Fundamentals of ENGINEERING MECHANICS

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John A. Hrones, *Editor*

Fundamentals of ENGINEERING MECHANICS

By

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TO F. G. S.

Preface

Engineering thought is a vector quantity—for it must be endowed not only with magnitude but with direction. Such thought cannot wander as fancy may dictate or whim suggest, but must be as precise, orderly, and efficient as the theorem of a Grecian geometer.

The development of challenging, unbiased, effective mental power is the primary responsibility of engineering pedagogy, and engineering education contributes most magnificently only when it recognizes its responsibility and determines its objective—honest training in fundamentals.

The store of factual information that the young engineer may carry with him into industrial practice is limited in range, can be retained more effectively in the literature than in the student's memory, and is subject to change as fundamental and applied research open new vistas or destroy dogmas.

The power of objective, analytical, coherent reasoning from hypothesis to conclusion is the equipment which his formal educational career must forge and refine in the engineering student

The vigor of the "free-body" technique of applied mechanics is the tangible embodiment of engineering *method* in thought. Here the student may reap the harvest which earlier courses have prepared.

The principles of Newtonian mechanics are first expounded in the freshmen physics course while simultaneous training in mathematics has been fashioning tools. In the engineering mechanics course, postulates and weapons of attack join forces to effect the solutions of problems of application to actual structure or machine. And here the rigid adherence to the setting of the problem—the crystallization of an awareness of its demands—finds expression when the student learns to isolate a body, freed of trappings which might obscure it and upon which his attention may be focused. In mechanics, as in all other spheres of mental activity, he who can set the problem is well along the road to its successful solution.

The author has, therefore, broken with the tradition of presenting in textual form only the axioms and techniques of elementary Statics and Dynamics. He believes that the fundamentals of engineering mechanics must also include the encouragement of devotion to the free-body philosophy—not mere lip service.

In such a venture, he believes that the student may be safely entrusted with the responsibility of a full partnership, and the classical phraseology of formal proofs has been abandoned in favor of a style that is intended to invite friendship and cooperation. The student is urged to use this text as a friendly guide, to accept its statements only after honest questioning, and never to assume that these statements come as divine revelations which must be accepted with humble submission. In engineering mechanics, the student will find encouragement to intellectual awakening, to the joys of scholarship, and to the satisfaction which comes with constructive essays into straight thinking. This is a very different approach to professional success from the search for formulas and type, or case, problems which, learned through the discipline of rote, make of any form of education a tragic farce.

In this insistance upon the building of a foundation of effective thinking, the subject matter usually covered in the texts of a first course in applied mechanics has not been neglected. In addition, the principle of virtual work, Mohr's circle as applied to moments of inertia and products of inertia, and a brief introduction to the subject of mechanical vibrations have been included as appropriate material in a first course in Statics and Dynamics. Another amplification of the usual subject matter has been the expansion of the amount of time devoted to the division of engineering kinematics, in order that entrance to the field of dynamics may rest upon a solid base.

The author wishes to express his appreciation of the encouragement and assistance received from his colleagues at the Massachusetts Institute of Technology, particularly to Professor Deane Lent and Mr. George L. Nelson of the staff of the Institute, who rendered invaluable aid in the preparation of many of the illustrations.

ALVIN SLOANE

Cambridge, Mass.

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CHAPTER I

The Free Body of Mechanics

1. The Study of Engineering Mechanics. The study of the subject of applied or engineering mechanics plays a dual role in the education of an engineer. This is the study of a science, as it presents direct or tool value to serve as a basis for the design of machines and of structures.

It is also a study of an art, as it presents a complementary opportunity for training in the development of mental power—the practice of the "straight thinking of the engineer."

An engineer must possess one characterizing tenet of personal philosophy—the vigorous search for truth. This aim is not unique. He shares it with searchers and researchers in all scientific fields—in medicine, biology, psychology, and the like.

The approach to the profession of engineering differs somewhat, however, from the preliminary skirmishes of the other fields of science.

Engineering education is determined to formally and directly focus attention upon the *method* of thinking—the forging and refinement of a weapon of attack to which the problems of engineering must yield.

The subject matter of engineering embraces an ever-expanding area which is already so extensive that no individual may become master of all the information contained in but one region—his professional branch of engineering. Engineering education, therefore, strives to build a common base, the trunk from which the branches may grow and from which they may be constantly nurtured.

This common base is the discipline of straight thinking—the crystallization of an objective, orderly, and effective system of reasoning. The procedure of such organization must be as formal and as vigorous as the successive stages of a theorem of Euclidean geometry, if it is to be effective and efficient. It must receive more than lip service from its devotees. Constant practice can then lead to grace and perfection in performance.

All intelligent men like to believe that they think clearly and directly, but the reasoning of all individuals tends to wander as personal bias or prejudice may dictate, or as tangents to the main avenue offer temptation.

The engineer must be trained to set his hypothesis, to develop the stages of a proof unswervingly, and to reach the goal of the conclusion.

The study of applied mechanics offers a glorious opportunity for training in orderly effective reasoning. Leonardo da Vinci paid tribute to mechanics as "the paradise of science wherein we reap the fruits of mathematics." That expression of the earliest mechanical engineer may be flowery, but it pungently appraises the value of the study of mechanics, in reaping for the practical use of man the harvest which the seeds of speculative philosophy and the tools of mathematical logic have prepared.

2. The Free Rigid Body. The initial stage in the routine of straight thinking appears in mechanics when attention is focused upon a free body upon which forces act. This establishing of a base of operations is apparently simple and, because of its apparent simplicity, is too frequently glossed over. Or, in the fervor to rush to apply the theorems of mechanics, this base of operations is recognized only casually. The novice bows or nods recognition to the free body, and then passes immediately to matters which seem to be more important because they require the use of mathematical tools or invite more profound speculation. Such haste is human, but may be disastrous.

An internal combustion engine is a complex machine. To study such an assembly of material bodies as a whole would present a formidable task. We find, as we cautiously start to analyze with ordered reasoning, that such an assembly is composed of parts—each of which, when segregated from its neighbors, presents a unit to which our attention may be devoted.

We may then proceed to discern qualitatively the nature of the system of forces that is acting upon the isolated unit to affect its motion. A logical base has been established, and the axioms of mechanics, as well as the tools of mathematics, may now be applied for a quantitative appraisal which will enable us to predict accurately the nature of the motion of the body.

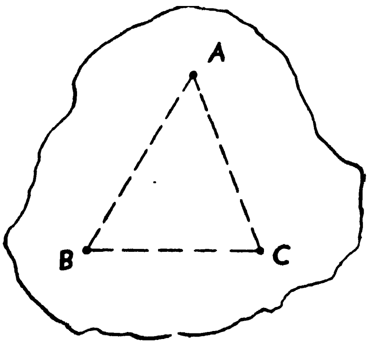


FIG. 1.

The free body of mechanics may be a single material body or a connected group of such material bodies. For the present, we shall take the single material body as our unit, and we shall make one assumption concerned with its nature.

This is the assumption that the free body is *rigid*—that is, nondeforming under the influence of the forces that act upon it.

Our free body takes the form shown in Fig. 1. To state that this body is rigid is to announce that the distances between A and B or between A and C, or B and C, or, indeed, between any two of its particles will not change, no matter what forces act upon the body or what motion it may then have.

Material bodies are never so rigid that these distances between particles remain constant no matter what forces act upon the entire body.

All material bodies do deform when forces are applied to them. A division of the science of mechanics, called the Strength of Materials, concerns itself very largely with such distortions. In our present studies, we are concerned with the motion of the body as a whole and may assume that the relative displacements of the particles with respect to one another are so small in their magnitude that the shape of the whole body is not significantly changed, and that such relative motion may be neglected.

This rigid body is to be freed of its neighbors for our convenience in studying its behavior We must painstakingly insure that in so segregating it, we truly represent the conditions under which this free body is operating as one unit of an assembly.

3. Force. When the free body is in its position as a member of a group of bodies, it is in physical contact with the neighboring bodies. Those neighbors act upon the free body—this action is known as *force*. *Contact force* is the name given to the action upon the free body at the point or region of contact of the neighboring bodies.

Our everyday experiences make us aware of the concept known as force. We slide a box along a surface by pushing with our hands or we raise an iron block by means of a rope passed over a pulley. The push of the hands upon the box is one example of neighboring bodies (the hands) exerting contact force on a free body (the box). The rope's pull on the block is another example of a contact force applied by a neighboring body (the rope) to a free body (the block).

In addition to contact forces supplied by contacting neighbors, other forces exist which may influence the motion of the free body. The weight of the iron block in the previous example is a pull or force which is not exerted through a direct contact between neighboring bodies, but by the gravitational attraction of the earth. Another example of a force arriving at the free body from a source at some distance is magnetic action which influences the motion of bodies placed in the magnetic field.

4. Vector Representation of the Force System. The identification of the system of forces that act upon the free body is dependent upon the nature of the neighboring bodies. With the exception of weight, all of the forces with which we shall usually have to deal are the contact actions of neighboring bodies.

When the engineer isolates a free body for his careful attention, he accounts for the action of each of the neighboring bodies by recording the presence of force at every contact with a neighboring body. To properly record these forces, we must examine the nature of force.

Force has *magnitude*—if we push harder upon the sliding box with our hands, we are exerting greater force, and we observe greater response from the box.

Force has *direction*—if we push vertically downward on the box, it will not move; if we push to the right, the box will move in that direction.

Quantities which possess both magnitude and direction are called *vector quantities*,* and forces should be represented on a drawing by *vectors*—the graphical translation of vector quantities.

A vector quantity is graphically represented by a directed straight line. This line, or vector, represents the magnitude of the force by its length at some convenient scale. The direction of the vector quantity has two aspects—its *inclination*, which is the angle between the line along which the force acts and any convenient and fixed axis of reference;

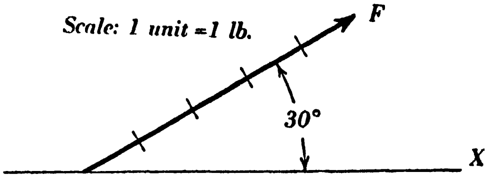


FIG. 2.

and *sense*, which distinguishes between the two possible quantities, having the same magnitude and inclination, directed opposite to each other.

As an example of vector representation, let us consider a force F, of five pounds making an angle of 30 degrees with a horizontal axis X, and so directed that its action upon a free body is upward and to the right. Fig. 2 shows the drawing, or vector representation, of this force. The vector is a straight line, five units long, making an angle of 30 degrees with axis X, and the sense is announced by placing an arrowhead at the end, or terminus, of the vector pointed in the direction of travel.†

* A further qualification of vector quantities demands that they be capable of addition by the parallelogram law, which is discussed in Article 7.

† The distinction which we are making between the two possible senses for any inclination is convenient in this subject where we deal, not with isolated vectors representing abstract quantities, but with forces applied to specific bodies. In such

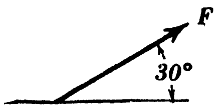


FIG. 3.

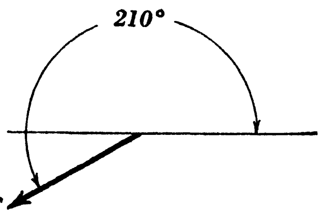


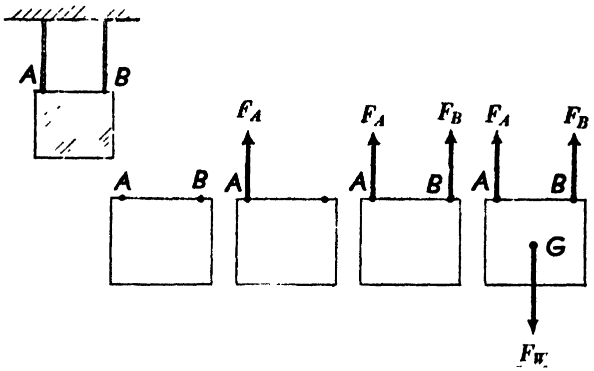
FIG. 4.

practical applications, we find it convenient to refer to the "inclination" and "sense" as two component parts of the concept of direction. Mathematically, no such division need be made, because we may arbitrarily announce that the arrowhead will always be placed in the direction of the force. We distinguish, then, between two forces that lie on the same line of action but are opposite in direction by giving their inclination in terms of the total angle (see, for example, Figs. 3 and 4). As we shall discover in

5. Qualitative Appraisal of the Force System. The appraisal of the force system that acts upon the free body must inquire into the nature of the neighboring bodies before contact forces can be given a qualitative evaluation.

It may be necessary to take many additional steps before a quantitative evaluation can also be made.

To illustrate the initial steps in the routine of force-system appraisal, let us observe Fig. 5. Here we find a block of metal suspended from a ceiling by cables at points A and B. The system of forces acting on the block is to be determined. Then the block itself is selected as a free body and isolated in a drawing (Fig. 5*a*) from its neighbors. We now



(*a*) (*b*) (*c*) (*d*)

FIG. 5.

examine every point of contact between the free body and a neighboring body, because at these points force is being exerted on the free body.

At point A, for example, we note the presence of a cable. Then vertical force is being exerted on the free body, and a vector F_A of vertical inclination is added to the drawing, which now appears as in Fig. 5*b*. The sense of this vector may be indicated as upward, for we note that force F_A is being supplied by a cable—a cable is incapable of exerting downward thrust because the cable would, in that event, collapse.

At point B, the neighboring body exerting force on the free body is another cable, and vector F_B, identical in at least inclination and sense to F_A, is added to the drawing, as in Fig. 5*c*.

Our appraisal has now recognized every possibility of contact force acting on the free body. Any forces other than contact forces must be those originating at a distance. In this case, the weight of the free body itself furnishes such action and is accounted for by noting that the pull

our investigation of actual engineering problems, this method, although being mathematically proper, causes awkwardness when we show the contact forces between neighboring bodies, and we shall use in this text the division of direction into inclination and sense.

of gravity is vertical in inclination, downward in sense, and has the weight of the body as its magnitude. A vector F_W is added to the drawing, as in Fig. 5*d* at the point *G*, which is its center of gravity. (See Article 37.)

There is, of course, force action within the body itself. Each particle of a body is in direct contact with the adjacent particles and exerts contact force on its neighbors, but these internal forces form a balanced system. This is a conclusion consistent with our acceptance of the definition of a rigid body. If a body suffers no deformation, but remains fixed in form, the location of particles relative to one another is not changing, and the forces of contact between the particles form a balanced system which cannot affect the motion of the body as a whole.

We are concerned with those contact forces which, like F_A, F_B, and F_W of Fig. 5*d* will influence the motion of the body as a whole. These forces will hereafter be referred to as *external forces*.

The free-body drawing is now complete in that we have a graphical picture of

1. The *isolated* body whose motion we are to study.
2. *All forces*, both known and unknown (as to magnitude, inclination, or sense) which are acting upon this body to influence its motion.

The solution for the unknowns may be made in a later stage of analysis Our present interest has been to illustrate the careful first step of analysis which permits our problem to crystallize in the form of a free-body diagram. Only when we have accomplished this first stage of isolation and organization of the available data are we given confidence to proceed to determine the unknown forces, and the following chapters of this text are devoted to fashioning the tools with which we can pursue such a solution.

Another example of problem organization is offered in the case of the machine part shown in Fig. 6.

This part is supported on a smooth or frictionless roller at *A* and a pin joint at *B*, and a thrust of 2000 pounds is exerted at *C* by a push rod (the thrust is assumed to be in the direction of the rod itself). The machine part will be assumed to be of negligible weight.

The part itself is selected as a free body and is shown, isolated from all neighboring bodies, in Fig. 6*a*. We now examine all points of contact with neighboring bodies to establish the system of contact or external forces acting on the free body.

At *A*, we find that the external force is applied by the roller. The contact between roller and machine part is contact along a geometrical line, because we are here assuming that all the bodies to which forces are applied are rigid and do not distort. Actually, when pressure is exerted such as that between roller and machine part, there is deformation of the material bodies, and the contact force is distributed over some finite area. It is sufficient for our present purposes to disregard such a development and to consider the external force at A to be concentrated at a point. In our later work, we shall recognize the possibilities involved when the area over which force is distributed becomes sufficiently great so that we can no longer consider such force to be localized or concentrated at a single point. Then the contact force which the roller exerts on the machine part is a vertical push upward, and is represented in Fig. 6*b* as F_A.

Examination of the contact between push rod and machine part at C reveals a thrust of 2000 pounds which we have already assumed to be

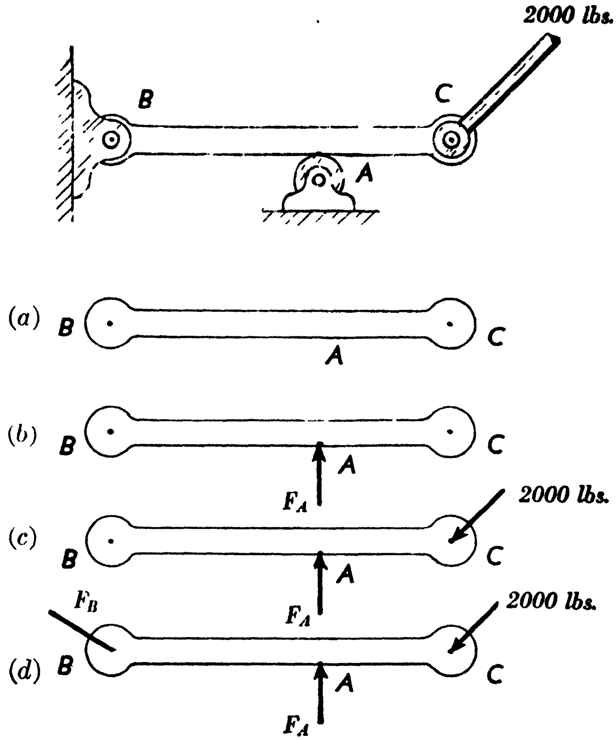


FIG. 6.

in the direction of the push rod. This contact force is added, as in Fig. 6*c*.

At B, we note the presence of a pin joint or hinge. Such a joint would permit the free body to rotate freely if no other constraint were present. (The possible frictional resistance of the pin surface will be considered, for the present, to be of negligible magnitude). Although the pin permits rotation, it prevents the body from moving either horizontally or vertically and is, therefore, exerting external force on the body. We cannot predict the inclination or sense of such a force, since it may have both horizontal and vertical components. Nor do we now have available any means of predicting the magnitude of such a force.

We therefore indicate, by means of an oblique vector F_B, of unstated inclination, sense, or magnitude, the presence of external force at point B. Fig. 6*d* shows the completed drawing of the free body and its external forces.

The contact or external forces thus far discussed have been assumed to act at a point. Not all neighboring bodies adjoin the free body at pins or knife-edges, which may be considered capable of concentrating these actions at points. Actual contact between material bodies must be contact over a finite area, no matter how small. When, for example, the elevator counterweight of Fig. 7 is isolated as a free body, and we seek to appraise the nature of the external forces acting upon that free body, we find that the vertical guides V_1 and V_2 act upon the counterweight over finite areas. These guides are capable of supplying horizontal

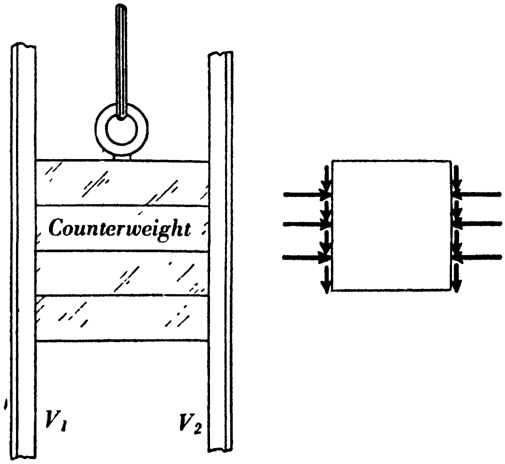


FIG. 7.

components of external force. In addition, the frictional resistance between guides and counterweight may be of appreciable magnitude.

Forces applied over an area so small that it may be considered to be a point will be called *concentrated forces* to distinguish them from distributed forces applied to areas which are obviously of appreciable finite extent, like those of the counterweight surfaces.

A complete investigation of a machine or structure may call for a series of selections of free bodies. In each case, however, no analysis of the force system, so vital to the designer who must fix the exact shape, size, and material, is possible until a painstaking survey is made, during which all pertinent free bodies have been isolated. The diagrams which reveal the force systems acting upon those bodies must then be drawn to present graphically to the analyst the problem he will face. Finally, the solution of the unknowns may proceed, now guided by a system which has clarity and coordination.

CHAPTER II

The Force System

6. The Resultant. Since forces are vector quantities, vector analyses must be made when we attempt to evaluate quantitatively the influence of external forces upon free bodies.

The **resultant** is the *simplest equivalent system* to which an original force system may be reduced.

7. Concurrent Forces. A system of external forces F_1 and F_2, both lying in a single plane, acts on a free body, as shown in Fig. 8*a*. Their

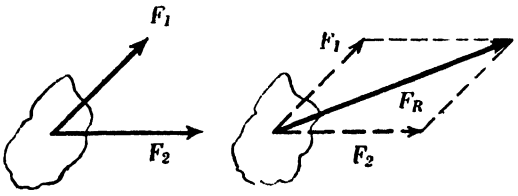


FIG. 8.

effect is to produce a change of motion of the body upward and to the right. An identical motion could be imparted to the body if a single force F_R, shown in Fig. 8*b*, provided that F_R is the diagonal of the parallelogram formed with F_1 and F_2 as sides. This axiom of mechanics was first formulated by Stevinius in the sixteenth century; it is capable of experimental proof, and forms the cornerstone of the science of engineering mechanics.

F_R is the resultant of the system of forces F_1 and F_2 because, as a single force equivalent in its action to the combined effect of F_1 and F_2, it conforms to the definition of the resultant as the simplest system to which the original system may be reduced.

When forces are applied to a free body so that they meet, or concur, at a single point, the system of forces is called a *concurrent* force system.

When we find a system of concurrent forces, as in Fig. 9*a*, we shall find it most convenient to first resolve each force into its components in directions of the X and Y axes.

The X component of F_1 is $F_1 \cos \theta_1$, the X component of F_2 is $F_2 \cos \theta_2$, and so forth. The Y component of F_1 is $F_1 \sin \theta_1$, and so forth. The Y component of F_2 is $F_2 \sin \theta_2$, and so forth.

Now, the total X component of the system is $\Sigma X = F_1 \cos \theta_1 + F_2 \cos \theta_2 + \cdots F_n \cos \theta_n$.

This sum of X components is a vector sum, since it is the sum of individual vector quantities. These individual vectors are, however, all of the same inclination, and if an arbitrary algebraic convention is adopted to describe sense, the sum of these components may be obtained algebraically. X components having sense to the right will be called *positive.* Then X components having sense to the left are *negative.* We investigate the Y components of the original system in identical manner and obtain their sum ΣY. The algebraic convention used in combining these components follows: Y components of upward sense are *positive;* those of downward sense *negative.*

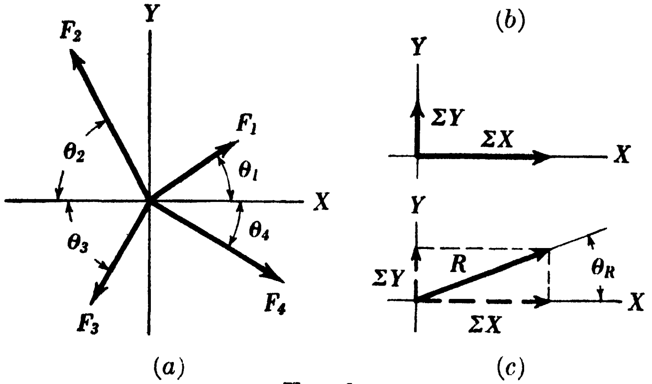


FIG. 9.

The summation of the X and Y components reduces the system to the pair of component forces, ΣX and ΣY shown in Fig. 9*b*.

To obtain the sum of these two forces, we find, as in Fig. 9*c*, the diagonal, R_1, of the parallelogram whose sides are ΣX and ΣY. The force R_1, as a single force, is the simplest system equivalent to the original system of forces and is their resultant.

Expressing these conclusions as equations, we note that

$$R = \sqrt{\overline{\Sigma X}^2 + \overline{\Sigma Y}^2}$$

The inclination of the resultant is determined by the angle:

$$\theta_R = \tan^{-1} \frac{\Sigma Y}{\Sigma X}$$

It should be noted that the angle θ_R may also be determined from the relationships

$$\theta_R = \sin^{-1} \frac{\Sigma Y}{R}$$

or

$$\theta_R = \cos^{-1} \frac{\Sigma X}{R}$$

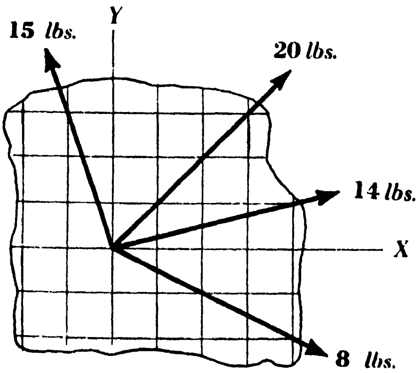
These relationships, written in the cool exactness of symbolical representation, are obviously literally true and mathematically exact, involving, as they do, only a trigonometric equivalence. The engineer cannot leave his problem when he has expressed the relationships only symbolically: he must perform the numerical operations following his substitution of numbers for symbols. It is at this stage of the solution that every precaution must be taken to insure the highest possible degree of accuracy in the results.

The method of determining the inclination of the resultant by using either $\sin^{-1} \Sigma Y/R$ or $\cos^{-1} \Sigma X/R$ is an illustration of vulnerability to greater possible inaccuracy than the use of $\tan^{-1} \dfrac{\Sigma Y}{\Sigma X}$. In the former case, the magnitude of R has already been fixed by a mathematical computation. If error or approximations due to rounding off of significant figures have been made in this operation, the succeeding operations are diminishing in accuracy. The direct use of the $\tan = \Sigma Y/\Sigma X$ avoids the interjection of one possible source of error or inaccuracy. We shall have occasion to note, as we proceed, similar opportunities to relieve a solution of possible error or inaccuracy by reducing the number of steps taken in the calculations.

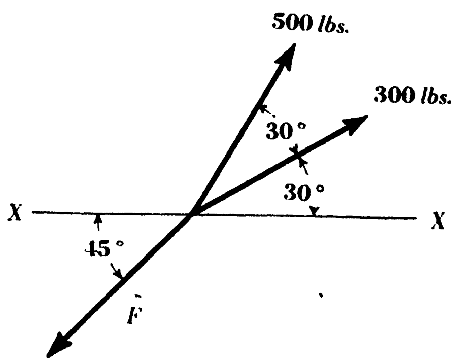
The sense of the resultant is fixed by making it consistent with the senses of its components, ΣX and ΣY.

Problems

1. A free body is acted upon by the system of concurrent forces shown. Determine the magnitude, inclination, and sense of the resultant. The indicated divisions are equal squares. *Ans.* $R = 41.4$ lb.; $\theta_x = 43.2°$.



Prob. 1

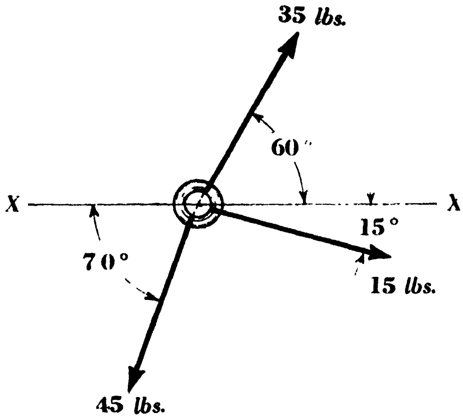


Prob. 2

2. The resultant of the system of forces shown has a magnitude of 488 lb. Determine the magnitude of force F and the inclination of the resultant.

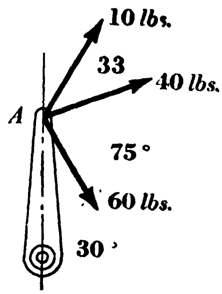
3. Three cables exert pulls on a ringbolt as shown. Determine the magnitude, inclination, and sense of the force which could be exerted by a single cable to produce a pull on the ringbolt equivalent to that of the three cables.

Ans. 23 lb.; $\theta_x = -43.8°$.



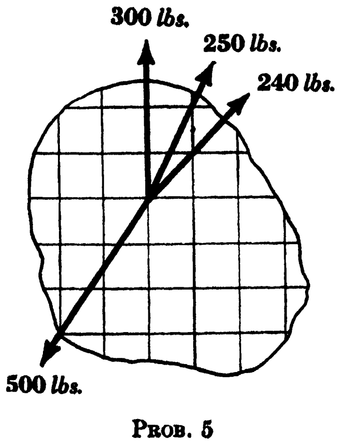
PROB. 3

4. Determine the magnitude, inclination, and sense of the resultant of the system of three forces shown acting at point A on the machine part.



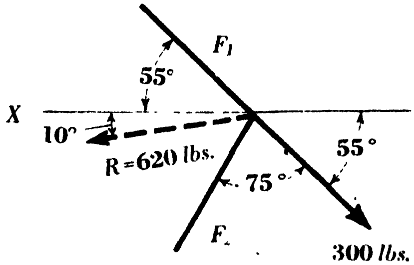
PROB. 4

5. Determine the magnitude, inclination, and sense of the resultant of the system of forces shown acting on the free body. The indicated divisions are equal squares.



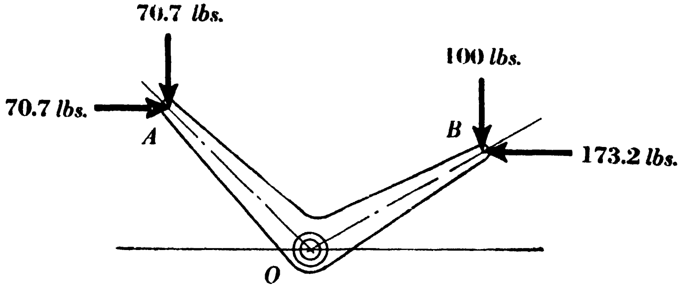
PROB. 5

6. The resultant of the system of forces is $R = 620$ lb. Determine the magnitudes and senses of F_1 and F_2. *Ans.* $F_1 = 712$ lb.; $F_2 = 582$ lb.



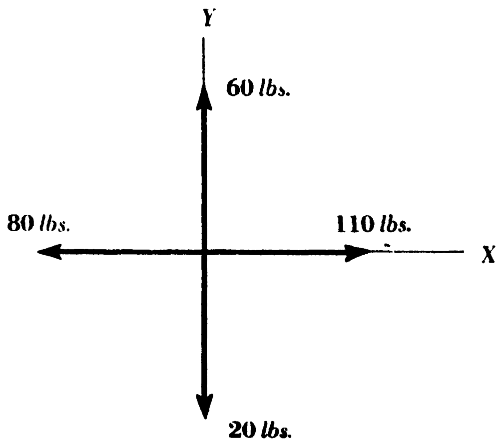
PROB. 6

7. The rocker arm is pinned to other members of a machine, so that the horizontal and vertical components exerted on the rocker arm are as shown. Determine the resultant force exerted on the shaft at O by the rocker arm.
Ans. $R = 199$ lb.; $\theta_x = 59°$.



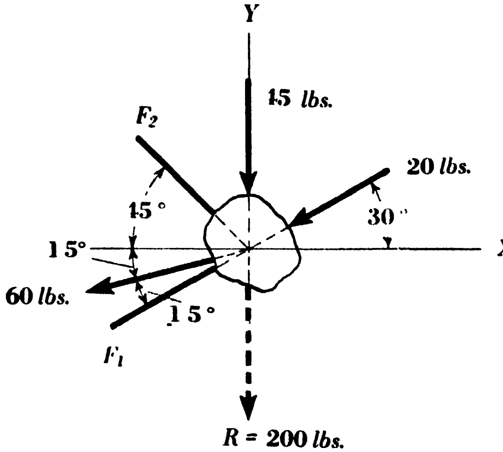
PROB. 7

8. Determine the resultant of the system of four forces shown.
Ans. $R = 50$ lb.; $\theta_x = 53.1°$.



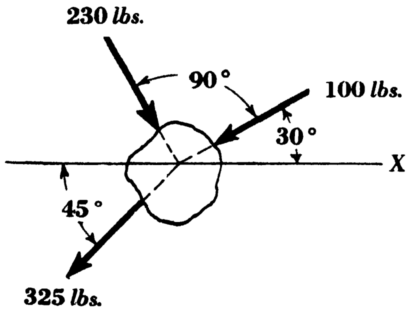
PROB. 8

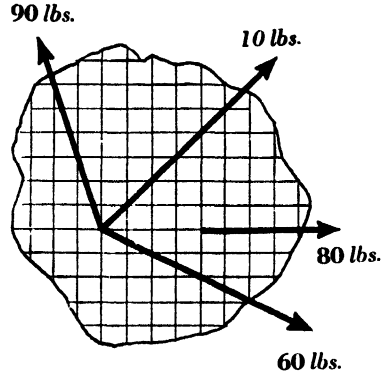
9. The resultant of the system of six concurrent forces shown is $R = 200$ lb., directed downward along the Y axis. Determine F_1 and F_2.



Prob 9

10. Determine the resultant of the system of three concurrent forces shown.





Prob. 10 Prob. 11

11. Determine the resultant of the system of four concurrent forces shown. The indicated divisions are equal squares. *Ans.* $R = 129$ lb.; $\theta_x = 30.3°$.

12. Determine the resultant of the system of four concurrent forces shown in Problem 11, if the 10-lb. force is increased to 100 lb.

13. Determine the resultant of the system of concurrent forces shown in Problem 11, if the 90-lb. force is removed.

8. Equilibrium of Concurrent Force System in a Plane. The system of forces, lying in a single plane and concurring at the same point, have yielded as their resultant a single force. The effect of any force is to alter the nature of the motion of the body upon which that force acts. If the body is originally at rest, the application of a force will cause it to move. If the body is already in motion, that motion will be changed by the applied force.

Equilibrium is broadly defined as a state of balance between opposing forces. The body to which a system of forces is applied is said to be in equilibrium if the system does not affect or change the motion of the body. If, for example, the body is originally at rest, it will remain at rest when a balanced system of forces is applied to it. And if the body were originally moving with constant velocity, it would continue to move with constant velocity when a balanced system of forces is applied to it.

In the case of force systems meeting at a single point, it is evident that such systems must be in balance if the bodies to which they may be applied are to be in equilibrium. This is equivalent to demanding that such systems yield no resultant or that when we reduce such systems to find the simplest equivalent system, we find the resultant is zero.

Expressing this statement as an equation, we have

$$\sqrt{\overline{\Sigma X}^2 + \overline{\Sigma Y}^2} = R = 0$$

Then $$\overline{\Sigma X}^2 + \overline{\Sigma Y}^2 = 0$$

And both $$\Sigma X = 0$$

and $$\Sigma Y = 0$$

These are the *conditions of equilibrium* which must be fulfilled when forces are applied at a single point of a free body, if that body is to suffer no change of motion.

Two conditions of equilibrium exist, and two simultaneous equations summarize the physical facts. Then it is possible to solve for two unknowns in such force systems.

In Fig. 10*a*, we find a system of forces applied to a free body at a point O. The body is in equilibrium. Forces F_1 and F_2 are known in inclination, unknown in magnitude and sense. (The vectors representing all forces have not been drawn to scale because the solution is to be analytical or algebraic.)

It is necessary to make two assumptions concerning the unknown forces. Since the solution rests upon two simultaneous equations (the conditions of equilibrium), we must insure its success by introducing but two unknowns. The senses of the forces F_1 and F_2 are, therefore, temporarily assumed to be as shown by the arrowheads of Fig. 10*b*.

No engineering problem has been completed until any assumptions which have been made have been checked for validity. The check available may be mathematical, as in the present instance, or it may be necessary to perform tests and to check by the resulting experience. In no case, however, does the engineer claim solution of a problem until all assumptions made during that solution have been checked.

The assumption of senses for F_1 and F_2 will be checked by noting whether the senses which are finally determined are those which were originally assumed.

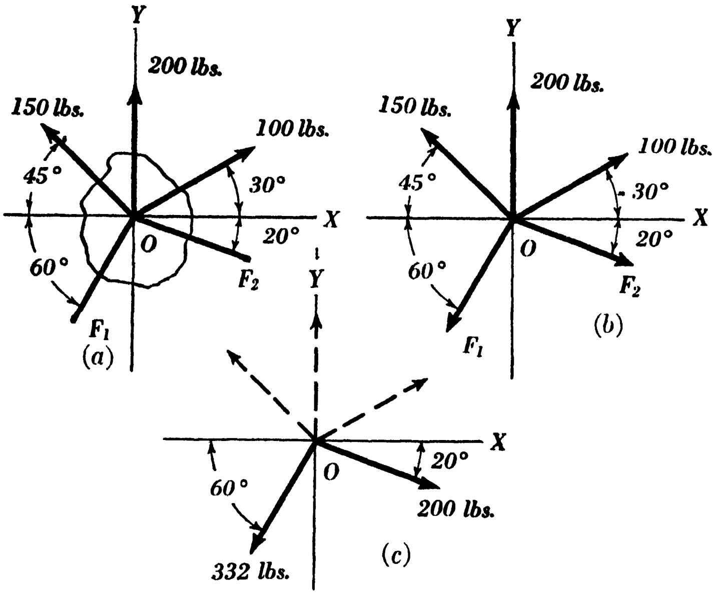


FIG. 10.

We proceed as follows:

$$\Sigma X = 0$$

$$+100 \cos 30° - 150 \cos 45° - F_1 \cos 60° + F_2 \cos 20° = 0$$

$$\Sigma Y = 0$$

$$+100 \sin 30° + 200 + 150 \sin 45° - F_1 \sin 60° - F_2 \sin 20° = 0$$

Solving these two equations simultaneously, we find that

$$F_1 = +332 \text{ pounds}$$
$$F_2 = +200 \text{ pounds}$$

The senses of F_1 and F_2 are revealed to have been correctly assumed, for the solution of the two simultaneous equations gives positive answers.

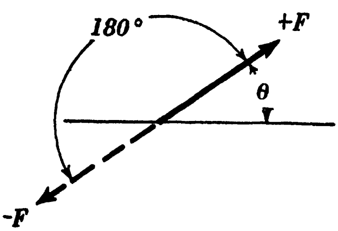


FIG. 11.

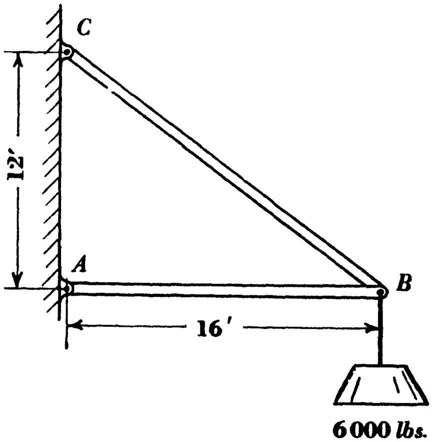
If an incorrect assumption had been made, that fact would have been revealed by the appearance of a minus sign before the magnitude of the force whose sense had been incorrectly assumed. The interpretation of this negative quantity may then be made. We are dealing with vector quantities. The negative of a vector quantity is a vector quantity having the same magnitude and inclination but opposite sense (Fig. 11). Our assumption may have revealed that we have solved for the negative of F. We can correct the assumption by reporting that the positive vector is the same in magnitude and inclination as the assumed one, but opposite in sense. (This trial by assumption may be avoided, as suggested in the footnote to page 6 but is justifiable because the correction of an invalid assumption is so simple.) The solution has now been completed. The validity of assumptions has been checked, and the results which have been obtained are to be reported.

A proper report is one which may be clearly and specifically understood by the reader. The graphical translation—a drawing of the results —is the engineer's typical objective and is a clear and exact method of reporting. We therefore plot or draw the results, as shown in Fig. 10c, where the answers for F_1 and F_2 are plotted by placing on the vectors representing F_1 and F_2 the magnitude, inclination, and sense.

Problems

14. The weight of 6000 lb. is suspended from a cable pinned to joint B. AB and CB are steel members, pinned to each other, and to the wall at A and C. Find the stresses in AB and CB.

Ans. AB = 8000 lb., compression; CB = 10,000 lb., tension.

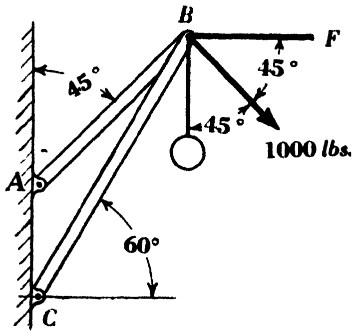


Prob. 14

15. Determine the horizontal and vertical components of the forces exerted by the wall (supporting forces) on CBA at C and A. Use the data given in Problem 14.

16. Two members, AB and BC, are pinned together at B, and pinned to a vertical wall at A and C. Three forces, F, 1000 lb., and a weight of 2000 lb. act on the pin at B as shown. Find the stress in BC and the magnitude and sense of force F. The stress in AB is 1500 lb., tension.

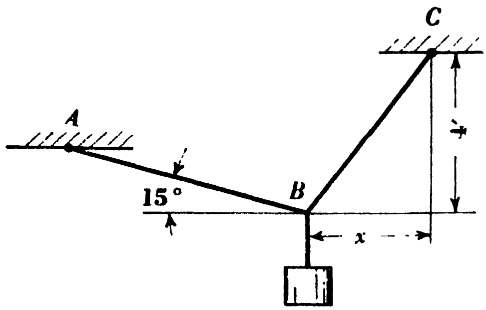
Ans. BC = 4350 lb., compression; F = 1820 lb.



Prob. 15

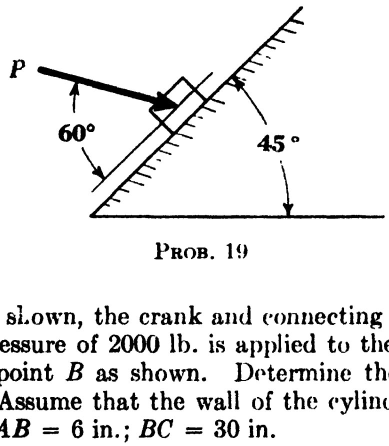
17. Determine the horizontal and vertical components of the supporting forces at A and C of Problem 16.

18. The weight of 300 lb. is supported by cables AB and BC. The stress in AB is 200 lb., tension. Determine the stress in BC and the horizontal distance x from B to C. *Ans.* BC = 315 lb., tension; x = 3.11 ft.



PROB. 18

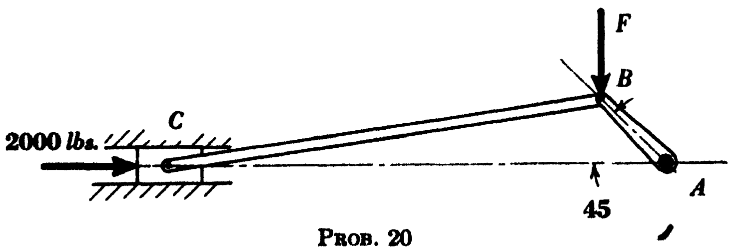
19. Determine the force P necessary to keep the block from sliding down the inclined plane if the block weighs 150 lb. and the inclined plane is assumed to be frictionless. The force P is inclined at 60° with the plane, and the plane at 45° with the horizontal.



PROB. 19

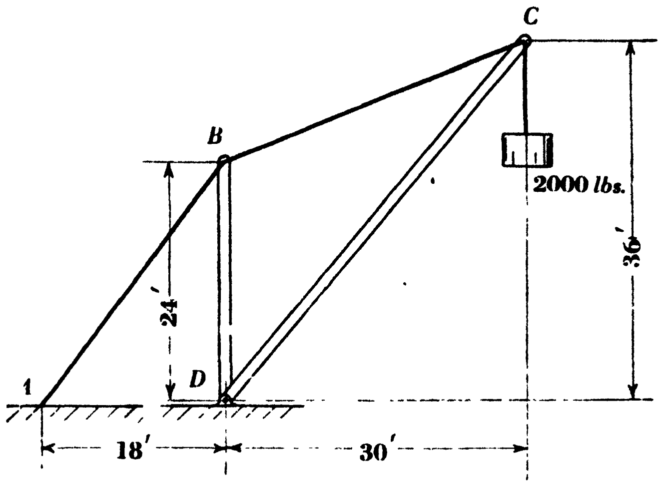
20. In the position shown, the crank and connecting rod of an engine are in equilibrium when a pressure of 2000 lb. is applied to the piston, and a vertical force F is applied at point B as shown. Determine the stresses in the crank and connecting rod. Assume that the wall of the cylinder in which the piston slides is frictionless. AB = 6 in.; BC = 30 in.

Ans. Connecting rod = 2020 lb., compression; crank = 2830 lb., compression.



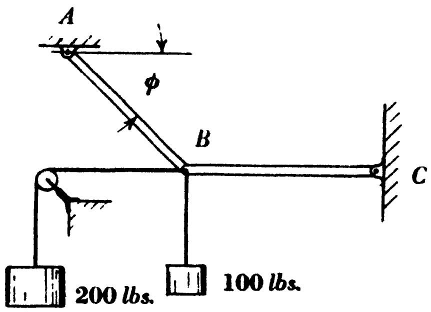
PROB. 20

21. Find the stresses in all members of the derrick shown.



PROB. 21

22. Two members, AB and BC, are pinned to each other at B, and to supports at A and C. The stress in BC is 300 lb, tension. A 100-lb. and a 200-lb. weight are supported on cables pinned to B. Determine angle ϕ. *Ans.* $\phi = 45°$.

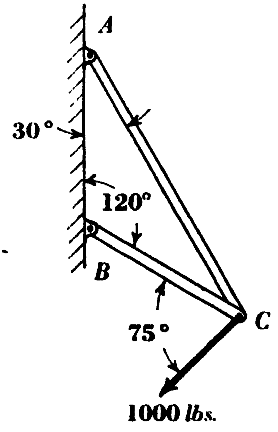


PROB. 22

23. Find the horizontal and vertical components of the supporting forces at A and C of Problem 22.

24. AC and BC are pinned together, and to the wall at A and B as shown. Determine the stresses in AC and BC due to the force of 1000 lb. applied at C.

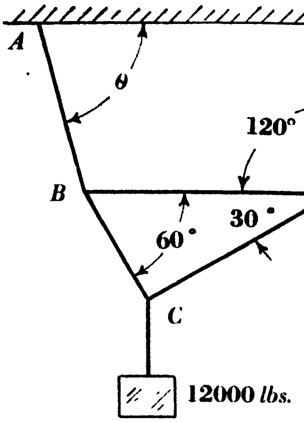
Ans. AC = 1930 lb., tension; BC = 1930 lb., compression.



PROB. 24

25. In Problem 24, determine the horizontal and vertical components of the supporting forces at A and B.

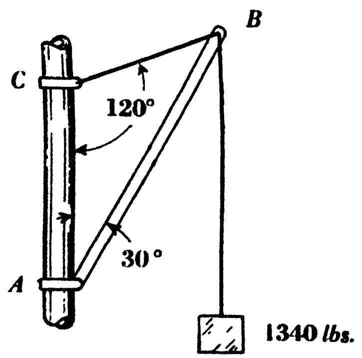
26. A system of members is to be fastened together to form a network supporting the load of 12,000 lb. Determine the horizontal and vertical components of the supporting forces at A and E.



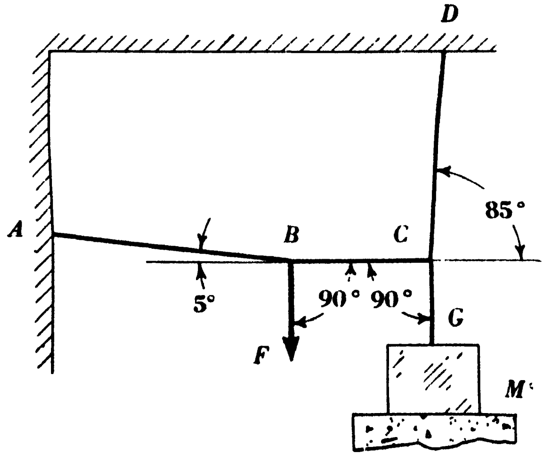
PROB. 26

27. In Problem 26, cables are to be used for tension members, and bars for compression members. The allowable unit stress in both cables and bars is 8000 psi. Determine the required cross-sectional area of all members, if this allowable stress is not to be exceeded, and state whether cable or bar is to be used in each case.

28. The crane supports a load of 1340 lb. Determine the stresses in AB and BC. *Ans.* AB = 2320 lb., compression; BC = 1340 lb., tension.



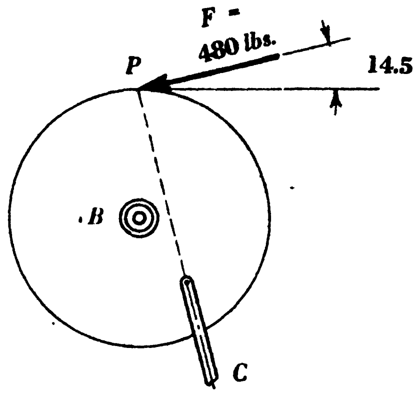
PROB. 28



PROB. 29

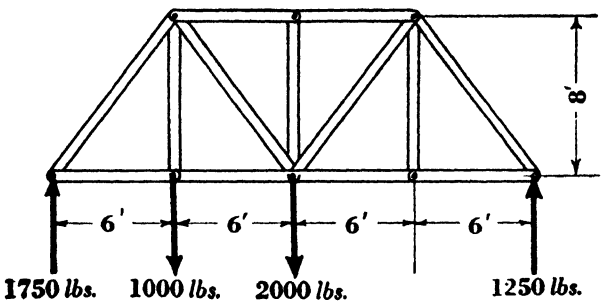
29. Cables AB, BC, CD, and CG are pinned together to form a rigger's sling, by means of which a machine M, weighing 20 tons, is to be raised from its foundation. What vertical force F must be applied at point B?

30. Force F exerts a pressure of 480 lb. at point P of a gear. The connecting rod C, at the instant shown, is perpendicular to the line of action of F. If the gear is in equilibrium and the bearing reaction on the gear at its center B is vertical, determine the stress in the connecting rod.



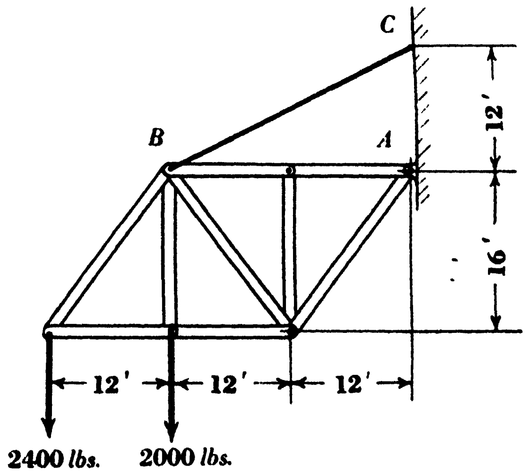
PROB. 30

31. Determine the nature and magnitude of the stresses in all members of the truss, which is built of individual members pinned together at their ends.



PROB. 31

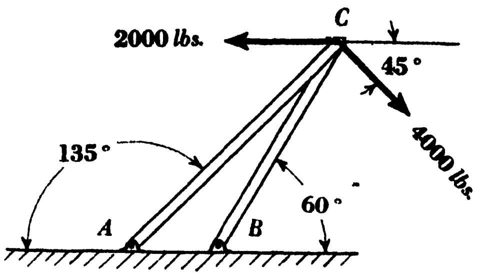
32. The truss is held in position by a pin joint at A and a cable BC, pinned to the wall at C, and to the truss at B. Determine the stresses in all members. The truss is built of individual members pinned together at their ends.



PROB. 32

33. The triangular frame ABC consists of two members AC and BC pinned together at C, and pinned to the foundation at A and B. Find the stress developed in both members due to the applied loads of 2000 and 4000 lb. at C.

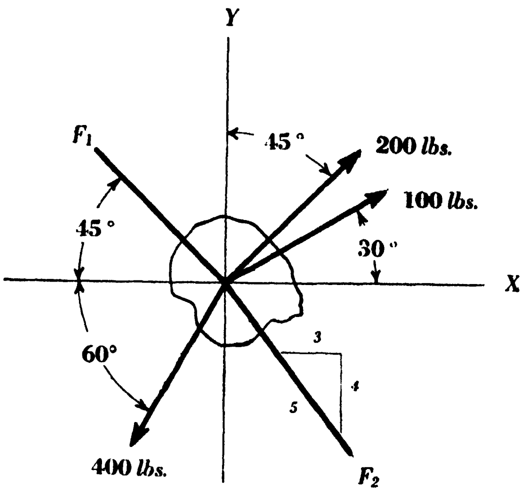
Ans. AC = 8240 lb., tension; BC = 10,000 lb., compression.



Prob 33

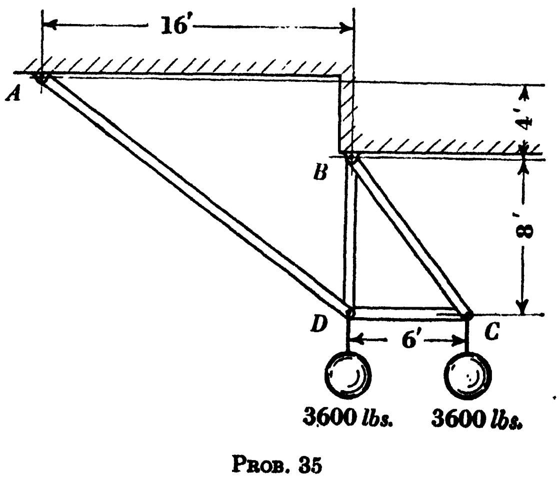
34. The system of five concurrent forces holds the free body in equilibrium. Determine F_1 and F_2.

Ans. F_1 = 500 lbs.; F_2 = 635 lbs.



Prob. 34

35. The frame shown supports two 3600-lb. loads applied at C and D. Determine the stresses in all members.



Prob. 35

36. Determine the horizontal and vertical components of the forces exerted on the ceiling by the frame of Problem 35.

Ans. $H_A = 2700$ lb.; $V_A = 2025$ lb.; $H_B = 2700$ lb.; $V_A = 9225$ lb.

9. Principle of Force Transmissibility. The *line of action* of a force is the straight line upon which the force is acting.

If a force is applied to a free body at point A, as shown in Fig. 12, it will influence the motion of the body. If the same force were to be applied at B, or C, or any other point along the line of action of the force, the same effect upon the motion of the body would be noted. This fact is confirmed by experience and receives formal expression as the *principle of force transmissibility,* which states that a force may be applied to a body at any point along its line of action, with no difference in its effect upon the motion of the body.

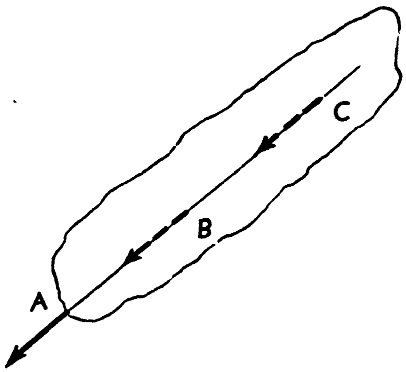


FIG. 12.

It will be noted that the rod of Fig. 13*a* will be in equilibrium if we apply two forces F_1 and F_2 having the same inclination (along the line of action AB) and the same magnitude, but *opposed* in sense. These forces

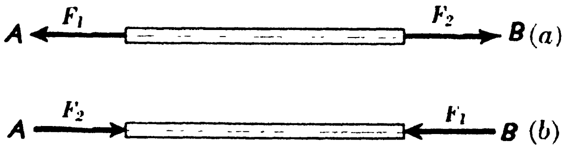


FIG. 13.

may be transferred as shown in Fig. 13*b*, and the equilibrium of the body is not disturbed by the change of location. The rod is affected, however, in that the forces, when applied as in Fig. 13*a*, are pulling on the rod and causing *tension;* whereas, when applied as in Fig. 13*b*, the forces push on the rod and cause *compression*. (See Article 20.)

The principle of transmissibility is of interest when we consider the motion of the body *as a whole*, but must not be applied when we are to consider the state of stress set up within the body. In the latter case, the external forces must be given points of application on the free body at exactly the point where the neighboring bodies make contact, if a correct analysis of the stress set up within the body is to be made.

10. Concurrent System of Three Forces. When three nonparallel forces, and only three, acting in the same plane are applied to a free body which is to be in equilibrium, the lines of action of these three forces must meet at a single point.

The lines of action of forces F_A and F_B of Fig. 14 intersect at point O. By applying the principle of transmissibility, these two forces may be considered to act at point O. Their resultant is a single force R. R must be opposed by an equal and opposite force acting through point O, if balance or equilibrium is to result. The only available force to serve in

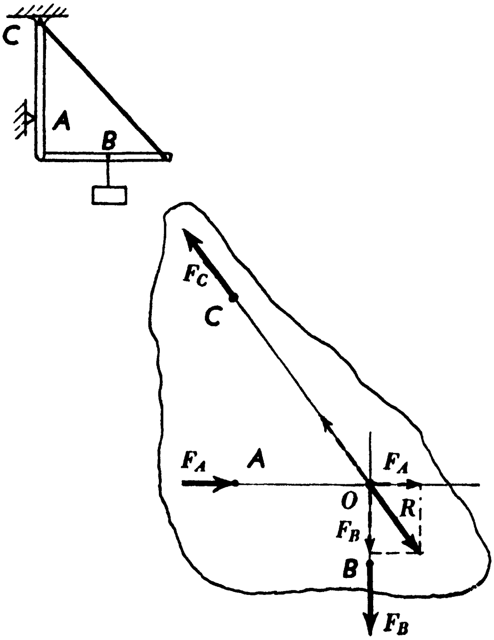


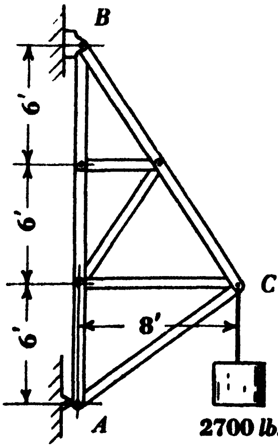
FIG. 14.

opposition to R is F_C, which must, therefore, have a line of action passing through point O. Then, when three forces, lying in the same plane, produce equilibrium, their lines of action must intersect at a common point.

PROBLEMS

37. The truss carries a load of 2700 lb. at C. The truss is pinned to a vertical wall at B, and supported on a knife-edge at A. Find the resultant forces exerted on the truss at A and B.

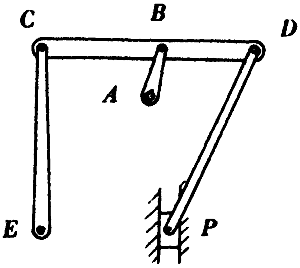
Ans. R_A = 1200 lb.; R_B = 2955 lb.



PROB. 37

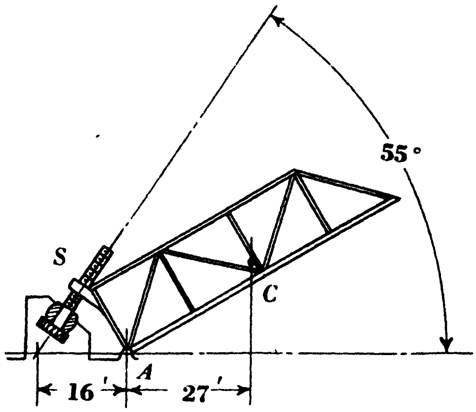
38. If the 2700-lb. load of Problem 37 has inclination of 45° to the left of the vertical, determine the resultant forces exerted on the truss at A and B.

39. The drive mechanism shown gives a long stroke of a piston at P when a short drawing crank AB is to be used. The beam CD is in equilibrium in the horizontal position shown. Crank CE is vertical and crank AB is inclined at 75° with the horizontal. Determine the angle that connecting rod DP makes with beam CD. Assume that CE, AB, and DP exert forces on the beam in the directions of those members. CB = 6.0 in.; BD = 4.6 in.



PROB. 39

40. The drawbridge is raised by a screw S. Determine the tension in the screw and the resultant force exerted by the bridge on the bearing at A. Assume that the total load of 60 tons is concentrated at point C.



PROB. 40

11. Moment of a Force. Very early in man's history, he discovered that a force F applied to a lever, as shown in Fig. 15, gave him a mechanical advantage in performing such operations as raising weights too heavy for the limited strength of his arms. He also observed that a force greater than, or equal to, F, applied at greater distance a from the support or fulcrum would yield even greater mechanical advantage. The basic action of force F, it will be noted, is to produce a rotation of the lever about an axis at the tip of the fulcrum. This effect is accomplished by a force acting at a distance from the axis of rotation. To evaluate such effect, we call this property, which is due to both the magnitude and direction

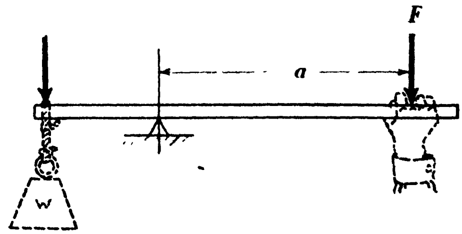


FIG. 15.

of the force as well as its location relative to an axis of rotation, the *moment* of a force. We define moment as the product of the force and the perpendicular distance (*moment arm*) from its line of action to an axis of reference, called the *moment axis.*

The quantity moment is a vector quantity because it possesses both magnitude and direction. The magnitude of the moment has already been defined as the product of the force and the perpendicular distance between the line of action of the force and the moment axis. Like all other vector quantities, the direction has two aspects—inclination and sense.

The inclination announces the plane in which the moment is acting. The sense of the moment tells whether it tends to produce clockwise

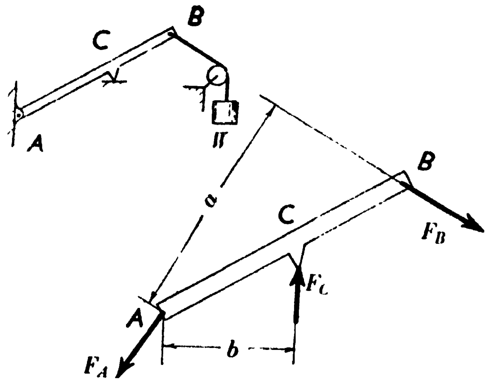


FIG. 16.

or counterclockwise rotation in the plane. When the force system is not confined to a single plane, vector representation and analysis of moments is useful. (See Art. 29.) We are at present confining our interest to force systems acting in a single plane, and only algebraic distinction need be made between the senses of moments, since they are all of the same inclination.

We shall, therefore, adopt an arbitrary convention to distinguish moments of opposite sense. Those moments tending to produce clockwise rotation will be considered positive, and those tending to produce counterclockwise rotation, negative.

Whether or not actual rotation of the free body will result when force is applied to a free body, is dependent upon the manner in which that body is supported and constrained by the neighboring bodies. The moment of force is present whether *actual* rotation, or only tendency to rotate, results from its application.

Figure 16 shows a beam fastened to a supporting member by means of a pin joint at A. A cable supporting weight W is tied to the beam at B, and a knife-edge supports the beam at C.

The beam itself may be taken as the free body, and the free-body drawing is shown in Fig. 16. The pull of the cable on the free body is shown as F_B, the thrust of the knife-edge is F_C, and the force exerted on the beam by the pin is F_A.

The moment of F_B about an axis at A is $+F_B \times a$. The moment of F_C about A is $-F_C \times b$. The moment of F_A about an axis at A is zero (since the force passes through A) and has, therefore, no moment arm.

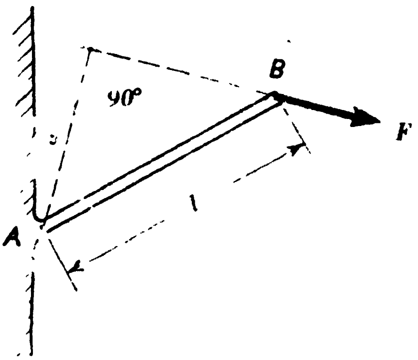


FIG. 17(a).

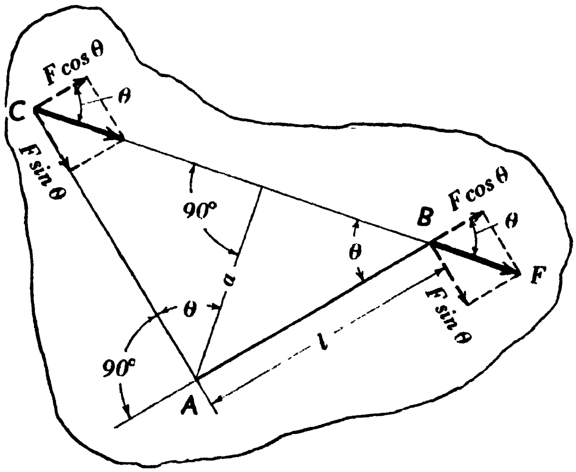


FIG. 17(b).

12. Varignon's Theorem. The force F (Fig. 17a) is oblique to the rod on which it is acting. In most machines or structures, dimensions of the members are given as direct distances along members like l. The moment arm, a, of force F about point A must be computed instead of being read directly from the given dimensions of the structure.

F is the resultant of the system of components $F \cos \theta$ and $F \sin \theta$ (Fig. 17b). The length of rod AB is l. The moment of F about an axis at A is $+F \times a$.

The sum of the moments about A of the equivalent system ($F \cos \theta$ and $F \sin \theta$) is

$$+F \sin \theta \times l + F \cos \theta \times 0$$
$$= F \sin \theta \times l = +F \times a$$

Or we may apply force F at point C (the intersection of the line of action of F) and a line AC perpendicular to AB at B, our authority being the principle of transmissibility. If F is now again resolved into components, the sum of the moments of the components about A is

$$+F \cos \theta \times AC = +F \times a$$

Then, the moment of any resultant about a given moment axis is equal to the sum of the moments of its components about that axis. The formal statement of this fact is known as *Varignon's Theorem*. Actually, the fact requires no such crystallization as a theorem because it only restates an axiom which we accepted when we defined the resultant as a system simpler than but equivalent to its components.

The advantage of the principle is that we may resolve any force into components along lines of members like AB and perpendicular to AB and make direct use of given dimensions in establishing moment arms.

13. Moment Equilibrium. An opportunity to add to our store of conditions of equilibrium is now presented. If we consider the beam of Fig. 18, we note that only when the moment of F_B about A is equal and opposite to the moment of F_C about A would there be no tendency for the beam to rotate about the pin joint at A.

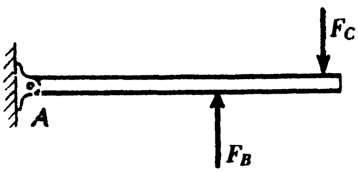


Fig. 18.

Then, for equilibrium, we conclude that the moments of the external forces about a moment axis must form a balanced system of moments, or

$$\Sigma M = 0$$

This balanced relationship must prevail, no matter what moment axis is chosen, for, if the sum of the moments about any axis is not equal to zero, the free body would rotate about that axis.

14. Couples. It is quite possible that a pair of forces may act upon a body in the manner shown in Fig. 19. F_1 and F_2 are parallel to each other in inclination and have the same magnitude, but are opposed in sense.

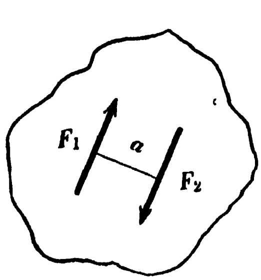


Fig. 19.

The X components of these two forces are equal and opposite, as are the Y components, or $\Sigma X = 0$ and $\Sigma Y = 0$. Then the body is in equilibrium as far as change of motion to the left or right, upward or downward, is concerned. If there is to be any change of motion at all, it must be one of rotation, since such a system possesses unbalanced moment. This pair of forces is its own resultant because no simpler, but equivalent, system may be found to replace it. This system is called a *couple.* The plane determined by the two parallel lines of action is called the plane of the couple, and the perpendicular distance a between the lines of action is called the arm of the couple.

The moment of a couple about all axes perpendicular to the plane of the couple is the same, no matter where these axes may be located.

In Fig. 20, we find a couple composed of two equal, parallel, opposite forces F_1 and F'_1, with an arm a.

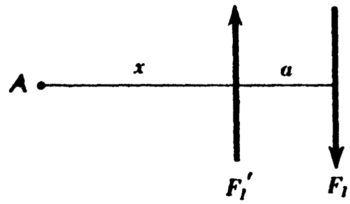


FIG. 20.

If we take moments of the forces about an axis at point A,

$$\Sigma M = +F_1(a + x) - F'_1(x)$$

but

$$F_1 = F'_1$$

$$\therefore \Sigma M = +F_1 a$$

The distance x may, therefore, be of any magnitude and in either direction from the forces of the couple, and the moment obtained for the couple will remain the same. We may also, as a corollary conclusion, note that if point A remains fixed and the couple be moved anywhere in its plane, it would still have the same moment, $+Fa$. Moment is the only property which a couple possesses that can affect the motion of the free body, and we conclude that *a couple may be moved anywhere in its own plane and still have the same effect on the free body.*

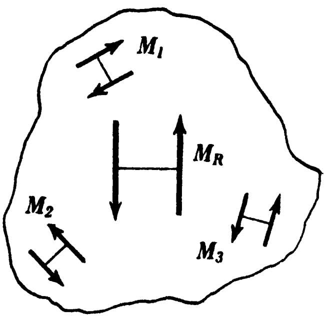


FIG. 21.

Since the only property which couples possess is moment, groups of couples may be added to find their combined effect by summation of their individual moments.

In Fig. 21, a free body is acted upon by a system of couples.

$$M_1 = +200 \text{ ft.-lb.}, \quad M_2 = -300 \text{ ft.-lb.}, \quad M_3 = -500 \text{ ft.-lb.}$$

If we add the moments of the couples, we obtain a summation

$$\begin{aligned} M_R &= \Sigma M \\ &= M_1 + M_2 + M_3 \\ &= +200 + (-300) + (-500) \\ &= -600 \text{ ft.-lb.} \end{aligned}$$

Then, a couple M_R, having counterclockwise moment of 600 foot-pounds, has an equivalent effect upon the body to that of the original system, and is their resultant.

Again, since moment is the only property which a couple possesses, *all couples of equal moment are equivalent.*

Figure 22 shows forces F_1 and F_2 forming a couple which has moment of +200 foot-pounds about an axis at point A or at any other point in the plane, and forces F_3 and F_4 which also have a moment of +200 foot-pounds about point A. Then, the couple formed by the first pair is equivalent to the couple formed by the second pair.

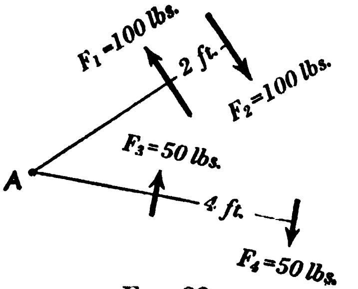
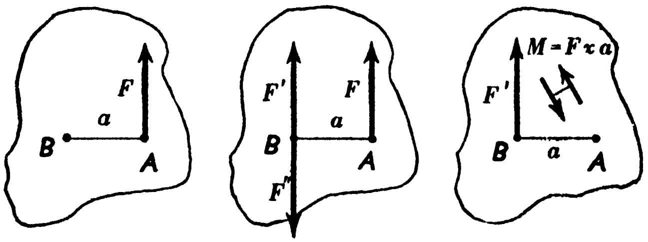


FIG. 22.

The individual pairs of forces may differ from each other, but if the product of force and moment arm in each case is the same, either system will furnish exactly the same tendency to rotate the free body.

15. Resolution of a Force into a Force and Couple. A force F is acting on a free body (Fig. 23a) at point A. The motion of the body will not be affected if we add a pair of forces that balance each other. If then, at point B, two forces, F' and F'', each of magnitude equal to F and each of the same inclination as F, but of opposed senses, are added, as in Fig. 23b, the body's motion has not been affected. We note, however, that the entire system of forces now acting on the body, although equivalent to the original system of Fig. 23a, is different in aspect, for it may be shown, as in Fig. 23c, as a single force F' acting at point B, and a couple $M = F \times a$, which may be shown acting anywhere in the plane.



(a) (b) (c)

FIG. 23.

This new system is identical in its properties with the original system containing only the single force F. This force has the same effect on the body as far as its motion in the vertical direction is concerned, and the new system has the same moment about an axis at point A or point B or at any other point in the plane, as the single force F.

This resolution of a single force into a force and a couple is a most useful device, and we shall find many opportunities for taking advantage of it.

16. The Parallel Force System in a Plane. When a system of forces, all acting in a single plane and all having the same inclination (as in

Fig. 24*a*), is applied to a free body, we observe that the motion of the body is affected in two ways. The body will tend to

1. Change its motion in a direction parallel to the forces.
2. Rotate about any axis which we may select—for example, an axis at point O.

A simpler, but equivalent, system could accomplish the same changes of motion. For example, a single force, having the same inclination as the original forces, equal in magnitude and sense to their sum, and so located that it has the same moment about O as the original system, could replace that system. It would then conform to the definition of the resultant of the system.

This qualitative appraisal of the relationship between the parallel-force system and its resultant, leads to the conclusion

$$R = \Sigma F$$
$$a = \frac{\Sigma M}{R}$$

in which a is the distance from any axis of moments and ΣM is the sum of the moments of the individual force about that axis.

The conclusions of the qualitative analysis are valid and may be confirmed by the technique of resolving individual forces into forces and accompanying couples, which was discussed in the previous article.

We add to the system (Fig. 24*b*) at point O a balanced pair of forces F_1' and F_1'', each equal and parallel to force F_1. The same operation is performed in the case of F_2, F_3, and F_4. When all the original forces have been so resolved into equivalent systems of forces and couples, we find (Fig. 24*c*) that we have

1. A single force F_R acting through point O, which is the vector sum of the original forces, or

$$F_R = \Sigma F$$

2. A single couple that is the sum of the individual moments about O of the original system, or

$$M_R = \Sigma M_o$$

M_R and F_R may now be combined to determine their resultant.

The couple M_R may be resolved into two opposite forces, each equal to F_R. The resulting moment arm between these forces will be

$$a = \frac{M_R}{F_R}$$

This system is shown in Fig. 24*d*. The two forces at point O balance and may, therefore, be removed from the system, and the final resultant is the single force shown in Fig. 24*e*.

It will be noted that the location of the resultant is determined by the senses of both F_R and M_R. It will be located on that side of O which insures that the sense of the resultant force is the same as the sense of the sum of the original forces. At the same time, the resultant force must

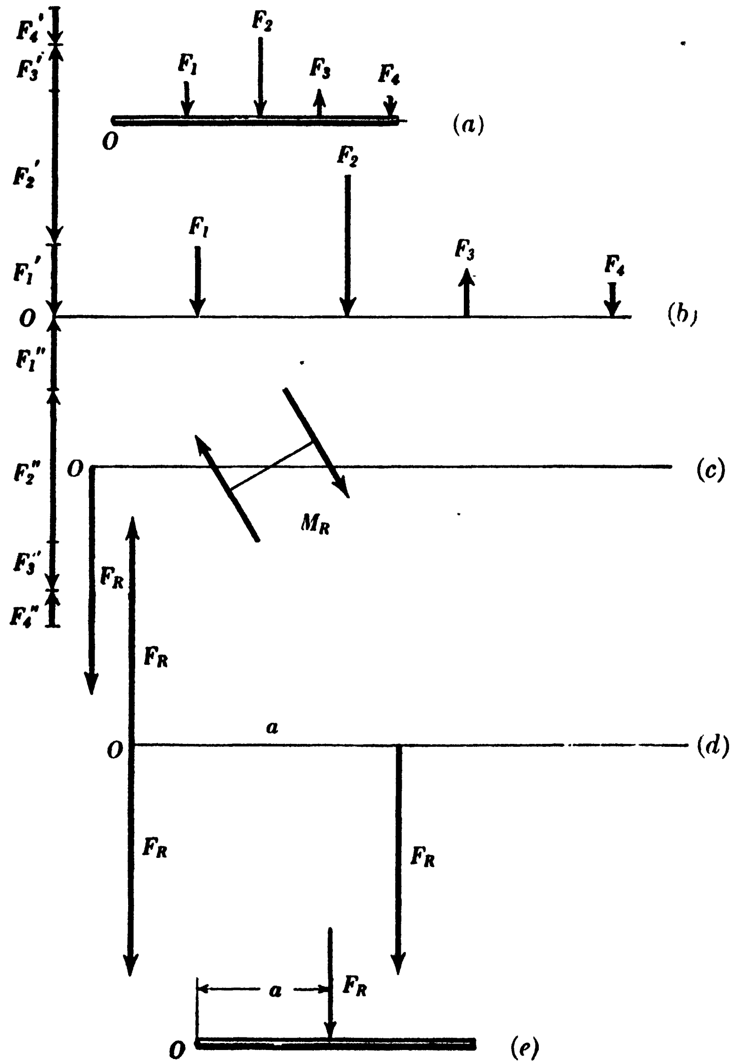


FIG. 24.

give a sense of moment about O identical with the sense of the sum of the original moments.

The algebraic signs employed in such operations as the solution of the above equation $a = \frac{M_R}{F_R}$ may be misleading if applied too vigorously. There are, for example, three combinations of positive and negative quantities (a, M, F), and we have opportunity in each case to adopt an arbitrary convention to differentiate between positive and negative. The + or − symbol, when assigned to distance, such as a, denotes a travel along a coordinate axis to one side or the other of the origin of axes. The + or − symbol, in the case of moment, refers to clockwise or counterclockwise rotation. Finally, the + or − symbol, applied to forces, is employed to conveniently add forces of opposite sense in determining their sum.

When these quantities are grouped in combinations to form equations like the present one, there is no value in stubbornly persisting in demanding that the quotient of $\frac{M_R}{F_R}$, involving independent conventions as to sign in both numerator and denominator, yield a correct sign for a that is consistent with the third independent convention adopted for that term. We should, instead, use such an equation to determine, quantitatively, the magnitude of moment arm a. Then, the resultant may be located on that side of the moment axis which makes the sense of R consistent with the sum of the individual forces, and the sense of the moment of R consistent with the sense of the sum of the moments of the individual forces about the moment axis.

The summary of the above technique for determining the resultant of a system of parallel forces in a plane is presented in the following equations:

$$R = \Sigma F$$

and

$$a = \frac{\Sigma M}{\Sigma F}$$

in which R is the resultant force

ΣF is the sum of the original forces

a is the moment arm, locating the resultant from an axis O

ΣM is the sum of the moments of the original forces about O.

It is quite possible that an original system will be found to possess the following properties:

$$\Sigma F = 0$$
$$\Sigma M \neq 0$$

Then the resultant of the system is a couple of moment ΣM.

Or, again, in a given system

$$\Sigma F \neq 0$$
$$\Sigma M = 0$$

Then, the resultant is a single force having its line of action through the point chosen as a moment axis.

17. Equilibrium of the Parallel Force System in a Plane. Another possibility presents itself when we investigate a system of parallel forces.

If the original system is already in balance, such equilibrium will be revealed when we find that

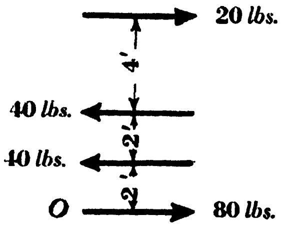
$$\Sigma F = 0$$
$$\Sigma M = 0$$

These, then, are the conditions which must be fulfilled by a system of parallel forces in a plane, if the free body upon which those forces act is to be in equilibrium.

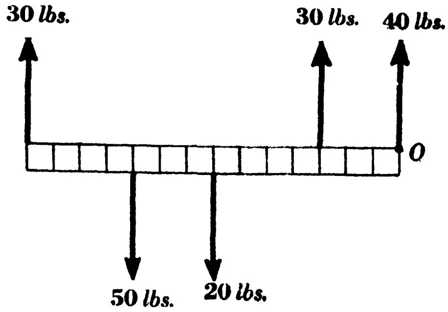
PROBLEMS

41. Determine the resultant of the system of parallel forces shown in Problem 41.
Ans. $R = 20$ lb.; $y_R = 4$ ft. below point O.

42. Determine the resultant of the system of forces given in Problem 41, if the 80-lb. force is decreased to 60 lb.



PROB. 41

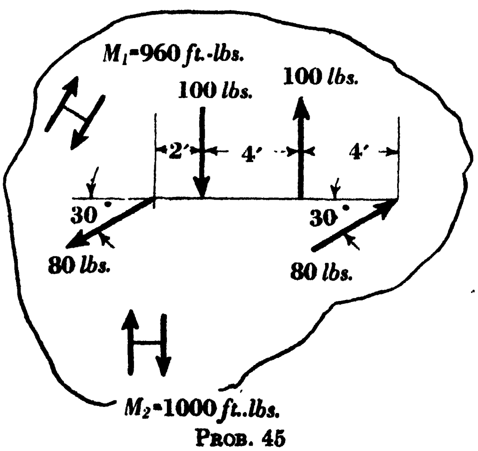


PROB. 43

43. Determine the resultant of the system of five parallel forces shown. Each unit represents 1 ft. *Ans.* $R = 30$ lb.; $x_R = 4.32$ ft. to right of point O.

44. Determine the resultant of the system of parallel forces given in Problem 43, if the 40-lb. force at point O is decreased to 10 lb.

45. Find the resultant of the system of parallel forces and couples shown.
Ans. 1160 ft.-lb., clockwise.

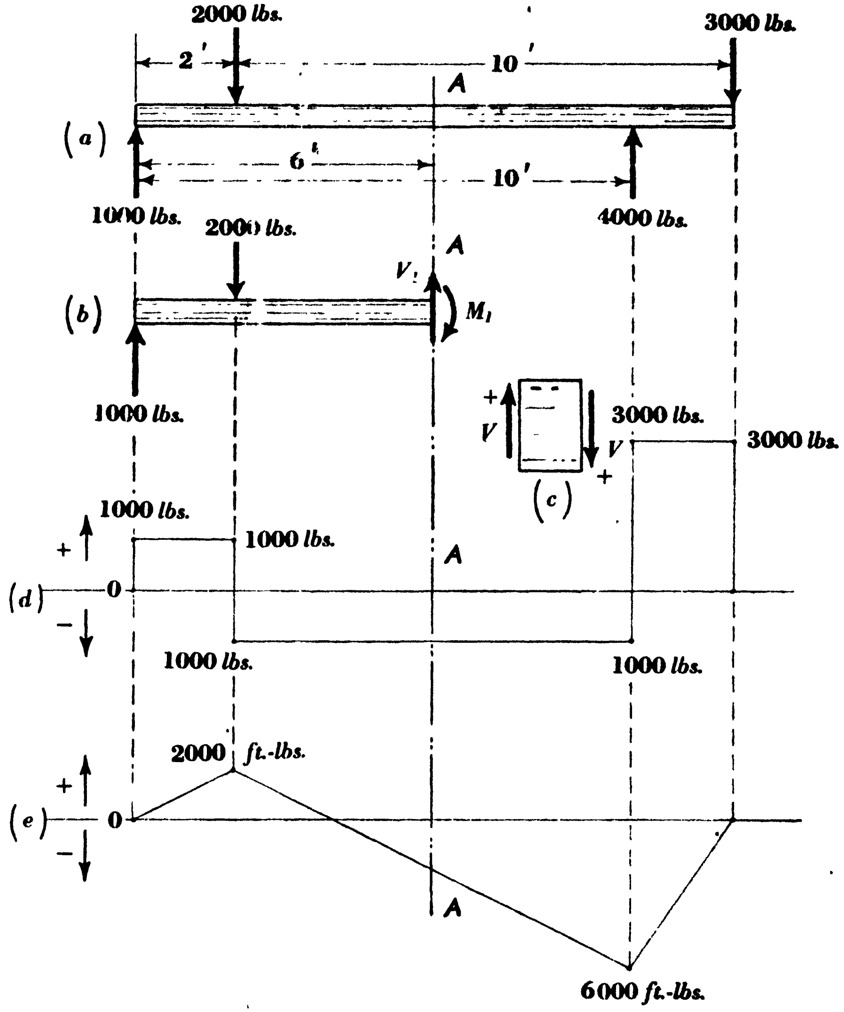


PROB. 45

46. In Problem 45, what change should be made in distance AB in order to produce equilibrium of the free body?

47. In the study of Strength of Materials, two properties of parallel force systems play an important part. These are the *shearing force* and the *bending moment*.

A loaded beam that is in equilibrium is shown in Fig. (*a*). The weight of the beam is assumed to be negligible. Figure (*b*) shows the portion of the beam lying



PROB. 47.

between the left end and section A-A isolated as a free body. The force system acting on the free body consists of the 1000 lb. supporting force, the 2000 lb. load, a vertical force V_1 and a moment M_1.

V_1 and M_1 represent the resultant action of the portion of the beam to the right of A-A on the isolated portion to the left. Such resultant action appears in the actual beam as internal force, or stress, distributed over the cross-section.

V_1 must be equal, in magnitude, to the resultant of all of the vertical forces acting to the left of section *A-A*. This resultant is $R = +1000 - 2000 = -1000$ lbs. Such a resultant is called the *shearing force* at section *A-A*.

The convention which is generally employed for the sense of shearing force is illustrated in Fig. (*c*). A small portion of a beam is shown. If the shearing force acting on the left face of such a portion is upward, the shearing force is *positive*. If the shearing force acting on the right face of such a portion is downward, the shearing force is *positive*. Both of the shearing forces shown are positive.

A survey of shearing force taken at every section along the span of the beam of Fig. (*a*) would reveal the shearing forces which have been plotted as a graph in Fig. (*d*). The ordinates of the graph represent shearing force, and the abscissae represent distances along the beam. Such a graph is called a *shearing force diagram*.

The sum of the moments of all forces acting to the left of *A-A* in Fig. (*b*) about an axis at *A-A* is:

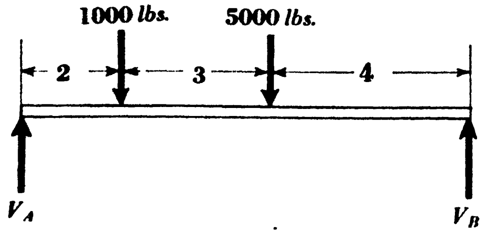
$$\Sigma M = +1000 \times 6 - 2000 \times 4 = -2000 \text{ ft.-lb.}$$

Such a resultant moment is called the *bending moment* at *A-A*, and is equal in magnitude to M_1.

The convention employed to establish the sense of a bending moment follows: if a moment tends to produce tension in the bottom fibers of the beam at the section being investigated, the bending moment is *positive*. Similarly, if a moment tends to produce compression in the bottom fibers, the bending moment is negative. For example, if we take the moment of the 1000 lb. supporting force about an axis at *A-A*, the bending moment produced by that force at *A-A* will be 6000 ft.-lb. Such moment would tend (if the remainder of the beam to the right of section *A-A* were held rigid) to bend the portion of the beam to the left of *A-A* so that the bottom fibers of the beam at *A-A* would be stretched, or placed in tension. Then the bending moment at section *A-A* caused by the 1000 lb. supporting force is +6000 ft.-lb. The 2000 lb. load will cause the bottom fibers at *A-A* to be placed in compression, and therefore produces a bending moment of −8000 ft.-lb. The net bending moment will be −2000 ft.-lb.

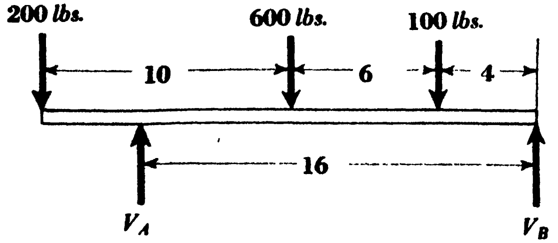
The graph in Fig. (*e*) shows the results of a survey of bending moment along the entire span of the beam, and is called the *bending moment diagram*.

48. The beam shown is supported by vertical supporting forces V_A and V_B, and carries two vertical loads of 1000 lb. and 5000 lb. Plot the shearing force and bending moment diagrams.



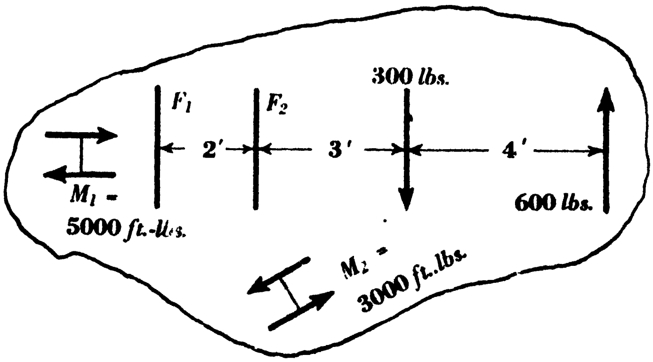
P-48.

49. Plot the shearing force and bending moment diagrams for the beam shown.



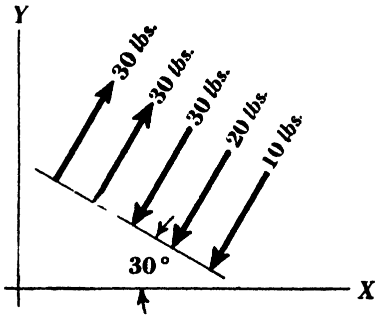
PROB. 49.

50. Find the magnitude and sense of F_1 and F_2 if the system of forces and couples shown is in equilibrium. *Ans.* $F_1 = 650$ lb.; $F_2 = 950$ lb.

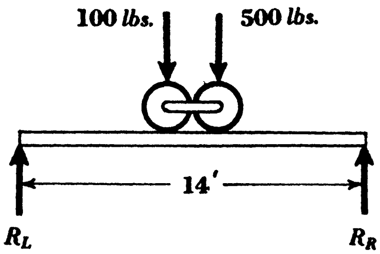


PROB. 50

51. Determine the resultant of the system of five parallel forces shown. The lines of action of the forces are 3 ft. apart.



PROB. 51

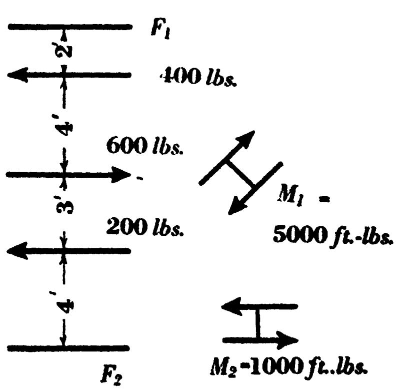


PROB. 52

52. Two loads of 100 and 500 lb. are placed on a horizontal beam 3 ft. apart. The supporting force at the right end of the beam (R_R) is three times as great as that at the left end (R_L). Determine the distance from the left end to the 100-lb. force.

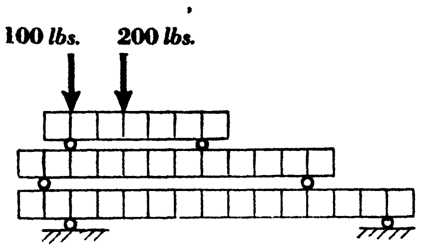
53. The system of parallel forces and couples shown is in equilibrium. Determine the senses and magnitudes of F_1 and F_2.

Ans. $F_1 = 230$ lb.; $F_2 = 230$ lb.

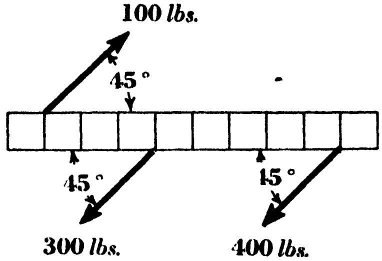


PROB. 53

54. Isolate each beam as a free body and determine the system of forces acting. Check results by selecting the entire system of beam as a free body and determining supporting forces at *A* and *B*. All rollers are frictionless.



PROB. 54



PROB. 55

55. What single force could be added to the system of forces shown to produce equilibrium of the beam? Each unit equals 1 ft.

18. General Case of Forces in a Plane. We have now established a base of axioms and techniques with which we are equipped to analyze the case of force systems that do not concur at a single point or do not all have the same inclination. Such a system, restricted only in that all of the forces lie in a single plane, is shown in Fig. 25*a*. If we resolve all of the forces into their *X* and *Y* components, we shall have the equivalent system shown in Fig. 25*b*.

The *X* components form a system of parallel forces in a plane. Then, this group of forces may be reduced, as in the preceding article, to a single force F_x acting at point *O*, and a single couple M_1, the sum of the moments of the *X* components about *O*.

The *Y* components also form a parallel force system and may be similarly resolved. They reduce to a single force F_Y acting through *O*, and a single couple M_2. The results of this stage of resolution are shown in Fig. 25*c*.

F_X and F_Y may be added to obtain their resultant F_R.

The couples M_1 and M_2 may be added, and their sum is M_R.

These two elements may be reduced to a single force F_R acting at distance a_R from point *O*, where $a_R = \frac{M_R}{F_R}$ (Fig. 25*d*).

In any problem that seeks to determine the resultant of a system of forces in a plane, it is unnecessarily laborious to trace each stage of the reduction of the original system to the final appearance of the simplest equivalent system, or resultant. We may, instead, employ the conclusions reached from that road of reduction. We summarize these conclusions as

$$R = \sqrt{\Sigma X^2 + \Sigma Y^2}$$

$$\theta_R = \tan^{-1} \frac{\Sigma Y}{\Sigma X}$$

$$a_R = \frac{\Sigma M}{R}$$

and proceed, more directly, in any given problem, with these equations as our weapons of attack.

Such a simplification, however, contains inherent possibilities of danger. These equations are the "formulas" applicable to this type of force system. The engineer must never permit himself to degenerate into a mere "formula-substituter." All formulas have limitations, and

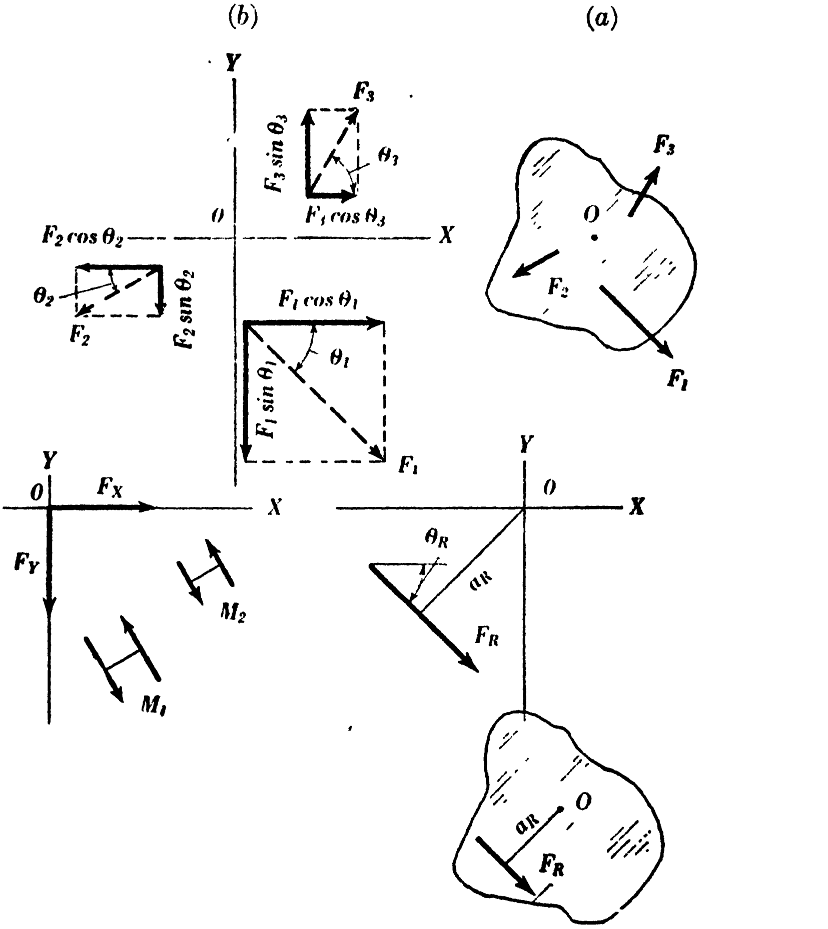
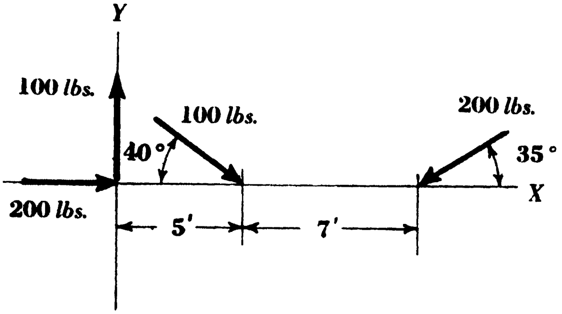


FIG. 25.

blind devotion to substitution may lead to abuse rather than proper use. Good engineering philosophy demands, rather, that we fully understand the background of every formula and, in such understanding, avoid exceeding the limitations of the formula. Formulas appear in the role of conveniently terse abbreviations of the statements of truth, which the engineer has previously vigorously examined and in which he may, therefore, place his confidence.

PROBLEMS

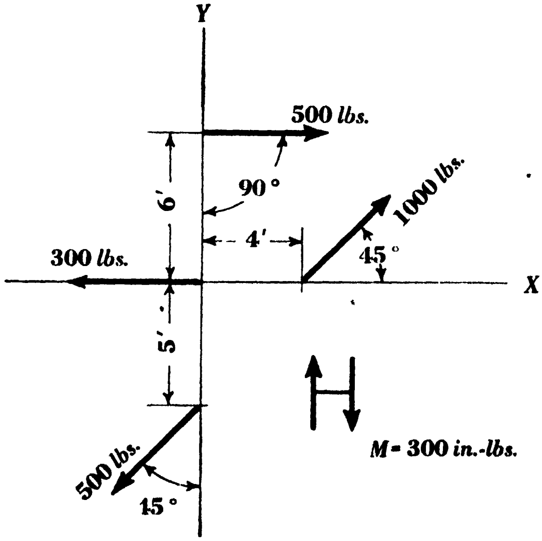
56. Determine the magnitude, sense, inclination, and location of the resultant of the system of forces shown.



PROB. 56

57. Find the X and Y intercepts of the line of action of the resultant of Problem 56.

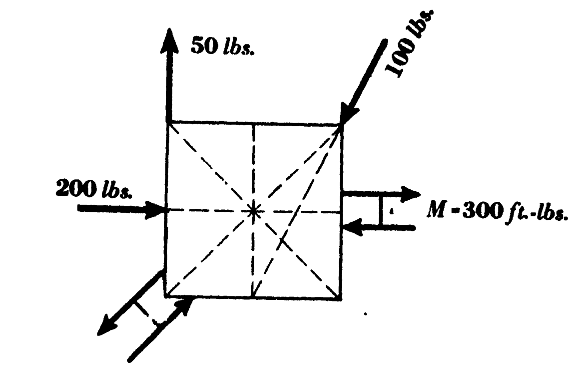
58. Given a coplanar system of four forces and a couple. Determine the magnitude, inclination, and sense of the resultant.



PROB. 58

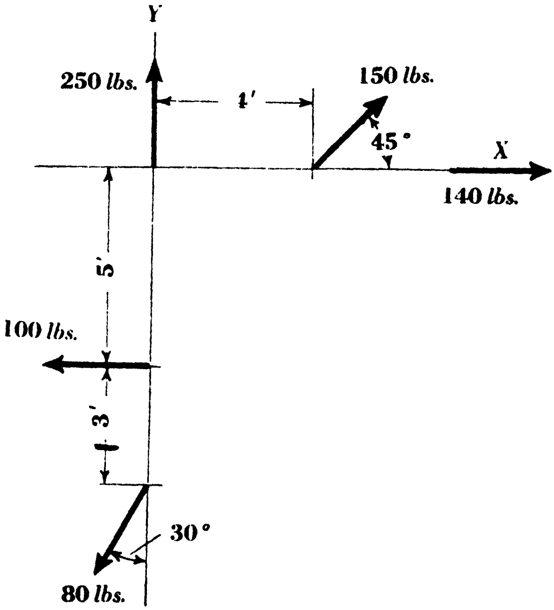
59. Find the X and Y intercepts of the line of action of the resultant of Problem 58.

60. The indicated free body is a 4 in. square. Determine the magnitude, sense, inclination, and location of a single force which, when added to the system, produces equilibrium of the free body.



PROB. 60

61. Determine the magnitude, inclination, sense, and location of the resultant of the system of five forces shown.



PROB. 61

62. Determine the X and Y intercepts of the line of action of the resultant of Problem 61.

19. Equilibrium Conditions for the General Case of Forces in a Plane. We have already defined the equilibrium of a free body as the condition in which the external force system applied to the body is in balance. An equivalent statement has also been indicated in our previous examinations of force systems—the force system must yield no resultant if there is to be equilibrium of the body. We have witnessed cases in which the original system of forces reduced to a single force. For equilibrium, then, we must find that

$$R = 0$$

which is also to demand that

$$\Sigma X = 0$$

and

$$\Sigma Y = 0$$

In addition, we have seen that systems of forces may reduce to a couple. Such moment must be eliminated if balance is to ensue and, therefore,

$$\Sigma M = 0$$

Then, the conditions of equilibrium for any system of forces in a plane are

$$\Sigma X = 0$$
$$\Sigma Y = 0$$
$$\Sigma M = 0$$

With three conditions of equilibrium presenting us with three simultaneous equations, it is possible to solve for three unknowns, and only three. The equation $\Sigma M = 0$ expresses but one fundamental fact. It is true that many equations of moment may be written for a given system of forces by selecting many moment axes. For one system of forces, such equations are only reiterations of one independent condition: we are writing identities as we write such equations, and only one unknown may be determined for each condition of equilibrium.

20. Axial Stress. The truss shown in Fig. 26*a* is a structural member composed of a series of flat bars pinned together at their ends.

The member AB has been isolated as a free body in Fig. 26*b*. The neighboring bodies EA, AC, BC, BH, BG, BD, and AD exert force on the free body, as does the load of 1000 pounds at pin A. If we neglect the weight of AB itself, we note that the force system acting on AB is that shown in Fig. 26*b*, which is acting on this member at its ends only. Such a member is sometimes called a *two-force member*, for when a member is loaded at its ends only, the system at each end is a concurrent force system and must have a single force as its resultant. Let us examine the nature of such a resultant.

Resultants R_1 and R_2 (the resultants of the concurrent force systems) are shown at the ends of the member in Fig. 26*c*.

If the member is to be in equilibrium, R_1 and R_2 must be equal, parallel, and opposed in sense, for only then will

$$\Sigma X = 0$$

and

$$\Sigma Y = 0$$

Another condition of equilibrium remains to be fulfilled—namely, $\Sigma M = 0$.

This condition can only be satisfied if R_1 and R_2 have the same line of action, because they would otherwise form a couple.

Then, R_1 and R_2 must be equal, parallel, opposite, and have the same line of action (the line AB, joining the points of application of R_1 and R_2).

Therefore, when a member is loaded at the ends only, the resultant force at each end acts along the line joining those ends.

We have assumed that the weight of the member itself might be neglected in order to establish the direction of the end resultants. This

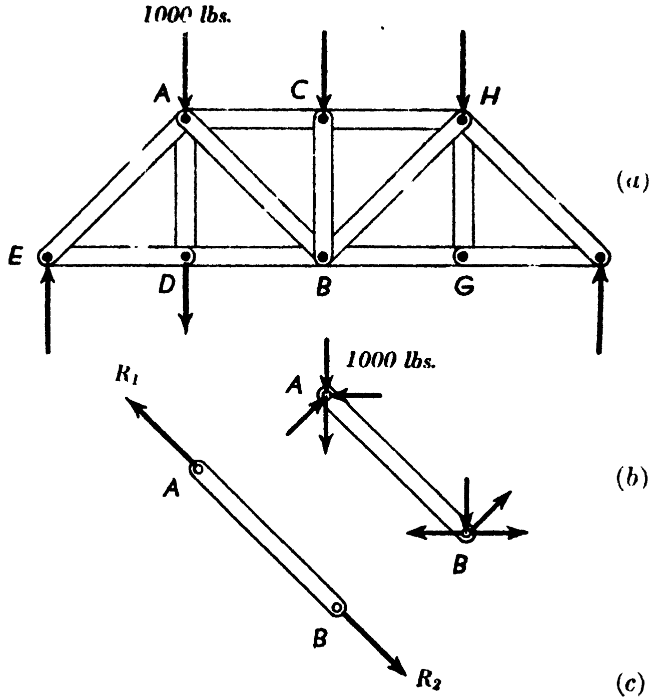


FIG. 26.

assumption is usually made during the first or primary investigation of force systems as they act upon such bodies. In later stages of the design of structural or machine members, the presence of weight is recognized, and proper modification of the calculations appears in the refined analysis. Under the influence of weight, which is distributed throughout the member, there is bending action, and the conclusion we have drawn concerning the direction of the end resultants cannot be defended. Investigations recognizing this bending action are made in the division of mechanics known as the Strength of Materials, and are beyond the limits of our present study of Statics.

Stress is internal force. The *total stress* is the amount of force transmitted from the body on one side of a plane section in a member, to the body on the other side of the plane section. In the case of the rod of

Fig. 27*a*, the system of external force F_1 and F_2 is applied at the ends only, and the rod is in equilibrium. Then the forces at the ends must lie in the direction of the axis of the rod. To ascertain the state of internal force, or stress, in the member, we must isolate a free body, for conditions of equilibrium are the tools with which we conduct our operations, and they apply only to the external forces acting on free bodies. We therefore isolate the body on one side of the plane section A-A, as shown in Fig. 27*b*. The force system is established as F_1 at the upper end (a known force) and F supplied by the neighboring body—the balance of the rod.

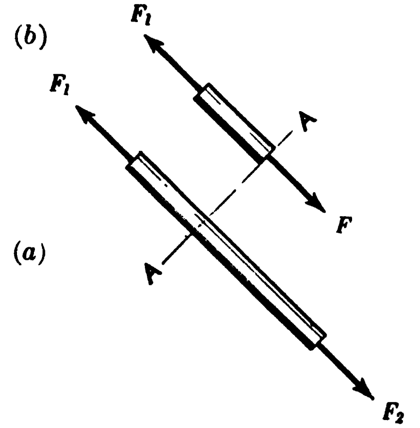


FIG. 27.

Applying the conditions of equilibrium we find that $F = F_1$. Then, F_1 is the total stress in the member, equal in magnitude to the end resultants. The nature of this stress is determined by noting that the external forces acting on the rod tend to stretch the member. Such stress is called *tension*. When the external forces tend to shorten the member, the stress is called *compression*.*

21. Illustrative Examples. The general system of forces in a plane is the type of force system most frequently encountered. We shall, therefore, proceed in the following examples to analyze in detail free bodies subjected to such force systems. At this time, we shall reiterate certain statements made earlier in this text.

Effectiveness in the solution of the problems of mechanics is primarily dependent upon the selection of a free body. Now our attention is focused upon a tangible and specific base. In order that this base may be solid, we should not only see it in our mind's eye, but must have a drawing. When the drawing of the free body is available, we can add vectors representing all known and unknown forces.

We may then proceed to apply the axioms that are pertinent to solve for the unknowns.

Two remaining stages in the proper solution of our problems have not previously been given the serious attention they merit. The first stage is the practice of confirming results by checking. A valid check should, if possible, be more than a mere repetition of the mathematical steps, taken in the solution to insure their accuracy. Whenever it is possible to make a confirming solution by means of material not previously used,

* In current engineering literature, the term *stress* is frequently used to denote unit stress, or stress acting on unit area. As used here, the resultant, or total force, is referred to.

the check has been freed of dependence upon the original solution and is, therefore, more effective in revealing possible error or in substantiating the original results. As we proceed with the analysis of illustrative examples, opportunity for effective checking will arise and will be employed.

A final stage in a proper problem solution is the presentation of a clear report of the results. Engineers understand one another most readily when they translate their thoughts or statements into a sketch or drawing. The language of the engineer is graphical. The report, in the case of mechanics problems involving force systems acting upon free bodies, should be a sketch of those free bodies with the forces which were the objective of the analysis shown on the sketch. Figures 32, 36, and 43 illustrate the proper method of rendering the report so that there can be no confusion in the mind of the reader as to the magnitude and direction of each force in relationship to the free body.

ILLUSTRATIVE EXAMPLE 1

The truss shown in Fig. 28 is supported on a frictionless roller at B and a pin joint at A. The truss consists of individual members, pinned

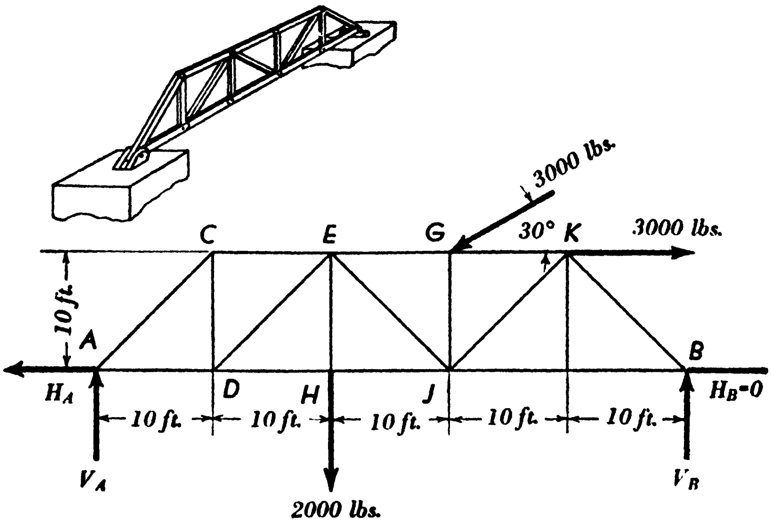


FIG. 28.

together at their ends, and all loads are placed at the pin joints. A truss constructed and loaded in this manner will be referred to as a *simple truss.*

Determine the following:

1. The horizontal and vertical components of the supporting forces at A and B. 2. The stresses in members AC and AD. 3. The stresses in members EG, EJ, and HJ.

1. We select a free body of such nature that the unknown forces (H_A, V_A, H_B, and V_B) are acting as contact, or external, forces upon the body, as well as some known force or forces (the loads).

The entire truss, free of its supports, conforms to this criterion of selection of a free body.

We next make a drawing of the free body. The known loads are shown as vectors (since the solution is to be analytical, such vectors need not be drawn to scale, but should have their magnitudes, senses, and inclinations noted on the drawing). The unknown forces are H_A, V_A, H_B, and V_B. These, too, are shown as vectors. In the case of H_B, we know that its magnitude is zero, for the roller is frictionless. V_B has sense upward, for the roller can only exert contact force in that direction. H_A and V_A are known in inclination only: we therefore assume their senses to be as shown. The appearance of a positive sign before the answer in each case will confirm the assumption—a negative sign will indicate that the assumption was improper and must be reversed.

Now that a tangible qualitative analysis is before us, we can decide whether the solution is *determinate*.

The system of forces is coplanar, nonparallel, and noncurrent.

Then, we have available three conditions of equilibrium—namely,

$$\Sigma X = 0$$
$$\Sigma Y = 0$$
$$\Sigma M = 0$$

There are three unknowns and three available simultaneous equations. Then, the problem is statically determinate, and we proceed with the solution.

It will be efficient to select first an equation in which only one unknown will appear, because we can then determine that unknown without the interdependence of two or more unknown quantities that would otherwise be involved.

If we select a moment axis at point A, H_A and V_A will not appear in the moment equation.

$$\Sigma M_A = +2000 \times 20 + 3000 \times 0.5000 \times 30 - 3000 \times 0.8660 \times 10 + 3000 \times 10 - 50V_B = 0$$

Then, $V_B = +1780$ lb.

The selection of a moment axis at B gives

$$\Sigma M_B = -2000 \times 30 - 3000 \times 0.5000 \times 20 - 3000 \times 0.8660 \times 10 + 3000 \times 10 + 50V_A = 0$$

Then, $V_A = +1720$ lb.

We have an opportunity here to check these two answers by applying

$$\Sigma Y = -2000 - 3000 \times 0.5000 + 1780 + 1720$$
$$\Sigma Y = 0$$

which checks V_A and V_B.

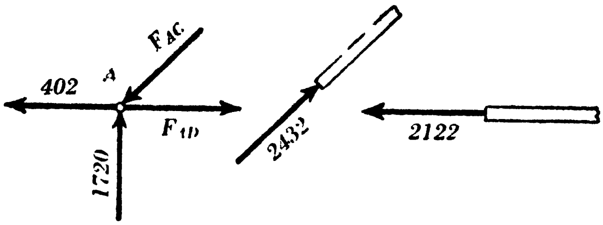
Next we apply

$$\Sigma X = -H_A - 3000 \times 0.8660 + 3000 = 0$$
$$H_A = +402 \text{ lb.}$$

The proper method of reporting the results of the solution of part 1 is shown in Fig. 32.

2. To determine the stresses in AC and AD, we select the pin at A as a free body.

The drawing of the free body and accompanying force system is shown in Fig. 29a. H_A and V_A are known forces, determined in part 1. The members AC and AD are two-force members, and the resultant forces that they exert on the free body have their lines of action in the directions of the members themselves. (See Article 20.) The senses of these two forces are assumed.



(a) (b) (c)

FIG. 29.

There are two unknowns and two simultaneous equations of equilibrium, and the problem is, therefore, determinate.

$$\Sigma Y = -0.7071 F_{AC} + 1720 = 0$$
$$F_{AC} = +2432 \text{ lb.}$$
$$\Sigma X = -402 - 2432 \times 0.7071 + F_{AD} = 0$$
$$F_{AD} = +2122 \text{ lb.}$$

F_{AC} and F_{AD} are forces acting on the free body—pin A. The member AC, if selected as a free body, will then have acting on it (Fig. 29b) a force of 2432 pounds as shown, and the member is being compressed. Then the stress in AC is 2432 pounds, compression. The member AD will have a force acting on it equal and opposite to F_{AD}, as shown in Fig. 29c. Then the stress in AD is 2122 pounds, tension.

The proper method of plotting these stresses is shown in Fig. 32.

3. It would be possible to determine the stresses in members EG, EJ, and HJ by selecting in turn the pins at C, D, E, and H as free bodies and analyzing at each of these joints the concurrent system of forces, as in part 1. We should finally arrive at the solution for the required stresses. This is a laborious undertaking and introduces a multiplicity of opportunities for error. If error is avoided, it presents at each stage

of the solution a necessary rounding off of significant figures and entails a questionable degree of accuracy in the final results.

A more judicious choice of free body is shown in Fig. 30.

Since a member such as *EG* is loaded at its ends only, the stress is axial. If, then, the member is cut at any section, as in Fig. 30, the force acting on the remaining portion of the member is in the direction of the member. This fact is justification for the choice of free body that is shown. The known forces acting on the force body are shown, as are forces F_{EG}, F_{EJ}, and F_{HJ}, which act, respectively, on this free body in

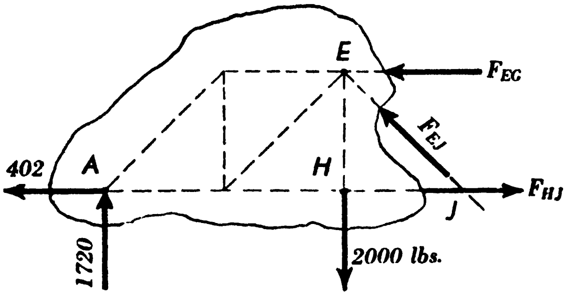


FIG. 30.

the inclinations of members *EG*, *EJ*, and *HJ*, respectively. The senses of these forces are assumed.

There are three unknowns and three simultaneous equations of equilibrium, and the problem is determinate. If a moment axis is selected at point *J*, two of the unknowns are eliminated from the resulting moment equation,

$$\Sigma M_J = -F_{EG} \times 10 - 2000 \times 10 + 1720 \times 30 = 0$$
$$F_{EG} = +3160 \text{ lb.}$$

Another moment axis is selected at *E* to eliminate two unknowns.

$$\Sigma M_E = -F_{HJ} \times 10 + 1720 \times 20 + 402 \times 10 = 0$$
$$F_{HJ} = +3842 \text{ lb.}$$

Finally, we may solve for the remaining unknown (F_{EJ}) without depending upon the calculations just made, by applying:

$$\Sigma Y = +0.7071 F_{EJ} - 2000 + 1720 = 0$$
$$F_{EJ} = +396 \text{ lb.}$$

These forces are translated in terms of the stresses which are produced in members *EG*, *HJ*, and *EJ*, and

Stress in *EG* = 3160 lb., compression
Stress in *HJ* = 3842 lb., tension
Stress in *EJ* = 396 lb., compression

The results are plotted in Fig. 32.

The method which we have just employed in isolating, as free body, a larger section of a given structure than a single pin or member, may be employed again to serve as a check on the results.

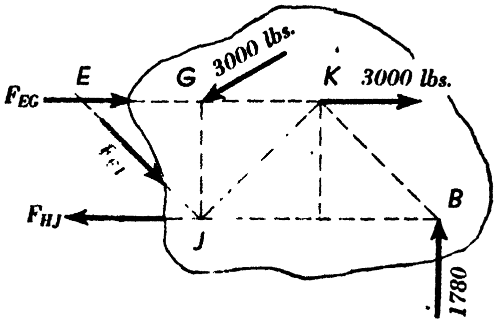


FIG. 31.

If we select as rigid body the section of the truss shown in Fig. 31, we note that the forces F_{EG}, F_{EJ}, and F_{HJ} appear as external forces.

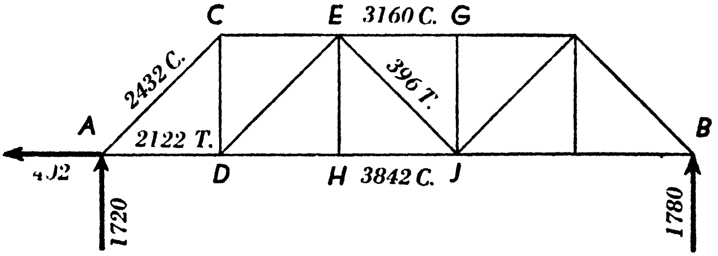


FIG. 32.

Taking moments about J,

$$\Sigma M_J = +F_{EG} \times 10 - 3000 \times 0.8660 \times 10 + 3000 \times 10 - 1780 \times 20 = 0$$

$$F_{EG} = +3160 \text{ lb.}$$

Taking moments about E,

$$\Sigma M_E = +F_{HJ} \times 10 + 3000 \times 0.5000 \times 10 - 1780 \times 30 = 0$$
$$F_{HJ} = +3840 \text{ lb.}$$

Applying ΣY,

$$\Sigma Y = -F_{EJ} \times 0.7071 - 3000 \times 0.5000 + 1780 = 0$$
$$F_{EJ} = +360 \text{ lb.}$$

These results have confirmed the previous answers. The technique of selecting as free body a different body than that used in the first solution, insures a check independent of the results of the former.

Illustrative Example 2

Given the framework shown in Fig. 33, consisting of a vertical mast *AB*, a horizontal boom *ST* (pinned to the mast at *S*), a cable *WU* (pinned to the boom and to the mast), and a cable *BC* (pinned to the mast and a supporting abutment). The mast is pinned to a supporting foundation at *A*, and the framework supports a vertical load of 4000 pounds.

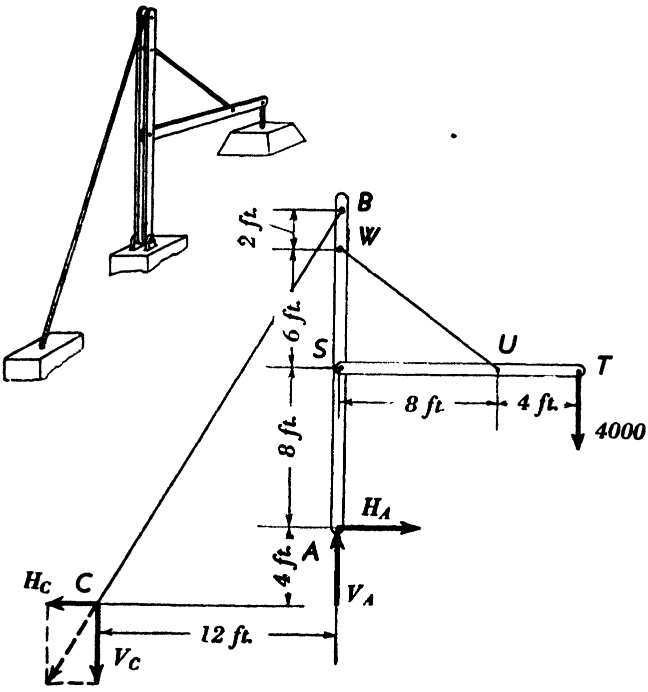


Fig. 33.

Determine the following: (1) The horizontal and vertical components of the supporting forces at *A* and *C*. (2) The stress in *WU*. (3) The horizontal and vertical component of the force acting on *ST* at *S*.

1. If the framework itself is selected as free body, as shown in Fig. 33, the unknown forces are H_A, V_A, H_C, and V_C. These are shown with their senses assumed.

Apparently, we are confronted with a system containing four unknowns.

We note, however, that the member *BC* is loaded at its ends only. In such a two-force member, the resultant force at each end has its line of action along the center line of the member. (See Article 20.) Then, the inclination of R_C, the resultant force at *C*, is known, and the relationship of the components is fixed as

$$\frac{H_C}{V_C} = \frac{12}{20} = \frac{3}{5}$$

The problem is, then, statically determinate, and we proceed with a moment equation, using point A as moment axis to eliminate two unknowns—H_A and V_A.

$$\Sigma M_A = +4H_C - 12V_C + 4000 \times 12 = 0$$

or

$$+4(\tfrac{3}{5}V_C) - 12V_C + 4000 \times 12 = 0$$

$$V_C = +5000 \text{ lb.}$$

$$H_C = \tfrac{3}{5} \times 5000 = 3000 \text{ lb.}$$

Applying

$$\Sigma X = -3000 + H_A = 0$$

$$H_A = +3000 \text{ lb.}$$

$$\Sigma Y = -5000 + V_A - 4000 = 0$$

$$V_A = +9000 \text{ lb.}$$

The proper plotting of these results is shown in Fig. 36.

2. In the selection of a proper free body, we always seek one upon which the unknown forces, which are to be determined, are acting. In addition, there must be some known force or forces to serve as evaluating material.

In the present case, when the stress in WU is to be determined, either the mast AB or the boom ST may be selected. The latter is to be preferred, since the solution can be made without dependence upon forces, originally unknown in this problem, and, therefore, determined during its progress. Such forces are as yet unchecked and present the possibility of interfering with the present solution, if they are either inaccurate or incorrect.

The drawing of the free body is shown as Fig. 34.

Applying

$$\Sigma M_S = -\tfrac{6}{10}F_{WU} \times 8 + 4000 \times 12 = 0$$

$$F_{WU} = 10{,}000 \text{ lb.}$$

Then the stress in WU is 10,000 pounds tension and is reported by plotting as in Fig. 36.

3. The free body that is selected and shown in Fig. 34 will also enable us to determine H_S and V_S.

$$\Sigma X = -\tfrac{8}{10} \times 10{,}000 + H_S = 0$$

$$H_S = +8000 \text{ lb.}$$

$$\Sigma Y = +\tfrac{6}{10} \times 10{,}000 - 4000 - V_S = 0$$

$$V_S = +2000 \text{ lb.}$$

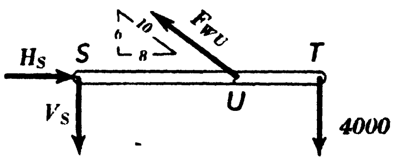


FIG. 34.

These forces cannot be properly shown in Fig. 36, where all results are being plotted as a report, because these forces are internal in the assemblage of members shown there. H_S and V_S are, for example, as shown in Fig. 34, the horizontal and vertical components of the force that member AB is exerting on ST at S.

The member ST is exerting equal and opposite components on AB, because the pin joining the two members is in equilibrium.

We therefore make a subordinate drawing on the main plot of results and show a small portion of ST to identify which member is the free body on which the components H_S and V_S, of the senses shown, are acting.

Check. In this problem, an excellent opportunity is afforded to accomplish a check by using a free body different from those selected in the original solution.

The mast AB is shown as Fig. 35, and the results of the previous solution are indicated as the system of external forces acting on AB.

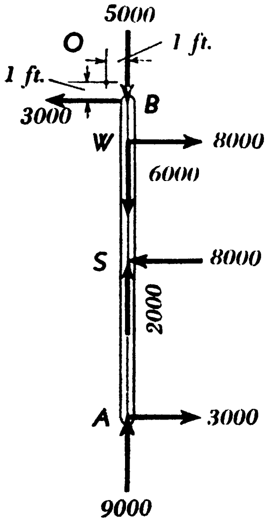


FIG. 35.

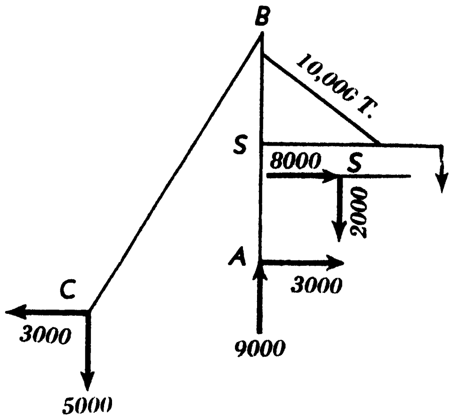


FIG. 36.

We apply the three appropriate conditions of equilibrium to AB.

$$\Sigma X = -3000 + 8000 - 8000 + 3000$$

which does equal zero.

$$\Sigma Y = -5000 - 6000 + 2000 + 9000$$

This sum also equals zero.

Point O has been selected as a moment axis to insure that all of the forces which have been determined in the solution will appear in the moment equation.

$$\Sigma M_o = +3000 \times 1 + 5000 \times 1 - 8000 \times 3 + 6000 \times 1 + 8000 \times 9 - 2000 \times 1 - 3000 \times 17 - 9000 \times 1 = 0$$

Then the original results have been confirmed.

3. The structure shown in Fig. 37 affords opportunity for an analysis in which we take advantage of "*simultaneous isolation*" of more than one free body.

Frequently, we encounter problems in which the usual isolation of a free body or the isolation, in turn, of a series of free bodies, fails to yield the necessary number of simultaneous equations based upon the conditions of equilibrium. This does not connote that the problem is necessarily statically indeterminate.

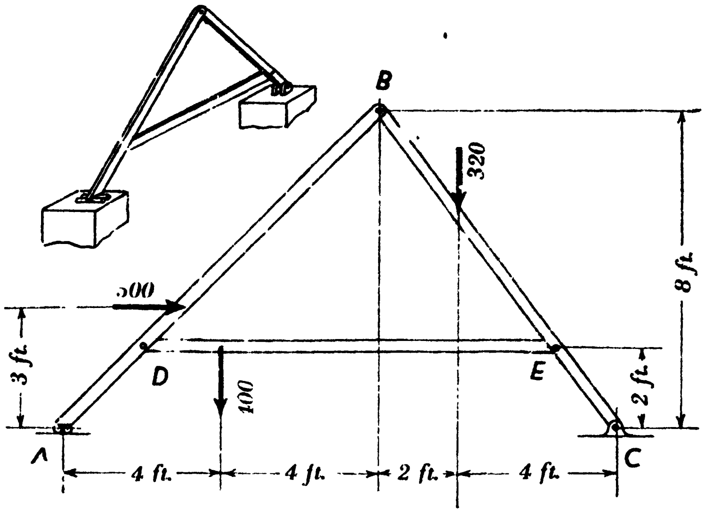


Fig. 37

If two bodies are joined by a pin joint, and we isolate the pin itself, assuming the pin to be frictionless, we know that the force system acting on the pin must satisfy the equations $\Sigma X = 0$ and $\Sigma Y = 0$, as in all concurring force systems in a plane.

For example, in the case of pin joint B (Fig. 37), two members AB and BC are joined by a pin, which is a body in equilibrium.

The pin B is shown as a free body in Fig. 38.

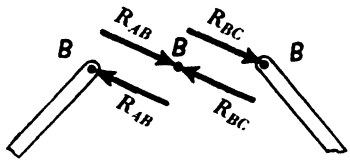


Fig. 38.

The resultant force R_{AB}, which member AB exerts on the pin, must be equal and opposite to the resultant force R_{BC}, exerted by member BC on pin B, because these are the only forces supplied by contacting bodies acting on the pin.

This opposition of forces acting at a pin joint is an example of equal and opposite action and reaction between contacting bodies.

We shall frequently find that isolation of individual free bodies, such as AB or BC, fails to present sufficient material for an attack on unknown forces. At such times, we can take advantage of the balance of the forces exerted by both members on the pin which joins them.

In such cases, we isolate both bodies and develop an attack on both free bodies simultaneously.

Let us illustrate such simultaneous isolation by determining, for the structure shown in Fig. 37, the following:

The horizontal and vertical components of the forces acting at A, B, C, D, and E.

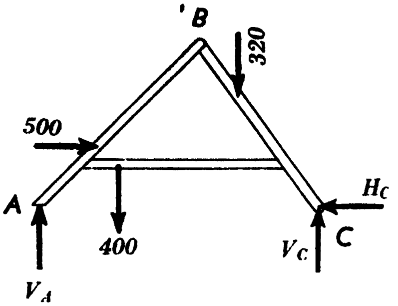


FIG. 39.

The supporting forces at A and C may be determined by selecting the entire frame as a free body. This selection agrees with the first criterion for a free body, which demands that we select a free body such that the unknowns we seek to determine are acting as external forces on the free body, as are some known forces, to serve as material for evaluation.

This free body is shown in Fig. 39.

$$\Sigma M_A = -14V_C + 500 \times 3 + 400 \times 4 + 320 \times 10 = 0$$
$$V_C = +450 \text{ lb.}$$
$$\Sigma M_C = +14V_A - 320 \times 4 - 400 \times 10 + 500 \times 3 = 0$$
$$V_A = +270 \text{ lb.}$$

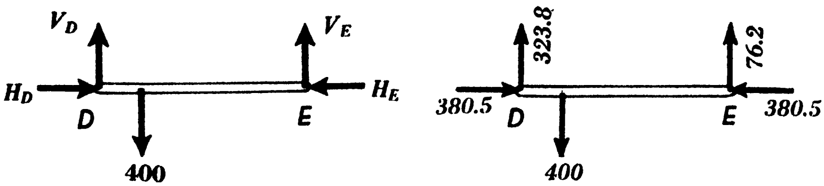
Applying ΣY to check these components,

$$\Sigma Y = +450 + 270 - 400 - 320 = 0$$

$$\Sigma X = +500 - H_C = 0$$
$$H_C = +500 \text{ lb.}$$

The proper plotting of these results is shown in Fig. 43.

To determine the horizontal and vertical components of the force acting at D, two free bodies are available—DE and AB.



(a) (b)

FIG. 40.

DE is shown in Fig. 40a. If we appraise DE qualitatively, we note that there are four unknowns acting, and since the system of external forces lies in one plane, only three conditions of equilibrium are available.

Then, we cannot determine all of the unknowns appearing on this free body.

AB is shown in Fig. 41. In this case, too, we encounter a number of unknowns which exceeds the number of available simultaneous equations.

If now we make a drawing, as in Fig. 42 of free body *BC*, we find an opportunity to deal simultaneously with two bodies and can take advantage of the nature of the action and reaction at pin *B*.

When we assume senses for H_B and V_B, acting on free body *AB*, as in Fig. 41, we have fixed the senses of H_B and V_B, acting on free body *BC* of Fig. 42. The latter must be respectively opposite to those senses assumed in Fig. 41, because they are the equal and opposite reactions to those components.

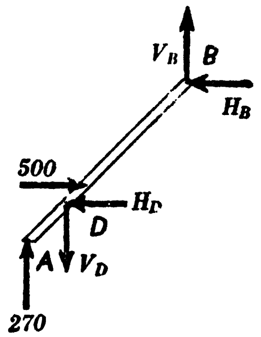


FIG. 41.

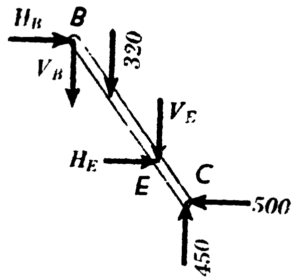


FIG. 42.

Now we may select point *D* (Fig. 41) as a moment axis.

$$\Sigma M_D = -6V_B - 6H_B + 500 \times 1 + 270 \times 2 = 0$$

$$-6V_B - 6H_B + 1040 = 0 \qquad \text{(a)}$$

Similarly, selecting point *E* of Fig. 42 as a moment axis,

$$\Sigma M_E = -4.5V_B + 6H_B - 320 \times 2.5 - 450 \times 1.5 + 500 \times 2 = 0$$

$$-4.5V_B + 6H_B - 175 = 0 \qquad \text{(b)}$$

Solving Equations (a) and (b) simultaneously, we have

$$V_B = +53.8 \text{ lb.}$$

$$H_B = +119.5 \text{ lb.}$$

On body *AB*,

$$\Sigma X = -H_D + 500 - 119.5 = 0$$

$$H_D = +380.5 \text{ lb.}$$

$$\Sigma Y = -V_D + 53.8 + 270 = 0$$

$$V_D = +323.8 \text{ lb.}$$

On body *BC*,

$$\Sigma X = +H_E + 119.5 - 500 = 0$$

$$H_E = +380.5 \text{ lb.}$$

$$\Sigma Y = -V_E - 53.8 - 320 + 450 = 0$$

$$V_E = +76.2 \text{ lb.}$$

We have here an opportunity to check the results by selecting *DE* as a free body (Fig. 40*b*) and showing the force system. All the forces acting on DE are known from the previous investigations of AB or, as in the case of the 400-pound load, these forces are given as original data.

On DE,

$$\Sigma X = +380.5 - 380.5 = 0$$
$$\Sigma Y = +323.8 - 400 + 76.2 = 0$$

The results of the investigation are plotted as Fig. 43. Such plotting reveals at once, not only the magnitudes of the forces acting at each pin

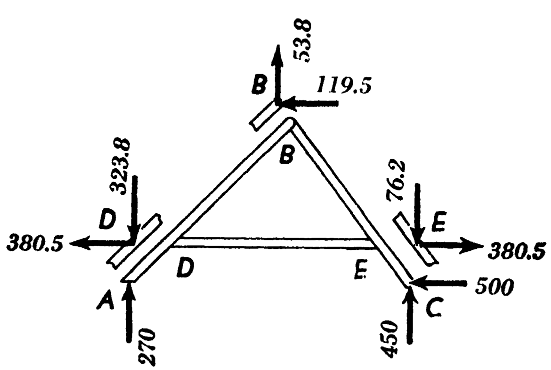


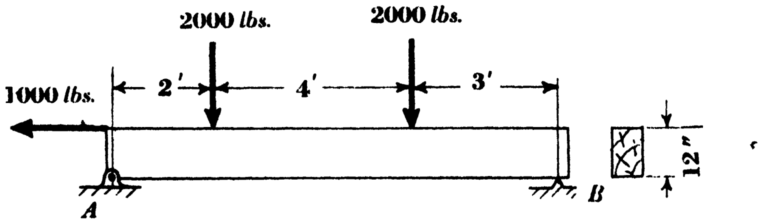
FIG. 43.

joint, but their directions relative to the specific bodies entering the joint. The small section of a member, shown as a subordinate drawing at each joint, clearly establishes which senses of the components are to be associated with the member whose section is shown.

The component forces acting on the adjacent member of the joint will be of opposite sense.

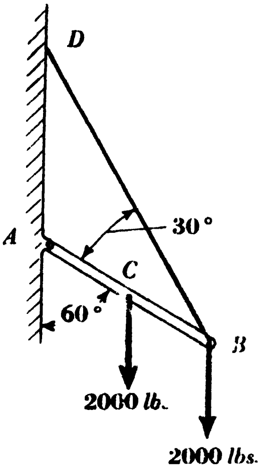
PROBLEMS

63. The 12-in. beam rests on a knife-edge at B, and is supported by a pin joint at A. Determine the resultant supporting forces exerted on the beam at A and B.

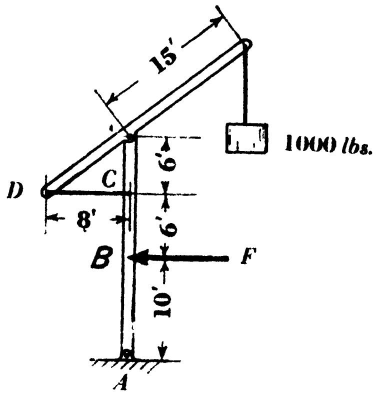


PROB. 63

64. The bar AB carries two loads of 2000 lb. each. C is the mid-point of AB. Determine the horizontal and vertical components of the supporting force at A, and the stress in cable BD. AB is pinned to the wall at A, and BD is tied to the wall at D. *Ans.* H_A = 2595 lb.; V_A = 500 lb.; BD = 5196 lb., tension.



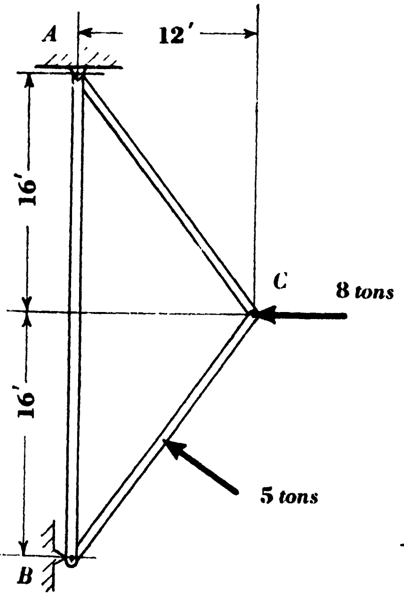
Prob. 64



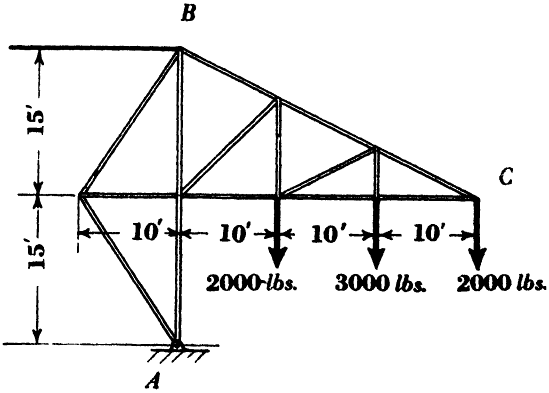
Prob. 65

65. The crane is supported on a pivot at A, and held in the upright position by a horizontal force F applied at point B. Find the force F and the stress in member CD.

66. The frame shown is supported on a pin joint at A, and rests on a knife-edge at B. The 8-ton force at C is horizontal, and the 5-ton force is perpendicular to BC at its mid-point. Determine the stresses in AC and AB.



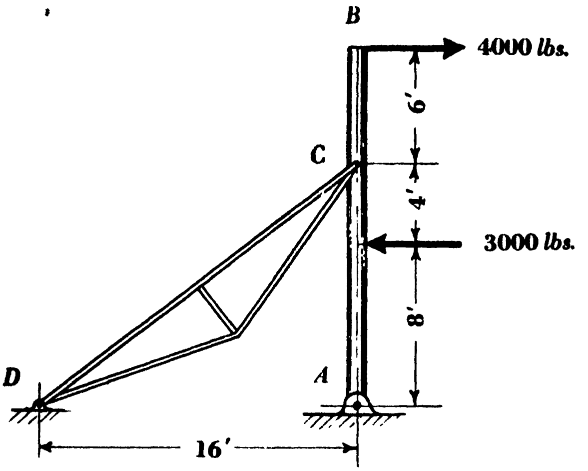
Prob. 66



Prob. 67

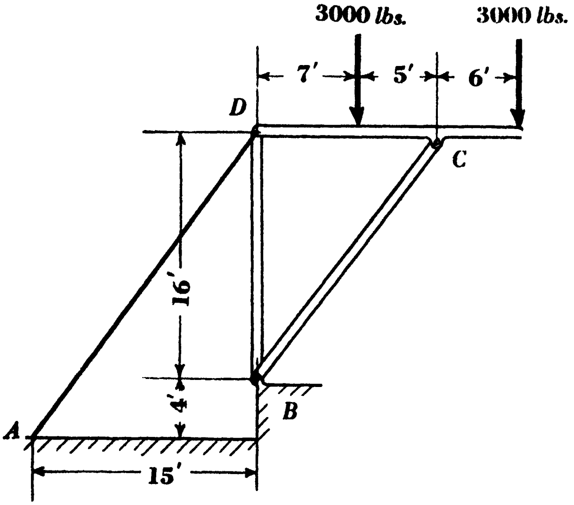
67. The truss is held in equilibrium by a horizontal cable at B. Determine the stress in the cable, and the horizontal and vertical components of the supporting force at A. *Ans.* Cable = 4670 lb., tension; H_A = 4670 lb.; V_A = 7000 lb.

68. The vertical mast AB is supported on a pin joint at A, and held in position by the bracing truss CD. Determine the horizontal and vertical components of the forces acting at A and C.



PROB. 68.

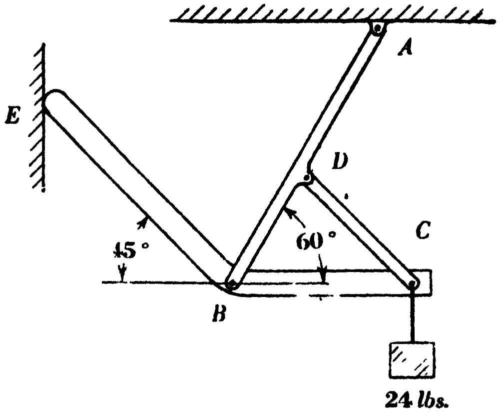
69. Determine the horizontal and vertical components of the supporting forces at A and B.



PROB. 69.

70. Determine the stresses in members BC and BD of the frame given in Problem 69.

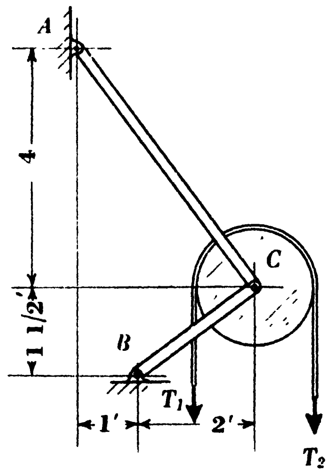
71. The trip mechanism shown is in equilibrium. A is a pin joint, and the vertical surface at E is frictionless. Determine the forces acting at A, B, and E. AB = 5.8 in.; BC = 3.6 in.; BE = 5.2 in.; BD = 2.6 in.



PROB. 71

72. Determine the stress in member CD of Problem 71.

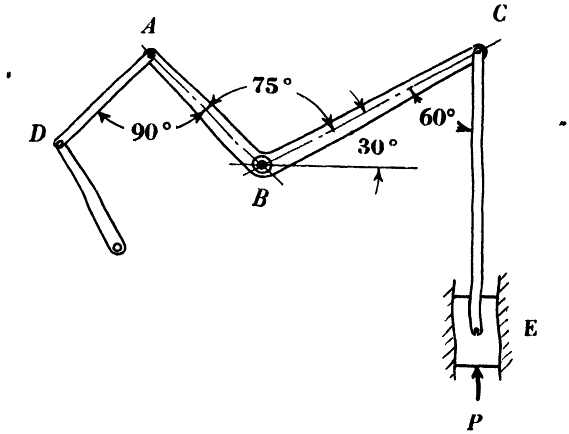
73. The pulley hanger is supported on pin joints at A and B. The pulley rotates about a shaft at C, and its diameter is 2 ft. The tensions in the belt are T_1 = 450 lb. and T_2 = 650 lb. Determine the horizontal and vertical components of the supporting forces at A and B.



PROB. 73.

74. Determine the horizontal and vertical components of the supporting forces at A and B in Problem 73 if the direction of rotation of the pulley is reversed by interchanging the belt tensions T_1 and T_2.

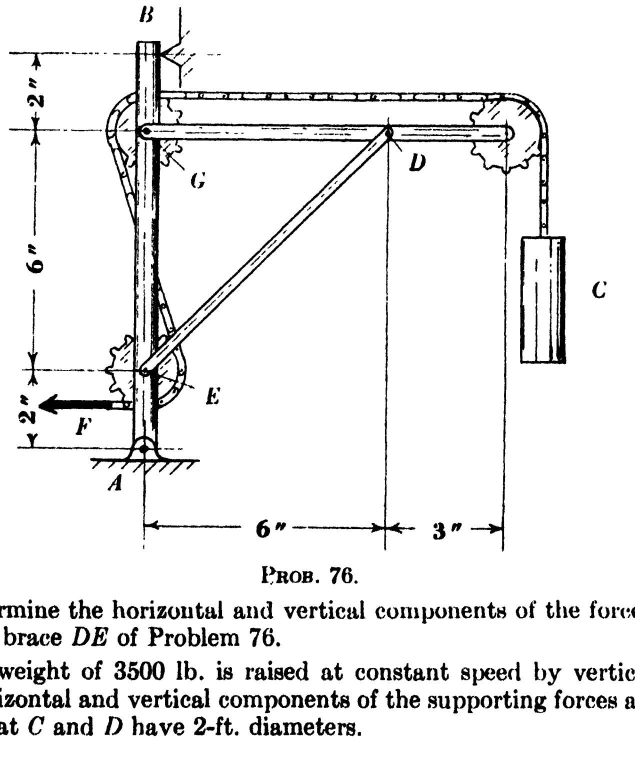
75. The rocker arm ABC rotates about a fixed axis at B, and is driven by a connecting rod AD. ABC is in equilibrium in the position shown. The force exerted by AD on the rocker arm is assumed to be in the direction of AD. Connecting rod CE is pinned to ABC at C and, at the instantaneous position shown, is vertical. If the total pressure on the position is P = 1000 lb., determine the stress AD, and the resultant force acting on the rocker arm at B. $AB = 8.0$ in.; $BC = 12.5$ in.



Prob. 75

76. In an automatic machine, a chain drive is used to raise cylinder C at constant speed. The framework is supported on a pin at A and rests against a knife-edge at B. Cylinder C weighs 40 lb. The pitch radius (effective radius to center line of chain) is 1.25 in. for each pulley. Determine the resultant supporting forces at A and B.

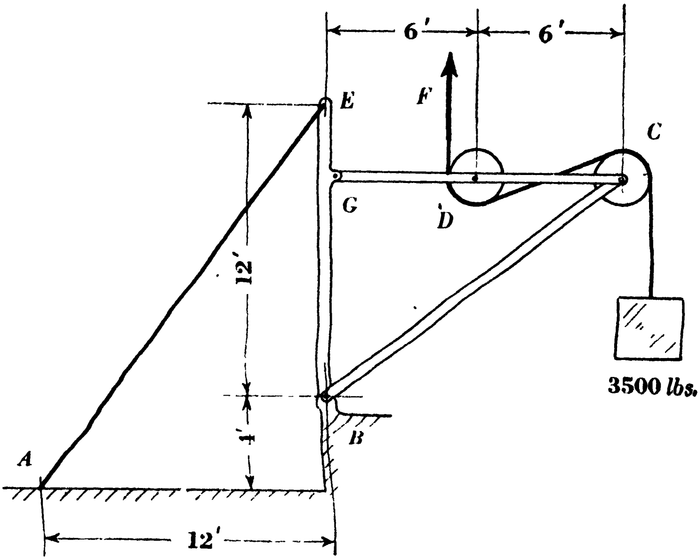
Ans. $H_B = 38$ lb.; $V_B = 0$; $H_A = 78$ lb.; $V_A = 40$ lb.



Prob. 76.

77. Determine the horizontal and vertical components of the force at G, and the stress in brace DE of Problem 76.

78. The weight of 3500 lb. is raised at constant speed by vertical force F. Find the horizontal and vertical components of the supporting forces at A and B. The pulleys at C and D have 2-ft. diameters.

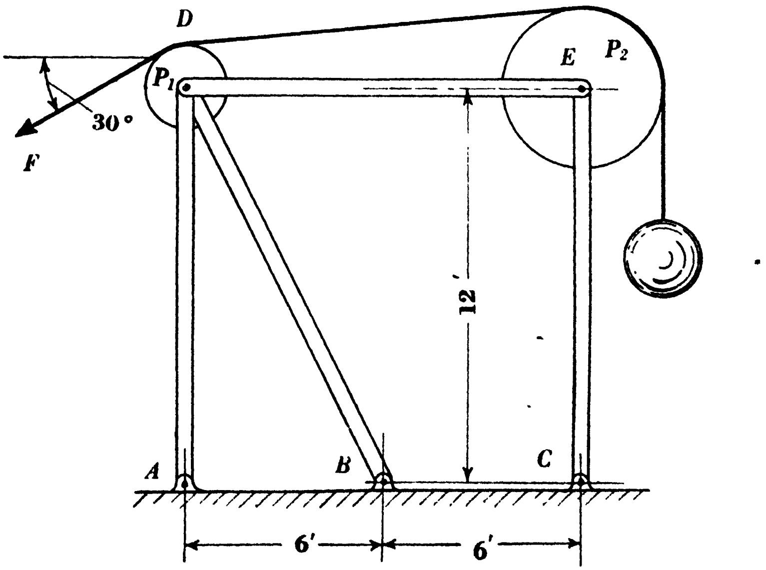


PROB 78

79. Determine the stress in member BC of Problem 78. $BG = 9$ ft.

80. If the line of action of force F in Problem 78 is inclined at 45° to the right, determine the horizontal and vertical components of the supporting force at B; the stress in AE; and the stress in BC.

81. The sphere weighing 3600 lb. is held in position by force F inclined at 30° with the horizontal. Determine the horizontal and vertical components of the supporting forces at A, B, and C. Pulley P_1 = 2 ft. diameter; pulley P_2 = 4 ft. diameter.



PROB. 81.

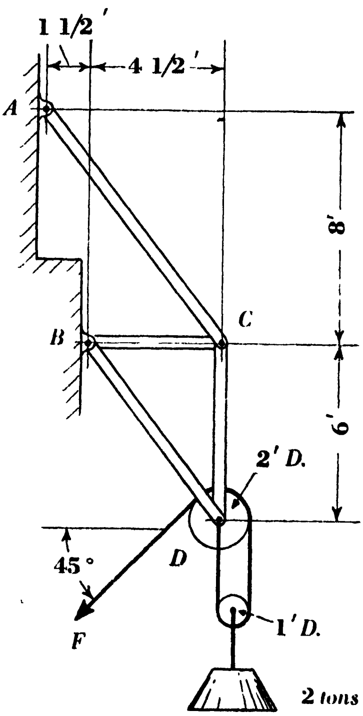
82. In Problem 81, determine the stresses in members *AD*, *BD*, and *CE* if force *F* is inclined at 60° with the horizontal.

83. The frame is pinned to the wall at *A* and *B*. If the load of 2 tons is being raised at constant speed by force *F*, find the horizontal and vertical components of the supporting forces at *A* and *B*.

Ans. $H_A = 2.74$ tons; $V_A = 3.65$ tons; $H_B = 3.45$ tons; $V_B = .94$ tons.

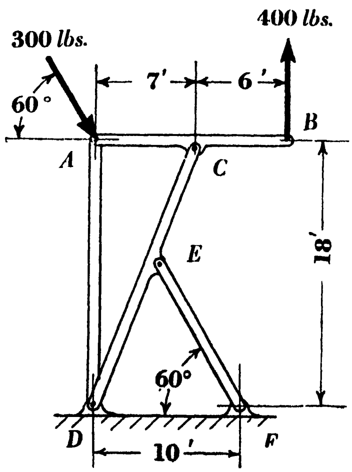
84. In Problem 83, determine the stresses in members *BC* and *BD*.

85. If the force *F* of Problem 83 is applied vertically, determine the stresses in members *BC*, *CD*, and *AC*.



PROB. 83.

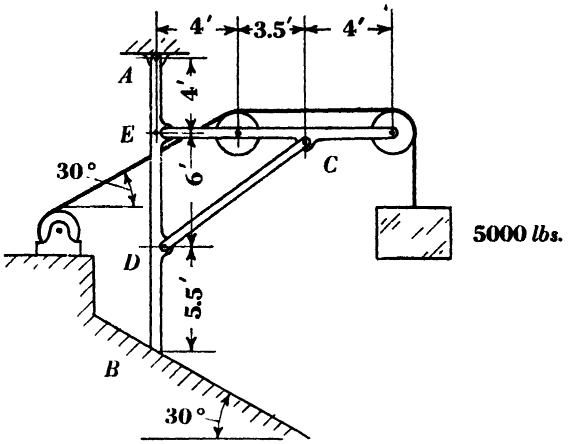
86. In the feed mechanism of an automatic machine, the member *AB* carries two loads of 300 lb. and 400 lb. as shown. If the mechanism is in equilibrium, determine the stress in *EF*.



PROB. 86.

87. In the mechanism of Problem 86, determine the stress in member *AD*, and the horizontal and vertical components of the supporting force at *D*.

88. The weight of 5000 lb. is being raised at constant speed by a motor-driven drum. The frame is supported by a pin joint at *A*, and rests on a frictionless surface at *B*. Determine the horizontal and vertical components of the supporting forces at *A* and *B*, and the stress in member *CD*. Pulley diameters are 2 ft.

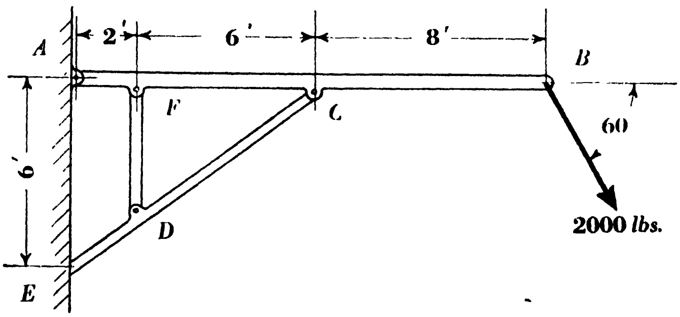


PROB. 88.

89. In Problem 88, find the horizontal and vertical components of the forces acting at *C* and *E*.

90. The horizontal member *AB* is pinned to a wall at *A*, and to members *CE* and *DF*, at *C* and *F*. *CE* rests on a frictionless surface at *E*. Determine the horizontal and vertical components of the supporting forces at *A* and *E*.

Ans. H_E = 4620 lb.; V_E = 0; H_A = 5620 lb.; V_A = 1730 lb.

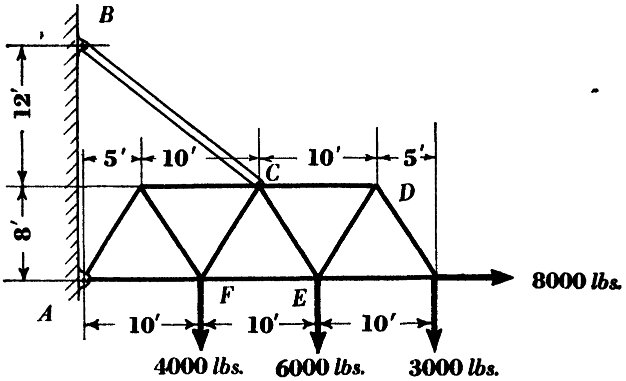


PROB. 90.

91. In Problem 90, determine the horizontal and vertical components of the forces acting at *C*, and the stress in the tie member *DF*.

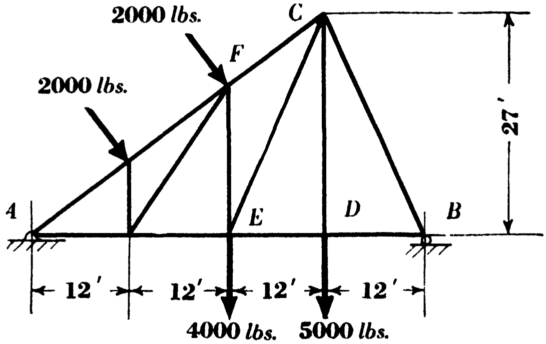
Note: In Problems 92–101, inclusive, the trusses are simple trusses—each member extends to the pin joints at its ends only, and all loads are applied directly to the pin joints. The truss members are shown as solid lines—the supporting members are outlined.

92. The truss is supported on a pin joint at A, and held in position by member BC, to which it is pinned. Determine the stresses in members CD, CE, and EF.



PROB. 92.

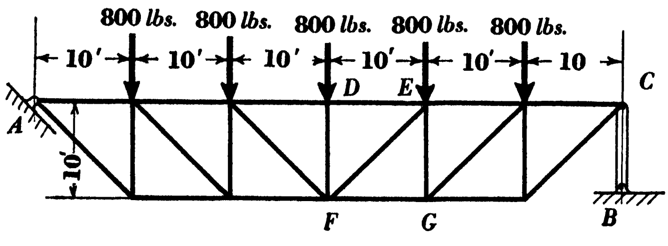
93. The truss is supported on a pin joint at A, and rests on a frictionless roller at B. Determine the stresses in members CD, CE, DE, and CF. The 2000-lb. loads are perpendicular to the line AC.



PROB. 93.

94. The truss is loaded with five vertical loads of 800 lb. each. It is supported by a pin joint at A, and pinned to the vertical member BC at C. Determine the stresses in members ED, EF, and FD.

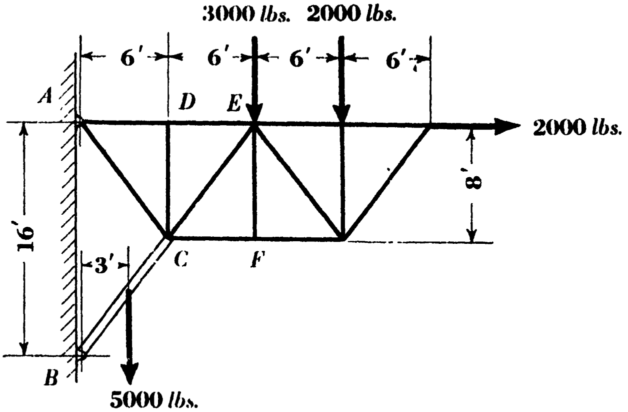
Ans. ED = 3600 lb., compression; EF = 566 lb., tension; FD = 800 lb., compression.



PROB. 94.

95. The truss is supported on a pin joint at A, and held in position by the brace BC, to which it is pinned. Determine the stresses in members CE, DE, EF, and CF.

Ans. $CE = 6250$ lb., compression; $DE = 7250$ lb., tension; $EF = 0$; $CF = 1500$ lb., compression.

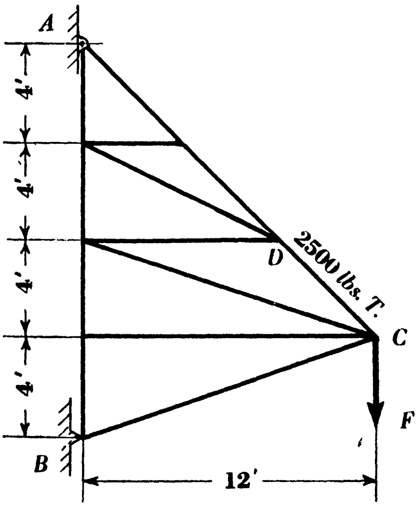


PROB. 95.

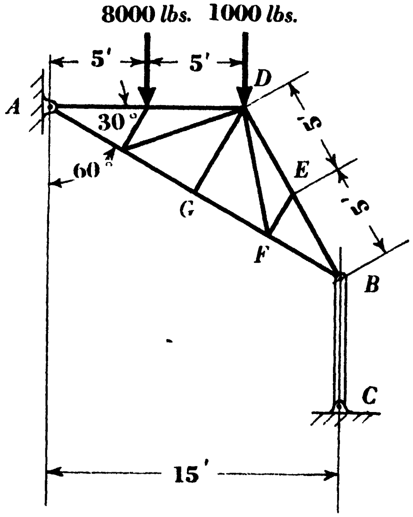
96. If, in the truss of Problem 95, the member CE must not be stressed to more than 5000 lb., compression, determine the magnitude to which the 3000-lb. vertical load at E must be reduced. (All other loads remain unchanged.)

97. If, in the truss of Problem 95, the stresses in DE and CF must not exceed 6000 and 1300 lb., respectively, determine the amount to which the 3000-lb. vertical load at E must be reduced, if no other load is changed.

98. The truss is pinned at A, and rests against a knife-edge at B. If the stress in member CD is 2500 lb., tension, determine the load F and the horizontal and vertical components of the supporting forces at A and B.



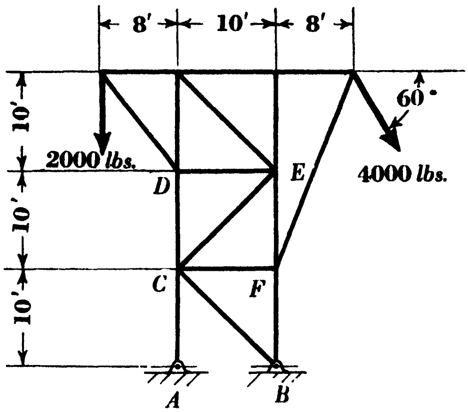
PROB. 98.



PROB. 99

99. The truss is supported by a pin joint at A, and a vertical member BC at B Determine the stresses in members, DE, DF, and FG.

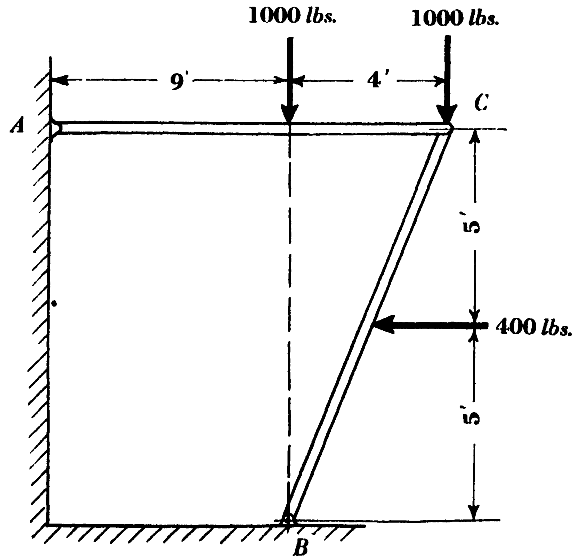
100. The tower truss is supported on pin joints at A and B. Determine the stresses in members CF, CE, and CD.



Prob. 100

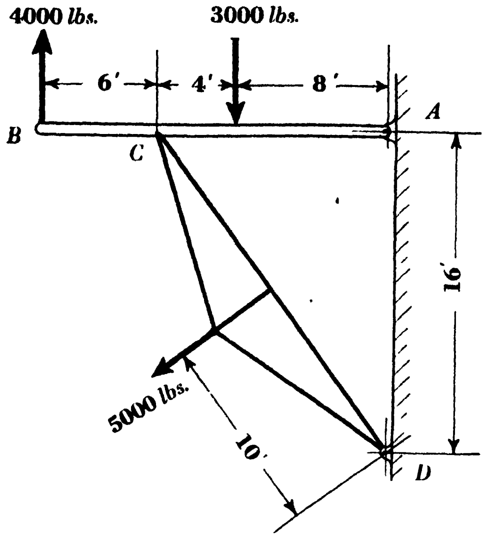
101. The members AC and BC are pinned together at C and to the supports at A and B. Determine the horizontal and vertical components of the supporting forces acting at A and B.

Ans. H_A = 477 lb.; V_A = 308 lb.; H_B = 877 lb.; V_B = 1692 lb.



Prob. 101

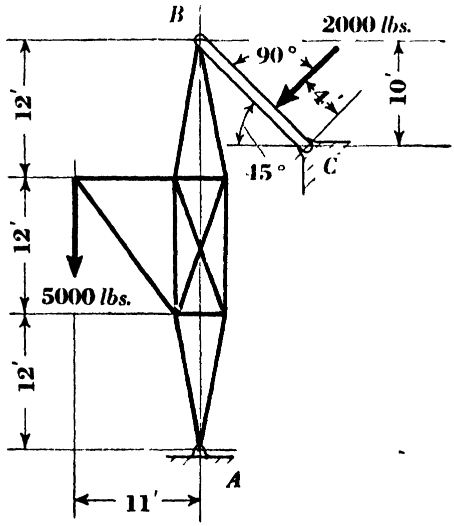
102. The horizontal member AB is pinned to the wall at A, and to a truss CD at C. The truss is pinned to the wall at D. Determine the horizontal and vertical components of the forces acting at A, C, and D. The 5000-lb. load is perpendicular to the line CD.



PROB. 102

103. The vertical truss AB is pinned to the ground at A, and to member BC at B. BC is pinned to an abutment at C. Determine the horizontal and vertical components of the supporting forces at A and C.

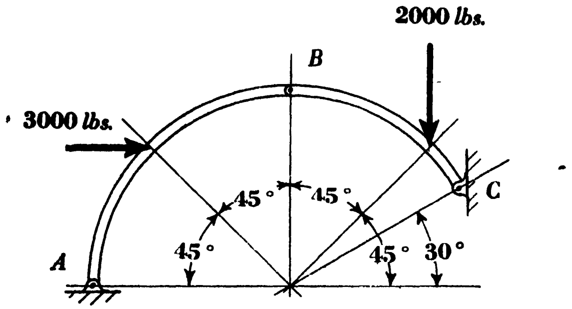
Ans. H_A = 1528 lb.; V_A = 7328 lb.; H_C = 2942 lb.; V_C = 914 lb.



PROB. 103

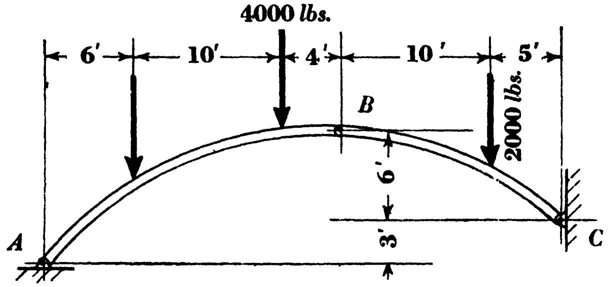
104. The three-hinged arch is a circular arc of 10-ft. radius. Members AB and BC are pinned together, and pinned to abutments at A and C. Determine the horizontal and vertical components of the supporting forces at A and C.

Ans. H_A = 1423 lb.; V_A = 544 lb.; H_C = 1577 lb.; V_C = 2544 lb.



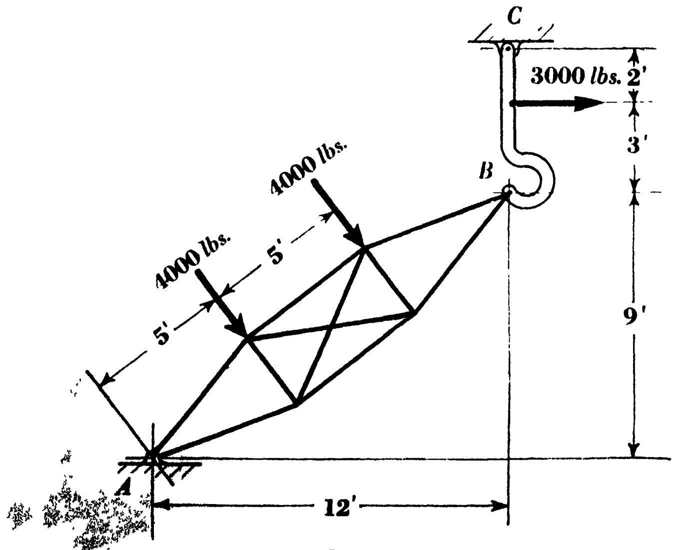
PROB. 104

105. The three-hinged arch consists of members AB and BC pinned to abutments at A and C, and to each other at B. Determine the horizontal and vertical components of the supporting forces at A and C.



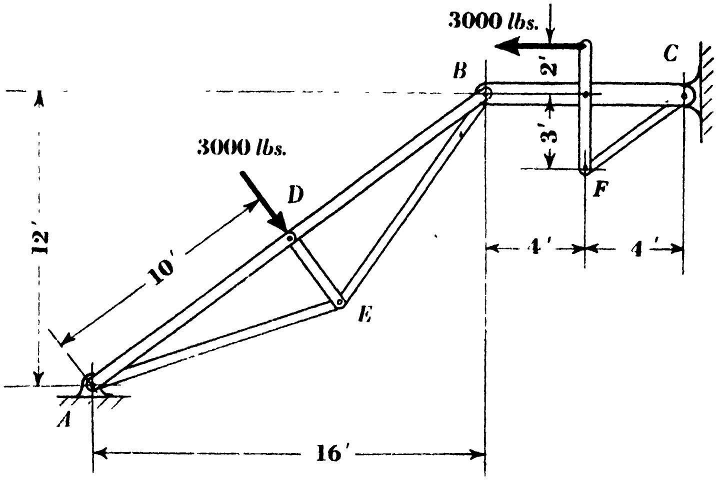
PROB. 105

106. The truss AB is pinned to a hook at B. The hook is pinned to a ceiling at C. Determine the horizontal and vertical components of the forces acting at A, B, and C. The 4000-lb. loads are perpendicular to the line AB.



PROB. 106

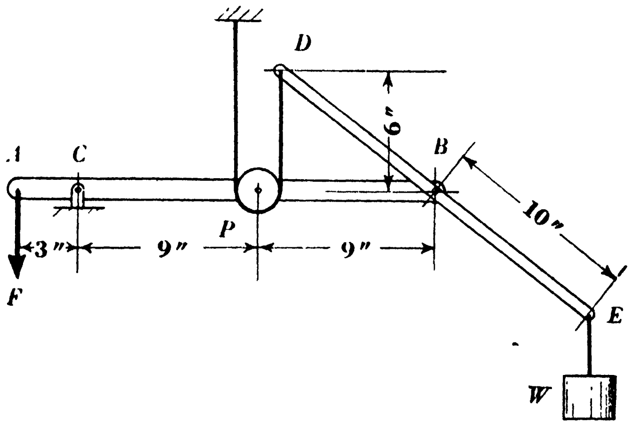
107. Determine the horizontal and vertical components of the supporting forces at A and C.



PROB. 107

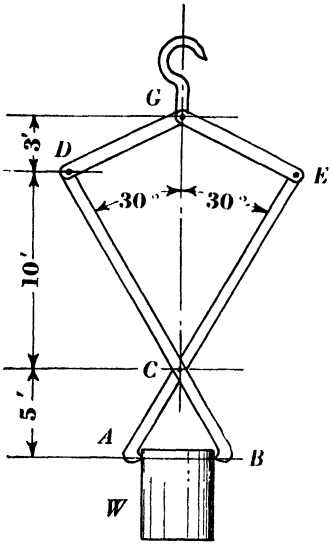
108. Find the stresses in members DE and CF of the structure shown in Problem 107.

109. The rod AB pivots at pin joint C, and carries pulley P and the bar DE. Determine the vertical force F which must be applied at A to hold the structure in equilibrium. The diameter of the pulley is 2 in. The cylinder W weighs 150 lb. *Ans.* $F = 900$ lb.



PROB. 109

110. The crane tongs carry a load W of 3200 lb. Determine the horizontal and vertical components of the forces acting at A, C, and E, and the stresses in DG and EG.



Prob. 110

CHAPTER III

Graphical Statics

22. Graphical Solutions. Graphical methods of solution of many forms of engineering problems have two outstanding features which recommend them to our attention. First, they frequently serve to present data in such form that the laws or facts or trends available from such data may be more readily comprehended than when any other method of data presentation is employed. Plotting a curve of the velocity of a moving body versus time or the cost of a process over a period of years, for example, presents a much more vivid picture of both performance and the nature of trends or changes, than we can read from columns of figures presenting equivalent data.

Second, graphical methods are frequently capable of reducing the amount of laborious computation of relationships which are complex and for which analytical equations necessitate many successive stages

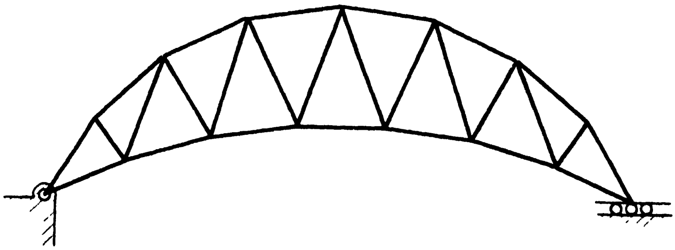


FIG. 44.

of substitution of known values before the unknowns may be determined. For example, in the case of the crescent-shaped truss shown in Fig. 44, the analytical determination of the stresses in the members will require an extensive amount of tedious calculation in establishing the geometric pattern of the members, before we can isolate free bodies and proceed with explorations of the accompanying force systems.

In such cases, graphical solutions are more efficient. They express the same basic equations as do their analytical analogies, but because any distances which are awkward to calculate may be scaled directly from the drawing, they reduce greatly the total amount of actual work necessary to achieve equivalent results.

Graphical solutions of the problems of statics possess advantages only when the force systems which they analyze lie in a single plane.

Although graphical solutions of statics problems in space could be used, the need for exploring many projections through dihedral angles generally precludes any advantage over analytical solutions. We shall, in the following, apply graphical methods only to the cases of single-plane force systems.

23. The Triangle of Forces. The parallelogram law has served as the cornerstone in the building of principles in our analytical investigation of force systems.

In graphical solutions of statics problems, it is the triangle, formed by the diagonal and two adjacent sides, which now assumes the function of the parallelogram.

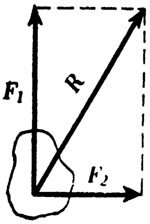


FIG. 45.

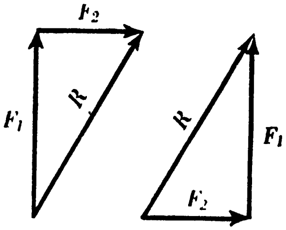


FIG. 46.

We know that if forces F_1 and F_2 of Fig. 45 are drawn to scale as sides of a parallelogram, the diagonal will represent, at the same scale, the resultant R of F_1 and F_2.

We can note that we need only draw forces F_1 and F_2, as in Fig. 46, with the origin of F_2 placed at the terminus of F_1, and the remaining side of triangle *abc* will be the resultant, because this triangle is simply one half of the parallelogram of Fig. 45, and it serves as well as the latter in establishing the magnitude and direction of the resultant.

The order in which F_1 and F_2 are drawn in establishing the triangle does not affect the resultant. The resultant is always the vector which originates at the origin of the first-drawn component and terminates at the terminus of the component drawn last.

The freedom in choice of the order in which component vectors are drawn in determining resultants is called the *commutative* law in vector addition. It should be noted that the sense of the resultant is in the direction of travel from the origin of the first vector to the terminus of the last vector.

If the resultant of more than two forces is to be found, the graphical method consists, in effect, of building successive triangles of forces.

Given the system of concurrent forces shown in Fig. 47*a*, we may first draw the triangle representing F_1 and F_2 (Fig. 47*b*) and determine R_1, the resultant of F_1 and F_2. If now we draw a triangle (Fig. 47*c*) with R_1 and F_3 as sides, we determine R, the resultant of R_1 and F_3, which is, therefore, the resultant of $F_1 \nrightarrow F_2 \nrightarrow F_3$. These triangles need not be drawn separately, but may be combined as has been done in Fig. 47*d*. Nor need we pause to establish the magnitude of a subordinate resultant,

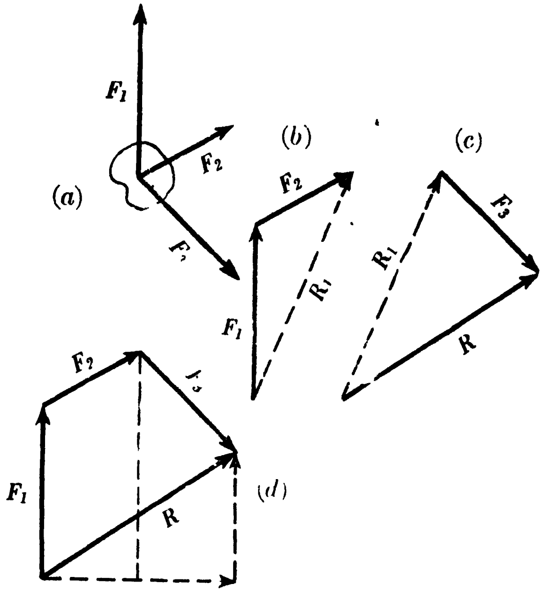


FIG. 47.

such as R_1. We need only draw vectors representing each force, with the origin of each placed at the terminus of the preceding. The final resultant of the system is the vector R drawn, as in the triangle of forces, from the origin of the first vector to the terminus of the last.

We now have a polygon which is called the *force polygon*.

It should be noted that addition of forces by the force polygon is a graphical translation of the analytical equations for the resultant of a concurrent force system,

$$R = \sqrt{\Sigma X^2 + \Sigma Y^2} \text{ (Fig. 47}d)$$

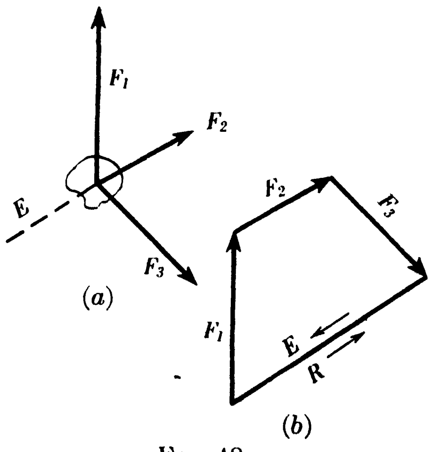


FIG. 48.

If a force system such as that shown in Fig. 48*a* acts upon a free body which is to be in equilibrium, a force E, which is equal, opposite, and collinear with the resultant, must be added to the system. This fact is recognized graphically in the force polygon Fig. 48*b*, if we reverse the sense of the final side, or resultant. The vector which is the equilibrant now appears as E in Fig. 48*b*, where it is the side that "closes" the polygon. E completes a circuit from the origin of the first vector through the mating termini and origins of the component vectors, to final return of terminus of the last vector or equilibrant at the origin of first vector.

Then, a graphical "condition of equilibrium" has been found which we may express as follows: *If a system of concurrent forces is to produce equilibrium, the force polygon must close.*

A system of notation, called *Bow's notation,* offers convenience in relating forces without the necessity of indicating their senses by arrowheads, and has merit in relating external forces, stresses, and the component members of such structures as trusses, where graphical solutions are most commonly applied.

This system of notation consists of lettering the spaces between the forces of a system with the capital letters, beginning with A. This

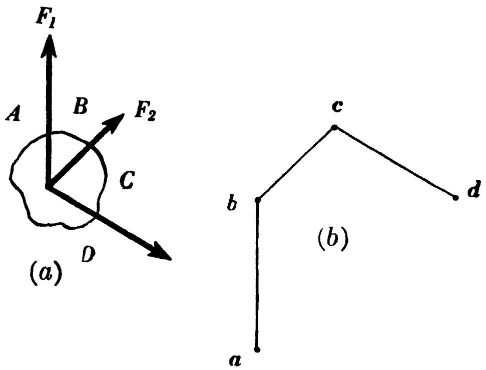


FIG. 49.

has been done in Fig. 49. The diagram which, like Fig. 49*a*, shows these forces as they act on a free body, is called the *space diagram.* We now adopt a conventional system in naming the forces, always proceeding to read clockwise around the system of forces. For example, F_1 is called the force AB, F_2 is BC, and F_3 is CD.

In drawing the force polygon, the points at the ends of the vectors representing forces AB, BC, and CD are given the corresponding small, or lower-case, letters. By placing these lower-case letters at the proper end, so that we travel along the vector in the direction from a-b (or the order in which the force was originally read in clockwise rotation in Fig. 49*a*), we announce the sense of force AB. As another illustration of the relationship of letters as they appear in the space diagram and force polygon, let us observe that force F_2 has sense upward and to the right in the space diagram. In Bow's notation, this is force BC (reading clockwise). Then, in the force polygon, we place b at the lower end of the vector and c at its upper end. In interpreting the sense of the vector, we note that to travel from b to c on the force polygon is to travel upward to the right. Force CD appears as cd in the force polygon, its sense being downward and to the right.

Problems

Before attacking the problems of this group, the first in which graphical methods are employed, a clear understanding should be established of the techniques of precise drafting.

When graphical analysis is to be used in solving a problem, the results must be obtained from a drawing. Such results are trustworthy only if the drawings are executed with a high degree of accuracy.

Engineering students are taught the principles and techniques of drafting early in their career, and we shall add but a few suggestions on precision drafting before proceeding with the applications.

First, the equipment of T squares and triangles should be tested for their accuracy before proceeding with graphical solutions. Tests for these instruments will be found in any of the textbooks or manuals of Engineering Drawing.

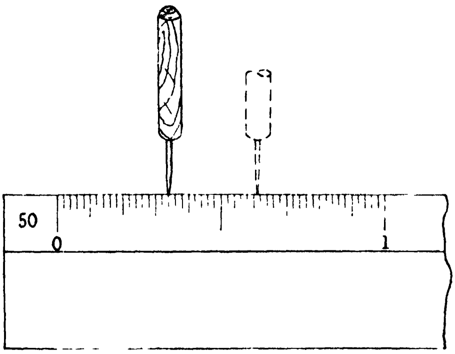


Fig. 50.

Next, the needle point should be used in marking off all measurements to insure greater precision than is possible with a pencil point. Drafting needle points, consisting of a needle mounted in a wooden handle, are available at all drafting-supply houses. The use of the needle point is illustrated in Fig. 50. It will be noted that measurements to the nearest one one hundredth of an inch may be made without difficulty with the use of the scale, commonly available, which is divided into fiftieths of an inch; and the needle point to split these divisions into halves, thereby yielding hundredths of an inch. In graphical solutions, the decimal system of inches with component parts reported as decimal fractions is standard practice. In using the needle point, it is advisable to close one eye when marking a distance to avoid parallax.

No measurement should be made "in the air"—that is, by pricking holes at the ends of a measured distance and then connecting these extremities with a line. Instead, measurements must always be made by first drawing a faint layout line, upon which the distance is measured. The layout line is made longer than the desired distance, the surplus line erased after measuring, and a sharp finish line applied exactly over the layout line. Since the needle point leaves a tiny hole, the erasure of extra layout line leaves the extreme points intact to limit the finish stroke. When these operations have been completed, the scale should be applied to the line, and the final length of line again measured as a check. The practise of measuring distances by setting the dividers on a wooden scale and then transferring the distance with the dividers to the paper, should be condemned: there is involved a probability of inaccuracy in transfer which the needle-point system completely avoids and, most important, the psychology of respect for the measured distance is disturbed.

In setting the compasses, the radius should be laid out on a drawn layout line, and the compasses set on the measured line.

Precision drafting is a matter of correct techniques, not inborn skill, and from the start, one should not be content with any devices but those which will insure the highest degree of accuracy. The use of such phrases as "the highest degree of accuracy" and the pungent word "precision" need scare no one, since no native skill or special talents are required. Mechanical routines readily mastered are the basis of operations, and when treated with the respect they merit, insure precision.

All line work, whether layout or finish, must be extremely sharp, or precision is sacrificed. In this type of drafting, the definition of Euclid, "a line has length but no breadth," is penetrating.

Hard, sharp drafting pencils must be used. The usual system of instructing user that a 4H pencil should be used for this type of line and the 2H for that type, is open to criticism. No two individuals have identical degrees of hand pressure on a drafting pencil, and the resulting lines yielded by the same pencil in the hands of different individuals may vary widely. The line itself should be used as guide, and each individual can determine the proper hardness of pencil that he must use to reach an objective of proper lines. Layout lines should be so faint that they can just be seen for measuring purposes; finish lines should be darker, but not broader. In both cases, the lines must be as sharp as constant use of sandpaper block or file can produce.

This discourse upon the character of the pencil point may seem unnecessarily insistent, but precision drafting rests upon the sharp line. The attitude of the user of sharp lines fosters a wholesome respect for the value of graphical solutions.

A further aid to extreme precision is the magnifying reading glass, mounted upon a small stand to leave the hands free, and used over the scale and needle point when measuring. This device is not always essential, but guarantees, at little additional effort, even higher degrees of accuracy.

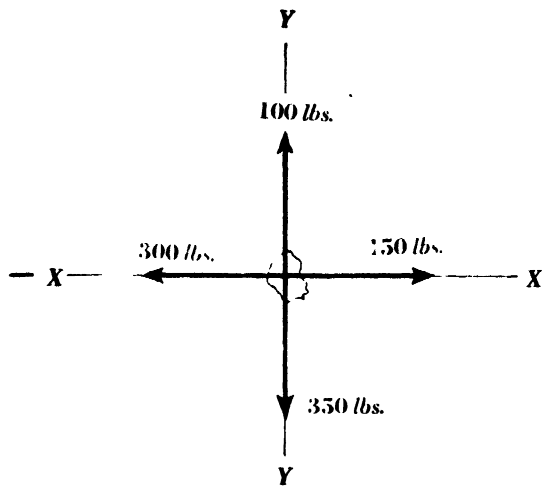
With proper equipment, techniques, and attitude, the accuracy of a graphical solution increases in direct proportion to the scale, or size, of a drawing, all other factors (such as the human one) being equal. It will be noted that if a line one inch long is to be measured to the nearest $\frac{1}{100}$ of an inch, a possible error of $\frac{1}{100}$ of the measured length, or one per cent, is being tolerated. If we measure a ten-inch line, which can still be measured to the nearest $\frac{1}{100}$ of an inch, then the possible error which is tolerated becomes $\frac{1}{1000}$ of the measured length, or 0.1 per cent. It follows that as large a scale as is convenient should be used in all graphical solutions.

The qualifying phrase, "factors like the human one," should be noted. Increasing the scale of the drawing presents no inviolable guarantee of accuracy. The proper construction of parallels and perpendiculars, the obtaining of sharp intersections between lines, are equally influential in determining the degree of success, but in all cases, since the drawing is mechanical or instrumental in its routine, a mastery of correct drafting technique will insure proper results.

In measuring angles, the use of the usual small, stamped protractors should be avoided, because inaccurate measurement of angles may nullify the care taken with all of the linear measurements. Such instruments are too crude for accurate measurement of angles.

The table of chords should be used, instead, and such a table with instructions for its use is given in the appendix.

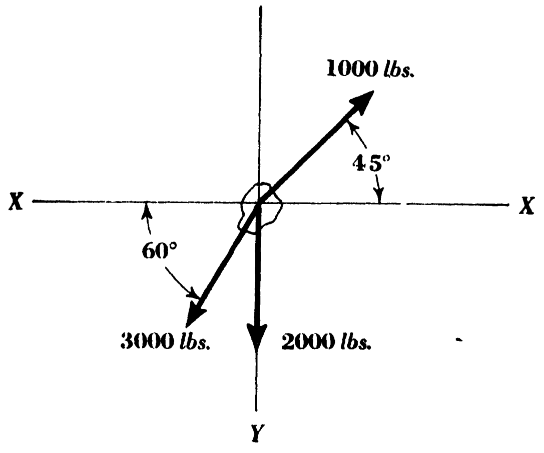
111. Determine the resultant of the system of four concurrent forces.



PROB. 111

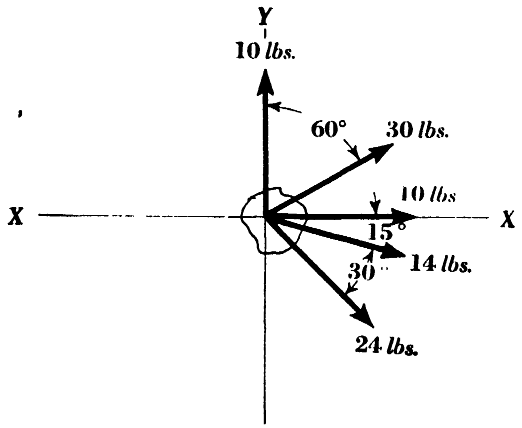
112. Determine the resultant of the system of three concurrent forces.

Ans. R = 3970 lb.; θ_X = 78.5°.



PROB. 112

113. Determine the resultant of the system of five concurrent forces.



PROB. 113

114. Solve Problem 1 graphically.
115. Solve Problem 3 graphically.
116. Solve Problem 9 graphically.
117. Solve Problem 14 graphically.
118. Solve Problem 16 graphically.

24. Trusses. The truss shown in Fig. 51 is composed of individual members, pinned together at the joints of the truss, and all loads exerted on the truss are applied at the pin joints. Every member is a two-

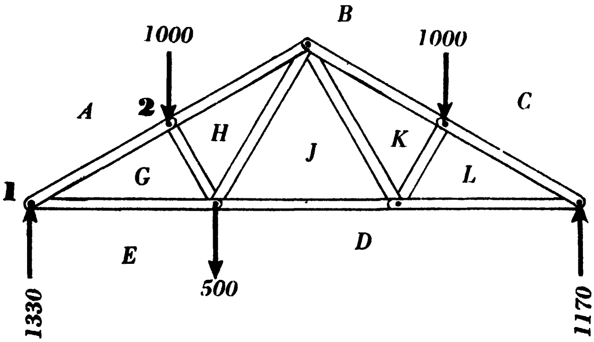


FIG. 51.

force member and exerts a force on the pins at its ends in the direction of the member itself. (See Article 20.)

Then, the force system at each pin joint is a concurrent force system, with the inclinations of all forces known. For example, if we isolate pin joint 1, as a free body, the external forces that act upon that body are those shown in Fig. 52.

In the graphical solution, we do not need to make individual drawings of each of the free bodies selected, but rather we draw the entire truss to any convenient scale, as was done in Fig. 51. The forces external to the truss are shown, and all spaces between external forces as well as the spaces bounded by the members of the truss are lettered as shown. The external forces need not be drawn to scale on the space diagram, for this diagram is used only in the solution to orient the members of the truss in their correct geometric inclinations and, therefore, only the actual dimensions of the truss play a part in the solution.

We seek to determine the magnitudes and natures of the stresses in all of the truss members.

The force polygon is started, as in Fig. 52, by drawing, to any convenient scale of forces, the known external force acting at joint 1. It should be noted that we have elected to make our first investigation at a joint, serving as a free body, upon which only *two* unknown forces are acting. A graphical solution does not bring into action new conditions

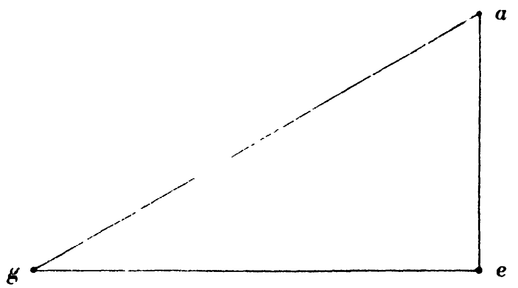


FIG. 52.

of equilibrium: it only translates our method of attack into the graphical language. If, then, we had elected to start our solution by considering joint 2 as a free body, we would have encountered three unknown forces. Since these groups of forces are concurrent systems in a plane, only two unknowns may be determined by the use of any method of attack—either graphical or analytical.

The forces acting on the pin at joint 1 are, reading clockwise about the joint, *EA*, *AG*, and *GE*.

Then *ea* of Fig. 52 is drawn to scale. The letter *e* is placed at its lower end and *a* at the upper end, because the sense of this force is known to be upward.

We now draw a line from point *a*, parallel to member *AG*. Point *g* must lie on this line, but its exact location is as yet unknown. We next draw a line through *e*, parallel to member *GE*. Point *g* must also lie on this line. Then point *g* is the intersection of the two lines, and has been so labelled. It is now possible to determine the stresses in the two members under consideration. We note that member *AG* of the space diagram is read as force *ag* on the force polygon. Then in so reading, we have proceeded, in going from *a* to *g*, downward and to the left. The force, therefore, has sense downward and to the left as it acts on the free-body pin joint 1. Then the member *AG* is in compression. Its magnitude is the length of line *ag*, and the stress has been reported on the space diagram, which is repeated as Fig. 54. The force *ge* reveals its

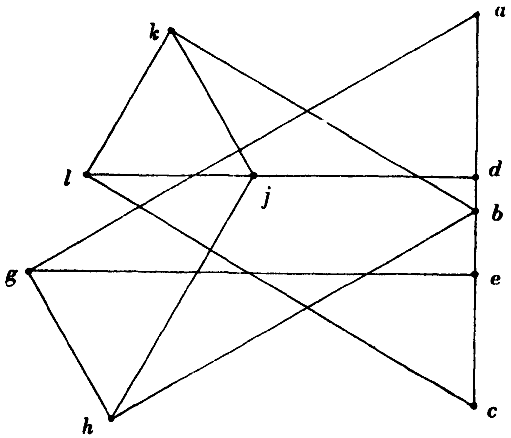


FIG. 53.

sense as to the right. Then the member is exerting a force to the right as it acts on the free body, and the nature of stress in the member itself in tension.

Individual force triangles, such as *eag* of Fig. 52, could be drawn to represent the solution of the force system at each joint. It is more

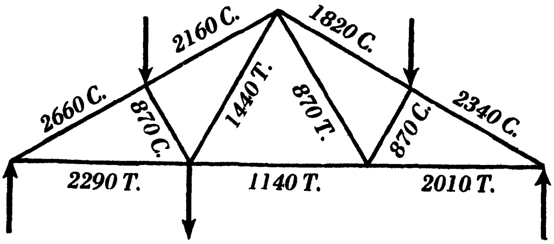


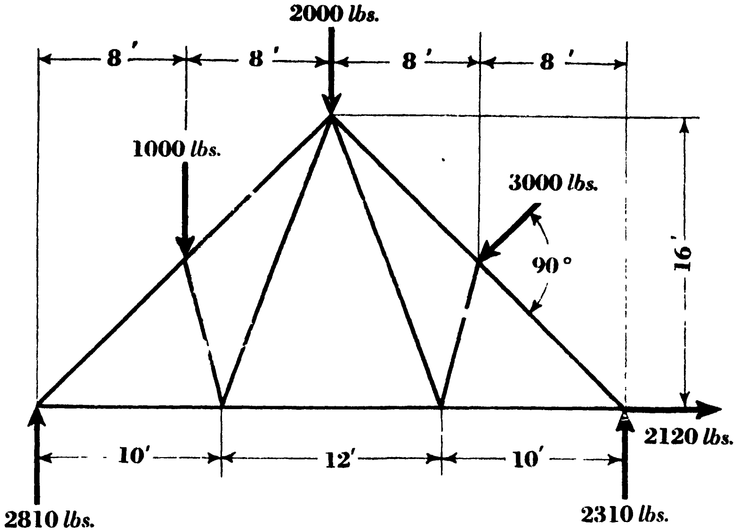
FIG. 54.

efficient, as well as more accurate, to make each force triangle representing a force system at a joint adjacent to the force triangle of the preceding joint, and a composite force polygon like that of Fig. 53 is the result.

Fig. 54 shows a proper report of the results of the complete investigation.

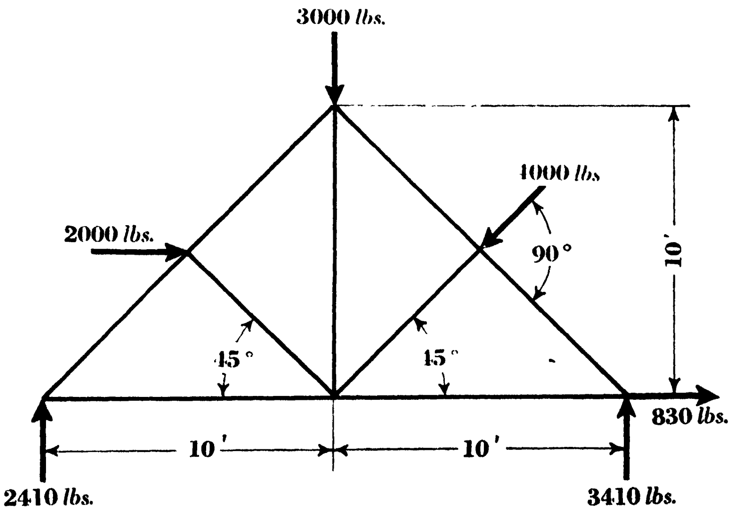
Problems

119. For the simple truss, loaded as shown, determine graphically the magnitude and nature of stress in each member.



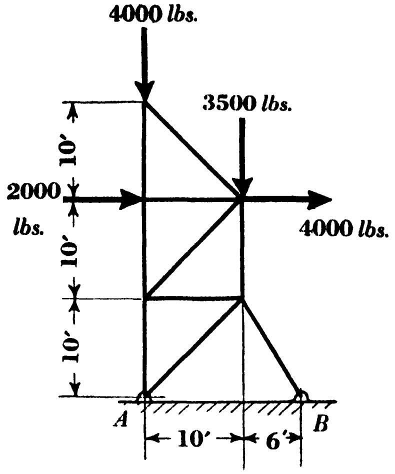
Prob 119

120. For the simple truss, loaded as shown, determine graphically the magnitude and nature of the stress in each member.



Prob. 120

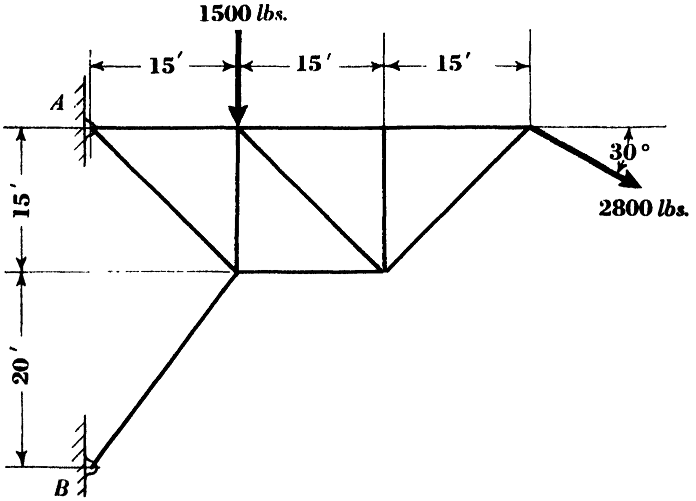
121. Determine the stresses in all members of the simple truss graphically.



PROB. 121

122. Determine the horizontal and vertical components of the supporting forces at A and B for the simple truss given in Problem 121. Check by determining these forces analytically.

123. Determine the stresses in all members of the simple truss graphically.



PROB. 123

124. Determine the horizontal and vertical components of the supporting forces at A and B for the simple truss given in Problem 123. Check by determining these forces analytically.

125. For the simple truss given in Problem 100, determine the stresses in all members graphically.

126. Determine, graphically, the stresses in all members of the simple truss of Problem 95.

127. Determine, graphically, the horizontal and vertical components of the supporting forces at A and C for the simple truss given in Problem 95.

25. Funicular Polygon. Another type of force polygon enables us to analyze graphically systems of forces that do not concur at a single point, but which lie anywhere in a plane.

If we pause to consider the triangle of forces shown in Fig. 55, we shall have a basis of method. Given any force such as F, we know that F is the resultant of an infinite number of possible combinations of component forces. F may, then, be resolved into any pair of components such as F_1 and F_2, the only limitation being that their sum must be the resultant force F. F_1 and F_2 are one such pair; F_3 and F_4, another.

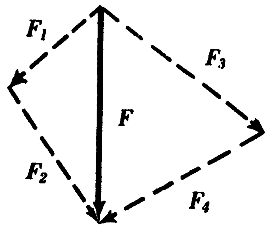


FIG. 55.

The force system of Fig. 56 contains three forces which may be given Bow's notation by sweeping clockwise around the system, lettering the spaces bounded by the forces. These spaces have been lettered A, B, C, and D.

A force polygon is next drawn, as in Fig. 57. The resultant of the system is *ad*.

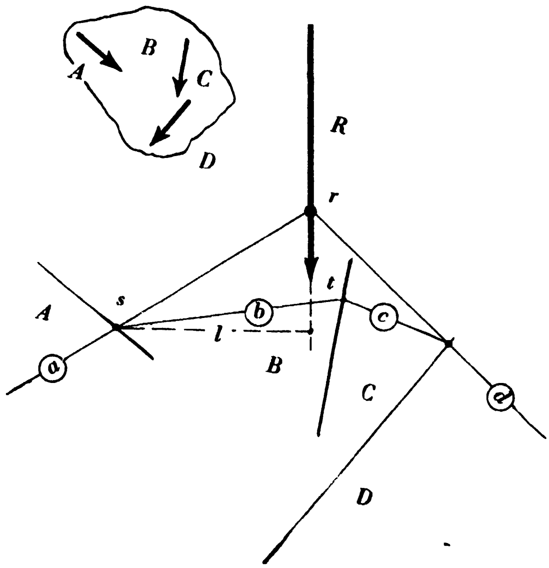


FIG. 56.

We can now expand the properties of our force polygon, so that it will determine the *location* of the resultant, serving in graphical solutions as moment equations have done in the analogous analytical analyses.

Any point P may be selected on the force polygon of Fig. 57, and lines drawn from P to each of the labeled points a, b, c, and d. The force ab, it will be noted, is a resultant of two component forces *aP* and *Pb*. The latter form an equivalent system and may, therefore, be substituted for *ab* in any application.

We now return to the space diagram of Fig. 56. There we draw a line (a) parallel to component force *aP* of the force polygon from any point *s*, on the line of action of force *AB*. A line (b) parallel to component *Pb* of the force polygon is also drawn through point *s*. *AB* is being resolved, on the space diagram, into two components, having the same inclinations as *aP* and *Pb* of the force polygon. The magnitudes of these components could be read from the force polygon, but we have, at present, interest only in their inclinations.

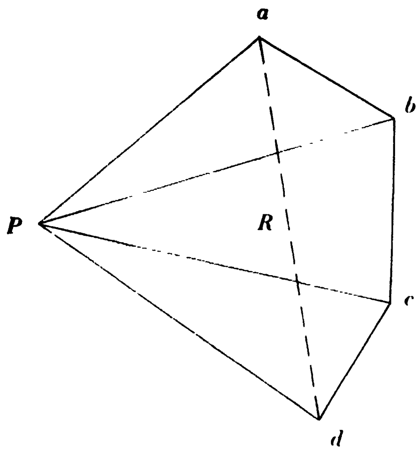


FIG. 57.

Now we proceed on the space diagram to the intersection *t* of (b) with force *BC*. Here, we repeat the process of resolving a force into components of known inclination. If we pause to observe the force polygon, we note that *bc* has as components *bP* and *Pc*. Then, at point *t* of the space diagram, we draw lines parallel to *bP* and *Pc*. One of these lines is already present as (b), and the other appears as (c).

Now, (b) is a line representing *Pb*, a component of *ab* of the force polygon. (b) also represents *bP*, a component of force *bc*. The only difference between these two components, both represented by distance *b-P*, is that they have opposite sense. Then, they must balance each other when they both appear as (b) on the space diagram.

If we repeat this process of replacing forces by components at *u*, the intersection of (c) with force *CD* on the space diagram, we again have a balancing pair of components represented by (c) and are left with an unbalanced component (d). The entire original system of forces has been replaced by an equivalent system, containing (b) and (c) as balanced pairs of components, and (a) and (d) as leftover, unbalanced components.

If now (a) and (d) are produced to their intersection at *r*, the resultant of the force system must pass through *r*, because only at the intersection of two such components can we combine them to obtain this resultant. The distance *l* is the moment arm of *R* about an axis at *s*. The moment arm relative to any axis may be measured on the space diagram.

We now have fully determined the resultant—its magnitude, inclination, and sense are established by the force polygon; its location has been determined by the series of resolutions into components on the space diagram.

It will be convenient to adopt terminology for the new lines of construction which have been added to the graphical solution. Points, such

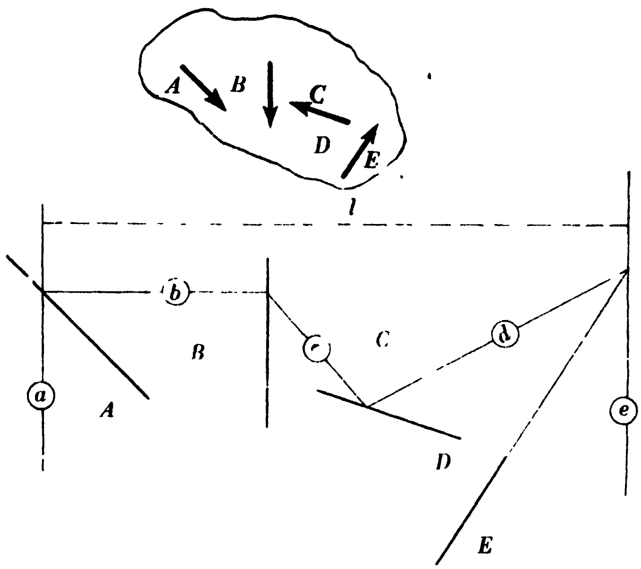


FIG. 58.

as P of the force polygon, are called *poles;* and lines such as Pa and Pb which radiate from the pole, are called *rays*.

The components on the space diagram, such as (a) and (b), are called *strings*, and the polygon formed by the strings is called the *string*, or *funicular*, polygon.

It is possible that a given system of forces may reduce to a resultant that is a couple.

To determine the resultant of the system of forces shown on the space diagram of Fig. 58, we proceed as in the preceding case. When the force polygon of Fig. 59 is drawn, we find that points e and a coincide. Then this system yields no resultant force.

We next select pole P, draw the rays, and proceed with the funicular polygon as in Fig. 58.

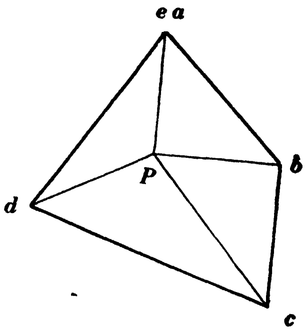


FIG. 59.

It will be noted that in this case, the leftover, or unbalanced, strings (a) and (e) are parallel and will, therefore, never intersect. We have reduced the original system to two parallel forces.

The string (a) of the space diagram represents component aP of the force polygon, and the string (e) represents component Pe of the force polygon. These components, aP and Pe, are the same line of the force polygon and have, therefore, equal magnitude. aP, a component of ab, has sense downward, and Pe, a component of de, has sense upward.

Then the resultant is a counterclockwise couple. Its magnitude is determined as the product of aP or Pe, measured to force scale on the force polygon, times the moment arm l measured to distance scale on the space diagram.

When the problem involves a system of forces in equilibrium, we should find that the force polygon must close, as it has done in the preceding case.

In addition, the funicular polygon must close, because otherwise a couple resultant would appear. The final strings of the funicular polygon are *always* parallel when the force polygon closes. For equilibrium, the final strings must also be collinear, because two forces can produce equilibrium only when they have the same magnitude, same inclination, opposite sense, and the same line of action.

Then, for *forces lying anywhere in a plane to produce equilibrium, both the force and funicular polygons must close.*

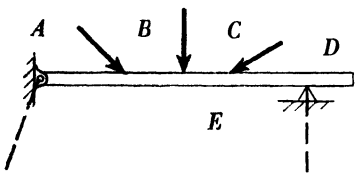


FIG. 60.

Let us apply the graphical method of analysis to the system of forces shown in Fig. 60. We are to determine the magnitude, inclination, and sense of EA, and the magnitude and sense of DE (the supporting forces of the beam).

The force polygon is started in Fig. 62, and it contains the known sides ab, bc, and cd. This polygon must eventually be closed for the system of forces is in equilibrium, but we do not at present have sufficient information available to complete it.

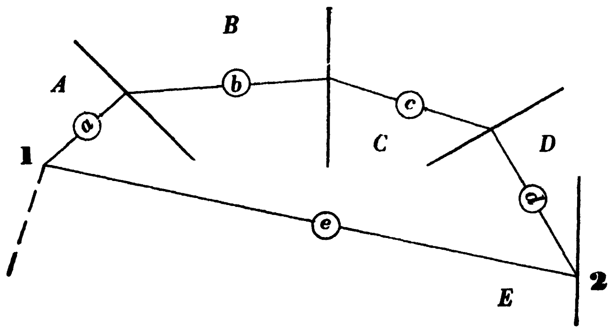


FIG. 61.

We therefore select any pole P, and draw rays to points a, b, c, and d.

Now the funicular polygon is started (Fig. 61). The (a) string must intersect the line of action of force EA. Only one point is at present known which lies on the line of action of EA. This is point 1 at the pin joint. Then the ⓐ string is drawn on the space diagram through point 1, and the ⓑ and ⓒ and ⓓ strings added to the diagram.

The ⓓ string intersects force DE at point 2. There is only one remaining string (ⓔ), and it must close the funicular polygon for equilibrium to ensue. Then a line drawn from point 2 to point 1 is the ⓔ string.

We can now return to the force polygon, our attack upon its solution strengthened by additional information. We draw a ray from pole P, parallel to the ⓔ string.

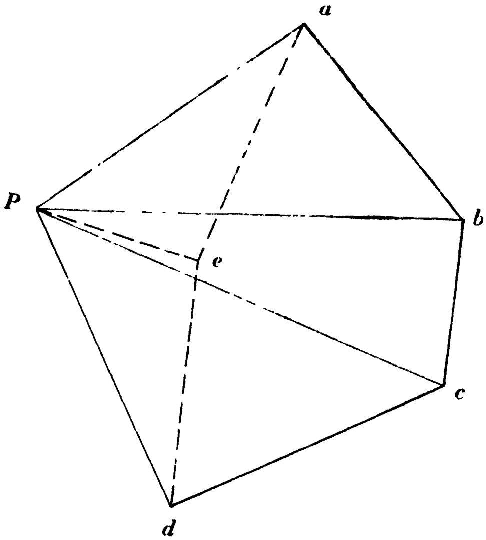


FIG. 62.

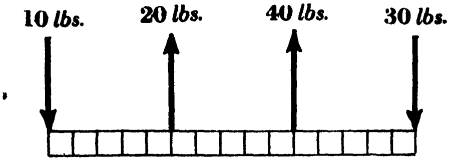
Point e of the force polygon must lie somewhere on this line, because rays are always parallel to the strings of similar letter.

We also know that point e must lie on a line of the force polygon through point d and must be parallel to force DE of the space diagram, because only such a line could properly represent force DE on the force polygon.

Then point e must lie at the intersection of those two lines. Now point e is located, and we measure de to obtain the magnitude of DE and observe that its sense, in the usual Bow's notation, is upward. The force at the pin joint is EA, which has now appeared as ca of the force polygon. It has sense upward and to the right. The magnitudes of these forces may be read from the force polygon.

PROBLEMS

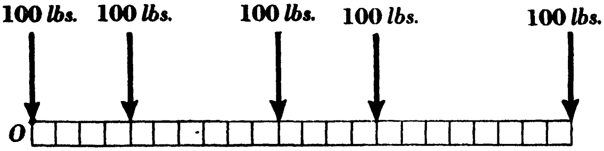
128. Determine graphically the resultant of the system of parallel forces shown. *Ans.* $R = 20$ lb.; $X = 2.5$ ft. from left end.



PROB. 128

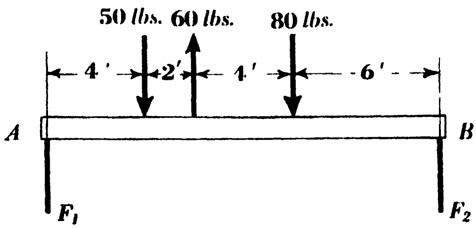
129. Determine graphically the resultant of the system of parallel forces given in Problem 128, if the 30-lb. force is increased to 50 lb.

130. Determine graphically the resultant of the system of parallel forces shown, and the moment of the system about point O.



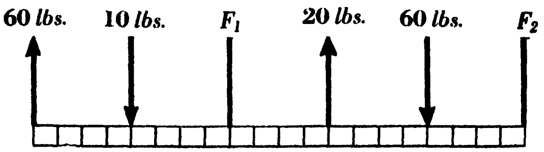
PROB. 130

131. Determine graphically F_1 and F_2 in the system of parallel forces shown, if the beam AB is in equilibrium.



PROB. 131

132. Determine graphically the forces F_1 and F_2. The parallel system of forces is in equilibrium. *Ans.* $F_1 = -80$ lb.; $F_2 = +70$ lb.



PROB. 132

133. Determine graphically the forces F_1 and F_2 of Problem 132, if the 10-lb. downward force is increased to 20 lb.

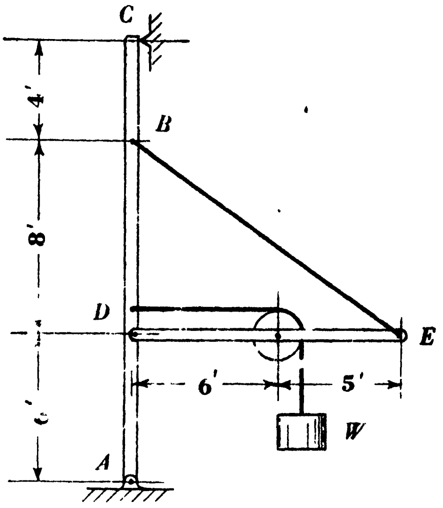
134. Solve Problem 56 graphically.

135. Solve Problem 60 graphically.

136. Solve Problem 64 graphically.

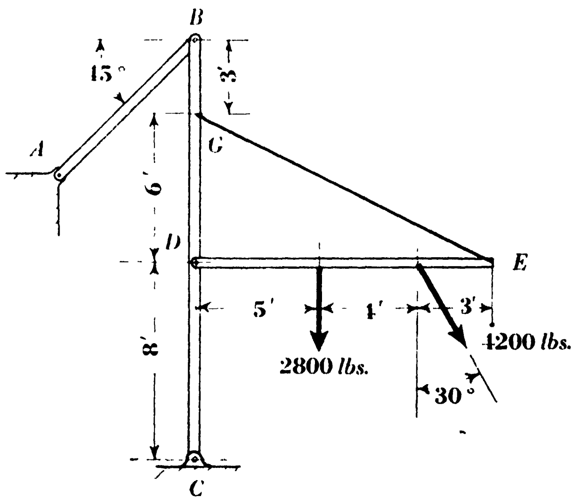
137. Solve Problem 67 graphically, and determine the stresses in all members of the simple truss.

138. Determine the stress in member *BE* graphically. $W = 1200$ lb. Pulley diameter = 2 ft. *Ans.* $BE = 1110$ tension.



PROB. 138

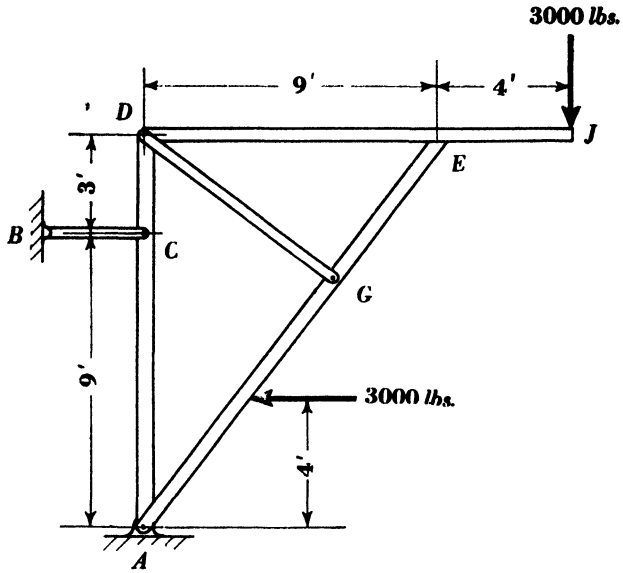
139. Determine graphically the horizontal and vertical components of the supporting forces at *A* and *C*.



PROB. 139

140. In the frame of Problem 139, determine graphically the stress in member *EG*.

141. Determine graphically the horizontal and vertical components of the supporting forces at *A* and *B*.



PROB. 141

142. In the frame of Problem 141, member DJ rests on member AE at E—the surface is frictionless. Angle $DGE = 90°$. Determine graphically the resultant force acting at E, and the stress in DG.

CHAPTER IV

The Three-Dimensional Force System

26. The Concurrent Force System. The basic principles which have been discussed and applied to forces in a plane may be used to analyze those systems of forces which are not confined to a single plane but may lie anywhere in space. Although some of those axioms and techniques may require amplification, we are not challenged to develop new methods of reasoning. The conditions that arise are corollary to the basic theorems of analysis which we have previously employed. We shall, as ever, be rigid in our insistence that we discern, by isolation, a free body upon which a system of forces acts.

We are challenged, in analyzing systems of three-dimensional scope, to visualize spatial conditions.

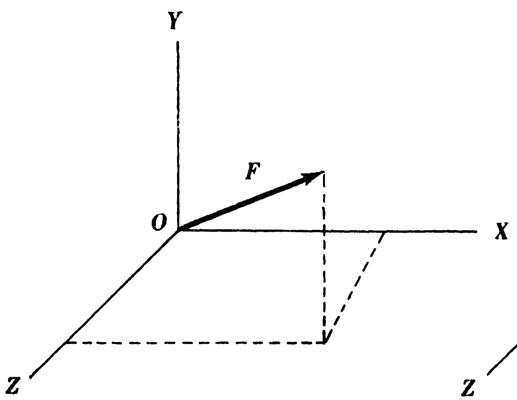


FIG. 63.

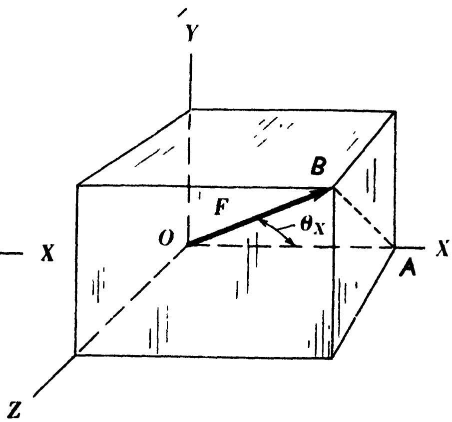


FIG. 64.

When we leave the single plane, the geometry becomes that of solids, and perspective drawings will assist in making clearer the conditions we face.

Let us, therefore, carefully observe that the force F of Fig. 63 has been placed in a framework consisting of three basic axes, X, Y, and Z.

If we imagine the force to have been applied at the corner of a box or rectangular parallelopiped (Fig. 64), we can note more readily the basic geometrical concepts which apply.

The angle between the force and the X axis lies in the plane OAB, which is determined by two intersecting lines—the line of action of the force and the X axis. The triangle OAB may be revolved about the X axis until it lies in the plane of the page, as in Fig. 65, and we observe that it is a right triangle.

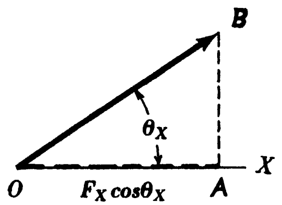


FIG. 65.

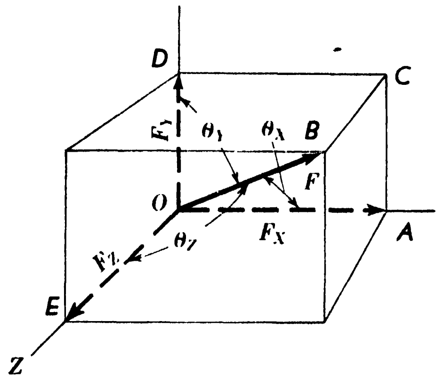


FIG. 66.

Then the X component is

$$F_X = F \cos \theta_X$$

Similarly,

$$F_Y = F \cos \theta_Y$$

and

$$F_Z = F \cos \theta_Z$$

We may confirm these relationships by combining these three components to obtain their resultant, which must be the force F (Fig. 66).

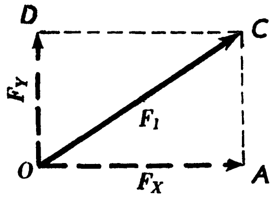


FIG. 67.

F_X and F_Y lie in the plane $OACD$ (Fig. 67), which is the rear face of the box. As concurrent forces acting in a single plane, these components reduce to a single force F_1, where

$$F_1 = \sqrt{F_X{}^2 + F_Y{}^2}$$

The original system has now been reduced to two concurrent forces F_1 and F_Z, whose lines of action intersect and, therefore, determine a plane ($OCBE$, Figs. 68 and 69).

Then

$$F = \sqrt{F_1{}^2 + F_Z{}^2}$$

But

$$\sqrt{F_1{}^2 + F_Z{}^2} = \sqrt{F_X{}^2 + F_Y{}^2 + F_Z{}^2}$$

Then

$$F = \sqrt{F_X{}^2 + F_Y{}^2 + F_Z{}^2}$$

The angle between F_X and F is

$$\theta_X = \cos^{-1} \frac{F_X}{F}$$

Likewise $$\theta_Y = \cos^{-1} \frac{F_Y}{F}.$$

And $$\theta_Z = \cos^{-1} \frac{F_Z}{F}.$$

No matter how many forces in space concur at a single point, they may each be reduced to X, Y, and Z components, and such components may be added to obtain the ΣX, ΣY, and ΣZ of the system. Then, these

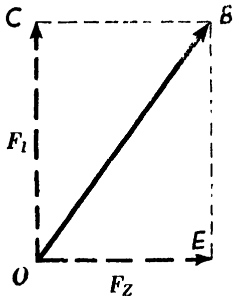


FIG. 68.

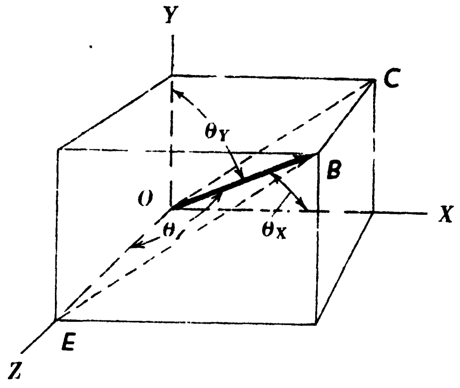


FIG. 69.

vectors may be added to obtain the resultant of the original system, which will be

$$R = \sqrt{\Sigma X^2 + \Sigma Y^2 + \Sigma Z^2}$$

This resultant will form, with the X axis, an angle

$$\theta_X = \cos^{-1} \frac{\Sigma X}{R}$$

Similarly, $$\theta_Y = \cos^{-1} \frac{\Sigma Y}{R}$$

And $$\theta_Z = \cos^{-1} \frac{\Sigma Z}{R}$$

Any two of these space angles fully determine the resultant, since

$$\sqrt{\Sigma X^2 + \Sigma Y^2 + \Sigma Z^2} = R$$

$$\Sigma X^2 + \Sigma Y^2 + \Sigma Z^2 = R^2$$

$$\frac{\Sigma X^2}{R^2} + \frac{\Sigma Y^2}{R^2} + \frac{\Sigma Z^2}{R^2} = 1$$

or $$\cos^2 \theta_X + \cos^2 \theta_Y + \cos^2 \theta_Z = 1$$

The sum of the squares of the cosines of the three angles between basic axes and resultant is, therefore, always equal to 1, and when two angles have been found, the third angle is determined. In reporting the inclination of a force in space, we need report but two of its space angles.

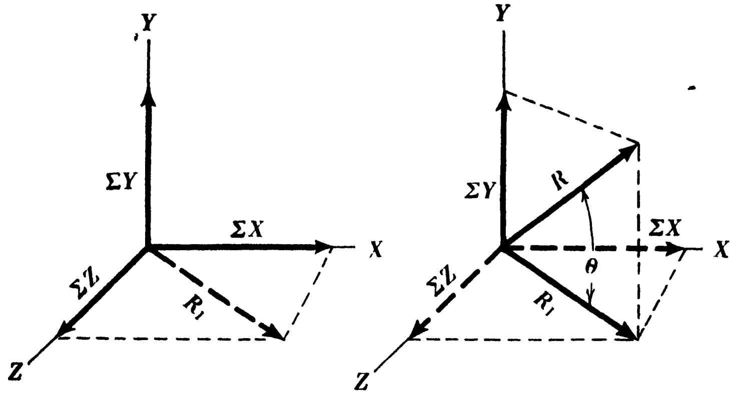


FIG. 70.

We can also note that if we first combine ΣX and ΣZ, we have, as in Fig. 70, a resultant R_1 of these two components.

$$R_1 = \sqrt{\overline{\Sigma X}^2 + \overline{\Sigma Z}^2}$$

The angle between the X axis and R_1 is

$$\theta_X = \tan^{-1}\frac{\Sigma Z}{\Sigma X}$$

Now, adding R_1 and ΣY, we have

$$R = \sqrt{R_1^2 + \overline{\Sigma Y}^2} = \sqrt{\overline{\Sigma X}^2 + \overline{\Sigma Z}^2 + \overline{\Sigma Y}^2}$$

as before.

And the angle between R_1 and R is

$$\theta = \tan^{-1}\frac{\Sigma Y}{R_1} = \frac{\Sigma Y}{\sqrt{\overline{\Sigma X}^2 + \overline{\Sigma Y}^2}}$$

27. Conditions of Equilibrium. The conditions of equilibrium for a concurrent force system in space follow directly from the expression for the resultant derived in the preceding article. For equilibrium, the system of forces must yield no resultant, and

$$R = \sqrt{\overline{\Sigma X}^2 + \overline{\Sigma Y}^2 + \overline{\Sigma Z}^2} = 0$$

Then $$\overline{\Sigma X}^2 + \overline{\Sigma Y}^2 + \overline{\Sigma Z}^2 = 0$$

And $$\Sigma X = 0$$

$$\Sigma Y = 0$$

$$\Sigma Z = 0$$

In selecting a free body upon which a concurrent system of forces in space acts, we may select one upon which not more than three unknowns appear in the force system.

28. Illustrative Example. The exploration of the three-dimensional concurrent forces in Articles 26 and 27 has been intended to establish basic spatial relationships. It is not only possible, but usually more effective, to make use in problem analysis, of simplification of such systems of forces. In Fig. 71, we find a beam supported at A, loaded with a force F_w at B, and tied to a wall by cables CD and CE. The line CS is the line of intersection of the plane DCE and a vertical plane V containing the axis of the beam (AB) and the line of action of force F_w. We wish to determine the stresses in cables CD and CE.

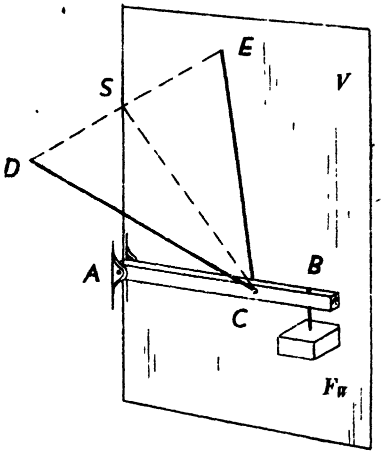


Fig. 71.

If we isolate the beam as a free body (Fig. 72), the system of external forces lying in plane V consists of F_w, F_A (the force applied at A by the support), and a force F_C acting at C. This force must be the resultant due to the combined effect of cables CD and CE. Such a resultant must have its line of action along CS, because it must be an element of force, which acts with the other forces in plane V to produce equilibrium of the beam. It must, therefore, lie in plane V. This force is the resultant

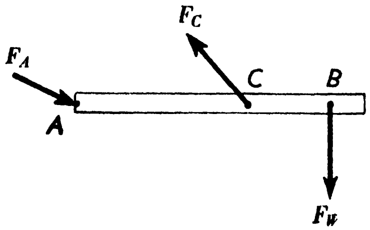


Fig. 72.

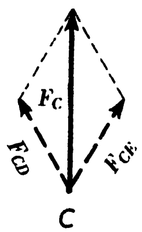


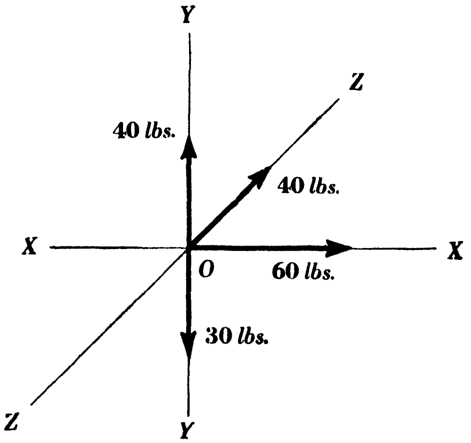
Fig. 73.

of two forces in the directions CD and CE. Then, it must also lie in plane DCE. If this force must lie in both planes simultaneously, it must have its line of action along CS, the intersection of both planes. We now solve for resultant F_C by analyzing the single plane system of Fig. 72. Then, with F_C known, the resultant F_C may be resolved into its components F_{CD} and F_{CE} (Fig. 73).

Problems

143. The system of four forces shown has a common origin at point O. Determine the resultant.

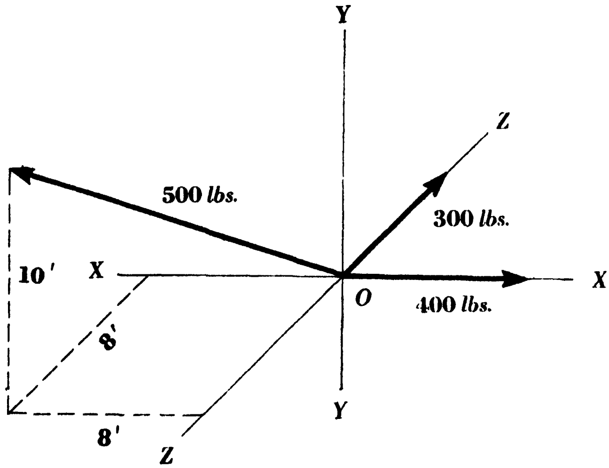
Ans. $R = 72.9$ lb.; $\theta_X = 34.6°$; $\theta_Y = 82.1°$; $\theta_Z = 56.7°$.



Prob. 143

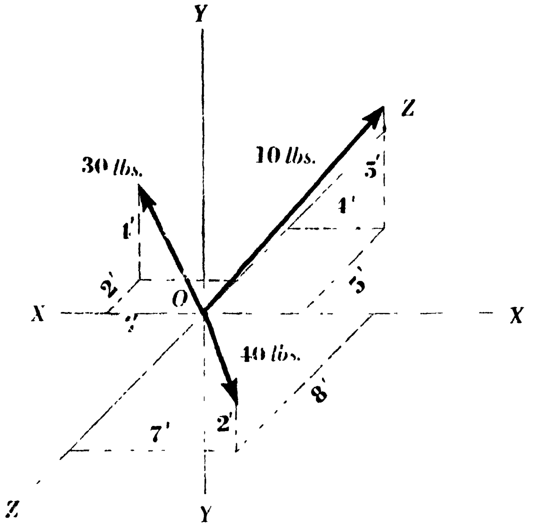
144. If the 30-lb. force of Problem 143 is increased to 70 lb., determine the resultant of the system.

145. The system of three forces shown has a common origin at point O. Determine the resultant.



Prob. 145

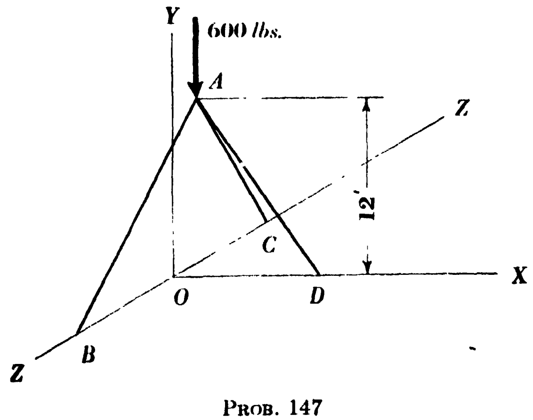
146. The system of three forces shown has a common origin at point O. Determine the resultant.



PROB. 146

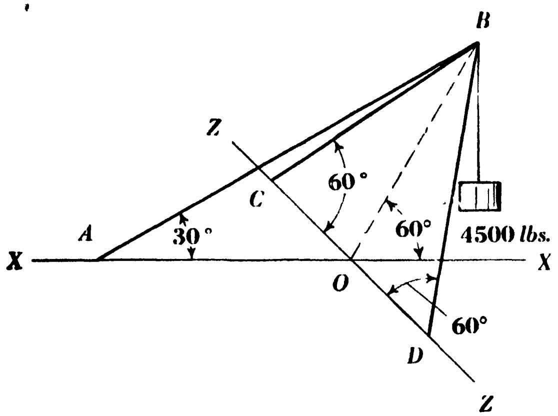
147. The shear-legs tripod shown carries a vertical load of 600 lb. at the vertex A. Points B, C, and D lie in the horizontal plane XOZ; $BO = OC = OD = 9$ ft. Determine the stresses in members AB, AC, and AD. Point A lies in the XOY plane, 3 ft. from the Y axis.

Ans. $AC = AB = 525$ lb. compression; $AD = 461$ lb. compression.



PROB. 147

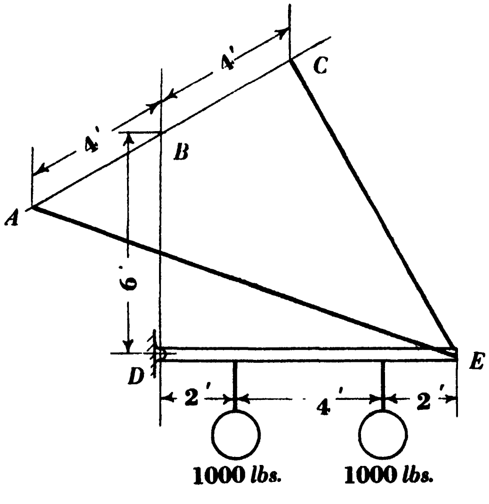
148. The shear legs CB and DB form an angle of 60° with the horizontal base line CD. The tie rod AB is inclined as shown. Determine the stresses in members AB, BC, and DB due to the vertical load of 4500 lb. at B.



PROB. 148

149. Points A, B, and C lie on the same horizontal line, and point B is vertically above point D. The beam DE is horizontal, and carries two vertical 1000-lb. loads. Determine the stresses in members AE and CE.

Ans. $AE = CE = 905$ lb. tension.

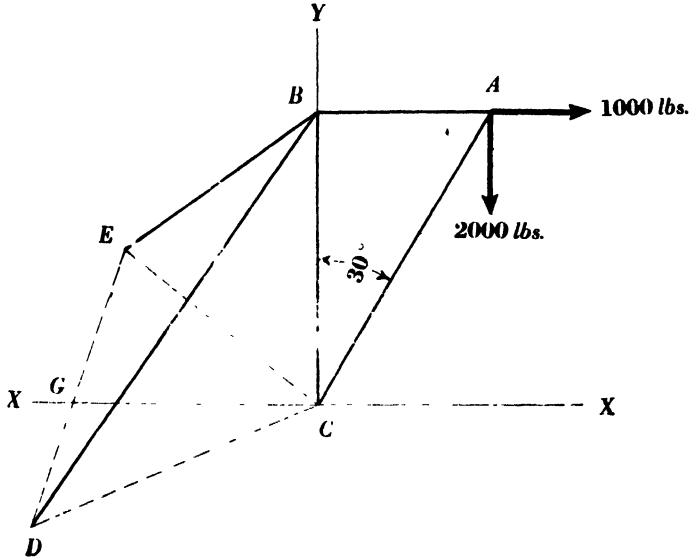


PROB. 149

150. If the distance AB of Problem 149 is decreased to 2 ft., determine the stresses in members AE and CE. Also determine the horizontal and vertical components of the force supporting the beam at point D.

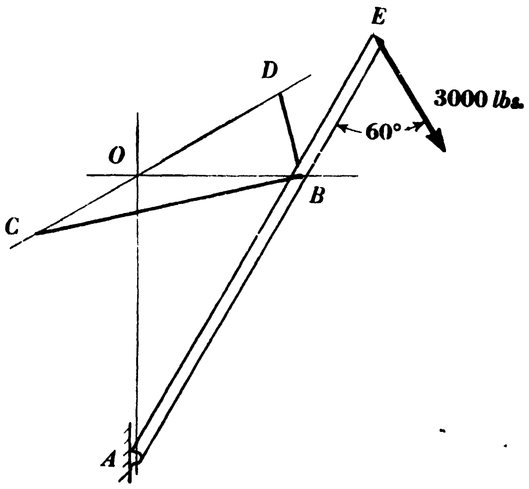
151. The derrick shown is subjected to a horizontal load of 1000 lb., and a vertical load of 2000 lb. Members AB, BC, AC, and the two loads all lie in the vertical plane XCY. Points C, D, E, and G lie in a horizontal plane. $BC = 12$ ft.;

$CE = CD = 12$ ft.; $GD = 4$ ft.; $EG = 8$ ft. Determine the stresses in all members of the derrick.



PROB. 151

152. The 24-ft. boom AE is pinned to a vertical wall at A, and held in position by the tie cables BC and BD, which lie in a horizontal plane $BCOD$.



PROB. 152

A load of 3000 lb. is applied at E, at an angle of 60° with AE. The 3000-lb. load, the boom AE, and line OA lie in the same vertical plane. Angle $OAB = 30°$; $AB = 16$ ft.; $BE = 8$ ft.; $OC = 6$ ft.; $OD = 8$ ft.

Determine the stresses in BC and BD, and the horizontal and vertical components of the supporting force at A.

29. Couples in Space. We have earlier noted (Article 14) that the moment of a couple is a vector quantity because it has both magnitude and direction. The conventional system of representing such vector quantities as vectors follows.

The inclination of the couple is the inclination of the plane in which the couple acts. Vectors representing couples are drawn in the direction of the moment axis or perpendicular to the plane of those couples. The magnitude, as in all vector representation, is announced by giving the vector scaled length. The sense of the couple tells whether it tends to cause clockwise or counterclockwise rotation in its plane when that plane is viewed from a specific side. This announcement of sense is made by observing the relationship of translation and rotation in the case of the right-handed screw (Fig. 74).

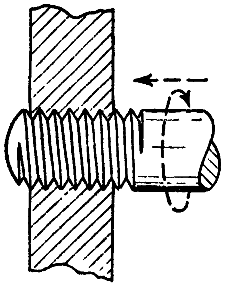


FIG. 74.

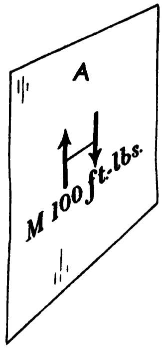


FIG. 75.

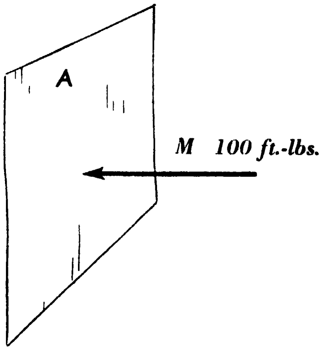


FIG. 76.

When the screw is turned clockwise when viewed from the right of the figure, there will be translation of the screw to the left, or more deeply into the nut. If turned counterclockwise, the screw will translate to the right, or tend to withdraw from the nut. If the sense of a couple (Figs. 75 and 76) is clockwise when viewed from the right, an arrowhead is placed on the left end of the vector, indicating translation to the left.

In Fig. 75, a couple of 100 foot-pounds, acting in plane *A* and with clockwise sense as viewed from the right, is shown symbolically, and the translation into vector representation is also shown in Fig. 76. The vector may be shown at any point in the plane, because a couple may be moved anywhere in its own plane. (See Article 14.)

Couples may also be moved into any parallel plane of the free body without differently affecting the motion of that body. If, for example, the jaws of a wrench are applied to a pipe, as shown at *A* in Fig. 77, they produce a certain amount of turning effort, or moment. An application of the same moment by a wrench applied at *B* in a parallel plane will produce the same rotation of the pipe. This change of location of the wrench is equivalent, in terms of force systems, to moving the applied couple into planes parallel to each other. For formal proof of this statement, we turn to the example of Fig. 78.

The planes P_1 and P_2 are parallel. Couple M_1 lying in plane P_1 is equal in moment to couple M_2 lying in plane P_2, but the two couples are of opposite sense. We may substitute for couple M_1 forces $F' = F$,

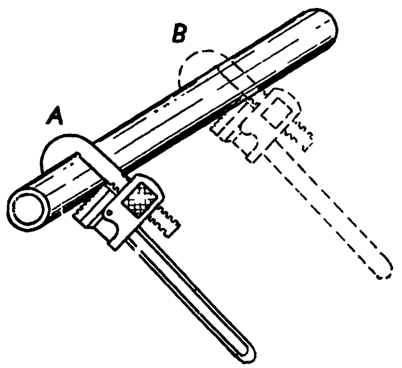


FIG. 77.

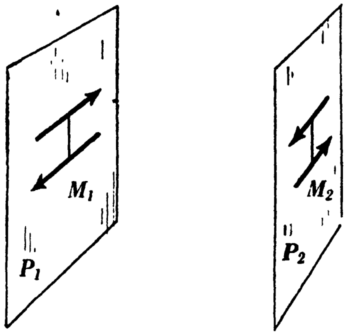


FIG. 78.

as shown in Fig. 79, having an arm $y = \frac{M_1}{(F = F')}$ between them. In plane P_2, we substitute forces $F' = F$ and arm $y = \frac{M_2}{F = F'}$ for couple M_2. The four forces are equal, as are the two moment arms.

Now the forces F lie in a plane $abcd$, and their resultant is the force $2F$ having its line of action along ef, the center line of the rectangle $abcd$.

Similarly, forces F' lie in a plane $ghij$, and their resultant is $2F'$ having its line of action along ef, the center line of rectangle $ghij$. These center lines are coincident because they lie in both planes and must, therefore, be the line of intersection of the planes.

Then, forces $2F$ and $2F'$ balance each other, and we may conclude that *couples of equal moment and of opposite sense that lie in parallel planes are equivalent.*

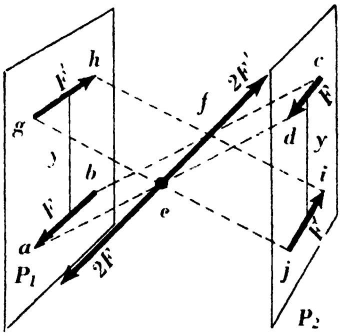


FIG. 79.

We could have effected balance had we opposed M_2 by another couple in plane P_2 which was equal in moment but of opposite sense. This moment, equal in its effect to couple M_1 of plane P_1 must, therefore, be equivalent to couple M_1. Then *couples of equal moment and of the same sense, lying in parallel planes, are equal,* or a couple may be moved from its own plane into any parallel plane.

30. Parallel Forces in Space. The system of forces shown in Fig. 80 consists of a number of forces, all having the same inclination. Such a system acting on a free body can tend to give the body a translation in the direction of the forces and may, in addition, tend to cause rotation.

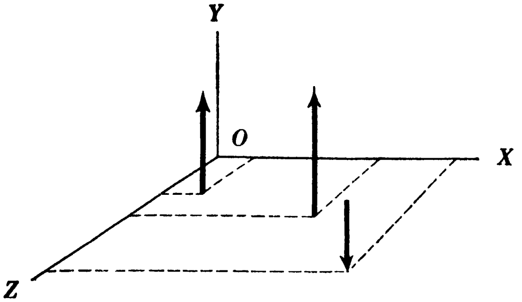


FIG. 80.

This qualitative examination indicates that the resultant, a simpler but equivalent system, could be a single force equal to the sum of the individual forces and be so located that it would have a moment about any given axis equivalent to the sum of the moments of the individual forces about that axis.

We can confirm the conclusions of the qualitative analysis by formal investigation.

In Fig. 81*a*, we find a single force F parallel to the Y axis at point A. If we add a balanced pair of forces at B (Fig. 81*b*), each force being equal

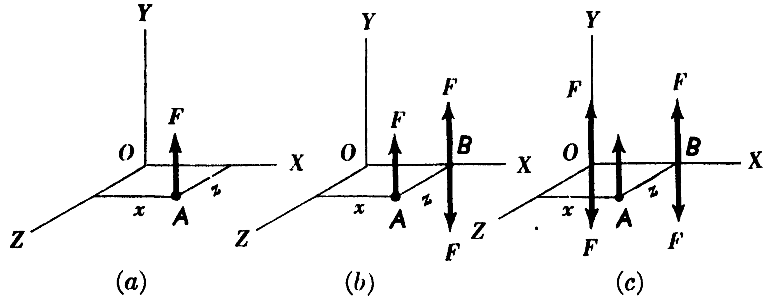


FIG. 81.

and parallel to F, we have resolved F into a single force F acting at point B and a couple $M_1 = F \times z$.

If now, as in Fig. 81*c*, we add to the system a balanced pair of forces at O, each force being equal and parallel to F, we have a single force F at the origin, a couple $M_1 = Fz$, and a couple $M_Z = Fx$.

If many parallel forces are present in the original system, each force may be resolved into a single force $R = \Sigma F$ at origin O, a couple ΣM_X equal to the sum of the moments of the original forces about the X axis, and a couple ΣM_Z equal to the sum of the moments of the original forces about the Z axis. Such a reduction of an original parallel force system appears in Fig. 82*a*. This system is capable of simplification: we reverse the procedure of resolution just employed. First, we decide that the resultant of the system R may be located by determining its point of application. Adding couple ΣM_X and force R, we have as their sum

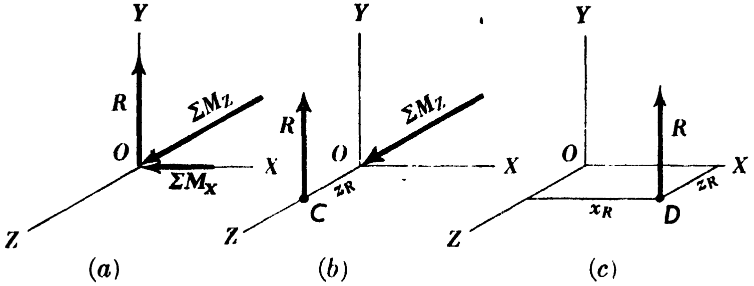


FIG. 82.

(Fig. 82*b*) a force R located at point C, with $z_R = \frac{\Sigma M_X}{R}$. We now add the remaining couple ΣM_Z to R, and have R, located at point D, with

$$x_R = -\frac{\Sigma M_Z}{R}$$

R, located at point D of Fig. 82*c*, is the final step of simplification and is the resultant, properly located, of the original system.

Summarizing the steps we have taken in the demonstration,

$$R = \Sigma F$$

$$x_R = -\frac{\Sigma M_Z}{R}$$

and

$$z_R = \frac{\Sigma M_X}{R}$$

We may find that in the case of a given system of forces,

$$R = 0$$
$$\Sigma M_Z \neq 0$$
$$\Sigma M_X \neq 0$$

Then this system yields as its resultant a couple whose components are ΣM_Z and ΣM_X. Such a case is shown in Fig. 83. The resultant couple has magnitude

$$M_R = \sqrt{\overline{\Sigma M_X}^2 + \overline{\Sigma M_Z}^2}$$

and the inclination of the vector M_R is

$$\theta_R = \tan^{-1} \frac{\Sigma M_Z}{\Sigma M_X}$$

The plane in which this couple acts is shown as plane *abcd*, which is any plane perpendicular to the couple vector M_R.

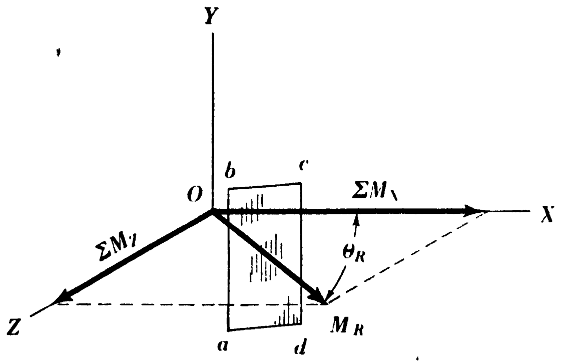


FIG. 83.

31. Conditions of Equilibrium. We may also find that in parallel force systems like those discussed in the preceding article,

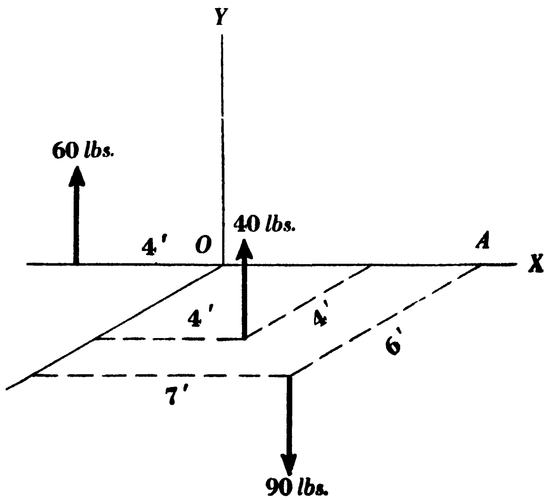
$$R = \Sigma F = 0$$
$$\Sigma M_X = 0$$
$$\Sigma M_Z = 0$$

Then the system yields no resultant and forms a system in equilibrium, with these equations serving as the conditions of equilibrium.

PROBLEMS

153. The three forces shown are parallel to the Y axis. Determine their resultant, and locate the point of application of the resultant.

Ans. $R = 10$ lb.; $x_R = 71$ ft.; $z_R = 38$ ft

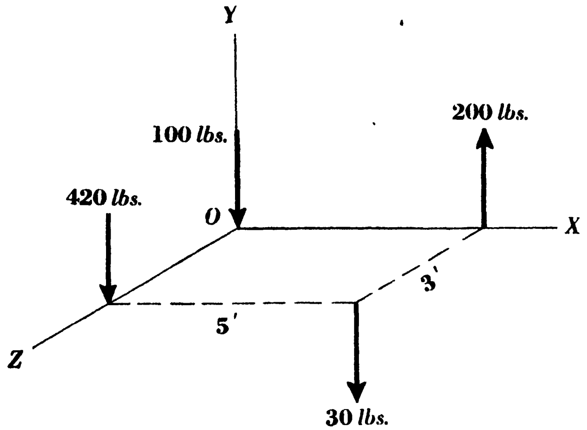


PROB. 153

154. If a 10-lb. force, parallel to the Y axis, and with sense downward, is added at point A to the parallel force system of Problem 153, determine the resultant of the system.

155. The four forces shown are perpendicular to plane XOZ. Determine their resultant, and locate the point of application of the resultant.

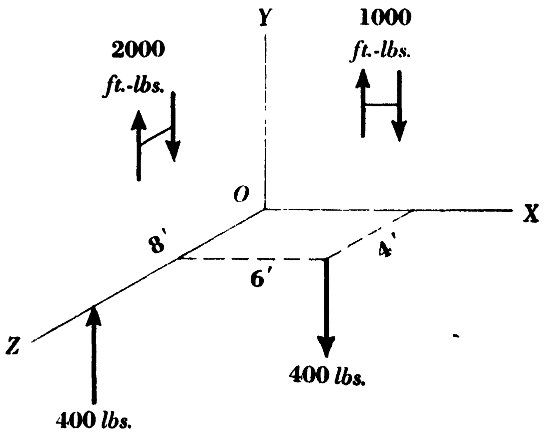
Ans. R = 350 lb.; x_R = 2.33 ft.; z_R = 3.86 ft.



PROB. 155

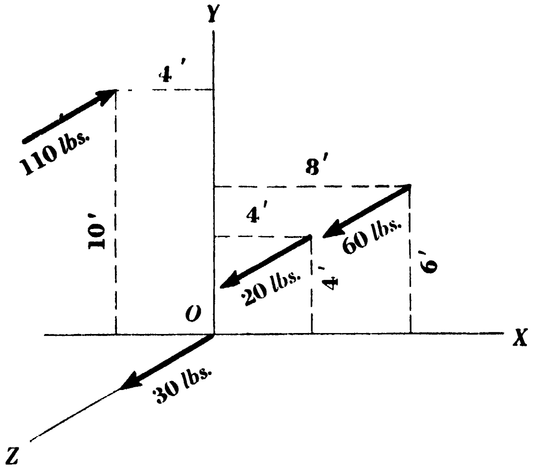
156. If the 30-lb. downward force given in Problem 155 is replaced by a force of 320 lb. upward, determine the resultant of the parallel force system.

157. The 400-lb. forces shown are parallel to the Y axis. The 2000-ft.-lb. couple acts in plane YOZ, and the 1000-ft.-lb. couple acts in plane XOY. Determine the resultant of the system. *Ans.* M_R = 4950 ft.-lb.; θ_X = 43.4°.



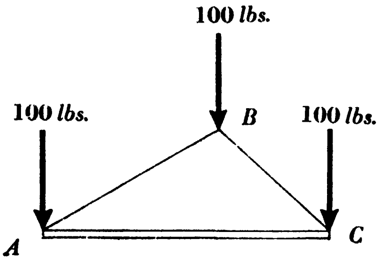
PROB. 157

158. The four forces shown are parallel to the Z axis. Determine the resultant of the parallel force system.



PROB. 158

159. The triangular plate ABC lies in a horizontal plane, and is acted upon by a system of the three equal vertical forces shown. Triangle ABC is a right triangle, with $BAC = BCA = 45°$. Hypotenuse $AC = 6$ ft.

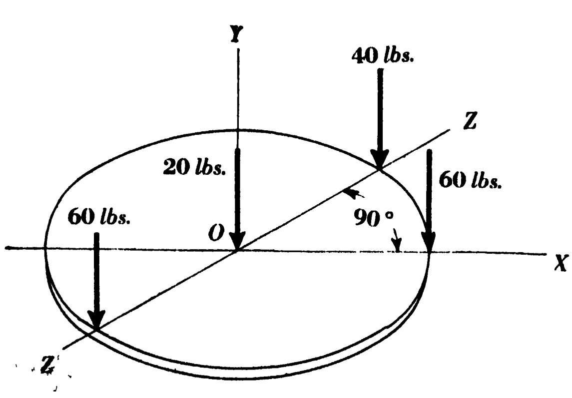


PROB. 159

Determine the resultant of the parallel force system, and locate its point of application.

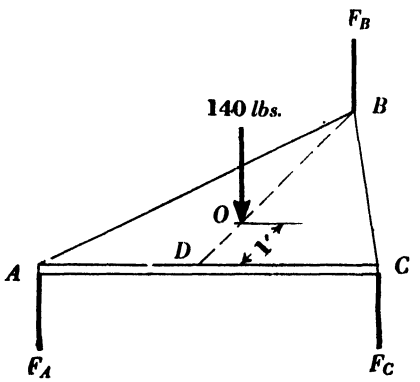
160. The system of four forces, parallel to the Y axis, acts on a horizontal circular plate. The diameter of the plate is 6 ft. Determine the resultant of the force system, and locate its point of application.

Ans. $R = -180$ lb.; $x_R = 1$ ft.; $z_R = .33$ ft.

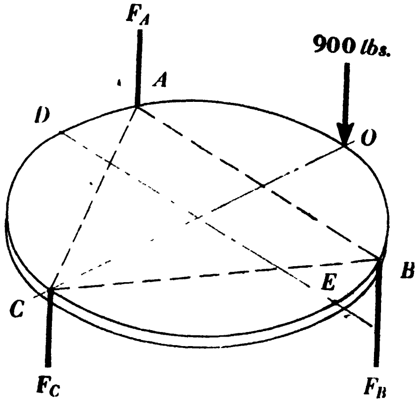


PROB. 160

161. The equilateral triangular plate ABC has 5-ft. sides. Point O lies on the median BD. Supporting vertical forces F_A, F_B, and F_C are applied at the vertices. A load of 140 lb. is applied at point O. Determine F_A, F_B, and F_C if the plate is in equilibrium.



Prob. 161

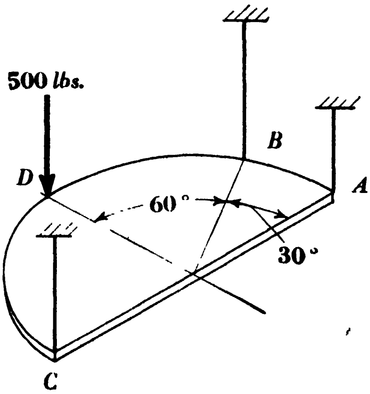


Prob. 162

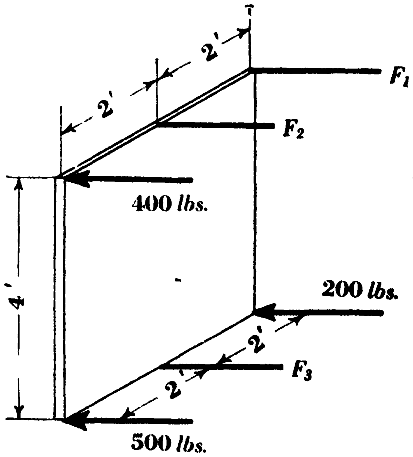
162. The horizontal circular plate shown has a diameter of 4 ft. The equilateral triangle ABC has its leg AB parallel to diameter DE. Diameter OC is perpendicular to AB.

If a vertical load of 900 lb. is applied at point O on the circumference of the plate, which is in equilibrium, determine the vertical supporting forces F_A, F_B, and F_C. *Ans.* $F_A = F_B = 600$ lb.; $F_C = 300$ lb.

163. The semicircular steel plate shown is supported in a horizontal plane by three vertical rods, placed at points A, B, and C. The radius of the plate is 3 ft. If a vertical load of 500 lb. is applied at point D, determine the stresses in the three rods.



Prob. 163



Prob. 164

164. The square plate of steel is placed in a vertical plane. A system of six horizontal parallel forces holds the plate in equilibrium. Determine F_1, F_2, and F_3. *Ans.* $F_1 = 700$ lb.; $F_2 = 1100$ lb.; $F_3 = 700$ lb.

32. The General Three-Dimensional Force System. Systems of forces which lie anywhere in space, such as that illustrated in Fig. 84,

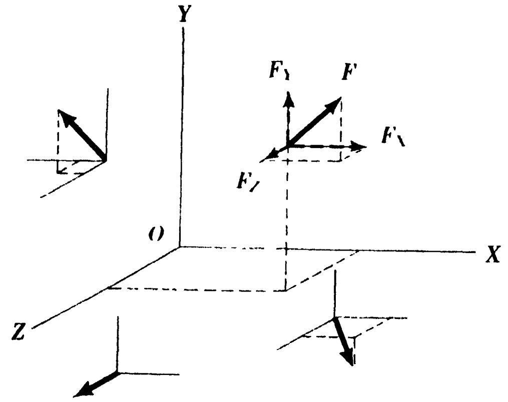


Fig. 84.

may be first resolved by determining the X, Y, and Z components of each force.

The X components will form a parallel force system, and the method employed in Article 30 will serve to reduce all of the X components to

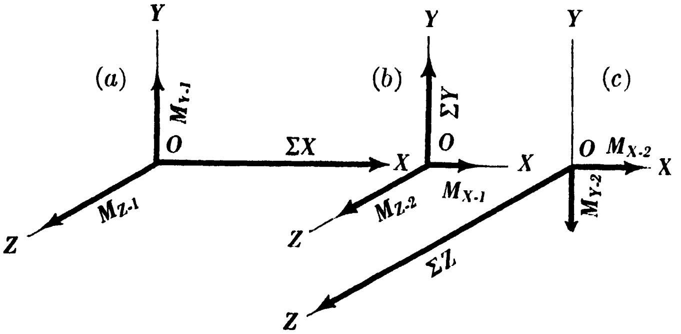


Fig. 85.

(1) a single force ΣX at origin O, (2) a couple M_{Y-1} equal to the sum of moments of these components about the Y axis, and (3) a couple M_{Z-1} equal to the sum of moments of these components about the Z axis. Vectors representing these three elements are shown in Fig. 85*a*.

The Y components, another parallel force system similarly resolved, will yield ΣY, M_{X-1}, and M_{Z-2} (Fig. 85*b*).

The Z components will yield ΣZ, M_{X-2}, and M_{Y-2} (Fig. 85*c*).

The vector summations may now be made, and we have at the origin a single force,

$$R = \sqrt{\Sigma X^2 + \Sigma Y^2 + \Sigma Z^2}$$

and a resultant couple

$$M_R = \sqrt{\Sigma M_X{}^2 + \Sigma M_Y{}^2 + \Sigma M_Z{}^2}$$

This stage of simplification is shown in Fig. 86.

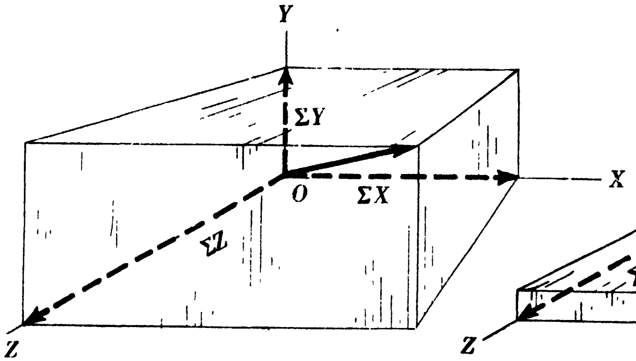


FIG. 86a.

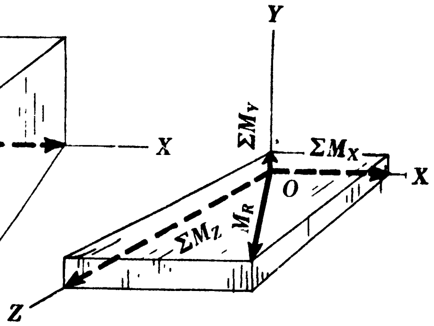


FIG. 86b.

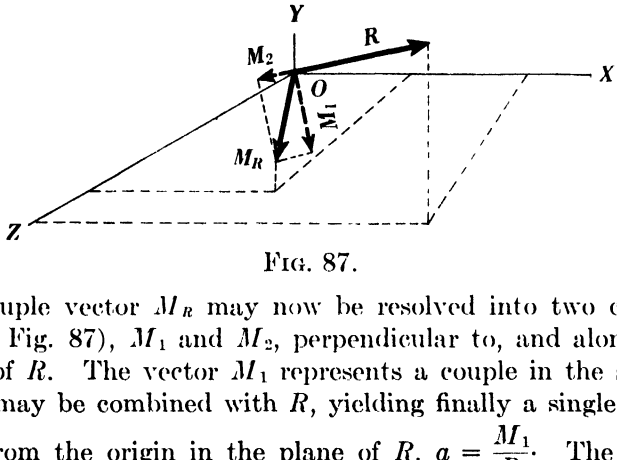


FIG. 87.

The couple vector M_R may now be resolved into two components (shown in Fig. 87), M_1 and M_2, perpendicular to, and along, the line of action of R. The vector M_1 represents a couple in the same plane as R and may be combined with R, yielding finally a single force R at distance from the origin in the plane of R, $a = \frac{M_1}{R}$. The vector M_2 represents a couple in a plane perpendicular to R and cannot be further combined with R. The final resultant, R and M_2, would cause a free body to translate in the direction of the line of action of R, rotating as it proceeded.

33. Conditions of Equilibrium. The preceding article has established the following equations for the resultant of the general three-dimensional force system:

$$R = \sqrt{\Sigma X^2 + \Sigma Y^2 + \Sigma Z^2}$$

For equilibrium, R must equal zero, and

$$\overline{\Sigma X}^2 + \overline{\Sigma Y}^2 + \overline{\Sigma Z}^2 = 0$$

Then
$$\Sigma X = 0$$
$$\Sigma Y = 0$$
and
$$\Sigma Z = 0$$

Nor may such a system reduce to a resultant couple or have moment about any axis if equilibrium is to ensue.

Then
$$M_R = \sqrt{\overline{\Sigma M_X}^2 + \overline{\Sigma M_Y}^2 + \overline{\Sigma M_Z}^2} = 0$$
$$\overline{\Sigma M_X}^2 + \overline{\Sigma M_Y}^2 + \overline{\Sigma M_Z}^2 = 0$$
and
$$\Sigma M_X = 0$$
$$\Sigma M_Y = 0$$
$$\Sigma M_Z = 0$$

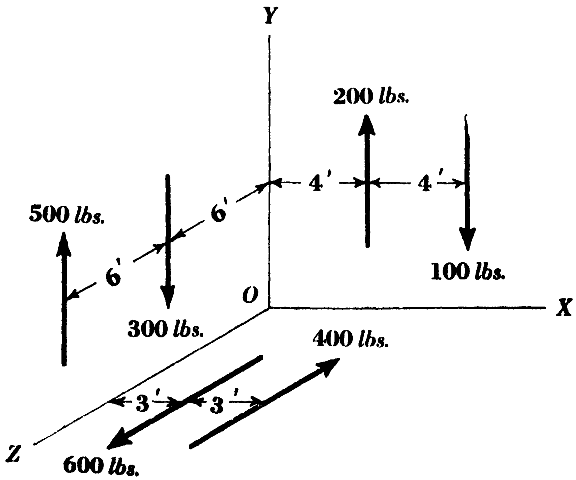
It is quite possible that a given system of forces may yield the following as we proceed with our analysis:

$$R = 0$$
$$M_R = \sqrt{\overline{\Sigma M_X}^2 + \overline{\Sigma M_Y}^2 + \overline{\Sigma M_Z}^2} \neq 0$$

Then such a system has as its resultant a single couple.

PROBLEMS

165. The 100-lb. and 200-lb. forces lie in the plane XOY, and are parallel to the X axis.



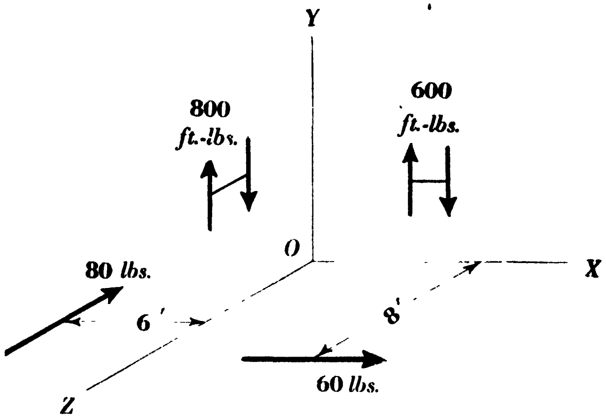
PROB. 165

The 300-lb. and 500-lb. forces lie in the plane YOZ, and are parallel to the Y axis.

The 600-lb. and 400-lb. forces lie in the plane XOZ, and are parallel to the Z axis.

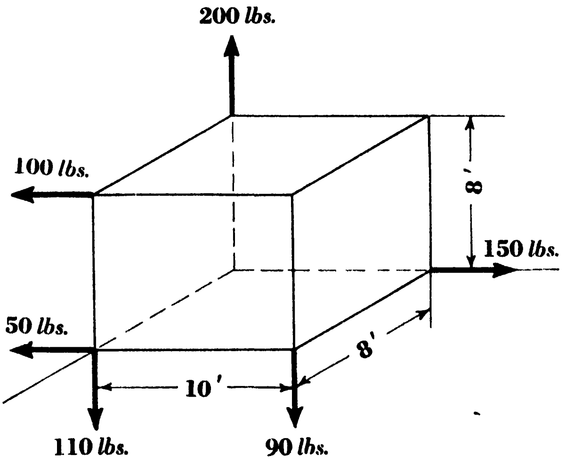
Determine the resultant of the system of forces.

166. The 800-ft.-lb. couple acts in plane ZOY and the 600-ft.-lb. couple acts in plane XOY. The 80-lb. and 60-lb. forces act in plane XOZ, and are parallel to the Z and X axes, respectively. Determine the resultant of the system of forces and couples.



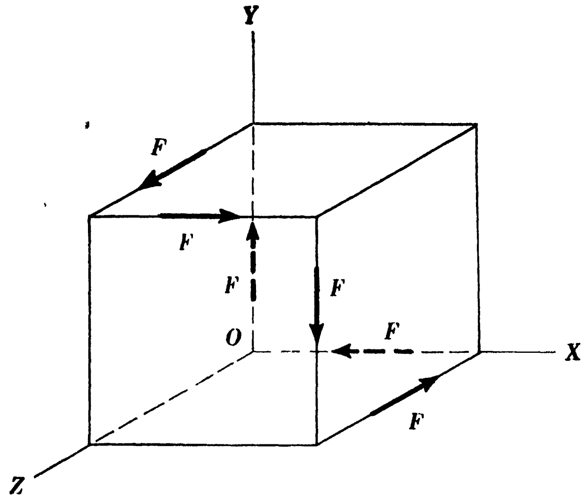
PROB. 166

167. The six forces shown act along the edges of a rectangular prism. Determine the resultant of the system. *Ans.* $M = 2000$ ft-lb.



PROB. 167

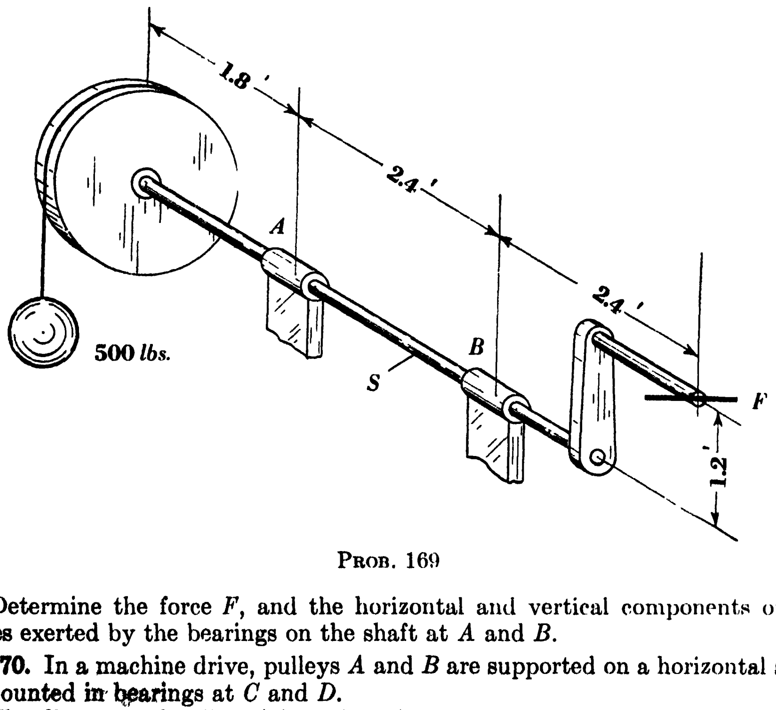
168. Six forces, F, act along the edges of a cube as shown. Determine the resultant. Length of side is a.



PROB. 168

169. The shaft S is horizontal, and is supported by bearings at A and B. A horizontal force F is applied at the crank handle, perpendicular to the axis of the handle. The system is in equilibrium as the 500-lb. weight is raised at constant speed, using a 2-ft. diameter drum.

Ans. $H_B = 833$ lb.; $V_B = 375$ lb.; $H_A = 417$ lb.; $V_A = 875$ lb.

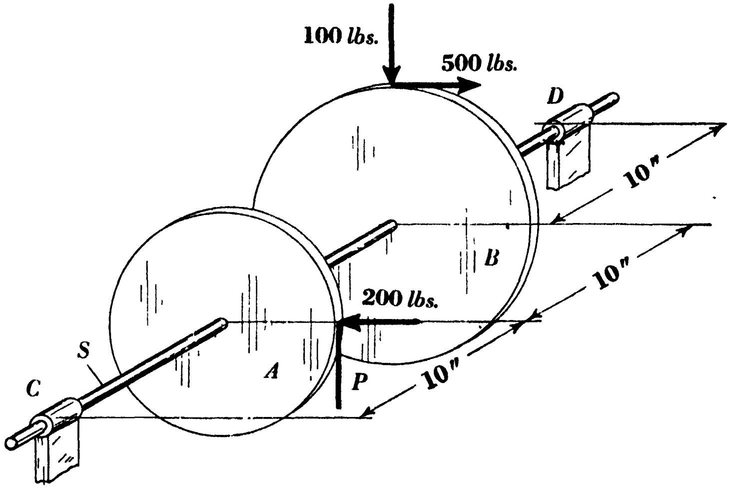


PROB. 169

Determine the force F, and the horizontal and vertical components of the forces exerted by the bearings on the shaft at A and B.

170. In a machine drive, pulleys A and B are supported on a horizontal shaft S, mounted in bearings at C and D.

The diameter of pulley A is 12 in.; the diameter of pulley B is 15 in. The 100-lb. force and force P are vertical; the 500-lb. and 200-lb. forces are horizontal.

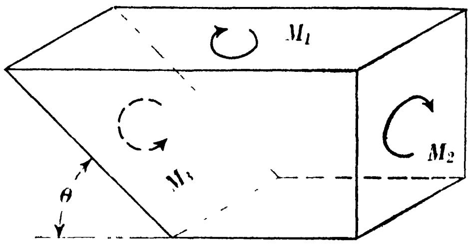


PROB. 170

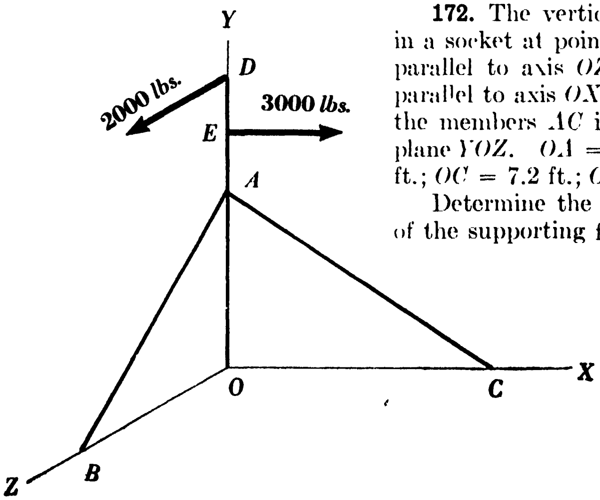
Determine force P, and the horizontal and vertical components of the forces exerted by the bearings on the shaft at C and D.

171. The truncated rectangular prism is in equilibrium when the three couples shown are applied. $M_1 = 200$ ft.-lb.; $M_2 = 300$ ft.-lb.

Determine the magnitude of M_3, and the angle θ between the sloping face and the base.



PROB. 171



PROB. 172

172. The vertical mast OD is supported in a socket at point O. The 2000-lb force is parallel to axis OZ, and the 3000-lb. force parallel to axis OX. The mast is braced by the members AC in plane XOY, and AB in plane YOZ. $OA = 6$ ft.; $AE = 2$ ft.; $DE = 2$ ft.; $OC = 7.2$ ft.; $OB = 6$ ft.

Determine the X, Y, and Z components of the supporting force at point O.

Ans. $F_X = 1000$ lb.; $F_Y = 7000$ lb.; $F_Z = 1670$ lb.

CHAPTER V

Distributed Forces

34. Resultant of Distributed Force System in a Plane. The forces which have thus far been considered have been concentrated—that is, forces whose contact with the free body may be considered to occur at a point. In actual contact between material bodies, a finite area exists over which force is applied. Even the point of a needle exerting pressure on a surface has finite contacting surface. All forces of contact between bodies are, therefore, distributed over some finite surface, even though that surface may be very limited in its extent.

The concentrated forces which have been discussed in the preceding chapters are actually resultants of distributed forces. In those cases, the region of application is so limited that no error is introduced by considering those areas to be zero and thereby conforming to the mathematical concept of a point.

As our explorations of force systems expand, we find cases in which the region at which contact force between neighboring bodies occurs is of appreciable magnitude.

For example, the pressure of the water on the surface of a dam cannot be considered to be a force so localized that it is applied at a single point.

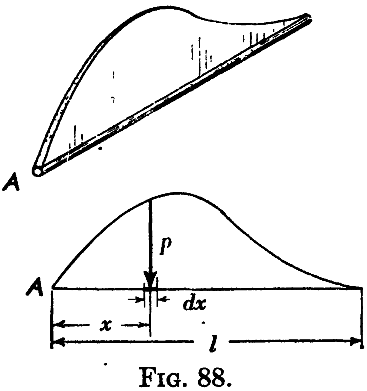


FIG. 88.

We must, therefore, recognize the expansion of the regions of contact at which force is applied.

In Fig. 88, we find a distributed force applied to a rod by the weight of a contacting, thin plate of the shape shown in the figure. The weight of the plate exerts contact force, which is an external force applied to the rod. The plate is assumed to be so thin that no error is introduced by assuming this external force to be applied in a single plane through the axis of the rod.

The intensity of force p at any distance x from the end A of the rod is the force per unit of length exerted by the thin plate at that distance.

Over a distance dx, a total force $p\,dx$ will be exerted, and the moment of this force about A will be $p\,dx\,x$.

On every other portion of length, dx, of the rod, a similar force $p\,dx$

will be exerted, which will be the product of the intensity p at that portion multiplied by length dx.

These $p\,dx$'s form a parallel force system in a plane. Then, the method of finding their resultant is the method which we discussed in Article 16, and

$$R = \Sigma F = \int p\,dx$$

The location of the point of application of this resultant is determined, as in all parallel forces systems, where

$$x_R = \frac{\Sigma M}{R} = \frac{\int px\,dx}{\int p\,dx}$$

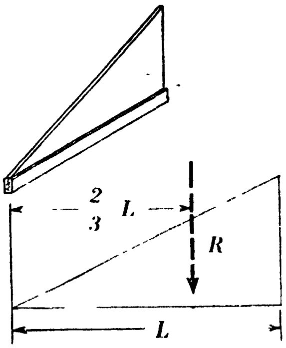


FIG. 89.

A *uniformly varying* force is one in which the intensity varies directly with distance from an origin, called the *neutral axis*, as in Fig. 89. In this case, $p = kx$, where k is a constant.

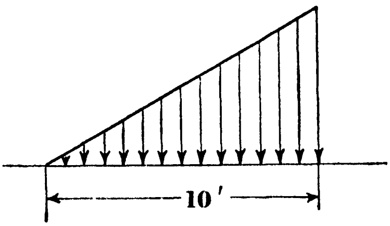
Then $$R = \int_0^L p\,dx = \int_0^L kx\,dx = \tfrac{1}{2}kL^2$$

and the point of application,

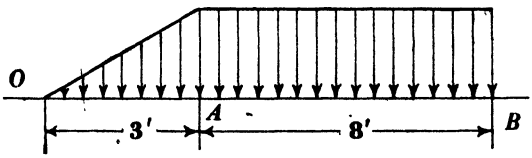
$$x_R = \frac{\int_0^L px\,dx}{\int_0^L p\,dx} = \frac{\int_0^L kx^2\,dx}{\int_0^L kx\,dx} = \frac{2L}{3}$$

PROBLEMS

173. Determine the resultant of the uniformly varying force shown. The maximum intensity of force is 120 lb. per ft. *Ans.* R = 600 lb.; x_R = 6.67 ft.



PROB. 173

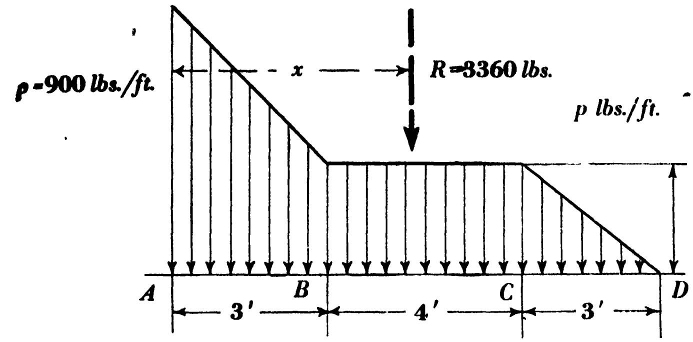


PROB. 174

174. The distributed force varies uniformly from zero at point O to 90 lb. per ft. at A, and is uniformly distributed from A to B. Determine the resultant force.

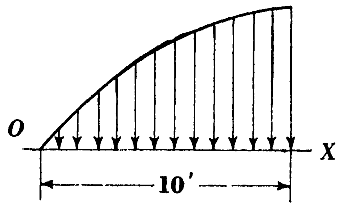
175. The resultant of the distributed force shown is $R = 3360$ lb., acting at x ft. from A. Determine the intensity of force at C and the distance x.

Ans. $p_C = 287$ lb. per ft.; $x = 6$ ft.



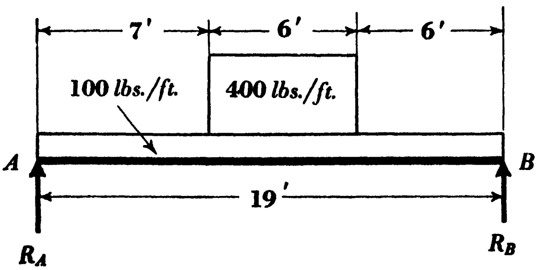
PROB. 175

176. The distributed force shown has an intensity $p = (50x - 2x^2)$ lb. per ft., when x is the distance from point O in feet. Determine the resultant force.



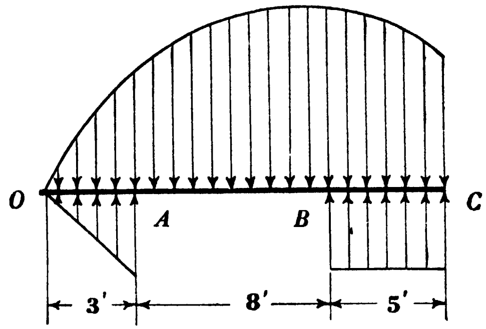
PROB. 176

177. The beam AB carries a uniformly distributed force of 100 lb. per ft. (including the weight of the beam) and a uniformly distributed force of 400 lb. per ft. as shown. If the beam is in equilibrium, determine the supporting forces R_A and R_B.



PROB. 177

178. The beam OC carries a distributed force whose intensity $p = \left(10x - \frac{x^2}{2}\right)$ lb. per ft., where x is the distance from point O in feet. The beam is supported on a distributed force from O to A which varies uniformly, and on a uniformly distributed supporting force from B to C.



PROB. 178

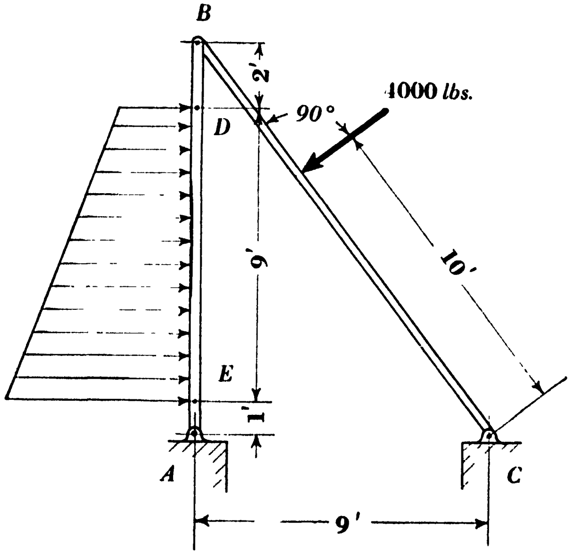
Determine the intensity of the supporting forces at A and B.

Ans. p_A = 151 lb. per ft.; p_B = 74 lb. per ft.

179. The three-hinged arch ABC consists of two members AB and AC pinned to each other at B, and pinned to abutments at A and C.

The distributed load on AB varies uniformly from 400 lb. per ft. at D to 1000 lb. per ft. at E.

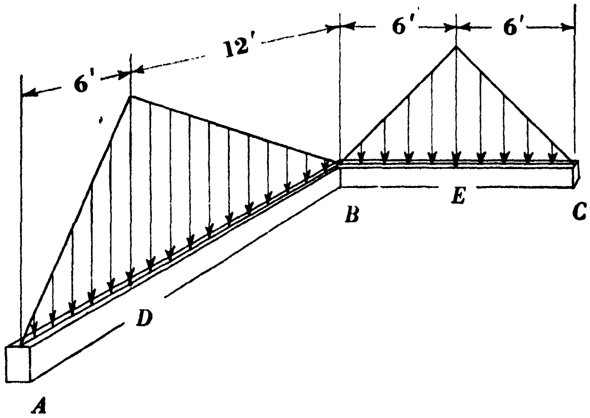
Determine the horizontal and vertical components of the supporting forces at A and C.



PROB. 179

180. Two beams AB and BC, perpendicular to each other, carry the distributed forces shown. The load on AB varies uniformly from zero at A to 1000 lb. per ft. at D, and varies uniformly from D to zero at B.

The load on beam BC varies uniformly from zero at B to 600 lb. per ft. at E, to zero at C.



PROB. 180

Determine the resultant of the entire system of distributed forces, and locate its point of application. It is assumed that the weight of the beams is negligible.

35. Parabolic Cables. The term *cable*, as it will be employed in the present discussion, is a generic and embraces ropes, wires as used in electric transmission lines, as well as the multistranded cables found in suspension bridges.

We shall assume that cables are perfectly flexible and, therefore, do not offer resistance to bending.

We shall also assume that cables are inextensible—that is, that their original length remains unchanged.

Such assumptions are approximations, but they afford opportunity to derive basic equations for the stresses in the cables which are sufficiently accurate to serve as a satisfactory basis of design.

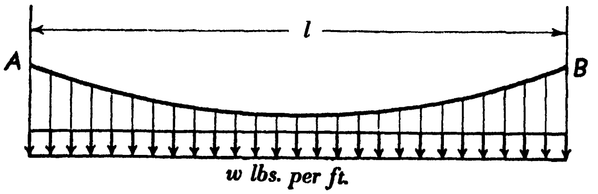


FIG. 90.

When these approximations are too crude—for example, when changes in temperature are of sufficient magnitude to cause expansions and contractions which must be taken into account—the theories are modified in practice to accommodate such changes. Such refinement is beyond our present interest, and we shall proceed, resting upon the assumptions of perfect flexibility and absolute inextensibility.

When the load applied to a cable may be considered to be uniformly distributed horizontally, the cable will assume the shape of a parabolic arc. This assumption cannot be made for the weight of the cable itself, because that is uniformly distributed (for a cable of constant cross section and homogeneous material) along the center line of the cable itself, rather than horizontally. If the load (w lb. per ft.), which is applied as in Fig. 90, is very much greater than the weight of the cable itself, we can safely make the assumption of uniform horizontal loading.

Fig. 91 shows a selection of one half of the cable as a free body, extending from the support at B to the center of the cable point O.

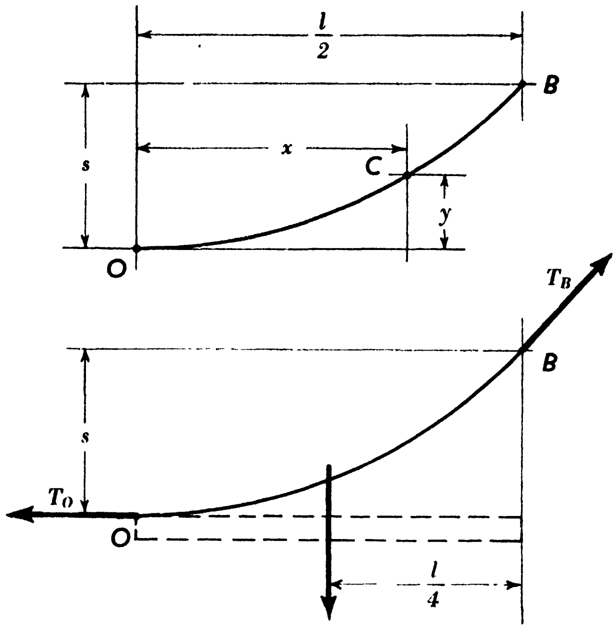


FIG. 91.

The distance s is called the *sag*, and s/l is the *sag ratio*. The load is w lb. per ft.

If T_o is the tension in the cable at O, we may evaluate T_o by taking moments about point B.

$$\Sigma M_B = T_o \times s - \frac{wl}{2} \times \frac{l}{4} = 0$$

Then $$T_o = \frac{wl^2}{8s}$$

If we now isolate a shorter portion of the cable as a free body, like that shown in Fig. 92, and take moments about point C,

$$\Sigma M_C = T_o \times y - \frac{wx^2}{2} = 0$$

Substituting the value of T_o obtained in the first equation, we have

$$\frac{wl^2}{8s} \times y - \frac{wx^2}{2} = 0$$

And
$$y = \frac{4x^2s}{l^2}$$

which is the equation of the curve assumed by the parabolic cable.

Now we can evaluate the tension at any point in the cable, such as C of Fig. 92.

$$T_C = \sqrt{T_o^2 + (wx)^2}$$

The angle which the tangent to the cable at any point makes with the horizontal will be

$$\theta_C = \tan^{-1}\frac{wx}{T_o}$$

The maximum tension in the cable will occur when the greatest value

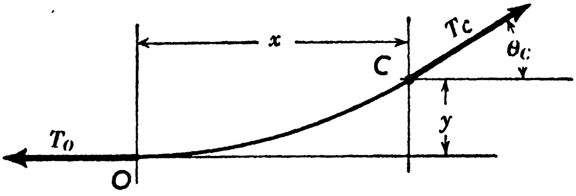


FIG. 92.

of x is substituted in the general equation for tension. If the cable is symmetrical, this will be the tension at the supports when $x = \frac{l}{2}$

$$T_{max} = T_B = w\sqrt{\frac{l^4}{64s^2} + \frac{l^2}{4}}$$
$$= \frac{wl}{2}\sqrt{\frac{l^2}{16s^2} + 1}$$

The angle θ at which the cable is inclined when it reaches the support is

$$\theta_B = \tan^{-1}\frac{8s\frac{l}{2}}{l^2} = \frac{4s}{l}$$

When the assumption of uniformly distributed horizontal loading upon which this derivation was based cannot be made safely, as when the sag ratio is very great or when we consider the problem of sag of a heavy cable, or one loaded with a heavy coating of ice, the curve of the cable is not a parabolic arc, but a catenary.

The properties of the catenary offer a favorite basis for problems in the calculus to illustrate the use of hyperbolic functions. The solutions are laborious, and recourse must generally be had to tables and diagrams to facilitate the mathematical computations.

Since we are here concerned with the fundamentals of engineering mechanics, there is no intrinsic gain in pursuing the problem of the catenary. Such discussion will be found in treatises on suspended cables.

Although many common engineering applications may be reduced to the parabolic cable, our attention is called to the limitations of the derivation of equations. Awareness of these limitations will prevent abuse of such formulas, as when they are applied to a cable that is itself the entire load and the sag ratio is large, or when a heavy coating of ice on a transmission line precludes the assumption of uniformly distributed horizontal loading.

PROBLEMS

181. A cable is suspended from two points on the same level. The points are 40 ft. apart. Determine the maximum and minimum tensions in the cable, if the sag is 6 ft., and the load is 30 lb. per ft. distributed uniformly horizontally.

182. At what angles does the cable of Problem 181 enter the supports?

183. The maximum allowable tension in a cable is 2000 lb. Determine the maximum intensity of horizontal loading which may be supported by the cable if the distance between the supports is 20 ft. and the cable is inclined at 12° with the horizontal as it enters the supports. *Ans.* 41.6 lbs. per ft.

184. In Problem 183 determine the sag ratio.

185. A cable of a suspension bridge carries a load of 1000 lb. per ft., uniformly distributed horizontally. The span of the bridge is 700 ft. and the sag is 40 ft. Determine the maximum tension in the cable.

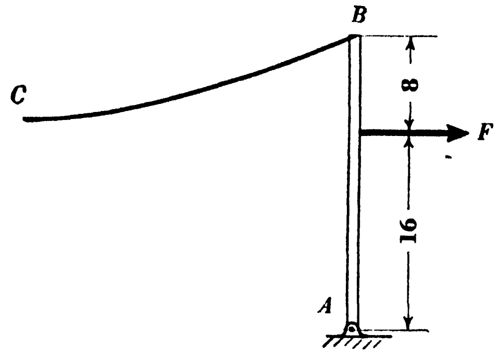
186. A wire weighing 0.08 lb. per ft. is to be strung between poles 135 ft. apart. The maximum allowable tension in the wire is 150 lb. Assuming that the weight of the wire may be assumed to be distributed uniformly horizontally, determine the minimum amount of sag which must be allowed. *Ans.* 1.21 ft.

187. If the sag of the wire in Problem 186 must be not less than 2 ft., determine at what distance apart the poles must be set.

188. A wire is stretched between three poles with the distance between the first two poles equal to 200 ft., and the sag equal to 3 ft., the distance between the second and third poles is 320 ft., and the sag is 4.5 ft.

Determine the magnitude and inclination of the resultant pull on the central pole if the wire weighs 0.6 lb. per ft. Assume the weight of the wire to be distributed uniformly horizontally.

189. A cable, weighing 2 lb. per horizontal foot is pinned to a vertical mast at point *B*.

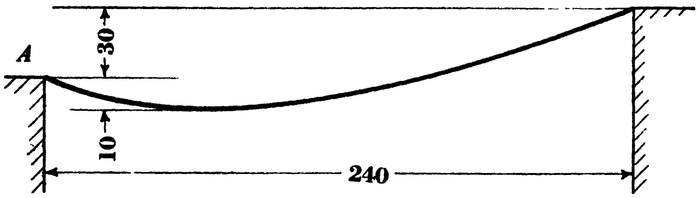


PROB. 189

The cable is horizontal at point C, which is at a horizontal distance of 200 ft. from A. The mast AB is pinned to the ground at A. A horizontal force F must be applied as shown to hold the mast in equilibrium. Determine the maximum tension in the cable, and the magnitude of force F.

190. A flexible cable is suspended from abutments at A and C. The cable weighs $\frac{1}{4}$ lb. per horizontal foot. Determine the maximum tension in the cable.

Ans. $T_{max.} \doteq 89.4$ lb.



PROB. 190

36. Force Distributed Over Area. In the case shown in Fig. 93, the force is applied to an area which has sufficient extent in both of its dimensions so that the assumption of force system in a single plane is invalid.

Now, we have an intensity of force p, defined as force per unit area. The total force acting on an area dA is then $p\,dA$; and a specimen $p\,dA$ is shown acting on an element of area dA located at distance x from the Z axis, and z from the X axis. Such forces form a parallel force system in space, and we employ the method discussed in Article 30, which applies to all parallel force systems in space.

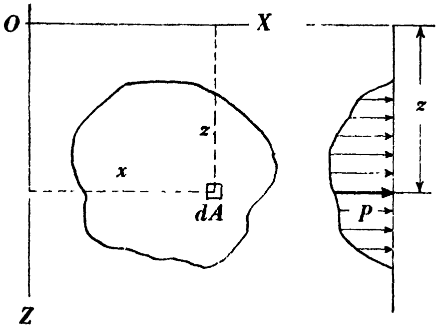


FIG. 93.

$$R = \Sigma F = \int p\,dA$$

and is so located that

$$x_R = \frac{\Sigma M_Z}{R} = \frac{\int px\,dA}{\int p\,dA}$$

and $$z_R = \frac{\Sigma M}{R} = \frac{\int pz\,dA}{\int p\,dA}$$

This type of distributed force is the kind most commonly encountered, and applications of it will be discussed in the following articles.

37. Center of Gravity. Centroid. A plate lying in an XOZ plane is shown in Fig. 94. The plate is of uniform thickness, and its density (weight per unit volume) is the same throughout the material. Then, we may evaluate the resultant of the force of weight and determine its point of application by treating the elements of weight of each portion of the plate as a parallel force system in space. This is a direct application of the method discussed in the preceding article.

The resultant weight,

$$R = \int p\, dA$$

Here, p is the intensity of force per unit area and is, therefore, δy, where δ is the density and y, the height of the plate.

Then
$$R = \int \delta y\, dA$$

But δ and y have been given as constants, and R is, therefore, $\delta y A$, the total weight of the plate.

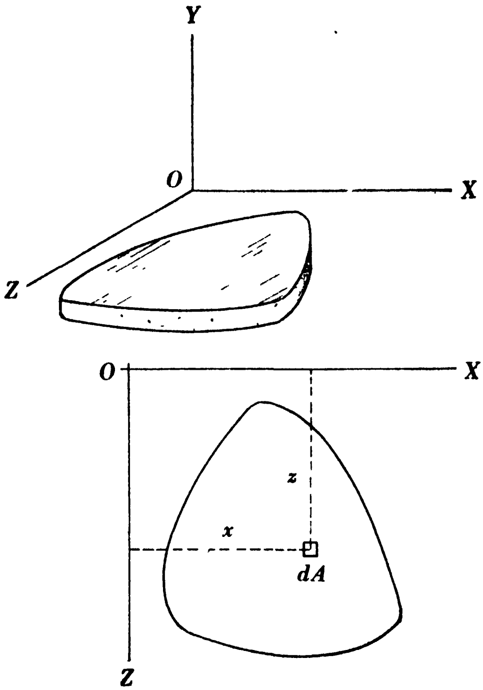


FIG. 94.

The location of the point of application is found, in this parallel force system, by establishing two coordinate distances from the basic axes of reference

$$x_R = \frac{\Sigma M_Z}{R} = \frac{\int \delta y\, dA\, z}{\int \delta y\, dA} = \frac{\int z\, dA}{\int dA}$$

and

$$z_R = \frac{\Sigma M_X}{R} = \frac{\int \delta y\, dA\, x}{\int \delta y\, dA} = \frac{\int x\, dA}{\int dA}$$

This point of application of the resultant weight is called the *center of gravity.*

It will be noted that the factors of density and height have vanished from our expressions. Then, they play no part in determining the location of the center of gravity of a plate of uniform height and material.

We might have started our investigation with a plane area, which could have been considered an example of a very thin plate of uniform height and density equal to zero, and we would have arrived at the same conclusion as to its center of gravity. Although such a point would have no significance as a point of application of a resultant weight, its counterpart for the weightless plane area is of great mathematical significance, and may be used as we proceed with the necessary mathematical stages of our analyses.

Integrals of the form contained in the numerators $\int x\,dA$ and $\int z\,dA$ are given the name *first* or *statical moments* of areas. The use of the term moment is justified by the analogy of their form to that of the moment of force, because such terms are products of area multiplied by perpendicular distance to axis.

The point of application of the resultant weight in the case of the material plate has been defined as its center of gravity. The analogous point in the case of the plane area is called its *centroid*.

The coordinates of the centroid of a plane area in the XOZ plane are, then

$$\bar{x} = \frac{\int x\,dA}{\int dA}$$

and

$$\bar{z} = \frac{\int z\,dA}{\int dA}$$

or, any coordinate of a centroid is determined by dividing the first moment of an area about an axis by the total area.

Since first moments are of the form $\int x\,dA$, they may be positive or negative depending upon the location of the area on one side or the other of an origin, and they may be zero. The first moment of an area about an axis through its centroid is

$$\int x\,dA = \bar{x}A$$

Here, $\bar{x}$ equals zero, because $\bar{x}$ is always the distance from axis of reference to centroidal axis. Then, the *first moment of an area about any of its centroidal axes is zero.*

38. Determination of Centroid of Common Areas. Engineering students encounter such mathematical properties of areas as $\int x\,dA$, $\int x^2\,dA$, or $\int xy\,dA$ in the class in the calculus. There, these properties of first moment, moment of inertia, and product of inertia appear as applications for exercise in integration. The use of formulas by the engineer without appreciative understanding of their background is condemned elsewhere in this book. The following derivations are included both for a review of the necessary calculus and to insure that proper limitations be imposed when the formulas are applied. The engineer's technique in calculations involving first moment and centroid is illustrated in Article 40.

Example 1

Locate the centroid of the triangle shown in Fig. 95.

$$dA = y\,dx = \frac{x}{b}\,h\,dx$$

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\frac{h}{b}\int_0^b x^2\,dx}{\frac{h}{b}\int_0^b x\,dx} = \frac{2}{3}b$$

Similarly

$$\bar{y} = \tfrac{1}{3}h$$

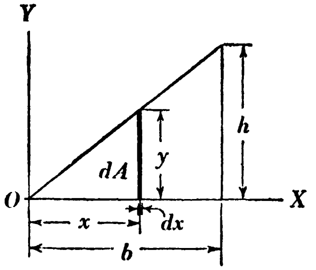


Fig. 95.

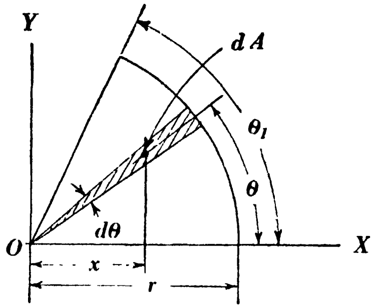


Fig. 96.

Example 2

Locate the centroid of the sector of a circle shown in Fig. 96.

We may proceed by selecting the elementary area shown in the figure, and using the results of Example 1.

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^{\theta_1} \frac{1}{2}r\,r\,d\theta\,\frac{2}{3}r\cos\theta}{\int_0^{\theta_1}\frac{1}{2}r\,r\,d\theta}$$

$$= \frac{2}{3}\,\frac{r\sin\theta_1}{\theta_1}$$

$$\bar{y} = \frac{\int y\,dA}{\int y\,dA} = \frac{\int_0^{\theta_1} \frac{1}{2}r\,r\,d\theta\,\frac{2}{3}r\sin\theta}{\int_0^{\theta_1}\frac{1}{2}r\,r\,d\theta}$$

$$= \frac{2}{3}\,\frac{r(1-\cos\theta_1)}{\theta_1}$$

For a quarter circle, $\theta_1 = 90°$. Then

$$\bar{x} = \frac{2}{3}\,\frac{r\sin\theta_1}{\theta_1} = \frac{4r}{3\pi}$$

and

$$\bar{y} = \frac{2r}{3}\,\frac{(1-\cos\theta_1)}{\theta_1} = \frac{4r}{3\pi}$$

Example 3

Locate the centroid of the area of the parabolic segment shown in Fig. 97, bounded by the axis of the parabola OX and the ordinate to

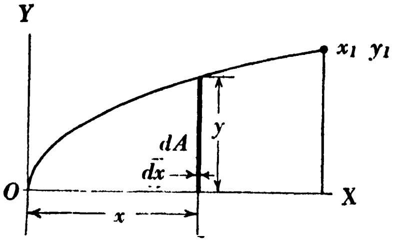


Fig. 97.

point x_1y_1. $y = \sqrt{cx}$ is the equation of the parabola.

$$dA = y\,dx = c^{\frac{1}{2}}x^{\frac{1}{2}}\,dx$$

$$= \frac{\int x\,dA}{\int dA} = \frac{\int_0^{x_1} xc^{\frac{1}{2}}x^{\frac{1}{2}}\,dx}{\int_0^{x_1} c^{\frac{1}{2}}x^{\frac{1}{2}}\,dx} = \frac{\int_0^{x_1} x^{\frac{3}{2}}\,dx}{\int_0^{x_1} x^{\frac{1}{2}}\,dx}$$

$$= \frac{3}{5}x_1$$

$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int_0^{x_1} \frac{c^{\frac{1}{2}}x^{\frac{1}{2}}}{2}\,c^{\frac{1}{2}}x^{\frac{1}{2}}\,dx}{\int_0^{x_1} c^{\frac{1}{2}}x^{\frac{1}{2}}\,dx} = \frac{c^{\frac{1}{2}}}{2}\,\frac{\int_0^{x_1} x\,dx}{\int_0^{x_1} x^{\frac{1}{2}}\,dx}$$

$$= \frac{3}{8}c^{\frac{1}{2}}x_1^{\frac{1}{2}}$$

$$= \frac{3}{8}y_1$$

Example 4

Locate the centroid of the parallelogram shown in Fig. 98.

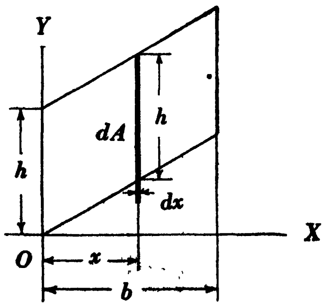


Fig. 98.

$$dA = h\,dx$$

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^b hx\,dx}{\int_0^b h\,dx} = \frac{b}{2}$$

The rectangle of base b and height h will similarly have its centroid at $\bar{x} = \frac{b}{2}$; $\bar{y} = \frac{h}{2}$.

EXAMPLE 5

Locate the centroid of the semicircular arc shown in Fig. 99.

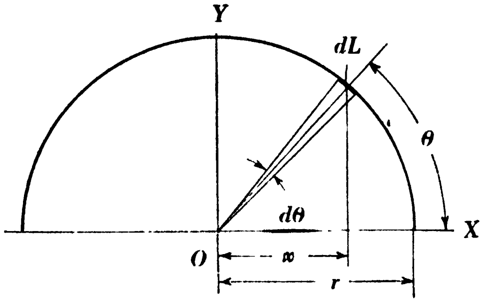


FIG. 99.

$$dL = r\,d\theta$$

$$\bar{x} = \frac{\int x\,dL}{\int dL} = \frac{\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} r\cos\theta r\,d\theta}{\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} r\,d\theta} = \frac{2r}{\pi}$$

$$\bar{y} = 0$$

39. Theorems of Pappus. The mathematical property of first moment may be conveniently employed to determine surfaces and volumes of revolution.

Theorem I. If a plane curve be revolved about any axis in its plane which does not intersect the curve, a surface of revolution will be generated which will be equal to 2π times the first moment of the plane curve about the axis.

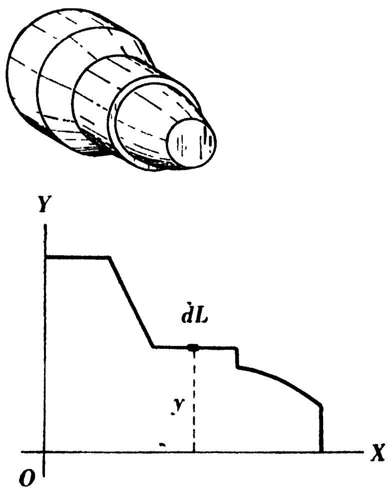


FIG. 100.

For example, in Fig. 100 the area generated by revolving dL about axis OX is $2\pi y\,dL$. Then, the total area generated by the entire curve is the sum of all such elementary areas, or

$$A = \int 2\pi y\,dL$$
$$= 2\pi \int y\,dL = 2\pi \bar{y} L$$

in which $\bar{y}L$ is the first moment of the plane curve about axis OX.

Theorem II. The solid, or volume, of revolution which will be generated by revolving a plane area about any axis lying in its plane which does not intersect the area is equal to 2π times the first moment of the plane area about the axis.

In Fig. 101, the volume generated by revolving dA about axis OX is $2\pi y\,dA$. Then the total volume generated by revolving the entire area is the sum of all of the elementary volumes,

or
$$V = \int 2\pi y\,dA$$
$$= 2\pi \int y\,dA = 2\pi \bar{y} A$$

in which $\bar{y}A$ is the first moment of the given area about the axis OX.

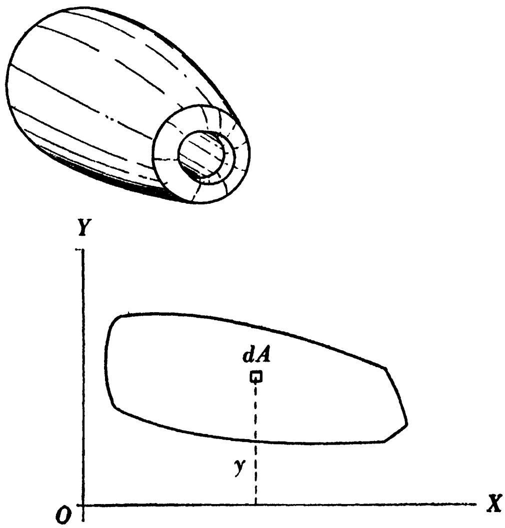


FIG. 101.

40. Determination of First Moment and Centroid of Areas by Division into Finite Parts. It is a characteristic of most of the areas encountered in engineering practice that they are composed of triangles, rectangles, and circles, or parts of those geometrical shapes. For example, the cross section of a beam may be a rectangle in the case of a wooden beam, a circle when the beam appears in the role of a shaft, or, as in the case of the common structural shapes, the cross-sectional area may be divided into rectangles and triangles, with the corner fillets discernible as parts of circles.

Since these shapes are so frequently encountered, it is efficient in the calculations of a problem to treat them by a technique of division into common geometrical shapes in determining such properties as the first moment or centroid.

In the preceding article, a first moment of an area about an X axis has been defined as $\int y\,dA$. We also noted that

$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int y\,dA}{A};\ \bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int x\,dA}{A}$$

Then
$$\int y\,dA = \bar{y}A;\ \int x\,dA = \bar{x}A$$

Or the first moment of any area about an axis is equal to the product of that area multiplied by the distance from the axis to the centroid of the area.

If we have a group of areas, the first moment of the group will be the sum of the first moments of the individual parts.

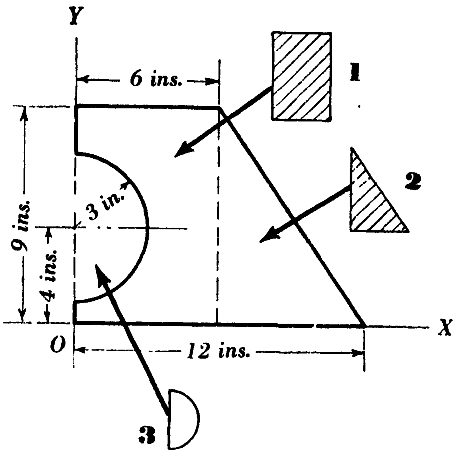


FIG. 102.

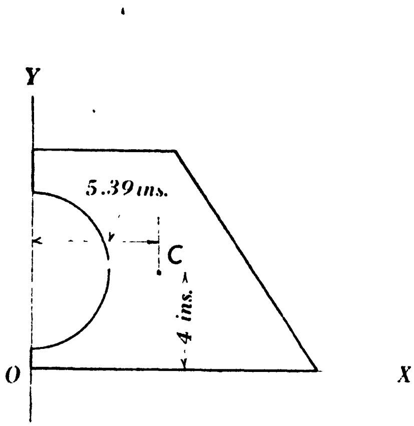


FIG. 103.

The area of Fig. 102 is composed of the sum of rectangle 1, triangle 2, minus semicircle 3. To locate the centroid of the entire area, we proceed as follows:

$$\bar{y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{6 \times 9 \times 4.5 + \frac{1}{2} \times 6 \times 9 \times 3 - \frac{\pi \times (3)^2}{2} \times 4}{6 \times 9 + \frac{1}{2} \times 6 \times 9 - \frac{\pi \times (3)^2}{2}}$$

$$= 4 \text{ in.}$$

$$\bar{x} = \frac{\Sigma \bar{x} A}{\Sigma A} = \frac{6 \times 9 \times 3 + \frac{1}{2} \times 6 \times 9 \times 8 - \frac{\pi \times (3)^2}{2} \times \frac{4 \times 3}{3 \times \pi}}{6 \times 9 + \frac{1}{2} \times 6 \times 9 - \frac{\pi \times (3)^2}{2}}$$

$$= 5.39 \text{ in.}$$

The location of the centroid C should be plotted by giving the coordinates as dimensions, as illustrated in Fig. 103.

PROBLEMS

191. Locate the centroid of the parabolic segment bounded by the X axis, the curve $y = 2x^3$, and the line $x = a$. *Ans.* $\bar{x} = \frac{4}{5}a$; $\bar{y} = \frac{4}{7}a^3$.

192. Locate the centroid of the area bounded by the curve $y = bx$, and lines $x = 2$ and $y = 1$.

193. Determine the first moments of the area given in Problem 191 about the X and Y axes.

194. If the area given in Problem 192 is rotated about the X axis, determine the volume generated in a complete revolution.

195. If the curve given in Problem 192 is revolved through an angle of 270° about the X axis, determine the generated surface area.

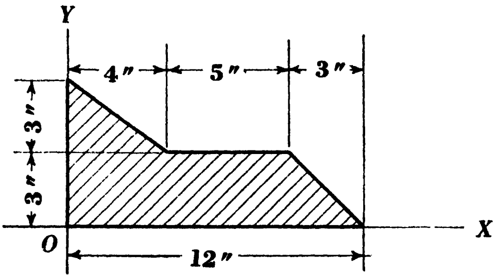
196. Locate the centroid of the area bounded by the curve $y^2 = 4x + 9$, the X axis, and the line $x = 3$.

197. If the area given in Problem 196 is rotated about the X axis, determine the volume generated in one revolution.

198. Locate the centroid of the area included between the curve $y^2 = 3x$ and the line $y = x$.

199. Determine the first moments of the area given in Problem 199 about the X and Y axes.

200. Determine the first moment of the shaded area shown about the X axis

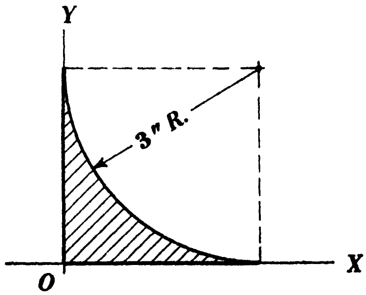


PROB. 200

201. Locate the centroid of the area given in Problem 200.

202. If the area given in Problem 200 is rotated about the X axis, determine the volume generated in one revolution.

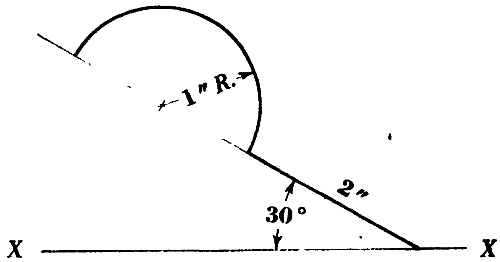
203. Determine the first moment of the shaded area about the X axis.



PROB. 203

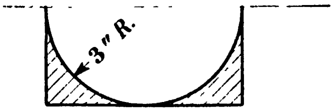
204. Locate the centroid of the area given in Problem 203.

205. The line shown is revolved about axis XX. Determine the surface rea generated in a complete revolution.



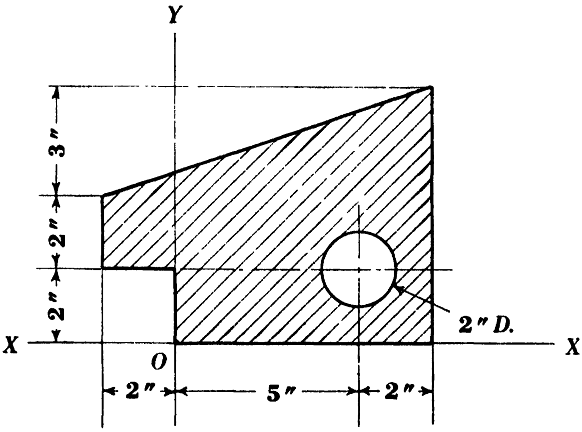
PROB. 205

206. Locate the centroid of the shaded area.



PROB. 206

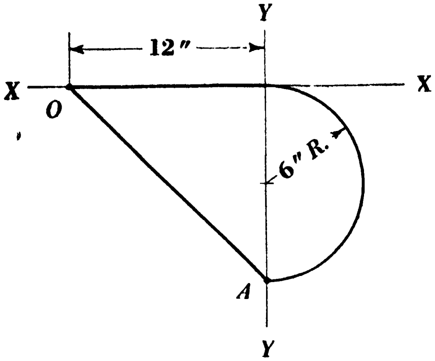
207. Determine the first moment of the shaded area about axis OY.
Ans. $\bar{x} = 3.13$ ins.; $\bar{y} = 3.05$ ins.



PROB. 207

208. Locate the centroid of the area given in Problem 207.

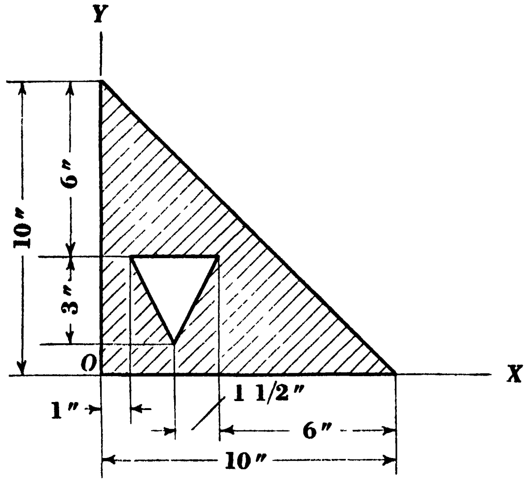
209. Locate the centroid of the area shown.
Ans. $\bar{x} = -1.1$ ins.; $\bar{y} = -4.87$ ins.



Prob. 209

210. Determine the first moment of the area given in Problem 209 about the Y axis.

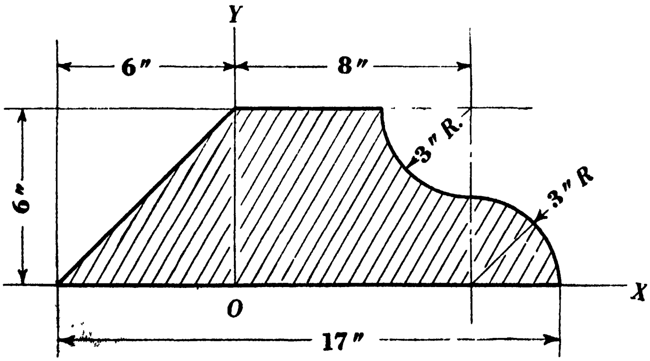
211. Locate the centroid of the shaded area shown.



Prob. 211

212. Determine the first moment of the shaded area shown about the Y axis.

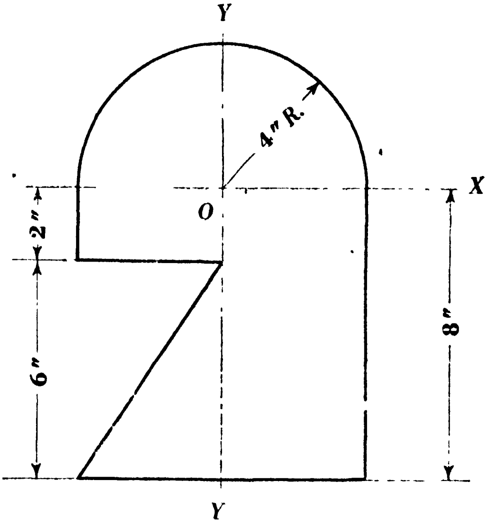
Ans. $\bar{x} = 2.64$ ins.; $\bar{y} = 2.36$ ins.



Prob. 212

213. Locate the centroid of the area given in Problem 212.

214. Determine the first moment of the area shown about XX.



PROB. 214

215. Locate the centroid of the area given in Problem 214.

41. Second Moment of Areas. Other mathematical properties of areas exist which, like the centroid and the first or statical moment, play important parts in the analysis of engineering problems.

The *second moment*, or *moment of inertia*, is such a mathematical property. It is defined as the product of an elementary area, such as dA of Fig. 104 and the square of the distance from that area to a given axis. The second moment of dA about the axis OX is $y^2\,dA$, and its second moment about OY is $x^2\,dA$. The moment of inertia of the entire area about an axis is the sum of the second moments of the elementary areas comprising the total area. For example, the moment of inertia of the area shown in the figure about OX is $\int y^2\,dA$ and about OY is $\int x^2\,dA$. The letter I is usually employed as the symbol for moment of inertia, and subscripts employed to announce the axis of reference—for example, I_{OX} is moment of inertia about axis OX.

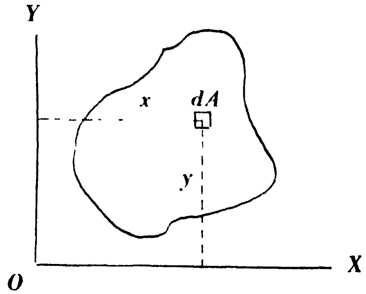


FIG. 104.

We should note that since a moment of inertia of an area is, by definition, a product of the square of a linear distance multiplied by an area, the units will be the fourth power of linear distance—that is, inches^4 or feet^4.

Since moments of inertia are of the form $\int x^2\,dA$, all moments of inertia are positive, because whether coordinate distances, such as x, are positive or negative, their squares must be positive.

The moment of inertia of an area about an axis perpendicular to the plane of the area is called the *polar moment* of inertia. In Fig. 105 an elementary area dA, lying in the plane XOY has $I_{OX} = y^2\,dA$ and $I_{OY} = x^2\,dA$. The moment of inertia about axis OZ is, by definition, $I_{OZ} = r^2\,dA$. By the Pythagorean theorem, $r^2 = x^2 + y^2$. Then, $r^2\,dA = x^2\,dA + y^2\,dA$. A polar moment of inertia of an area is, therefore, equal to the sum of the moments of inertia of the area about two mutually perpendicular axes—that is,

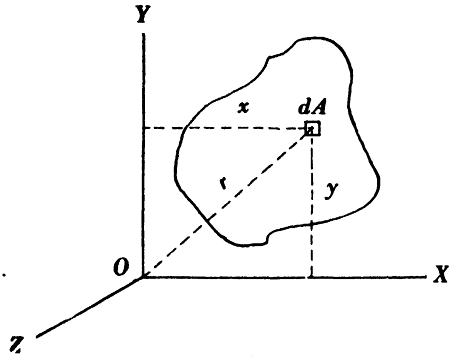


FIG. 105.

$$I_{OZ} = I_{OX} + I_{OY}$$

A mathematical property used in some engineering analyses, particularly those dealing with the design of columns, is the radius of gyration of an area relative to an axis, which is the square root of the quotient obtained by dividing the moment of inertia of the area about that axis by the area itself, or

$$\rho = \sqrt{\frac{I}{A}}$$

To establish the units of radii of gyration, we note that

$$\sqrt{\frac{(\text{distance})^4}{(\text{distance})^2}} = \text{distance}^1$$

and the units are inches or feet.

The axes about which we desire to establish the moment of inertia of areas rarely occur where expressions for their magnitude are known. (See Article 42 for the derivation of the most common moments of inertia.)

We could, of course, derive by integration a new expression for each new axis we encounter.

It is much less laborious to have available a "transfer," or transformation, technique which will enable us to establish moments of inertia about any axes when we have been equipped with one basic expression. The calculus is employed to establish a basic expression, usually for moment of inertia about a centroidal axis. The transformation technique is then used to determine moment of inertia about any other axis.

For example, we are given the moment of inertia of the area shown in Fig. 106 about an axis $\overline{XX}$ through its centroid C.

We wish to obtain the moment of inertia about a parallel axis XX.

$$\begin{aligned} I_{xx} &= \int y^2\, dA \\ &= (y_1 + \bar{y})^2\, dA \\ &= \int y_1^2\, dA + 2\int y_1 \bar{y}\, dA + \int \bar{y}^2\, dA \end{aligned}$$

The term y is a constant—it is the fixed distance between the parallel axes.

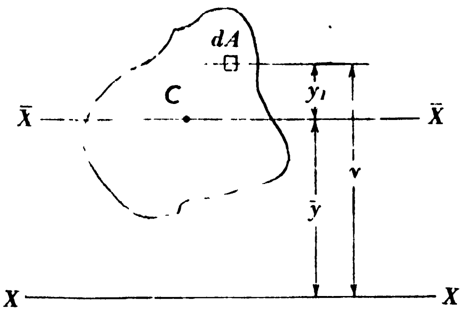


FIG. 106.

The second term is, then, $2y_1\int \bar{y}\, dA$. But $\int \bar{y}\, dA$ is the first moment of the total area about its centroidal axis and is, therefore, equal to zero.

The $\int y_1^2\, dA$ is the moment of inertia of the total area about its centroidal axis.

Then, *the moment of inertia of any area about any axis is equal to the moment of inertia of the area about a parallel axis through its centroid plus the product of the square of the distance between the two parallel axes multiplied by the area,* or

$$I_{XX} = I_{\overline{XX}} + \bar{y}^2 A$$

This is a formula and may be used, but not abused by failing to recognize its limitations. The formula can accomplish transformation of moments of inertia of plane areas only between *parallel* axes, and one of the two parallel axes *must pass through the centroid.*

42. Derivation of Moments of Inertia of Common Areas.

1. *Rectangle About Axis Through Its Centroid* (Fig. 107).

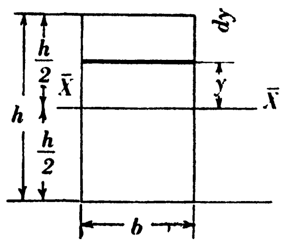


FIG. 107.

$$dA = b\, dy$$

$$I_{\overline{XX}} = \int y^2\, dA = \int_{-\frac{h}{2}}^{+\frac{h}{2}} y^2 b\, dy = \frac{bh^3}{12}$$

By using the transformation expression, we obtain the moment of inertia of the rectangle about its base

$$I_{XX} = \frac{bh^3}{12} + \left[\frac{(h)}{(2)}\right]^2 bh = \frac{bh^3}{3}$$

2. *Triangle About Axis Through Its Centroid* (Fig. 108).

$$dA = \left(\frac{2}{3}b - \frac{by}{h}\right) dy$$

$$I_{\overline{XX}} = \int y^2\, dA = \int_{-\frac{h}{3}}^{+\frac{2}{3}h} y^2 \left(\frac{2}{3}b - by\right) dy = \frac{bh^3}{36}$$

$$I_{\overline{XX}} = \frac{bh^3}{36} + \left(\frac{h}{3}\right)^2 \frac{1}{2} bh = \frac{bh^3}{12}$$

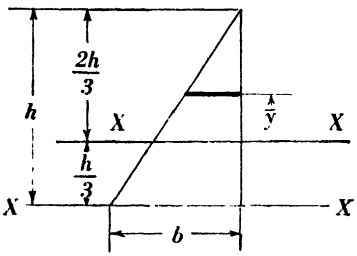


FIG. 108.

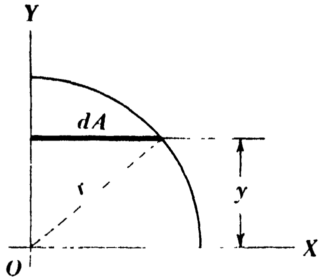


FIG. 109.

3. *Quadrant of a Circle About Diameter* (Fig. 109).

$$dA = \sqrt{r^2 - y^2}\, dy$$

$$\int y^2\, dA = \int_0^r y^2 \sqrt{r^2 - y^2}\, dy = \frac{\pi r^4}{16}$$

The transformation expression may be used to establish the moment of inertia about the centroidal axis (Fig. 110*a*) for the quadrant.

$$I_{\overline{XX}} = \frac{\pi r^4}{16} - \left(\frac{4r}{3\pi}\right)^2 \frac{\pi r^2}{4} = \frac{r^4(9\pi^2 - 64)}{144\pi}$$

A semicircle about its base diameter (Fig. 110*b*) would have $I_{XX} = \frac{\pi r^4}{8}$, and the complete circle $I_{XX} = \frac{\pi r^4}{4}$ (Fig. 110*c*).

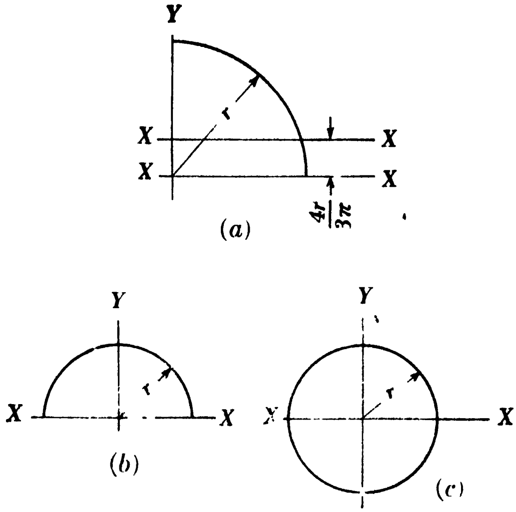


FIG. 110.

43. Moments of Inertia of Areas by Division into Finite Parts. We noted, in the discussion of first moments and centroids, that the engineer encounters most frequently cross-sectional areas of machine and structural parts which are composed of rectangles, triangles, and circles. The same technique employed there of division of the composite area into the common geometrical areas may be employed in determining the moment of inertia of the composite area.

Moments of inertia are always positive, and the total moment of inertia of any area about a given axis will be the sum of the moments of inertia of the component areas about that axis.

ILLUSTRATIVE EXAMPLE

The area shown in Fig. 111 is composed of a triangle 1, a rectangle 2, and a semicircle 3.

The moment of inertia of the composite area about axis XX is the sum of the moments of inertia about XX, of areas 1, 2, and 3.

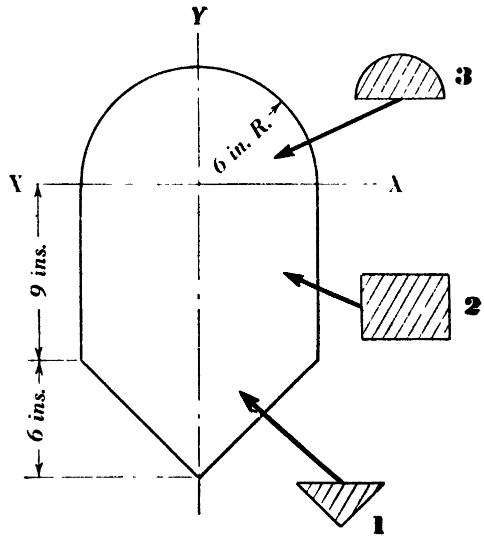


FIG. 111.

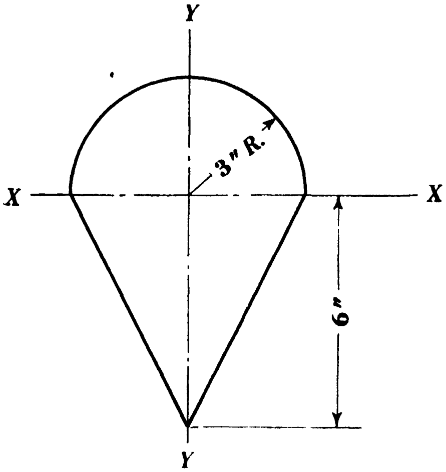
$$\int y^2\,dA = \Sigma(I_{XX}(\text{area 1}) + I_{XX}(\text{area 2}) + I_{XX}(\text{area 3})) =$$

$$I_{XX} = \frac{12 \times (6)^3}{36} + (11)^2 \times \frac{1}{2} \times 12 \times 6 + \frac{12 \times (9)^3}{3} + \frac{\pi \times (6)^4}{8}$$

$$= 7853 \text{ in.}^4$$

PROBLEMS

216. Determine the moment of inertia of the area shown about the X axis.

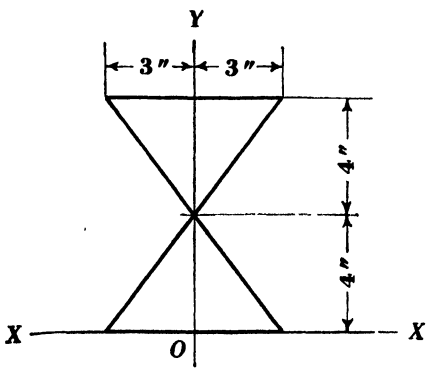


PROB. 216

217. Determine the moment of inertia of the area given in Problem 216 about the centroidal axis of the area parallel to XX.

218. For the area given in Problem 216, determine the radius of gyration relative to axis YY.

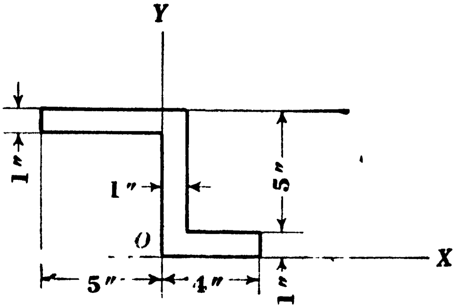
219. Determine the moment of inertia of the area shown about XX.



PROB. 219

220. For the area given in Problem 219, determine the polar moment of inertia about an axis perpendicular to the area at O.

221. Determine the polar moment of inertia of the area about an axis perpendicular to the area at O.



Prob. 221

222. For the area given in Problem 221, determine the radius of gyration relative to axis OY.

223. For the area given in Problem 222, determine the moments of inertia about centroidal axes of the area parallel to OX and OY.

Ans. $I_{\bar{X}} = 170$ ins.[4]; $I_{\bar{Y}} = 959$ ins.[4]

224. For the shaded area given in Problem 211, determine the moment of inertia about the X axis. *Ans.* $I_X = 153$ ins.[4]

225. For the shaded area given in Problem 207, determine the following: (a) I_X and I_Y, (b) Polar moment of inertia at O, and (c) Radii of gyration about OX and OY.

226. For the area given in Problem 214, determine the following: (a) I_X and I_Y, (b) Polar moment of inertia at O, and (c) Radii of gyration about XX and YY.

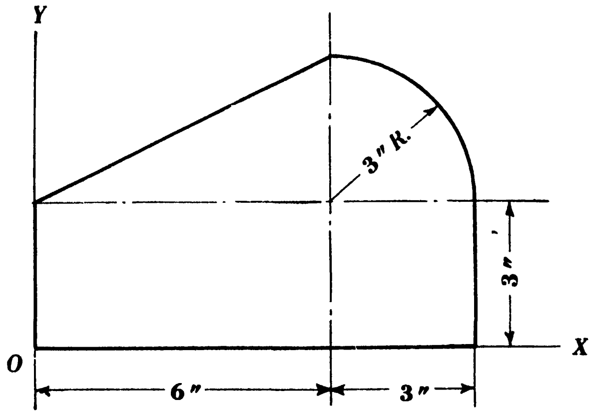
227. For the area given in Problem 212, determine the polar moment of inertia about an axis perpendicular to the plane of the area at point O.

Ans. $I_P = 1957$ ins.[4]

228. For the area given in Problem 209, determine the polar moment of inertia about an axis perpendicular to the plane of the area at point A.

229. Determine the moment of inertia of the area about axis OX.

Ans. $I_X = 334$ ins.[4]



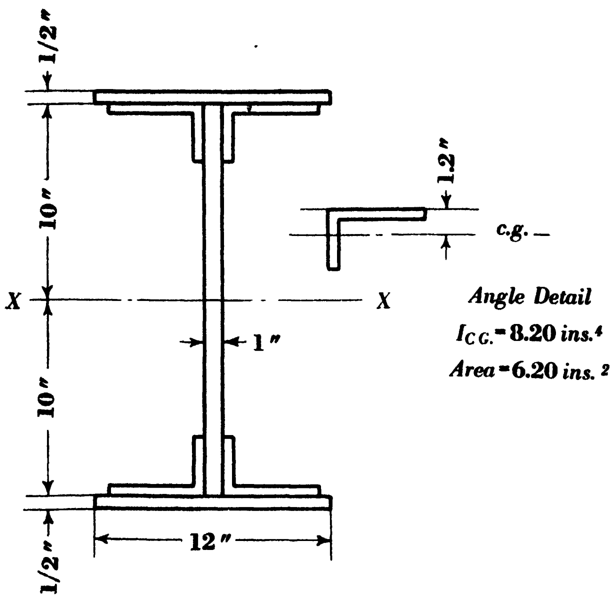
Prob. 229

230. Determine the moment of inertia of the area given in Problem 229 about axis OY.

231. Determine the polar moment of inertia of the area giveñ in Problem 229 about an axis perpendicular to the plane of the area at its centroid.

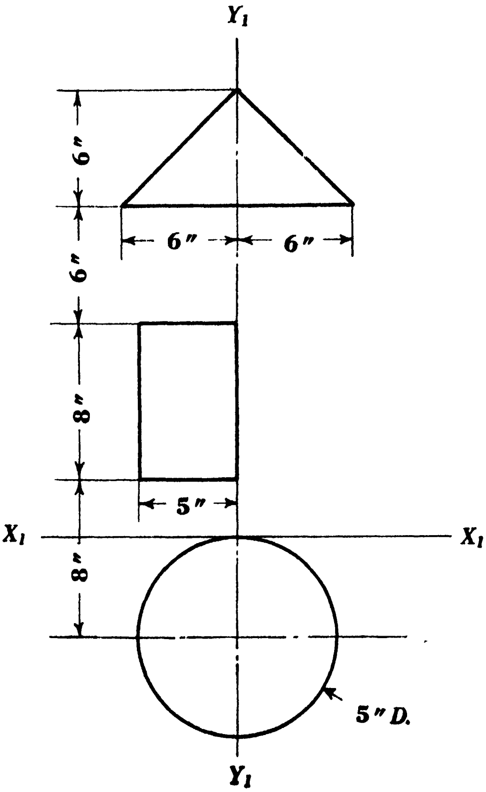
232. The area shown in the cross section of a built-up beam, which consists of 12 by $\frac{1}{2}$ in. flange plates, a 20 by 1 in. web plate, and 4 angles.

Determine the moment of inertia of the cross-sectional area of the beam about axis XX located at the centroid.



PROB. 232

233. Determine the moments of inertia for the system of areas shown about X and Y axes through the centroid of the system.



PROB. 233

234. Determine the moment of inertia with respect to the X axis of the parabolic segment bounded by the curve $y^2 = 4x$ and the line $x = 9$ in.

Ans. $I_X = 518$ ins.^{4}

235. Determine the moment of inertia of the parabolic segment given in Problem 234 about the line $x = 5$ in.

44. Product of Inertia. Areas possess one additional mathematical property of value in engineering calculations.

The elementary area dA of Fig. 112 is located at distance y from the OX axis, and a distance x from the OY axis. The product of the area multiplied by the coordinate distances is, then, $xy\,dA$, and this product is called the *product of inertia*. The product of inertia is, then, a mathematical property that is dependent upon three factors: the area itself and its location relative to two mutually perpendicular axes. The product of inertia of the entire area will be $\int xy\,dA$.

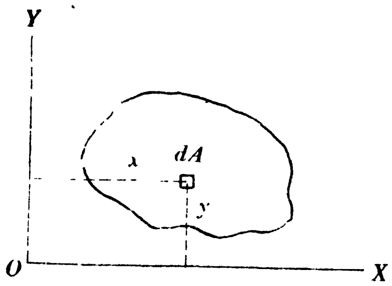


FIG. 112.

The letter I_{XY} is used as a symbol for product of inertia, with the subscripts serving to announce the pair of axes of reference.

The units of products of inertia are inches^4 or feet^4, because, like moments of inertia, they are products of two linear distances (which are the coordinates) and the two linear dimensions involved in the magnitude of the area. Unlike moments of inertia, however, products of inertia involve only the first powers of the coordinate distances, and such products may be positive, negative, or equal to zero.

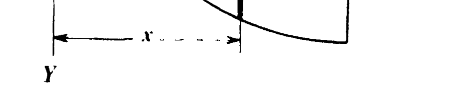


FIG. 113.

When an area, such as that of Fig. 113, has as one axis of reference, an axis of symmetry, such as XX, I_{XY} about this axis and any perpendicular axis, such as YY, will be equal to zero.

We observe that for the elementary area dA shown in the figure, the product $xy\,dA$ is a product of x times $y\,dA$. ($y\,dA$ is the first moment of dA about an axis through its centroid.) The first moment of any area about a centroidal axis is zero. Then $xy\,dA$ is zero.

We have found that the product of inertia of any area about a pair of axes, one of which is an axis of symmetry of the area, is equal to zero.

Products of inertia, like moments of inertia, must be transferred

between axes of known I_{XY} to other axes. A transformation expression may be derived.

In Fig. 114, axes $\overline{XX}$ and $\overline{YY}$ are centroidal, and XX and YY are axes, respectively parallel to the centroidal axes. Then,

$$\begin{aligned} I_{XY} &= \int xy\,dA = \int (x_1 + \bar{x})(y_1 + \bar{y})\,dA \\ &= \int x_1 y_1\,dA + \int x_1 \bar{y}\,dA + \int \bar{x} y_1\,dA + \int \bar{x}\bar{y}\,dA \\ &= \int x_1 y_1\,dA + \bar{y}\int x_1\,dA + \bar{x}\int y_1\,dA + \bar{x}\bar{y}\int dA \end{aligned}$$

The second and third terms each contain the first moments of the area about a centroidal axis, which is zero, and those terms vanish.

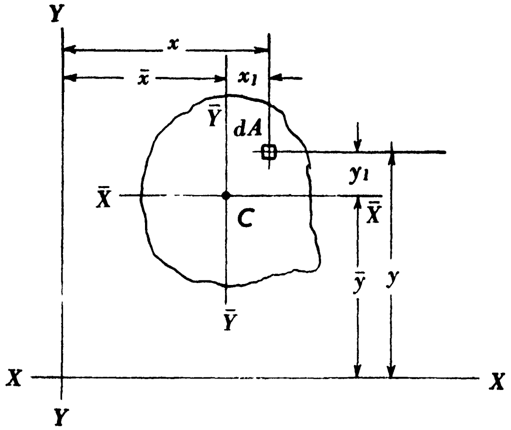


FIG. 114.

The first term is the product of inertia of the area about the centroidal axes, and the last term is the product of the transfer distances between the axes and the total area.

Then the *product of inertia of an area about any pair of mutually perpendicular axes is equal to the sum of the products of inertia of that area about a parallel mutually perpendicular pair through the centroid plus the product of the distances between the axes times the area,* or

$$I_{XY} = I_{\overline{XY}} + \bar{x}\bar{y}A$$

when $\bar{x}$ and $\bar{y}$ are the distances between the parallel pairs of axes.

45. Determination of Products of Inertia of Common Areas.

1. *Rectangle About Axes XX and YY* (Fig. 115).

$$I_{XY} = \int_0^b \int_0^h xy\,dx\,dy = \frac{b^2h^2}{4}$$

Another method of deriving the product of inertia in this case would be the use of the transformation expression, which is always most efficient when one or both of the centroidal axes of the given area are axes of symmetry (Fig. 116).

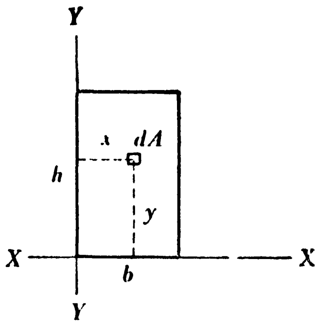


FIG. 115.

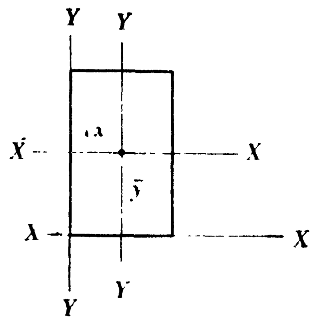


FIG. 116.

Then:

$$I_{XY} = I_{\bar{X}\bar{Y}} + \bar{x}\bar{y}A = 0 + \left(\frac{b}{2}\right)\left(\frac{h}{2}\right)bh = \frac{b^2h^2}{4}$$

2. *Triangle About Axes XX and YY* (Fig. 117).

$$y = h - \frac{x}{b}h$$

$$I_{XY} = \int_0^b \int_0^{h-\frac{x}{b}h} xy\,dx\,dy = \frac{b^2h^2}{24}$$

The product of inertia of this area relative to axes $\overline{XX}$ and $\overline{YY}$ (Fig. 118) may be found by the transformation expression. The origin

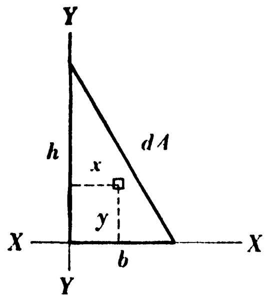


FIG. 117.

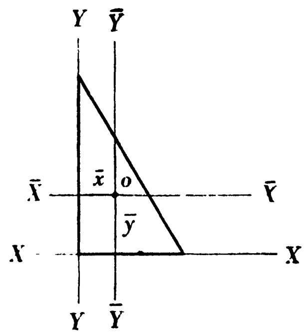


FIG. 118.

of these axes is point O at the centroid of the triangle. Abscissas to the right of O are considered positive, and abscissas to the left, negative.

Ordinates above O are positive, and ordinates below O, negative. Then,

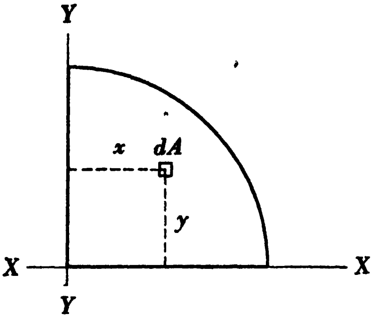


FIG. 119.

$$I_{\overline{XY}} = I_{XY} - \bar{x}\bar{y}A$$
$$= \frac{b^2h^2}{24} - \left(-\frac{b}{3}\right)\left(-\frac{h}{3}\right)\left(\frac{bh}{2}\right)$$
$$= -\frac{b^2h^2}{72}$$

3. *Quadrant of a Circle About Axes XX and YY* (Fig. 119).

$$x^2 + y' = r^2$$

$$I_{XY} = \int_0^r \int_0^{\sqrt{r^2+y^2}} xy\,dx\,dy = \frac{r^4}{8}$$

PROBLEMS

236. Determine the product of inertia of the system of areas given in Problem 233 about the X_1 and Y_1 axes shown.

237. Determine the product of inertia of the shaded area of Problem 212 about the X and Y axes shown. *Ans.* $I_{XY} = 381$ ins.^{4}

238. Determine the product of inertia of the shaded area given in Problem 211 about the X and Y axes shown.

239. Determine the product of inertia of the shaded area given in Problem 200 about centroidal axes parallel to the X and Y axes shown. *Ans.* $I_{XY} = 256$ ins.^{4}

240. Determine the product of inertia of the area given in Problem 214 about centroidal axes parallel to the X and Y axes shown.

241. Determine the product of inertia of the shaded area given in Problem 229 about the X and Y axes shown.

242. Determine the product of inertia of the shaded area given in Problem 207 about the X and Y axes shown.

243. Determine the product of inertia of the shaded area given in Problem 207 about centroidal axes parallel to the X and Y axes shown.

46. Moments of Inertia about Inclined Axes. Mohr's Circle of Inertia. The elementary area dA of Fig. 120 is located at coordinate distances x and y from axes OY and OX, respectively.

Then, for the elementary area,

$$dI_X = y^2\,dA;\ dI_Y = x^2\,dA;\ \text{and}\ dI_{XY} = xy\,dA$$

If axes OX_1 and OY_1 are inclined at θ with the axes OX and OY, the coordinate distances from dA to these inclined axes will be x_1 and y_1.

$$y_1 = y\cos\theta - x\sin\theta (ae = ad - bc)$$

Then,

$$I_{X_1} = \int y_1^2\,dA = \int (y\cos\theta - x\sin\theta)^2\,dA$$
$$= \int y^2\cos^2\theta\,dA - \int 2xy\sin\theta\cos\theta\,dA + \int x^2\sin^2\theta\,dA$$
$$= I_X\cos^2\theta - 2I_{XY}\sin\theta\cos\theta + I_Y\sin^2\theta$$

This expression may be used to determine the moment of inertia of a given area about an axis, such as OX, inclined at angle θ from a basic axis, such as OX, when I_X, I_Y, and I_{XY} are known.

We observe from the form of the equation that there is sinusoidal variation in I_X, as the angle θ changes, and that I_X must have maximum and minimum values.

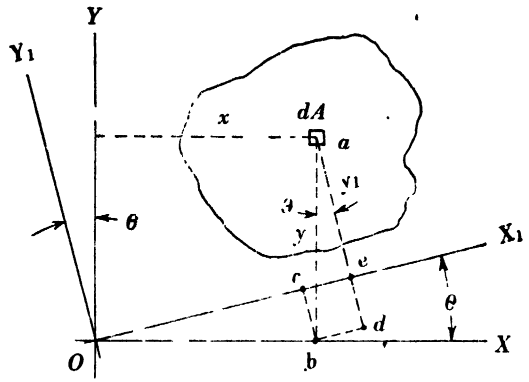


FIG. 120.

To determine such values, let us differentiate the basic equation with respect to θ and set the first derivative equal to zero.

$$\frac{dI_{x_1}}{d\theta} = -2I_X \sin\theta\cos\theta - 2I_{XY}(\cos^2\theta - \sin^2\theta) + 2I_Y \sin\theta\cos\theta$$

$$= 2(I_Y - I_X)\sin\theta\cos\theta - 2I_{XY}(\cos^2\theta - \sin^2\theta) = 0$$

Then

$$\frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \tan 2\theta = \frac{2I_{XY}}{I_Y - I_X}$$

There are two values of 2θ, differing by 180 degrees, having the same tangent. Then there are two values for θ, differing by 90 degrees. About one of the axes at angle θ from the original X axis, the value of the moment of inertia will be *maximum;* about another axis, inclined at 90 degrees with the first, the value of the moment of inertia will be *minimum.*

These two axes are called *principal axes of inertia,* and the moments of inertia about them (the maximum and minimum values of moment of inertia about axes through a given point) are called *principal moments of inertia.* The sum of the principal moments of inertia, like the sum of any two moments of inertia about mutually perpendicular axes, is the polar moment of inertia. (See Article 41.)

In Fig. 121, an elementary area dA is located at coordinate distances x and y from the axes OY and OX, respectively. Axes OX_1 and OY_1 are inclined at θ from the original axes OX and OY. The coordinate distances of dA from the inclined axes are x_1 and y_1:

$$x_1 = x \cos\theta + y \sin\theta (am + mn)$$

and
$$y_1 = y \cos\theta - x \sin\theta (ad - de)$$

The product of inertia about the axes OX_1 and OY_1 is

$$\begin{aligned} I_{X_1Y_1} &= \int x_1 y_1\, dA = \int (x \cos\theta + y \sin\theta)(y \cos\theta - x \sin\theta)\, dA \\ &= \int xy \cos^2\theta\, dA - \int x^2 \sin\theta \cos\theta\, dA + \int y^2 \sin\theta \cos\theta\, dA \\ &\qquad - \int xy \sin^2\theta\, dA \\ &= I_{XY} \cos^2\theta - I_Y \sin\theta \cos\theta + I_X \sin\theta \cos\theta - I_{XY} \sin^2\theta \\ &= I_{XY}(\cos^2\theta - \sin^2\theta) + (I_X - I_Y) \sin\theta \cos\theta \end{aligned}$$

But we have found that when we arrive at the principal axes (page 147)

$$I_{XY}(\cos^2\theta - \sin^2\theta) + (I_X - I_Y)(\sin\theta \cos\theta) = 0$$

Therefore the product of inertia of an area about a pair of axes which are principal axes of inertia is zero.

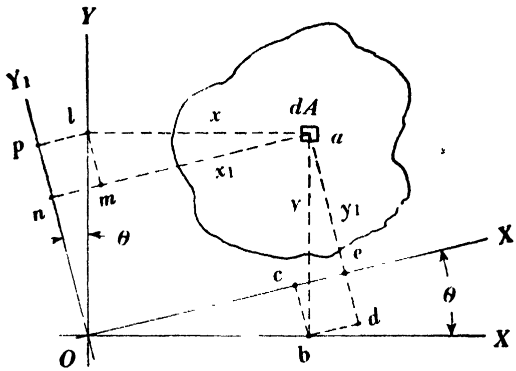


FIG. 121.

The product of inertia about a pair of axes, at least one of which is an axis of symmetry, is also equal to zero. Then *axes of symmetry are principal axes of inertia.*

Mohr's circle is a graphical method of solving equations of the form presented by the relationship between the moments and products of inertia of an area about all of the axes passing through any given point in the plane of the area.

As a very general mathematical tool, the Mohr's circle will reappear in other stages of the training of an engineer—as, for example, when we study the relationships of the stresses on the planes which may be passed through a given point in a body subjected to loading.

In Fig. 122, we have the point P, any point of a given area.

We have already discussed the methods of obtaining the moments of inertia I_X and I_Y as well as the product of inertia I_{XY}, which are moments related to the X and Y axes.

We also determined the relationships between these mathematical properties and the principal moments of inertia.

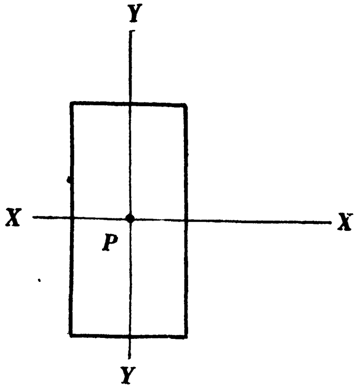


FIG. 122.

Let us assume that X and Y are principal axes of inertia. Then I_X and I_Y are principal moments of inertia and $I_{XY} = 0$.

We draw a graph (Fig. 123) with the magnitudes of I_X and I_Y as abscissas, laying out these values to scale along the axis OM. Now, a circle is drawn with $I_X - I_Y$ as diameter: this is Mohr's circle. If now a point, such as A, be taken anywhere on the circumference of the circle, its coordinates will be

$$\text{Abscissa} = I_A$$
$$\text{Ordinate} = I_{AB}$$

The point A lies on a radius of the circle inclined at angle 2θ from the M axis.

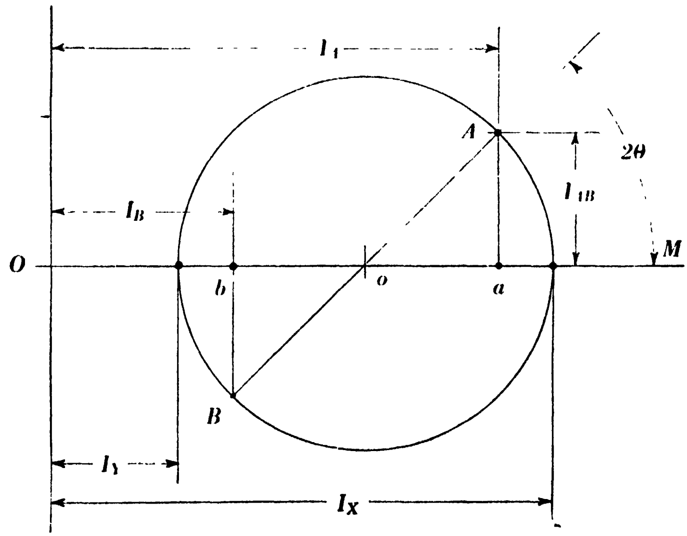


FIG. 123.

If now we set up axes A and B, as in Fig. 124, we shall find it possible to prove that the abscissa I_A of Fig. 123 is the moment of inertia of the given area about axis PA inclined at θ with the X axis and that the ordinate I_{AB} is the product of inertia of the given area relative to the mutually perpendicular axes A and B.

The general relationship between moments of inertia and products of inertia has previously been expressed as

$$I_A = I_X \cos^2 \theta + I_Y \sin^2 \theta - 2I_{XY} \sin \theta \cos \theta.$$

In the present case, our X and Y axes are principal axes, and $I_{XY} = 0$

Then, $$I_A = I_X \cos^2 \theta + I_Y \sin^2 \theta$$

In Fig. 123, we note that

$$I_A = Oo + [oa = \text{radius} \times \cos 2\theta]$$

in which $$Oo = \frac{I_X + I_Y}{2}$$

$$oa = \frac{I_X - I_Y}{2}$$

Then, $$I_A = \frac{I_X + I_Y}{2} + \frac{I_X - I_Y}{2} (\cos 2\theta = \cos^2 \theta - \sin^2 \theta)$$

Substituting $\cos^2 \theta + \sin^2 \theta = 1$ in the first term,

$$I_A = I_X \cos^2 \theta + I_Y \sin^2 \theta$$

Then the abscissa I_A of the Mohr's circle does yield the correct value for the moment of inertia about the inclined axis PA of Fig. 124.

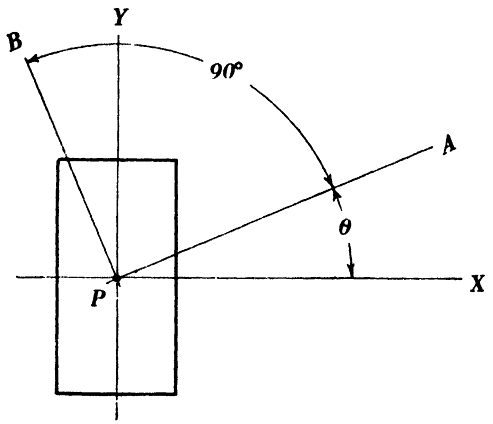


FIG. 124.

The moment of inertia of the given area (Fig. 124) about axis PB is given by the analytical expression as

$$\begin{aligned} I_B &= I_X \cos^2 (\theta + 90°) + I_Y \sin^2 (\theta + 90°) \\ &= I_X \sin^2 \theta + I_Y \cos^2 \theta \end{aligned}$$

This moment of inertia is reported by the Mohr's circle as

$$I_B = Oo - ob$$

in which

$$Oo = \frac{I_X + I_Y}{2}$$

$$ob = \frac{I_X - I_Y}{2}$$

$$I_B = \frac{I_X + I_Y}{2} - \frac{I_X - I_Y}{2}\cos 2\theta$$
$$= I_X \sin^2\theta + I_Y \cos^2\theta$$

which is in agreement with the analytical expression. It should be noted that axes PA and PB are inclined at 90 degrees with each other in the drawing of the area itself (Fig. 121). On the Mohr's circle, double angles are operative and, therefore points A and B are 180 degrees apart.

Similarly, let us note that the expression for product of inertia (see page 148) of an area about axes through any point inclined at angle θ with the original X and Y axes is

$$I_{AB} = I_{XY}(\cos^2\theta - \sin^2\theta) + (I_X - I_Y)\sin\theta\cos\theta$$

In our illustration, then,

$$I_{AB} = (I_X - I_Y)\sin\theta\cos\theta$$

because our X and Y axes are principal axes, and $I_{XY} = 0$.

In Fig. 123, we note that

$$I_{AB} = \text{distance } aA = \text{radius} \times \sin 2\theta$$
$$= \frac{I_X - I_Y}{2}\sin 2\theta$$
$$= \frac{(I_X - I_Y)}{2}\, 2\sin\theta\cos\theta$$
$$= (I_X - I_Y)\sin\theta\cos\theta$$

and the ordinate I_{AB} is properly presented by the circle as the product of inertia of the given area relative to the mutually perpendicular axes PA and PB.

In Article 41, we found that a polar moment of inertia (the moment of inertia of a given area about an axis perpendicular to the plane of the area at any point) is equal to the sum of the moments of inertia of the given area relative to any pair of mutually perpendicular axes.

In the Mohr's circle of Fig. 123, we find that

$$\frac{I_x + I_y}{2} = \text{distance } Oo$$

But

$$\frac{I_A + I_B}{2} = \text{distance } Oo$$

Then, $I_X + I_Y = I_A + I_B$, and such sums of moments of inertia are reported as always equal to the same constant (the polar moment of inertia) by the Mohr's circle, which is further evidence of the agreement of this graphical presentation with the basic relationships we found by analytical derivation.

We shall note one additional factor of agreement between the Mohr's circle and the analytical expressions, before proceeding to make use of the Mohr's circle in practical application.

The basic equation which we have derived to determine the location of principal axes of inertia (see page 147) is

$$\tan 2\theta = \frac{2I_{XY}}{I_X - I_Y}$$

in which θ is the angle of inclination of the principal axes with the original X and Y axes.

In Fig. 123, we note that Mohr's circle yields:

$$\tan 2\theta = \frac{Aa}{oa} = \frac{I_{AB}}{\frac{I_A - I_B}{2}} = \frac{2I_{AB}}{I_A - I_B}$$

which is additional evidence that Mohr's circle is a correct graphical translation of the relationships between moments and products of inertia.

We shall employ the Mohr's circle in the following example to illustrate the technique of its use.

This circle could be drawn to scale and, therefore, used as a completely graphical solution of the problem. Such a graphical solution, in common with all graphical solutions, must be drawn at large scale, if values trustworthy in their accuracy are to be obtained. It will be noted, however, as we proceed with the problem, that by using the Mohr's circle as guide to a routine of calculation, the computations themselves become simple arithmetical operations, and the advantage which a completely graphical solution might have is questionable.

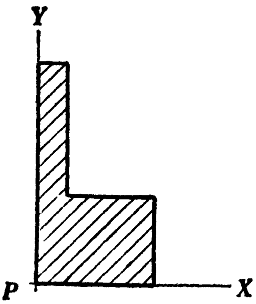


FIG. 125.

ILLUSTRATIVE EXAMPLE

Given the plane area shown in Fig. 125. We are to determine the location of the principal axes of inertia, the magnitudes of the principal moments of inertia, and the least radius of gyration at point P.

The moment of inertia about $PX - (I_X)$ is given as 2680 inches. $I_Y = 1050$ in.^{4} $I_{XY} = +765$ in.^{4} The area = 66 in.^{2}

The Mohr's circle is started, as in Fig. 126, from an origin O. I_X and I_Y are laid out at any convenient scale as abscissas Ox and Oy

respectively. At point x, we erect a perpendicular to Ox, xs, equal to I_{XY} at the same scale as that used for I_X and I_Y. $yt = xs$ is erected perpendicular to Oy (yt represents the negative of I_{XY}). Points s and t determine the Mohr's circle, which is now drawn with st as its diameter. Points a and b, the intersections of the Mohr's circle with axis OM, determine the principal moments of inertia, because we note that such points as a and b have abscissas equal to moments of inertia, and ordinates

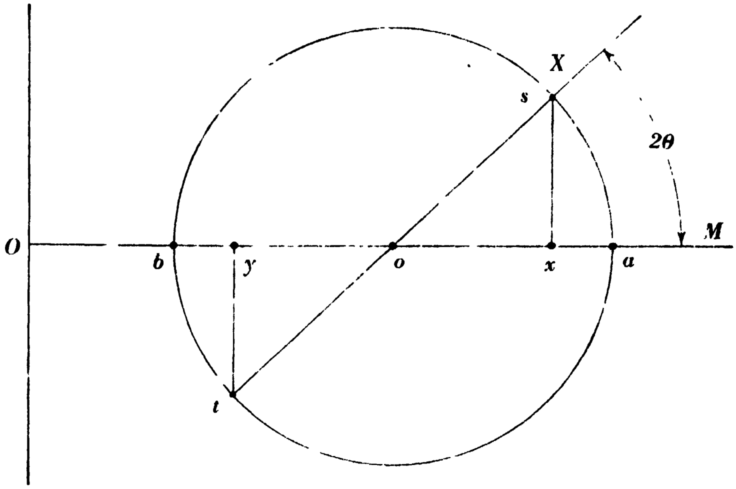


FIG. 126.

equal to products of inertia. At points a and b, the ordinates are equal to zero. Then, the products of inertia are equal to zero, and we have reached principal axes of inertia. The distance oa is the maximum moment of inertia, and ob is the minimum moment of inertia.

To evaluate θ, I_A, and I_B, we first determine the radius os.

$$os = \sqrt{\overline{ox}^2 + \overline{xs}^2}$$
$$= \sqrt{\overline{815}^2 + \overline{765}^2} = 1118 \text{ in.}^4$$

Then

$$I_A = Oa = Oo + oa$$
$$= 1865 + 1118 = 2983 \text{ in.}^4$$
$$I_B = Ob = Oo - ob$$
$$= 1865 - 1118 = 747 \text{ in.}^4$$
$$2\theta = \tan^{-1}\left[\frac{xs}{ox} = \frac{765}{815} = 0.9387\right]$$
$$= 43.2^\circ$$
$$\theta = 21.6^\circ$$

In plotting these results, as in Fig. 127, it is necessary to note that, on the Mohr's circle, we have proceeded clockwise in going from point s to point a through an angular distance 2θ.

Then, in plotting θ on the area, as in Fig. 127, we must also proceed clockwise from the X axis.

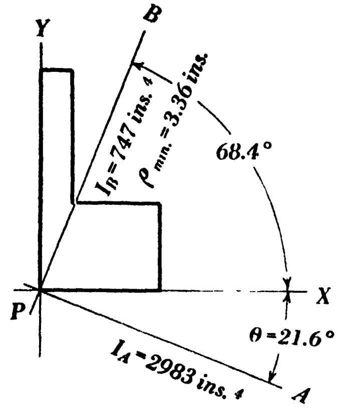


FIG. 127.

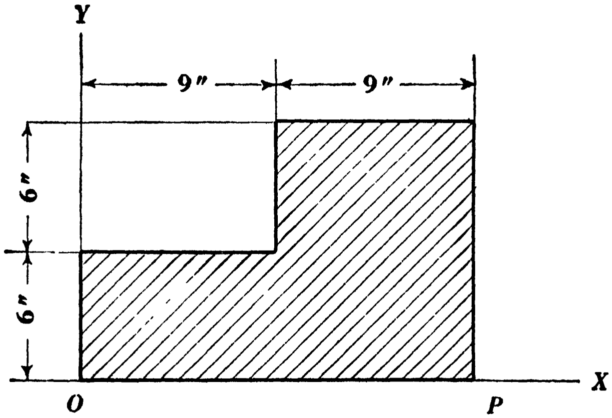
The least radius of gyration, an important factor in the design of columns, is

$$\rho_{min} = \sqrt{\frac{I_B}{\text{Area}}}$$
$$= \sqrt{\tfrac{747}{66}} = 3.36 \text{ in.}$$

PROBLEMS

244. Determine the following at origin O, for the shaded area shown. (a) Location of the principal axes of inertia. (b) Magnitude of the principal moments of inertia, and (c) Least radius of gyration.

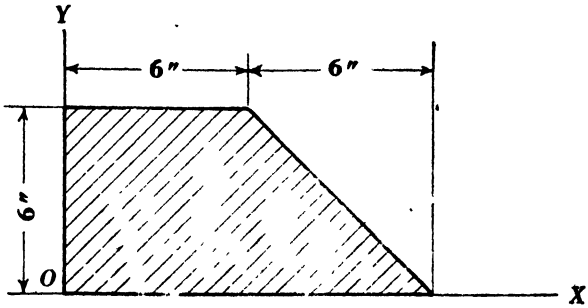
Ans. $\theta_x = 24.9°$ and $114.9°$; $I_{max.} = 26{,}250$ ins.^{4}; $I_{min} = 1450$ ins.^{4}; $\rho_{min} = 9$ ins.



PROB. 244

245. At point P of the shaded area given in Problem 244, determine the following: (a) Location of the principal axes of inertia. (b) Magnitude of the principal moments of inertia, and (c) Least radius of gyration.

246. Determine the location of the axis through point O, for which the moment of inertia the shaded area will be minimum. *Ans.* $\theta X = 13.3°$



PROB. 246

247. Determine the least radius of gyration of the beam section given in Problem 232.

248. For the area given in Problem 229, determine the principal moments of inertia at the centroid of the area.

249. For the area given in Problem 209, determine the principal moments of inertia and the least radius of gyration at the centroid.

250. Determine the least radius of gyration of the shaded area given in Problem 206.

251. For the area given in Problem 211, determine the least radius of gyration.

252. Determine the following, for the shaded area given in Problem 212. (a) Location of the principal axes at point O and (b) Magnitude of the principal moments of inertia at point O.

253. Determine the principal moments of inertia at the centroid of the area given in Problem 214.

254. For the area given in Problem 200, determine the following: (a) Location of the principal axes of inertia at point O and (b) Magnitude of the principal moments of inertia at point O.

47. Center of Gravity of Solids. The concepts of first moment and center of gravity apply to solids as well as to plane areas. The first moment of an element of volume relative to a line or plane is the product of the volume, multiplied by the perpendicular distance to the line or plane.

The first moment of an element of volume dV of the body shown in Fig. 128 relative to the plane YOZ is $x\,dV$.

Then, the first moment of the total volume relative to the plane YOZ is $\int x\,dV$—relative to the plane YOX, it is $\int z\,dV$, and relative to the plane XOZ, it is $\int y\,dV$.

In locating the centroid of the volume, we proceed as we have done in the case of plane areas, and

$$\bar{x} = \frac{\int x\,dV}{\int dV}$$

$$\bar{y} = \frac{\int y\,dV}{\int dV}$$

$$\bar{z} = \frac{\int z\,dV}{\int dV}$$

When the volume leaves its geometrical status and becomes a physical

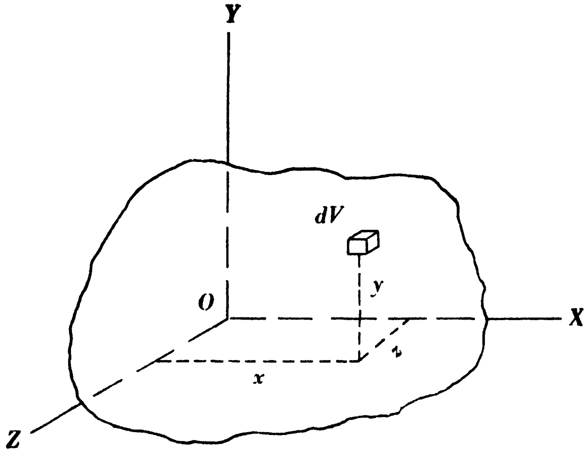


FIG. 128.

solid endowed with weight, the expressions for the location of the center of gravity are

$$\bar{x} = \frac{\int x\,dW}{\int dW}$$

$$\bar{y} = \frac{\int y\,dW}{\int dW}$$

$$\bar{z} = \frac{\int z\,dW}{\int dW}$$

48. Derivation of Centroids of Common Geometrical Solids.

1. *Cylinder (Bases Parallel) or Prism* (Fig. 129). The volume of the elementary slice, perpendicular to axis OY, is $dV = A\,dy$.

Then
$$\bar{y} = \frac{\int y\,dV}{\int dV}$$

$$= \frac{\int y\,A\,dy}{\int A\,dy} = \frac{h}{2}$$

The centroid of each elementary slice must lie on the axis OB of the cylinder or prism. If θ is the angle between OB and the X axis,

$$\bar{x} = \frac{h}{2\tan\theta}, \text{ and when } \theta = 90°,\ \bar{x} = 0$$

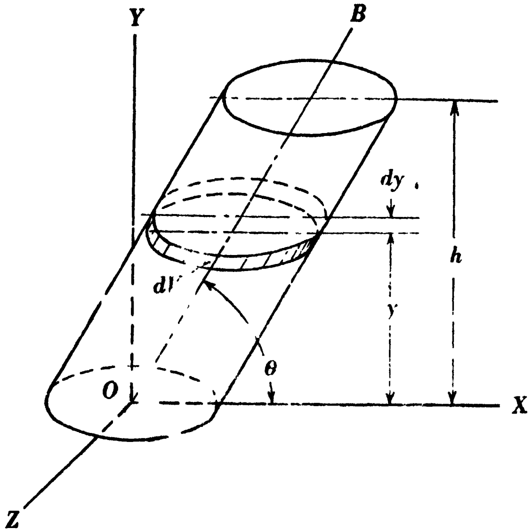


FIG. 129.

2. *Cone or Pyramid* (Fig. 130). The volume of the elementary slice, perpendicular to axis OY, is $dV = A\,dy$, and $\dfrac{A_1}{A} = \dfrac{(h-y)^2}{h^2}$

Then
$$\bar{y} = \frac{\int y\,dV}{\int dV} = \frac{\int \dfrac{y(h-y)^2 A\,dy}{h^2}}{\int \dfrac{(h-y)^2 A\,dy}{h^2}} = \frac{h}{4}$$

The centroid of each elementary slice must lie on the axis OB of the cone.

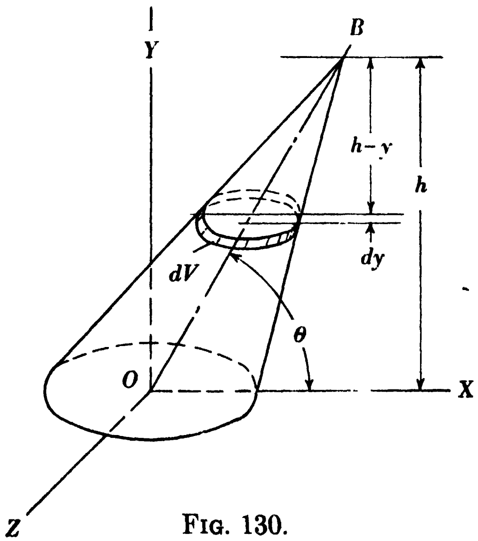


FIG. 130.

If θ is the angle between OB and the X axis,

$$\bar{x} = \frac{h}{4} \tan \theta, \text{ and when } \theta = 90°, \bar{x} = 0$$

The centroid of the truncated cone or pyramid may be located by introducing the corresponding limits of y in the above integration.

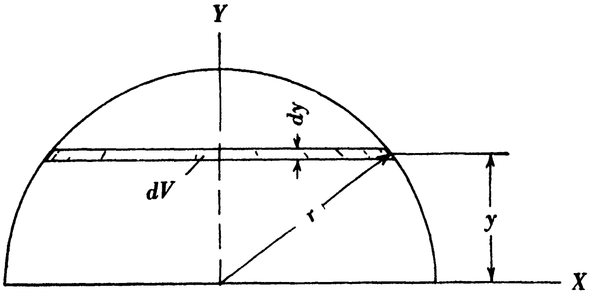


FIG. 131.

3. *Hemisphere* (Fig. 131). The volume of the elementary slice, perpendicular to axis OY is $dV = A\,dy = \pi(r^2 - y^2)\,dy$

$$\bar{y} = \frac{\int y\,dV}{\int dV} = \frac{\int y\pi(r^2 - y^2)\,dy}{\int \pi(r^2 - y^2)\,dy} = \frac{3}{8}r$$

and
$$\bar{x} = 0$$

49. Centroid of Solids by Division into Finite Parts. The connecting rod of Fig. 132 is composed of two hollow cylinders (1 and 2) and a rectangular prism 3. The sections are aligned on central axis XX. The

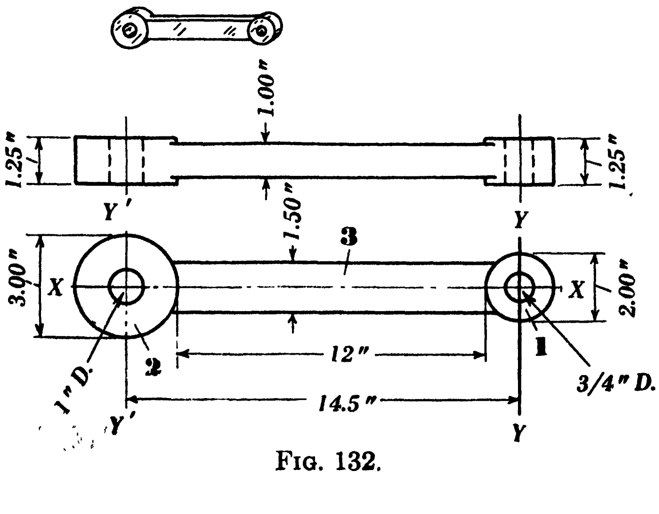


FIG. 132.

material of which sections 1 and 2 are made weighs 0.1 pounds per cubic inch, and that of part 3 weighs 0.2 pounds per cubic inch.

We are to locate the center of gravity of the total weight, relative to axis YY.

$$W_1 = 0.1 \times [\pi \times 1 - \pi \times (\tfrac{3}{8})^2]1.25 = 0.338 \text{ lb.}$$

$$W_2 = 0.1 \times [\pi \times (1.5)^2 - \pi \times (\tfrac{1}{2})^2]1.25 = 0.784 \text{ lb.}$$

$$W_3 = 0.2 \times 12 \times 1.5 \times 1 = 3.60 \text{ lb.}$$

$$\bar{x} = \frac{\Sigma(\bar{x}_1 W_1 + \bar{x}_2 W_2 + \bar{x}_3 W_3)}{\Sigma(W_1 + W_2 + W_3)}$$

$$= \frac{0 + 7 \times 3.60 + 14.5 \times 0.784}{0.338 + 3.60 + 0.784} = 7.75 \text{ in.}$$

Such results should be confirmed by checking the location of the centroid from a different axis—for example, from Y_1Y_1.

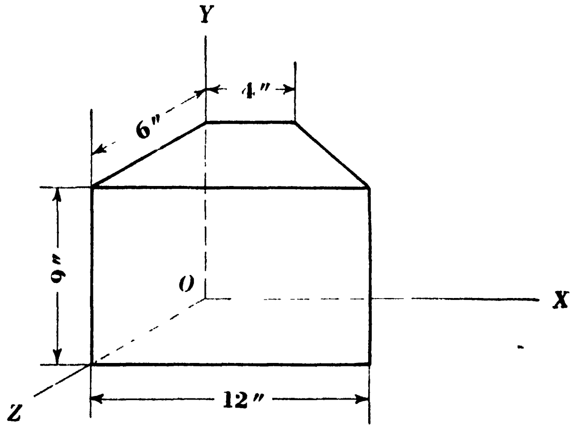
The coordinate of the centroid from Y_1Y_1 is

$$\bar{x} = \frac{0 + 7.5 \times 3.60 + 14.5 \times 0.338}{0.784 + 3.60 + 0.338} = 6.75 \text{ in.}$$

PROBLEMS

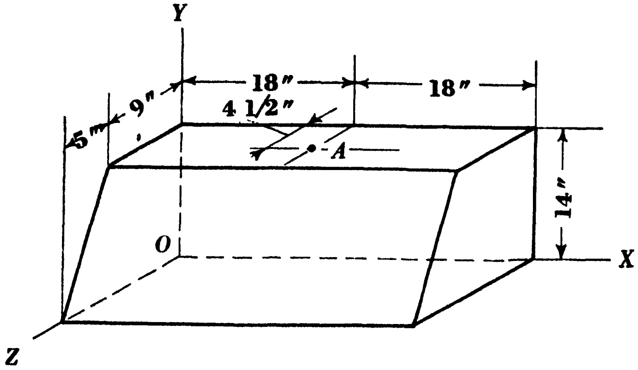
255. Locate the centroid of the solid shown.

Ans. $\bar{x} = 4.33$ ins.; $\bar{y} = 4.5$ ins.; $\bar{z} = 3.50$ ins.



PROB. 255

256. The casting shown weighs 0.3 lb. per cu. in. Locate the center of gravity. *Ans.* $\bar{x} = 18$ ins.; $\bar{y} = 6.49$ ins.; $\bar{z} = 5.85$ ins.

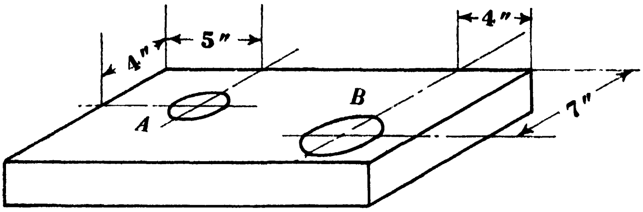


PROB. 256

257. If a hole of 3 in. diameter is cored out at point A through the casting of Problem 256, locate the center of gravity.

258. The machine bed plate is 19 by 10 by $2\frac{1}{2}$ in. and made of homogeneous material. The holes at A and B run through the plate, and have diameters of 3 and 4 in., respectively.

Locate the center of gravity of the plate.

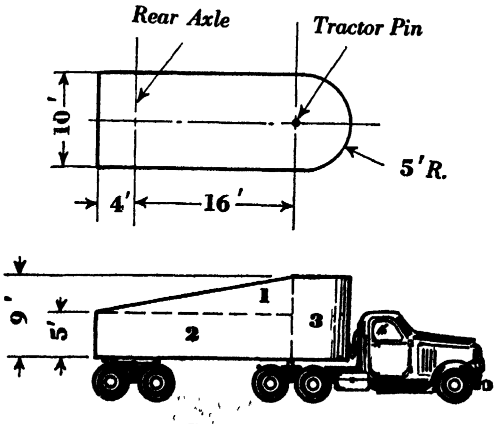


PROB. 258

259. The trailer load completely fills the volume of the trailer. The load in volume 1 weighs 15 lb. per cu. ft.; in volume 2, it weighs 20 lb. per cu. ft.; in volume 3, the load weighs 30 lb. per cu. ft.

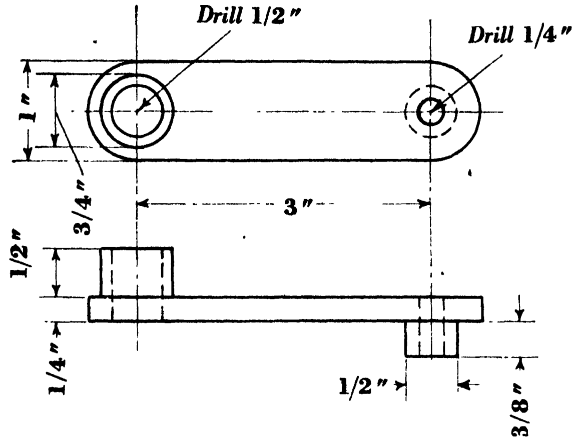
Determine the resultant force exerted on the tractor pin by the loading, assuming it to be concentrated.

Determine the intensity of the distributed force exerted on the rear axle, assuming that the length of the axle is 10 ft., and that the force is uniformly distributed along the axle.



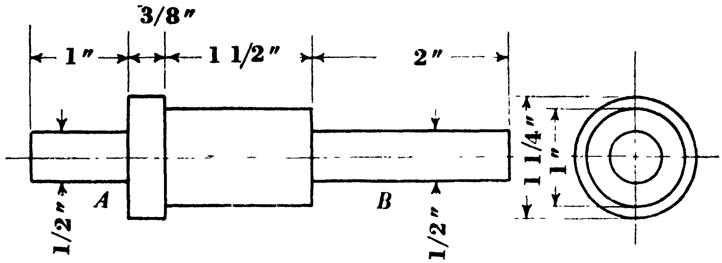
PROB. 259

260. Locate the center of gravity of the connecting rod shown. The material is homogeneous.



PROB. 260

261. Locate the center of gravity of the stud shown. The material is homogeneous.

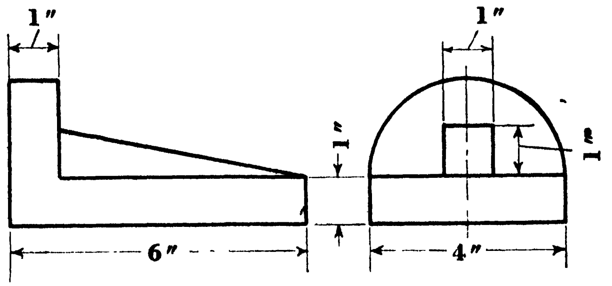


PROB. 261

262. The stud of Problem 261 is to be redesigned so that different metals will be used. Sections A and B are to be made of an aluminum alloy weighing 0.10 lb. per cu. in., and the balance of the stud is to be made of bronze, weighing 0.295 lb. per cu. in.

Locate the center of gravity of the new model.

263. The machine part shown is made of homogeneous material. Locate its center of gravity.



PROB. 263

264. The machine part shown in Problem 263 is made of the following materials: The semicylindrical disk material weighs 0.10 lb. per cu. in. The rectangular prism serving as base weighs 0.20 lb. per cu. in. The triangular prism used as a rib weighs 0.18 lb. per cu. in.

Locate the center of gravity of the machine part.

50. Moments of Inertia of Solids. The moment of inertia of a solid is a second moment and conforms to the definition of second moment given, in the case of plane areas, in Article 41.

For example, the moment of inertia of the element of volume, dV, shown in Fig. 133 relative to plane XOZ is $y^2\,dV$; and the moment of inertia of the total volume relative to plane XOZ is $\int y^2\,dV$.

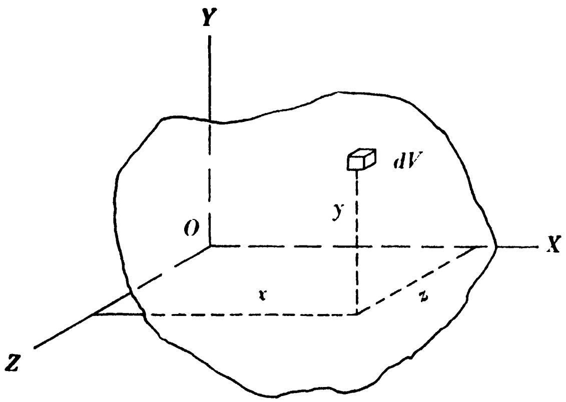


FIG. 133.

The most important use of this mathematical property arises in the field of dynamics, when we deal with solids having weight, or the derived property of mass.

In the derivation of expressions for moments of inertia of solid bodies, we shall, therefore, establish such second moments in terms of weight.

The units of moment of inertia are established from direct substitution in the defining expression: there are no derived or abbreviated terms in common usage.

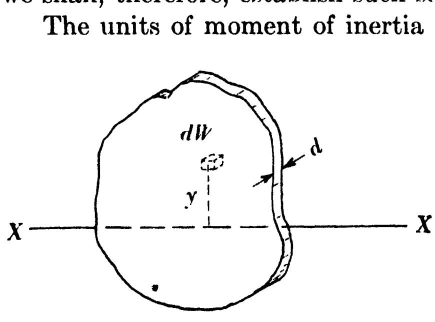


FIG. 134.

Since these second moments are of the form $\int y^2\,dW$ in which y is a linear distance and W a weight, the units of moment of inertia are lb.-ft.^{2} (or lb.-in.^{2}).

We can now take advantage of the technique which we employed in deriving expressions for the second moments of plane areas.

The plate shown in Fig. 134 is assumed to be very thin, of uniform depth d, and of uniform density δ.

Then, the moment of inertia of the element of weight dW about an axis XX lying in the central plane of the plate is

$$I_{XX} = y^2\,dW$$

in which $dW = \delta d\,dA$.

Then, for the entire plate,

$$I_{XX} = \int y^2 \delta d\,dA$$

But δ and d have been anounced as constant, and

$$I_{XX} = \delta d \int y^2\,dA.$$

We recognize the integral as the moment of inertia of the plane area of the thin plate about XX, and conclude that *the moment of inertia of a thin plate about any axis in its plane is equal to the moment of inertia of the area of the plate about the same axis multiplied by the product of the density and the thickness of the plate.*

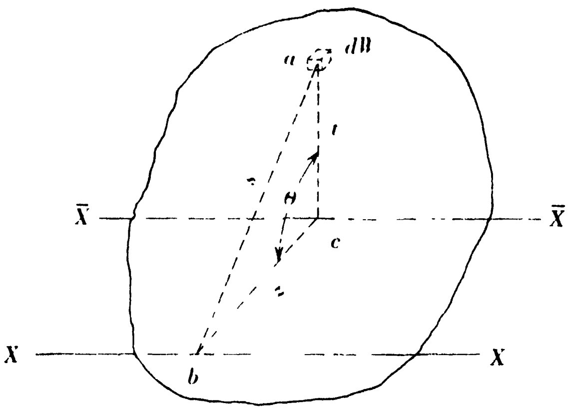


FIG. 135.

The same relationship between moment of inertia of area and plate will prevail for any axis perpendicular to the thin plate. (This would be the polar moment of inertia.)

We have discussed the relationship between moment of inertia of area and of thin plate because in the case of the moments of inertia of most of the geometrical solids, we can note that such solids are an aggregate volume composed of thin plates or layers.

We may then derive their moments of inertia by direct use of the moments of inertia of plane areas which we have previously established.

Before proceeding with such derivations, we require, as in the case of plane areas, a transformation expression which will allow us to transfer moments of inertia between parallel axes.

In Fig. 135, $\bar{X}\bar{X}$ is a centroidal axis of the solid. XX is any axis parallel to $\bar{X}\bar{X}$ at distance z.

The perpendicular distance from XX to the element of weight dW is s, and the perpendicular distance from dW to $\bar{X}\bar{X}$ is t. The angle between z and t is θ.

The moment of inertia of dW about axis XX is $s^2\,dW$, and for the entire body,

$$I_{XX} = \int s^2\,dW$$

In the triangle abc,

$$s^2 = z^2 + t^2 - 2zt\cos\theta$$

Then

$$I_{XX} = \int(z^2 + t^2 - 2zt\cos\theta)\,dW$$

$$= z^2\int dW + \int t^2\,dW - 2z\int t\,dW$$

The third term contains the first moment of the solid about its centroidal axis, which is zero, and therefore vanishes.

The $\int t^2\,dW$ is the moment of inertia of the solid about its centroidal axis.

The first term is the product of the square of the distance between the parallel axes multiplied by the total weight.

Then the transformation expression becomes

$$I_{XX} = I_{\bar{X}\bar{X}} + z^2\,dW$$

This form is directly analogous to the transformation expression for plane areas and it is subject to the same limitations. It can only transfer moments of inertia between *parallel* axes, and one of those axes *must* pass through the centroid of the solid.

51. Derivation of Moments of Inertia for Common Geometrical Solids.

1. *Right Cylinder About Its Axis* (Fig. 136). For the elementary slice dW (equivalent to the thin plate of the preceding article), the

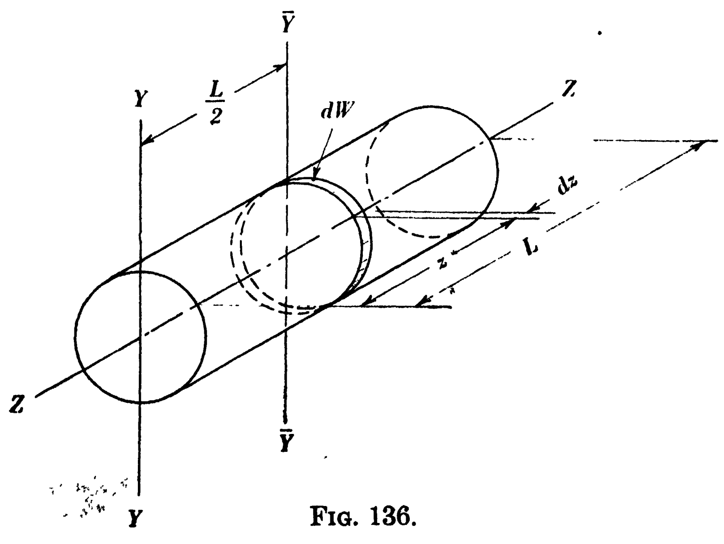


FIG. 136.

moment of inertia about axis ZZ will be

$$I_{ZZ} = \int_0^L \delta\, dz \frac{\pi r^4}{2}$$

in which $\frac{\pi r^4}{2}$ is the polar moment of inertia of the cross-sectional area of the cylinder about axis ZZ, and δ is the density of the material of the cylinder.

$$I_{ZZ} = \delta L \frac{\pi r^4}{2} = (\delta L \pi r^2) \frac{r^2}{2} = \frac{Wr^2}{2}$$

in which W is the total weight of the cylinder.

The axis YY is a diameter of the cylinder at its end.

Then, using the same thin slice as an element, and the transformation expression, we have

$$I_{YY} = \int_0^L \delta\, dz \frac{\pi r^4}{4} + \int_0^L (z)^2 \delta\, dz\, \pi r^2$$
$$= \frac{Wr^2}{4} + \frac{WL^2}{3}$$

The axis $\bar{Y}\bar{Y}$ is parallel to YY through the centroid of the volume.

Then
$$I_{\bar{Y}\bar{Y}} = \left(\frac{Wr^2}{4} + \frac{WL^2}{3}\right) - \left(\frac{L}{2}\right)^2 W$$
$$= \frac{Wr^2}{4} + \frac{WL^2}{12}$$

2. *Thin Rod* (Fig. 137). The thin rod is one in which the cross-sectional area is so small in comparison with its length that we may, without appreciable error, assume that the term r which appeared in the expressions for the right cylinder of the preceding example may be neglected.

Then, for a diametral axis YY perpendicular to the rod at its centroid, the expression $I_{YY} = \frac{Wr^2}{4} + \frac{WL^2}{3}$ of the preceding article becomes

$$I_{YY} = \frac{WL^2}{3}$$

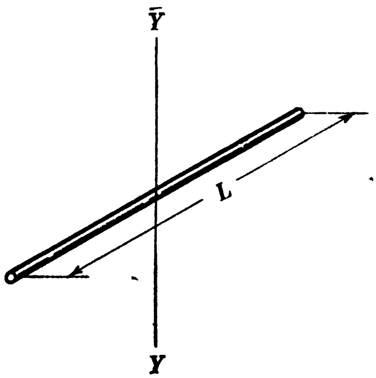


FIG. 137.

When the rod is inclined at an angle θ with the axis of reference, as in

Fig. 138, and the density δ is the weight per unit of length, we have

$$I_{YY} = \int x^2\,dW = \int_0^L (l \sin\theta)^2 \delta\,dl$$
$$= \frac{\delta L^3}{3}\sin^2\theta = \frac{WL^2}{3}\sin^2\theta$$

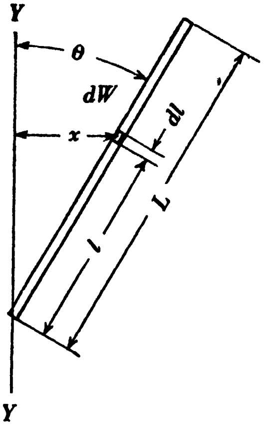


FIG. 138.

in which W is the total weight of the rod and L, its length.

When $\theta = 90°$, we have

$$I_{YY} = \frac{WL^2}{3}$$

which confirms the conclusion reached above.

3. *Rectangular Prism* (Fig. 139). The method of borrowing the expressions previously derived from plane areas may again be employed here by considering the prism to be composed of elementary slices, or very thin plates.

The axis XX is perpendicular to the side of length b of the prism at its end, and passes through the centroid. δ is the density per unit of volume.

$$I_{XX} = \int_0^L \delta dz\,\frac{ab^3}{12} + \int_0^L z^2\delta\,dz\,ab$$
$$= \frac{ab^3}{12}\delta L + ab\delta\frac{L^3}{3}$$
$$= \frac{Wb^2}{12} + \frac{WL^2}{3}$$

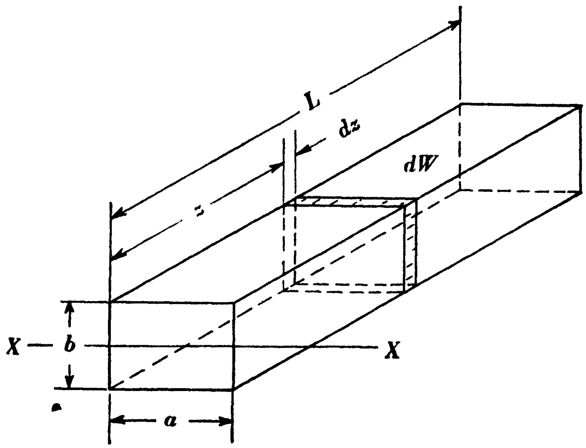


FIG. 139.

Axis $\bar{X}\bar{X}$ passes through the centroid of the parallelopiped and is parallel to XX.

Then,
$$I_{\bar{x}\bar{x}} = \frac{Wb^2}{12} + \frac{WL^2}{3} - \left(\frac{L}{2}\right)^2 W$$
$$= \frac{Wb^2}{12} + \frac{WL^2}{12} = \frac{W}{12}(b^2 + L^2)$$

4. *Hemisphere* (Fig. 140). The axis XX is a diametral axis at the base of the hemisphere. δ is the density per unit of volume. The elementary slice dW parallel to the base has a radius $x = \sqrt{r^2 - y^2}$

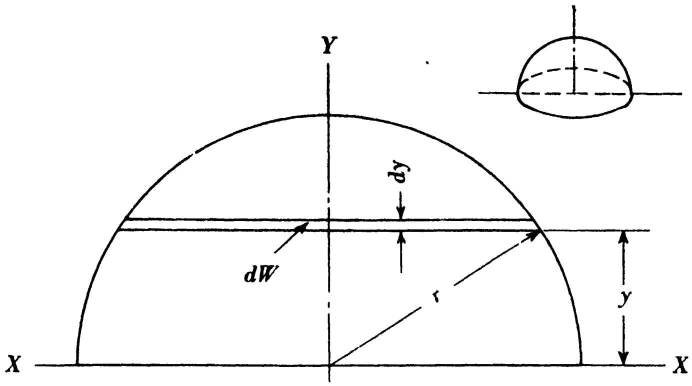


FIG. 140.

Then,
$$I_{XX} = \int_0^r \delta\, dy \frac{\pi}{4}(r^4 - 2r^2y^2 + y^4) + \int_0^L y^2\delta\, dy\, \pi(r^2 - y^2)$$
$$= \frac{\pi\delta}{4}\left(r^5 - \frac{2r^5}{3} + \frac{r^5}{5}\right) + \pi\delta\left(\frac{r^5}{3} - \frac{r^5}{5}\right)$$
$$= \frac{\pi\delta}{4}\left(\frac{8}{15}r^5\right) + \pi\delta\left(\frac{2}{15}r^5\right)$$
$$= \frac{4\pi\delta r^5}{15} = \frac{2}{5}Wr^2$$
(W is the weight of the hemisphere)

Then a sphere will have, about any diameter such as XX,
$$I_{XX} = \tfrac{2}{5}Wr^2$$
(W is the weight of the sphere)

52. Derivation of Moments of Inertia of Solids by Finite Parts. As in the case of plane areas (Article 43), the moments of inertia of the common engineering solids may be most conveniently established by dividing those solids into finite parts.

The moments of inertia of the parts comprising the total solid are then established by using the results of the integration of the preceding article, and, whenever necessary, the transformation expression.

For example, the machine element of Fig. 141 is composed of a sphere (1), a cylindrical shaft (2), and a cylindrical disk (3). All elements are aligned on a central axis XX, which passes through their centroids.

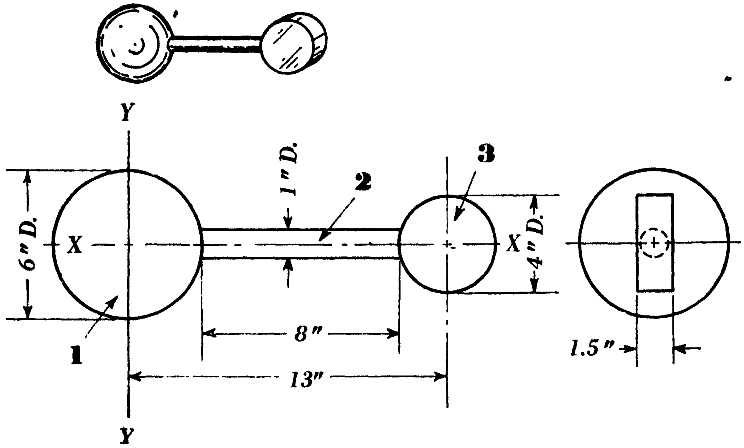


FIG. 141.

The density of the material of sections 1 and 2 is 0.2 pounds per cubic inch, and the density of the material of section 3 is 0.1 pounds per cubic inch. We are to determine the moment of inertia of the entire part about axis YY.

The total weight of each section is

$$W_1 = \tfrac{4}{3}\pi \times (3)^3 \times 0.2 = 22.61 \text{ lb.}$$
$$W_2 = \pi \times (\tfrac{1}{2})^2 \times 8 \times 0.2 = 1.26 \text{ lb.}$$
$$W_3 = \pi \times (2)^2 \times 1.5 \times 0.1 = 1.88 \text{ lb.}$$

Then, for section 1,

$$I_{YY} = \tfrac{2}{5}W_1r_1^2 = \tfrac{2}{5} \times 22.61 \times (3)^2 = 81.41 \text{ lb.-in.}^2$$

For section 2,

$$I_{YY} = \frac{W_2r_2^2}{4} + \frac{W_2L_2^2}{12} + (x_2)^2W_2$$
$$= \frac{1.26 \times (\frac{1}{2})^2}{4} + \frac{1.26 \times (8)^2}{12} + (7)^2 1.26 = 68.54 \text{ lb.-in.}^2$$

For section 3,

$$I_{YY} = \frac{W_3r_3^2}{4} + \frac{W_3L_3^2}{12} + (x_3)^2W_3$$
$$= \frac{1.88 \times (2)^2}{4} + \frac{1.88 \times (1.5)^2}{12} + (13)^2 1.88 = 319.95 \text{ lb.-in.}^2$$

Then, the moment of inertia of the entire machine element about axis YY is

$$I_{YY} = 81.41 + 68.54 + 319.95 = 469.9 \text{ lb.-in.}^2$$

Problems

265. Derive the expression for the moment of inertia of a right circular cone of weight W, base radius r, and altitude h, with respect to its own axis.

Ans. $I = \frac{3}{10}Wr^2$.

266. Derive the expression for the moment of inertia of a right circular cone of weight W, base radius r, and altitude h, with respect to a diameter of the base.

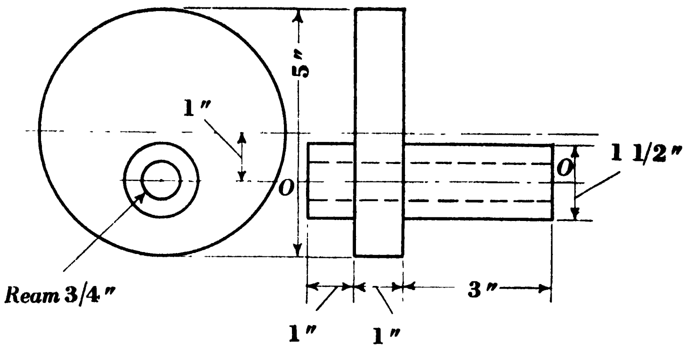
Ans. $I = \frac{3}{20}Wr^2 + \frac{Wh^2}{10}$,

267. Determine the moment of inertia of a steel cylinder which has a diameter of 5 in. and a height of 20 in. about its axis. Weight of steel is 490 lb. pe· cu. ft.

268. Determine the moment of inertia of à thin rod, 20 in. long weighing 0.10 lb. per linear foot, about an axis at its end which is inclined at 30° with the axis of the rod.

269. Determine the moment of inertia of a 10-in. diameter sphere, weighing 490 lb. per cu. ft., about a centroidal axis.

270. The eccentric cam shown rotates about axis OO. Determine its moment of inertia about the axis of rotation.



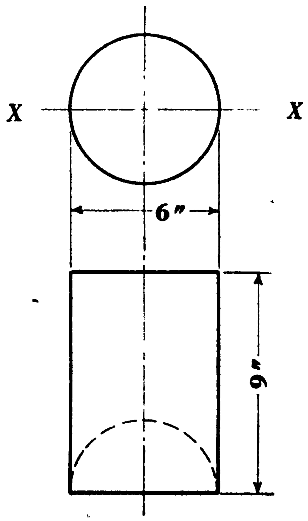
Prob. 270

271. The connecting rod given in Problem 260 is made of steel weighing 0.285 lb. per cu. in. Determine its moment of inertia about the axis of the ½-in. hole.

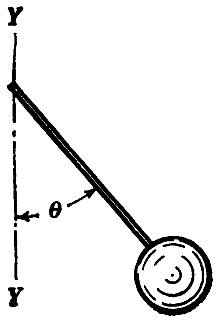
272. Determine the moment of inertia of the stud given in Problem 261 about its axis. The material weighs 0.28 lb. per cu. in.

Ans. 0.072 lbs. ins.^{2}

273. Determine the moment of inertia of the homogeneous solid shown about axis XX, which is a diameter of the top circular face. The hollow cup at the bottom is a hemisphere.



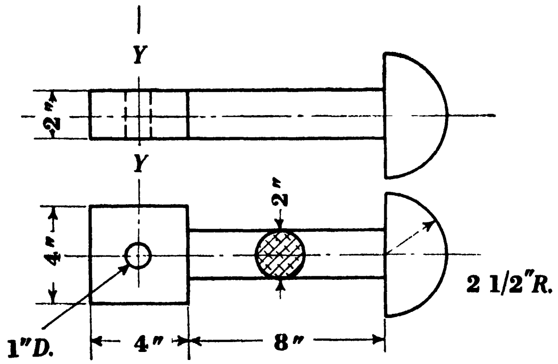
Prob. 273



PROB. 274

274. The control arm of a speed governor consists of a sphere with a diameter of 3 in., supported at the end of a 10-in. slender rod, so that it revolves about axis YY. Determine the moment of inertia of the control arm relative to axis YY, when θ, the angle of inclination, is 40°. The rod and sphere are made of steel, weighing 0.29 lb. per cu. in.

275. The arm shown revolves about axis YY. The rectangular prism is made of bronze, weighing 0.29 lb. per cu. in., and the cylinder and hemisphere of an aluminum alloy weighing 0.12 lb. per cu. in. Determine the moment of inertia of the arm about the YY axis.



PROB. 275

53. Uniformly Varying Stress and Pressure Distributed Over Area. In Fig. 142, the plan view of an area is shown together with a side view showing the force distribution over that area when the force varies as some function of z.

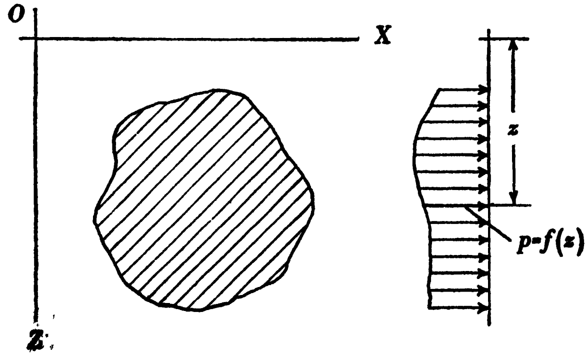


FIG. 142.

The common occurrence of a particular type of such variable distributed force in engineering problems warrants our analyzing that type—the *uniformly varying force,* where the intensity at any distance is

$$p = cz$$

in which c is a constant. The uniformly varying force is encountered in the case of surfaces, such as those of tank walls or dams immersed in fluids, and in beams in bending where the distribution of the stress across a right section of the beam is found to be uniformly varying.

An example of uniformly varying force exerted on an area is shown in Fig. 143.

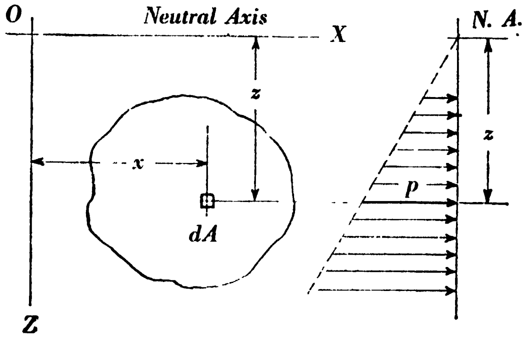


FIG. 143.

The force varies uniformly, and the value of its intensity at any distance z from the zero intensity level, usually called the *neutral axis* (axis OX), is

$$p = cz$$

The total force on an elementary area dA is

$$dF = p\,dA = cz\,dA$$

All of these forces dF are perpendicular to the area and, therefore, form a parallel force system in space.

In Article 25, we found that in such a parallel force system

$$R = \Sigma F$$

$$x_R = \frac{\Sigma M_z}{R}$$

and

$$z_R = \frac{\Sigma M_x}{R}$$

Then our resultant force in the present instance is

$$R = \Sigma\, dF = \int cz\,dA = c\int z\,dA = c\bar{z}A$$

or the product of the constant c multiplied by the first moment of the area over which the force is distributed, relative to the neutral axis.

To determine the point of application of this resultant, we find that

$$z_R = \frac{\Sigma M_X}{R} = \frac{\int pz\,dA}{\int p\,dA} = \frac{\int cz^2\,dA}{\int cz\,dA} = \frac{cI_X}{c\bar{z}A}$$

$$= \frac{I_{NA}}{\bar{z}A}$$

or, the second moment of the area, relative to the neutral axis divided by the first moment of the area about the same axis.

We now have one coordinate dimension of the point of application of the resultant force. If we set up any axis OZ perpendicular to OX, we can locate the point of application of the resultant by determining

$$x_R = \frac{\Sigma M_Z}{R} = \frac{\int px\,da}{\int p\,dA} = \frac{\int czx\,dA}{\int cz\,dA} = \frac{cI_{XZ}}{c\bar{z}A}$$

$$= \frac{I_{XZ}}{\bar{z}A}$$

or, the product of inertia of the area relative to a pair of axes, mutually perpendicular, with the neutral axis as one of the pair, divided by the first moment.

We learned in Article 30 that the resultant of a parallel force system in space may be a couple.

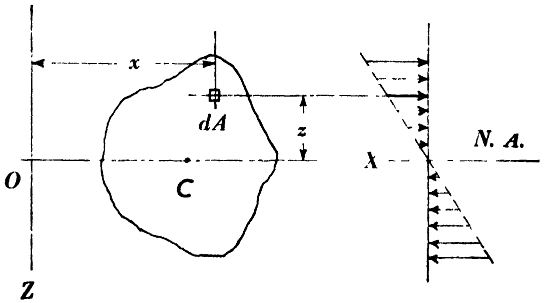


FIG. 144.

If, as in Fig. 144, the neutral axis of a uniformly varying force passes through the centroid of the area C, the resultant will be a couple, for:

$$R = \Sigma F = c\bar{z}A = 0$$

$$\Sigma M_X = \int pz\,dA = \int cz^2\,dA = cI_X$$

$$\Sigma M_Z = \int px\,dA = \int czx\,dA = cI_{XZ}$$

These component couples are shown as couple vectors in Fig. 145. Their resultant is

$$M_R = \sqrt{\overline{\Sigma M_X}^2 + \overline{\Sigma M_Z}^2}$$

and the inclination of this resultant couple vector with the X axis is determined as

$$\theta_X = \tan^{-1} \frac{\Sigma M_Z}{\Sigma M_X}$$

The couple M_R lies in a plane perpendicular to the couple vector representing it.

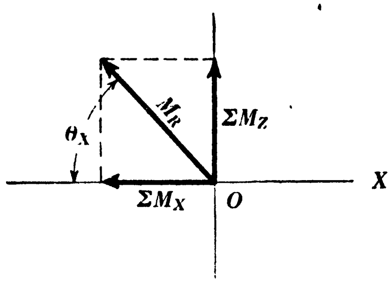


FIG. 145.

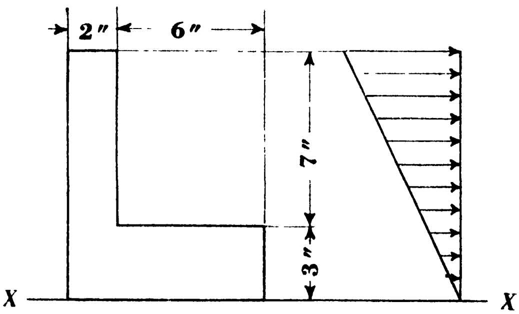
If OX and OZ are principal axes, as, for example, in the case where one of them is an axis of symmetry, the couple $\Sigma M_Z = cI_{XZ}$ will vanish, and the resultant couple M_R will equal cI_X.

PROBLEMS

276. The area shown is subjected to a uniformly varying pressure with the neutral axis at XX. The intensity of pressure at 1 in. from the neutral axis is 100 psi.

Determine the resultant pressure and locate its point of application.

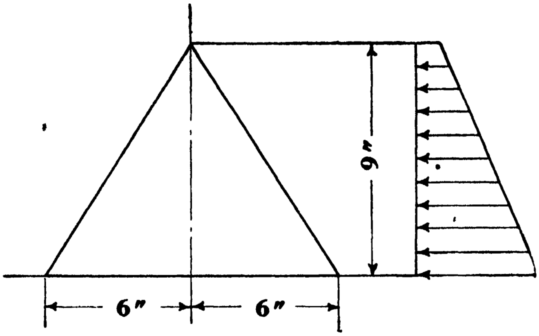
Ans. $R = 12{,}700$ lb.; $x_R = 5.67$ ins. above N.A.; $y_R = 1.85$ ins. from left side of area.



PROB. 276

277. The triangular area shown is subjected to stress which varies uniformly from an intensity of 100 psi at the apex of the triangle to an intensity of 1000 psi at its base.

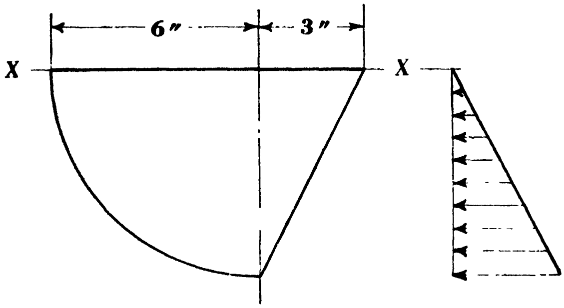
Determine the resultant stress and locate its point of application.



PROB. 277

278. The area shown is subjected to a pressure which varies uniformly from the XX axis to a maximum intensity of 1200 psi.

Determine the resultant pressure on the area, and locate its point of application.

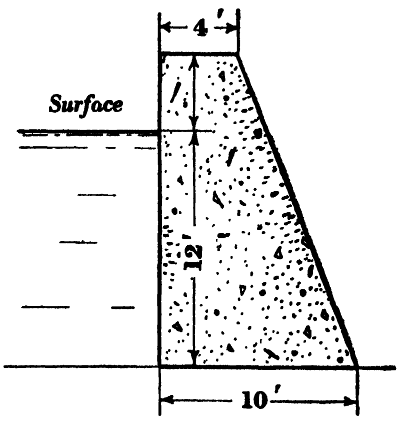


PROB. 278

279. The depth of water on the surface of a dam is 12 ft. The cross section of the dam is shown.

Locate the point at which the resultant thrust due to water pressure and the weight of masonry intersects the base of the dam.

Weight of water = 62.5 lb. per cu. ft.; weight of masonry = 150 lb. per cu. ft.

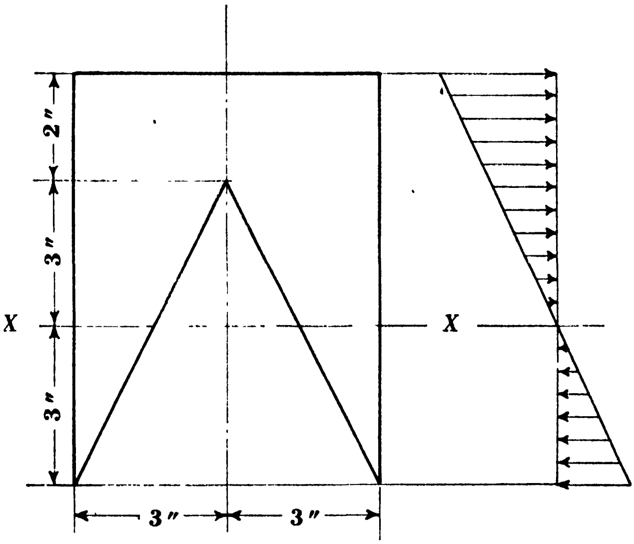


PROB. 279

280. The area shown is subjected to a uniformly varying stress, so that the neutral axis is at axis XX. The maximum intensity of stress is 1600 psi.

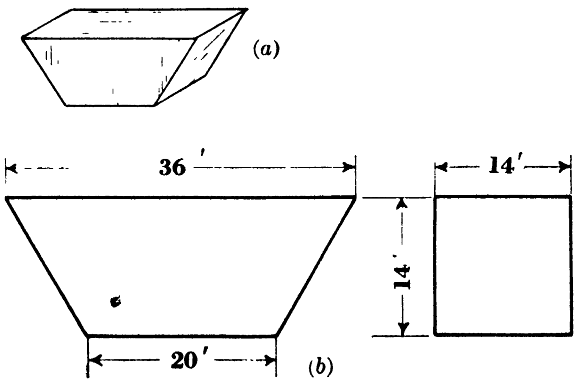
Determine the resultant stress and locate its point of application.

Ans. R = 13,200 lbs.; y_R = 3.8 ins. above XX.



PROB. 280

281. The tank shown contains oil weighing 60 lb. per cu. ft. Determine the resultant pressure on each vertical wall of the tank.

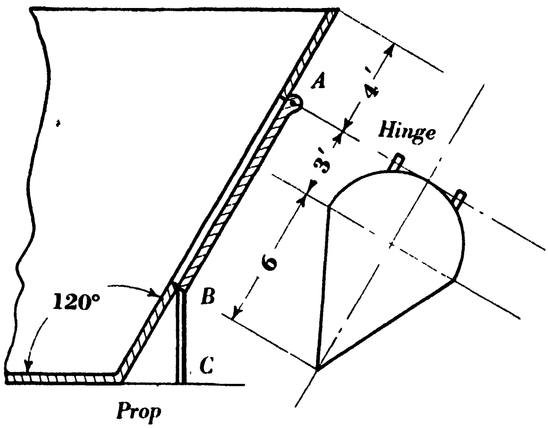


PROB. 281

282. Two views of a gate placed in the side of a sand hopper are shown. The gate is supported by a hinge at A and is held in position by a vertical member BC, which is pinned to the ground at C.

Determine the stress in BC. Assume the behavior of the sand to be equivalent to that of a fluid weighing 110 lb. per cu. ft.

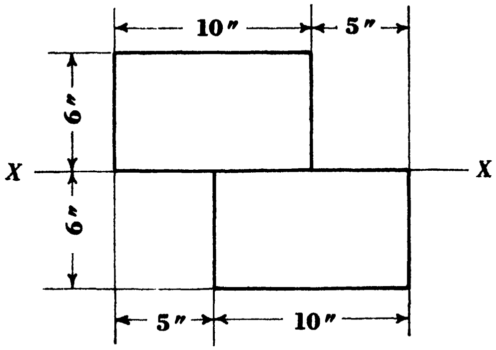
Ans. BC = 21,300 lb. compression.



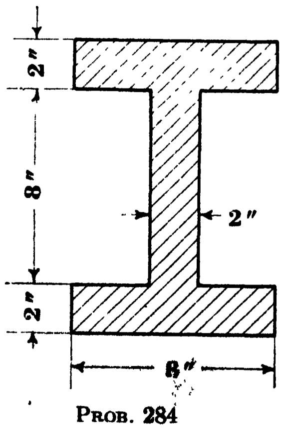
PROB. 282

283. The area shown is subjected to a uniformly varying stress, with neutral axis at XX. The maximum intensity of stress is 300 psi.

Determine the resultant of the stress.



PROB. 283



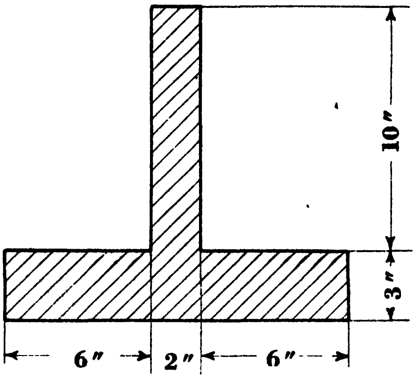
PROB. 284

284. The cross section of a beam is shown. The common theory of beams rests upon the assumption that the normal stress on a section of the beam perpendicular to its axis is uniformly varying, and that its resultant is a couple.

If the resultant of the normal stress on the beam section shown is 20,000 ft.-lb., determine the maximum intensity of stress. The neutral axis is parallel to the 8-in. base. *Ans.* 1340 psi.

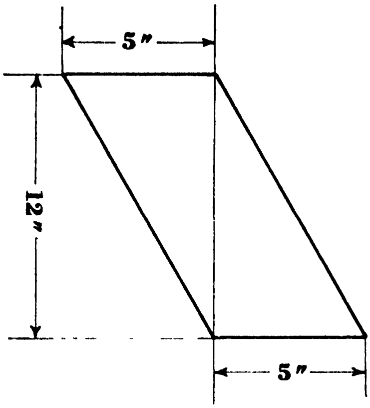
285. The cross section of a T-beam shown is subjected to a uniformly varying stress whose resultant is a couple of 10,000 ft.-lb.

Determine the intensity of stress at the top and bottom of the beam section. The neutral axis is parallel to the 14-in. base.



PROB. 285

286. The area shown is subjected to a uniformly varying stress whose resultant is 20,000 in.-lb. Determine the maximum intensity of stress. The neutral axis is horizontal.



PROB. 286

CHAPTER VI

Friction

54. Friction. When we attempt to slide one body over another, the contacting surfaces offer resistance to the motion. This action is called *frictional resistance* or, more simply, *friction*. We observe that very rough or very dry surfaces offer greater resistance to sliding than very smooth or lubricated surfaces.

The laws governing the behavior of surfaces subjected to frictional forces are not thoroughly understood, and they are the subject of extensive research.

In our present explorations of contacting forces, we shall find that we may summarize the influence of frictional force no matter what laws govern the intimate details of its behavior or the mechanism involved in the lubrication of the surfaces.

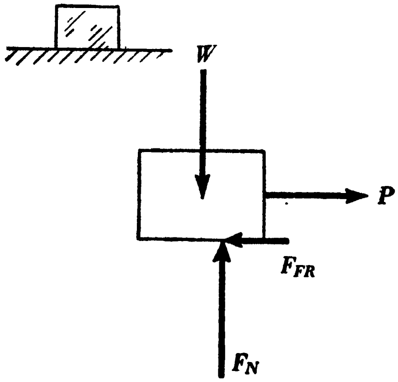


FIG. 146.

When a block rests upon a surface, as in Fig. 146, we can ascertain the relationship of the contacting forces by selecting, as a free body, the block itself. We find that the force system consists of the weight W, which may be assumed to be concentrated at the center of gravity (see Article 37) and the resultant F_N of the normal pressure of the supporting surface on the block. If now we add a pull P to the block, this external force will tend to cause motion of the block to the right, and a force of frictional resistance, F_{FR}, will begin to act. F_{FR} will be directed opposite to force P, for it resists the attempt of P to cause motion.

The block may, we find, remain at rest, or in equilibrium, under the influence of this system of forces. If, however, the force P is increased sufficiently, the block will begin to slide, for the force F_{FR} has a limiting value.

This limiting value is used as the criterion for establishing a coefficient μ, which is called the *coefficient of static friction*.

$$\mu = \frac{F_{FR}}{F_N}$$

in which F_{FR} is the value of frictional force when motion is just impending.

The coefficient μ has been empirically established for many kinds of sliding surfaces and is available in engineering handbooks.

The resultant reaction of the surface upon the free body is R (Fig. 147).

The angle θ between the normal pressure and the resultant is called the *angle of friction.*

Then,
$$\tan \theta = \frac{F_{FR}}{F_N} = \mu$$

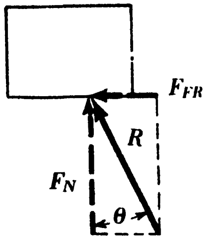


FIG. 147.

A block rests upon a plane, inclined at angle ϕ with the horizontal, as shown in Fig. 148. The block will tend to slide at the instant that the component of its weight ($W \sin \phi$) equals the frictional force F_{FR}. F_N is equal to $W \cos \phi$. At that instant, with motion just impending,

$$\frac{F_{FR}}{F_N} = \mu = \frac{W \sin \phi}{W \cos \phi} = \tan \phi$$

But $\mu = \tan \theta$, in which θ is the angle of friction. Then, as motion impends, the plane is inclined to the horizontal at an angle equal to the

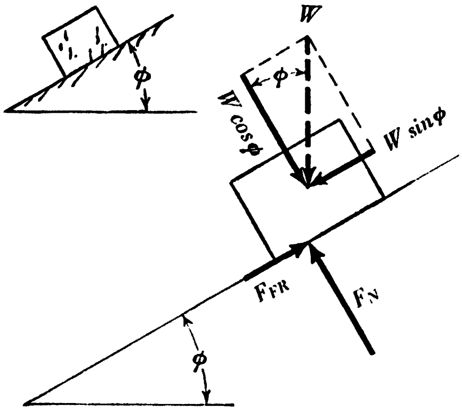


FIG. 148.

angle of friction. The angle ϕ for this limiting case is called the *angle of repose.*

If the angle between inclined plane and horizontal is increased beyond the angle of repose, the block will begin to slide down the plane. This fact is the basis of a simple experimental method for determining coefficients of friction, in which inclined plane and block are given surfaces of the kind of material and degree of smoothness under observation. The angle of repose is then measured, and the coefficient determined by substitution in the expression $\mu = \tan \phi$.

When a three-dimensional system of forces whose resultant is a single force, R, acts on a body, as shown in Fig. 149, the resultant of the system of forces (excluding the reaction of the plane) must be balanced by an equal, opposite, and collinear force R_1 if the body is to be in equilibrium. R_1 is limited by the coefficient of friction, because it must act within a cone having a cone angle equal to 2θ, where θ is the angle of friction.

It should be noted that μ expresses the ratio of frictional force to normal pressure only in the limiting case when motion is impending, and that $F_{FR} = \mu F_N$ is the maximum value of frictional force. In static equilibrium, any value of frictional force up to this limiting value may be active. For example, if we return to Fig. 146, we note that F_{FR} is always equal to P in static equilibrium and, therefore, may have a value less than the value of P which causes actual sliding.

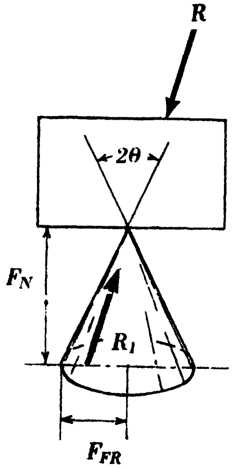


FIG. 149.

When sliding does take place, a definite relationship exists between the frictional force and the normal pressure. This coefficient is usually referred to as the *coefficient of kinetic friction* because of its analogy to the coefficient of static friction. It is not, however, equivalent to the latter, and involves variables, such as the relative velocity of the sliding bodies, and the lubrication and temperature of the sliding surfaces. In the absence of the more exact data, which will be revealed as research in this field continues, coefficients of kinetic friction are taken as those yielded by present experimentation. In the analyses of mechanics problems, coefficients of kinetic friction are used in the same manner as the coefficient of static friction to establish relationships in force systems.

55. The Inclined Plane. The block of Fig. 150 has been isolated, and the drawing shows this free body with the accompanying system of external forces. The weight of the block is W, and force P is a force inclined at angle β with the plane. The reaction of the plane is represented by components normal and parallel to the inclined plane. These components are F_N, the normal pressure, and F_{FR}, the frictional resistance.

If we select an X axis in the direction of the inclined plane and assume that force P is to draw the block up the plane at constant speed, the conditions of equilibrium may be applied.

$$\Sigma X = +P \cos \beta - W \sin \phi - F_{FR} = 0$$
$$\Sigma Y = -P \sin \beta - W \cos \phi + F_N = 0$$

But
$$F_{FR} = \mu F_N$$

Then,
$$\frac{P}{W} = \frac{\sin \phi + \mu \cos \phi}{\cos \beta - \mu \sin \beta}$$

The position at which the block will remain in equilibrium without having any force P in action ($P = 0$) is indicated in Fig. 151. This is the position when motion down the plane is impending and the sense of F_{FR} is reversed.

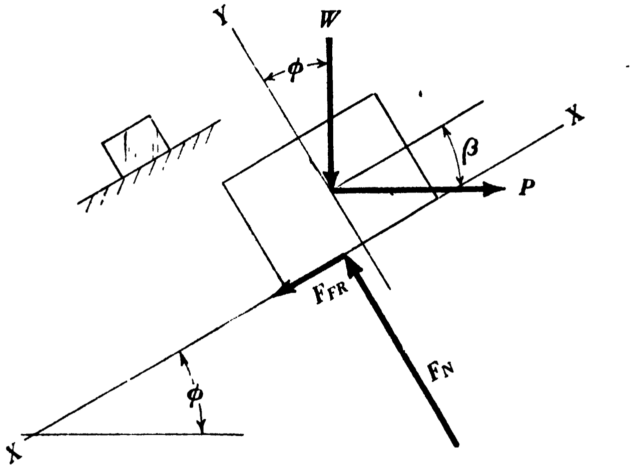


FIG. 150.

Then, applying the conditions of equilibrium,

$$\Sigma X = +F_{FR} - W \sin \phi = 0$$
$$\Sigma Y = +F_{N} - W \cos \phi = 0$$

Then,
$$\tan \phi = \frac{F_{FR}}{F_N}$$

If $\tan \phi$ is less than μ, the block will remain in position unaided by a force P. Such a condition is called *self-locking*. If $\tan \phi$ is greater than μ, a force P is required to prevent the block from sliding. When $\tan \phi = \mu$, the plane has been inclined at the angle of repose.

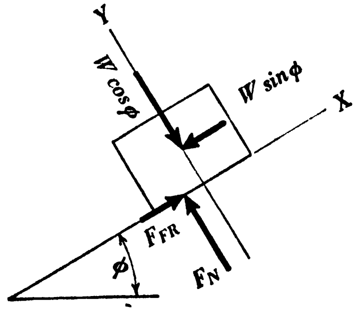


FIG. 151.

56. The Screw Thread. If the inclined plane discussed in the previous article is visualized as a thin sheet that is wrapped about a cylinder, as shown in Fig. 152, the screw thread is formed. The pointed rod, or follower, indicates the relationship of screw and nut.

If we parallel the free-body analysis made in the preceding article, we have the condition shown in Fig. 153. In this case, $\tan \phi = \frac{p}{2\pi r}$, where p is the pitch of the screw thread and r is taken as its mean, or pitch, radius.

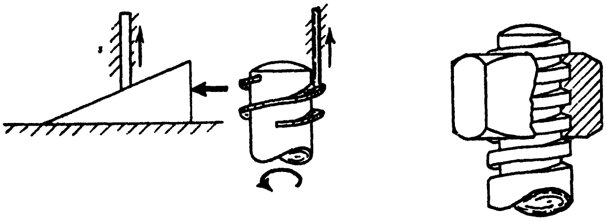


FIG. 152.

The force P, in this case, is applied horizontally, and $\beta = \phi$.

Then,
$$\frac{P}{W} = \frac{\sin \phi + \mu \cos \phi}{\cos \phi - \mu \sin \phi}$$

If $\tan \theta = \mu$ is substituted,

$$\frac{P}{W} = \frac{\sin \phi + \tan \theta \cos \phi}{\cos \phi - \tan \theta \sin \phi}$$

Dividing both numerator and denominator by $\cos \phi$, we have

$$\frac{P}{W} = \frac{\tan \phi + \tan \theta}{1 - \tan \theta \tan \phi} = \tan (\phi + \theta)$$

The self-locking position of a block on an inclined plane was discussed in the previous article and may be applied here to ascertain the self-

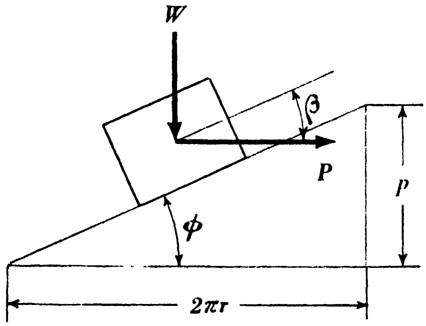


FIG. 153.

locking position of the screw—that is, the position in which no force P is needed to prevent the weight W from dropping when held by the screw.

In this case,

$$\tan \phi = \frac{p}{2\pi r}$$

When, therefore, $\frac{p}{2\pi r}$ is less than the angle of friction, the screw thread is self-locking. If the screw is used in a jack, as shown in Fig. 154, and force F is applied at a distance a from the axis of the screw,

$$F \times a = P \times r$$

Then,

$$F \times a = W \times r \times \tan(\phi + \theta)$$

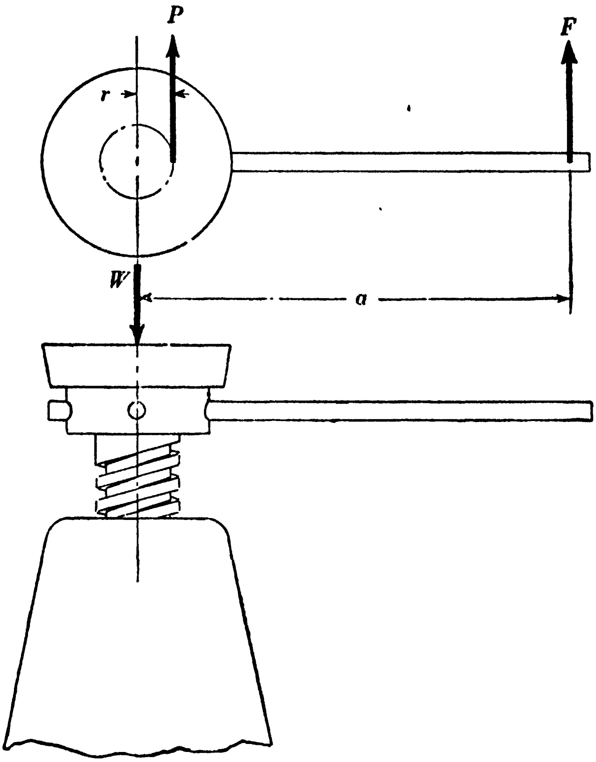


FIG. 154.

57. Pivot and Disc Clutch Friction. The shaft of Fig. 155 rests on a horizontal bearing surface, which is a circle of radius r. The normal pressure is assumed to be uniformly distributed over the entire circular area and, therefore, the normal pressure per unit of area is $\frac{W}{A}$.

If we examine an elementary area dA, the normal pressure on dA is

$$dF_N = \frac{W}{A}\,dA$$

The friction force is

$$dF_{FR} = \mu \frac{W}{A}\,dA$$

and has a moment about an axis perpendicular to the circular area at its center

$$dM_{FR} = \rho\mu \frac{W}{A}\,dA$$

The total moment of frictional resistance is

$$M_{FR} = \int dM_{FR} = \int \rho\mu \frac{W}{A} dA$$

$$dA = \rho \, d\theta \, d\rho$$

Then

$$M_{FR} = \int_0^{2\pi} \int_0^r \rho\mu \frac{W}{A} \rho \, d\theta \, d\rho$$

$$= \tfrac{2}{3}\mu Wr$$

In the disk clutch, the moment of friction is equal to that just derived, for the disk is a bearing surface subjected to conditions identical with those of the horizontal bearing surface shown in Fig. 155.

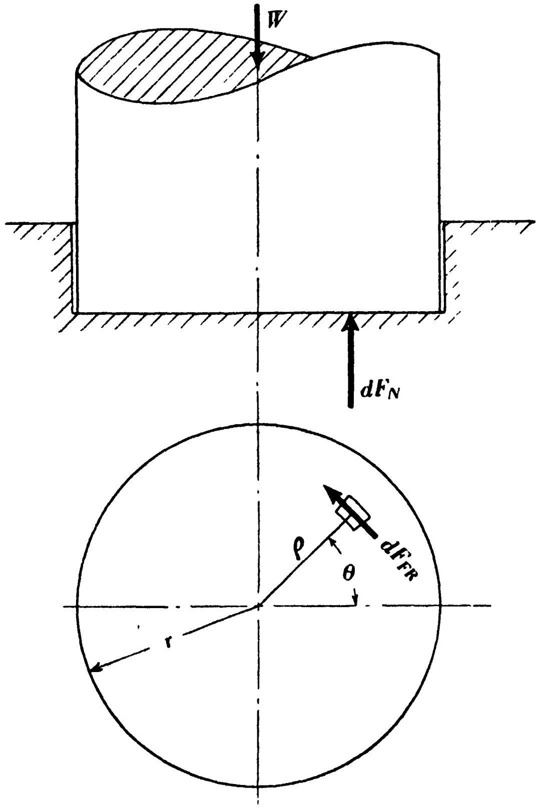


FIG. 155.

58. Brake-Band and Belt Friction. Although frictional resistance is undesirable in cases where its presence means dissipation of energy through heat, and consequent power losses, in many applications we depend upon friction to accomplish our purposes. In brakes, friction drives, and the like, frictional resistance is depended upon for the successful functioning of the mechanism. In the belt drive, illustrated in

Fig. 156, the force of friction enables the belt to produce torque and thus cause rotation of the pulley.

As a free body, an elementary length of belt has been isolated, and this body is shown in Fig. 157.

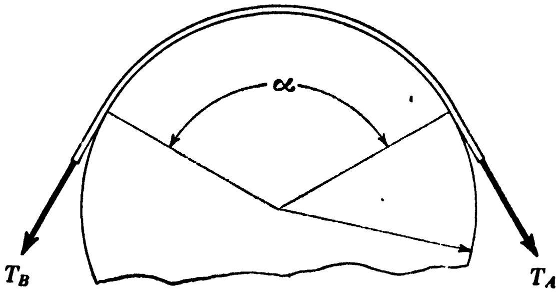


FIG. 156.

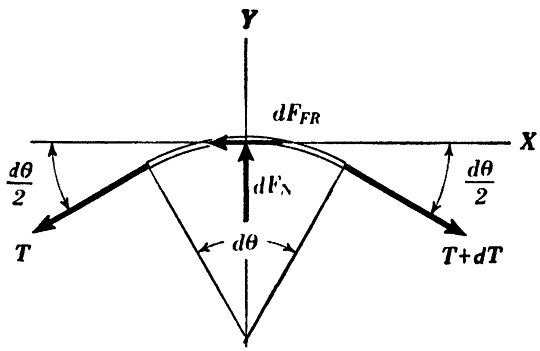


FIG. 157.

If p is the intensity of normal pressure between pulley and belt

$$dF_N = pr\,d\theta$$
$$dF_{FR} = \mu\,dF_N = \mu pr\,d\theta$$

If we apply the conditions of equilibrium, assuming the belt to be running at constant speed, then,

$$\Sigma X = +(T + dT)\cos\frac{d\theta}{2} - T\cos\frac{d\theta}{2} - \mu pr\,d\theta = 0$$

$$\Sigma Y = +pr\,d\theta - (T + dT)\sin\frac{d\theta}{2} - T\sin\frac{d\theta}{2} = 0$$

Since $\frac{d\theta}{2}$ is a small angle, $\cos\frac{d\theta}{2}$ approximates unity, and $\sin\frac{d\theta}{2}$ approximates $\frac{d\theta}{2}$.

The term $dT\,\frac{\sin d\theta}{2}$ is a small quantity of the second order and may be neglected. Then,

$$\Sigma X = dT - \mu pr\, d\theta = 0$$

$$\Sigma Y = pr\, d\theta - 2T\frac{d\theta}{2} = 0$$

Then,

$$\frac{dT}{T\, d\theta} = \frac{\mu pr\, d\theta}{pr\, d\theta} = \mu$$

We now sum up the forces acting on all of the elementary lengths of the belt in contact with the pulley over the angle α, usually called the *angle of wrap* and measured in radians.

$$\int_{T_B}^{T_A} \frac{dT}{T} = \int_0^{\alpha} \mu\, d\theta$$

$$\text{Log}_e \frac{T_A}{T_B} = \mu\alpha$$

$$\frac{T_A}{T_B} = e^{\mu\alpha}$$

This derivation of the relationship of the tensions in the sides of the belt is valid only when the speed of the pulley is moderate. At high speeds, centrifugal force will tend to diminish the normal pressure and, with it, the frictional resistance between belt and pulley. The derived expression must then be refined to properly express the relationship between tensions. The effect of high speed on such centrifugal force will be discussed in the chapters devoted to the subject of kinetics.

Brake-band friction. The preceding discussion may be applied with equal validity to the form of brake in which a band serves in the same manner as the belt of the previous example. Or, again, in the case of lines or hawsers wrapped around snubbing posts, the force required to restrain a much greater pull—as when a boat is held from moving with the current of a river—may be calculated from the derived expression.

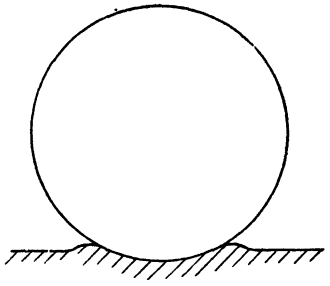


FIG. 158.

59. Rolling Resistance. When a cylindrical roller moves on a surface, the surface will deform or yield, as suggested in Fig. 158. P is the horizontal force required to move the roller at constant speed. W is the load carried by the roller, and R is the resultant force which the surface exerts on the roller. Fig. 159 shows the roller isolated as a free body.

If we select point B, the point of application of R, as a moment-axis,

$$\Sigma M_B = +P \times b - W \times a$$

and

$$P = \frac{a}{b} \times W$$

The distortion of the surface is slight, and the radius of the roller r may be substituted for the moment-arm b.

Then,
$$P = \frac{a}{r} W$$

The term a is usually called the *coefficient of rolling resistance*, and it is not a pure number like the μ we have previously employed as a coefficient of friction, but is a linear distance, and its value is, therefore, expressed in inches.

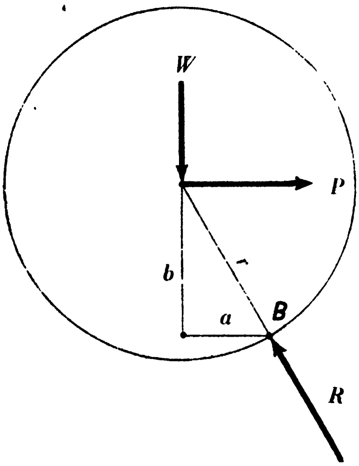


FIG. 159.

The laws of rolling resistance are not thoroughly understood, but some coefficients of rolling resistance obtained from experiment have been published.

60. Journal Bearings. Friction Circle. The detailed analysis of the frictional resistance when a shaft rotates in a bearing is involved because of the difficulty in establishing the exact surface of contact. When a shaft is newly fitted in a bearing, there may be a considerable region of contact between the neighboring bodies. This region will vary with the allowance made in the cylindrical fit. As the bearing wears, the contact will be confined to a more limited region.

An approximate solution may be effected by assuming that, in a bearing which has been worn in, the region of contact is so limited that it is confined to a line. This condition is illustrated in Fig. 160.

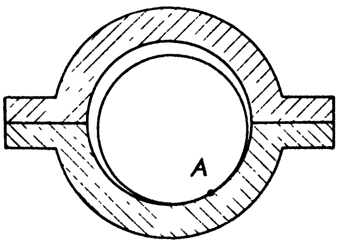


FIG. 160.

Point A is the mid-point of the line of contact. The shaft has been isolated as a free body in Fig. 161, and the resultant force R exerted by the bearing on the shaft at point B is shown. The direction of rotation of the shaft is clockwise.

Then R has components F_N directed radially toward the center of the shaft, and F_{FR}, with sense opposing the motion. R is tangent to a circle of radius r_1 whose center is the center of the shaft.

$$r_1 = r \sin \theta$$

where r is the radius of the shaft.

θ is the angle between F_N and R and is, therefore, the angle whose tangent is μ, the coefficient of friction. The circle of radius r_1 is called the *friction circle.* When the shaft is lubricated, μ is small and tan θ may be substituted for sin θ. Then,

$$r_1 = r\mu$$

The connecting rod of Fig. 162 joins the pins A and B, which are rotating relative to the connecting rod in the indicated senses. If the connecting rod is in compression, the normal pressure exerted on the con-

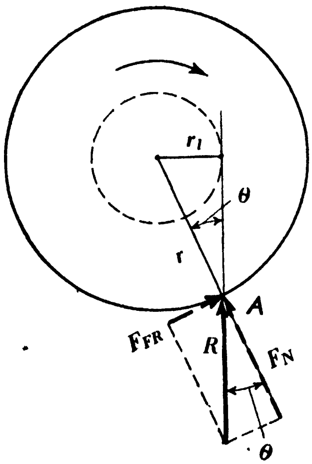


FIG. 161.

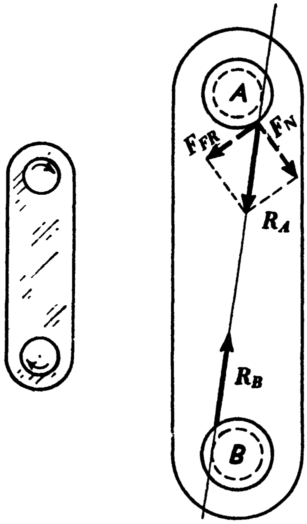


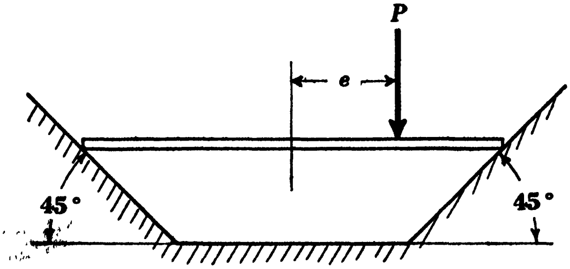
FIG. 162.

necting rod by the pin at A will be F_N, and the friction force will be F_{FR}. The resultant R_A of these forces will act tangent to the friction circle, as shown. The resultant force acting on the connecting rod at point B will be R_B, which must be equal to, opposite to, and collinear with R_A; R_B will, therefore, be tangent to the friction circle at B.

PROBLEMS

287. The 20-ft. beam is horizontal, and the eccentricity e of the load P is measured from the center of the beam. The coefficient of friction between walls and beam is $\mu = 0.3$.

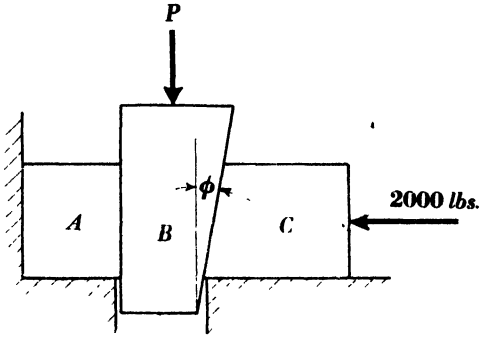
Neglecting the weight of the beam, determine the maximum value of e to prevent sliding of the beam.



PROB. 287

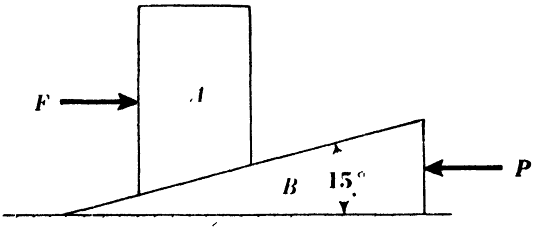
288. In the cotter joint shown, angle $\phi = 13°$. The coefficient of friction for all surfaces is $\mu = 0.18$. Determine the value of force P required to cause motion to impend against a resistance of 2000 lb., applied as shown.

Ans. $P = 1320$ lb.



PROB. 288

289. Block A weighs 1000 lb. Force F is applied by the vertical side guide. The coefficient of friction for all sliding surfaces is $\mu = 0.20$. Determine forces P and F for equilibrium of the system.



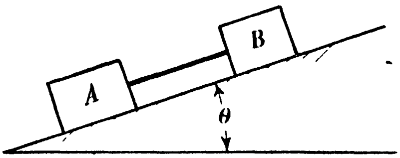
PROB. 289

290. A homogeneous beam of length L and weight W rests on a horizontal floor and is supported by a vertical wall. The coefficient of friction between the floor and beam is $\mu_1 = 0.30$ and between beam and wall is $\mu_2 = 0.35$.

Determine the angle between the floor and the beam when motion impends.

291. Block A weighs 200 lb., and block B weighs 100 lb. The coefficient of friction between A and the inclined plane is $\mu_A = 0.30$; and the coefficient of friction between B and the inclined plane is $\mu_B = 0.60$.

Determine the limiting value of θ for equilibrium, and the stress in the rod connecting A and B at that position. *Ans.* $\theta = 21.8°$.



PROB. 291

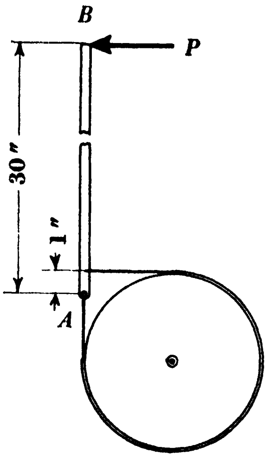
292. The pitch diameter of the thread on a jackscrew is 1.5 in. The screw has 4 threads per inch. If a force $P = 50$ lb. is applied on the handle, 24 in.

from the center, and the coefficient of friction of the screw-thread is $\mu = 0.10$, determine the weight W which may be raised by the jack.

What weight W may be lowered with $P = 50$ lb.? *Ans.* $W = 8750$ lb.

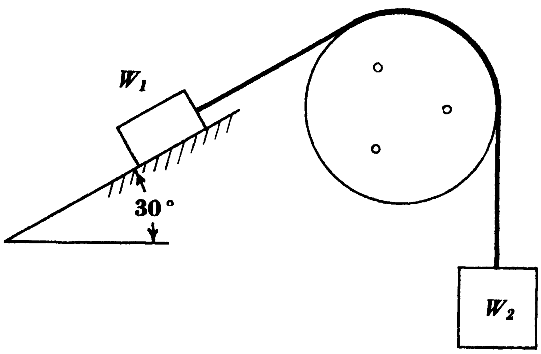
293. A rope is wrapped around a snubbing post. If a 50-lb. pull will resist a 1000-lb. load, determine the number of turns of the rope on the snubbing post. The coefficient of friction is $\mu = 0.15$.

294. The brake band shown has a coefficient of friction $\mu = 0.2$. The arm AB is 30 in. long and rotates about a fixed axis at A. Determine the force P necessary to produce a frictional moment of 30,000 in.-lb. on the brake drum if the drum is rotating counterclockwise. Also determine the tension in the band. Diameter of band = 18 in.



PROB. 294

295. A drum is attached to a wall so that it cannot rotate. The weights W_1 and W_2 are attached by a belt passing over the drum. $W_2 = 100$ lb. Determine the maximum and minimum values of W_1 if the system is to remain in equilibrium. The coefficient of friction between W_2 and the inclined plane is $\mu_2 = 0.20$, and between the drum and belt is $\mu = 0.25$.



PROB. 295

CHAPTER VII

Virtual Work

61. Definition of Work. When a force is applied to a body, as illustrated in Fig. 163, so that the body moves, the work done by the force is defined as the product of the force multiplied by the distance that the point of application of the force moves, or

$$\text{Work} = F \times s$$

The block of Fig. 164 is pushed along the plane surface by force R, which is oblique to the plane surface. We may replace R by the equiv-

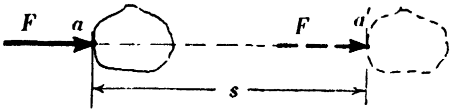


FIG. 163.

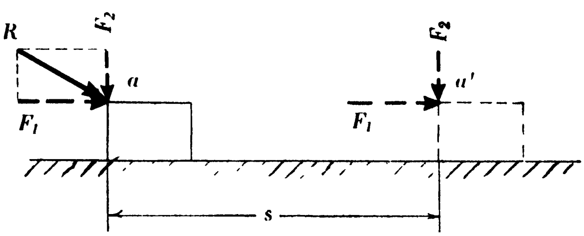


FIG. 164.

alent system of component forces, F_1 and F_2. In terms of the definition of work, the work done by force F_1 is then,

$$W = F_1 \times s$$

The work done by component force F_2 is, by our definition, zero, because there has been no displacement of the point of application a in the direction that F_2 is acting.

$$F_1 \times s = R \cos \theta \times s$$

and we conclude that a force does work only in the direction in which its point of application moves.

We may note that F_{FR}, the frictional resistance of the plane surface, is doing work equal to $F_{FR} \times s$, and the normal pressure of the plane surface on the block F_N is doing no work as the body moves.

We shall encounter the concept of work done by forces as they act upon moving bodies in greater detail in the problems of dynamics.

62. Virtual Work. At present, we are concerned with a limited form of work—that done when a body is assumed to move through a very short distance. We shall, then, establish the relationship of the displacements of the points of application of the external forces acting on the body. For example, in the case of the lever shown in Fig. 165, F_1, F_2, and F_3 are acting as external forces. F_3 is the force exerted by the fulcrum about which the lever is pivoting.

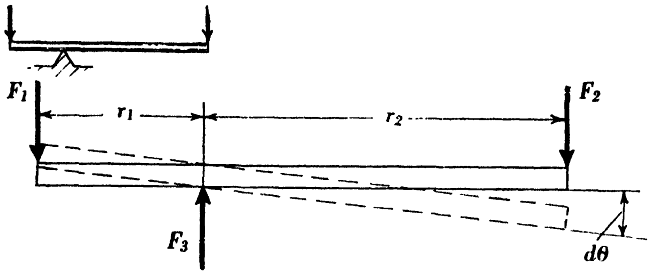


FIG. 165.

Let us assume that the lever is rotated through a small angle $d\theta$. Then the point of application of force F_1 moves through the distance $r_1\,d\theta$, and the work done by F_1 is

$$W_1 = F_1 \times r_1\,d\theta$$

This statement is literally true only in the limit as $d\theta$ approaches zero, for only in the limit is the actual displacement of the point of application of the force moving through a distance which may be represented by $r_1\,d\theta$. By assuming such motion, we also find that the work done by F_2 is

$$W_2 = F_2 r_2\,d\theta$$

The work done by force F_3 is zero, for its point of application has not moved at all. The point of application of F_1 has been raised, while the point of application of F_2 was lowered.

To properly summarize the effect of the total work done, we note that there is *opposition* of the work done by the two forces—if the work done by F_2 is considered positive work or effort; the work done by F_1 has been negative work or resistance.

Then, $$\Sigma W = W_1 + W_2 = -F_1 r_1\,d\theta + F_2 r_2\,d\theta$$

We have thus far assumed that the lever was given a very small angular displacement to determine the relationship between the terms representing work done by the forces. If the lever does not rotate at all (when $d\theta$ has reached its limit—zero), the system of forces acting upon it must be in equilibrium—no point of application of a force actually moves, and the sum of all work done on the body is zero.

Then, $$\Sigma W = -F_1r_1\,d\theta + F_2r_2\,d\theta = 0$$

The term $d\theta$ may be canceled, and

$$-F_1r_1 + F_2r_2 = 0$$

This is a conclusion identical with that we should have reached had we used the condition of equilibrium, discussed in Article 17 for a system of parallel forces in a plane.

The technique which we have employed differs very markedly, however, from the method of taking moments. It is the type of philosophical speculation which enabled Galileo and others to make valuable contributions to the founding of the science of mechanics.

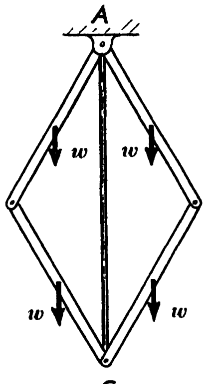


FIG. 166.

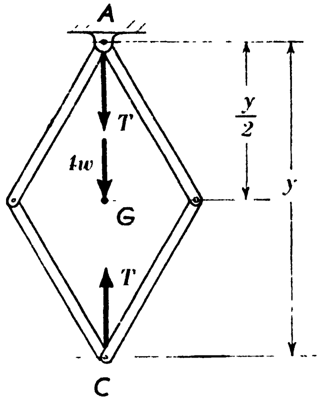


FIG. 167.

Such speculative analysis has direct value in applications of mechanics when the analysis by the usual technique of statics is cumbersome or difficult.

The principle we have employed is called the *principle* of *virtual work*. The term *virtual* appears here as an adjective, denoting that it is present in spirit or virtue, but not in fact, which is a significance quite consistent with the definition we find in the dictionary. We have assumed, for example, rotation of the lever to discover the relationship which would prevail between the terms representing work, no matter how small the angle of rotation, and have then reduced such an angle to its ultimate limit of zero without doing violence to the relationship which was discovered in the process.

Fig. 166 shows four rods of equal weight w and equal length, pinned together to form a rhombus.

The entire frame is suspended from a pin joint at A. A thin rod AC, whose weight is so small that it may be neglected, joins the pins A and C, and prevents the rhombus from collapsing under its own weight.

It is desired to find the tension in rod AC.

To use the principle of virtual work, we assume that rod AC is removed and replaced by forces T, as shown in Fig. 167.

If distance AC is called y, the distance from A to G (the center of gravity of the four rods) is $\frac{y}{2}$. If point C is given a small vertical displacement dy, the positive work done by the lower force T will be $T\,dy$. The work done by the upper force T will be zero, for its point of applica-

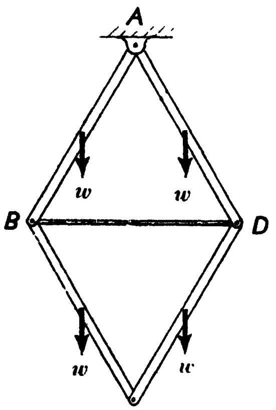


FIG. 168.

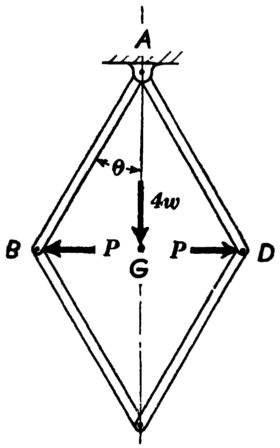


FIG. 169.

tion has not been displaced. The point G will rise through a displacement $\frac{dy}{2}$, and the resistance work done by the combined weights of the four rods is $4w\frac{dy}{2}$.

Then, applying the principle of virtual work,

$$\Sigma W = +T\,dy - 4w\frac{dy}{2} = 0$$

and

$$T = 2w$$

which is the tension in rod AC.

Four rods of equal weight w and equal length are shown in Fig. 168. The rhombus which is formed is braced against collapsing by a rod BD (of negligible weight), and we wish to determine the stress in rod BD.

BD is removed and replaced by the forces P shown (Fig. 169). If we assume that angle θ is increased by $d\theta$, the point of application of each force P will move horizontally through a displacement $AB\,d\theta\cos\theta$. Then the positive work done by the two forces P will be $2PAB\,d\theta\cos\theta$. At the same time, the weight of $4w$ will be raised through a vertical displacement $AB\,d\theta \sin\theta$, and the resistance work done will be $4wAB\,d\theta \sin\theta$.

If we apply the principle of virtual work,

$$\Sigma W = 2PAB\,d\theta \cos\theta - 4wAB\,d\theta \sin\theta = 0$$
$$P = 2w \tan\theta,$$

which is the compression in rod BD.

In the crank-and-connecting-rod mechanism of Fig. 170, AB is a crank which rotates about fixed axis A, and BC is a connecting rod, pinned to the crank and to a piston at C. The piston slides in fixed guides, and we shall assume that all surfaces are frictionless.

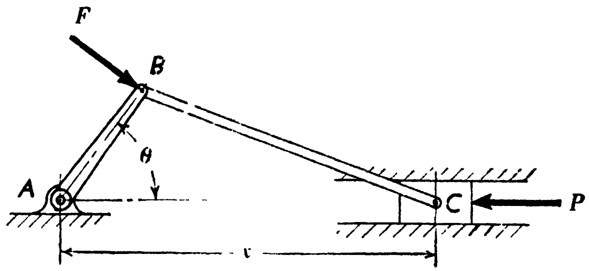


FIG. 170.

The length of AB is r; of BC, l; and the distance AC may be called x The total pressure on the piston is P, and it is required to find the force F, perpendicular to the crank at B, which will hold the system in equilibrium.

If the crank is turned clockwise from its indicated position through angle $d\theta$, the positive work done by force F is $Fr\,d\theta$. The negative work done on the piston by force P will be $P(-dx)$.

Then, by the principle of virtual work,

$$\Sigma W = +Fr\,d\theta - P(-dx) = 0$$

or

$$F = -\frac{P\,dx}{r\,d\theta}$$

The relationship between $d\theta$ and dx may be determined by applying the law of cosines,

$$l^2 = r^2 + x^2 - 2rx\cos\theta.$$

Differentiating with respect to θ, we have

$$0 = 0 + 2x\frac{dx}{d\theta} - 2r\left(-x\sin\theta\frac{d\theta}{d\theta} + \cos\theta\frac{dx}{d\theta}\right)$$

Then,

$$\frac{dx}{d\theta}(2x - 2r\cos\theta) = -2rx\sin\theta$$

and

$$\frac{dx}{d\theta} = -\frac{rx\sin\theta}{x - r\cos\theta}$$

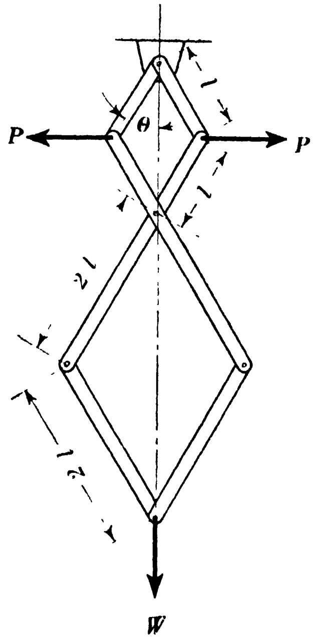
Therefore,

$$F = \frac{Px\sin\theta}{x - r\cos\theta}.$$

Problems

296. The members are symmetrical about the vertical center line. Determine the value of P in terms of W and θ for equilibrium.

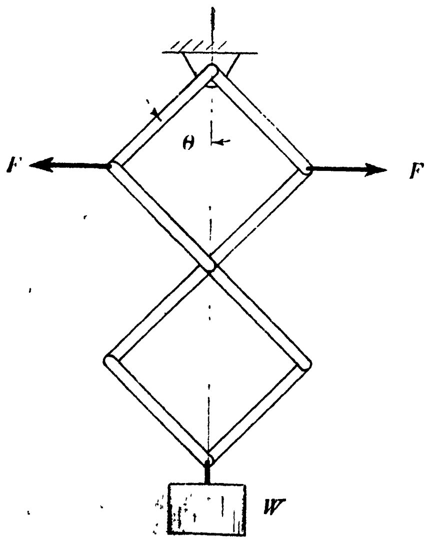
Ans. $P = 3\,W \tan \theta$.



Prob. 296

297. The members are of equal length and are symmetrical about the center line.

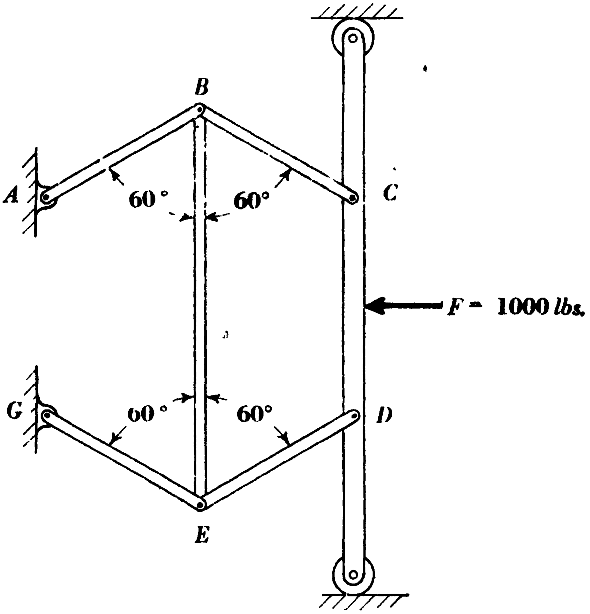
Determine the value of θ when $F = W$.



Prob. 297

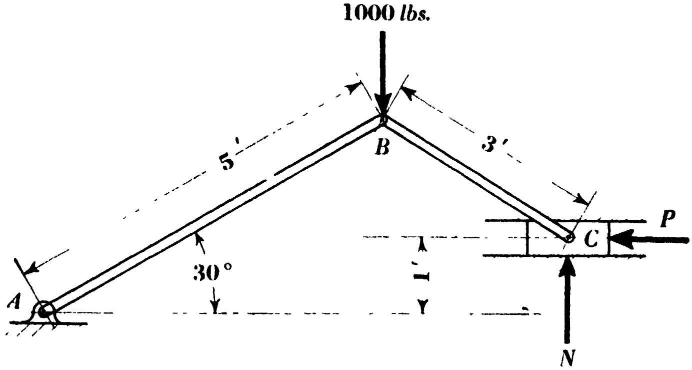
298. The frame shown consists of four members AB, BC, DE, and EG of equal length and the tie rod BE.

If a force $F = 1000$ lb. is applied as shown, determine the stress in BE when the system is in equilibrium, assuming no friction. F is applied at the mid-point of CD. All joints are pin joints.



Prob 298

299. Determine the forces P and N acting on the sliding block at C if the system is in equilibrium. Assume that all members are weightless, and that no other external forces act on the sliding block.



Prob. 299

Part II

DYNAMICS

CHAPTER VIII

Velocity

63. Kinematics* is the branch of mechanics that investigates the motion of bodies, without accompanying analyses of the force systems which must be present to produce the motion, or of the mass properties of the bodies themselves. Kinematics is, then, a study of the geometry of motion and of the time element involved in the motion, as fundamental concepts, and of velocity and acceleration, which are derived from these fundamentals.

We have concerned ourselves, in the portion of our work devoted to statics, very largely with studying the nature of systems of forces—generally when such forces produced no change of motion of the free body upon which they act.

We shall now add a study of the subject of kinematics, so that we may be equipped with the axioms and techniques which changes of motion, as they indicate themselves in the properties of velocity and acceleration, demand. Later, we shall merge our growing store of the fundamental principles concerning systems of forces and resulting changes of the motion of the free bodies upon which they act. This integration will occur when we enter the field of kinetics, which merges the study of force systems, mass properties of the free bodies, and acceleration.

64. Orthogonal Components. The vectors discussed thus far have been of general interest and of universal application as representations of such vector quantities as force. In the study of kinematics, one class of component vectors are of particular interest.

We have, in the past, dealt frequently with rectangular components of vectors, which were the components obtained by resolution when the axes of the components are at right angles.

In our studies of velocity and acceleration, we shall find that constant use may be made of *one* of the mates of a pair of rectangular components. The complementary, or mated, component is, of course, always present, but an individual rectangular component, by itself, becomes a very effective means of attack. Since we shall use such an isolated component constantly, it is convenient to give it a distinguishing name—the *orthogonal* (right-angled) *component.*

An orthogonal component is obtained by projecting a vector upon

* In the chapter devoted to Kinematics, the author has presented some of the material previously published in his *Engineering Kinematics.* The Macmillan Company, New York: 1941.

any desired axis. The projection is orthographic projection—that is, the projector is always perpendicular to the axis of the component. In Fig. 171, A is the orthogonal component of R on the axis OX, and B is the orthogonal component of R on the axis OS. The heavy arcs represent angles of 90 degrees.

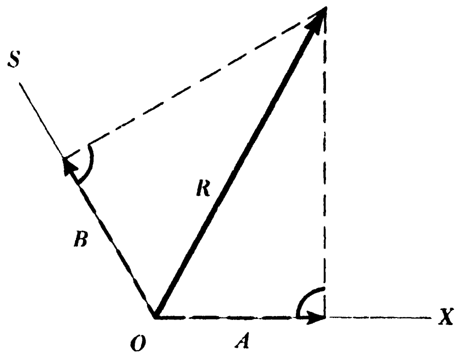


FIG. 171.

65. Theorems of Orthogonal Components. From the definition of orthogonal components, it follows that many resultant vectors have the same orthogonal component along a given axis. In Fig. 172, R_1, R_2, and R_3 are all vectors which, originating from the same origin O, will have the vector A as their orthogonal component along axis OX.

This statement leads to a closely related thought. If it is known that a vector, such as B of Fig. 173, is an orthogonal component of some

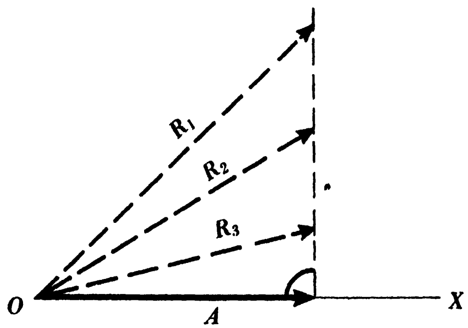


FIG. 172.

resultant vector, then the resultant must have an origin at O, and a terminus which lies somewhere in the line ab, perpendicular to the line of action of B.

Now if, in addition, the inclination of the resultant is known, that resultant may be determined. Given the inclination (as 30 degrees with the X axis) of the resultant, we construct, as in Fig. 174, a line making an angle of 30 degrees with the X axis from origin O. The resultant has now been fixed at its proper inclination. Next, a perpendicular to OX at a terminus of B is erected, and this perpendicular is extended to meet

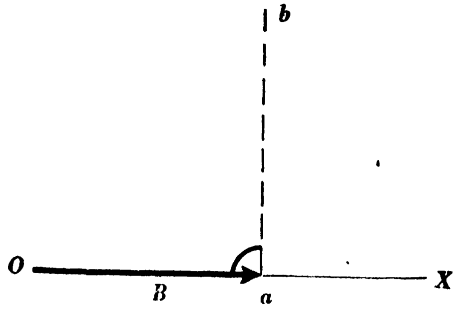


FIG. 173.

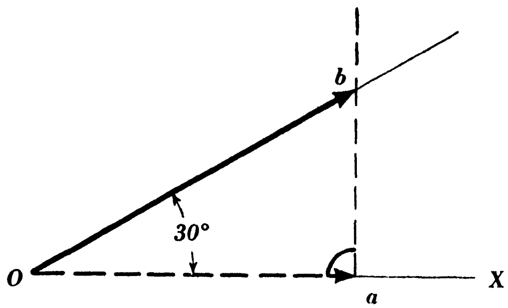


FIG. 174.

the line of action of R at point b. Then Ob is the magnitude of the resultant R. R must have a sense upward to the right, as indicated by the arrowhead, in order that the sense of resultant and given component may be consistent. We summarize this conclusion as follows:

Theorem I. One orthogonal component and the inclination of the resultant vector determine the resultant vector.

We can also determine a resultant vector if given another source of information.

The components A and B of Fig. 175 are given as orthogonal, components of the same resultant vector.

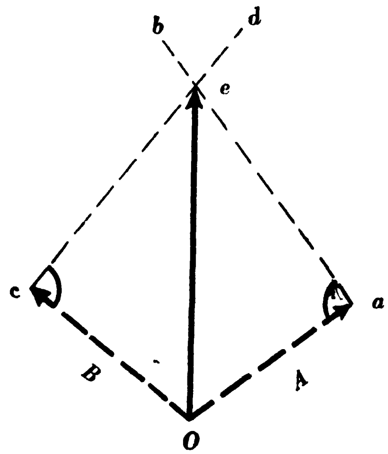


FIG. 175.

Any resultant vector of which A is an orthogonal component must have an origin at O and a terminus in the line ab.

Similarly, any resultant vector of which B is an orthogonal component must have an origin at O and a terminus in the line cd. c is the only point that can lie in both ab and cd and is, therefore, the terminus of the resultant R.

This information crystallizes as a second theorem:

Theorem II. Two orthogonal components of any vector determine that resultant vector.

66. Linear Displacement. We are accustomed to thinking of the motion of a body as a *change of position.* If a particle moves from point A to point B (Fig. 176), the change of position is defined as the displacement Δs. This definition does not describe the exact path over which the particle has traveled in going from A to B, but tells us only that the particle started at A and finally arrived at B. It might have traveled along any one of an unlimited number of paths, as suggested by the dotted lines of the figure Δs, the displacement or announcement of change of position remaining the same in all cases. This displacement is a vector quantity, for it has both magnitude and direction.

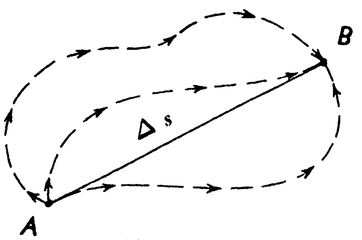


FIG. 176.

67. Absolute and Relative Displacement. Δs is independent of any framework of reference axes that might surround it. We could place Δs in any framework, such as axes OX and OY of Fig. 177, but any other framework would serve equally well in defining the displacement. In each case, we would describe the displacement as relative to the framework of axes. Since displacement is a vector quantity, we could give information as to the resultant displacement or we could give equivalent information by announcing the X and Y components of the displacement.

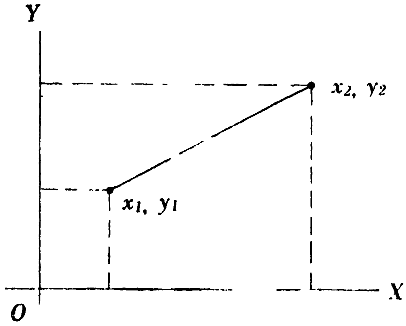


FIG. 177.

All displacements, then, are described in terms of an origin and framework of axes. Now, this framework may be fixed or attached to the earth's surface. We appreciate the fact that the earth's surface is in motion, but in our analysis of machines and structures, we usually encounter cases where the machine or structure has a foundation that is either stationary on the earth's surface or is moving relative to the earth's surface. The earth's surface is, therefore, taken as reference, and displacement relative to the earth is called *absolute displacement.*

When the origin of axes, accompanied by its framework, is in motion relative to the earth, it becomes necessary to divide a description of displacement into two contributing factors. In Fig. 178, we note a point A on a wheel which is in turn mounted upon a car. A might be a point on the crank of an engine which drives an automobile. If the car is stationary, we can set up an origin, O', and axes $O'X'$ and $O'Y'$, as shown in Fig. 178. Now we can describe the displacement, or change of position, of point A as the wheel turns. In such a description in terms of relationship of A to $O'X'$ and $O'Y'$, we are giving displacement relative to axes which are themselves stationary. We are, therefore, announcing an absolute displacement.

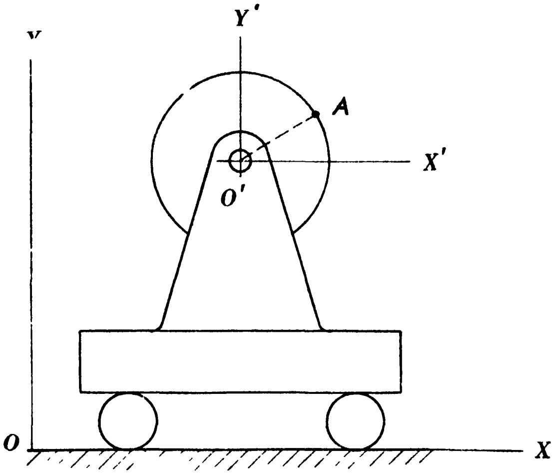


FIG. 178.

If the car itself is now set in motion relative to the earth's surface, the origin of our axes $O'X'$ and $O'Y'$ will be carried along with the car. To complete the description of the absolute displacement of point A in such a case, we add an origin O and axes OX and OY. The absolute displacement of origin O' is its displacement relative to origin O. The absolute displacement of point A, in this case, is the *sum* of its displacement relative to origin O', plus the absolute displacement of origin O'.

This analysis has led us to a conclusion which we shall express as our third theorem of kinematics:

Theorem III. The absolute displacement of a moving point is equal to the sum of its displacement relative to a second moving point, plus the absolute displacement of the second moving point.

The quantity displacement is a vector quantity. We shall find it convenient to use symbols to express vector addition and subtraction. The symbol $+\!\!\!\!\rightarrow$ will denote plus in vector notation, and $\rightarrow$ indicates minus.

We shall also adopt terminology to distinguish between absolute and relative concepts. The subscript A/B will be used as an abbreviation of the phrase "A relative to B," and a single letter will be used to indicate an absolute property.

Abbreviating Theorem III, for example, we have

$$s_A = s_{A/B} \nrightarrow s_B$$

68. Degrees of Freedom. When we relate the motion of a particle to a fixed point, we must recognize the quality of the various possibilities involved in such motion. If a complete description of the motion may be given by specifying only one dimension in describing the displacement, the particle has but one *degree of freedom.* A particle that is free to move only along a straight line has but one degree of freedom, and only one dimension in space—distance along the straight line—describes the motion. A particle that is forced to move in a circular path about a fixed reference point also has but one degree of freedom, since giving the angle which the rotating radius containing the point makes with a fixed axis through the reference point in the plane of the motion determines the location of the particle.

When a particle is free to move anywhere in a plane, two coordinate dimensions must be given in specifying the location of the moving point at any instant, and the particle has two degrees of freedom. If a particle is free to move anywhere in space, three describing dimensions are necessary, and the particle has three degrees of freedom.

This concept of *degrees of freedom* is the kinematical counterpart of geometrical descriptions. When we expand the concept to enable us to discuss bodies, we note that a body may have angular motion about an axis at the same time that it is free to move in the direction of the axis.

Then a free body may have as many as six degrees of freedom—it may have angular motion relative to each of the three coordinate axes, and it may have, at the same time, linear motion in the direction of each of the three axes.

69. Translation. If a body moves so that no straight line of the body changes its inclination, the motion is called *translation.* If the body shown in Fig. 179 moves from an initial position (shown by the solid outline) to a final position (indicated by the dotted outline), so that line AB has constant inclination θ with the X axis, the body has moved in translation. All points lying in the line AB or, indeed, anywhere on the body, have moved along parallel paths. When these parallel paths are straight lines, the translation is said to be *rectilinear,* and when the paths are curved, the translation is *curvilinear.*

Then, if we describe the motion of one particle of a body in translation, we have given a description common to all particles.

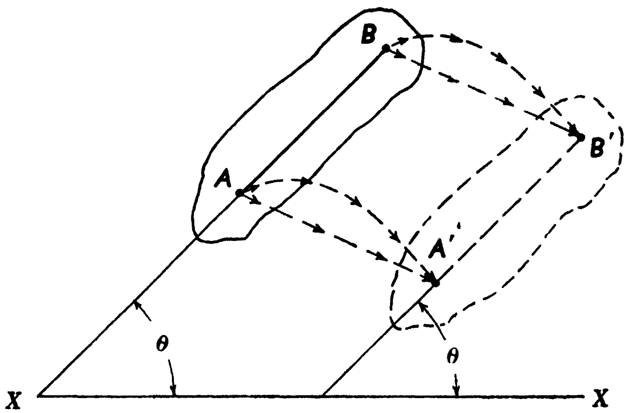


FIG. 179.

70. Rotation. During a motion of rotation, the lines of the body change their inclination with a fixed axis as the body is displaced. In the displacement illustrated in Fig. 180, the inclination of line AB has changed from angle θ with the X axis to angle θ_1 with the same axis. Then the motion is rotation.

Pure rotation is defined as motion of the body when an axis, called the *axis of rotation*, through one point of the body remains fixed, while all other particles describe circular paths about the fixed axis.

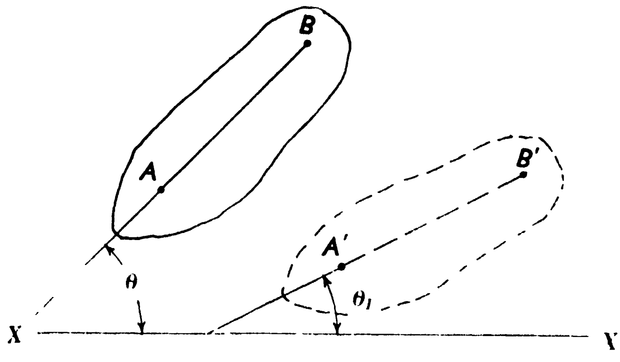


FIG. 180.

71. Velocity. The preceding discussion of motion has established certain geometrical descriptions associated with change of position, or displacement. In that discussion, we have avoided considering the time which is involved in these changes of position. The first property of motion that amplifies our knowledge of change of position beyond the geometry of the motion is *velocity*. Velocity adds a measure of the time involved in a change of position and is defined as the rate of change of position with respect to time or, more simply, the time rate of displacement.

72. Linear Velocity. When a point or particle changes its position, time is consumed in making that change. During this time, the particle may be moving continuously in one direction (as in rectilinear motion) or it may be changing its direction (as in curvilinear motion). In either case, the displacement is a directed, or vector, quantity, and the velocity, which adds the concept of time, is also a vector quantity whose direction is identical with that of displacement.

When a point travels along a path so that equal amounts of distance are covered in equal intervals of time, the point is said to have constant speed. The speed will then be the quotient obtained by dividing any distance Δs by the corresponding time interval, Δt. If the point is traveling along a path which is a straight line, the direction of displacement remains unchanged, and the point has constant velocity.

When the point travels along a curved path, the speed may be constant, but the velocity is changing with the change in direction of displacement, and hence is variable. In the case of a point traveling along the circumference of a circle at constant speed, it will traverse equal lengths of arc in equal time intervals, but its velocity will be constantly changing in direction.

When a point travels along a path so that its speed is variable, we must measure its speed instantaneously, since there is no constant or general expression which prevails.

If, in this event, we select a small interval of time Δt during which the point travels a distance Δs, and then set

$$v = \frac{\Delta s}{\Delta t}$$

v will be the average value of speed during the time interval. By decreasing the interval indefinitely, we have

$$v = \operatorname*{Limit}_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

Speed is then the first derivative of displacement with respect to time, and we must turn to the calculus for analysis of problems involving variable or nonuniform motion.

As we plan those attacks upon problems in which we must make use of the calculus as a tool of analysis, two avenues of approach confront us.

When displacement-time relationships are available in the form of equations expressing directly the displacement as a function of time—for example, when

$$s = 6t^2 + 3t + 4$$

the analytical differentiation yields

$$v = \frac{ds}{dt} = 12t + 3$$

and speed at any desired time may be computed.

More often, as in mechanisms, we encounter displacements that are in such relationships with time that no direct equation, or only an approximate empirical one, may be available. In these cases, the calculus is again employed, but now, in place of analytical operations, we turn to graphical ones.

PROBLEMS

300. A cutting tool has a length of stroke equal to 4.6 in. If the time per stroke is 0.2 sec., find the average speed of the tool in feet per minute.

301. A mechanism follower travels along a straight path with

$$s = 3t^2 + 4t$$

where s = displacement in inches
t = time in seconds

Determine the speed at the end of 4 sec. in feet per minute. *Ans.* 140 fpm.

302. A cam follower moves in a straight line with

$$s = t^3 - 8t^2 + 13t$$

with s in feet and t in seconds.

What is the speed of the follower when $t = 0.25$ sec.?

How long does it take the follower to come to rest from $t = 0$?

303. The velocity of a cam follower is

$$v = 1.6t + 0.4$$

with v in inches per second and t in seconds.

How long will it take the follower to travel 5 in. from a start at $t = 0$?

73. Graphical Calculus. Differentiation. The graphical calculus translates, as do many other graphical techniques, a mathematical or analytical procedure into the language of the drawing.

Having noted that speed is the derivative of displacement with respect to time, let us translate the definition of a derivative into graphical terms.

If Δs is any increment (it may be an increase or decrease) given to displacement and Δt is the corresponding increment in time, then the derivative of displacement with respect to time (or speed) is the limit of the ratio of Δs to Δt as Δt approaches zero or, more tersely,

$$v = \underset{\Delta t \doteq 0}{\text{Limit}} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

On the curve of Fig. 181, where it is assumed that a curve of the displacement-time relationship is known, it is desired that the speed at any point, as A, of the curve be determined.

It will be noted that in any finite time (Δt), represented by coordinate distance (Ac), the corresponding increment in displacement (Δs) is the coordinate distance bc, and the ratio $\frac{\Delta s}{\Delta t}$ is the tangent of the angle bAc.

As Δt or Ac diminishes, approaching zero as its limit, the chord Ab approaches, as its limiting direction, the line Ab'' tangent to the curve at A. At the same time, the ratio $\frac{\Delta s}{\Delta t}$ approaches the tangent of the angle $b''Ac''$. In the limit, a tangent to the curve at A will establish the limiting value of $\frac{\Delta s}{\Delta t}$, or the speed.

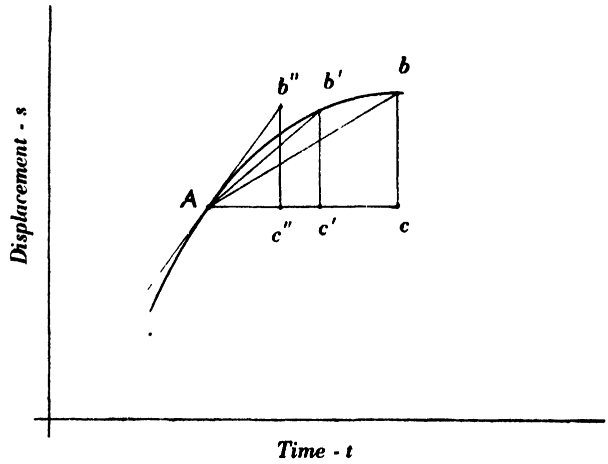


FIG. 181.

In drawing the tangent to a curve, optical aids may be employed, but in most work, these aids are unnecessary. The correct position of the tangent to a drawn curve may be found by placing a straight-edge on the curve and finding, by eye alone, the position when the straight-edge is just tangent to the curve, recalling that when a line is tangent to a curve, it just touches the curve, but does not cut into it. The eye is a surprisingly trustworthy agent for fixing the position of the tangent.

One simple optical aid to fixing the position of the tangent is worthy of mention, since its provision will enhance the accuracy of the results without involving the use of costly equipment. If a mirror or the polished surface of a strip of steel is used to fix the normal to the curve at any desired point, the tangent is readily drawn. The polished surface is placed at any point, such as A, of a drawn curve so that it will reflect the curve, as in Fig. 182. When the image or reflection lines up exactly with the drawn curve, the strip of steel is normal to the curve. At either side of this position, there will be a break in the continuity of curve and reflection. The tangent is drawn where the normal, established by reflection, intersects the curve.

The tangent line should be drawn as long as convenient since, as has already been noted in the discussions of precision drafting, the longer the line, the greater is its potential accuracy of measurement.

In Fig. 183, the speed at point A is determined. The line bc is drawn tangent to the given curve at point A. Line bd (parallel to the time axis) and line cd (parallel to the displacement axis) are next drawn, and these intersect at point d.

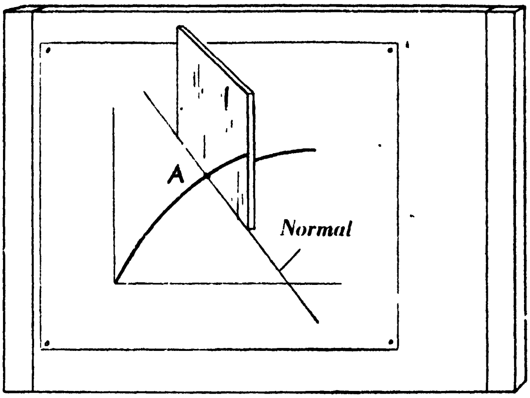


FIG. 182.

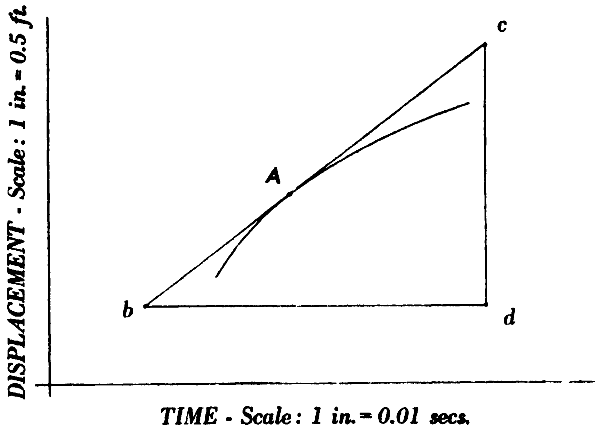


FIG. 183.

$cd = 1.34$ in. when measured, and $bd = 1.73$ in. Then the speed at point A is

$$v_A = \frac{\Delta s}{\Delta t} = \frac{cd}{bd} = \frac{1.34 \text{ actual inches}}{1.73 \text{ actual inches}}$$

$$= \frac{1.34 \times 0.5 \text{ ft. (by given scale)}}{1.73 \times 0.01 \text{ sec. (by given scale)}}$$

$$= \frac{0.67 \text{ ft.}}{0.0173 \text{ sec.}} = 38.7 \text{ feet per second}$$

The process of differentiation may be repeated at other points than A, and a sufficient number of points obtained to establish a curve showing the speed-time relationship.

A word concerning the number of points required to properly plot a curve is pertinent here. The infinite variety of shapes of curves makes any fixed rule impracticable. One point orients a straight line whose inclination is also known; two points determine any straight line; and some curves require an extremely large number of points. The rapidity of change of curvature is the only basis upon which intelligent decision may be made, and a sufficient number of points must be taken so that no significant value of the curve is lost. All plotted curves are approximations, and the portions of curve lying between plotted points are capable of concealing deviations from the *faired-in*, or approximated, curves. A safe procedure is to select a large number of points in regions of rapid change, and smaller numbers when the curvature is gradual or slow. Whenever there is suspicion of concealed change, additional points should be taken in the questionable region, until possibilities of unusual or unexpected change are exhausted.

The process of graphical differentiation is not an exact method, but an approximate one whose potential value is dependent upon the accuracy of setting the tangent. When the method is employed, therefore, it is wise that we have available some means of checking the solution. We shall next turn to another graphical method—graphical integration; which, in addition to serving in its own right, affords a means of checking the graphical differentiation of a given curve.

PROBLEMS

304–305. Given the value of displacement for each value of time. Plot the curve of displacement-time relationship. Differentiate graphically to obtain the velocity-time curve, which is to be plotted.

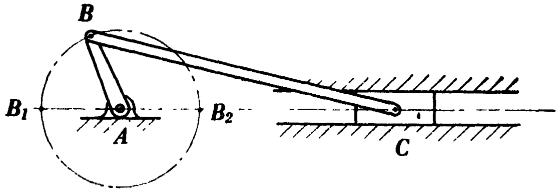
304.

DISPLACEMENT (*In.*)	TIME (*Sec.*)
0	0
0.40	0 01
1 26	0 02
2.80	0 03
4 94	0 04
7 06	0 05
8 44	0 06
9 12	0 07
9 36 (max.)	0 08
9 22	0 09
8 74	0 10
7 64	0 11
6 26	0 12
4.72	0.13
3.24	0.14
2.00	0 15
1.00	0.16
0.32	0 17
0	0.18

305.

DISPLACEMENT (*In.*)	TIME (*Sec.*)
0 00	0 0
1 29	0 1
1 93	0 2
2 30	0 3
2 50	0 4
2 54	0 5
2 37	0 6
1 98	0 7
1 44	0 8
0 72	0 9
0.00	1.0

306. The crank-and-connecting-rod mechanism shown in the figure has a driving crank AB which rotates clockwise with angular speed equal to 1000 r.p.m. $AB = 2$ in.; $BC = 8$ in.

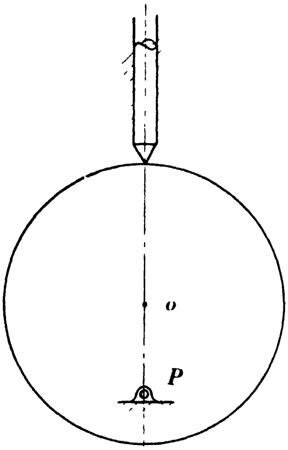


PROB. 306

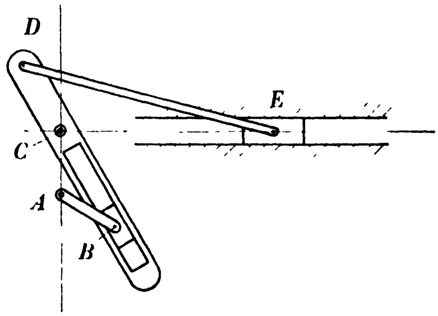
For each 15° position of AB, from starting position at B_1 to end of stroke at B_2, draw the line BC to determine the displacement of point C.

Plot the displacement-time curve, and differentiate, graphically. Plot the velocity-time curve.

307. The eccentric wheel cam shown in the figure has a constant angular velocity of 300 r.p.m. clockwise. The follower is pointed. Determine the displacement schedule for the follower. Plot the displacement-time curve. Differentiate graphically, and plot the velocity-time curve of the follower. Diameter of cam is 7 in., with center at O. Fixed axis is at P. $OP = 2.3$ in.



PROB. 307



PROB. 308

308. The Whitworth Quick-Return Linkage shown has a driving crank AB which rotates clockwise at 200 r.p.m. Determine the displacement of point E from start at the beginning of the stroke to the right, for a complete revolution of the driving crank. Differentiate the displacement-time curve graphically and plot the resulting velocity-time relationship. $AB = 3.2$ in.; $AC = 2.7$ in.; $CD = 4$ in.; $DE = 13.5$ in. A and C are fixed axes.

74. Graphical Integration is the reverse process to the differentiation just discussed. As in analytical integration, this process may be employed to investigate the character of the velocity when the nature of the acceleration is known or to determine the displacement when the known data give the description of the velocity.

Here, we may borrow from the calculus the concept that obtaining an integral is analogous to finding the area under a given curve—for example, displacement $s = \int v\, dt$ is graphically the area under the curve of relationship between velocity v and time t.

In Fig. 184, a known curve of velocity is plotted with the values of velocity as ordinates, and of time as abscissas. The area *abcd* represents the integral of the velocity with respect to time for time interval Δt. Since

$$s = \int v\, dt$$

this area represents the increment of displacement which has been added during this interval. A simple method for determining the amount of

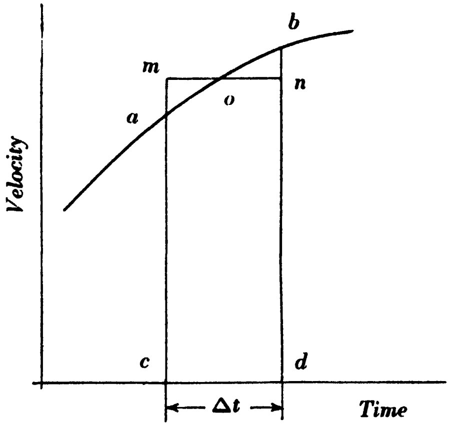


FIG. 184.

area *abcd* is to substitute a rectangle, *mncd*, which is equivalent in area to figure *abcd*. The altitude of the equivalent rectangle is found by moving a straightedge up or down, always parallel to the axis of abscissas, until line *mn* is established at such a level that the small area *bno* appears, by eye, to be equal to area *moa*. In matching these areas, faint trial lines, such as *mn*, are drawn until, by constant trial to reduce error, a final level is established. Here again the trustworthiness of the eye as a judge of matched areas can be relied upon to produce a reasonable accuracy.

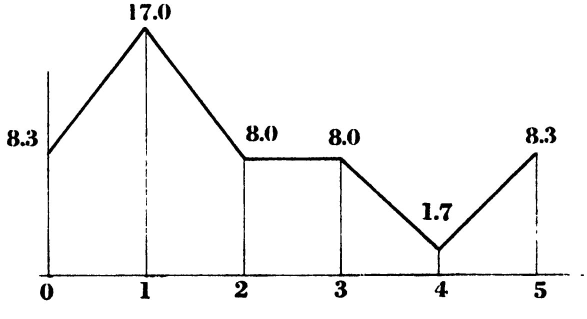
With *mn* finally placed, the altitude of rectangle *mncd* multiplied by the base Δt gives the increment of displacement. This displacement must be added to the displacement of the body at the beginning of time interval to obtain the final displacement at the end of time Δt. The initial displacement is thus operating in the same fashion as in analytical integration, where such an initial displacement would appear as the constant of integration; or the graphical integration has given only the change of displacement between the limits of the time interval Δt.

It will be noted that the accuracy of the displacement curve obtained will be increased in proportion to the number of time intervals into which the total motion is divided for study. The approximation of areas by substitution of rectangles is coarse if the time interval selected, Δt, is great. The approximation becomes refined as we operate with lesser time intervals.

Various devices are used to enhance the accuracy of graphical integration. One such devide is *Simpson's Rule*, which may be found in any of the textbooks of the calculus. In applying the routine operations of graphical differentiation or integration, it is dangerous to make readings at uniform time intervals, unless the curvature of the curve which is being treated is fairly uniform. Instead, in regions where there is rapid change of the character of the curve, a greater number of readings should be taken than in regions of more constant nature.

Problems

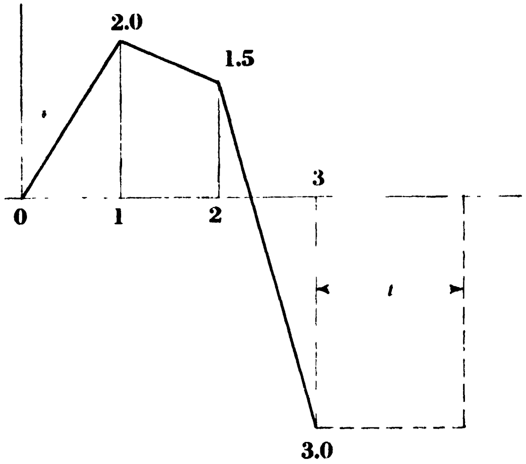
309. The curve giving the velocity-time relationship gives values of velocity v as ordinates in feet per second, and values of time t as abscissas, in seconds. Determine the displacement from O at the end of each second. The portions of the curve are straight lines, and displacements are to be determined analytically by calculating the areas. *Ans.* 12.65 ft. (1 sec.)
25.15 ft. (2 sec.)
33.15 ft. (3 sec.)
38.00 ft. (4 sec.)
43.00 ft. (5 sec.)



Prob. 309

310. A particle travels in a straight line, starting at an origin O, with the velocity-time curve shown. Determine analytically the displacement at the end of each second. Velocities (ordinates) are given in feet per second, and time intervals (abscissas) in seconds. Velocities plotted as positive values are of one sense; those plotted as negative values are of opposite sense.

How long (time t) will it take the particle to return to the origin if it continues to travel at constant velocity of 3 f.p.s. after the 3-sec. mark?



Prob. 310

311. Given a table showing the velocity-time relationship for a moving particle whose path is a straight line. Plot the curve, and integrate graphically. Plot the displacement-time curve.

Velocity (*In. Per Sec.*)	Time (*Sec.*)
0	0
1 91	1
3 58	2
4 92	3
5 76	4
6 07 (max.)	5
5 89	6
5 30	7
4 47	8
3 44	9
2 31	10
1.16	11
0	12

75. Angular Velocity. Angular velocity is the time rate of angular displacement. Here again we may have uniform motion, and the angular velocity ω is the quotient obtained by dividing the angular displacement $\Delta\theta$ during a time interval Δt by that time interval

$$\omega = \frac{\Delta\theta}{\Delta t}$$

If the angular velocity is variable, we must, as in the case of the linear counterpart, deal with instantaneous values by averaging the velocity over an increasingly smaller interval of time, until in the limit we establish the instantaneous value

$$\omega = \operatorname*{Limit}_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

76. Relation Between Linear and Angular Velocity. Point B in Fig. 185 is moving as line AB rotates about axis A. The linear speed of point B at any instant is $v = \frac{ds}{dt}$, where ds is the linear displacement in time dt. During this same interval of time, the line AB undergoes angular displacement $d\theta$.

The linear displacement ds is equal to $r\,d\theta$

Then,

$$v = \frac{r\,d\theta}{dt}$$

But, as defined in Article 75,

$$\frac{d\theta}{dt} = \omega$$

Therefore,

$$v = \omega r$$

The direction of the linear velocity is perpendicular to AB, since the

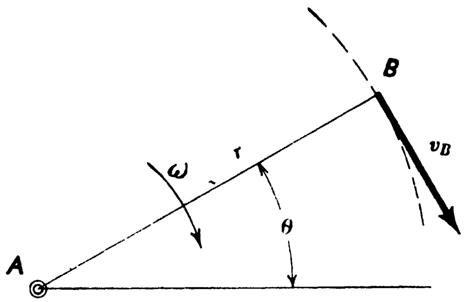


FIG. 185.

direction of B's displacement is always tangential to the path, and at any instant the direction of velocity must agree with the direction of displacement.

77. Rotation About Fixed Axis. The wheel W rotates about a fixed axis O, as in Fig. 186, with constant angular velocity ω.

Any point on the wheel has a linear velocity v whose magnitude is ωr (where r is the radius from the fixed axis to the point in question) and whose direction is perpendicular to the moving radius at whose end the point lies. Two such velocities are shown as the velocity of A, $v_A = \omega r_1$, and the velocity of B, $v_B = \omega r_2$. These velocities have directions perpendicular to r_1 and r_2, respectively.

The relationship between v_A and v_B is typical of the relationship between the linear velocities of all points on the rotating body. Since

$$v_A = \omega r_1$$

and

$$v_B = \omega r_2$$

then

$$\frac{v_A}{v_B} = \frac{r_1}{r_2}$$

or, the *magnitudes of the linear velocities of points on a rotating body are directly proportional to the distances of the points from the axis of rotation,* and *the direction of all linear velocities is perpendicular to the radius to the axis of rotation.*

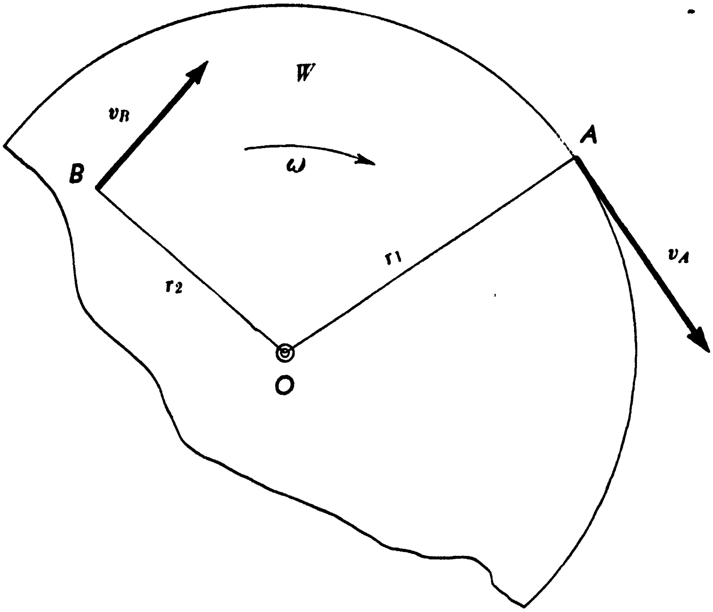


FIG. 186.

PROBLEMS

312. A cylinder whose diameter is 2 ft. rotates about a fixed axis at its center with angular velocity of 60 r.p.m. Calculate
(a) the speed of a point A on its circumference, in feet per minute.
(b) the speed of point B, which is 9 in. from the axis, in inches per second.
Ans. (a) 377 f.p.m. (b) 56.5 in. per sec.

313. A twist drill, $\frac{3}{8}$ in. in diameter, has an angular velocity of 300 r.p.m. Calculate the speed of a point on the surface of the drill in feet per second.

314. A point on a wheel rotating about its own axis has a speed of 1500 f.p.m. If the point is 6 in. from the axis, calculate the angular velocity of the wheel and the speed of a point 8.6 in. from the axis.

315. Two cylinders are keyed to the same shaft. If the surface speed of cylinder A is 2400 f.p.m., calculate the surface speed of cylinder B. The diameter of cylinder A = 3.40 ft., and the diameter of cylinder B = 1.92 ft.
Ans. 1355 f.p.m.

316. A gear has an angular velocity of 500 r.p.m. If a point A on the gear has a speed of 4000 in. per min., how far is point A from the axis of the gear?

317. Two points, A and B, on a radius of a pulley have speeds of 1500 and 2000 f.p.s., respectively. If the angular velocity of the pulley is 1850 r.p.m., how far apart are the points?

78. Graphical Analysis of Velocity Vectors. In graphical solutions, ratios and proportions are established through the medium of similar triangles. For example, if the linear velocity of a point A, Fig. 187, is known, the velocity of any other point, such as B, may be found by constructing similar triangles to yield the proper proportion.

Let the known velocity of A be Aa' (equal to the angular velocity of the rotating body ω times the radius OA).

Draw line $a'O$ from the terminus of A's velocity to the axis of rotation O. Erect, at origin B, a vector whose direction is perpendicular to radius OB. The intersection of this vector with $a'O$ will be its terminus, and Bb' is the linear velocity of point B.

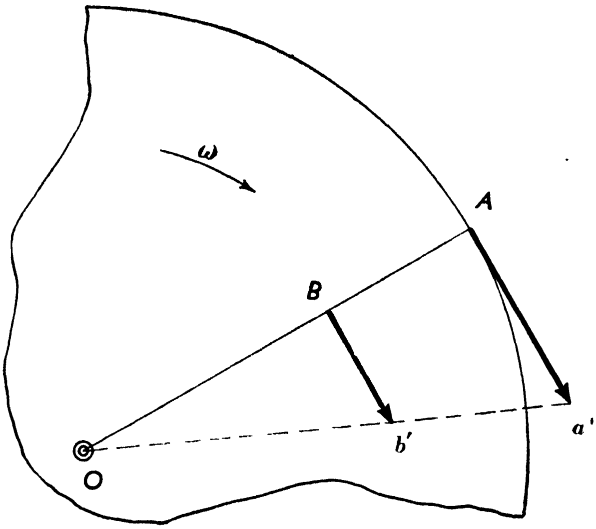


FIG. 187.

We note that Bb' is in the correct ratio with Aa'—the two velocities are to each other as the distances from center O, since triangles OBb' and OAa' are similar. Further, the directions of both velocities are perpendicular to the moving radial line, which conforms with the definition of direction of linear velocity.

When the points A and B do not lie on the same radial line, we have the condition illustrated in Fig. 188.

Given the linear velocity of point A as Aa', it is desired that the linear velocity of point B be found.

Velocity has two properties: magnitude, or speed, and direction.

The magnitude of B's velocity may be obtained by swinging an arc of radius OB and center O until it meets radius OA at B_1.

The speed of point B_1 is obtained as in the previous case, and is B_1b_1'. This quantity is also the speed of point B, since points which are equally distant from center O will have the same speed. Returning now to point B, we erect a vector perpendicular to OB, which establishes the direction of B's velocity, and of length equal to B_1b_1', which establishes its magnitude. The completed vector Bb' is the linear velocity of point B.

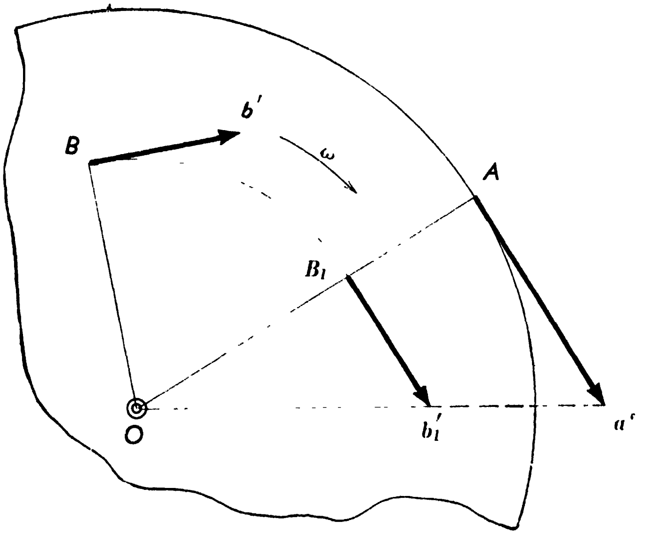


Fig. 188.

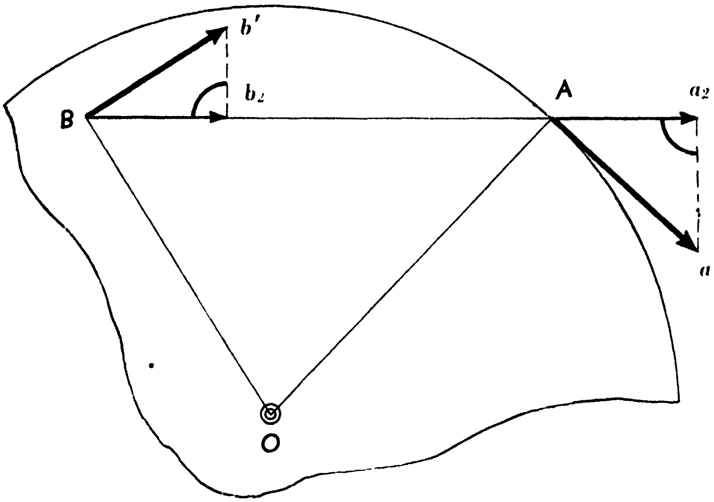


Fig. 189.

Additional information on the relationship of the linear velocities of points on the same body may be obtained by recalling the concept of the rigid bodies of mechanics, which have already been described as nondeformable. If, for example, the wheel of Fig. 189 is not deformed during its motion, points O, A, and B will remain constantly at the same distance from one another.

Since the distance AB remains unchanged during any motion, it follows that points A and B must have the same orthogonal component of velocity in the direction $AB(Bb_2 = Aa_2)$. We can readily prove this to be true by noting that if B has a greater or lesser orthogonal component of velocity in the direction AB than point A, it would be drawing nearer to, or receding from, point A, and the distance AB would be a changing one.

We conclude that *any two points on a rigid body have the same orthogonal component of linear velocity in the direction connecting the two points.*

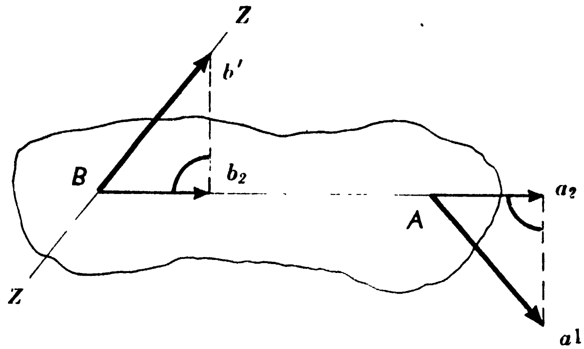


FIG. 190.

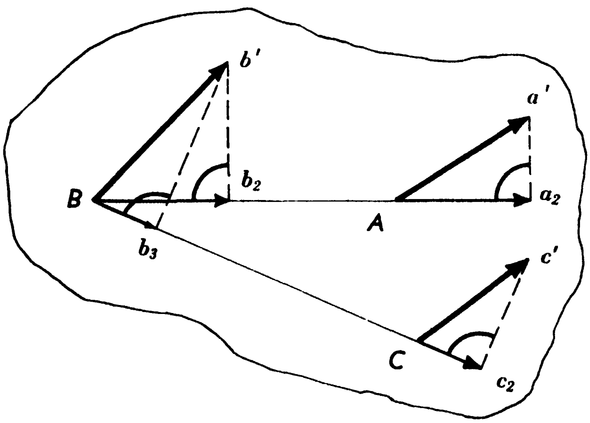


FIG. 191.

This relationship is of value. Together with the theorems of orthogonal components (see Article 65), we have an additional weapon of analysis for attacking problems of linear velocity.

Application of Theorem I. Given, as in Fig. 190, two points A and B which lie on the same rigid body. Given also the linear velocity of point A, as Aa', and a known inclination of the velocity of point B, as the direction ZZ. It is desired to find the linear velocity of point B.

We first find the orthogonal component of A's linear velocity in the direction AB. This is Aa_2. Next, the orthogonal component of B's velocity in direction AB, Bb_2, is made equal to Aa_2.

We now have, for point B, one orthogonal component and a known inclination of resultant. By erecting b_2b' perpendicular to AB to intersect known direction ZZ, we obtain Bb', the linear velocity of point B.

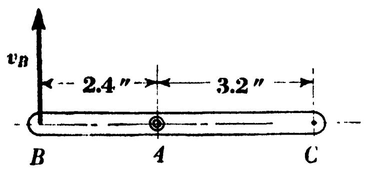
Application of Theorem II. We are given, as in Fig. 191, three points A, B, and C which lie on the same rigid body, and hence are at fixed distances, AB, BC, and CA from one another. The velocity of point A is Aa', which is known. The velocity of point C is Cc', which is also known. We are to find the velocity of point B.

Points A and B must have the same orthogonal component in direction $AB(Aa_2 = Bb_2)$. Similarly, points C and B must have the same orthogonal component in direction $CB(Cc_2 = Bb_3)$.

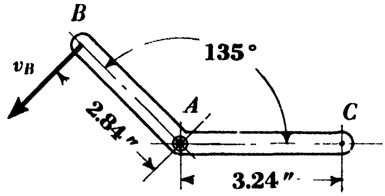
Then, two orthogonal components of B's velocity are known, and the resultant linear velocity Bb' is obtained by erecting perpendiculars to Bb_2 and Bb_3 at their termini, which will intersect at b', which, in turn, becomes the terminus of the resultant velocity of point $B(Bb')$ by the principle summarized in Theorem II.

Problems

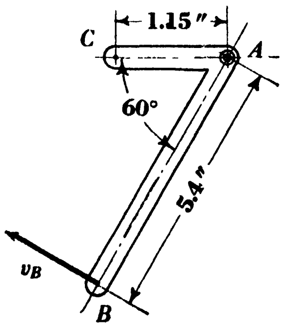
318–321. The body shown in the figure rotates about a fixed axis at A. If the velocity of point B is 2 in. per sec., find the velocity of point C, graphically. Calculate the angular velocity of the body in r.p.m.



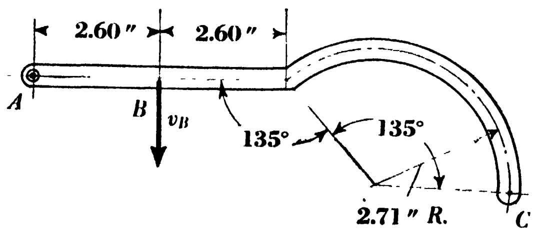
Prob. 318



Prob. 319

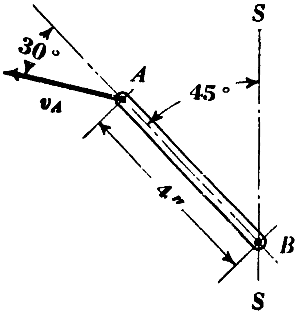


Prob. 320

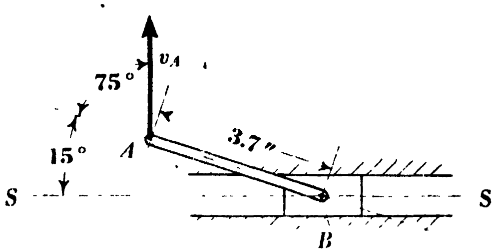


Prob. 321

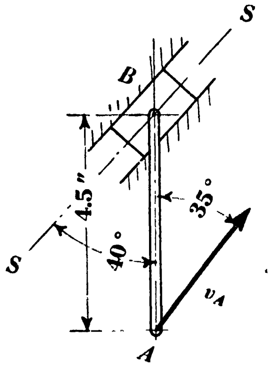
322–325. The velocity of point A is 25 f.p.m. The inclination of B's velocity is known and is given as the axis SS. Find, graphically, the magnitude and sense of B's velocity by applying Theorem I.



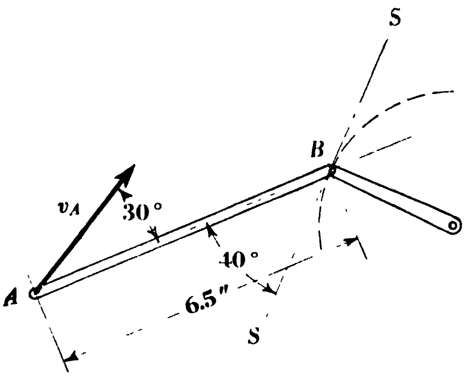
PROB. 322



PROB. 323

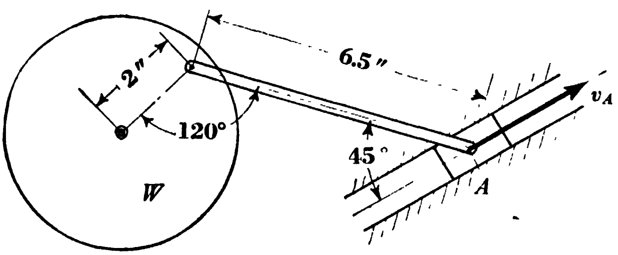


PROB. 324



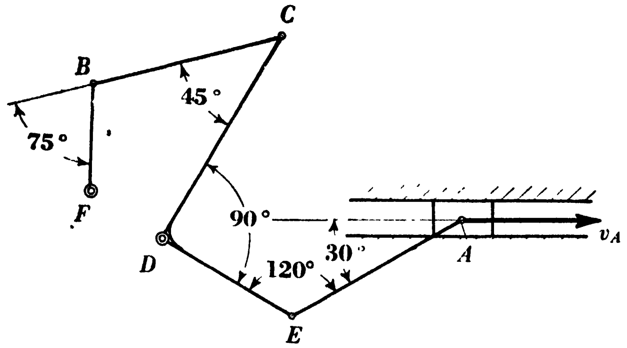
PROB. 325

326. The velocity of point A is $v_A = 50$ f.p.m. Find the surface speed of the wheel W, graphically, and calculate its angular velocity in r.p.m. The diameter of the wheel is 5 in. *Ans.* 51 f.p.m.; 39 r.p.m.



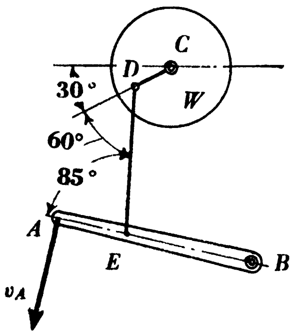
PROB. 326

327. In the series of links shown, point A has a velocity of 200 f.p.m. Find the velocity of crank BF. Links CD and DE are fastened to each other and form one rigid body. FB = 2.2 in.; BC = 4.0 in.; CD = 4.8 in.; DE = 3.1 in.; EA = 4.0 in.

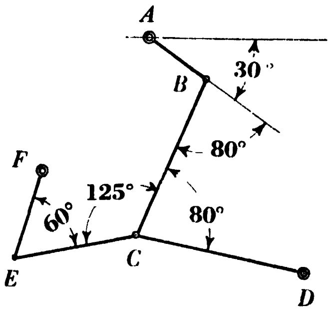


Prob. 327

328. Point A of the grindstone treadle has a velocity of 65 f.p.m. in the position shown. Find the angular velocity of the grindstone W, and its surface speed. The diameter of the grindstone is 2 ft. $CD = 8$ in.; $DE = 30$ in.; $AE = 14$ in.; $AB = 40$ in. *Ans.* 11.5 r.p.m.; 72 f.p.m.



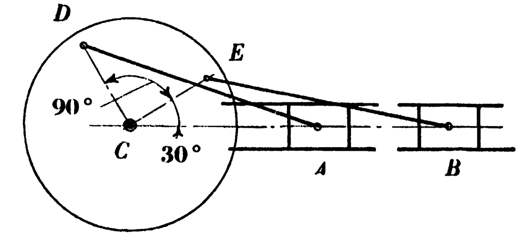
Prob. 328



Prob. 329

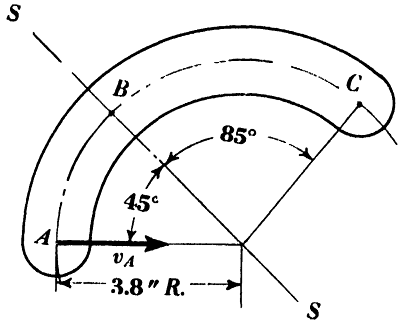
329. The toggle mechanism shown consists of a series of links fastened to each other by pins. If the angular velocity of crank AB is 2 radians per sec. counterclockwise, determine the angular velocity of crank EF at the position shown. $AB = 7$ in.; $BC = 17$ in.; $CE = 13$ in.; $CD = 18$ in.; $EF = 9$ in.

330. In the engine shown, piston A has a velocity $v_A = 300$ f.p.m. to the right. Find, graphically, the velocity of piston B. Cranks CD and CE are 4.5 in. long, and connecting rods DA and EB are 12.5 in. long.

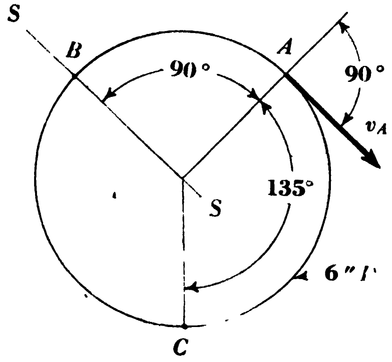


Prob. 330

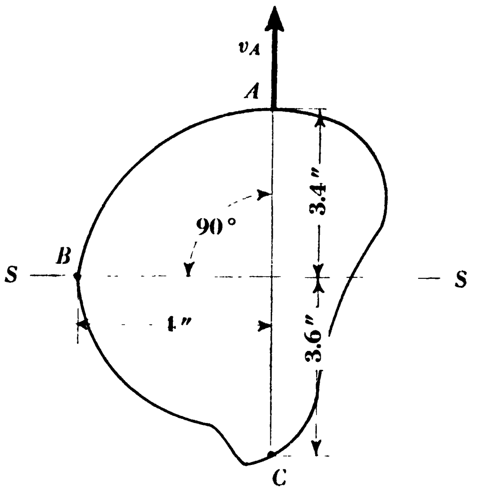
331–334. The velocity of point A and the inclination of the velocity of point B are known. $v_A = 2$ in. per sec. The inclination of B's velocity is the



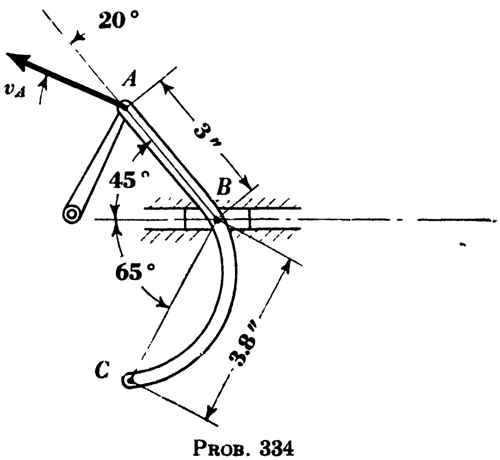
PROB. 331



PROB. 332



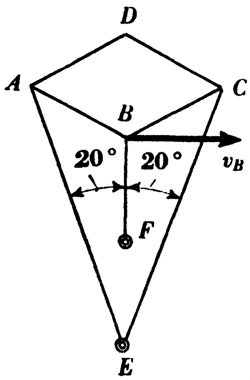
PROB. 333



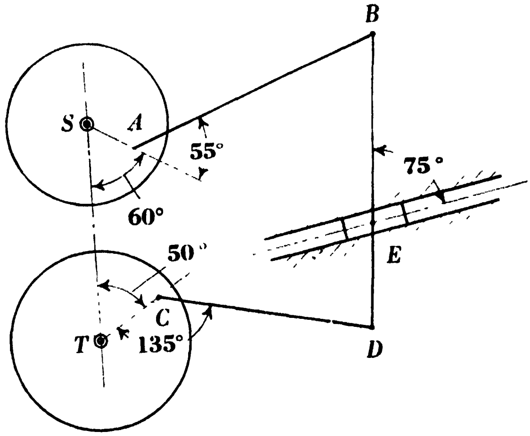
PROB. 334

axis SS. Find the resultant velocity of point B by applying Theorem I, and the velocity of point C by applying Theorem II. The outline of the machine part containing points A, B, and C is shown in the figure, but need not be drawn in the graphical solution.

335. Point B of a Peaucellier's Cell Linkage has a velocity of 3 in. per sec. Find the velocity of point C. The cell is composed of four equal links, AB, BD, DC, and CA. Cranks AE and DE are 8.1 in. long. Crank BF is 3 in. long. The distance between fixed axes E and F is 3 in. *Ans.* 4.61 in. per sec.



PROB. 335



PROB. 336

336. Wheels W_1 and W_2 have an angular velocity of 2 radians per sec. counter-clockwise. The arm BD is attached to W_1 by connecting rod BA, and to W_2 by connecting rod DC. BED is one continuous member. Point E of arm BD is pinned to a block which slides in fixed guides. Find the velocity of point E. $SA = 10$ in.; $AB = 5.4$ in.; $BE = 4.0$ in.; $CD = 4.4$ in.; $TC = 1.4$ in.; $ST = 4.4$ in.

79. The Instantaneous Axis of Velocities. When a body, such as BC of Fig. 192, has a determinate motion, the various particles of the body must have motion that is in some definite relationship.

The definition of rigid body given in the first chapter demanded only that the distances between the particles of the body remain unchanged no matter how the body may move under the influence of force systems. But for our present purposes in the field of kinematics, we shall require a somewhat different concept than the usual description of a solid body of fixed dimensions.

The wheel, shown in Fig. 193, has definite dimensions: diameter and thickness, which establish its physical extent. In kinematics, however, we may consider the wheel to be but part of a body of indefinite extent.

It will be noted that any particle of the wheel, such as A, is forced to turn in a circular path about the axis O, at a speed which is determined by the speed of the wheel.

If, as suggested by the dotted outline, the wheel were to be increased in size, so that it contained a particle, such as B, then B would also be forced to turn in a circular path about O, and its speed would be fixed by the motion given to the wheel. The distance of particle B from A and from O would remain unchanged as the wheel turned, because we have assumed such bodies to be rigid.

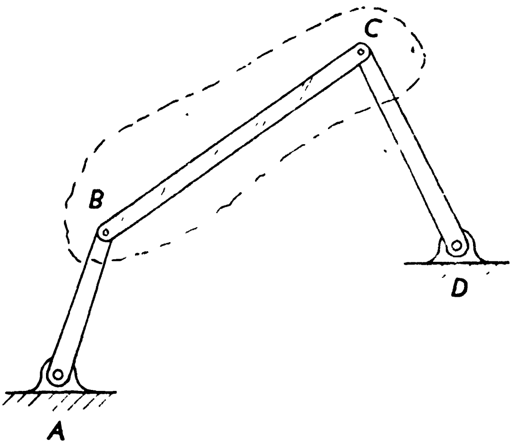


Fig. 192.

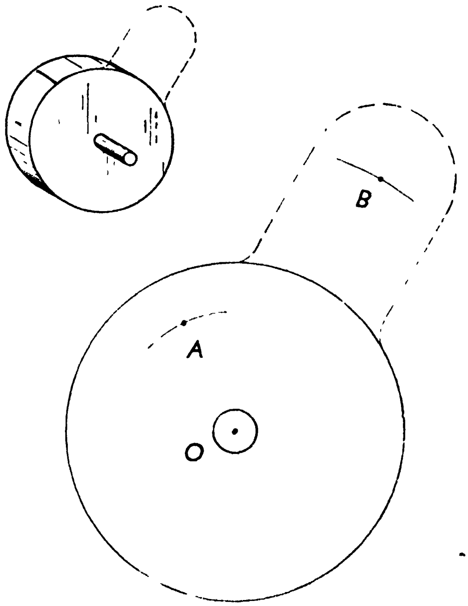


Fig. 193.

Now, the physical size of the wheel need not be increased to enable it "kinematically" to contain particle B. If any particle, such as B, is so moving that its distance from A and from O remains constant, then this particle, as far as motion is concerned, belongs to the wheel as truly as if it were physically attached. This "kinematic expansion" of the rigid body is subject to no limits of size, but is restricted in that every particle assigned to the rigid body must have the proper motion properties possessed by particles of the rigid body.

The body *BC* of Fig. 192 is, then, unlimited in extent. Points that lie directly along the line *BC* all have resultant velocity of some finite magnitude. If we leave the line *BC*, but are careful to deal always with points of the body *BC*, we can find a point which has, at this instant, zero resultant velocity. Since *B* and *C* rotate in circular paths about *A*

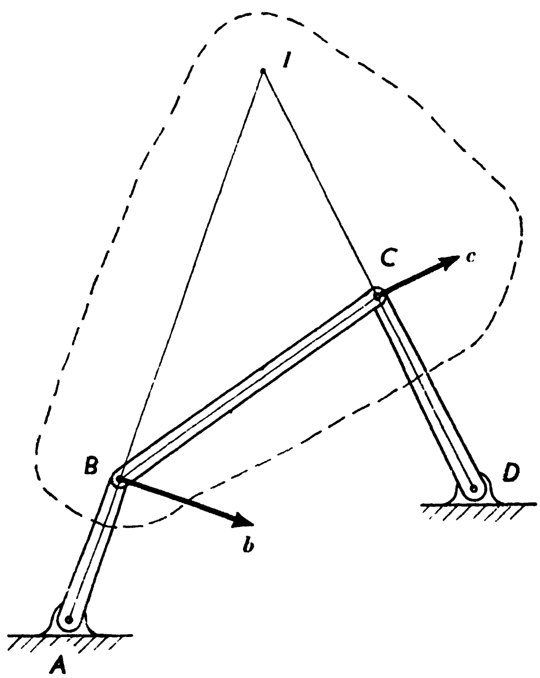


FIG. 194.

and *D*, respectively, the inclinations of their resultant velocities are known.

In Fig. 194, the location of the point on body *BC* which has zero velocity has been determined. A line *BI* perpendicular to *Bb* (the resultant velocity of point *B*) and another line *CI* perpendicular to *Cc* (the resultant velocity of point *C*) are drawn. These lines intersect at point *I*, which will be called the *instantaneous center* of body *BC*. An axis through this instantaneous center, perpendicular to the plane of motion, will be called the *instantaneous axis of velocities* or, more briefly, the *instantaneous axis*. This point *I* has zero velocity.

Since points *B* and *I* lie on the same rigid body *BC*, the orthogonal components of velocity in the direction *IB* connecting them must be the same. But the resultant velocity of point *B* is perpendicular to *IB* and has no orthogonal component in this direction. If I has any resultant velocity, that velocity must then be perpendicular to IB. Then, if I has any velocity at all, that velocity must have an orthogonal component in the direction IC; and point C, on the same rigid body as I, will have the same orthogonal component in the direction IC.

But we now note that IC is perpendicular to Cc, the resultant velocity of point C. C cannot have an orthogonal component of velocity in this direction. It follows that I cannot have such an orthogonal component either. Then I must have zero velocity.

A wheel rotating about its own center as a fixed axis is an example of pure rotation. This axis is also a point of zero velocity. In the present

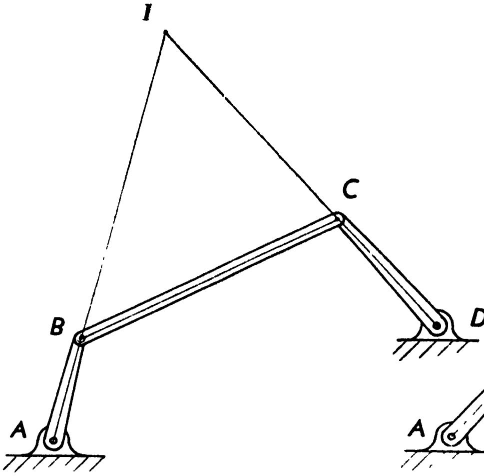


Fig. 195.

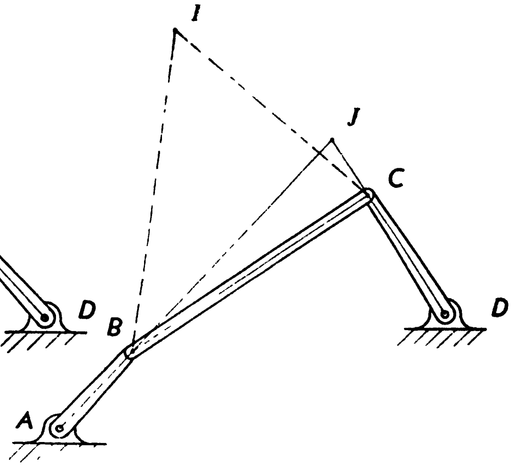


Fig. 196.

case, we have reproduced the motion of pure rotation by finding a point on the body BC that at the instant has zero velocity, and hence is acting, just as in the case of the wheel, as an axis of rotation. A vital distinction exists between the two cases which must be observed. In the case of the wheel in pure rotation, the zero-velocity axis is a *fixed*, or permanent, axis. In the case of body BC, the zero-velocity axis is not fixed, but is operative as an axis of rotation *at the given instant only.*

We can reinforce this thought by noting that when a body BC is displaced from the position shown in Fig. 195 to a new position, as in Fig. 196, particle I of body BC is no longer at the intersection of lines perpendicular to the resultant velocities Bb and Cc, but is elsewhere on the body, and a different particle of the rigid body—namely J, is now at the intersection and may be shown, as before, to be the point of zero velocity, or the instantaneous center for the new position of BC.

When a body is moving in plane motion, then, we may consider it, at any given instant and at that instant only, to be moving in pure rotation about an instantaneous axis of velocities. We may, here, as in any case of pure rotation, apply the principles of velocity relationship between points on the same rotating body.

Illustrative Example 1

Given the mechanism shown in Fig. 197, with the angular velocity ω_1 of wheel W_1 known. We are to locate the instantaneous axis of body BC and then to find the velocity of any point which, like point S, lies on body BC. Wheel W_1 is rotating about a fixed axis A, and wheel W_2 is rotating about a fixed axis D.

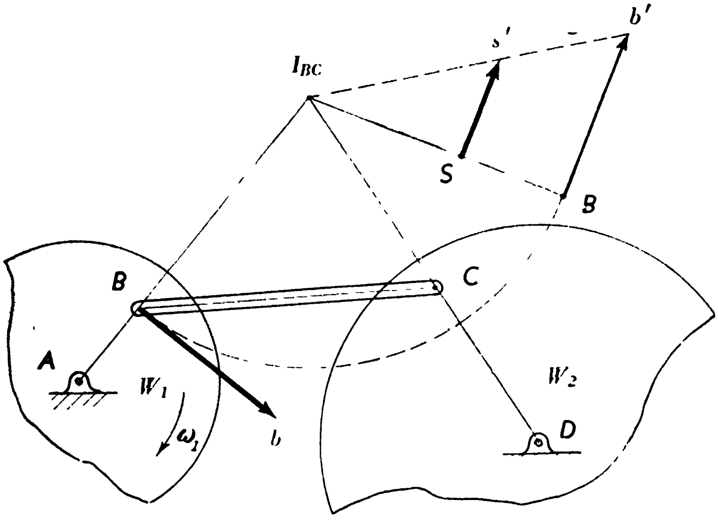


Fig. 197.

The direction of B's resultant velocity is fixed, since B must travel in a circular path about point A. The inclination of its resultant velocity is also known as perpendicular to AB. The inclination of C's resultant velocity is also known as perpendicular to CD. If we erect perpendiculars to these resultant velocities (these perpendiculars are, of course, extensions of AB and CD), their intersection I_{BC} is the instantaneous center of BC. At this instant, the entire body BC has a motion of pure rotation about I_{BC}.

Point S is moving, at this instant, in a circular path about I_{BC}, as is every other point on body BC. Then the direction of the resultant velocity of S is perpendicular to $I_{BC}S$. We can establish the magnitude of S's resultant velocity by proportion. The point B has resultant velocity $Bb' = \omega_1 AB$. The magnitude of Bb' is used to establish the magnitude of S's velocity by similar triangles, as in the previous analyses of rotation. Ss' is obtained as the resultant velocity of point S.

We have available other methods for finding such velocities as that of point S. For example, we may turn to Theorem I and establish, as in Fig. 198, the orthogonal component Ss_2 of S's velocity in the direction of BS. The instantaneous axis I_{BC} of body BC may then be used to

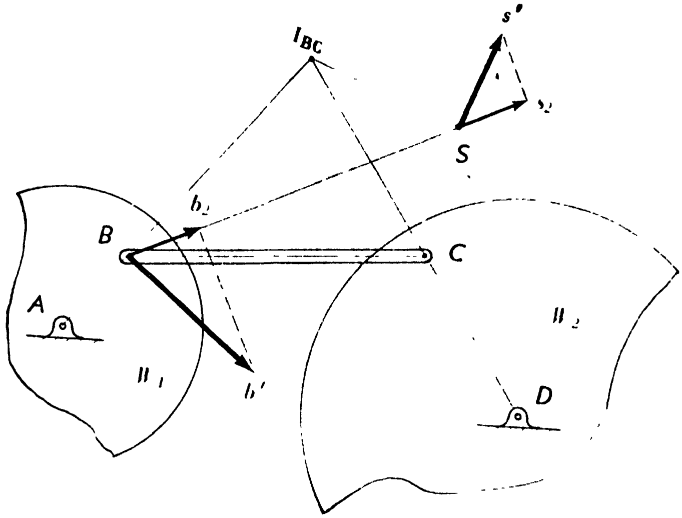


FIG. 198.

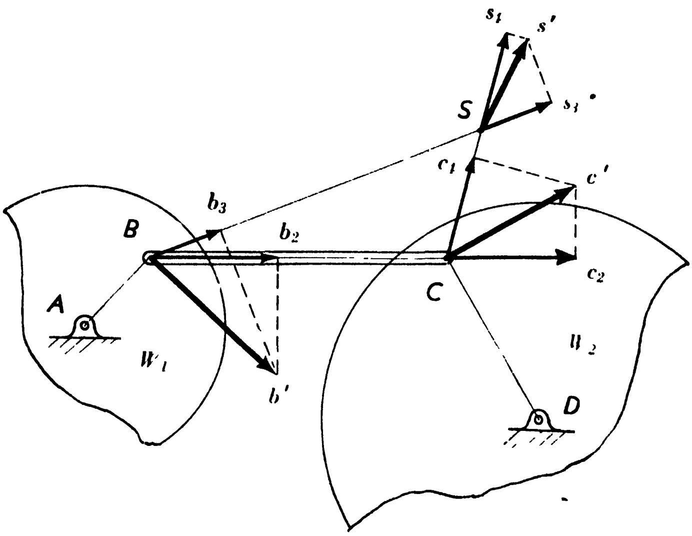


FIG. 199.

fix the inclination of S's resultant velocity. Having one orthogonal component and the inclination of the resultant velocity, the resultant velocity is determined.

Or, again, we may resort to Theorem II, as in Fig. 199. Bb' is known, and Cc', the resultant velocity of C, may first be determined by setting $Cc_2 = Bb_2$ (the orthogonal component) in direction BC, and establishing C's direction of velocity as perpendicular to CD, since C travels in a circular path about fixed axis D. This is another application of Theorem I.

With two resultant velocities Bb' and Cc' now known, we observe that S has orthogonal component $Ss_3 = Bb_3$ in the direction BS, and orthogonal component $Ss_4 = Cc_4$ in direction CS. This procedure yields two known orthogonal components of the resultant velocity of point S, and, by Theorem II, the resultant velocity of $S = Ss'$ is now determined.

The three methods of attack outlined above yield equivalent results. It is desirable to take advantage of more than one method, since opportunity is afforded to check the results. Apart from the mental stimulus we feel in accomplishing a valid check of our work, we go forward into use of these procedures in further analysis with the confidence that they have been proven trustworthy. In addition, a multiplicity of instruments of analysis enables us to face a broader range of situations than would be possible with more limited equipment.

Two more examples of plane motion are frequently encountered. These follow, and will be pursued only to the point of locating the instantaneous axis, at which time the method of solution merges with the method outlined above.

Let us first summarize the equipment needed thus far for the location of an instantaneous center. The instantaneous center of a body may be determined when the inclinations of the resultant velocities of two points are known, provided that these resultant velocities are not parallel. Let us now see what material is necessary when two points of the body do have parallel velocities.

ILLUSTRATIVE EXAMPLE 2

In Fig. 200, body AB is moving so that velocities Bb' and Aa' are parallel. Here, we find that we must know the magnitudes of the velocities in addition to their directions in order to locate the instan-

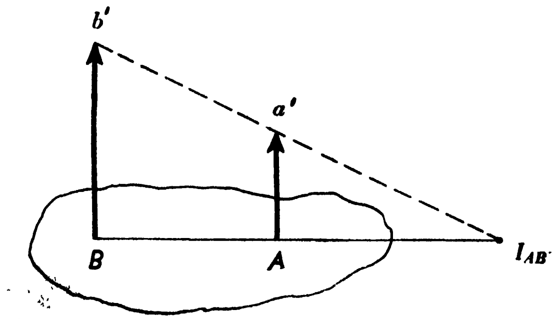


FIG. 200.

taneous center. With magnitudes known, we may draw a straight line through b' and a', meeting AB produced at I_{AB}, which is then the instantaneous center of the body AB. The resultant velocities of A and B are both perpendicular to the radius from the instantaneous center, and the magnitudes of these velocities are in direct proportion to the distances $I_{AB}A$ and $I_{AB}B$ from the axis to the respective points.

When a body is constrained to move with translation, all points have equal and parallel velocity (Fig. 201). Then perpendiculars to the individual velocities are parallel, and will not converge. The instantaneous axis must, in that event, be located at an infinite distance from the body.

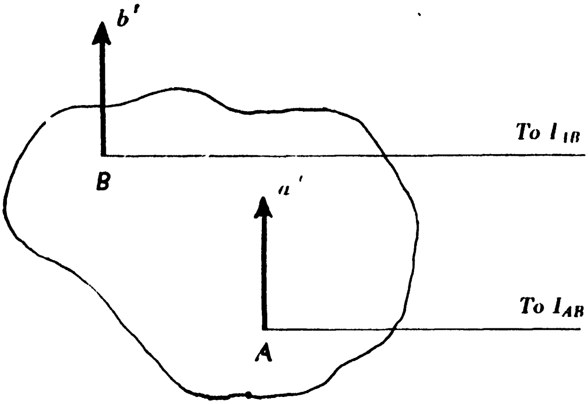


FIG. 201.

A word of caution may be quite proper here, even though it introduces no new concept but insists only upon careful attention to those already developed. The instantaneous center of velocity of a body is a point that definitely belongs to that rigid body. When velocity analyses of connected bodies are made, we must assign each instantaneous axis to the particular body to which it belongs, and to no other. In this text, each instantaneous axis is identified by denoting the body to which it belongs as a subscript—that is, I_{BC} signifies the instantaneous axis of body BC.

ILLUSTRATIVE EXAMPLE 3

In Fig. 202, a series of connected bodies is shown. The wheel W is moving with pure rotation about an axis, which is permanently fixed at point A. This is an example of an axis of rotation that remains the same at all instants. It is a point of body W and serves as the rotational center for all resultant velocities of points on W.

Connecting rod BC has plane motion, with its instantaneous axis (for the instant shown) at I_{BC}, which is a point of body BC, and the resultant velocities of points on BC have their axis here.

Finally, sliding block CS has a motion of pure translation, with its instantaneous axis, I_{CS}, at infinity.

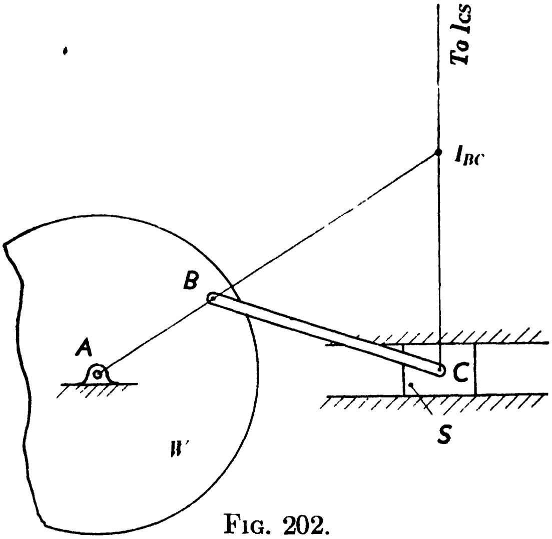
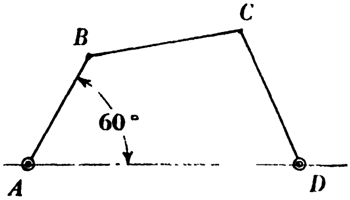


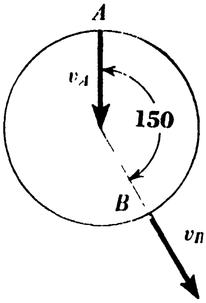
Fig. 202.

Problems

337. Locate graphically the instantaneous axis of BC. $AB = 2.6$ in.; $BC = 3.2$ in.; $CD = 3.0$ in.; $AD = 5.6$ in.



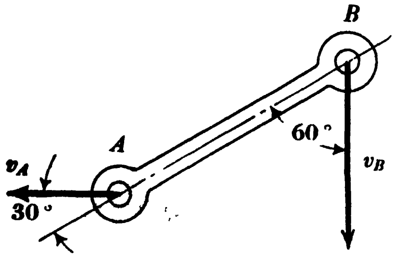
Prob. 337



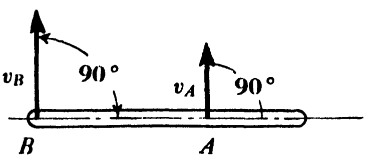
Prob. 338

338. Locate the instantaneous axis of the cylinder, which has a diameter of 4 in.

339. Locate the instantaneous axis of the connecting rod. $AB = 5.4$ in.



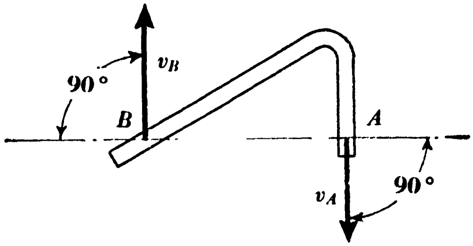
Prob. 339



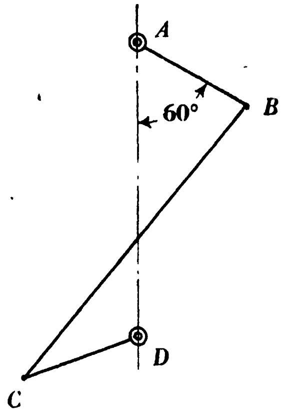
Prob. 340

340. The velocity of point A is 2 f.p.m., and the velocity of point B is 3.2 f.p.m. Locate the instantaneous axis of body AB. $AB = 3.4$ in.

341. The velocity of point A is 1.54 in. per sec. Point B has a velocity of 2.32 in. per sec. Locate the instantaneous axis of body AB. $AB = 6$ in.



PROB. 341



PROB. 342

342. The angular velocity of the body BC is 4 radians per sec. Find the angular velocities of cranks AB and CD. $AB = 2.6$ in.; $BC = 7.2$ in.; $CD = 2.5$ in.; $AD = 6$ in.

80. Velocity Analysis of Plane Motion. Combined Translation and Rotation. The instantaneous axis has furnished one road of attack on the analysis of plane motion. Of equally important fundamental value is the division of such motion into its two component bases: translation and rotation.

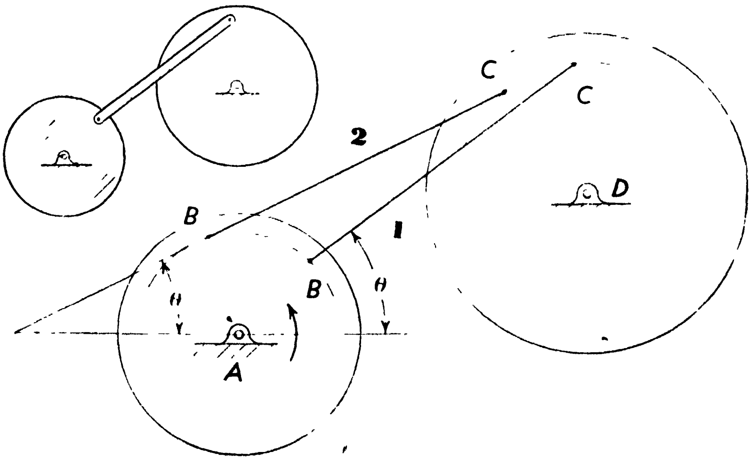


FIG. 203.

Let us, therefore, explore the mechanism shown in Fig. 203, where body BC illustrates a typical case of plane motion. We note that if we rotate the wheels from position 1 to position 2, the inclination of the line BC (note the angle which it makes with the X axis) has changed. Then this motion cannot be one of pure translation. Next, we note that while point B is turning about a fixed axis A, point C is turning about fixed axis D. Then the motion cannot be pure rotation, which demands that all points in BC be turning about the same fixed axis.

We may, however, analyze the motion by "breaking it down" into two parts.

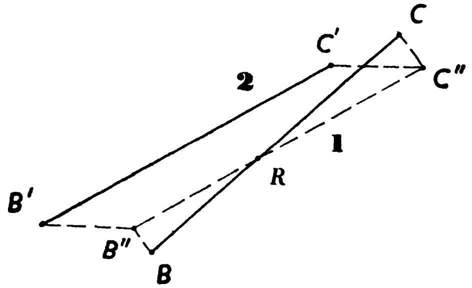


FIG. 204.

We can reproduce the resultant displacement of link BC from position 1 to 2 by traveling in two successive stages, as in Fig. 204.

First Stage. Rotate line BC about any point, such as R, in BC (or BC produced) until it lies parallel to the final position $B'C'$. This intermediate position is $B''C''$. During the time interval of this first stage, the motion has been one of pure rotation since the line has rotated about a fixed axis.

Second Stage. Now move the line so that B'' goes to B' and C'' goes to C'. This motion is one of pure translation, since the line is remaining constant in its inclination while moving to the final position, and all points are traveling in parallel paths.

We have now established a most useful division of the two elements in a plane motion: we have found that any plane motion of a body may be divided into stages—a rotation of the body about any point plus a translation of that point (and hence of the entire body). The sum of these two stages is equivalent to the resultant plane motion, and the division frequently forms a more convenient method of attack than dealing with the resultant motion as a single stage. This type of analysis is usually described as *combined translation and rotation.*

In the previous examination of the nature of plane motion, we have concerned ourselves with viewing only the two stages of displacement which are involved. Let us now analyze the factors of velocity which make up the plane motion.

A slightly more elaborate path of travel will be useful here. Let the line BC (Fig. 205) be moved from starting position 1 to final position 4, or $B'C'$ as follows:

1. Move line BC along its own path produced until point C is at R, the intersection of BC and $B'C'$. The line is now in position 2. This element of motion has been pure translation.

2. Now rotate the line from position 2 about R as an axis through angle θ, until it coincides in direction with $B'C'$. The line is now in position 3. This motion has, of course, been pure rotation.

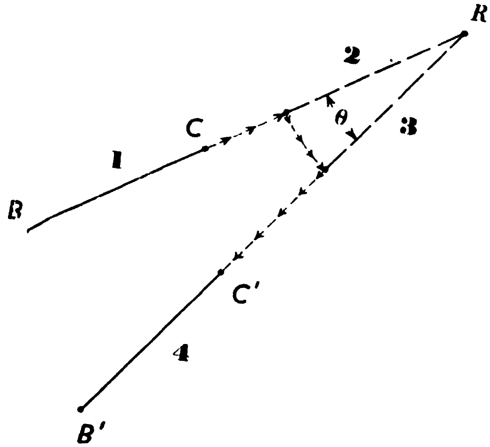


FIG. 205.

3. Finally, move the line along $B'C'$ produced until it occupies the desired final position 4. This has again been a motion of pure translation.

Let us analyze the velocities involved in the complete motion.

During (1), all points of BC have a velocity whose direction is along BC, and these velocities are all equal.

During (2), all points of BC have velocities which are constantly perpendicular to BC and in the ratio of their respective distances from R.

During (3), all points of the line will again have velocities in the direction of the line, and these velocities will again be equal to one another.

Now we shall consider increasingly smaller amounts of angular displacement θ. As the angle becomes smaller, position 3 approaches position 2 as a limit. The linear velocities of the line approach perpendiculars to position 1 of BC as their limiting direction. In the limit, the linear velocities of the rotation are perpendicular to BC, and their magnitudes are proportional to their respective distances from R.

At the same time, the equal velocities involved in the translation from position 3 to position 4 are approaching in direction the line BC itself. In the limit, all of the translation velocities will be in direction BC.

Now, the approach which we have made to a limit has established, in reality, the nature of the velocity of the points in line BC at any instant in a plane motion. This approach reveals the fact that the instantaneous velocity of such points has two components: a component of translation (which lies along the line itself) and a component of rotation (which is perpendicular to the line). The components of translation of all points have equal magnitudes; the components of rotation are proportional in magnitude to their distance from some point which, like R of Fig. 205, is serving as an axis of rotation. The components of translation and rotation are mated rectangular components.

We have added another means of analysis to our growing store. Let us apply this principle to the example which is shown in Fig. 206.

At the instant, point B has known velocity Bb'. Then, C's velocity Cc' may be found, as previously, through combining one known orthogonal component Cc_2 $(= Bb_2)$ and known inclination (perpendicular to

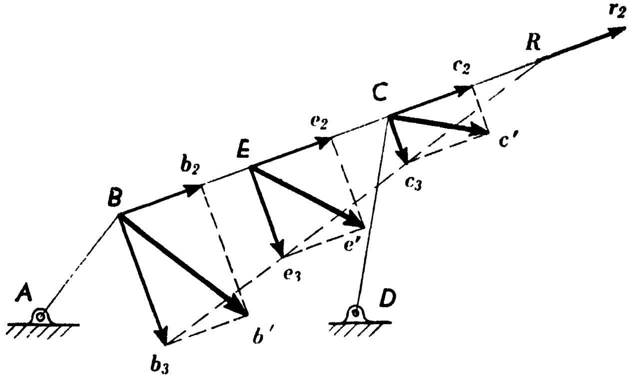


FIG. 206.

CD). It is desired that we find the velocity of point E, a point on body BC.

Body BC has plane motion. At the instantaneous position shown, all points in BC have components of translation along BC. Bb_2, the orthogonal component of Bb' in direction BC, is one such component of translation, and E must have the same component of translation, which is established as $Ee_2 = Bb_2$.

B's component of rotation must be rectangular component Bb_3, since that is the mated rectangular component perpendicular to BC. C's component of rotation is Cc_3. We know that these components of rotation are in direct proportion to their distances from an axis of rotation, and we can again resort to similar triangles to set the proportion graphically. Therefore, we draw a line from terminus b_3 through c_3 to intersect BC produced at point R. Then we erect a vector perpendicular to line BC at E, having its terminus e_3 in line b_3c_3R. Then Ee_3 is E's component of rotation, for

$$\frac{Bb_3}{Ee_3} = \frac{BR}{ER}$$

We now have Ee_2 as component of translation and Ee_3 as component of rotation, and we may add them to obtain vector Ee', which is the desired resultant velocity of point E.

We should note, before leaving this example, that point R itself has velocity. Since it lies along the line BC, it must have the same component of translation as have points such as B, C, and E. Then R's component of translation is $Rr_2 = Bb_2$ or Cc_2 or Ee_2. But R is also the center, or axis, of the components of rotation and has, therefore, no component of rotation. Rr_2 is, then, the resultant velocity of point R.

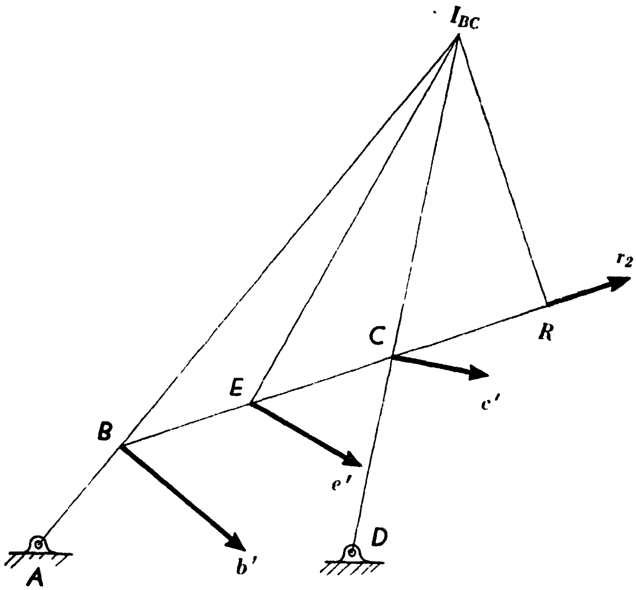


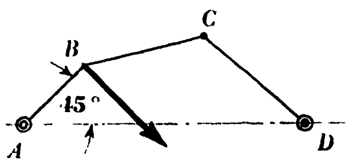
FIG. 207.

The resultant velocities of points B, E, C, and R are all resultant velocities of points of one body that is in plane motion. They may be checked, therefore, as in Fig. 207, by finding the instantaneous axis of the body BC.

I_{BC} will lie at the intersection of line $I_{BC}B$ and $I_{BC}C$ drawn perpendicular to Bb' and Cc', respectively. If now lines $I_{BC}E$ and $I_{BC}R$ be drawn, they must be perpendicular to Ec' and Rr_2, respectively.

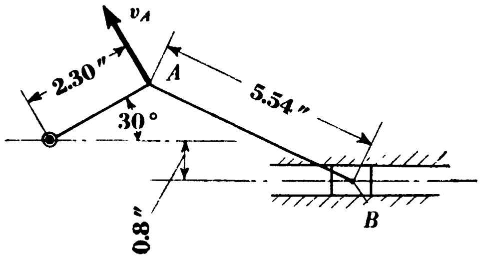
PROBLEMS

343. Determine the components of translation and rotation of points B and C along and perpendicular to line BC. Also locate R, the axis of the components of rotation, and determine its velocity. Resultant velocity of B = 3 in. per sec. AB = 2.70 in.; BC = 3.80 in.; CD = 4.22 in.; AD = 8.82 in.

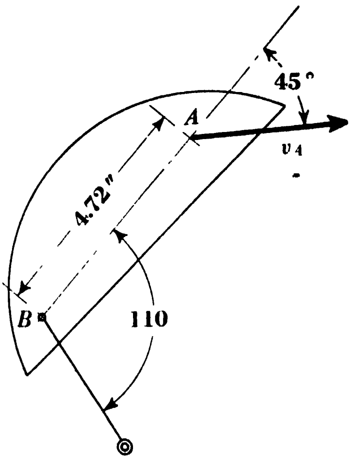


PROB. 343

344–345. Find the components of translation and rotation of points A and B along and perpendicular to line AB, respectively. Locate R, the axis of the components of rotation, and determine its velocity. Resultant velocity of A = 3 in. per sec.

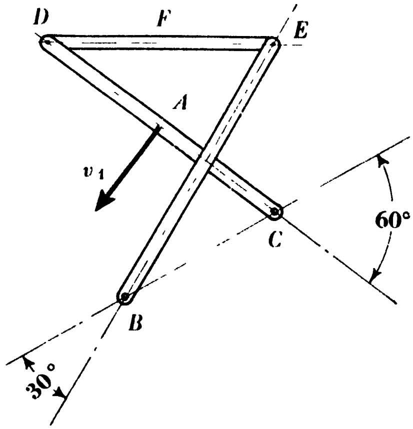


PROB. 344



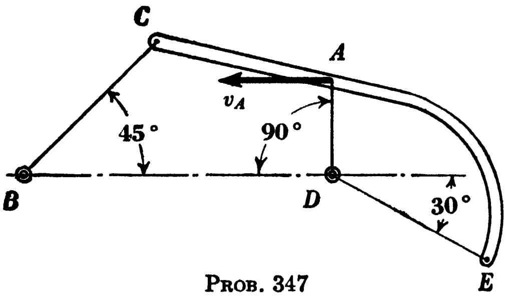
PROB. 345

346. Determine the velocity of point E, given the velocity of point A = 2 f.p.m. (a) Divide the plane motion of the body containing F into components of translation and rotation and (b) Check results by using the instantaneous-axis method. BC = 3.5 in.; BE = 6 in.; CD = 6 in.; DA = 2.8 in ; EF = 2.2 in.



PROB. 346

347. Determine the velocity of point E, given the velocity of point A, v_A = 100 f.p.m. (a) Divide the plane motion of the body containing point E into components of translation and rotation and (b) Check the results by using the instantaneous-axis method. BC = 3.8 in.; BD = 6.4 in ; DE = 3.6 in.; DA = 1.8 in.



PROB. 347

81. Absolute and Relative Velocity. Velocity, a property of motion, may be classified as absolute or relative, depending, as in the general motion definitions, upon the choice of reference point or axis. When a point or axis fixed to the earth is used as reference, velocity becomes *absolute* velocity. When the velocity of one point is referred or related to any point other than one fixed upon the earth's surface, that velocity is a relative velocity. If, as in Fig. 208, a body is moving so that the velocity of point A, relative to point C, which is a fixed point on the earth, is v_A, then v_A is an absolute velocity.

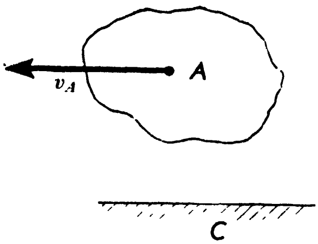


FIG. 208.

We have already noted (Theorem III) that the absolute motion of a point may be analyzed by the less direct method of relating it, first to a second point serving as a reference, and then considering the motion of the second point relative to the earth. An illustration in the form of rectilinear motion will furnish a simple exposition of absolute and relative velocity and help in crystallizing our grasp of their relationship.

Car A in Fig. 209 has a velocity at the instant of observation of 20 miles per hour to the right. This is its absolute velocity, and car A will approach point C, a fixed point on the road, at the rate of 20 miles per hour.

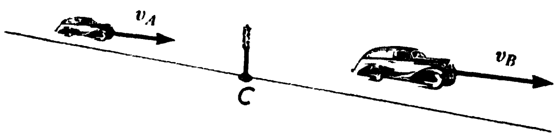


FIG. 209.

Car B has at the same instant an absolute velocity in the same direction of 45 miles per hour and is, therefore, traveling away from point C at that rate.

The difference between B's velocity and A's velocity is the velocity of B relative to A, or 25 miles per hour, to the right. This difference is the difference between two vector quantities.

In algebraic solutions, a difference between two quantities may be established by the direct process of subtraction, as $a - b = c$. The difference may also be established by adding to the first quantity the negative of the second quantity, or $a + (-b) = c$.

In vector subtraction, the process of *addition* is always used: we obtain the difference between two vectors by adding to the first, the negative of the subtrahend. The negative of a vector quantity is a vector quantity of the same magnitude and inclination, but of opposite sense.

In the present case, the velocity of B relative to A is

$$v_{B/A} = v_B \rightarrow v_A$$

Then, as in Fig. 210, we add to the absolute velocity of B, (v_B), the negative of the absolute velocity of A, ($-v_A$), or

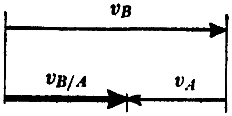


FIG. 210.

$$v_{B/A} = v_B \nrightarrow (-v_A)$$

It will be noted that in obtaining the velocity of B relative to A, we subtract the velocity of the point to which we are referring as an axis from the velocity of the point we are relating to it. This order is important, since the vector solution must be relied upon to fix the sense of the relative velocity as well as its magnitude and inclination.

A further illustration will reveal the need for care in the order of setting the terms in the subtraction operation. The velocity of "A relative to B" indicates that we are interested in subtracting B's absolute velocity from A's absolute velocity: $v_{A/B} = v_A \rightarrow v_B$. Fig. 211 is the required solution, and the vectors representing the terms having been set up in their proper order, the sense of the vector difference is correctly obtained.

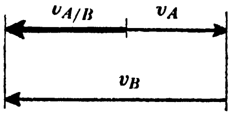


FIG. 211.

The principle applied in the above example need not be confined to absolute and relative velocities of the same inclination, but is universally applicable.

Fig. 212 shows a car which carries a wheel W supported so that it is free to rotate about center A while the car is either at rest or in motion.

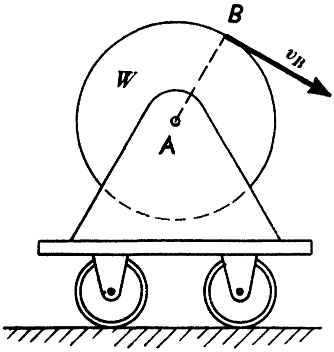


FIG. 212.

Let the wheel turn about center A with angular velocity ω directed clockwise while the car is at rest. Then, point B will have linear velocity v_B relative to A equal to ωAB and at right angles to radius AB. Since point A is at rest relative to the track, this linear velocity of B is absolute. If the car is now set in motion, as represented in Fig. 213, so that point A has an absolute linear velocity v_A, then the previous velocity of B is no longer absolute, but is relative to a moving point.

To find the absolute velocity of point B, v_B, we must add the absolute velocity of point A, v_A, to the velocity of B relative to A, or $v_B = v_{B/A} \nrightarrow v_A$ (Fig. 214).

Let us note that the velocity of B relative to A is alwa perpendicular to AB—that is, tangential to its only possible path about A.

We can check this observation by recalling that A and B are two points of the same rigid body and that, therefore, the distance between them remains unchanged regardless of the motion of the body. It then follows directly that B can have no component of relative velocity with respect to A in the direction connecting them.

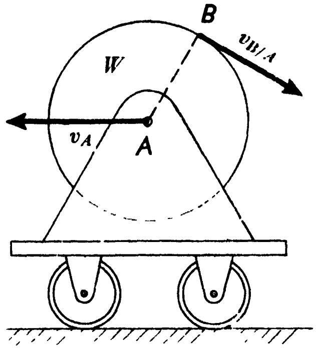


FIG. 213.

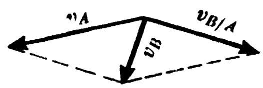


FIG. 214.

The velocity analysis of plane motion of Article 80 rested upon segregation of the translation and rotational components whose sum was equivalent to the total plane motion.

We may again view plane motion to observe how such a combination of translation and rotation is an application of the principles of absolute and relative velocity.

If, as in Fig. 215, a body is given plane motion, the relationship of the velocities of any two points A and B has been fixed. If point A is

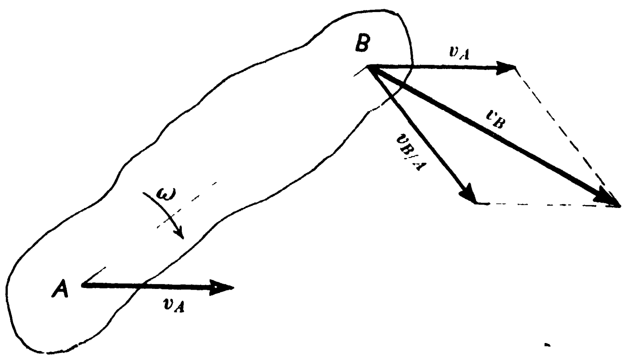


FIG. 215.

chosen as a reference, the absolute velocity of point B is equal to the sum of the absolute velocity of A plus the velocity of B relative to A, or $v_B = v_A \nrightarrow v_{B/A}$.

The velocity of B relative to A is

$$v_{B/A} = \omega r$$

where ω is the angular velocity of the body at the given instant, and r is the distance AB.

This would be the velocity of B relative to A whether the point A were itself in motion or at rest. We have, in this relative motion, a rotation about A as an axis. We next add to $v_{B/A}$ the absolute velocity of A to obtain the absolute velocity of B. This is equivalent to giving the entire body a motion of translation, imparting to every particle a velocity identical with the absolute velocity of A.

The absolute velocity of point B, the sum of its velocity relative to A plus the absolute velocity of A, is a combination of a translation plus a rotation.

In breaking down this velocity analysis into translation and rotation, points A and B have been taken anywhere on the body, and enjoy no

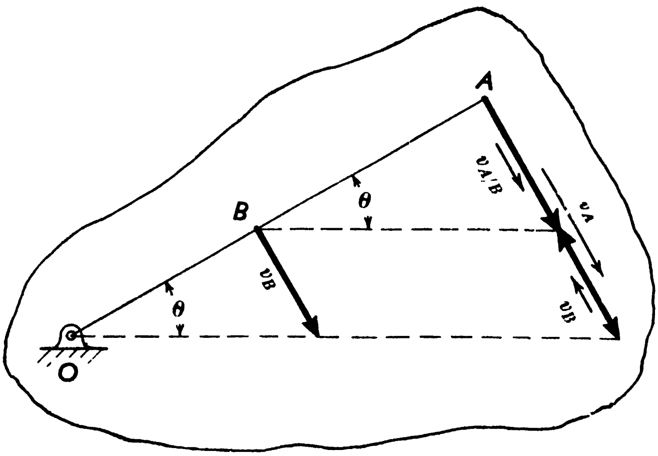


Fig. 216.

special properties. We may, therefore, always analyze a plane motion of a body by dividing it into the two elements of translation and rotation. In such a division, we may select as a reference any point of the body, and relate other points to it, provided that we know the velocity of the reference point, and the angular velocity of the body.

Frequently, we shall find that the known data consist of the absolute linear velocities of two points on the body. It is desired that the angular velocity ω of the body be found. Then we take the difference of the two absolute velocities. This difference is the relative velocity between the given points, and equals ωr, where ω is the angular velocity sought, and r is the distance between the two points.

For example, the absolute velocity of point A, Fig. 216, is v_A, and the absolute velocity of B is v_B. Then,

$$v_{A/B} = v_A \rightarrow v_B$$

This vector difference is shown on the drawing.

The angular velocity of the body is $\omega = \frac{v_A}{AO}$ or $\frac{v_B}{BO} = \tan\theta$. We note that $\frac{v_{A/B}}{AB} = \tan\theta$. Then $\omega = \frac{v_{A/B}}{AB}$.

When two points lie on different bodies, the relative velocity may again be obtained as the vector difference. In this case, we have no clue as to direction of relative velocity, since the two points may now have a component of relative velocity in the direction connecting the two points, which is no longer forced to remain rigid.

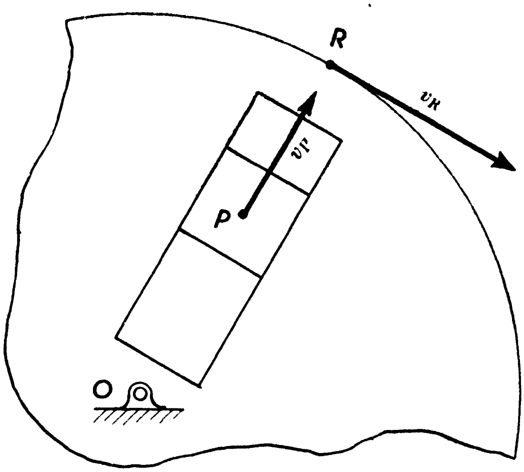


FIG. 217.

In Fig. 217, for example, we find a wheel whose axis O is fixed. Inserted in a slot in the wheel is a small block P. The block is free to move in the guides whether the wheel moves or not, and points on the block may have an absolute velocity in the indicated direction v_p.

Points on the wheel, however, are constrained, so that they must move in a circular path about O, and a point R can only have absolute linear velocity in a tangential direction, like v_R. Then points P and R may have entirely different components of velocity in the direction connecting them.

The discussion of absolute and relative velocity has thus far dealt with points, and the velocities we have studied have been linear. The translation element of plane motion is covered by such a discussion, since velocities of translation are equivalent for all points of the moving bodies. A more adequate background for the study of velocity and acceleration should equip us with the ability to deal with relative velocity properties of bodies in rotation. This requires that we note such properties with lines, rather than points, as our basis of study.

In Fig. 218, we find two rotating bodies, A and B, mounted so that they have a common and fixed axis. These two bodies are independent of each other, and either may turn while the other remains at rest or rotates with the same or different angular velocity as the first. If lines, such as *ab* and *cd*, are marked upon the bodies, we shall be able to observe the performance of these lines and draw conclusions as to the angular velocities of the rotating bodies.

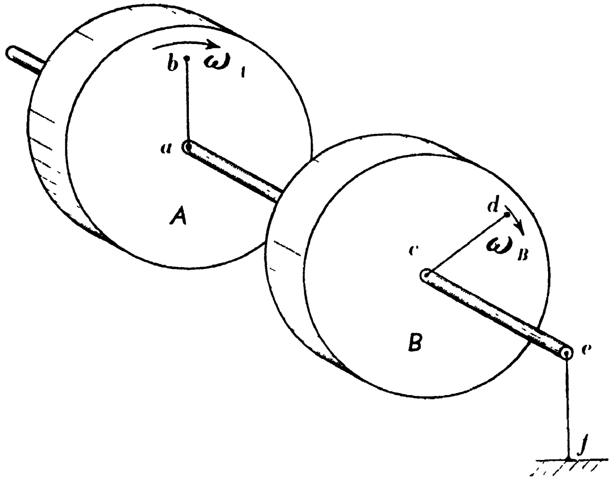


FIG. 218.

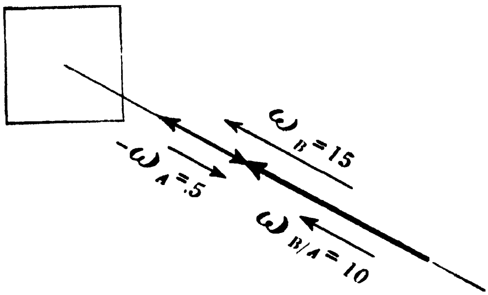


FIG. 219.

Line *ab* has the angular velocity of body A, ω_A; and line *cd* has the angular velocity of body B, ω_B.

If we cause both bodies to rotate so that their angular velocities are measured relative to a fixed axis *ef*, both ω_A and ω_B will be absolute angular velocities.

The angular velocity of body B relative to A will be the difference between the absolute angular velocity of B and the absolute angular velocity of A, or

ω_B (relative to A) $= \omega_B$ (absolute) $- \omega_A$ (absolute), or $\omega_{B/A} = \omega_B \rightarrow \omega_A$

This is a difference of two vector quantities, and we may subtract vectors representing these angular velocities to obtain the answer.

Angular velocities, like couples, are represented by vectors perpendicular to the plane of motion. The right-hand screw convention is used in announcing the sense of such vectors. (See Article 29.)

Illustrative Example

Line ab on wheel A has absolute angular velocity of five radians per second, clockwise as we look at the wheel from point C (Fig. 218). Line cd, on wheel B, has absolute angular velocity of 15 radians per second, also clockwise as we look from point C.

Proceeding with vector representation, we draw vectors ω_A and ω_B as in Fig. 219. These vectors represent absolute angular velocities. The angular velocity of B relative to A will be their vector difference $\omega_{B/A}$, and we note that the angular velocity of B relative to A is ten radians per second, clockwise as we look toward the cylinders from point C.

The result may be verified by analyzing analytically without the use of vectors. The line ef is perpendicular to the axis (Fig. 218) and fixed in space. Line ab is rotating clockwise, relative to ef, at the rate of five radians per second. Line cd is going faster, since it is rotating, also clockwise, relative to ef, at the rate of 15 radians per second.

If cd is rotating relative to ef at 15 radians per second clockwise, and ab is rotating relative to ef at 5 radians per second clockwise, then cd must be rotating relative to ab at the rate of ten radians per second, clockwise.

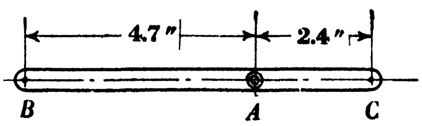
This analysis is quite simply made without recourse to vectors when the two wheels A and B have parallel planes of motion. If these planes of rotation are oblique to each other, the vector solution will have the advantage, since the subtraction of two vectors is as simple a process when the vectors are oblique as when they are parallel.

Problems

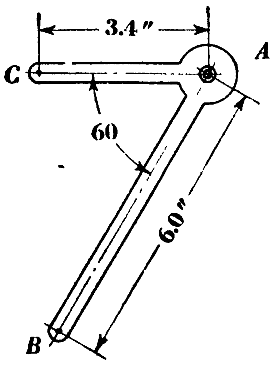
348. A car A is traveling east at the rate of 40 m.p.h. Another car B is traveling west at the rate of 55 m.p.h. Find (a) the velocity of A relative to B and (b) the velocity of B relative to A.

349. A particle A travels north with an absolute velocity of 13 f.p.s. A second particle B travels southeast with an absolute velocity of 10.8 f.p.s. Find the velocity of B relative to A. *Ans.* $v_{B/A}$ (magnitude) = 22 f.p.s.

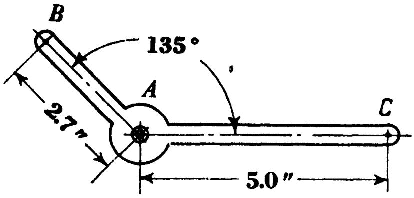
350–353. The body ABC rotates about fixed axis A with an absolute angular velocity of 1 radian per sec. clockwise. Find the absolute velocities of points B and C, the velocity of C relative to B, and the velocity of B relative to C.



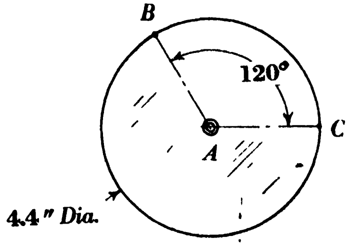
Prob. 350



Prob. 351



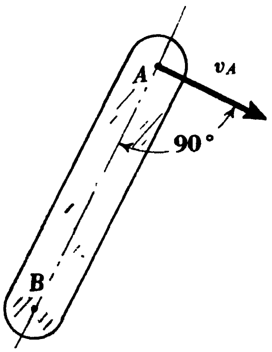
PROB. 352



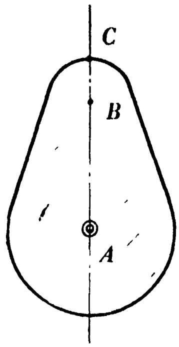
PROB. 353

354. Point A has an absolute velocity $v_A = 3.5$ in. per sec. The vel city of point B relative to A is 2.1 in. per sec. in the same sense as v_A. Find the absolute velocity of point B and the angular velocity of body AB. $AB = 5$ in.

Ans. 1.4 in. per sec.; 0.42 radians per sec.



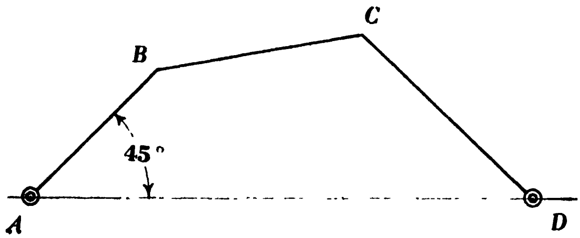
PROB. 354



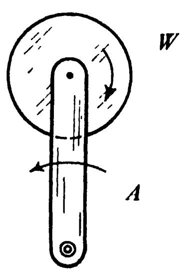
PROB. 355

355. Point C of the cam shown has a velocity relative to point B of 1.6 in. per sec. If the angular velocity of the cam is 2 radians per sec., find the distance BC and the absolute velocity of point C. $AB = 2.6$ in.

356. The angular velocity of crank AB is 10 radians per min. counterclockwise. Find the velocity of the instantaneous axis of body BC relative to C. $AB = 2.8$ in.; $BC = 3.2$ in.; $CD = 3.6$ in.; $AD = 7.7$ in.



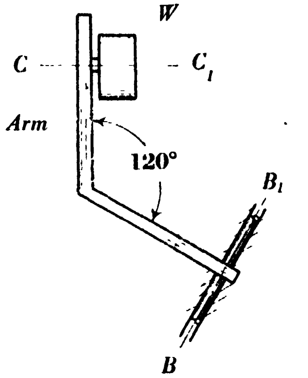
PROB. 356



PROB. 357

357. Wheel W has an absolute angular velocity of 300 r.p.m. clockwise. The arm A carrying the axis of the wheel has absolute angular velocity of 500 r.p.m. counterclockwise. Determine the angular velocity of W relative to A.

358. The arm rotates about fixed axis B_1B carrying with it the axis C_1C supporting wheel W. If the absolute angular velocity of W is 2 radians per sec. clockwise when viewed from C_1, and the absolute angular velocity of the arm is 3.6 radians per sec. clockwise when viewed from B, determine (a) the angular velocity of W relative to the arm and (b) the angular velocity of the arm relative to W.



PROB. 358

82. Sliding Contact. When bodies are in contact so that one constrains or determines the motion of another, it is possible to define the contact as one of two general types. Fig. 220 illustrates *sliding contact*. A block A has been inserted in a slot cut in body B so that the block may slide freely in the slot.

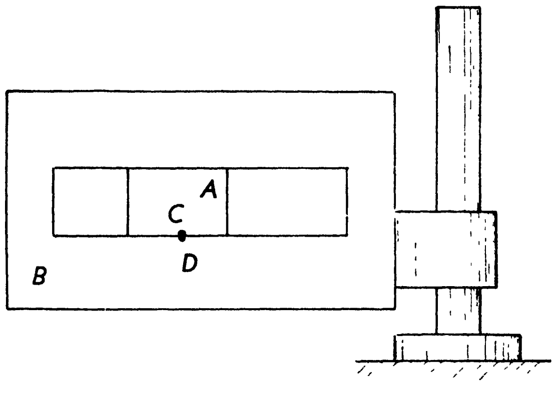


FIG. 220.

Consider two points which are at the instant in contact, like C of block A and D of body B. If, as in Fig. 221, point C is given a velocity Cc' while D remains at rest, then Cc' is the velocity of C relative to D, which is also its absolute velocity. The only velocity which C may have relative to D is in a direction parallel to the sides of the slot which constrain the block, and which we shall call the *sliding surfaces*. This velocity of C relative to D is called the *rate of sliding*, or occasionally *C's slip with reference to D*.

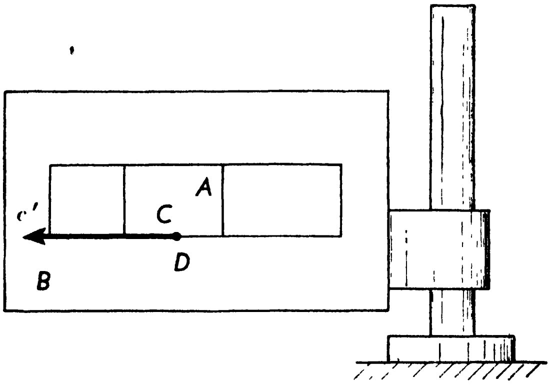


FIG. 221.

If we introduce a new element of motion by allowing body B to move with velocity v_B, as in Fig. 222, point D now has velocity, $Dd_3 = v_B$. C has velocity Cc', which may be resolved into two components: one parallel to the sliding surfaces and another perpendicular to the sliding surfaces. The component of C's velocity parallel to the sliding surfaces

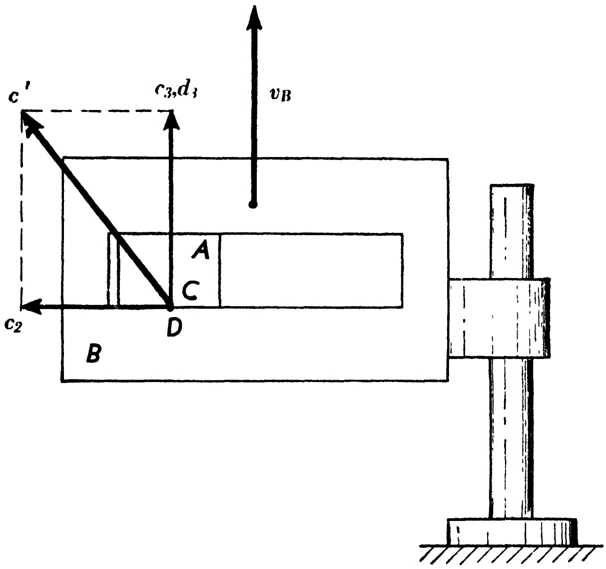


FIG. 222.

Cc_2 is as independent of the motion of point D as it was when body B was at rest. The component of C's velocity perpendicular to the sliding surfaces, Cc_3 is, however, forced to remain equal to the component of D's velocity in that same direction, Dd_3, for C and D may have no relative velocity in a direction perpendicular to the sliding surfaces, since there is no freedom between them in that direction.

Fig. 223 shows a mechanism which involves sliding contact. The large piece M rotates about fixed axis O. A slot is cut in M, and a block inserted which may move freely in the slot. An arm or crank AB is attached to the block by a pin joint at A, and rotates about fixed axis B. The angular velocity ω_{AB} of the arm AB is known, and it is desired that we determine the angular velocity of the piece M.

Aa', the linear velocity of point A on the arm, is readily obtained in direction (perpendicular to AB) and magnitude ($\omega_{AB}AB$).

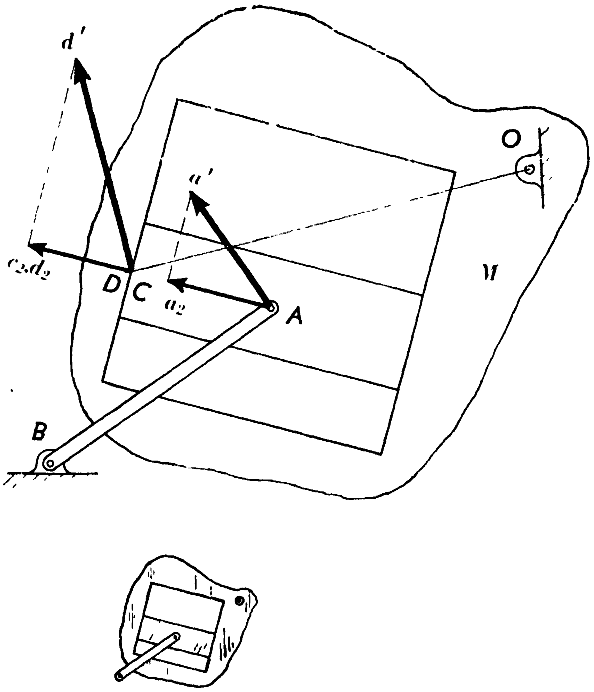


FIG. 223.

The orthogonal component of A's velocity perpendicular to the sliding surface is Aa_2. Point C, which, like point A, is on the small block, must have an orthogonal component in the direction AC which is $Cc_2 = Aa_2$, for these are two points of the same rigid body and must, therefore, have the same orthogonal component in the direction connecting them. Point D on body M must have an orthogonal component perpendicular to the sliding surface which is $Dd_2 = Cc_2$, since there can be no relative velocity between points C and D in this direction.

We now have for point D, one orthogonal component. Since D is a point on body M, the inclination of its resultant velocity must be perpendicular to DO. Then the resultant velocity of point D becomes known

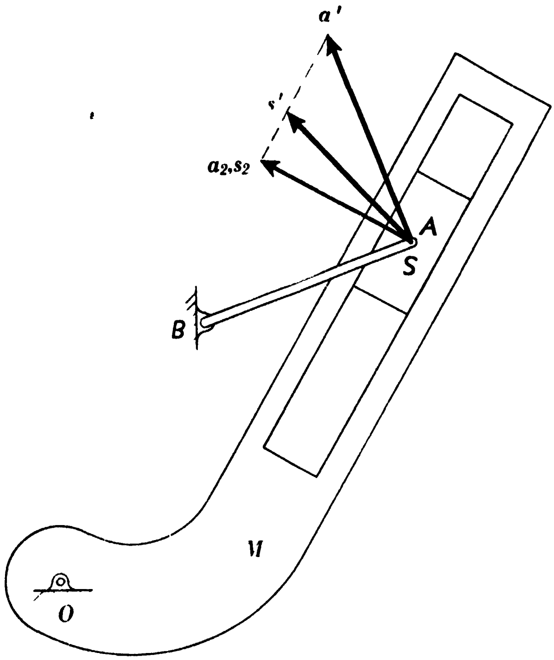


FIG. 224.

(Theorem I), and is Dd'. If we divide the magnitude of linear velocity Dd' by radius DO, we have the angular velocity of body M.

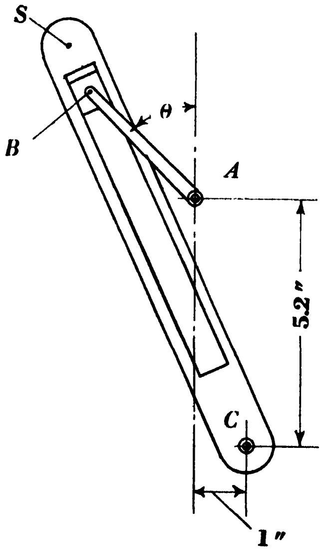
In most cases of sliding contact, the resolution of velocities is made directly at the pin A of the block, as in Fig. 224.

This solution makes use of rigid-body properties, as follows: The piece M, from the concept of the rigid body of mechanics, is unlimited in extent. There is then a point S of body M which is located at the same position in space as point A of the block, and such a point may be used to establish velocity relationships. This point of body M must have an orthogonal component perpendicular to the sliding surface Ss_2 which is equal to Aa_2, and a resultant velocity Ss', perpendicular to SO.

PROBLEMS

359. The angular velocity of crank AB is 2 radians per sec. clockwise. Determine the angular velocity of beam CS when $\theta = 45°$. $AB = 3$ in.; $CS = 8.8$ in.

Ans. 0.7 radians per sec.



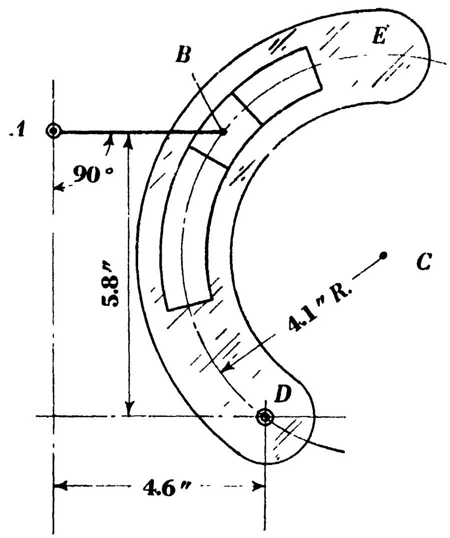
PROB. 359

360. Using the dimensions of the oscillating-beam mechanism shown in Problem 359, determine the velocity of point S when $\theta = 60°$, and the angular velocity of AB is 300 r.p.m. counterclockwise.

361. In Problem 360, find the rate of sliding of point B on the small block relative to point B on the piece SC.

362. Crank AB has an angular velocity of 1 radian per sec. clockwise. Point C is the center of curvature of the slot. Determine the velocity of point C and the angular velocity of body DE. $AB = 3.6$ in.

Ans. 2.12 in. per sec.; 0.52 radians per sec.



PROB. 362

363. In Problem 362, determine the rate of sliding of point B on crank AB relative to point B on the piece DE.

83. Rolling Contact. When a body rolls upon another, the contact may be such that there is no relative motion between the two points, one lying on each body, which are in contact. This type of motion is called *pure rolling contact.*

To clearly appreciate such a rolling action, let us consider the pair of equal cylinders shown in Fig. 225. If both cylinders are rotating with the same angular speed but in opposite directions, lines a and b will become the contacting lines at the same instant and will then depart, allowing other mated lines to come up to contact and depart. Exactly one revolution after the time that a and b are in contact they will again meet, and will continue to do so as long as the wheels turn. At all times, lines a and b have the same speed, and at the instant of contact, since their direction of motion is the same, they will have the same velocity. There will then be no relative velocity between the contacting lines, and the bodies are in pure rolling contact.

If we now interrupt this smooth, regular, rolling action by holding the follower still while the driver rotates, line *a* of the driver will have an absolute velocity while it passes the contacting position, but the contacting line of the follower, being at rest, will have zero absolute velocity. There is now relative velocity between the contacting lines of the two bodies, and these lines are sliding, or *slipping*, by each other. In this case, the slipping is complete, and the relative velocity between the contact lines is equal to the entire absolute velocity of the moving line.

If the driver is now given an angular velocity which is different from the angular velocity of the equal-size follower, the two lines which come into contact will have different velocities at the instant of contact. The difference of the velocities—the relative velocity—will be the rate of

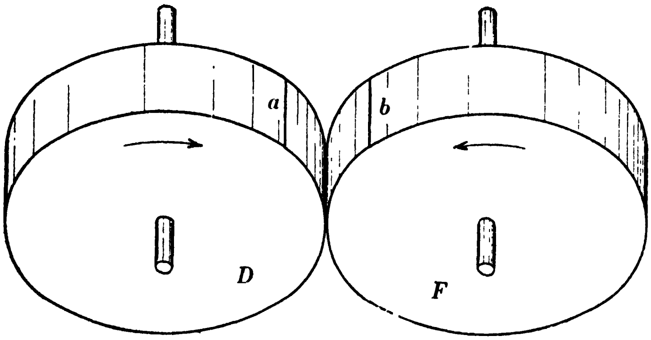
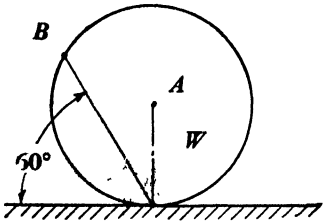


Fig. 225.

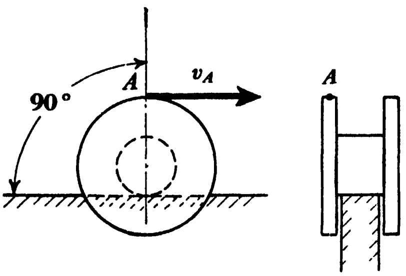
sliding which will not be a complete sliding, as in the previous illustration, but a fractional, or partial, slip.

The velocity analysis of such slip is a case to be explored with the methods outlined in Article 82 for any sliding contact. The contacting bodies need not be equal cylinders; whenever two bodies are in contact so that there is no relative velocity between contacting lines or points, they are in pure rolling contact.

364. Cylinder *W*, 4 in. in diameter, is in pure rolling contact with a fixed track. The angular velocity of the cylinder is 4 radians per sec. clockwise. Find the velocity of point *A*, the center of the cylinder, and of point *B* on its surface *Ans.* 8 in. per sec.; 13.9 in. per sec.



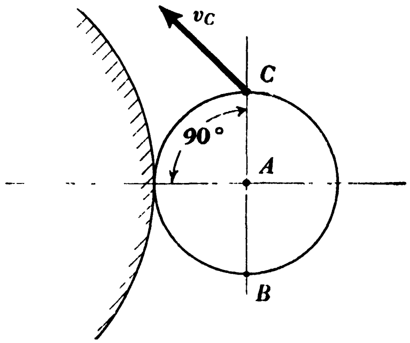
Prob. 364



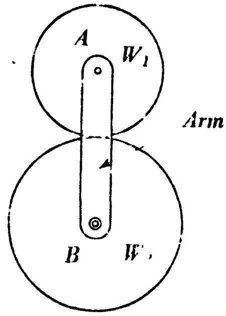
Prob. 365

365. The drum, consisting of a central shaft and disks fastened to the shaft, rolls without slip on the fixed track. The diameter of the shaft is 1.8 in. and the diameter of the disks is 2.72 in. Given the velocity of point A, $v_A = 4$ in. per sec., determine the angular velocity of the drum and the velocity of the top of the shaft.

366. The disk, 6 in. in diameter, rolls without slip on the curved fixed track. Given the velocity of point C, $v_c = 2$ f.p.m., determine the velocities of points A and B.



PROB. 366



PROB. 367

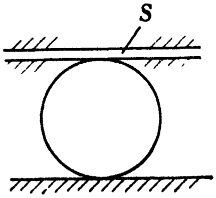
367. Wheel W_1, 4 in. in diameter, is supported on axis A carried by the arm, and is in pure rolling contact with W_2, 5.3 in. in diameter. The arm and W_2 are mounted upon fixed axis B. The angular velocity of the arm is 2 radians per sec. clockwise, and the angular velocity of W_2 is 1 radian per sec. clockwise.

(a) Locate the instantaneous axis of W_1 and determine its absolute angular velocity.

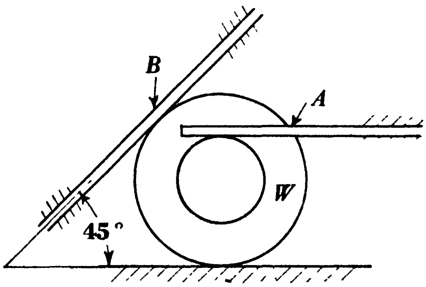
(b) Determine the angular velocity of W_1 relative to the arm.

368. The wheel is in pure rolling contact with the fixed track and with the bar S, which slides parallel to the track. The diameter of the wheel is 4 in., and its angular velocity is 150 r.p.m. counterclockwise. Find the velocity of S.

Ans. 314 f.p.m.



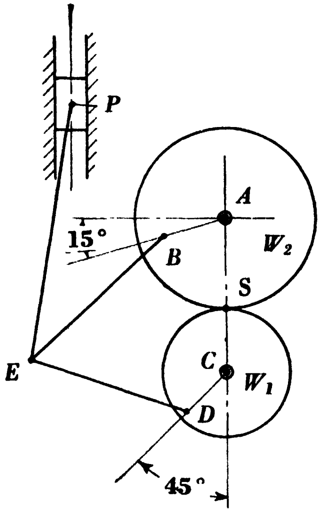
PROB. 368



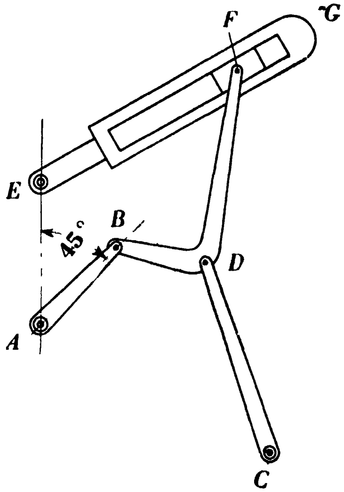
PROB. 369

369. Wheel W is in pure rolling contact with sliding bars A and B. A moves parallel to the track, and B moves at an angle of 45° with the track. If the velocity of A is 2.5 in. per sec. to the left, find the velocity of B, the angular velocity of W, and the rate of sliding of W on the fixed horizontal track. The diameter of the wheel is 5.3 in., and the diameter of the concentric projection which is in contact with A is 2.7 in.

370. The mechanism of a differential stroke engine is shown. Wheels W_1 and W_2 are in pure rolling contact at S. $AB = 1.9$ in.; $AS = 2.7$ in.; $SC = 2$ in.; $CD = 1.7$ in.; $DE = 5$ in.; $BE = 5.7$ in.; $PE = 7.7$ in. The center line of P is 4.5 in. to the left of A. If W_1 has an angular velocity of 1 radian per sec. clockwise, determine the linear velocity of the piston P.



PROB. 370



PROB. 372

371. If the piston P of Problem 370 has a linear velocity of 20 in. per sec., in the position shown, determine the angular velocities of wheels W_1 and W_2.

372. The mechanism is driven by crank AB. A, C, and E are fixed axes. E is 4.2 in. above A. C is 3.9 in. below and 7 in. to the right of A. $AB = 3.3$ in.; $BD = 2.7$ in.; $CD = 6$ in.; $DF = 6$ in. Angle BDF of the rocker arm is 90°. The angular velocity of $AB = 600$ r.p.m. counterclockwise. For the position shown,

(a) Locate the instantaneous axis of the rocker arm.

(b) Determine the angular velocity of EG.

(c) Determine the rate of sliding of the block at F relative to the slot.

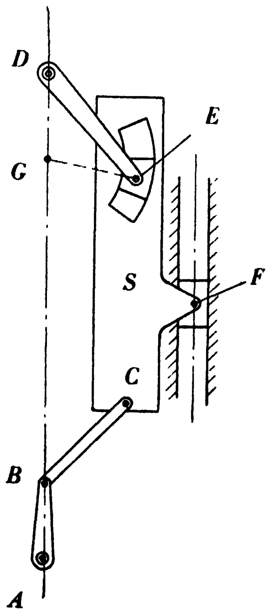
373. Cranks AB and DE are geared together (gearing not shown) so that AB makes ten revolutions while DE makes three in the opposite direction. The block at E slides in a slot, cut in piece S, with center of curvature at G. The block at F slides in vertical fixed guides, and is pinned to S. $AD = 10$ in.; $AB = 1.4$ in.; $BC = 2.30$ in.; $DE = 2.8$ in.; $GE = 1.8$ in. in radius; $CF = 2.4$ in.

The center line of the block at F is parallel to AD, and 3 in. to the right of it.

In the position shown, AB is on AD, $DG = 1.8$ in.; $EC = 4.5$ in.

If the angular velocity of AB is 60 r.p.m. clockwise,

(a) Locate the instantaneous axis of S.



PROB. 373

(b) Determine the linear velocity of F.
(c) Determine the absolute angular velocity of S.
(d) Determine the rate of sliding of the block at E in the curved slot.

84. Instantaneous Axis of Rolling Contact. When a wheel W has a motion of pure rolling contact with its track, as in Fig. 226, the points of contact A on W_1 and B on the track must have the same velocity.

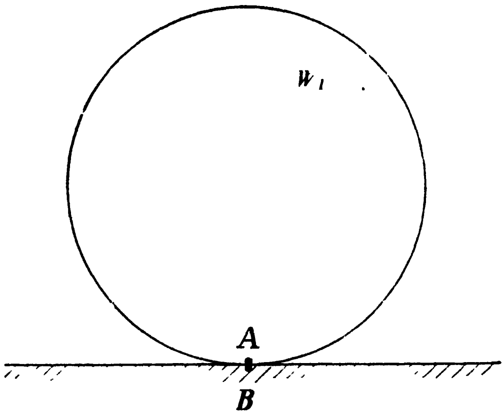


FIG. 226.

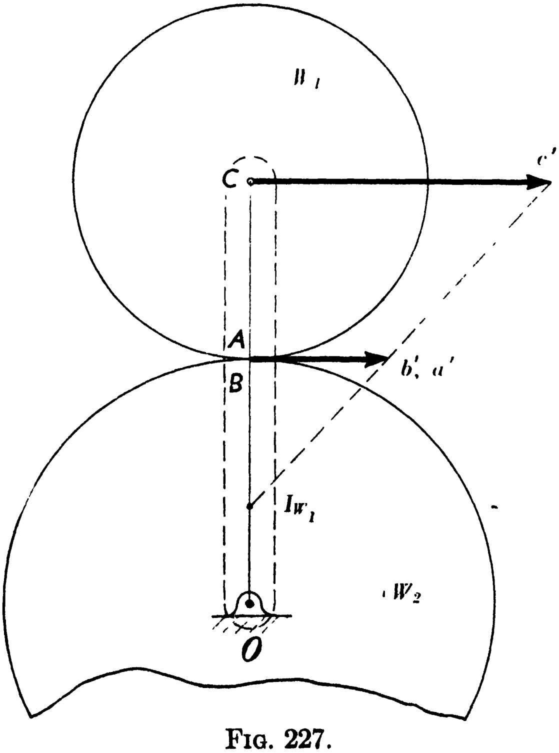


FIG. 227.

If the track is at rest, as when it is fixed to the earth, point B has no velocity. Point A then becomes a point of zero velocity on body W_1; hence its instantaneous center.

If the track or body on which W_1 rolls is itself in motion, as in Fig. 227, and the motion is one of pure rolling contact, the instantaneous center of W_1 is no longer at point A, which now has linear velocity.

Wheel W_2 is turning about fixed axis O with angular velocity ω_2. Then point B of body W_2 has linear velocity $Bb' = \omega_2 OB$. Point A has linear velocity $Aa' = Bb'$, since there is no slip. The arm which carries the axis of W_1, point C, is turning about fixed axis O with angular velocity ω_A. Then point C has linear velocity $Cc' = \omega_A OC$. Since the velocities of two points A and C on body W_1, are known, the instantaneous axis may be found, and is determined as I_{W_1}.

CHAPTER IX

Acceleration

85. Acceleration. The preceding investigations of kinematic properties have been concerned with change of position—displacement—and the time rate of change of position—velocity. Velocity may itself be variable, and changes in velocity with respect to time may now be explored.

Acceleration is defined as the time rate of change of velocity.

Velocity is a vector quantity. Then changes in velocity may be changes in magnitude, in direction, or in both magnitude and direction. These changes must be described by noting the degree of change of both a magnitude and a direction. Acceleration is, therefore, a vector quantity.

86. Linear Acceleration. The simplest form in which the kinematic property of acceleration is encountered arises in the case of a particle which has rectilinear motion. Such a particle will have but one degree of freedom; its inclination is therefore fixed, and its velocity may vary in magnitude or in sense only.

When a change of velocity Δv takes place in time Δt, the average acceleration, or rate of change with respect to time, then must be $a = \frac{\Delta v}{\Delta t}$ and, as indefinitely shorter intervals of time are considered,

$$a = \underset{\Delta t \doteq 0}{\text{Limit}} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

or, acceleration is the first derivative of velocity with respect to time.

It has already been noted that

$$v = \frac{ds}{dt}$$

Then,
$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

or, acceleration is the second derivative of displacement with respect to time.

These expressions furnish a basis for relating a, v, s, and t. It should be noted that the process of differentiation employed yields only the *magnitude* of the acceleration. A modified form of the relationship is also useful. Since

$$v = \frac{ds}{dt}$$

and
$$a = \frac{dv}{dt}$$

$$dt = \frac{ds}{v} = \frac{dv}{a}$$

and

$$v\,dv = a\,ds$$

These basic relationships become simplified when the acceleration is *constant*. If, in this case, the velocity at the beginning of a time interval is called v_0, and the velocity at the end of the time interval v_f, the acceleration,

$$a = \frac{v_f - v_0}{t}$$

or

$$v_f = v_0 + at.$$

The distance s traveled during the time interval t, with acceleration constant, will be the product of the average velocity during the time interval and the time, or

$$s = \frac{v_0 + v_f}{2}\,t$$

Since the velocity is changing uniformly,

$$s = \frac{v_0 + v_f}{2}\,t = \frac{v_0 + (v_0 + at)}{2}\,t$$

Then,

$$s = v_0 t + \tfrac{1}{2}at^2$$

Another relationship of constant acceleration may be derived by eliminating the time element t.

$$v_f = v_0 + at$$

and

$$v_f^2 = v_0^2 + 2v_0 at + a^2t^2$$

$$s = v_0 t + \tfrac{1}{2}at^2$$

and

$$2as = 2av_0 t + a^2t^2$$

Then,

$$v_f^2 = v_0^2 + 2as$$

When the linear acceleration is *variable*, other expressions than those given above must, of course, be used. In general, the calculus must be employed. If the equation for velocity-time is known, we differentiate and have an expression for acceleration. Or, if the known material presents an equation of relationship between displacement and time, a first differentiation yields the velocity-time relationship, and the second differentiation establishes the acceleration.

Many mechanisms, particularly of the linkage type, produce a linear motion in which the equation expressing relationship between displacement and time cannot readily be obtained. In such cases, there is available the graphical equivalent for analytical differentiation. This method was discussed in the chapter on Velocities (see Article 73) and may be employed to obtain acceleration.

In the crank-and-connecting-rod mechanism of Fig. 228, the dimensions of the linkage and the angular velocity of the driving crank AB are known. Then, by drawing the mechanism in several positions, a curve may be established which gives the relationship between the displacement of point C and time.

Setting tangents to this curve will produce the velocity-time relationship that is shown in Fig. 229. If this latter curve is again differentiated by the tangent method, the acceleration-time curve may be plotted. Fig. 230 shows the resulting acceleration-time curve. This method of establishing the acceleration is basic and simple. It is capable of universal application and is the most adequate method for general investigations of acceleration in mechanism applications.

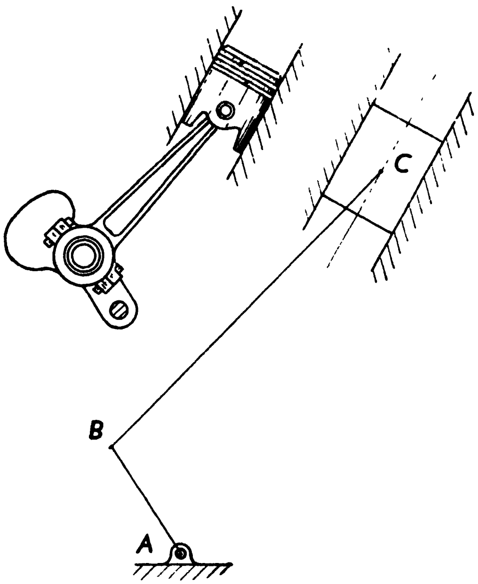


FIG. 228.

Simple Harmonic Motion. A special form of variable acceleration is encountered frequently, both in linkage applications and as a basic factor in vibratory motions. This is *simple harmonic motion*. In this case, a particle moves along a straight line so that its acceleration is always proportional to its displacement x from any fixed point in the line which may be used as a reference point, and the sense of the acceleration is always towards the reference point.

Expressed mathematically,

$$a = \frac{d^2x}{dt^2} = -Kx$$

where K is a constant and the minus sign indicates that the sense of the acceleration is always opposite to that of the displacement x. For example, in Fig. 231, point O has been selected as an origin. Then the point A is displaced a distance x from O, which is a positive displacement by the conventions set along the axis. The acceleration will be in the negative direction, or its sense will always be directed towards point O.

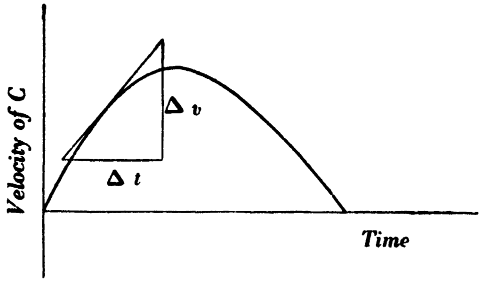


FIG. 229.

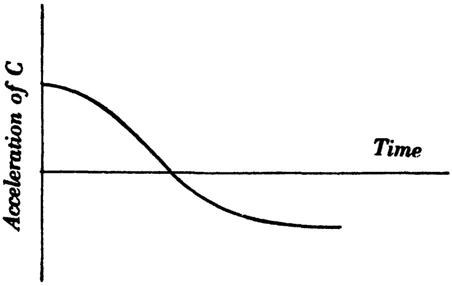


FIG. 230.

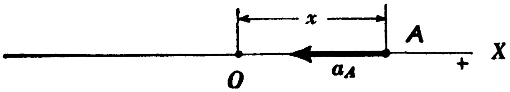


FIG. 231.

When a particle B is moving along a circular path at the end of a radius r, which has constant angular velocity, ω, as in Fig. 232, then the motion of the point A (which is the projection of point B on the diameter of the circle) is simple harmonic motion. If t is the time interval during which the point A moves from position O to position A, then $\theta = \omega t$

and $$x = r \sin \theta = r \sin \omega t$$

The velocity of point A

$$v_A = \frac{dx}{dt} = \omega r \cos \omega t$$

and A's acceleration,

$$a_A = \frac{d^2x}{dt^2} = -\omega^2 r \sin \omega t = -\omega^2 x$$

Then the motion of A is simple harmonic, and the constant K of the basic expression $a = -Kx$ is ω^2, which is the square of the constant angular velocity of the rotating radius.

The circle of the preceding discussion, whose radius is moving with constant angular velocity, is called the *auxiliary circle* of this simple harmonic motion, and its radius is called the *amplitude* of the motion. The time for the radius to complete one revolution during which point A will travel forward and back to starting position, or one oscillation, is called the *period* of the motion, and is usually expressed in seconds. Then if the period is called T,

$$T = \frac{2\pi}{\omega}$$

in which ω is the constant angular velocity of the rotating radius in radians per second.

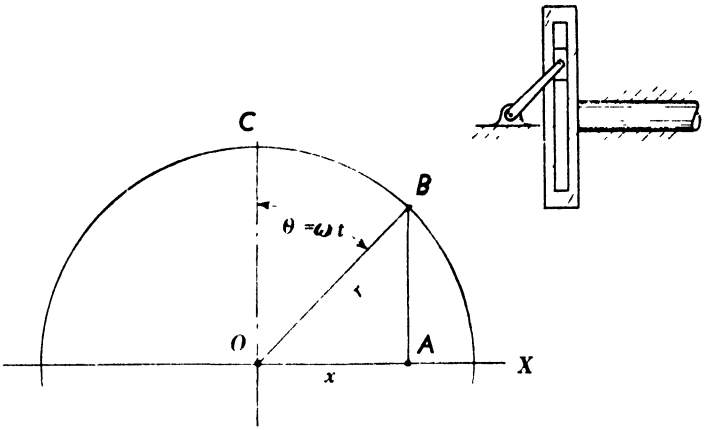


FIG. 232.

Frequency is the number of oscillations per second. Therefore, it is the reciprocal of the period, or

$$f = \frac{\omega}{2\pi}$$

A very convenient form of graphical representation is available to show the relationship between displacement, velocity, and acceleration, in simple harmonic motion (Fig. 233). This method of representation is known as the position-vector, or rotating-vector, system.

We are to consider the motion of a point a along the XX axis.

From the previous discussion, we have the following relationships, expressed analytically, in simple harmonic motion:

$$x = r \sin \omega t$$

$$v = \frac{dx}{dt} = \omega r \cos \omega t$$

$$a = \frac{d^2x}{dt^2} = -\omega^2 r \sin \omega t = -\omega^2 x$$

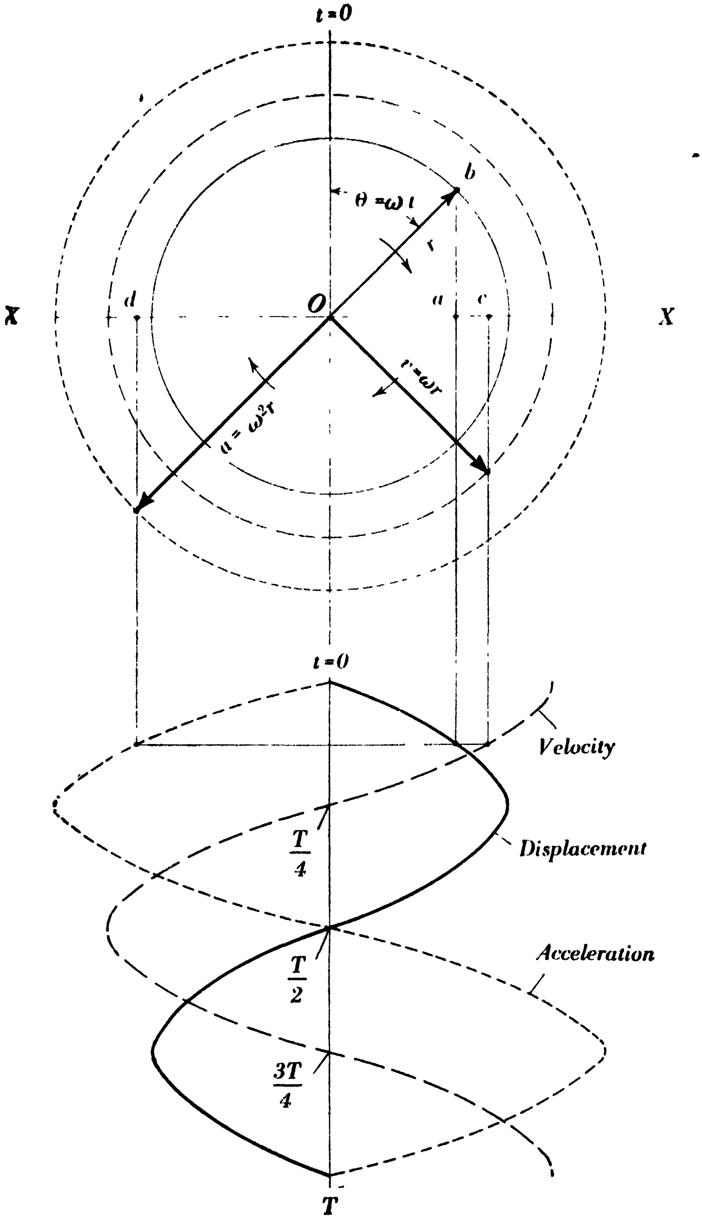


Fig. 233.

Graphically, we proceed by drawing an auxiliary circle of radius r, equal to the amplitude of the motion. The point a, always at the foot of an orthographic projector from point b (the end of the rotating or position vector r) has simple harmonic motion. O is the center of its path.

It will be noted that distance $Oa = r \sin \omega t$ and, therefore, represents at all times the displacement. In the graph, this displacement is ordinate, and abscissas represent time. If, now, a vector $v = \omega r$ (which is constant in magnitude) is drawn as shown, at right angles to r, this vector will rotate with r to properly represent the inclination and sense of velocity of point b. The projection of the end point of this vector on the diameter of the auxiliary circle is point c, and $Oc = \omega r \cos \omega t$, which matches the analytical expression for the velocity of point a. These projections are plotted on the graph, and their locus is the velocity-time curve.

Finally, a vector $a = \omega^2 r$ (which is constant in magnitude) is drawn as shown, having the same inclination as r to properly represent the inclination and sense of acceleration of point b. The projection of the end point of this vector on the diameter of the auxiliary circle is point d, and $Od = -\omega^2 r \sin \omega t$, which matches the analytical expression for the acceleration of point a. These projections are plotted on the graph, and their locus is the acceleration-time curve.

This rotating-vector projection is useful in the analysis of mechanical vibrations and will be employed in the analysis of such motions in Article 100.

Problems

374. A particle travels along a straight path according to the following equation of motion:

$$s = 6t^2 - 2t^3$$

where s is the displacement in inches, and t is time in seconds.

Plot the displacement-time curve for the first 3 sec. starting at $t = 0$.

Plot the acceleration-time curve, determining values by graphical differentiation, for the first 3 sec. starting at $t = 0$.

375. Solve Problem 374, changing the equation of motion to

$$s = 3t^3 - 5t^2 + 3t$$

376. The following table gives the velocity-time relationship of a particle which moves in a straight line. Determine the acceleration-time curve.

Velocity (*In. Per Sec.*)	Time (*Sec.*)
0	0
1	1.91
2	3.58
3	4.92
4	5.76
5	6.07 (max.)
6	5.89
7	5.30
8	4.47
9	3.44
10	2.31
11	1.16
12	0

87. Acceleration in Curvilinear Motion. When a particle moves with curvilinear motion, the velocity is changing in direction, and its magnitude of velocity, or speed, may be constant or changing. In the example shown in Fig. 234, a particle travels in a circular path about O with constant speed. The direction of its velocity will be constantly changing.

At A, the velocity of the particle is v_A, and at B, it is v_B. The difference between these two velocities is $\Delta v = v_B \rightarrow v_A$, the vector difference

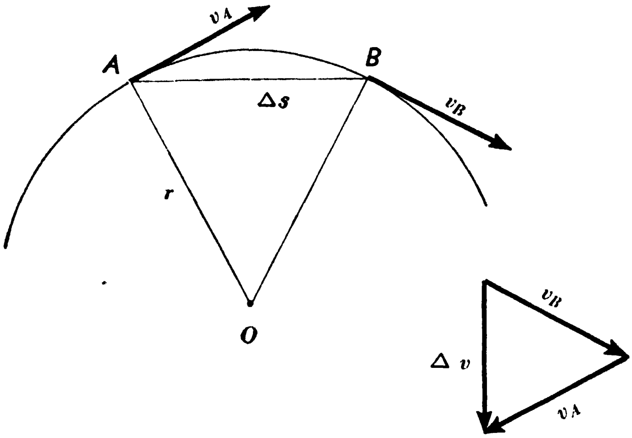


FIG. 234.

obtained as shown. The vector triangle of Fig. 234 is similar to triangle OAB because corresponding angles are equal.

Then,
$$\frac{\Delta v}{v_A} = \frac{\Delta s}{r}$$

As angle AOB is indefinitely decreased, Δv approaches as its limit dv, and Δs approaches ds. In the limit

$$\frac{dv}{v} = \frac{ds}{r}$$

But
$$ds = vdt$$

and
$$\frac{dv}{v} = \frac{vdt}{r}$$

or
$$\frac{dv}{dt} = \frac{v^2}{r}$$

That is,
$$a = \frac{v^2}{r}$$

This acceleration is in the direction of radius r. Note that the chord Δs will approach a perpendicular to radius OA as its limiting direction, and Δv in the similar triangle will become perpendicular to v_A at its limit. This acceleration in a radial direction is called *normal acceleration*, and will be symbolized as a_n.

When the particle moves in a circular path and has a normal acceleration $a_n = \frac{v^2}{r}$, but is no longer moving with constant speed, the expression for a_n becomes an instantaneous expression, valid only at the instant when the velocity is v. If the path is curvilinear but not circular, the value of r in the expression is the distance from the point at which velocity is v, to the center of curvature at the instant, or the instantaneous radius of curvature.

An equivalent expression for normal acceleration may be derived in terms of the angular velocity of the moving radius.

Since $$a_n = \frac{v^2}{r}$$

and $$v = \omega r$$

Then, $$a_n = \frac{\omega^2 r^2}{r} = \omega^2 r$$

The values of ω and r are instantaneous values if the angular velocity or the radius of curvature (or both) are changing quantities.

At any instant, then, there is a normal acceleration which is dependent only upon the instantaneous value of the velocity and the radius of curvature. This acceleration is present at such an instant whether the speed later changes or remains constant.

If, however, the velocity does change in magnitude, such a change must be in the direction of the velocity, which is tangential to the path.

We shall call this type of acceleration the tangential component of acceleration a_t. Its magnitude may be obtained from the basic definition of acceleration,

$$a_t = \frac{dv}{dt}$$

To evaluate this term, it is convenient to turn to the relationship between linear and angular accelerations, which will be discussed in the next article.

88. Linear and Angular Acceleration. Angular acceleration α is the rate of change of angular velocity of a line with respect to time, or the time rate of change of angular velocity.

Then, $$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

The tangential component of linear acceleration

$$a_t = \frac{dv}{dt}$$

may be most readily evaluated by noting that since

$$v = \omega r$$

$$a_t = \frac{d(r\omega)}{dt} = r\frac{d\omega}{dt} = r\alpha$$

This component of acceleration depends then upon the angular acceleration of the moving line as well as on the instantaneous or permanent radius of curvature.

89. Resultant Acceleration. A complete investigation of the linear acceleration of a point moving with curvilinear motion and with variable speed necessitates establishing two vector quantities: normal and tangential accelerations. The resultant acceleration is the vector sum of these two, as in Fig. 235.

A particle at A on radius AB is constrained to rotate about center B. Line AB is two inches long and has angular velocity ω of three revolu-

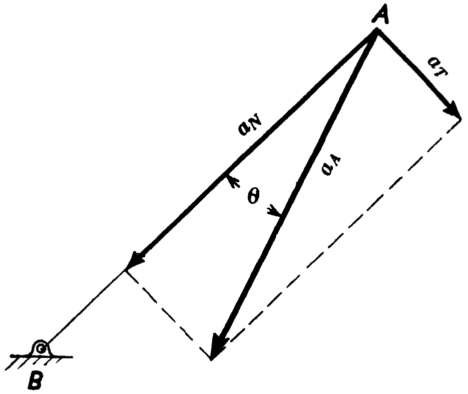


FIG. 235.

tions per second, clockwise, and angular acceleration α of 20 revolutions per second per second, clockwise. The normal component

$$a_N = \omega^2 r^* = (3 \times 2\pi)^2 2 = 711 \text{ in. per sec.}^2$$

The tangential component,

$$a_T = \alpha r = (20 \times 2\pi)2 = 251 \text{ in. per sec.}^2$$

Then the resultant acceleration of point A is $a_A = a_t \nrightarrow a_n$

or, $$a_A = \sqrt{a_T^2 + a_N^2} = 754 \text{ in. per sec.}^2$$

The resultant is inclined so that it makes an angle

$$\theta = \tan^{-1} \tfrac{251}{711} = 19.4°$$

with the radius AB. The sense of the resultant is determined from the senses of the two components.

PROBLEMS

377. The following table gives the values of the acceleration of a piston against crank angles. The crank has a constant angular speed of 240 r.p.m.

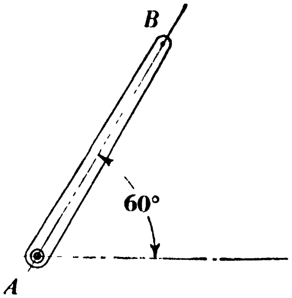
Plot the acceleration-time curve and determine the stroke of the piston by graphical integration.

* Note that ω must be expressed in radians per unit of time.

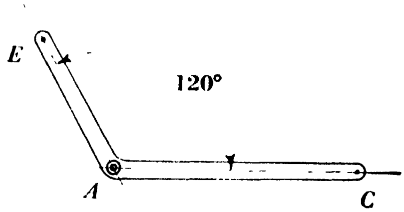
Acceleration (*Ft. per Sec.*²)	Crank Angle (°)
329	0
305	15
262	30
190	45
99	60
8	75
−68	90
−124	105
−165	120
−185	135
−194	150
−196	165
−197	180

378. The angular velocity of crank AB is 2 radians per min. clockwise, and its angular acceleration is 4 radians per min.² counterclockwise in the position shown. Determine the resultant acceleration of point B. $AB = 4$ in.

Ans. 22.6 in. per min.²



Prob. 378

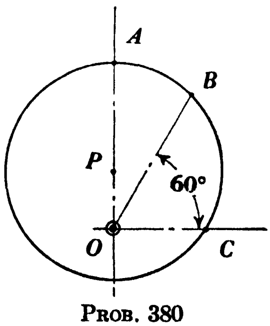


Prob. 379

379. Determine the resultant acceleration of points C and E if the rocker arm CAE rotates about fixed axis A, and has an angular velocity of 6 radians per min. clockwise, and an angular acceleration of 30 radians per min.² clockwise, in the position shown. $AC = 7.5$ in.; $AE = 4.5$ in.

380. An eccentric wheel rotates about axis O, and has an angular velocity of 4 radians per sec. counterclockwise, and an angular acceleration of 12 radians per sec.² clockwise in the position shown.

Determine the resultant accelerations of points A, B, and C. Diameter of wheel is 8 in., with center at P. $OP = 2.3$ in.



Prob. 380

90. Absolute and Relative Accelerations. The basic statement for all concepts of absolute and relative motion, as outlined in Theorem III, applies to accelerations. The *absolute* acceleration of a particle, then, is the sum of the acceleration of that particle relative to any other particle and the absolute acceleration of the other particle. If, as in Fig. 236, the acceleration of point B relative to point A is known as $a_{B/A}$ and the absolute acceleration of point A (that is, its acceleration relative to a point C on the earth's surface) is known as a_A, the absolute acceleration of B is a_B, the vector sum of a_A and $a_{B/A}$, or $a_B = a_A \mathrel{+\!\!\!\!\rightarrow} a_{B/A}$ (Fig. 237).

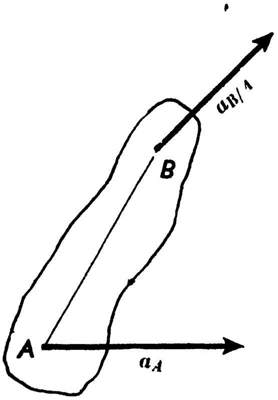


FIG. 236.

This method of finding B's absolute acceleration may seem, at first, to be indirect; in reality, it forms the most effective means of exploring for acceleration relationships.

The principle is equally valid when applied to components of acceleration. If we obtain the orthogonal component of A's absolute acceleration in the direction of AB, we have, in either Fig. 238 or 239, Aa_3. If we obtain the orthogonal component in this same direction of B's acceleration relative to A, we have Bb_3.

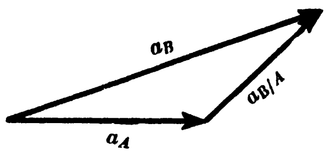


FIG. 237.

It will be seen in Fig. 239 that Bb_4 which is the orthogonal component of B's absolute acceleration in direction AB, is the vector sum of these orthogonal components Aa_3 and Bb_3. This conclusion may be stated as follows:

The orthogonal component of the absolute acceleration of a particle in a given direction is equal to the sum of the orthogonal component in that

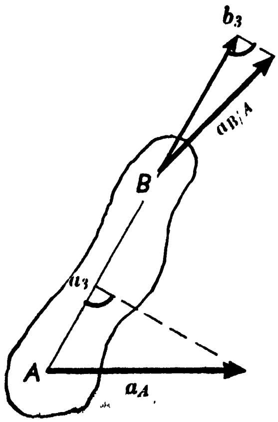


FIG. 238.

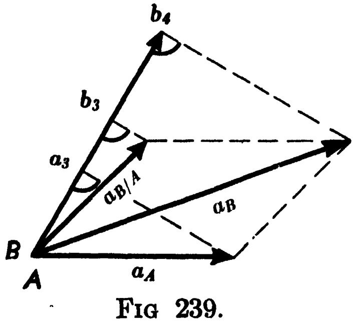


FIG 239.

direction of the acceleration of that particle relative to another particle, plus the orthogonal component in the same direction of the absolute acceleration of the second particle.

91. Acceleration in Pin-Connected Bodies. A typical analysis of acceleration relationships is involved in the case of the four-bar linkage shown in Figs. 240 and 242. It will be assumed that the dimensions of all members are known and that the kinematic properties—velocity and acceleration—of AB are also known.

An orderly and effective solution of the problem will require a complete knowledge of the angular and linear velocities of the several bodies and their particles.

Fig. 240 is the velocity analysis. This repeats the application of principles with which we are now familiar from the earlier work in

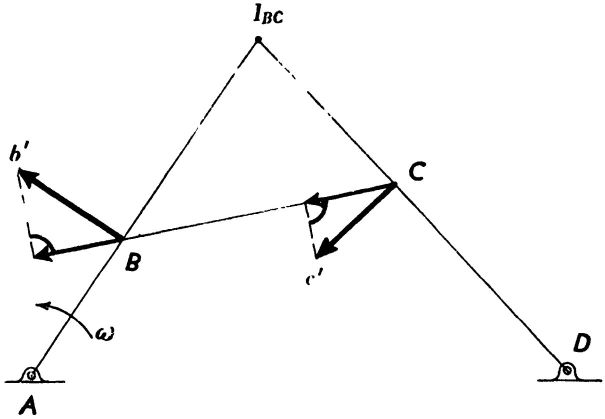


FIG. 240.

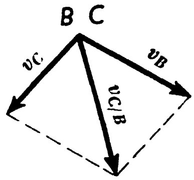


FIG. 241.

velocity. Bb' is obtained as the resultant (and absolute) velocity of point B, and Cc' is the resultant absolute velocity of point C.

The angular velocity of body AB, ω_{AB}, is given in the original data, and $Bb' = \omega_{AB}AB$. The angular velocity of body BC may be obtained by dividing linear velocity Bb' by radius $I_{BC}B$, the distance to the instantaneous axis of velocities, or Cc' may be divided by $I_{BC}C$ to give ω_{BC}.

The linear velocity of C, Cc', may be divided by radius CD, to obtain the angular velocity of CD, ω_{CD}.

The velocity of C relative to B, $v_{C/B}$, is obtained as a vector difference in Fig. 241 and must be perpendicular to BC; $v_{C/B} = v_C \rightarrow v_B$. These values are all instantaneous; they will change continuously as the crank AB is rotated to new positions, and in each position of the driving crank, a solution of velocities, valid for that instant only, may be obtained by the methods outlined.

A complete description of linear and angular velocity relationships in the bodies comprising the mechanism is now available to initiate the attack upon accelerations.

The linear acceleration of point B is a_B (Fig. 242). The components of this acceleration are

$$a_T = \alpha_{AB} AB$$
$$a_N = (\omega_{AB})^2 AB$$

and
$$a_B = a_T \nrightarrow a_N$$

The acceleration of point C may now be found by using the principle of absolute and relative accelerations.

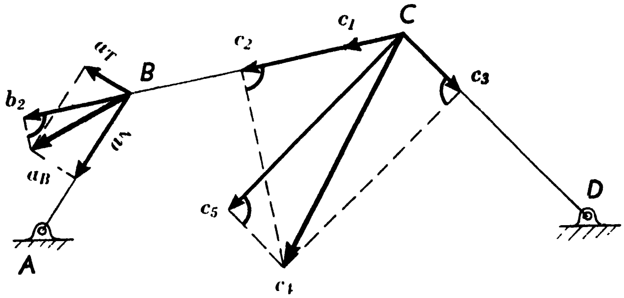


FIG. 242.

The component acceleration of point C relative to point B in the direction BC is a normal component of acceleration. This may be appreciated fully by noting that the only motion which C may have, *relative to B*, is pure rotation about B. Two points on the same rigid body may have no relative velocity in the direction connecting them. Therefore, any velocity which C may have relative to B must be in a direction normal to BC. Then the relative velocity relationship between C and B is the same in nature as would be obtained if B were considered, for the instant only, an axis of rotation, and C were moving in a circular path about B.

$$v_{C/B} = v_C \rightarrow v_B.$$

This appraisal of the nature of C's motion has been made before. (See Article 80.) When a rigid body, such as BC, has plane motion, the absolute motion of any point, such as C, may be broken up into two contributing elements—the motion of C relative to any other point of the body, such as B, and the absolute motion of the second point.

The normal component of C's acceleration relative to B is then $a_n = \frac{v^2}{r}$, where the velocity v is the relative velocity of C with respect to B, which was obtained in the analysis of velocities as $v_{C/B}$; and r is the distance BC. C must also have a tangential component of relative acceleration with respect to B. This component is not needed in the present analysis, but attention is called to it so that the normal component of C's acceleration relative to B will not be interpreted as a resultant acceleration. The normal component of acceleration of C relative to B just obtained is called Cc_1 in Fig. 242.

$c_1c_2 = Bb_2$, which is the orthogonal component of B's absolute acceleration in direction BC, is now added to Cc_1. The vector sum of Cc_1 and $c_1c_2 = Cc_2$ is the orthogonal component of C's absolute acceleration in direction BC.

The principles of Theorems I and II of orthogonal components demand that in finding a resultant vector we must have either (1) an orthogonal component and a direction of resultant or (2) two orthogonal components.

One orthogonal component (Cc_2) of C's absolute acceleration is now determined, but no basis of prediction of its inclination is readily available, and another orthogonal component must, therefore, be found.

The absolute velocity of C is already known as Cc' (of Fig. 240) from the previous velocity analysis. Then C has a normal component of acceleration relative to D, $a_n = \frac{v^2}{r}$, where v is Cc' and r is CD. This is also an orthogonal component of C's absolute acceleration in direction CD, because D, which is fixed, has zero orthogonal component of absolute acceleration in this or any other direction.

This orthogonal component of acceleration is recorded as Cc_3 in Fig. 242 and is combined with orthogonal component Cc_2 to find the resultant Cc_4, which is the absolute linear acceleration of point C.

The angular acceleration of crank CD may now be found. Since Cc_4 is an absolute acceleration, its orthogonal component (Fig. 242) perpendicular to CD, Cc_5, must be the tangential component of C's acceleration with respect to fixed point D. Then, dividing Cc_5 by CD, the angular acceleration of crank CD is obtained.

$$\alpha_{CD} = \frac{a_t}{r} = \frac{Cc_5}{CD}$$

The connecting rod BC has angular acceleration α_{BC} which may be found by considering again the fact that, relative to B, radius BC has a motion of pure rotation at the instant with B serving as a center of rotation. Then, points along BC have tangential accelerations relative to B which are perpendicular to BC and equal in magnitude to the angular acceleration of BC, α_{BC}, times the radius from B to the point.

This relative motion may be utilized in finding α_{BC}. In Fig. 243, point B has absolute acceleration Bb' and an absolute orthogonal component of acceleration Bb_2, perpendicular to BC. Point C, whose absolute acceleration is Cc', has an absolute orthogonal component Cc_2 perpendicular to BC.

The vector difference between the orthogonal components Cc_2 and

$Bb_2 = Cb_2$ of Fig. 244 is the relative acceleration of C with respect to B in a direction perpendicular to BC, or the tangential acceleration of C relative to B. If this tangential acceleration is divided by BC, the quotient will be the magnitude of the angular acceleration of connecting rod BC.

This analysis of acceleration in the four-bar linkage has rested upon bases which may be stoutly defended. We have used the definitions of

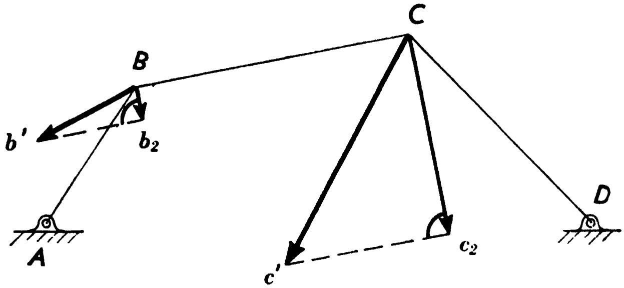


FIG. 243.

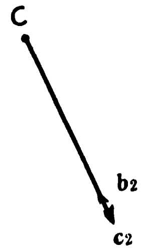


FIG. 244.

acceleration and of component accelerations, the theorems of absolute and relative motions, and the theorems of orthogonal components.

The analysis may be conveniently summarized in equation form.

$$a_C = a_{C/B} \nrightarrow a_B$$

or,

$$(a_C)_t \nrightarrow (a_C)_n = (a_{C/B})_t \nrightarrow (a_{C/B})_n \nrightarrow (a_B)_t \nrightarrow (a_B)_n$$

There are available in treatises on kinematics specialized *constructions* which will also solve the problem of obtaining accelerations. Construction methods are objectionable in that they may be used, like other formulas, without basic understanding; and they may suffer, as do all formulas, from abuse. The one advantage of construction methods rests in the reduction of the number of lines which must be drawn to accomplish a solution. Such geometrical devices, in short-cutting the stages of solution, defeat the purpose of the student of acceleration who can only feel secure in an analytical exploration towards an objective when he has truly mastered each stage.

The mastery of the acceleration principles involved in the analysis of point C of the connecting rod BC of Fig. 242 enables one to attack confidently the problem of finding the acceleration of any point on that rigid body. For example, the acceleration of any point on rigid body BC may now be determined. It will be observed, as the analysis is developed, that no new weapons of attack must be supplied—this basic method is thoroughly penetrating and will be adequate for a complete solution. In other examples which follow, an equipment of definitions and basic theorems will again be found to be powerful enough to accomplish the solution; acceleration problems, when faced with such equipment, yield readily and satisfactorily, and mastery of fundamental principle will avoid the pitfalls of formula substitution.

It is assumed in the problem shown in Fig. 245 that the investigation has been carried through the stage shown in Fig. 243 and that the kinematic properties ω_{BC} and α_{BC} of body BC are known. The absolute acceleration of point S, any point on the line BC, is to be determined.

Every point on line BC has a tangential component which is perpendicular to BC, and of magnitude equal to αr, where α is α_{BC} and r is the radius from a center, in this case point R, of the tangential components. Then a line joining b_2 and c_2 establishes s_2, and the tangential component Ss_2 of the absolute acceleration of S has been found.

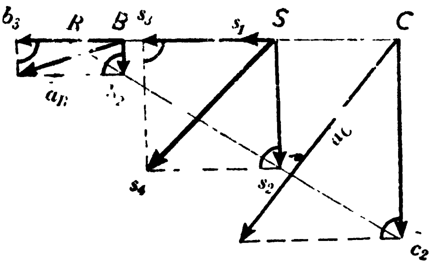


Fig. 245.

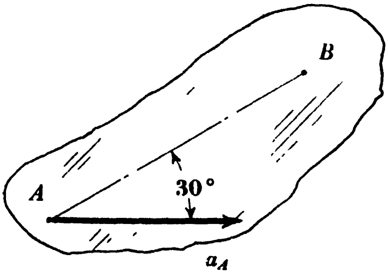
Ss_1, the acceleration of S relative to B in the direction BS, is next determined as $\omega^2 r$, where ω is ω_{BC} and r is the length of BS. If $s_1s_3 = Bb_3$, the orthogonal component of B's absolute acceleration is added to Ss_1, the sum Ss_3 is the orthogonal component of the absolute acceleration of S in the direction BS.

Now two orthogonal components of S's absolute acceleration are known, and their resultant is Ss_4, the absolute acceleration of point S.

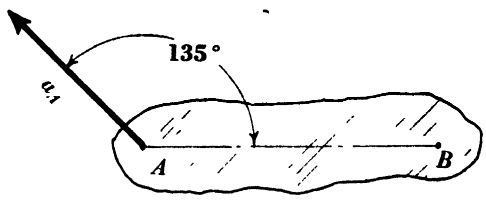
$$a_S = Ss_4 = Ss_2 \nrightarrow Ss_3.$$

Problems

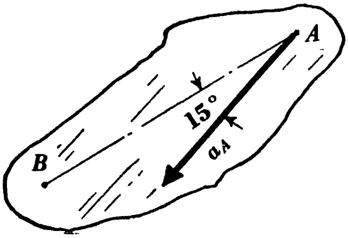
381–384. The body shown has constant absolute angular velocity $\omega = 1$ radian per sec. The absolute acceleration of point A is a_A, which is shown in direction and has magnitude of 4 in. per sec.^{2} Determine the resultant acceleration of B relative to A, and the absolute acceleration of B. $AB = 6$ in.



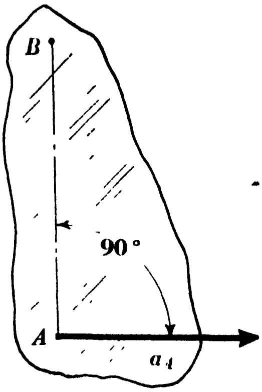
Prob. 381



Prob. 382

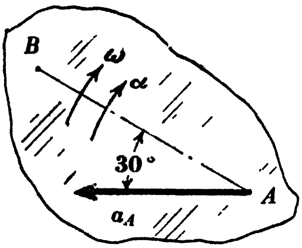


Prob. 383

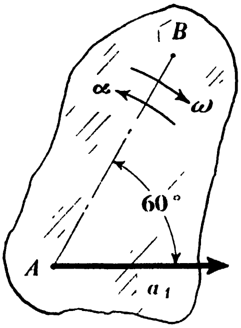


Prob. 384

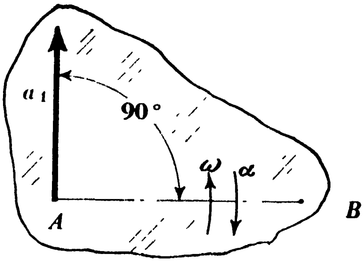
385–387. The body shown has angular velocity $\omega = 2$ radians per sec. and angular acceleration $\alpha = 5$ radians per sec.[2] (senses shown). The absolute acceleration of A is $a_A = 10$ in. per sec.[2] $AB = 5$ in. Determine the resultant acceleration of B relative to A, and the absolute acceleration of B.



Prob. 385

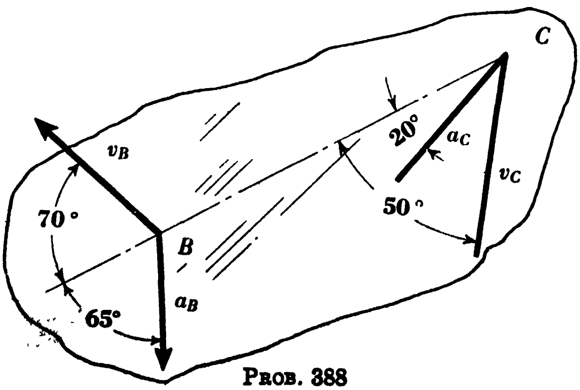


Prob. 386

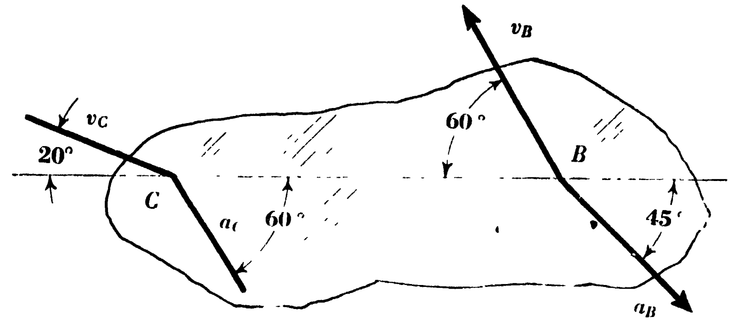


Prob. 387

388–389. The inclinations of the absolute velocities and accelerations of two points, B and C, are shown. The absolute speed of B is 2 in. per sec., and the absolute acceleration of B is 2 in. per sec.[2] in the senses shown. Determine the absolute acceleration of point C. $BC = 3$ in.



Prob. 388



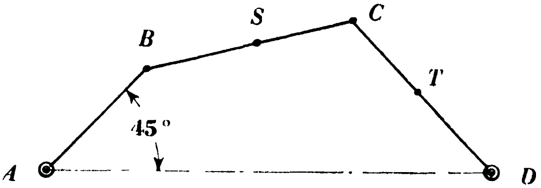
Prob. 389

390. Determine the angular velocity and the angular acceleration of the body described in Problem 388.

391. Determine the angular velocity and the angular acceleration of the body described in Problem 389.

392. The driving crank AB has a constant angular velocity of 1 radian per sec. clockwise. Determine the absolute acceleration of point C for the position shown. $AB = 4.5$ in.; $BC = 6.5$ in.; $CD = 6.3$ in.; $AD = 13.8$ in.

Ans. A^C (magnitude) = 10 in. per sec.²

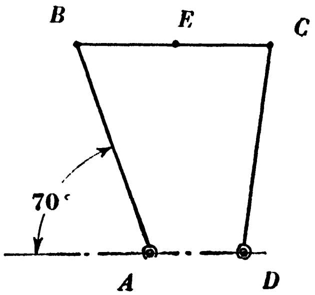


Prob. 392

393. Determine the absolute accelerations of points S and T of Problem 392. $CS = 3$ in.; $CT = 3$ in.

394. Determine the angular acceleration of bodies BC and CD of Problem 392.

395. Crank AB has angular velocity of 4 radians per sec. clockwise, and angular acceleration of 10 radians per sec.² counterclockwise. Determine the absolute accelerations of points C and E for the position shown. $AB = 6.9$ in.; $BC = 6.0$ in.; $CD = 6.5$ in.; $AD = 2.9$ in.; $CE = 3$ in.



Prob. 395

396. Determine the angular acceleration of crank CD of Problem 395.

92. Acceleration of the Instantaneous Axis of Velocities. The instantaneous axis of velocities is a point of zero velocity. We have already noted in Article 79 that the particle of a rigid body which serves at a given instant as its instantaneous axis will at any later instant lie in such a position on the body that it will now have velocity, and a different particle of the rigid body will have become the point of zero velocity. Then there has been change in the velocity of the particle which originally served as instantaneous axis. A change in velocity involves acceleration. A particle serving as an instantaneous axis of velocities has, therefore, an acceleration.

The absolute accelerations of B and C (Fig. 246) are known, and are Bb' and Cc', respectively.

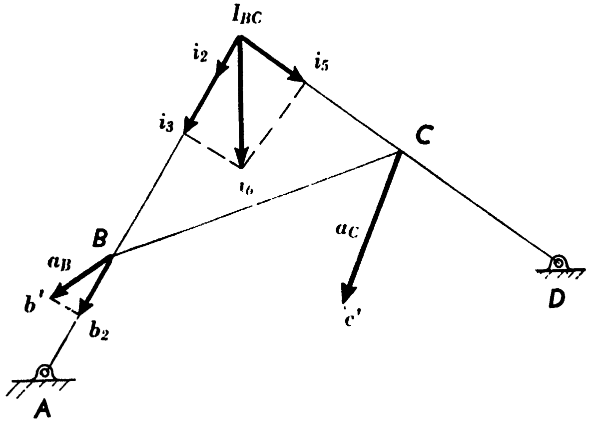


FIG. 246.

As before, ground may be broken by a velocity study in which we should determine the velocity of I_{BC} relative to B and its velocity relative to C.

The normal component of acceleration of I_{BC} relative to B will again be $a_n = \frac{v^2}{r}$, where v is the velocity of I_{BC} relative to B, and r is the distance $I_{BC}B$. This acceleration component is $I_{BC}i_2$ in the figure. To it must be added $i_2i_3 = Bb_2$, the orthogonal component in direction $I_{BC}B$ of B's absolute acceleration.

Then $I_{BC}i_3$, the vector sum of these components, is one orthogonal component of I_{BC}'s absolute acceleration, in the direction $I_{BC}B$.

$I_{BC}i_5$ is obtained similarly by relating I_{BC} to point C, and two orthogonal components are now available which may be combined by application of Theorem II to yield a resultant $I_{BC}i_6$, which is the absolute acceleration of the instantaneous axis of velocities I_{BC}.

93. The Instantaneous Center of Accelerations. The instantaneous center of accelerations is defined as the point on an accelerating body which has, at a given instant, zero absolute acceleration. If a body such as is shown in Fig. 247 has at a given instant angular velocity ω and angular acceleration α, which are both known, a basis of method is available which may be used in a search for the instantaneous center of accelerations.

A and *B* are two particles, and the distance between them is *r*. Then the acceleration of *A* relative to *B* has a tangential component

$$a_t = \alpha r$$

and a normal component

$$a_n = \omega^2 r$$

The angle θ is $\tan^{-1} \dfrac{\alpha r}{\omega^2 r} = \dfrac{\alpha}{\omega^2}$ and forms the angle between radius *AB* and the resultant acceleration of *A* relative to *B*.

If *B* were a point of zero absolute acceleration, then θ would be the angle between radius *AB* and the *resultant absolute* acceleration of point *A*. The same angle θ would then be formed by the radius *BC* and the *resultant absolute* acceleration of *C*, where *C* is any point on the rigid body. Then if we know the absolute acceleration of two points on a rigid body, we may determine the instantaneous center of accelerations.

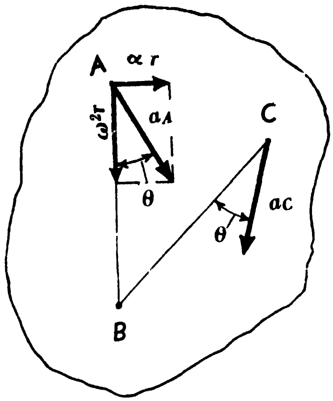


FIG. 247.

A specific application will serve to clarify the procedure. On the body of Fig. 248, particle *A* has, at a given instant, absolute acceleration Aa'. At the same instant, the absolute acceleration of particle *C* is Cc'. Then the acceleration of *A* relative to *C* is Aa_2, obtained as the vector difference between Aa' and Cc' ($a_{A/C} = a_A \rightarrow a_C$). The angle between Aa_2 and line *AC* will be θ. The component of Aa_2 in the direction of *AC* is Aa_3, which is a normal component of the acceleration of *A* relative to *C*.

Then,
$$Aa_3 = (\omega_{AC})^2 \times AC$$

Aa_4 is the component of Aa_2 perpendicular to *AC* and is, therefore, the tangential component of the acceleration of *A* relative to *C*.

Then,
$$Aa_4 = \alpha_{AC} \times AC$$

$$\theta = \tan^{-1}\left[\frac{\alpha_{AC} \times AC}{\omega_{AC}^2 \times AC} = \frac{\alpha_{AC}}{\omega_{AC}^2}\right]$$

Since α and ω are, at any instant, the same for all lines of the body, this angle θ will lie between the absolute acceleration of any point and the line joining that point and the instantaneous center of accelerations of the body.

If, now, as in Fig. 248, we draw a line AZ making an angle θ with Aa' (the absolute acceleration of A) and another line CZ (making an angle θ with Cc'), the absolute acceleration of C, AZ and CZ will intersect at point Z, the *instantaneous center of accelerations* for the body containing A and C.

Two lines exist which, like AZ, will make an angle θ with absolute acceleration Aa'; and there are two which, like CZ, will make an angle θ with Cc'. The correct pair may be isolated by trial of the four possibilities, noting that Z must be so located that the senses of the normal and

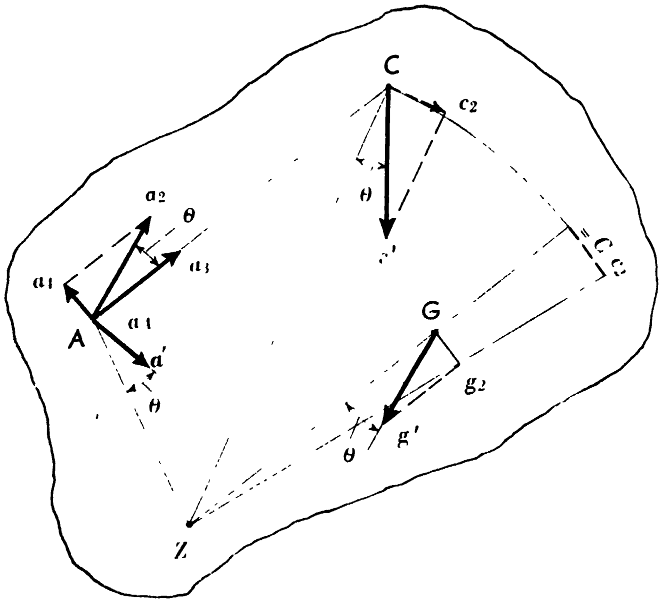


FIG. 248.

tangential components of the absolute accelerations of A and C are consistent.

The acceleration of point G, another point on the body, may now be found by noting that it has an inclination of θ with line ZG. To establish its magnitude, one orthogonal component is necessary—for example,

$$Gg_2 = a_t = \alpha_{AC} \times ZG$$

The value of α_{AC} became available when Aa_2, the acceleration of A relative to C, was established, for the orthogonal component of Aa_2 at right angles to $AC(Aa_4)$ is the tangential component of A's acceleration relative to C and is, therefore,

$$Aa_4 = a_t = \alpha_{AC}AC$$

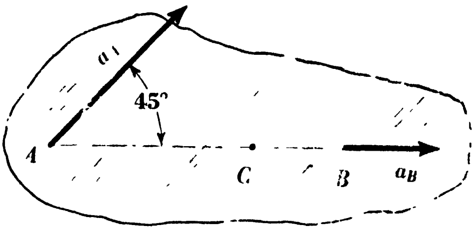
Then,

$$\frac{Aa_4}{AC} = \alpha_{AC}$$

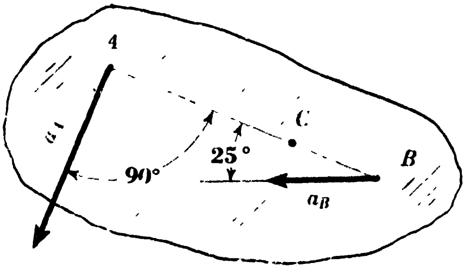
Gg_2 may also be established, graphically, by proportion from Cc_2, as in the figure.

Problems

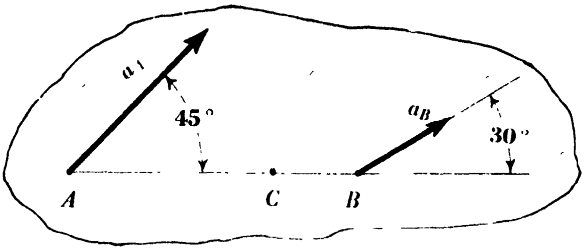
397–399. The magnitude of the absolute acceleration of point A is 4 in. per sec.^{2}, and that of B is 2 in. per sec.^{2} The inclinations and sense are shown. Locate the instantaneous axis of accelerations of the body, and determine the absolute acceleration of point C. $AB = 6$ in.; $AC = 4.2$ in.



Prob. 397



Prob. 398



Prob. 399

400. Locate the instantaneous axis of accelerations of body BC of Problem 395, and determine the absolute acceleration of point E.

94. Acceleration in Rolling Contact. The cylinder shown in Fig. 249 is in pure rolling contact with a fixed track, and has angular velocity ω,

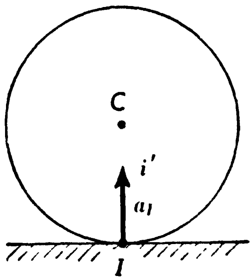


Fig. 249.

which is constant. The acceleration of any point on the cylinder may be found by applying, as in previous cases, the basic theorems.

For example, I, the point of contact with the track, has zero velocity, ∴ I is the instantaneous axis of velocities of the cylinder. The velocity of I relative to C, the center of the cylinder, is $v_1 = \omega \times IC$, and the acceleration of I relative to C in the direction connecting I and C is a normal component of relative acceleration. This is $Ii' = \frac{v_1^2}{IC}$, or $Ii' = \omega^2 IC$. This normal component is the resultant acceleration of I relative to C, for with ω announced as constant, $\alpha = 0$.

To find the absolute acceleration of I, we must add to its acceleration relative to C, the absolute acceleration of C, which in this case is zero,

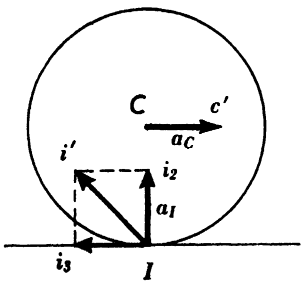


FIG. 250.

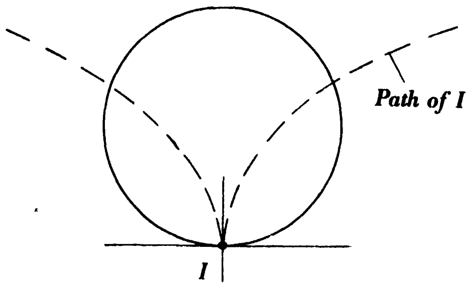


FIG. 251.

for point C has rectilinear motion with constant speed. Ii' is the absolute acceleration of I.

If the cylinder has an angular acceleration α (Fig. 250), the acceleration of I relative to C will have a tangential component $Ii_3 = a_t = \alpha IC$ and a normal component $Ii_2 = a_n = \omega^2 IC$. Ii' will be the resultant acceleration of I relative to C. C will also have an absolute acceleration Cc' parallel to the track and equal to αIC. $Cc' = Ii_3$; then the absolute acceleration of I will be Ii_2, the sum of Ii' and Cc', in the direction of radius IC, as it was when the rolling body had constant angular velocity.

$$a_I = a_{I/C} \nrightarrow a_C$$

This conclusion may be verified if we investigate the motion from another approach.

The path of the particle of the cylinder which is, at the instant shown, at I, is cycloidal. This path is illustrated in Fig. 251. The cycloidal path is normal to the track at the instant of contact, and I must, therefore, have a resultant absolute acceleration normal to the track.

Another example of an acceleration study in pure rolling contact is illustrated in Fig. 252. In this epicyclic wheel train, the arm A has absolute angular velocity ω_A and absolute angular acceleration α_A. The driving wheel has constant angular velocity ω_D. The resultant absolute acceleration of point f on the follower is ff'. This is determined from the given data concerning the arm, because point f is on the arm as well as on the follower.

$$ff_2 = \omega_A{}^2 \times df$$
$$ff_3 = \alpha_A \times df$$
$$ff_1 = ff_2{}^2 \mathrel{+\!\!\!\!\rightarrow} ff_3{}^2$$

The point of contact p (on the follower) has a normal component of acceleration pp_2 relative to f which may be determined as follows:

The absolute angular velocity of the follower ω_F is determined by dividing the velocity of f relative to p by the radius fp. Then, pp_2

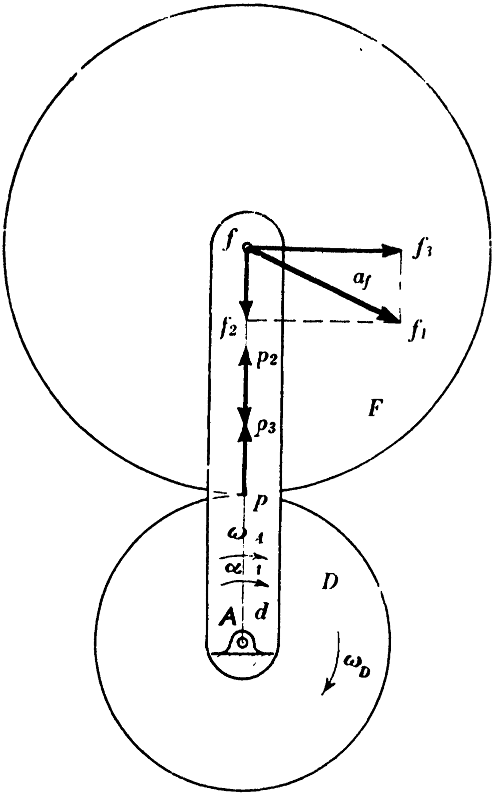


FIG. 252.

$= (\omega_F)^2 fp$. To pp_2 we next add p_2p_3 equal to the component of f's absolute acceleration in direction fp (ff_2). The sum of pp_2 and p_2p_3 is pp_3, which is the orthogonal component in direction fp of p's absolute acceleration. pp_3 is also the resultant absolute acceleration of point p, because point p has no horizontal component of acceleration. This statement may be confirmed by noting that point p on the driver has no horizontal component of acceleration (ω_D constant) and point p on the follower is in pure rolling contact with it.

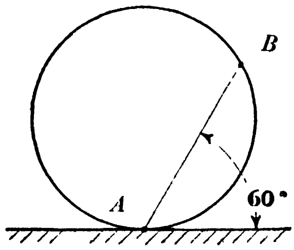
The acceleration of any other point on the follower may be obtained now that the absolute accelerations of two points on that body have been determined.

Problems

401. The 6-in. diameter cylinder is in pure rolling contact with the fixed track at A, and has a constant angular velocity of 1 radian per sec. counterclockwise. Determine

(a) The acceleration of point B, relative to point A.
(b) The absolute acceleration of point A.
(c) The location of the instantaneous axis of accelerations.

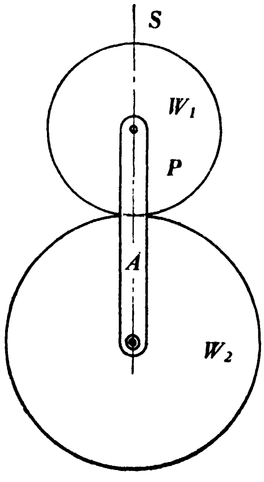
Ans. (a) 5.20 in. per sec.^{2}; (b) 3 in. per sec.^{2}



Prob. 401

402. Solve Problem 401 if the cylinder has an absolute angular velocity of 1 radian per sec. clockwise, and an absolute angular acceleration of 1 radian per sec.^{2} clockwise.

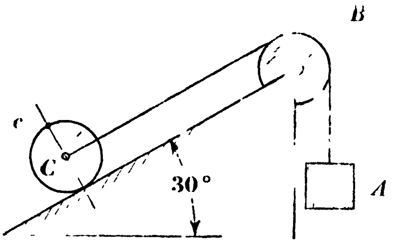
403. In the epicyclic wheel train, W_1 has a 7.5-in. diameter, and W_2 has a 5.2-in. diameter. The wheels are in pure rolling contact. W_1 has constant angular velocity of 1 radian per sec. clockwise. The arm A has angular velocity of 2 radians per sec. clockwise, and angular acceleration of 3 radians per sec.^{2} clockwise for the position shown. Determine the acceleration of point P, the point of contact on W_2.



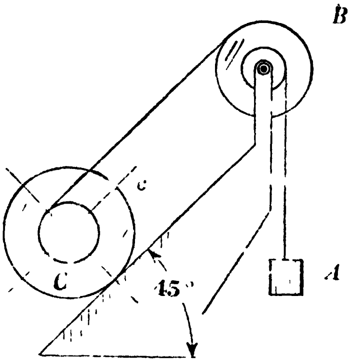
Prob. 403

404. Locate the instantaneous axis of accelerations of wheel W_2 in Problem 403, and determine the acceleration of point S.

405. The weight A has an absolute velocity of 5 ft. per sec. and an absolute acceleration of 10 ft. per sec.², both downward in the position shown. The cord joining A with the center of C passes over pulley B, which is mounted on a fixed axis. C is in pure rolling contact with the inclined plane. The position of cord from B to C is parallel to the plane. Determine the absolute acceleration of point c on the rolling cylinder. Diameter of C = 4 ft. *Ans.* 23.6 ft. per sec.²



PROB. 405



PROB. 406

406. Weight A has an absolute velocity of 6 ft. per sec. and an absolute acceleration of 14 ft. per sec.², both downward. The two-step pulley B is mounted on a fixed axis. Inner diameter = 1.5 ft Outer diameter = 3 ft.

The cylinder C is in pure rolling contact with the inclined plane. C has an inner diameter of 1.8 ft., and outer diameter of 4 ft. The cord from B to C is parallel to the plane. Determine the absolute acceleration of point c on the rolling cylinder.

95. Acceleration in Sliding Contact. *Fixed Guides.* In Fig. 253, AB is a driving crank, and BC a connecting rod which is pinned to AB at B, and to a block at C which is constrained to move in fixed guides. The dimensions of the links and the kinematic properties of the driving crank are known.

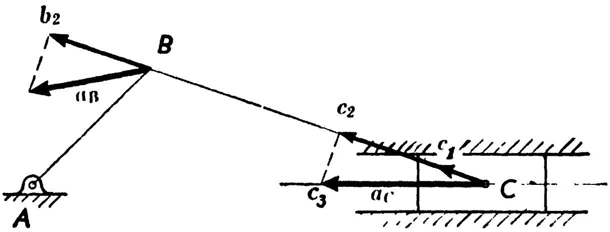


FIG. 253.

The orthogonal component of C's absolute acceleration in direction BC is Cc_2, which is the vector sum of the normal component of C's acceleration relative to B, in direction BC, Cc_1, and the orthogonal component of B's absolute acceleration in the same direction, $c_1c_2 = Bb_2$.

The inclination of C's absolute acceleration is known, since it must be parallel to the sliding surfaces. One orthogonal component and inclination of the resultant vector are then known, and the application of Theorem I yields the resultant Cc_3, which is the absolute acceleration of point C, and, sinte the block is moving with pure translation, is the absolute acceleration of any point on the block.

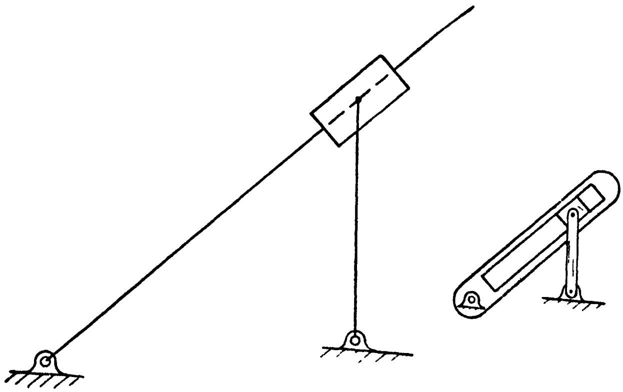


FIG. 254.

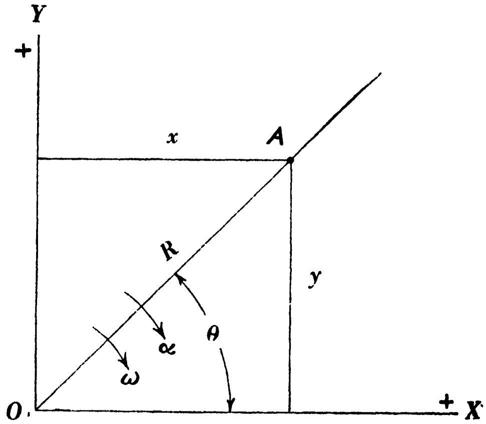


FIG. 255.

Moving Guides. When the sliding contact takes place in guides carried by a moving body, the acceleration analysis becomes somewhat more complex. Fig. 254 illustrates such a case.

The problem may be viewed in its simplest form by considering a particle A of Fig. 255. The motion of the particle is described as follows: Particle A is moving outward (away from O) on radius R at the same time that radius R turns about the origin O with angular velocity ω and angular acceleration α. The problem is to find the absolute acceleration of A.

In previous studies of acceleration, radii, such as R, were of fixed length. We now are faced with the problem of accelerations when the radial distance is varying. Expressions for the X and Y components of the absolute acceleration of A may be obtained by differentiation.

$$x = R\cos\theta \qquad \text{and} \qquad y = R\sin\theta$$

$$v_X = \frac{dx}{dt} = \frac{dR}{dt}\cos\theta - R\sin\theta\frac{d\theta}{dt}$$

$$a_X = \frac{d^2x}{dt^2} = \frac{d^2R}{dt^2}\cos\theta - \frac{dR}{dt}\sin\theta\frac{d\theta}{dt} - \frac{dR}{dt}\sin\theta\frac{d\theta}{dt} - R\cos\theta\left(\frac{d\theta}{dt}\right)^2 - R\sin\theta\frac{d^2\theta}{dt^2}$$

$$= \cos\theta\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] - 2\frac{dR}{dt}\sin\theta\frac{d\theta}{dt} - R\sin\theta\frac{d^2\theta}{dt^2}$$

$$v_Y = \frac{dy}{dt} = \frac{dR}{dt}\sin\theta + R\cos\theta\frac{d\theta}{dt}$$

$$a_Y = \frac{d^2y}{dt^2} = \frac{dR}{dt}\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{d^2R}{dt^2} + \frac{dR}{dt}\cos\theta\frac{d\theta}{dt} - R\sin\theta\left(\frac{d\theta}{dt}\right)^2 + R\cos\theta\frac{d^2\theta}{dt^2}$$

$$= \sin\theta\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] + 2\frac{dR}{dt}\cos\theta\frac{d\theta}{dt} + R\cos\theta\frac{d^2\theta}{dt^2}$$

These are expressions for the X and Y components of A's absolute acceleration. From them we may obtain expressions for the normal and

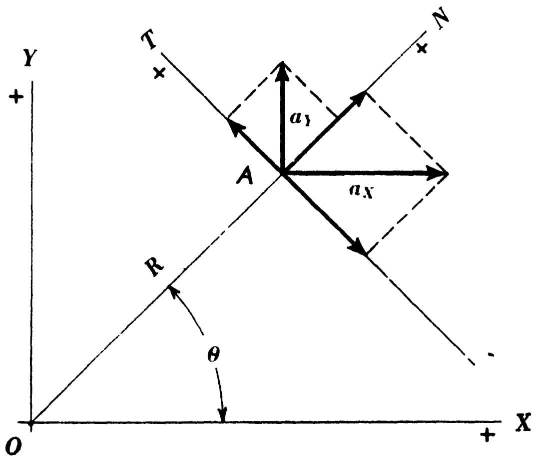


Fig. 256.

tangential components of A's acceleration, a_n and a_t, along and perpendicular to R, respectively (Fig. 256).

$$a_n = a_X \cos\theta + a_Y \sin\theta$$

$$a_n = \cos^2\theta\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] - 2\frac{dR}{dt}\sin\theta\frac{d\theta}{dt}\cos\theta - R\sin\theta\cos\theta\frac{d^2\theta}{dt^2}$$
$$+ \sin^2\theta\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] + 2\frac{dR}{dt}\cos\theta\frac{d\theta}{dt}\sin\theta + R\cos\theta\sin\theta\frac{d^2\theta}{dt^2}$$
$$= (\cos^2\theta + \sin^2\theta)\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right]$$
$$= \frac{d^2R}{dt^2} - R\omega^2$$

$$a_t = a_Y \cos\theta - a_X \sin\theta$$

$$a_t = \sin\theta\cos\theta\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] + 2\frac{dR}{dt}\cos^2\theta\frac{d\theta}{dt} + R\cos^2\theta\frac{d^2\theta}{dt^2}$$
$$- \sin\theta\cos\theta\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] + 2\frac{dR}{dt}\sin^2\theta\frac{d\theta}{dt} + R\sin^2\theta\frac{d^2\theta}{dt^2}$$
$$= 2\frac{dR}{dt}\frac{d\theta}{dt}(\cos^2\theta + \sin^2\theta) + R\frac{d^2\theta}{dt^2}(\cos^2\theta + \sin^2\theta)$$
$$= 2\frac{dR}{dt}\frac{d\theta}{dt} + R\frac{d^2\theta}{dt^2}$$

$\frac{dR}{dt} = v_R$, where v_R is the component of the velocity of point A relative to the coinciding point on the moving radius, in the direction of R.

Then, $$a_t = 2v_R\omega + R\alpha$$

If the radius were constant, the expression for normal acceleration would reduce to $a_n = \omega^2 R$, as in the past, and a_t would equal αR.

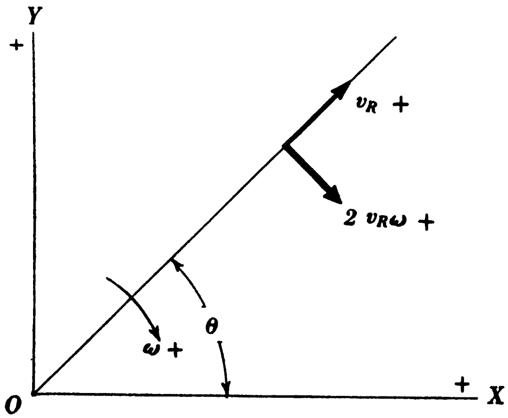


Fig. 257.

The term $2v_R\omega$ which has now appeared is known as the *Coriolis'* acceleration. This term has resulted from the sliding contact in moving guides, with changing radial distance. Such an acceleration always appears when a particle slides relative to a rotating path.

The sense of the vector $2v_R\omega$ may be determined by comparison with the assumed senses of the basic derivation.

In Fig. 257, the senses assumed in the basic derivation and the resulting sense of the $2v_R\omega$ term are shown. If either v_R or ω is opposed to the sense shown, the $2v_R\omega$ term will be of opposite sense. If both are reversed, the $2v_R\omega$ term will again be of the sense shown in Fig. 257.

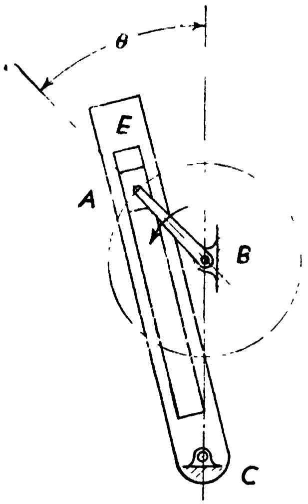


FIG. 258.

ILLUSTRATIVE PROBLEM

The swinging-block quick-return mechanism of Fig. 258 has a driving crank BA which is one inch long and has constant angular velocity $\omega_{BA} = 2$ radians per sec. counterclockwise. $BC = 2$ in.

In the position shown, with $\theta = 45°$, the angular acceleration α of the beam CE is to be found. The velocity of point A (Fig. 259) on crank BA has magnitude $v_1 = \omega_{BA} \times BA = 2 \times 1 = 2$ in. per sec. and the direction shown. The velocity of the coinciding point A on CE has magnitude $v_2 = 1.73$ in. per sec. and the direction shown. The velocity of A on BA relative to A on CE has magnitude $v_R = 1.01$ in. per sec and the direction shown.

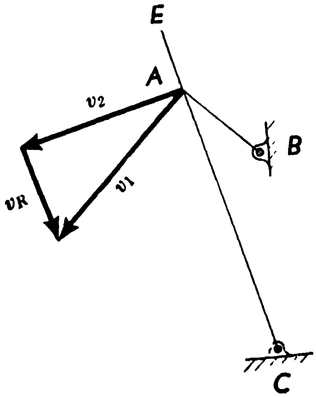


FIG. 259.

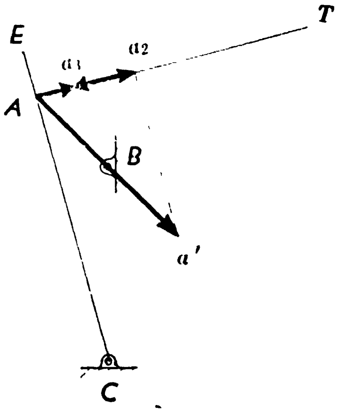


FIG. 260.

The angular velocity of CE, $\omega_{CE} = \frac{1.73}{2.8} = 0.62$ radians per sec. counterclockwise.

The analysis of accelerations is shown in Fig. 260.

The T axis is perpendicular to CE at A.

In this component direction, the basic theorem of absolute and relative motion may be employed: the orthogonal component of the absolute acceleration of point A on BA is equal to the sum of the orthogonal component of the absolute acceleration of point A on CE and the orthogonal component of the acceleration of A on BA relative to A on CE.

A on BA has a resultant absolute acceleration $Aa^1 = (\omega_{BA})^2 \times BA = 2^2 \times 1 = 4$ in. per sec.^{2} in the direction shown. The orthogonal component on the T axis of this acceleration is $Aa_2 = 2.02$ in. per sec.^{2} in the direction shown.

Aa_2 equals the vector sum

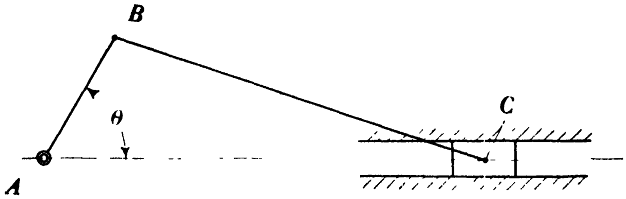
$$2v_R\omega \nrightarrow R\alpha = 2v_R\omega_{CE} \nrightarrow (CA \times \alpha_{CE})$$
$$CA \times \alpha_{CE} = Aa_2 \rightarrow 2v_R\omega_{CE}$$

or, $$2v_R\omega_{CE} = 2 \times 1.01 \times 0.62 = 1.25 \text{ in. per sec. per sec.}$$

The sense of this vector is from A toward T, and its magnitude is the distance a_3a_2.

The vector difference is $Aa_3 = R\alpha = CA \times \alpha_{CE} = 0.77$ in. per sec.^{2} Aa_3 is the tangential component of the acceleration of A on CE, and $\alpha_{CE} = \frac{a_t}{R} = \frac{0.77}{2.8} = 0.275$ radians per sec.^{2} clockwise.

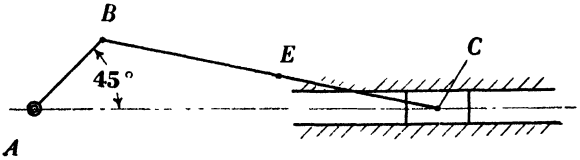
407. AB has constant angular velocity of 100 r.p.m. clockwise. Determine the absolute acceleration of point C when $\theta = 60°$. $AB = 3$ in.; $BC = 8$ in.



PROB. 407

408. Solve Problem 407 with $\theta = 45°$.

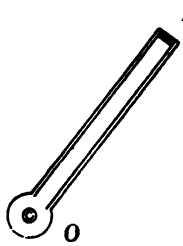
409. Point C has an absolute velocity of 2 in. per sec., and an absolute acceleration of 2 in. per sec.^{2}, both to the right. Determine the angular velocity and angular acceleration of AB. $AB = 2$ in.; $BC = 7$ in.



PROB. 409

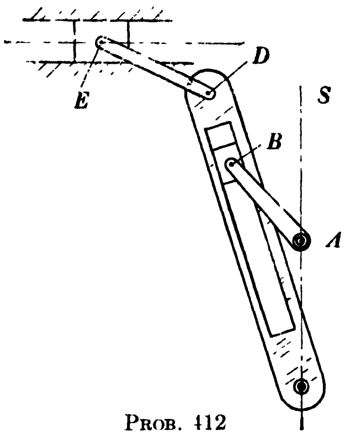
410. Locate the instantaneous axis of accelerations of body BC of Problem 409, and determine the acceleration of point E, the mid-point on BC.

411. A particle P leaves the tube with a relative velocity (between tube and particle) of 2 f.p.s. The tube rotates about fixed axis O with constant angular velocity of 60 r.p.m. clockwise. Determine the absolute acceleration of the particle at the instant it leaves the tube. $OP = 6$ in.



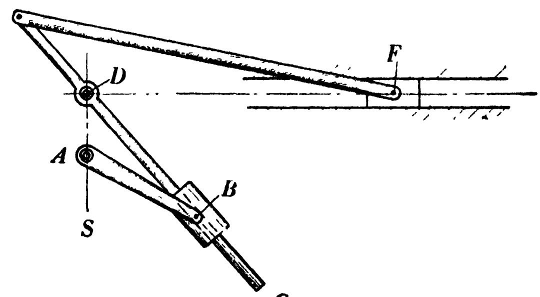
PROB. 411

412. Crank AB has constant angular velocity of 60 r.p.m. clockwise. Determine the absolute accelerations of points D and E when AB makes an angle of 30° with the line marked AS. (The angle is to be measured counterclockwise from AS about A.) $AB = 5.2$ in.; $CD = 16$ in.; $DE = 6$ in.; $AC = 7.8$ in.; Center line of E is 6.5 in. above A.



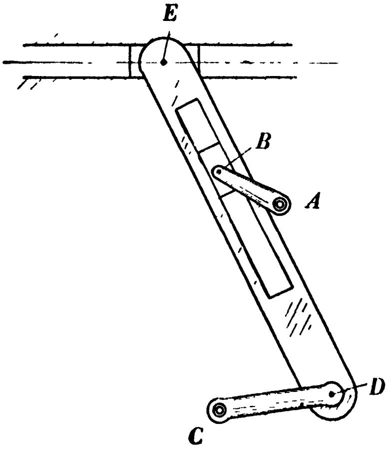
PROB. 412

413. If crank AB has constant angular velocity of 150 r.p.m. clockwise, determine the absolute accelerations of points E and F, when AB makes an angle of 60° with AS. (The angle is to be measured counterclockwise from AS about A.) A and D are fixed axes. $AB = 5.2$ in.; $DE = 8$ in.; $EF = 20$ in.; $AD = 2.6$ in.



PROB. 413

414. If crank AB has constant angular velocity of 1 radian per min. clockwise, determine the absolute acceleration of point E when AB is in its left horizontal position. A and C are fixed axes. The center line of the sliding block at E is 8.5 in. above A. C is 12.5 in. below, and 2.4 in. to the left of A. $AB = 4.2$ in.; $CD = 7.0$ in.; $DE = 23$ in.



PROB. 414

CHAPTER X

Translation

96. Basic Concepts and Laws. In our explorations of the field of statics, we were concerned with free bodies in equilibrium. As we developed methods of attack there, the combined effect of all of the external forces which might act upon the free body crystallized in the concept of the resultant—a simpler, but equivalent system. The use of such a technique made it possible for us to more readily realize, as simplification always does, the combined effect of complex systems. At that time we were primarily interested in systems of external forces which were in balance. Their balanced nature was revealed when, upon seeking the resultant, we found that the original system reduced itself to zero—the condition in which no force or couple remained. This conclusion led, in turn, to the formulation of the equations of equilibrium—all of these postulates arose as original systems of force, when summarized, yielded zero resultant.

The field of dynamics is similarly concerned with the effect of systems of external forces applied to free bodies. In this division of mechanics, too, we make use of the technique of reducing original systems of forces to their resultants. Here, we find that the resultant is not zero, but has finite magnitude either as a force or couple. Then, dynamics is the study of the behavior of free bodies when the system of external forces is not a balanced system.

The formulation of law in this field is primarily due to Sir Isaac Newton, who codified both his own observations and those of earlier philosophers. The studies of Newton were based upon the motion of planets. The range of motion of a planet is so great in comparison with the size of the planet that his conclusions are most properly applied to the motion of particles.

The free bodies which we are to consider, however, are composed of particles, and the extension of Newtonian principle to include bodies is a natural and valid extension.

We may summarize the results of such observations as follows.

If an unbalanced force F_1 is applied to a free body, there will be a change of motion of the body which may be observed as acceleration a_1.

If a different force F_2 is applied to the same body, there will be a different acceleration a_2.

The unbalanced forces and the accompanying accelerations may be measured, and it will be found that forces and accelerations are varying directly, or

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \frac{F_n}{a_n} = C$$

in which C is a constant.

The value of this constant quotient may be established by allowing the body to fall freely under the influence of its own weight W acting as the unbalanced force.

Now, we find that the acceleration of the body is g (gravity acceleration), and we conclude that

$$C = \frac{W}{g}$$

Then,
$$F = \frac{W}{g} a$$

The constant of proportionality $\frac{W}{g}$ is defined as the *mass* of the body and is abbreviated m. The value of g employed by engineers is 32.2 ft.

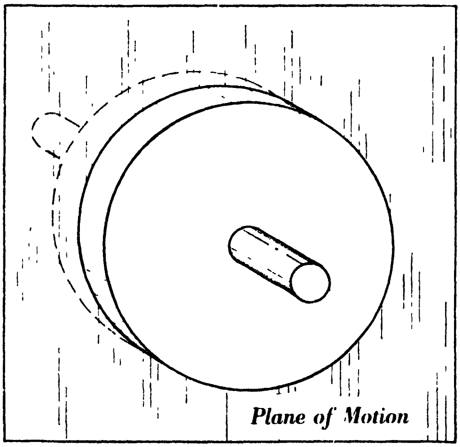


FIG. 261.

per sec². Various names are given to the units of mass—slugs, g-pounds, and the like. All these units are derived from the usual stock of the engineer of the English-speaking countries—pounds, feet, and seconds, or their multiples. The derived units must be reset in terms of pounds, feet, and seconds in engineering usage, and hence serve no useful purpose. In this text, concerning itself with engineering application of fundamental principle, we shall, therefore, use no derived units for mass.

When we considered the motion of bodies, in kinematics, we made one expansion of the size of the free body. All particles that move with a body so that they remain ever at constant distance from the other particles of the body may be considered to belong to the body in analysis of motion. Such particles need not be physically joined to the material free body—if they have the motion properties of particles of the body, they belong to that body. The instantaneous center of velocities, for example, is such a particle or point, and we found it convenient to use the device of "kinematic expansion" in velocity studies.

We shall now make one convenient contraction of the size of the free body. The greater portion of our work in the fundamentals of dynamics concerns itself with *coplanar motion*—that is, motion in which all particles of the body move in the same or in parallel planes. For example, in the wheel of Fig. 261, the particles that lie in any plane perpendicular to the axis move in that plane constantly. The description of motion which is developed by considering the particles of any one such plane is repeated exactly in all of the planes parallel to it. We may, therefore, simplify our analysis by assuming that the wheel is of one-plane thickness and investigate the story of motion in that one plane, reporting, if necessary, identical conditions in all parallel planes.

The plane to which the breadth of the free body is thus reduced will be called the *plane of motion*, and we shall select the plane containing the mass center of the body as the plane of motion.

97. External Forces and Acceleration-Forces. The form of equation to which the basic postulate of our study of dynamics has reduced is

$$R = ma$$

in which R is the resultant of the original system of external forces applied to a free body.

This term is the left-side member of an equation.

Then the right-side member must be equivalent to R in all respects. R is a force—it follows that ma must also be a force.

R has occasionally been distinguished by referring to it as a real force, whereas ma is called an imaginary force. These concepts are inadequate evasions of conditions which we must face. R is no more real than ma, if reality means physical existence; "force" is judged by its effect—we do not touch, see, or smell force.

Nor is ma imaginary, for it exists as a summarizing expression of change of motion based on two very sound concepts: mass and acceleration.

There is distinction, however, between our approaches to the two sides of the basic equation. R is determined, as in Statics, by observing the neighboring bodies and recording their action upon the free body. ma may be determined by measuring the acceleration of a body, and multiplying it by its mass.

In accepting the basic law of Newton, we have accepted the equality of both terms. In the solutions of our problems of dynamics, we shall be making the distinguishing approaches to the two sides of our equation suggested above. It will be, therefore, convenient to adopt distinguishing and characterizing names for the two forces.

R is already familiar from our study of statics as the resultant of the *external* forces (contact actions of the neighboring bodies).

ma may be called the *acceleration force*, a term which will distinguish it from external forces and associate it directly with the acceleration of the body. ma is also known as the *effective*, or *inertia*, force. These terms, however, have no connotation of the acceleration source of such force and are, therefore, susceptible to confusion. All forces are "effective"—we know of force only through its effect. The term acceleration force will, therefore, be used in this text.

The nicety of balance between external force and acceleration-force, expressed in $R = ma$, makes it possible for us to direct our attacks in the same exact and organized manner which makes the theorems of statics clear and powerful.

98. D'Alembert's Principle. We have accepted, as a basic equation, the Newtonian law

$$R = ma$$

If we transpose the terms, we have

$$R + (-ma) = 0$$

$-ma$ is the negative of the acceleration force, and may be called the *reversed acceleration force*.

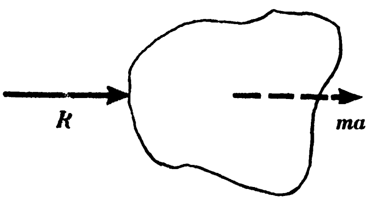


FIG. 262.

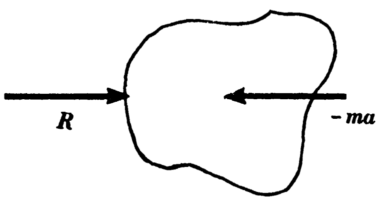


FIG. 263.

The significance of such a transposition of terms is important, for it has reduced our basic equation to the form in which all of the equations of equilibrium appeared in our discussion of statics—the sum of forces is now equal to zero. If, as illustrated in Fig. 262, an unbalanced external force R is applied to a free body, there will be acceleration of that body. The presence of acceleration is noted by adding to the drawing a vector representing ma, the acceleration force.

R and ma are equivalent.

The existence of R as an external force acting on the free body and supplied by a contacting or neighboring body has been denoted by drawing that vector with solid lines, and ma has been shown with dotted lines, to differentiate this acceleration-force from the external force.

If, as in Fig. 263, we were to add another force R_1 also exerted by a contacting body, the free body would be in equilibrium if

$$R_1 = -ma$$

Then, adding a reversed acceleration force endows the problem with the nature of a statics problem.

In this technique, we employ ma only to establish its magnitude, inclination, and sense. When we have evaluated these properties, we assume that there is added to the system of external forces an additional external force which is the negative of the acceleration force. We may now apply the statical equations of equilibrium, because any free body which did have such a system of external forces acting on it would be in equilibrium.

This analytical device is due to the philosophical speculation of an eighteenth-century engineer named D'Alembert, and the principle of changing the aspect of our attack from a dynamic to a static one is known as *D'Alembert's Principle*. It is, in reality, only a transposition of the terms of the basic Newtonian law, but possesses virtue in many of the problems of dynamics where the clarity and vigor of statical equations may enhance the solution.

99. Translation. When a body moves so that no straight line of the body changes its inclination with a fixed axis, the motion is said to be *translation*.

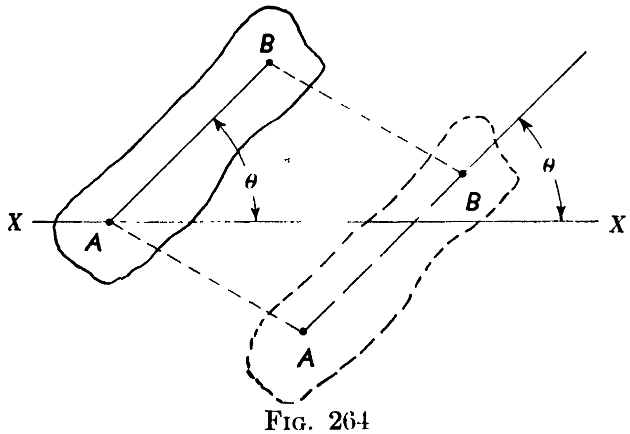


FIG. 264

If the body of Fig. 264 moves from the position shown by the solid outline to the position indicated by the dotted outline, so that the line AB (or any other line of the body) remains ever at constant inclination θ with the X axis, the body has moved in translation. Then, at any instant, all of the particles of the body must have the same velocity and, if that velocity is changing, the same acceleration.

We note that all of the points lying in line AB (or, indeed, anywhere on the body) have moved along parallel paths. When these parallel paths are straight lines, the translation is said to be *rectilinear;* when the parallel paths are curved, the translation becomes *curvilinear.*

If a body, like that of Fig. 265, is given a motion of translation, all particles will have equal accelerations. The particle dm, for example, will have acceleration a, the acceleration common to all particles. Then the acceleration force of dm is $dm\ a$. All particles of mass dm will likewise have acceleration-forces equal to $dm\ a$.

These forces form a parallel force system in a plane, and we can establish their resultant ma_R as we have done when systems of parallel forces in a plane appeared in the problems of statics

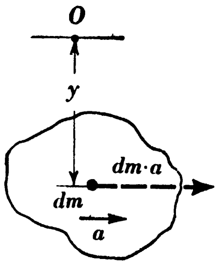


FIG. 265.

$$ma_R = \Sigma dm\ a = ma$$

in which m is the total mass of the body $\left(\frac{W}{g}\right)$ and a is the acceleration of the body.

This resultant may be located, as in all parallel force systems, by taking moments about an axis through any point, such as O. The resultant moment

$$M = \int y\ dm\ a = a\int y\ dm$$

We recognize $\int y\ dm$ as the first moment of the mass about axis O.

If, again as in any parallel force system, we divide the resultant moment by the resultant force, we shall have, as quotient, the distance from the axis of moments.

Then,
$$y_R = \frac{a\int y\ dm}{a\int dm}$$

But
$$\bar{y} = \frac{\int y\ dm}{\int dm}$$

in which $\bar{y}$ is the distance from axis of reference to center of gravity.

Then the *resultant of the acceleration-force system passes through the center of gravity of a body in pure translation.* The resultant of the acceleration-forces, however, is equivalent to the resultant of the external forces. We can conclude, therefore, that if a free body is in translation, the resultant of the system of external forces acting on that body has its line of action through its center of gravity.

The equation of motion may now be written for the case of a free body in translation. For the form of such equations, we shall use the balanced form of the Newtonian equation, in which the left side is the resultant of the external force system, and the right side is the resultant of the acceleration forces, or

$$\Sigma F_{\text{EXTERNAL}}{}^* = \Sigma F_{\text{ACCELERATION}}.\dagger$$

* Abbreviated EXT.
† Abbreviated ACC.

It is convenient to select as an X axis the direction of the acceleration in an investigation of translation.

Then,

$$\Sigma X_{EXT} = ma_X$$
$$\Sigma Y_{EXT} = ma_Y = 0$$
$$\Sigma M_{EXT} = \Sigma M_{ACC}$$

In the latter case of the moment equation, the same moment axis must be used in taking either the moments of the external forces or the moments of the acceleration-forces, if we are not to violate the law stating that those two systems are equivalent.

Whenever convenient, we may take moments about the center of gravity, and our third equation becomes

$$\Sigma M_{EXT} = \Sigma M_{ACC} = 0$$

ILLUSTRATIVE PROBLEMS

In dynamics, as in all branches of mechanics, an orderly, objective, and accurate attack upon a problem demands that we have, first, a tangible base of operations—the isolated free body. Second, we should, as in our explorations of statics, qualitatively appraise the problem by making a drawing of the isolated free body with its accompanying system of external forces shown as vectors. Third, the acceleration-force should likewise be shown as a vector—in this text acceleration-forces are shown by dotted lines. Finally, the attack upon the unknowns is made by establishing the equivalence of the external-force system and the acceleration-force, if we elect to use the Newtonian equation $R = ma$ directly.

If we elect to apply D'Alembert's principle, the addition of the reversed acceleration-force is made to the external system, so that statical equations may be written.

ILLUSTRATIVE PROBLEM 1

In an extrusion machine, a plunger A (Fig. 266) weighing 10 pounds is drawn up a plane, inclined at 30 degrees with the horizontal, by a force $P = 10$ lb., parallel to the plane. The coefficient of friction between plunger and plane is $\mu = 0.20$. The center of gravity of the plunger is at G. We are to determine the acceleration of the plunger. If it is necessary to prevent tipping of the plunger, determine the range of values of y (the distance from the plane to force P) at which P may be applied.

$$\Sigma X_{EXT} = \Sigma X_{ACC}$$
$$+10 - 5 - \mu N = +\frac{10}{32.2}a_x$$
$$\Sigma Y_{EXT} = 0$$
$$-8.66 + N = 0$$

Then,
$$N = 8.66$$
and
$$a_X = 10.5 \text{ ft. per sec.}^2$$

When tipping of the plunger impends, the plunger will tend to pivot about either point A or B. At that instant, forces N and FR will be concentrated at one of these points instead of being distributed over the surface.

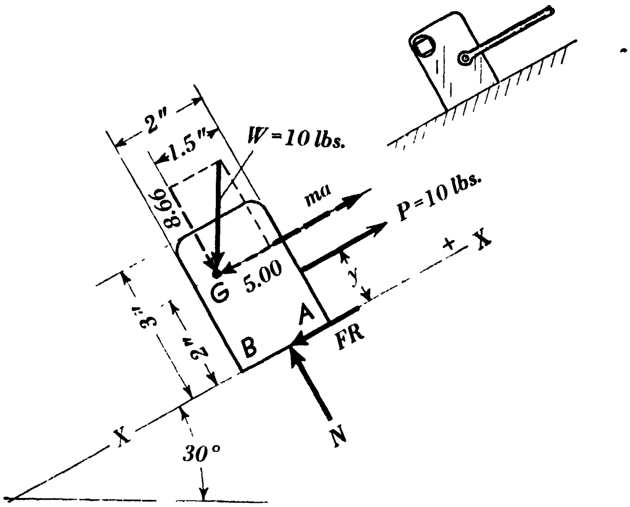


FIG. 266.

Taking point A as an axis of moments,

$$\Sigma M_{EXT} = \Sigma M_{ACC}$$

$$+10y - 5 \times 2 - 8.66 \times 1.5 = + \frac{10}{32.2} \times 10.5 \times 2$$

$$y = 2.95 \text{ in.}$$

Taking point B as an axis of moments,

$$\Sigma M_{EXT} = \Sigma M_{ACC}$$

$$+10y - 5 \times 2 + 8.66 \times \frac{1}{2} = + \frac{10}{32.2} \times 10.5 \times 2$$

$$y = 0.22 \text{ in.}$$

Force P, therefore, may be applied between the limits of $y = 0.22$ in. and $y = 2.95$ in. from the plane, and the plunger will slide, but not tip, over.

Illustrative Problem 2

A car C weighing 500 pounds (Fig. 267) is drawn along the track by a cable attached to counterweight D weighing 1000 pounds. The coefficient of friction between the car and track is $\mu = 0.20$. The center of gravity of C is at point G. The car starts form rest. Determine the following:

(a) The velocity of the car at the end of two seconds.

(b) The tension in the cable.

(c) The normal and frictional components of the reaction of the track on the car.

Part (a): In selecting a free body, we use the same criteria which governed the selection of a free body in statics—the external force system acting on the body should contain the unknowns we seek to determine, as well as some knowns to serve as evaluating material.

Figure 268 illustrates the choice of the car C as a free body. It will be

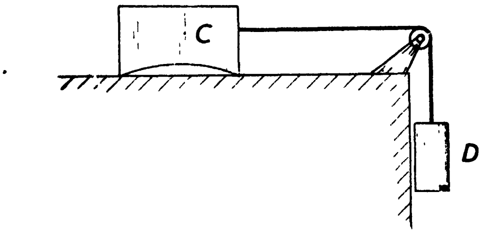


FIG. 267.

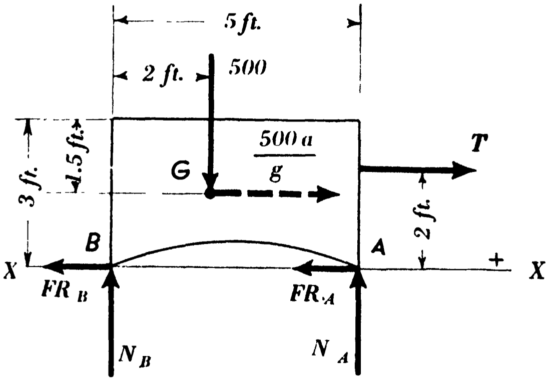


FIG. 268.

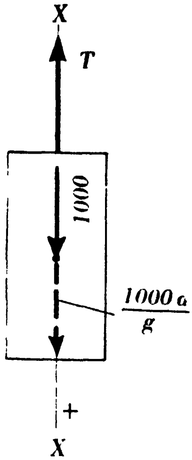


FIG. 269.

noted that the external force system contains the following unknowns: N_A, FR_A, N_B, FR_B, and T. In addition, as we appraise the acceleration-force system, we find that the acceleration force ma is unknown, for a has not yet been determined.

Applying $\mu = \frac{FR}{N}$, we reduce the number of unknowns, but have remaining an excessive number for the number of equations available. The unknowns are N_A, N_B, T, and a; there are but three equations of motion available. We can reinforce the solution by making available an additional equation: the equation of motion for body B. Therefore, we simultaneously isolate body B as a free body, as in Fig. 269. Now we may proceed.

For body B (with X axis chosen, as shown, in the direction of motion)

$$\Sigma X_{EXT.} = \Sigma X_{ACC.}$$

$$+1000 - T = \frac{1000}{g} a \qquad (1)$$

For body A (with X axis again chosen in the direction of motion)

$$\Sigma X_{EXT.} = \Sigma X_{ACC.}$$

$$+T - FR_A - FR_B = \frac{500}{g} a$$

$$\Sigma Y_{EXT.} = 0$$

$$-500 + N_A + N_B = 0$$

Then,
$$+T - 0.2(500) = \frac{500}{g} a \qquad (2)$$

Solving Equations (1) and (2) simultaneously,

$$a = \frac{900g}{1500} = 0.6 \times 32.2 = 19.32 \text{ ft. per sec.}^2$$

Then, the velocity at the end of two seconds will be

$$v_2 = v_0 + at = 0 + 19.32 \times 2 = 38.64 \text{ ft. per sec.}$$

Part (b): The tension in the cable is determined from Equation (1) and is

$$T = 1000 - \frac{1000}{g} \times \frac{900g}{1500} = 1000 - 600 = 400 \text{ lb.}$$

and is checked in Equation (2) as

$$T = 100 + \frac{500}{g} \times \frac{900g}{1500} = 100 + 300 = 400 \text{ lb.}$$

Part (c): We may select point A of Fig. 268 as an axis of moments

$$\Sigma M_{EXT.} = \Sigma M_{ACC.}$$

$$+400 \times 2 - 500 \times 3 + N_B \times 5 = \frac{500}{g} \times \frac{900g}{1500} \times 1.5$$

$$N_B = 230 \text{ lb.}$$

$$FR_B = 0.2 \times 230 = 46 \text{ lb.}$$

Now, using point B as an axis of moments,

$$\Sigma M_{EXT.} = \Sigma M_{ACC.}$$

$$+400 \times 2 + 500 \times 2 - N_A \times 5 = \frac{500}{g} \times \frac{900g}{1500} \times 1.5$$

$$N_A = 270 \text{ lb.}$$

$$FR_B = 0.2 \times 270 = 54 \text{ lb.}$$

$\Sigma Y = 0$ will serve as a tentative check on the normal forces.

$$\Sigma Y = +230 + 270 - 500 = 0.$$

$\Sigma X_{EXT.} = \Sigma X_{ACC.}$ will similarly serve in tentatively checking the frictional forces

$$\Sigma X_{EXT} = +400 - 46 - 54 = +300 \text{ lb.}$$
$$\Sigma X_{ACC.} = \frac{500}{32.2} \times \frac{900 \times 32.2}{1500} = +300 \text{ lb.}$$

A much more effective check involves the work and energy relationships of the problem and is discussed later in Article 113. The results of the solution are plotted in Fig. 270.

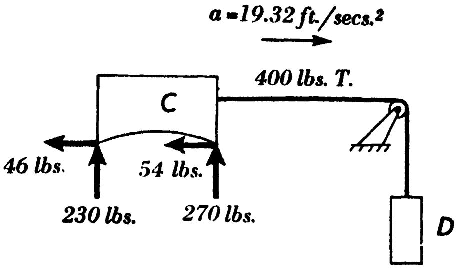


FIG. 270.

ILLUSTRATIVE PROBLEM 3

In the four-bar linkage $CABD$ shown in Fig. 271, the connecting rod AB weighs 40 pounds. Cranks AC and BD are parallel and of equal

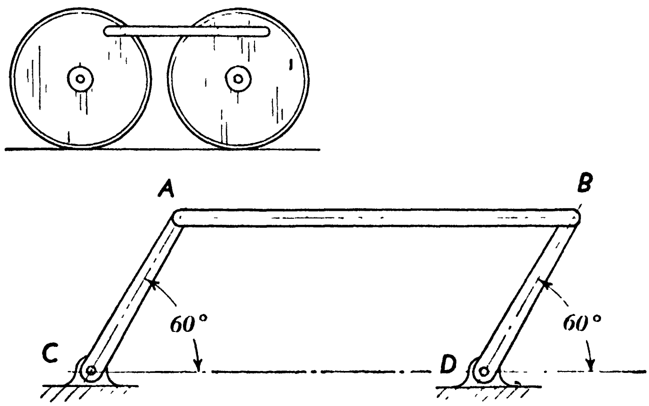


FIG. 271.

length. The motion of AB is, therefore, an example of curvilinear translation. If the speed of the cranks is constant and equal to 60 r.p.m., determine the forces acting at pins A and B when the cranks are inclined at 60 degrees with the horizontal. Assume that the force exerted by CA at A is perpendicular to CA. $CA = BD = 9$ in. $AB = 20$ in.

The acceleration of a particle at A on crank AC is

$$a_N = \omega^2 r = \left(\frac{60 \times 2\pi}{60}\right)^2 \times \frac{9}{12} = 29.6 \text{ ft. per sec.}^2$$

Since AB has a motion of translation, *all* particles have the same acceleration.

In Fig. 272, the connecting rod is shown as a free body. The system of external forces contains F_{CA} at the inclination shown and with sense assumed. The weight, $W = 40$ lb., is also an external force and

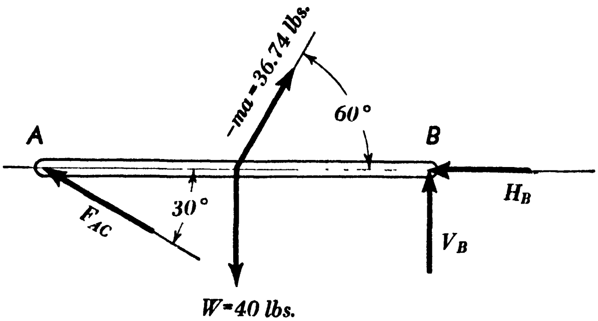


FIG. 272.

acts at the center of gravity. The horizontal and vertical components of F_{BD}, the force exerted by crank BD, are shown, with senses assumed. The acceleration force, acting at the center of gravity, is

$$ma = \frac{40}{32.2} \times 29.6 = 36.74 \text{ lb.}$$

Then, applying D'Alembert's Principle, we add a reversed acceleration force of 36.74 pounds (acting at the center of gravity) to the

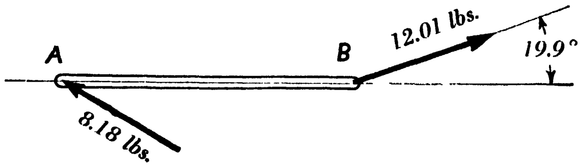


FIG. 273.

external force system, and apply conditions of equilibrium to solve for the unknowns.

$$\Sigma M_B = +F_{AC} \times 0.5000 \times 20 - 40 \times 10 + 36.74 \times 0.8660 \times 10 = 0$$

$$F_{AC} = 8.18 \text{ lb.}$$

$$\Sigma X = -8.18 \times 0.8660 + 36.74 \times 0.5000 - H_B = 0$$

$$H_B = 11.29 \text{ lb.}$$

$$\Sigma Y = +8.18 \times 0.5000 + 36.74 \times 0.8660 - 40 +$$

$$V_B = 4.09 \text{ lb.}$$

Then, $F_{BD} = \sqrt{(11.29)^2 + (4.09)^2} = 12.01$ lb.

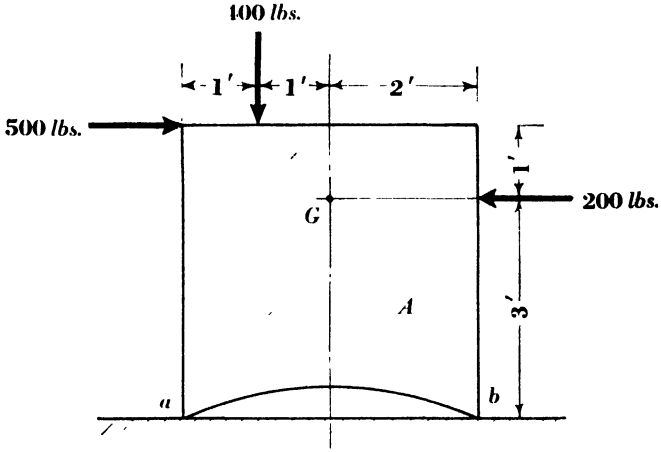
$$\theta = \tan^{-1}\left(\frac{4.09}{11.29}\right) = 19.9°$$

The results are plotted as Fig. 273.

Problems

415. A block slides down a 45° inclined plane. The coefficient of friction $\mu = 0.20$. How many seconds are required to increase the velocity of the body from 10 ft. per sec. to 30 ft. per sec.? *Ans.* $t = 1.1$ secs.

416. A body, A, weighing 966 lb. is moved along a horizontal plane by the two horizontal forces of 500 and 200 lb., respectively, and the vertical force of 400 lb. shown.



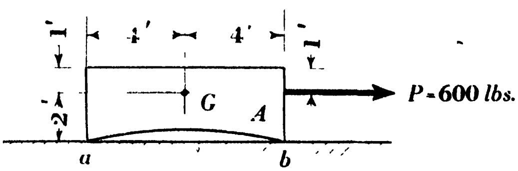
Prob. 416

The coefficient of friction between the plane and A is $\mu = 0.10$. The center of gravity is point G.

If the body starts from rest, determine its velocity at the end of 10 sec.

417. Determine, for the sliding body A of Problem 416, the normal and frictional components of the forces exerted by the horizontal plane on A at points a and b.

418 The car A is drawn along a horizontal track by a constant force $P = 600$ lb. A weighs 1932 lb. The coefficient of friction is $\mu = 0.125$. The center of gravity is at G.

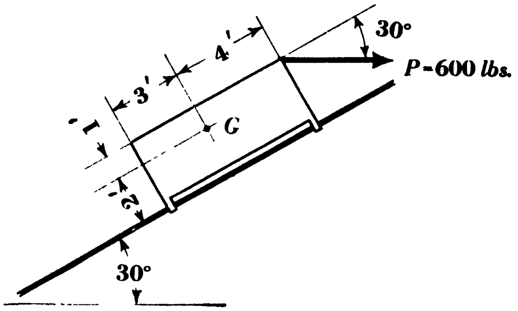


Prob. 418

If the initial velocity is 6 ft. per sec. to the right, determine the normal and frictional components of the forces exerted by the track on the car at points a

and b. Also determine the velocity of the car at the end of 10 sec., and the total distance it has traveled.

419. Car A, weighing 2000 lb., is acted upon by a constant force $P = 600$ lb. The coefficient of friction $\mu = 0.10$. The center of gravity of the car is at point G. Determine the distance which the car travels in 10 sec., if its initial velocity is 10 ft. per sec. down the plane. *Ans.* $s = 323$ ft.



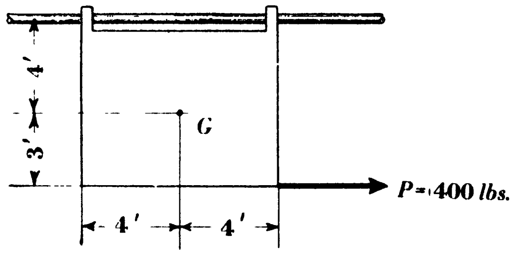
PROB. 419

420. Bodies A, B, and C weigh 100, 200, and 300 lb., respectively. The cable joining B and C passes over a weightless and frictionless pulley. The coefficient of friction between the horizontal plane and bodies A and B is $\mu = 0.20$.

(a) Find the acceleration of the bodies.

(b) Determine the stress in the cable joining A and B.

421. The tramway car shown weighs 1000 lb., and slides on a horizontal guide rail, the coefficient of friction between rail and car being $\mu = 0.25$. The center of gravity of the car is point G. The car has an initial velocity of 10 ft. per sec. to the right. Determine



PROB. 421

(a) The magnitude and direction of the normal and frictional forces exerted on the car by the rail.

(b) The velocity of the car at the end of 15 sec.

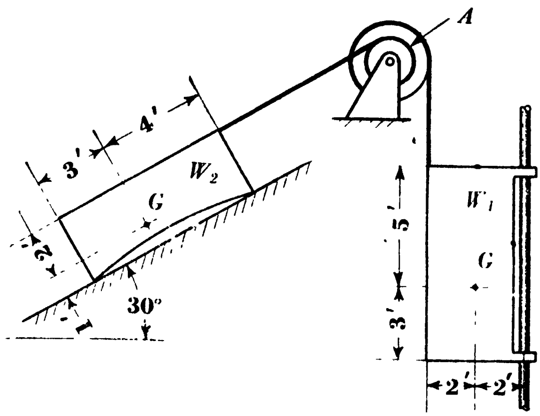
(c) The distance which the car travels in 15 sec.

422. A train weighing 32.2 tons moves with constant acceleration along a horizontal track. The total resistance (air resistance plus frictional resistance) is 0.004 times the weight of the train.

Determine the drawbar pull of the locomotive on the train if the train travels 1 mile, starting from rest, in the first 2 min.

423. The elevator W_1 shown weighs 2400 lb. when loaded. A counterweight W_2 weighs 1800 lb., and slides on an inclined plane. The weight of the pulley A

may be neglected. The coefficient of friction $\mu = 0.20$ for both sliding surfaces. Inner diameter of pulley $A = 2$ ft.; outer diameter $= 3$ ft.



PROB. 423

Determine the following, when the elevator has started from rest and has traveled for 10 sec.

(a) The acceleration of W_1.

(b) The acceleration of W_2.

(c) The tension in the cables.

(d) The velocity of each weight.

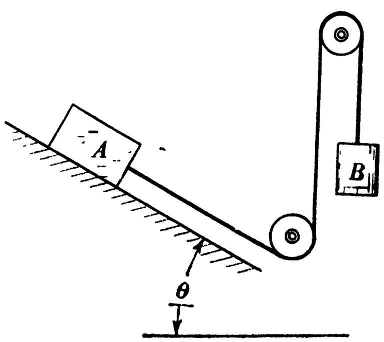
(e) The normal and frictional components of the forces exerted on the elevator by the guide rail.

424. A freight car weighing 48.3 tons travels on a siding which has a 2-per-cent downgrade. The resistance of the car to motion is 10 lb. per ton. The velocity of the car as it enters the siding is 5 m.p.h. If the siding is 100 ft. long, determine the force with which the car will strike a bumper spring at its end.

425. A railroad car is detached from the rest of the train at the top of a grade whose coordinates are 1 vertical to 18 horizontal. The resistance to motion is 15 lb. per ton, and the weight of car and contents is 50 tons. How fast will the car be moving after it has gone $\frac{1}{2}$ mile?

426. Body A weighs 150 lb.; body B weighs 150 lb.; Coefficient of friction $\mu = 0.3$. If the system starts from rest, determine the tension in the cable, and the acceleration of the bodies and their velocity at the end of 3 sec. The weight of the small pulleys is negligible. Tan $\theta = \frac{3}{4}$.

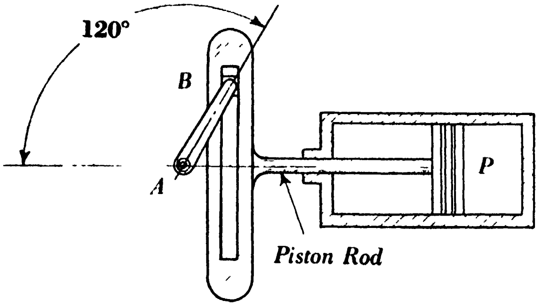
Ans. $T = 48$ lb.; $a = 21.9$ ft./sec.^{2}; $v = 65.7$ ft./sec.



PROB. 426

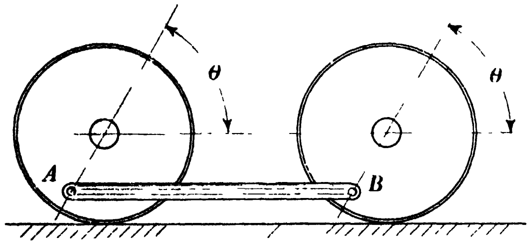
427. In the air compressor shown, the piston P and piston rod weigh 70 lb. The piston has a diameter of 8 in. Crank AB is turning clockwise, and has a length of 8 in. The crank speed is 100 r.p.m. The pressure on the piston is 60 p.s.i. for the crank position shown.

Determine the force exerted by the crank pin B on the piston rod for the position shown.

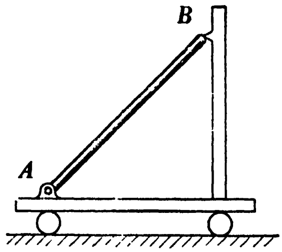


PROB. 427

428. The parallel rod of a locomotive AB weighs 322 lb. The wheels are revolving at a constant speed of 100 r.p.m. Determine the force at pins A and B when $\theta = 60°$.



PROB. 428



PROB. 429

429. The uniform, slender bar AB weighs 32.2 lb., is supported on a pin joint at A, and rests against a frictionless surface at B. Determine the acceleration of the car at which the bar will start to swing away from the upright.

100. Mechanical Vibrations. Mechanical vibrations are an important source of concern in the design of machines and structures, because such motion is frequently responsible for excessive wear, repeated stresses which may cause fatigue failure in metals, and objectionable noises or discomfort in vehicles. The factor of vibration is so important and so extensive a field of exploration in dynamics that constant research is devoted to it. A growing literature is being devoted to the subject,* and we shall confine our discussion to the manner in which simple vibration problems are attacked.

Such an introduction lies within the limits of our present concern

* The present discussion of mechanical vibration is intended only to furnish an additional example of force versus displacement relationships. For a more comprehensive treatment, the reader is referred to J. P. Den Hartog, *Mechanical Vibrations*, 3rd Ed. New York: McGraw-Hill Book Company, 1947.

with the fundamentals of engineering mechanics, because we shall find that the tools and techniques which are to be employed are those we have already developed in the other applications of fundamental dynamics.

Vibrating masses are free bodies in accelerated motion. External and acceleration forces act, and the axioms, or equations of motion, apply. The only distinctions which need be made in considering these free bodies are those considerations which involve the repetition of the motion—a feature which has been absent from the moving free bodies we have already encountered.

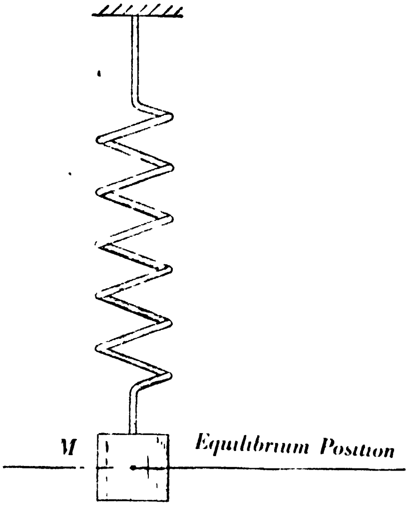


FIG. 274.

A *vibration* is a periodic motion which repeats itself after a definite interval of time. This time interval devoted to a single typical motion is called the *period* of the vibration, and is usually measured in seconds. Each repetition of the typical motion is called a *cycle*, and the number of cycles per second is the *frequency*.

Many of the most essential factors in vibratory motion may be studied by considering the system to be replaced by a small body, such as M of Fig. 274 which is suspended from a spring. If this body were drawn downward from the equilibrium position shown and then released, the spring would exert an unbalanced external force on the body. Such unbalanced force, as in all of our previous cases, will cause the body to have an acceleration which, in this case, acts in the direction of its equilibrium position.

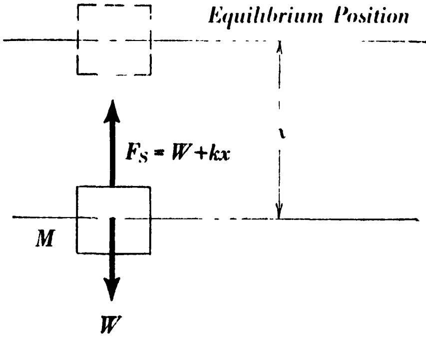


FIG. 275.

Such a condition is illustrated in Fig. 275. The body M is shown, isolated as a free body. The system of external forces consists of F_s, the force exerted by the spring, and W, the weight of the body. The direction of motion, as in all of our previous cases of translation, is an X axis, and the position of static equilibrium is the origin. x distances downward from the equilibrium position are being considered positive, and the displacement for the position shown in the figure is $+x$.

The force exerted by the spring (F_s) may be expressed by assuming the spring to be perfectly elastic, in which case the tension or compression in the spring is directly proportional to the amount of deformation. Expressed as an equation,

$$F_s = kx$$

where k is the constant of proportionality between spring force and displacement, usually called the *spring constant*.

The spring, in our example, has been stretched from the position of equilibrium (at which time it was exerting a force equal to W, the weight of the body) to displacement x, where it will exert a force $W + kx$. If, now, we apply the equation of motion,

$$\Sigma X_{\text{EXT.}} = \Sigma X_{\text{ACC.}}$$

$$+W - (W + kx) = \frac{W}{g} a_x = \frac{W}{g} \frac{d^2x}{dt^2}$$

Then,

$$-kx = \frac{W}{g} \frac{d^2x}{dt^2}$$

or

$$\frac{d^2x}{dt^2} = -\frac{kg}{W} x$$

We recognize this form as the equation which we used in defining simple harmonic motion in Article 86.

This vibratory motion is, therefore, simple harmonic. We have already assumed an ideal spring. If the resistance of the air is neglected, the body will continue to vibrate indefinitely under the influence of the variable force exerted by the spring.

Such a vibration is called *free vibration*. The only other force involved, in addition to that of the spring and the weight of the body, has drawn the body downward from its equilibrium position, and after so disturbing it, has vanished.

The differential equation which we have derived for free vibration, like any simple harmonic motion, may be written,

$$\frac{d^2x}{dt^2} = -\omega^2 x \left(\omega = \sqrt{\frac{kg}{W}}\right)$$

in which ω is the constant angular velocity of the radius of the auxiliary circle of simple harmonic motion. (See Article 86.)

To solve this equation we note that x is a function of time, a function whose second derivative with respect to time equals x multiplied by a negative constant $(-\omega^2)$. From the differential calculus, it is learned that sine and cosine functions repeat themselves in such fashion.

Then, substitution of $x = \sin \omega t$ or $x = \cos \omega t$ will give solutions of the differential equation. Substitution of a more general value.

$$x = C_1 \sin \omega t + C_2 \cos \omega t$$

will give a more complete or general solution. The arbitrary constants C_1 and C_2 may be determined from the conditions of the given problem.

If, for example, the body of Fig. 275 has initial displacement x_0 and initial velocity v_0 when it is released by the disturbing force at time t = zero;

$$x = C_1 \sin \omega t + C_2 \cos \omega t$$

$$x_0 = C_1 \times 0 + C_2 \times 1 ,$$

and $$C_2 = x_0$$

Differentiating the equations of displacement,

$$x = C_1 \sin \omega t + C_2 \cos \omega t$$

with respect to time, we have

$$\frac{dx}{dt} = v = \omega C_1 \cos \omega t - \omega C_2 \sin \omega t$$

and substituting $v = v_0$ at time t = zero,

$$v_0 = \omega C_1 \times 1 - \omega C_2 \times 0$$

and $$C_1 = \frac{v_0}{\omega}$$

Then, $$x = \frac{v_0}{\omega} \sin \omega t + x_0 \cos \omega t$$

This equation fully describes free vibration: it has rested upon a general solution of the differential equation to embrace all possible initial conditions.

The x displacement of the resultant displacement is, by this method of solution, the sum of two terms representing two individual vibrations of the same frequency and differing in phase by 90 degrees.

The first vibration is

$$x = \frac{v_0}{\omega} \sin \omega t$$

and has existence only if the body is given initial velocity when the disturbing force is released.

The second vibration is

$$x = x_0 \cos \omega t$$

and occurs only if there is displacement from the equilibrium position of the body when the disturbing force is released.

The period of the vibration is

$$T = \frac{2\pi}{\omega} = \frac{2}{\sqrt{\frac{kg}{W}}} = 2\pi \sqrt{\frac{W}{kg}}$$

The frequency is

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{kg}{W}}$$

Both expressions may be somewhat simplified by noting that at the position of static equilibrium, the deformation of the spring is $x_s = \frac{W}{k}$.

Then,
$$T = 2\pi\sqrt{\frac{x_s}{g}}$$

and
$$f = \frac{1}{2\pi}\sqrt{\frac{g}{x_s}}$$

It should be noted that both the period and frequency of a free vibration are dependent only upon the deformation of the spring in the equilibrium position and are independent of the initial displacement and velocity.

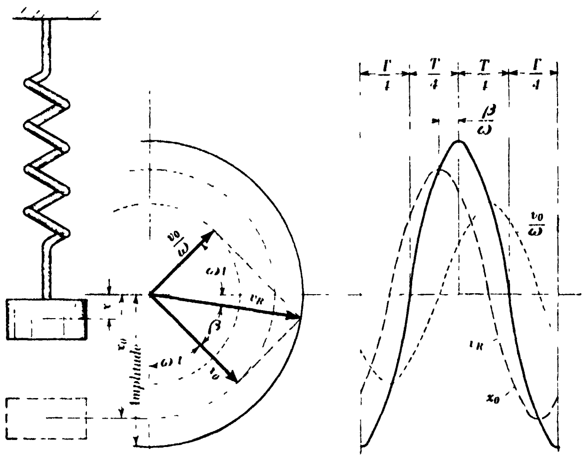


FIG. 276.

The resultant vibration represented by the two terms $\frac{v_0}{\omega}\sin\omega t + x_0\cos\omega t$ may be viewed, for increasing our familiarity with the relationship involved, by using the position vector system described in Article 86.

Figure 276 is the graphical representation. Each of the vibrations is represented by a rotating vector of magnitudes $\frac{v_0}{\omega}$ and x_0, respectively.

The resultant vibration is represented by a vector

$$v_R = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

v_R will lag behind x_0 as the vectors rotate by an angle β, usually called the *phase angle*, whose magnitude is:

$$\beta = \tan^{-1}\left(\frac{v_0}{x_0\omega}\right)$$

It will be seen from the graphical representation of the motion that the total displacement of the vibrating body at any instant has magnitude

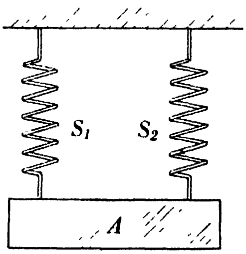
$$x = v_R \cos(\omega t + \beta)$$

We note that the resultant vibration will reach its maximum value of displacement at time $\frac{\beta}{\omega}$ seconds after the component vibration $x_0 \cos \omega t$ has reached its maximum value.

The resultant vibration will also reach its maximum value of velocity and acceleration at time $\frac{\beta}{\omega}$ seconds after the component vibration $x_0 \cos \omega t$ has reached maximum velocity and acceleration.

Problems

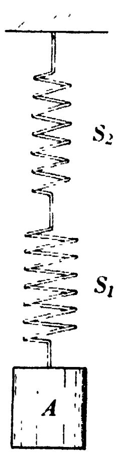
430. The body A weighing W lb. is supported by a parallel system of two springs, S_1 and S_2. If the spring constants of the springs are k_1 and k_2, respectively, determine the equivalent spring constant of the system.



Prob. 430

431. Body A weighing W lb. is suspended from a series system of two springs S_1 and S_2. If the spring constants of the springs are k_1 and k_2, respectively, determine the equivalent spring constant of the system.

Ans. $k = \frac{k_1 k_2}{k_1 + k_2}$.

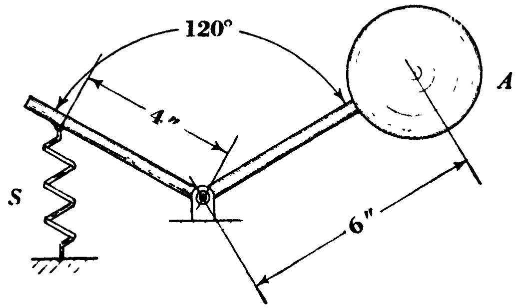


Prob. 431

432. A weight of 100 lb. falls from a height of 2 in. upon a spring, whose constant is 225 lb. per in. Determine the period, frequency, and amplitude of the free vibration of the body, assuming that it remains attached to the upper surface of the spring.

433. A load of 50 lb. causes a cantilever beam to deflect $\frac{1}{10}$ in. at its free end. If a weight of 100 lb. is dropped on the free end of the beam from a height of 3 in., determine the frequency, period, and amplitude of the resulting free vibration, assuming that the 100-lb. weight remains on the beam after striking it.

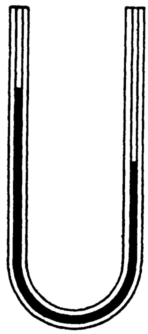
434. Body A weighs 50 lb., and the spring constant of spring S is 30 lb. per in. Determine the frequency of the system. Assume that the weight of the rocker arm may be neglected.



PROB. 434

435. A beam AB deflects 0.2 in. when a load of 1000 lb. is applied at the center of the beam. Determine the frequency of vibration of the beam, when carrying a central load of 20,000 lb. *Ans.* $f = 1.56$ cycles per sec.

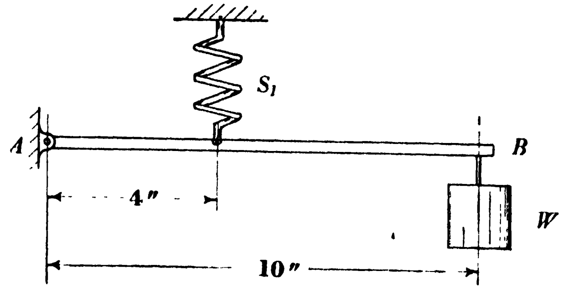
436. If liquid is placed in a U tube, as shown, and one side of the liquid is depressed and then suddenly released, determine the period of vibration of the column of liquid. The total length of the column of liquid is l, and the friction may be assumed to be negligible.



PROB. 436

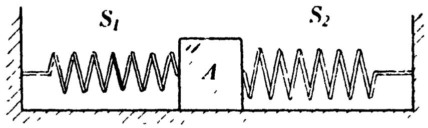
437. A spring deflects 1 in. when supporting a weight of 450 lb. If the magnitude of the weight is tripled, then displaced from its equilibrium and released, determine the period and frequency of the resulting free vibration.

438. The arm AB is pinned to a support at A, and carries a load $W = 30$ lb. The spring constant for spring S_1 is 20 lb. per in. Determine the frequency of vibration of the arm.



PROB. 438

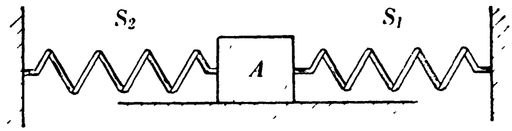
439. A block A weighing 10 lb. is resting on a frictionless surface as shown. The spring constants for S_1 and S_2 are 6 lb. per in. and 7 lb per in., respectively. Determine the frequency and period of the vibration of A if it is drawn 1 in. away from its equilibrium position, and then released.



PROB. 439

440. For block A of Problem 439, determine its maximum velocity while it is vibrating freely.

441. The block A weighing 4 lb. slides on a frictionless surface. The spring constant of spring S_1 is 5 lb. per in. If block A is displaced to the right from its equilibrium position by a force of 20 lb. and then released with no initial velocity, it will have maximum velocity of 5 ft. per sec. as it vibrates.



PROB. 441

Determine the distance from the mid-point of the travel of A which it was displaced by the original disturbing force of 20 lb., and the spring constant of spring S_2.

442. Determine the frequency of vibration of block A of Problem 441.

CHAPTER XI

Rotation

101. Rotation. In the chapter devoted to kinematics, we found that all bodies having motion in a plane could be considered to be rotating bodies. The axis of rotation may be fixed or instantaneous. The velocities of the particles varied in direct proportion to their respective distances from the axis of rotation—as that axis was located at greater distances from the particles comprising the body, the velocities of the several particles approached one another in magnitude and direction. In the limit, when the axis of rotation has receded to an infinite distance from the body, the velocities of all particles were found to be equal to one another, and this motion has been called *translation.*

It will now be convenient to divide the remaining forms of coplanar rotation into two categories. When the axis of rotation is fixed, the motion is generally called *pure rotation.* When the axis of rotation is constantly changing and we must seek a series of instantaneous axes, the motion is generally described as combined translation and rotation, or *general plane motion.*

102. Pure Rotation. The free body shown in Fig. 277 has a motion of pure rotation about a fixed axis (OZ) perpendicular to the plane of motion (XOY) at O.

The angular velocity of the body is ω, and its angular acceleration is α. The axis OX passes through the mass center of the body point G. dm is an element of mass, whose distance from the OX axis is y; from the OY axis, x; and from the axis of rotation, r.

Let us first establish the resultant of the acceleration-force system which as a simpler equivalent, will make appraisal of the problems of rotation which we face more penetrating.

In establishing this resultant, we note that the system of forces is coplanar, but not parallel and not concurrent. We use, then, the method of attack which was developed in Article 18 for finding the resultant of any force system in a plane.

Whether the system of forces is a system of external forces or one of acceleration-forces, the procedure is general. For such a system, the resultant is

$$R = \sqrt{\Sigma X^2 + \Sigma Y^2}$$

The acceleration-force component of dm in the direction of r is a normal component and has magnitude

$$dF_N = \omega^2 \, r \, dm$$

The acceleration-force component perpendicular to r is a tangential component and has magnitude

$$dF_T = \alpha\, r\, dm$$

The resolution of dF_N and dF_T into components parallel to the X and Y axes is shown in the subordinate drawings.

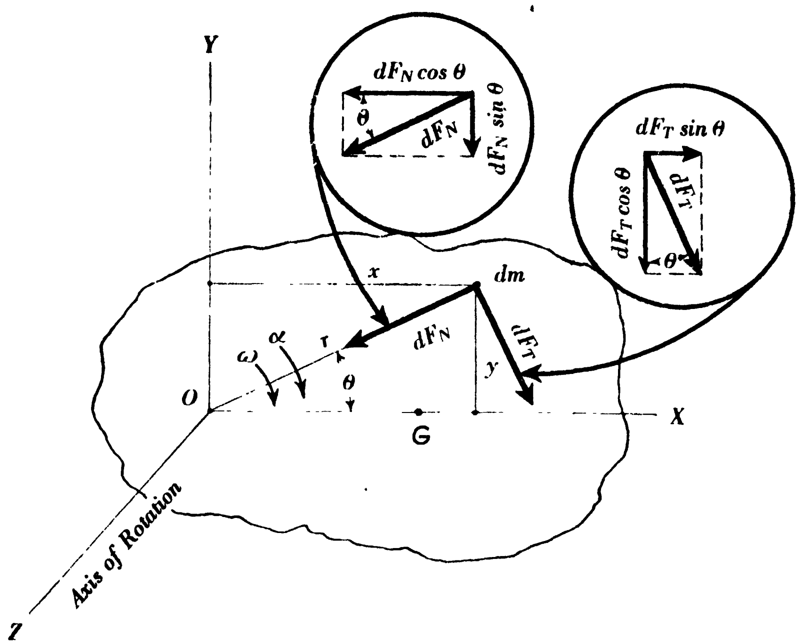


FIG. 277.

Summarizing the X components of the acceleration forces, we have

$$\Sigma X = \int dF_N \cos\theta + \int dF_T \sin\theta$$
$$= \int \omega^2 r\, dm \cos\theta + \int \alpha\, r\, dm \sin\theta$$

But $\quad r\cos\theta = x$ and $r\sin\theta = y$

Then, $\quad \Sigma X = \int \omega^2 x\, dm + \int \alpha\, y\, dm$

Since, at any instant, ω and α are constant for all of the elementary particles comprising the body,

$$\Sigma X = \omega^2 \int x\, dm + \alpha \int y\, dm$$

The $\int y\, dm$ is the first moment of the entire mass about the X axis which was set as a centroidal axis. Then $\int y\, dm = 0$, and

$$\Sigma X = \omega^2\, \bar{x}\, m$$

Now let us summarize the components of the acceleration forces in the direction of the Y axis.

$$\begin{aligned} \Sigma Y &= \int dF_N \sin\theta + \int dF_T \cos\theta \\ &= \int \omega^2 r\, dm \sin\theta + \int \alpha r\, dm \cos\theta \\ &= \omega^2 \int y\, dm + \alpha \int x\, dm \end{aligned}$$

Therefore, $$\Sigma Y = \alpha\, \bar{x}\, m$$

These equations have established the magnitude of the X and Y components of the resultant acceleration-force. The resultant itself would be the vector sum of these two components. In most of the problems of dynamics, however, the rectangular components just derived play the leading role, and further reduction of the system is unnecessary.

For convenience in the derivation of the values of ΣX and ΣY, we have used the customary X and Y axes. In the problems of rotation, the most convenient axes are the N and T illustrated in Fig. 278.

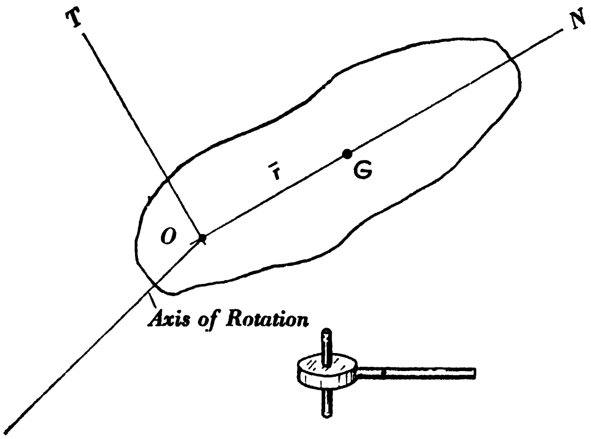


FIG. 278.

The N axis passes through the mass center of the body and the fixed axis at O; the T axis is perpendicular to N at the axis of rotation. $\bar{r}$ is the distance along the N axis from the axis of rotation at O to the mass center at G.

Since we have chosen to base our problem attack upon such axes, it will be wise to make the necessary changes of nomenclature in the expressions for ΣX and ΣY previously derived. These expressions will, therefore, become

$$\Sigma N = \omega^2\, \bar{r}\, m$$

and $$\Sigma T = \alpha\, \bar{r}\, m$$

We shall need, in addition to the magnitudes and inclinations of these components of the resultant acceleration force, the location of the resultant.

Again we proceed as in all force systems by summarizing the moments. Selecting point O as a moment axis (Fig. 277),

$$\begin{aligned}\Sigma M_o &= \int dF_T\, r \\ &= \int \alpha r^2\, dm = \alpha \int r^2\, dm \\ &= \alpha I_o\end{aligned}$$

Then the distance along the N axis from the axis of rotation to the point of application of the resultant acceleration force will be

$$l = \frac{\Sigma M_o}{\Sigma T} = \frac{\alpha I_o}{\alpha \bar{r} m} = \frac{I_o}{\bar{r} m}$$

in which I_o is the polar moment of inertia of the entire body about the axis of rotation and $\bar{r}m$ is the first moment of the body about the same axis.

The point of application of the resultant acceleration force is called the *center of percussion* of the body

The equations developed in this article have been expressions concerning the acceleration force.

We need to take one additional step to draw our conclusions as to equations of motion for bodies in pure rotation.

We have already accepted the Newtonian law of equivalence of external force and acceleration force. Then, our equations of motion are

$$\begin{aligned}\Sigma N_{\text{EXT}} &= \omega^2 \bar{r}\,\frac{W}{g} \\ \Sigma T_{\text{EXT}} &= \alpha \bar{r}\,\frac{W}{g} \\ \Sigma M_{\text{EXT}} &= \alpha\,\frac{I_W}{g}{}^{*}\end{aligned}$$

When the axis of rotation O coincides with the center of gravity of the body, the equations of motion become

$$\begin{aligned}\Sigma N_{\text{EXT.}} &= 0 \\ \Sigma T_{\text{EXT.}} &= 0 \\ \Sigma M_{\text{EXT.}} &= \frac{\alpha I_{C.G.}}{g}\end{aligned}$$

Illustrative Problem 1

The crank OB (Fig. 279) is rotating in a horizontal plane about vertical axis YY. The weight of the crank is 100 pounds, $I_{YY} = 700$ lb.-ft.^{2} and the center of gravity is at point G. At the instant when the crank is in the position shown, a force of 200 pounds is acting in the plane

* In the moment equation, it should be noted that the moment of inertia I_W, calculated in terms of the usual data available, is a weight moment of inertia (lb.-ft.^{2} or lb.-in.^{2}). It must, therefore, be divided by g to preserve the balance of units.

of motion and is inclined at 30 degrees with the axis of the crank at point B. The angular velocity ω at this instant is five radians per second. We are to determine the components of the bearing reactions at O, O_N, and O_T.

Isolating the crank as a free body and drawing a plan view as free-body diagram (Fig. 280), we note that the external forces acting in the plane of motion are the 200-pound load, O_N, and O_T. The senses of O_N and O_T are assumed.

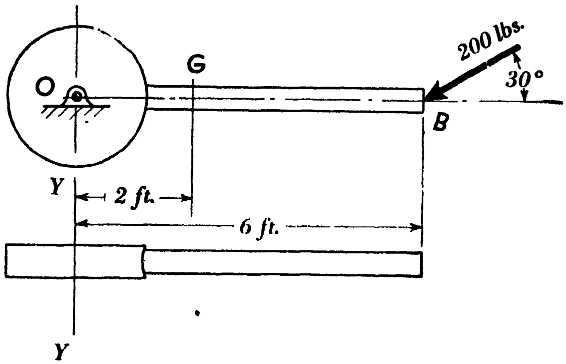


FIG. 279.

We next examine the acceleration-force components.

$$\Sigma N_{ACC.} = \omega^2 \bar{r} \frac{W}{g}$$

$$= (5)^2 \times 2 \times \frac{100}{32.2} = 155.3 \text{ lb.}$$

$$\Sigma T_{ACC.} = \alpha \bar{r} \frac{W}{g}$$

To find α, we must turn to the equation of motion.

$$\Sigma M_o = \alpha \frac{I_o}{g}$$

$$200 \times 0.5000 \times 6 = \alpha \frac{700}{32.2}$$

$$\alpha = 27.6 \text{ radians per sec.}^2$$

Then, $$\Sigma T_{ACC.} = 27.6 \times 2 \times \frac{100}{32.2} = 171.4 \text{ lb.}$$

The acceleration-force components are plotted as dotted vectors in the free body diagram (Fig. 280) at point p. The location of p need not be determined, because only the inclinations of the acceleration forces play a part in the present solution.

To evaluate the components of the force acting at the bearing, we may apply

$$\Sigma N_{\text{EXT.}} = \Sigma N_{\text{ACC}}$$
$$+O_N - 173.2 = -155.3$$
$$O_N = 17.9 \text{ lb.}$$

Then the sense assumed for O_N has been confirmed as correct.

$$\Sigma T_{\text{EXT.}} = \Sigma T_{\text{ACC.}}$$
$$-O_T - 100 = -171.4$$
$$O_T = 71.4 \text{ lb.}$$

The assumed sense of O_T is therefore correct

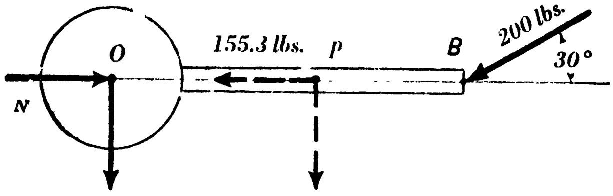


FIG. 280.

Another approach to evaluating O_T may be made by using the center of percussion p.

$$Op = \frac{I_W}{\bar{r}w} = \frac{700}{200} = 3.5 \text{ ft.}$$

Taking moments about point p,

$$\Sigma M_{\text{EXT.}} = \Sigma M_{\text{ACC}}$$
$$\Sigma M_p = -O_T \times 3.5 + 100 \times 2.5 = 0$$
$$O_T = \frac{100 \times 2.5}{3.5} = 71.4 \text{ lb.}$$

The results of the investigation are shown properly plotted in Fig. 281. It is quite as necessary in the case of dynamic problems, as in their

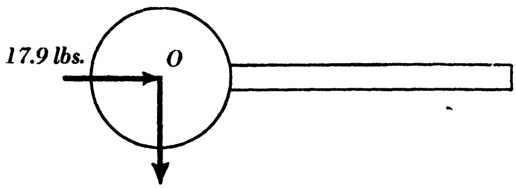


FIG. 281.

statical counterparts, to present the results of our analyses so that the reader may have clear and accurate information as to the magnitudes

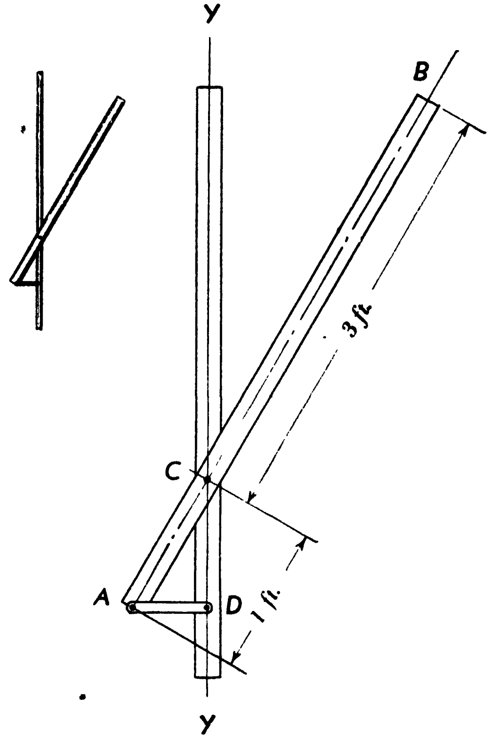


FIG. 282.

and directions of the evaluated forces. The forces O_N and O_T must be associated with one of the contacting bodies—the crank or the supporting shaft at the bearing. The diagram shows those forces that are exerted on the crank.

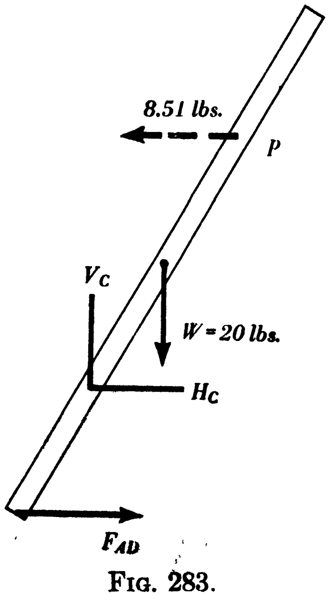


FIG. 283.

ILLUSTRATIVE PROBLEM 2

The thin rod AB of uniform section and homogeneous material (Fig. 282) rotates about the vertical axis YY at constant speed of 50 r.p.m. The weight of the rod is 20 pounds. A rod AD holds the rod AB at a 30-degree inclination. Determine the stress in AD.

The rod AB is isolated and shown as a free body in Fig. 283. The system of external forces comprises H_C, V_C, the weight W, and the force F_{AD}.

The acceleration-force system consists of

$$\Sigma N = \omega^2 \bar{r} \frac{W}{g} = \left(\frac{50 \times 2\pi}{60}\right)^2 \times (1 \times 0.5000) \times \frac{20}{32.2}$$
$$= 8.51 \text{ lb.}$$

This force will have its point of application at the center of percussion p.

The X coordinate of the point p from axis YY is

$$x = \frac{I_{YY}}{\bar{x}W} = \frac{\frac{WL^2}{12}\sin^2\theta + (\bar{x})^2 W}{\bar{x}W}$$
$$= \frac{\frac{W \times 16}{12} \times (0.5000)^2 + \left(\frac{1}{2}\right)^2 W}{\frac{W}{2}}$$
$$= 1.167 \text{ ft.}$$

The Y coordinate from point C is $\frac{1.167}{\tan 30°} = \frac{1.167}{0.5774} = 2.021$ ft.

The resultant acceleration force of 8.51 pounds is shown as a dotted vector at point p.

Now, taking moments about point C,

$$\Sigma M_{EXT} = \Sigma M_{ACC}$$
$$20 \times \tfrac{1}{2} - F_{AD} \times 0.8660 = -8.51 \times 2.021$$
$$F_{AD} = 31.4 \text{ lb.}$$

Then the stress in AD is 31.4 lb., tension.

Illustrative Problem 3

The cylindrical two-stage pulley C shown in Fig. 284 has a moment of inertia about the axis of rotation at O, $I_o = 400$ lb.-ft.^{2} The pulley is symmetrical about the axis of rotation. If weight $A = 150$ lb., and weight $B = 200$ lb., determine the angular velocity of the pulley at the end of five seconds, if the pulley starts from rest. Also, determine the tension in each cable.

In this case, the selection of either A, B, or C as a free body individually will fail to supply a sufficient number of equations of motion to permit us to complete the solution. We turn, therefore, to "simultaneous isolation" to take advantage of the interdependence of the elements of the force system.

Figure 285 shows the three bodies segregated as individual free bodies.

On body A, the external forces are the tension in the supporting cable T_1, and the weight of 150 pounds. The acceleration force is $ma = \frac{150}{32.2} a_A$ lb.

Then, $$\Sigma X_{EXT.} = \Sigma X_{ACC.}$$

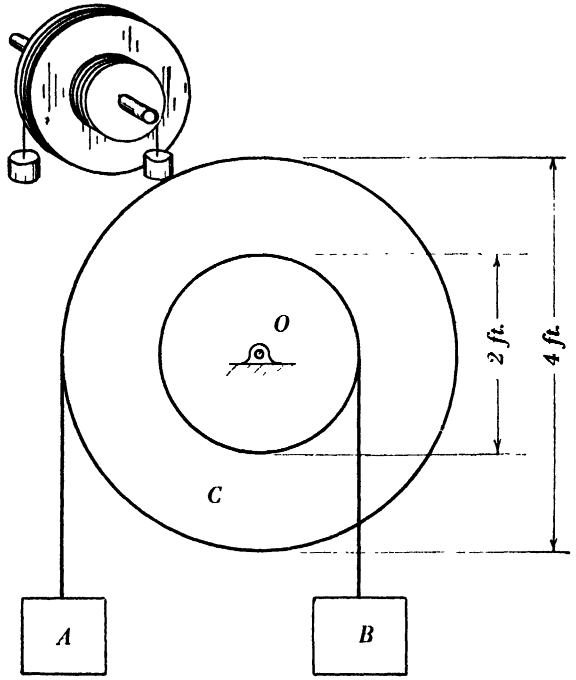


FIG. 284

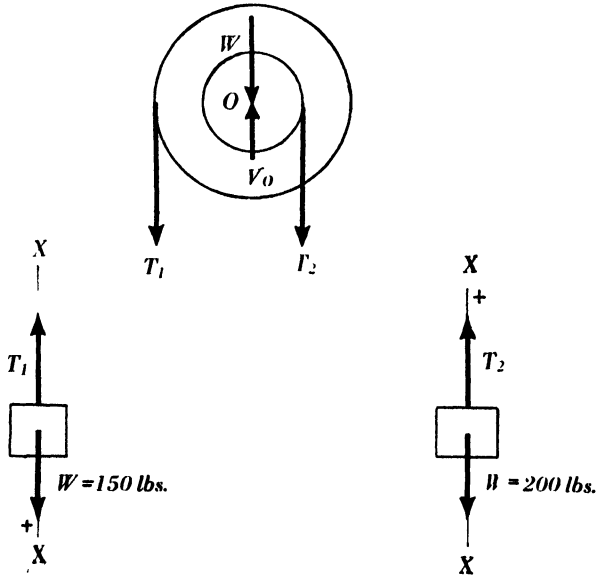


FIG. 285.

(As in all cases of translation, we have selected an X axis in the direction of motion, with the positive side of the axis as indicated.)

$$150 - T_1 = \frac{150}{32.2} a_A$$

Body B, the other translating body, has been similarly treated.

$$\Sigma X_{\text{EXT.}} = \Sigma X_{\text{ACC.}}$$
$$T_2 - 200 = \frac{200}{32.2} a_B$$

Body C having a motion of pure rotation about its center of gravity, yields

$$\Sigma M_{\text{EXT.}} = \Sigma M_{\text{ACC.}}$$
$$T_1 \times 2 - T_2 \times 1 = \alpha \frac{400}{32.2}$$

We have at present five unknowns: T_1, T_2, a_A, a_B, and α. From kinematics, we borrow the equation of relationship between angular and linear accelerations

$$a = \alpha r$$
$$a_A = \alpha \times 2(\text{ft. per sec.}^2)$$
$$a_B = \alpha \times 1 \text{ ft. per sec.}$$

Now, solving the three equations yielded by the three bodies simultaneously

$$150 - T_1 = \frac{150}{32.2} \times \alpha \times 2$$
$$-200 + T_2 = \frac{200}{32.2} \times \alpha \times 1$$
$$2T_1 - T_2 = \frac{400}{32.2} \alpha$$

We obtain

$$\alpha = 2.68 \text{ radians per sec.}^2$$

Then, $\omega_{5\text{sec}} = \omega_0 + \alpha t = 0 + 2.68 \times 5 = 13.40$ radians per sec.
$T_1 = 125$ lb.
$T_2 = 216.7$ lb.

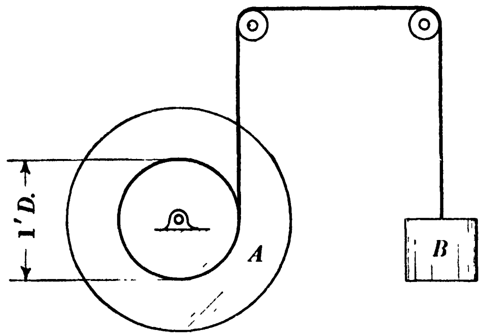
PROBLEMS

443. A cylinder weighing 161 lb. and having a diameter of 2 ft. rotates about its geometrical axis. What constant moment M must be applied to the cylinder to bring it up to a speed of 50 radians per sec. in 5 sec., starting from rest?
Ans. $M = 100$ ft.-lb.

444. The initial velocity of the wheel A is 5 radians per sec. counterclockwise. The moment of frictional resistance at the bearing of A is 10 ft.-lb. Wheel A weighs 400 lb. and has a moment of inertia about axis O of 600 lb.-ft.² Neglect the weight of the small pulleys and the friction at their bearings. Weight of $B = 200$ lb.

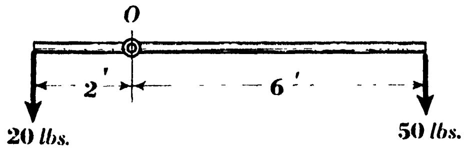
Determine the tension in the cable, and the speed of A at time $t = 10$ sec. after the start.

Ans. $T = 186$ lb.; $\omega = 49.5$ radians per sec.



PROB. 444

445. The rod shown revolves in a horizontal plane about a fixed axis at O. Determine its velocity at the end of 5 sec., if the rod starts from rest. Neglect friction at the axis. The rod weighs 80 lb.



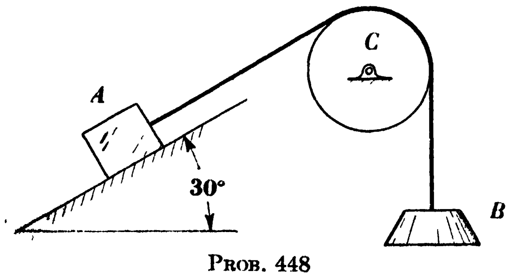
PROB. 445

446. If the rod given in Problem 445 has rotated for 5 sec., starting with an initial velocity of 10 radians per sec., determine the resultant force exerted by the axis on the rod.

447. A 10-ft. rod weighing 10 lb. per foot rotates in a horizontal plane about a vertical axis placed 2 ft. from one end of the rod. The angular speed of the rod is 120 r.p.m. Determine the maximum tension in the rod, and the pull which the rod exerts on the axis. Pull on axis is 1470 lb. *Ans.* $T_{max} = 1568$ lb.

448. Two bodies, A, weighing 300 lb., and B, weighing 200 lb., are connected by a cable running over pulley C, which weighs 100 lb. and is supported on fixed bearings. The moment of inertia of pulley C about its axis is 400 lb.-ft.^{2}, and its diameter is 4 ft. Determine the tension in the cable, and the velocity of each body at the end of 10 sec., if the system starts from rest.

Neglect the weight of the cable, and assume that all surfaces are frictionless.

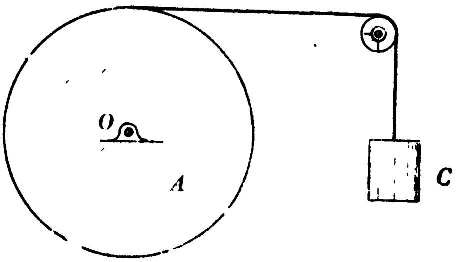


PROB. 448

449. Solve Problem 449, if the coefficient of friction between A and the inclined plane is $\mu = 0.25$.

450. A solid cylinder A, 4 ft. in diameter, is mounted on a fixed axis at its center O. The moment of inertia of the cylinder about O is 400 lb.-ft.^{2} and the moment of frictional resistance at its bearing is 8 ft.-lb. The weight of C is 100 lb.

If the system starts from rest, determine the tension in the cable, and the velocity of each body at the end of 2 sec. Neglect the weight of the small guide pulley. *Ans.* $T = 52$ lb.; $\omega_A = 15.4$ radians per sec.

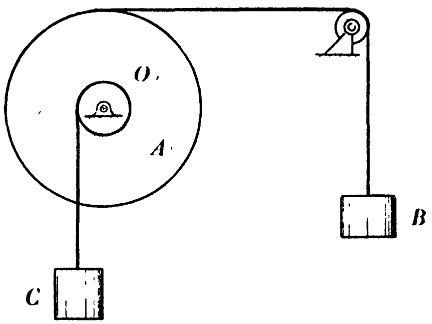


PROB. 450

451. The system shown starts from rest. The moment of inertia of wheel A about its axis is $I = 322$ lb.-ft.^{2} Body B weighs 48.3 lb., and body C weighs 64.4 lb. The moment of frictional resistance at the axis of the wheel is 60 in.-lb. The mass of the small pulley may be neglected. The inner diameter of A is 12 in.; the outer diameter is 48 in.

Determine the tensions in the connecting cables, and the horizontal and vertical components of the supporting force at axis O.

Determine the velocity of each body at the end of 3 sec.



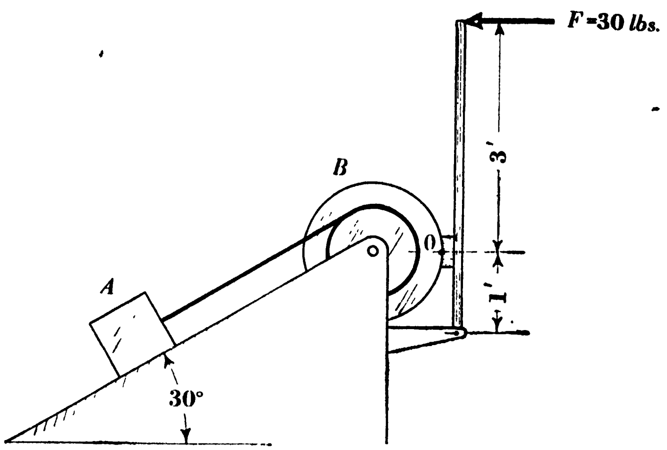
PROB. 451

452. The weight A has an initial velocity of 30 ft. per sec. down the inclined plane. A controlling brake is used to regulate the speed of A.

The weight of A is 161 lb., and the weight of drum B is 322 lb. The radius of gyration of B about its axis is 3 ft. The inner diameter of drum B is 3 ft.; the outer diameter is 4.5 ft.

Assume that the frictional force exerted by the brake is concentrated at point O, and that the friction between the plane and A may be neglected. The coefficient of friction between the brake shoe and drum B is $\mu = 0.3$.

If a force $F = 30$ lb. is applied, determine the distance that weight A will travel before stopping.



Prob. 452

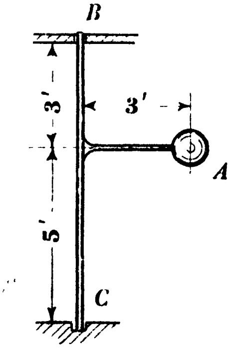
453. For the brake mechanism of Problem 452, determine what force F must be applied to bring the weight A to a stop in 3 sec., if its initial velocity is 10 ft. per sec.

454. A sphere A, 1 ft. in diameter and weighing 250 lb., rotates in a horizontal plane about the shaft BC. The sphere is attached to the shaft by means of a rigid horizontal arm. Speed of the system is 60 r.p.m.

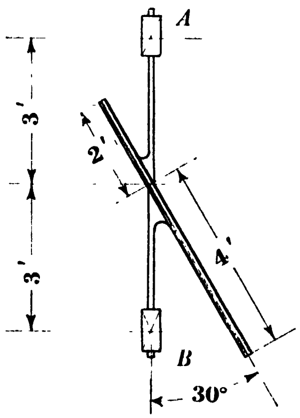
Determine the horizontal and vertical components of the forces exerted on the shaft at B and C.

Assume that the resultant force at B is horizontal. Neglect the weight of the shaft and horizontal arm, and the friction of the bearings.

Ans. $H_B = 670$ lb.; $H_C = 250$ lb.; $V_C = 250$ lb.



Prob. 454



Prob. 455

455. A 6-foot rod rigidly attached to a supporting vertical shaft rotates about the vertical axis AB at constant speed of 100 r.p.m. The rod weighs 10 lb. per ft.

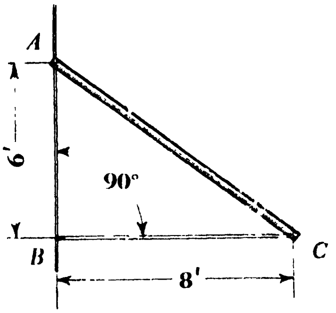
Determine the horizontal and vertical components of the forces exerted on the vertical shaft by the bearings at A and B, assuming that the resultant force at A is horizontal.

Neglect the weight of the vertical shaft and friction.

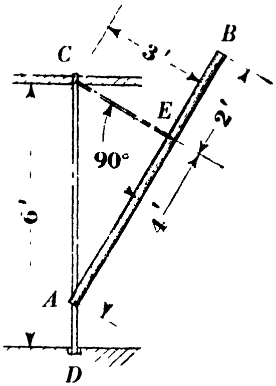
456. A slender rod AC weighing 180 lb. rotates at constant angular speed of 240 r.p.m. about vertical axis AB.

The rod AB is held in position by a rod BC, which rotates with AB. The weight of BC is negligible.

Determine the stress in BC. *Ans.* 9300 lb., tension.



PROB. 456



PROB. 457

457. A thin rod AB, weighing 5 lb. per ft., rotates about axis CD at constant angular speed of 4 radians per sec. Determine the stress in CE, and the horizontal and vertical components of the force exerted on the vertical shaft at C and D, assuming that the resultant force at C is horizontal.

The weights of CE and CD are negligible.

103. Simple Pendulum. A simple circular pendulum (Fig. 286) consists of a particle suspended at the end of a weightless cord, so that it swings, or vibrates, in a vertical plane with a path which is an arc of a

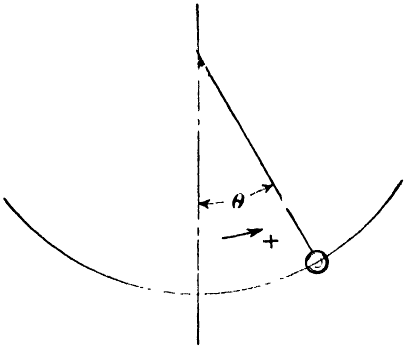


FIG. 286.

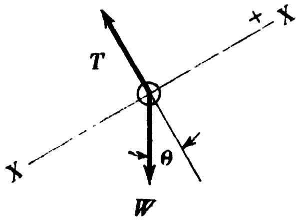


FIG. 287.

circle. If we isolate the particle as a free body, as shown in Fig. 287, the system of external forces acting on the free body consists of the weight W and the tension exerted by the cord T.

If we set our X axis in the tangential direction with the positive direction of displacement as indicated, we have

$$\Sigma X_{\text{EXT.}} = \Sigma X_{\text{ACC.}}$$
$$-W \sin\theta = \frac{W}{g} a_x$$
$$a_x = -g \sin\theta$$

When the angle through which the particle vibrates is restricted to a small angle, $\sin\theta = \theta$, and

$$a_x = \frac{d^2x}{dt^2} = -g\theta = -\frac{g}{r}\,dx$$

Comparing our result with the equation of simple harmonic motion ($a = -kx$), we note that such motion is simple harmonic, having the period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{g}{r}}} = 2\pi\sqrt{\frac{r}{g}}$$

and the frequency,

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{r}}$$

PROBLEMS

458. Two simple pendulums are geometrically similar, with the size of the larger equal to ten times that of the smaller. Determine the ratio of their periods.

459. If the period of a simple pendulum is 1.2 sec., determine its length.

460. The frequency of a simple pendulum is 0.7 cycles per second. Determine the length of the pendulum. *Ans.* 20 in.

104. Compound Pendulum. When a body which, unlike the simple pendulum particle, has finite dimensions (Fig. 288) and rotates in a vertical plane about a horizontal axis of rotation (point O), the body is known as a *compound pendulum.* In the position shown, the forces acting on the free body are its weight W and the supporting force at the axis. It is assumed that there is no friction at the axis.

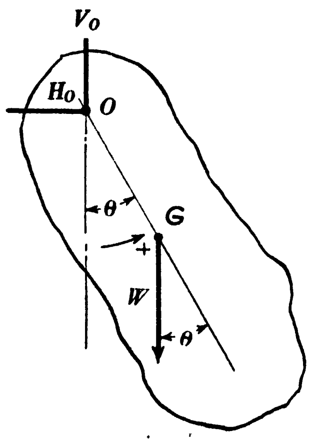


FIG. 288.

If we apply the equation of motion,

$$\Sigma M_{o\text{-EXT.}} = \Sigma M_{o\text{-ACC.}}$$
$$-W\,\bar{r}\sin\theta = I_o\,\alpha = \frac{\rho^2 W}{g}\,\alpha$$

in which ρ is the radius of gyration of the free body about axis O. The sign of the moment is taken as positive when it is in the same sense as angular displacement θ.

When the angular displacements are confined to small angles, $\sin\theta = \theta$, and

$$\alpha = -\frac{\bar{r}g}{\rho^2}\theta$$

which is of the form of a simple harmonic motion, with the period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{\bar{r}g}{\rho^2}}} = 2\pi\sqrt{\frac{\rho^2}{\bar{r}g}}$$

and

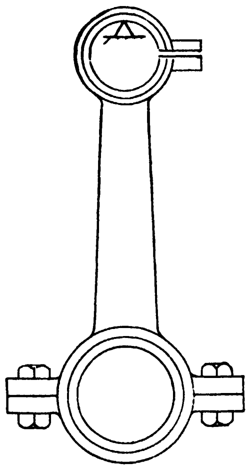
$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\bar{r}g}{\rho^2}}$$

PROBLEMS

461. A compound pendulum consists of a thin rod 20 in. long weighing 3 lb., and a cylindrical disk 8 in. in diameter weighing 6 lb. The pendulum swings about an axis at the end of the thin rod. Determine the frequency of the pendulum if the angle of oscillation is small. *Ans.* .645 cycles/sec.

462. A method of determining the moment of inertia of a body of awkward cross section makes use of the properties of the compound pendulum. For example, if the connecting rod shown is supported on a knife-edge, and given a small angle of oscillation, the period is timed as $T = 1$ sec.

The connecting rod weighs 6 lb. and the distance from the knife-edge to the center of gravity of the connecting rod is 6 in. Determine its moment of inertia relative to the knife-edge.

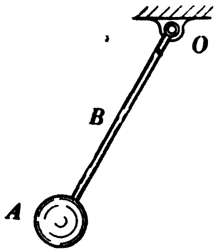


PROB. 462

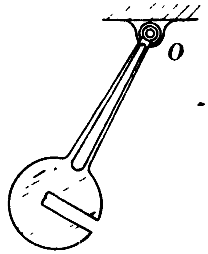
463. A machine part weighing 100 lb. is suspended from a knife-edge and allowed to oscillate as a compound pendulum. The observed period is 2.5 sec. Determine the moment of inertia of the machine part. The distance from the axis of oscillation to the center of gravity of the machine part is 3 in.

Ans. $I = 127.5$ lbs. ft.^{2}

464. The compound pendulum shown consists of a sphere A, weighing 50 lb. and having a diameter of 6 in., and a thin rod B, 2 ft. long, weighing 30 lb. and supported at axis O. Determine the period of the compound pendulum.



PROB. 464



PROB. 465

465. The pendulum of a Charpy impact testing machine, shown, weighs 50 lb., and the distance from the axis of suspension O to the center of gravity is 25 in.

If the pendulum is allowed to oscillate freely, it is observed to have a frequency of $\frac{1}{2}$ cycle per second. Determine its moment of inertia about the axis of suspension.

105. Torsional Free Vibration. The disk D of Fig. 289 is rigidly attached to a thin vertical rod of length l.

If the disk is given an angular displacement θ by a disturbing moment and then released, the disk will oscillate. The moment necessary to cause the oscillation is exerted by the elastic, slender supporting rod, in similar manner to the force exerted by the ideal springs of our previous discussions of free vibration. In this case, the torsional spring constant is k, which is the moment required to produce a unit angular twisting of the rod.

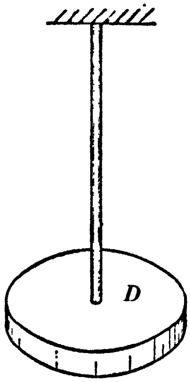


FIG. 289.

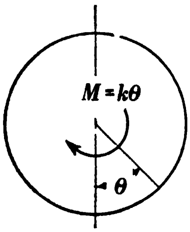


FIG. 290.

The disk is shown in plan view as a free body in Fig. 290.

The external moment exerted on the free body consists of the rod torque or moment $M = k\theta$. The sense of this moment is opposite to that of the angular displacement, θ.

Applying the equation of moment for a rotating body, we have

$$-M = -k\theta = \frac{I_W}{g}\alpha = \frac{I_W}{g}\frac{d^2\theta}{dt^2}$$

Then, $$\frac{d^2\theta}{dt^2} = -\frac{kg}{I_W}\theta$$

This differential equation is of the typical form we have previously found to be basic in harmonic motion. Then, its solution follows the pattern of the previous solutions of such an equation, and

$$\theta = \frac{\omega_o}{\omega}\sin\omega t + \theta_o\cos\omega t$$

in which θ is the total angular displacement, ω_o is the initial angular velocity, ω is the angular velocity of the rotating position vector of the auxiliary circle, and θ_o is the initial angular displacement

$$\omega = \sqrt{\frac{kg}{I_W}}$$

The period of oscillation will therefore be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I_W}{gk}}$$

To evaluate k, we must borrow from the subject of Strength of Materials the expression

$$k = \frac{\pi r^4 G}{2l}$$

in which r is the radius of the rod, l its length, and G is the modulus of elasticity in shear.

Then, $$T = 2\pi\sqrt{\frac{I_W}{g\frac{r^4G}{2}}}$$

PROBLEMS

466. A cylindrical disk weighing 16.1 lb., with a diameter of 24 in., is suspended from a thin steel rod as shown in Fig. 289. The rod is 24 in. long and 0.20 in. in diameter. The modulus of elasticity of steel in shear is 12×10^6 psi. Determine the period of the torsional vibration.

467. A cylindrical disk, having a diameter of 1 ft. and a weight of 40 lb., is suspended from a thin rod and allowed to oscillate as a torsional pendulum. The observed period is $T_1 = 1$ sec.

Another disk is suspended from the same shaft and similarly oscillated, and the observed period is found to be 1.2 sec. Determine the moment of inertia of the second body relative to the axis of the shaft.

106. Balancing of Rotating Masses. When a body, such as W of Fig. 291, is rotating at constant speed about an axis XX, there will be an acceleration force $ma = \omega^2\bar{r}\frac{W}{g}$ in which ω is the angular velocity of the rotating body and $\bar{r}$ is the distance from the axis of rotation to the center of gravity of the body.

Reactions such as F_1 and F_2 are set up at the bearings, which are due to the rotation and consequent normal acceleration of body W. Such reactions will rotate with the body, and their horizontal and vertical components will vary periodically. Such periodically varying forces will produce vibrations of the supporting framework which may become severe, particularly if the natural frequency of the support is the same as the frequency of the periodically varying reactions.

If the axis of rotation of the revolving mass could be made a principal axis of inertia, there would be no such development of pulsating force. In general, however, it is difficult to design rotating mass with the axis of rotation a principal axis of inertia, and we must turn to some means of producing balance.

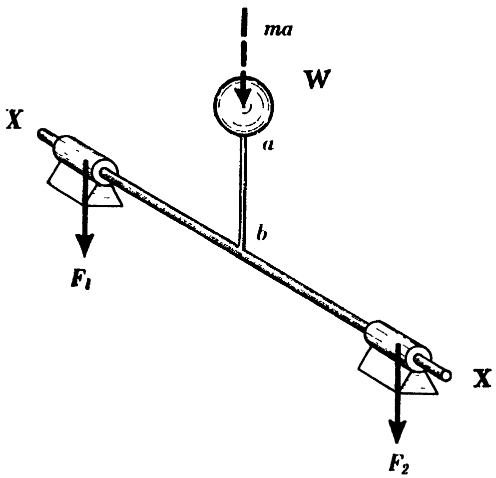


FIG. 291.

Such reactions as F_1 and F_2 of Fig. 291, which are due to the acceleration, are called *dynamic reactions*. The objective of dynamic balancing is to reduce such bearing reactions to zero. Of course, the only manner in which forces, such as these bearing reactions, may be reduced to zero, is to oppose them with equal and opposite forces.

The practice of dynamic balancing, therefore, consists of introducing additional rotating masses into the system. The additional bodies are of such magnitude and location that they will rotate with the system and set up bearing reactions which will oppose those caused by the acceleration forces of the original system. Then, at any angular position of the shaft, there will be dynamical balance.

In our illustrative examples, we shall consider only the dynamic reactions. We shall assume that the shaft is perfectly rigid and that weights of any supporting member, such as the crank ab of Fig. 291, is negligible.

W_1 (Fig. 292) is rotating about axis XX with angular velocity ω. The distance from XX to the center of gravity of W_1 is r_1. To effect dynamic balance, a weight W_2 is placed, as shown, in the plane of W_1 at distance r_2 from the axis of rotation. The acceleration force of W_1 is $\omega^2 \frac{W_1}{g} r_1$; that of W_2 is $\omega^2 \frac{W_2}{g} r_2$.

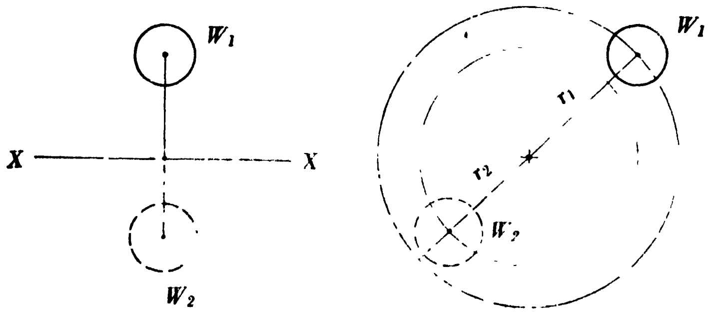


Fig. 292.

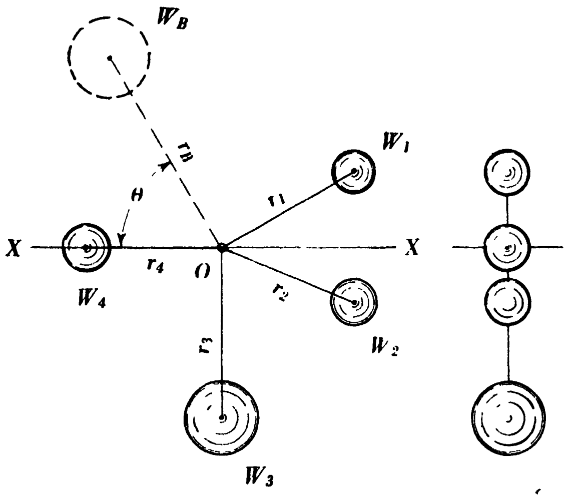


Fig. 293.

If these acceleration forces are equal, there will be dynamic balance for any angular position of the shaft.

Then,
$$\omega^2 \frac{W_1}{g} r_1 = \omega^2 \frac{W_2}{g} r_2$$
or
$$W_1 r_1 = W_2 r_2$$

It is necessary, therefore, only to insure that the product $W_2 r_2$ of the balancing weight be made equal to the product of the original weight times its distance from the axis ($W_1 r_1$.) Any combination of balancing weight and radial distance will effect balance if the *products* of weight and distance are equivalent.

When a group of bodies W_1, W_2, W_3, and W_4 are rotating in the same plane, as indicated in Fig. 293, their acceleration forces form a concurrent system of forces in a plane.

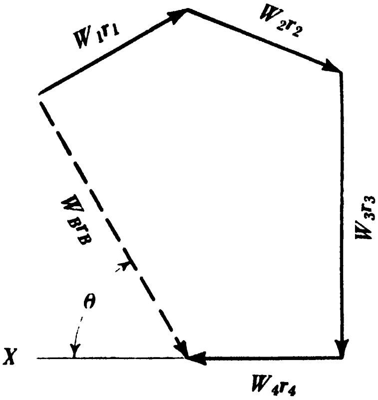


FIG. 294.

We recall, from our studies of statics, that such a system of forces yields, as resultant, a single force.

It is effective to reduce such a problem to a problem in statics by using D'Alembert's principle, and it is efficient to use the graphical attack on the problem of statics discussed in Chapter III.

Then a force polygon could be constructed, representing the reversed acceleration-forces, shown in Fig. 294. We note that the term $\frac{\omega^2}{g}$ appears in the value of each force, and is the same for all. There is no advantage, therefore, in carrying this term into a force polygon. Instead, we plot the polygon with products of $W \times r$ representing the reversed acceleration forces, which are drawn to scale.

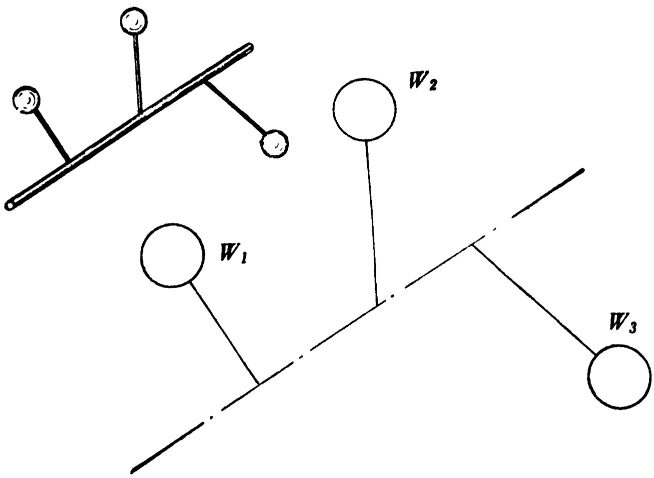


FIG. 295.

For equilibrium, force polygons must close. The closing side is W_Br_B. The inclination of this closing side with the horizontal is θ.

Then we conclude that any weight W_B may be placed along a line inclined at θ with axis XX of Fig. 293.

If this weight is so located at a distance r_B such that the product W_Br_B

is equal to the closing side of the polygon, there will be dynamic balance.

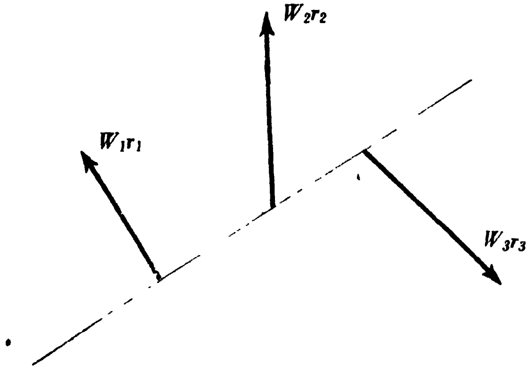


FIG. 296.

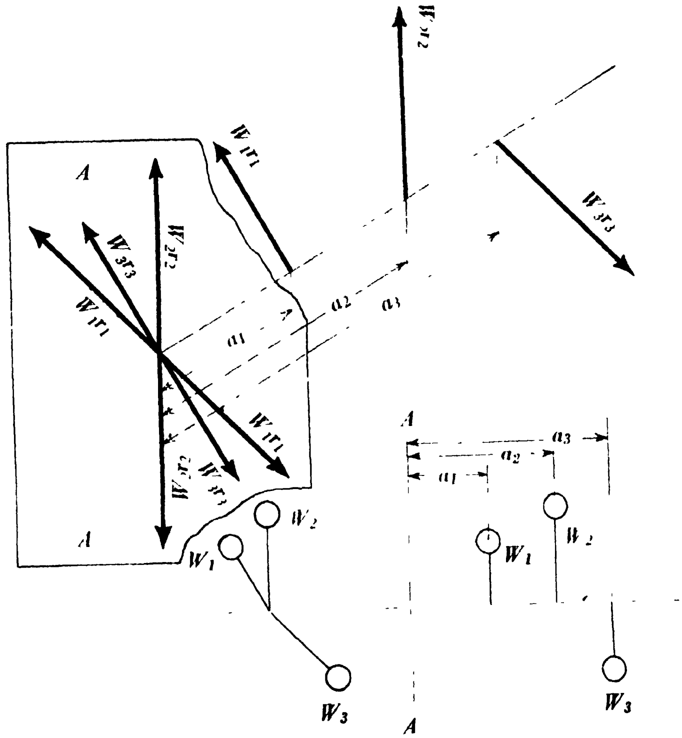


FIG. 297.

When the system of rotating bodies is not confined to a single plane, we have the condition illustrated in Fig. 295.

In this case, as in the preceding one, we can make use of D'Alembert's principle to reduce the problem to one of statics, and the reversed acceleration forces, which are to be balanced, are shown in Fig. 296. We note that we are now confronted with the general case of force system, for these forces act in different planes, and are neither concurrent nor parallel.

We shall again employ the technique discussed in Article 15, of resolving each force into a force and a couple. This resolution is shown in Fig. 297, where the forces are shown as a concurrent system in any plane of reference A-A. In addition, we have a system of couples containing $W_1r_1a_1$, $W_2r_2a_2$, and $W_3r_3a_3$. (It will be noted that the constant term

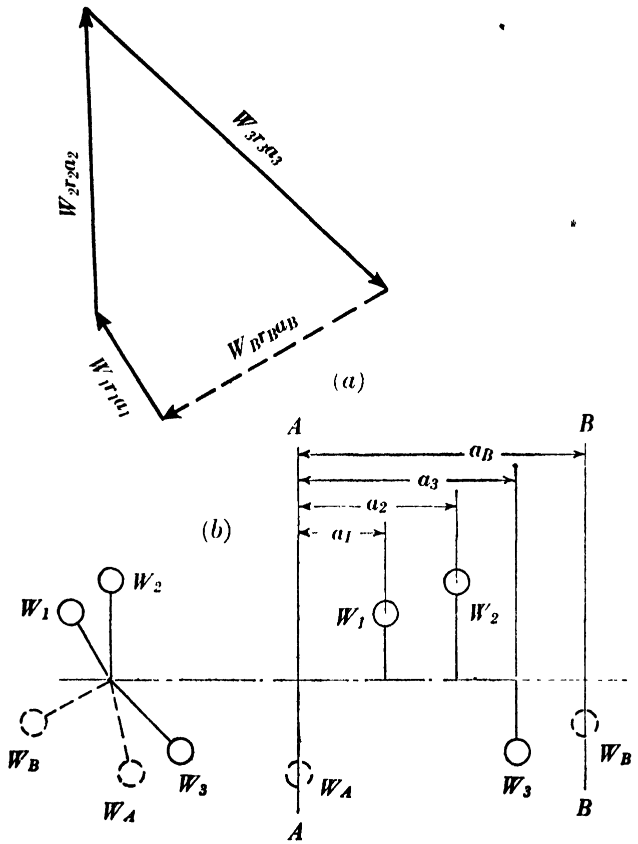


FIG. 298.

$\frac{\omega^2}{g}$, common to all of the forces and couples, has not been carried along since, as before, it serves no useful purpose.)

The original force system has now been reduced to a single force, acting in plane A-A, which is the resultant of the concurrent force system there; and a single resultant couple in a plane having an axis coincident with the axis of rotation.

The dynamic balancing of the system can be effected by adding two rotating bodies. First, we evaluate the effect of the resultant couple by constructing a vector polygon, known as the *moment polygon* and shown in Fig. 298*a*. The Wra products are laid off to scale in this polygon, using the right-hand-screw convention to indicate their senses. The closing side, $W_B r_B a_B$ of this polygon reveals the magnitude, inclination, and sense of the resultant couple of the original system.

If we select a transverse plane *B-B* (Fig. 298*b*) at any arbitrarily chosen distance a_B, and place a weight in that plane, and in an axial plane making an angle θ with the horizontal, selecting the product of weight W_B, radial distance r_B, and a_B equal to the product $W_B r_B a_B$ of the moment polygon, we shall have effected a dynamic balancing of the couple of the original system.

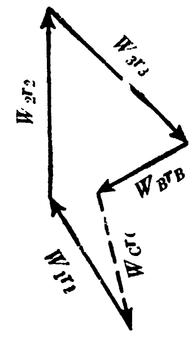


FIG 299.

This weight W_B, however, has not balanced the resultant of the system of concurrent forces in plane *A-A*.

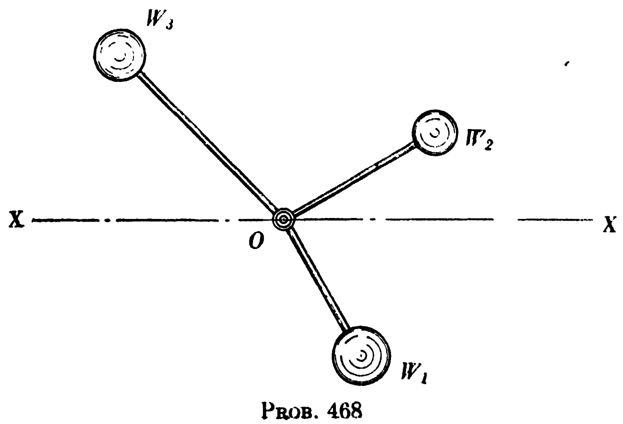
Now we construct a force polygon for the system of concurrent forces in plane *A-A*, plus the new force $W_B r_B$. This polygon is shown in Fig. 299. The closing side of the force polygon is $W_C r_C$. Therefore, we add in plane *A-A* a weight W_C located at radial distance r_C. Either W_C or r_C may be arbitrarily chosen, but their product must equal $W_C r_C$, the closing side of the force polygon. Since W_C has been placed in plane *A-A*, it will not disturb the previously accomplished balance of couples, for its moment relative to plane *A-A* is zero.

We have now effected complete dynamic balance of the system.

PROBLEMS

468. The three weights, $W_1 = 20$ lb., $W_2 = 12$ lb., and $W_3 = 16$ lb., rotate in the same plane, which is perpendicular to a horizontal axis at O. The radial distances to the centers of the weights are, $r_1 = 16$ in.; $r_2 = 18$ in.; $r_3 = 24$ in.

Determine the value of a balancing weight, and its inclination relative to the X axis which must be placed at a radial distance of 10 in. to produce dynamic balance.



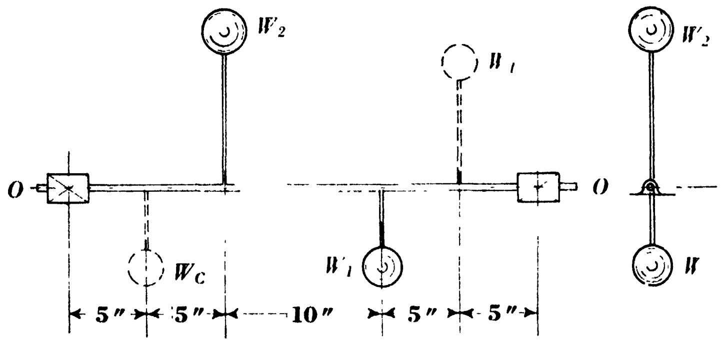
PROB. 468

469. If the system described in Problem 468 is to be put in dynamic balance by a weight W_B at radial distance of 12 in., determine W_B and the inclination of its axis.

470. If the system of Problem 468 is to be placed in dynamic balance by a weight W_B of 5 lb., determine its radial distance and the inclination of its axis from the X axis.

471. Two weights, $W_1 = 40$ lb. at $r_1 = 5$ in., and $W_2 = 50$ lb. at $r_2 = 10$ in., rotate about axis OO, as shown.

Determine the magnitude of W_C at $r_C = 5$ in., and W_D at $r_D = 8$ in., acting in the indicated planes to produce dynamic balance.

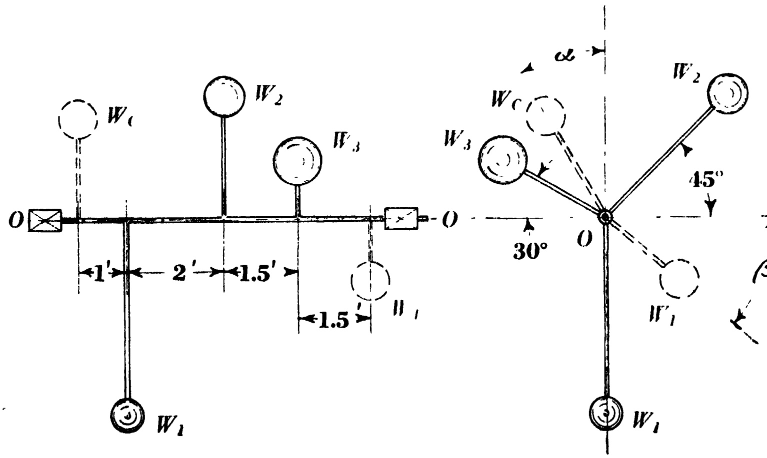


PROB. 471

472. Three weights, $W_1 = 20$ lb., $W_2 = 28$ lb., and $W_3 = 30$ lb., rotate about axis OO in the indicated planes $r_1 = 10$ in.; $r_2 = 9$ in.; $r_C = 6$ in.

The system is to be placed in dynamic balance by two weights, W_C and W_D, located in the planes shown at radial distances $r_C = 6$ in. and $r_D = 5$.

Determine the magnitudes of W_C and W_D, and angles α and β.



PROB. 472

107. General Plane Motion. Combined Translation and Rotation. In Article 80, we dealt with the kinematics of plane motion. We found then that all plane motions are capable of division into a translation plus a rotation of the free body. The cases with which we have already concerned ourselves in dynamics have been, in reality, special cases of this general concept. In the case of pure translation, the element of rotation was absent from the motion. In the case of rotation about a fixed axis, there was no element of translation.

In those cases, however, we have laid a foundation of equations of motion which we may use in combination when we analyze the general case of plane motion—that is, the case when the free body moves in a plane, so that its elements have combined motions of translation and rotation.

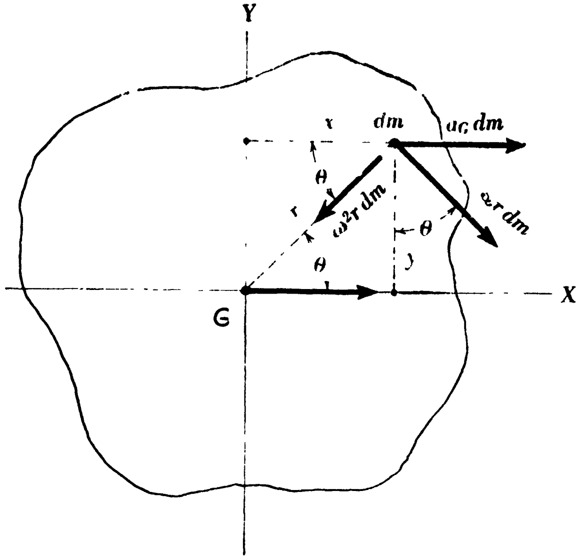


FIG. 300.

In the kinematical analysis of Article 80, we found that we could divide the combined motion into two parts,

1. A motion of rotation about any axis perpendicular to the plane of motion, and

2. A motion of translation of the axis selected as axis of rotation.

All of the particles of the body then had an absolute motion which was the sum of the contribution made by these two parts. For example, the free body of Fig. 300, which has plane motion, may be considered to have a motion of rotation about any point, plus the translation of that point. To illustrate, point G, the center of gravity of the body, has been selected to serve as the axis of the rotational element of motion. Then, any particle, such as dm, at distance r from G, will have acceleration-forces $\omega^2 r\,dm$, in which ω is the angular velocity of the body; and $\alpha r\,dm$, in which α is the angular acceleration of the body. In addition, the motion of translation of the point G and of all particles of the body, will give to dm an acceleration force $a_G\,dm$.

To summarize the effect of these acceleration-forces, an X and Y axis have been selected, as shown, parallel and perpendicular to the direction of a_G, respectively.

The sum of the X components of the acceleration-forces will be

$$\Sigma X = \int \omega^2 r\, dm \cos\theta + \int \alpha\, r\, dm \sin\theta + \int a_G\, dm$$

But $$\cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}$$

Then, $$\Sigma X = \int \omega^2 r\, dm \frac{x}{r} + \int \alpha\, r\, dm \frac{y}{r} + \int a_G\, dm$$

$\int x\, dm$ is the first moment of the body about a centroidal axis and $\int y\, dm$ is another first moment about a centroidal axis. Such first moments are equal to zero, and the first and second terms of the right side of the equation vanish, leaving

$$\Sigma X = m a_G$$

in which m is the total mass of the body and a_G is the acceleration of the center of gravity.

Summarizing the Y components of the acceleration forces, we have

$$\begin{aligned}\Sigma Y &= \int \omega^2 r\, dm \sin\theta + \int \alpha\, r\, dm \sin\theta \\ &= \int \omega^2 y\, dm + \int \alpha\, x\, dm = 0\end{aligned}$$

Then, $$\Sigma Y = 0$$

Now we turn to the consideration of moments, with G as moment axis,

$$\Sigma M_G = \int \alpha\, r^2\, dm + \int a_G\, dm\, r \sin\theta$$

But as before, $\int r\, dm \sin\theta = \int y\, dm$ and the second term of the right side of the equation vanishes, leaving

$$\Sigma M_G = \alpha \int r^2\, dm = \alpha I_G$$

The moment equation would have been similarly simplified had we chosen, as an axis of rotation and axis of moment, any point of the body which had zero acceleration or any point lying in the X axis selected.

Now we may establish our equations of motion for the general case of plane motion by equating external and acceleration-force systems.

$$\begin{aligned}\Sigma X_{\text{EXT}} &= m a_G{}^* \\ \Sigma Y_{\text{EXT}} &= 0^* \\ \Sigma M_G &= \alpha I_G\end{aligned}$$

Illustrative Problem

The cylinder (Fig. 301) weighing 300 pounds, is drawn up the plane by force P, which is parallel to the plane and equal to 200 pounds.

* It must be noted that in the derivation of these equations, the X axis has been selected in the direction of a_G.

There is pure rolling contact between cylinder and plane. The diameter of the cylinder is four feet.

Determine the velocity of the center of gravity G at the end of ten seconds, if the cylinder starts from rest and determine the coefficient of friction necessary to prevent slipping. The diagram of the cylinder, isolated as a free body, is shown in Fig. 302.

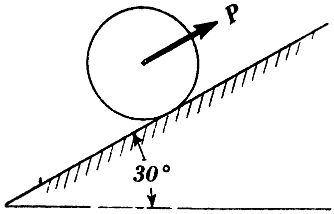


FIG. 301.

Since this free body is an example of the general case of plane motion, the equations of motion are

$$\Sigma X_{\text{EXT}} = m\,a_G; \qquad \Sigma Y_{\text{EXT}} = 0; \qquad \Sigma M_G = \alpha\, I_G$$

$$\Sigma X = +200 - 150 - FR = \frac{300}{32.2} a_x$$

$$\Sigma Y = +N - 259.8 = 0$$

$$\Sigma M_G = +FR \times 2 = +\alpha \frac{300 \times 4}{2 \times 32.2}$$

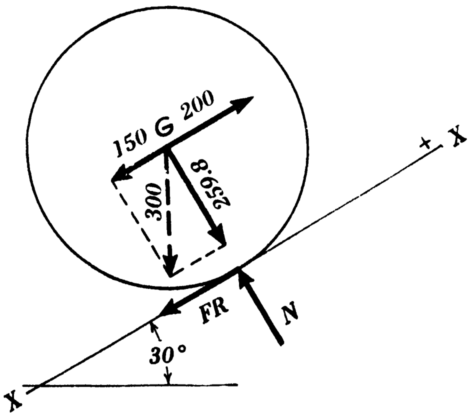


FIG. 302.

We note the presence of four unknowns; hence we must bring another simultaneous equation into the solution. From kinematics, we have

$$a_x = \alpha r$$

Now, solving the simultaneous equations

$$\Sigma X = 50 - FR = \frac{300}{32.2} \times 2\alpha$$

$$\Sigma M = 2 \times FR = \frac{600}{32.2} \alpha$$

and

$$\alpha = \frac{50 \times 32.2}{900}$$

$$a_x = \frac{50 \times 32.2}{900} \times 2$$

$$v_{10\,\text{sec.}} = v_o + a_x t = 0 + \frac{50 \times 32.2 \times 2}{900} \times 10$$

$$= 35.8 \text{ ft. per sec.}$$

From the same simultaneous equations,

$$FR = 16.67 \text{ lb.}$$

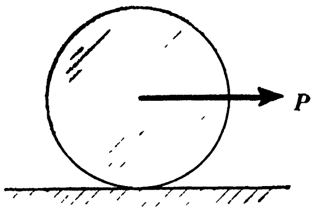
This is the frictional resistance necessary to insure pure rolling contact, and any lesser value will not prevent slipping.

From ΣY, $N = 259.8$ lb.

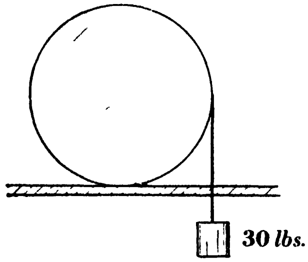
Then, $$\mu = \frac{16.67}{259.8} = 0.064.$$

PROBLEMS

473. The cylindrical disk shown weighs 322 lb., and has a diameter of 5 ft. If force $P = 150$ lb., determine the force of frictional resistance necessary to permit rolling without slip. Also determine the acceleration of the center of gravity of the cylinder. *Ans.* $FR = 50$ lb.; $a_G = 10$ ft./sec.^{2}



PROB. 473



PROB. 474

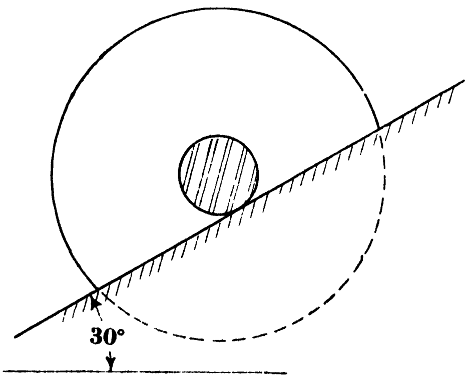
474. A 30-lb. weight is suspended from a line wrapped around a cylinder rolling on a horizontal track. The cylinder has a diameter of 4 ft. and weighs 161 lb.

If the system starts from rest and there is pure rolling contact between the disk and track, determine the angular velocity of the disk and the linear velocity of the weight at the end of 3 sec.

475. The wheel has a diameter of 3 ft., and its axle has a diameter of 8 in. The wheel weighs 161 lb., and its moment of inertia about its axis is 322 lb.-ft.^{2}

Determine the frictional force necessary to prevent slipping.

Ans. $FR = 60.4$ lb.



PROB. 475

476. If the wheel of Problem 475 starts down the plane with an initial angular velocity of 10 radians per sec., determine its angular velocity at the end of 10 sec.

477. Two cylindrical disks, A and B, are fixed to an axle C, as shown. Each disk weighs 40 lb., and has a diameter of 2 ft. The axle weighs 15 lb., and has a diameter of 6 in.

Force P = 15 lb. is applied parallel to the track.

If the wheels roll without slipping, determine the angular acceleration of the disks, and the linear acceleration of the central axis.

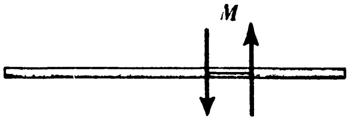
478. If the system given in Problem 477 starts from rest, find the angular velocity of the wheels at the end of 5 sec.

479. A solid cylinder weighing 1610 lb. is drawn along a horizontal plane without slip by a force F = 15[illegible] lb., applied at its axis, parallel to the track.

Determine the coefficient of friction necessary to prevent slipping.

480. How far will the cylinder of Problem 479 roll in 10 sec. if its axis has an initial velocity of 15 ft. per sec.?

481. A thin rod, weighing 48.3 lb. and 4 ft long, rests on a horizontal frictionless surface. If a couple M = [illegible]00 in.-lb. is applied to the rod for 2 sec., determine the angle through which the rod will rotate.



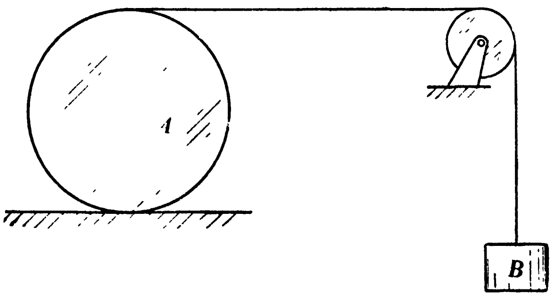
Prob. 481

482. What is the angular velocity of the rod of Problem 481 at the end of 1 sec.? The rod starts from rest.

483. A cylinder A, weighing 322 lb. and having a diameter of 2 ft., starts from rest and rolls on a horizontal plane as shown with pure rolling contact. The weight of block B is 100 lb. Neglect the mass of the small pulley and the friction at its bearings.

Determine the acceleration of the centers of gravity of A and B.

Ans. a_A = 7.3 ft./sec.^{2}; a_B = 14.6 ft./sec.^{2}

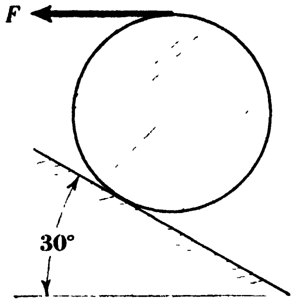


Prob. 483

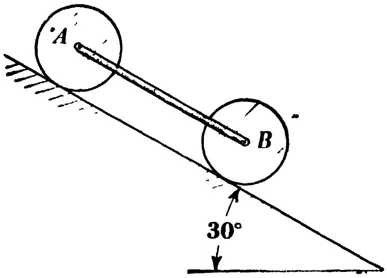
484. If the system of Problem 483 starts from rest, determine the velocity of the centers of gravity of A and B at the end of 10 sec.

485. The cylindrical disk shown weighs 322 lb. and has a diameter of 2 ft. The force F = 240 lb. is constant and is applied horizontally. Initial velocity of disk is 0.

Determine the velocity of the body at the end of 15 sec.



Prob. 485



Prob. 486

486. The rolling cylinders A and B are connected by a rod pinned to their centers. If the system starts from rest and the wheels roll without slip, determine the stress in the rod. Neglect the weight of the rod.

Weight of A = 300 lb. and weight of B = 600 lb. Diameter of A and B is 4 ft.

CHAPTER XII

Work and Energy

108. Introduction to Work and Energy. Newtonian law has served thus far to enable us to develop basic equations of motion. With such equations, we have been able to face many of the problems of dynamics—in particular, when the free bodies under consideration were assumed to be rigid. The study of nonrigid bodies encountered in the consideration of fluids and of gases may also make use of such equations of motion. In general, however, these studies are based upon considerations of energy. In addition, in many of the problems encountered in rigid-body mechanics, consideration of energy plays an important role.

The development of the principles of application of work and of energy may be based upon the same axioms of Newtonian law we have

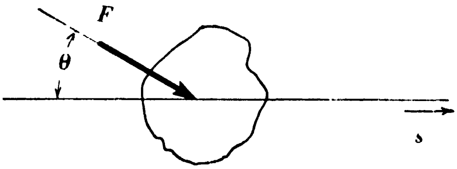


FIG. 303.

already used. Force, mass, and acceleration have been the quantities which, since they are directly involved in our equations, we have thus far studied. Acceleration also involves the consideration of displacement, velocity, and time.

When we establish a relationship between force and displacement we are dealing with work.

Work has already been defined (See Article 61) as the product of force and distance or, in more accurate detail, as the product of a force times the component of the displacement of the point of application of that force along the line of action of the force.

The presence of a force F acting upon a free body is indicated in Fig. 303. If F remains constant in magnitude as the free body is displaced through a distance s, the work done by F will be

$$W = F s \cos \theta$$

in which θ is the angle between the line of action of force F and displacement s.

If the displacement s is perpendicular to the line of action of F, θ will be equal to 90 degrees, and

$$W = F s \cos \theta = 0$$

When the displacement s is parallel to the line of action of the force, θ will be equal to zero, and

$$W = F s$$

When the force varies in magnitude, but its line of action is parallel to the displacement, we must consider the elements of work done in infinitesimal displacements ds. Now, the work W is equal to $F\,ds$, and the total work will be the sum of all elements of work, or

$$W = \int F\,ds$$

In this case, F must be expressed as a function of displacement s, so that the necessary integration may be performed. If the force acts,

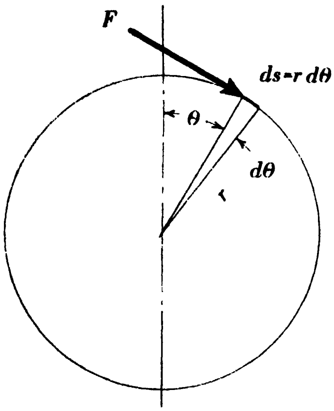


FIG. 304.

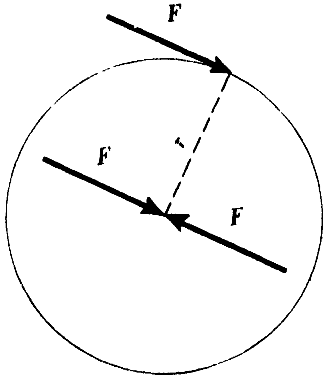


FIG. 305.

as in Fig. 304, remaining ever tangential to a rotating body, the work done by the force during an angular displacement is

$$W = \int F\,ds = \int F\,r\,d\theta$$

The product Fr is the moment of the force about the axis of rotation. If this moment M remains constant,

$$W = \int_0^\theta M\,d\theta = M\theta$$

When a tangential force such as that which is applied in Fig. 304 is exerted upon a body, it may be resolved into a force and couple. Such a resolution is illustrated in Fig. 305. The force F at the axis of rotation does no work during the angular displacement, because its point of application suffers no displacement.

A rotating couple, therefore, does work

$$W = \int F\,r\,d\theta = \int_0^\theta M\,d\theta = M\theta$$

during an angular displacement θ.

Work is a scalar quantity and is expressed in the direct units of the basic expression

$$W = Fs = \text{lb.} \times \text{ft. (or in.)} = \text{ft.-lb. (or in.-lb.)}$$

It is desirable to distinguish between the two types of work which contacting forces may do upon a free body. When the force and displacement are in the same sense, we describe the work as *positive*, or, sometimes, *effort*. An example of positive work is given in the case of the free body of Fig. 306. The body is moving to the right, and force P has sense to the right. Then $W = \int P\,ds$ is positive.

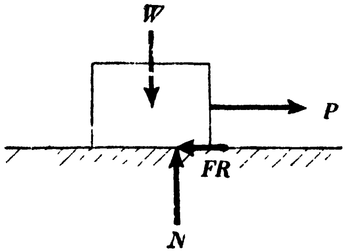


FIG. 306.

The force of friction FR has its sense opposite to the sense of the displacement. Such work as that done by FR is called *resistance* work. W, the weight of the body, and N, the normal force exerted by the plane surface, do no work, because there has been no displacement of their respective points of application.

When a body A is moved up an inclined plane such as that of Fig. 307 the force P is doing positive work; the component of the weight being

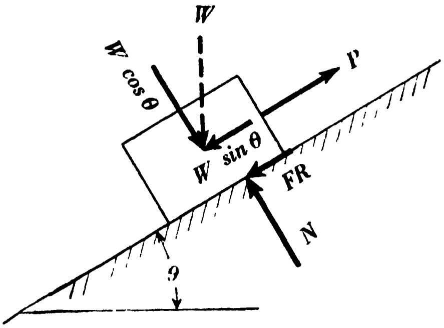


FIG. 307.

raised ($W \sin\theta$) is doing resistance work and FR is doing resistance work.

Such a distinction makes it possible to algebraically summarize the combined work done upon a free body when a system of external forces is acting. We call the terms representing positive work *positive*, those representing resistance work *negative*, and add algebraically to determine the combined effect. Such a technique of addition has already been employed in the chapter devoted to virtual work, when the involved displacements were infinitesimals. We shall also employ it in relating work and energy when the displacements are finite (Article 113).

109. Power. Power is defined as the rate of work with respect to time. In translation, power is, in accordance with the definition,

$$P = \frac{F\,ds}{dt} = Fv$$

when $\frac{ds}{dt}$ or v is the velocity of the body at the instant when power is being determined.

Power is a scalar quantity, and its units follow directly from the form of the defining equation

$$P = Fv = \text{lb.-ft. per sec.}$$

In most of the engineering applications involving discussion of power, it is convenient to use a larger unit of power than that above. *Horsepower* is the conventional unit and is defined as

1 horsepower = 550 lb.-ft. per sec. or 33,000 lb.-ft. per min.

If a force F acts on a body in the direction of the displacement of its point of application, the horsepower exerted by the force will, at any instant, be

$$\text{H.P.} = \frac{Fv}{550}$$

where v is the velocity of the point of application of the force.

The power involved in rotation similarly follows from the basic definition

$$P = \frac{M\,d\theta}{dt} = M\omega$$

where ω is the angular velocity in radians per unit of time. The horsepower exerted by a rotating couple, therefore, will be

$$\text{H.P.} = \frac{M\omega}{550}$$

where M is the moment of the couple in foot-pounds, and ω the angular velocity in radians per second.

110. Energy. The energy of a body is the measure of its ability to do work. Several forms of energy exist. A body may possess potential energy by virtue of its position or configuration; kinetic energy because it has velocity; thermal energy when its temperature is elevated and its molecules given molecular kinetic energy; chemical energy by virtue of its chemical composition. The ability of a body to do work is usually manifested when one of these forms is changed into another.

The first two forms of energy—potential and kinetic—are customarily grouped as mechanical energy, and it is with these forms that we are concerned in engineering mechanics.

111. Potential Energy. Potential energy of a body exists by virtue of the body's position. This may be the position of the particles of the body relative to one another (called the *configuration* of the body) or the position of the body itself in some field of force, like that of gravity or of electromagnetic action.

When an elastic spring is compressed or an elastic beam is bent or twisted, the particles of the body store potential energy, frequently called *strain energy*. When the spring is released or the bending or twisting loads removed from the beam, such potential energy is released.

If a weight is raised to greater elevation above the earth's surface, the weight will possess greater potential energy by virtue of its new position. (Actually, the potential energy in this case is a property of the earth as well as the elevated body. We assume, in engineering mechanics, that the earth is fixed, and attribute such potential energy to the elevated body.)

Such potential energies are dependent only upon the configuration or relative positions—in the case of the spring, of its particles; in the case of the elevated weight, of the relative positions of weight and earth.

The stress in the spring for a given configuration is the same whether one happens to be shortening or elongating the spring. The potential energy of the elevated weight is the same at a given distance from the earth's surface whether the body is rising or falling.

When, as in the above cases, the potential energy is dependent only upon the relative positions of the particles, or bodies, comprising a system, that system is called a *conservative* system, and the forces acting upon the particles or bodies are called *conservative forces*.

A nonconservative force depends upon the motion of the body, and is not always the same for the same configuration. For example, the direction of friction is always opposed to the direction of motion of a sliding body, and changes its direction when the direction of motion of the body changes. In our present study of fundamentals of engineering mechanics, the consideration of potential energy is important only in the case of conservative systems. In most of our problems in dynamics we shall find that for non-conservative systems, the property of kinetic energy plays the more important part.

There is one concept, applicable in conservative systems, which is worthy of our attention. We have, in our study of the principles of statics, defined equilibrium as a state of balance between opposing forces. A free body in equilibrium simply suffered no change of motion. In developing the conditions of equilibrium, we took pains to insure that the force systems, when combined to derive their total effect, would reduce to zero resultants. We can now enhance our observations of equilibrium by turning to the rod shown in Fig. 308. Three different positions have been indicated for pins serving as possible axes of rotation.

In the first case, with an axis at point A, a slight angular displacement could be given to the rod. If it is then released, the rod will return to its original position. Such equilibrium is called *stable equilibrium.* In the case of a pin at point B, serving as an axis, a slight angular displacement of the rod from the indicated position will cause it to increase the displacement, and we speak of the initial position as one of *unstable equilibrium.* If now we place the pin at C, which is its center of gravity, and again give it slight angular displacement, the rod will remain at the new position. Such equilibrium is spoken of as *indifferent* or *neutral equilibrium.*

In terms of potential energy, we note that in the first case, that of stable equilibrium, the particles of the rod were in such relative position

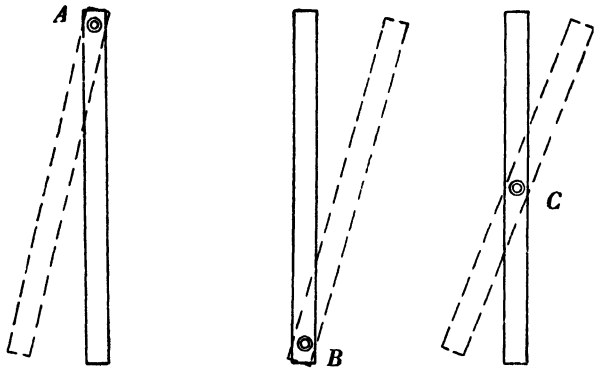


FIG. 308.

that the system of particles, or the rod, possessed a minimum of potential energy. In unstable equilibrium, conditions were reversed, and the particles so disposed that their system possessed a maximum of potential energy. In neutral equilibrium, the system of particles will possess the same potential energy in any position of the rod.

Then *minimum potential energy* is a criterion of stable equilibrium.

112. Kinetic Energy. Kinetic energy has already been defined as the capacity which a body possesses for doing work by virtue of its motion. The measure of motion involved here is the velocity of the body. We know that if a particle, or element of the body, dm, is acted upon by an unbalanced force F, the element will be given an acceleration a and some displacement ds. Since

$$F = dm\, a$$

the work done on the particle by force F will be

$$F\, ds = dm\, a\, ds$$

$$= dm \frac{dv}{dt} ds = dm \frac{ds}{dt} dv$$

$$= dm\, v\, dv$$

Then the work done on the elementary mass dm in increasing its velocity from rest to a velocity v to a final velocity as the displacement is changed from s_1 to s_2, will be

$$\int_{s_2}^{s_1} F\,ds = \int_0^v dm\,v\,dv = dm\frac{v^2}{2}$$

The left side of the equation represents the work done on the elementary mass. The right side is the kinetic energy of the elementary mass.

Work and energy are scalar quantities.

For an entire free body, then, the total kinetic energy is the arithmetical sum of the kinetic energies of the individual elementary masses comprising the body, or

$$\text{K.E.} = \tfrac{1}{2}\,m\,v^2$$

where m is the mass of the entire body and v is its velocity at the instant when K.E. is its kinetic energy.

The units of kinetic energy must be equivalent to those of work, for this energy is "stored" work. Then the units of energy are foot-pounds or inch-pounds.

In translation, all elements of the free body have the same velocity v; therefore,

$$\mathbf{K.E.} = \tfrac{1}{2}\,m\,v^2$$

where v is the common velocity.

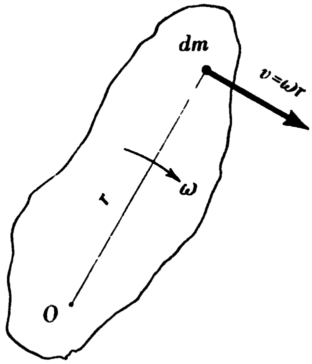


FIG. 309.

In rotation, all elements have different velocities. However, in rotation, the angular velocity of any line joining an element dm with the axis of rotation is ω (Fig. 309). All lines of the body have the same angular velocity ω.

The kinetic energy of dm is

$$\text{K.E.} = \tfrac{1}{2}\,dm\,v^2 = \tfrac{1}{2}\,dm(\omega r)^2 = \tfrac{1}{2}\,dm\,\omega^2 r^2$$

We note that the term $dm\,r^2$ is the moment of inertia of the element dm about the axis of rotation. Then,

$$\text{K.E.} = \tfrac{1}{2}\,I\,\omega^2$$

For an entire body, which is an assembly of such dm's, the kinetic energy will be the sum of the kinetic energies of the individual elementary masses, or

$$\begin{aligned}\mathbf{K.E.} &= \textstyle\int \tfrac{1}{2}\,dm\,r^2\\ &= \tfrac{1}{2}\,I\,\omega^2\end{aligned}$$

in which I is the moment of inertia of the entire free body about the axis of rotation, and ω is the angular velocity of the body.

When we consider the general case of plane motion, we recall that such motion may always be divided into two component parts: one component of rotation about any axis, plus a component of translation of that axis.

An elementary mass dm is shown at point A in Fig. 310. The velocity of the element is v_A. The body has plane motion, which may be divided into a rotation about G, the center of gravity of the body, and a trans-

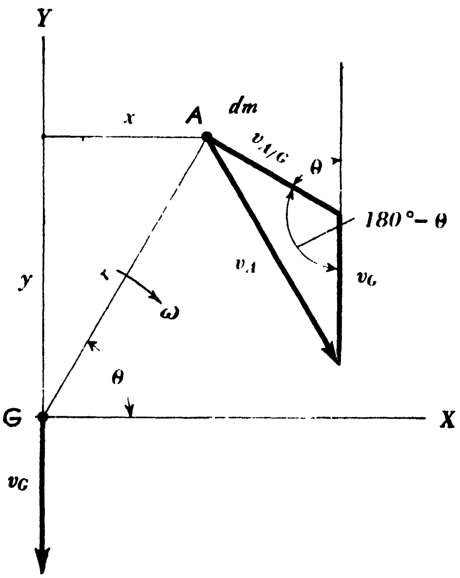


FIG. 310.

lation of point G. The velocity of G is v_G, and the angular velocity of the body is ω. Then,

$$v_A = v_G \nrightarrow v_{A/G}$$

The relative velocity of A with respect to G, $v_{A/G}$ is perpendicular to AG and has magnitude ωr. (See Article 81.) Then,

$$v_A{}^2 = v_G{}^2 + (\omega r)^2 - 2\, v_G\, (\omega r) \cos\, (180° - \theta)$$

The kinetic energy of dm is

$$\begin{aligned} \text{K.E.} &= \tfrac{1}{2}\, dm\, v_A{}^2 \\ &= \tfrac{1}{2}\, dm[v_G{}^2 + (\omega r)^2 + 2\, v_G\, \omega\, r \cos\, \theta] \end{aligned}$$

The kinetic energy of the entire body will be

$$\begin{aligned} \text{K.E.} &= \int \tfrac{1}{2}\, dm[v_G{}^2 + (\omega r)^2 + 2\, v_G\, \omega\, r \cos\, \theta] \\ &= \tfrac{1}{2}\int dm\, v_G{}^2 + \tfrac{1}{2}\int dm\, \omega^2 r^2 + 2\int dm\, v_G\, \omega\, r \cos\, \theta \end{aligned}$$

The third term contains $dm\,r \cos\theta$, which (Fig. 310) is equal to $dm\,x$. Then, the third term vanishes, for $\int x\,dm$ is a first moment of a body about a centroidal axis and is, therefore, equal to zero.

The expression for kinetic energy reduces, when the plane motion is resolved, into a rotation about the center of gravity and a translation of that center, to

$$\text{K.E.} = \tfrac{1}{2} m v_G^2 + \tfrac{1}{2} I_G \omega^2$$

113. Work vs. Energy Methods. The foregoing concepts of work and energy may be used to advantage in the problems of dynamics. They yield methods of attack which are of great value in problems involving variable forces, and they can operate with effectiveness in checking the results of attacks made upon problems in which the equations of motion have been used as tools.

Let us first summarize the conclusions reached in the foregoing article, as

$$\int F\,ds = m\frac{v_F^2 - v_0^2}{2}$$

The integral on the left side represents the total work done on the body by force F or, more completely, the total work done by the component of force F in the direction of the displacement ds. The right side of the equation represents the change in kinetic energy of a body of mass m as its velocity is changed from some initial velocity, v_0, to a final velocity, v_F. Then, a work-energy theorem may be established, as a codified expression of the equation.

The positive work, or effort, done upon a body has already been defined as the work done by the contacting forces acting on a free body which agree in sense with the sense of the displacement. The negative or resistance work has also been previously defined—it is the work done by those forces which have sense opposite to the sense of the displacement.

The $\int F\,ds$ is the sum of all work, both positive and negative, done on the body during the displacement. Then, our work-energy theorem becomes:

Positive Work − Resistance Work = Change in Kinetic Energy

and is abbreviated

P.W. − Res. W. = ΔK.E.

In the case of the sliding block A illustrated in Fig. 311, which is moving up the plane, we can appraise the force system by drawing the usual free-body diagram. Such an appraisal permits us to identify those forces doing positive and resistance work so that we may make proper entries in using the work-energy theorem.

The sense of P agrees with the sense of displacement: then P is doing positive work. FR, the friction force, is always of sense opposite to that of the displacement and is, therefore, doing resistance work. $W \sin \theta$ has sense opposite to that of the displacement, and is therefore doing resistance work.

Applying the work-energy theorem for displacement s,

$$\text{P.W.} - \text{Res. W.} = \Delta\text{K.E.}$$

$$\int_0^s P\,ds - \int_0^s FR\,ds - \int_0^s W \sin \theta\,ds = \Delta\text{K.E.} = \tfrac{1}{2}m(v_F^2 - v_0^2)$$

If the displacement of the block is directed down the plane, as illus-

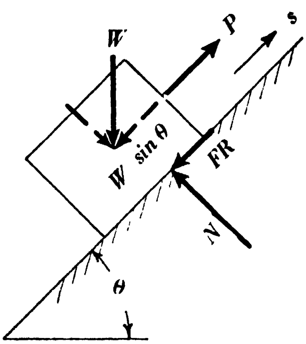


FIG. 311.

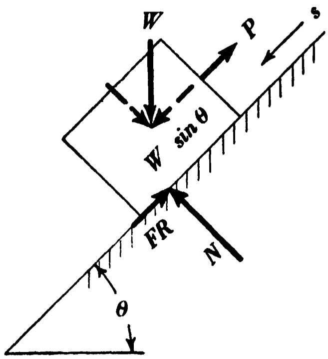


FIG. 312.

trated in Fig. 312, the application of the work-energy theorem yields

$$\text{P.W.} - \text{Res. W.} = \Delta\text{K.E.}$$

$$+ \int_0^s W \sin \theta\,ds - \int_0^s P\,ds - \int_0^s FR\,ds = \Delta\text{K.E.}$$

If the three forces in the above equation are constant, the statement becomes, in a displacement s,

$$W \sin \theta\, s - F\, s - FR\, s = \Delta\text{K.E.}$$

When either of the forces is a variable, it must be expressed as a function of displacement, in order that the necessary integration may be performed.

If an equation, such as $F = 4s^2 + 3s$ may be written to give the relationship, the integration may be performed analytically. In many cases which arise in the design of machines, no such equation of relationship is readily available. Frequently, we may establish such a relationship experimentally—as, for example, in taking an indicator card of a steam engine by means of an instrument specifically designed to draw a curve of relationship between force and displacement.

Our attack upon such problems may then be made if we can establish the $\int F\,ds$. In Fig. 313, a curve of force vs. displacement has been plotted. Then, the $\int F\,ds$ is the area under the curve and may be determined by measurement of area with instruments, such as planimeters, designed for that purpose, or by the method of graphical integration which was discussed in Article 74.

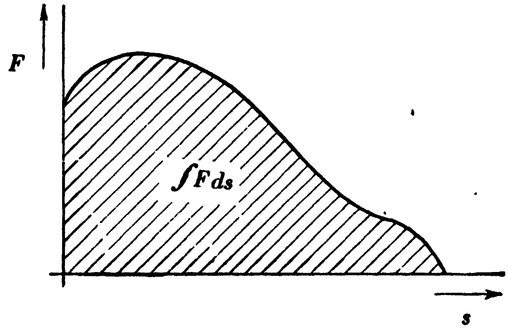


FIG. 313.

Another opportunity for effective use of the work-energy theorem is suggested in Fig. 314. The body AB rotates in a vertical plane about a horizontal axis at A. If body AB started its rotation from the position shown, with an initial angular velocity $\omega_o = 0$, it is required that we determine the angular velocity ω_θ when the line AB is inclined at angle θ with the horizontal. The pin at A is assumed to be frictionless. W

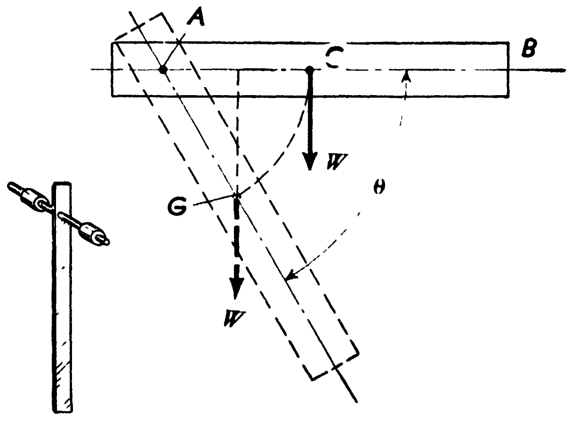


FIG. 314.

is the weight of the body, and no other external forces doing work are exerted on the body.

The free-body diagram is shown. Point G is the center of gravity. Then,

$$\text{P.W.} - \text{Res. W.} = \Delta\text{K.E.}$$

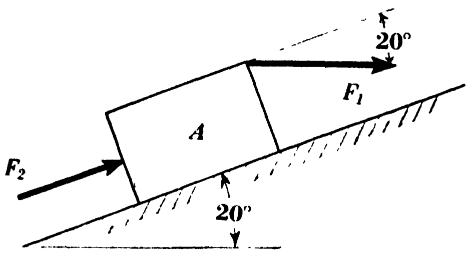
$$W \times AG \sin\theta - 0 = \tfrac{1}{2} I_A \omega_\theta^2$$

from which ω_θ may be determined.

Problems

487. Body A weighing 50 lb. is moved up an inclined plane by forces F_1 and F_2, equal to 40 and 80 lb., respectively. The frictional resistance is 15 lb. The body weighs 50 lb.

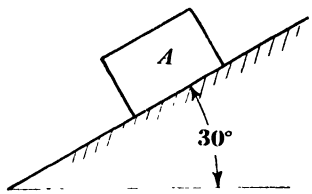
Determine the work done by each external force acting on the body.



Prob. 487

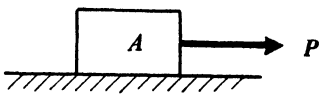
488. A car weighing 2000 lb. is traveling at a speed of 50 mph. Determine the kinetic energy of the car. *Ans.* $K.E. = 167{,}000$ ft.-lb.

489. The block A weighing 10 lb. is given an initial velocity of 8 ft. per sec. up the plane. The coefficient of friction $\mu = 0.10$. What is the kinetic energy of the block when it returns to its starting position?



Prob. 489

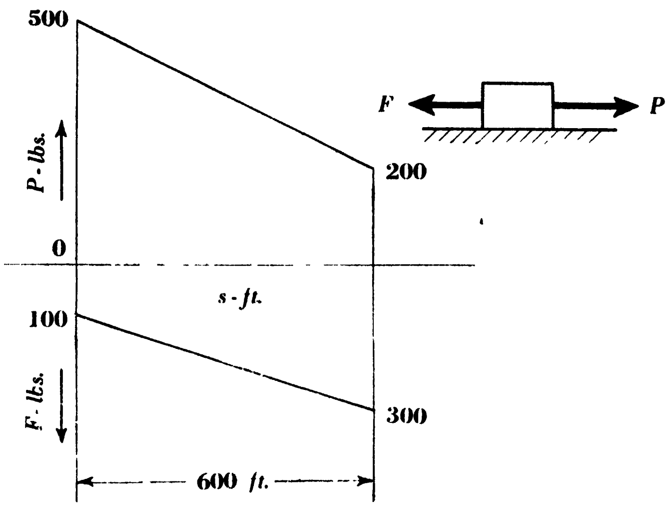
490. Body A weighing 2000 lb. is drawn along a horizontal plane by force P, which varies uniformly from 1200 to 1800 lb. in a distance of 90 ft. The coefficient of friction is $\mu = 0.20$. If the body starts from rest, determine the maximum horsepower exerted by force P.



Prob. 490

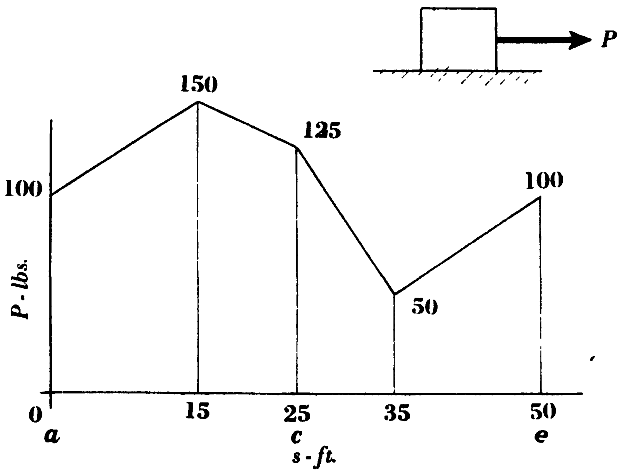
491. A body weighing 16,100 lb. is drawn along a horizontal plane by force P, which varies uniformly as shown. All resistances are represented by force F which also varies uniformly.

If the initial velocity of the body is 5 ft. per sec., determine the maximum velocity of the body during the 600-ft. distance, and the horsepower exerted by force P at the point of maximum velocity.



PROB. 491

492. A body weighing 966 lb. is drawn along a horizontal plane by a force P which varies as shown. The coefficient of friction $\mu = 0.10$ and the initial velocity of the body at point a is 10 ft. per sec. Determine the velocity of the body as it passes points c and e. Also determine the horsepower exerted by force P at point c. *Ans.* $v_c = 12.46$ ft./sec.; $v_e = 11.31$ ft./sec.; H.P. = 2.83.



PROB. 492

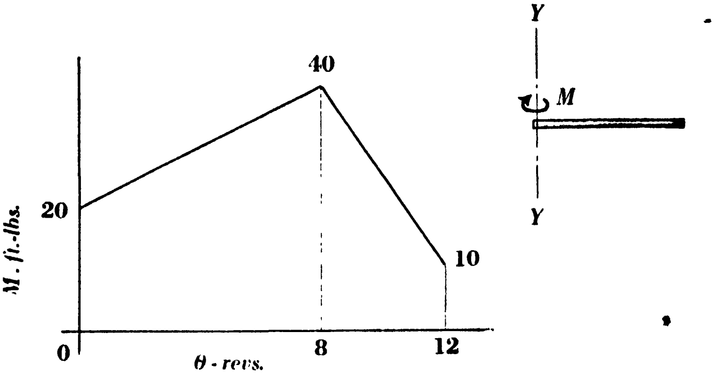
493. Determine the change in kinetic energy of body A of Problem 416, if the body has an initial velocity of 10 ft. per sec. and travels for 10 sec.

494. Determine the kinetic energy of body A of Problem 426 at the end of 3 sec., if the system starts from rest.

495. Determine the angular speed of wheel A of Problem 444, using the Work-Energy Theorem.

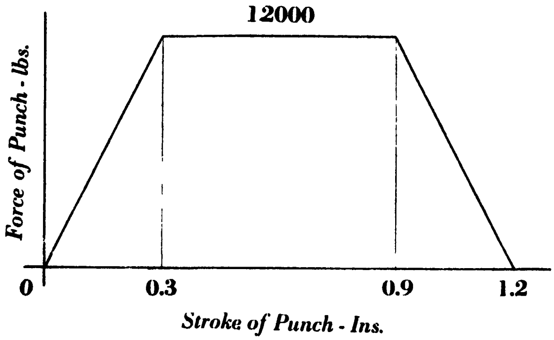
496. Determine the kinetic energy of the rod of Problem 447.

497. A rod, 20 in. long and weighing 15 lb., is rotated in a horizontal plane about the vertical axis YY by a couple M, which varies uniformly as shown. The initial velocity of the rod is 80 r.p.m. The moment of frictional resistance at the axis is 5 ft.-lb. If the total angular displacement is 12 revolutions, determine the kinetic energy of the rod and the horsepower exerted by M.



Prob. 497

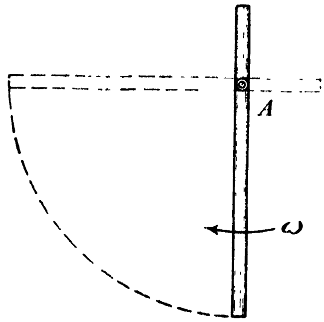
498. In a punch press, the force of the punch as it strikes the stock varies as shown. The punch is driven by a flywheel which has a moment of inertia about its axis of rotation of 300 lb.-ft.^{2} and a speed of 40 r.p.m. when the punch strikes. Assuming that all energy is supplied by the flywheel, determine its speed at the end of the cut.



Prob. 498

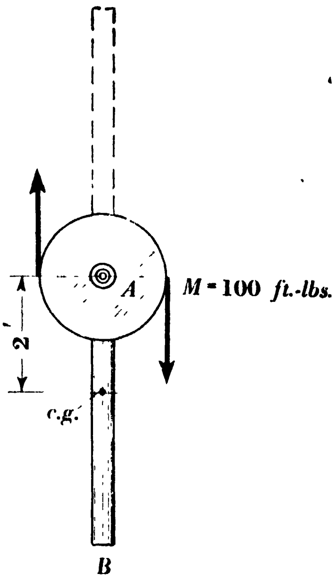
499. The reduction in kinetic energy of a flywheel is 100 ft.-tons while the speed is reduced from 60 to 50 r.p.m. How much will the kinetic energy be increased if the speed is increased from 60 to 65 r.p.m.? *Ans.* 57.8 ft.-tons.

500. A straight rod weighing 400 lb. is supported on a horizontal axis at A. The angular velocity of the rod when it is in the vertical position shown is $\omega = 5$ radians per sec. clockwise. Determine the angular velocity of the rod as it passes the horizontal position indicated by the dotted outline.



Prob. 500

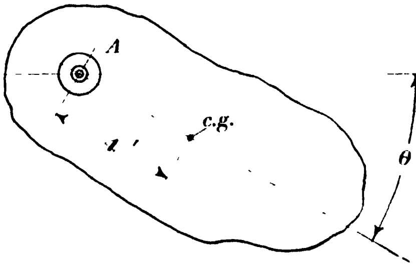
501. The rod AB weighs 64.4 lb., and is rotated by a constant couple $M = 100$ ft.-lb. If friction is neglected and the rod starts from rest in the vertical position shown, determine the speed of the rod as it passes the upper vertical position indicated by the dashed outline. The moment of inertia of the rod about the axis at A is 322 lb.-ft.^{2} *Ans.* $\omega = 3.36$ rads./sec.



PROB. 501

502. The body shown rotates about a horizontal axis at A, and starts from rest when $\theta = 0$.

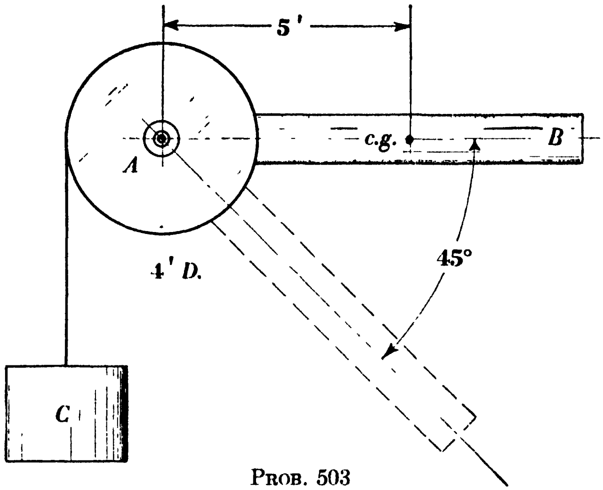
When $\theta = 30°$, determine the angular acceleration and the angular velocity of the rod. Weight is 100 lb.; moment of inertia about axis at A is 3220 lb.-ft.^{2} *Ans.* $\alpha = 3.46$ rads./sec.^{2}; $\omega = 2$ rads./sec.



PROB. 502

503. The body AB supports a 50-lb. weight C by means of a rope. AB weighs 100 lb., and its radius of gyration relative to the horizontal fixed axis at A is 6 ft.

If AB starts from rest in the horizontal position shown, determine its angular acceleration and angular velocity as it passes the 45° position indicated by the dashed outline.



PROB. 503

504. The weight A starts from rest in the horizontal position shown and swings down to the indicated 60° position. The center of gravity of A is 4 ft. from the horizontal axis of rotation at O. A weighs 322 lb., and its moment of inertia about O is 16,100 lb.-ft.² B weighs 200 lb.

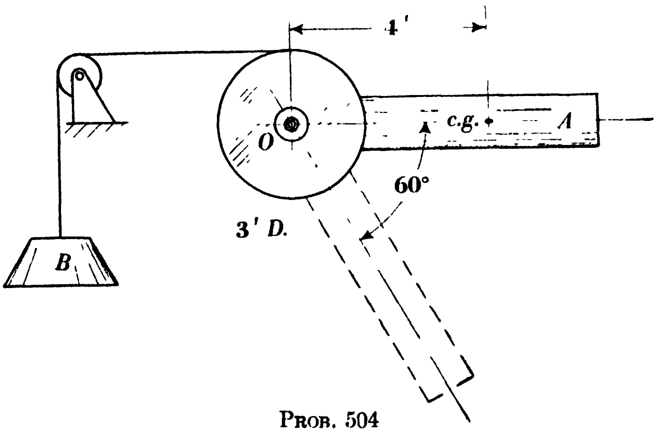
Determine, for the 60° position, the following:

(a) Tension in the cable.

(b) Kinetic energy of bodies A and B.

(c) The horizontal and vertical components of the force exerted on body A at the axis of rotation.

The weight of the small pulley, and all bearing friction is negligible.



PROB. 504

505. Determine the kinetic energy of the disk of Problem 473 at the end of 5 sec., if its initial velocity is zero.

506. Determine the kinetic energy of the cylinder of Problem 480 at the end of 10 sec.

507. Determine the kinetic energy of the system of bodies given in Problem 483 at the end of 10 sec. *Ans.* $K.E. = 73{,}000$ ft.-lbs.

CHAPTER XIII

Impulse and Momentum

114. Linear Impulse and Momentum. Impulse-Momentum Theorem. The basic Newtonian law may be expressed in terms of velocity as

$$F = m\,a = m\,\frac{dv}{dt}$$

If we separate the variables, we shall have

$$F\,dt = m\,dv$$

The term $F\,dt$, which presents us with a measure of the time effect of force, is called the *linear impulse* of the force. The term $m\,dv$ appearing on the right side of the basic equation is called the *linear momentum* of the mass. Both linear impulse and linear momentum, like the force and velocity which are their source, are vector quantities. Their units are the units directly obtained from the definition.

$F\,dt$ is the product of a force and time, and the units of impulses are, therefore, the product of pounds and seconds, or *pound-seconds*. $m\,dv$ is the product of mass and velocity. Then,

$$m\,dv = \frac{W}{g}\,dv$$
$$= \frac{\text{lb.} \times \text{ft./sec.}}{\text{ft. sec.}^2} = \text{lb. sec.}$$

These fundamentals—impulse and momentum—play an important role in dynamics problems, particularly in problems involving the flow of fluids, such as the action of a steam jet on the blades of a turbine or the action of an airplane propeller on the slip stream of air.

In other cases, when the available data present us with information as to the variation of force with respect to time, we again have opportunity to make a more effective attack than with the basic Newtonian law. Such an attack is particularly effective when the variable force is exerted over a very small interval of time, as in the case of collision of two bodies, the discharge of a projectile from a gun barrel, the motion of a rocket, or jet propulsion.

Integrating the basic equation given above for a time interval $t_2 - t_1$ during which the velocity of the mass changes from v_1 to v_2, we obtain the statement which we shall call the *Impulse-Momentum Theorem*.

$$\int_{t_1}^{t_2} F\,dt = \int_{v_1}^{v_2} m\,dv = m(v_2 - v_1)$$

Since linear impulse is a vector quantity, the component of such impulse along any axis is the vector sum of the components of the individual linear impulses along that axis contributed by the individual forces. The total linear momentum of a body is the sum of the linear momentums of all of the elementary masses comprising the body.

We have already found that any plane motion of a free body may be divided into two component motions: one of translation and one of rotation.

In the discussion of such division of motion into its constituent parts, we found it most convenient to select the center of gravity as an axis of rotation. (See Article 107.) Then the velocity of translation common to all particles at any instant is the velocity of the center of gravity.

It follows that the linear momentum of any body is the product of the mass of the entire body multiplied by the velocity of the center of gravity. Expressed mathematically,

$$\Sigma_{\text{linear momentum}} = \int dm\, v_G = m\, v_G$$

in which m is the total mass of the body and v_G is the velocity of the center of gravity. This form of momentum will be the entire momentum of the body if its plane motion is confined to translation.

115. Angular Momentum and Impulse. When in addition to the linear momentum of translation discussed in the preceding article, rotation is present, the particles of the body have additional velocity and, accompanying the velocity, additional momentum. Since the momentum of a particle is a vector quantity, it may be treated as are all other vector quantities.

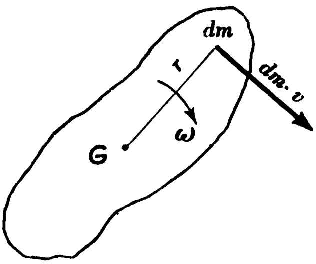


Fig. 315.

For example, the elementary mass dm of Fig. 315 has velocity v due to its rotation about center of gravity G. (It should be recalled that in the discussion of the preceding article, we elected to make use of the *center of gravity* as an axis of rotation in dividing plane motion into translation and rotation.) The momentum of the particle due to the rotation is, then, $dm\, v$.

A vector representing this momentum is shown in the figure. The distance from G to dm is r, and $dm\, v = dm\, \omega\, r$. The moment of this localized vector about the axis of rotation at G is $dm\, \omega\, r^2$, when ω is the angular velocity of the body. $dm\, \omega\, r^2$ is, then, a *moment of momentum*, and is called the *angular momentum*. From the form of the term, we note that the units of angular momentum are

$$dm\, \omega\, r^2 = \frac{dW}{g}\, \omega\, r^2 = \frac{\text{lb.} \times \text{radians} \times \text{ft.}^2}{\text{ft./sec.}^2 \times \text{sec.}}$$

$$= \text{lb. ft. secs.}$$

A body composed of such elementary masses will have a total angular momentum, which may be found, as in the case of any system of moments lying in the same or parallel planes, by addition of the individual moments. Then, the angular momentum of a body is

$$\Sigma_{\text{angular momentum}} = \int dm\, \omega\, r^2$$
$$\omega \int r^2\, dm$$

for ω is the same for all of the elementary masses and, therefore, appears as a constant in our integration.

The integral $\int r^2\, dm$ is the moment of inertia of the entire body about the axis of rotation at its center of gravity. Then,

Angular Momentum $= I_G\, \omega$

It should be observed that this equation has been simplified (as in Article 107) by referring the rotational element of plane motion to an axis of rotation at the center of gravity of the body. The expression would be similarly simplified if the instantaneous axis of velocities of the body were to be used as an axis of rotation.

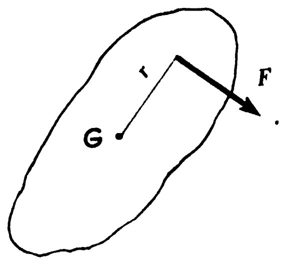


FIG. 316.

Angular impulse is defined as the moment of the linear impulse, or impulse of a force, about the axis of rotation. Then, any force, such as F of Fig. 316, will have an impulse $F\, dt$, and its moment about an axis of rotation at G will be $F\, dt\, r$, or $M\, dt$, where M is the moment of the force with respect to G.

The Impulse-Momentum Theorem, in the case of rotation, may be expressed in the same manner as its linear counterpart. Starting with the Newtonian law expressed for rotation (in Article 101) as

$$M = I_G\, \alpha$$
$$= I_G \frac{d\omega}{dt}$$

Separating the variables, as before, and integrating, we have,

$$M\, dt = I_G\, d\omega$$

and for a time interval $t_2 - t_1$ during which the angular velocity of the body changes from ω_1 to ω_2, we have, upon integrating

$$\int_{t_1}^{t_2} M\, dt = \int_{\omega_1}^{\omega_2} I_G\, d\omega = I_G(\omega_2 - \omega_1)$$

116. Location of Resultant Linear Momentum. The resultant linear momentum of a body in plane motion is $M\, v_G$, where M is the total mass of the body and v_G the linear velocity of its center of gravity G. The total moment of momentum has been found to be $I_G\, \omega$.

These vector quantities may be combined by the technique which we employed in statics as we considered the analogous combination of two vector quantities: the force and couple.

In Fig. 317, the linear momentum of the body is $M\,v_G$, and its moment of momentum is $I_G\,\omega$.

The resultant linear momentum will act at a distance from G which is

$$r_M = \frac{I_G\,\omega}{M\,v_G} = \frac{\rho^2 M\omega}{M\,v_G} = \frac{\rho^2\omega}{v_G}$$

in which ρ is the radius of gyration with respect to the axis of rotation at G.

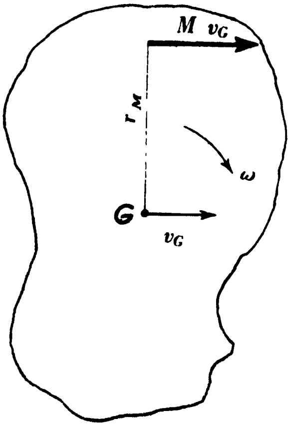


FIG. 317.

This resultant linear momentum would become a constant if the system of external forces acting upon the body is in equilibrium, for in

$$\int_{t_1}^{t_2} F\,dt = M(v_2 - v_1)$$

the left side of the equation would become equal to zero. Then,

$$M(v_2 - v_1) = 0$$

and the linear momentum of the body remains constant.

This equation expresses the principle of the *conservation* of *momentum*, which states: *If the resultant of the external forces which act upon a free body is equal to zero, the linear momentum of the body remains constant.*

Such a statement is of great value for it involves no demand that our free body remain rigid. Therefore, it furnishes a powerful tool with which to attack problems of dynamics which are concerned with free bodies comprising nonrigid systems of elementary masses or particles.

The equivalent statement, applied with respect to angular momentum would be

$$\int M\,dt = 0$$

Then,
$$I_G(\omega_2 - \omega_1) = 0$$

and again there is no change of angular momentum.

For example, the small cylinder A of Fig. 318 is free to slide along the frictionless surface of rod R, which is rotating about a fixed axis at G with angular velocity ω. If the system containing cylinder A and rod B is isolated as a free body, we note that it is non-rigid, for A may change its position relative to G. As A slides outward, the moment of inertia of the system relative to the axis of rotation at G is increased.

Since no external forces are acting on the free body to produce moment about G, the angular momentum of the system is, by the principle of the conservation of momentum, a constant. With I_G increasing, the angular velocity ω must decrease. If A is forced to slide inward along the rod, I_G will decrease and ω must increase.

117. Impact. Another opportunity for the use of the principle of the conservation of momentum arises when the collision of two bodies is to be studied. Impact is defined as the impulse of a force $\int_{t_1}^{t_2} F\,dt$ when the time interval $t_2 - t_1$ is extremely short.

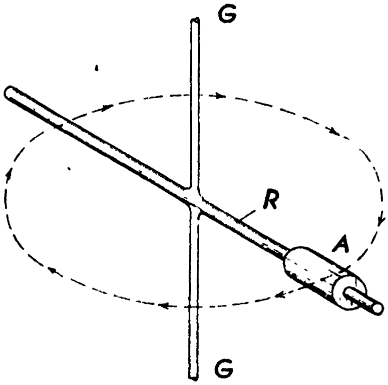


FIG. 318.

Since the time element is so small, the nature of variation in the force of reaction between the colliding bodies cannot be exactly determined, even though high-speed photography has revealed many of its features.

We are, in general concerned with the effect of the impact upon the velocities of the bodies after collision. Other forces than those exerted by the colliding bodies upon each other are usually very small in comparison with the impulsive force, or impact, and may be neglected.

In Fig. 319, we note that spheres A and B of weight W_A and W_B, respectively, are moving so that the absolute velocities of their centers of gravity before collision are v_A and v_B, which lie on an axis joining the centers of the spheres. Then, the impact will lie along this axis. Such impact is called *direct central impact.*

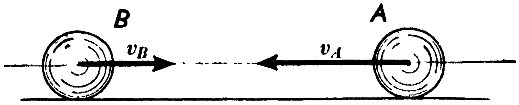


FIG. 319.

The system chosen as free consists of the two spheres. Such a free body is non-rigid. Then, we base our attack upon the principle of the conservation of momentum. The force of impact is internal to the system, and all external forces have been assumed to be negligible.

In accordance with the principle, the momentum of the system is constant, before and after impact. If we call the velocities of the bodies after impact v_{A_1} and v_{B_1}, respectively,

$$\frac{W_A}{g} v_A \leftrightarrow \frac{W_B}{g} v_B = \frac{W_A}{g} v_{A_1} \leftrightarrow \frac{W_B}{g} v_{B_1}$$

This equation does not segregate the terms v_{A_1} and v_{B_1} and we must bring other and simultaneous information into action.

Such information, experiment has shown, depends upon the materials of which the bodies are made.

The period of impact consists of a period of compression, during which the bodies deform, and a period of separation, during which the bodies recover from the deformation as they separate. In the case of perfectly inelastic bodies there would be completely permanent set or deformation —in the case of perfectly elastic bodies, there would be no permanent set. In actual bodies the deformation lies between these two limits.

During the compression, the velocity of A, v_A will change to some value, which we shall call v, at the instant of greatest compression.

At that instant B will also have velocity v.

As recovery of their shape takes place, the velocity of the bodies will approach v_{A_1} and v_{B_1}, and when they finally do separate, those values will be their velocities.

The ratio of the relative velocities of the bodies before and after impact is expressed as a coefficient,

$$e = -\frac{v_{A1} - v_{B1}}{v_A - v_B}$$

called the *coefficient of restitution*.

The negative sign indicates that the relative velocities are oppositely directed.

We now have an equation which, together with that previously based upon the conservation of momentum of the system, furnishes us with two simultaneous equations, and v_{A_1} and v_{B_1}, the two unknowns involved, may be determined.

508. A horizontal force $F = 0.04t$ (F in pounds, and t in seconds) acts on a body. Determine the linear impulse of F for the time interval between $t = 0.1$ sec. and $t = 1.2$ sec.

509. A body weighing 161 lb. rests on a horizontal surface. If a constant horizontal force $F = 100$ lb. is applied to the body, determine its velocity at the end of 10 sec. The coefficient of friction $\mu = 0.20$.

510. A body weighing 1000 lb. moves on a horizontal plane under the influence of the force system shown, which consists of a force P, acting in the direction of motion; a force R opposing the motion; frictional resistance; and normal pressure of the plane surface. The coefficient of friction is $\mu = 0.20$.

Determine, for the body at the end of 4 secs., the following.

(a) The velocity.

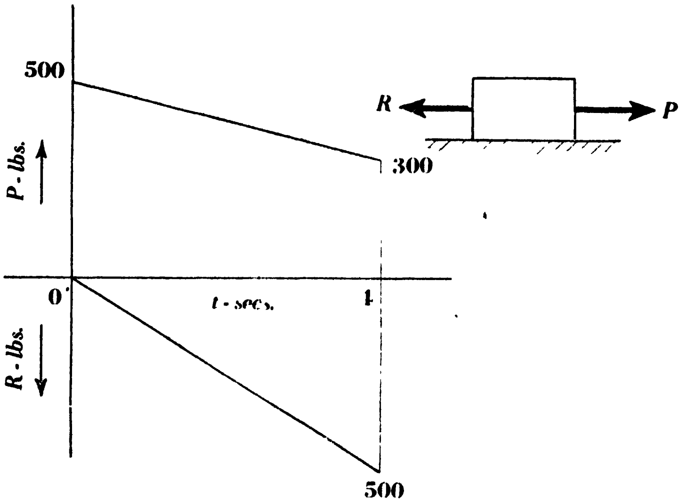
(b) The linear momentum.

(c) The kinetic energy.

(d) The horsepower exerted by force P.

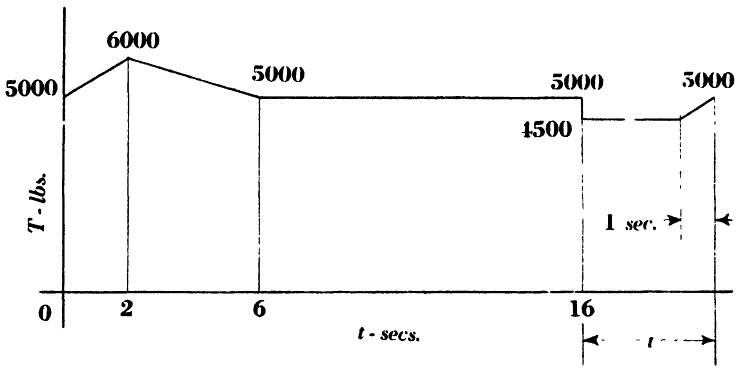
The initial velocity of the body is 15 ft. per sec. to the right.

Ans. (*a*) 8.56 ft./sec.; (*b*) 266 lb. ft. sec.; (*c*) 1136 ft.-lb.; (*d*) 4.67.



PROB. 510

511. The tension T in the cable of a 5000-lb. elevator varies with time as shown. If the car starts from rest, determine its maximum velocity. Determine the time t required to stop the elevator after deceleration begins.

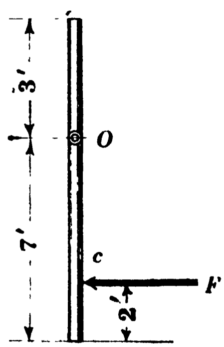


PROB. 511

512. A uniform, slender rod weighing 100 lb. is suspended, at rest, from a horizontal axis at O. If a force $F = 400$ lb. is applied at point c and acts for 0.05 sec. during which it is assumed to remain constant and normal to the rod, determine the following:

(a) The angular velocity of the rod at the end of the 0.05-sec. interval.

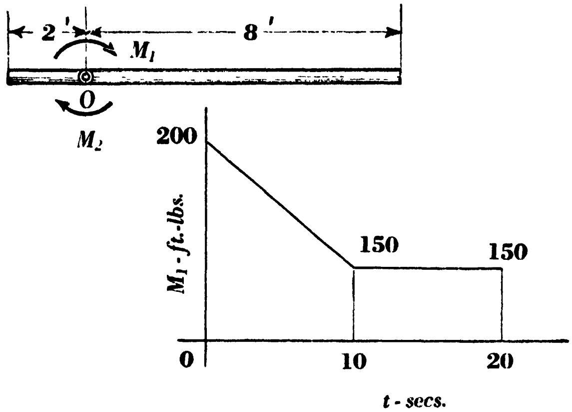
(b) The angle through which the rod swings upward from the vertical position



PROB. 512

513. A uniform, slender rod weighing 161 lb. rotates in a horizontal plane about a fixed axis at O under the influence of couple M_1, which varies as shown. A constant moment of resistance, $M_2 = 160$ ft.-lb., is applied at the axis. Determine the maximum velocity of the rod. Rod starts from rest.

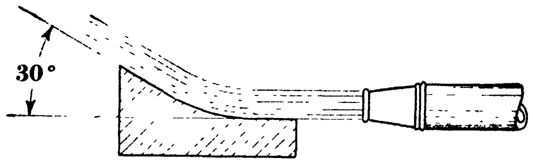
Ans. $\omega_{max} = 1.84$ rads./sec.



Prob. 513

514. A jet of water 1 in. in diameter impinges, at a velocity of 30 ft. per sec., on a blade inclined at an angle of 30° with the line of action of the jet.

Determine the horizontal and vertical components of the pressure of the water on the blade.



Prob. 514

515. A cylindrical jet of water 4 in. in diameter, having a velocity of 60 ft. per sec., impinges on a stationary blade. The blade deflects the jet through an angle of 65°. Determine the resultant pressure on the blade.

516. Two bodies, of masses m_1 and m_2, moving in opposite directions with velocities of 30 ft. per sec. and 15 ft. per sec., respectively, collide. Determine the distance between m_1 and m_2 5 seconds after impact.

TABLE 1

Trigonometric Functions

SINE **COSINE**

	.9	.8	.7	.6	.5	.4	.3	.2	.1	.0		
	.0	**.1**	**.2**	**.3**	**.4**	**.5**	**.6**	**.7**	**.8**	**.9**		
0°	0.000000	00175	00349	00524	00698	00873	01047	01222	01396	01571	01745	**89°**
1	01745	01920	02094	02269	02443	02618	02792	02967	03141	03316	03490	88
2	03490	03664	03839	04013	04188	04362	04536	04711	04885	05059	05234	87
3	05234	05408	05582	05756	05931	06105	06279	06453	06627	06802	06976	86
4	06976	07150	07324	07498	07672	07846	08020	08194	08368	08542	08716	85
5	08716	03889	09063	09237	09411	09585	09758	09932	10106	10279	10453	84
6	10453	10626	10800	10973	11147	11320	11494	11667	11840	12014	12187	83
7	12187	12360	12533	12706	12880	13053	13226	13399	13572	13744	13917	82
8	13917	14090	14263	14436	14608	14781	14954	15126	15299	15471	15643	81
9	15643	15816	15988	16160	16333	16505	16677	16849	17021	17193	0.17365	**80°**
10°	0.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	1908	79
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	2079	78
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	2250	77
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	2419	76
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	2588	75
15	2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	2756	74
16	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	2924	73
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3090	72
18	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3256	71
19	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	0.3420	**70°**
20°	0.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3584	69
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3746	68
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3907	67
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	4067	66
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	4226	65
25	4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	4384	64
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	4540	63
27	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	4695	62
28	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	4848	61
29	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	0.5000	**60°**
30°	0.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	5150	59
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	5299	58
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	5446	57
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	5592	56
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	5736	55
35	5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	5878	54
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	6018	53
37	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	6157	52
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	6293	51
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	0.6428	**50°**
40°	0.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	6561	49
41	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	6691	48
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	6820	47
43	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	6947	46
44°	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	0.7071	**45°**

TRIGONOMETRIC FUNCTIONS (*Cont.*)

SINE **COSINE**

		.9	*.8*	*.7*	*.6*	*.5*	*.4*	*.3*	*.2*	*.1*	*.0*	
	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9		
45°	0.7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	7193	**44°**
46	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	7314	43
47	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	7431	42
48	7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	7547	41
49	7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	7660	**40°**
50°	0.7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	0.7771	39
51	7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	7880	38
52	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	7986	37
53	7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	8090	36
54	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	8192	35
55	8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	8290	34
56	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	8387	33
57	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	8480	32
58	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	8572	31
59	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	0.8660	**30°**
60°	0.8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	8746	29
61	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	8829	28
62	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	8910	27
63	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	8988	26
64	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	9063	25
65	9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	9135	24
66	9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	9205	23
67	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	9272	22
68	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	9336	21
69	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	0.9397	**20°**
70°	0.9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	9455	19
71	9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	9511	18
72	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	9563	17
73	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	9613	16
74	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	9659	15
75	9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	9703	14
76	9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	9744	13
77	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	9781	12
78	9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	9816	11
79	9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	0.9848	**10°**
80°	0.9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	9877	9
81	9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	9903	8
82	9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	9925	7
83	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	9945	6
84	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	9962	5
85	9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	9976	4
86	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	9986	3
87	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	9994	2
88	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0.9998	1
89°	0.9998	9999	9999	9999	9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	**0°**

Trigonometric Functions (*Cont.*)

TANGENT **COTANGENT**

		.9	*.8*	*.7*	*.6*	*.5*	*.4*	*.3*	*.2*	*.1*	*.0*	
	.0	**.1**	**.2**	**.3**	**.4**	**.5**	**.6**	**.7**	**.8**	**.9**		
0°	0.000000	00175	00349	00524	00698	00873	01047	01222	01396	01571	0.17455	**89°**
1	01746	01920	02095	02269	02444	02619	02793	02968	03143	03317	03492	**88**
2	03492	03667	03842	04016	04191	04366	04541	04716	04891	05066	05241	**87**
3	05241	05416	05591	05766	05941	06116	06291	06467	06642	06817	06993	**86**
4	06993	07168	07344	07519	07695	07870	08046	08221	08397	08573	08749	**85**
5	08749	08925	09101	09277	09453	09629	09805	09981	10158	10334	10510	**84**
6	10510	10687	10863	11040	11217	11394	11570	11747	11924	12101	12278	**83**
7	12278	12456	12633	12810	12988	13165	13343	13521	13698	13876	14054	**82**
8	14054	14232	14410	14588	14767	14945	15124	15302	15481	15660	15838	**81**
9	15838	16017	16196	16376	16555	16734	16914	17093	17273	17453	0.17633	**80°**
10°	0.1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	1944	**79**
11	1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	2126	**78**
12	2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	2309	**77**
13	2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	2493	**76**
14	2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	2679	**75**
15	2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	2867	**74**
16	2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3057	**73**
17	3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3249	**72**
18	3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3443	**71**
19	3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	0.3640	**70°**
20°	0.3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3839	**69**
21	3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	4040	**68**
22	4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	4245	**67**
23	4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	4452	**66**
24	4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4663	**65**
25	4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4877	**64**
26	4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	5095	**63**
27	5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	5317	**62**
28	5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	5543	**61**
29	5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	0.5774	**60°**
30°	0.5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	6009	**59**
31	6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	6249	**58**
32	6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	6494	**57**
33	6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	6745	**56**
34	6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	7002	**55**
35	7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	7265	**54**
36	7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	7536	**53**
37	7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	7813	**52**
38	7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	8098	**51**
39	8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	0.8391	**50°**
40°	0.8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	8693	**49**
41	8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	9004	**48**
42	9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	9325	**47**
43	9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	0.9657	**46**
44°	0.9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	1.0000	**45°**

Trigonometric Functions (*Cont.*)

TANGENT **COTANGENT**

		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	
	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	-	
45°	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	0355	44°
46	0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	0724	43
47	0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	1106	42
48	1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	1504	41
49	1504	1544	1585	1626	1677	1708	1750	1792	1833	1875	1.1918	40°
50°	1.1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	2349	39
51	2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	2799	38
52	2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	3270	37
53	3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	3764	36
54	3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	1.4281	35
55	1.4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	4826	34
56	4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	5399	33
57	5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	6003	32
58	6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	6643	31
59	1.6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	1.7321	30°
60°	1.732	1.739	1.746	1.753	1.760	1.767	1.775	1.782	1.789	1.797	1.804	29
61	1.804	1.811	1.819	1.827	1.834	1.842	1.849	1.857	1.865	1.873	1.881	28
62	1.881	1.889	1.897	1.905	1.913	1.921	1.929	1.937	1.946	1.954	1.963	27
63	1.963	1.971	1.980	1.988	1.997	2.006	2.014	2.023	2.032	2.041	2.050	26
64	2.050	2.059	2.069	2.078	2.087	2.097	2.106	2.116	2.125	2.135	2.145	25
65	2.145	2.154	2.164	2.174	2.184	2.194	2.204	2.215	2.225	2.236	2.246	24
66	2.246	2.257	2.267	2.278	2.289	2.300	2.311	2.322	2.333	2.344	2.356	23
67	2.356	2.367	2.379	2.391	2.402	2.414	2.426	2.438	2.450	2.463	2.475	22
68	2.475	2.488	2.500	2.513	2.526	2.539	2.552	2.565	2.578	2.592	2.605	21
69	2.605	2.619	2.633	2.646	2.660	2.675	2.689	2.703	2.718	2.733	2.747	20°
70°	2.747	2.762	2.778	2.793	2.808	2.824	2.840	2.856	2.872	2.888	2.904	19
71	2.904	2.921	2.937	2.954	2.971	2.989	3.006	3.024	3.042	3.060	3.078	18
72	3.078	3.096	3.115	3.133	3.152	3.172	3.191	3.211	3.230	3.251	3.271	17
73	3.271	3.291	3.312	3.333	3.354	3.376	3.398	3.420	3.442	3.465	3.487	16
74	3.487	3.511	3.534	3.558	3.582	3.606	3.630	3.655	3.681	3.706	3.732	15
75	3.732	3.758	3.785	3.812	3.839	3.867	3.895	3.923	3.952	3.981	4.011	14
76	4.011	4.041	4.071	4.102	4.134	4.165	4.198	4.230	4.264	4.297	4.331	13
77	4.331	4.366	4.402	4.437	4.474	4.511	4.548	4.586	4.625	4.665	4.705	12
78	4.705	4.745	4.787	4.829	4.872	4.915	4.959	5.005	5.050	5.097	5.145	11
79	5.145	5.193	5.242	5.292	5.343	5.396	5.449	5.503	5.558	5.614	5.671	10°
80°	5.671	5.730	5.789	5.850	5.912	5.976	6.041	6.107	6.174	6.243	6.314	9
81	6.314	6.386	6.460	6.535	6.612	6.691	6.772	6.855	6.940	7.026	7.115	8
82	7.115	7.207	7.300	7.396	7.495	7.596	7.700	7.806	7.916	8.028	8.144	7
83	8.144	8.264	8.386	8.513	8.643	8.777	8.915	9.058	9.205	9.357	9.514	6
84	9.514	9.677	9.845	10.019	10.199	10.385	10.579	10.780	10.988	11.205	11.430	5
85	11.430	11.664	11.909	12.163	12.429	12.706	12.996	13.300	13.617	13.951	14.301	4
86	14.301	14.669	15.056	15.464	15.895	16.350	16.832	17.343	17.886	18.464	19.081	3
87	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27	28.64	2
88	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08	57.29	1
89°	57.29	63.66	71.62	81.85	95.49	114.59	143.24	191.0	286.5	573.0	∞	0°

TABLE 2

CHORDS

Note: Tabulated values give chord for unit radius. In graphical solutions, multiples of unit radius and corresponding chord should be used for greater accuracy.

Deg.	0′	10′	20′	30′	40′	50′	60′
0	0000	.0029	0058	.0087	0116	0145	0174
1	.0174	.0204	.0233	0262	0291	0320	0349
2	.0349	0378	0407	0436	0465	0494	0523
3	.0523	0552	.0582	0611	0640	0669	0698
4	.0698	0727	0756	0785	0814	0843	0872
5	.0872	.0901	.0930	0959	0988	1017	1047
6	.1047	.1076	1105	.1134	1163	1192	.1221
7	.1221	1250	.1279	1308	.1337	.1366	1395
8	.1395	.1424	1453	.1482	1511	1540	1569
9	1569	.1598	1627	1656	1685	1714	1743
10	.1743	1772	1801	1830	.1859	1888	1917
11	1917	1946	1975	2004	2033	2062	2090
12	2090	2119	.2148	2177	2206	2235	2264
13	2264	2293	2322	2351	2380	2409	2437
14	2437	.2466	2495	2524	.2553	2582	2610
15	2610	2639	2668	2697	.2726	2755	2783
16	2783	2812	2841	2870	2899	2927	2956
17	2956	2985	3014	3042	3071	3100	3129
18	.3129	.3157	3186	.3215	3243	.3272	3301
19	.3301	3330	3358	3387	3416	.3444	3473
20	.3473	3502	3530	3559	3587	3616	3645
21	3645	3673	3702	3730	3759	3788	3816
22	3816	3845	.3873	.3902	3930	3959	3987
23	.3987	.4016	4044	4073	.4101	4130	4158
24	4158	.4187	4215	.4243	4272	4300	4329
25	.4329	.4357	.4385	4414	4442	.4471	4499
26	.4499	.4527	4556	4584	4612	4641	.4669
27	4669	4697	4725	4754	4782	4810	4838
28	.4838	4867	4895	4923	4951	4979	5008
29	.5008	.5036	5064	5092	.5120	.5148	.5176
30	.5176	5204	5232	5261	.5289	.5317	.5345
31	5345	.5373	5401	.5429	.5457	.5485	.5513
32	.5513	5541	.5569	5596	.5624	5652	5680
33	.5680	.5708	.5736	5764	.5792	5820	5847
34	.5847	.5875	.5903	.5931	.5959	.5986	6014
35	.6014	.6042	.6069	6097	.6125	.6153	.6180
36	.6180	.6208	.6236	6263	.6291	.6318	6346
37	.6346	.6374	.6401	6429	.6456	.6484	.6511
38	6511	.6539	.6566	6594	.6621	.6649	.6676
39	.6676	6703	.6731	6758	.6786	.6813	.6840
40	6840	.6868	6895	.6922	.6950	.6977	.7004
41	7004	.7031	.7059	.7086	.7113	.7140	.7167
42	7167	7194	.7222	.7249	.7276	.7303	7330
43	.7330	.7357	7384	.7411	.7438	.7465	7492
44	.7492	.7519	.7546	7573	.7600	.7627	.7654

CHORDS (*Cont.*)

Deg.	0′	10′	20′	30′	40′	50′	60′
45	.7654	.7680	.7707	.7734	.7761	.7788	.7815
46	.7815	.7841	.7686	7895	7921	.7948	7975
47	.7975	.8001	.8028	.8055	.8081	.8108	8135
48	.8135	.8161	.8188	.8214	.8241	.8267	8294
49	.8294	.8320	.8347	.8373	.8400	.8426	.8452
50	.8452	.8479	.8505	.8531	.8558	.8584	8610
51	.8610	.8636	.8663	.8689	.8715	.8741	.8767
52	.8767	.8793	.8820	.8846	.8872	.8898	.8921
53	8924	.8950	.8976	.9002	.9028	.9054	.9080
54	.9080	.9106	.9132	.9157	.9183	.9209	.9235
55	9235	.9261	.9286	.9312	.9338	.9364	9389
56	.9389	9415	.9441	.9466	.9492	.9518	9543
57	9543	.9569	.9594	.9620	9645	.9671	.9696
58	.9696	.9722	.9747	.9772	.9798	.9823	9848
59	.9848	.9874	.9899	.9924	.9949	.9975	1.0000
60	1.0000	1.0025	1.0050	1.0075	1.0100	1.0126	1 0151
61	1.0151	1 0176	1.0201	1.0226	1.0251	1.0276	1.0301
62	1.0301	1.0326	1.0350	1.0375	1.0400	1.0425	1 0450
63	1.0450	1.0475	1.0500	1.0524	1.0550	1.0574	1.0598
64	1.0598	1.0623	1.0648	1.0672	1.0697	1.0721	1 0746
65	1.0746	1.0770	1 0795	1.0819	1.0844	1.0868	1.0893
66	1.0893	1 0917	1.0941	1.0966	1.0990	1.1014	1 1039
67	1.1039	1.1063	1.1087	1.1111	1 1135	1.1159	1 1184
68	1.1184	1.1208	1.1232	1.1256	1.1280	1.1304	1 1328
69	1.1328	1.1352	1.1376	1.1400	1.1424	1.1448	1 1471
70	1.1471	1.1495	1.1519	1.1543	1.1567	1.1590	1 1614
71	1.1614	1.1638	1.1661	1.1685	1.1708	1.1732	1 1756
72	1.1756	1.1780	1.1803	1.1862	1.1850	1.1873	1 1896
73	1.1896	1.1920	1.1943	1.1966	1.1990	1.2013	1 2036
74	1.2036	1.2059	1.2083	1.2106	1.2129	1.2152	1 2175
75	1.2175	1.2198	1.2221	1.2244	1.2276	1.2290	1 2313
76	1.2313	1.2336	1.2360	1.2382	1.2405	1.2427	1 2450
77	1.2450	1.2473	1.2496	1.2518	1.2541	1.2564	1.2586
78	1.2586	1.2609	1.2631	1.2654	1.2677	1.2699	1.2721
79	1.2721	1.2744	1.2766	1.2789	1.2811	1.2833	1.2856
80	1.2856	1.2878	1.2900	1.2922	1.2945	1.2967	1.2989
81	1.2989	1.3011	1.3033	1.3055	1.3077	1.3099	1 3121
82	1.3121	1.3143	1.3165	1.3187	1.3209	1.3231	1.3252
83	1.3252	1.3274	1.3296	1.3318	1.3340	1.3361	1.3383
84	1.3383	1.3404	1.3426	1.3447	1.3469	1.3490	1.3510
85	1.3512	1.3533	1.3555	1.3576	1.3597	1.3619	1.3642
86	1.3640	1.3661	1.3682	1.3704	1.3725	1.3746	1.3767
87	1.3767	1.3788	1.3809	1.3830	1.3851	1.3872	1.3893
88	1.3893	1.3914	1.3935	1.3956	1.3977	1.3997	1.4018
89	1.4018	1.4039	1.4060	1.4080	1.4101	1.4121	1.4142
90	1.4142						

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