformations

3. (a) $C_3 : l_c < l_i + l_j$, $l_c < l_i$, $l_c < l_j$ and $h_i = h_j$ (see Fig. 5.6q).

   (b) T_3 :.

      i. $h_c \to h_c + h_{i,j}$, $l_i \to l_i - l_{i,c}$, $l_j \to l_j - l_{j,c}$, $(x_c, y_c) \to (x_c, y_c - l_i)$ and $(x_j, y_j) \to (x_j + l_{j,c}, y_j)$ (see Fig. 5.6r)

Algorithm 4 TFunction(m_i, m_j, m_c, RD$_1$)

Input: (m_i, m_j, m_c, RD$_1$)
Output: (Boolean flag, RD$_1$)

```
if C_1 then
    newRD_1 = T_1(RD_1)
    return (true, newRD_1)
else if C_2 then
    newRD_1 = T_2(RD_1)
    return (true, newRD_1)
else if C_3 then
    newRD_1 = T_3(RD_1)
    return (true, newRD_1)
else if C_4 then
    newRD_1 = T_4(RD_1)
    return (true, newRD_1)
else if C_5 then
    newRD_1 = T_5(RD_1)
    return (true, newRD_1)
else if C_6 then
    newRD_1 = T_6(RD_1)
else if C_7 then
    newRD_1 = T_7(RD_1)
    return (true, newRD_1)
else if C_8 then
    newRD_1 = T_8(RD_1)
    return (true, newRD_1)
else if C_9 then
    newRD_1 = T_9(RD_1)
    return (true, newRD_1)
else if C_10 then
    newRD_1 = T_10(RD_1)
    return (true, newRD_1)
else if C_11 then
    newRD_1 = T_11(RD_1)
    return (true, newRD_1)
else if C_12 then
    newRD_1 = T_12(RD_1)
    return (true, newRD_1)
else
    return (false, RD_1)
end if
```

4. (a) $C_4 : w_c = w_i$, $h_i < h_j$, m_i is adjacent to m_a, m_b $(a, b \neq j, c)$ and m_t, $1 \leq t \leq p, t \neq i$ (Fig. 5.6e).

   (b) T_4 :

      i. $w_i \rightarrow w_i + w_b$, $w_a \rightarrow w_a - w_b$, $h_b \rightarrow h_b - c_{i,b}$, $(x_a, y_a) \rightarrow (x_a + h_b, y_a)$ (Fig. 5.6f),

ii. $h_a \to h_a + c_{i,b} + h_c/2$ and $h_t \to h_t - c_{i,b} - h_c/2$, $(x_t, y_t) \to (x_t, y_t + c_{i,b} + h_c/2)$ for $1 \leq t \leq p$ (Fig. 5.6f).

iii. T_2 (Fig. 5.6g).

Algorithm 5 Transformation of an RD$_1(n)$ to an another RD$_2(n)$.

Input: A RD$_1$, G$_n$, $X = X_1 \cup X_2 \cdots \cup X_k$.

Output: A RD$_2(n)$ for H.

```
for all t from t = 1 to k do
    for all (v_i, v_j) ∈ X_t do
        v_c = getNeighbor(v_i, v_j, G_n)
        (flag, RD_1)←TFunction(m_i, m_j, m_c, RD_1)
        if flag = true then
            continue
        else
            ROTATE RD_1 90° anticlockwise
            (flag, RD_1)←TFunction(m_i, m_j, m_c, RD_1)
            if flag = true then
                continue
            else
                ROTATE RD_1 by 180° anticlockwise
                (flag, RD_1)←TFunction(m_i, m_j, m_c, RD_1)
                if flag = true then
                    continue
                else
                    FLIP RD_1 right to left
                    (flag, RD_1)←TFunction(m_i, m_j, m_c, RD_1)
                    if flag = true then
                        continue
                    else
                        FLIP RD_1 upper to lower
                        (flag, RD_1)←TFunction(m_i, m_j, m_c, RD_1)
                        if flag = true then
                            continue
                        else
                            ROTATE RD_1 90° clockwise
                            (flag, RD_1)←TFunction(m_i, m_j, m_c, RD_1)
                        end if
                    end if
                end if
            end if
        end if
    end for
end for
```

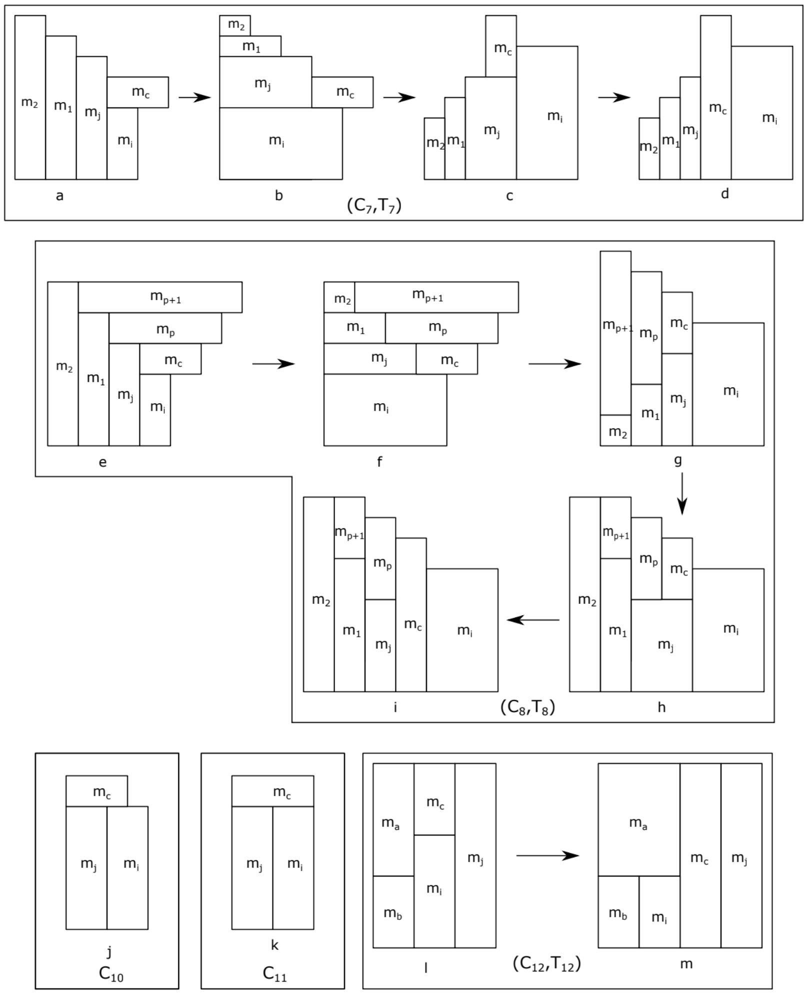

Figure 5.7: Geometric demonstrations of conditions and transformations

5. (a) $C_5 : w_c = w_i$, $h_i < h_j$ and both m_c and m_i are adjacent to the same side of the exterior (Fig. 5.6h). Here we assume that exterior has four sides, i.e., east, west, north and south.

   (b) T_5 :

      i. $w_i \to 2w_i, (x_i, y_i) \to (x_i, y_i)$ (Fig. 5.6i),

      ii. $h_c \to h_c + c_{i,j}, w_i \to w_i/2, (x_c, y_c) \to (x_c, y_i - h_i)$, and

      iii. $h_i \to h_c - h_i$ and $(x_i, y_i) \to (x_i + w_c, y_i)$ (Fig. 5.6j).

6. (a) $C_6 : w_c = w_i, h_i < h_j$, and m_i is adjacent to the exterior rectangle $m_a, a \neq j, c$ (Fig. 5.6k).

(b) T_6 :

i. $h_a \to h_a - h_i$, $(x_a, y_a) \to (x_a, y_a + c_{i,a})$ and $w_i \to w_i + w_a$ (Fig. 5.6l),

ii. T_2 (Fig. 5.6m).

7. (a) $C_7 : w_c > w_i$, $h_i + h_c < h_j$ and m_j is adjacent to m_1 vertically, m_k is adjacent to m_{k+1} vertically, $(1 \leq k \leq (p-1))$ and $h_j < h_1, h_k < h_{k+1}, (1 \leq k \leq (p-1))$ (Fig. 5.7a).

(b) T_7 :

i. $h_k \to h_k - h_{k-1}, (x_{k-1}, y_{k-1}) \to (x_{k-1} - w_k, y_{k-1})$ and $w_{k-1} \to w_{k-1} + w_k$, repeat it for every k, where $k = p, p-1, \ldots, 2$, in succession (Fig. 5.7b),

ii. $h_1 \to h_1 - h_j$, $w_j \to w_j + w_1$ and $(x_j, y_j) \to (x_j - w_1, y_j)$ (Fig. 5.7b),

iii. $w_i \to w_i + w_j$, $h_j \to h_j - h_i$ and $(x_i, y_i) \to (x_i - w_j, y_i)$ (Fig. 5.7b),

iv. T_1 (Fig. 5.7c),

v. T_2 (Fig. 5.7d).

8. (a) $C_8 : w_c > w_i$, $h_i + h_c = h_j$, $m_j, m_1, m_2, ..., m_{p-1}$ be such that m_j is adjacent to m_1 vertically, m_k is adjacent to m_{k+1} vertically such that $h_j < h_1, h_i < c_{i+1}$, $(1 \leq k \leq (p-2))$ and $m_p, m_{p+1}, ..., m_t$ be such that m_j and m_c are adjacent to m_p horizontally, m_l is adjacent to m_{l+1} horizontally such that $w_j + w_c < w_p, w_l < w_{l+1}$, $(p \leq l \leq (t-1))$ (Fig. 5.7e).

(b) T_8 :

i. $h_t \to h_t - h_{t-1}$, $w_{t-1} \to w_{t-1} + w_t$ and $(x_{t-1}, y_{t-1}) \to (x_{t-1} - w_t, y_{t-1})$. Perform it for every t where $t = p, p-1, \ldots, 3, 2$ in succession (Fig. 5.7f),

ii. $h_1 \to h_1 - h_j$, $w_j \to w_j + w_1$ and $(x_j, y_j) \to (x_j - w_1, y_j)$ (Fig. 5.7f),

iii. $h_j \to h_j + 1$, $w_1 \to w_j + 1$, $(x_j, y_j) \to (x_j, y_j + 1)$ and $w_t \to w_{t-1} + 1$ for $2 \leq q \leq t$ (Fig. 5.7f),

iv. $w_i \to w_i + w_j$, $h_j \to h_j - h_i$, and $(x_i, y_i) \to (x_i - w_j, y_i)$ (Fig. 5.7f),

v. T_1 (Fig. 5.7g),

vi. $w_j \to w_j + w_1, (x_j, y_j) \to (x_j - w_1, y_j), h_p \to h_p - c_{j,p}, (x_p, y_p) \to (x_p, y_p + h_{j,p})$ (Fig. 5.7h),

vii. $h_k \to h_j + h_l/2, (x_k, y_k) \to (x_k - l_{k+1}, y_k), (p \leq l \leq t), (1 \leq k \leq (p-1))$ (Fig. 5.7h),

viii. $h_p \to h_{p+1} + h_{p-1}, (x_p, y_p) \to (x_p - l_p, y_p)$ (Fig. 5.7h),

ix. T_2 (Fig. 5.7i).

9. (a) $C_9 : w_c = w_i$, $h_i < h_j$, $h_i > h_a$ and m_i is adjacent to both the exterior rectangles m_a and m_b, $a, b \neq j, c$ (Fig. 5.6n).

   (b) T_9 :

   i. $h_i \to h_i + h_a$, $w_b \to w_b - w_{a,i}$, and $(x_b, y_b) \to (x_b, y_b - h_{i,a})$ (Fig. 5.6o),

   ii. $h_a \to h_b + h_i$, $(x_a, y_a) \to (x_a + w_b, y_a)$ (Fig. 5.6p),

   iii. T_2 (Fig. 5.6m).

10. (a) $C_{10} : w_c = w_i + w_j$, $h_i = h_j$, m_i is adjacent to rectangles m_t, $(1 \leq t \leq p)$, $(t \neq j, c)$ such that $h_i < \sum_{t=1}^{p} h_t$, and $c_{i,p} < h_p$ and m_j is adjacent to rectangles m_k, $(1 \leq k \leq q)$, $(k \neq i, c)$ such that $h_j < \sum_{k=1}^{q} h_k$ and $c_{j,q} < h_q$ (Fig. 5.7j),

   (b) T_{10} :.

   i. $h_p \to h_p - c_{i,p}$, $w_i \to w_i + w_1/2$, and $w_t \to w_t - w_t/2$, $(1 \leq t \leq (p - 1))$,

   ii. $h_q \to h_q - c_{i,q}$, $w_j \to w_j + w_1/2$, and $w_k \to w_k - w_t/2$, $(1 \leq k \leq (q - 1))$,

   iii. T_2.

11. (a) C_{11} :

   $w_c = w_i + w_j$, $h_i = h_j$, $w_{i,c} < w_i$, and m_j is adjacent to rectangles m_k, $(1 \leq k \leq q)$, $(k \neq i, c)$ such that $h_j < \sum_{k=1}^{q} h_k$ and $c_{j,q} < h_q$ (Fig. 5.7k).

   (b) T_{11} :

   i. $h_q \to h_q - c_{i,q}$, $w_j \to w_j + w_1/2$, and $w_k \to w_k - w_t/2$, $(1 \leq k \leq (q - 1))$,

   ii. T_2.

12. (a) C_{12} :

   $w_c = w_i$, $h_i < h_j$, $w_{i,c} = w_j$, m_j is through rectangle[3], and m_a is adjacent to both m_i and m_c (Fig. 5.7l).

   (b) T_{12} :

[3]A rectangle in a rectangular dual is called through rectangle if its two opposite sides are adjacent with exterior.

i. $(x_j, y_j) \to (x_{j+1}, y_{j+1})$ and $(x_k, y_k) \to (x_{k+1}, y_{k+1})$, where every rectangle m_k lies on the right side of m_j, $(1 \leq k \leq q)$ (Fig. 5.7m),

ii. $h_i \to h_i - h_{i,a}$, $w_a \to w_a + w_c$, $h_c \to h_j$, and $(x_c, y_c) \to (x_{i+1}, y_{i+1})$ (Fig. 5.7m).

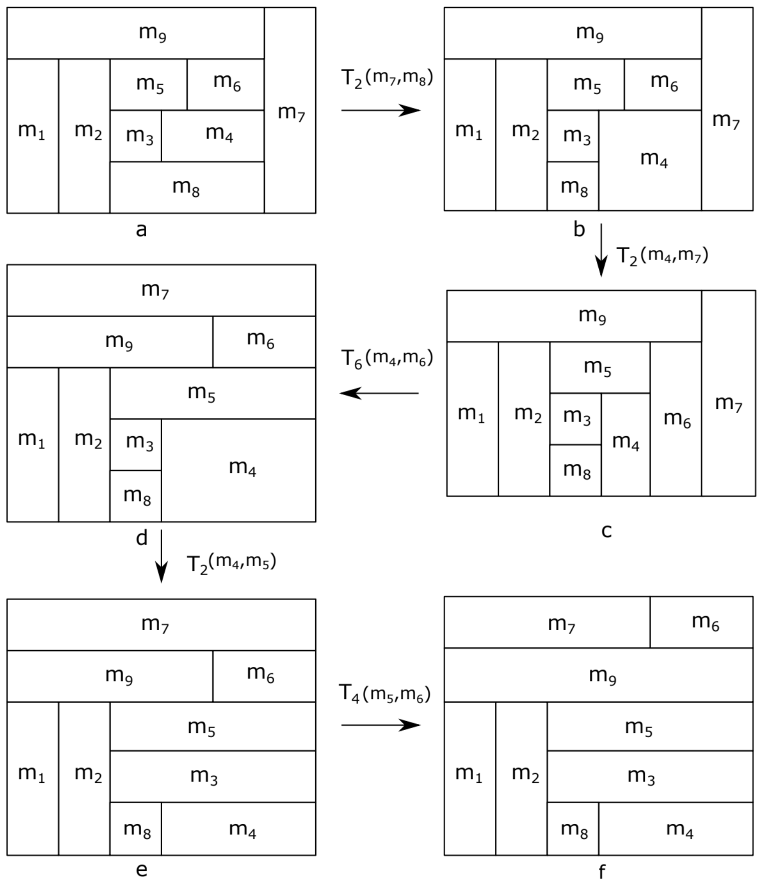

Figure 5.8: (a) An existing rectangular dual, (b-i) intermediate steps of the transformations, and (i) a reduced rectangular dual.

Theorem 5.5.1. Algorithm 5 can be implemented in O(n)-time.

Proof. Note that

(a) v_c can be found in linear time (line 3).

(b) All $(v_i, v_j) \in X_t$ can be found in $n|X_t|$ time.

(c) Time complexity of the lines 1-30 is $n \sum_{t=1}^{k} |X_t| = n|X|$.

Hence, the complexity of Algorithm 5 is $O(n)$. □

For instance, consider an RDG G and its rectangular dual $RD_1(9)$ shown in Fig. 5.8b and Fig. 5.8a respectively. Using Algorithm 5, the rectangular dual $RD_1(9)$ can be transformed to another rectangular dual $RD_2(9)$ shown in Fig. 5.8i.

5.6 Concluding Remarks

We studied the method of transformations among rectangular duals from graph notion. We derived a necessary and sufficient condition for an RDG to be edge-reducible to another RDG and implemented it in polynomial time. It is useful to deal with boundary constraint in rectangular floorplans. The crux of this approach is to identify the maximum size of the exterior of a rectangular dual. In other words, whenever an RDG $G_2 = (V, E_2)$ is a super graph of any edge-irreducible RDG $G_1 = (V, E_1)$ such that $E_1 \subset E_2$, its rectangular dual can be constructed with desired number of rectangles on its boundary.

We also derived a necessary and sufficient condition for an RDG to be edge-reducible RDG and to be edge-irreducible RDG. We also showed that no subgraph H (except Hamiltonian path) of each of edge-irreducible RDGs is an RDG.

We also showed that an edge-reducible RDG can be restored to a minimal one (an edge-irreducible RDG) and presented an algorithm (Algorithm 3) to restore the first one to the minimal one. The removal of an edge from a reducible RDG takes an interior vertex to the exterior.