Chapter 7

A Class of Area-Universal Rectangular Duals

In this chapter, we describe a class of RDGs wherein each RDG can be realized by an area-universal weak rectangular dual upto combinatorial equivalence in polynomial time. We also show that every induced rectangularly dualizable subgraph of graphs in this class can also be realized by an area-universal weak rectangular dual.

7.1 Introduction

Recall that a rectangular dual is *area-universal* [22] if any assignment of areas to each of its rectangles can be realized by a *combinatorially weak equivalent* rectangular dual. If there assign area to each of its rectangles, then the rectangular dual is known as a *rectangular cartogram*. More precisely, a rectangular cartogram is a map in which each region is a rectangle enclosing a rectangular area where the size of each rectangle relies on the area assigned to it.

The rectangular cartograms [49] are used to visualize spatial information (it may be economic strength, population etc.) of geographic regions, i.e. they are used to display more than one quantity associated with the same set of geographic regions (in [49], the population, land area and wealth within the United States were shown as cartograms). The visual comparison of multiple cartograms corresponding to the same set of geographic regions can be made easier if each of the cartograms is area-universal. Area-universal rectangular duals are useful in controlling aspect ratios [30, 44, 79]. At an early stage, the areas of soft modules are not yet known, however the relative positions of modules of a VLSI circuit are known. Constructing an area-universal rectangular dual enables us to assign areas to its rectangles (modules) at the later stage. Thus, the ability of constructing an area-universal rectangular dual at an early stage optimizes circuit's area at the later stage.

Not every RDG can be realized by an area-universal rectangular dual [22, 51]. Rinsma [51] described a vertex-weighted outer planar G (area is assigned to each of its vertex) such that there exists no rectangular dual for G having these weights as rectangles' areas. Thus it is interesting to know when a rectangularly dualizable graph can be realized by an area-universal rectangular dual. Recently, Eppstein et al. [22] derived the following necessary and sufficient condition for a rectangular dual to be area-universal.

Theorem 7.1.1. [22, Theorem 2] A rectangular dual R is area-universal if and only if every maximal internal line segment of R is a side of atleast one of its component rectangles.

Eppstein et al. [22] described an algorithm that constructs an area-universal weal rectangular dual for an RDG G, if it exists. The computational complexity of this algorithm is $O(2^{O(K^2)}n^{O(1)})$ where K is the maximum number of 4 degree vertices in any minimal separation component. This algorithm is not fully polynomial. For instance, if K is fixed, it runs in a polynomial time but in general, it runs in an exponential time.

The chapter is structured as follows: any efficient algorithm to construct an area-universal weak rectangular dual for a given RDG, if it exists, still unknown. We describe an important class of RDGs in which each RDG can be realized by an area-universal weal rectangular dual in polynomial time. We also show that every induced rectangular dualizable subgraph of graphs in the class can also be realized by an area-universal rectangular dual. This work can be seen in Section 7.2. In Section 7.3, we conclude our contribution.

7.2 Constructive Algorithm for Area-Universal Weak Rectangular Duals

In this section, we first identify a class $\mathscr{L}$ of planar graphs in which every graph admits an area universal rectangular dual. Then we present four algorithms. Algorithm 6 computes a set P of 4 degree vertices forming a path in a given RDG G which is the input for Algorithm 7 as well as Algorithm 8. Algorithm 7 decides whether the given RDG belongs to $\mathscr{L}$. Algorithm 8 constructs an area-universal rectangular dual for the given RDG that belongs to $\mathscr{L}$. Algorithm 9 computes the position of rectangles to be inserted in the rectangular dual. It is used as a call function by Algorithm 8. Then we analyze the computation complexity of these algorithms. The proof of correctness of Algorithm 8 follows from Theorem 7.2.1 and the characterization of $\mathscr{L}$ follows from Theorem 7.2.2. It is noted that Algorithm 7 works for biconnected graphs only which may not be seen as a restriction since a separable connected graph can be partitioned into biconnected subgraphs.

The class $\mathscr{L}$ of RDGs is defined as follows: Let G be an RDG. Let L_1 be a set of 4 degree vertices forming a path in G such that $|L_1| > 1$. Let L_i $(i \geq 2)$ be the sets of vertices of G such that $L_{i-1} \subset L_i$ and has exactly one more vertex v_s other than the vertices of L_{i-1} with the property $|N(v_s) - (N(v_s) \cap L_{i-1})| \leq 3$.

Algorithm 6 getDegree4AdjacentVertices

Input: An RDG $G = (V, E)$

Output: A set P of 4-degree vertices forming a path

```
P = ϕ
checked = ϕ
chainStarted = true
for all v ∈ V do
    if d(v) == 4 then
        while chainStarted == true do
            P = P ∪ {v}
            checked=checked ∪{v}
            for u ∈ (N(v)− checked) do
                chainStarted = true
                if d(u) == 4 then
                    v = u
                    break
                end if
            end for
            chainStarted = false
        end while
        if |P| > 1 then
            return P
        else
            print G does not belongs to ℒ.
        end if
    end if
end for
```

As the elements of P forms a path, therefore they can be indexed by another set $P_1 = \{v_1, v_2, \ldots, v_k\}$ in such a way that v_i is adjacent to v_{i+1} where v_1 is the initial point and v_k is the end point of the path. We use this indexed set P_1 as the input requirement of Algorithm 8.

Algorithm 7 Checking whether G belongs to $\mathscr{L}$

Input: An RDG $G = (V, E)$ and P (computed by Algorithm 6),
Output: Either G belongs to $\mathscr{L}$ or not

$L = P$
for all $v_i \in V - L$ **do**
  if v_i is adjacent to a vertex of L such that $|N(v_i) - (N(v_i) \cap L)| \leq 3$ **then**
    $L = L \cup \{v_i\}$
  else
    continue
  end if
end for
if $L == V$ **then**
  return G belongs to $\mathscr{L}$
else
  print G does not belong to $\mathscr{L}$
end if

For an example, consider an RDG G_1 shown in Fig. 7.1. For G_1, we have
$L_1 = \{v_2, v_3\}$,
$L_2 = \{v_1, v_2, v_3\}$ and $|N(v_1) - (N(v_1) \cap L_1)| = 2$,
$L_3 = \{v_1, v_2, v_3, v_4\}$ and $|N(v_4) - (N(v_4) \cap L_2)| = 2$,
$L_4 = \{v_1, v_2, v_3, v_4, v_5\}$ and $|N(v_5) - (N(v_5) \cap L_3)| = 3$,
$L_5 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $|N(v_6) - (N(v_6) \cap L_4)| = 2$,
$L_6 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $|N(v_7) - (N(v_7) \cap L_5)| = 3$,
$L_7 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $|N(v_8) - (N(v_8) \cap L_6)| = 1$,
$L_8 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and $|N(v_9) - (N(v_9) \cap L_7)| = 1$,
$L_9 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and $|N(v_{10}) - (N(v_{10}) \cap L_8)| = 0$.

This shows that G_1 in Fig. 7.1 belongs to $\mathscr{L}$. Also, for G_1, Algorithm 6 computes the set $P = \{v_2, v_3\}$ of 4 degree vertices, which is the only set of 4 degree vertices. But there can be more than one path consisting of 4 degree vertices in G. In such case, using Algorithm 7, we first check whether G belongs to $\mathscr{L}$ for each such set. If we have more than one such set for which G belongs to $\mathscr{L}$, we can derive an area-universal rectangular dual for each such set. If for every such set, G fails to belong to $\mathscr{L}$, then it is inconclusive whether G admits an area-universal rectangular dual. Another case for inconclusiveness is that each set is either empty or singleton (line 20, Algorithm 6).

Let R denote a rectangular dual and $R_i((x_i, y_i), l_i, h_i)$ denote the i^{th} component rectangle corresponding to v_i with bottom left coordinate (x_i, y_i), length l_i and height h_i.

In Algorithm 8, the input graph is same as that of Algorithm 7.

Algorithm 8 Constructing an area-universal weak rectangular dual

Input: $G=(V,E)$, P_1, $N(v_i)$, $\forall v_i \in V$, k and l are integers such that $k=|P_1|$ and $l=|N(v_t)\cap L|$, a fix point (x_0,y_0) in the plane

Output: An area-universal rectangular dual of G

$L=\phi$
$F=\phi$
for all i from $i=1$ to k **do**
    for all $v_i \in P_1$ **do**
        $x_0=x_i, y_0=y_i$
        InsertRectangle $R_i((x_i,y_i),1,1)$ in R
        $x_0 \leftarrow x_0+1$
        $L=L\cup\{v_i\}$
    end for
end for
for all $v_t \in (V-L)$ **do**
    if $|N(v_t)-(N(v_t)\cap L)| \leq 3$ **then**
        if $N(v_t)\cap L == \{v_k, 1\leq k\leq l\}$ **then**
            InsertRectangleFunction$(F,v_t,L,N(v_t))$
            $L=L\cup\{v_t\}$
        end if
    end if
end for
return R

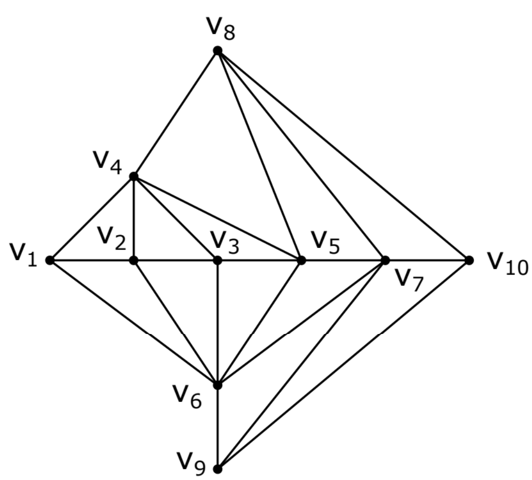

Figure 7.1: An RDG G_1 that belongs to $\mathscr{L}$.

Algorithm 9 InsertRectangleFunction$(F, v_t, L, N(v_t))$

Input: $F, v_t, L, N(v_t)$

Output: R is updated with the addition of rectangle R_t for some vertex v_t.

if $y_1 == y_2 == \ldots == y_l$ **then**
    $x_p = \min\{x_k, 1 \le k \le l\}$
    insertRectangle $R_t((x_p, y_p - 1), \sum_{i=1}^{l} l_i, 1)$ in R
else if $x_1 == x_2 == \ldots == x_l$ **then**
    $y_p = \min\{y_k, 1 \le k \le l\}$
    insertRectangle $R_t((x_p - 1, y_p), 1, \sum_{i=1}^{l} h_i)$ in R
else if there exists $x_i \ne x_j, i \ne j, 1 \le i, j \le l$ **then**
    $y_p = \min\{y_k, 1 \le k \le l\}$
    insertRectangle $R_t((x_p + l_p, y_p), 1, \sum_{i=1}^{l} h_i)$ in R
else
    $x_p = \min\{x_k, 1 \le k \le l\}$
    insertRectangle $R_t((x_p, y_p + h_p), \sum_{i=1}^{l} l_i, 1)$ in R
end if

Now we construct an area-universal rectangular dual for the graph G_1 in Fig. 7.1 using Algorithm 8. Algorithm 8 first updates R by inserting rectangles R_2 and R_3 dual to the vertices of P as shown in Fig. 7.2a. Again, Algorithm 8 recursively updates R by inserting rectangles dual to the vertices of $V - P = \{v_1, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ such that $|N(v_i) - (N(v_i) \cap L)| \le 3$ as shown in Fig. 7.2b-7.2i. Thus, Algorithm 8 constructs an area-universal rectangular dual as shown in Fig. 7.2i for G_1.

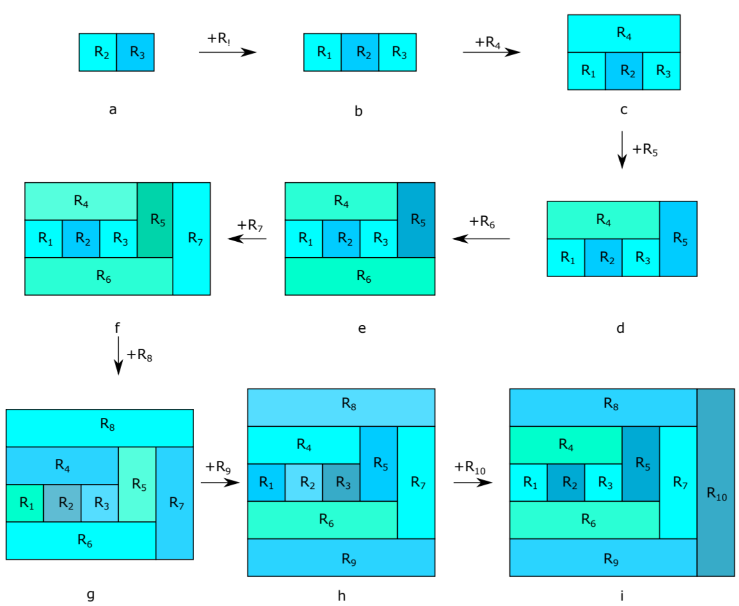

Figure 7.2: Constructing an area-universal rectangular dual for G_1 in Fig. 7.1

Analysis of computational complexity

For Algorithm 6, to search P, the entire list of vertices requires scanning in worst case. This implies that the computational complexity of Algorithm 6 is $|P||N(v_s)| \cong O(n)$ where v_s is a vertex of the largest degree.

For Algorithm 7, it is easy to note that $|V-L|$ is large if and only if $|L|$ is small. This implies that both of them can not approach to $|V|$ simultaneously. Therefore, the computational complexity of Algorithm 7 is $|V-L||(N(v_i)\cap L)| = |N(v_i)||L||V-L| \cong |N(v_s)||L||V| \cong O(n)$ where v_s is the vertex of the largest degree. Hence, the computational complexity of Algorithm 7 is linear.

The computational complexity of lines $3-9$ and lines $10-17$ in Algorithm 8 are $k|P_1|$ and $|V-L|.|(N(v_i)\cap L)|$ respectively. The complexity of $|(N(v_i)\cap L)|$ is $|N(v_s)|\times|L|$ where v_s is the vertex of the largest degree. Now $\max\{k|P_1|, |V-L|.|(N(v_s)\cap L)|\} \cong |V|.|N(v_s)||L| \cong O(n)$. Hence the computational complexity of Algorithm 8 is $O(n)$.

Remark 7.2.1. The case when $|N(v_s)|$ is so large that it approaches to $|V|$ for some graphs, then the computational complexity of Algorithm 6, Algorithm 7 and Algorithm 8 is $O(n^2)$. However in design problems, such graphs do not appear quite often. Using binary search method, this complexity can be reduced to $n\log n$ but at present, we are not considering this method.

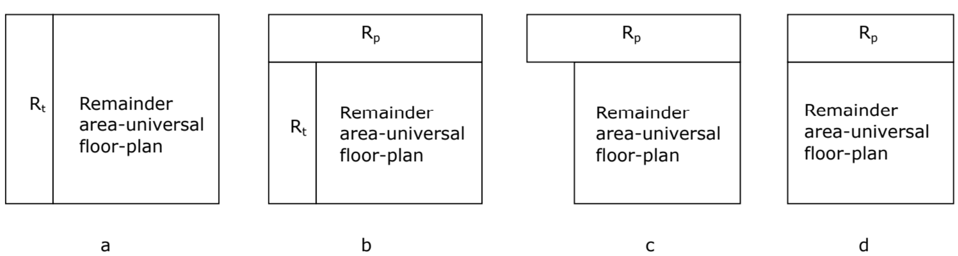

Figure 7.3: (a-b) Possible positions for an exterior rectangle R_t in the rectangular dual R of G, (c) dual R' after deleting R_t from R and (d) adjusting R_p so that the resultant dual is a rectangular dual for H_1.

Theorem 7.2.1. Every RDG G in $\mathscr{L}$, admits an area-universal rectangular dual.

Proof. We know that a rectangular dual is area-universal if and only if every internal line-segment is maximal. It is noted that Algorithm 8 updates an empty dual R by inserting rectangles one by one for each vertex of G such that every resultant dual is an rectangular dual and there generates a new internal line segment (one of the sides of a rectangle) while inserting each rectangle (except the first one) in R. We prove that every generated internal line segment is maximal and is the side of atleast one of the rectangles of the rectangular dual for G, i.e., we need to prove that each internal line segment is not extendable at its endpoints.

As R is updated by inserting a rectangle R_i (except for the first one) by Algorithm 8, a new internal line segment l_i is generated in R. We claim that l_i does not make an angle of 180° with any of the preceding internal line segments (segment). There arise the following two possibilities:

i. l_i is perpendicular to some preceding internal line segments (segment) meeting at one of the endpoints of these preceding internal line segments (segment) (for instance, the new generated line segment l_6 (say) due to the insertion of R_6 in Fig. 7.2e is perpendicular to the preceding internal line segments lying at the common borders of R_1 and R_2, R_2 and R_3, and R_3 and R_5),

ii. none of the endpoints of any of the preceding internal line segments (segment) is intersected by l_i (for instance, the new generated line segment l_3 (say) due to the insertion of R_3 in Fig. 7.2b does not meet to the preceding internal line segment l_2).

From above two possibilities, it is clear that l_i does not allow to extend the preceding internal line segments (segment). Since l_i is arbitrary. Hence every internal line segment of R is not extendable and is one of the side of rectangles of R. Hence the proof. □

Theorem 7.2.2. Every induced rectangularly dualizable subgraph of the graphs in $\mathscr{L}$ admits an area-universal rectangular dual.

Proof. Suppose that H_k is an induced rectangularly dualizable subgraph of any of the graph $G = (V,E)$ of $\mathscr{L}$. Since H_k is a rectangularly dualizable graph, it admits a rectangular dual where no four component rectangles meet at a point. This implies that the interior regions of H_k are triangular. Since H_k is an induced subgraph of G, H_k is the graph whose vertex set $V' \subsetneq V$ and the edge set E' contains the edges of G whose endpoints are in V'. Suppose that $V' = V - S$. Clearly, S is nonempty. First we claim that S contains an exterior vertex of G. To the contrary, suppose that S has no exterior vertex of G. This implies that S only contains the interior vertices (vertex) of G and hence H_k has atleast one region of length atleast 4 which is a contradiction to the fact that every interior region of H_k is triangular. Therefore, S contains atleast one exterior vertex of G. Let $v_t \in S$ be an exterior vertex of G. Consider an induced graph H_1 of G obtained by deleting v_t together with all the edges incident to v_t. There arise two possibilities for deletion of the rectangle R_t (corresponding to v_t) in the rectangular dual R for G as shown in Fig. 7.3a and 7.3b. Note that the remaining possibilities can be covered by flipping or rotating R. The first possibility shows that on deleting R_t from R, the remaining part is still an area universal rectangular dual for H_1 while the second possibility shows that on deleting R_t from R, the remaining part is not a rectangular dual (see Fig. 7.3c) for H_1, but can be transformed to an area universal rectangular dual by adjusting the dimension of the rectangle R_p as shown in 7.3d.

Proceeding as before, we can find an exterior vertex $v_k \in S - v_t$ since H_1 being an induced subgraph of G admits an area-universal rectangular dual and the induced subgraph H_2 of H_1 obtained by deleting v_k from H_1 admits an area-universal rectangular dual. Continuing in this way until $S = \phi$, we find that H_k admits an area-universal rectangular dual. Hence the proof. □

7.3 Concluding Remarks

In this chapter, we have identified a class $\mathscr{L}$ of RDGs wherein each RDG can be realized by an area-universal rectangular dual. The algorithm described by Eppstein et al.[22] for the construction of an area-universal weak rectangular dual for an RDG if it exists is, in general, not fully polynomial. We have described a polynomial time algorithm to construct an area-universal weak rectangular dual for every graph in $\mathscr{L}$. Further every induced rectangularly dualizable subgraph of graphs in $\mathscr{L}$ admits an area-universal weak rectangular dual.

Recently the most challenging problem is to find an efficient algorithm to construct an area-universal rectangular dual for a given RDG, if it exists.

It can be observed that the importance of class $\mathscr{L}$ lies in the fact that if a given RDG G can be split into the union U of graphs in $\mathscr{L}$, then G also admits an area-universal weak rectangular dual. In fact, area-universal weak rectangular duals for RDGs from U can be glued to obtain an area-universal weak rectangular dual for G. Contrarily, it is interesting to find out in future that if a given RDG can not be split into the union of graphs in $\mathscr{L}$, then whether this RDG can be realized by an area-universal weak rectangular dual.