MECHANISM AND THE KINEMATICS OF MACHINES

By the same Author

ENGINEERING MATERIALS, MACHINE TOOLS AND PROCESSES. With Diagrams and Illustrations

MECHANISM
AND THE
KINEMATICS OF MACHINES

BY

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THE MILITARY COLLEGE OF SCIENCE

WITH DIAGRAMS

LONGMANS, GREEN AND CO.

LONDON • NEW YORK • TORONTO

LONGMANS, GREEN AND CO. LTD.
OF PATERNOSTER ROW
43 ALBERT DRIVE, LONDON, S.W. 19
NICOL ROAD, BOMBAY
17 CHITTARANJAN AVENUE, CALCUTTA
36A MOUNT ROAD, MADRAS

LONGMANS, GREEN AND CO.
55 FIFTH AVENUE, NEW YORK

LONGMANS, GREEN AND CO.
215 VICTORIA STREET, TORONTO

FIRST PUBLISHED 1940
REPRINTED BY REPLIKA PROCESS 1945

CODE NUMBER 86689

PRINTED IN GREAT BRITAIN BY THE REPLIKA PROCESS BY
LUND HUMPHRIES & CO LTD. • LONDON & BRADFORD

PREFACE

THE kinematics of machines and the study of mechanisms are subjects that are usually dealt with in books on the "theory of machines," a title that covers a very wide field, including, as it does, not only the kinematics, but also the statics and dynamics of machines. Consequently the treatment afforded to kinematics and mechanism in such books is unduly curtailed. On the other hand, such books as Reuleaux's *Kinematics of Machines* and *The Constructor* besides being out of print and difficult to obtain, are too comprehensive and detailed to be of much use to students as textbooks. In this book the author has endeavoured to provide an adequate treatment of the kinematics of machines and the study of mechanisms as mechanical contrivances, while avoiding excessive detail and encyclopædic comprehensiveness; he has endeavoured to steer a middle course between unduly academic treatment on the one hand and excessively detailed descriptive treatment on the other hand. It is hoped, therefore, that the book will be of use not only to students at engineering colleges and technical institutes, but also to practising engineers, designers and draughtsmen.

Considerable space has been devoted to the consideration of the freedom and constraint of bodies and the principles of geometric or kinematic design. The latter is a subject that is seldom dealt with in engineering textbooks—a defect that the author thinks should be remedied. The principles of geometric design can be applied not only to instruments where the forces acting are small, but also to machines where large forces are encountered, and a knowledge of those principles should be part of the mental equipment of all engineers engaged in the design of machines.

The determination of the velocities and accelerations of points of mechanisms is of fundamental importance and receives what is thought to be adequate treatment. The derivation and application of Coriolis's law are explained at some length, this being, in the author's experience, one of the stumbling-blocks of most students.

The theory of toothed gearing has been given fuller treatment than it receives in most textbooks other than those devoted solely to it, this being considered desirable in view of the very wide use of such gearing and the dearth of books on the subject.

Numerous exercises are included throughout the book, and in many cases they are made to supplement the text by introducing variations of the mechanisms described in the text, and thus make the book more comprehensive without increasing its size.

The scope of the book should make it suitable for use by students studying for the examinations of the British universities up to final examination standard, but it is hoped that less-advanced students will also be able to use it with advantage.

Every endeavour has been made to avoid errors, but it is realised that complete absence of mistakes is improbable, and the author will be pleased if readers will notify him of any errors they may discover.

W. S.

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CHAPTER I

MOTION OF A POINT

1. Rest and Motion.—These are essentially relative terms, as it is not possible to tell whether any body is at rest in an absolute sense ; all that can be said is that one body is at rest relative to another body when the position of the one relative to the other remains unchanged. It is therefore necessary to consider how the position of one body relative to another may be determined and specified. For simplicity the position and motion of a point will first be considered.

2. Position of a Point.—The position of one point relative to another may be specified by fixing one point at the intersection of three mutually perpendicular planes and then giving the perpendicular distances between the other point and the planes as shown

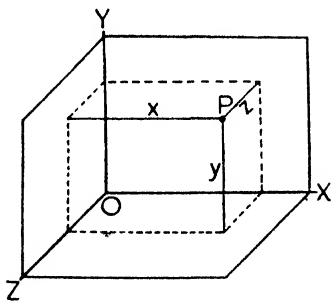


Fig. 1

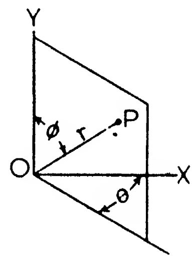


Fig. 2

in Fig. 1. The distances x, y and z are the *rectangular co-ordinates* of the point P. The point O is the *origin* and OX, OY and OZ are the co-ordinate axes. The distance OP is clearly given by

$$OP=\sqrt{x^2+y^2+z^2} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Alternatively the *polar co-ordinates* r, θ and ϕ may be specified as shown in Fig. 2, where r is the distance OP between the points,

ϕ is the angle between the line OP and the axis OY and θ is the angle between the plane POY, containing OP, and the plane XOY. Clearly the polar co-ordinates are related to the rectangular co-ordinates by the equations

$$x=r \operatorname{Sin} \phi \operatorname{Cos} \theta \quad \text{(2)}$$
$$y=r \operatorname{Cos} \phi \quad \text{(3)}$$
$$z=r \operatorname{Sin} \phi \operatorname{Sin} \theta \quad \text{(4)}$$

Whichever co-ordinates are used three quantities have to be specified in order to specify the position of the point P relative to the point O.

When the point P moves relative to O its motion may be regarded as consisting of three component motions parallel respectively to OX, OY and OZ. If these component motions are independent of each other, the point P is perfectly free and is said to possess *three degrees of freedom.*

3. Plane Motion of a Point.—If the point P lies always in the plane XOY, Fig. 1, its z co-ordinate is always zero and its position may be specified by giving only the x and y co-ordinates. The motion of P, in the plane XOY, may be regarded as consisting of two component motions parallel respectively to OX and OY. If these component motions are quite independent, then the point is perfectly free in the plane and it possesses *two* degrees of freedom.

Thus a point that is confined to a plane possesses only two degrees of freedom ; this applies also to a point that is confined to *any* surface, for although the motion of a point that moves on a curved surface may be regarded as consisting of three component motions parallel to the three co-ordinate axes, yet those component motions are *not independent*, but are related in some way depending on the shape of the surface. Looked at in another way, if the axes OX and OY are imagined to be tangent to the surface at the point O, then a point situated at O can move in the direction of either OX or OY, but not in the direction of OZ, that is, in the direction of a normal to the surface.

4. Motion of a Point along a Line.—Lastly let the point P be confined to the line OX ; then its position is specified by the single co-ordinate x. The point possesses only *one* degree of freedom, and this applies also to a point that is confined to any line, straight or otherwise, for although the motion of a point along a curved line may be regarded as consisting of three component motions, yet those component motions are not independent, but are related in some way depending on the shape of the line.

Or, if the axis OX is imagined to be tangent to the line at the origin O, then a point situated at O can move only in the direction OX and not in the direction of either OY or OZ, which are normals to the line.

5. Frames of Reference.—The three mutually perpendicular planes intersecting in the axes OX, OY, OZ form a convenient *frame of reference* relative to which the position of a point or body may be specified. It is frequently useful to regard each of two bodies whose relative motion is to be studied as having such a frame fixed in it and then to study the relative motion of those frames.

6. Displacement of a Point.—When the position of a point, relative to a frame of reference, changes, then the point is said to receive a displacement. This can be specified by giving the coordinates of the first and last positions of the point or by giving its magnitude and direction, that is, the length and direction of the line joining the first and last positions of the point. A displacement can also be represented by a line drawn parallel to it and of a length proportional to its magnitude. For example, if P_1 and P_2, Fig. 3, are respectively the first and last positions of a point, then the line p_1p_2 represents the displacement of the point, and it should be noted that an arrowhead is placed on the line in order to show the *sense* of the displacement—that it is from P_1 to P_2 and not from P_2 to P_1. The sense of a displacement can be indicated by the order of naming of the points; thus p_1p_2 represents a displacement from P_1 to P_2, while p_2p_1 would represent a displacement from P_2 to P_1.

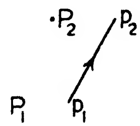


Fig. 3

7. Relative Displacements.—Suppose now that P_1 and P_2 represent two separate points both of which were originally at P_1, then the line p_1p_2 represents the displacement of P_2 relative to P_1, that is, the displacement P_2 would appear to have to an observer fixed to P_1. To an observer fixed to P_2 the point P_1 would appear to receive a displacement p_2p_1. Thus a line p_1p_2 represents either the displacement of P_2 relative to P_1, or the displacement of P_1 relative to P_2, according to the sense in which the line is traversed. Thus p_1p_2 with an arrowhead pointing from p_1 to p_2 represents the displacement of p_2 relative to p_1, while p_2p_1 with an arrowhead pointing from p_2 to p_1 represents the displacement of p_1 relative to p_2.

8. Successive Displacements.—If a point receives successively two displacements, relative to the same frame of reference, first from A to B and then from B to C, then clearly the final result is the same as if the point had received a single displacement from A to C. The displacement AC is equivalent to, and is called the *resultant* of, the successive displacements AB, BC. If the point receives more than two successive displacements, then the resultant is found by setting out the displacements in order and joining the first and last points as in Fig. 4, where *ae* is the resultant of *ab*, *bc*, *cd* and *de*. The polygon *abcde* is a *polygon of displacements*, and it is essential that the sides of the polygon be set out so that the arrows point in the directions in which the successive displacements are actually made, and that when this is done they point in the same direction round the polygon, that is, if the sides of the polygon are traversed in the directions of the arrows, a continuous circuit of the polygon will be made, except that the arrow on the side representing the resultant will be in the opposite direction to all the others.

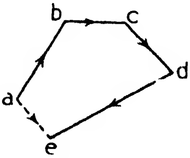


FIG. 4

It should be noted that the same resultant will be obtained if the displacements are set out in a different order from that in which they actually occur, provided that the condition with regard to the direction of the arrows round the polygon is complied with. This is shown in Fig. 5, where the sides 1, 2, 3 and 4 represent the displacements *ab*, *cd*, *de* and *bc* respectively, but are placed in a different order. Clearly the displacements need not all lie in one plane.

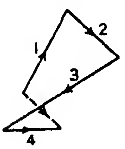


FIG. 5

9. Simultaneous Displacements.—In the previous article the displacements considered were all relative to the same frame of reference, but a point may receive a displacement relative to one frame of reference while that frame itself receives a displacement relative to a second frame. Thus a man might walk across the deck of a steamer and receive a displacement relative to the steamer while the steamer moved through the water, thus receiving a displacement relative to the water. With this meaning the man may be said to receive two simultaneous displacements, and clearly in the same manner it is possible for him to receive any number of simultaneous displacements; these, however, are relative to different frames of reference. Relative to the steamer the man receives a perfectly definite displacement which in Fig. 6 is represented by AB, and relative to the water the steamer receives a displacement AC. Clearly the first and last positions of the man relative to the water are A and D, and AD represents his displacement relative to the water. AD is the *resultant* of AB and AC, and so we have the following rule.

10. The Parallelogram Law.—If the adjacent sides AB and AC of a parallelogram ABCD represent respectively the displacement of a body L relative to a body M, and the displacement of the body M relative to a third body N, then the diagonal AD represents the displacement of the body L relative to the body N.

11. The Polygon Law.—Since CD in Fig. 6 is equal and parallel to AB, it may be taken to represent the displacement AB, and it will be seen that the resultant AD is obtained by drawing the triangle ACD, that is, by treating the displacements as if they were successive and not simultaneous. This method of finding the resultant is very convenient when the number of simultaneous displacements is greater than two, and gives rise to the following rule.

If the sides *ab*, *bc*, *cd* and *de* of a polygon represent the displacements of a body B relative to a body A, a body C relative to B, a

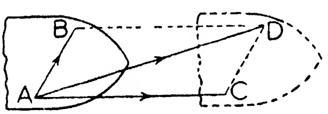


Fig. 6

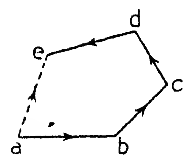


Fig. 7

body D relative to C and a body E relative to D respectively, then the closing side *ae* represents the displacement of E relative to A. This is shown in Fig. 7 ; as with successive displacements, care must be taken to get the arrows on the sides of the polygon pointing in the direction of the displacements and pointing, with the exception of the resultant, the same way round the polygon. Again the sides of the polygon need not all lie in one plane.

12. Vectors.—Displacements, as has been seen, possess both magnitude, direction and sense, and the resultant of two displacements is a third displacement which can be found by means of either the parallelogram or polygon law ; these characteristics are also possessed by a number of other quantities such as velocities, forces, etc., and such quantities are called *Vectors*, whereas quantities such as areas, volumes, energy, etc., which possess magnitude but not direction, are called *Scalars*.

Two vectors are said to be *added* when their resultant, called their *vector sum*, is found by means of the parallelogram or polygon *law. It should hardly be necessary to say that only vectors of the same kind can be added together.*

In printed works vector quantities are often distinguished by the use of Clarendon type; thus **AB** represents the "vector AB" and **AB**+**CD** the "vector sum of AB and CD." In written works the distinction is often made thus, $\overline{AB}$ and $\overline{AB}+\overline{CD}$.

13. Speed of a Point.—Consider a point which is moving along a line OPQ and which at a particular instant occupies the position P. After an interval of time (t) the point arrives at the position Q, having travelled a distance (s) along the line. Then the ratio $\dfrac{\text{distance travelled}}{\text{time taken}}$, that is $\dfrac{s}{t}$, is defined as the *average speed* of the point along the line during that interval of time. It is convenient to make unit speed correspond to the traversing of unit distance in unit time, and clearly the unit of speed will depend on the units adopted for distance and time. If these are feet and seconds, then speeds will be measured in $\dfrac{\text{feet}}{\text{seconds}}$ or feet per second; this may be written ft./sec. or f.s. If miles and hours are adopted, then speeds will be in $\dfrac{\text{miles}}{\text{hours}}$ or miles per hour, which may be written miles/hr. or m.p.h. The change from one set of units to another is easily effected, thus:

$$30 \text{ m.p.h.} = 30 \times \frac{\text{mile}}{\text{hour}} = 30 \times \frac{5280 \text{ feet}}{60 \times 60 \text{ seconds}} = 44 \text{ ft./sec.}$$

14. Constant Speed.—When the magnitude of the average speed is the same whether the time interval considered is large or small, the speed is said to be constant or uniform. If a point is moving with a constant speed, then the distance traversed in any time interval is proportional to that interval, and the graph obtained by plotting the distance s against the time t is a straight line as shown in Fig. 8. The speed is then equal to the slope of the line, $v=\dfrac{s_1}{t_1}=\dfrac{s_2}{t_2}$, the distances and times being, of course, measured to the appropriate scales. Thus we have, for constant speed, the following relations between the distance traversed, s, the time taken, t, and the speed, v:

$$s = v \times t, \quad v = \frac{s}{t}, \quad t = \frac{s}{v} \qquad (5)$$

provided the units used are consistent, that is, if v is in ft./sec., s must be in feet and t in seconds.

15. Variable Speed.—When the speed is variable the graph corresponding to Fig. 8 will not be a straight line, but some curve as shown in Fig. 9, and the value of the average speed will be

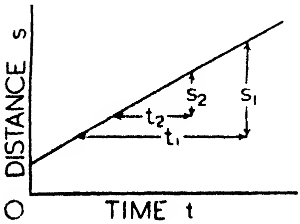


Fig. 8

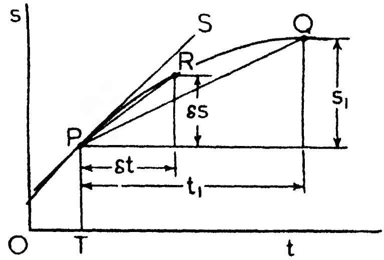


Fig. 9

different for different time intervals. The actual speed at any instant during the interval t_1 might be widely different from the average value $\frac{s_1}{t_1}$, but if the time interval is made smaller and smaller, then the likelihood of the actual speed at any instant during the interval being much different from the average speed for the interval will become less and less. Thus if a very small interval of time, δt, is considered, during which the distance traversed is δs, then the average speed $\frac{\delta s}{\delta t}$ for that interval will not be very widely different from the actual speed at any instant during the interval. The ratios $\frac{s_1}{t_1}$ and $\frac{\delta s}{\delta t}$ are actually the slopes of chords PQ and PR of the distance-time curve, and as the time interval is made smaller and smaller so the slope of the chord gradually approaches that of the tangent PS; and, finally, when the time interval is made indefinitely small the interval becomes an instant and the average speed becomes the actual speed at the instant. This cannot now be found from the expression $\frac{\delta s}{\delta t}$, since this ratio assumes the indefinite form of $\frac{o}{o}$, but it can be found since it is now equal to the slope of the tangent PS. Thus the speed at any particular instant during a variable motion may be found by drawing a tangent to the distance or space-time curve, at the instant, and measuring its slope. The process of drawing a tangent to a curve is not usually susceptible of very great accuracy, and so the result obtained by this method will be only an approximation to the true speed; but for a great many practical purposes the accuracy is sufficient.

Example.—The table below gives corresponding values of the distance (s) traversed and the time (t) taken for the motion of a motor car. Find the average speeds for the intervals $t=2$ to $t=8$, 6 and 4 respectively, and the instantaenous speed when $t=2$.

t, secs. . .	0	1	2	3	4	5	6	7	8
s, feet . .	25	32·5	52	81	119	162	210	265	325

The distance traversed in the interval $t=2$ to $t=8$ is 325−52 =273 feet; hence the average speed for this interval is $\frac{273}{6}=45{\cdot}5$ ft./sec. Similarly for the other intervals the average speeds are $\frac{210-52}{4}=39{\cdot}5$ ft./sec. and $\frac{119-52}{2}=33{\cdot}5$ ft./sec. On plotting against t the graph, Fig. 10, is obtained, and by drawing the

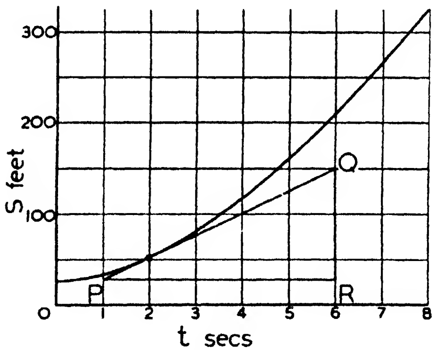


Fig. 10

tangent PQ at the time $t=2$ the instantaneous speed is obtained as $\frac{QR}{PR}=\frac{120}{5}=24$ ft./sec. (The measurement of the slope of this tangent was actually made on a graph drawn to much larger scales than in the figure.)

In the language of the differential calculus the limiting value of the ratio $\frac{\delta s}{\delta t}$, when δt is made indefinitely small, is called the differential coefficient of s with respect to t and is written $\frac{ds}{dt}$. If the equation to the space-time curve, i.e. the equation connecting s and t, is known, then the value of $\frac{ds}{dt}$ can be found, by differentiation, to any desired degree of accuracy. The differential coefficient $\frac{ds}{dt}$ is the rate of change of the distance s with respect to the time t or, shortly, the time rate of change of position. It is sometimes more convenient to use the Newtonian notation and to denote $\frac{ds}{dt}$ by $\dot{s}$ (read as *s dot*). Thus $v=\frac{ds}{dt}=\dot{s}=$the slope of the space-time graph.

Example.—The height, h ft., fallen through by a body falling from rest under the action of gravity is related to the time, t secs., by the equation $h=16{\cdot}1\ t^2$. What is the speed of the body after 5 seconds ?

$$\text{Speed}=\text{Time rate of displacement}=\frac{dh}{dt}$$

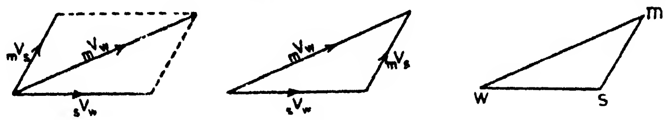
$$=\frac{d}{dt}(16{\cdot}1\ t^2)=32{\cdot}2\ t$$

$$\text{When } t=5\quad \frac{dh}{dt}=32{\cdot}2\times 5=161 \text{ ft./sec.}$$

16. Velocity.—The ratio *distance moved/time taken* does not take any account of the direction in which the motion takes place ; in the *velocity* of a point, however, the direction is considered. The velocity of a point is its speed *in a stated direction*, and it is fully specified only when both the speed and the direction are stated, e.g. 20 m.p.h. S.E. to N.W.

Velocities are vectors and may be represented by straight lines in a similar manner to displacements, and also, being rates of displacement, they are, like displacements, relative. The statement that the velocity of a train is 20 m.p.h. S. to N. means that to an observer fixed to the earth the train appears to be moving from S. to N. at a speed of 20 m.p.h. To an observer fixed to the train the earth would appear to be moving from N. to S. at the same speed. Thus the velocity of the earth relative to the train is equal and opposite to the velocity of the train relative to the earth, and, as with displacements, both can be represented by the same line traversed in opposite directions. Thus if *et*, with the arrow pointing from e to t, represents the velocity of the train relative to the earth, then *te*, with the arrow pointing from t to e, represents the velocity of the earth relative to the train. It is convenient to write these velocities by ${}_tv_e$ and ${}_ev_t$ respectively, and we can write ${}_tv_e=-{}_ev_t$, the minus sign indicating that the two velocities are opposite in sense.

17. Simultaneous Velocities.—A point can have two or more simultaneous velocities, in the same manner as it can have two or more simultaneous displacements, and these can be dealt with by means of the parallelogram and polygon laws. Thus, using the same illustration as before, if ${}_mv_s$ represents the velocity of the man relative to the steamer and ${}_sv_w$ that of the steamer relative to the water, then the velocity of the man relative to the water is the resultant, or vector sum, of ${}_mv_s$ and ${}_sv_w$ and is obtained as shown in Fig. 11 (*a*). This operation may be



(*a*) (*b*) (*c*)

FIG. 11

represented symbolically by the equation ${}_m\boldsymbol{v}_w = {}_m\boldsymbol{v}_s + {}_s\boldsymbol{v}_w$, and when there are more than two component velocities this becomes ${}_a\boldsymbol{v}_n = {}_a\boldsymbol{v}_b + {}_b\boldsymbol{v}_c + {}_c\boldsymbol{v}_d + \ldots {}_m\boldsymbol{v}_n$ and the vector addition is most conveniently performed by means of the polygon law. Fig. 12 shows the operation when there are four components. As with displacements, the order in which the sides of the polygon are set out is immaterial, but care must be used to ensure that the arrow on each side does point in the direction of the velocity that

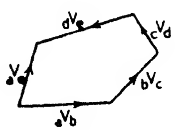


FIG. 12

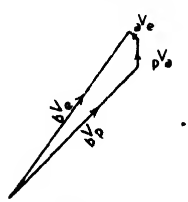


FIG. 13

side represents, and to ensure that the arrows follow on continuously round the polygon except for the closing side. The sides of the polygon need not all lie in one plane.

Example.—The velocity of an aeroplane (*p*) relative to the air (*a*) is 150 ft./sec. due N., while that of the air relative to the earth (*e*) is 50 ft./sec. due N.W. A bullet (*b*) is fired from the aeroplane at a speed of 1000 ft./sec. in a direction due N.E. relative to the aeroplane. What is the velocity of the bullet relative to the earth?

We have ${}_b\boldsymbol{v}_e = {}_b\boldsymbol{v}_p + {}_p\boldsymbol{v}_a + {}_a\boldsymbol{v}_e$, and on drawing the polygon shown in Fig. 13 it will be found that ${}_bv_e = 1115$ ft./sec. at an angle of 37° E. of N. (By calculation ${}_bv_e = 1117$ f.s. at 36° 58′.)

18. Relative Velocities.—Referring to Fig. 11, it is clear that if ${}_mv_w$ and ${}_sv_w$ are given, then it is easy to find ${}_mv_s$; it is merely necessary to set out ${}_mv_w$ and ${}_sv_w$ *from a common point* and to join their ends; the joining line then represents ${}_mv_s$. Some difficulty may be experienced in deciding which way the arrow on the joining line should be placed, but this difficulty disappears if the velocities are labelled as described in Art. 16. If this is done as shown in Fig. 11 (*c*), then ${}_mv_w$ is represented by the line *wm* and ${}_sv_w$ by the line *ws*, so that both sides must be set out from the point *w*; the ends of the joining line will then be labelled *m* and *s*, and *ms* with the arrow pointing from *m* to *s* represents the velocity of *s* relative to *m*. If the equation ${}_m\boldsymbol{v}_w = {}_m\boldsymbol{v}_s + {}_s\boldsymbol{v}_w$ is treated algebraically, we may write ${}_m\boldsymbol{v}_s = {}_m\boldsymbol{v}_w - {}_s\boldsymbol{v}_w$, the velocity of the man relative to the steamer being equal to the vector difference of the velocities of the man and the steamer relative to the water, so that *sm* in Fig. 11 (*c*) represents the vector difference of *wm* and *ws*. Thus if two vectors are set out *from a common point*, then the line joining their ends represents their vector *difference*. This should be compared with the parallelogram law for finding the vector *sum* of two vectors.

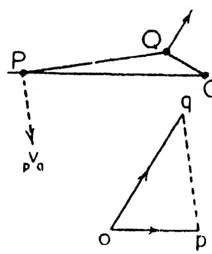


FIG. 14

Example.—The velocities of the ends P and Q of the connecting-rod of an engine (relative to any point, say O, of the frame) are respectively 190 ft./min. in the direction PO and 300 ft./min. at right-angles to QO as shown in Fig. 14. What is the velocity of P relative to Q?

In the velocity diagram *opq* draw *op* parallel to PO and equal to 190 ft./min. to a convenient scale, also draw *oq* perpendicular to OQ and equal to 300 ft./min. to the same scale. Then *qp* represents the velocity of P relative to Q, and by measurement this will be found to be 264 ft./min. in a direction perpendicular to QP, as indicated by the dotted line.

19. Resolution of a Velocity.—By performing the reverse operation to that of finding the resultant of two velocities it is possible to replace a single velocity by two other velocities, called its *components*, in any two chosen directions provided that those directions and that of the original velocity all lie in one plane. Let OR, Fig. 15, be the velocity to be resolved and OA and OB the directions of the components; then on drawing from R lines parallel to OA and OB to intersect OB and OA respectively

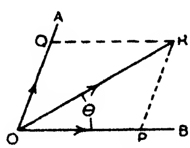


FIG. 15

in P and Q the desired components are obtained as OP and OQ. When the angle AOB is a right angle, as is most usual, then $OP = OR \cos \theta$ and $OQ = OR \sin \theta$.

This process can be extended to include the resolution of a vector in three directions not lying in one plane. The resultant OR of three velocities OP, PQ and QR as shown in Fig. 16 is obtained in the usual way by drawing the vector polygon OPQR (called a *gauche* polygon, since the sides do not all lie in one plane). If it is required to resolve OR into three components in the directions OX, OY and OZ, then clearly it is merely necessary to draw, on OR as diagonal, the parallelepiped OPQSTURV, whose sides are parallel to OX, OY and OZ, and the components are obtained as OP, OS and OT. When the directions OX, OY and OZ are mutually at right-angles, as shown in Fig. 17,

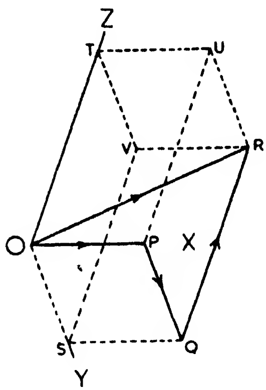


Fig. 16

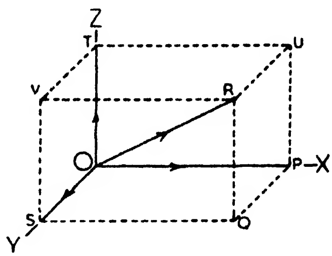


Fig. 17

then $OP = OR \cos \angle ROX$, since $\angle RPO$ is a right angle; similarly $OS = OR \cos \angle ROY$ and $OT = OR \cos \angle ROT$; these cosines are called the *direction cosines* of OR and are usually denoted by l, m and n respectively, thus $OP = OR \times l$, $OS = OR \times m$ and $OT = OR \times n$.

20. Acceleration.—When the velocity of a point is not constant the ratio $\dfrac{\text{change of velocity}}{\text{time taken}}$ is called the *average acceleration* of the point. Accelerations are therefore measured in $\dfrac{\text{units of velocity}}{\text{units of time}}$, that is, as so many units of velocity per unit time. If the unit velocity is one foot per second and the unit of time one second, then the unit of acceleration is one $\dfrac{\text{foot per sec.}}{\text{sec.}}$ or one foot per second per second. This may be written ft./sec.². If m.p.h. and seconds are the units, then the unit of acceleration is one m.p.h. per sec., but it is usual to have the same unit of time for the acceleration as for the velocity. The change from one set of units to another is easily effected, thus :

$$60 \text{ m.p.h. per min.} = 60 \frac{\text{mile}}{\text{hour}} \Big/ \text{min.}$$

$$= 60 \times \frac{5280 \text{ ft.}}{60 \times 60 \text{ sec.}} \Big/ 60 \text{ sec.}$$

$$= 1{\cdot}467 \text{ ft./sec.}^2$$

21. Constant Acceleration.—The acceleration of a point that moves in a straight line is due solely to the change of speed that occurs, and in this article and the next this kind of acceleration only is considered. It may be constant or variable; if it is constant, then the graph of the speed plotted against the time will be a straight line as in Fig. 18. The average acceleration $\left(\frac{v}{t}\right)$ is the same whether the time interval is large or small and is equal to

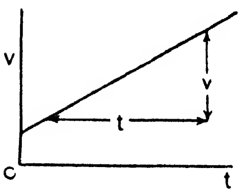


Fig. 18

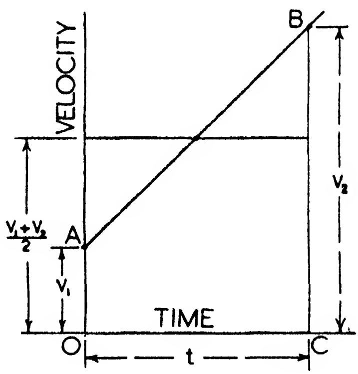


Fig. 19

the slope of the line. If the constant acceleration is denoted by f, then we have

$$f = \frac{v}{t}, \; v = ft, \; t = \frac{v}{f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where v is the change of speed that occurs in the time t. If the speed at the beginning of an interval of time t is v_1, then the speed at the end of that interval will be $v_1 + ft = v_2$; the speed-time graph is as shown in Fig. 19 and the average speed during the interval is $\frac{v_1 + v_2}{2}$, hence the distance s traversed in the interval is

$\left(\frac{v_1+v_2}{2}\right)\times t=v_1t+\frac{1}{2}ft^2$. Thus we have, for constant accelerations,

$$v_2=v_1+ft \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$s=v_1t+\tfrac{1}{2}ft^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$v_2=v_1{}^2+2fs \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

the last being obtained by eliminating t_1 between (7) and (8).

22. Variable Acceleration.—When the acceleration is variable, then the speed-time graph will be a curve and the actual acceleration at any instant may be obtained by drawing the tangent to the curve at that instant and measuring its slope. This gives the value of $\frac{dv}{dt}$, the time rate of change of velocity, that is, the acceleration, at the instant. If the equation to the curve is known, then $\frac{dv}{dt}$ may be obtained by differentiation. In the Newtonian notation $\frac{dv}{dt}$, being $\frac{d\left(\frac{ds}{dt}\right)}{dt}$, is written as $\ddot{s}$ (read as s *double dot*).

Referring to Fig. 19, it will be seen that the area of the quadrilateral OABC is given by $\frac{(v_1+v_2)\times t}{2}=v_1t+\frac{1}{2}ft^2$, so that that area, to some scale, is equal to the distance traversed by the point in

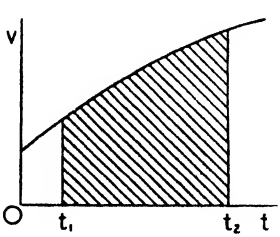


FIG. 20

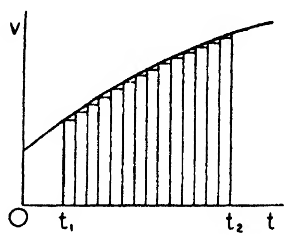


FIG. 21

the time t. The same holds when the speed-time graph is a curve, as in Fig. 20, where the shaded area, to some scale, is equal to the distance traversed during the time interval t_2-t_1. This will be understood if the curve is regarded as being of the stepped form shown in Fig. 21, the speed during each of the small time intervals represented by the width of the strips being constant. The distance travelled during each small interval will then be equal to the area of the rectangular strip, and thus the distance traversed in the interval t_2-t_1 (the sum of the small intervals) is equal to the sum of the areas of the strips, that is, to the area under the stepped curve. If now the number of strips is made infinitely large the stepped curve will coincide with the actual curve and the distance travelled is equal to the area under the curve.

If in Fig. 21 there are n strips each of width δt and the heights of the strips are $v_1, v_2 \ldots v_n$, then the distance traversed, s, is given by $\Sigma\ v_1\delta t + v_2\delta t + \ldots v_n\delta t$, where Σ stands for "the sum of." This operation of finding the sum of a number of quantities of the same type, such as $v_1\delta t$, $v_2\delta t$, etc., is sometimes denoted symbolically by $\sum_1^n v\delta t$. As long as the number of terms is finite their sum can be found by actual addition, but when the number becomes infinite this is no longer possible, but the sum can then be found by means of the integral calculus. Thus the shaded area of Fig. 20 is given by the integral of v with respect to t between the limits t_1 and t_2, and this is written $\int_{t_1}^{t_2} vdt$. When the equation connecting v and t is known, the value of this integral can usually be calculated.

The scale to which the area under the speed-time curve represents the distance traversed may be obtained thus: Let the scale to which the speeds are set out be 1 inch to m units of velocity and that for the times be 1 inch to n units of time, then the scale for the area is 1 sq. in. to $m \times n$ (units of velocity) $\times$ (units of time), that is, 1 sq. in. to $m \times n \dfrac{\text{units of distance}}{\text{units of time}} \times$ units of time, that is, $m \times n$ units of distance. If the scale for speeds is 1 inch to 20 ft./sec., and for times 1 inch to 10 seconds, then the scale for areas is 1 sq. in. to $20 \dfrac{\text{ft.}}{\text{sec.}} \times 10$ sec., that is, 1 sq. in. to 200 ft.

23. In the previous two articles the changes of velocity considered were changes of magnitude or of speed, since the direction of the motion, being along a straight line, was unaltered. A change of velocity, and therefore an acceleration, occurs, however, if the direction of the motion of a point changes, even though the speed remains constant. If *ea*, Fig. 22, is the velocity of a point A relative to a frame of reference E at one instant, and ea_1 is the corresponding velocity after an interval of time t, then the change of velocity is represented by aa_1, this being the velocity that when added to *ea* produces ea_1. The magnitude of the average acceleration for the interval is then $\dfrac{aa_1}{t}$ and its direction is from a to a_1.

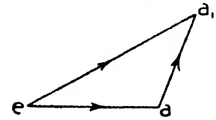


FIG. 22

Example.—At one instant the velocity of a ship is 20 m.p.h. due N.E.. and 10 seconds later it is 10 m.p.h. due E. What is the average acceleration during the interval ?

In Fig. 23 draw $os_1=20$ m.p.h. to any convenient scale and in the N.E. direction, and $os_2=10$ m.p.h. to the same scale; then s_1s_2 is the change of velocity during the interval. By measurement $s_1s_2=16{\cdot}7$ m.p.h., hence the average acceleration is

$$\frac{16{\cdot}7 \text{ m.p.h.}}{10 \text{ secs.}}=16{\cdot}7\times\frac{5280}{60\times60}\times\frac{1}{10}\,\frac{\text{ft./sec.}}{\text{sec.}}=0{\cdot}22 \text{ ft./sec.}^2$$

in a direction 25° W. of S.

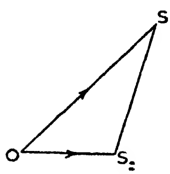


FIG. 23

When a point is moving along a curved path the velocity at any instant is in the direction of the tangent to the curve at the point occupied by the moving point at the instant, and as this direction will be continually changing, the point will be continually accelerated, even though the speed along the curve is constant. If that speed is changing, then the acceleration of the point is due partly to the changing speed and partly to the changing direction. This is considered more fully later.

24. Simultaneous Accelerations.—A point may have two or more simultaneous accelerations in the same manner as it may have two or more simultaneous displacements or velocities ; these may be combined by means of the parallelogram or polygon law. Also, referring to Fig. 11 (*c*), if *ws* and *wm* represent the accelerations of *s* and *m* relative to *w*, then *sm* represents the acceleration of *m* relative to *s* and *ms* that of *s* relative to *m*.* An acceleration may be resolved into components in the same way as is described for velocities in Art. 19.

EXERCISES I

1. The co-ordinates of a moving point are x, y and z, and the motion satisfies the single equation $y=f(x)$. How many degrees of freedom does the point possess ?

2. A point moves in such a way that its co-ordinates x, y and z satisfy the equation $z=x+y$. How many degrees of freedom does the point possess ?

3. A rod has a ball end which engages a fixed spherical socket. How many degrees of freedom does any particular point of the rod possess ?

4. A nut engages a fixed screw on which it is free to turn. How many degrees of freedom does any point of the nut possess and what is the shape of the path described by the point in space ?

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* As will be seen later, this applies only to accelerations relative to frames of reference that remain always parallel to each other. See Arts. 106 and 107.

5. If the screw in the previous question instead of being fixed is connected by a ball-and-socket joint to a fixed body, how many degrees of freedom does any point of the nut now possess ?

6. A point moves in such a way that it always lies in the surfaces of two fixed spheres. How many degrees of freedom does it possess ?

7. How many degrees of freedom does a point on the axle of a railway wagon possess, supposing the wheels to remain always in contact with the rails and slip to be absent ?

8. The co-ordinates x, y and z of a moving point always satisfy the two equations $y=f(x)$ and $z=F(x)$. How many degrees of freedom does the point possess ?

9. A point receives successively the following displacements, all in the horizontal plane : 1 yard from S.W. to N.E., $1\frac{1}{2}$ yards from S.E. to N.W. and 1 yard from E. to W. What is the final displacement of the point ?

10. An aeroplane receives a displacement relative to the air of 1 mile from S. to N., and simultaneously the air receives a displacement relative to the earth of 0·5 mile from W. to E. What is the displacement of the aeroplane relative to the earth ?

11. A wheel 2 ft. dia. rolls without slip along a horizontal line running W. to E. If the wheel turns through an angle of 45°, what is the displacement of that point of its rim which was initially in contact with the ground ?

12. The table below gives the distances (s) of a motor car from a given point at various times (t). Plot a displacement-time curve and find, graphically, the instantaneous speed of the car when $t=4$ secs.

t, secs.	0	1	2	3	4	5	6	7
s, ft.	0	41	76	105	128	145	156	161

13. Plot the displacement-time curve corresponding to the table below and find the instantaneous speeds when $t=2$, $t=5$ and $t=7·5$ secs.

t, secs.	0	1	2	3	4	5	6	7	8	9	10
s, ft.	2	+1·62	+0·62	−0·62	−1·62	−2	−1·62	−0·62	+0·62	+1·62	+2

14. A body A has a velocity of 20 m.p.h. from S. to N. relative to a body B, while the latter has a velocity of 30 m.p.h. from S.W. to N.E. relative to a body C. What is the velocity of A relative to C ?

15. A body A has a velocity of 10 f.s. from S.W. to N.E. relative to the earth, while a body B has a velocity of 10 f.s. from S.E. to N.W. relative to the earth. What is the velocity of A relative to B ?

16. A body which moves in a straight line has a constant acceleration of 1 ft. per sec. per sec. If it starts from rest, what is its speed in m.p.h. at the end of 2 mins. and what distance will it have covered ?

17. A body which moves in a straight line has a velocity at a given moment of 18 m.p.h. If it has a constant acceleration of 60 yards per min. per sec., what is its speed after 20 secs., and what distance will it have covered in that time ?

18. The table below gives the velocity (v) of a body, which moves in a straight line, at times t. Plot a speed-time curve and find (a) the acceleration when $t=3$, (b) the distance covered between the times $t=1$ and $t=5$, and (c) the average speed over this latter period.

t, secs.	0	1	2	3	4	5	6	7	8
v, ft./sec.	5	5·6	7·8	12·2	19·4	30·0	44·6	63·8	88·2

19. At one moment a body has a velocity of 10 m.p.h. from E. to W., and 10 secs. later it is moving with a velocity of 15 m.p.h. from S.E. to N.W. What was the average acceleration during the period ?

20. A train starts from rest at a station A and moves with a constant acceleration f for a certain time, and then with a constant deceleration f_1, so that it comes to rest at station B, which is at a distance d from A. Prove that the time taken for the journey is given by $t=\sqrt{\dfrac{2d(f+f_1)}{ff_1}}$.

21. If in the previous question the maximum speed is limited to V, prove that the time for the journey is given by $t=\dfrac{V(f+f_1)}{2ff_1}+\dfrac{d}{V}$, provided $d \geqq \dfrac{V^2 . (f+f_1)}{2ff_1}$.

CHAPTER II

ANGULAR MOTION

25. Angular Speed.—Consider a point which moves in a circle, centre O, Fig. 24, and which at a given instant is at P and after a time t is at Q. Then the angle POQ $=\theta$ measures the angular displacement of the point about the centre O ; similarly the angle PO_1Q would measure the angular displacement about any other point O_1. The ratio $\frac{\theta}{t}$ is the average angular speed about O, and the unit by which this is measured is $\frac{\text{unit angle}}{\text{unit time}}$, so that we commonly have radians per second (rads./sec.) or revolutions per minute (r.p.m.) as units of angular speed. The conversion from one unit to another is easily effected, thus :

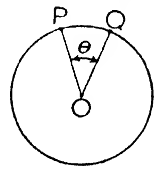


FIG. 24

$$60 \text{ r.p.m.} = 60 \times \frac{\text{revolution}}{\text{minute}} = 60 \times \frac{2\pi \text{ radians}}{60 \text{ seconds}} = 2\pi \text{ rads./sec.}$$

If the average angular speed is the same for all time intervals, then the angular speed is constant and, denoting it by ω, we have

$$\omega = \frac{\theta}{t},\ \theta = \omega t,\ t = \frac{\theta}{\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

These relations should be compared with equations (5), page 6. The graph of θ against t is a straight line the slope of which is equal to the angular speed.

If the angular speed is not constant, then the angular-displacement-time graph is a curve and the angular speed at any instant may be obtained by measuring the slope of the tangent to the curve at the instant, or this may be obtained, if the equation connecting θ and t is known, by differentiation, thus $\omega = \frac{d\theta}{dt} = \dot{\theta}$.

26. Relation between Angular and Linear Speed.—Referring to Fig. 24, the linear displacement of the point consequent on the angular displacement θ is PQ and the average linear velocity of the point is consequently $\frac{PQ}{t}$ in the direction of the chord PQ.

If the time interval is made smaller and smaller until ultimately it is indefinitely small, then the direction of the chord PQ will approach and ultimately coincide with that of the tangent to the curve at P, and thus the direction of the velocity of the point when it is at P is in the direction of the tangent at P, that is, perpendicular to the radius OP. If the time interval is very small, the chord PQ is approximately equal to the arc PQ, and since, if θ is in radians, this is equal to $OP.\theta$, the average linear velocity of the point is $\frac{OP.\theta}{t}$. When the time interval is indefinitely small the approximation becomes exact and the average angular speed $\left(\frac{\theta}{t}\right)$ becomes the actual speed ω at the instant. Hence we have the relation

$$v = r\omega \qquad \qquad (2)$$

between the linear velocity v, the angular velocity ω and the radius r, for circular motion. In using this relation the units employed must be consistent; thus if ω is in radians per second and r is in feet, then v will be in feet $\times \frac{\text{radians}}{\text{secs.}}$, i.e. in feet per sec., since a radian is merely a ratio of two lengths.

27. Angular Acceleration.—When the angular speed is not constant the ratio $\frac{\text{change of angular speed}}{\text{time taken}}$ is the average angular acceleration during the interval. The most convenient unit for this is one radian per second per second (rad./sec.^{2}), but r.p.m. per minute is sometimes convenient. The change from one unit to another is easily effected, thus :

$$\begin{aligned} 60 \text{ r.p.m. per min.} &= 60 \times \frac{\left(\frac{\text{revolution}}{\text{minute}}\right)}{\text{minute}} \\ &= 60 \times \frac{\left(\frac{2\pi \text{ radians}}{60 \text{ secs.}}\right)}{60 \text{ secs.}} \\ &= \frac{2\pi}{60} \text{ radians/sec.}^2 \end{aligned}$$

If the average angular acceleration is the same for all time intervals, then the angular acceleration is constant and is equal to the slope of the angular speed-time graph, which is a straight line. Denoting the constant angular acceleration by a, we have the relations

$$a = \frac{\omega}{t}, \ \omega = at, \ t = \frac{\omega}{a} \qquad \qquad (3)$$

where ω is the change of angular speed in the time interval t. Again the units used must be consistent. If the angular speed at the beginning of a time interval t is ω_1, then the angular speed at the end of the interval is $\omega_2 = \omega_1 + at$, the average angular speed is $\frac{\omega_1+\omega_2}{2}$ and the angle turned through in the interval is $\theta = \left(\frac{\omega_1+\omega_2}{2}\right)t = \omega_1 t + \frac{1}{2}at^2$. Thus, for constant angular acceleration, we have the relations

$$\omega_2 = \omega_1 + at \quad \cdots \quad (4)$$

$$\theta = \omega_1 t + \tfrac{1}{2}at^2 \quad \cdots \quad (5)$$

$$\omega_2^2 = \omega_1^2 + 2a\theta \quad \cdots \quad (6)$$

the last being obtained by eliminating t between (4) and (5).

28. Variable Acceleration.—If the angular acceleration is variable, then the graph of the angular speed against the time is a curve, and the angular acceleration at any instant may be obtained by measuring the slope of the tangent to the curve at that instant, or if the equation connecting the angular speed and the time is known, the angular acceleration at any instant may be obtained by differentiation. Thus $a = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}$.

When the angular acceleration is variable, then the angle turned through in any time interval may be obtained by measuring the area under the angular speed-time curve in a similar manner to that described, for linear motion, in Art. 22. If the scale for angular speed is 1 inch to m rads./sec. and that for time is 1 inch to n seconds, then the scale for the area is 1 sq. in. to $mn \frac{\text{rads.}}{\text{secs.}} \times \text{secs.}$, that is, 1 sq. in. to mn radians.

29. Linear Acceleration of a Point having Circular Motion.—Consider a point which moves in a circle with an accelerated motion; let its angular speed when it is at P (Fig. 25) be ω_1 and,

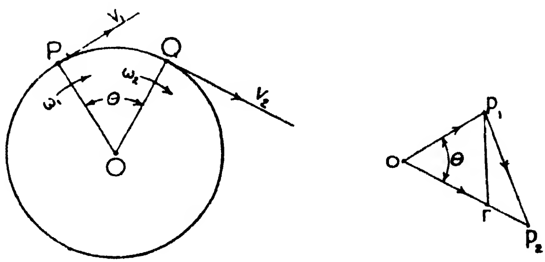


FIG 25

after an interval t, when it is at Q, be ω_2. The corresponding linear velocities are $v_1=r\omega_1$ and $v_2=r\omega_2$, and on setting these out from a common point o the change of velocity is represented by the line p_1p_2 joining their ends (see Art. 23). On op_2 take a point r such that $or=op_1$; then the change of velocity p_1p_2 may be resolved into the components p_1r and rp_2, the latter being equal to v_2-v_1. The acceleration of the point may be regarded as being composed of two components corresponding to the two components of the change of velocity. Taking the component p_1r first, if the time interval t is very small, then the chord p_1r equals the arc p_1r approximately, hence the average acceleration due to the component p_1r is $\frac{p_1r}{t}=\frac{op_1 \cdot \theta}{t}=\frac{v_1\theta}{t}$, and its direction is along p_1r. When the time interval is made indefinitely small the approximation above becomes exact, and the direction of p_1r becomes perpendicular to op_1, that is, parallel to OP, and $\frac{\theta}{t}$ becomes ω_1, the angular speed at the instant when the point is at P. Hence one component of the acceleration of the point is equal to $v_1\omega_1$ and is directed towards the centre of rotation; this component may be called the *normal* or centripetal component of the acceleration and, clearly, it is due simply to the changing *direction* of the velocity of the point. Since $v_1=r\omega_1$, the normal acceleration may be written $r\omega_1^2$ or $\frac{v_1^2}{r}$.

Coming now to the other component of the change of velocity, the average acceleration due to this is $\frac{v_2-v_1}{t}=\frac{r(\omega_2-\omega_1)}{t}$ in the direction op_2. When the time interval is made indefinitely small this direction coincides with op_1, that is, it becomes perpendicular to OP, and the average acceleration becomes the actual acceleration at the instant. Since the value of the ratio $\frac{\omega_2-\omega_1}{t}$ when the interval t is made indefinitely small is the angular acceleration a at the instant when the point is at P, the acceleration of the point due to the component rp_2 is equal to ra. This component is called the *tangential* component and clearly is due to the changing *magnitude* of the velocity of the point.

30. Motion along any Curve.—These results can be applied to the motion of a point along any curve. Thus if O is the centre of curvature of the curve APB at the point P, Fig. 26, then OP($=r$) is the radius of curvature at that point, and if v is the velocity of the point at the instant it is at P, then the angular speed of the point about O at the instant is $\dfrac{v}{r}=\omega$. The normal acceleration of the point is $\dfrac{v^2}{r}=r\omega^2$ and is directed along PO. The tangential acceleration is in the direction of the tangent at P and is given by $\ddot{s}$, s being the distance of the point measured along the curve from any point on the curve as origin.

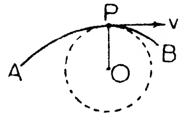


FIG. 26

Numerically $\dot{s}=v$, the velocity at any instant, and hence the tangential acceleration $\ddot{s}$, being equal to $\dfrac{d(\dot{s})}{dt}$, is equal to $\dfrac{dv}{dt}$.

EXERCISES II

1. A flywheel 2 ft. dia. rotates about a fixed axis at 1000 r.p.m. What is the speed in ft. per sec. of a point on its circumference ?

2. A point on a wheel 3 ft. dia., whose axis is fixed, has a linear speed of 1000 ft./min. What is the angular speed in rads./sec. ?

3. A wheel 30 in. dia. rolls without slip along the ground and its angular speed is 200 r.p.m. What is the speed in ft./sec. at any instant of (*a*) the point of the wheel that is in contact with the ground, (*b*) the point on the circumference that is vertically above the axis, and (*c*) a point on the axis ?

4. A shaft rotates about a fixed axis, and at a given instant its angular speed is 100 r.p.m. Ten seconds later its speed is 500 r.p.m. What is the average angular acceleration, in rads./sec.^{2}, during the interval ?

5. A flywheel is rotating with a constant angular acceleration of 2 rads./sec.^{2}. What will be its angular speed at the end of 5 secs. if its angular speed at the commencement is 150 r.p.m. ? What angular displacement will have occurred during the interval ?

6. A weight is attached to a cord which is coiled round a pulley 3 in. dia. If the weight falls 8 ft. in 10 secs., starting from rest and having constant acceleration, find (*a*) the angular acceleration of the shaft, (*b*) the angular speed at the end of 8 secs., and (*c*) the angle turned through in the 10 secs.

7. A flywheel 2 ft. dia. rotates at 1000 r.p.m. What is the acceleration of a point on its rim ?

8. A wheel 1 ft. dia. starts from rest with a constant angular acceleration of 1 rad./sec.^{2}. What is (*a*) the normal component and (*b*) the tangential component of the acceleration of a point on its rim 5 secs. after the start ?

9. A car moves along a curved path with a speed of 40 m.p.h. If at a given moment the radius of curvature of the path is 120 ft., what is the normal acceleration ?

CHAPTER III

(This chapter may be omitted on a first reading.)

ANALYTICAL KINEMATICS OF A POINT

31. Axial Components of Velocity and Acceleration.—When the equation to the path of a point is given in terms of the rectangular co-ordinates of the point it is convenient to obtain the velocity and acceleration of the point in terms of its component velocities and accelerations parallel to the axes. These components are sometimes called the *axial* components. Consider a point that moves along a fixed curve as in Fig. 27. Let its position at time t be P and at time $t+\delta t$ be Q. Then the change of position in the interval δt is $PQ=\delta s$ and the average velocity during the interval is $\frac{\delta s}{\delta t}$ in the direction PQ. When the interval δt is made indefinitely small the ratio $\frac{\delta s}{\delta t}$ becomes $\frac{ds}{dt}$, the actual velocity at the time t, and its direction is that of the tangent to the curve at P. The displacement δs may be resolved into the components δx and δy, and since these occur in the time δt, the average velocities in the directions OX and OY respectively are $\frac{\delta x}{\delta t}$ and $\frac{\delta y}{\delta t}$. When δt is made indefinitely small these velocities become $\frac{dx}{dt}$ and $\frac{dy}{dt}$, the components, in the directions OX and OY, of the velocity $\frac{ds}{dt}$ of the point at the instant. Since $\delta s^2=\delta x^2+\delta y^2$, we have

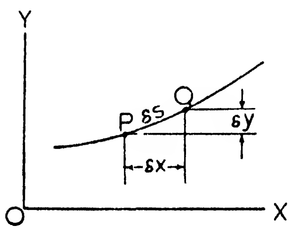


FIG. 27

$$\left(\frac{\delta s}{\delta t}\right)^2=\left(\frac{\delta x}{\delta t}\right)^2+\left(\frac{\delta y}{\delta t}\right)^2$$

and when δt is made indefinitely small this becomes

$$\left(\frac{ds}{dt}\right)^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$$

from which the velocity $\frac{ds}{dt}$ may be found in terms of its components $\frac{dx}{dt}$ and $\frac{dy}{dt}$. The direction of the velocity $\frac{ds}{dt}$ is given by

$$\begin{aligned}\text{Tan}\ \theta &= \text{Lim}_{\ \delta t\to 0}\ \frac{\delta y}{\delta t} \div \frac{\delta x}{\delta t}\\ &= \text{Lim.}\ \frac{\delta y}{\delta x}\\ &= \frac{dy}{dx}\end{aligned}$$

θ being the angle between the velocity and the x axis.

32. Similarly the components, parallel to OX and OY, of the acceleration of the point, being the time rates of change of the component velocities $\frac{dx}{dt}$ and $\frac{dy}{dt}$, are given by $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ respectively. The acceleration of the point is $\sqrt{\left(\frac{d^2y}{dt^2}\right)^2+\left(\frac{d^2x}{dt^2}\right)^2}$ and its direction is given by $\text{Tan}\ \phi=\frac{d^2y}{dt^2}\div\frac{d^2x}{dt^2}$, ϕ being the angle between the direction of the acceleration and the x axis. It should be noted that the acceleration of the point is *not* $\frac{d^2s}{dt^2}$, which is only one component, the tangential, of its acceleration.

33. Radial and Transverse Components.—When the equation to the path is given in polar co-ordinates it is convenient to obtain the velocity and acceleration of the point in terms of their components along, and perpendicular to, the radius vector of the point at any instant. In Fig. 28 let the position of the moving point at time t be P, determined by the co-ordinates r, θ. After an interval of time δt let the point have moved a distance δs along the curve to the position Q (co-ordinates $r+\delta r$ and $\theta+\delta\theta$). Draw the arc PR with centre O; then OR$=r$, RQ$=\delta r$. The displacement PQ may be resolved into the components PR and RQ. Now if δt, and therefore $\delta\theta$, is very small, the chords PQ and PR may be assumed equal to the corresponding arcs; the displacement of the point is then δs in the direction PQ and its components are PR$=r\delta\theta$ in the direction PR and δr in the direction RQ.

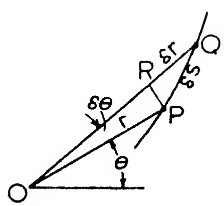


FIG 28

Hence the average velocity of the point during the interval is $\frac{\delta s}{\delta t}$ and its components are $\frac{r\delta\theta}{\delta t}$ and $\frac{\delta r}{\delta t}$. When the time interval is made indefinitely small the approximations made above become exact, $\frac{\delta s}{\delta t}$ becomes $\frac{ds}{dt}$, the actual velocity at the time t, and its components are $r\frac{d\theta}{dt}$ perpendicular to OP and $\frac{dr}{dt}$ along OP. These are respectively the *transverse* and *radial* components.

34. Turning now to the acceleration of the point, let the radial and transverse components of the velocity of the point at time t be denoted by u and v respectively as in Fig. 29. After an interval δt these components have changed their magnitudes to $u+\delta u$ and $v+\delta v$ and their directions have changed by the angle $\delta\theta$. Through Q draw QL and QM parallel and perpendicular respectively to OP. Then at time $t+\delta t$ the component, in the direction QL, of the velocity of the point is

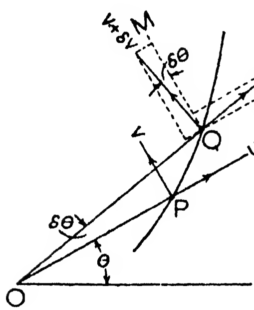


FIG. 29

$$(u+\delta u)\ \mathrm{Cos}\ \delta\theta-(v+\delta v)\ \mathrm{Sin}\ \delta\theta\ ;$$

hence the change of velocity in this direction is $(u+\delta u)\ \mathrm{Cos}\ \delta\theta-(v+\delta v)\ \mathrm{Sin}\ \delta\theta-u$. Now when $\delta\theta$ is very small $\mathrm{Cos}\ \delta\theta=1$ and $\mathrm{Sin}\ \delta\theta=\delta\theta$ approximately; thus the change of velocity along QL is $\delta u-v\delta\theta-\delta v\delta\theta$ and the last term is negligible in comparison with the others. Hence the average acceleration in the direction QL is $\frac{\delta u-v\delta\theta}{\delta t}=\frac{\delta u}{\delta t}-v\frac{\delta\theta}{\delta t}$, and when δt is made indefinitely small the approximation becomes exact and the average acceleration in the direction of QL becomes the actual acceleration at the time t, along OP; its value is then $\frac{du}{dt}-v\frac{d\theta}{dt}$. The first term is the rate of change of the radial velocity u and the second term is the acceleration due to the changing direction of the velocity v; this will be clear if $\frac{d\theta}{dt}$ is written as ω and reference is made to Art. 29. Since (by Art. 33) $u=\frac{dr}{dt}$ and

$v=r\dfrac{d\theta}{dt}$, we obtain for the radial component of the acceleration of the point the expression $\dfrac{d^2r}{dt^2}-r\left(\dfrac{d\theta}{dt}\right)^2$, or $\ddot{r}-r(\dot{\theta})^2$.

Similarly for the acceleration in the direction of QM, this is given by the limiting value, when $\delta t=o$, of the ratio

$$\frac{\text{change of velocity in the direction QM}}{\delta t}$$

that is by

$$\text{Lim.}_{\delta t=o}\frac{(v+\delta v)\text{ Cos }\delta\theta+(u+\delta u)\text{ Sin }\delta\theta-v}{\delta t}$$

which is equal to $\dfrac{dv}{dt}+u\dfrac{d\theta}{dt}$. The first term of this is the rate of change of the velocity v and the second term is the acceleration due to the changing direction of the velocity u. Since $u=\dfrac{dr}{dt}$ and $v=r\dfrac{d\theta}{dt}$, we obtain for the transverse component of the acceleration of the point the expression

$$\frac{dr}{dt}\cdot\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}+\frac{dr}{dt}\cdot\frac{d\theta}{dt}$$

$$=r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\cdot\frac{d\theta}{dt},\text{ or }r\ddot{\theta}+2\dot{r}\dot{\theta}$$

35. Motion of a Point that Moves in a Rotating Plane.—Consider a point P, Fig. 30, which is moving along the curve LM while the plane in which that curve lies rotates about the axis OY. Let the position of the plane XOY be specified by the angle θ between it and a fixed plane AOY, and let the position of the point in the plane be specified by the rectangular co-ordinates x, y. The velocity and acceleration of the point P at any instant can be resolved into three components, one parallel to OY, one parallel to OX and one perpendicular to the plane XOY. The components parallel to OY are unaffected by the rotation of the plane, and hence are $\dot{y}$ and $\ddot{y}$ respectively. The other components are the same as the corresponding components of the velocity and acceleration of the projection Q of the point. The motion of Q being defined by the polar co-ordinates x, θ, we have, by Arts. 33 and 34, radial velocity parallel to OX$=\dot{x}$, radial

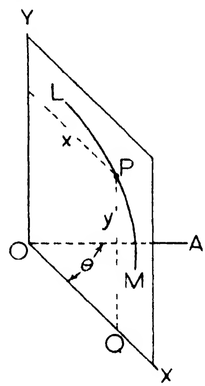


FIG. 30

acceleration $=\ddot{x}-x(\dot{\theta})^2$, transverse velocity perpendicular to plane XOY $=x\dot{\theta}$, transverse acceleration $=x\ddot{\theta}+2\dot{x}\dot{\theta}$.

If x is constant, then the point P moves on the surface of a cylinder whose axis is OY and it is said to have *cylindrical motion.* Then if $x=r$, the radius of the cylinder, $\dot{x}$ and $\ddot{x}$ are zero, and on substituting these values in the expressions above the components of the velocity and acceleration in this type of motion are obtained. Again, if $x=y$ Tan a, where a is a constant, then the point moves on the surface of a cone having O as apex and OY as axis. Then

$$\dot{x}=\frac{dx}{dy}\cdot\frac{dy}{dt}=\text{Tan}\ a\ .\ \dot{y}$$

and

$$\ddot{x}=\frac{d}{dt}(\dot{y}\ \text{Tan}\ a)=\ddot{y}\ \text{Tan}\ a$$

and on making these substitutions the components for this type of motion, *conical motion*, are obtained. Similarly if the curve LM is a circle, centre O, the point P moves on the surface of a sphere and has *spherical motion.* The components may be obtained in a similar manner to that employed for conical motion, but this is left as an exercise for the student.

36. Moving Axes.—Consider a point P, Fig. 31, that is moving along a curve LM, which is fixed relatively to the frame $X_1O_1Y_1$, while that frame itself moves relatively to the fixed frame XOY, but remains always in the plane XOY. Then the velocity and acceleration of P relative to the frame XOY may be found as follows. Let the position of P relative to $X_1O_1Y_1$ be determined by the co-ordinates x_1, y_1, and let the position of $X_1O_1Y_1$ relative to XOY be determined by $\bar{x}$, $\bar{y}$, the co-ordinates of O_1, and the angle θ. Then the co-ordinates of P relative to XOY are given by

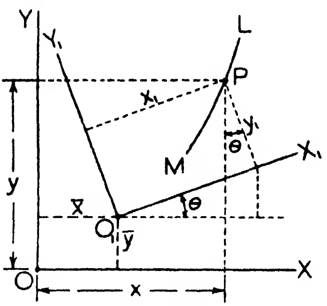


FIG. 31

$$x=\bar{x}+x_1\ \text{Cos}\ \theta-y_1\ \text{Sin}\ \theta$$
$$y=\bar{y}+x_1\ \text{Sin}\ \theta+y_1\ \text{Cos}\ \theta$$

and on differentiating these expressions with respect to time we obtain $\frac{dx}{dt}$ and $\frac{dy}{dt}$, the components of the velocity of P relative to the frame XOY.

Thus, $$\frac{dx}{dt}=\frac{d\bar{x}}{dt}+\frac{dx_1}{dt}\ \text{Cos}\ \theta-x_1\ \text{Sin}\ \theta\ \frac{d\theta}{dt}-\frac{dy_1}{dt}\ \text{Sin}\ \theta-y_1\ \text{Cos}\ \theta\ \frac{d\theta}{dt}$$

$$=\frac{d\bar{x}}{dt}+\frac{dx_1}{dt}\text{ Cos }\theta-\frac{dy_1}{dt}\text{ Sin }\theta-(x_1\text{ Sin }\theta+y_1\text{ Cos }\theta)\frac{d\theta}{dt}$$

and $$\frac{dy}{dt}=\frac{d\bar{y}}{dt}+\frac{dy_1}{dt}\text{ Cos }\theta+\frac{dx_1}{dt}\text{ Sin }\theta-(y_1\text{ Sin }\theta-x_1\text{ Cos }\theta)\frac{d\theta}{dt}.$$

The term $\frac{d\bar{x}}{dt}$ is the component, parallel to OX, of the velocity of O_1 relative to O. The terms $\frac{dx_1}{dt}\text{ Cos }\theta-\frac{dy_1}{dt}\text{ Sin }\theta$ are the components parallel to OX of the velocity of P relative to the frame $X_1O_1Y_1$, while the remaining term $-(x_1\text{ Sin }\theta+y_1\text{ Cos }\theta)\frac{d\theta}{dt}$ is the component parallel to OX of the velocity of P relative to O_1 due to the angular velocity of the axes about O_1. Similarly with the expression for $\frac{dy}{dt}$.

Thus it is seen that the velocity of P relative to XOY is the vector sum of the velocity P would have if the frame $X_1O_1Y_1$ were fixed and P moved along LM, and the velocity P would have if it were fixed relative to the frame $X_1O_1Y_1$ while that frame performed its motion relative to XOY. Expressed in another way the velocity of P relative to XOY is the vector sum of the velocity of P relative to a frame $X'O_1Y'$ that, while moving with the point O_1, does not rotate, and the velocity of O_1 relative to XOY.

37. On differentiating the expressions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ with respect to time we shall obtain $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$, the components of the acceleration of P relative to the frame XOY. Thus,

$$\begin{aligned}\frac{d^2x}{dt^2}=\frac{d^2\bar{x}}{dt^2}&+\frac{d^2x_1}{dt^2}\text{ Cos }\theta-\frac{dx_1}{dt}\text{ Sin }\theta\,.\,\frac{d\theta}{dt}-\frac{d^2y_1}{dt^2}\text{ Sin }\theta-\frac{dy_1}{dt}\text{ Cos }\theta\frac{d\theta}{dt}\\&-\left[\frac{dx_1}{dt}\text{ Sin }\theta+x_1\text{ Cos }\theta\,.\,\frac{d\theta}{dt}+\frac{dy_1}{dt}\text{ Cos }\theta-y_1\text{ Sin }\theta\,.\,\frac{d\theta}{dt}\right]\frac{d\theta}{dt}\\&-(x_1\text{ Sin }\theta+y_1\text{ Cos }\theta)\frac{d^2\theta}{dt^2}\end{aligned}$$

$$=\underbrace{\frac{d^2\bar{x}}{dt^2}}_{(1)}+\underbrace{\frac{d^2x_1}{dt^2}\text{ Cos }\theta-\frac{d^2y_1}{dt^2}\text{ Sin }\theta}_{(2)}\underbrace{-(x_1\text{ Sin }\theta+y_1\text{ Cos }\theta)\frac{d^2\theta}{dt^2}}_{(3)}$$

$$\underbrace{-(x_1\text{ Cos }\theta-y_1\text{ Sin }\theta)\left(\frac{d\theta}{dt}\right)^2}_{(4)}\underbrace{-2\frac{d\theta}{dt}\left(\frac{dx_1}{dt}\text{ Sin }\theta+\frac{dy_1}{dt}\text{ Cos }\theta\right)}_{(5)}$$

Similarly,

$$\frac{d^2y}{dt^2}=\underset{(1)}{\frac{d^2\bar{y}}{dt^2}}+\underbrace{\frac{d^2y_1}{dt^2}\text{ Cos }\theta+\frac{d^2x_1}{dt^2}\text{ Sin }\theta}_{(2)}\underbrace{-(y_1\text{ Sin }\theta-x_1\text{ Cos }\theta)\frac{d^2\theta}{dt^2}}_{(3)}$$

$$\underbrace{-(y_1\text{ Cos }\theta+x_1\text{ Sin }\theta)\left(\frac{d\theta}{dt}\right)^2}_{(4)}\underbrace{-2\frac{d\theta}{dt}\left(\frac{dy_1}{dt}\text{ Sin }\theta-\frac{dx_1}{dt}\text{ Cos }\theta\right)}_{(5)}$$

Terms (1), (3) and (4) of these expressions together give the acceleration P would have if it were fixed to the frame $X_1O_1Y_1$ while that frame performed its motion relative to XOY, while terms (2) give the acceleration P would have if the curve LM and its frame $X_1O_1Y_1$ were fixed relative to XOY and the point performed its motion along the curve. Thus it is not true to say that the acceleration of P relative to XOY is the vector sum of the acceleration P would have if $X_1O_1Y_1$ were fixed and the acceleration P would have if it were fixed to $X_1O_1Y_1$, since this does not bring in the acceleration represented by terms (5). These terms represent an acceleration $2u\omega$ where u is the velocity of the point P along the curve LM and ω $\left(\text{equal to } \frac{d\theta}{dt}\right)$ is the angular velocity of that curve relative to XOY, and this acceleration is sometimes called the *compound supplementary acceleration*. Thus we have

The acceleration of P relative to XOX

=The acceleration P would have if the curve LM were fixed to XOX.

+The acceleration P would have if it were fixed to the curve while that curve performed its motion relative to XOX.

+The compound supplementary acceleration $2u\omega$.

This is usually called *Coriolis's Law*. The direction of the compound supplementary acceleration is perpendicular to that of the velocity u and its sense may be determined as described in Art. 107.

38. Rotating Axes.—The results of the two previous articles are considerably simplified when the frame $X_1O_1Y_1$ merely rotates, with a constant angular velocity, about the point O, especially if the fixed axes OX, OY are chosen, as they usually can be, so that they coincide, at the instant under consideration, with the moving axes. Then $\bar{x}$, $\bar{y}$, $\frac{d\bar{x}}{dt}$, $\frac{d\bar{y}}{dt}$, $\frac{d^2\bar{x}}{dt^2}$, $\frac{d^2\bar{y}}{dt^2}$, θ and $\frac{d^2\theta}{dt^2}$ are all zero and the expressions for the component velocities reduce to

$$\frac{dx}{dt}=\frac{dx_1}{dt}-y_1\frac{d\theta}{dt}$$
$$=u$$

and
$$\frac{dy}{dt}=\frac{dy_1}{dt}+x_1\frac{d\theta}{dt}$$
$$=v$$

while the expressions for the component accelerations become

$$\frac{d^2x}{dt^2}=\frac{d^2x_1}{dt^2}-2\frac{dy_1}{dt}\cdot\frac{d\theta}{dt}-x_1\left(\frac{d\theta}{dt}\right)^2$$
$$=\frac{du}{dt}-v\frac{d\theta}{dt} \; *$$

and
$$\frac{d^2y}{dt^2}=\frac{d^2y_1}{dt^2}+2\frac{dx_1}{dt}\cdot\frac{d\theta}{dt}-y_1\left(\frac{d\theta}{dt}\right)^2$$
$$=\frac{dv}{dt}+u\frac{d\theta}{dt} \; *$$

which brings out clearly the fact that the acceleration of P is due partly to the changing magnitude and partly to the changing direction (consequent on the rotation of the axes) of the velocity of P.

EXERCISES III

1. The ends A and B of a rod 2 ft. long are constrained to lie in two lines OX and OY respectively, the angle YOX being 90°. If the end A moves with a constant velocity of 1 f.s. towards O, find, for the instant that OA = 1 ft., the components parallel to OX and OY of the velocity and acceleration of the point of the rod distant 6 in. from A.

2. A rod AB moves relative to rectangular axes OX, OY, and at any instant the component velocities of its ends A and B are $\dot{x}_a, \dot{x}_b, \dot{y}_a$ and $\dot{y}_b$. Prove that the components of the velocity of a point P distant a from A and b from B are $\dot{x}_P=\dfrac{a\dot{x}_b+b\dot{x}_a}{a+b}$ and $\dot{y}_P=\dfrac{a\dot{y}_b+b\dot{y}_a}{a+b}$.

3. A particle moves along the curve $y=x^2$ with a constant speed of 10 f.s. What are the components parallel to the axes of the velocity and acceleration of the point when $x=2$ ft. ?

4. A particle moves along a curve whose equation relative to rectangular axes Ox, Oy is $y=x^2$ with constant speed of 10 f.s., while those axes rotate about O with constant angular speed of 1 rad./sec. anticlockwise. At a given moment $x=2$ ft.; what are then the components, parallel to fixed axes which at that moment coincide with the moving ones, of the velocity and acceleration of the particle ?

5. A particle moves round a circle radius 1 ft. with a constant speed of 10 f.s., while the circle rotates with constant speed of 5 rads./sec. about a vertical diameter. Taking the latter to be the y axis and the centre of the circle as origin,

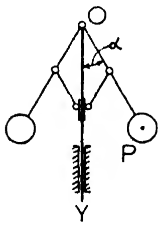
---

* These expressions cannot be obtained by differentiating those for u and v, because in those expressions terms whose values are zero have been omitted, but the differential coefficients of these zero terms are not necessarily zero.

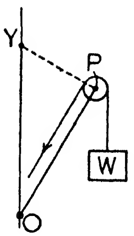
find the x, y and z components of the velocity of the particle when the plane of the circle makes an angle of 45° with the xy plane and the radius to the particle makes an angle of 45° with the xz plane. Take the sense of rotation of the circle to be from Ox to Oz and the sense of the rotation of the particle from Oy towards the xz plane.

6. A particle slides with a constant acceleration of 2 ft./sec.^{2} down the generator of a cone, semi-apex angle 30°, starting from rest at the apex, while the generator rotates about the axis with an angular acceleration of 0·5 rads./sec.^{2}. If the generator starts from rest in the xy plane, at the same moment as the particle, and rotates towards the yz plane, find the x, y and z components of the velocity and acceleration of the particle at the end of 2 secs.

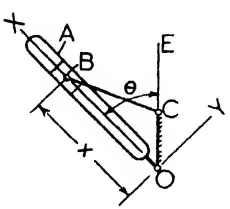
7. A point P moves round a circle centre O, radius r, while the plane of the circle rotates about a vertical radius OY (Y being above O). Taking OY as the axis of y and O as the origin, derive expressions for the radial, transverse and axial components of the velocity and acceleration of the point. Let angle YOP be a and take the angular speed and acceleration of the circle about OY to be ω and $\dot{\omega}$ respectively.



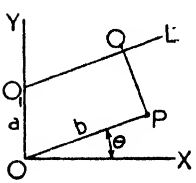
8. The figure shows a simple form of governor, and the whole mechanism is rotating about the axis OY with an angular velocity of 10 rads./sec. and an angular acceleration of 2 rads./sec.^{2}. If OP = 1 ft., $a = 30°$, $\dot{a} = 1$ rad./sec. and $\ddot{a} = -0·5$ rads./sec.^{2}, find the components of the acceleration of the point P.



9. In the jib crane shown diagrammatically in the figure the jib OP is 20 ft. long and is being raised with an angular velocity of 0·05 rads./sec. and is rotating about the axis OY at an angular speed of 0·2 rads./sec. If the rope is being hauled up with an acceleration of 2 ft./sec.^{2} and the load W starts from rest when the distance PW is 10 ft. and the angle YOP is 45°, find the components of the acceleration of the load W. Neglect the diameter of the pulley at P.



10. The figure shows the mechanism of a rotary engine (see Art. 122), and relative to axes OX, OY, fixed to the cylinder A, the position of the piston B is given approximately by $x = l + r \operatorname{Cos} \theta - \frac{r^2}{2l^2} \operatorname{Sin}^2 \theta$. If the cylinder is rotating about O with an angular speed of 1000 r.p.m., find the components along and perpendicular to OE when the angle $\theta = 20°$. Assume $l = 1$ ft. and $r = 0·25$ ft.



11. A rod OP, length b, rotates with constant angular speed ω about a fixed centre O. A second rod O_1L also rotates about a fixed centre O_1 and is always parallel to OP. If PQ is drawn perpendicular to OP, prove that the components of the velocity of Q relative to the fixed frame XOY are $\omega a \operatorname{Sin}^2 \theta - \omega a \operatorname{Cos}^2 \theta - \omega b \operatorname{Sin} \theta$ alon OX and $b\omega \operatorname{Cos} \theta - 2a\omega \operatorname{Sin} \theta \operatorname{Cos} \theta$ along OY.

CHAPTER IV

MOTION OF A LINE. PLANE MOTION OF A BODY

39. This chapter commences with the consideration of the position and motion of a line; but this also covers a type of motion of a body which is of very frequent occurrence in mechanisms—namely, *plane motion.*

A body is said to have plane motion when three of its points that form a triangle lie always in a fixed plane, this plane being called the plane of motion. A cube which always keeps the same face in contact with a fixed flat table has plane motion. The plane of motion, or one parallel to it, will then always intersect the body in the same section, and that section may be used to represent the body. Further, since the position of every point of such a section is fixed relatively to the ends P and Q of any line of the section, the line PQ may be used to represent the section and therefore the body.

40. Position of a Line.—The position of a line is fixed relative to a rectangular frame of reference when the positions of any two points of the line are fixed; six co-ordinates must be given to specify the positions of these two points, but these co-ordinates are not independent. They are related by an equation of the form $(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2=l^2$, where l is the length of the line between the points $x_1y_1z_1$ and $x_2y_2z_2$.

Alternatively the position of a line may be specified by giving the three co-ordinates of any point on it together with the three angles that it makes with the three axes. Again six quantities must be given, but again they are related by an equation.

A line which is free in space has thus only five degrees of freedom.

41. Motion of a Line.—This may be of two distinct kinds, namely, *Translation* or *Rotation*, although it may have both kinds simultaneously.

In motion of translation the line always remains parallel to its original position; the paths described by all the points of the line are similar, as indicated in Fig. 32.

In motion of rotation the paths of all the points of the line are concentric circles whose centres lie on a fixed straight line which is the axis of rotation. In Fig. 33 this axis is seen as the point O, since it is perpendicular to the plane of the paper because the first, last and all the intermediate positions of the line lie in that

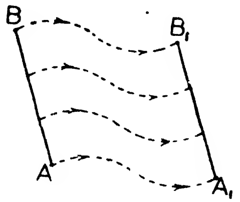


FIG. 32

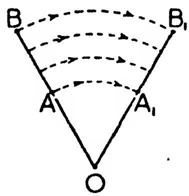


FIG. 33

plane. The point O is called the centre of rotation; it should, however, always be remembered that rotations are about *axes* and not about *points*.

A line that is free in space may have *three* independent translations, parallel to the three axes. It can have, however, only *two* independent rotations, about any two of the axes. Any rotation about the third axis could be produced by giving the line suitable rotations about the other two axes, and so is not independent. These three translations and two rotations constitute the five degrees of freedom which, as stated in Art. 40, a line that is free in space possesses.

If a line is confined to a particular plane, say that of the paper, then it has only three degrees of freedom, two translations and one rotation, the latter about any axis perpendicular to the plane of motion.

If one point of a line is fixed the line has only two degrees of freedom, two rotations about any two axes passing through the fixed point.

42. Translation produced by two Rotations.—The change of position produced by a translation can also be produced by means of two rotations, about two centres, one of which may be chosen arbitrarily. The translation from AB to A_1B_1 in Fig. 34 may be produced by a rotation about any point O_1 on the perpendicular through the mid-point of AA_1 and lying in the plane of AB and A_1B_1 (which brings the line to A_1B_2) together with a rotation, about A_1, through the same angle as that about O_1, but in the opposite sense.

Alternatively, as shown in Fig. 35, the first rotation may be about any point O_1, bringing the line to A_2B_2. The second rotation is again equal in magnitude, but opposite in sense to the first, and it must be about one particular point O_2 lying on the perpendicular bisector of A_2A_1 and chosen so that $\angle A_2O_2A_1 = \angle AO_1A_2$.

Fig. 35 shows also that a rotation, as, for example, from AB to

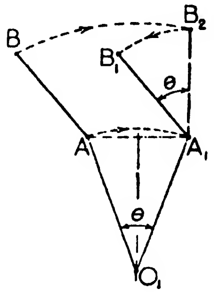


Fig. 34

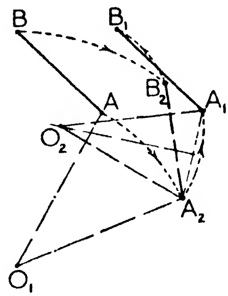


Fig. 35

A_2B_2, may be produced by a translation, from AB to A_1B_1, together with a rotation, about another axis O_2.

43. Virtual Centres.—When a line is confined to a plane, then any displacement it may receive can also be produced by a single rotation about some particular centre, called the *virtual centre* for the displacement, which may be found as follows :

Let the displacement be from AB to A_1B_1 in Fig. 36. Join AA_1 and BB_1, and from their mid-points draw perpendiculars to intersect in O, which is the required centre. Actually, of course, the rotation is about an axis, through O, perpendicular to the plane of motion. This axis may be called the virtual axis, and such an axis may be found for any displacement of a line, whether in a plane or not. Thus let AB and A_1B_1 be the two positions of the line. Join A to A_1 and at the mid-point L erect a plane perpendicular to AA_1. Join B to B_1 and at the mid-point M erect a plane perpendicular to BB_1. These two planes will then intersect in a line which is the required axis. Should the two planes be parallel to each other, then the virtual axis lies at infinity and the displacement is a translation.

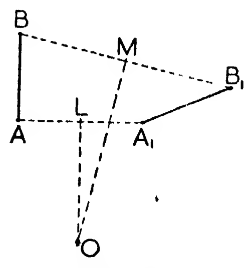


Fig. 36

44. Instantaneous Centres. Plane Motion.—A line that is moving in any manner may be considered at any particular instant to be rotating about some centre. This centre is called the instantaneous centre and is the virtual centre for the movement of the line during an indefinitely small time interval including the instant under consideration. It cannot, however, be found by the methods used for virtual centres, but consideration will show that if the displacement AB to A_1B_1 (Fig. 36) is indefinitely small, then the lines LO and MO become the normals to the paths of A and B, respectively, at the positions occupied by those points at the instant, so that the instantaneous centre may now be found by drawing the normals as shown in Fig. 37.

Since the velocities of the points A and B are tangential to their respective paths, the normals AO and BO are perpendicular to those velocities ; hence if the directions of those velocities alone are known, the instantaneous centre may be found, as shown in Fig. 38, by drawing the perpendiculars AO and BO.

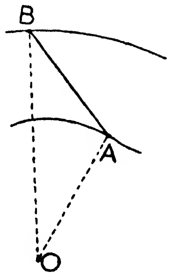


FIG. 37

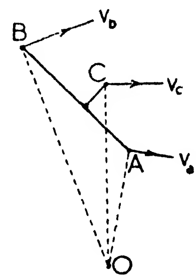


FIG. 38

If the instantaneous angular velocity of AB about O is Ω rads./sec., then

$$\Omega=\frac{v_a}{\mathrm{OA}}=\frac{v_b}{\mathrm{OB}}.$$

Hence if the magnitude of v_a is known, that of v_b can be calculated after OA and OB have been measured.

The velocity of any other point C of, or attached to, the line can also be found, thus $v_c=\Omega\,.\,\mathrm{OC}=v_a\,.\,\frac{\mathrm{OC}}{\mathrm{OA}}$. The direction of v_c is, of course, perpendicular to OC and its sense is obtained by inspection.

It should be noted, however, that the accelerations of the points A, B, etc., are not the same as those the points would have if O were a *fixed* centre ; instantaneous centres should not be used to determine accelerations.

The constructions of Figs. 37 and 38 fail when the velocities v_a and v_b are parallel. In this case, however, the velocities must either be equal, when the motion is one of translation, or, if they are unequal, must both be perpendicular to the line, otherwise it will be found that the components, along AB, of v_a and v_b are unequal, which is impossible, since the distance AB is fixed. When v_a and v_b are parallel and unequal the instantaneous centre can be found, if the magnitudes of v_a and v_b, or at least their ratio, are known, as indicated in Fig. 39.

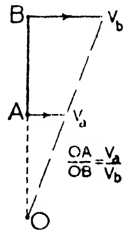


Fig. 39

45. Centrodes.—In Fig. 40, let the line AB be drawn on a sheet of paper 1, which is free to move relatively to the sheet 2, underneath it, and let the ends A and B of the line move along the paths *aa* and *bb*, respectively, drawn on the sheet 2, which is regarded as being fixed. At the instant that the line occupies the position shown the instantaneous centre of AB will be some point O. Let this point O be marked on a third sheet 3, which, for the present, is at rest relative to 2. An instant later the instantaneous centre will be some new point O_1, and as the line AB moves so the instantaneous centre traces out a path XX on the sheet 3. Let this sheet carry a line CD, then the curve XX is the *centrode of AB relative to CD*, and it may be regarded as being fixed to CD. Suppose now that AB is brought to rest in the position shown, but that by giving a suitable motion to the sheet 3 the relative motion between AB and CD is kept the same as before. The ends of the line CD will then trace out paths *cc* and *dd* relative to the fixed sheet 2, and at any instant there will be an instantaneous centre for CD. This centre will have a locus, some curve YY, which may be drawn on the sheet 1, which is now at rest. The curve YY is the centrode of CD relative to AB and it may be regarded as being fixed to AB. At any instant the two centrodes touch in a point which is the instantaneous centre at the instant, and the relative motion between AB

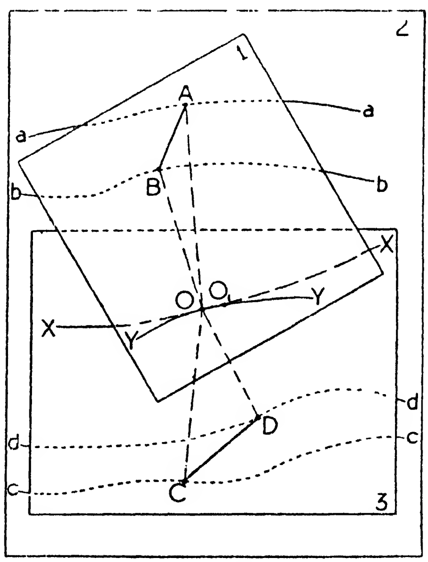


Fig. 40

and CD may be produced by rolling the one centrode, without slip, on the other.

When considering the motion of a line in space the centrode XX which is fixed relatively to the space is sometimes called the *space-centrode*, while the centrode YY which is fixed relative to the line or body AB is called the *body-centrode*.

It is instructive to derive the centrodes in another way, as follows: Referring to Fig. 41, let AB, A_1B_1, A_2B_2, etc., be a number of consecutive positions of a line moving in a plane, and let O_1, O_2, O_3, etc., be the virtual centres for the displacements AB to A_1B_1, A_1B_1 to A_2B_2, etc. Then O_1, O_2, O_3, etc., form a polygon which may be regarded as being fixed to the plane of motion. Now let a second polygon O_1, O_2', O_3', etc., be constructed such that $O_1O_2'=O_1O_2$, $\alpha=\angle AOA_1$; $O_2'O_3'=O_2O_3$, $\theta=\angle A_1O_2A_2-\phi$, etc. Then if the second polygon is imagined to be fixed to the line and to rotate about O_1 until O_1O_2' coincides with O_1O_2, then about O_2 until $O_2'O_3'$ coincides with O_2O_3, etc., the line will come successively into the positions A_1B_1, A_2B_2, etc. If now the successive positions of the line are taken closer and closer together, then the polygons will ultimately become smooth continuous curves and the motion of the line relative to the plane of motion could be produced by the rolling of the curve that is fixed to the line on that which is fixed to the plane of motion; it will be seen that the curves are respectively the body and space centrodes.

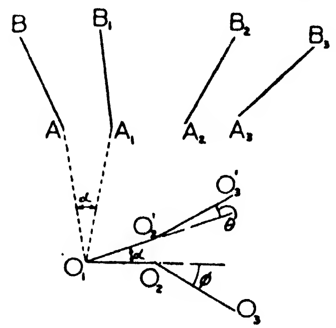


FIG. 41

46. Axodes.—The instantaneous centres, O_1, O_2, etc., are, of course, only the end views of instantaneous axes. These axes form a pair of ruled surfaces (surfaces which can be swept out by the continuous movement of a straight line), which are called axodes. The centrodes are thus merely the intersections of the axodes with the plane of motion. In plane motion all the instantaneous axes are perpendicular to the plane of motion and form a cylindrical surface, but for other types of motion the axodes are either conical or of a general type of ruled surface, neither conical nor cylindrical. (It should be noted that the terms cylindrical and conical do not imply that the surfaces are *circular* cylinders or cones, although in many cases they are circular.)

47. Instantaneous Axes. Non-plane Motion.—When the velocities of the ends of a line are not co-planar, the instantaneous axis may be found as follows : Let the line be labelled AB, and at A and B erect planes perpendicular respectively to the velocities of the points A and B. These planes will intersect in a line which is the instantaneous axis.

The construction fails when the velocities are both perpendicular to the line, but, provided the magnitudes of the velocities are known, the instantaneous axis may be found as follows : Choose a plane (see Fig. 42) containing the line AB and such that the resolved parts of the velocities v_a and v_b in this plane are equal in magnitude and the same in sense. The resolved parts of v_a and v_b in a plane, also containing AB, but being perpendicular to the first plane, will then be either unequal in magnitude but the same in sense, as shown in the figure, or they may have opposite senses and can then be either equal or unequal in magnitude. Join the ends C and D of these last velocity vectors and produce the line CD to intersect the line AB, produced, if necessary, in O. Then a rotation about an axis OO_1 parallel to the equal components of v_a and v_b will give A and B the unequal component velocities, while a translation along the axis through O will give A and B the equal component velocities. Hence a screw motion about O_1O will give A and B their actual velocities The axis OO_1 is the instantaneous axis of the motion of AB.

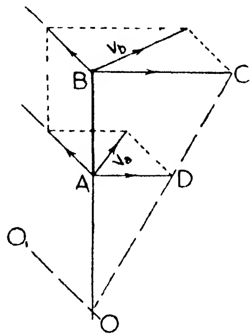


FIG. 42

A screw motion about an instantaneous axis is the most complicated motion a line (or a rigid body) can have.

The locus of the axis OO_1 is the space-axode of the motion of AB and corresponding to it there is the body-axode fixed to the line. The motion of the line is then produced at any instant by a screw motion of the body-axode about the line in which, at the instant, it touches the space-axode.

48. Spherical Motion.—A body is said to have spherical motion when it moves so that a fixed spherical surface always intersects the body in the same section as indicated in Fig. 43. Since the position of any point of that section is fixed relatively to the ends, A and B, of a portion of a great circle * of the sphere, the arc AB may be used to represent the section and thus the body, just as a straight line may be used to represent a body having plane

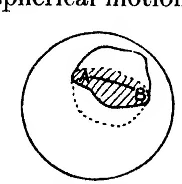


FIG. 43

* A "great circle" of a sphere is any circle, lying on the surface of the sphere, whose plane contains the centre of the sphere.

motion; in fact, the latter is simply a special case of spherical motion, the sphere being infinitely large. Thus what has been said in Arts. 42 to 46 has a counterpart in spherical motion. For any displacement of the line AB a virtual axis may be found about which a single rotation will produce the displacement. The method of finding this axis is analogous to that described in Art. 43. In Fig. 44, let the displacement be from AB to A_1B_1, and join A to A_1 and B to B_1 by arcs of great circles. Through the mid-point L of the arc AA_1 draw a plane OLR containing the centre O of the sphere and being perpendicular to the plane AOA_1. A suitable rotation about any axis containing O and lying in the plane OLR will bring A to A_1. Similarly through the mid-point of the arc BB_1 draw a plane OMR containing O and being perpendicular to the plane BOB_1. A rotation about any axis containing O and lying in the plane OMR will bring B to B_1. The planes OLR and OMR will intersect in a line OR which is the required virtual axis. To prove this join R to A, A_1, B and B_1 by means of arcs of great circles; then the spherical triangles ARB and A_1RB_1 are equal, having their corresponding sides equal, hence the angles ARB and A_1RB_1 are equal, and on adding to each the angle BRA_1 the angles ARA_1 and BRB_1 are seen to be equal. Thus a rotation about OR through the angle ARA_1 which will bring A to A_1 will also bring B to B_1.

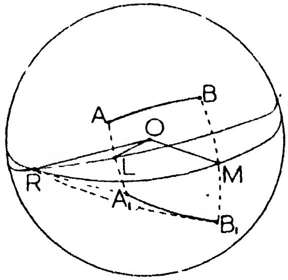


Fig. 44

When the displacement of the line is made indefinitely small the virtual axis becomes the instantaneous axis, and it may be found by drawing through A and B planes perpendicular to the velocities of A and B respectively. These planes will intersect in the instantaneous axis OR, which can always be found, without exception. The locus of the instantaneous axis OR is the space-axode and is a conical surface whose apex is at O. This conical surface intersects the sphere in a curve which is the space-centrode. Corresponding to the space-axode there is a conical surface having its apex at O and being fixed to the line AB; this is the body-axode, and the motion of the line may be produced by the rolling of the body-axode on the space-axode.

49. Angular Velocity.—Just as the linear *velocity* of a point is its speed in a stated direction, so the angular *velocity* of a line or body is its angular speed in a stated direction, the direction being that of the axis about which the motion takes place. Thus angular velocities, like linear velocities, possess magnitude, direction and sense, and they also can be represented by lines, but whereas a portion of any line parallel to the direction of a linear velocity may be used to represent that velocity, with angular velocities portions of the axes of the motions only may be used. This difference is sometimes expressed by the statement that a linear velocity is an *unlocalised* or *free* vector, while an angular velocity is *localised in a line*, the axis of the motion, and is called a *rotor* or *locor*. An angular velocity is also called a *sliding* vector, since any portion of the axis may be used to represent it.

To indicate the sense of an angular velocity, clockwise or anti-clockwise, an arrowhead is placed on the line representing it, according to the following convention. The arrow-head is placed so that it points in the direction a right-handed screw would travel if it turned in a fixed nut in the same sense as the angular velocity, as shown in Fig. 45. The method of labelling linear velocities; stated in Art. 16 may also be used for angular velocities; thus ab is the angular velocity of B relative to A, and ba is the angular velocity of A relative to B.

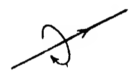


FIG. 45

50. Simultaneous Angular Velocities. Intersecting Axes.—A line or body may have two or more simultaneous angular velocities; this is illustrated by the arrangement shown in Fig. 46, where the bevel wheel Q is free to turn on the arm R of an axle S that rotates in fixed bearings. The wheel Q meshes with a fixed bevel wheel T. When the axle S is turned, then Q turns on the arm R about the axis YY and the axis YY rotates about XX.

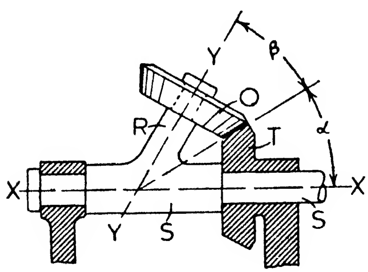


FIG. 46

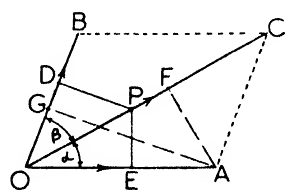


FIG. 47

When the axes of these simultaneous motions intersect, the motions may be compounded by the parallelogram law. Thus if OA, Fig. 47, represents the angular velocity ω_a of a body about an axis OA and OB represents the angular velocity ω_b of the axis OA about an axis OB, which is fixed relatively to a frame of reference (X), then OC represents the angular velocity Ω of the body relative to the frame of reference (X). This may be proved as follows: Let P be any point on OC and draw PE and PD perpendicular to OA and OB respectively. Then due to the angular velocity OA the point P has a linear velocity OA × PE perpendicular to the paper and directed upwards, while due to the angular velocity OB the point P has a linear velocity OB × PD perpendicular to the paper and directed downwards. But OA × PE = OA × OP Sin α and OB × PD = OB × OP Sin β and since OA Sin α = AC Sin β = OB Sin β, the two linear velocities of P are equal. Hence P is at rest and OC is the axis of the resultant motion. Now consider the point A and draw AG and AF perpendicular to OB and OC respectively. The linear velocity of A is due solely to the angular velocity OB and is equal to OB × GA, but this is equal to OC × AF, since each is equal to the area of the parallelogram OABC. Now OC × AF is equal to the linear velocity A would have due to an angular velocity OC. Hence OC is equal to the resultant angular velocity.

Since OC = OF + FC = OA Cos α + AC Cos β, we have the relation $\Omega = \omega_a \text{ Cos } \alpha + \omega_b \text{ Cos } \beta$, giving the magnitude of the resultant angular velocity, and since OA Sin α = OB Sin β, the position of the axis of the resultant is given by the relation $\dfrac{\text{Sin } \alpha}{\text{Sin } \beta} = \dfrac{\omega_b}{\omega_a}$.

51. Parallel Axes.—In Fig. 48, let AB represent the angular velocity ω_a of a body about the axis AB and let CD represent the angular velocity of the axis AB about a fixed axis CD. These motions are realised in the arrangement shown on the right in the figure, where the wheel Q turns on a pin carried by an arm R that rotates about an axis O. The wheel Q meshes with a fixed wheel S whose axis is also O. Then the resultant angular velocity of the body relative to the fixed frame of reference is equal to $\omega_a + \omega_b$ and is about an axis RR situated as shown. The sense of the resultant is the same as those of the components. When the components are opposite in sense the position of the resultant depends on the relative magnitudes of ω_a and ω_b. When ω_a is greater than ω_b the resultant is equal to $\omega_a - \omega_b$ and is situated as in Fig. 49, its sense being the same as that of ω_a. The sketch on the right in Fig. 49 shows a realisation of this case, the wheel S now having internal teeth. When ω_a is less than ω_b the resultant is equal to $\omega_b - \omega_a$,

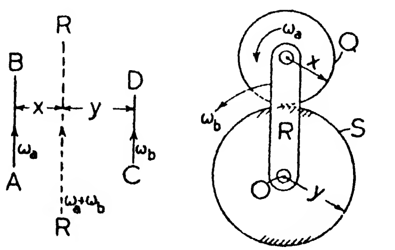


Fig. 48

and is situated as in Fig. 50, its sense being the same as that of ω_b. The sketch on the right of the figure shows a realisation of this

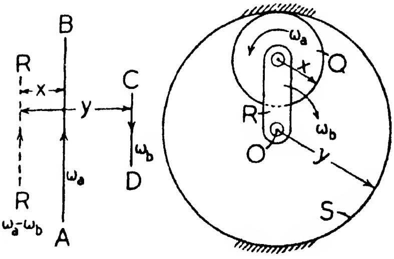


FIG. 49

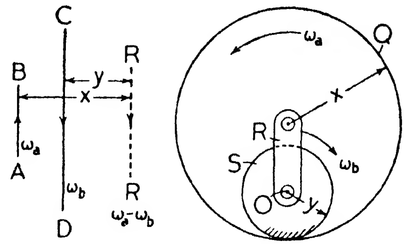


FIG. 50

case, the wheel Q now being an internal one. The student should easily be able to verify these results for himself.

52. Skew Axes.—When the axes AA and BB of the component motions are not parallel and do not intersect, then the resultant motion is a screw motion about an axis RR which may be found as follows: Fig. 51 shows the axes in elevation and plan, ST being the shortest distance between the axes; O is the plan view of ST. Consider a line RR which intersects ST at right-angles and which makes angles α and β with OA and OB as shown. Let the shortest distances between RR and AA and BB be l and m respectively.

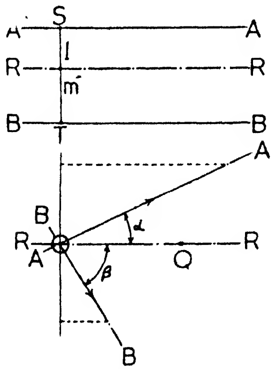


FIG. 51

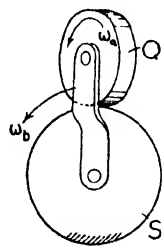


FIG. 52

Resolve ω_a and ω_b parallel and perpendicular to RR. The perpendicular components are ω_a Sin α and ω_b Sin β, and because of these any point Q of RR will have vertical velocities equal to $\omega_a \text{ Sin } \alpha \times OQ$ and $\omega_b \text{ Sin } \beta \times OQ$, the former being directed downwards and the latter upwards. Let α and β be chosen so that these vertical velocities are equal; to do this we must have $\frac{\text{Sin } \alpha}{\text{Sin } \beta}=\frac{\omega_b}{\omega_a}$; then the point Q and thus the line RR will have no vertical velocity. The components of ω_a and ω_b parallel to RR are ω_a Cos α and ω_b Cos β, and because of these components the line RR will have linear velocities, perpendicular to RR and in the horizontal plane, of magnitudes $\omega_a \text{ Cos } \alpha \times l$ and $\omega_b \text{ Cos } \beta \times m$, the senses of these velocities being opposite. Let l and m be chosen so that these velocities are equal; to do this we must have $\frac{l}{m}=\frac{\omega_b \text{ Cos } \beta}{\omega_a \text{ Cos } \alpha}=\frac{\text{Tan } \alpha}{\text{Tan } \beta}$; then the line RR will have no velocity perpendicular to itself in the horizontal plane Returning to the components ω_a Sin α and ω_b Sin β, because of these components any point of RR will have velocities $\omega_a \text{ Sin } \alpha \times l$ and $\omega_b \text{ Sin } \beta \times m$ along OR. Thus the resultant motion is a screw motion about RR as axis, the angular velocity of this screw motion being equal to $\omega_a \text{ Cos } \alpha + \omega_b \text{ Cos } \beta$ and its linear velocity being equal to $l\omega_a \text{ Sin } \alpha + m\omega_b \text{ Sin } \beta$. The student should verify these last results for himself by showing that the linear velocity of the point S, which is due solely to the angular velocity ω_b, is also given by the resultant motion about RR. Fig. 52 shows a realisation of this case, the gears S and Q now being "skew" gears (see Art. 186).

53. Angular Acceleration.—The angular acceleration due to the changing angular speed about an axis whose direction is fixed has been considered in Arts. 27 and 28; there is, however, an angular

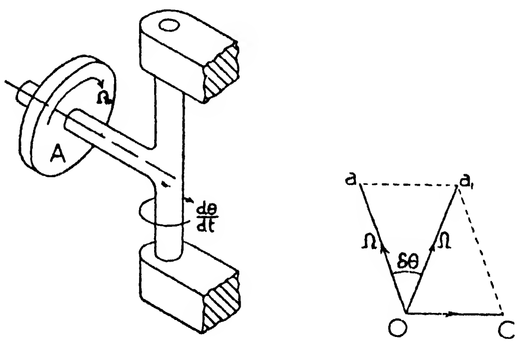


FIG. 53

acceleration when the direction of the axis of an angular motion is changed, and this will now be considered. In Fig. 53 let Oa represent the angular velocity, Ω, of a body A at one instant, and let Oa_1 represent the angular velocity after an interval of time δt.

The magnitude of the angular velocity has not been changed, but the direction has been changed by the angle $\delta\theta$. Complete the rectangle Oaa_1C, then the change in angular velocity during the interval δt is represented by OC, since $\mathbf{OC}+\mathbf{O}\boldsymbol{a}=\mathbf{O}\boldsymbol{a}_1$. Hence the average angular acceleration during the interval is $\frac{\text{OC}}{\delta t}$ about OC as axis. Now $\text{OC}=aa_1=\text{O}a\,.\,\delta\theta$ approximately, so that the average angular acceleration is $\Omega\,.\,\frac{\delta\theta}{\delta t}$. When the interval δt is made indefinitely small the approximation becomes exact and the angular acceleration at the instant is $\Omega\,.\,\frac{d\theta}{dt}$ about an axis perpendicular to Oa, that is, perpendicular to the axis of the angular velocity. This result should be compared with that of Art. 29, where it is shown that if a linear velocity v is changing its direction, i.e. rotating, at a rate $\frac{d\theta}{dt}=\omega$, then there is a normal acceleration perpendicular to the velocity and equal to $v\omega=v\frac{d\theta}{dt}$. The analogy is illustrated in Fig. 54, where the rotating velocities are in full lines and the corresponding accelerations in dotted lines.

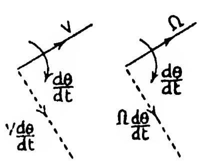


FIG. 54

EXERCISES IV

1. A line AB moves subject to the following restraints; state for each case how many degrees of freedom the line possesses in general and what those freedoms are.

(a) The end A lies always in a fixed plane.
(b) It makes constant angles with two fixed lines that are not parallel to each other.
(c) The ends A and B each lie on a fixed surface.
(d) The end A lies always on a fixed line, and the end B always on a fixed surface.

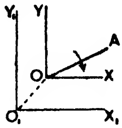
2. The ends A and B of a rod slide along two fixed intersecting lines OX and OY respectively. Find the instantaneous centre of the motion for any one position of the rod.

3. What do you understand by the terms *virtual centre, instantaneous axis, centrode of* A *relative to* B and *axode*?

4. A line AB 6 in. long moves so that A describes a circle radius 2 in., centre O, with constant angular speed of 100 r.p.m., while B moves along a produced diameter of the circle. Find graphically the speed of a point of the rod distant 1 in. from A when the angle AOB = 60°.

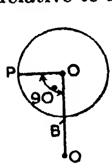
5. A rod moves so that a point P on it slides along a line OY, while the rod always passes through a fixed point Q on a line OX perpendicular to OY. Find the I.C. for several positions of the rod, and hence sketch the shape of the centrode of the rod relative to the frame XOY. If when OP = 6 in. and the angle OPQ = 60° the velocity of P along OY is 10 f.s., what is the angular speed of the rod?

6. One end P of a rod PQ is constrained to lie always on a line OY and the rod is in contact with a disc, radius 1 in., pivoted on a fixed pivot at a point R. The angle POR $=90°$ and OR $=4$ in. If there is no slip between the rod and the disc, find the velocity of P when the disc rotates at 1 r.p.s. and the angle OPQ is 45°. Take the point of contact of the rod and disc on the same side of OR as P.

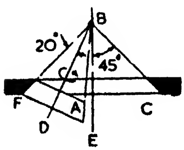


7. In the figure the rod OA is 6 in. long and is rotating relative to XOY at 50 r.p.m. clockwise about O. The frame XOY has a motion of translation relative to the fixed frame $X_1O_1Y_1$ such that O describes a circle 3 in. radius, centre O_1, at a speed of 100 r.p.m. anticlockwise. What is the velocity of A relative to $X_1O_1Y_1$ when O_1OA is a straight line ?

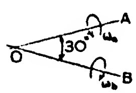
8. A rod OA 6 in. long rotates clockwise about O at a speed of 50 r.p.m. relative to a frame of reference XOY. The latter, however, rotates at a speed of 100 r.p.m. clockwise about O relative to a fixed frame X_1OY_1. What is the velocity of A relative to X_1OY_1 ?



9. The figure shows a disc A, radius 1 in., which is rotating about an axis O_1 relative to an arm B, 3in. long, which is rotating about O_2, both rotations being anticlockwise. If the speed of A relative to B is 200 r.p.m. and that of B relative to the earth is 100 r.p.m., find the magnitude of the velocity of the point P on the edge of the disc and its inclination to the line O_1O_2. Verify your result by finding the resultant of the velocity of P relative to the arm and the velocity of P regarded as a point fixed to the arm.



10. In the grinding-mill mechanism indicated in the figure A is a conical roller carried in the fixed frame by a ball-and-socket joint at B, and it rolls without slip on the fixed conical surface C as shown, thus rotating about its axis BD while that axis rotates about BE. If the semi-apex angles are as shown and the speed of A about BD is 100 r.p.m., find the speed of rotation of the axis BD about BE and the angular speed of A relative to the fixed frame.



11. A body A rotates about an axis OA as shown in the figure while that axis rotates about OB. If $\omega_a=100$ r.p.m. and $\omega_b=50$ r.p.m. and the shortest distance between the axes is 3 in., find the magnitude and the position of the axis of the resultant angular velocity of A and the magnitude of the velocity of translation along that axis.

12. The flywheel of a motor car rotates about a horizontal axis at 2000 r.p.m. while the car moves in a horizontal circle radius 100 ft. at 60 m.p.h. Find (*a*) the resultant angular speed of the flywheel and (*b*) its angular acceleration.

CHAPTER V

MOTION OF A BODY. GEOMETRIC DESIGN

54. Position of a Body.—To specify the position of a material body relative to a frame of reference it is necessary to specify the positions of three points (not lying in a straight line) of the body. If only two points of the body were fixed, the body could pivot about the line joining those points, but if a third point, not lying on that line, is fixed, then no motion is possible. It would thus appear that nine co-ordinates must be given in order to fix the positions of three points of the body, and thus the body itself, but actually these nine co-ordinates are not all independent, because there are three equations connecting them with the distances between the three points of the body, and so the body has only six degrees of freedom. It may be noted that since the position of every point of the body is fixed relative to any three points of the body that form a triangle, this triangle may be used to represent the body.

55. Motion of a Body.—The motion of a body in general is composed of motion of translation and motion of rotation. In the former any two lines (not being parallel to each other) fixed in the body remain always parallel to their initial positions. In the latter all the points of the body describe circles whose centres lie on a fixed straight line. A body perfectly free in space is able to have three independent translations parallel to three mutually perpendicular axes, and three independent rotations about those axes, and thus possesses six degrees of freedom.

56. Motion of Body with One Point Fixed.—If one point of a body is fixed, then a sphere having that point as centre will always intersect the body in the same section, and this section has spheric motion. The position of any point of this section may be specified relative to the ends P and Q of the arc of a great circle by drawing a spherical triangle having the point as apex and PQ as base ; hence this arc may be used to represent the section and thus the body. Hence what has been said about the spheric motion of a line applies to the motion of a body of which one point is fixed ; in particular any displacement can be effected by a single rotation about some virtual axis passing through the fixed point.

57. Coming now to the most general motion a body can have, when no point is fixed, it can easily be shown that any displacement can be produced by a simple translation together with a single rotation. Let the first and last positions of the body be represented by the triangles PQR, $P_1Q_1R_1$ in Fig. 55. Join any two corresponding points, say P and P_1; then a translation parallel to PP_1 will bring the body to the position $P_1Q_2R_2$ and the change of position from this to the final position $P_1Q_1R_1$ is one in which one point of the body is fixed. Hence a virtual axis OO can be found about which a rotation will effect this change of position. Clearly there is an infinite number of ways in which this displacement, from PQR to $P_1Q_1R_1$, may be effected, but one of these ways is of special interest. It is that way for which the virtual axis is parallel to the direction of the translation; the displacement is then effected by a translation along, and a rotation about, a line, i.e. by a *screw* motion.

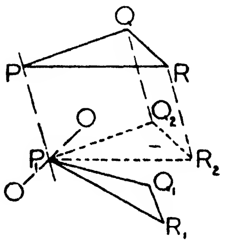


FIG. 55

If the displacement is made indefinitely small, the line about which a screw motion will effect the displacement becomes the instantaneous axis. As the body moves, so the position and direction of the instantaneous axis change, and it will sweep out a surface which is fixed in space; this is the axode of the body relative to space, or the space-axode. Corresponding to it there is another surface, which is fixed relative to the body; this is the axode of the space relative to the body, or the body-axode. The two axodes will at any instant be in contact along a line which is the instantaneous axis at that instant. Thus the most general motion a body may have at any instant may be produced by a screw motion of an axode that is fixed to the body about the line in which, at the instant, that axode touches one that is fixed in space. If now a second body is substituted for the space, then the space-axode becomes an axode fixed in the second body, and the relative motion of the two bodies at any instant is a screw motion about the line in which the two axodes touch at the instant.

58. The Constraint due to Contact between Two Bodies.—In practice bodies are never perfectly free, but are always constrained in some way; for example, the links or parts of a machine almost invariably can move only in one way, that is, they have only one degree of freedom relative to any other link, while in structures the constraint is complete and the links have no degrees of freedom relative to each other. Now the simplest, and almost the only, method of restricting the freedom of one body relative to another is to bring them into contact at a number of points, and so the constraint brought about by such contact will now be considered. At first the constraint of a body having only plane motion will be examined. The section of such a body by the plane of motion will always remain in that plane and may be taken to represent the body; it has three degrees of freedom, two translations and one rotation. Suppose now that the body A in Fig. 56 is brought into contact, at a single point, with a body B that is regarded as being fixed, then, *so long as contact is maintained at one point*, one degree of freedom of A relative to B is destroyed. A can now have a translation parallel to the outline of B, or it can move with a rolling motion on B; this last motion may be resolved into three components, but the components are not independent and so

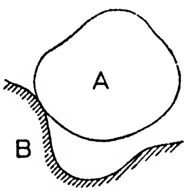


FIG 56

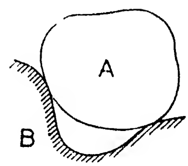


FIG. 57

constitute only one degree of freedom. Thus a single contact, *maintained*, between two bodies destroys one degree of freedom between them, leaving two.

If a second contact is arranged and maintained between A and B as in Fig. 57, then a second degree of freedom is destroyed and only one remains; A can now move in only one way, and although its motion can be resolved into three components, these are not independent and constitute only one degree of freedom. Thus two contacts maintained between two bodies destroy two degrees of freedom between them.

If now a third contact is arranged and maintained between A and B, then, in general, a third degree of freedom will be destroyed and A will be fixed relative to B. The reader is advised to cut out cardboard shapes to represent A and B and to prove the above statements by trial. Thus in general three contacts maintained between two bodies that are confined to a plane will destroy all three degrees of freedom between them and any further contacts are unnecessary or redundant as regards destroying the relative freedom between the bodies.

Suppose now that, due to distortion of the body B, the third contact in Fig. 58 moves, relative to the other two, to the dotted position; then clearly the position of the body A will change slightly, but the body A will not be distorted in any way. But if a fourth contact is arranged as in Fig. 59, then clearly a movement of this fourth contact will result in the body A being distorted. Also in order to obtain the fourth contact the body B must be made accurately to fit the body A. Thus redundant constraints necessitate accurate workmanship and may cause distortion of the body whose freedom is restricted. Therefore unless some advantage is obtained from their use which outweighs the disadvantages mentioned, redundant constraints should be avoided.

Although in general three contacts will destroy three degrees of freedom, yet this is not always so; let the body A be arranged with two contacts, P and Q, as in Fig. 60, and underneath it, fixed to the

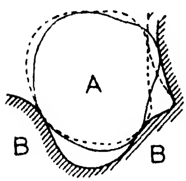


FIG. 58

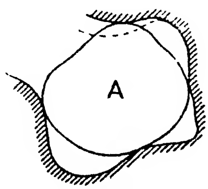


FIG. 59

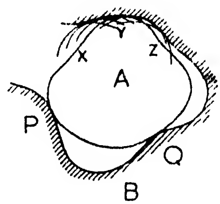


FIG. 60

body B, let a sheet of paper be placed; let the outline of the corner XYZ of A be traced on this paper and then let the body A be moved slightly, while maintaining two contacts with B, and the corner be traced again, and so on. Then the successive curves drawn on the paper will in general have an envelope and, clearly, if the body B is made to the shape of this envelope, then a third contact can be arranged without destroying the remaining degree of freedom of the body A. Similarly a fourth and other contacts could be arranged while still leaving one degree of freedom, so that although three contacts are in general sufficient to destroy three degrees of freedom, yet it does not follow that because there are three contacts three degrees of freedom are destroyed. Clearly, if the shape of the body A is circular, then any number of contacts will still leave one degree of freedom. It should be clear, however, that all but two of these contacts are redundant and, besides necessitating accurate workmanship, introduce a likelihood of distortion.

59. Extending this to three dimensions, it will be found again that, in general, each contact between two bodies will destroy one degree of freedom between them, and so six contacts, maintained, are in general sufficient to destroy all six degrees of freedom between the bodies. Again, it does not follow that because there are six contacts six degrees of freedom are destroyed. Also, if a body is fixed relative to another body by means of six contacts, then any relative movement between the contact points of the one body, due to its distortion, will not involve distortion of the other body, but only a slight change of position ; but if there are more than six contacts, then all but six are redundant and, besides necessitating accurate workmanship, may involve distortion of the bodies.

60. Geometric Design.—The principles considered in the previous article find their application in the design of scientific instruments, the advantages obtained over ordinary design being, first a great reduction in the accuracy of the workmanship required in the manufacture of the instruments, with a consequent saving in the cost and time of manufacture, and, secondly, freedom from errors in the working of the instruments. Errors are frequently caused, in instruments of ordinary design, by slackness in the fit of mating parts, by wear of those parts and by the distortion of parts that have to be clamped to other parts, and these sources of error are eliminated in geometric design. The underlying principle of geometric design is that only as many contacts are provided between two mating parts as are necessary to destroy the required number of degrees of freedom between them ; for example, if one part is to have only one degree of freedom relative to another part, then five contacts will be arranged between them. Also the forms of the contacting portions of the parts are made such that the contacts occur at definite points.

61. Examples of Geometric Design.—The "slot, hole and plane" method of fixing a stand to a base is an example of geometric design. The stand is provided with three legs, the ends of which are approximately hemispherical and the base is provided with a trihedral hole, a vee slot and a plane surface as shown in Fig. 61. The hole gives three contacts with one leg, the slot two more with a second leg, and the plane surface and the third leg provide the sixth contact. The stand can then be removed from the base as often as is desired, and when it is replaced it will always occupy the same position relative to the base. An alternative form of base is one having three radially disposed vee slots each of which provides two contacts with a leg of the stand. Such stands and bases are much used in physics laboratories, and although unmachined castings are employed, perfect "fitting" is obtained between any stand and any base.

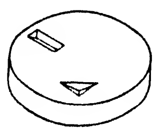


Fig. 61

The optical bench used in laboratories affords another example. It consists of a long rectangular base provided with a vee groove and a flat surface along its whole length. Three-legged stands are again used, two legs resting in the vee groove and the third on the flat surface ; thus five contacts are obtained and the stand is left with one degree of freedom, a translation parallel to the vee groove.

62. The Conditioning of Contacts.—There is a right and a wrong way of arranging contacts, and these are shown in Fig. 62*a* and *b*.

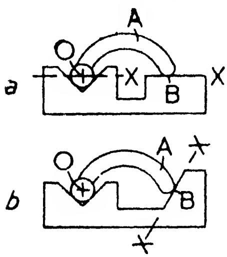


Fig. 62

The body A is supposed to have four contacts with a vee groove, leaving two degrees of freedom, and the fifth contact, at B, is required to destroy a fifth degree, namely, freedom to rotate about the axis O. In *a* the fifth contact is "well conditioned," while in *b* it is badly conditioned. In the former the common tangent plane XX at the point of contact B is perpendicular to the direction in which the point B would move if the surface XX were removed, whereas in the latter this tangent plane is at an acute angle to that direction. The force exerted between the surfaces in a badly conditioned contact will be greater than that in a well-conditioned one, and also the displacement of the body A due to a layer of dirt or rust on the surface XX will be greater.

63. Force and Body Closure.—The contacts necessary between two bodies in order to limit their relative freedom can be maintained in two ways, by using a force such as the weight of one of the bodies or the elastic force of a spring, or by providing further contacts to maintain the necessary ones. The first method is called "Force-closure" *and is always used in geometric design,* while the second method is called *Body-closure,* and is used, as will be seen later, in the joints of mechanisms.

64. Further Examples of Geometric Design (1).—Another example is given in Fig. 63, which shows some features of a measuring microscope manufactured by the Cambridge Instrument Company, to whom the author is indebted for supplying the drawings from which this figure has been prepared. The microscope, indicated at A, is held in vees in a head which has vees to fit the tube B, suitable clamps being provided in both places to maintain the contacts. The tube B rests in vees in the base casting C, contact being maintained chiefly by the weight of the microscope, but partly by small spring-loaded plungers bearing on the top of the tube immediately above the vees. The tube B is prevented from rotating by the arm D, which is fixed to it and which bears against the steel strip E fixed to the base. Contact is maintained by the overhanging weight of the microscope. The axial position

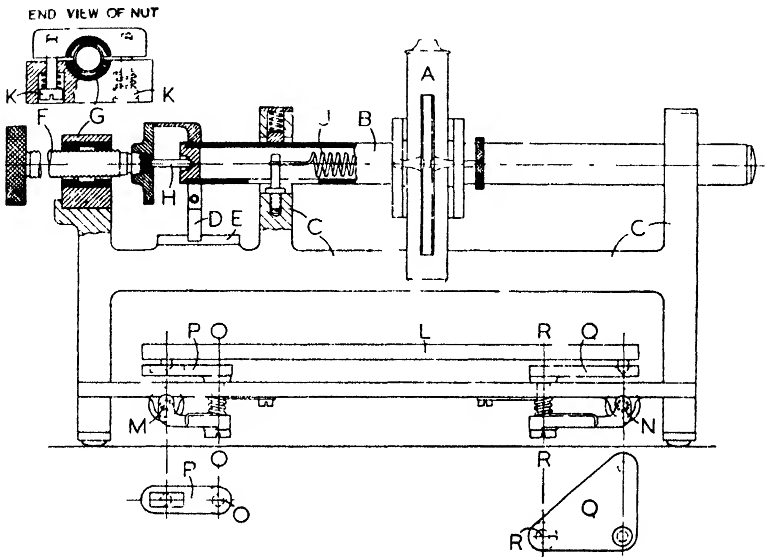


Fig. 63

of the tube B is determined by the micrometer screw F. This works in a specially formed nut G and is connected at its end to the tube B by the round-ended strut H, which bears in conical holes in B and F. This avoids any unwanted constraint being applied to the tube B by the screw. Contact is maintained by the spring J inside the tube. The nut G bears on the screw F at six places only, three places spaced 120° apart at each end, and is made in halves which are held together by spring-loaded screws K. This arrangement gives a connexion in which slackness of fit is absent, but which is also quite free from undue friction ; it is sometimes called a "geometric nut."

The object being measured rests on a table L, and in order to bring the line of the object, along which the measurement is to be made, parallel to the travel of the microscope, the table is provided with adjustments as indicated at M and N. Thus a round-ended pin fixed to the table rests in a vee groove in an arm P, which can be turned about the axis OO by the screw indicated. At N there are two round-ended pins, one resting in a conical hole in the plate Q and the other on a flat surface on that plate. The plate Q can be rotated about the axis RR. The six contacts necessary to fix the table are thus obtained; while each end of the table can be moved independently in a direction perpendicular to the travel of the tube B and microscope.

The clamp shown in Fig. 64 gives another example of the

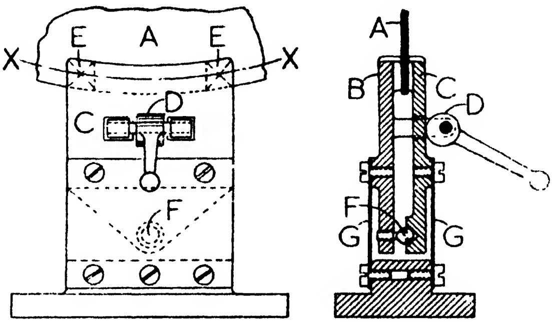


Fig. 64

application of the fundamental principle of geometric design, i.e. the avoidance of redundant constraint. It is used on a machine, designed by Mr. G. A. Tomlinson, of the Metrology Department of the National Physical Laboratory, for the accurate measurement of the teeth of gear wheels. It was required to clamp a spindle against rotation, and so a disc, a portion of which is seen at A, was fixed to the spindle, and the clamp grips the rim of this disc when the two parts B and C are drawn together by turning the eccentric D. The part B carries a ball-ended pin F which engages a conical hole formed in the part C, and both B and C engage the disc A over small areas EE at each corner. The parts B and C are secured to the base by thin strips of steel GG and are thus firmly held against any motion in the direction XX, but are not constrained against small movements in other directions. The disc A can thus be firmly held against rotation, but is not constrained in any unwanted manner.

A second example from the same machine is shown in Fig. 65. The ball-ended stylus A, which is to bear on a tooth that is to be measured, is carried in a lever B that is connected to a frame C by a hinge formed of two pairs of thin flexible metal strips DD and EE secured at their ends to the frame C and to the lever B, and arranged to lie in perpendicular planes as shown. The strips destroy all freedom except that of rotation through small angles about the line of intersection of the planes of the strips, thus giving a hinge that is free from backlash and unwanted motions. The frame C is carried on the base casting H on a kinematic slide consisting of two balls F and G running in vees in both the frame C and the base casting H, and one ball J running in a vee on H and a flat surface on C. Contact is maintained by the weight of the lever K, which applies a force, through a round-ended strut, to the frame C, so as to keep it in contact not only with the three

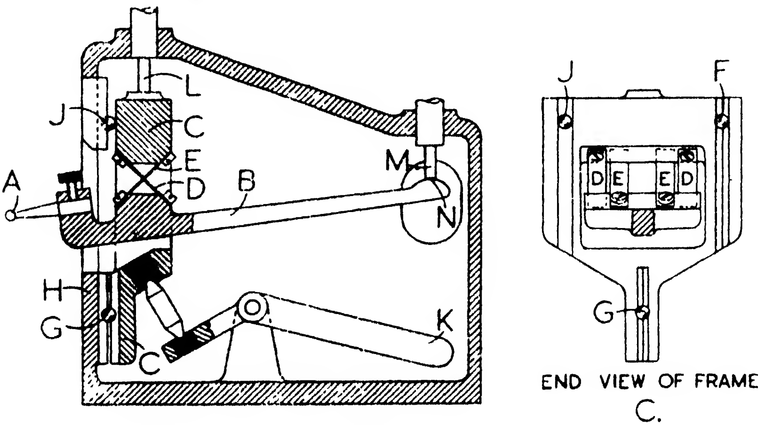


FIG. 65

balls between it and the base casting, but also with the micrometer spindle L. Measurements with an accuracy of 0·00002 in. are taken by bringing the micrometer spindle M into contact with the knife-edge end of the lever B, the moment of contact being determined by observing the gap at N against an illuminated ground-glass screen.

65. Further Examples of Geometric Design (2).—Extensive use is made of geometric design in the microtomes manufactured by the Cambridge Instrument Company, and one of them has been chosen as an example. (*N.B.*—A microtome is an instrument for cutting very thin slices off specimens of physiological objects so that their cross-sectional structure may be examined by viewing the slices in a microscope.)

Views showing the main features where geometric design is used are given in Fig. 66, which has been prepared from drawings kindly supplied by the manufacturers. All details that are not part of the geometric features under discussion have been omitted.

The specimen block A is held in a suitable holder on the leg of a T-shaped casting B, the cylindrical trunnions C and D of which rest in vees formed in the base casting E. The member B is left therefore with two degrees of freedom, rotation about and translation along the axis XX of the trunnions. The rotation is used

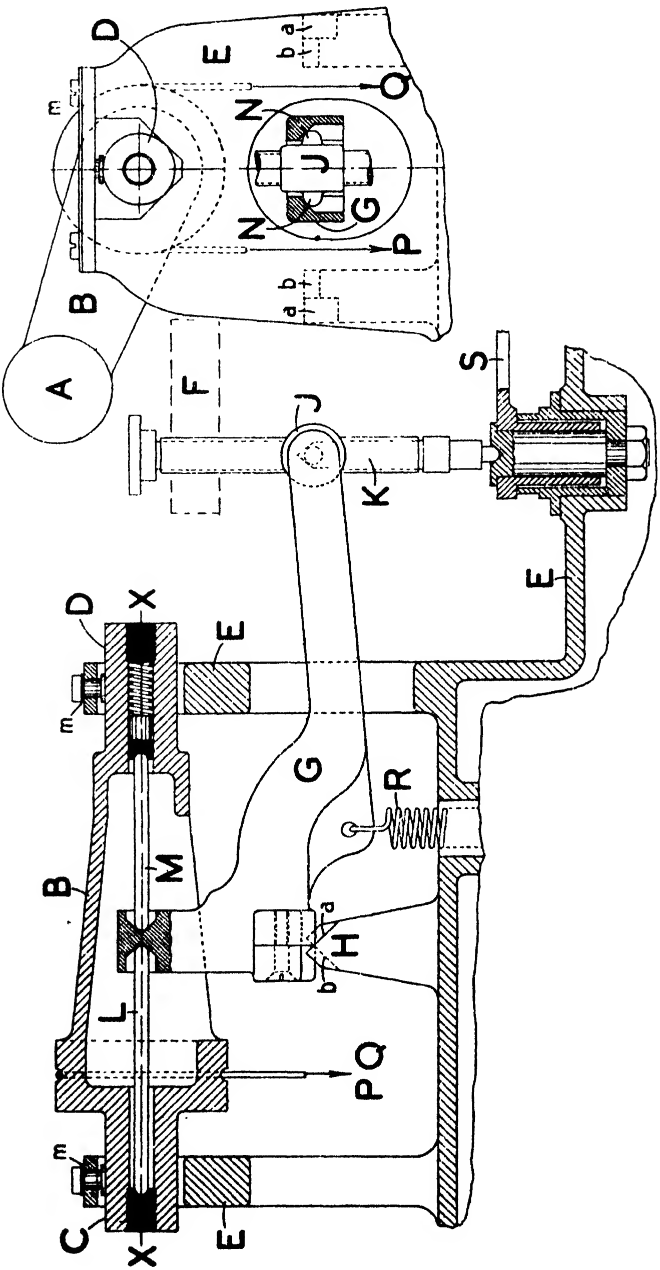


Fig. 66

to cause the specimen to move past a stationary knife whose position is indicated at F, which enables slices to be cut off the specimen. The translation is used, when the leg has been returned so that the specimen is clear of the knife, to move the whole member B along the axis XX by an amount equal to the thickness of slice required.

The trunnions of B are kept in contact with the vees by a spring which applies a pull P to a wire that encircles and is fixed to a circular portion of B. The other end Q of this wire goes round suitable guide pulleys and is attached to an actuating arm S which pulls the leg of B up by overcoming the pull P. The springy strips, *m*, *m*, also assist in maintaining contact with the vees.

The axial position of B is determined by the bell-crank G, which is pivoted at H on a geometric hinge or knife-edge (described in detail below). The long arm of this bell-crank is attached by a geometric connexion (also described below) to a nut J working on a screw K, the lower end of which is hemispherical and bears on a conical hole in a part that is fixed to the base casting. Contact between the bell-crank and the knife-edge seating, between the bell-crank and the nut J and between the end of the screw K and its seating is maintained by the spring R. The short arm of the bell-crank is connected to the member B by hemispherically ended struts L and M, the strut M being spring-loaded. The last part of the return motion of the actuating arm S is arranged to give, through an adjustable ratchet mechanism that is not shown, suitable small rotations to the screw K, thus feeding the member B and the specimen along as required.

The principle of the geometric hinge mentioned as being used at H is indicated in Fig. 67 (*a*) and (*b*). The base casting is formed with upwardly projecting lugs shaped, as shown at (*a*), so that there are two flat surfaces *aa* at right-angles to two flat surfaces *bb*. The bell-crank (G of Fig. 66) is provided with " knife-edge " strips as indicated at (*b*). These are two pieces *c* and *d* screwed together so that the corners *ee* and *ff* (see inverted view) lie in one straight line. This is easily done by grinding the face *ggg* after the pieces *c* and *d* have been fixed together. When this assembly is placed on the lugs of the base casting the corners *ee* rest on the surfaces *aa* and the corners *ff* on the surfaces *bb*. Thus there are four contacts, and the piece *cd* (and the bell-crank which carries them) will have two degrees of freedom relative to the base, rotation about the line *effe* and translation along it. The latter may have to be destroyed by a fifth contact, but usually this is not necessary. The arrangement gives a pivot having a very precise motion and is comparatively easy to manufacture. If the faces *a* and *b* were slightly rounded, then point contact would be obtained; they are, however, usually flat, thus giving (with careful machining) line contact, which, although not strictly geometric design, is better for many practical purposes.

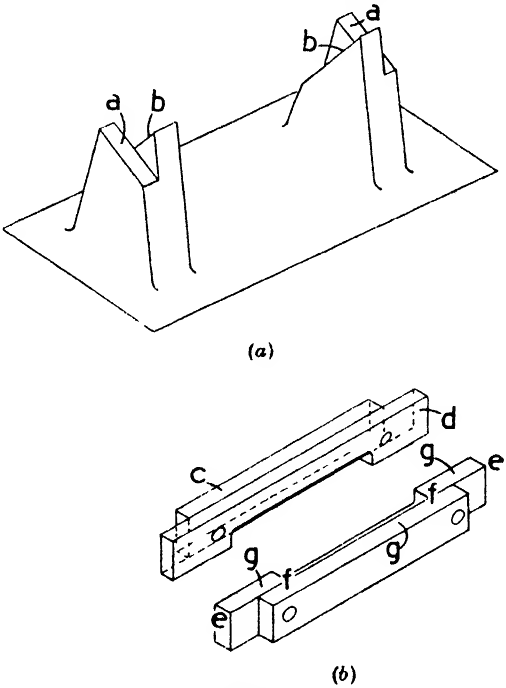


FIG. 67

The connexion between the nut (J, Fig. 66) and the end of the bell-crank G takes the form of two rounded projections NN integral with the nut and which engage two vee-shaped grooves cut, on the slant relative to the axis of the screw K, in the end of the bell-crank G, giving four contacts. The nut thus has two degrees of freedom relative to G, freedom to rotate about two axes perpendicular to each other and to the axis of the screw K, and is thus able to take up a position of repose while being held against turning about the axis of the screw.

66. The Application of the Principle of Geometric Design when the Loads are Heavy.—Geometric design, involving as it does contact at points, is suitable only for instruments, etc., where the weights of the parts and the forces acting are not large, but the principle that redundant constraint should be avoided finds a wider application. Thus if redundant constraint is avoided by using suitable connexions, each of which destroys a definite number of degrees of freedom, then relative movements between the points of support of a body will not distort the body.

Consider an ordinary shaft carried at its ends in plain journal bearings; there is redundant constraint, and if one bearing moves slightly relative to the other, the bearings may bind or seize. If,

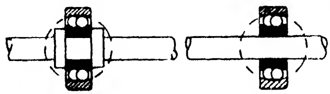


FIG. 68

however, the bearings are self-aligning bearings arranged as shown in Fig. 68, then the constraint is only just sufficient and there will be no danger of binding or seizing if one bearing moves relative to the other.

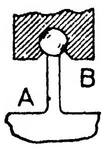


FIG. 69

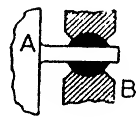


FIG. 70

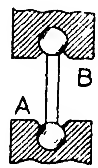


FIG. 71

Among the connexions available are the ball-and-socket joint, Fig. 69; the ball-and-socket slide, Fig. 70; and the ball-and-socket shackle, Fig. 71. These connexions destroy respectively three, two and one degrees of freedom between the bodies A and B that they connect. By using three ball-and-socket slides, as in Fig. 72, a motor-car engine can be securely held in the frame, but distortion of the latter, within limits, will not produce any distortion of the engine. Alternatively a ball-and-socket joint, a ball-and-socket slide and a ball-ended shackle arranged as in Fig. 73 may be used. In either case redundant constraint is avoided.

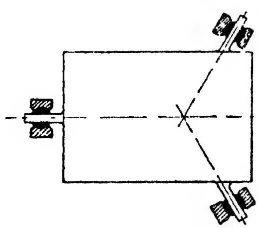


FIG. 72

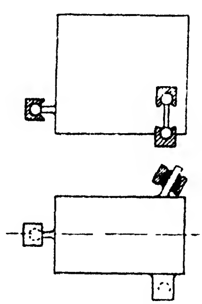


FIG. 73

A further example is afforded by the arrangement, indicated diagrammatically in Fig. 74, of a chuck used on an axle-turning

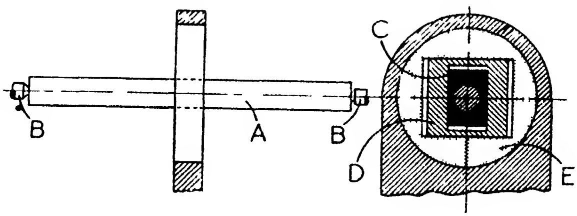


Fig. 74

lathe. The axle A is held between centres BB and is operated on at both ends simultaneously; it is therefore driven from the centre, and to avoid redundant constraint the chuck C is left free to slide in a block D, which is itself free to slide, in a direction perpendicular to the first slide, in the driving wheel E. An alternative arrangement is shown in Fig. 75, where the chuck C is attached by the links FF to the floating ring D, which is itself attached to the driving wheel E by the links GG.

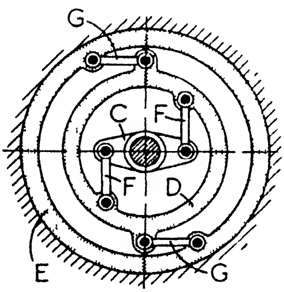


Fig. 75

A last example is the "floating frame" used in the Melville-Macalpine system of reduction gearing for large turbine-driven ships. Double helical gearing (see Art. 166) is used, and the two toothed portions of the pinion are seen at A and A_1 in Fig. 76. They are integral with the shaft which is carried in bearings in the floating frame B. The shaft is allowed freedom to move axially. The toothed portions A and A_1 engage toothed portions of the

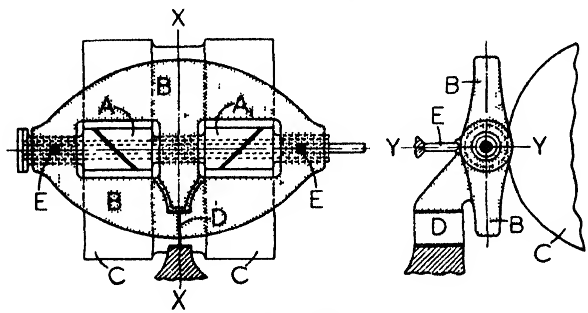


Fig. 76

wheel C, which is carried in bearings in the main casing surrounding the gears, and which is not allowed any axial freedom. The frame B is connected to the main casing by a short length of I joist D which, while being comparatively flexible as regards rotation of B about the axes XX and YY, is comparatively rigid against all other motions. The freedom of rotation about YY is desired in order that the teeth of the pinion shall be free to align themselves with those of the wheel, thus ensuring that contact shall occur along the whole length of the teeth, and not merely at the ends, which is what happens when the pinion is carried in bearings directly mounted in the main casing, and that casing distorts. The freedom of rotation about XX is not wanted and is destroyed by round-ended struts EE that are provided between the ends of the frame B and portions of the main casing. In order that the stiffness of the pinion shaft, which has of course to be coupled to the turbine, shall not destroy or restrict unduly the freedom of rotation about YY, the connexion is made by a long shaft of small diameter. In order to save space this shaft is sometimes arranged to be inside the pinion shaft, which is made hollow. For a detailed description of this gear the reader is referred to *Engineering*, November 28th, 1919.

It may be remarked that gearing using this floating frame can be designed to carry higher tooth loads than gearing arranged in the ordinary way; nevertheless, the arrangement is little used.

EXERCISES V

1. A body moves subject to the following constraints; state in each case how many degrees of freedom the body possesses and what those freedoms are.

(*a*) A line fixed in the body always makes a constant angle with a fixed plane.

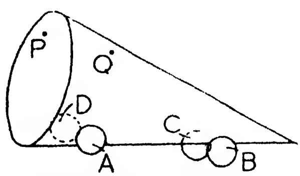
(*b*) A point P of the body lies always on a fixed line AB, and a line of the body is always parallel to a fixed line CD.

(*c*) A point P of the body lies always in one fixed surface, and a second point Q lies always in a second fixed surface.

(*d*) One point P of the body is fixed, and a second point Q lies always in a fixed surface.

(*e*) One point P of the body is fixed, and a second point Q lies always in a fixed line.

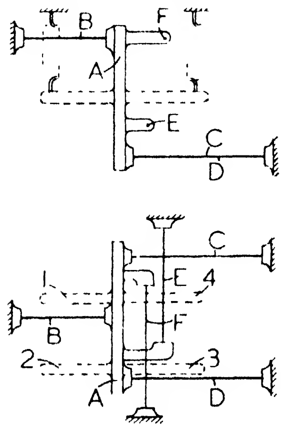
2. A cone is maintained in contact with four fixed balls A, B, C and D, as shown in the figure. How many degrees of freedom does the cone possess? If it is desired to eliminate the freedom of translation, which of the points P and Q would be the better position for the fifth contact?



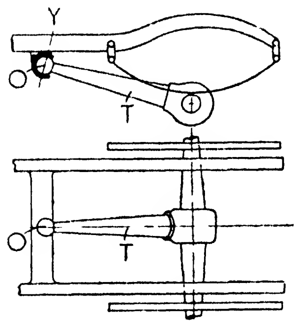
3. A circular cone rests in a vertical position (apex downwards) in a square hole. How many degrees of freedom does it possess and how many redundant constraints are there?

4. A cylindrical bar is to rest in vee-shaped recesses so as to have two degrees of freedom, translation along and rotation about its axis. What, in general, would be the best angle for the vees ?

5. Describe briefly the principles of geometric design and give its advantages and disadvantages in comparison with ordinary design. Explain clearly the difference between " well-conditioned " and " badly conditioned " contacts and state the drawbacks of the latter.



6. The figure shows an arrangement known as Mallock's vibrator which has been used for the study of vibrations. The body A is to be supported by a single spring acting on an arm fixed to the body in such a position that the force of the spring in conjunction with the weight of the body keeps all the ball-ended struts B, C, D, E and F in contact with sockets on the body and on the fixed frame. Four alternative positions are shown, in dotted lines, for this arm. Which one is suitable for the attachment of the spring and, when the spring is attached to that arm, what freedom (for small displacements) will the body possess ?



7. The figure shows diagrammatically the rear axle of a motor car. There is a ball-and-socket connexion between the torque tube (T) of the axle and the frame at O, and it is required to use the spring connexions to the axle to eliminate the freedom of rotation about the axis OY which at present the axle possesses, but redundant constraint is to be avoided. The connexions between the springs and frame are to be the usual pin shackles. Devise suitable forms of connexion between the springs and axle and state whether pivots or shackles should be used for the spring to frame connexions.

8. In Sarrut's straight-line mechanism (see Art. 134) each of the six hinges destroys five degrees of freedom. How many redundant constraints does the body B (Fig. 177) suffer, and which of its degrees of freedom are destroyed more than once ? Sketch a modification of the mechanism in which redundant constraint is avoided.

CHAPTER VI

THE KINEMATICS OF MACHINES

67. Definition of a Machine.—Reuleaux in his *Kinematics of Machinery** defines a machine as "A combination of resistant bodies so arranged that by their means the mechanical forces of Nature can be compelled to do work accompanied by certain determinate motions." In the Kinematics of Machines we consider machines from the point of view of the motions of the various parts, without regard to the forces that produce those motions or that arise from them. In doing this the actual forms of the parts of a machine may, to a great extent, be neglected, the parts being represented by geometric lines so that only the skeleton of the machine is considered.

68. Kinematic Pairs.—A machine is made up of a number of members or *links* which are usually bars that are rigid in the sense that their deformations under the forces that act on them are negligibly small, but flexible bands, springs and fluids may be used in some circumstances. Each link is connected to at least two other links, and the connexions may conveniently be called *joints*, so that a machine may be said to be composed of links and joints. The form of the joint between a pair of links is usually such that there is only one degree of freedom between them, and such a pair of links is called a *kinematic pair*. As will be seen later, joints which leave two or more degrees of freedom between the links they connect are sometimes used, but then the arrangement of the machine itself destroys some of those freedoms, so that, in effect, only one remains. Kinematic pairs are divided into two classes, *Lower* and *Higher*, the difference between which will now be explained.

69. Lower and Higher Pairs.—If the member A of the kinematic pair shown in Fig. 77 is regarded as fixed, then the member B has one degree of freedom, and when it moves any point P of it will describe some line ; this line is the *point path* of P relative

* *The Kinematics of Machinery*, translated by Alex. B. W. Kennedy, C.E., Macmillan & Co., 1876.

to B, and in the example shown is a circle. Now, with the links in the positions shown in the figure, suppose that B is fixed and A is moved. Then the end of a pointer attached to A and which was coincident with P will describe a point path relative to B. When the point path of P relative to A is coincident with that of P relative to B, as in the example shown, the pair is a lower pair. When the point paths are not coincident, as in Fig. 78, the pair is

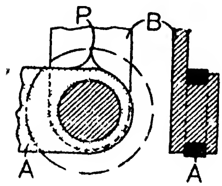


Fig. 77

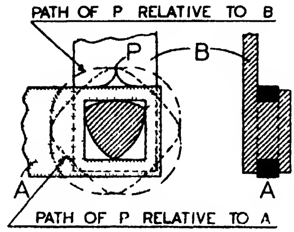


Fig. 78

a higher pair. Usually the contact between the links of a higher pair is at a number of points or along a number of lines, whereas in the lower pairs the contact is usually over a surface, the geometrical forms of the surfaces forming the joint being identical; but higher pairs sometimes have surface contact and lower pairs may have point or line contact.

70. The Lower Pairs.—There are only three kinds of lower pairs, namely, the *Turning Pair*, Fig. 79, in which the relative motion is one of rotation about an axis; the *Sliding Pair*, Fig. 80, in

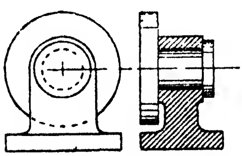


Fig. 79

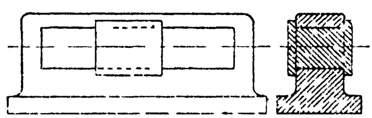


Fig. 80

which the relative motion is a translation along a straight line; and the *Screw Pair*, Fig. 81, in which the relative motion is a screw motion.

It will be seen that each of these lower pairs consists of a solid member fitting inside a hollow member. It is purely a matter of convenience which of the two members is made the hollow one and which the solid one. Thus in Fig. 77 B has a solid shaft fitting in the hole in A, but the relative motion would be the same if A had a cylindrical projection fitting inside a hole in B. When the hollow and the solid members of a kinematic pair are interchanged in this way the pair is said to be *inverted*. Also the constructional arrangement of a kinematic pair does not affect the relative motion that is possible. Thus a turning pair may consist of a block fitting in a circular groove as shown in Fig. 82, and the

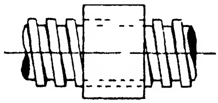


FIG. 81

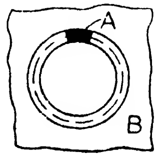


FIG. 82

extent of the slot may be limited as in Fig. 83, when the construction resembles that of a sliding pair. It may be noted here that a block, or slider, cannot be made to fit a slot in all positions unless the slot is either straight or circular. A cylindrical block or pin, however, may be made to touch the two sides of a slot having any shape, but the freedom between the members is not then limited to one degree, since the pin can move bodily along the slot and can, quite independently, turn about its own axis. Pins fitting in slots in this manner are used frequently in machines, but the arrangement of the machine is then such that one of the freedoms is destroyed and only one remains. A body carrying two pins, both of which fit a slot, as in Fig. 84, is a true kinematic pair, there being only one degree of freedom between the members.

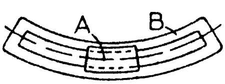


FIG. 83

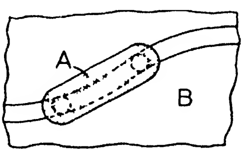


FIG. 84

The principal higher kinematic pairs are Cams, Toothed Gears, Belts and Chains, and these are considered subsequently.

The fashioning of the pairs so as to give them the best form as regards cost of manufacture, ease of adjustment and replacement, etc., is part of the subject of Machine Design and Construction and is beyond the scope of this book.

71. Kinematic Chains.—When a number of links A, B, C, etc., are connected by joints so that A and B, B and C, etc., form kinematic pairs the result is a *kinematic chain*. To be of any use as a mechanism or machine a kinematic chain must be *closed*, that is, the last link must be joined to the first and the arrangement must be such that *any link has only one degree of freedom relative to any other link.* If in a kinematic chain some of the links have more than one degree of freedom relative to any other link, the chain is said to be *incompletely constrained* and the relative motion of the links is indeterminate.

72. Mechanisms.—A mechanism is simply a kinematic chain, one link of which is regarded as being fixed. In general the mechanism obtained by fixing any link of a kinematic chain is different from that obtained by fixing any of the other links, so that, in general, as many different mechanisms can be obtained from a closed kinematic chain as that chain has links. Any one of these mechanisms is called an *inversion* of any of the others.

73. Machines.—When the parts of a mechanism are so made that they are able to withstand the forces that act on them when the mechanism is put to work, the mechanism becomes a useful machine. It will be found that, whenever it is possible, lower pairs are used rather than higher pairs, because not only are they easier to manufacture, but also they are less subject to wear. The mechanisms therefore that will first be considered are those using lower pairs.

74. Mechanisms Using Only Turning Pairs.—Clearly a closed chain of only three links joined by turning joints is rigid, and is thus not capable of forming a mechanism, although it is well adapted to form part of a structure. The simplest chain, using only turning joints, that can form a mechanism is one of four links joined by turning joints whose axes are parallel to each other. Such a chain is called a four-bar chain and will now be considered.

75. The Four-Bar Chain.—In Fig. 85 the link AB is taken to be the fixed link and the proportions are such that $AD+DC<AB+BC$ and $BC+CD>BA+AD$. The link AD can then make complete revolutions, and is called a *crank*, while CB can only oscillate between the positions BC_1 and BC_2, and is called a *lever*. Clearly C_1 is found by drawing an arc with centre A and radius equal to $DC-AD$ to cut the circle of motion of C. Similarly for C_2, except that the radius of the arc will be $AD+DC$. This mechanism is very widely used in all branches of engineering practice; the relative velocities and accelerations of its parts are considered subsequently.

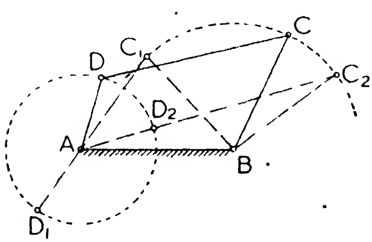


Fig. 85

76. The Inversions of the Four-Bar Chain.—When the link BC of the above chain is made the fixed link, as shown in Fig. 86, then, the proportions being such that $CD+DA<CB+BA$ and $BA+AD<BC+CD$, neither AB nor CD can make complete revolutions and the mechanism is a *double-lever* one. The extreme positions are determined by drawing arcs from B and C as centres and with radii $BA+AD$ and $CD+AD$, respectively, to intersect the paths of D and A.

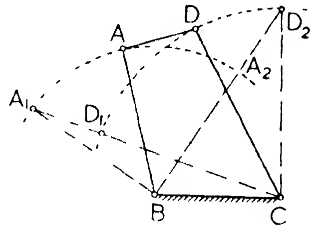


FIG. 86

Fixing the link CD gives another lever-crank mechanism similar to that of Fig. 85.

Lastly, if AD is made the fixed link, as in Fig. 87, then, since $AB+BC>AD+DC$ and $DC+CB>DA+AB$, both AB and DC can make complete revolutions and the mechanism is a *double-crank* mechanism. To avoid fouling between the links the actual construction must be on the lines indicated in Fig. 88.

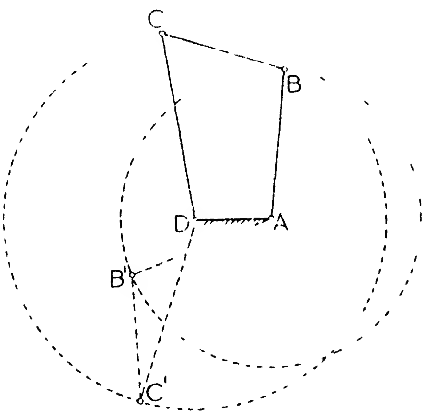


FIG. 87

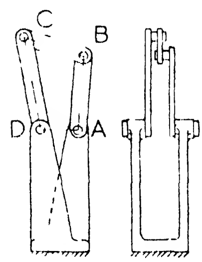


FIG. 88

These different mechanisms may be regarded as being obtained by altering the proportions of the links of a mechanism whose nature remains unchanged, and some people prefer to do this.

Thus if the chain is expressed by means of the formula

$$AB-T.P.-BC-T.P.-CD-T.P.-DA-T.P.-AB,$$

where T.P. stands for "turning pair," then whichever link is regarded as being fixed the formula is of exactly the same form, and so the mechanisms can be regarded as being the same, their different properties being due to the different proportions of the links.

77. Opposite Links Equal in Length.—When the opposite pairs of links are equal in length, as in Fig. 89, the chain assumes the familiar form in which it is widely used in locomotives to enable one driving shaft or axle to drive a second axle whose axis is parallel to the first. The coupling-rod CD has then a motion of translation alone, and any point attached to it will therefore move in a path parallel to those of C or D, thus tracing out a circle.

Three shafts whose axes are parallel may therefore be connected by a single link as shown in Fig. 90, and this arrangement has

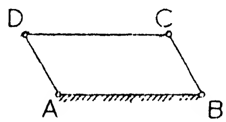


Fig. 89

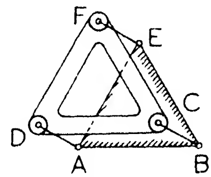


Fig. 90

been used in some petrol engines to enable two camshafts to be driven from a single shaft. Clearly any number of shafts might be connected in this manner. Similarly, when E lies in the same straight line as A and B a single coupling-rod DCF might be used as shown in Fig. 91, but it will be noticed that in both of these arrangements the constraint of the point F is redundant, since that point is constrained by being a point of DCF and also by being a point of EF, and, as pointed out in Chapter V, redundant constraint involves distortion of some of the links if the other links either alter their size or shape from any cause or are made to incorrect dimensions. Thus unequal expansion of the link DCF and the frame ABE would set up severe stresses in the links, and this was found to be a serious trouble in the petrol-engine application. The redundant constraint is avoided in locomotive practice by the use of two separate coupling-rods DC and CF.

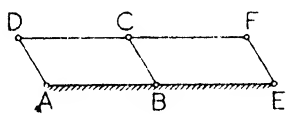


Fig. 91

78. Dead-Points.—Suppose the crank AD in Fig. 92 is being turned by some agency and is thus causing the crank BC to turn, and let the mechanism be in the position AD_1C_1B, where all the links lie in one straight line. Then, in that position, the crank AD_1 is unable to cause the crank BC to turn, because AD can only act on BC through the coupling-rod, and can only apply to BC a force whose direction is along DC, which in this position of the mechanism passes through the axis B. Hence the force transmitted by the coupling-rod has no tendency, in this position, to turn the crank BC and the mechanism is said to be at a *dead-point* or *dead-centre*. The position AD_1C_1B is clearly a dead-point, whether AD or BC is the driver, but in Fig. 85 the position ABC_1D_1 is a dead-centre for BC as driver, but not for AD as driver.

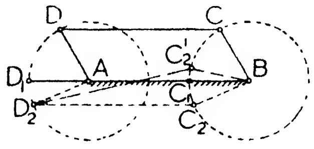


FIG. 92

79. Change-Points.—When the mechanism of Fig. 92 arrives at the position AD_1C_1B, then, if AD_1 turns onwards to AD_2, the crank BC is usually required to turn onwards to BC_2, but clearly there is a possibility of its turning backwards to BC_2', the mechanism then becoming a " crossed " one, whereas previously it was an open chain. Hence the position AD_1C_1B is called a *change-point*. Dead-points and change-points frequently, but not necessarily, occur together, and a mechanism cannot, strictly speaking, be considered perfect kinematically while such points exist. There are two methods of enabling a mechanism to pass through its dead-points and change-points. The first is to use the momentum of some part of the mechanism; thus a flywheel might be fixed to the crank BC, but this remedy is of no use at starting, when the speed, and therefore the momentum, of the flywheel is zero. The second method is to duplicate the chain as shown in Fig. 93 and to arrange that when one chain is at a dead-point the other is not. These two methods are sometimes referred to as *force-closure* and *chain-closure* respectively.

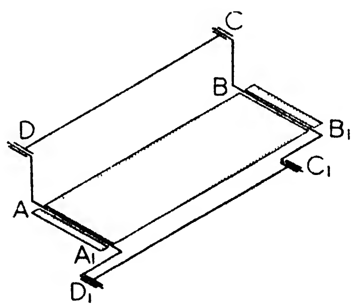


FIG. 93

Other examples of kinematic chains composed of links connected solely by turning pairs are given later in connexion with straight-line motions.

80. Alteration of the Four-Bar Chain.—In the four-bar chain shown in Fig. 94 the point C moves in the arc XX of a circle whose centre is at B, and the motion will be unchanged if the link 2 is replaced by a block 2 sliding in a guide whose axis is the arc XX, the links 3 and 4 being unchanged. If now the slot in which the block 2 slides is made straight, as in Fig. 95, the mechanism on which practically all reciprocating engines, compressors, pumps, etc., are based is obtained. This mechanism is usually called the *slider-crank chain* or the *direct-acting engine mechanism* and is extremely important; it is considered in some detail in a later chapter. It is the equivalent of the four-bar chain of Fig. 94, the link BC having been made infinitely long.

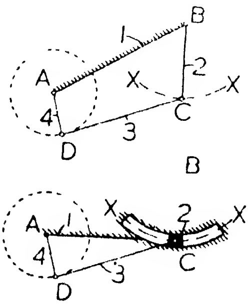


Fig. 94

81. The Double-Slider-Crank Chain.—In Fig. 95 the motion of the point D relative to the block 2 is in the arc of a circle, centre C, radius CD, and if the link CD is replaced, as shown in Fig. 96, by a block 3 sliding in a guide formed in the link 2 and having C as centre and a radius equal to the length of the *connecting-rod* CD of Fig. 95, the motion will be unchanged.

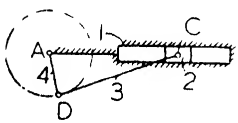


Fig. 95

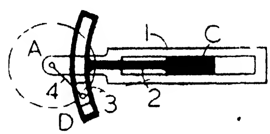


Fig. 96

Again the slot may be made straight, which is equivalent to making the connecting-rod CD of Fig. 95 infinitely long; the mechanism then appears as in Fig. 97, and is called either the

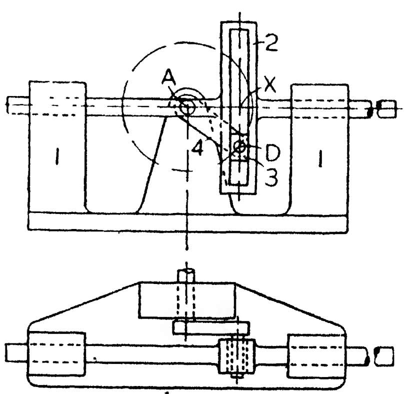


Fig. 97

infinitely long connecting-rod mechanism or the *double-slider-crank chain.* In most of the actual constructional forms of this mechanism the member 2 is supported on both sides of the crankshaft axis A as shown.

The motion of the slider 2 is the same as that of the point X, the projection of the crank-pin axis D upon the line of stroke through A. If the crank AD rotates with *constant angular speed,* then the motion of the point X is the simplest type of vibratory motion and is called *Simple Harmonic Motion.*

82. Simple Harmonic Motion.—This type of motion is of great importance and will now be considered in some detail. In Fig. 98 let the crank AD (length r ft.) rotate about A with constant angular speed (ω rads. per sec.), and let X be the projection of D upon the diameter LM ; then the motion of X is simple harmonic. The displacement of the point X measured from A as origin is

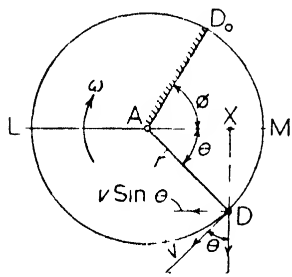


Fig. 98

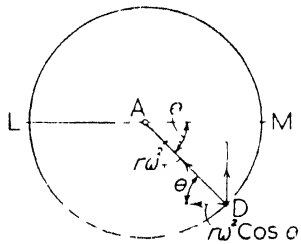


Fig. 99

$x = AX = r \text{ Cos } \theta$, being positive when to the right of A and negative when to the left. The velocity of X is at every instant equal to the component, parallel to LM, of the velocity D, which is always perpendicular to AD and equal to $r\omega$. Hence we have : Velocity of $X = -v \text{ Sin } \theta = -r\omega \text{ Sin } \theta$, the minus sign being introduced because the velocity of X is to the left, while the displacement is positive when to the right.

Similarly the acceleration of X is equal to the component, parallel to LM, of the acceleration of D, which is along DA towards A and equal to $r\omega^2$; hence, from Fig. 99, we have : Acceleration of $X = -r\omega^2 \text{ Cos } \theta$, the minus sign again being introduced because the acceleration is in the opposite direction to the displacement.

Suppose now that the angular displacement of the crank is measured from some initial position AD_o (Fig. 98), and that we begin to measure time from the moment when the crank is at AD_o.

Let t=the time required for the crank to turn from AD_o to AD.

$\angle D_oAM=\phi$

Then
$$t=\frac{\theta+\phi}{\omega}$$
$$\theta=\omega t-\phi$$

Hence the displacement of X $=r\ \mathrm{Cos}\ (\omega t-\phi)$

,, ,, velocity of X $=-r\omega\ \mathrm{Sin}\ (\omega t-\phi)$

,, ,, acceleration of X $=-r\omega^2\ \mathrm{Cos}\ (\omega t-\phi)$

These results may be obtained more directly, by differentiation, as follows :

We have
$$x=r\ \mathrm{Cos}\ \theta=r\ \mathrm{Cos}\ (\omega t-\phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$\therefore\ \frac{dx}{dt}=-r\ \mathrm{Sin}\ \theta\ .\ \frac{d\theta}{dt}=-r\omega\ \mathrm{Sin}\ (\omega t-\phi) \quad . \quad . \quad . \quad (2)$$
$$\therefore\ \frac{d^2x}{dt^2}=-r\omega\ \mathrm{Cos}\ \theta\ .\ \frac{d\theta}{dt}=-r\omega^2\ \mathrm{Cos}\ (\omega t-\phi) \quad . \quad . \quad . \quad (3)$$

If the displacement x is plotted against the time t, the graph obtained is as in Fig. 100. The maximum displacement from the

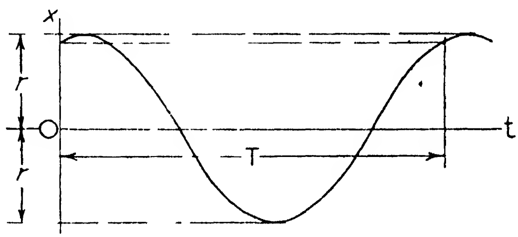


FIG. 100

centre A is clearly equal to the length r of the crank, and this is called the *amplitude* of the motion. The time T required for one complete vibration of the point X is the time required for one revolution of the crank and is equal to $\frac{2\pi}{\omega}$; this is called the *periodic time*. The reciprocal of the periodic time, i.e. $\frac{1}{T}$, is the number of vibrations per unit time and is called the frequency. The angle ϕ, or its time equivalent $\frac{\phi}{\omega}$, is called the *epoch*.

83. The Inversions of the Double-Slider-Crank Chain.—The formula for this chain is 1—S.J.—2—S.J.—3—T.J.—4—T.J.—1, and inspection shows that only three different mechanisms can be obtained from it, since the same mechanism is obtained by fixing link 3 as by fixing link 1, the fixed link in either case having a turning joint at one end and a sliding joint at the other.

Fixing link 2 gives what is known as the *elliptic trammels* shown in Fig. 101. Any point on link 4 traces out an ellipse relative

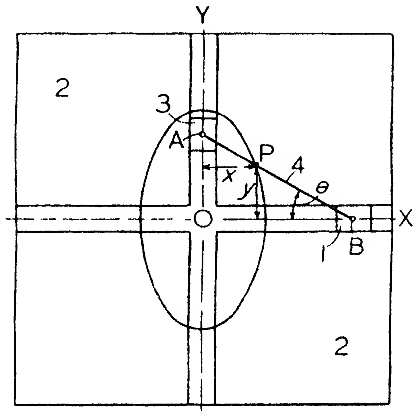


FIG. 101

to link 2. To prove this take OX and OY as axes and let $\angle OBA = \theta$. Then the co-ordinates of P are

$$x = AP \cdot \text{Cos}\ \theta$$

and
$$y = BP \cdot \text{Sin}\ \theta$$

$$\therefore \frac{x^2}{AP^2} + \frac{y^2}{BP^2} = \text{Cos}^2\ \theta + \text{Sin}^2\ \theta = 1$$

and since AP and BP are constant, this is the equation to an ellipse. Clearly the semi-axes are AP and BP. Similarly any point on AB produced will trace out an ellipse. Also it can be shown that an ellipse is obtained when the angle between the slots has any value.

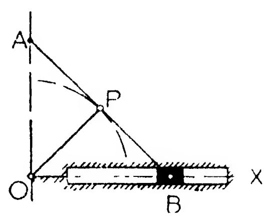


FIG. 102

If AP = BP, then the axes of the ellipse are equal and the ellipse becomes a circle; the point P could then be joined to the point O by a link. It follows therefore that in the mechanism shown in Fig. 102, if AP = BP = OP, then the point A will describe a straight line, provided that B is confined to the straight line OX. This mechanism is known as the *Scott-Russel* straight-line mechanism.

Fixing link 4 gives the *Oldham coupling*, used to connect shafts whose axes are parallel but not coincident, and shown in Fig. 103. The shafts 1 and 3 (corresponding to the blocks 1 and 3 of Fig. 101) turn about their axes in the frame 4 and are provided with tongues

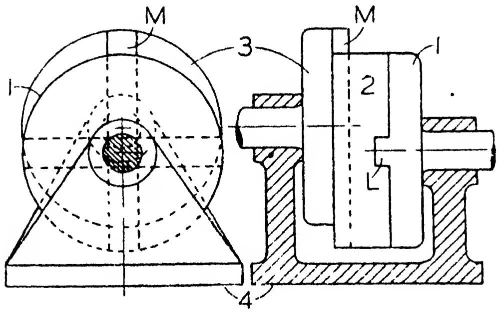


FIG. 103

L and M. The tongues engage grooves in the disc 2, the grooves being on opposite faces of the disc and at right-angles to each other. Clearly, if shaft 1 turns through an angle θ, then because of the tongues L and M the disc 2 and shaft 3 must turn through the same angle ; hence the angular speed of 3 is always equal to that of 1. As the shafts turn, so the disc 2 has to slide to and fro along the tongues L and M, and if this motion is not to be excessive, the distance between the axes of the shafts must be kept small.

An examination of Fig. 101 will show that, since the angle AOB is constant and equal to 90°, if the link 4 is fixed, then the point O will describe a circle about AB as diameter, and no difficulty should be experienced in showing that in the Oldham coupling the centre of the disc 2 describes a complete circle for each half-turn of the shafts 1 and 3.

84. The Crossed-Slide-Crank Chain.—The double-slider-crank chain was obtained from the slider-crank chain by replacing the turning joint between the links 2 and 3 by a sliding joint. If, instead, the turning joint between links 3 and 4 is replaced by a sliding joint, then the crossed-slide-crank chain is obtained. The derivation is shown in Fig. 104 (*a*), (*b*) and (*c*) ; to get the same extent of motion in each of the kinematically equivalent mechanisms, Fig. 104 (*a*) and (*b*), the slot in the link 4 of the latter must be a complete circle, and when this slot is made straight this is impossible and the motion is limited. The chain shown in Fig. 104 (*c*), with the link 4 fixed, has been used as a tiller actuating mechanism known as Rapson's slide. The block 3 is actuated by chains or wire ropes and thus causes the tiller-bar 1 to be turned.

As the tiller moves away from the middle position shown, the turning moment acting on the tiller becomes greater in relation to the force applied to the block 3. The mechanism is also used in the steering-boxes of motor cars, the block 3 then being actuated

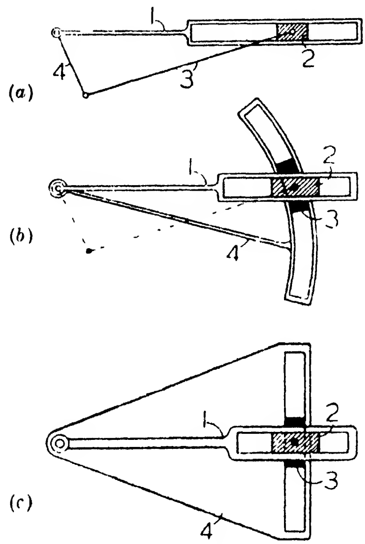


FIG. 104

by a screw passing through it and carried in bearings in the link 4. The formula for this mechanism is

$$1-\text{S.J.}-2-\text{T.J.}-3-\text{S.J.}-4-\text{T.J.}-1,$$

and inspection will show that all the inversions are similar mechanisms.

EXERCISES VI

1. Explain the difference between lower and higher kinematic pairs, and what is meant by the inversion of a kinematic pair. Enumerate the lower pairs and give two or three examples of higher pairs.

2. What is meant by the terms "dead-point" and "change-point"? State the methods by which they can be obviated.

3. By means of simple diagrams show how the single-slider-crank chain can be evolved from a four-bar chain.

4. Sketch to scale a double-crank form of four-bar chain in which the cranks are unequal in length and longer than the fixed link, and show the position of the links for several positions of the driving crank.

5. Describe briefly, with diagrammatic sketches, the inversions of (*a*) the single-slider-crank chain and (*b*) the double-slider-crank chain.

6. A point moving with S.H.M. has a period of 2 secs. and an amplitude of 1 ft. What are its accelerations 0·25, 0·5 and 0·75 secs. after the moment when it has its greatest positive displacement?

CHAPTER VII

THE VELOCITIES OF POINTS IN MECHANISMS

There are several ways in which the velocity of any point in a mechanism may be found when that of some other point is known. The chief of these methods are :

1. By deriving a displacement-time graph and drawing tangents to it ;
2. By means of instantaneous centres ;
3. By means of *velocity diagrams* ;
4. Analytically ;

the choice of method depending on the nature of the mechanism and the accuracy desired. The methods will be considered in the order named.

85. Example of Displacement-Time Curve Method.—As an example of the first method, suppose it is desired to find the angular velocity of the link OD of the mechanism shown in Fig. 105 when the angle θ equals 180°, the constant angular velocity ω of the crank AB being given.

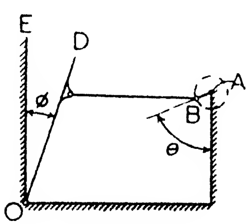


Fig. 105

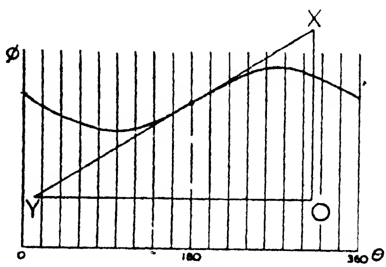


Fig. 106

Draw the mechanism to scale for a series of values of θ increasing from 0° to 360° by equal steps. Measure off each drawing the value of the angle ϕ and plot these values against the corresponding values of θ, thus obtaining the graph shown in Fig. 106. Since the crank rotates at a constant speed the angles θ are proportional to the times taken for the crank to turn through them.

and the graph can be converted into a displacement-time graph simply by converting the θ scale into a time one; this, however, is not really necessary, as will be seen. At $\theta=180°$ draw the tangent XY and measure its slope $\frac{OX}{OY}$. If OX is measured on the ϕ scale and OY on the time scale, then this slope gives $\frac{d\phi}{dt}$, the required angular velocity, directly. If, however, OY is measured on the θ scale, then $\frac{OX}{OY}$ gives $\frac{d\phi}{d\theta}$, and to get $\frac{d\phi}{dt}$ we must use the relationship $\frac{d\phi}{dt}=\frac{d\phi}{d\theta}\cdot\frac{d\theta}{dt}=\omega\cdot\frac{d\phi}{d\theta}$, since $\frac{d\theta}{dt}=\omega$. Clearly the greater number of values chosen for θ the greater the accuracy of the displacement-time graph.

86. Example of the Use of Instantaneous Centres.—In the four-bar chain ABCD (Fig. 107), if the velocity of the point D, relative to the fixed link, is of known magnitude ${}_Dv_A$, then the velocity ${}_Cv_B$ of the point C, also relative to the fixed link, may be found thus. The direction of ${}_Dv_A$ is, of course, perpendicular to AD and that of ${}_Cv_B$ is perpendicular to BC; hence the intersection ${}_3O_1$ of AD and BC produced is the instantaneous centre of link 3 relative to link 1 (see Art. 44). Let Ω be the instantaneous angular velocity of link 3 about ${}_3O_1$,

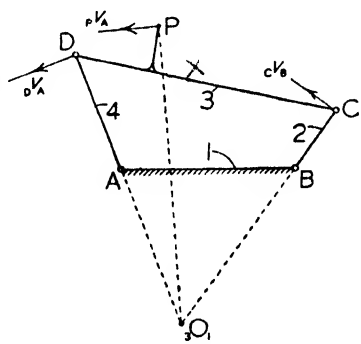


FIG. 107

then
$$\Omega=\frac{{}_Dv_A}{{}_3O_1D}=\frac{{}_Cv_B}{{}_3O_1C}$$

hence
$$ {}_Cv_B={}_Dv_A\times\frac{{}_3O_1C}{{}_3O_1D}$$

and so the magnitude of ${}_cv_b$ may be found, the lengths of ${}_3O_1C$ and ${}_3O_1D$ being measured off the diagram, which must, of course, be drawn to scale.

The actual lengths of the radii ${}_3O_1C$ and ${}_3O_1D$ need not be known, only their ratio being necessary. Thus if the centre ${}_3O_1$ lies at an inconvenient distance it may be dispensed with, the ratio $\frac{{}_3O_1C}{{}_3O_1D}$ being found by drawing from A a line parallel to BC to intersect DC in X. Then triangles ${}_3O_1DC$ and ADX are similar and $\frac{{}_3O_1C}{{}_3O_1D} = \frac{AX}{AD}$.

Similarly the velocity of any point P that is attached to CD is at the instant perpendicular to ${}_3O_1P$ and equal to $\Omega \times {}_3O_1P$ or ${}_Dv_A \times \frac{{}_3O_1P}{{}_3O_1D}$.

The point A in Fig. 107 is the instantaneous centre ${}_4O_1$ of link 4 relative to link 1, in this case a permanent centre, while D is the instantaneous centre ${}_3O_4$ of link 3 relative to link 4, also a permanent centre. Thus the three instantaneous centres ${}_3O_1$, ${}_4O_1$ and ${}_3O_4$ associated with the links 1, 3 and 4 lie on a straight line. That this must be so is readily proved, as follows.

87. The Principle of Three Centres.—Let the three outlines 1, 2 and 3, in Fig. 108, represent three bodies moving in the plane of

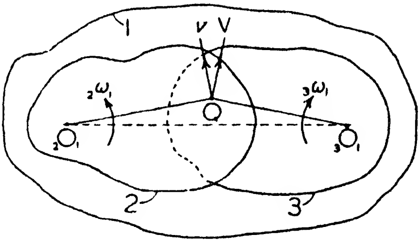


FIG. 108

the paper, let 1 be regarded as fixed and let ${}_2O_1$ and ${}_3O_1$ be the instantaneous centres of 2 and 3, respectively, relative to 1. Then it is required to show that the instantaneous centre of 3 relative to 2 lies on the line ${}_3O_1\,{}_2O_1$. Now the instantaneous centre of 3 relative to 2 is some point of 3 which coincides with *and is at rest relative to* some point of 2, and it follows that if this point of 3 has a velocity relative to 1, then the coincident point of 2 must have the same velocity relative to 1, in magnitude, direction and sense. Let Q be any point; then Q may be regarded as belonging either to 2 or to 3; let it be labelled Q_2 or Q_3 accordingly. At the instant under consideration Q_2 has a velocity v perpendicular to ${}_2O_1Q$, and Q_3 a velocity V perpendicular to ${}_3O_1Q$, both velocities being relative to 1. Now if Q is to be the instantaneous centre of 3 relative to 2, the velocities V and v must have the same direction, and clearly this can only occur if ${}_2O_1Q$ and ${}_3O_1Q$ coincide in direction, i.e. if ${}_2O_1\,Q\,{}_3O_1$ is a straight line; hence the three instantaneous centres lie on a straight line.

If $_2\omega_1$ and $_3\omega_1$ are respectively the instantaneous angular velocities of 2 and 3 relative to 1, then since v must be equal to V in magnitude, $_2\omega_1 \times {}_2O_1Q = {}_3\omega_1 \times {}_3O_1Q$, so that

$$\frac{_3\omega_1}{_2\omega_1} = \frac{_2O_1Q}{_3O_1Q}$$

The instantaneous centre of 3 relative to 2 (the point Q) thus divides the distance $_2O_1\, {}_3O_1$ in the inverse ratio of the angular velocities. Clearly if Q lies between $_2O_1$ and $_3O_1$, then the angular velocities must be opposite in sense as shown, while if the angular velocities have the same sense, then Q divides $_2O_1\, {}_3O_1$ externally.

The principle of three centres is of great help with complex mechanisms. As an example consider the mechanism shown in Fig. 109. This is the pencil mechanism of the Crosby steam-engine indicator, and it is arranged so that the end A of the link

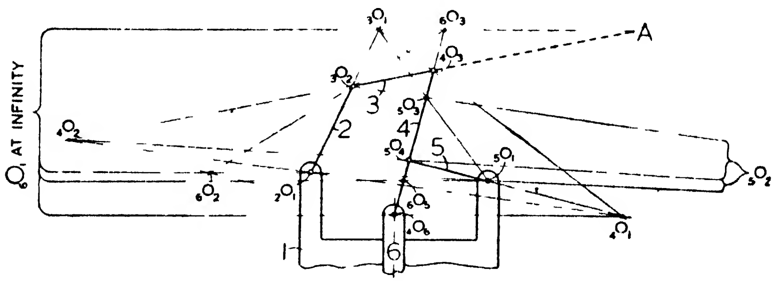


FIG. 109

3 reproduces, with a sufficient degree of accuracy and to an enlarged scale, the motion of the piston rod 6. The instantaneous centres are :

$$\begin{array}{lllll}
{}_2\mathit{O}_1 & {}_3\mathit{O}_2 & {}_4\mathit{O}_3 & {}_5\mathit{O}_4 & {}_6O_5 \\
{}_3O_1 & {}_4O_2 & {}_5O_3 & {}_6\mathit{O}_4 & \\
{}_4O_1 & {}_5O_2 & {}_6O_3 & & \\
{}_5\mathit{O}_1 & {}_6O_2 & & & \\
{}_6O_1 & & & &
\end{array}$$

That is, $5+4+3+2+1=15$ in all. Six of these (those in italics above) are permanent centres and their positions are obvious ; the others may be found as follows. Because the piston 6 moves in a straight line relative to 1, the centre $_6O_1$ will lie at infinity, and any line drawn to pass through it will be perpendicular to the line of stroke of 6, i.e. will be horizontal. Then the centre $_4O_1$ must lie on $_4O_6\, {}_6O_1$, that is, on a horizontal through $_4O_6$; it must also lie on $_5O_4\, {}_5O_1$; hence the intersection of these lines gives $_4O_1$. Then

${}_3O_1$ must lie on ${}_2O_1\,{}_3O_2$ and on ${}_4O_1\,{}_4O_3$ and is thus determined. The centre ${}_4O_2$ must lie on ${}_4O_3\,{}_3O_2$ and on ${}_4O_1\,{}_2O_1$, being therefore at their intersection ; ${}_5O_2$ must lie on ${}_4O_2\,{}_5O_4$ and on ${}_2O_1\,{}_5O_1$ and is thus determined; ${}_6O_2$ lies at the intersection of ${}_4O_6\,{}_4O_2$ and ${}_6O_1\,{}_2O_1$, the latter being horizontal, while ${}_5O_3$ lies at the intersection of ${}_5O_1\,{}_3O_1$ and ${}_5O_4\,{}_3O_4$; ${}_6O_3$ lies on ${}_4O_6\,{}_4O_3$ and on ${}_3O_1\,{}_6O_1$, the latter being horizontal; and lastly ${}_6O_5$ lies at the intersection of ${}_4O_6\,{}_5O_4$ and ${}_5O_1\,{}_6O_1$, the latter being horizontal.

In many mechanisms some of the centres may be found by means of alternative intersections, and it is then possible to check the accuracy of the work. Thus ${}_6O_2$, ${}_6O_3$ and ${}_3O_2$ should lie on a straight line, and if the work is accurate they will do so.

Clearly if the pencil A is to describe a straight line parallel to the line of stroke of the piston, then the line joining A to the instantaneous centre ${}_3O_1$ must be horizontal for all positions of the mechanism. This condition is not satisfied rigidly. (Note : In the figure the proportions of the links have been modified in order to bring as many as possible of the centres into the space available.)

88. Another Example on the Use of Instantaneous Centres.—In the example given in Art. 86 the points C, D and P all belonged to the same link ; when they are on different links the process of finding the unknown velocities is a little longer. Suppose ${}_Pv_A$ (Fig. 110) is known and ${}_Qv_B$ is to be found. Then the velocity of

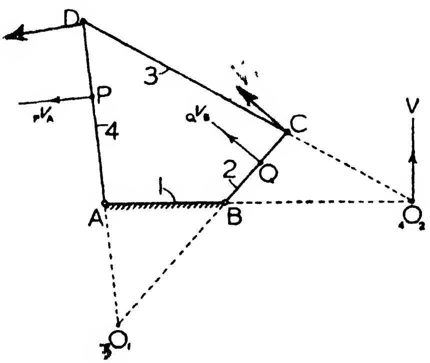


FIG. 110

D is easily found and, by the use of the instantaneous centre ${}_3O_1$, that of C may be found, and hence that of Q, thus

$${}_Qv_B = {}_Pv_A \times \frac{AD}{AP} \times \frac{{}_3O_1C}{{}_3O_1D} \times \frac{BQ}{BC};$$

but a slightly shorter method may be used. The instantaneous centre ${}_4O_2$ may be regarded as being attached to either of the links 2 or 4. Regarding it as belonging to 4, then it will have a velocity V, perpendicular to A_4O_2 and equal to ${}_Pv_A \times \frac{A_4O_2}{AP}$, relative to link 1. Now regarding it as belonging to 2 its velocity relative to 1 must again be equal to V and we have the relation

$$\therefore \quad V = {}_Qv_B \times \frac{B_4O_2}{BQ}; \quad \text{hence, } {}_Qv_B = {}_Pv_A \times \frac{A_4O_2}{AP} \times \frac{BQ}{B_4O_2}.$$

89. Centrodes.—It has been stated that the locus of the instantaneous centre of one body relative to another is called the centrode of the one body relative to the other. In Fig. 111 the

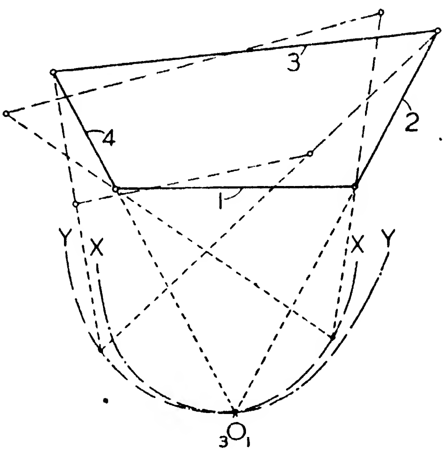


FIG. 111

point ${}_3O_1$ is the instantaneous centre of 3 relative to 1 for the full-line configuration of the mechanism. As the mechanism moves to new configurations so the instantaneous centre traces out the curve XX, which is therefore the centrode of 3 relative to 1. The curve XX having been drawn with the link 1 fixed may be regarded as being fixed to or carried by the link 1.

Suppose now that the link 3 is taken to be the fixed link of the mechanism, then in the full-line configuration ${}_3O_1$ is also the instantaneous centre of 1 relative to 3, but as the mechanism moves to new configurations this instantaneous centre will not trace out the curve XX, but some other curve YY. This is the centrode of 1 relative to 3, and may be regarded as being fixed to or carried by the link 3. If now the centrode YY, with the link 3 attached to it, rolls without slip on the curve XX (regarded as at rest), then the link 3 will have the same motion relative to 1 as is given to it by the connexion afforded by the links 2 and 4. The latter could therefore be dispensed with if the centrodes are rolled together without slip.

90. The Centrodes for the Elliptic Trammel Mechanism.—As an example consider the elliptic trammel mechanism shown in Fig. 112. The instantaneous centre of AB relative to the frame

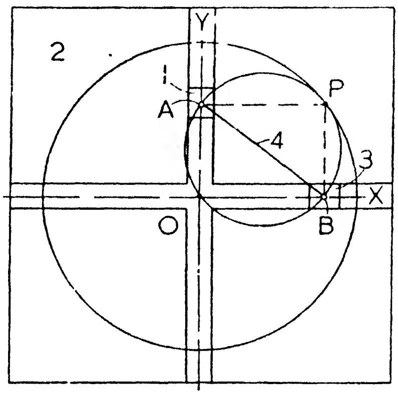


FIG. 112

XY is the point P, where AP is perpendicular to OY and BP to OX. Clearly the distance OP is always equal to AB; hence the point P will trace out a circle, centre O, radius equal to AB, when AB moves. This circle is the centrode of AB relative to XY. If now AB is fixed and XOY is moved, then, since AP is always perpendicular to OY and BP to OX, $\angle APB = \angle XOY = 90°$. Hence P will trace out a circle on AB as diameter. This circle is the centrode of XOY relative to AB.

As another example of centrodes consider the four-bar chain shown in Fig. 113, where opposite links are equal in length and the chain is crossed. Then if link 1 is fixed the centrode of 3 relative to 1 is found to be the two branches AB and CD of a hyperbola, and when 3 is fixed the centrode of 1 relative to 3 is found to be the two branches LM and NO of an exactly similar hyperbola.

91. As a further example consider two bodies A and B (Fig. 114) that rotate about fixed axes O_A and O_B with constant angular velocities ω_A and ω_B. The instantaneous centre is then some point P lying on the line joining O_AO_B and such that $\frac{O_AP}{O_BP}=\frac{\omega_B}{\omega_A}$, and since this ratio is constant, P is a fixed point on O_AO_B. If now B is fixed and the relative motion between A and B kept the same, then the line O_BO_A will turn about O_B as centre with an angular velocity equal to $-\omega_B$, and the locus of P

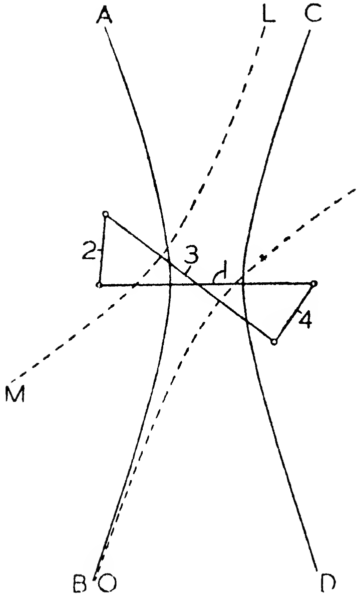


FIG. 113

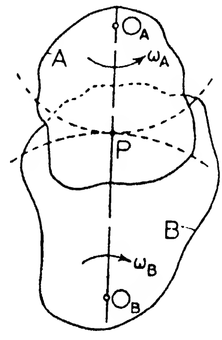


FIG. 114

will be a circle, centre O_B, radius O_BP, as shown. This is the centrode of A relative to B. Similarly the centrode of B relative to A is a circle, centre O_A and radius O_AP. If the ratio of the angular velocities had not been constant, then the loci of P (the centrodes) would not have been circles, but some other curves; they would, however, always touch each other at some point, such as P, lying on the line of centres O_AO_B. Since the instantaneous centre is only the point of intersection of the instantaneous axis with the plane of the paper, it follows that the centrodes are only the intersections of the surfaces composed of all the successive instantaneous axes with the plane of the paper. These surfaces are, of course, the axodes. In the case under consideration, and in all cases of plane motion, the instantaneous axis is always parallel to the axes O_AO_B or perpendicular to the plane of motion. It follows that the axodes are cylinders, in the general sense, i.e. surfaces composed of straight lines all of which are perpendicular to the plane of motion. When the angular velocity ratio is constant the centrodes are circles and the axodes are thus circular cylinders.

92. Velocity Diagrams.—The method of drawing velocity diagrams will be explained by taking a number of examples, but it will be convenient to prove three simple propositions first. They are :

1. *If* AB *is a rigid link, then the velocity of* B *relative to* A *must be perpendicular to* AB. For if it were in any other direction, as in Fig. 115, then it would have a component in the direction AB, and the distance AB would therefore be changing, but, since the link is rigid, this is impossible; hence the proposition is proved.

2. If A, Fig. 116, is a block constrained to slide along the slot in the link B, then the point P may be regarded as belonging to both A and B and may be denoted by P_a or P_b accordingly. *Then the velocity of* P_a *relative to* P_b *is along the tangent* (PX) *to the slot as shown.* For if it were in any other direction (PY) it would have a component perpendicular to the slot, and this is impossible.

3. If two rigid bodies A and B (Fig. 117) move in the plane of the paper and are in contact at any instant at some point P, then,

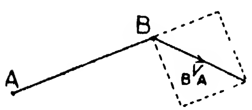


Fig. 115

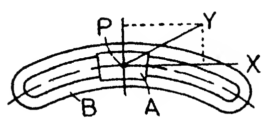


Fig. 116

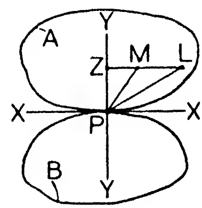


Fig. 117

in general, there is a common tangent XPX to the profiles of the bodies and also a common normal YPY perpendicular to XPX. The point P may be regarded as belonging to both A and B and may be denoted by P_a or P_b accordingly. *Then, if the profiles of the bodies remain always in contact, the velocity of* P_a *relative to* P_b *must be perpendicular to the common normal* YPY. For if it were in any other direction it would have a component along YPY, and the profiles would either be separating or the one would be penetrating the other, and since neither of these actions is permitted the proposition is proved. It follows that if PL and PM represent the velocities of P_a and P_b respectively, relative to a third body, say the paper, then the components along the common normal of these velocities must be equal in magnitude and sense; hence LMZ is perpendicular to YPY.

It may be remarked that XPX is the intersection of the *plane* that is tangent to both A and B at P with the plane of the paper. Any line lying in this common tangent plane and passing through P is a common tangent to A and B; hence there is an infinite number of common tangents at P. There is, however, only the one common normal. The proposition stated above, and its corollary, thus hold for any motions of the bodies, plane or non-plane.

93. Example of a Velocity Diagram.—As a very simple example consider the four-bar chain in Fig. 118. Suppose it is required to find the velocity ${}_{C}v_{B}$, the velocity ${}_{D}v_{A}$ being known. From any convenient point *a* as pole set out *ad* parallel to ${}_{D}v_{A}$ (and hence perpendicular to AD) and equal to it to any convenient scale. (In order to get the senses of the vectors correct it is helpful to memorise the form of statement used in Art. 16, thus "*ad* with the arrow pointing *from a to d* is the velocity of *D relative to A*.") Since B is at rest relative to A, the point *a* also represents B and may be labelled *b* as shown. Now the velocity of C relative to B, if it were known, would be represented by a vector *bc* perpendicular to BC, and then the vector *cd* would represent the velocity of D relative to C; but this is perpendicular to CD, hence the point *c* may be determined by drawing from *d* a line perpendicular to CD to intersect the line drawn through *b* parallel to the direction of the velocity ${}_{C}v_{B}$.

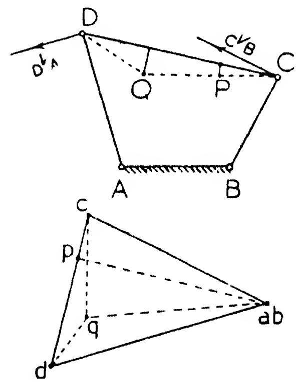


FIG. 118

A line joining the pole *a* to any point *p* of *cd* represents the velocity, relative to A, of the corresponding point P of the link CD; hence *cd* is called the velocity image of CD. The position of *p* is determined from the relation $\frac{cp}{cd}=\frac{\mathrm{CP}}{\mathrm{CD}}$. Similarly to determine the velocity image of a point Q that is attached to CD it is merely necessary to erect on *cd* a triangle *cdq* similar to CDQ. Care must be taken to place the triangle *cdq* properly on *cd*, and this is facilitated if the triangle CDQ be imagined drawn on a piece of tracing paper and transferred, without being lifted up, so that C coincides with *c* and D with *d*, the size of the triangle being reduced or increased as may be necessary.

94. Another Example of a Velocity Diagram.—Fig. 119 shows a mechanism that is used to operate the slide-valves of locomotives and which is known as *Joy's Valve Gear.* ABC is a slider-crank chain forming the main engine mechanism, AB being the crank, BC the connecting-rod and C the cross-head. At a point D of the connecting-rod is pivoted a link DE, the end E of which is connected to a link EF hinged at F to a point on the frame of the locomotive. In a similar way the link GH is pivoted at G to a

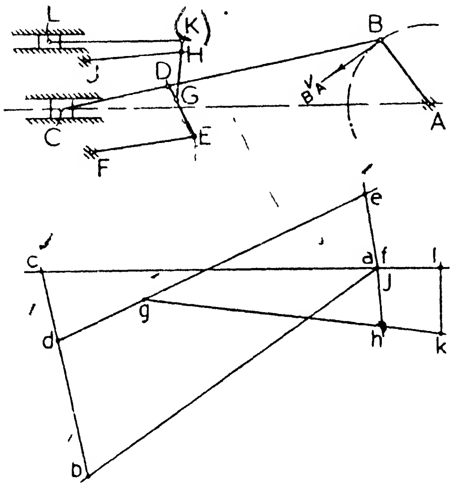


Fig. 119

point on DE and at H to the link HJ, which in turn is pivoted at J to the frame. (Actually the point H is guided in a curved slot of radius JH, but this is, kinematically, the same as the construction shown. The inclination of the slot can, however, be changed in order to reverse the direction of rotation of the engine, which is not practicable when the rod JH is used.) The link GH is extended to K, which point is connected by the rod KL to the valve-rod, which moves in a guide in the frame.

Supposing the velocity $_{B}v_{A}$ of the crank-pin B to be known, let it be required to find, by means of a velocity diagram, the velocity of the valve-rod, i.e. of the point L.

Choose any convenient point a to represent the fixed centre A; then this point will also represent the fixed centres F and J. Set out ab perpendicular to AB and equal to $_{B}v_{A}$ to any convenient scale, taking care to get the sense of the vector the same as that of the velocity it represents. Through a draw a line parallel to the line of stroke of the cross-head C; then the point c, the velocity image of C, lies somewhere on this line. Since the velocity of C relative to B must be perpendicular to BC, c must also lie on a line drawn through b perpendicular to BC, and its position is thus determined. Then bc is the velocity image of the connecting-rod BC and a point d chosen such that $\frac{cd}{cb}=\frac{CD}{CB}$ will be the image of D. The velocity of E relative to D must be perpendicular to DE, hence e must lie on a line through d drawn perpendicular to DE. Since the velocity of E relative to F must be perpendicular to EF, e must also lie on a line drawn through f perpendicular to EF and is thus determined; de is then the velocity image of DE and g is fixed by making $\frac{dg}{de}=\frac{DG}{DE}$. Then h must lie on a line drawn through g perpendicular to GH, and also on a line drawn through j perpendicular to JH, being thus determined; k is found by making $\frac{gk}{gh}=\frac{GK}{GH}$, and finally l must lie on a line drawn through k perpendicular to KL and also on a line drawn through a parallel to the valve-rod guide. Then al is the velocity of the valve-rod to the velocity scale chosen. In the configuration of the mechanism shown the valve-rod is thus moving in the opposite direction to the cross-head C.

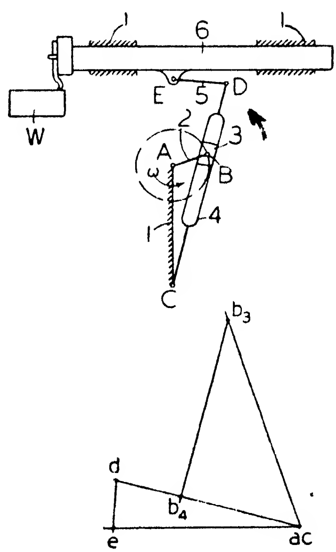


FIG. 120

95. Third Example of a Velocity Diagram.—In Fig. 120 is shown a mechanism that is commonly used to give a reciprocating motion to the ram, 6, of shaping machines. (The ram carries a tool as indicated which operates on the work W on the forward, or right to left, stroke.) The links 1, 2, 3 and 4 comprise an inversion of the slider-crank chain (see Art. 80), added to which are the links 5 and 6. If the crank 2 is rotating with an instantaneous angular speed ω, supposed known, then the instantaneous velocity of the ram may be found by drawing a velocity diagram. The linear velocity of B_3 (B regarded as belonging to link 3) is ω . AB, and this may be set out from any pole a to any convenient scale, giving the vector ab_3 perpendicular to AB. The velocity of B_4 relative to C is perpendicular to CB; hence from c (which is coincident with a, both C and A being at rest) draw a line perpendicular to CB. The velocity of B_4 relative to B_3 is along the slot in 4; hence from b_3 draw a line parallel to this slot, thus settling the point b_4. Then cb_4 is the velocity image of CB and d is obtained by producing cb_4 to d such that $\frac{cd}{cb_4}=\frac{\text{CD}}{\text{CB}}$. Since the velocity of E relative to D is perpendicular to DE, draw from d a line perpendicular to DE. Finally, since the velocity of E relative to A is along the ram-guide, draw from a a line parallel to this guide, thus obtaining the point e. Then ae, to the velocity scale chosen, is the velocity of E relative to A.

96. The Angular Velocities of Links.—If a velocity diagram be drawn for any configuration of a mechanism, then each link of the mechanism will have its image in the diagram, this image representing the velocity of one end of the link relative to the other end. Thus in Fig. 120 cd represents the velocity of D relative to C. Clearly then the magnitude of the instantaneous angular velocity of the link relative to either end is given by $\frac{cd}{\text{CD}}$, cd being measured *to the velocity scale* and CD being the *actual length* of the link CD. If this angular velocity is to be obtained in radians per unit time, as is desirable, the units used must be consistent, i.e. cd in say *feet* per sec. and CD in *feet*, the angular velocity being then in radians per sec. The sense of the angular velocity of the link relative to one end will of course be opposite to that of the angular velocity relative to the other end; the sense, however, is usually unimportant. The magnitudes of the instantaneous angular velocities of links relative to their ends are required when the acceleration diagrams of mechanisms are required, as will be seen later.

97. Special Method for when the Above Method Fails.—With some mechanisms it will be found that the velocity diagram cannot be drawn by the straightforward method adopted in the previous examples, and special methods must be used. Consider the Stephenson link motion shown in Fig. 121. This cannot be set out on the drawing-board by ordinary line and circle constructions; thus, the position of the crank AOB being known, the points C and D must lie on circles centres B and A respectively, also E must lie on a circle centre F, but in order to find the position of the link CDE it is necessary to use a template cut out to the shape CED and to adjust it until each of the points C, E and D lies on its respective locus. Similarly when drawing the velocity diagram the vectors oa and ob can be set out and then the lines ad^1 and bc^1, drawn respectively perpendicular to AD and BC, are the loci of the velocity images of C and D; also a line through o perpendicular to EF is the locus of the velocity image of E, but the points c, d and e cannot be determined in the ordinary way. Suppose, however, the position of the image c is guessed to be c_1, then the position d_1 of the image of D may be determined by drawing c_1d_1 perpendicular to CD to intersect ad^1 (the locus of d) in d_1, and then the position e_1 of the image e is determined by

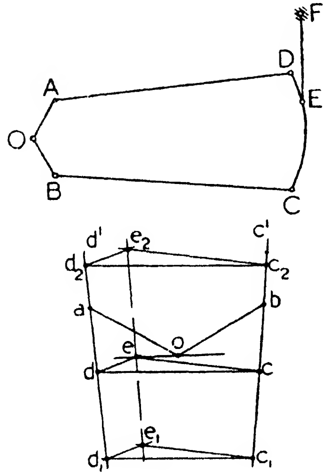


FIG. 121

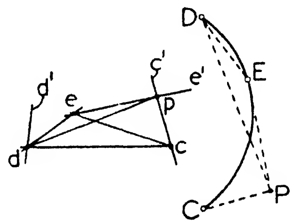


FIG. 122

drawing a triangle $c_1d_1e_1$ similar to CDE. Then e_1 should lie on the line oe, and if it does not the position selected for c_1 is wrong. Choose another position for c, say c_2, and repeat the process, thus obtaining e_2; then a line drawn through e_1e_2 will intersect the line oe in e, the true position of the velocity image of E, and c and d may be determined by drawing ec and ed perpendicular respectively to EC and ED.

In the above problem, and in similar ones, where the loci of the velocity images of three points of a link are known, the actual positions of those images may be determined by means of the following construction. Suppose CED (Fig. 122) is a link, and that the velocity images of C, D and E are known to lie on the lines c^1, d^1 and e^1 respectively. To find the positions of c, d and e choose the intersection of any two of the lines c^1, d^1 and e^1, say the point p, the intersection of c^1 and e^1; then through C draw CP perpendicular to c^1 and through E draw EP perpendicular to

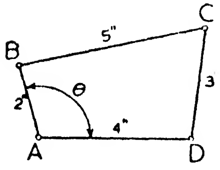
e^1, thus determining the point P. Join PD. From p draw pd perpendicular to PD to intersect the line d^1 in d. Then d is the velocity image of D, and c and e may be determined by drawing dc and de perpendicular respectively to DC and DE to intersect the lines c^1 and e^1 as shown. (Note: If the work has been accurately done ec should be perpendicular to EC.)

The construction, which is sometimes called the *three-line construction*, may be proved thus. If the point P is regarded as being attached to the link CED, then clearly its velocity image is p, since, by the construction, pc is perpendicular to PC and pe is perpendicular to PE. Then the velocity of D relative to P must be perpendicular to PD, and hence must lie at d. The points c and e are then determined in the usual manner.

98. Revolved Velocity Diagrams.—If a velocity diagram, drawn in the usual way, is turned round through a right angle, then the velocity images of the various links will be parallel to those links instead of perpendicular to them as in the diagrams given above. It is considered by some that the diagrams are easier to draw when thus revolved, but, in the writer's opinion, it is better not to revolve the diagrams. Velocity diagrams are also sometimes called velocity polygons.

99. Velocities by the Analytical Method.—An example of the analytical method has been given in Art. 82, from which it will be seen that the method consists in deriving, by algebraic and trigonometrical means, an expression for the displacement of the point under consideration and then differentiating this expression with respect to time. The method is also illustrated in the chapter on the direct-acting engine mechanism. The drawback of this method is that with all but simple mechanisms the expressions for the displacement are extremely cumbersome to use and tedious to derive, but when workable expressions can be obtained the method is probably the best of all.

EXERCISES VII



1. In the four-bar chain shown in the figure the link AB oscillates so that the angle θ varies from 20° to 180°. Plot a graph showing the displacement of the point C against θ and hence find the speed of that point when $\theta = 60°$, assuming that the speed of AB is then 50 r.p.m. Check your result by using the instantaneous centre.

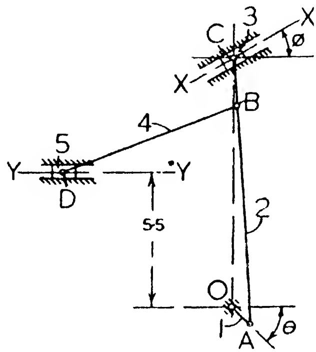
2. Solve Question 1 by drawing the velocity diagram.

3. Find all the instantaneous centres belonging to the single-slider-crank chain.

4. In a slider-crank chain the crank is 3 in. long and rotates at 1000 r.p.m. and the connecting-rod is 12 in. long. Find, by drawing the velocity diagram, the

velocity of the piston when the crank has turned through an angle of 60° from the inner dead-centre. Find also the angular velocity of the connecting-rod relative to the fixed frame and, taking the diameter of the crank-pin to be 2 in., the rubbing speed between it and the connecting-rod.

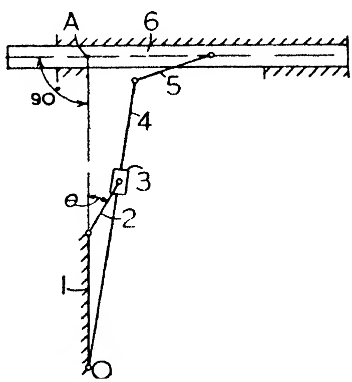
5. In the Marshall valve gear indicated in the figure the crank 1 rotates about a fixed centre O and the blocks 3 and 5 are guided along lines XX and YY



respectively. The fixed frame (link 6) is not shown. Find all the fifteen instantaneous centres belonging to the mechanism.

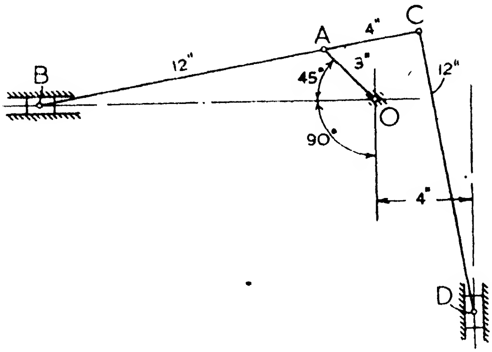
6. In the mechanism of Question 5 the crank OA is 2 in. long and rotates at 100 r.p.m. anticlockwise. The angle ϕ is 30°, C is vertically above O, AB = 9 in., AC = 11 in. and BD = 7·5 in. Draw the velocity diagram for $\theta = 45°$ and find the velocity of the slider 5 and the angular velocity of link 2 relative to the frame 6.

7. Find all the fifteen instantaneous centres belonging to the shaper mechanism shown in the figure.

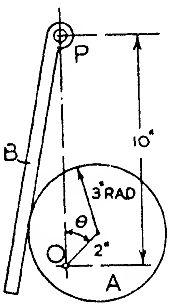


8. In the mechanism of Question 7 the crank 2 is 3 in. long and rotates at 20 r.p.m. anticlockwise. Draw the velocity diagram for $\theta = 30°$ and find the velocity of the ram 6, the angular velocity of link 4 about O and the velocity of sliding of the block 3 along the link 4. The dimensions are link 1 = 7 in., link 4 = 15 in., link 5 = 4 in. and OA = 16 in.

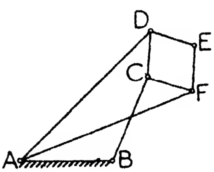
9. If in the mechanism shown in the figure the crank OA rotates at 100 r.p.m.



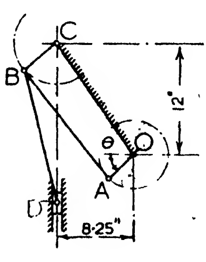
clockwise, find, by drawing the velocity diagram, the velocity of the slider D, which is actuated by the rod CD from the link BAC, which is rigid.



10. The eccentric A in the figure rotates about the fixed centre O at 100 r.p.m. and actuates the follower B. Find, when $\theta = 45°$, the angular velocity of B about P and the rubbing speed of B on A.



11. In the Peaucellier straight-line motion shown in the figure AB = BC = 3 in., AD = AF = 6 in. and CD = DE = EF = FC = $1\frac{1}{2}$ in. Taking the angle DAB to be 45° and the velocity of D relative to AB to be unity, draw the velocity diagram and thus verify that the velocity of E relative to AB is perpendicular to AB.



12. The figure shows a mechanism used in a moulding press to obtain great pressures at the ram D. The crank OA rotates about the fixed centre O and gives an oscillatory motion to the crank CB, which turns about the fixed centre C and operates the ram D through the connecting-rod BD. Draw the velocity diagram and find the velocity of the ram when $\theta = 45°$ and the speed of rotation of the crank OA is 100 r.p.m. OA = 3·75 in., CB = 4·5 in., AB = 15 in., BD = 15 in.

CHAPTER VIII

(Articles 105 to 110 of this chapter may be omitted on a first reading.)

THE ACCELERATIONS OF POINTS IN MECHANISMS

100. There are two principal methods of finding the acceleration of any point of a mechanism the motion of one link of which is known; they are:

1. By means of *acceleration diagrams*.
2. Analytically.

The analytical method is similar to that used for determining velocities and is illustrated in subsequent chapters. The present chapter is concerned with the first of the above methods.

Examples will be used to explain the method of drawing acceleration diagrams, but a few preliminary remarks will be made.

101. If AB, Fig. 123, is a rigid link, moving in any manner, the acceleration of B relative to A is, of course, the acceleration B would have if A were fixed, and if A were fixed, then the only motion possible to B is one of rotation about A as centre. In general, this motion will be a variable one, so that at any instant AB will have an angular velocity (ω) and an angular acceleration (α) about A. Because of the angular velocity, B will have a *normal acceleration* equal to AB . ω^2 directed from B towards A and, because of the angular acceleration, B will have a *tangential acceleration* equal to AB . α in a direction perpendicular to AB. The sense of the tangential acceleration must conform to that of the angular acceleration. The acceleration of B relative to A is the vector sum of these two component accelerations, but usually this sum is not required, the components being kept separate. The following notation will be used in connexion with accelerations:

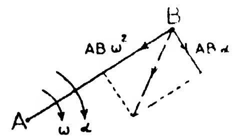


FIG. 123

$$ {}^{n}_{A}a_B = \text{normal acceleration of A relative to B} $$
$$ {}^{t}_{A}a_B = \text{tangential acceleration of A relative to B} $$

$$_{\mathrm{A}}a_{\mathrm{B}} = \text{acceleration of A relative to B}$$
$$= {}^{n}_{a}a_{b} + {}^{t}_{a}a_{b}$$

If now A is joined to some point O by a second rigid link, then the acceleration of A relative to O will also, in general, be composed of a normal and a tangential component directed respectively along and perpendicular to AO. The acceleration of B relative to O is then given by the vector sum of the acceleration of B relative to A and of A relative to O, each of these accelerations being composed of two components, as shown in Fig. 124, where $\boldsymbol{oa}_1$ * is the normal component and $\boldsymbol{a}_1\boldsymbol{a}$ the tangential component of the acceleration of A relative to O, and $\boldsymbol{ab}_1$ and $\boldsymbol{b}_1\boldsymbol{b}$ are the corresponding components of the acceleration of B relative to A; $\boldsymbol{ob}$ is then the acceleration of B relative to O.

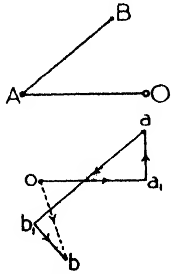


FIG. 124

When a link AB is part of a mechanism, the magnitude of the normal acceleration of B relative to A can always be found if the velocity diagram can be drawn. Thus ω, the angular velocity of AB about A, is equal to $\frac{_{\mathrm{B}}v_{\mathrm{A}}}{\mathrm{AB}} = \frac{ab}{\mathrm{AB}}$, where ab is the velocity image of AB, and hence the normal acceleration of B relative to A is given by $\mathrm{AB}\,.\,\omega^2 = \frac{_{\mathrm{B}}v_{\mathrm{A}}^2}{\mathrm{AB}} = \frac{(ab)^2}{\mathrm{AB}}$, it being understood that ab is measured off the velocity diagram to the proper scale; consistent units must be used, e.g. if ab is in ft./sec., AB must be in feet, the acceleration then being in $\frac{(\text{ft./sec.})^2}{\text{ft.}} = \text{ft./sec.}^2$. The magnitudes of the tangential components are usually unknown (being found from the acceleration diagrams), but their directions are always known, being perpendicular to the respective links.

102. The acceleration of a point B of a block that slides along a slot in a fixed link ECF is composed of a normal component directed from B towards the centre O and equal to $\frac{_{\mathrm{B}}v_{\mathrm{C}}^2}{r}$, where r is the radius of curvature of the slot, and of a tangential component perpendicular to the radius BO and equal to $_{\mathrm{B}}\dot{v}_{\mathrm{C}}$. If the slot is a straight one, then the radius r is infinite and the normal acceleration is zero; hence the acceleration of a point of a block that slides along a straight slot in a fixed link is parallel to the slot.

* Throughout this chapter heavy type is used to indicate vectors appearing in the *acceleration* diagrams.

When the link ECF is not fixed, but has a motion of rotation about some centre, the acceleration of the point B relative to any fixed frame of reference must be determined as shown in Art. 106, so that, for the present, mechanisms involving rotating slotted links will not be considered.

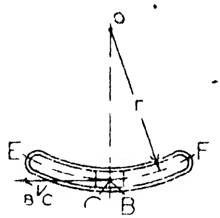


FIG 125

103. Example of an Acceleration Diagram.--In the four-bar chain ABCD, Fig. 126, the link AB is fixed and AD rotates with a constant angular velocity ω radians per sec. What is the angular acceleration of CB about B when the mechanism occupies the position shown ?

Having drawn the mechanism to scale in the given configuration, the first step is to draw the velocity diagram ; this is shown in Fig. 127. The acceleration diagram can now be drawn, the underlying principle being that the acceleration of the point C relative

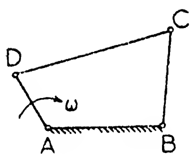


FIG. 126

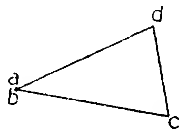


FIG. 127

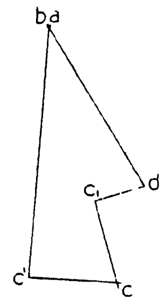


FIG. 128

to the fixed link can be obtained by considering C as a point of the link BC, when its acceleration will consist simply of a normal and a tangential component, or by considering C as a point of the link DC, when its acceleration relative to the fixed link will be given by the vector sum of its acceleration relative to D and the acceleration of D relative to A.

Thus take any point a (Fig. 128) to represent A (and thus also B, since both are fixed points) and set out ad parallel to DA and equal to AD . ω^2 ft./sec.^{2} to any convenient scale, to represent the acceleration of D relative to A. Since ω is constant, D has no tangential acceleration relative to A. Next the acceleration of C relative to D must be added to that of D ; from d draw dc_1 parallel to CD and equal to $\frac{{}_cv_d{}^2}{CD}=\frac{(dc)^2}{CD}$ ft./sec.^{2} to represent the normal acceleration of C relative to D. From c_1 draw a line perpendicular to CD, then c, the acceleration image of C, must lie on this line. Turning now to the link BC, the normal acceleration of C relative to B may be set out parallel to CB as $bc' = \frac{(bc)^2}{\mathrm{BC}}$ ft./sec.^{2}. From c' draw a line perpendicular to BC ; then c, the acceleration image of C, must lie on this line also and must therefore lie at the intersection c.

The components of the acceleration of C relative to B are bc' ft./sec.^{2}, the normal component, and $c'c$ ft./sec.^{2}, the tangential component. The angular acceleration of BC about B is then given by

$$\alpha = \frac{\text{Tangential acceleration of C relative to B}}{\mathrm{CB}}$$

$$= \frac{c'c}{\mathrm{CB}} \frac{\text{ft./sec.}^2}{\text{ft.}}$$

$$= \frac{c'c}{\mathrm{CB}} \text{ rads./sec.}^2$$

The sense of this angular acceleration must conform to that of the tangential acceleration $c'c$; thus $c'c$ with the sense c' to c is the tangential acceleration of C relative to B ; hence the sense of the angular acceleration α is clockwise, i.e. the angular velocity of BC about B is increasing.

Similarly the angular acceleration of DC relative to C is given by $\beta = \frac{c_1c}{\mathrm{CD}}$ rads./sec.^{2} in a clockwise sense.

104. Another Example.—Fig. 129 shows diagrammatically a valve gear used for operating the slide-valves of steam engines. The crank AB may be taken to revolve at a constant speed. The end C of the link BC is constrained to move along a line XX the inclination of which is fixed, except that it can be altered for certain purposes such as reversing the direction of rotation of the engine. A point D of BC is coupled by a rod DE to a slider E guided in a fixed guide YY. The slider E is connected rigidly to the slide-valve. To find the acceleration of the valve, for the given configuration of the mechanism, being given the speed of the crank AB, the velocity diagram is first drawn, as shown at the bottom of the figure.

The first step in drawing the acceleration diagram, which is shown in Fig. 130, is to set out ab, the acceleration of B relative to A. This is equal to $\frac{{}_{\mathrm{B}}v_{\mathrm{A}}^2}{\mathrm{AB}}\left(=\frac{ab^2}{\mathrm{AB}}\right)$ and is parallel to AB, and in sense is directed from B to A. Since the speed of AB is constant, there is no tangential component. The normal acceleration of C relative to B is next set out as $bc_1 = \frac{{}_{C}v_{B}^2}{BC} = \frac{(bc)^2}{BC}$ and c_1G is drawn perpendicular to BC and of indefinite length. The acceleration image (c) of C lies on c_1G. But the acceleration of C relative to

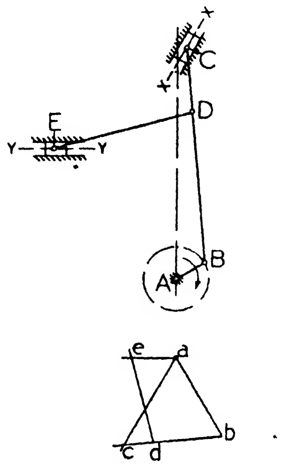


FIG. 129

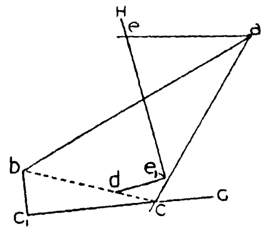


FIG. 130

the fixed frame, i.e. relative to A, is parallel to XX ; hence from a draw a line parallel to XX to intersect c_1G in c. Join bc ; this is now the acceleration image of BC and the image of D is found by taking d such that $\frac{bd}{bc} = \frac{BD}{BC}$. The normal component of the acceleration of E relative to D is next set out as $de_1 = \frac{{}_{E}v_{D}^2}{DE} = \frac{(de)^2}{DE}$ and e_1H is drawn perpendicular to ED and of indefinite length. Then e, the acceleration image of E, lies on e_1H. But the acceleration of E relative to A is along YY ; hence from a draw a line parallel to YY to intersect e_1H in e. Then ae is the acceleration of E relative to the fixed frame. Since, from the velocity diagram, E is moving from right to left, the velocity of E is, at the moment, increasing.

105. Special Method for when the above Method Fails.—When the special method described in Art. 97 has to be used to draw the velocity diagram it becomes necessary to use a special method for the acceleration diagram. Considering the mechanism shown in

Fig. 131, the following accelerations can be calculated when the velocity diagram (shown at the bottom of the figure) has been drawn, ${}_{A}a_{O}$, ${}_{B}a_{O}$, ${}^{n}_{C}a_{B}$, ${}^{n}_{D}a_{A}$ and ${}^{n}_{E}a_{F}$. These may be set out as in Fig. 132, but to proceed any further the following method must

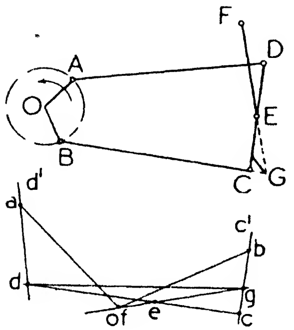


Fig. 131

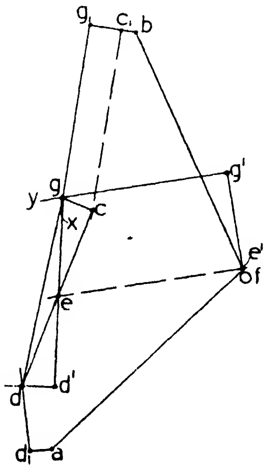


Fig. 132

be adopted. Produce BC and FE to intersect at G and regard G as a point attached to DEC. Now

$$\begin{aligned}{}_{G}a_{O} &= {}_{G}a_{C} + {}_{C}a_{B} + {}_{B}a_{O}\\ &= {}^{n}_{G}a_{C} + {}^{t}_{G}a_{C} + {}^{n}_{C}a_{B} + {}^{t}_{C}a_{B} + {}_{B}a_{O}\\ &= {}^{n}_{G}a_{C} + {}^{n}_{C}a_{B} + {}_{B}a_{O} + {}^{t}_{G}a_{C} + {}^{t}_{C}a_{B}\end{aligned}$$

and of these components the first three are known in magnitude and direction $\left({}^{n}_{G}a_{C} = \frac{{}_{G}v_{C}^{2}}{GC}, \text{ etc.}\right)$, and, because G has been chosen to lie on BC produced, the directions of the last two components coincide. Hence from b set out $bc_1 = \frac{{}_{B}v_{C}^{2}}{BC}$ parallel to BC and from c_1 set out $c_1g_1 = \frac{{}_{C}v_{G}^{2}}{CG}$ also parallel to BC. From g_1 draw g_1x perpendicular to BC. Then g, the image of G, lies on g_1x. But

$$\begin{aligned}{}_{G}a_{F} &= {}_{G}a_{E} + {}_{E}a_{F}\\ &= {}^{n}_{G}a_{E} + {}^{t}_{G}a_{E} + {}^{n}_{E}a_{F} + {}^{t}_{E}a_{F}\\ &= {}^{n}_{G}a_{E} + {}^{n}_{E}a_{F} + {}^{t}_{G}a_{E} + {}^{t}_{E}a_{F}\end{aligned}$$

and the first two components are known completely, while, again, because of the choice of the point G, the directions of the last two components coincide. Hence from f set out $fe' = \frac{{}_{E}v^2_{F}}{EF}$ parallel to EF and from e' set out $e'g' = \frac{{}_{G}v^2_{E}}{GE}$ also parallel to EF. Draw $g'y$ perpendicular to EF to intersect g_1x in g; then g is the acceleration image of G. The image of D may then be found, because

$${}_{D}a_{O} = {}_{G}a_{O} + {}^{n}_{D}a_{G} + {}^{t}_{D}a_{G}$$

and

$${}_{D}a_{O} = {}_{A}a_{O} + {}^{n}_{D}a_{A} + {}^{t}_{D}a_{A}$$

and in each case the first two components are known completely, while the directions of the last components are known. Thus from g set out $gd' = \frac{{}_{G}v^2_{D}}{GD}$ parallel to GD and through d' draw a line perpendicular to GD; d then lies on that line. Also, from a set out $ad_1 = \frac{{}_{D}v^2_{A}}{AD}$ parallel to AD and through d_1 draw a line perpendicular to AD; then d lies on this line and is established at the intersection. The remainder of the diagram is straightforward. Thus on gd a triangle similar to GCD is drawn and c is established and hence e. As checks on the accuracy of the drawing it should be noticed that c should lie on a perpendicular to BC drawn through c_1 and e should lie on a perpendicular to FE drawn through e'.

106. Rotating Slotted Links.—Let B, Fig. 133, be any point of a block that is sliding along the slot in the link DECF while that link itself is rotating about the fixed centre D. The point C of the link coincides with B at the moment under consideration. Let the velocity of B along the slot be u; this is the velocity of B relative to a frame of reference YCY fixed to the link and hence rotating with it. The acceleration of B relative to the link, i.e. relative to YCY, then consists of a normal component $\frac{u^2}{r}$ directed from B towards the centre of curvature O and of a tangential component $\dot{u}$ parallel to u. These components constitute the acceleration B would have if the link were fixed and the block moved along the slot, i.e. the acceleration of B relative to the link.

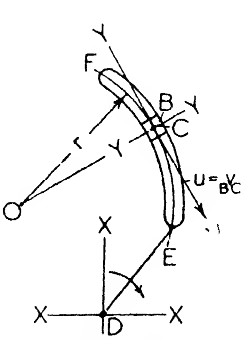


Fig. 133

Let the angular velocity and acceleration of the link DECF be Ω and α respectively. Then the acceleration of C relative to any fixed frame of reference XDX is composed of a normal component directed from C towards D and equal to CD . Ω^2 and of a tangential component equal to CD . a perpendicular to CD. These components constitute the acceleration B would have if it were fixed to the link while the link rotated. Then, as shown analytically in Art. 37,

$$\begin{matrix}\text{The acceleration of}\\ \text{B relative to D}\end{matrix} = \left\{\begin{array}{l}\text{The acceleration of B relative to the link}\\ \quad +\text{the acceleration of C relative to D}\\ \quad +2u\Omega\end{array}\right.$$

the term $2u\Omega$ being the compound supplementary acceleration due to the rotation of the frame YCY relative to the frame XDX. It is instructive to obtain this result by a non-analytical method, as follows.

107. Coriolis's Law.—Let EF and E_1F_1 (Fig. 134) be the positions of the centre-line of the slot of the link at the beginning

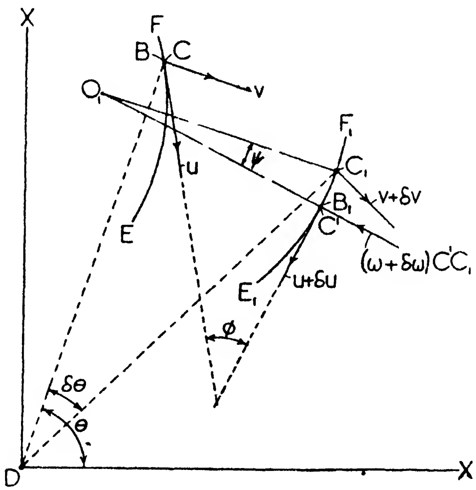


Fig. 134

and end, respectively, of a small interval of time δt. During this interval the link will have rotated through the angle $CDC_1 = \delta\theta$, while the point B will have moved down the slot a little and will be at B_1. In the first position the velocity of B relative to XDX is the vector sum of its velocity u relative to the link and the velocity v of the coincident point C of the link relative to XDX. In the second position the velocity of B_1 is the vector sum of its velocity relative to C_1 and the velocity $v+\delta v$ of C_1 relative to XDX. The velocity of B_1 relative to C_1 is equal to the vector sum of the velocity $u+\delta u$ of B_1 relative to the coincident point C′ and the velocity of C′ relative to C. In the first position the angular velocity of EF about D was ω, and in the second position it is $\omega+\delta\omega$. The velocity of C′ relative to C is thus $C_1C' \times (\omega+\delta\omega)$ perpendicular to C_1C', while $v=\omega$. CD and $v+\delta v=(\omega+\delta\omega) . C_1D =(\omega+\delta\omega) . CD$, so that $\delta v=CD . \delta\omega$. The changes in the velocity of the point B relative to XDX are thus :

1. An increase δu.
2. The velocity u has been turned through the angle ϕ.
3. An increase δv.
4. The velocity v has been turned through the angle $CDC_1=\delta\theta$.
5. The velocity $(\omega+\delta\omega) . C_1C'$ has been added.

The corresponding average accelerations are then :

1. $\dfrac{\delta u}{\delta t}$.
2. $\dfrac{u\phi}{\delta t}$ (see Art. 29)$=\dfrac{u . (\delta\theta+\psi)}{\delta t}=\dfrac{u\delta\theta}{\delta t}+u . \dfrac{C_1C'}{r .} . \dfrac{1}{\delta t}$ where r is the radius of the slot.
3. $\dfrac{\delta v}{\delta t}$.
4. $\dfrac{v . \delta\theta}{\delta t}$ (see Art. 29).
5. $\dfrac{\omega . C_1C'}{\delta t}+\dfrac{\delta\omega . C_1C'}{\delta t}$.

Let the interval δt be made indefinitely small, then the actual accelerations become :

1. $\dfrac{du}{dt}=\dot{u}$.
2. $u\dfrac{d\theta}{dt}+\dfrac{u^2}{r}=u . \omega+\dfrac{u^2}{r}$, since the limit of $\dfrac{C_1C'}{\delta t}$ when δt is indefinitely small is u.
3. $\dfrac{dv}{dt}=CD . \dfrac{dw}{dt}=CD . a$.
4. $v . \dfrac{d\theta}{dt}=v . \omega . =CD . \omega^2$.
5. $\omega . u+\dfrac{d\omega}{dt} . 0=u\omega$.

The directions of these are as follows :

1. Parallel to u.
2. Perpendicular to u and from B towards the centre of curvature of the slot.
3. Perpendicular to CD.
4. From C towards D.
5. From B towards the centre of curvature of the slot (because the change of velocity $(\omega+\delta\omega)C_1C'$ is perpendicular to C_1C' and in the limit this is perpendicular to u).

These accelerations may be grouped as follows :

(a) $\dot{u}$ and $\frac{u^2}{r}$.

(b) CD . ω^2 and CD . a.

(c). $2u\omega$.

Group (a) is then seen to comprise the acceleration of B relative to the link ; group (b) comprises the acceleration of the coincident point C of the link relative to XDX, while (c) is the compound supplementary acceleration.

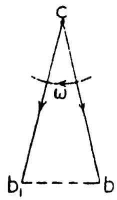


Fig. 135

The direction of the compound supplementary acceleration is perpendicular to that of the velocity u ; its sense may be determined by using the following rule.

Set out cb (Fig. 135) to represent the velocity u. Rotate this vector, in the same sense as that of the angular velocity ω, to the position cb_1. Then bb_1 is the sense of the compound supplementary acceleration.

108. Equivalent Mechanisms.—Since the two mechanisms shown in Fig. 136 are kinematically identical (see Art. 80), the

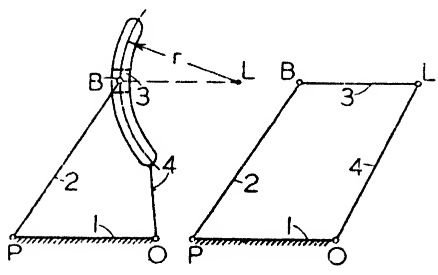


Fig. 136

block and slotted link may be replaced by two links and a mechanism be thus obtained for which the acceleration diagram presents no difficulty. When the radius r is reasonably small this is undoubtedly the best way to proceed, but when it is large, in particular when it is infinite and the slot is straight, the substitution is impracticable and the method of the following article must be used.

109. Example of Mechanism with Rotating Slotted Link.—Let the crank AB of the shaping machine mechanism shown in Fig. 137 be rotating at a constant angular speed Ω, and let it be required to draw the acceleration diagram for the given configuration. (The slot in the link CD has been placed at an angle to the line CD in order to clarify the acceleration diagram.)

The velocity diagram is first drawn. It is shown, on the left, in Fig. 138. To draw the acceleration diagram choose any point a (Fig. 138, right) and set out ab=AB . Ω^2 to represent the

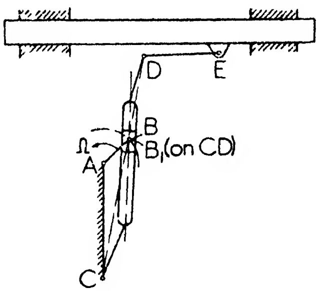


Fig. 137

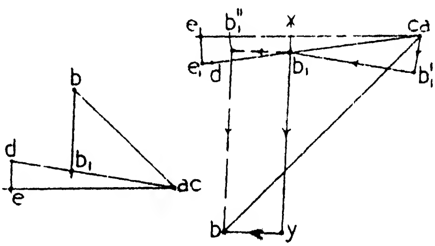


Fig. 138

acceleration of B relative to A. The point b, the acceleration image of B, may also be arrived at by considering the acceleration of B relative to C, which consists of the vector sum of the acceleration of B relative to B_1, the acceleration of B_1 relative to C and the compound supplementary acceleration; this is indicated in Fig. 138, where cb_1' and $b_1'b_1$ are respectively the normal and the tangential components of the acceleration of B_1 relative to C; b_1b_1'' is the compound supplementary acceleration (equal to $2\,{}_{B}v_{B_1}\omega$) and $b_1''b$ is the acceleration of B relative to B_1. The difficulty, however, is that the magnitude of $b_1'b_1$ is unknown, and so the position of b_1 is unknown. The difficulty may, however, be circumvented by setting out by equal in magnitude, but opposite in sense, to the compound supplementary acceleration and from y drawing a line yx (parallel to bb_1'', the acceleration of B relative to B_1, and hence parallel to the slot in the link CD) to intersect a line drawn through b_1' perpendicular to CB_1 in b_1.

Then $\boldsymbol{cb}_1$ is the acceleration image of CB_1 and the image of D may be obtained by extending $\boldsymbol{cb}_1$ to $\boldsymbol{d}$ such that $\frac{\boldsymbol{cb}_1}{\boldsymbol{cd}}=\frac{CB_1}{CD}$.

Next the normal acceleration of E relative to D may be set out as $\boldsymbol{de}_1=\frac{{}_E v_D}{DE}=\frac{(de)^2}{DE}$ parallel to ED and directed from E to D. Then $\boldsymbol{e}$, the acceleration image of E, must lie on a line drawn through $\boldsymbol{e}_1$ perpendicular to DE ; also $\boldsymbol{e}_1$ must lie on a line drawn through $\boldsymbol{a}$ parallel to the ram slide, and hence is determined

Then $\boldsymbol{ae}$ is the acceleration of E, which is thus speeding up. The angular acceleration of the link CD about C is given by $\frac{\text{Tangential acceleration of } B_1 \text{ relative to } C}{CB_1}=\frac{\boldsymbol{b}_1'\boldsymbol{b}_1}{CB_1}$; its sense is obtained from that of $\boldsymbol{b}_1'\boldsymbol{b}_1$, and hence is anticlockwise. The angular speed of CD is thus increasing.

It should be noted that the acceleration of B relative to B_1 is represented by $\boldsymbol{b}_1''\boldsymbol{b}$ or by $\boldsymbol{b}_1\boldsymbol{y}$, and *not* by $\boldsymbol{b}_1\boldsymbol{b}$.

110. Accelerations in Cams.—When two bodies are in contact and one has a combined rolling and sliding motion relative to the other, a motion which commonly occurs in cams (see Chap. XIX), the acceleration of any point of the one body relative to the other body depends on the radii of curvature of the bodies at the point of contact, and it is not usually practicable to draw an acceleration diagram for a mechanism in which such contacts occur. The accelerations may be found either by plotting a space-time curve and differentiating it graphically twice or by the analytical method. The sliding-rolling contact can, however, be replaced at any instant by a link joining the two centres of curvature O_1, O_2 (Fig. 139), when, in the example shown, a simple four-bar chain CO_1O_2D is obtained, for which the acceleration diagram can easily be drawn.

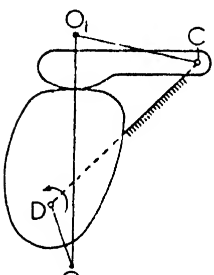


FIG. 139

EXERCISES VIII

1. Draw the acceleration diagram for the mechanism of Question 4, Exercises VII, and, assuming the crank speed to be constant, find the acceleration of the piston. Check your result by using Klein's construction and also by calculation, using the approximate expression for the piston acceleration.

2. Draw the acceleration diagram for the mechanism of Question 6, Exercises VII, and find the acceleration of the slider 5, and the angular acceleration of link 4.

3. Draw the acceleration diagram for the mechanism of Question 9, Exercises VII, and find the acceleration of the slider D.

4. Draw the acceleration diagram for the Peaucellier mechanism of Question 11, Exercises VII, and verify that the acceleration of E is perpendicular to the link AB.

5. Draw the acceleration diagram for the shaper mechanism of Question 8, Exercises VII, and find the acceleration of the ram, the acceleration of the slider 3 relative to link 4 and the angular acceleration of the latter.

6. Draw the acceleration diagram for the Whitworth quick-return motion of Question 16, Exercises VII, and find the acceleration of the ram D.

7. Draw the acceleration diagram for the mechanism of Question 17, Exercises VII, and find the acceleration of the door.

8. Draw the acceleration diagram for the mechanism of Question 1,2 Exercises VII, and find the acceleration of the ram D.

CHAPTER IX

THE DIRECT-ACTING ENGINE MECHANISM

111. The Piston Velocity.—In Fig. 140 the point O is clearly the instantaneous centre of the connecting-rod BC relative to the fixed frame; hence $\frac{{}_cv_a}{{}_bv_a}=\frac{\text{OC}}{\text{OB}}=\frac{\text{AP}}{\text{AB}}$, where P is the intersection of BC, produced, with a line drawn through A perpendicular to the line of stroke XX. This gives a simple graphical method of finding the velocity of the piston in terms of the velocity of the crank-pin. Thus,

$$\text{Piston velocity } {}_cv_a=\text{Crank-pin velocity } {}_bv_a\times\frac{\text{AP}}{\text{AB}}$$

and this is true whether the line of stroke XX passes through the centre A or not.

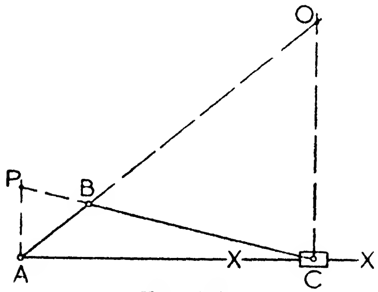


Fig. 140

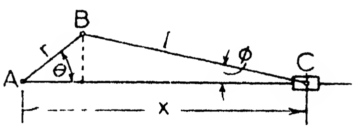


Fig. 141

112. An approximate expression for the velocity of the piston may be found analytically as follows. In Fig. 141, let AB$=r$, BC$=l$, $\angle$CAB$=\theta$, $\angle$BCA$=\phi$. Then the distance of the piston C from the centre A is given by

$$x=r\,\text{Cos}\,\theta+l\,\text{Cos}\,\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

but $$r\,\text{Sin}\,\theta=l\,\text{Sin}\,\phi$$

$$\therefore\ \text{Sin}\,\phi=\frac{r}{l}\,\text{Sin}\,\theta$$

$$\therefore\ \text{Cos}\,\phi=\sqrt{1-\left(\frac{r}{l}\,\text{Sin}\,\theta\right)^2}=\left[1-\left(\frac{r}{l}\,\text{Sin}\,\theta\right)^2\right]^{\frac{1}{2}} \quad . \quad (2)$$

and on expanding this by means of the binomial theorem we obtain

$$\operatorname{Cos}\phi=1-\tfrac{1}{2}\left(\frac{r}{l}\operatorname{Sin}\theta\right)^2-\tfrac{1}{8}\left(\frac{r}{l}\operatorname{Sin}\theta\right)^4-\tfrac{1}{16}\left(\frac{r}{l}\operatorname{Sin}\theta\right)^6-\text{etc.}\quad . \quad (3)$$

When r is small compared with l the third and subsequent terms of this expression for Cos ϕ may be neglected without introducing any appreciable inaccuracy. Thus, $\operatorname{Cos}\phi=1-\tfrac{1}{2}\left(\frac{r}{l}\operatorname{Sin}\theta\right)^2$ approx.; hence

$$x=r\operatorname{Cos}\theta+l\left[1-\frac{r^2}{2l^2}\cdot\operatorname{Sin}^2\theta\right]\text{ approx.}\quad . \quad . \quad . \quad (4)$$

and on differentiating this with respect to time we shall obtain an expression for $\frac{dx}{dt}=\dot{x}$, the velocity of the piston. Thus,

$$\frac{dx}{dt}=\frac{dx}{d\theta}\cdot\frac{d\theta}{dt}=\left[-r\operatorname{Sin}\theta-\frac{r^2}{l}\operatorname{Sin}\theta\operatorname{Cos}\theta\right]\frac{d\theta}{dt}$$

$$=-r\omega\left[\operatorname{Sin}\theta+\frac{r}{2l}\operatorname{Sin}2\theta\right]\text{ approx.}\quad . \quad (5)$$

since $\frac{d\theta}{dt}=\omega$, the angular velocity of the crank AB. The minus sign appears because the velocity of the piston in the position of the mechanism shown in the figure is to the left, towards A, while the displacement x is to the right, from A.

113. By substituting the exact expression, Eq. (2), for Cos ϕ in the expression for the displacement x, and then differentiating with respect to time, an exact expression for the piston velocity can be obtained. Thus,

$$x=r\operatorname{Cos}\theta+l\left[1-\left(\frac{r}{l}\operatorname{Sin}\theta\right)^2\right]^{\frac{1}{2}}$$

$$\therefore\ \frac{dx}{dt}=\frac{dx}{d\theta}\cdot\frac{d\theta}{dt}$$

$$=\left[-r\operatorname{Sin}\theta+\frac{l}{2}\left\{1-\left(\frac{r}{l}\operatorname{Sin}\theta\right)^2\right\}^{-\frac{1}{2}}\times\left\{-2\frac{r}{l}\operatorname{Sin}\theta\times\frac{r}{l}\operatorname{Cos}\theta\right\}\right]\frac{d\theta}{dt}$$

$$=-r\omega\left[\operatorname{Sin}\theta-\frac{r\operatorname{Sin}2\theta}{2l\sqrt{1-\left(\frac{r}{l}\operatorname{Sin}\theta\right)^2}}\right]\quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When the ratio $\frac{r}{l}$ is less than about $\frac{1}{3}$, however, the error involved in the approximate expression is quite small, and for most practical purposes that expression is used.

114. The piston velocity is plotted on a basis of piston displacement for various values of the ratio $\frac{r}{l}$, in Fig. 142, while Fig. 143 gives similar curves, but on a crank-angle basis.

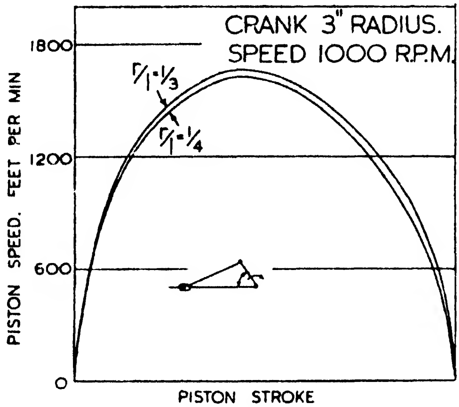


FIG. 142

A consideration of Fig. 140 will show that the piston velocity, which is proportional to AP, is a maximum approximately when the angle ABC between the crank and the connecting-rod is a right angle and is equal to the crank-pin velocity when the crank angle CAB is a right angle.

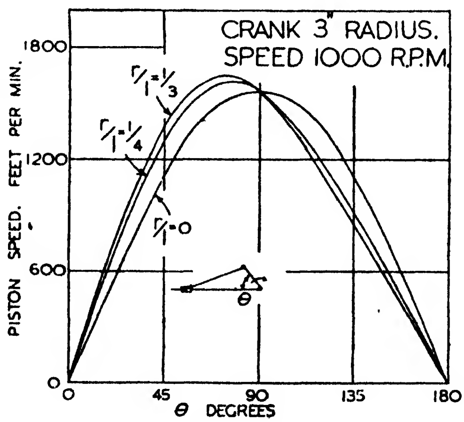


FIG. 143

115. The Piston Acceleration.—There are several graphical methods of determining the piston acceleration, the best known being probably *Klein's construction*, which is shown in Fig. 144, and is as follows. Produce CB to intersect a line through A perpendicular to the line of stroke XX in the point P, and with B as centre and BP as radius describe a circle. On BC as diameter

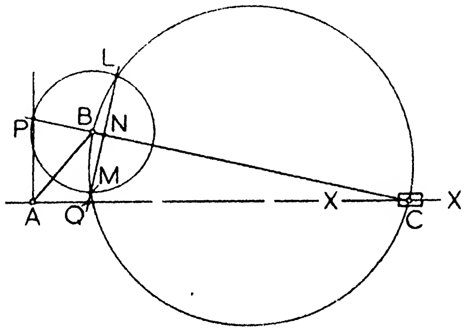


FIG. 144

describe a circle to intersect the first circle in L and M. Join LM and produce it, if necessary, to intersect a line drawn through A parallel to the line of stroke XX, in the point Q. Then AQ is proportional to the acceleration of the piston at the given instant. This may be proved as follows,

$$a_a = {}_b a_a + {}^n_c a_b + {}^t_c a_b$$

Now in the quadrilateral ABQN, AB is parallel to ${}_b a_a$, BN to ${}^n_c a_b$, NQ to ${}^t_c a_b$ and AQ to ${}_c a_a$; hence if it can be shown that $\frac{AB}{{}_b a_a} = \frac{BN}{{}^n_c a_b}$ it follows that ABNQ is the acceleration diagram, but with the vectors all reversed in direction, and AQ is proportional to ${}_c a_a$.

Now $${}^n_c a_b = \frac{{}_c v_b{}^2}{CB}$$

$$\therefore \frac{BN}{{}^n_c a_b} = \frac{BN \, . \, BC}{{}_c v_b{}^2}$$

but $$BN \, . \, BC = BN(BN + NC)$$
$$= BN^2 + BN \, . \, NC$$

also $$BN \, . \, NC = NL^2$$
$$= BL^2 - BN^2$$
$$= BP^2 - BN^2$$

hence $$BN \, . \, BC = BN^2 + BP^2 - BN^2$$
$$= BP^2$$

$$\therefore \frac{BN}{{}_c^n a_b} = \frac{BP^2}{{}_c v_b{}^2}$$

but $\dfrac{{}_c v_b}{{}_b v_a} = \dfrac{BP}{AB}$ since $\triangle APB$ is similar to the velocity diagram for the mechanism

$$\therefore \frac{BP^2}{{}_c v_b{}^2} = \frac{AB^2}{{}_b v_a{}^2}$$

$$\therefore \frac{BN}{{}_c^n a_b} = \frac{AB^2}{{}_b v_a{}^2} = \frac{AB}{{}_b v_a{}^2/AB} = \frac{AB}{{}_b a_a}$$

hence ABNQ is the acceleration diagram for the mechanism and AQ is proportional to ${}_c a_a$. Thus,

$$\frac{AB}{{}_b a_a} = \frac{AQ}{{}_c a_a}$$

$${}_c a_a = AQ \times \frac{{}_b a_a}{AB}$$

and if the acceleration scale is chosen so that AB is equal to the acceleration ${}_b a_a$, then AQ is equal to the acceleration of the piston; the acceleration scale is thus 1″ to $\dfrac{{}_b a_a}{AB}$ ft./sec.^{2}.

116. Bennet's Construction.—Another construction is Bennet's, which is as follows. Set out the mechanism with the crank at right-angles to the line of stroke, as shown at AB_1C_1 in Fig. 145,

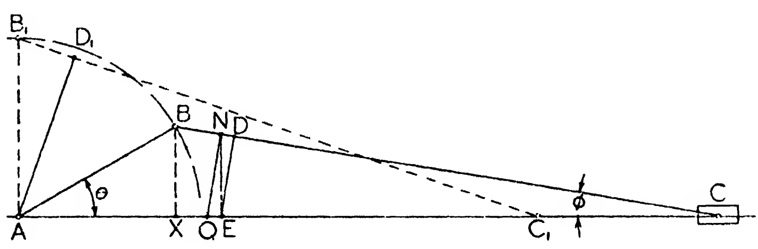


FIG. 145

and draw AD_1 perpendicular to B_1C_1 to determine the point D_1 on the connecting-rod. With the mechanism in the position for which the piston acceleration is required draw DE perpendicular to BC, draw EN perpendicular to AC and draw NQ perpendicular to BC. Then AQ is proportional to the acceleration of the piston, ABNQ again being the acceleration diagram reversed. Again, it is necessary only to prove that

$$\frac{BN}{AB}=\frac{\text{Normal accn. of C rel. B}}{\text{Normal accn. of B rel. A}}$$

i.e.

$$\frac{BN}{AB}=\frac{BC\dot{\phi}^2}{AB\dot{\theta}^2}$$

i.e.

$$BN\dot{\theta}^2=BC\dot{\phi}^2$$

Now the preliminary construction gives

$$BD\,.\,DC=AD^2$$

$$\therefore\ BD(BC-BD)=AB^2=BD^2$$

$$\therefore\ BD\,.\,BC=AB^2$$

$$\therefore\ BD=\frac{AB^2}{BC}$$

In any position of the mechanism

$${}_bv_a\ \text{Cos}\ \theta={}_cv_b\ \text{Cos}\ \phi$$

$$\therefore\ AB\dot{\theta}\ \text{Cos}\ \theta=BC\dot{\phi}\ \text{Cos}\ \phi$$

$$\therefore\ AX\dot{\theta}=CX\dot{\phi}$$

$$\therefore\ \dot{\phi}=\frac{AX}{CX}\,.\,\dot{\theta}$$

$$\therefore\ BC\dot{\phi}^2=BC\,.\,\dot{\theta}^2\,.\,\frac{AX^2}{CX^2}$$

$$=BC\,.\,\dot{\theta}^2\frac{(AB^2-BX^2)}{CX^2}$$

$$=BC\,.\,\dot{\theta}^2\frac{(AB^2-BC^2+CX^2)}{CX^2}$$

$$=\dot{\theta}^2\left[BC-\frac{(BC^2-AB^2)BC}{CX^2}\right]$$

$$=\dot{\theta}^2\left[BC-\frac{BC^2}{CX^2}\left(BC-\frac{AB^2}{BC}\right)\right]$$

$$=\dot{\theta}^2\left[BC-\frac{BC^2\,.\,CD}{CX^2}\right]$$

$$=\dot{\theta}^2\left[BC-\frac{EC}{DC}\,.\,\frac{NC}{EC}\,.\,CD\right]$$

$$=\dot{\theta}^2[BC-NC]$$

$$=\dot{\theta}^2\,.\,BN$$

117. Ritterhaus's Construction.—Yet another construction is that due to Ritterhaus, which is as follows. Produce CB, Fig. 146, to intersect a perpendicular to the line of stroke through

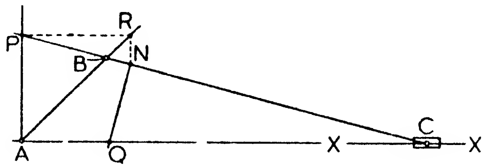


FIG. 146

A in P. Draw PR parallel to the line of stroke to intersect the crank AB, produced, in R. Draw RN parallel to AP to meet the connecting-rod in N. Draw NQ perpendicular to BC to intersect a line through A parallel to the line of stroke XX in Q. Then AQ is proportional to the acceleration of the piston, ABNQ again being the acceleration diagram reversed. The scale for the acceleration is thus the same as in the previous constructions. The proof is simple ; as in the previous constructions it is merely necessary to show that

$$\frac{{}_b a_a}{\text{AB}} = \frac{{}^n_c a_b}{\text{BN}}$$

Now $$ {}_b a_a = \frac{{}_b v_a{}^2}{\text{AB}}$$

$$\therefore \frac{{}_b a_a}{\text{AB}} = \frac{{}_b v_a{}^2}{\text{AB}^2} = \frac{{}_b v_a{}^2}{\text{BP}^2}, \quad \text{since } \frac{{}_b v_a}{\text{AB}} = \frac{{}_b v_c}{\text{BP}}$$

But triangles APB and BRN are similar

$$\therefore \frac{\text{BN}}{\text{BP}} = \frac{\text{BR}}{\text{AB}}$$

also triangles ABC and BPR are similar

$$\therefore \frac{\text{BR}}{\text{AB}} = \frac{\text{BP}}{\text{BC}}$$

$$\therefore \frac{\text{BN}}{\text{BP}} = \frac{\text{BP}}{\text{BC}}$$

$$\text{BP}^2 = \text{BN} \cdot \text{BC}$$

$$\therefore \frac{{}_b a_a}{\text{AB}} = \frac{{}_b v_c{}^2}{\text{BP}^2} = \frac{{}_c v_b{}^2}{\text{BN} \cdot \text{BC}} = \frac{{}^n_c a_b}{\text{BN}}$$

118. An approximate expression for the acceleration of the piston may be obtained by differentiating the approximate expression, Eq. (5), with respect to time, thus:

$$\frac{dx}{dt} = -r\omega\left[\text{Sin }\theta + \frac{r}{2l}\text{ Sin }2\theta\right] \text{ approx.}$$

$$\frac{d^2x}{dt^2} = \frac{d\left(\frac{dx}{dt}\right)}{d\theta} \cdot \frac{d\theta}{dt}$$

and, assuming $\omega \left(= \frac{d\theta}{dt}\right)$ to be constant, we have

$$\frac{d^2x}{dt^2} = -r\omega\left[\text{Cos }\theta + \frac{2r}{2l}\text{ Cos }2\theta\right]\frac{d\theta}{dt}$$

$$= -r\omega^2\left[\text{Cos }\theta + \frac{r}{l}\text{ Cos }2\theta\right] \quad . \quad . \quad . \quad . \quad . \quad (7)$$

As with the approximate expression for the velocity, this expression is sufficiently accurate for all practical purposes for

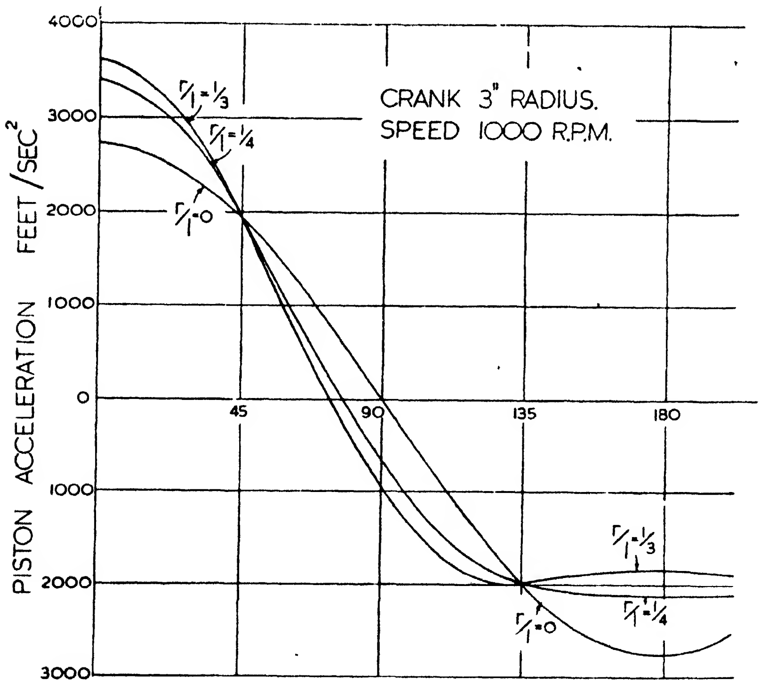


FIG. 147

ratios $\frac{r}{l}$ less than $\frac{1}{3}$; thus when $\frac{r}{l}=\frac{1}{3}$ and $\theta=45^\circ$ the error is only 1·4 per cent., and when $\theta=0$ or 180 the error is nil for all values of $\frac{r}{l}$. Fig. 147 shows the piston acceleration, as given by the approximate expression, Eq. (7), for various values of the ratio $\frac{r}{l}$.

An exact expression for the piston acceleration is obtained by differentiating the exact expression, Eq. (6), with respect to time, thus :

$$\frac{dx}{dt}=-r\omega\left[\operatorname{Sin}\theta-\frac{r\operatorname{Sin}2\theta}{2l\sqrt{1-\left(\frac{r}{l}\operatorname{Sin}\theta\right)^2}}\right]$$

and again assuming ω to be constant, we obtain

$$\frac{d^2x}{dt^2}=-r\omega^2\left[\operatorname{Cos}\theta+\frac{\frac{r}{l}\operatorname{Cos}2\theta+\left(\frac{r}{l}\right)^3\operatorname{Sin}^4\theta}{\left\{1-\left(\frac{r}{l}\right)^2\operatorname{Sin}^2\theta\right\}^{3/2}}\right]\quad . \quad . \quad . \quad (8)$$

119. The Piston Motion as the Sum of Two S.H.M.s.—It is easily seen that the motion of the piston in the slider-crank chain is, to a close approximation, composed of two simple harmonic motions and could be produced by means of a combination of two double-slider crank chains, as shown in Fig. 148. The crank AB of

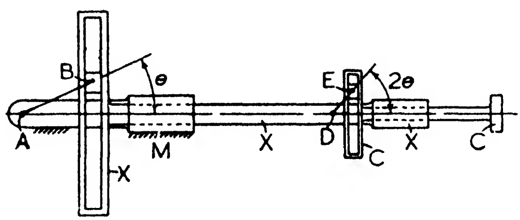


FIG. 148

length r rotates about the fixed centre A with an angular velocity ω, thus giving simple harmonic motion to the slider XXX. The latter has mounted on it a second crank DE which rotates about the centre D with an angular velocity 2ω. The crank DE is of a length $\frac{r^2}{4l}$ and gives simple harmonic motion to the slider CC relative to its " frame " XXX. Let the cranks both start from the same inner dead-centre position at the same moment ; then when the crank AB has turned through an angle θ the crank DE will have turned through an angle 2θ. The acceleration of the slider XXX relative to the fixed frame AM is ${}_x a_a = -r\omega^2 \text{ Cos } \theta$, while that of the slider CC' relative to its frame XXX is

$$ {}_c a_x = -\left(\frac{r^2}{4l}\right)(2\omega)^2 \text{ Cos } 2\theta = -\frac{r^2\omega^2}{l} \text{ Cos } 2\theta $$

Then the acceleration of the slider CC' relative to the fixed frame is given by ${}_c a_a = {}_c a_x + {}_x a_a$, and, since the accelerations are all in the same direction, the vectorial addition becomes merely algebraic addition, so that

$$ {}_c a_a = -r\omega^2 \text{ Cos } \theta + \left(-\frac{r^2\omega^2}{l} \text{ Cos } 2\theta\right) $$

$$ = -r\omega^2\left[\text{Cos } \theta + \frac{r}{l} \text{ Cos } 2\theta\right] $$

which is, approximately, the acceleration of the piston of a slider-crank chain having a crank of length r rotating at an angular speed ω and a connecting-rod of length l.

That component of the piston acceleration which is due to the crank AB is called the *primary* component, while that due to the crank DE is called the *secondary* component. The cranks themselves may be referred to as the primary and secondary cranks respectively. It should be noted that the secondary crank rotates at twice the speed of the primary crank and that both cranks start from the inner dead-centre position at the same moment.

The primary and secondary components and their resultant are shown in Fig. 149, in which the resultant is the alegebraic sum of the two components.

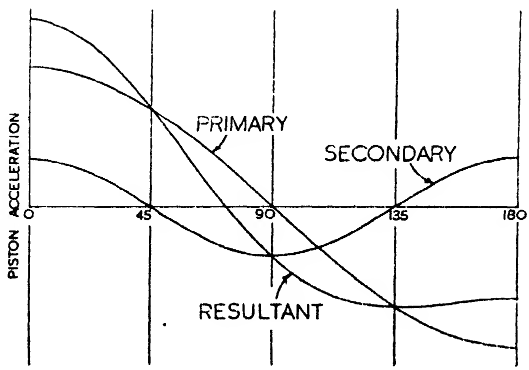


Fig. 149

120. The Higher Harmonics in the Piston Motion.—The piston acceleration can be represented by a series of the form

$$\ddot{x} = -r\omega^2[\text{A Cos } \theta + \text{B Cos } 2\theta + \text{C Cos } 4\theta + \text{D Cos } 6\theta +, \text{ etc.}]$$

and the values of the constants AB, etc., may be found as follows.

Since $$x = r \text{ Cos } \theta + l \text{ Cos } \phi$$

and $$\text{Cos } \phi = \left[1 - \frac{r^2}{l^2} \text{ Sin } ^2\theta\right]^{\frac{1}{2}}$$

we have, on expanding Cos ϕ by means of the binomial theorem,

$$x = r \text{ Cos } \theta + l\ [1 - \tfrac{1}{2}m^2 \text{ Sin}^2 \theta - \tfrac{1}{8}m^4 \text{ Sin}^4 \theta - \tfrac{1}{16}m^6 \text{ Sin}^6 \theta -, \text{ etc.}]$$

where $m = \frac{r}{l}$, and, using the relations

$$\text{Sin}^2 \theta = \tfrac{1}{2} - \tfrac{1}{2} \text{ Cos } 2\theta$$
$$\text{Sin}^4 \theta = \tfrac{3}{8} - \tfrac{1}{2} \text{ Cos } 2\theta + \tfrac{1}{8} \text{ Cos } 4\theta$$
$$\text{Sin}^6 \theta = \tfrac{5}{16} - \tfrac{15}{32} \text{ Cos } 2\theta + \tfrac{3}{16} \text{ Cos } 4\theta - \tfrac{1}{32} \text{ Cos } 6\theta, \text{ etc.,}$$

we get

$$\begin{aligned} x - r \text{ Cos } \theta + l[1 - \tfrac{1}{4}m^2 - \tfrac{3}{64}m^4 - \tfrac{5}{256}m^6 -, \text{ etc.}] \\ + l \text{ Cos } 2\theta[\tfrac{1}{4}m^2 + \tfrac{1}{16}m^4 + \tfrac{15}{512}m^6 +, \text{ etc.}] \\ + l \text{ Cos } 4\theta[-\tfrac{1}{64}m^4 - \tfrac{3}{256}m^6 -, \text{ etc.}] \\ + l \text{ Cos } 6\theta[\tfrac{1}{512}m^6, \text{ etc.}] \end{aligned} \qquad (9)$$

and on rearranging

$$x = \text{K} + \text{A Cos } \theta + \text{B Cos } 2\theta + \text{C Cos } 4\theta + \text{D Cos } 6\theta + \text{ etc.} \qquad (10)$$

where
$$\text{K} = r\left[\frac{1}{m} - \tfrac{1}{4}m - \tfrac{3}{64}m^3 - \tfrac{5}{256}m^5 -, \text{ etc.}\right]$$
$$\text{A} = r$$
$$\text{B} = r[\tfrac{1}{4}m + \tfrac{1}{16}m^3 + \tfrac{5}{512}m^5 +, \text{ etc.}]$$
$$\text{C} = -r[\tfrac{1}{64}m^3 + \tfrac{3}{256}m^5 +, \text{ etc.}]$$
$$\text{D} = r[\tfrac{1}{512}m^6 +, \text{ etc.}]$$

Hence

$$\dot{x} = \omega[-\text{A Sin } \theta - 2\text{B Sin } 2\theta - 4\text{C Sin } 4\theta - 6\text{D Sin } 6\theta -, \text{ etc.}] \qquad (11)$$

and

$$\ddot{x} = -\omega^2[\text{A Cos } \theta + 4\text{B Cos } 2\theta + 16\text{C Cos } 4\theta + 36\text{D Cos } 6\theta +, \text{etc.}] \qquad (12)$$
$$= -r\omega^2[\text{Cos } \theta + \text{B}_1 \text{ Cos } 2\theta + \text{C}_1 \text{ Cos } 4\theta + \text{D}_1 \text{ Cos } 6\theta +, \text{ etc.}]$$

where
$$\text{B}_1 = m + \tfrac{1}{4}m^3 + \tfrac{5}{128}m^5 +, \text{ etc.}$$
$$\text{C}_1 = -[\tfrac{1}{4}m^3 + \tfrac{3}{16}m^5 +, \text{ etc.}]$$
$$\text{D}_1 = \tfrac{1}{32}m^6 +, \text{ etc.}$$

Numerical values of B_1, C_1 and D_1 for various values of m are given in the table below :

$\frac{l}{r}=\frac{1}{m}$.	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
B_1 . .	0·5325	0·4164	0·3428	0·2916	0·2540	0·2250	0·2020
C_1 (−)	0·0371	0·0178	0·0100	0·0062	0·0041	0·0028	0·0021
D_1 . .	0·0005	0·0001	0·0000	0·0000	0·0000	0·0000	0·0000

It will be seen that even for comparatively high values of the ratio $\frac{r}{l}$ the fourth and higher harmonics are negligibly small, and in practice they are almost invariably neglected. Occasionally, however, when resonance occurs, the higher harmonics, though very small in magnitude, may produce appreciable vibrations and may therefore have to be taken into consideration.

121. Offset Cylinders.—Internal-combustion engines are often arranged with their cylinders offset in relation to the crankshaft axis, i.e. the cylinder axis does not pass through the crankshaft axis. Usually the amount of offset, the dimension e in Fig. 150,

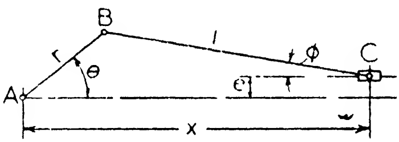


FIG. 150

is small compared with the crank radius, being of the order of one-quarter of that radius. The construction given in Art. 111 for the piston velocity, and those of Arts. 115, 116 and 117 for the piston acceleration, are true for offset cylinders. Analytical expressions for the piston velocity and acceleration may easily be obtained ; thus, referring to Fig. 150, we have

$$x=r\ \mathrm{Cos}\ \theta+l\ \mathrm{Cos}\ \phi$$

$$r\ \mathrm{Sin}\ \theta=l\ \mathrm{Sin}\ \phi+e$$

$$\therefore\ \mathrm{Sin}\ \phi=\frac{r\ \mathrm{Sin}\ \theta-e}{l}$$

$$\therefore\ \mathrm{Cos}\ \phi=\left\{1-\left(\frac{r\ \mathrm{Sin}\ \theta-e}{l}\right)^2\right\}^{\frac{1}{2}}$$

$$=1-\frac{(r\ \mathrm{Sin}\ \theta-e)^2}{2l^2}\ \text{approx.}$$

$$\therefore\ x=r\ \mathrm{Cos}\ \theta+l-\frac{(r\ \mathrm{Sin}\ \theta-e)^2}{2l}\quad .\ .\ .\ .\ .\ .\quad (13)$$

$$\therefore \frac{dx}{dt} = \left[-r \operatorname{Sin} \theta - \frac{(r \operatorname{Sin} \theta - e)(r \operatorname{Cos} \theta)}{l} \right] \frac{d\theta}{dt}$$

$$= -r\omega \left[\operatorname{Sin} \theta + \frac{r \operatorname{Sin} 2\theta}{2l} - \frac{e}{l} \operatorname{Cos} \theta \right] \quad . \quad . \quad . \quad (14)$$

$$\therefore \frac{d^2x}{dt^2} = -r\omega^2 \left| \operatorname{Cos} \theta + \frac{r}{l} \operatorname{Cos} 2\theta + \frac{e}{l} \operatorname{Sin} \theta \right| \quad . \quad . \quad . \quad (15)$$

and it is seen that the effect of the offset is to modify the primary component of the acceleration by the addition of the term $\frac{e}{l}$ Sin θ. The curves given in Fig. 151 are the piston velocity, for various

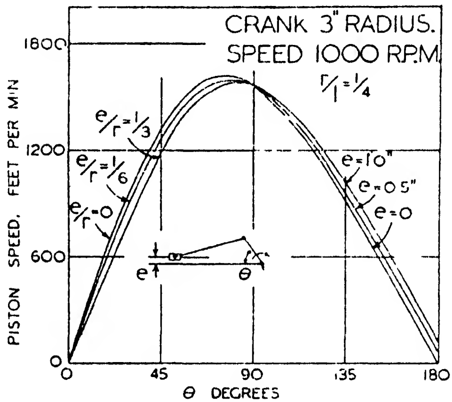


FIG. 151

amounts of offset, plotted on a crank-angle base, and they show that the magnitude of the piston velocity is not much affected by offsetting the cylinder. The corresponding curves for the piston acceleration are so close together as to be indistinguishable.

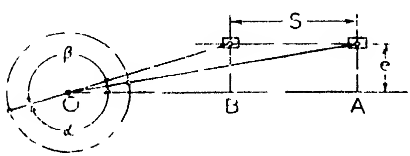


FIG. 152

When the cylinder is offset the piston stroke is no longer equal to twice the crank radius, and the inner and outer dead-centres no longer occur when the crank-angle θ is respectively 0° and 180°. Referring to Fig. 152, the inner dead-centre is obtained by striking an arc with radius $r+l$ from the crankshaft axis to intersect the line of stroke, similarly for the outer dead-centre, except that the radius of the arc is $l-r$. The piston stroke is given by

$$S = OA - OB = \sqrt{(r+l)^2 - e^2} - \sqrt{(r-l)^2 - e^2}$$

The angle turned through by the crank while the piston performs its outward stroke is α, and for the inward stroke β; hence, assuming the crank to rotate at a constant angular speed, we have

$$\frac{\text{Time taken for outward stroke}}{\text{Time taken for inward stroke}} = \frac{\alpha}{\beta}$$

The mechanism has thus been used as a quick-return motion for machine tools, the cutting tool being held in a tool-head which corresponds to the piston; the mechanism is not a good quick-return motion, however, and is no longer used.

122. The Inversions of the Slider-Crank Chain.—Four mechanisms are obtainable from the slider-crank chain by fixing each of its links in turn; the most important of these has been considered above, and the remaining ones will now be considered. By fixing the link 2, Fig. 153, a mechanism is obtained which is used for

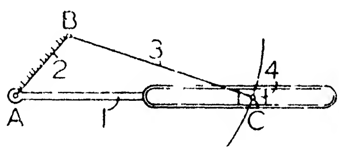


FIG. 153

two widely differing purposes, namely, as a *rotary engine* and as a quick-return motion for machine tools. In the rotary engine the link 1, i.e. the cylinders, rotates about the centre A with approximately constant speed, while links 3 and 4, i.e. the connecting-rod and piston, rotate about B with a variable angular speed. The motion of the piston relative to the cylinder is, of course, unchanged. Such engines were at one time extensively used in aircraft, and in order to secure a regular firing sequence and proper balance of the engine an odd number of cylinders was used, usually seven or nine, all in one plane and all working on one crank-pin. Only one connecting-rod, the "master-rod," M, had its big-end bearing actually on the crank-pin B, the other rods N (only one of which is shown) having their big-ends pivoted on pins C carried by the big-end of the master-rod as shown in Fig. 154. The advantages of the rotary engine over other types no longer obtain, and such engines are now practically obsolete. The *radial engine*,

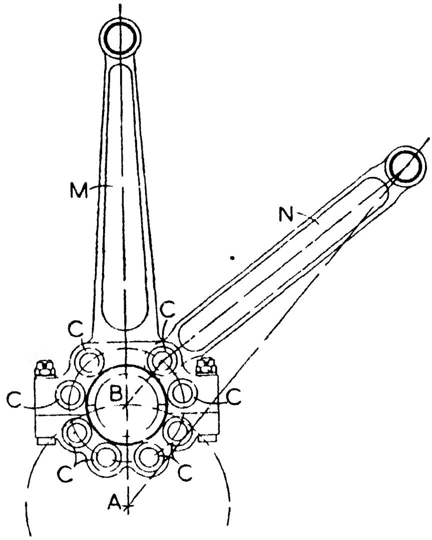


Fig 154

which is of similar construction, but in which the cylinders are fixed, is still widely used.

123. The Whitworth Quick-Return Motion.—Referring to Fig. 155, the link 3 is driven at a constant angular velocity and imparts a variable motion to the link 1. The latter is extended to D and is coupled by a link DE to the tool-head of the machine. Clearly, when the line of stroke of the tool-head passes through the centre A, as shown, the extreme positions of the tool-head

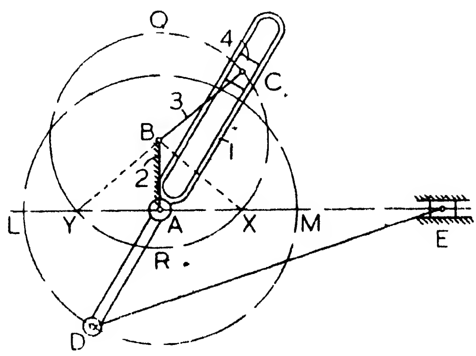


Fig. 155

occur when D is at L and M, that is when the pin C is at X and Y. Hence we have

$$\frac{\text{Time taken for cutting stroke}}{\text{Time taken for return stroke}}=\frac{\text{Arc } XQY}{\text{Arc } YRX}$$

If the proportions of the links are changed, another quick-return motion is obtained. It is used extensively in shaping machines and is shown in Fig. 156. Clearly, we have

$$\frac{\text{Time for cutting stroke}}{\text{Time for return stroke}}=\frac{\text{Arc } XQY}{\text{Arc } YRX}$$

The difference is that link 3 is now shorter than link 2, whereas in the Whitworth mechanism it was longer. Expressions for the angular velocity and acceleration of the slotted link are given in Question 7, Exercises IX.

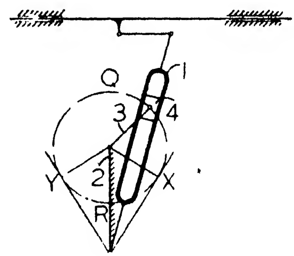


FIG. 156

This inversion of the slider-crank chain also gives the "Geneva stop" mechanism described in Art. 273, and numerous blowers for pumping air and liquids. One of the latter is shown in Fig. 157. It consists of a cylindrical casing A inside which is placed a rotor B provided with a number of slots to accommodate vanes C. The rotor is eccentric to the casing A, as shown, and the vanes are guided by blocks D sliding in slots, concentric with A, formed in the end covers. Inlet and outlet ports are formed as indicated at I and O, and the pumping action is due to the variation in the volume

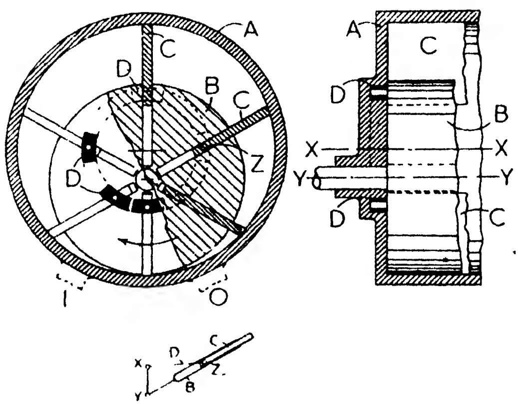
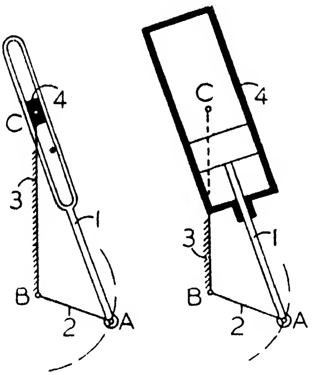


FIG. 157

of the spaces enclosed between the casing, the rotor and a pair of adjacent vanes. Consideration will show that this is identical, kinematically, with the Whitworth motion.

For a comprehensive historical account of the use of this and other mechanisms as rotary engines, blowers and pumps, the reader should consult a series of articles published in 1939 in *The Engineer*.

The mechanism resulting from fixing link 3 has been used as an engine having an oscillating cylinder and is sometimes known as the oscillating cylinder engine mechanism. It is now used as an engine only in toys and a few special applications, but it is still employed for minor purposes; for example, as a pump to circulate the lubricating oil in internal-combustion engines. As will be seen from Fig. 158, the links 1 and 4 have their forms inverted, the male member, block 4, of (*a*) becoming the female member, cylinder 4, of (*b*). The mechanism is used for door-stops, link 2 being the door, by filling the cylinder with fluid and arranging a small passage from one side of the piston to the other. It is also used as a quick throw-over mechanism as shown in Fig. 159. As



(*a*) (*b*)

FIG. 158

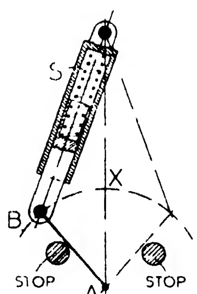


FIG. 159

the lever AB is moved towards the dead-centre position AX the spring S is compressed, and as soon as the lever gets over the dead-centre position the spring moves it rapidly to the extreme limit of its travel.

On fixing the remaining link, 4, the pendulum-pump mechanism (Fig. 160) is obtained, the name arising from the motion of the link 3, which now swings to and fro about C as centre. Again the forms of links 1 and 4 are inverted, while link 1 is extended to

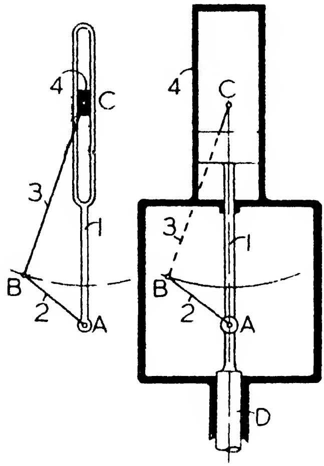


FIG. 160

form the pump plunger D. The mechanism is no longer used as an engine mechanism, but finds occasional use in other forms.

124. Adjustable-Throw Cranks.—In some applications of the slider-crank chain and its inversions it is desirable to be able to vary the throw of the crank and thus the stroke of the slider. Two methods of doing this are shown in Figs. 161 and 162. In

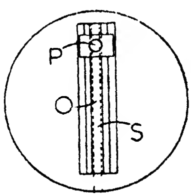


FIG. 161

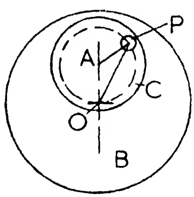


FIG. 162

the former the crank-pin P is part of a block that can be traversed along a slot formed in the face of a disc, by means of a screw S carried by the disc and engaging a nut fixed to the block. The effective length of the crank OP can thus be varied, and it will be noticed that its angular position relative to the crankshaft is not altered when this is done. In the second method the crank-pin P is made part of a disc or cylinder C which is eccentrically mounted in the disc B. By rotating C relative to B the effective length of the crank OP can be varied between the limits $OA \pm AP$.

In this method the angular position of the crank relative to the crankshaft changes as the throw of the crank is varied. With both methods it is not difficult to arrange for the alteration of throw to be made while the crank rotates.

EXERCISES IX

1. Prove that the piston velocity and acceleration in the single-slider-crank chain are given approximately by the expressions

$$\dot{x} = -r\omega\left[\operatorname{Sin}\theta + \frac{r}{2l}\operatorname{Sin}2\theta\right]$$

and

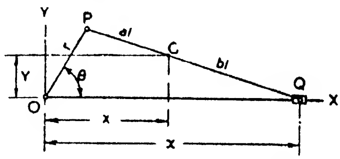
$$\ddot{x} = -r\omega^2\left[\operatorname{Cos}\theta + \frac{r}{l}\operatorname{Cos}2\theta\right]$$

respectively. r — length of crank, l — length of connecting-rod.

2. Using a graphical method of finding the piston velocity draw, on a stroke basis, a diagram showing the variation of the piston velocity in a slider-crank chain having a crank/con.-rod ratio of 1/3.

3. Repeat Question 2, taking the line of stroke to be offset by an amount equal to one-sixth of the crank radius.

4. Prove that the components, in the directions OX and OY (see figure), of



the velocity of the point G of the connecting-rod are

$$\dot{X} = a\dot{x} - r\omega b\operatorname{Sin}\theta$$

and

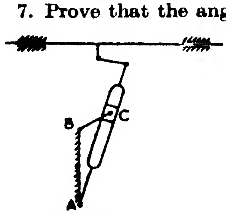
$$\dot{Y} = r\omega b\operatorname{Cos}\theta$$

respectively. l — length of connecting-rod $=$ PQ, and $\omega = \frac{d\theta}{dt}$.

5. Derive expressions, similar in form to those of Question 4, for the corresponding components of the acceleration of the point G.

6. In a Whitworth quick-return motion the fixed link is 4 in. long and is at right-angles to the line of stroke of the ram which passes through the centre of rotation of the slotted link. If the ratio $\frac{\text{Cutting time}}{\text{Return time}} = 2$, find the length of the driving crank and the value of the ratio $\frac{\text{Max. return speed}}{\text{Max. cutting speed}}$ if the stroke of the ram is 6 in. Neglect the effect of the obliquity of the connecting-rod.

7. Prove that the angular velocity and acceleration of the slotted link of the shaper mechanism shown in the figure is given respectively by



$$\dot{\phi} = \frac{a(b\operatorname{Cos}\theta - a)}{(a^2 + b^2 - 2ab\operatorname{Cos}\theta)}\cdot\dot{\theta}$$

and

$$\ddot{\phi} = \frac{\operatorname{Sin}\theta(a^3b - ab^3)}{(a^2 + b^2 - 2ab\operatorname{Cos}\theta)^2}\cdot(\dot{\theta})^2$$

where $a =$ BC, $b =$ AB, $\theta = \angle$ABC, $\phi = \angle$BAC.

8. Prove that the speed of rubbing at the crank-pin of a direct-acting engine mechanism is given by

$$R\omega\left[1 + \frac{r \operatorname{Cos} \theta}{\sqrt{l^2 - r^2 \operatorname{Sin}^2 \theta}}\right]$$

where R = radius of crank pin, r = radius of crank, l = length of connecting rod, ω = angular speed of crank and θ = angle turned through by crank from inner dead-centre.

9. Find the magnitude of the ratio of the maximum values of the primary and secondary components of the piston acceleration for an engine having a crank/con.-rod ratio of 1/3.

CHAPTER X

(This chapter may be omitted on a first reading.)

STRAIGHT-LINE MOTIONS AND THE PANTOGRAPH

125. The term "straight-line motion" is used to describe those mechanisms in which the paths of one or more points, not being directly guided by means of sliding pairs, are exactly, or to a close approximation, straight lines; alternatively such mechanisms are called "parallel motions." They may be divided into two classes:

1. Those in which the line is mathematically straight.
2. Those in which the line is only approximately straight.

They may also be classified according as to whether the mechanism contains one or more sliding pairs of which the straight-line path is more or less directly a copy, or whether it is composed wholly of turning pairs, the straight line then being said to be "generated" as opposed to being "copied."

In the early days before the advent of really accurate machine tools the production of accurate sliding pairs to give straight-line motions was difficult, whereas turning pairs could be produced comparatively easily; hence there was an incentive towards the invention of straight-line motions using only turning pairs. Nowadays sliding pairs can be produced so accurately and easily that they are used in preference to straight-line motions, which are now of little practical importance. It is, however, instructive to consider them briefly; those who wish to go more deeply into this matter are referred to a paper by Mr. A. B. Kempe entitled "On a General Method of Obtaining Exact Rectilinear Motion by Linkwork" in the *Proc. Roy. Soc.*, 1875, also to his lectures "How to Draw a Straight Line" (Macmillan, 1877).

126. Peaucellier's Cell.—This mechanism was invented in 1864 by M. Peaucellier, a French engineer officer, and is shown in Figs. 163 and 164. The following equalities must hold between the various links, CD=DE=EF=FC, AD=AF and BC=AB, the latter being the fixed link. Then the path of the point E is a straight line perpendicular to AB. The proof is as follows. By symmetry the points A, C and E always lie on a straight line, also the diagonals CE and DF bisect each other at right-angles at L. Draw EM perpendicular to AB produced and describe the circle

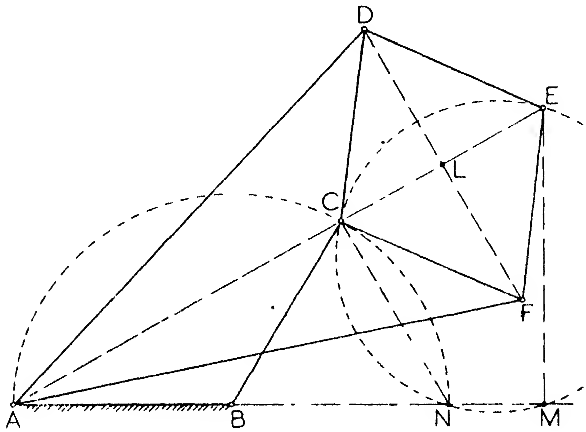


FIG. 163

in which C moves to intersect AB produced in N. Then ACN (and hence $\angle$NCE) is a right angle; hence in the quadrilateral CEMN the angles NCE and NME are right angles; hence a circle may be drawn through C, E, M and N,

$$\therefore\ AM \times AN = AC \times AE$$

$$AM \times (AB + BN) = (AL - LC)(AL + LE)$$

$$AM \times 2AB = AL^2 - LC^2, \text{ since } LE = LC.$$

$$= (AD^2 - DL^2) - (CD^2 - DL^2)$$

$$= AD^2 - CD^2$$

$$\therefore\ AM = \frac{AD^2 - CD^2}{2AB} = \text{constant};$$

hence E describes a straight line perpendicular to AB. An alternative arrangement of the cell is shown in Fig. 164. It may be noted that if the links AB and BC are *not* quite equal, then E will describe the arc of a circle having a very large radius, a property of the mechanism which may be valuable.

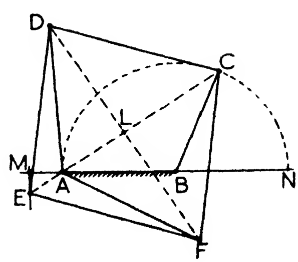


FIG. 164

127. Fig. 165 shows a mechanism which is a particular case of a more general mechanism described by Mr. Kempe in the paper mentioned above, to which the reader is referred for details of its derivation and a proof of its accuracy. The quadrilaterals ABCD and A′B′C′D′ are similar, so that $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\frac{DA}{D'A'}$, and they are symmetrically placed with regard to the bisector of the angle

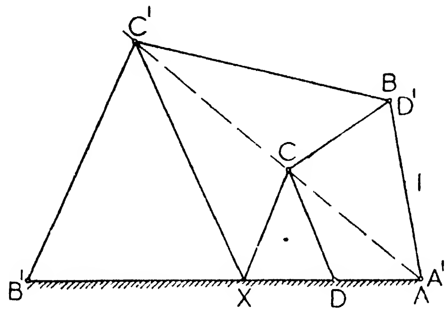


FIG. 165

BAD. Also BC=CD (hence B′C′=C′D′), CX=CD and C′X=C′B′. The joint at X is between the links C′X and CX, there being no connection at this point with the fixed link ADB′. Then X describes a straight line coinciding with ADB′.

128. Hart's Motion.—This consists essentially of a "crossed parallelogram" DEFG (Fig. 166) in which DE=FG and EF=DG,

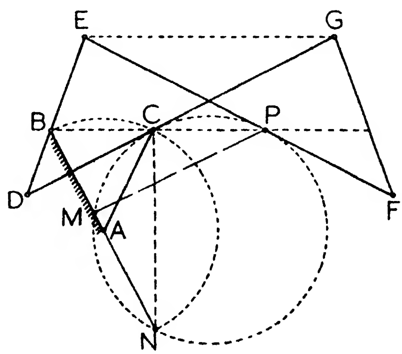


FIG. 166

and of two equal links AB and AC, of which AB is fixed. The point B may be any point on DE, but C must be such that BC is parallel to EG and DF, which are clearly always parallel. The point P, the intersection of BC, produced, with the link EF, will then describe a straight line perpendicular to AB. The proof is as follows. Draw PM perpendicular to AB and let the circle in which C moves intersect AB produced in N. Then $\angle BCN$ (and hence $\angle PCN$) and $\angle PMN$ are right angles; hence the quadrilateral MCPN may be circumscribed by a circle.

$$\therefore\ BM \times BN = BC \times BP$$

$$\therefore\ BM \times 2BA = BC \times BP$$

but
$$\frac{BC}{EG} = \frac{DB}{DE} \text{ and } \frac{BP}{DF} = \frac{EB}{ED}$$

$$\therefore\ BC \times BP = EG \times DF \times \frac{DB \times EB}{DE^2}$$

but, since a circle may be drawn through DEGF

$$DG \times EF = DE \times GF + EG \times DF$$

$$\therefore\ EG \times DF = DG^2 - DE^2 = \text{constant};$$

hence
$$BC \times BP = \text{constant}$$

$$\therefore\ BM = \text{constant, being} = \frac{BC \times BP}{2BA};$$

hence P describes a straight line perpendicular to AB.

129. The "Grasshopper" Motion.—This is a modification of the Scott-Russel motion described in Art. 83. Referring to Fig. 167, if the ends of the link AP are guided so as to move along the lines OX, OY respectively, then any point B of that link will describe an ellipse (see Art. 83). Conversely, if the point A is made to move along OX and B to move in the ellipse, then P will describe the straight line OY. It is simpler, instead of guiding A in a straight line, to guide it in the arc of a circle of large radius, by means of a link AC, the point C being fixed, and similarly instead of guiding B in the ellipse it is simpler to approximate to the ellipse by a portion of a circle; thus B is connected to the fixed pivot D by a link. The line described by P is then only approximately straight. The link DB should be made equal in length to the radius of curvature of the ellipse, in which B should move, at the end of the major axis, and this is equal to $\frac{(\text{major axis})^2}{\text{minor axis}}$, i.e. to $\frac{BP^2}{AB}$. If AB is made equal to BP, as in the

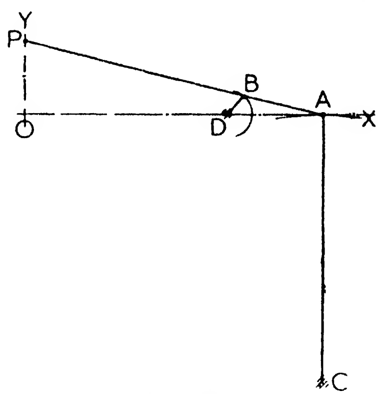


FIG. 167

Scott-Russel motion, then the only inaccuracy in the line described by P is that due to A moving in the arc of a circle instead of in a straight line.

130. Tchebicheff's Motion.—This is shown in Fig. 168. The links AD and BC are of equal length and are crossed as shown. The path of the point P is then approximately in a straight line parallel to the fixed link AB.

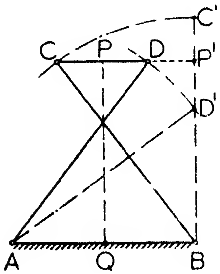


FIG. 168

If when the linkage occupies the position AD′C′B (i.e. when the link BC is perpendicular to AB) the distance P′B is to be equal to the distance PQ, then it can easily be shown that the following relations must hold:

$$CD = \tfrac{1}{2}AB \text{ and } AD = BC = 1\tfrac{1}{4}AB.$$

131. Watt's Motion.—This is a four-bar chain consisting of two links AB and CD (Fig. 169) which are pivoted to the frame link AC (not shown in the figure) and are coupled by a link BD. If the mechanism is given a small displacement, then B will deviate to the left of the vertical BD, and D to the right; hence some point P of the coupler BD will not deviate at all. It can be shown that this point P is determined by the relationship $\frac{BP}{DP} = \frac{CD}{AB}$. The path of the point P is a lemniscoid, part of which is shown dotted, and it will be seen that the portion BD is very closely a straight line. The links AB and CD may, if convenient,

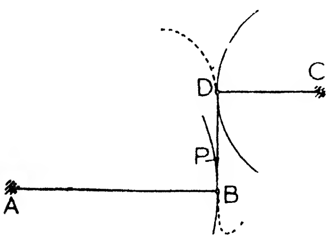


FIG. 169

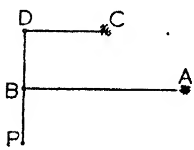


FIG. 170

both be situated on the same side of the coupler as shown in Fig. 170. The closest approximation to a straight line for the path of the point P can be shown to result from making AB parallel to CD when P is in the middle of its stroke and the inclination of the coupler to the line of stroke in the middle position

(θ_1 in Fig. 171) equal, but of opposite sign, to the inclinations (θ) in the extreme positions. These conditions can be shown to require that the line of stroke SS shall bisect the versed sines

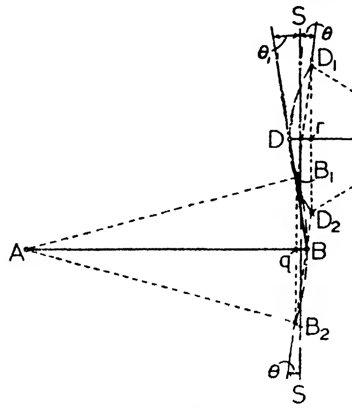


FIG. 171

Bq and Dr of the arcs B_1BB_2 and D_1DD_2 in which the ends of the links AB and CD travel. For further information about the design of this motion the reader is referred to *Machinery and Millwork,* by Professor Rankine, published by Charles Griffin & Co., London.

132. Roberts's Motion.—This is shown in Fig. 172. The links AB and CD are equal in length and the coupler BD is made half the length of the fixed link AC. The tracing point P is attached to the coupler and is the apex of an isosceles triangle with the coupler as base. It follows that AB=BP=DP=DC. The point P then traces out an approximate straight line coincident with

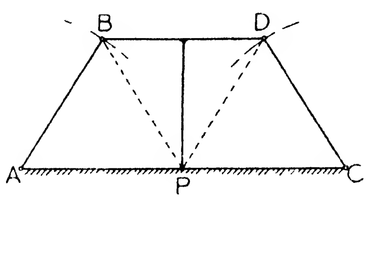


FIG. 172

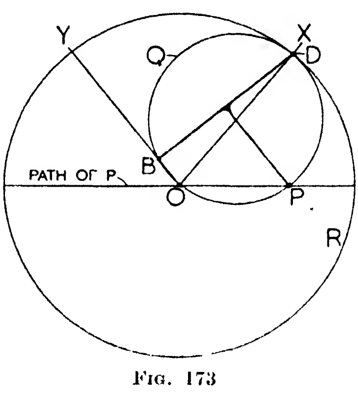


FIG. 173

AC. The links AB and CD should be as long as possible in relation with the semi-bases AP and CP, the accuracy of the line described by P being thereby enhanced. It may be noted that if B and D (or any two points of the coupler) were guided in straight lines OY, OX (Fig. 173), then the path of P would be exactly straight provided that P was chosen so as to lie on the circle Q, which is the centrode of the coupler BD relative to the fixed space. This follows since the centrode Q rolls on the other centrode, a circle R having a diameter twice that of Q; hence any point of the circle Q describes a hypocycloid which is a diameter of R.

133. Kempe's Motions.—Fig. 174 shows a mechanism in which the link A′B′ lies in and moves exactly along the axis of the link AB. The quadrilateral ABCD has the adjacent sides AB and BC equal, and also AD is equal to DC. Such a quadrilateral has been called a *kite*. A second and similar kite ADFE is combined with the first and then the whole mechanism is duplicated, the links FDC′ and CDF′ being rigid. The links AB and BC are made twice the length of the links AD and DC; hence it follows that

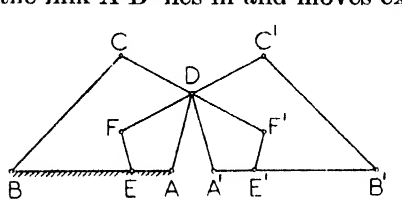


FIG. 174

$$AE = EF = \tfrac{1}{2}AD = \tfrac{1}{2}FD = \tfrac{1}{4}AB = \tfrac{1}{4}BC \text{ and } A'B' = AB, \text{ etc.}$$

Fig. 175 shows a mechanism in which the link GKH is parallel to and moves exactly perpendicular to the fixed link AEB. The mechanism again consists of two double kites. ABCD and GHFD are equal kites, so that AB = GH = BC = HF and AD = GD = DC = DF = ½AB; also AEFD and GKCD are equal kites, so that AD = GD = DF = DC and AE = GK = EF = KC = ½AD = ¼AB.

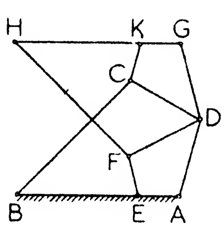


FIG. 175

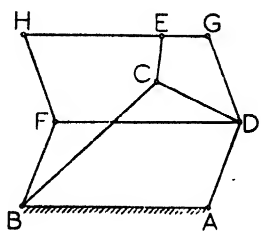


FIG. 176

Exactly the same relative motion is obtained between the links AB and GH of the mechanism shown in Fig. 176 as between the corresponding links in Fig. 175. The kite ABCD is just twice the size of the kite GECD and also BF=FH=AD and DF=GH=AB. These three mechanisms are due to Mr. Kempe.

134. Sarrut's Motion.—This motion is reputed to have been invented in 1853 and is probably the earliest exact straight-line motion as well as being one of the simplest. It is shown in Fig. 177. The piece B has a straight-line motion perpendicular

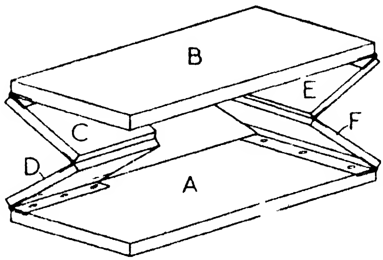


FIG. 177

to the plane of the piece A. Two pieces C and D connect B to A at one end, all the axes of the joints being mutually parallel; similarly two other pieces E and F connect A and B at the other end, the axes of the joints of this second connexion being mutually parallel and all perpendicular to those of the first connexion. The first connexion, alone, would confine any point of B to a plane perpendicular to the axes of that connexion, while the second connexion, alone, would confine the same point of B to a plane perpendicular to the axes of the second connexion. The two connexions, together, therefore confine any point of B to the intersection of two planes, and this intersection is a line perpendicular to A. Clearly it is not necessary, though it may be desirable, to have the axes of the two sets of joints exactly perpendicular to one another; they might be placed at any angle provided it is not zero. This mechanism has been shown to be a special form of a more general mechanism, for details of which the reader is referred to *Mechanism*, by Dunkerley, published by Longmans, Green & Co. It has recently been patented as an independent suspension for the wheels of motor cars.

135. The Pantograph.—The pantograph is a mechanism having the property that if one point of it is made to trace a given outline, then some other point will trace an exactly similar outline, usually to a different scale, i.e. it is a *copying* mechanism. One arrangement of it is shown in Fig. 178, from which it is seen to consist of a four-bar chain ABCD having opposite links equal and the link DC extended to P. The point A is fixed. If P traces out

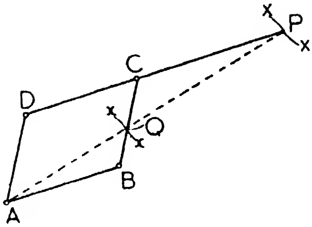


FIG. 178

the curve XX, then the point Q (the intersection of AP and the link BC) will trace out a similar curve xx, but reduced in the ratio $\frac{AQ}{AP}$. This follows from the fact that, since the triangles PCQ and PDA are always similar and the ratio of their corresponding sides is constant, the points A, Q and P are always co-linear and any movement of P in the direction of AP produces a proportionate movement of Q in the same direction, and also any movement of P perpendicular to AP merely rotates the whole mechanism about A and produces a proportionate movement of Q also perpendicular to AP. Alternative arrangements are shown in Fig. 179.

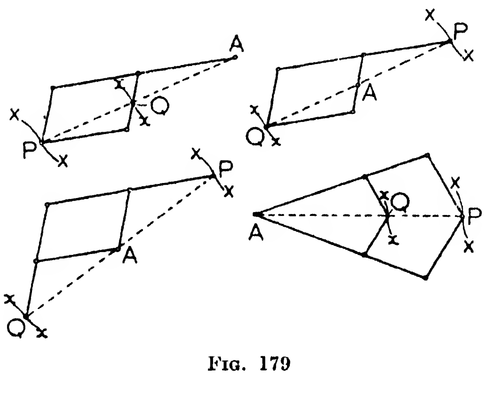


FIG. 179

CHAPTER XI

TOOTHED GEARING

136. General.—Toothed gearing is used to transmit motion of rotation from one shaft to another and forms the most important example of the use of higher kinematic pairing in engineering practice, the only other examples of any importance being cams and belts. At the outset it is useful to classify toothed gearing according to the relative disposition of the axes of the shafts connected by the gearing. Thus,

Spur gears connect shafts whose axes are parallel.

Bevel gears connect shafts whose axes intersect.

Skew, Skew-bevel, Hypoid, Screw and Worm gears connect shafts whose axes are neither parallel nor intersecting.

All these types of gearing may also be divided into the following classes :

1. Constant velocity ratio gearing
2. Variable velocity ratio gearing

according as to whether the ratio of the angular velocities of the shafts connected is constant or variable. In the majority of toothed gearing the velocity ratio is required to be constant, and most of what follows is confined to such gearing. It is instructive to begin with a consideration of *friction gearing*, from which toothed gearing was probably developed, and spur gearing will first be dealt with, the other types being taken in the order given above.

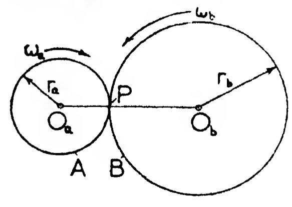


FIG. 180

137. Friction Gearing.—Let O_a and O_b (Fig. 180) be the end view of the parallel axes of two shafts, and let A and B represent circular cylindrical drums, radii r_a and r_b respectively, which are mounted on the shafts and pressed together by some means. If one of the drums is rotated, then the friction between the drums will cause the other drum to rotate also. Provided the resistance to the motion of the driven drum does not exceed a certain limit the drums will roll together with very little slip occurring. Hence, if ω_a and ω_b are the angular speeds of the drums about their axes, we have

$$\frac{\omega_a}{\omega_b}=\frac{r_b}{r_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

since the linear velocity of the point P of the drum A, viz. $r_a\omega_a$, must be equal to that of the point P of the drum B, viz. $r_b\omega_b$, if no slip occurs. The speeds of the drums are thus inversely as their radii. Clearly the drums turn in opposite directions.

In addition to equation (1) we have also

$$r_a+r_b=O_aO_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and these two equations enable r_a and r_b to be calculated when the velocity ratio $\frac{\omega_a}{\omega_b}$ and the centre distance O_aO_b are given.

It has been shown in Art. 91 that P is the instantaneous centre of A relative to B and that the circles A and B are the centrodes or, more accurately, P is the trace of the instantaneous axis, and the circles A and B are the traces of the axodes, on the plane of the paper. Thus in friction gearing we have an example of the rolling together of axodes.

Friction gearing has only a limited use, its chief practical drawbacks being first that if the slip is to be kept reasonably small in amount the effort that can be transmitted is extremely limited and, secondly, that since a small amount of slip always does occur, and also because of the distortion of the drums, the velocity ratio is often not sufficiently constant. In order to avoid such slip it seems an obvious step to form projections or teeth on the one drum and to make them engage corresponding teeth on the other drum, the effort that can be transmitted being then limited only by the strength of the teeth. The shapes of the teeth, however, become of importance and their action will now be examined.

138. The Fundamental Action of Gear Teeth.—Let O_a and O_b (Fig. 181) be the traces on the plane of the paper of two fixed parallel axes about which two bodies A and B respectively can revolve, and let the rotation of the one body be transmitted to the other by means of the contact between their outlines *xx* and *yy*. These outlines are the traces of corresponding surfaces of the bodies A and B on the plane of the paper and may be quite arbitrarily chosen. Let *xx* and *yy* be in contact at Q. Then at Q there will be a common normal NN, and a common tangent

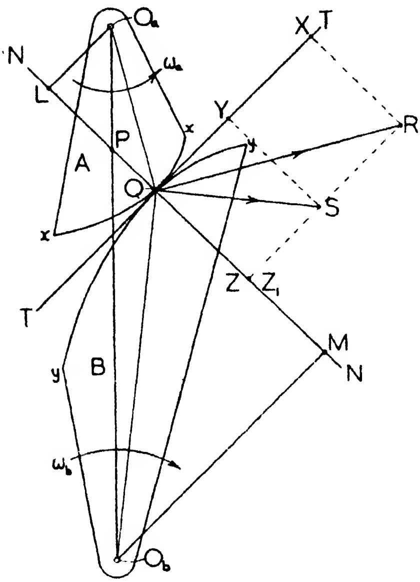


FIG. 181

TT, to the curves. Let NN intersect the line of centres O_aO_b in P and let the instantaneous angular speeds of A and B about O_a and O_b respectively be ω_a and ω_b. Then it will be proved that $\frac{\omega_a}{\omega_b}=\frac{O_bP}{O_aP}$.

The point Q may be considered to belong either to A or to B and may be labelled Q_a or Q_b accordingly. The velocity of Q_a is perpendicular to O_aQ and equal to $O_aQ\times\omega_a$. Similarly the velocity of Q_b is perpendicular to O_bQ and equal to $O_bQ\times\omega_b$. Let these velocities be represented by the vectors QR and QS respectively, and let them be resolved along and perpendicular to the common normal, the components being respectively QZ and QZ_1 along the common normal and QX and QY perpendicular to it. Then, as has been stated in Art. 92, since the outlines *xx* and *yy* are in contact at the instant under consideration, the components QZ and QZ_1 along the common normal must be equal and Z must coincide with Z_1, as shown.

Then
$$\frac{v_{qa}}{v_{qb}}=\frac{O_aQ\times\omega_a}{O_bQ\times\omega_b}=\frac{QR}{QS}$$

$$\therefore \frac{\omega_a}{\omega_b}=\frac{O_bQ}{O_aQ}\times\frac{QR}{QS}$$

Draw O_aL and O_bM perpendicular to NN. Then $\triangle$'s O_bLQ and QZR are similar

$$\therefore \frac{QR}{QZ}=\frac{O_aQ}{O_aL}$$

Also $\triangle$'s O_bMQ and QZS are similar

$$\therefore \frac{QS}{QZ}=\frac{O_bQ}{O_bM}$$

$$\therefore \frac{QR}{QS}=\frac{O_aQ}{O_aL}\times\frac{O_bM}{O_bQ}$$

$$\therefore \frac{\omega_a}{\omega_b}=\frac{O_bQ}{O_aQ}\times\frac{QR}{QS}=\frac{O_bM}{O_aL}$$

but $\triangle$'s O_aLP and O_bMP are similar

$$\therefore \frac{O_bM}{O_aL}=\frac{O_bP}{O_aP}$$

$$\therefore \frac{\omega_a}{\omega_b}=\frac{O_bP}{O_aP}$$

This result is of fundamental importance in connexion with toothed gears and may be stated in words thus. When two bodies are free to rotate about two fixed parallel axes, and one body drives the other by means of the contact between them, then the common normal at the point of contact of the traces of the teeth on any plane perpendicular to the axes divides the line joining the axes and lying in that plane inversely in the ratio of the angular speeds of the bodies.

Now on any plane perpendicular to the axes the tooth surfaces will have traces corresponding to, but not necessarily the same as, *xx* and *yy*, and the above proof may be applied. But since the ratio of the angular speeds can have only one value at any instant, it follows that the points corresponding to P are at the same distances from the axes for all such planes. Hence the locus of P (in the plane containing the axes) is a line lying in the plane of the axes and parallel to them, and this line divides any line joining the axes in the inverse ratio of the angular speeds of the bodies.

It can be shown that QP (Fig. 181) is the projection, on the plane of the paper, of the common normal to the tooth *surfaces* at Q, and hence this common normal intersects the line through P parallel to the axes. This line is, of course, the instantaneous axis of the one body relative to the other. We may now summarise the above arguments thus :

When two bodies are free to rotate about two fixed parallel axes, and one body drives the other by means of the contact between them, then, at every instant, the common normal at any point of contact between them must intersect the instantaneous axis of the one body relative to the other.

In general, if the traces xx and yy of Fig. 181 are any arbitrary curves, the point P will not be a fixed point on O_aO_b, but will move up and down as the traces xx and yy engage at different points, and the velocity ratio will not be constant.

139. Condition for Constant Velocity Ratio.—If a constant velocity ratio is required, then the traces xx and yy must be such that *for any point of contact between them the point* P *is the same point of the line of centres* O_aO_b.

The point P is called the *pitch point,* and thus we may say that in constant velocity ratio spur gearing the traces of the teeth on any plane perpendicular to the axes must be such that the common normal at the point of contact between them always passes through the pitch point.

When each of the bodies has only one "tooth," as in Fig. 181, the above condition is the only one that has to be satisfied. For complete revolutions of the bodies to be possible, however, a number of teeth must be used, and then some minor conditions have to be satisfied as well; these will be dealt with later.

In what follows, the action of the teeth is examined by means of the traces of the teeth on a plane perpendicular to the axes. The arguments and conclusions apply to any such plane, but for the present it is not desirable to consider the forms of the teeth in a direction parallel to the axes. We may, therefore, to fix our ideas, imagine that we are dealing with very thin discs.

140. The Possible Shapes for Gear Teeth.—Theoretically any shape may be chosen for one of a pair of mating teeth and then the proper shape for the other tooth, in order that the fundamental condition shall be satisfied, may be found as follows.

Let O_a and O_b (Fig. 182) be the fixed centres of rotation of the two bodies to be connected by the teeth, and let the constant velocity ratio to be maintained between them be $\dfrac{\omega_a}{\omega_b}=k$. Then the required relative motion between the bodies may be produced by rolling a disc A, centre O_a, radius r, without slip on a disc B, centre O_b, radius R, the radii of the discs being such that $\dfrac{R}{r}=\dfrac{\omega_a}{\omega_b}=k$ and $r+R=O_aO_b$.

Let the outline of the tooth of A be chosen quite arbitrarily as the curve aa, and let this curve be cut out in cardboard and mounted on the disc A. Next let a disc of stiff paper L (shown shaded) be mounted on the disc B and be arranged to lie between the disc A and the tooth *aa* as indicated.

Now, starting with the tooth *aa* in the position shown in full line, draw round the curve *aa* with a pencil, thus marking that

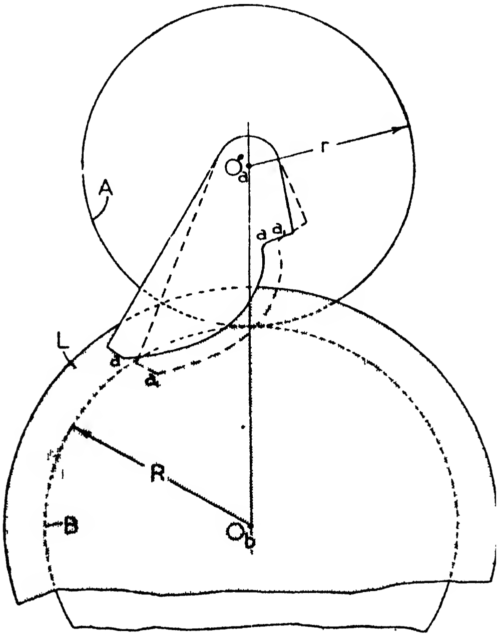


FIG. 182

curve on the disc of paper M. Then turn the disc A through a small angle (say θ) until the tooth *aa* comes to the dotted position a_1a_1, *taking care that during this motion the discs* A *and* B *roll together without slip*. (This may be checked by measuring the angle turned through by disc B, which should be equal to $\theta \times \frac{r}{R}$.) In the new position draw round the tooth (now occupying the position a_1a_1) again. Repeat the process a large number of times, turning the disc A (and B in unison) forward through a small angle each time. The result will be that a large number of similar curves will have been drawn on the paper M, and it will be found that these curves have an envelope *bb* as shown in Fig. 183. This envelope *bb* is the proper shape for the tooth of the body B to mate with the arbitrarily chosen tooth *aa* of A in order to satisfy the fundamental condition and to ensure that the constant velocity ratio *k* is obtained between the bodies.

Since there will be only the one envelope *bb* for the curves drawn on the disc M, it follows that there is only one possible shape for the tooth of B if it is to mesh properly with the given tooth of A. This one tooth shape for B is called the *conjugate*

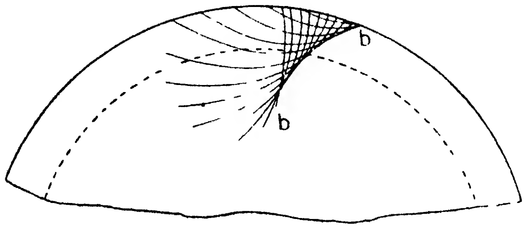


FIG. 183

tooth to the tooth of A. Thus any given tooth has only one conjugate tooth for given conditions of velocity ratio and centre distance. Since, however, the tooth of A is quite arbitrary, there is clearly an infinite number of possible tooth shapes available; many of these, however, will not be practicable, since they will be found to give looped shapes for the conjugate teeth, and clearly a looped tooth is not practicable. The method described above of deriving the conjugate tooth to any given tooth will be referred to as the *envelope method.* The method cannot be carried out with any high degree of accuracy in practice, and the errors of draughtsmanship render the result of no practical use and thus the method is not used as a drafting process. *It does, however, form the basic principle of all the gear-generating machines actually used in practice for cutting gear teeth, and the student ought therefore to understand it thoroughly before proceeding.*

The circumferences of the discs A and B, which are of course the centrodes, are often called the pitch lines; in constant velocity gearing they are circles, but even if they are not circles the method described above can still be used to find the conjugate tooth to any arbitrarily chosen one.

141. Another Method of Obtaining Mating Tooth Shapes.—This method is really a double application of the envelope method. Let *ab* and *cd* (Fig. 184) be the pitch lines or centrodes of the wheels for which teeth are to be found, and let them be in contact at P. Let *xy* be any curve in contact with both *ab* and *cd* at P,

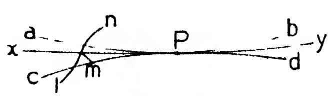


FIG. 184

and let *lmn* be a curve fixed to *xy* at *m*. Then, regarding *lmn* as a tooth outline as shown in Fig. 185, the conjugate tooth of *ab* can be found by the envelope method. Let it be the curve *st*. Similarly the conjugate tooth of *cd* may be found; let it be the curve *uv* (Fig. 186). Thus both *st* and *uv* are conjugate to *lmn*,

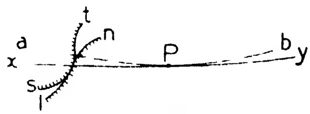


FIG. 185

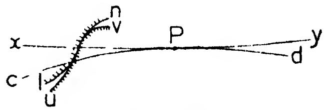


FIG. 186

and the consideration that both *st* and *uv* may be derived simultaneously by rolling *ab* and *cd* on *xy* so that all three curves are always in contact at one point at any instant will show that *st* and *uv* are mutually conjugate. It will be noticed that when deriving *uv* the solid portion of the tooth *lmn* must be placed on the opposite side of *lmn* to that adopted when deriving *st*, and this means that two cutters will be required, one to cut *st* and the other to cut *uv*. If, however, *xy* is made a straight line, and if the curve *lmn* is chosen so that when it is revolved about *m* through an angle of 180° it coincides with its original position, then the two cutters will be identical and only one will be required to cut both *st* and *uv*. This extension of the envelope method of obtaining mating tooth outlines will be found of assistance when gear-cutting processes are considered.

142. Roulettes as Tooth Shapes.—[A "roulette" is the path traced out by any point of, or attached to, any curve which rolls without slip on any other curve.] A third method of obtaining mating tooth shapes is the following. Let *ab* and *cd* (Fig. 187) be the pitch lines of two wheels in contact at P, and let *fg* be any curve in contact with both *ab* and *cd* at P. Then if *fg* rolls without slip on *ab*, any point Q of *fg* will trace out a path *st* relative to *ab*. Similarly if *fg* rolls on *cd*, Q will trace out a path *uv* relative to *cd*. Then if the curves *st* and *uv* are taken as the outlines of the teeth of *ab* and *cd* respectively, they will mate properly. This may be proved thus: since, in the position shown (which may be any position), *fg* is rolling about P as instantaneous centre, it follows that QP is the normal to *st* at Q; similarly QP is the normal to *uv* at Q. Hence QP is the common normal to the tooth outlines *st* and *uv* at their point of contact Q, and this common normal passes through the pitch point P, the point of contact of the pitch lines *ab* and *cd*, so that the curves *st* and *uv* satisfy the fundamental condition requisite for correct mating.

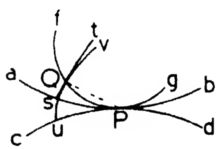


FIG. 187

This method has been described for the general case when *ab* and *cd* are any curves, but one particular case is of special interest, namely, when *ab*, *cd* and *fg* are all circles. The teeth then obtained are known as cycloidal teeth, and at one time they were almost exclusively used; they have, however, been almost entirely displaced by involute teeth, and the latter will therefore be considered first.

143. Tooth Outlines by Means of Secondary Centrodes.—Suppose A and B, Fig. 188, are the centrodes of two bodies which are rotating about fixed centres C and D with angular velocities ω_a and ω_b respectively. Then $\frac{\omega_a}{\omega_b}=\frac{DP}{CP}$, where P is the point of contact of the centrodes, which point of course lies on CD. Through P draw a line LM making an angle θ with CPD, and draw CM and DL perpendicular to LPM. Then $\frac{DL}{CM}=\frac{DP}{CP}=\frac{\omega_a}{\omega_b}$. Let the centrodes be rotated about C and D through a small angle, and let the above process be repeated, using the same value of θ as before,

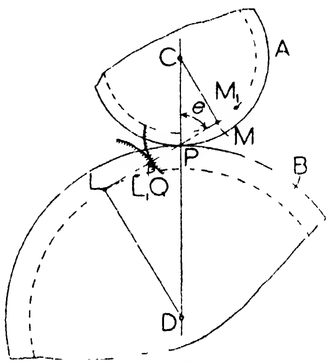


FIG. 188

thus establishing the new points L_1M_1 corresponding to L and M. Then the points L and L_1 and M and M_1, and all points established in the same way, form curves, shown dotted, which have been called *secondary centrodes*, since they possess the property that if they are supposed to be connected by an inextensible cord (LM), then the relative motion between the bodies due to the constraint of the cord wrapped round the secondary centrodes will be identical with that produced by the rolling of the primary centrodes.

If now any point, say Q, of the tangent LM be chosen as a describing point and be made to trace out paths on the bodies A and B as they move, then those paths will be the outlines of teeth which, when mating together, will give the bodies the same relative motion as they have when the primary centrodes roll together. For clearly, in the position shown, LQ is the normal to the tooth of B at Q and MQ is the normal to the tooth of A at Q. Hence LQM is the common normal to the teeth at their point of contact and, by the construction, LQM always passes through P, the point of contact of the centrodes.

Secondary centrodes may also be derived by drawing through each point of the primary centrode a line making a fixed angle with the tangent to the primary centrode at the point. The envelope of all the lines is a secondary centrode. In the particular case where the centrodes are circles the secondary centrodes are also circles, and the tooth outlines become the involutes of circles, giving "involute teeth." These will now be considered in detail.

144. Involutes.—Imagine an inextensible cord AB to be wrapped round the outside of a body as in Fig. 189, the end A being fixed to the body. Let the cord be now unwrapped, care being taken to keep it always taut, then the end of the cord will trace out some path BCDE and any other point F of the cord will trace out a path FGHI. The curves BCDE and FGHI are *involutes* of the *base curve* which is the outline of the body from off which the cord is unwrapped.

It should be clear that if at any point, say C, of an involute a line, CT, is drawn tangent to the base curve, that line is the normal to the involute at that point.

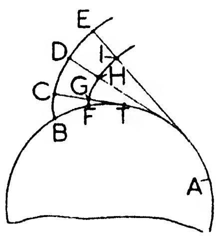


FIG. 189

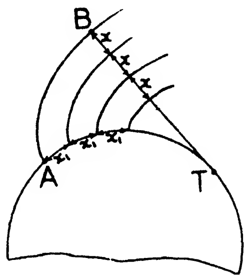


FIG. 190

It may be noted that the base curve of any involute is the envelope of all the normals which may be drawn to the involute. The base curve is called the *evolute* of the curve from which it is derived. Clearly any base curve will give an infinite number of involutes corresponding to the infinite number of points, on the cord, which may be selected as the describing point. On the other hand, any given curve will give only one base curve or evolute.

The base curves of involute toothed gearing are almost invariably circles and are called *base circles*. The involutes described by any two points of a cord which is unwrapped from a base circle are identical in shape, and the cord in any position is normal to both the involutes. If describing points are taken at equal intervals along the cord, that interval is the *normal pitch* of the involutes described and is equal to the distance, measured along the arc of the base circle, between consecutive involutes. This distance is called the *base circle pitch* of the involutes. Thus the base circle pitch x_1 in Fig. 190 is equal to the normal pitch x.

The radius of curvature of an involute at any point is equal to the length of the line drawn from the point tangent to the base curve. In Fig. 190 TB is the radius of curvature of the involute AB at B.

It will now be shown that gear teeth whose outlines are portions of involutes of base circles satisfy the fundamental condition for the transmission of motion with a constant velocity ratio.

145. Involute Teeth.—Let O_a and O_b (Fig. 191) be the fixed centres of rotation of two gears A and B, and let the teeth, *ef* and *gh*, of those gears be made portions of involutes of the base circles *a* and *b*. Let the teeth be in contact at *any* point Q. Draw the tangents QL and QM. Then QL is the normal to the involute *ef* at Q and QM is the normal to the involute *gh* at Q. Hence LQM is the *common* normal of the teeth at their point of contact, and hence LQM is a straight line and is the common tangent to the base circles. Join O_aO_b. Then LQM intersects the line of centres O_aO_b in the fixed point P; hence the fundamental condition is satisfied.

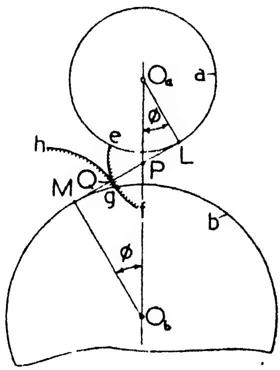


Fig. 191

Now $$\frac{\omega_a}{\omega_b}=\frac{O_bP}{O_aP}.$$

P is the pitch point and circles, centres O_a and O_b respectively, passing through P are the pitch circles. The radii of the pitch circles are therefore $O_aP=R_a$ and $O_bP=R_b$. Draw O_aL and O_bM perpendicular to LM. Then triangles O_aLP and O_bMP are similar.

$$\therefore \frac{O_aL}{O_bM}=\frac{O_aP}{O_bP}=\frac{R_a}{R_b}=\frac{\omega_b}{\omega_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus the ratio of the radii of the base circles is equal to the ratio of the radii of the pitch circles and to the inverse ratio of the angular speeds.

Clearly $$O_aL=O_aP \text{ Cos } \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and $$O_bM=O_bP \text{ Cos } \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Since the point of contact, Q, between the teeth always lies somewhere on the line ML, that line is called the *line of action* of the teeth.

146. The Motion Unaffected by Alteration of the Centre Distance.—If O_a and O_b are moved apart, or towards each other, slightly, then the teeth will come into contact at some new point

Q_1. The argument used above applies equally well, however, and it is still true that the ratio of the angular speeds is the inverse ratio of the base circle radii. As the latter have not been altered, the velocity ratio is unchanged. The pitch point will, however, occupy a new position P_1, and the pitch circles will have different radii (O_aP_1 and O_bP_1); the angle ϕ will also have a different value.

This property of involute toothed gears gives them a distinct advantage over cycloidal teeth (see Chap. XIII), with which, if the centre distance is altered, the fundamental condition for the transmission of motion with a constant velocity ratio is no longer satisfied and the teeth will not work together properly.

147. The Pressure Angle or Angle of Obliquity.—If the centre distance O_aO_b and the velocity ratio $\frac{\omega_a}{\omega_b}$ are fixed, then the pitch point and pitch circle radii are determined by the equations

$$\frac{R_a}{R_b}=\frac{O_aP}{O_bP}=\frac{\omega_b}{\omega_a} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$R_a+R_b=O_aO_b \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The radii of the base circles, however, are still open to choice and may have any values provided their ratio is equal to that of the pitch circle radii. If, however, a value is chosen for the angle ϕ, Fig. 191, then the base circle radii are settled, being given by $O_aL=O_aP \operatorname{Cos} \phi$ and $O_bM=O_bP \operatorname{Cos} \phi$. The angle ϕ is called the *pressure angle* or the *angle of obliquity*. The name arises because, in the absence of friction, the force or pressure between the teeth acts along the common normal LM at the angle ϕ to the ideal direction for the pressure, which is along the common tangent of the pitch circles.

148. The First and Last Points of Contact between a Pair of Teeth.—Fig. 192 shows two involute teeth, *abc* and *def*, which will engage each other if the wheels are turned through suitable angles. The point of contact between the teeth will always lie on the line of action LM, and since (with the directions of rotation shown) *d* will be the first point of the profile *def* to intersect LM, it follows that *d* will be the first point of the tooth *def* to engage with any point of *abc*. This engagement will occur at R, where a circle, centre O_2 and passing through *d*, intersects LM. Similarly, since *c* will be the last point of the profile *abc* to intersect LM, and will do so at S, where a circle, centre O_1 and passing through *c*, intersects LM, it follows that S is the last point of contact between the teeth. Thus contact between the teeth occurs between R and S and RS is the length of the path of contact of the teeth.

If the directions of rotation of the wheels be reversed, then S is the first, and R the last, point of contact between the teeth *abc* and *def*, and the latter is driving the former instead of vice versa. If with the original directions of rotation the wheel O_2 is required to drive O_1, then the profile $d'e'f'$ must be made to engage $a'b'c'$, and the line of action will be L'M'; the first and last contacts are obtained as before.

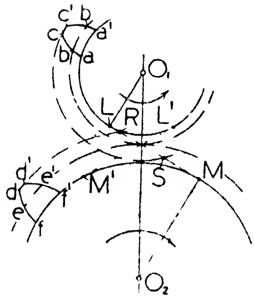


FIG. 192

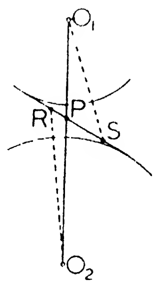


FIG. 193

The positions of the points R and S, and thus the length RS, are determined solely by the heights of the teeth above the pitch circles, which dimensions are called the *addenda* of the teeth, assuming the pitch circle diameters and pressure angle to be constant. The length RS may be calculated by solving the two triangles O_1SP and O_2RP (Fig. 193), of which the sides O_1P, O_2P, O_1S and O_2R are known (being respectively the pitch circle radii and the pitch circle radii plus the addenda), in addition to the angles O_1PS, O_2PR, which are equal to the pressure angle plus a right angle.

149. The Arcs of Approach and Recess.—Fig. 194 shows *one* pair of teeth in engagement in three different positions, at the beginning (R) and end (S) of the engagement, and when the point of contact between the teeth is at the pitch point P. Clearly, while the point of contact moves from R to P the point *b* of the pitch circle 22 moves from *b* to P. During this time the teeth are approaching each other and the arc *b*P is thus called the *arc of approach*. Similarly *c*P, which is equal to *b*P, is the arc of approach. Again, while the point of contact moves from P to S, the point P of the pitch circle 22 moves from P to b_1, and during this time the teeth are receding from each other. The arc Pb_1 is thus the *arc of recess*; similarly Pc_1 is also the arc of recess. Let d and d_1 be the positions, corresponding to b and P, of the intersection of the tooth outline Rbd with the base circle zz, then

$$\angle dO_2b = \angle d_1O_2P\,;$$

hence $$\angle dO_2d_1 = \angle bO_2P\,;$$

hence $\therefore$ $$\frac{\text{arc } dd_1}{\text{arc } bP} = \frac{O_2d}{O_2b} = \frac{R_2 \operatorname{Cos} \phi}{R_2} = \operatorname{Cos} \phi$$

ϕ being the pressure angle ;

but $$\text{arc } dd_1 = RP \text{ (see Art. 144)}$$

$\therefore$ $$\text{arc } bP = \frac{\text{arc } dd_1}{\operatorname{Cos} \phi} = \frac{RP}{\operatorname{Cos} \phi}$$

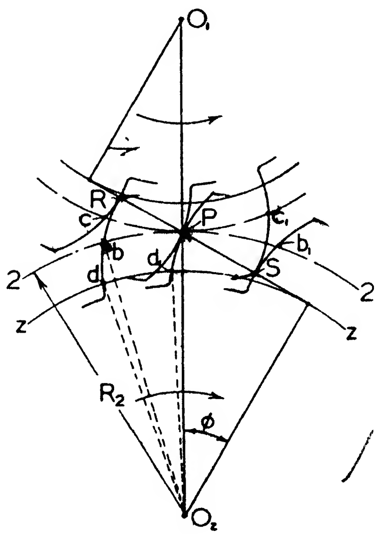


FIG. 194

The angle bO_2P is called the *angle of approach* or *angle of incidence* of the wheel 2, while the angle cO_1P (not equal to $\angle bO_2P$) is the angle of approach or incidence for the wheel 1.

150. Undercutting and Interference.—When either of the points R and S (Fig. 192) approaches the corresponding end L or M of the line of action, the teeth of one or both of the wheels will be undercut as shown in Fig. 195. When R (or S) lies beyond L (or M) the undercutting will be such that a portion of the involute flank of the tooth, which is required during the action of the teeth, will be cut away. In Fig. 196 the portion ab of the involute flank (which should extend down to the base circle) has been cut away by the tips of the mating teeth. (Actually it would probably be cut away by the cutter that produced the teeth, and if it were not it would cause the gears to jam when they were meshed together. This last condition is known as *interference*, and the points L and M are called the interference points. Interference

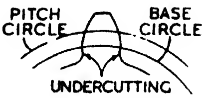


FIG. 195

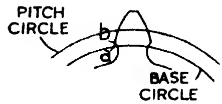


FIG. 196

can generally be avoided by one of the methods described in Art. 160. It occurs chiefly when wheels having few teeth are made to work together or, more frequently, when a small wheel works with a large one, in which case it is the teeth of the small wheel which are affected.

EXERCISES XI

1. The axes of two shafts are parallel and are 9 in. apart. They are to be connected by friction gears giving a ratio of 2 to 1. What are the diameters of the wheels (*a*) if the shafts are to rotate in opposite directions and (*b*) if they are to rotate in the same directions.

2. Prove that the fundamental condition to be satisfied by the tooth profiles of spur gears having a constant velocity ratio is that the common normal at the point of contact of the teeth must always pass through a fixed point on the line of centres.

3. Being given the pitch lines and centres of rotation of two mating spur gears and the shape of the teeth of one of them, describe how the proper shape for the teeth of the other gear may be determined.

4. When the conjugate tooth shapes for two spur gears are generated simultaneously by means of an arbitrary tooth shape fixed to an auxiliary pitch line, what is the condition that must be fulfilled by the arbitrary tooth shape if a single cutter is to be sufficient to cut the teeth of both wheels ?

5. Describe the method of obtaining conjugate tooth shapes by the use of a rolling curve.

6. What do you understand by a "secondary centrode" ? Describe how a secondary centrode may be used to derive the conjugate tooth shapes for a pair of spur gears rotating about fixed axes.

7. Find the radius of curvature of the involute tooth profiles of a wheel at the point of intersection of the profiles with the pitch circle whose diameter is 6 in. Pressure angle 20°.

8. Show that teeth which are the involutes of circles will satisfy the fundamental condition for the transmission of motion with a constant velocity ratio. Show also that the velocity ratio is unaffected by variation of the centre distance between the axes of rotation.

CHAPTER XII

TOOTHED GEARING—CONTINUED. TOOTHED WHEELS

151. The Normal Pitches of Involute Toothed Wheels that Mesh Together must be Equal.—In the previous chapter we have been concerned with the action of one tooth on another, but it is evident that when wheels are required to make complete revolutions they must be provided with a number of teeth, each of which must conform to the principles evolved in the last chapter. It is now necessary, however, to consider wheels having a number of teeth.

Assuming that the teeth are involutes of base circles, then the fundamental condition that must be satisfied in order that two wheels shall mesh properly is that *their normal or base circle pitches shall be the same and shall be equal to or less than the length of the path of contact*. This will be clear on considering Fig. 197, which shows a pair of teeth just going out of engagement at S, it being evident that, if the motion is to be continued, a second pair of teeth must at least have just come into engagement at R. SR is thus the maximum permissible normal pitch, and it is evident that the normal pitches of the two wheels must be the same if the second pair of teeth are to engage properly.

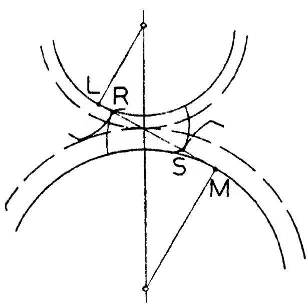


FIG. 197

152. The Number of Teeth in Engagement.—The quotient

$$\frac{\text{Length of path of engagement}}{\text{Normal pitch}} = \frac{\text{RS}}{\text{N.P.}}$$

gives the maximum and minimum number of pairs of teeth in engagement at any instant. When the normal pitch equals RS the quotient is unity, and the number of teeth in engagement varies from 2 to 1; if $\frac{\text{RS}}{\text{N.P.}} = 2{\cdot}7$ say, then the number of teeth in engagement varies between 3 and 2. It is generally considered desirable to have as many pairs of teeth in engagement as possible, and this end is attained by making RS as long as possible and the normal pitch as small as possible. The absolute limit to RS is LM, and in practice this limit is not always attainable, since it may involve excessive undercutting of the teeth ; the addenda should, however, be proportioned so that R and S approach as close to L and M as is possible without excessive undercutting. The normal pitch also cannot be made smaller than a certain amount, otherwise the teeth will be pointed and the addendum desirable from the first consideration will be unattainable. The addenda and other proportions of gear teeth have, however, to a large extent become standardised, and it will be convenient to consider these standard proportions.

153. Definitions, Circular and Diametral Pitch.—Fig. 198 shows the commonly used terms for the parts of gear-wheel teeth. The

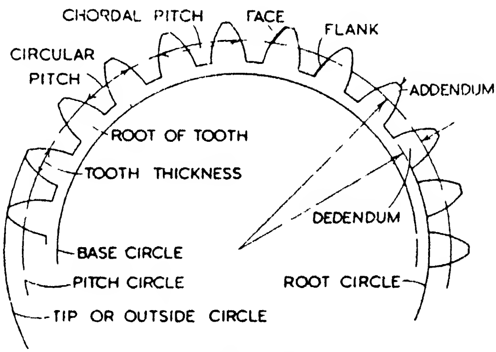


FIG. 198

circular pitch is the distance, measured round the arc of the pitch circle, between consecutive teeth, while *chordal pitch* is the chord corresponding to the circular pitch. It should be noted that the term " face " is used in two senses, to denote that portion of the tooth profile lying outside the pitch circle and to denote the axial length of the teeth. The following abbreviations will be used :

P.C.D.=Pitch circle diameter. B.C.D.=Base circle diameter. O.D.=Outside or blank diameter. C.P.=Circular pitch. N.P. =Normal or base circle pitch. D.P.=Diametral pitch. M= Module. N=Number of teeth in the larger of a pair of wheels ; n=Number of teeth in the smaller wheel (pinion). C.D.=Centre distance between the axes of the shafts of a pair of wheels. Add^m=Addendum ; Ded^m=Dedendum.

The C.P. is determined chiefly by considerations of strength of the teeth and the number of pairs of teeth in engagement, but even so is to a great extent arbitrary ; the addenda and dedenda are even more arbitrary, although considerations of the length of the path of contact and of interference cannot be neglected. At one time the addendum and dedendum were made definite proportions of the circular pitch, although almost every works chose a different proportion, and this practice is still quite common, though the proportions are expressed indirectly, since diametral pitch is now generally used instead of circular pitch. In modern practice the addenda and dedenda are frequently not made fixed proportions of the circular pitch, but are chosen to suit the conditions in each individual design. When circular pitches are used they range from about $\frac{1}{8}$ in. up to 3 in. or more, usually advancing in steps of $\frac{1}{16}$ in. up to 1 in., in steps of $\frac{1}{8}$ in. from 1 in. to 2 in. and in steps of $\frac{1}{4}$ in. from 2 in. to 3 in. ; at least, these are regarded as standard pitches. Then

$$\text{P.C.D.}=\frac{\text{N}\times\text{C.P.}}{\pi}$$

$$\text{C.D.}=\frac{(\text{N}+n)\times\text{C.P.}}{2\pi};$$

hence the pitch circle diameters and the centre distance always involve division by π, and cannot therefore be given exactly, but only to as many decimal places as we choose. This difficulty is avoided by the use of the module, which is defined by the relation

$$\text{Module}=\frac{\text{P.C.D.}}{\text{No. teeth in the gear}}$$

and it is always made a whole number ; hence, since the number of teeth in a wheel must be a whole number, it follows that pitch circle diameters and centre distances will always work out to exact numbers. It follows, of course, that the circular pitch will then be obtained in terms of π, but since the circular pitch need never be measured, whereas the centre distances must be, it is advantageous to employ the module, and this is the standard practice on the Continent, the pitch circle diameters being expressed, of course, in millimetres. In England the module was used many years ago under the name Manchester Pitch, but is now little used ; instead diametral pitch is used, this being the reciprocal of the module. Thus

$$\text{Diametral pitch}=\frac{\text{No. teeth in the wheel}}{\text{P.C.D. in inches}}$$

$$\text{D.P.}=\frac{\text{N}}{\text{P.C.D.}}$$

$$=\frac{\pi}{\text{C.P.}}$$

The following are regarded as standard diametral pitches : 1, $1\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$, 2, $2\frac{1}{4}$, $2\frac{1}{2}$, $2\frac{3}{4}$, 3, $3\frac{1}{2}$, 4, 5, etc. Since the P.C.D. $=\frac{N}{D.P.}$, pitch circle diameters will not always work out exactly, although they often will do so. Thus diametral pitch is not so convenient as the module, and the use of the latter is worthy of encouragement. However, diametral pitch is firmly established and is not likely to be displaced easily.

154. Tooth Proportions.—Although, as stated above, the addenda and dedenda of gear teeth are nowadays frequently chosen arbitrarily and do not conform to any fixed standard, yet there are some advantages to be obtained from using fixed toothed proportions, e.g. the standardisation of cutters and the consequent reduction in the number that must be carried in stock ; facility in ordering gears and in interchangeability. Thus the old and established Brown and Sharpe standard tooth proportions are widely used. They are

$$\text{Addendum}=\frac{1}{D.P.},\ \text{Dedendum}=\frac{1{\cdot}15708}{D.P.}$$

and pressure angle $=14\frac{1}{2}^{\circ}$.

Hence, the outside diameter is given by

$$\text{Outside or blank dia.}=\frac{N+2}{D.P.}$$

and as far as the teeth are concerned the only data that need be given in order that the wheel may be cut are the blank diameter, the diametral pitch and the number of teeth ; the pitch circle diameter is always given in addition, and other data are frequently supplied to facilitate manufacture and inspection.

There is another standard for tooth proportions which is fairly widely used; it is the Fellow's Stub Tooth. In this the pressure angle is 20° and two numbers are used to designate any gear ; thus a gear may be said to be $^{6}/_{8}$ (read as six-eight) pitch. The first number (6) is the true diametral pitch as defined by the relation D.P. $=\frac{N}{P.C.D.}$, while the second number (8 in the example) is a false diametral pitch which is used only to give the addenda and dedenda from the relations

$$\text{Add}^{m}=\frac{1}{D.P.},\ \text{Ded}^{m}=\frac{1{\cdot}157}{D.P.}$$

The tooth thickness, measured along the pitch circle, is in all gears equal to half the circular pitch less half the *backlash*, and the latter varies from less than $\frac{1}{1000}$ in. in accurately cut and mounted gears of fine pitch up to $\frac{1}{10}$ in., or even more, in unmachined gears; the latter are not now much used, however.

155. The Minimum Number of Teeth in a Wheel.—Consider two equal wheels having teeth of a fixed pitch and which mesh together. As the number of teeth is reduced the length of the line of contact is reduced also, and there is thus a limit to the number of teeth that can be used. This limit is set by the condition: Length of line of contact must not be less than the normal pitch of the teeth. Referring to Fig. 199,

$$LM = LP + MP = 2O_1P \operatorname{Sin} \phi,$$

but
$$O_1P = \frac{n}{2 \times \text{D.P.}}$$

$$\therefore \text{ L.M.} = \frac{n}{\text{D.P.}} \times \operatorname{Sin} \phi$$

Now the normal pitch
$$= \text{C.P.} \times \operatorname{Cos} \phi$$
$$= \frac{\pi}{\text{D.P.}} \times \operatorname{Cos} \phi$$

hence
$$\frac{\pi}{\text{D.P.}} \times \operatorname{Cos} \phi \leqq \frac{n}{\text{D.P.}} \times \operatorname{Sin} \phi$$
$$n \geqq \pi \operatorname{Cot} \phi$$

when
$$\phi = 14\tfrac{1}{2}^\circ \quad n_{min} = 12{\cdot}1, \text{ say } 13$$
$$\phi = 20^\circ \quad n_{min} = 8{\cdot}6, \text{ say } 9$$

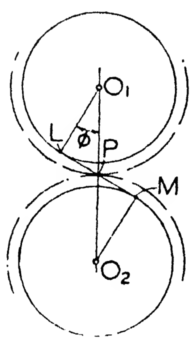


Fig. 199

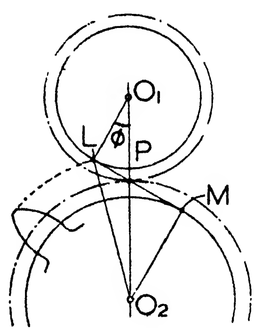


Fig. 200

When one wheel is larger than the other then the pinion could have fewer teeth than these numbers if the above condition were the only limitation, but in practice interference will occur and sets a higher limit to the size of the pinion, which limit is easily found in any given case as follows: Let the wheel have 50 teeth of 5 D.P. and be to the B. & S. standard. Then the P.C.D. $=10$ in., add$^m=0{\cdot}2$ in. In triangle O_2PL (Fig. 200) $O_2P=5$ in., $O_2L=5{\cdot}2$ in., $\angle O_2PL=104\frac{1}{2}°$;

$$\therefore\ O_2L^2=O_2P^2+PL^2-2O_2P\times PL\times \text{Cos}\ \widehat{O_2PL}$$

$$5{\cdot}2^2=5^2+PL^2-2\times 5\times PL\times \text{Cos}\ 104\tfrac{1}{2}°$$

Solving this for PL, we find

$$PL=0{\cdot}6474''$$

$$\therefore\ O_1P=\frac{PL}{\text{Sin}\ 14\frac{1}{2}}=2{\cdot}586''$$

$$\therefore\ \text{P.C.D. of pinion}=5{\cdot}172''$$

$$\therefore\ \text{No. of teeth in pinion}=25{\cdot}86$$

$$\text{say } 26$$

In practice the smallest number of teeth used is generally 12, the teeth being corrected for interference as described later.

156. Example of the Design of a Pair of Gears.—Except for the diametral pitch, which is determined chiefly by considerations of strength, we can now design completely a pair of gears for a given duty; for example, the following. Gear ratio 4 to 1, centre distance 10 in., D.P. 5.

Let n = number of teeth in pinion

and N = ,, ,, wheel

Then P.C.D. of pinion $=\frac{n}{\text{D.P.}}=\frac{n}{5}$

and ,, wheel $=\frac{N}{\text{D.P.}}=\frac{N}{5}$

$$\therefore\ \text{Centre distance}=\frac{n+N}{2\times 5}$$

but the gear ratio $=\frac{N}{n}=4$

$$\therefore\ N=4n$$

$$\therefore\ \text{Centre distance}=10''=\frac{5n}{10}$$

$$\therefore\ n=20$$

$$\therefore\ N=80$$

Thus the wheels have respectively 20 and 80 teeth; their pitch circle diameters are 4 in. and 16 in. and their blank diameters (if B. & S. teeth are used) are 4·4 in. and 16·4 in.

157. Another Example.—Let the data be as above except that the gear ratio is to be 4·5 to 1.

Then the centre distance $=10'' = \dfrac{n+4{\cdot}5n}{2\times 5}$

$$n=18{\cdot}2 \text{ approx.}$$

But n must be a whole number. Let it be taken as 18. Then $N=4{\cdot}5n=81$, and the centre distance becomes $\dfrac{18+81}{10}=9{\cdot}9$ in. instead of 10 in. Thus a compromise must be made, and either the centre distance or the gear ratio must be varied slightly from that specified. As worked above the centre distance was varied and the gear ratio maintained exact. If the centre distance must be exact while the gear ratio need be only approximate, then, as before, we find $n=18{\cdot}2$ and make it 18, but, since the C.D. must be exact, we now have $\dfrac{n+N}{2\times 5}=10$; hence $N=82$, and the gear ratio is $\frac{82}{18}=4{\cdot}556$ instead of 4·5.

In some cases it will be impossible to obtain either the gear ratio or the centre distance exactly *if a standard diametral pitch is to be used.* If a non-standard pitch is permissible, then any ratio can be obtained at any centre distance, but non-standard cutters will be required, and these generally imply increased costs.

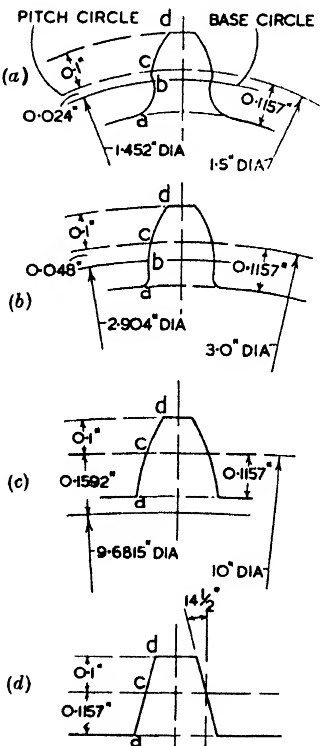


Fig. 201

158. Effect of Number of Teeth in a Wheel on the Tooth Shape.—Consider a number of wheels having different numbers of teeth, say 15, 30, 100 and ∞, but all having a pressure angle of $14\frac{1}{2}°$, a diametral pitch of 10 and B. & S. tooth proportions. The teeth of the 15-tooth wheel will be as shown in Fig. 201 (*a*). The P.C.D. $=1{\cdot}5$ and the base circle diameter $=1{\cdot}5$ Cos $14\frac{1}{2}=1{\cdot}452$ in. The radial distance between pitch circle and base circle is 0·024 in., while the dedendum is 0·1157 in. Since an involute cannot extend inside its base curve, it follows that the portion ab of the profile is not an involute ; its shape is more or less arbitrary and usually it plays no part in the action of the teeth. The working profile is *bcd*, of which *bc* is inside the pitch circle and *cd* outside. If the wheel meshes with a similar 15-tooth wheel, then *bc* will engage with a portion similar to *cd*; hence there will be considerable sliding between the teeth. The teeth are thinner at the root than at the pitch circle and are consequently weak. The involute portion *bcd* has a relatively large average curvature. The 30-tooth wheel is shown in Fig. 201 (*b*). The P.C.D. = 3·0 in. and the base circle diameter = 2·904 in. The portion *bc* is now more nearly equal to *cd*; hence when meshing with a similar wheel the sliding will be less than with 15-tooth wheels. The form of the tooth and the curvature of its profile are also better. The 100-tooth wheel is shown at *c*; its P.C.D. = 10 in. and its base circle diameter = 9·6815 in., so that the whole of the tooth profile is involute. The length of the working portion of *ac* is nearly equal to the length of *cd*, so that, with equal wheels, sliding will be less. The tooth form is stronger than before and the curvature is less.

When the number of teeth is infinite the pitch circle diameter is infinite and the pitch circle becomes a straight line —the *pitch line*. The wheel is then called a *rack*, and the tooth profile, as shown in Fig. 201 (*d*), becomes a straight line inclined at the pressure angle, $14\frac{1}{2}°$, to the normal to the pitch line.

Thus in general the greater the number of teeth in a wheel the better the tooth shape and strength and the less the sliding between mating teeth.

159. Effect of Pressure Angle on Tooth Shape.—Fig. 202 (*a* to *d*) shows the teeth of wheels having 15, 30, 100 and ∞ teeth, of 10 D.P. and B. & S. standard proportions, except that the pressure angle is 20°. The pitch circle diameters are the same as in the corresponding $14\frac{1}{2}°$ wheels, but the base circle diameters are smaller. The involute portion of the flank is more nearly equal to the face portion, the tooth is a better shape and

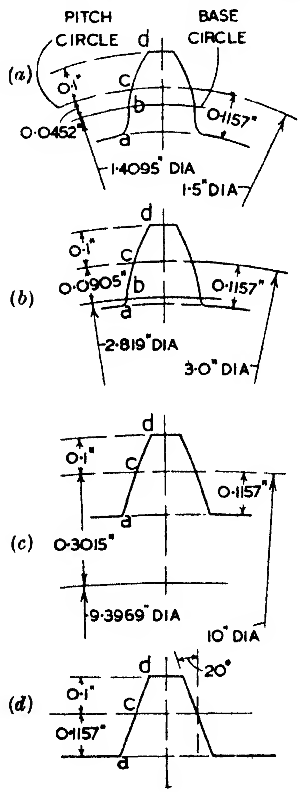


FIG. 202

the average curvature less. Thus an increase in pressure angle, other factors being unchanged, gives better-shaped teeth and less sliding between mating teeth. Angles greater than $14\frac{1}{2}°$ are being used to an increasing extent, but any increase beyond about 20° leads to difficulties, as will be seen later.

160. Methods of Avoiding Interference.—The simplest method is to reduce the addenda of the teeth until the points R and S (Fig. 192) lie between L and M (alternatively the corners of the interfering teeth may be rounded off). These methods are indicated in Fig. 203; they have the disadvantage that the number of pairs of teeth in mesh may be reduced and that to avoid interference and undercutting during the manufacture of the teeth non-standard cutters with shortened addenda or having rounded-off corners must be used. Both methods are, however, used. A third method is to increase the pressure angle of the teeth; this results in moving the points L and M so that the points R and S are less likely to come outside them; this is shown in Fig. 204.

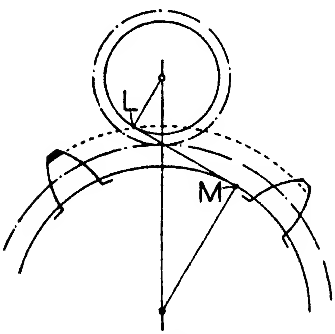


Fig. 203

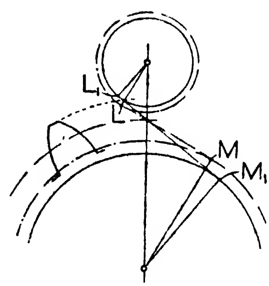


Fig. 204

In the Fellow's stub teeth, as compared with the Brown and Sharpe standard, the risk of interference is lessened by reducing the addenda as well as by increasing the pressure angle. The fourth method is by the use of *corrected teeth*, and these will now be dealt with.

161. Corrected Teeth.—To understand the principle involved imagine the teeth of the two mating wheels to be generated simultaneously by a straight-sided rack by the method described in Art. 141, and as indicated on the left in Fig. 205. Now let this generating rack be shifted away from the centre of the pinion, on which the interference occurs, by an amount y, and at the same time let the blank radius of the pinion be increased and that of the wheel decreased by the same amount y. The teeth generated will then be *corrected teeth*, as shown on the right in Fig. 205, the pinion having its addendum increased and its dedendum decreased, and the wheel having its addendum decreased and its dedendum increased, all by the amount y. The thickness of the pinion teeth has been increased and that of the wheel teeth decreased, but the pressure angle, pitch and base circles and centre distance

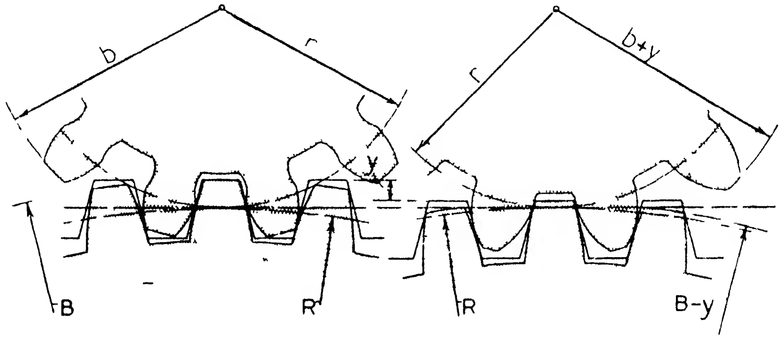


Fig. 205

are all unaltered. If the amount of correction y is sufficient, then the interference will be eliminated and the pinion teeth will be considerably stronger than the uncorrected teeth. The number of pairs of teeth in engagement will be unchanged. The method can only be used when the interference occurs on one wheel only, and then only when the shifting of the generating rack to avoid interference on that wheel does not introduce it on the other. These conditions are only satisfied when a small wheel meshes with a considerably larger one. The method has the advantage that standard cutters can be used.

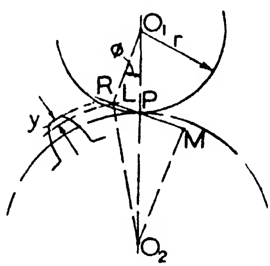


Fig. 206

The necessary correction y may be found graphically by drawing a circle, centre O_2 (Fig. 206), to pass through L and then measuring the radial distance between this circle, and one passing through the tips of the standard teeth. It may be calculated by solving the triangle O_2PL (in which $LP = O_1P \text{ Sin } \phi = r \text{ Sin } \phi$ and is thus known, ϕ being the pressure angle) for O_2L, which is then subtracted from O_2R.

11

Messrs. David Brown & Sons, of Huddersfield, make $y = \frac{k}{\text{D.P.}}$ where k is the *correction coefficient*, and is standardised by them in steps of 0·02. In their book on gear teeth (entitled *Spur, Spiral and Bevel Gearing*) they give charts from which the minimum correction necessary to avoid interference in any combination of wheels may quickly be found.

162. Internal Gears.—When one of the pitch circles rolls on the inside of the other, as in Fig. 207, the larger wheel must have internal teeth, thus becoming an *internal* or *ring* gear. The pinion is an ordinary external gear, and the profiles of the internal teeth are exactly the same as those of an external gear having the same pitch circle and pressure angle. With internal gears, however, the metal is now situated outside the tooth profiles instead of inside, and the addendum is measured inside and the dedendum outside the pitch circle. The line of action is the common tangent (PLM, Fig. 208) to the base circles, passing, of course,

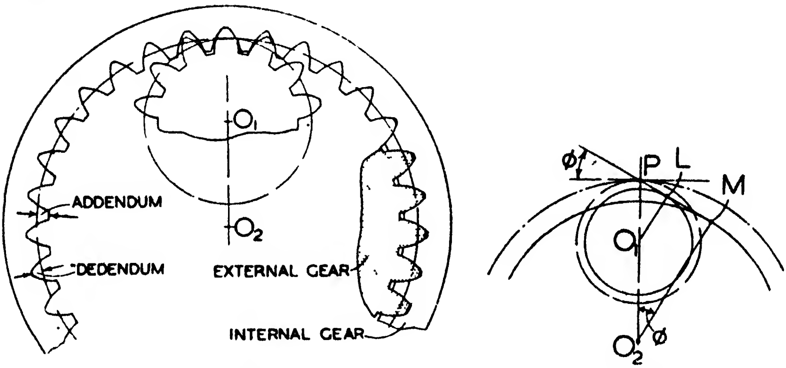


Fig. 207 Fig. 208

through the pitch point. The relations between the pitch circle radii and the base circle radii are unaltered, so that

$$\frac{O_1L}{O_2M} = \frac{O_1P}{O_2P} = \frac{\omega_2}{\omega_1}$$

and

$$O_1L = O_1P \operatorname{Cos} \phi$$

$$O_2M = O_2P \operatorname{Cos} \phi$$

ϕ being the pressure angle.

The chief advantage of internal gearing is that for a given gear ratio and pinion diameter the centre distance between the shafts is much less than with external gearing, being equal to the difference, instead of the sum, of the pitch circle radii. There is also less sliding between the teeth and, since a convex tooth engages with a concave one, the surface stresses in the teeth are less. The disadvantages are that a gear ratio of less than about $2\frac{1}{2}$ to 1 is impracticable, and that it is a little more difficult to arrange the bearings supporting the shafts. The latter difficulty becomes serious when two or more pairs of gears are required between the same pair of shafts, as, for example, in a change-speed gear-box.

Interference occurs on the pinion teeth if the inside circle of the internal gear intersects the line of action on that side of L opposite to P. It can be dealt with as in external gearing. Another form of interference arises if the pinion has too many teeth relative to the wheel, since the paths of the corners of the pinion teeth as the latter rolls inside the wheel then intersect the corners of the wheel teeth, so that unless these were removed during manufacture the gears would foul, and if they are removed the tooth action is affected.

163. Helical-Toothed Spur Gears.—All the figures used to illustrate the preceding articles on toothed gearing are views taken parallel to the axes of the gears, i.e. they are projections on planes perpendicular to those axes, the action of the teeth thus being examined by means of the action between the profiles given by the intersections of the teeth and a surface which is everywhere perpendicular to the pitch surfaces; it has not been necessary to

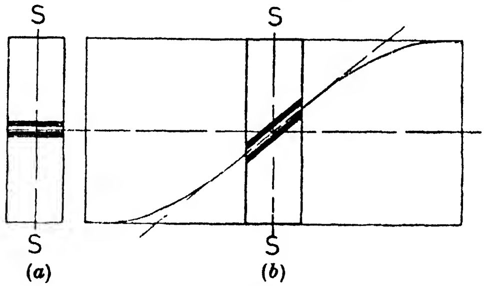


Fig. 209

consider the axial thickness of the gears. When the latter is taken into account, however, it is at once apparent that spur gears are of two types. In the first type the teeth are disposed parallel to the axes of the gears as in Fig. 209 (*a*), while in the second type they form parts of helices as indicated in Fig. 209 (*b*), although the sections of the teeth by the planes SS might be identical. The two types are known as *Straight-toothed* and *Helical-toothed* spur gears or, shortly, as Spur and Helical gears. The latter are sometimes referred to as "spiral" gears, but the term is a bad one and should not be used.

Helical gears are used because quiet, smooth running at high speeds can be more easily obtained than with straight teeth. This is chiefly because the engagement between any two teeth is a gradual process, starting at one end and gradually spreading across the whole tooth, while disengagement is equally gradual; whereas with straight teeth engagement and disengagement occur instantaneously across the whole of the teeth, also, as will be seen later, it is possible to use finer pitches (higher D.P. numbers) with helical teeth than with straight teeth, and this is conducive to smooth running. They are generally supposed to be evolved from the *stepped gear*, originated by Dr. Hooke, which is, in effect, a straight-toothed gear cut into a number of slices which are then turned about their axis so that each one is slightly in advance of the preceding one.

164. Spiral Angle, Real and Normal Pitches.—In Fig. 210 b_1b_1' is the intersection of the helical tooth seen in the end view

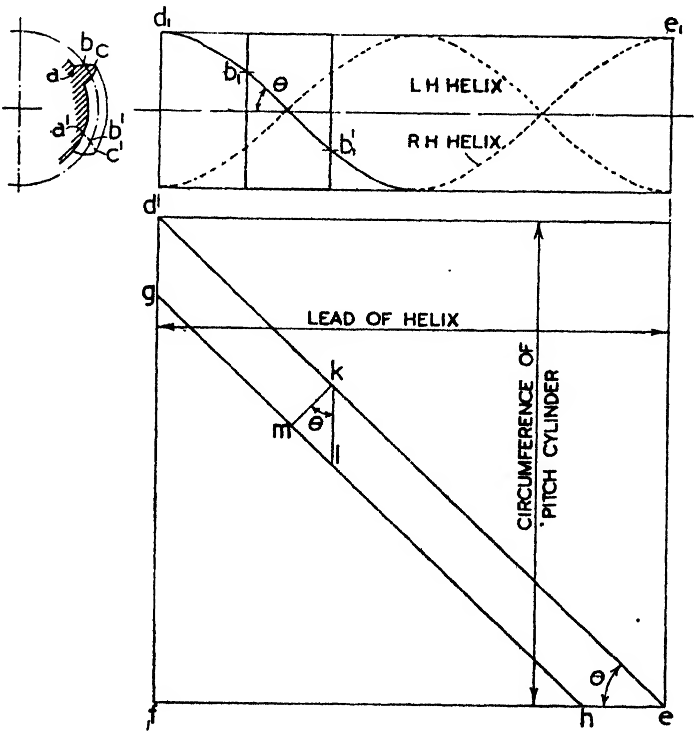


FIG. 210

as *abc*, *a'b'c'* with the pitch cylinder, and is part of the right-handed helix $d_1b_1b_1'e_1$. The angle θ between the axis of the gear and a tangent to this helix at any point is called the *Spiral angle* of the tooth, though a better name would be helix angle, and a little consideration will show that a tooth that is to mate with the tooth *abc* must have a spiral angle also equal to θ and must be of the opposite hand, i.e. left-handed, as indicated by the dotted line, thus *the teeth of mating helical wheels must have the same spiral angles and must be of opposite hands.* If the pitch cylinder be developed out into a plane surface the helix $d_1b_1b_1'e_1$ will appear as the straight line *de*, and in the right-angled triangle *dfe*, *df* is equal to the circumference of the pitch cylinder, while *fe* ($=d_1e_1$) is the distance a point would advance, parallel to the axis of the helix, in moving round the helix for one revolution, and is called the *lead* (pronounced leed) of the helix Then

$$\text{Tan } \theta = \frac{df}{fe} = \frac{\pi D}{L}$$

where D = pitch diameter

L = lead of helix

The tooth next to *abc* would develop in a similar way into the line *gh*, and then *kl*, drawn perpendicular to *ef*, is the (developed) distance, measured along a pitch circle, between two consecutive teeth, i.e. kl = the circular pitch. It is convenient to call *kl* the *Real Circular Pitch* (abbreviated to R.C.P.).

Then $kl = \dfrac{\pi D}{N}$ where N is the number of teeth in the wheel.

The pitch of the teeth can, however, be measured along a line drawn perpendicular to the developed tooth lines *de*, *gh*, being then represented by *km*. The pitch *km* is the development of the pitch of the teeth as measured in the pitch cylinder along a helix that intersects the tooth helix at right-angles, i.e. normally; it is therefore called the *Normal Circular Pitch* or shortly the Normal Pitch. (This normal pitch is an entirely different thing from the normal or base circle pitch mentioned in Arts. 144 and 151, and the distinction must always be borne in mind.)

Then $km = kl \times \text{Cos } \theta$.

Normal Circular Pitch

= Real Circular Pitch × Cosine of Spiral Angle.

Now $\dfrac{N}{D}$ = the Diametral Pitch of the teeth = P

and $$P = \frac{\pi}{\text{Circular Pitch}} = \frac{\pi}{kl}$$

Corresponding to the diametral pitch given by this relation, and which will be called the *Real Diametral Pitch*, there is a diametral pitch given by the relation

$$\text{D.P.} = \frac{\pi}{\text{Normal Circular Pitch}} = P_n$$

this will be called the *Normal Diametral Pitch* (N.D.P.).

Then $$P_n = \text{N.D.P.} = \frac{\pi}{\text{N.C.P.}} = \frac{\pi}{\text{R.C.P.} \times \text{Cos}\ \theta} = \frac{\text{R.D.P.}}{\text{Cos}\ \theta}$$

The Normal Diametral Pitch =

The Real Diametral Pitch ÷ Cosine of Spiral Angle.

Now, in order that the same cutters may be used to cut helical teeth as are used for straight teeth, it is frequently necessary to make the R.D.P. a standard pitch, and this is usually done by adjusting the spiral angle to a suitable value. The spiral angle is otherwise settled either quite arbitrarily or, in conjunction with the axial thickness of the gear, is chosen to give an arbitrary number of teeth always in engagement at the pitch point. This latter consideration will now be dealt with.

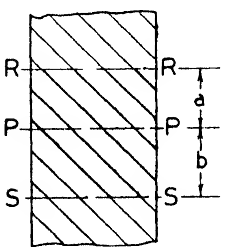


FIG. 211

The pitch point is merely the end view of the line of contact of the pitch cylinders, and this is shown in Fig. 211, as PP. At every point of intersection of a tooth line with PP there is a pair of teeth in engagement at the pitch point; thus in the figure there are four such engagements, and the number can be increased at will by increasing the axial thickness of the gear. Let the pitch of the tooth lines measured along PP be called the *axial pitch* and be denoted by A. Then

$$\text{No. teeth in engagement at the pitch point} = \frac{\text{PP}}{\text{A}}$$

$$= \frac{\text{PP Tan}\ \theta}{\text{R.C.P.}}$$

$$= \frac{\text{PP Sin}\ \theta}{\text{N.C.P.}} = \frac{\text{PP Sin}\ \theta \times \text{N.D.P.}}{\pi}$$

since $$\text{A} = \frac{\text{R.C.P.}}{\text{Tan}\ \theta} = \frac{\text{N.C.P.}}{\text{Sin}\ \theta} = \frac{\pi}{\text{N.D.P. Sin}\ \theta}$$

If a line RR is drawn at a distance a equal to the length of the arc of approach, and another line SS at a distance b equal to the arc of recess, then all the teeth or portions of teeth whose tooth lines are between RR and SS are in engagement.

165. Example of Calculation of Helical-Toothed Gears.—Let two wheels be required to work at a centre distance of 18 in. with a gear ratio of 5 to 1 and to be cut with a 5 D.P. cutter at a spiral angle of about 23°, and let the number of teeth in engagement at the pitch point be required to be 4.

Let D = P.C.D. of wheel N = No. teeth in wheel
 d = ,, ,, pinion n = ,, ,, pinion

Then
$$\frac{d}{D}=\frac{1}{5}$$

and the centre distance $=\dfrac{d+D}{2}=18$ in

$$\therefore\ 3d=18 \quad d=6 \text{ in.}$$
$$\therefore\ D=30 \text{ in.}$$

Now the N.D.P. must be 5 to enable a 5 D.P. cutter to be used; hence R.D.P. = N.D.P. × Cos spiral angle

$$=5\times 0{\cdot}9205=4{\cdot}6025 \text{ approx.}$$
$$\therefore\ n=6\times 4{\cdot}6=27{\cdot}6, \text{ say } 28$$
$$N=5n=140.$$

The actual R.D.P. is thus

$$\tfrac{28}{6}=4{\cdot}666'$$
$$\therefore\ \text{N.D.P.}=5=\frac{\text{R.D.P.}}{\text{Cos }\theta}=\frac{4{\cdot}666'}{\text{Cos }\theta}$$
$$\text{Cos }\theta=\frac{4{\cdot}666'}{5}=0{\cdot}933\dot{3}'$$
$$\theta=21^\circ\ 2\tfrac{1}{3}'.$$

$$\text{The axial pitch}=\frac{\pi}{\text{N.D.P. Sin }\theta}=\frac{\pi}{5\times 0{\cdot}3914}=1{\cdot}6 \text{ approx.}$$

∴ Face width of wheel $=3\times 1{\cdot}6=4{\cdot}8$ in., say 5 in.

Thus the cutting data for the wheels are: pitch circle diameters = 6 in. and 30 in., numbers of teeth = 28 and 140, spiral angle = 21° $2\frac{1}{3}'$, face width of wheels = 5 in., blank diameters = 6·4 in. and 30·4 in. (for B. & S. tooth proportions, since the addenda $=\dfrac{1}{\text{N.D.P.}}=\frac{1}{5}$ in.).

166. Double Helical-Toothed Gears.—When helical-toothed gears are transmitting power the force between the teeth has an axial component tending to move the gears along their axes, and which necessitates the provision of suitable thrust bearings. To avoid this necessity double helical teeth are used, the axial components of the tooth forces then cancelling each other. The two halves of the teeth may be one continuous piece of metal or may be separated by a gap at the centre or be made separately and bolted together. The spiral angles of the two halves are of course equal. To ensure equal contact between the two halves of the teeth of wheel and pinion the latter (usually) is left free axially, being then positioned by the contact between the teeth.

EXERCISES XII

1. With gears having involute teeth what is the essential condition that must be satisfied if consecutive teeth are to engage properly ?

2. A gear has 20 teeth of 5 D.P. What is the pitch circle diameter ? If the teeth are B. & S. standard, what is the outside diameter ?

3. Two parallel shafts are to be connected by gears of 5 D.P. with a gear ratio of 4 to 1. If the centre distance between the shaft axes is 10 in., find the numbers of teeth in the wheels, their pitch circle diameters and, assuming B. & S. standard tooth proportions, their blank diameters.

4. Two shafts are to be connected by spur gears, the gear ratio and centre distance being respectively $2\frac{1}{2}$ to 1 and $7\frac{1}{2}$ in. as nearly as possible. If the D.P. is 8, find the best numbers of teeth for the wheels and the exact centre distance or gear ratio.

5. Describe the methods by which interference in gear teeth may be obviated.

6. Prove that Base Circle Pitch = Circular Pitch × Cos (Pressure angle).

7. Two wheels having respectively 20 and 70 teeth mesh together at a centre distance of $4\frac{1}{2}$ in. with a pressure angle of $14\frac{1}{2}°$. If the centre distance is increased to 4·6 in., find the new pressure angle.

8. Two spur gears, having 30 and 50 teeth respectively, mesh together at the standard centre distance corresponding to a D.P. of 10. If the teeth are to B. & S. standard proportions, find (*a*) the length of the path of contact, (*b*) the base circle pitch and (*c*) the maximum and minimum numbers of teeth in contact at any moment.

9. Find the number of teeth in the smallest wheel that can mesh, without interference occurring, with a wheel having 60 teeth if the teeth are of B. & S. standard proportions.

10. Repeat Question 9, but take the teeth to be Fellow's stub tooth standard and of $^6/_8$ pitch.

11. Two wheels having respectively 30 and 50 teeth of B. & S. standard proportions mesh together at the standard centre distance of 4 in. If the smaller wheel is the driver and rotates at 1000 r.p.m., find the speed of sliding between a pair of teeth when they first come into contact.

12. A wheel having 30 teeth is cut with a cutter of 10 D.P. and $14\frac{1}{2}°$ pressure angle, at the standard centre distance. It is meshed with a rack having straight-sided teeth at an angle of 70° to the pitch line. If the wheel rotates at 60 r.p.m., find the linear speed of the rack.

13. Find the amount of correction necessary just to obviate interference when a wheel having 60 teeth meshes, at the standard centre distance, with a pinion having 15 teeth, the D.P. being 6 and the tooth proportions being B. & S. Find also the blank diameters of the corrected gears.

14. Two helical-toothed gears have respectively 20 and 60 teeth, the normal diametral pitch being 10 and the spiral angle 20°. Find the centre distance between the shaft axes and the blank diameters of the gears if B. and S. tooth proportions are used. If the axial thickness of the gears is 3 in., find how many teeth are always in engagement at the pitch point.

15. Using the relevant data of Question 14, find the value of the spiral angle if the gears are required to work at a centre distance of 6 in. exactly.

CHAPTER XIII

(This chapter may be omitted on a first reading.)

CYCLOIDAL TEETH. SPECIAL FORMS OF GEAR

167. Definitions.—A $\left\{\begin{array}{l}\textit{Cycloid}\\ \textit{Epicycloid}\\ \textit{Hypocycloid}\end{array}\right\}$ is the curve described by any point of a circle which rolls without slip on $\left\{\begin{array}{l}\text{a straight line}\\ \text{the outside of a circle}\\ \text{the inside of a circle}\end{array}\right\}$.
Fig. 212 shows the three types of curve. The epicycloid EPI may be described by a point of either the full line rolling circle r (dia. d)

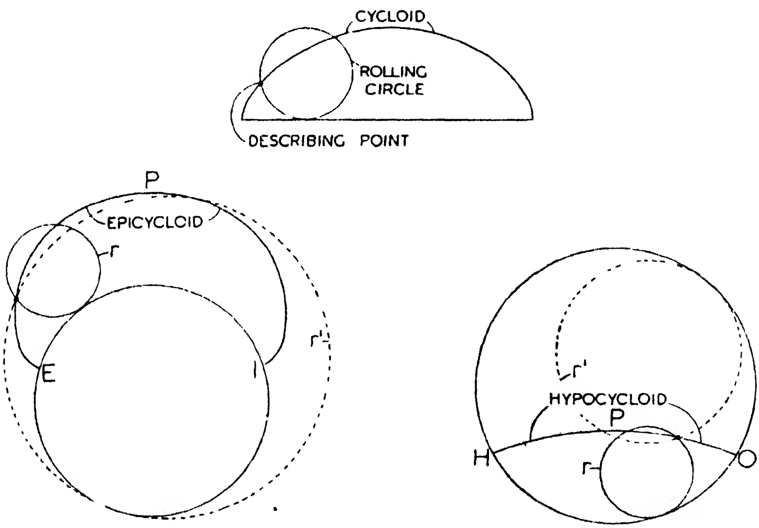


Fig. 212

or the dotted rolling circle r' (dia. d'), the former touching the fixed circle (dia. D) externally and the latter internally, while

$$d' = d + D \quad \text{or} \quad D = d' - d.$$

If the diameter of the rolling circle be made infinitely large, the rolling circle becomes a straight line and the epicycloid becomes an involute. If the diameter of the rolling circle r be made zero, the rolling circle becomes a point and the epicycloid becomes a point also.

Similarly the hypocycloid HPO may be described either by the rolling circle r (dia. d) or the rolling circle r' (dia. d') and

$$D = d + d'$$

If d is made equal to $\frac{D}{2}$, the hypocycloid becomes a straight line, a diameter of the fixed circle. If $d = o$ (or $d' = D$), then the hypocycloid becomes a point.

168. Cycloidal Teeth.—In Fig. 213 O_a and O_b are the centres of the pitch circles A and B of two wheels, P is the pitch point and r is a rolling circle which when rolled on the inside of A gives a hypocycloid, a portion eQf of which is used as the outline of a tooth of A, while when r is rolled on the outside of B it gives an epicycloid, a portion gQh of which is used as the outline of a tooth of B. The two teeth are in contact at Q. Let the two pitch

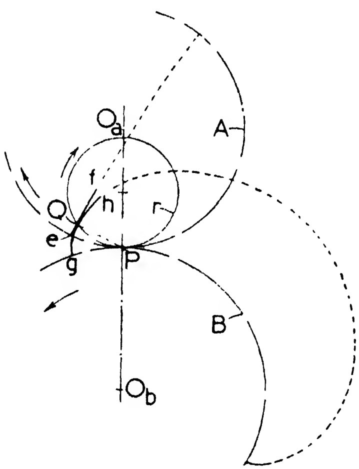


FIG. 213

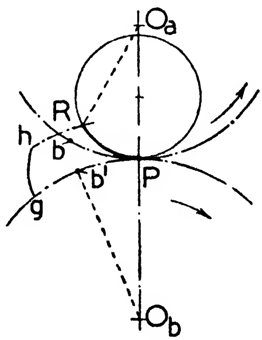


FIG. 214

circles and the rolling circle be revolved about their respective centres so that all three roll together without slip, then Q will trace out the two tooth outlines simultaneously. It follows that the point of contact between the teeth is always somewhere on the circumference of the circle r, and that circle is thus the *path of contact*. That the teeth will satisfy the fundamental condition for the transmission of motion with a constant velocity ratio will be clear when it is observed that at any instant P is the instantaneous centre of the rolling circle r relative to either pitch circle, and hence PQ is the normal to the epicycloid eQf and to the hypocycloid gQh at Q; PQ is thus the common normal to the teeth at their point of contact and it always passes through the pitch point P.

The first point of contact (for the directions of rotation shown) is at R (Fig. 214), the point of intersection of a circle, centre O_b and passing through h, the tip of the tooth gh, with the path of contact. Clearly the last point of contact is at P. Thus contact lasts while the point of contact between the teeth moves round the arc RP. During this time both pitch circles will move through

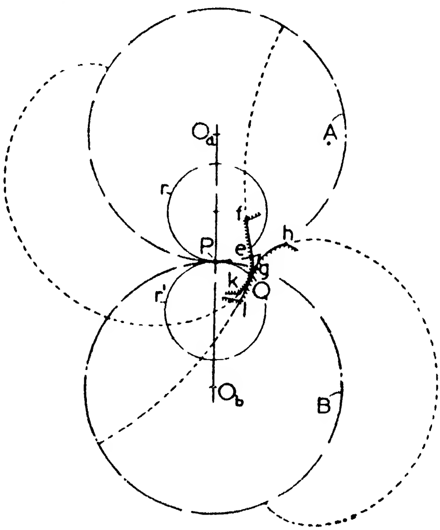


FIG. 215

angles subtending arcs equal to RP, thus the arcs Pb and Pb' (made equal to RP) are the arcs of approach and $\angle bO_aP$ and $\angle b'O_bP$ are the (unequal) angles of approach. If the directions of rotation are reversed the action is wholly during recess and Pb and Pb' are the arcs of recess. With teeth as shown, therefore, the action is wholly during either approach or recess; to obtain action during both approach and recess the teeth must be made to project outside as well as to lie inside the pitch circles. To do this a second rolling circle r' must be used, as shown in Fig. 215. This is arranged to roll outside A and inside B and portions eQk and gQl of the resulting epicycloid and hypocycloid are used as the continuations of the teeth of A and B respectively. The circle r' need not be the same size as r. Thus one rolling circle generates the face of the tooth of one wheel and the flank of the tooth of the other wheel, while the second rolling circle generates the flank of the tooth of the first wheel and the face of that of the second wheel.

169. Interchangeable Wheels must have a Common Rolling Circle.—Suppose any two of three wheels A, B and C have to mesh together, then since the faces of the teeth of A will engage the flanks of the teeth of B and C, it follows that the latter must be generated by equal rolling circles, and since the flank of B engages with the face of C, these must be generated by equal rolling circles; hence it follows that the face and flank of the teeth of C must be generated by equal rolling circles and similarly for A and B. In practice the diameter of the rolling circle common to a set of interchangeable wheels was usually made equal to the radius of the pitch circle of the smallest wheel in the set. The flanks of the teeth of that wheel were then radial lines.

170. The Condition for Continuity of Action.—For continuity of action a second pair of teeth must come into contact at least just as the previous pair go out of contact, and a little consideration will show that this implies that the circular pitch of the teeth must be not greater than the sum of the arcs of approach and recess.

171. The Line of Action and the Pressure Angle.—The common normal at the point of contact between the teeth is the direction in which the force between the teeth acts (excluding friction) and may be termed the line of action. Clearly it is not fixed in position, but changes as the point of contact between the teeth changes. The angle between the line of action and the common tangent to the pitch circles is the pressure angle, and this also is a variable quantity, its maximum values being at the beginning and end of the contact, while when the teeth are in contact at the pitch point its value is zero. The maximum value of the pressure angle in cycloidal teeth may be made greater than the constant value adopted in involute teeth.

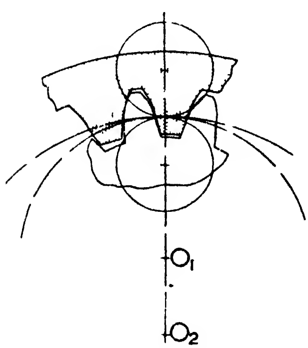


Fig. 216

172. Internal Teeth.—Fig. 216 shows a pinion meshing with an internally toothed wheel. The face of the tooth of the latter is now inside the pitch circle and the flank outside. The mating portions of the teeth are now both epicycloids or both hypocycloids. The path of contact is composed of those portions of the rolling circles that are cut off by the addendum circles as with external gears.

173. Secondary Contact.—If, keeping the size of the pinion fixed, the size of the annular wheel is gradually reduced, a point will be reached when what is called *secondary contact* will occur. This secondary contact arises because of the fact, mentioned in Art. 167, that any epicycloid or hypocycloid can be generated by two different rolling circles. In Fig. 217 O_1 and O_2 are the

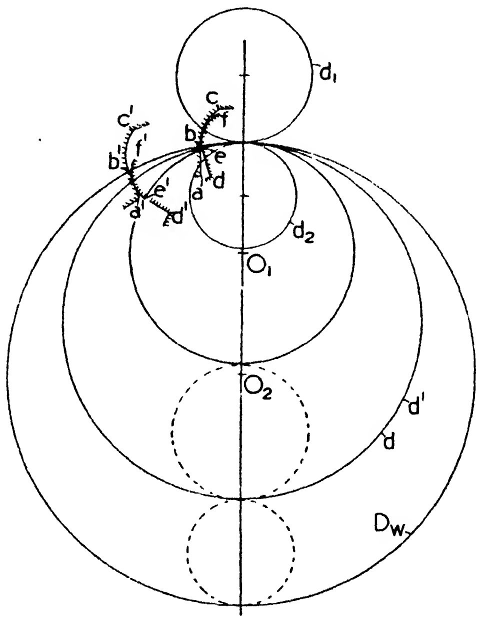


FIG. 217

centres of the pitch circles of the wheels. The tooth outline *abc* of the tooth of the wheel is obtained by rolling d_1 on the outside and d_2 on the inside of the pitch circle giving respectively the flank *bc* and the face *ab*. The latter, however, could equally well be generated by the rolling circle *d*, provided that the condition $d+d_2$=pitch circle diameter of wheel$=D_w$ is satisfied. The same rolling circles d_1 and d_2 are, of course, used to generate the tooth outline *def* of the pinion, d_2 giving the flank *de* and d_1 the face *ef*. The latter, however, could equally well be generated by the rolling circle d' provided that the condition $d'-d_1$=pitch circle diameter of pinion=D_p is satisfied. And if d and d' are made to coincide, then the *faces ab* and *ef* will both be generated by the same rolling circle and will mate together. For this to happen we must have

$$d=d'=D_p+d_1=D_w-d_2$$

$$\therefore D_w-D_p=d_1+d_2$$

and
$$O_1O_2=\frac{D_w-D_p}{2}=\frac{d_1+d_2}{2}$$

Secondary contact is shown between the teeth $a'b'c'$ and $d'e'f'$; it will commence at the point of intersection of the addendum circle of either the pinion or the wheel and the path of secondary contact (the rolling circle d), whichever intersection is nearer to the pitch point, and it will continue (theoretically) up to the pitch point. It follows that so soon as primary contact commences there will be (theoretically) two contacts between the teeth. Since secondary contact is between the faces of the teeth and primary contact is between face and flank, if the flanks of the teeth of both pinion and wheel are made to lie inside the hypocycloids giving the primary contact, the latter will be eliminated and secondary contact alone will remain.

If the centre distance between the wheels is reduced beyond the value given by $O_1O_2=\frac{d_1+d_2}{2}$ (which gives secondary contact), then interference will occur between the faces of the teeth and the gears will not turn.

174. Pin Gearing.—Pin teeth may be regarded as being an arbitrary tooth shape for which the conjugate tooth can be found by the envelope method of Art. 140. Let A (Fig. 218) be the pitch circle of a pinion having pin teeth, one of which is shown (shaded). The outline of the teeth of the mating wheel B must then be the envelope of the circles representing the successive positions of the pin as the pitch circle A rolls on that of the wheel B. Clearly the path of the centre C of the pin is the epicycloid *dCe* and the wheel tooth outline is therefore the envelope *fg* of circles equal in diameter to the pin and having their centres on *dCe*. In general this envelope will be of the form shown at *klm*, coming down inside the pitch circle to *l* and then up again to *m*. The portion *lm* will therefore be eliminated by interference. Reference should be made to Art. 249, dealing with interference in cams.

The path of contact is the locus of the point of intersection *q* (Fig. 219) of the circle representing the pin in any position and the line CP joining the centre of the pin to the pitch point. CP is the common normal at the point of contact between the teeth and also, of course, is the normal to the epicycloid described by the centre C. The path of contact thus lies inside the pitch circle of the pinion ; it may conveniently be traced by means of a template consisting of a straight-edge having the points C and q marked on

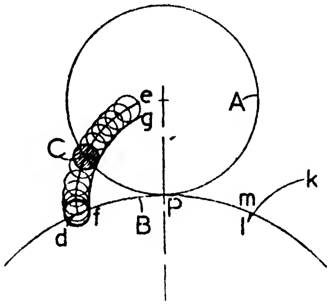


FIG. 218

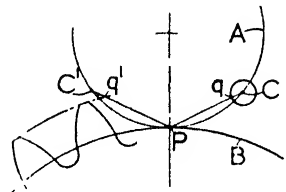


FIG. 219

it. This template is placed with C lying on the pitch circle A and its edge passing through P and then q is marked off, the process being repeated in a number of different positions. The same template will serve to determine the first (or last) point of contact q' by setting it so that C lies on the pitch circle A, q on the addendum circle of B and the edge passes through P. The last (or first) contact would occur when the pin centre reached P were it not for the interference mentioned in the preceding paragraph. Because of that interference it actually occurs when the pin centre is slightly nearer to C''. The arc of approach or recess is the arc PC'', and this must be not less than the circular path of the teeth.

Pin teeth may also be derived from cycloidal teeth by using particular sizes of rolling circles. Thus let the rolling circle used to generate the faces of the teeth of B be made equal in diameter to the pitch circle A. The hypocycloidal flanks of the latter will then degenerate into points lying on the pitch circle, and these points will engage the epicycloidal faces of B, being actually the tracing points of the latter. Since actual points cannot be used, they are expanded into pins and the teeth of B are made " parallels " to the original epicycloids.

Pin teeth are used fairly extensively in clocks and mechanisms where the loads and speeds are low ; they are usually arranged with the pin teeth driven, since then the action is wholly during recess and is smoother than when it is wholly during approach.

If the diameter of the pin wheel is made infinite the pitch circle becomes a straight line and the pin wheel becomes a pin rack as shown in Fig. 220. The epicycloids described by the pin centres become *involutes* of the *pitch* circle B and the teeth of B are parallels to those involutes.

Pin gearing can be arranged as internal gearing, the wheel teeth then becoming parallels to hypocycloids. If the gear ratio is made 1 to 2 these hypocycloids become straight lines, diameters of

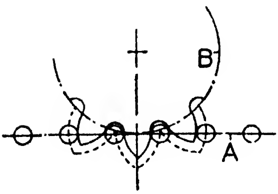


Fig. 220

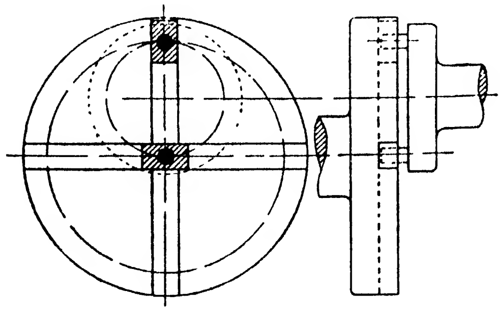


Fig. 221

the pitch circle of the wheel, and this enables blocks to be pivoted on the pins as shown in Fig. 221, where only two pins are employed. The teeth of the wheel then take the form of grooves cut in the face of a disc. Since surface contact is now obtained, the mechanism can be designed to deal with heavy loads. It is used, epicyclically, in the " Burn " reduction gear which has been successfully used in many land and marine applications. The mechanism has also been used as a camshaft drive in motor-car engines. It should not be confused with the Oldham coupling (Art. 83).

CHAPTER XIV

(Articles 176 to 183 of this chapter may be omitted on a first reading.)

BEVEL GEARING

175. As with spur gearing so with bevel gearing it is convenient to begin with the equivalent friction gearing. To transmit motion from one shaft to another, when their axes intersect, conical friction wheels must be used, as shown in Fig. 222.

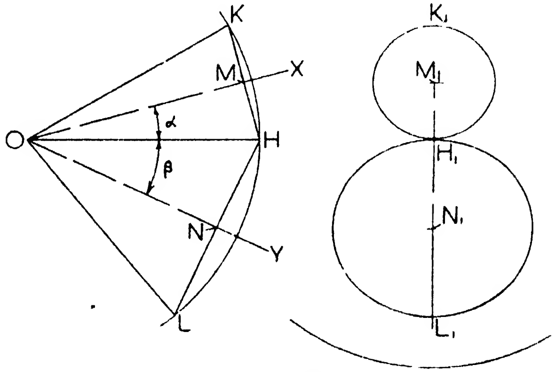


Fig. 222

If the velocity ratio between the shafts is variable, then the wheels will be cones in the general sense, but if, as is usually required, the velocity ratio is to be constant, then the cones must be circular. These cones are of course the axodes. Imagine a sphere to be described with the intersection O of the axes as centre; this sphere will intersect the two cones in the circles HK and HL, which are in contact at H. If the cones roll together without slip, then the circles HK and HL also roll without slip. Hence if ω is the angular velocity of the cone OHK about its axis OX and Ω that of OHL about it's axis OY, we have,

$$\frac{\omega}{\Omega}=\frac{HL}{HK}=\frac{HN}{HM}=\frac{OH\ \mathrm{Sin}\,\beta}{OH\ \mathrm{Sin}\,\alpha}=\frac{\mathrm{Sin}\,\beta}{\mathrm{Sin}\,\alpha} \qquad (1)$$

where α and β are the semi-apex angles of the cones. Also if θ is the angle between the shafts

$$\alpha+\beta=\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These two equations enable α and β to be determined when the gear ratio $\frac{\omega}{\Omega}$ and the angle θ are given.

In practice frustrums only of the cones would be used, but as such friction gearing suffers from the same limitations as that between parallel shafts, it is little used. Instead, teeth are formed on the cones, thus giving bevel gearing. Now, in examining the action of spur gear teeth the intersections of the teeth with a surface which intersected the pitch surfaces everywhere at right-angles were used, and a similar method is adopted with bevel gear teeth. The surface which intersects the pitch surfaces everywhere at right angles is clearly a sphere having its centre at the intersection O of the axes. Such a sphere may be called a *sphere of reference*, and provided that everything we do is done, or imagined to be done, on the surface of this sphere, we can apply most of what has been done in connexion with spur gears to bevel gears also. For the present we shall confine our attention to constant-velocity bevel gearing and shall not consider the forms of the teeth in the directions of the axes.

176. The Fundamental Condition to be Satisfied by Bevel Gear Teeth.—This is the same as with spur gears—namely, that at any point of contact between the tooth surfaces the common normal to those surfaces shall intersect the instantaneous axis. Referring to Fig. 222, it follows that in the view along the line HO, which is the instantaneous axis, the common normal at any point of contact will appear as a line which must pass through the point H_1; this point on the surface of the sphere of reference may thus be called the *pitch point* by analogy with spur gears.

This line, the projection of the common normal, is also the projection of a great circle passing through H, The condition may therefore be stated thus: In any section of a pair of bevel gear teeth by a sphere having its centre at the intersection of the axes, a great circle of that sphere, drawn through the point of contact of the teeth, and whose plane contains the common normal to the tooth outlines, must pass through a fixed point on that great circle of the sphere which joins the axes.

177. The Possible Shapes for Bevel Gear Teeth.—As with spur gears so with bevel gears any shape may be chosen for one of a pair of mating teeth and the conjugate tooth may be found by the envelope method, provided this is carried out on the surface of the sphere of reference. Since the surface of a sphere cannot be developed on to a plane, the process cannot be performed practically, but only in theory; again, however, the basic principle underlying the action of bevel gear generating machines is this envelope method of deriving the conjugate tooth.

Suitable tooth shapes may also be found by a method which is analogous to that described in Art. 141, and it is useful to describe this method. Thus let *ab* and *cd* (Fig. 184) be the intersections of the centrodes with the sphere of reference; these curves may by analogy be called the pitch lines, and in the case of constant velocity gearing will be circles. Let them be in contact at P and let *xy* be any curve lying in the surface of the sphere of reference and being in contact with both *ab* and *cd* at P. Let *lmn* be a curve lying in the sphere of reference and fixed to *xy* at *m*. Then regarding *lmn* as a tooth outline, the conjugate tooth of *ab* may be found by the envelope method; let it be the curve *st* (Fig. 185). Similarly the conjugate tooth of *cd* may be found; let it be the curve *uv* (Fig. 186). Then, since both *st* and *uv* could be generated simultaneously by rolling all three curves *ab*, *xy* and *cd* together without slip and so that they are all in contact at a single point at all times, it follows that *st* and *uv* are conjugate tooth outlines. If the curve *xy* be made a portion of a great circle of the sphere of reference, and if the curve *lmn* be made a portion of a second great circle of the sphere of reference, then the resulting tooth outlines will be of the type that is actually used for bevel gear teeth at the present time. Such teeth are known by the name *Octoid teeth.* The cutter corresponding to the portion of a great circle is actually a straight-sided one, and thus one cutter can be used to cut both of the wheels. When it is remembered that the pitch lines *ab* and *cd* are merely the intersections of the pitch cones or axodes with the sphere of reference it will be seen that the curve *xy* is the intersection of some conical surface, having its apex at the centre of the sphere of reference, with that sphere. When *xy* is made a portion of a great circle the conical surface whose trace it is becomes a flat disc whose centre coincides with that of the sphere of reference. Similarly the curve *lmn* when it is made a portion of a great circle is the trace of a similar disc. Fig. 223 shows in perspective the relative disposition of the pitch cones and the discs corresponding to *xy* and *lmn*. As the pitch cones turn about their axes OX and OY so the disc O*xy* turns about its axis VV, and the portion O*lmn* of another disc fixed to the disc O*xy* generates the tooth surfaces of the two wheels.

It should be clear that the angle between the discs O*lmn* and O*xy* is the pressure angle of the resultant teeth, which angle is thus constant.

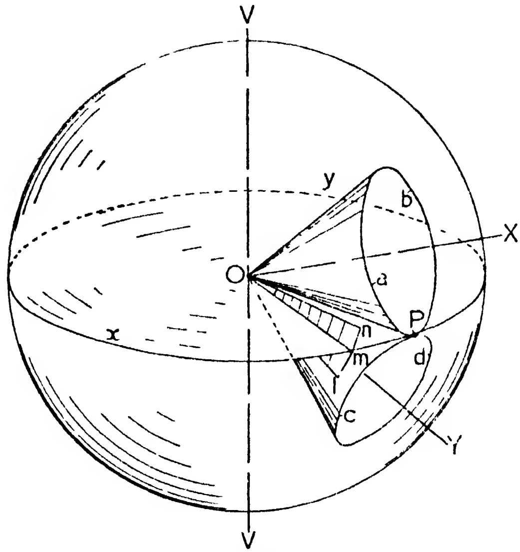


FIG 223

178. Roulettes as Bevel Gear Teeth Shapes.—No difficulty should be experienced in seeing that the process described in Art. 144, and which was there supposed to be carried out on a plane surface, can be carried out on the sphere of reference, thus giving pairs of conjugate bevel gear tooth outlines. In particular when the rolling curves *fg* are made circles the resulting tooth outlines are similar to the family of cycloidal curves; they may be called spherical cycloidal curves. The rolling curves are, of course, only the traces on the sphere of reference of rolling cones which, when the curves are circles, are circular cones. Fig. 224 indicates in perspective the process of generation of cycloidal bevel gear teeth. OX and OY are the axes of the pitch cones OHK and OHL, and OZ the axis of the rolling cone OH*f*. All three of these cones are in contact along the line OH. If now all three cones are revolved about their respective axes in such a manner that they all remain in contact along OH and that no slip

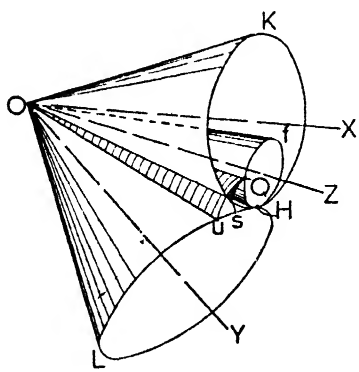


FIG. 224

occurs, then any line OQ of the rolling cone will trace out the spherical epicycloidal tooth surface OuQ relative to the pitch cone OHL, and the same line will trace out the spherical hypocycloidal tooth surface OsQ relative to the pitch cone OHK. The two tooth surfaces will be in contact along the line OQ, and since OH is the instantaneous axis of the rolling cone relative to either pitch cone, it should be clear that the normals to both tooth surfaces at any point of OQ must pass through OH, and the fundamental condition is satisfied. To obtain tooth surfaces that lie partly inside and partly outside the pitch cones two rolling cones must be used; these need not be equal cones unless the teeth belong to wheels any one of which must mesh with any other, in which case all the rolling cones generating the teeth must be equal. Some years ago cycloidal bevel gear teeth were much used and they are still used to some extent.

179. Bevel Tooth Outlines by means of Secondary Centrodes.—By an analogous method to that described in Art. 143 it is possible to derive, from the axodes corresponding to the relative motion of two bodies that revolve about fixed intersecting axes, secondary axodes. These, like the primary axodes, will be cones, in the general sense, having their apexes at the intersection of the axes. Let the curves A and B (Fig. 225) be the traces, on any sphere of

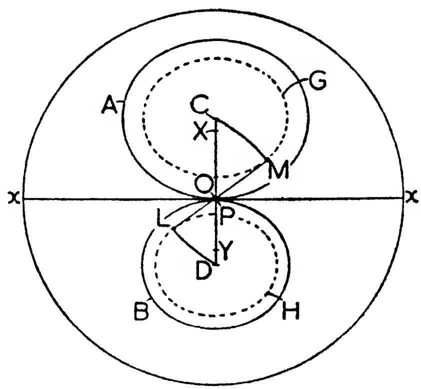


FIG. 225

reference, of the primary axodes which are in contact along a radius OP of that sphere. The view being along the radius OP, the latter is seen as a point. Then at each point of the line OP the axode A will have a definite centre of curvature and these centres will lie on an axis of curvature which will be a radius OC of the sphere of reference and will intersect that sphere in some point C. Similarly along the line OP the axode B will have an axis of curvature OD which will intersect the sphere of reference in D. It should scarcely need pointing out that OC, OP and OD are co-planar. Also through OP may be erected the common tangent plane of the axodes A and B ; this plane will intersect the sphere of reference in a great circle which is seen as the line xx, and may be thought of as a disc bounded by that circle and with its centre at O. Through OP erect a plane OLM making any convenient angle ϕ with the plane Oxx. This plane will intersect the sphere of reference in a great circle which is seen as the line LPM and may also be thought of as a disc. Through the axes of curvature OC and OP erect planes perpendicular to the plane OLM. These planes will intersect the sphere of reference in great circles of which the portions CM and DL are shown, and they will intersect the plane OLM in lines OL and OM. Then these lines are lines of the secondary axodes given by the plane OLM. By turning the primary axodes through small angles and repeating the whole process the secondary axodes may be derived as conical surfaces, whose traces on the sphere of reference will be curves such as G and H. They have the property that if they are supposed to roll together with the disc OLM without slip then the relative motion between them will be exactly the same as that between the primary axodes. If the traces on the sphere of reference of the primary axodes, i.e. the curves A and B, are called the primary centrodes, then the corresponding traces of the secondary axodes, i.e. the curves G and H, may be called the secondary centrodes. Also the portion LM of the trace of the disc OLM on the sphere of reference may be thought of as an inextensible string that is wrapped round the secondary centrodes and which passes from the one to the other along a portion LM of a great circle of the sphere of reference. The relative motion of the secondary centrodes is then communicated by this string. If any point of this string be chosen as a tracing point, then as the secondary centrodes revolve about the fixed axes this tracing point will trace out, on the surface of the sphere of reference, a pair of conjugate tooth outlines. These outlines will always be in contact at the point of LM that is occupied by the tracing point at the instant under consideration. Thus LM is the path of contact. When the velocity ratio is constant the primary and secondary axodes are circular cones and the corresponding centrodes are circles. The tooth outlines are then spherical involutes. These outlines are of course only the traces on the sphere of reference of corresponding tooth surfaces which may be obtained by joining every point of the outlines to the centre of the sphere of reference. It is, however, instructive to derive these tooth surfaces directly instead of first deriving their traces on the sphere of reference. In Fig. 226, OGM and OHL are the secondary axodes and $OaLMb$ is a portion of the disc that rolls together with them to give the required relative motion. OQ is any line drawn on the disc. Then as the axodes and the disc are rolled together OQ will sweep out the tooth surfaces Ost and Ouv.

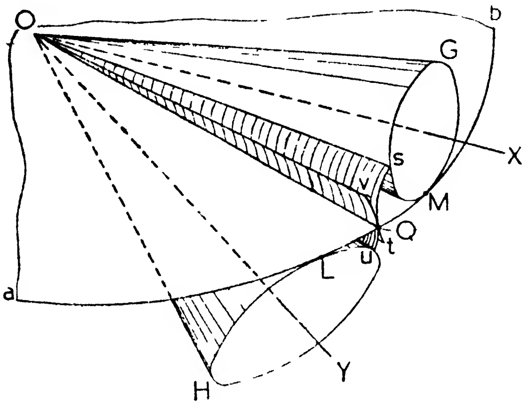


FIG. 226

These surfaces will always be in contact along the line OQ wherever that line may be. Thus contact is always along a line lying in the common tangent plane of the secondary centrodes. When the latter are circular cones they are called the *base cones*, and the reader should have no difficulty in proving that $\frac{r_b}{R_b}=\frac{r}{R}$ where r_b and R_b are the radii of the base circles given by any sphere of reference and r and R are the pitch circles given by the same sphere. Some people find it easier to imagine a flexible sheet wrapping off one base cone and on to the other, as those cones revolve, rather than to imagine a disc rolling with the cones as described above. The line OQ drawn on this flexible sheet will then describe the spherical involute tooth surface.

The spherical involute tooth outlines obtained in this way are very similar to the octoid tooth outlines obtained by the method of Art. 177, and many writers state that bevel gear teeth as used at the present are spherical involute teeth, but that is not so. The spherical involute tooth is not quite the same as the octoid tooth, although the difference is usually extremely small, and modern bevel gear teeth are undoubtedly octoid teeth.*

* In motor-car axles bevel gears are now being used in which the teeth of the wheel (which is not a true crown wheel) are made straight-sided and the teeth of the pinion are generated to the conjugate shape. Such teeth are neither involute nor octoid.

180. Crown Wheels.—As the semi-apex angle of the pitch cone or axode of a bevel gear is gradually increased from zero up to 90°, so the pitch cone itself gradually changes from a mere line to a flat disc. A bevel wheel whose pitch cone is a flat disc is called a crown wheel, and it corresponds, in bevel gearing, to the rack in spur gearing. The name arises from the resemblance of such a wheel to a crown, and it is often misapplied to any large bevel wheel whose semi-apex pitch cone angle approaches 90°. The term should be, however, and in this book is, restricted to wheels having discs as pitch cones. It will shortly be seen that the traces on a sphere of reference of the teeth of crown wheels when of the spherical involute form are very nearly portions of great circles, and it should be clear from a consideration of Art. 177 that in octoid teeth the traces of the teeth of a crown wheel are definitely made portions of great circles.

181. Definitions.—Having settled on the form of tooth outline to be used, it becomes necessary to consider such matters as continuity of action, the number of teeth in mesh, interference, etc., and as these things depend upon the proportions adopted for the teeth, it is necessary to deal with these. It will be convenient, however, to give definitions of some of the practical terms used in connexion with bevel gears. Let HKVW (Fig. 227)

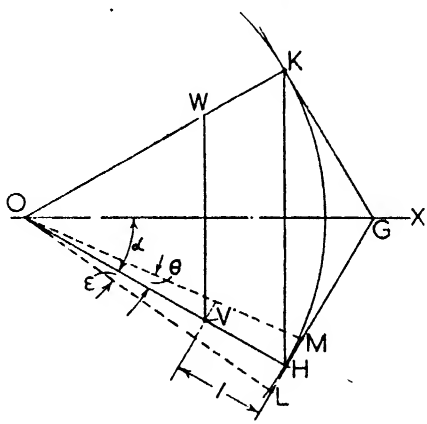


FIG. 227

be a frustrum of a cone, axis OX, that forms the pitch cone of a bevel gear. Then the circle HK is taken as the *pitch circle.* Let the number of teeth in the wheel be N, then $\frac{N}{HK}$ is the *diametral pitch,* HK being measured in inches. As with spur gears, this is usually made a whole number. The pitch of the teeth measured round the pitch circle is the *circular pitch* (C.P.) and clearly $C.P.=\frac{\pi}{D.P.}$. The semi-apex angle α is called the *pitch cone angle* and the length OH the *cone distance*. Clearly OH is the radius of the sphere of reference that intersects the pitch cone in the pitch circle HK. Draw HG and KG perpendicular to OH and OK respectively. Then GKH represents a cone, called the *back cone*, which intersects the pitch cone normally in the pitch circle, and it should be clear that this back cone is tangent to the sphere of reference that intersects the pitch cone in the pitch circle HK. The teeth of the gear will extend, on the pitch cone, from H to V, and this length l is called the *face width*. Let HL, measured on the surface of the back cone, be the height of the teeth above the pitch cone, then HL is the addendum; similarly HM is the dedendum. The angle ϵ subtended by HL is called the *addendum or top angle*, and the angle θ subtended by the dedendum is called the *dedendum or block angle*. The angle $LOX=\alpha+\epsilon$ is called the *face cone angle*. Clearly $\text{Tan}\,\epsilon=\frac{HL}{OH}=\frac{2\times HL\times \text{Sin}\,\alpha}{HK}$ $=\frac{2\times HL\times \text{Sin}\,\alpha}{N}\times(D.P.)$. Similarly for Tan θ. The thickness of the teeth measured round the pitch circle is made, as in spur gears, equal to half the circular pitch less a very small clearance. The addenda and dedenda and pressure angles of bevel gears are usually made the same as those of spur gears of the same pitch. Thus if Brown & Sharpe proportions are used the addendum is $\frac{1}{D.P.}$ and the dedendum $\frac{1{\cdot}157}{D.P.}$, while the pressure angle is $14\frac{1}{2}°$. Since bevel gears practically never have to be interchangeable in the sense that any one of a set shall work with any other of the set, there is even less reason for having fixed standard tooth proportions than there is with spur gears, and departures from the recognised standards are much more frequent. The face widths of bevel gears are usually made not greater than one-third of the cone distance.

182. Tredgold's Approximation.—In order to draw out the proper shapes for templates by means of which bevel gear teeth could be marked out on the patterns from which the gears were to be cast, Tredgold assumed that the surfaces of the back cones could be taken to coincide with the surface of the sphere of reference in the vicinity of the pitch circles. By means of this assumption Tredgold was able to derive for any pair of bevel gears a pair of *equivalent spur gears* the tooth action of which was for all practical purposes identical with that of the bevel gears. Such matters as the number of teeth in contact at any instant, the amount of interference and the correction necessary to avoid it, etc., could then be examined, by the methods described in Chapter XII, as if the action was between a pair of spur gears instead of between bevel gears. The method of deriving the equivalent spur gears will now be explained.

In Fig. 228 let OHK and OHL be the pitch cones of a pair of bevel gears, and let XHK and YHL be the corresponding back

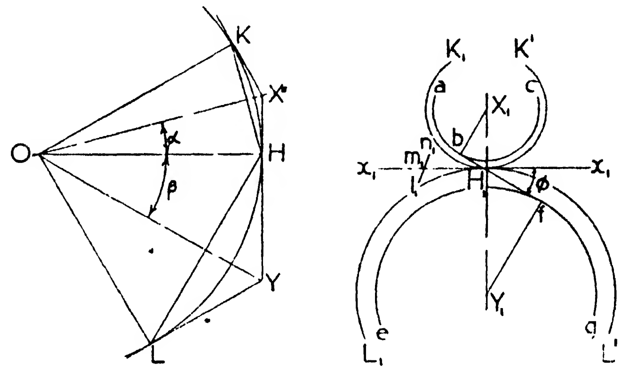


Fig. 228

cones, HK and HL being thus the pitch circles, while the arc KHL represents the sphere of reference. Now, by assuming that in the region of the pitch circles the back cones are coincident with the sphere of reference, the traces of the teeth on the latter may be transferred to the back cones. The surfaces of the latter may then be developed out into a plane, when they will become sectors of flat discs. Thus in the view on the right of Fig. 228 the arcs K_1H_1K' and L_1H_1L' are the developments of the pitch circles HK and HL, and these arcs are the pitch circles of the equivalent spur gears. Clearly the circular pitch (and hence the diametral pitch also) of the equivalent spur gears is the same as that of the bevel gears. Also since the addendum and dedendum of the bevel gears are measured on the back cones, the addendum and dedendum of the equivalent spur gears will be equal to those of the bevel gears. It should also be clear that the straight line x_1x_1 is the development of the circumference of the disc Oxy (Fig. 223) used in obtaining octoid teeth and the straight line $l_1m_1n_1$ is the development of the portion lmn of the great circle shown in that figure and which gave the octoid teeth. When the pitch lines K_1H_1K' and L_1H_1L' are rolled together with the line x_1x_1 the line $l_1m_1n_1$ will generate involute teeth on the equivalent spur gears. The base circles of these involutes will be circles *abc* and *def*, which are tangent to the line bH_1f, which is perpendicular to $l_1m_1n_1$. These base circles are approximately the developments of the base circles of spherical involute teeth of the bevel gears whose pressure angle is ϕ. Thus in the equivalent spur gear involute tooth outlines represent either octoid or spherical involute bevel gear teeth.

The pitch diameters of the equivalent spur gears are

$$2X_1H_1 = 2XH = \frac{KH}{\operatorname{Cos} \alpha}$$

and

$$2Y_1H_1 = 2YH = \frac{HL}{\operatorname{Cos} \beta}$$

Since the diametral pitch is the same as in the bevel gears, the numbers of teeth in the equivalent spur gears are

$$\frac{HK}{\operatorname{Cos} \alpha} \times \text{D.P.} = \frac{n}{\operatorname{Cos} \alpha} = n_1$$

and

$$\frac{HL}{\operatorname{Cos} \beta} \times \text{D.P.} = \frac{N}{\operatorname{Cos} \beta} = N_1$$

n and N being the numbers of teeth in the bevel gears. N_1 and n_1 are called the equivalent or virtual numbers of teeth, and as they are rarely whole numbers, the nearest whole number is usually taken.

Clearly for continuity of action in a pair of bevel gears the *base circle* pitch of the equivalent spur gears must not be greater than the length of the path of contact of those gears. Similarly if interference occurs in the equivalent spur gears it will occur in the bevel gears, and if the addenda and dedenda of the equivalent spur gears are altered by certain amounts in correcting the teeth for interference, the same alterations will be required in the bevel gears. Lastly the strengths of bevel gear teeth are expressed in terms of the strengths of the equivalent spur gear teeth.

183. Spiral Bevel Gear Teeth.—In Fig. 223 the portion of the disc *Olmn* that generated the octoid teeth intersects the disc *Oxy* in a radial line. No difficulty should be experienced in seeing that the surface *Olmn* could be some other shape than a flat disc without affecting the generation of the teeth. Of course, if the teeth are to be octoid teeth any sphere of reference must intersect the surface *Olmn* in a portion of a great circle, but this is not incompatible with the surface *Olmn* being such as to intersect the disc *Oxy* in a curve, in particular in a circular arc. If the latter condition holds, then the teeth of the bevel gears will be curved in the direction of a generating line of the pitch cone. Gears using such teeth are called *Spiral Bevel Gears* and are used very extensively, so much so that they tend to displace straight bevel gears altogether. The reason is chiefly that it is actually easier and cheaper to manufacture spiral bevel gears than straight bevel gears and in addition the former are stronger and quieter.

It should be clear that by taking a curved line on the rolling cone OHf in Fig. 224 to be the describing line of the teeth, spiral spherical cycloidal teeth will result. Similarly, by taking a curved line of the disc Oab in Fig. 226 as the describing line, spiral spherical involute teeth will result.

EXERCISES XIV

1. Two shafts whose axes intersect at right-angles are to be connected by gears with a ratio of 3 to 1. Find the semi-apex angles of the pitch cones.

2. Two shafts whose axes intersect at an angle of 70° are to be connected by gears, meshing externally, and having a ratio of 3 to 1. Find the semi-apex angles of the pitch cones.

3. Describe briefly the various methods by which pairs of conjugate teeth may be obtained for bevel gears.

4. If the gears of Question 1 are of 5 D.P. and the pinion has 20 teeth, find the pitch circle diameters and the numbers of teeth in the equivalent spur gears.

5. If the gears of Question 2 have 24 and 72 teeth of 8 D.P. and are to B. & S. proportions, what are the face cone angles of the gears ?

CHAPTER XV

(This chapter may be omitted on a first reading.)

GEARING CONNECTING NON-PARALLEL NON-INTERSECTING AXES

Axes that are not parallel and do not intersect may conveniently be called *skew* axes, and although in practice several different types of gearing are used to connect such axes, these different types are kinematically identical. What follows therefore applies to all such gearing.

184. Suppose that bodies A and B rotate about axes AA and BB (Fig. 229) with angular speeds ω_a and $-\omega_b$ respectively ; then the motion of A relative to B is obtained by bringing B to rest by giving the whole system an angular velocity $+\omega_b$ about the axis BB. The body A will then have a motion of rotation ω_a about AA, while that axis will have a motion of rotation $+\omega_b$ about BB. Now, it has been proved in Art. 52 that the resultant motion of A under these conditions is a screw motion about an axis RR as shown and which satisfies the conditions

$$\frac{\text{Sin } \alpha}{\text{Sin } \beta}=\frac{\omega_b}{\omega_a}$$

$$\frac{l}{m}=\frac{\text{Tan } \alpha}{\text{Tan } \beta}.$$

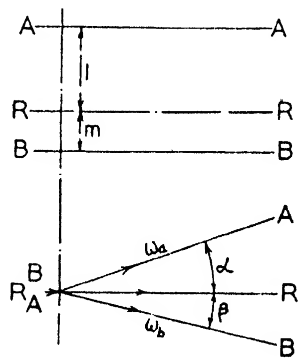


FIG. 229

The line RR is, of course, the instantaneous axis of A relative to B or of B relative to A. If the ratio of the angular speeds remains constant, then the position of RR relative to AA and BB will be fixed. It follows that when B is fixed and AA rotates about BB, then RR rotates about BB also, thus sweeping out an hyperboloid of revolution having BB as axis. This surface is the axode of A relative to B. Similarly, when A is fixed the axis RR will rotate about AA, thus sweeping out another hyperboloid of revolution, but having AA as axis. This is the axode of B relative to A.

The two axodes will touch at any instant along the line RR; the instantaneous axis and the relative motion between A and B is obtained by a screw motion of the one axode relative to the other about the line in which those axodes touch at any instant.

185. Possible Shapes for Skew Gear Teeth.—In investigating the possible shapes for the teeth of spur and bevel gears it was possible to find a surface which intersected the pitch surfaces of the wheels normally at every point, and it was thus possible to reduce the problem from one concerning the action of two surfaces in contact to one concerning the action of two lines in contact, a great simplification. With skew axes this is no longer possible, and so the investigation of the possible shapes for skew gears is much more difficult than the corresponding problem with spur or bevel gears; so much more difficult, in fact, that, in view of the limited field of usefulness of skew gears, it will not here be attempted. The subject has been considered in detail by Prof. Charles William MacCord in his book *Kinematics*, published by John Wiley & Sons, New York, which is a most stimulating book and one that should be read by all who wish to pursue the subject of toothed gearing further than can be done in this book. In engineering practice, however, when skew axes are connected by a pair of toothed wheels, the gears employed are either identical with helical-toothed spur gears or are worm gears, and it will suffice to consider these from a practical standpoint. Helical-toothed spur wheels when used to connect skew axes are generally called *skew gears* or *spiral gears*, and these will now be dealt with.

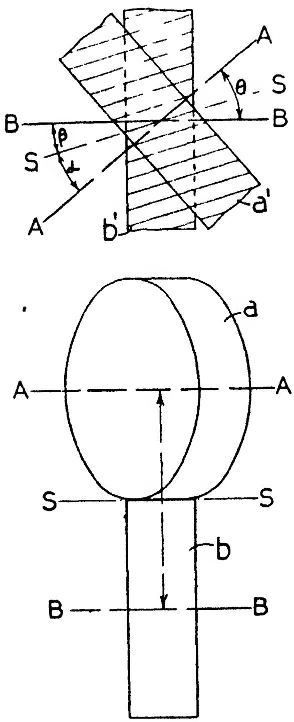


Fig. 230

186. Skew Gears.—Let AA and BB (Fig. 230) be a pair of skew axes and let the outlines a and b be the pitch cylinders of a pair of helical-toothed spur wheels. It should be noted, at the outset, that these cylinders are not intended to be portions of the hyperboloidal axodes, a fact which will become evident shortly when it will be seen that another pair of helical-toothed spur gears, having quite different pitch cylinders, could be used to connect the axes AA and BB without any alteration in the velocity ratio or the relative motion, i.e. without any change in the axodes. Now imagine these pitch cylinders to be developed out into the plane of the paper, which is to be imagined as their common tangent plane, seen as the line SS in the lower view. They will develop into the rectangles a' and b', portions only of which are shown. Now, if the teeth of the wheel A had had a spiral angle a their traces on the pitch cylinder would develop into straight lines inclined at the angle a to the axis AA, as shown. Now let the spiral angle β of the teeth of B be made equal to $\theta-a$; then the developed traces of the teeth of B will coincide with those of A where the pitch cylinders overlap, as shown. The wheels A and B would then mesh correctly and transmit motion between the shafts with a constant velocity ratio. Clearly the normal circular pitches (and hence the N.D.P.s) of the two gears must be the same. When it is remembered that the pitch cylinder of A has been rolled down on to the upper suface of the paper while that of B has been rolled up on to the under surface it will be clear that the teeth of A are *left-handed* and the teeth of B are also left-handed. Suppose, however, that the teeth of A had been made right-handed; the developed pitch cylinders would then be as in Fig. 231, and it will be seen that the teeth of B are still left-handed, but that we now have the relationship $\beta=\theta+a$ between the spiral angles and the shaft angle. Thus we may say that when the wheel teeth have *the same* hand the *sum* of the spiral angles must equal the shaft angle, and when the wheel teeth have *opposite* hands then the *difference* of the spiral angles must equal the shaft angle.

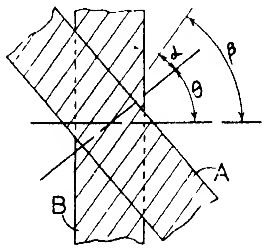


FIG. 231

Now, by Art. 164 the real diametral pitches of the wheels are given by

$$\text{R.D.P. of A}=\text{N.D.P.}\times\text{Cos } a$$
$$\text{R.D.P. of B}=\text{N.D.P.}\times\text{Cos } \beta$$

N.D.P. being the common normal diametral pitch. Let the numbers of teeth in the wheels be N_a and N_b respectively, then the pitch cylinder diameters D_a and D_b are given by

$$D_a=\frac{N_a}{\text{R.D.P. of A}}=\frac{N_a}{\text{N.D.P. Cos } a}$$

$$D_b=\frac{N_b}{\text{R.D.P. of B}}=\frac{N_b}{\text{N.D.P. Cos } \beta}$$

and the gear ratio is

$$\frac{N_a}{N_b}=\frac{D_a \operatorname{Cos} \alpha}{D_b \operatorname{Cos} \beta}$$

N_a and N_b must, of course, be whole numbers.

Also, if L is the shortest distance between the shaft axes, then

$$D_a+D_b=2L.$$

For practical reasons it is generally necessary that the N.D.P. shall be a standard pitch. Collecting the equations, we have

$$\alpha+\beta=\theta \text{ or } \alpha\sim\beta=\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{\omega_a}{\omega_b}=\frac{N_b}{N_a}=\frac{D_b \operatorname{Cos} \beta}{D_a \operatorname{Cos} \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$D_a=\frac{N_a}{\text{N.D.P. } \operatorname{Cos} \alpha} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$D_b=\frac{N_b}{\text{N.D.P. } \operatorname{Cos} \beta} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$D_a+D_b=2L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now, in any practical problem the shaft angle θ will be specified, and also gear ratio, the distance L, and the N.D.P., and the problem is to determine the numbers of teeth in the wheels, the spiral angles, and the pitch cylinder diameters. As there are six quantities to be determined and only five equations, it is necessary to assume a value for one of the unknown quantities. Usually it is best to assume a value for the number of teeth in the pinion. The other quantities may then be determined. Obviously by selecting different values for the arbitrarily assumed quantity other solutions could be found. As an example, let us determine the dimensions of two skew gears to connect shafts at an angle of 50°, the gear ratio being 2 to 1, the shortest distance between the shafts $3\frac{1}{2}$ in., and the N.D.P. 10.

Assume $N_a=20$. Then $N_b=40$.

From (3), (4) and (5), $\quad D_a+D_b=\dfrac{20}{10 \operatorname{Cos} \alpha}+\dfrac{40}{10 \operatorname{Cos} \beta}=2L=7$

$$\therefore \operatorname{Cos} \beta+2 \operatorname{Cos} \alpha=3{\cdot}5 \operatorname{Cos} \alpha \operatorname{Cos} \beta$$

And by (1) $\quad \operatorname{Cos}(50-\alpha)+2 \operatorname{Cos} \alpha=3{\cdot}5 \operatorname{Cos} \alpha \operatorname{Cos}(50-\alpha)$

This equation determines α; it is best solved by plotting the two sides against α and finding the intersection of the resulting curves, and to reduce the amount of computation it is advisable to derive the approximate value of α by a graphical method to be described later. The value of α is thus found to be 14° 6′, whence $\beta=35°\ 54'$.

$$\therefore D_a = \frac{2}{\text{Cos } 14^\circ 6'} = 2{\cdot}062 \text{ in.}$$

$$D_b = \frac{4}{\text{Cos } 35^\circ 54'} = 4{\cdot}938 \text{ in.}$$

Since the N.D.P. is 10 the addenda (assuming B. & S. standard proportions) are 0·1 in. and the outside or blank diameters are thus 2·262 in. and 5·138 in.

Assuming $N_a = 21$ gives, as the equation for a,

$$2{\cdot}1 \text{ Cos } (50 - a) + 4{\cdot}2 \text{ Cos } a - 7 \text{ Cos } (50 - a) \text{ Cos } a,$$

from which a is found to be 39° 42′. Hence $\beta = 10^\circ$ 18′. The pitch cylinder diameters then are $D_a = 2{\cdot}729$ in., $D_b = 4{\cdot}271$ in., and by taking $N_a = 19$ a third pair of gears can be found for which the spiral angles are 9° and 41° approximately.

Any of these pairs of gears will connect the shafts at the specified centre distance, gear ratio and N.D.P., and the choice between them turns on the amount of sliding between the teeth, which will be dealt with later, or on some other requirement, such as that the gears must be approximately the same size.

187. Graphical Determination of the Spiral Angles.—Probably the best graphical method of determining the spiral angles is the following: Referring to Fig. 232, the lines OA, OB represent the shaft axes; thus $\angle BOA = \theta$. The line OC is then drawn such that $\frac{CD}{CE} = \frac{N_a}{N_b}$, C being any point on OC, and CD and CE being perpendiculars on to OA and OB respectively. Then, having assumed a value for N_a, the value of $\frac{N_a}{\text{N.D.P.}}$ is calculated. This represents the pitch diameter of a spur wheel having N_a teeth and having a real diametral pitch equal to the N.D.P. of the skew gear; this spur gear is sometimes called the equivalent spur gear and its diameter $\left(\frac{N_a}{\text{N.D.P.}}\right)$ the equivalent diameter. A point C is then found on OC such that $CD = \frac{N_a}{\text{N.D.P.}}$. Next an accurately divided scale is taken and is manipulated until its zero lies on OA, its edge passes through C, and the reading where it intersects OB is equal to twice the required shortest distance between the

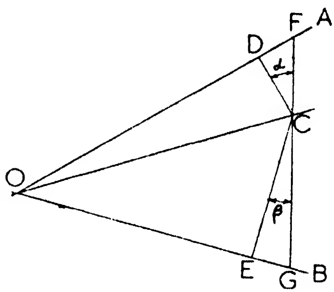


FIG. 232

shafts. Let FCG be the position found for the straight edge. Then CF is the pitch cylinder diameter of the pinion, $\angle$DCF its spiral angle α, while CG is the pitch cylinder diameter of the wheel and $\angle$ECG its spiral angle. The construction is easily proved. Thus CD=CF Cos α and CE=CG Cos β; thus $\frac{CD}{CE}=\frac{CF \text{ Cos } \alpha}{CG \text{ Cos } \beta}=\frac{N_a}{N_b}$ by construction. Thus the lengths CF and CG and the angles α and β satisfy equation (2). Also by construction FG=CF+CG= twice the required centre distance=2L. Thus CF and CG satisfy equation (5). Also $\angle$ACD+$\angle$ECG=180°−$\angle$ECD=$\angle$DOE=θ; thus α and β satisfy equation (1). Lastly, by construction,

$$CD=\frac{N_a}{\text{N.D.P.}} \quad \therefore \quad \frac{CD}{\text{Cos } \alpha}=\frac{N_a}{\text{N.D.P.} \times \text{Cos } \alpha}=D_a=CF$$

188. The Sliding of the Teeth.—In Fig. 233, OA and OB are the axes of two skew gears whose velocity ratio $\frac{\omega_a}{\omega_b}=r$. The shortest distance between the shafts is L and the spiral angles are α and β respectively. Then O is the point of contact of the pitch cylinders and, considering it as a point of A, it has a velocity $OL=\frac{\omega_a D_a}{2}$, as shown. Similarly, when considered as a point of B it has a velocity $OM=\frac{\omega_b D_b}{2}$. Now by equation (2)

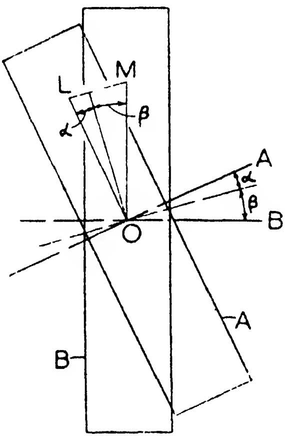


Fig. 233

$$\frac{OL}{OM}=\frac{D_a \omega_a}{D_b \omega_b}=\frac{\text{Cos } \beta}{\text{Cos } \alpha}$$

Let OL and OM be drawn in this proportion, and resolved along and perpendicular to the developed trace of the teeth, drawn as a chain-dotted line in the figure. The perpendicular components will then be equal (an obvious necessity), while the vector difference of the components along the tooth line is the velocity of sliding of the teeth in the direction of their length. Thus,

$$\text{Velocity of sliding } v=\omega_a D_a \text{ Sin } \alpha+\omega_b D_b \text{ Sin } \beta.$$

189. The Spiral Angle for Least Sliding.—It has been seen that for a given gear ratio, centre distance and N.D.P. many pairs of gears can be found, having different spiral angles and pitch diameters ; now, one of these pairs will have a lower velocity of sliding between the teeth than any of the others, and the spiral angles of this pair will now be found.

Since $$\omega_b D_b = \omega_a D_a \frac{\text{Cos } a}{\text{Cos } \beta}$$

$$v = \omega_a D_a \left\{ \frac{\text{Sin } a \text{ Cos } \beta + \text{Sin } \beta \text{ Cos } a}{\text{Cos } \beta} \right\}$$

$$= \frac{\omega_a D_a \text{ Sin } (a+\beta)}{\text{Cos } \beta}$$

$$= \frac{\omega_a D_a \text{ Sin } \theta}{\text{Cos } \beta}$$

Now $$D_a = \frac{N_a}{\text{N.D.P. Cos } a}$$

and $$D_b = \frac{N_b}{\text{N.D.P. Cos } \beta} = \frac{rN_a}{\text{N.D.P. Cos } (\theta - a)}$$

also $$D_a + D_b = 2L.$$

$$\therefore \frac{N_a}{\text{N.D.P.}} \left\{ \frac{1}{\text{Cos } a} + \frac{r}{\text{Cos } (\theta - a)} \right\} = 2L$$

$$\therefore N_a = \frac{2 \times \text{N.D.P.} \times L \times \{\text{Cos } a \text{ Cos } (\theta - a)\}}{\text{Cos } (\theta - a) + r \text{ Cos } a} \quad . \quad . \quad . \quad (6)$$

$$\therefore D_a = \frac{2L \text{ Cos } (\theta - a)}{\text{Cos } (\theta - a) + r \text{ Cos } a} = \frac{2L \text{ Cos } \beta}{\text{Cos } (\theta - a) + r \text{ Cos } a}$$

$$\therefore v = \frac{2\omega_a L \text{ Sin } \theta}{\text{Cos } (\theta - a) + r \text{ Cos } a}$$

Now, the numerator of this is independent of the spiral angles ; hence for the minimum sliding the denominator must be a maximum.

$$\therefore \frac{d}{da}\{\text{Cos } (\theta - a) + r \text{ Cos } a\} = o$$

$$\therefore \text{Sin } (\theta - a) = r \text{ Sin } a$$

$$\text{Sin } \theta \text{ Cot } a - \text{Cos } \theta = r$$

$$\text{Cot } a = \frac{r + \text{Cos } \theta}{\text{Sin } \theta} \quad . \quad . \quad . \quad (7)$$

and this gives a maximum for the denominator, since the second derivative is negative. Hence the spiral angle for least sliding is given by equation (7).

It may be pointed out that *if the pitch diameters are assumed to* *be constant*, then the minimum sliding occurs when $a=\frac{\theta}{2}$, and this value for a is given in many textbooks as resulting in the minimum sliding; but this assumption is not in accordance with practical considerations, since it implies variations in the gear ratio and this has a fixed value.

The spiral angle given by equation (7) may result in the number of teeth in the pinion being fractional, and then, of course, all that can be done is to take the nearest whole number. In the example considered above equation (7) gives $\text{Cot } a=\frac{2+\text{Cos } 50°}{\text{Sin } 50°}$, whence $a=16° 9'$, and on putting this value in equation (6) N_a is found to be 20·3; hence the solution found by assuming $N_a=20$ happens to be the one giving least sliding.

Now, the axodes of the relative motion are given by the equations $\frac{\text{Sin } a}{\text{Sin } \beta}=\frac{\omega_b}{\omega_a}$; $\frac{l}{m}=\frac{\text{Tan } a}{\text{Tan } \beta}$; $a+\beta=\theta$ and $l+m=L$, a and β being the angles between the respective generating lines of the axodes and the axes and l and m being the radii of those axodes at their gorges. Then,

$$\frac{\text{Sin } a}{\text{Sin } \beta}=\frac{\omega_b}{\omega_a}=\frac{1}{r}$$

$$\therefore \text{Sin } \beta=r \text{ Sin } a$$

$$\text{Sin } (\theta-a)=r \text{ Sin } a$$

$$\therefore \text{Cot } a=\frac{r+\text{Cos } \theta}{\text{Sin } \theta}$$

and the solution that gives the minimum sliding is that one for which the pitch surfaces of the wheels coincide at the gorges with the axodes of the motion.

By the methods given above the principal dimensions (that is, the numbers of teeth, pitch diameters and spiral angles) of a pair of skew gears may be determined. Each of the wheels may then be regarded as a helical-toothed spur gear, and the addenda, dedenda, etc., may be made the standard amounts according to the system in use for spur gears. The wheels may then be cut in exactly the same way as helical-toothed spur gears, and it is not until the two gears are meshed together with their axes in the skew relationship that they can be distinguished from helical-toothed gears and definitely said to be "skew gears." The wheels may, however, be cut by the method adopted for worm gears, which will be described later; gears cut in this manner should, however, be regarded as worm gears, and these will now be considered.

190. Worm Gears.—There are two types of worm gearing in general use; namely,

1. Straight or Parallel worm gears,
2. Globoidal or Hour-glass worm gears,

and the fundamental difference between the types lies in the form of the worm and can best be explained by describing how the worms could be produced. (In practice the method adopted is not that described, but is the same in principle.) Dealing first with the parallel worm, this is cut like an ordinary screw thread; thus a cylinder (the " blank ") is mounted so that it can revolve about its axis, and a cutting tool of suitable shape is mounted so as to be able to slide parallel to that axis. The relative motion between the cutting tool and the blank is controlled by external gearing between them in such a way that the cutting tool travels a fixed distance for a given angular motion of the blank. The motion of the cutting tool across the face of the blank in conjunction with the rotation of the latter results in a helical thread on the blank.

The globoidal worm is made by turning a blank to the shape shown in Fig. 234, the surface *aa a'a'* of which is a surface of revolution obtained by revolving the curve *aa* about the axis XX. The curve *aa* is an arc of a circle whose centre O lies on the axis of the worm wheel which ultimately will mesh with the worm being cut. This blank is then mounted so that it can revolve about its axis XX and a cutting tool of suitable shape is mounted so that it can revolve about the centre O. The relative motion between the cutting tool and the blank is controlled by external gearing so that the angular motion of the cutting tool is proportional to the angular motion of the blank. The motion of the cutting tool across the face of the blank in conjunction with the rotation of the blank results in a thread being formed on the blank.

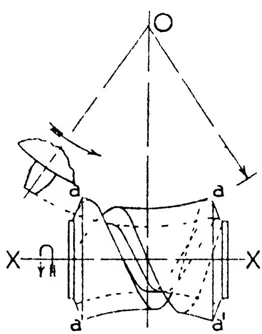


FIG. 234

For both types of gear the worm wheel is made by making a cutter or " hob " which is, in all except minor details, a replica of the worm. This hob is gashed so as to form a number of cutting edges round its threads and is, of course, suitably hardened and tempered. The method of using the hob differs, however, for the two types. Considering the globoidal type first, the hob is mounted with its axis in the same angular position relative to the wheel blank as the axis of worm will ultimately be when working with the wheel after the latter has been cut. The perpendicular distance between the hob axis and the blank axis is made, however, greater than the corresponding distance between the worm and wheel axes. The hob is connected to the blank by external gearing having the same ratio as that of the worm and wheel under consideration. The hob is then rotated at a suitable speed (thus causing the wheel blank to rotate at the proper speed also) and is gradually moved towards the blank in a direction perpendicular to the blank axis until the shortest distance between the hob and blank axes is equal to the required distance between the worm and wheel axes. As the rotating hob is moved towards the blank it gradually cuts the tooth spaces of the latter and ultimately leaves the teeth the right shape to engage the worm threads.

There is sometimes a difficulty arising from the fact that while the hob is being fed inwards towards the blank it cuts away some of the metal of the latter which would not be removed if the hob axis could be maintained at all times at the correct distance from the blank axis. Where metal has been removed in this way the wheel teeth cannot engage the worm threads when the two are meshed together, and unless this interference between the hob and the wheel is kept within reasonable limits, the contact between the worm and wheel may be reduced to such an extent that the load the gears can safely carry will be unduly diminished. This generally necessitates the use of a comparatively short worm.

The parallel type worm hob is used in a similar manner to the above except that the hob axis is, at all times, at the same distance from the blank axis as the worm axis is when the worm ultimately meshes with the wheel. The hob starts in the position 1 in Fig. 235 and is fed tangentially, relative to the blank, until it reaches position 2. The hob and blank are of course geared together externally and the hob is rotated at a suitable speed. The gear ratio between the hob and blank is now not the same as that of the worm and wheel, but must allow for the axial motion of the hob. Clearly either rotation or axial translation of the hob, separately, will result in rotation of the wheel, and when the hob both rotates and moves axially the wheel must receive both rotations simultaneously, and the gear ratio between hob and blank must be arranged accordingly.

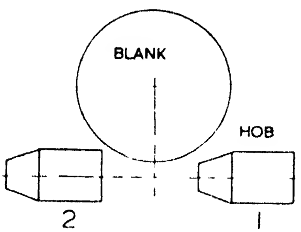


FIG. 235

The hob is usually tapered, as shown, to facilitate the commencement of the cutting action.

The axial feed of the hob eliminates interference between the hob and the blank, thus giving the best possible contact between the worm and wheel.

The parallel type of worm is much more widely used than the globoidal type, and the next few articles deal more fully with the parallel type.

191. Single and Multiple Thread Worms.—A worm having two threads is shown in Fig. 236. If, starting from the point A, a

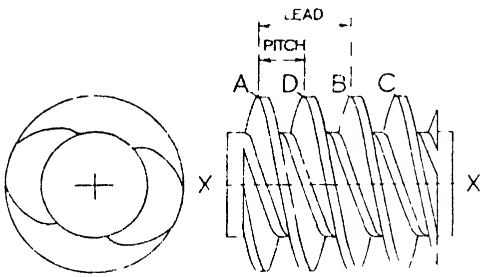


FIG. 236

thread is followed for one complete revolution about the axis XX, the point B will be reached. It will be seen that between A and B, which are corresponding points of the same thread, a second thread is situated. If, starting from A and making one revolution, the point C had been reached, then the worm would have been a three-thread worm. Multi-thread worms are commonly referred to as multi-*start* worms, since in the end view the start of each thread is clearly seen.

The distance AB, measured parallel to the axis XX, is called the *lead* of the thread or worm, while the distance AD, between corresponding points of *consecutive* threads, is called the pitch. Thus Lead=Number of threads × Pitch. On a single-start worm the lead and pitch are equal.

192. Thread Shapes and Proportions.—The section of a parallel worm by a plane containing the axis has the appearance of a rack, as shown in Fig. 237, and in the early days of worm gearing this rack section was made identical with the racks used in spur gearing; that is, the threads were made straight-sided as shown and the pressure angle ϕ was made either $14\frac{1}{2}°$ according to the B. & S. standard or 20° according to the Fellow's standard. Similarly, the height of the thread was given by means of the addendum and dedendum measured respectively above and below a line corresponding to the pitch line of the rack, as shown. The cylinder (diameter d) of which this pitch line is a generator is usually called the *pitch cylinder* of the worm, but, as in skew gears, the pitch cylinder is merely a surface on which the pitch of the teeth is measured and is not an axode. The addendum and dedendum were usually given the same values as for spur gears and thus were given in terms of a diametral pitch. Nowadays the threads of worms are frequently not made straight-sided, and even when straight-sided are not made similar to spur gear racks; in particular the pressure angle is made greater than is usual with spur gears, and the dedendum is made smaller than the addendum, or may be made zero.

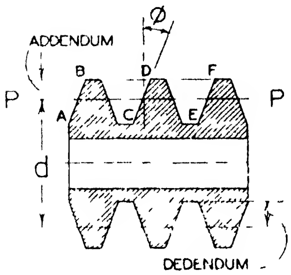


FIG. 237

193. The Action of a Worm with a Wheel.—Consideration will show that if the worm of Fig. 237 is revolved about its axis the rack section ABCDEF given by the intersection of the worm threads with the plane of the paper will appear to travel along the pitch line PP. If the worm threads are right-handed and the worm is rotated in the clockwise direction when viewed from its right-hand end, then the rack section will appear to travel from left to right. If the direction of rotation of the worm *or* the hand of its threads is reversed, then the direction of travel of the rack section will also be reversed. The speed of travel of the rack section is given by

$$\text{Speed of travel (inches/min.)} = \text{R.p.m. of worm} \times \text{Lead of worm threads (inches)},$$

and if the angular speeds of the worm and its lead are constant, then the speed of travel of the rack section will also be constant. Suppose the worm to mesh with a very thin wheel; then, provided the teeth of this wheel are conjugate teeth to the rack section, the travel of the latter at constant speed will result in rotation of the wheel with constant angular speed. The number of teeth of the rack section that will pass the pitch point per minute is given by

$$\text{Number of teeth per min.} = \frac{\text{Speed of travel of rack section}}{\text{Pitch of worm teeth}} = \frac{\text{R.p.m. of worm} \times \text{Lead of worm}}{\text{Pitch of worm teeth}} = \text{R.p.m. of worm} \times \text{Number of threads of worm}$$

Also the number of teeth of the wheel passing the pitch point per minute is given by—R.p.m. of wheel × Number of teeth in wheel—and this number must be the same as for the rack section. Hence,

$$\frac{\text{R.p.m. of wheel}}{\text{R.p.m. of worm}}=\frac{\text{Number of threads of worm}}{\text{Number of teeth of wheel}}$$

and the worm may be regarded as a wheel having a number of teeth equal to the number of its threads.

The action of the worm and wheel as examined above by means of the *central section* (given by the *central plane* containing the worm axis and being perpendicular to the wheel axis) is seen to be that of a rack with a wheel. Thus any shape may be chosen for the rack teeth, the choice being restricted, of course, by considerations of interference, as with spur gears. The proper shape for the wheel teeth to mesh with the arbitrarily chosen central rack section of the worm is obtained automatically by reason of the hobbing process used to cut the wheel teeth, in which process the wheel-tooth shape is generated by the rack section to which it has to be conjugate.

If the action of the worm and wheel is examined by means of a section by any plane parallel to the central plane, the action will again be found to be that of a rack and wheel. This is shown in Fig. 238, where ABCDEF is the rack section given by the intersection of the plane MN with the worm. Rotation of the worm

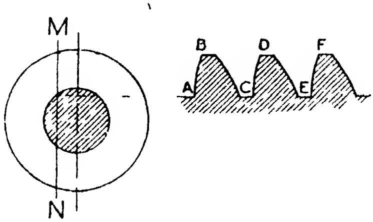


FIG. 238

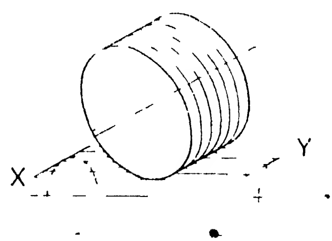


FIG. 239

produces a translation of this rack section equal to that of the central section. The section of the wheel by the plane MN will give teeth that are conjugate to those of the rack, which, of course, actually generates them. The actual shape of the teeth of the rack given by a side section such as MN depends on the shape of the teeth of the central rack and on the pitch diameter and lead of the threads. If these factors are settled, then the shape of the side-section rack teeth is automatically settled also. Alternatively the shape of the teeth of the side-section rack can be chosen arbitrarily, thus settling automatically the shape of the central rack teeth. In practice the central section is usually the arbitrarily chosen one, but some makers choose the side section.

The "pitch lines" of all the rack sections given by planes like MN may be thought of as rolling with corresponding "pitch circles" of the wheel sections. Since all the pitch lines of the racks travel at the same linear speed, it follows that all the "pitch circles" have the same diameter. The rack pitch lines will form a plane XY (Fig. 239), parallel to both the worm and the wheel axes, and the pitch circles will form a cylinder, concentric with the wheel axis and tangent to the plane XY as shown. These surfaces are sometimes referred to as pitch surfaces; they are not, of course, axodes.

194. The Contact between the Teeth.—Each of the teeth of the rack section given by a plane such as MN in Fig. 238 will in general make contact with the conjugate tooth of the wheel section at one point, which point will, of course, move along the tooth profiles as the action proceeds. The sum of all these point-contacts between the teeth is a line of contact whose shape and position at a particular moment might be as shown at *ab* in Fig. 240. If the worm, and thus, of course, the wheel also, be rotated through a small

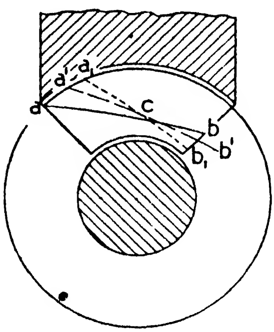


FIG. 240

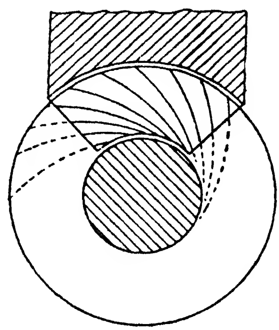


FIG. 241

angle, the new line of contact might be such as a_1b_1. If the line *ab* had been marked on the worm in the original position, it would be seen, after the worm had been rotated through the small angle, as *a′b′*. Then the point of intersection *c* of the lines a_1b_1 and *a′b′* is a point which has contact with the wheel teeth at two separate instants. It is considered by some authorities that such double contacts should be avoided, since the first contact tends to disperse the lubricant from the region of the contact and the subsequent second contact then occurs without the presence of the lubricant, which distributes the pressure between the surfaces over an area sufficiently large to keep the intensity of pressure within safe limits and which reduces the friction and consequent heating. The result is that the surface of the metal of the wheel teeth is overstressed and breaks down and the gear fails. It would follow therefore that one of the primary considerations in settling on a suitable thread section for a worm is the avoidance of these multiple contacts, the shape being chosen so that the successive lines of contact are as shown in Fig. 241, and some manufacturers claim to achieve this result.

The determination of these lines of contact is not difficult for a parallel worm if a graphical method, or a combination of analytical and graphical methods, is used, but it has not been thought advisable to give the methods here. For further information the reader is referred to a paper by W. Abbott entitled "Worm Gear Contacts," *Proc. I.Mech.E.*, Vol. 133, 1936.

195. Tooth Contact with Globoidal Worms.—The central section of a globoidal type worm is as shown in Fig. 242, having the appearance of a rack lying round the arc xy of a circle. If the worm be rotated slightly, then this section will move round the

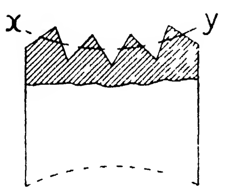


FIG. 242

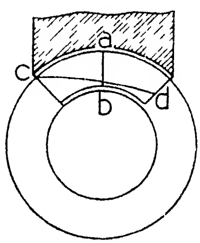


FIG. 243

arc xy a small amount, but will be otherwise unchanged. Since the wheel teeth can be made the exact counterpart of the thread section, it follows that on the central section there is line contact between any thread of the worm and its mating wheel tooth.

A side section of the worm will also have the appearance of a rack, with unsymmetrical teeth, lying round a circular arc, but if the worm is rotated through a small angle, then not only will this rack section move round the arc by a small amount, but the actual shape of its teeth will change. If the tooth sections are drawn for a large number of successive small rotations of the worm, it will be found in some cases that the successive tooth shapes, when "set back" to allow for the rotation of the rack section round the arc of the circle, will have an envelope. When this is so this envelope will be the shape of the wheel tooth in the plane of the section, and this wheel tooth will, in general, touch the worm rack section in a point for every position of the worm. Then, considering all the possible side sections on which a similar contact occurs, it will be seen that the worm threads will be in contact with the wheel teeth along a second line of contact (*cd*, Fig. 243) roughly at right-angles to the line of contact *ab* given by the central section. The position of this second line of contact will change as the worm rotates. It is claimed by some upholders of the globoidal type of worm that this double line contact enables a greater load to be carried than is possible with the parallel type worm under comparable conditions as to size, speed, etc. In some investigations made by the author the successive tooth sections of the worm did not have an envelope at all, one section lying outside all the others. This being so, it followed that the second line of contact *cd* was entirely absent.

The load-carrying capacity of a worm gear cannot, however, be predicted, with any degree of certainty, from an investigation of the tooth contact, but must be settled by experiment; and experiments show that there is little to choose between the types on the score of efficiency and that the globoidal type can carry rather higher loads than the parallel type. Consideration of this aspect of worm gearing is beyond the scope of this book.

EXERCISES XV

1. The axes of two shafts are at right-angles, and the shortest distance between them is $4\frac{1}{2}$ in. The shafts are to be connected by skew gears having a ratio of 2 to 1. If the N.D.P. is 10 and the pinion has 20 teeth, find the pitch circle diameters and the spiral angles of the teeth, which are to be of opposite hands.

2. If the pinion in Question 1 rotates at 1000 r.p.m., what is the speed of rubbing between the teeth ?

3. Using the relevant data of Question 1, find the numbers of teeth in the wheels, the pitch circle diameters, spiral angles and exact centre distance when the gears are designed for minimum sliding.

4. Describe briefly the essential differences between the parallel and globoidal types of worm.

5. A parallel worm has three starts and its lead is 6 in. It meshes with a wheel having 30 teeth. If the pitch cylinder diameter of the worm is 4 in., what is the centre distance between the shaft axes ?

6. Describe briefly the action that occurs between (*a*) a parallel worm and worm wheel and (*b*) a globoidal worm and wheel. What are the three most important factors to be considered in choosing the shape of thread section for a worm ?

7. A two-start parallel worm has a lead of 3 in. It rotates at 5 r.p.m. and meshes with a wheel having 20 teeth. Simultaneously it travels in the direction of its axis at a linear speed of 6 in./min. Find the speed of rotation of the wheel if the linear motion of the worm (*a*) increases and (*b*) decreases the rotation of the wheel.

CHAPTER XVI

GEAR TRAINS

196. Definition.—Two or more gear wheels of any type when used to transmit motion from one shaft to another constitute a train of gears. It is convenient at the outset to divide gear trains into two classes: *ordinary trains* in which all the wheels merely revolve about their own axes, which are fixed, and *epicyclic trains* in which some of the wheels besides revolving about their own axes have also a bodily motion about some other axis.

197. Ordinary Gear Trains.—These may be subdivided into two classes, *simple* and *compound*, the difference being shown by Fig. 244 (*a*) and (*b*). In the simple train (*a*) the wheel A on the driving shaft drives the wheel B, which in turn drives C, which in turn drives the wheel D on the driven shaft. The velocity ratio between A and D is easily seen to be $\frac{\omega_a}{\omega_b}=\frac{D}{A}$, D and A representing the numbers of teeth in the respective wheels. The wheels B and C are *idlers*, the numbers of teeth in them do not affect the velocity ratio at all, and they are used either because the centre distance between the shafts A and D is too great for the wheels A and D to be meshed directly together, if those wheels are to be of a reasonable size, or because direct meshing of A and D would result in the wrong direction of rotation of the driven wheel, or for both of these reasons. In a simple train of gearing all the wheels are usually of the same type, but when helical-toothed gears are used it becomes possible to arrange some of them as spur gears and some as skew gears. In the compound train (*b*) the wheel A on the driving shaft drives the wheel B on an intermediate shaft or *layshaft*, which also carries a second wheel C, which meshes with the wheel D carried by the second layshaft, to which is also fixed the wheel E, which drives the wheel F on the driven shaft. Each intermediate shaft carries two wheels, a

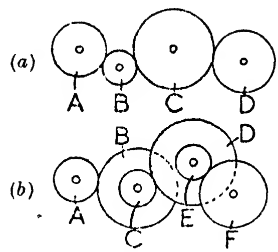


Fig. 244

driven and a driving wheel, both being fixed to the shaft. The gears need not all be of the same type, and, if they are, need not be all of the same pitch, etc.

The velocity ratio between A and F is

$$\frac{\omega_a}{\omega_f}=\frac{B\times D\times F}{A\times C\times E}, \text{ that is,}$$

$$\frac{\text{Speed of 1st driving wheel}}{\text{Speed of last driven wheel}}=\frac{\text{Product of teeth in driven wheels}}{\text{Product of teeth in driving wheels}}$$

The intermediate wheels thus affect the velocity ratio.

197a. Reverted or Co-axial Trains.—When the axes of the first and last wheels of a gear train are made to coincide the train is sometimes referred to as a *reverted* or *co-axial* train. An example is given in Fig. 245. Since if A, B, C and D are the numbers of teeth

$$L=\frac{A+B}{2D.P._a}=\frac{C+D}{2D.P._c}$$

it follows that if the wheels are to be all of the same pitch then

$$A+B=C+D$$

and this equation, together with that for the centre distance L and that for the gear ratio $\left(\frac{\omega_a}{\omega_d}=\frac{B\times D}{A\times C}\right)$, will enable the numbers of teeth to be determined (for the given centre distance, gear ratio and D.P.) only when the number of teeth in one wheel has been chosen arbitrarily.

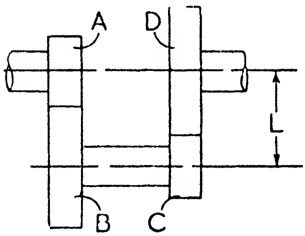


Fig. 245

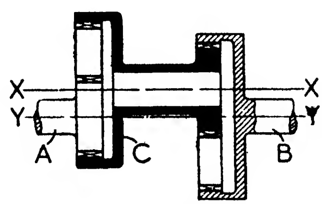


Fig. 246

It is not always possible to get exactly the gear ratio required at the exact centre distance if standard pitches only are used, and an approximate solution must then be accepted.

A very compact form of co-axial drive is obtained by using internal gears as shown in Fig. 246. The intermediate shaft C revolves in fixed bearings (not shown) about the axis XX, and the shafts A and B have the common axis YY.

198. Epicyclic Trains.—These also may be divided into *simple* and *compound* trains. A simple epicyclic train is shown in Fig. 247. It consists of an annulus A, having internal teeth, an

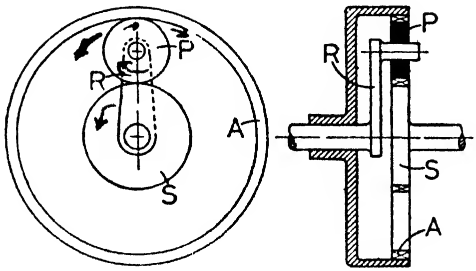


FIG. 247

"arm" R free to revolve independently of, but coaxially with, the annulus, a "planet" wheel P carried by, but free to revolve on, the pin of the arm, and lastly a "sun" wheel S coaxial with the annulus and arm, but independent of them. The frame which serves to support the members A, R and S is not shown.

Such a train may be employed in several ways, a common one being to couple the sun S to the driving motor or engine and the arm R to the machine to be driven and to hold the annulus stationary. The arm will then revolve in the same direction as the sun, but at a lower speed.

The velocity ratio may be found by considering the motions of the various members *relative to the arm* as follows. Let the numbers of teeth in the annulus and sun be respectively A and S. Then

Speed of annulus relative to arm = Speed of annulus relative to earth − Speed of arm relative to earth,

i.e. $${}_a\omega_r = {}_a\omega_e - {}_r\omega_e$$

and since $${}_a\omega_e = 0$$

this gives $${}_a\omega_r = -{}_r\omega_e$$

Now $${}_s\omega_r = -{}_a\omega_r \times \frac{A}{S}$$

because, both motions being relative to the arm, the relations between them are as in ordinary gearing. The minus sign indicates that, relative to the arm, the sun turns in the opposite sense to the annulus. Substituting for ${}_a\omega_r$ we have

$${}_s\omega_r = +{}_r\omega_e \times \frac{A}{S}$$

but $${}_s\omega_e = {}_s\omega_r + {}_r\omega_e$$

$$\therefore\ {}_s\omega_e = {}_r\omega_e \times \frac{A}{S} + {}_r\omega_e$$

$$= {}_r\omega_e\left(\frac{A}{S} + 1\right)$$

and $$\text{Velocity ratio} = \frac{{}_s\omega_e}{{}_r\omega_e} = \frac{A}{S} + 1$$

The velocity ratio may also be found very quickly by a tabular method in which the proper relative motion between the members (annulus, sun and arm in the example) is arrived at as the sum of two component motions given in two separate steps.

In the first step the arm is given the motion it is to have in the result *and all the other members are given the same motion.* It follows that in the second step the arm receives no motion, i.e it is fixed; hence in the second step we have to deal only with ordinary gearing. The motion that must be given to *one* of the remaining members (sun or annulus) is now settled, since it must be such that when added to the motion given in the first step the result is the required motion for that member. The motion received by the remaining member in the second step can now be calculated as for ordinary gearing (because in the second step the arm is fixed), and on adding this motion to that given in the first step the resultant motion is obtained.

Suppose it is required to find, in the example given above, what motion the sun receives when the arm is turned once while the annulus is held stationary. The working is as follows:

Member	Arm R	Annulus A	Sun S	Remarks
1st Step .	$+1$	$+1$	$+1$	All members receive the same motion.
2nd Step	0	-1	$+\frac{A}{S}$	The arm is fixed
Result	$+1$	0	$\frac{A}{S}+1$	

Mistakes will be avoided if the result, so far as it is known, is written down *first*, and if secondly a nought is placed for the motion of the *arm* in the second step.

199. Another Example.—Fig. 248 shows a type of epicyclic gear in which two sun wheels S_1 and S_2 are employed. They mesh with the toothed portions P_1 and P_2 of the compound planet, which is free to turn on the pin of the arm R. In one common use of such a train the arm is coupled to the driving motor or engine, one sun is coupled to the driven machine and the other sun is fixed. (The frame supporting the members S_1, S_2 and R is not shown.)

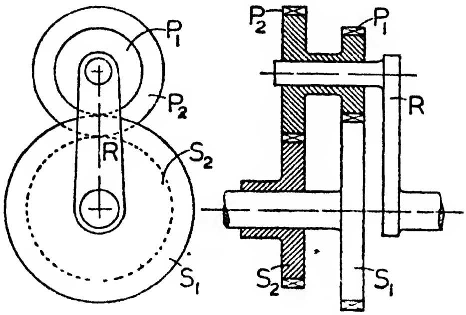


FIG 248

Suppose the arm R to be driven at 1000 r p.m and the sun S_2 to be fixed, and let the numbers of teeth be $S_1=30$, $P_1=20$, $P_2=26$, $S_2=24$. Then the speed of S_1 is found, using the tabular method, as shown in the table below.

R	S_2	S_1	
1000	+1000	1000	
0	−1000	$1000 \times \frac{24}{26} \times \frac{20}{30}$	$615\frac{5}{13}$
+1000	0	$+384\frac{8}{13}$	

Thus the sun S_1 is driven in the same sense as the arm, but at a lower speed.

If, in the same gear, the sun S_1 is made the fixed member, so that S_2 is driven, it will be found that S_2 is driven in the opposite sense to the arm, the working being given below.

R	S_1	S_2	
+1000	+1000	+1000	
0	−1000	$-1000 \times \frac{30}{20} \times \frac{26}{24}$	−1625
+1000	0	625	

This form of epicyclic gear is thus available as a forward or a reverse gear according as to whether the fixed sun is smaller or bigger than the driven sun.

200. Example When Motion of Arm is Unknown.—In this case the procedure has to be modified slightly. The simplest method is to find the gear ratio of the train by giving the arm one turn and finding, by the tabular method, the motion of the other moving member. The motion of the arm consequent to the actual motion of that member may then be found by simple proportion.

Thus, considering the double sun train of Fig. 248, let the sun S_2 be fixed and S_1 be rotated at 1000 r.p.m. What is then the speed of the arm? The gear ratio is found thus :

R	S_2	S_1
+1	+1	+1
0	−1	$\frac{24}{26} \times \frac{20}{30} \quad \frac{8}{13}$
1	0	$+\frac{5}{13}$

Hence, if S_1 rotates at 1000 r.p.m., the arm R rotates at $1000 \times \frac{13}{5}$ $=2600$ r.p.m. in the same sense.

201. Epicyclic Trains having No Fixed Member.—It is not essential that one member of an epicyclic train should be fixed; all the members may rotate. Thus in the train shown in Fig. 247 the annulus A might be driven from an external source, as well as the arm R. The resulting motion of the remaining member, the sun S in the example, may then be thought of as the sum of two component motions, one due to the motion of the arm R, the annulus being regarded as fixed, and the other due to the motion of the annulus, the arm being regarded as fixed.

As an example let S have 50 teeth, let the arm R rotate at 1000 r.p.m. clockwise and let A have 100 teeth and rotate at 500 r.p.m. anticlockwise. Then the motion of S consequent on +1000 r.p.m. of R, the annulus being regarded as fixed, is found to be +3000 r.p.m. and the motion of S consequent on −500 r.p.m. of the annulus, the arm being regarded as fixed, is +1000 r.p.m. The total motion of S is thus +4000 r.p.m. This may be found directly thus :

R	A	S
1000	+1000	+1000
0	−1500	$+1500 \times \frac{100}{50}$
+1000	−500	+4000

When the motion of the arm is to be found it will generally be simplest to find the component motions separately and to add them afterwards.

202. Bevel Epicyclic Trains.—These are frequently used, an example being shown in Fig. 249. The bevel wheel A meshes with the planet bevel wheel B, which is free to turn on the arm R. The planet B also meshes with a bevel wheel C. In one use of the train the wheel C is held stationary, A is the driving member and the arm is the driven member. The gear ratio is easily found by the tabular method.

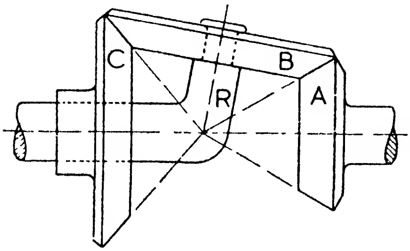


Fig. 249

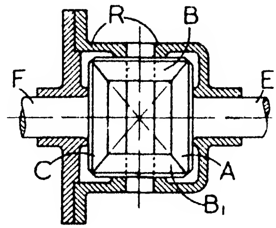


Fig. 250

203. The Differential.—Referring to Fig. 249, if the wheels A and C are made equal in size, the train becomes the common differential used on motor cars and in many machines. As used in motor cars the arm R is driven by the engine and A and C are coupled to the driving wheels, the construction being on the lines of Fig. 250. The arm R takes the form of a drum-like casing, usually made in two parts for convenience in manufacture and assembly. The planets B and B_1 are carried by a pin running radially across the casing, two planets (and sometimes three or four) being used to reduce the loads on the teeth and to give rotational balance. The planets mesh with the wheels A and C, which are fixed rotationally to the road-wheel shafts E and F.

It will easily be seen that if the casing is revolving at say 400 r.p.m. and A is revolving in the same sense at say 390 r.p.m., then the speed of C will be 410 r.p.m. in the same sense also. This is the action when rounding a corner; when the car is going in a straight line A, C and R all revolve at the same speed, and there is no relative motion between the planets BB_1 and the pin that carries them.

204. The Differential as an Adding Mechanism.—A differential forms a convenient mechanism for adding or subtracting two motions, and it is used for this purpose in gear-cutting machines and in various forms of calculating machines. Referring to Fig. 250, if the C is held stationary and A is turned through an angle θ, then the arm R will be turned in the same direction through $\frac{\theta}{2}$. If now A is held stationary and C is turned through an angle ϕ, then the arm R will be turned through $\frac{\phi}{2}$ and its total motion will be $\frac{\theta+\phi}{2}$, being thus proportional to the sum of the motions of A and C. The motions of A and C may of course be given simultaneously instead of consecutively.

A differential also affords a convenient method of changing the phase relationship between a driving and a driven shaft whilst they are running. If the casing R in Fig. 250 is held stationary, then rotation of A will drive C at the same speed, but in the opposite direction. If now R is turned through any angle the phase relationship between A and C will be altered by twice that angle.

205. Compound Epicyclic Trains.—A compound epicyclic train consists of a combination of two simple epicyclic trains, some of the members of one train being integral with some of the members of the other train. Usually one of the trains may be regarded as the main train and the other as the auxiliary train, the function of the latter being to give motion to some member of the main train. An example is shown in Fig. 251, the main train being numbered 1 and the auxiliary train 2, both trains being of the sun and annulus type. They are compounded by making the sun S_2 integral with the annulus A_1 and the arm R_2 integral with the arm R_1. The annulus A_2 is held stationary, and then rotation of S_1 causes R_1R_2 to rotate, and vice versa. The main train 1 may be regarded as having two driving members, the sun S_1, which is coupled to some external motor, and the annulus A_1, which receives its motion from the auxiliary train; the arm R_1 then receives simultaneously the sum of two motions, one that due to the motion of the sun S_1 and the other that due to the motion of the annulus A_1. The gear ratio between the sun S_1 and the arm R_1 may be found by giving the arm R_2 of the auxiliary train (the only train having a fixed member) one turn. The resulting motion of S_2 may then be found. Using the tabular method and taking the numbers of teeth to be $S_2=40$, $A_2=80$, $S_1=30$, $A_1=90$, the working is shown in Table 1 and the motion of S_2 is seen to be $+3$ turns. Next consider the main train. In this the arm R_1,

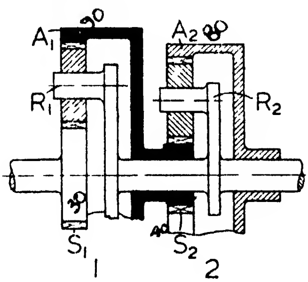


FIG 251

R_2	S_2	A_2
$+1$	$+1$	$+1$
0	$+\frac{80}{40}$	-1
$+1$	$+3$	0

TABLE 1

being integral with R_2, has received +1 turn, and the annulus A_1, being integral with S_2, has received +3 turns; the resultant motion of S_1 may therefore be found. Using the tabular method, the working is shown in Table 2. It is seen to be −5 turns; hence the gear ratio between S_1 and R_1 is −5 to 1 and the train gives a reverse drive.

R_1 R_2	A_1 S_2	S_1
$+1$	$+1$	$+1$
0	$+2$	$-2 \times \frac{90}{30}$
$+1$	$+3$	-5

TABLE 2

In the above example the driving member of the auxiliary train, the arm R_2, received motion because it was made integral with the driven member, the arm R_1, of the main train; in the example shown in Fig. 252 the driving member of the auxiliary train is the sun S_2, and this receives its motion by being integral with the sun S_1, the driving member of the main train. The annulus A_1 of the main train is driven by the auxiliary train, with the arm R_2 of which it is integral. The annulus A_2 is fixed. The gear ratio may be found as follows: Let the numbers of teeth be $S_1=25$, $A_1=75$, $S_2=30$, $A_2=90$, and let the arm of the auxiliary train be given one turn. The resulting motion of S_2 is then found by the tabular method, thus:

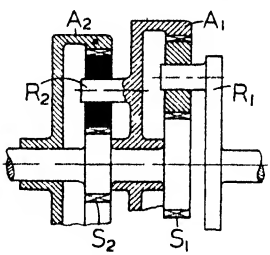


FIG. 252

R_2	A_2	S_2
$+1$	$+1$	$+1$
0	-1	$+\frac{90}{30}$
$+1$	0	$+4$

Therefore, in the main train, the motions of A_1 and S_1 are known, being respectively $+1$ and $+4$ turns. The resulting motion of R_1 may then be found, thus. Let it be x turns and let the motion of S_1 when R_1 receives x turns and A_1 receives $+1$ turns be found (in terms of x); then on equating the result to the actual motion of S_1, i.e. $+4$ turns, x may be found. In tabular form we have:

R_1	A_1	S_1
x	x	x
0	$-x+1$	$-(-x+1)\times\frac{75}{25}$
x	$+1$	$4x-3$

Then $4x-3=4$, $\therefore x=1\frac{3}{4}$, and the gear ratio between S_1 and R_1 is 4 to $1\frac{3}{4}$ or 16 to 7.

206. Doubly Compounded Trains.—The process of compounding epicyclic trains may be carried on indefinitely, thus we might add to the gear of Fig. 252 a third train whose function is to give motion to the annulus of the second train. The resulting gear might then be arranged as in Fig. 253. The sun S_3 is the fixed member, being held stationary by a brake applied to the drum D fixed to it. The gear ratio may be found by giving the arm R_3 of the auxiliary train having a fixed member one turn, and finding

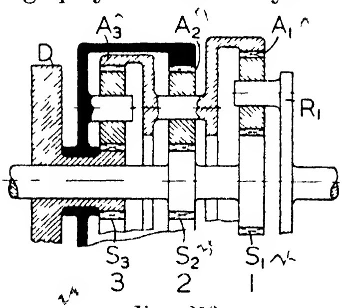


FIG 253

the resulting motion of A_3. If the numbers of teeth in S_3 and A_3 are 25 and 75 respectively, the motion of A_3 will be $+\frac{4}{3}$. Then in train No. 2 the arm R_2, being integral with A_3, has been given $+\frac{4}{3}$ turns, and the annulus A_2, being integral with R_3, has been given $+1$ turn; the resulting motion of S_2 is then found to be $+\frac{7}{3}$. Lastly, in train No. 1, the annulus A_1, being integral with R_2 and A_3, has received $+\frac{4}{3}$ turns, and the sun S_1, being integral with S_2, has received $+\frac{7}{3}$ turns; the resulting motion of the arm R_1 is then found to be $+\frac{19}{12}$ turns. The gear ratio between S_1 and R_1 is therefore $+\frac{7}{3}$ to $+\frac{19}{12}$ or $1\frac{9}{19}$ to 1.

207. Condition for the Assembly of Epicyclic Trains having more than One Set of Planet Pinions.—It has been mentioned in connexion with differentials that two, three or more planet wheels or sets of planet wheels are generally used, principally to reduce the tooth loads and secondarily to obtain rotational balance, and this is also done in most epicyclic trains. This introduces a limitation in the design because the numbers of teeth in the various wheels must satisfy a certain condition if all the planet gears are to mesh properly or even be assembled. This condition for a sun and annulus train will now be established. Suppose the train in Fig. 254 is to have n planet gears spaced at equal intervals (equal spacing is assumed throughout this section), then the angle between the arms R and R_2 carrying any two consecutive planets is $\frac{360°}{n}$. With the arm R in the position shown in full line let the planet P_1 be meshed with the annulus and sun; then, keeping the sun stationary, let the arm R_1 be turned through the angle $\frac{360°}{n}$ so that the arm R_2 comes to the full-line position. It will obviously now be possible to mesh the planet P_2 with the sun, but it will only be possible to mesh it with the annulus if the latter has moved through a whole number of pitches so that one of its tooth spaces now occupies the same position relative to the sun and arm as one did when the first planet was meshed. The annulus has, however, moved through an angle $\frac{360}{n}\left(1+\frac{S}{A}\right)$ which corresponds to $\frac{360}{n}\left(1+\frac{S}{A}\right)\div\frac{360}{n}$ pitches. Therefore $1+\frac{S}{A}$ must equal a whole number or $A+S=kA$, where k is an integer, and this is the condition. It can also be expressed in the form $A+S=k_1S$, where k_1 is an integer. Another condition is that if D_a, D_s and D_p are respectively the pitch diameters of the annulus, sun and planet, then $D_a=D_s+2D_p$, which on multiplying by the diametral pitch becomes $A=S+2P$. If the amount of backlash between the teeth can be varied, this condition need only be satisfied approximately. A third condition of course is that $1+\frac{A}{S}$ or $1+\frac{S}{A}$ (according as to whether the annulus or sun is the fixed member) shall equal either exactly or approximately, the required gear ratio. Lastly, the diametral pitch of the teeth is determined by considerations of the power to be transmitted, the speeds and the space available, etc.

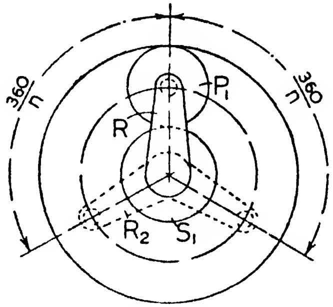
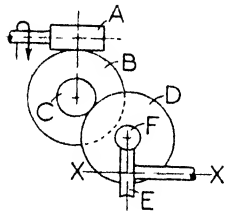


FIG. 254

In a manner similar to that used above the corresponding conditions for a double-sun type epicyclic train can be shown to be first $\frac{S_1}{S_2}=k\frac{P_1}{P_2}$, k being an integer and $S_1S_2P_1$ and P_2 the numbers of teeth in the suns and planets; secondly, $S_1+P_1=S_2+P_2$ if all the teeth are of the same pitch; and thirdly, that the numbers of teeth give the required gear ratio. It may be noted that the first condition is based on the assumption that the position of one toothed position of a planet wheel relative to the other toothed portion is the same for all the planets. If the toothed portions of the planets can be moved relatively, then the condition need not be satisfied.

EXERCISES XVI

1. In a simple gear train the driver has 20 teeth and the final driven gear has 40 teeth. There are three idlers. If the speed of the driver is 100 r.p.m. clockwise, what are the speed and direction of rotation of the driven gear ?



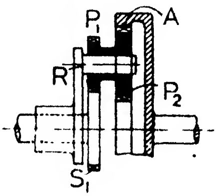
2. In the compound train shown in the figure the worm A has 2 starts and is right-handed, and the wheels B, C, D and E have respectively 30, 20, 50 and 40 teeth. The worm F has 3 starts and is left-handed. If A rotates at 100 r.p.m. in the direction shown by the arrow, what are the speed and direction of rotation of E ? Give the direction as seen when viewed from right to left.

3. A reverted train in which all the teeth are to be of 10 D.P. is to have an overall ratio of 3 to 1. If the centre distance between the driving and layshaft axes is to be 4 in. and the gear ratio between them is to be $1\frac{1}{2}$ to 1, find the numbers of teeth for all the wheels, (*a*) if the centre distance is to be kept exact, and (*b*) if the gear ratio is to be exact. Find also the actual gear ratio in case (*a*) and the actual centre distance in case (*b*).

4. A sun and annulus type of epicyclic train is used with the annulus fixed. If the arm is rotated at +1000 r.p.m., what will be the speed of the sun if it has 25 teeth and the annulus has 100 teeth ?

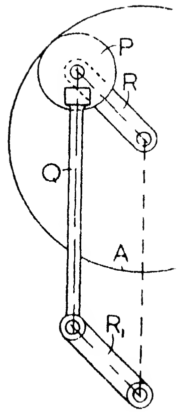
5. The annulus of an epicyclic train has 80 teeth and rotates at +500 r.p.m., the sun being fixed and having 20 teeth. What is the speed of the arm ?

6. In a double-sun type epicyclic gear the suns S_1 and S_2 have 28 and 35 teeth and are of the same pitch. S_1 is fixed and meshes with a planet having 30 teeth. What is the gear ratio between arm and S_2 ?

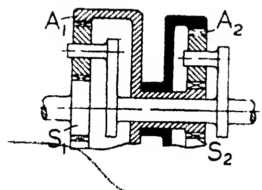


7. The arm R of the epicyclic train shown in the figure rotates at 1000 r.p.m. and the sun S rotates in the same direction at 500 r.p.m. If the numbers of teeth are as shown, what is the motion of the annulus ? $S_1=25$, $P_1=20$, $P_2=30$, A 75.

8. In the mechanism shown in the figure the planet P which meshes with the annulus A is prevented from rotating by the link Q and crank R_1. If the arm R rotates at 100 r.p.m., what is the speed of rotation of the annulus? Numbers of teeth: $A = 100$, $P = 30$.

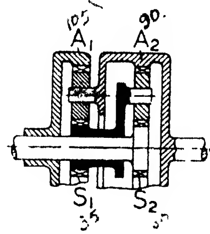


9. Find the gear ratio of the compound epicyclic gear shown in the figure. The annulus A_2 is fixed. Numbers of teeth: $S_1 = 25$, $A_1 = 100$, $S_2 = 30$, A_2 90.



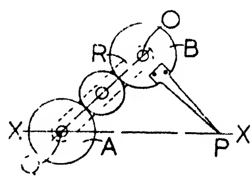
10. Find the speed of the annulus A_2 in the compound epicyclic gear shown when the sun S_2 rotates at $+1000$ r.p.m. and the annulus A_1 rotates at -500 r.p.m. Numbers of teeth:

$$S_2 = 30, \ A_2 = 90, \ S_1 = 35, \ A_1 = 105.$$



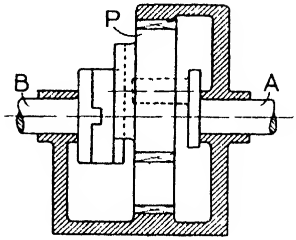
11. Deduce the relationship that must obtain between the numbers of teeth in the bevel gears A and C of Fig. 249 if n equally spaced planet wheels are to be used.

12. The wheels A and B in the figure have the same number of teeth, and A is fixed. B carries a pointer whose length OP equals the centre distance OQ between A and B. Prove that when the arm R rotates the point P describes a straight line XX with simple harmonic motion.



13. Find the gear ratio of the doubly compounded epicyclic gear shown in Fig. 253. The sun S_1 is fixed by means of the brake drum D and the numbers of teeth are: $S_1 = 31$, $A_1 = 93$, $S_2 = 25$, $A_2 = 75$, S_3 20, $A_3 = 60$. N.B. The arm R_2 is integral with the annuli A_1 and A_3.

14. The figure shows the " Avamore " reduction gear in which the planet P is coupled to the input shaft B by means of an Oldham coupling. Find the gear ratio. Numbers of teeth : planet 40, annulus 90.



CHAPTER XVII

WRAPPING CONNECTORS—BELTS, ROPES AND CHAINS

208. General.—A wrapping connector is one which is flexible enough to wrap round two or more pulleys or wheels, and so can be used to transmit motion between them. When the connector takes the form of a band of material such as leather, fabric, rubber or steel whose thickness is small in comparison with its width it is called a **belt** or **strap**; if its cross-section is circular, or approximately circular, it is a **cord** or **rope**; while if it is composed of links hinged together it is generally called a **chain.** Chains are generally used in conjunction with wheels of such a form that no reliance has to be placed on friction to prevent slip between them, while belts and ropes generally rely on friction to prevent slip. Chains will be considered more particularly at the end of this chapter.

The pulleys that are connected by belts and ropes are usually, but not necessarily, circular, and the belts and ropes themselves are usually endless, either by virtue of their manufacture or because their ends are joined by a fastener.

209. The Velocity Ratio of Wrapping Connectors.—Fig. 255 shows two non-circular pulleys pivoted respectively at A and B and connected by a belt or cord which (if the motion of the pulleys is not required to be continuous) may be secured to the pulleys at points such as C and D, thus making the connexion independent of friction between the belt and the pulleys. If continuous motion is required the belt must be endless; it cannot, of course, be

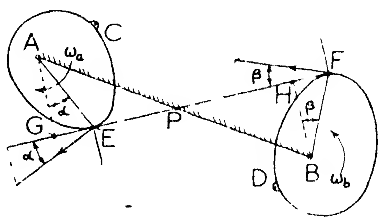


FIG. 255

secured to the pulleys, and it will generally have to be passed over some spring or gravity-loaded tightening pulley in order to keep the tension in it approximately constant and to maintain sufficient frictional forces between it and the pulleys to prevent slip. Let the instantaneous angular velocities of the pulleys be ω_a and ω_b

as indicated, and let E and F be the points of tangency of the belt and pulleys. From A and B drop perpendiculars AG and BH on to EF produced if necessary. Let the angles GAE and FBH be α and β respectively. The instantaneous velocity of E is AE . ω_a perpendicular to AE, and the component of this along EF is AE . ω_a Cos α = AG . ω_a. Similarly the component along EF of the velocity of F is BH . ω_b. Now, if the belt is regarded as inextensible, these components must be equal; hence

$$AG \,.\, \omega_a = BH \,.\, \omega_b$$

$$\therefore \frac{\omega_a}{\omega_b} = \frac{BH}{AG} = \frac{BP}{AP}$$

where P is the intersection of EF with the line joining the centres A and B. The connexion is thus equivalent, at the instant under consideration, to a four-bar chain AEFB.

It follows that if the velocity ratio is required to be constant the point P must be fixed. This condition can be complied with by non-circular pulleys, but when a constant velocity ratio is required circular pulleys are always used, and these only will be considered in what follows.

210. Constant Velocity Ratio Belt Gearing.—The condition for constant velocity ratio is obviously fulfilled by a belt connecting circular pulleys as in Fig. 256, the ratio being

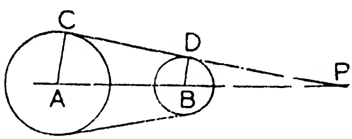


Fig. 256

$$\frac{\omega_a}{\omega_b} = \frac{BP}{AP} = \frac{BD}{AC} = \frac{D_b}{D_a}$$

where D_a and D_b are the pulley diameters. The ratio may also be found from the consideration that if the belt is inextensible its speed in the direction of its length at every point must be constant, and if no slip occurs between the belt and the pulleys, and if the belt is thin in relation to the pulley diameters, the speed of the belt must be the same as the peripheral speeds of the pulleys. Hence these peripheral speeds are equal and

$$\pi D_a N_a = \pi D_b N_b$$

so that

$$\frac{N_a}{N_b} = \frac{D_b}{D_a} = \frac{\omega_a}{\omega_b}$$

211. The Effect of the Thickness of a Belt.—When a thick belt is wrapped round a pulley the outer layers of its fibres are extended and the inner layers are compressed, while one layer at the middle is unaltered in length, and it is the speed of this layer which is constant throughout its length. Let this speed be v, and let the thickness of the belt be t. Then Fig. 257 shows that

$$v=\frac{D_a+t}{2}\cdot\omega_a$$

and similarly $$v=\frac{D_b+t}{2}\cdot\omega_b$$

hence $$\frac{\omega_a}{\omega_b}=\frac{N_a}{N_b}=\frac{D_b+t}{D_a+t}$$

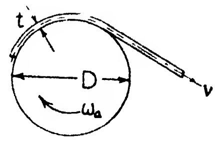


FIG. 257

In practice this correction for the belt thickness is hardly ever applied.

212. The Run of a Cord on a Pulley.- Fig. 258 shows a portion of a cord running round a cylindrical pulley: the part AB is approaching, the part BCD is in contact with, and DE has left the pulley. Clearly DE can have no influence on the way the cord wraps on to the pulley if the friction between the part BCD and the pulley is sufficient to prevent slip, but the part AB has a controlling influence. If AB is perpendicular to the axis of the pulley, then the cord will wrap itself on to the pulley in a circle and the belt will run steadily, the portion ABCD always lying in the same plane; but if the approaching portion of the belt makes an angle $90-\phi$ with the pulley axis as shown by the dotted line, then each

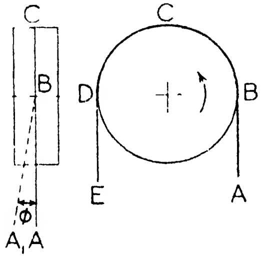


FIG. 258

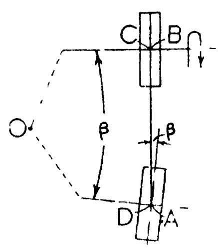


FIG. 259

successive portion of the cord will come on to the pulley a little to one side of the preceding portion and the cord will wrap on to the pulley in a helix whose spiral angle is $90-\phi$. If the speed of the cord is v, then the cord will travel along the pulley in the direction of the axis with a speed $v\ \text{Sin}\ \phi$. Thus if a cord runs round two pulleys whose axes are co-planar but not parallel, as in Fig. 259, it will travel towards the point of intersection of the axes unless it is prevented by guides, because if the portion AB at the top runs on to the top pulley perpendicular to the axis, the portion CD underneath must necessarily run on to the bottom pulley at the angle β, equal to the angle between the axes. If, however, the axes do not intersect, i.e. are skew axes, it is possible to arrange the pulleys so that a cord will run steadily on them for one direction of rotation.

213. Cords Connecting Pulleys on Skew Axes.—Let the axes be 1 1 and 2 2 in plan and 1′ 1′ and 2′ 2′ in elevation, the plan view being taken along the common normal or shortest distance between the shafts, as in Fig. 260. The pulleys must be arranged so that each portion of the cord that is advancing towards a pulley shall lie in the central plane of that pulley. Thus the portion AB which is advancing towards the upper pulley must lie in the plane of that pulley, and the portion CD which is advancing towards the lower pulley must lie in the plane of that pulley. This condition for steady running of the cord can be stated in the form : *the point of delivery of the cord from the first pulley must lie in the central plane of the second pulley and the point of delivery of the cord from the second pulley must lie in the plane of the first pulley.* Alternatively that the central planes of the pulleys must intersect in the line joining the points of delivery of the pulleys.

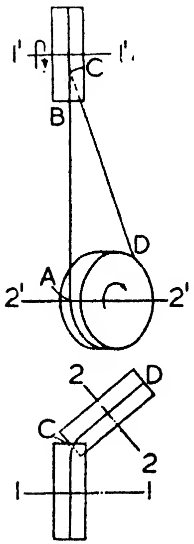


FIG. 260

Cylindrical pulleys are, however, little used with cords, for which V pulleys are generally provided, and in this case it becomes possible to arrange a skew drive that will run steadily in both directions. Fig. 261 shows a portion of a cord embracing a pulley having a V-shaped rim. The portion AB of

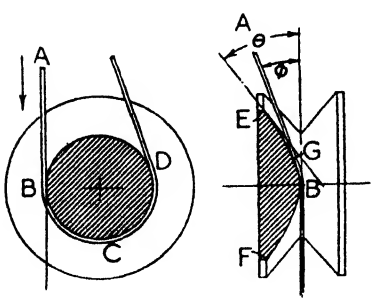


FIG. 261

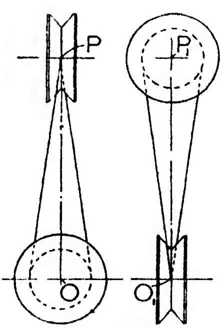


FIG. 262

the cord which is advancing towards the pulley lies in a plane tangential to the bottom of the groove at B, and this plane intersects the V surfaces of the pulley in a hyperbola, one branch of which, EBF, is shown. The cord AB is straight between A and G, at which point it touches the pulley; between G and B the cord follows the hyperbolic section of the pulley. Provided the angle φ at which the cord approaches the pulley is less than the limiting value θ for which the point G coincides with E, the cord will run steadily on to the pulley. A skew drive arranged as in Fig. 262, being symmetrical, will therefore run equally well in either direction. The arrangement brings the common normal or shortest distance (OP) between the axes to lie in the central planes of both pulleys.

214. Pulley Camber or Crowning.—What has been said in Arts. 212 and 213 about the run of a cord on a pulley applies also to flat belts, but the latter possess considerable lateral stiffness, which a cord does not, and this stiffness is utilised to help a belt to run steadily. Thus the pulleys used with flat belts are frequently *cambered* or *crowned* as indicated in Fig. 263, being made

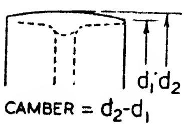


FIG. 263

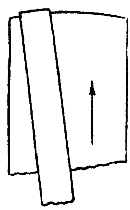


FIG. 264

larger in diameter at the centre than at the edges. The manner in which this camber helps the belt to run truly is shown in Fig. 264, where the belt is shown lying on one side of the pulley. Because of the lateral stiffness of the belt, and because the natural path of a flat inextensible belt which is wrapped on to a conical surface is a spiral, the free portion of the belt is deflected towards the centre of the pulley, and as the belt advances on to the pulley it moves towards the centre until finally it runs truly on the middle of the pulley. This action can only occur if no slip occurs between the belt and the pulley over an arc towards the advancing side, since if slip did occur the portion of the belt on the pulley could not control the free advancing portion. It has been shown* that slip or creep occurs between a belt and a pulley over an arc starting at the point of delivery of the belt and extending over an

* "Power Transmission by Belts: An Investigation of Fundamentals," by H. W. Swift, M.A., D.Sc., *Proc. I.Mech.E.*, No. 3, 1928.

angle which depends on the ratio of the forces in the advancing and receding parts of the belt and on the coefficient of friction. As long as the arc over which creep occurs is less than the whole arc of contact of the belt and pulley there will be an arc, at the advancing side, over which creep does not occur, and the self-centring action described above will take place if the belt is displaced to one side of the pulley. When, however, the ratio of the forces in the two sides of the belt reaches a certain limit, such that the arc over which creep occurs is approximately equal to the whole arc of contact, there will be no " idle " arc, and the self-centring action will not take place. The belt will then probably slip bodily off the pulley. For a full discussion of pulley camber the reader is referred to a paper entitled "Cambers for Belt Pulleys," by H. W. Swift, M.A., D.Sc., *Proc. I.Mech.E.*, Vol. 122, p. 627.

215. Fast and Loose Pulleys.—When a belt is used to drive a machine which may have to be stopped while the driving shaft continues to revolve, fast and loose pulleys are frequently used either on the machine itself or, more often, on an intermediate or *counter* shaft. The arrangement is indicated in Fig. 265, where A is a wide cylindrical pulley fixed to the driving shaft, B is a pulley which is free to rotate on the driven shaft, being provided with either a phosphor bronze bush or some form of ball or roller bearing to reduce friction and wear, and C is a pulley that is fixed to the driven shaft D. Pulley B is fixed axially and is sometimes made slightly smaller in diameter than the fixed pulley in order to reduce the pull in the belt and the loads on the bearings when the belt is idling. Both pulleys B and C are frequently crowned. In the position shown the belt is idling and the shaft D is at rest; if, however, the belt is moved along to the dotted position, D will be driven by the belt. To shift the belt some form of belt-shifting or striking mechanism is provided.

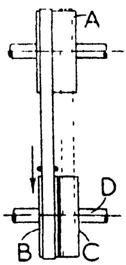


Fig. 265

216. Belt-Shifting Mechanisms.—The simplest shifting mechanism consists of a rectangular-section bar carried in guides so that it can slide parallel to the axis of the pulley on which the belt to be shifted runs. This bar carries two prongs or fingers in between which the belt lies, the fingers being arranged to be as close to the pulley as is convenient and, of course, on the advancing side of the belt. When the bar is slid along the belt is shifted. This simple arrangement can only be used when the fast and loose pulleys are at a convenient height. Usually they are near to the ceiling, and some lever or other arrangement must be provided to enable the sliding bar carrying the shifting fork or fingers to be moved along.

Two pairs of fast and loose pulleys are frequently used to enable a countershaft or machine to be driven at two different speeds or in different directions, and Fig. 266 shows a form of striking gear

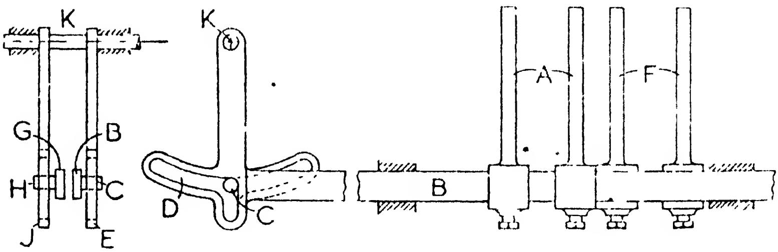


FIG. 266

used in this connexion to ensure that one belt is on its loose pulley before the other can be shifted on to its fast pulley, thus obviating trouble due to the belts coming off because both are trying to drive the driven shaft. The fingers A controlling one belt are secured to a sliding bar B which has a projecting pin C engaging the slot D in the lever E. and similarly the fingers F controlling the other belt are secured to another sliding bar G which also has a projecting pin H engaging a second lever J. Both levers E and J are secured to a shaft K which can be turned through an angle of from 30° to 60° by a lever. The slots in the levers are similar, consisting of a radial portion merging into a portion concentric with the axis of the shaft K. In the position shown both belts are on their loose pulleys, and, clearly, if the shaft K is turned in the clockwise direction the lever E will cause the pin C and thus the fingers A to move to the left, thus bringing one belt on to its fast pulley. The other pin H is not moved. since it is engaged with the circular portion of the slot in the lever J, and it cannot be moved without first bringing the pin C, and thus the belt controlled by the fingers A, back to the neutral position shown.

217. Speed Cones and Stepped Pulleys.—These are used to enable a shaft running at a constant speed to drive another shaft at any desired speed within a certain range. By varying the position of the belt on the conical pulleys shown in Fig. 267 the speed of the driven shaft can be varied from a minimum, when the belt is in the position shown, to a maximum when it occupies the dotted position. Owing to the tendency of a belt to move towards the

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larger diameter of a coned pulley, guides must be provided to control the belt, and, unless some tensioning device is used, the profile of the pulleys must be such that the tension in the belt is maintained approximately constant. The arrangement is not

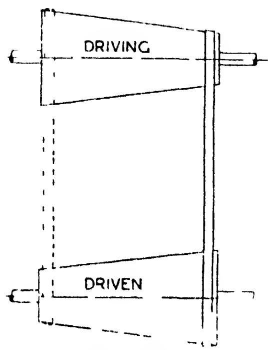


FIG. 267

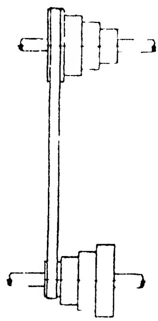


FIG. 268

very satisfactory except for very light loads and is rarely used; its chief attraction is that it gives an infinite number of speeds between the maximum and minimum.

Fig. 268 shows an example of stepped pulleys. The number of speeds available at the driven shaft is equal to the number of steps, namely four, so that the fine adjustment possible with coned pulleys is lost, but the arrangement is nevertheless much more satisfactory and consequently much more widely used, although it is being displaced by gear-boxes such as are described in the next chapter. The diameters of the steps must be arranged so that the tension in the belt remains approximately constant, which implies that the length of the belt required for each step must be the same, and this will now be considered, expressions being derived for the length of a belt in terms of the pulley diameters and the centre distance between the shaft axes. Of course, if the length of a belt to run on a pair of existing pulleys is required the practical method of obtaining the necessary length for the belt is to wrap a cord round the pulleys and then to measure the cord.

218. The Length of a Belt. **(1) Open.**—Let the diameters of the pulleys be d and D, as in Fig. 269, and let the distance between the shaft axes be L. Draw AC parallel to the belt DE. Then,

$$\text{Sin } \phi = \frac{BC}{AB}$$

$$= \frac{D-d}{2L}$$

Now the total length of the belt is given by

$$
\begin{aligned}
l &= \mathrm{ED} + \text{Arc DF} + \mathrm{FG} + \text{Arc GE} \\
&= 2\mathrm{L} \operatorname{Cos} \phi + \text{Arc DF} + \text{Arc GE} \\
&= 2\mathrm{L} \operatorname{Cos} \phi + \frac{d}{2}(\pi - 2\phi) + \frac{\mathrm{D}}{2}(\pi + 2\phi) \\
&= \sqrt{4\mathrm{L}^2 - (\mathrm{D}-d)^2} + \frac{\pi}{2}(\mathrm{D}+d) + (\mathrm{D}-d) \operatorname{Sin}^{-1}\frac{(\mathrm{D}-d)}{2\mathrm{L}} \qquad (1)
\end{aligned}
$$

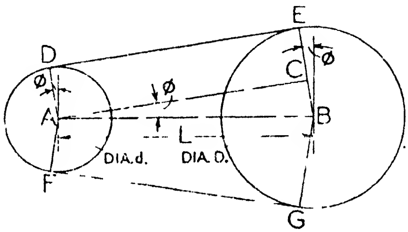


FIG. 269

When ϕ is small, that is, when $\mathrm{D}-d$ is small in comparison with L, the above expression may be simplified by writing Sin ϕ instead of ϕ and by expanding the expression under the root sign and retaining only the first two terms. Thus :

$$
\begin{aligned}
l &= 2\mathrm{L}\left\{1 - \left(\frac{\mathrm{D}-d}{2\mathrm{L}}\right)^2\right\}^{\frac{1}{2}} + \frac{\pi}{2}(\mathrm{D}+d) + (\mathrm{D}-d) \operatorname{Sin} \phi \\
&= 2\mathrm{L}\left\{1 - \tfrac{1}{2}\left(\frac{\mathrm{D}-d}{2\mathrm{L}}\right)^2\right\} + \frac{\pi}{2}(\mathrm{D}+d) + \frac{(\mathrm{D}-d)^2}{2\mathrm{L}} \\
&= 2\mathrm{L} + \frac{(\mathrm{D}-d)^2}{4\mathrm{L}} + \frac{\pi}{2}(\mathrm{D}+d) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)
\end{aligned}
$$

In the design of a stepped pulley one step may be settled arbitrarily and the necessary length of belt calculated by means of the above expressions. The remaining steps must then be determined by solving the two equations giving respectively the required velocity ratio and the required length of belt, in terms of the two unknown pulley diameters.

If the approximate expression, Eq. (2), for the length of belt will suffice, the solution is simple, but if the accurate expression, Eq. (1), has to be used the solution is involved. Reuleaux in his *The Constructor*, published by H. H. Suplee, Philadelphia, U.S.A., page 189, gives a graphical method of solution to which the reader is referred for further information. As Reuleaux's book is out of print, it may be mentioned that his construction is given in R. J. Durley's *Kinematics of Machines*, John Wiley & Sons, New York, page 247.

219. (2) Crossed Belts.— Fig. 270 shows that $\text{Sin}\ \phi = \frac{D+d}{2L}$ and the length of belt is given by

$$l = \text{Arc DF} + \text{Arc GE} + 2\text{DE}$$

$$= \frac{d}{2}(\pi + 2\phi) + \frac{D}{2}(\pi + 2\phi) + 2\text{L Cos}\ \phi$$

$$= (\pi + 2\phi)\frac{(D+d)}{2} + 2\text{L Cos}\ \phi;$$

hence the length of belt is constant if $D+d$ is constant, and a

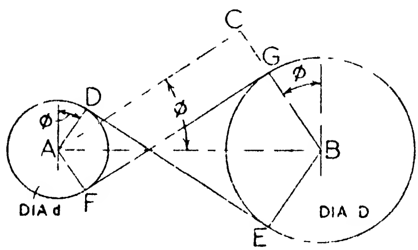


FIG. 270

stepped pulley for use with a crossed belt may be designed by solving the equations $D+d$=constant and $\frac{D}{d}$=required velocity ratio.

It may be mentioned, however, that stepped pulleys are generally designed for open belts.

220. Jockey Pulleys.—It can be shown that the difference that can be maintained between the tensions in the advancing and receding parts of a belt without serious slip occurring between the belt and the pulley depends on the length of the arc over which the belt makes contact with the pulley, and this *arc of contact* should be as large as possible. In general it should never be less than about 120°. Now, when the velocity ratio is high, so that the pulleys are very dissimilar in size, and when the centre distance is small in comparison with the larger pulley diameter, it is often not possible to obtain a sufficiently long arc of contact with open belts without resorting to the use of a *jockey* pulley as indicated in Fig. 271, where A is the jockey pulley, carried on bearings on the end of a lever pivoted at B, in such a manner that the jockey pulley runs on the slack side of the belt and increases the arc of contact of the belt on the smaller pulley. Sometimes the weight of the jockey pulley and lever may be sufficient to give the necessary tensioning effect, but often additional dead weights, or a spring as shown, must be used for this purpose. Obviously if a belt fastener is used it must be of such a type that the jockey pulley can run properly on the back of the belt. A jockey pulley should never be made to act on the tight side of a belt.

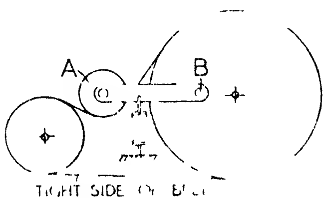


FIG. 271

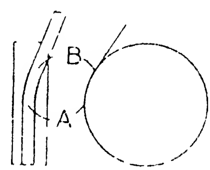


FIG. 272

221. Skew Belt Drives.—Skew drives are as practicable with flat belts as with cords, and the same fundamental condition must be fulfilled; namely, that the point of departure of the belt from one pulley must lie in the central plane of the other pulley and vice versa. Some slight complication arises because the belt deviates from the central planes of the pulleys before losing contact with the pulley surfaces, as is shown in Fig. 272, where B is the actual point of departure of the belt and is the point which must lie in the central plane of the other pulley. Prof. J. B. Webb has shown * that the portion AB of the belt would be a catenary if the pulley surface were developed into a plane, and has shown how to determine the proper relative positions for the pulleys when allowance is made for this deviation of the belt. With most skew drives in practice the points of departure may be assumed to lie in the central planes of the pulleys in order to arrive at a first approximation to the proper positions for the pulleys whose final positions may be determined by trial after the drive has been erected, when it will generally be found that only small adjustments of the pulleys will be required in order to make the belt run truly.

222. Guide Pulleys.—By the use of suitably placed guide pulleys a skew drive may be made reversible. Thus referring to Fig. 273, two points, *a* and *b*, are chosen in the line of intersection XX of the planes of the pulleys and the tangents *ac*, *ad*, *be* and *bf* are drawn. The guide pulleys are then placed so that their planes

* *Trans. Am. Soc. Mech. Eng.*, Vol. IV, 1882 3.

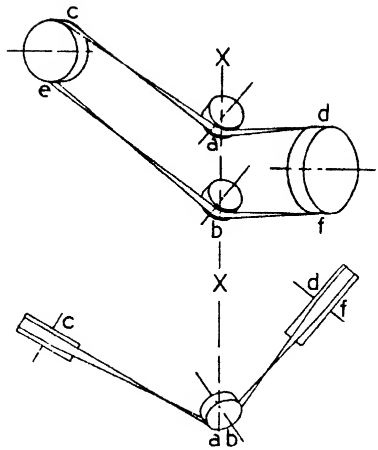


FIG. 273

contain respectively the tangents *ac*, *ad* and *bc*, *bf*. The use of guide pulleys often simplifies the design of skew drives which are not required to be reversible.

223. V Belts.—Belts of V section are sometimes used; they are generally made either from a number of links, usually of metal faced with leather, or rubber, which are hinged together, or in the form of a continuous endless band of rubberised fabric. Such belts can transmit somewhat higher powers than flat belts. They can, of course, be used only for open drives between parallel shafts.

224. Rope Drives.—Ropes made from hemp or similar material were once extensively used for transmitting high powers, but they are now not so widely used. Two systems, known respectively as the *Multiple rope* or *English* system and the *Single rope* or *American* system, have been evolved. In the former the pulley surfaces are provided with a number of V grooves and an equal number of separate endless ropes are used. In the latter a single rope is used, as indicated in Fig. 274, its path from the point A being round the end groove in the pulley B across to the end groove of C, back to the second groove of B, and so on until it leaves the last groove of C and passes round the guide pulley D to the tensioning pulley E, and thence round the guide pulley F back to A again. The tensioning pulley E is mounted in a carriage which is free to slide in fixed guides and which is pulled on by a weight W so as to maintain the correct tension. Fig. 275 (*b*) shows the rope in a groove of a driving or driven pulley, the contact being on

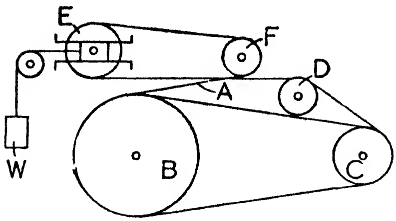


FIG. 274

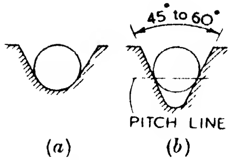


FIG. 275

the sides of the V so as to secure a wedging action, while (*a*) shows the groove of a guide pulley in which the rope seats on the bottom. When the ropes are made of wire they are always made to seat on the bottom of the groove, which is then usually lined with some material which will provide a high coefficient of friction.

225. Chain Drives.—The common oval-link chain is not suitable for transmitting motion and power at any but extremely low speeds, and is consequently used only in such applications as in hoisting tackle, etc., where the speeds are low. The chain shown in Fig. 276, which is commonly known as the *malleable* chain (its links being malleable iron castings), is also suitable only for low speeds. It is used in agricultural and in conveying machinery. The *block* chain of Fig. 277 can be used at moderate speeds, but

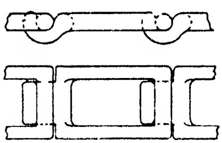


FIG. 276

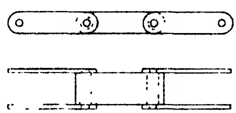


FIG. 277

for high speeds either the *roller* chain (Fig. 278) or the *inverted-tooth* or *silent* chain (Fig. 279) must be used. In the latter the link

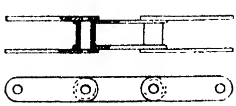


FIG. 278

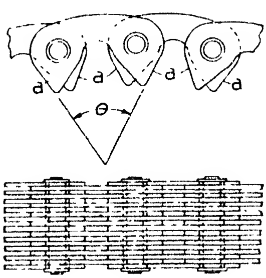


FIG. 279

plates are all exactly alike, the working faces, a, a, being straight, and the chain may be made up in various widths by using more or less plates side by side. The angle θ between the faces aa is usually either 60° or 75°. This type of chain has the property of automatically accommodating any increase in pitch due to wear in the pivot pins. It does this by seating itself higher on the wheel teeth and, as a result, it is not normally much noisier when worn than when new, whereas the roller chain is usually very noisy when worn. In the Renold inverted-tooth chain the pins connecting the links engage half-bushes instead of complete bushes; the initial fit of pins and bushes can consequently be made closer and much less initial "stretch" occurs. In the *Morse* chain the type of joint shown in Fig. 280 is used, with the object of substituting rolling friction for the sliding friction that occurs in the ordinary pin joint. The half-pin a is fixed in the link A and rolls on the half-pin b which is fixed in the link B, holes of suitable shape being provided in the links so that a clears B and b clears A.

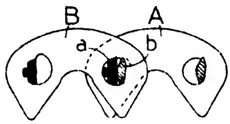


FIG. 280

226. Variation of Velocity Ratio in Chain Drives.—A chain drive can be considered to be, at any instant, a four-bar chain, if the free portion of chain between the sprockets is assumed to be rigid or inextensible. It follows that the angular velocity ratio between the sprockets is continually varying between the maximum when the relative position of the sprocket teeth is as in Fig. 281 (a) and

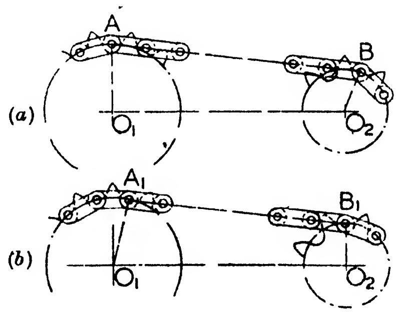


FIG. 281

the minimum when the teeth are as in Fig. 281 (b). O_1ABO_2 and $O_1A_1B_1O_2$ are the equivalent four-bar chains, AB and A_1B_1 being the free portions of the chain. Unless wheels having very few teeth are used this variation is generally negligible, and probably is less than the variation due to the flap of the free portion of the chain.

EXERCISES XVII

1. Derive an approximate expression for the length of an open belt in terms of the pulley diameters and the centre distance.

2. The smallest steps of a four-step pulley are to be 6 in. dia. and the largest 12 in. dia. If the intermediate steps are such that the four ratios form a geometric progression, find the diameters of the intermediate steps. Centre distance 8 ft. open belt.

3. Repeat Question 2, but assuming the belt to be crossed.

4. State the principle governing the laying out of skew belt drives and make a sketch showing the application of the principle when the shafts connected are at right-angles.

5. The diameters of the steps of a pulley are d_1, d_2, d_3 and d_4 and the diameters of the corresponding steps of a second pulley are D_1, D_2, D_3 and D_4. If $d_1 = D_4$ and $d_4 = D_1$ and the four ratios form a geometric progression, prove that $d_2 = D_3$ and $d_3 = D_2$ are necessary conditions if an open belt is to have the same tension on all four steps.

CHAPTER XVIII

MECHANICAL VARIABLE-SPEED GEARS

It is often necessary to be able to adjust the velocity ratio or "gear ratio" between a driving and a driven shaft over a range of values. For example, in machine tools the driving motor or pulley generally runs at a constant speed, while the speed of the work or cutting tool has to be altered to suit the prevailing conditions. For this purpose variable-speed mechanisms or gears are used, and these may be divided into two classes :

(1) Mechanisms capable of giving an infinite number of ratios ; and

(2) Mechanisms giving only a finite number of ratios.

Variable-speed gears may also be classified according to the principle of operation ; thus : (*a*) Mechanical, (*b*) Hydraulic, and (*c*) Electric. The last type is beyond the scope of this book. The mechanical aspects of the mechanisms used in the second type are considered in other chapters.

227. Mechanical Infinitely Variable Gears.—Very many forms of mechanical infinitely variable gear have been invented, but few have proved successful in practice, and then usually only for low powers. One of the simplest is the ordinary friction gear as shown diagrammatically in Fig. 282. The disc A is fixed rotationally to its shaft, but can be slid along so as to make contact with the disc B at any point on the diameter CD, so that the velocity ratio, which is clearly equal to $\frac{r}{R_A}$, can be varied by infinitely small steps between the limits $-\frac{R_B}{R_A}$ to $+\frac{R_B}{R_A}$. Since the drive is by means of the friction between the discs, the latter have to be pressed together by a force, usually that of a spring, and the reactions of this force

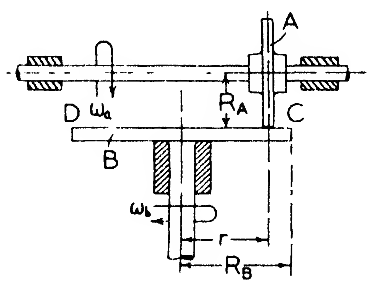


FIG. 282

come upon the bearings of the shafts. The gear thus tends to be somewhat massive even for small powers. Also it is generally not possible to slide the disc A while it is pressed against B, and the discs must be separated when it is required to alter the velocity ratio. A small amount of slip always occurs with friction gears however small the loads transmitted may be, but if the force pressing the discs together is not great enough, or if the coefficient of friction between them is reduced by any cause, serious slip may occur and this usually results in irregular wear, flats being worn on the disc A and depressions on the face of B, and these set up vibration and noise. Theoretically the disc A should be infinitely thin, or its edge should be crowned slightly so as to give point contact with the disc B, but actually, of course, contact always occurs over an area and some slip must always occur. The discs are sometimes both of metal, but often the disc A (and/or the disc B) is lined with a fabric or composition lining which affords a high coefficient of friction.

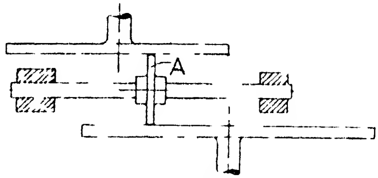


FIG. 283

A variation of the above gear is shown in Fig. 283. Theoretically the velocity ratio can be varied between zero and infinity according to the position of the disc A. Practically the upper limit is restricted to about 20.

228. Other Forms of Friction Gear.—Much ingenuity has been spent in trying to obviate the difficulties that arise when the shaft bearings have to support the force which presses the friction members together, and in many of the gears now to be described one of the objects of the inventor has been to free the bearings of this force. This has been done successfully in the Dorman gear which is shown in Fig. 284. The disc A is fixed to one of the shafts to be connected and drives the disc B, which is fixed to the other shaft, directly through the frictional contact at F and indirectly through the idler C, the ring D and the disc E. The force necessary to produce the frictional force required is provided initially by making the internal diameter of the ring D a little less than the sum of the diameters of the discs A, C and E and the thickness of the disc B, so that the ring D has to be sprung into position. When the gear is running the tangential forces between the discs C and E and the ring D tend to move the latter into a position such as is shown by the dotted lines in Fig. 285, thereby increasing the pressure between the discs. This increase of pressure is approximately proportional to the torque acting on the disc A so that the pressure between the discs is adjusted automatically in

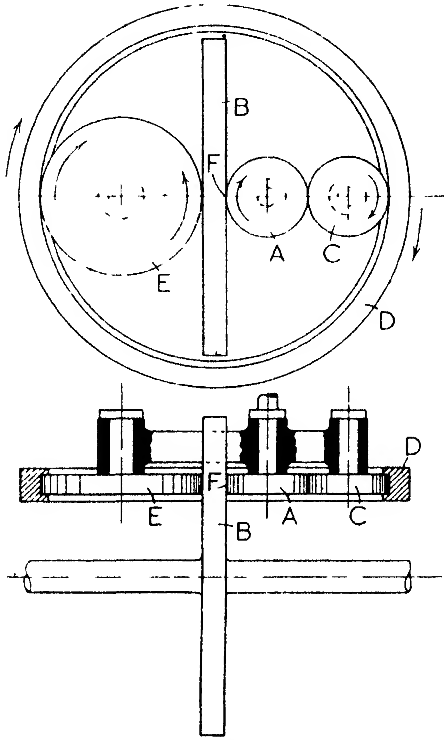


FIG. 284

accordance with the torque to be transmitted. The discs A, C and E are carried by a member, shown in black in the plan view of Fig. 284, which can be moved in or out in order to adjust the velocity ratio to the required value. The mean value of the velocity ratio may be varied by varying the size of the disc A or by making E the driving disc.

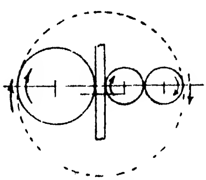


FIG. 285

The gear shown in Fig. 286 is known as the Sellers gear. The discs A and B are fixed to their shafts, and their slightly thickened rims make contact with the coned discs C and D through which the drive is transmitted from A to B or vice versa. The discs C and D are pressed together by the springs shown, spherical seatings being provided at E and F to enable the discs C and D to tilt as the gear ratio is varied. This is done by moving the axis of the discs CD parallel to itself so as to cause A to make contact nearer to that axis and B to make contact farther from it or vice versa.

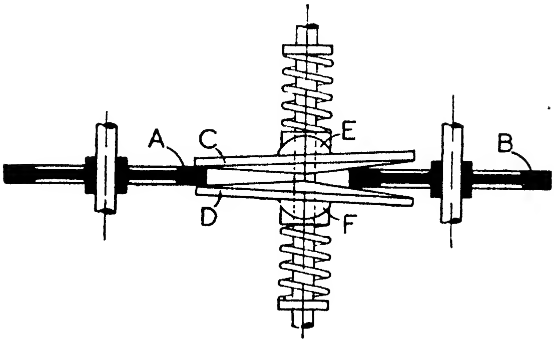


FIG. 286

229. The Hayes Friction Gear.—This consists essentially of a friction gear whose principle is indicated in Fig. 287. The driving shaft A has two discs B and C mounted on splines on it so that they are fixed rotationally but free to slide axially upon it. Between these discs is a third one, D, which is fixed to the driven

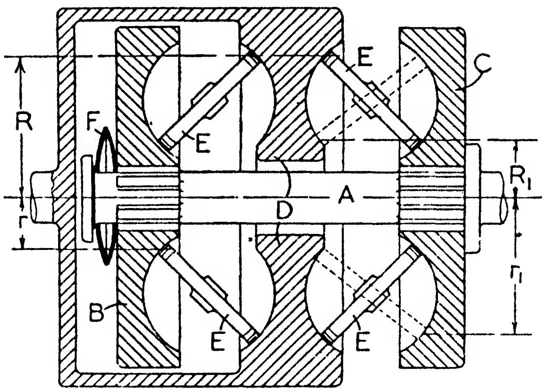


FIG. 287

shaft. The faces of all three discs have suitable tracks formed on them, and rollers EEEE run on these tracks and transmit the drive from the discs B and C to the disc D. To provide the necessary friction the discs B and C are pressed towards each other by a spring F. The rollers are ground spherical on their working faces and revolve on bearings on pins which are fixed for any particular velocity ratio, but whose inclination can be varied in order to vary the velocity ratio. Clearly when the rollers are in the full line positions the ratio is $\frac{r}{R}$, while when they are in the dotted positions it is $\frac{r_1}{R_1}$. In the application of this gear to the motor car the inclination of the rollers is controlled partly by the speed of the driving shaft and partly by the torque acting on the driven shaft so as to give an " automatic " transmission, but the details of this part of the mechanism do not now concern us. The method of altering the inclination of the rollers is, however, rather ingenious and may be described. It consists in moving the axes of the rollers slightly so that the rollers occupy positions as shown in Fig. 288, their natural path relative to the discs then being a spiral as indicated by the dotted line. Relative to one of the discs the roller will move along this spiral so that its point of contact with the disc gets farther and farther away from the axis of the disc, but relative to the other disc the roller will move along the spiral so that its point of contact approaches the axis and thus the result is that the roller is tilted. When the alteration of inclination thus brought about is sufficient to give the required alteration in velocity ratio the axis of the roller is restored to its normal position in which its axis intersects the axis of the discs. Comparatively small forces are required to move the rollers in this way, whereas very large ones would be required to effect a direct alteration of the inclination of the rollers. Another feature of this gear is that the force that presses the discs into contact with the rollers is arranged to be proportional to the torque acting on the driving shaft. This is done by transmitting the driving torque to the disc C (and thus through the splines and shaft A to the disc B) through the connexion shown in Fig. 289. The driving shaft G engages the ring H by dogs so that H is driven, and it in turn drives the disc C through the balls K, which run on tracks on the ring H that are helices of small pitch. Clearly if relatively to the disc C the ring H turns slightly, the balls in rolling on their helical tracks will tend to force the ring H and the disc C apart axially, but as the ring H abuts against the thrust bearing L and nut M, which is screwed on to the shaft A, the forcing apart of C and H results in the rollers being pressed against the discs as is required. Since the axial force acting on the disc C is the component, parallel to the axis of the discs, of the force transmitted

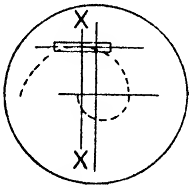


Fig. 288

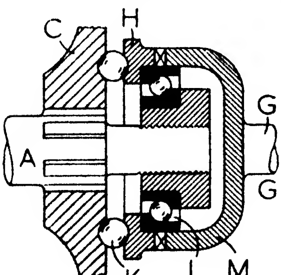


Fig. 289

by the balls K and the torque transmitted to the disc C is proportional to the component, perpendicular to that axis, of the same force, the force pressing the discs into contact with the rollers is proportional to the torque acting on the driving shaft G.

230. Fig. 290 shows a type of mechanism that has had a considerable amount of success in practice. The shafts are connected

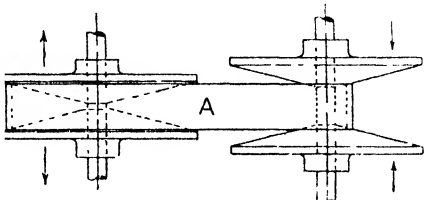


FIG. 290

by an endless belt A which runs on V pulleys whose effective diameters can be varied by moving their flanges relatively to each other in an axial direction as indicated by the arrows, one pair moving together and the other pair apart. If the axial movements of the flanges of the two pulleys are of equal magnitude, some tensioning device will be required to maintain the necessary tension in the belt, since the sum of the effective diameters will be constant and, as explained in Art. 218, this condition does not give a constant tension with an open belt. The difficulty is sometimes overcome by controlling only one pair of flanges positively, the other pair being controlled by springs which tend to force them together. Alternatively the flanges are controlled by a cam mechanism so that their effective diameters satisfy the condition for a constant belt tension. Occasionally only one of the pulleys is adjustable, and the belt tension is maintained by varying the distance between the shaft axes. In a recent application the belt is replaced by a hardened steel ring.

231. The P.I.V. Gear.—In this gear, which is manufactured by J. Stone & Co., Ltd., of Deptford, London, to whom the writer is indebted for the supply of drawings, etc., the variation of velocity ratio is obtained by expanding pulleys on the same principle as with the gear just described, but the drive is made a positive one by using a special form of chain whose "teeth" engage grooves cut in the faces of the pulley flanges. The principal pieces which go to form a link of the chain are shown separately in Fig. 291 and assembled in Fig. 292. The link plates A are hinged together by pins B on the lines of an inverted-tooth

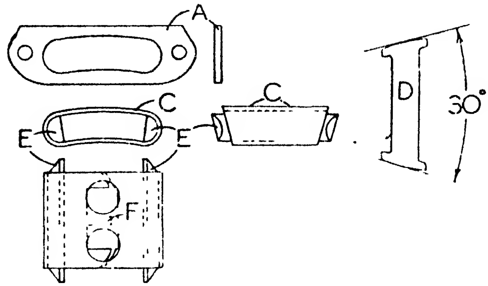


Fig. 291

chain, the holes in the plates A being provided with hardened bushes. These link plates, however, are slotted and house the liner C, three views of which are given and which is formed of a piece of sheet metal bent up into the shape shown. It is kept in place by the tongue F, which is bent up between the link plates after the liner has been inserted. The liner C serves to carry a pack of thin, slightly tapered sliding plates D. Finally the curved ends of the slots in the link plates are filled by the end plates EE. The thin plates D are quite free to slide relative to each other and also relative to the link plates, in a direction perpendicular to the length of the chain, and in Fig. 292 some are seen to be slid over to one side and some to the other side. By sliding transversely in this way the sliding plates accommodate themselves to the grooved faces of the pulley flanges, a developed view of the outer edges of which is shown in Fig. 293, from which it will be seen that the recesses in one flange are placed opposite the raised portions of the other flange, these relative positions being maintained because both flanges are keyed to their shaft, although, of course, being free to slide along that shaft. The profiles of both of the pulley flanges being the same, the distances xx, yy, zz are all equal and the sliding plates can be accommodated satisfactorily between the flanges. Thus a positive drive is obtained at all times. Theoretically the radius of curvature of the slots in the link plates A should be variable and made equal to the effective radius of the pulleys in the position occupied by the chain at any instant, but this, of course, is not possible, and so the slot radius is made equal to the mean effective radius of the pulleys, and the ends of the sliding plates are slightly curved

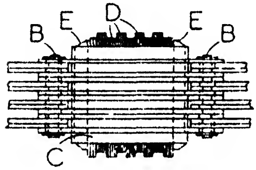


Fig. 292

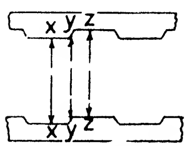


Fig. 293

as shown in the figure. Also the sliding plates should, ideally, be infinitely thin, but since this is not possible the grooves in the pulley flanges are made slightly wider than the raised portions. (The axial movements of the pulley flanges are controlled by two levers, one at each side, which are fulcrumed on fixed cam surfaces midway between the pulley axes, the cam surfaces being such that the tension in the chain, when the gear is just idling, is approximately constant for all positions of the chain. Spring-loaded slippers are also arranged to bear on the back of the chain so as to prevent any flapping.) This gear has been very successful in practice and large numbers are in use.

232. Mechanical Gears Giving a Finite Number of Velocity Ratios.—The majority of these are "gear-boxes" using toothed gears, or sometimes chains, which provide a number of trains of gears which can be brought into operation either separately or in various combinations in order to give the required number of velocity ratios. The gear-boxes that are now to be described have been selected to illustrate the various ways in which the trains of gears can be arranged and the methods adopted for bringing the trains into operation when required.

233. Gear-boxes Using Sliding Keys.—The essential features of such a gear-box are indicated in Fig. 294. One of the shafts has a number of gears fixed to it, and these mesh constantly with an equal number of gears which are free rotationally on the other shaft, but which can be brought into driving connexion with it by means of the sliding key A. This is free to slide in the keyway in the shaft and is shown locking the gear B to the shaft so that the middle pair of gears is transmitting the drive. To prevent two gears being engaged simultaneously as the key is slid from one gear to the next, rings C are fitted between the gears and the end of the key is pivoted as shown so that on moving it endways it is pressed down into the keyway and held there until it is clear of the one gear and it is safe for it to be engaged with the next. The speed of the wheel B relative to its shaft at the moment of engagement of the key must be kept low, otherwise the parts will be damaged by the impact on engagement, and in practice it is necessary to bring the shafts to rest before sliding the key to change the gear ratio.

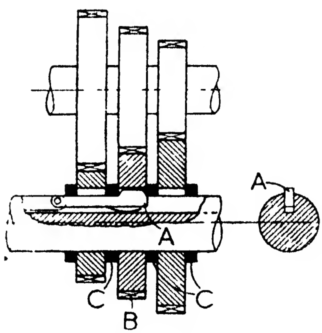


Fig. 294

This type of gear still finds a place in many machine tools, generally in the feed mechanism, but is not so widely used as formerly, because of the loss of time in changing from one ratio to another.

234. The Tumbler Type of Gear.—This is shown in Fig. 295, and comprises a nest of gears of gradually increasing size, which

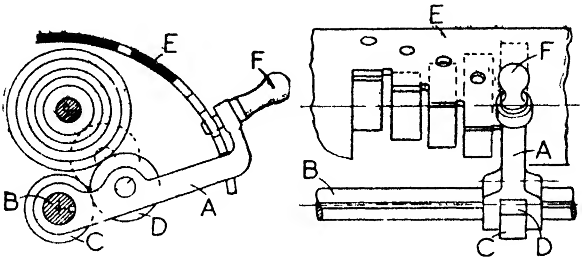


FIG. 295

are fixed to the shaft that carries them, and the tumbler A which is free to slide along the shaft B along which the gear C is free to slide, though it is fixed rotationally by a feather or spline. The gear C meshes constantly with an idler gear D carried by the tumbler, and this idler gear can be made to engage with any of the gears of the nest of gears by sliding the tumbler along the shaft B to the appropriate position and then pivoting it about that shaft until the end of the tumbler frame comes up against the stepped edge of the cover plate E which determines the correct meshing of the gears. The tumbler is provided with a spring-loaded knob F, a pin on which engages holes drilled in the cover plate, and this serves to keep the tumbler in position when a gear has been engaged. The change from one gear to another can only be made satisfactorily at low speeds.

235. Motor-Car Type Gear-boxes—The Sliding-Mesh Type.—Gear-boxes have been developed along special lines for motor cars, and it is beyond the scope of this book to do more than describe the principal types used. They may be classified as sliding-mesh, constant-mesh and epicyclic gear-boxes, but many gear-boxes are a combination of the first two types. A sliding-mesh gear-box is shown diagrammatically in Fig. 296. The shaft A is coupled to the engine and is free to revolve in bearings in the casing, which is not shown. Integral with this shaft or fixed to it is a pinion B which meshes constantly with the larger wheel C. This gear is fixed to the " layshaft " D to which the gears E, F and G are also fixed, and the whole assembly is free to turn about the axis XX, suitable bearings being provided. Co-axial with the engine shaft A is a third shaft H, which is coupled to the propeller shaft of the car. This shaft is supported in bearings in the casing at the right-hand end, and at the left-hand end has a short cylindrical spigot which is free to revolve in a bearing inside the shaft A. The shaft H is splined throughout its length and carries the sliding gears J and K, which are thus fixed rotationally to the shaft, but

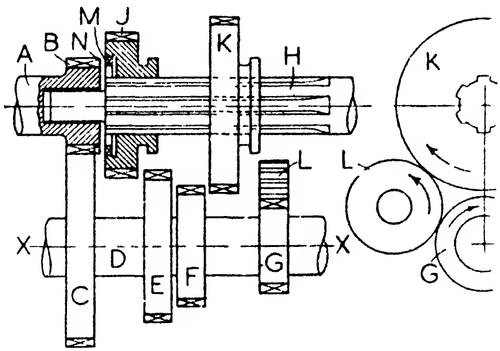


FIG. 296

are free to slide along it. If K is slid to the left so that it meshes with F, a drive is obtained between A and H *via* B, C, F and K. Another gear ratio is obtained by sliding J along to mesh with E, while a direct drive is obtained between A and H by sliding J to the left until the dog-clutch teeth M engage those (N) on the wheel B. A reverse gear is obtained by sliding K to the right until it comes into line with G. As G is made smaller than F, it does not mesh directly with K, but drives K through the medium of the reverse idler L, which is constantly in mesh with G and positioned so as to engage with K. The reverse idler is free to revolve on a shaft lying slightly below the plane of the other shafts as shown in the end view, and its effect, of course, is to reverse the direction of rotation of H relative to A.

236. The Constant-Mesh Type.—The principle of this type will be clear on consideration of Fig. 297. The three shafts are situated exactly as in the gear-box just described and the gears are similar on both the engine shaft A and the layshaft D, but those on H are free rotationally and fixed endways so that they are always in mesh with the corresponding gears on the layshaft. To obtain a drive one of the dog-clutch members M or N is slid to right or left so as to engage with the corresponding dog-clutch teeth on one of the wheels B, J, K or P, thereby fixing that wheel to the shaft H rotationally and causing the latter to be driven by A. A reverse idler connects G to P and gives a reverse gear.

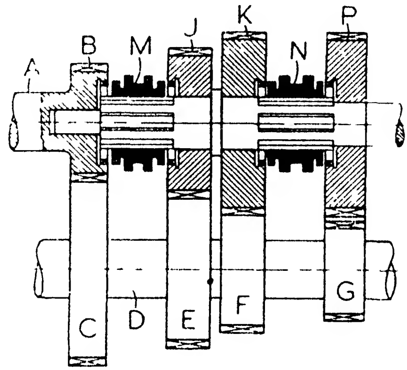


FIG. 297

237. Epicyclic Gear-boxes.—There are two methods by which a gear-box giving the choice of a number of ratios may be obtained when epicyclic trains are used. The first method is to use separate epicyclic trains for each ratio and to arrange to bring each one into action when required. Usually some of the members of these trains can be made common to all the trains, thus simplifying the construction. The other method is to use a simple train to give one ratio and to compound it with other trains in various ways to obtain the other ratios. With both methods a direct drive is obtained by locking the gear so that it can only rotate "solid," i.e. as a whole.

238. The first method is illustrated in Fig. 298. The arm of the epicyclic trains is formed by the web A of the engine flywheel,

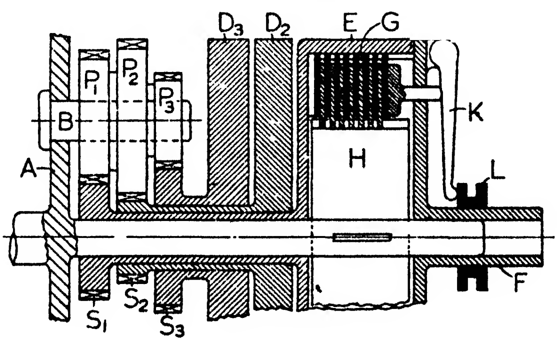


FIG. 298

into which are screwed pins B which carry the planet gears P_1P_2 and P_3. These are fixed together. The sun S_1 is, in effect, integral with the output shaft F. There are two epicyclic trains, each of the double-sun type. One train, consisting of $S_1P_1P_2$ and S_2, gives the low * forward gear when its sun S_2, which is smaller than S_1, is held stationary by applying a brake to the drum D_2. The second train, consisting of $S_1P_1P_3$ and S_3, gives the reverse gear when its sun S_3, which is bigger than S_1, is held stationary by applying a brake to the drum D_3. The gears S_1 and P_1 and the arm are thus common to both trains.

A direct drive is obtained by engaging the plate clutch G, the inner plates of which are splined to the drum H. The latter is keyed to an extension of the crankshaft. The outer plates are splined to the drum E and the plates are pressed together, to engage the clutch, by the levers K actuated by the sleeve L.

The control of the brakes and the sleeve L must be such that two brakes, or a brake and the clutch, can never be actuated simultaneously.

239. An example of the second method is the Wilson gear used on many motor vehicles. The epicyclic part of this gear is shown, diagrammatically, in Fig. 299. The drive enters at the shaft A,

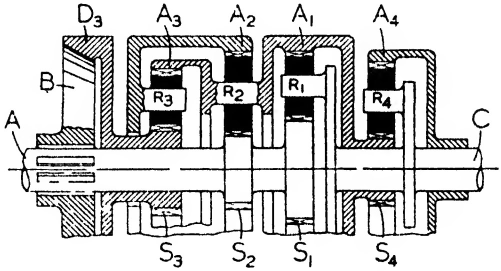


FIG. 299

to which are fixed the two sun wheels S_1 and S_2 and the cone-clutch member B, the latter being free to slide axially. The sun S_1 meshes with planets carried by the arm R_1 which is the driven member and which is fixed to the output shaft C. The planets mesh with an annulus A_1 which can be held stationary by means of a brake. If this is done then S_1, A_1 and R_1 form a simple train and R_1 will rotate in the same direction as S_1, but at a lower speed. This gives the lowest gear. The next higher gear is obtained by compounding the train $S_1P_1R_1$ with the train $S_2R_2A_2$, whose arm

* In motor-car practice a "low gear" is one which makes the speed of the output shaft low relative to that of the input shaft.

R_2 is fixed to the annulus A_1. If A_2 is held at rest by a brake, then the train $S_2R_2A_2$ will cause the arm R_2, and thus the annulus A_1, to turn in the same direction as S_1S_2. The arm R_1 will now receive motion not only from S_1, but also from A_1, both components being in the same sense and therefore additive. Thus a higher gear is obtained. The third gear is obtained by compounding the train $S_2R_2A_2$ with the train $S_3R_3A_3$ of which the arm R_3 is integral with A_2 and the annulus A_3 is integral with the arm R_2 and thus with A_1. When the sun S_3 is fixed by applying a brake to the drum D_3 the train $S_3R_3A_3$ constrains the annulus A_2 to rotate in the same direction as S_1S_2 so that in train $S_2R_2A_2$ the arm is receiving motion from both S_2 and A_2 and rotates faster (for a given speed of A) than when the annulus A_2 was fixed. Since R_2 is fixed to A_1 the latter is also moving faster than it did on second gear, and so R_1 also moves faster; thus a higher gear than second is obtained. A direct drive is obtained by engaging the cone-clutch member B with the female cone formed in D_3. This locks the gear so that it can only rotate "solid." The reverse gear is obtained by compounding the train $S_1R_1A_1$ with the train $S_4R_4A_4$ of which the sun S_4 is integral with the annulus A_1 and the arm R_4 with R_1. On fixing A_4 the train $S_4R_4A_4$ causes the sun S_4 and thus the annulus A_1 to rotate in the opposite sense to the sun S_1, and the numbers of teeth are chosen so that the backward component of the motion of R_1 due to the backward motion of A_1 is greater than the forward component due to S_1, so that R_1 rotates backwards and a reverse gear is obtained.

240. Synchronising Devices.—In sliding-mesh and in constant-mesh gear-boxes using dog clutches it is possible to change from one ratio to another while the shafts are rotating at high speeds only if the actual movement bringing about the engagement of the gears or dog clutches is timed correctly so that it coincides with the instant at which the members being engaged are moving at approximately the proper relative speed (i.e. the speed they would have immediately after the engagement had been effected). The necessary manipulative skill for this is quickly acquired by most people. When, for any reason, the engagement is attempted at the wrong moment great stresses are set up by the resulting impact and damage often results. To eliminate this trouble devices to ensure proper synchronisation of the members prior to their engagement have been developed and one design is shown in Fig. 300. It is for use with a constant-mesh box and the wheels A and B are free to revolve (but are fixed axially) on the shaft C and are in permanent mesh with gears on the layshaft. To obtain a drive through either gear the dog-clutch member D must be slid along the splines (that fix it rotationally to C) until the teeth

E or F engage the teeth G or H. The member D is shifted by means of the ring K, which is fixed by screws to the projecting portions L (usually three in number). The projecting portions L work in slots M formed in a sleeve N, which is free to slide on the outside of the member D except that spring-loaded balls P, engaging recesses in the sleeve, tend to prevent any relative movement. The sleeve N has cones formed in its ends and there is a small clearance between these cones and the male cones formed on the gears A and B. The action is as follows:

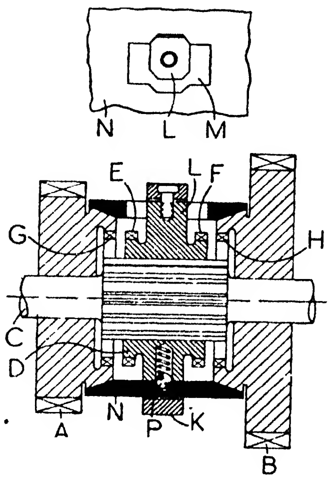


FIG. 300

Suppose the shaft C is stationary and gear A is rotating, and that it is desired to engage the dog teeth E and G. The ring K is pressed to the left, thus bringing the cone in N into contact with the cone on A (because the balls P tend to make N, D and K move as one). If a pressure towards the left is maintained on K, then the friction set up between the cones will tend to bring A and N (which is the same thing as D) to the same speed. When such synchronism has been obtained a somewhat greater pressure on K will overcome the resistance of the balls P and will move D relative to N, thereby enabling the dog teeth to be engaged.

As described above the mechanism is not foolproof, since if K is pressed too hard D will move along and the dog teeth may engage before synchronisation has been established. To obviate this the slots in which the projections L slide are sometimes shaped as shown in the part plan view. Until synchronisation is established either gear A is accelerating the sleeve N (and D and C) or it is slowing it down, and in either event N will tend to move rotationally relative to L and the latter will enter the recesses in the slots M, and this will prevent axial movement of L. At the moment of synchronisation the tendency to relative rotation between N and L will reverse in direction and consequently the projections L will tend to move across from one side of the slot in M to the other side, and as there is an axial pressure on L, when it comes to the central position it will move along to the end of the slot.

241. Pre-selective Gear-boxes.—These are gear-boxes in which any particular ratio can be *selected* at any instant simply by moving a lever or dial to the appropriate position, but in which the train of gears giving that ratio is not brought into action until an operating lever or pedal is actuated. Two examples will be described.

In the Wilson epicyclic gear-box the gears (except the direct drive) are obtained by applying brakes to the annuli, each of which is provided with a set of parts such as are shown in Fig. 301.

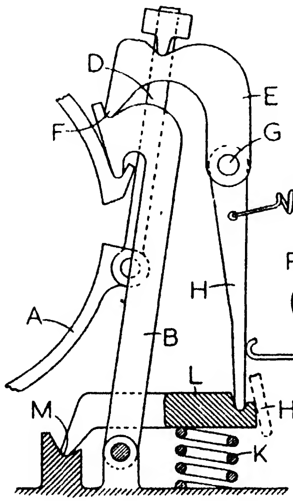


Fig. 301

There is a similar set of parts, with the exception of the brake bands A, for the direct-drive clutch. The brake bands A are anchored by the links B and their free ends are pulled up by the rods D when the levers E are turned about the fulcra F. This is done by a spring, shown diagrammatically at K, through the medium of a " bus-bar " lever L, common to all the sets of parts, and struts H hinged at G to the levers. The bus-bar pivots on a knife-edge at M. The light springs N tend to pull the struts out of engagement with the bus-bar and the more powerful springs O, through the links U, tend to push them in. All the springs O except one are prevented from acting, however, by cams P on a shaft that is connected to the pre-selector lever. The operating pedal, which engages a gear after it has been selected, merely controls the bus-bar L.

The action is as follows: Suppose the brake A is " on," and a gear is thus engaged, and the pre-selector lever is turned to select another gear of which the mechanism will be denoted by letters to correspond to Fig. 301, but with the suffix 1. All that happens is that the camshaft is turned to a new position, the cam P presses the link U back and the cam P_1 allows the link U_1 to move forward. This latter movement does not, however, bring the strut H_1 into engagement with the bus-bar, because the relative positions are as shown by the dotted outline. Nor is the spring N strong enough to pull the strut H out of engagement. Nothing more happens until the operating pedal is actuated and the bus-bar thereby pressed down against the action of the spring K. When this is done the strut H is lowered and the brake A released, and when the bus-bar is right down the spring N pulls that strut out of engagement. Simultaneously the spring O_1 pushes the strut H_1 into engagement, so that when the bus-bar is released the selected brake A_1 is applied. For more detailed description of the Wilson box the reader is referred to a paper by W. G. Wilson, *Proc. I.A.E.*, Vol. XXVI.

242. The Herbert Pre-optive Gear-box.—This is a constant-mesh gear-box the gear trains of which are brought into action by means of friction clutches. It was developed by Alfred Herbert,

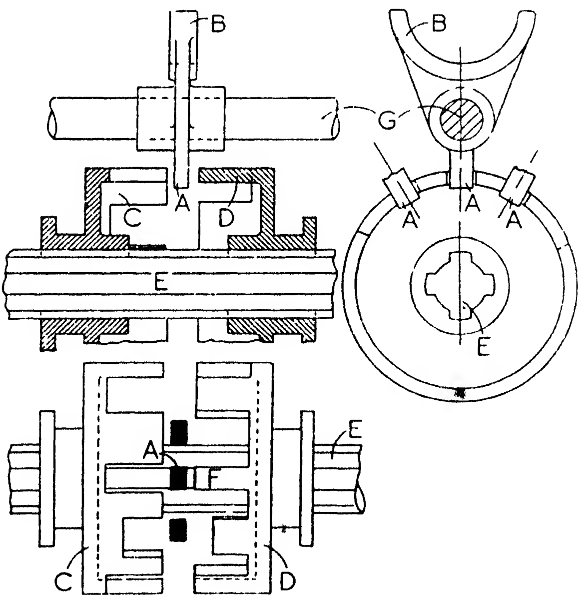


FIG. 302

Ltd., and is used on many of their machine tools. The principle of the pre-selective mechanism is illustrated by Fig. 302. AB is a " striking fork " of which there are as many as there are friction clutches to be operated. The clutches are of the type that once engaged remain so until forcibly disengaged. The striking forks move along suitable fixed guide rods G and the fork parts B engage the actuating parts of the clutches. The arms A project between the ends of two cylindrical members C and D which are formed with a number of projecting fingers and corresponding slots as shown. The drums C and D can slide along the shaft E, but must rotate with the latter when it is rotated by the pre-selection lever, and the action of pre-selecting a gear merely rotates the drums C and D to the proper position. The operating lever, by means of which the selected gear is actually obtained, enables the drums C and D to be slid towards each other along the shaft E. When this is done the finger F will force the arm A to the left, thus engaging the clutch to which AB belongs. Simultaneously other fingers similar to F will engage any of the other clutches as may be necessary in order to bring the required train of gears into operation, while the arms (A) of any clutches that may have been engaged previously, and which are no longer required to be engaged, will be brought back to the central or disengaged position. Having in this way engaged the selected ratio, if the operating lever is released a spring will bring the drums C and D back to the position shown, when they may again be rotated to a new position in order to pre-select another ratio.

CHAPTER XIX

(Arts. 252 to 257 of this chapter may be omitted on a first reading.)

CAMS

243. A cam is a piece which is given a rotary or reciprocating motion and which, because of the shape of its edge, or face, or of a groove formed in it, causes a *follower*, which bears against it, to move in the required manner. They are very widely used, being the most convenient method of producing irregular and intermittent motions.

Examples of cams are shown in Fig. 303, which also serves to

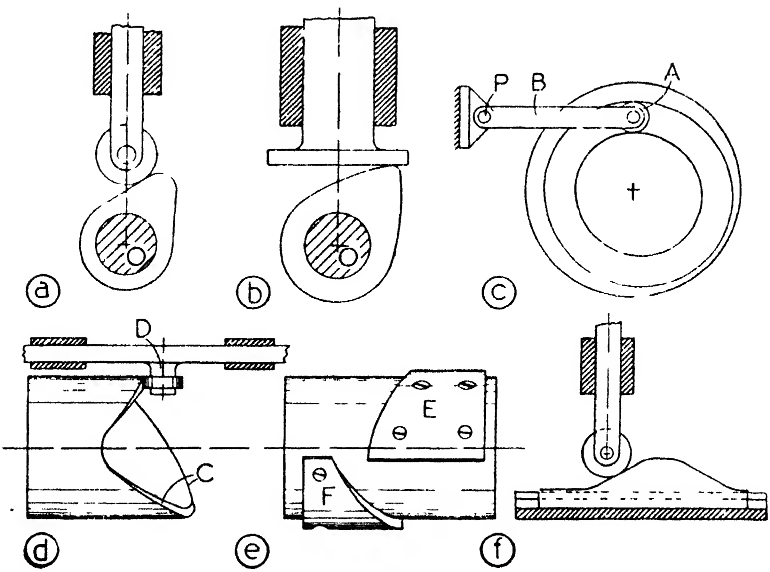


Fig. 303

show different types of follower; most of the latter, however, could be adapted for use with any of the cams shown.

At *a* is shown a *disc* cam with a *roller* follower constrained to move in a straight line which in this example passes through the axis of rotation, O, of the follower.

The cam shown at *b* is also a disc cam, but now the follower is a flat-footed one. Clearly with such a follower the cam should be entirely convex.

The cam shown at *c* is a disc in the face of which a groove is cut to receive the roller A, which is carried on the end of the follower B, which is now an arm pivoted on a fixed centre at P.

The next example *d* is a *face* cam, the surface C being formed on the end of a hollow cylinder which rotates about its axis. The follower D is free to slide in guides parallel to the axis of the cylinder. A similar cam of slightly different construction is shown at *e*; here the cam surface is formed by pieces E which may be bolted to the surface of the cylinder and thus be easily changed for others giving a different motion. Such cams are extensively used in automatic machine tools.

At *f* is shown a cam that has a reciprocating motion in a straight line.

It will be noticed that in all the examples, except *c*, force closure is used, the weight of the follower, or more often the force of a spring, being used to keep the follower in contact with the cam, and this is common practice, body closure as shown at *c* being generally used only in lightly loaded slow-speed machines.

The object of using roller followers is to substitute rolling for sliding motion between the cam and follower. The motion of the follower would be quite unaffected if the roller were prevented from rotating.

244. Form of Roller for Cylindrical Cams.—It is common practice with cylindrical cams of large size to use cylindrical rollers for the followers, but it is obvious that the action cannot be one of pure rolling, for the peripheral speeds of the ends of the roller are equal, whereas those of the corresponding portions of the cam are unequal. It can be shown * that if the cam has a constant pitch then the proper form of roller to obtain pure rolling is part of an hyperboloid of revolution and that the axis of the roller must be offset from that of the cam. If the pitch of the cam is not constant, then no form of roller can give pure rolling and the best compromise is probably a conical roller.

245. Multi-turn Cylindrical Cams.—Such cams consist of a shaft or drum in which is formed a groove that, starting at one end of the shaft travels to the other end and there, reversing its direction, returns to the starting-point, the return groove thus intersecting

* See *Applied Mechanics*, by D. A. Low, published by Longmans, Green & Co., Ltd.

the forward groove at a number of points. In order to ensure that the follower shall continue in the proper groove at the intersection points it has to be made longer than the gap formed by the groove intersected, as is indicated in Fig. 304, which shows an

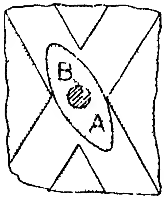


FIG. 304

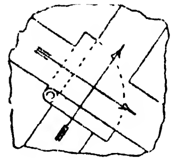


FIG. 305

intersection. The follower shoe A is pivoted on a pin B, and the motion between it and the cam is purely sliding. In such a case a roller follower is not practicable. It is sometimes feasible, however, to use cylindrical rollers in conjunction with spring-loaded latches at the intersections as shown in Fig. 305.

246. Almost any motion can be produced by means of a cam, the only limitation being that the acceleration of the follower must not at any moment be too high or the forces between the cam and follower will be so great as to damage the parts or, alternatively, the force of the spring or other constraint keeping the follower in contact with the cam may be insufficient and the follower may lose contact with the cam, when its motion will not be the required motion. In slow-speed machinery this limitation does not have to be considered very much, provided that it is kept in mind that the velocity of the follower cannot be changed instantaneously, either in magnitude or direction, because such a change would involve infinite acceleration and correspondingly infinite forces. In high-speed machinery the acceleration will usually have to be determined in order to design the spring for keeping the follower in contact with the cam, or so that if it is too high the design may be altered.

Thus the problems that present themselves in connexion with cams are roughly of two kinds: Firstly the motion of the follower is specified and the shape of the cam has to be determined so that it will produce this motion; and secondly, the cam shape and speed being given, the acceleration of the follower at any moment has to be determined. These problems will now be considered.

247. Design of Cam to Produce a Specified Motion.—Suppose that the follower A (Fig. 306) is required to occupy consecutively, and at equal intervals of time, the positions numbered 0–12 as the shaft B, which is to carry the cam, turns at constant speed through one complete revolution about its axis O. Let the smallest radius of the cam be settled as R, the radius of the follower roller as r and its line of stroke as OX. Then clearly the distance 0O must be $R+r$ and the centre O can be marked off along XO at that distance from the position 0.

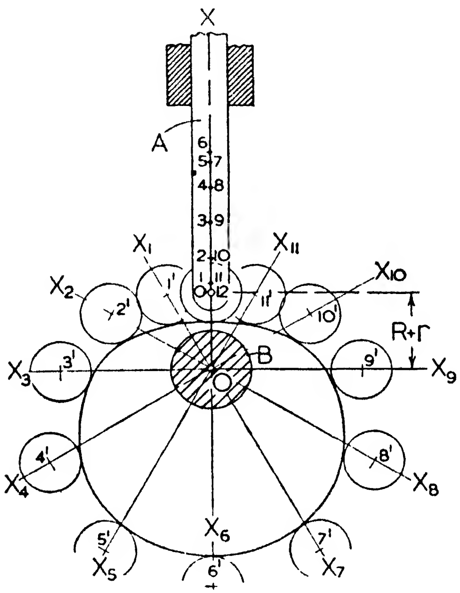


FIG 306

Now, since the camshaft is to rotate at constant speed it will turn through equal angles in equal intervals of time. Thus it would turn through an angle $\frac{360°}{12}=30°$ while the follower moved from position 0 to position 1. It is, however, more convenient to imagine the cam to be stationary and the follower to rotate round it. If this is done, then the consecutive positions occupied by the line of stroke OX will be the lines OX_1, OX_2, etc., which are set out at equal angles of 30°. But when the cam has turned through 30° relative to the line of stroke OX the follower will have reached position 1, and since we are imagining the cam to be stationary, the follower will be at 1′, where O1′=O1. Similarly the positions 2′, 3′, 4′, etc., corresponding to 2, 3, 4, etc., are found by marking off along OX_1, OX_2, OX_3, etc., distances equal respectively to 02, 03, 04, etc.

Next, with the points 1′, 2′, 3′, etc., as centres, circles, radii r, may be drawn to represent the follower roller, and lastly a curve drawn to touch all these circles as shown. This is the shape to which the cam must be formed to give to the follower the required motion.

The process is essentially the same as the envelope method of determining the conjugate tooth to a given tooth described in Art. 140.

248. Design of Cam with Pivoted Follower.—The principle is essentially the same as that just described, but the process is slightly more complicated. Let the consecutive positions of the centre of the follower roller be the points numbered 0 to 12 on the arc YY (Fig. 307), whose centre is at P and let O be the centre

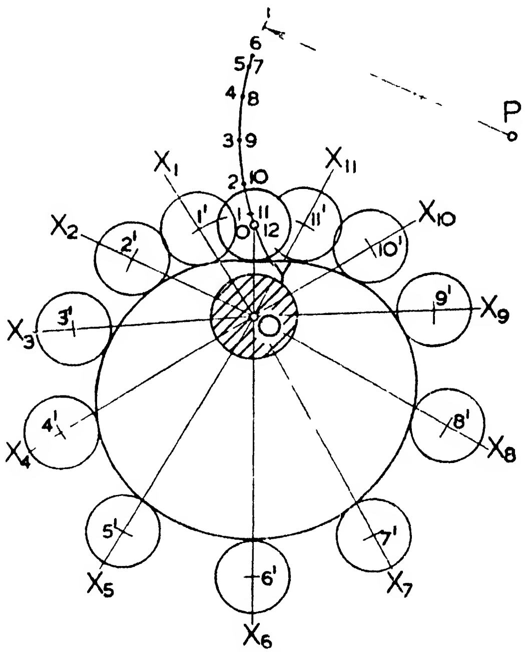


FIG. 307

of rotation of the cam, the shape of which is then found as follows:

Join O to 1 and set out OX_1 such that the angle $1OX_1$ equals 30° (i.e. 360°/12). Make O1′ equal to O1. Then 1′ is the position of the centre of the follower roller when the centre of rotation P has rotated through an angle of 30° relative to the cam. Join O to 2 and set out OX_2 such that the angle $2OX_2$ equals 60°. Make $O2'$ equal to $O2$. Proceed in a similar manner for all the other points. The circles representing the follower roller can then be drawn with the points $1'$, $2'$, etc., as centres, and a curve drawn to touch them as shown. This curve is the required cam shape.

249. Interference in Cams.—It will sometimes be found that the envelope to all the circles representing the consecutive positions of the follower roller is a looped curve. As an example of this suppose the curve AB (Fig. 308) is the locus of the centre of the follower roller, then, when the roller circles are drawn in, if the radius of the roller exceeds a certain amount it will be found that the envelope is looped as shown by CDEF. The cam could not, of course, be made this shape, and if it were made to the shape CGF there would be a discontinuity in the motion of the follower due to the absence of the position GDEG. This interference could be obviated in the example shown by using a smaller roller.

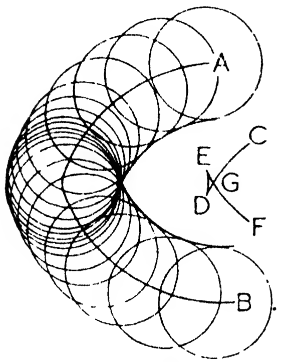


FIG. 308

250. Determination of the Acceleration of the Follower.—There are several methods that may be used to determine the velocity and acceleration of the follower when the shape and speed of rotation of the cam are known and the actual method to be adopted will depend upon personal preferences, the accuracy required, etc. First, the process described in Arts. 247 and 248 may be reversed and the positions of the follower for assumed rotations of the cam may be found. A displacement-time curve may thus be drawn for the follower, and this may be differentiated graphically by drawing tangents, thus giving the velocity of the follower at a number of instants. The resulting velocity-time curve may then be differentiated graphically to give the acceleration-time curve. This method, however, is not usually a very accurate one—on the contrary, it is usually most inaccurate, the curves obtained from most cams being such as do not lend themselves to the accurate drawing of tangents. Second, if the centres of curvature of the cam and follower at the point of contact can be determined accurately, then an equivalent four-bar chain or slider-crank chain may be substituted for the cam mechanism ; the velocity and acceleration of the follower may then be found by any of the methods described in Chapters VII and VIII respectively. As an example of this, suppose the centre of curvature of the flank *ab* of the cam shown in Fig. 309 is O_2 and that O_3 is the centre of the roller follower. Join O_2O_3 and O_1O_2. Then a slider-crank chain having O_1O_2 as crank and O_2O_3 as connecting-rod would give the follower, at the moment under consideration, exactly the same motion as it receives from the cam. The velocity and acceleration are thus easily found. It should be noted, however, that the approximate analytical expressions for the velocity and acceleration of the slider of a slider-crank chain derived in Arts. 112 and 118 cannot be used here, because the "crank" O_1O_2 will usually be approximately the same length as, or it may be greater than, the length of the connecting-rod O_2O_3. Quite accurate results can, however, be obtained from graphical methods. This method is almost the only one available when a pivoted follower or rocker follower is used, since then the analytical method, which is the chief alternative, becomes hopelessly cumbersome.

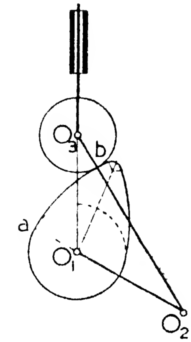


FIG. 309

Thirdly, if the common normal at the point of contact of cam and follower can be drawn with sufficient accuracy, the velocity of the follower may sometimes be found by using the proposition proved in Art. 138. The acceleration remains, however, to be found by the other methods given.

Lastly, if the cam is made up of definite geometric curves such as arcs of circles, straight lines, logarithmic spirals, etc., the lift. velocity and acceleration may be found analytically, although the expressions obtained may often be somewhat unwieldy. The cams used for operating the valves of internal-combustion engines are usually of this type, and thus may be used as examples of the method.

251. Internal Combustion Engine Cams.—The symbols and notation that will be used are as follows: Referring to Fig. 310,

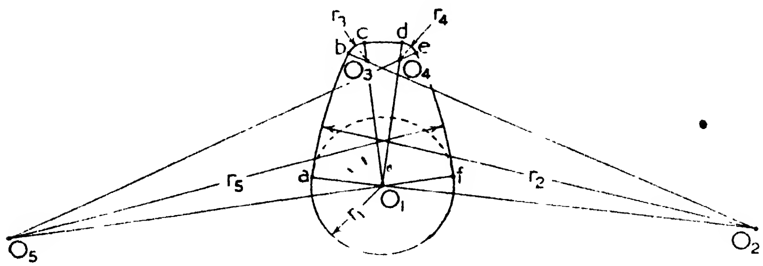


FIG. 310

fa is the *base circle*, radius r_1 ; *ab* is the *flank*, radius r_2 ; *bc* is the *nose*, radius r_3 ; *cd* is the *dwell* and is concentric with O_1 ; *de* (radius r_4) and *ef* (radius r_5) are also respectively nose and flank, but may be distinguished from *ab* and *bc* by the use of the adjectives *advancing* and *receding* according to the direction of rotation, e.g. for the counter-clockwise direction of rotation *ab* is the advancing flank and *de* is the receding nose. If the cam is symmetrical, as is often the case, then $r_2=r_5$ and $r_3=r_4$. Also the dwell *cd* is frequently dispensed with, so that O_3 and O_4 coincide. It is convenient to measure the angular rotation of the cam in terms of the angle between O_1a and a line through O_1 parallel to the line of stroke of the follower. If the latter passes through O_1, then the cam is a *central* cam, otherwise it is an *offset* cam. When the point of contact between the cam and follower is on the base circle the follower is at rest, when it is on the advancing flank the follower is moving upwards and has an upwards or positive acceleration, i.e. its velocity is increasing. When the contact is on the advancing nose the follower, although still moving upwards, has a negative or downwards acceleration, i.e., its velocity is decreasing. When the contact is on the dwell the follower is at rest, and when it is on the receding nose the follower again has a negative acceleration, i.e. its downwards velocity is increasing. Finally, when the contact is on the receding flank the follower once more has a positive acceleration, i.e. its downwards velocity is decreasing. The upwards or positive accelerations are produced by the force exerted on the follower by the cam, while the negative or downwards accelerations are produced by the force exerted on the follower by the valve spring.

For an offset cam, symmetrical or otherwise, four sets of equations will have to be found, each set applying while the contact is on the portion of the cam for which it was derived, e.g. advancing or receding flank or nose. For a symmetrical central cam the equations for the receding side are identical with those for the advancing side.

252. Convex Cam with Offset Roller Follower.—(1) *Contact on advancing flank.*—Referring to Fig. 311, we have

$$\text{Lift}=l=PO_6-PR$$

R being the position occupied by the roller centre O_6 when the contact is on the base circle.

$$PR=O_1R\ \text{Cos}\ a=(r_1+r_6)\ \text{Cos}\ a$$

where a is given by

$$\text{Sin}\ a=\frac{e}{r_1+r_6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

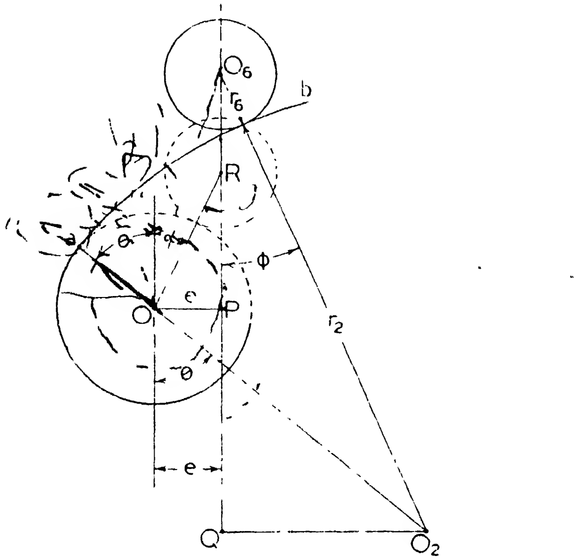


FIG. 311

Now $\quad PO_6 = QO_6 - QP$

$$= (r_2 + r_6) \operatorname{Cos} \phi - (r_2 - r_1) \operatorname{Cos} \theta$$

$$= b \operatorname{Cos} \phi - a \operatorname{Cos} \theta$$

where $\quad b = r_2 + r_6$ and $a = r_2 - r_1$.

Also $\quad b \operatorname{Sin} \phi + e = a \operatorname{Sin} \theta,$

e being the offset of the line of stroke of the follower.

Thus
$$\operatorname{Sin} \phi = \frac{a \operatorname{Sin} \theta - e}{b}$$

and
$$\operatorname{Cos} \phi = \left\{\frac{b^2 - (a \operatorname{Sin} \theta - e)^2}{b^2}\right\}^{\frac{1}{2}}.$$

Hence

$$l = \{b^2 - (a \operatorname{Sin} \theta - e)^2\}^{\frac{1}{2}} - a \operatorname{Cos} \theta - (r_1 + r_6) \operatorname{Cos} \alpha \quad . \quad . \quad . \quad (2)$$

$$\therefore \frac{dl}{dt} = \frac{dl}{d\theta} \cdot \frac{d\theta}{dt} = \left[\frac{-a \operatorname{Cos} \theta (a \operatorname{Sin} \theta - e)}{\{b^2 - (a \operatorname{Sin} \theta - e)^2\}^{\frac{1}{2}}} + a \operatorname{Sin} \theta\right] \cdot \frac{d\theta}{dt} \quad . \quad . \quad . \quad (3)$$

$$\frac{d^2 l}{dt^2} = \left[\frac{a \operatorname{Sin} \theta (a \operatorname{Sin} \theta - e) - a^2 \operatorname{Cos}^2 \theta}{\{b^2 - (a \operatorname{Sin} \theta - e)^2\}^{\frac{1}{2}}} - \frac{a^2 \operatorname{Cos}^2 \theta (a \operatorname{Sin} \theta - e)^2}{\{b^2 - (a \operatorname{Sin} \theta - e)^2\}^{3/2}} + a \operatorname{Cos} \theta\right] \left(\frac{d\theta}{dt}\right)^2$$

$$= -\left[\frac{a^2 b^2 \operatorname{Cos} 2\theta + a b^2 e \operatorname{Sin} \theta + a \operatorname{Sin} \theta (a \operatorname{Sin} \theta - e)^3}{\{b^2 - (a \operatorname{Sin} \theta - e)^2\}^{3/2}} - a \operatorname{Cos} \theta\right] \left(\frac{d\theta}{dt}\right)^2 \quad (4)$$

These equations hold only while the contact is on the flank *ab*, and the corresponding limiting values of θ must now be found. When contact is at *a* then clearly $\theta = -\alpha$, and when contact is at *b*, as shown in Fig. 312, then $\theta = \theta_1 - \zeta - \beta$, where θ_1 depends on

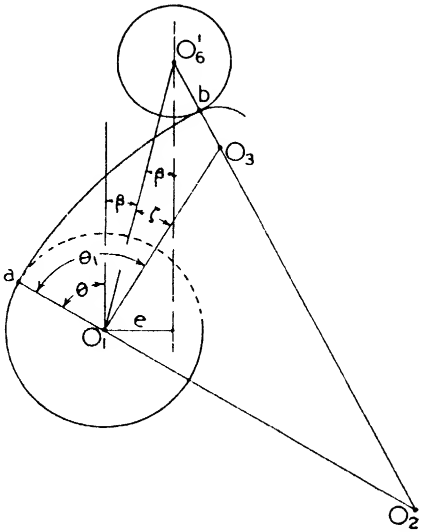


FIG. 312

the geometry of the cam alone and is either given, or may be settled more or less arbitrarily, while ζ can be found by solving the triangle $O_1O_3O_6'$ of which the sides O_1O_3 and O_3O_6' are known, and the angle between them can be found by solving the triangle $O_1O_2O_3$ (of which the three sides are known), for $\widehat{O_1O_3O_2}$; and lastly β is given by $\text{Sin}\,\beta = \dfrac{e}{O_1O_6'}$, and O_1O_6' is found when the triangle $O_1O_3O_6'$ is solved.

However, it is probably easier to determine α, β, θ and ζ from carefully made drawings.

Thus the limits of θ for contact to be on the flank of the cam are $\theta = -\alpha$ to $\theta = \theta_1 - \zeta - \beta$.

(2) *Contact on advancing nose.*—From Fig. 313 it will be seen that

$$l = PO_6 - (r_1 + r_6)\ \text{Cos}\ \alpha$$

and

$$PO_6 = O_1O_3\ \text{Cos}\ (\theta_1 - \theta) + O_3O_6\ \text{Cos}\ \psi$$

$$= d\ \text{Cos}\ \epsilon + f\ \text{Cos}\ \psi$$

where $\qquad d = O_1O_3,\ f = r_3 + r_6$

and $\qquad \epsilon = \theta_1 - \theta \ (\therefore \frac{d\epsilon}{d\theta} = -1)$

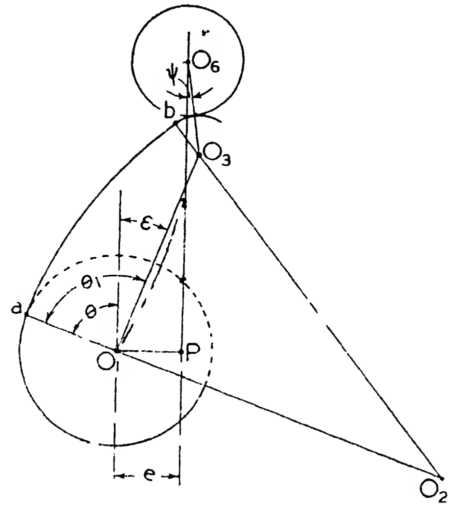


FIG. 313

Now $\qquad f \operatorname{Sin} \psi + e = d \operatorname{Sin} \epsilon$

$$\therefore \operatorname{Cos} \psi = \left\{ \frac{f^2 - (d \operatorname{Sin} \epsilon - e)^2}{f^2} \right\}$$

Hence

$$l = d \operatorname{Cos} \epsilon + \{f^2 - (d \operatorname{Sin} \epsilon - e)^2\}^{\frac{1}{2}} - (r_1 + r_6) \operatorname{Cos} a \quad . \quad . \quad . \quad (5)$$

$$\frac{dl}{dt} = \frac{dl}{d\epsilon} \cdot \frac{d\epsilon}{d\theta} \cdot \frac{d\theta}{dt} = \left[d \operatorname{Sin} \epsilon + \frac{(d \operatorname{Sin} \epsilon - e) d \ . \operatorname{Cos} \epsilon}{\{f^2 - (d \operatorname{Sin} \epsilon - e)^2\}^{\frac{1}{2}}} \right] \left(\frac{d\theta}{dt} \right) \quad . \quad . \quad (6)$$

$$\therefore \frac{d^2 l}{dt^2} = \left[-d \operatorname{Cos} \epsilon + \frac{-d^2 \operatorname{Cos}^2 \epsilon + d \operatorname{Sin} \epsilon (d \operatorname{Sin} \epsilon - e)}{\{f^2 - (d \operatorname{Sin} \epsilon - e)^2\}^{\frac{1}{2}}} \right.$$

$$\left. - \frac{(d \operatorname{Sin} \epsilon - e)^2 d^2 \operatorname{Cos}^2 \epsilon}{\{f^2 - (d \operatorname{Sin} \epsilon - e)^2\}^{3/2}} \right] \left(\frac{d\theta}{dt} \right)^2$$

$$= -\left[d \operatorname{Cos} \epsilon + \frac{f^2 d^2 \operatorname{Cos} 2\epsilon + f^2 ed \operatorname{Sin} \epsilon + d \operatorname{Sin} \epsilon (d \operatorname{Sin} \epsilon - e)^3}{\{f^2 - (d \operatorname{Sin} \epsilon - e)^2\}^{3/2}} \right] \left(\frac{d\theta}{dt} \right)^2 . \quad (7)$$

These equations hold while contact is on the nose, and the corresponding limits for θ are $\theta = \theta_1 - \zeta - \beta$ to $\theta = \theta_1 - \delta$ where δ, as will be seen from Fig. 314, is given by

$$\operatorname{Sin} \delta = \frac{e}{d + r_3 + r_6}$$

This figure also shows that the maximum lift of the follower is given by

$$l_{max} = O_1O_6 \operatorname{Cos} \delta - (r_1 + r_6) \operatorname{Cos} \alpha$$
$$= (d + r_3 + r_6) \operatorname{Cos} \delta - (r_1 + r_6) \operatorname{Cos} \alpha$$

which on substituting for Cos δ and Cos α and writing k for $d + r_3 + r_6$ becomes

$$l_{max} = (k^2 - e^2)^{\frac{1}{2}} - \{(r_1 + r_6)^2 - e^2\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (8)$$

(3) *Contact on receding nose.*—If an analysis similar to the above is carried out when the contact is on the receding nose it will be found that the equations giving the lift and the acceleration are identical with equations (5) and (7) respectively, except that ϵ is replaced by η, where $\eta = \theta_1 + \mu - \theta$ (μ being the angle of dwell), and that if the receding nose radius differs from the advancing nose radius, then the constants d and f will have different numerical values. If there is no dwell, then

$$\mu = o \text{ and } \eta = \theta_1 - \theta = \epsilon,$$

so that the expressions are identical. The equation giving the velocity of the follower when the contact is on the receding nose differs from equation (6) above, not only in having η instead of ϵ and possibly different numerical values for the constants d and f, but also in that the second term has a minus sign instead of a plus sign in front of it.

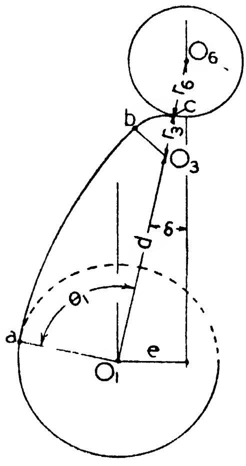


Fig. 314

With these differences the equations then hold while contact is on the receding nose, i.e. for values of θ from $\theta = \theta_2 - \delta + \mu$ to $\theta = \theta_2 - \theta_3 + \xi - \gamma$ (Fig. 315), where θ_2 and θ_3 are either given or can be settled more or less arbitrarily; ξ is found by solving triangle $O_1O_4O_6''$ of which the sides O_1O_4 and O_4O_6 are known, and the angle between them can be found by solving the triangle $O_1O_5O_4$, of which the three sides are known, for angle $O_1O_4O_5$. In a symmetrical cam $\xi = \zeta$ and $\gamma = \beta$, so that the limits can be written down without further computation.

(4) *Contact on receding flank.*—Analysis will show that the equations giving the lift, velocity and acceleration are identical with equations (2), (3) and (4) respectively, except that $-e$ has to be written for e wherever the latter occurs, that $\theta_2 - \theta$ replaces θ, and that a minus sign must be placed in front of the whole

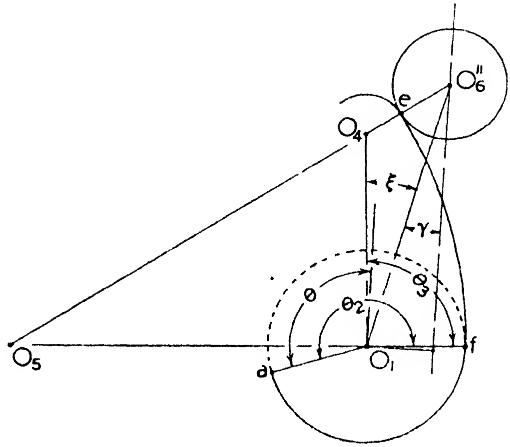


FIG. 315

expression for the velocity. The equations then hold for values of θ from $\theta=-\theta_2-\theta_3+\xi-\gamma$ to $\theta=-\theta_2-\alpha$.

253. Central Convex Cam with Roller Follower.—The equations for a central convex cam can be obtained by putting $e=0$ in the equations derived above. If this is done we get :

1. *Contact on flank.*

$$\text{Lift}=l=\{b^2-a^2\,\text{Sin}^2\,\theta\}^{\frac{1}{2}}-a\,\text{Cos}\,\theta-(r_1+r_6) \quad . \quad . \quad . \quad (9)$$

$$\text{Velocity}=\frac{dl}{dt}=\left[\frac{-a^2\,\text{Sin}\,2\theta}{2\{b^2-a^2\,\text{Sin}^2\,\theta\}^{\frac{1}{2}}}+a\,\text{Sin}\,\theta\right]\left(\frac{d\theta}{dt}\right) \quad . \quad . \quad . \quad (10)$$

$$\text{Accleration}=\frac{d^2l}{dt^2}=-\left[\frac{a^2b^2\,\text{Cos}\,2\theta+a^4\,\text{Sin}^4\,\theta}{\{b^2-a^2\,\text{Sin}^2\,\theta\}^{3/2}}-a\,\text{Cos}\,\theta\right]\left(\frac{d\theta}{dt}\right)^2 \quad . \quad (11)$$

which equations hold for values of θ from $\theta=0$ to $\theta=\theta_1-\zeta$.

2. *Contact on nose.*

$$\text{Lift}=l=d\,\text{Cos}\,\epsilon+\{f^2-d^2\,\text{Sin}^2\,\epsilon\}^{\frac{1}{2}}-(r_1+r_6) \quad . \quad . \quad . \quad (12)$$

$$\text{Velocity}=\frac{dl}{dt}=\left[d\,\text{Sin}\,\epsilon+\frac{d^2\,\text{Sin}\,2\epsilon}{2\{f^2-d^2\,\text{Sin}^2\,\epsilon\}^{\frac{1}{2}}}\right]\left(\frac{d\theta}{dt}\right) \quad . \quad . \quad . \quad (13)$$

$$\text{Acceleration}=\frac{d^2l}{dt^2}=-\left[d\,\text{Cos}\,\epsilon+\frac{f^2d^2\,\text{Cos}\,2\epsilon+d^4\,\text{Sin}^4\,\epsilon}{\{f^2-d^2\,\text{Sin}^2\,\epsilon\}^{3/2}}\right]\left(\frac{d\theta}{dt}\right)^2 \quad . \quad (14)$$

where $\epsilon=\theta_1-\theta$. These equations hold for values of θ from $\theta=\theta_1-\zeta$ to $\theta=\theta_2$.

The same equations will hold for the receding side as for the advancing side, except that for an unsymmetrical cam the numerical values of the constants will be different, and that a minus sign must precede the expression for the velocity in any case.

254. Concave Cam.—The equations for a concave cam may be obtained from those derived for the convex cam by treating the radius r_3 as negative (which has the effect of making the constants a and b negative) and by taking the negative root whenever roots have to be extracted. The equations when the contact is on the nose are, of course, the same as for the convex cam.

255. Straight or Tangential Cam.—When the contact is on the flank, as in Fig. 316, then

$$\text{Lift}=l=\text{PO}_6-(r_1+r_6)\,\text{Cos}\,a$$

where a, as before, is given by

$$\text{Sin}\,a=\frac{e}{r_1+r_6}$$

Now
$$\text{PO}_6=\frac{\text{O}_1a}{\text{Cos}\,\theta}+e\,\text{Tan}\,\theta+\frac{\text{O}_6\text{R}}{\text{Cos}\,\theta}$$
$$=\frac{r_1+r_6}{\text{Cos}\,\theta}+e\,\text{Tan}\,\theta$$

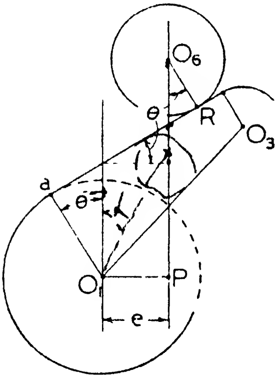


Fig. 316

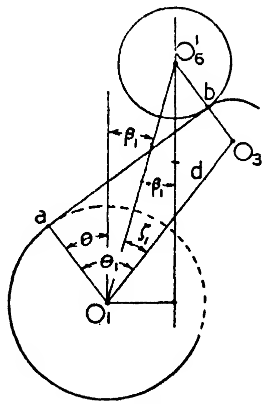


Fig. 317

Hence
$$l=\frac{r_1+r_6}{\text{Cos}\,\theta}+e\,\text{Tan}\,\theta-(r_1+r_6)\,\text{Cos}\,a \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\frac{dl}{dt}=\left[\frac{(r_1+r_6)\,\text{Sin}\,\theta}{\text{Cos}^2\,\theta}+\frac{e}{\text{Cos}^2\,\theta}\right]\left(\frac{d\theta}{dt}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\frac{d^2l}{dt^2}=\left[(r_1+r_6)\left\{\frac{1}{\text{Cos}\,\theta}+\frac{2\,\text{Sin}^2\,\theta}{\text{Cos}^3\,\theta}\right\}+\frac{2e\,\text{Sin}\,\theta}{\text{Cos}^3\,\theta}\right]\left(\frac{d\theta}{dt}\right)^2$$
$$=\left[\frac{(r_1+r_6)(1+\text{Sin}^2\theta)+2e\,\text{Sin}\,\theta}{\text{Cos}^3\,\theta}\right]\left(\frac{d\theta}{dt}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

These equations hold for values of θ from $\theta=-a$ to $\theta=\theta_1-\zeta_1-\beta_1$, where ζ_1, as is seen from Fig. 317, is found by solving the triangle $O_1O_3O_6'$, of which the sides O_1O_3 and O_3O_6' are known, and the angle between them is equal to $180-\theta$, while β_1 is given by $\text{Sin}\ \beta_1=\frac{e}{O_1O_6'}$, and O_1O_6' is also obtained from triangle $O_1O_3O_6'$.

Again the equations when contact is on the nose are identical with those for the convex cam.

When contact is on the receding flank the equations giving the lift and acceleration are obtained by putting $e=-e$ in (15) and (17), while the equation for the velocity is obtained by putting $e=-e$ in (16) and placing a minus sign in front of the whole expression.

256. Convex Cam with Flat-Footed Follower.—The amount the follower has lifted from its lowest position when the cam occupies the position shown in Fig. 318 is given by

$$l=O_2P-QO_1-r_1$$
$$=r_2-O_1O_2\ \text{Cos}\ \theta-r_1$$

but $O_1O_2=r_2-r_1$

$$\therefore\ l=(r_2-r_1)(1-\text{Cos}\ \theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

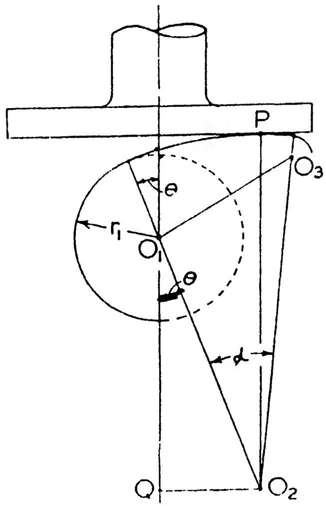


Fig. 318

Hence the follower velocity is given by

$$\frac{dl}{dt}=\frac{dl}{d\theta}\cdot\frac{d\theta}{dt}=(r_2-r_1)\ \text{Sin}\ \theta\ .\left(\frac{d\theta}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and the acceleration by

$$\frac{d^2l}{dt^2}=(r_3-r_1)\ \text{Cos}\ \theta\ .\left(\frac{d\theta}{dt}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (20)$$

These expressions hold while the contact is on the flank, i.e. for values of θ from $\theta=0$ to $\theta=\alpha$. The angle α is found by solving the triangle $O_1O_2O_3$, of which the sides $O_1O_2(=r_3-r_1)$, $O_2O_3(=r_2-r_3)$ and O_1O_3 are known; this may be done quite accurately enough by drawing the triangle to an enlarged scale.

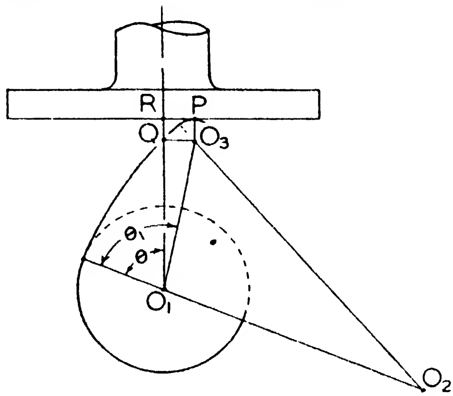


FIG. 319

When the contact is on the nose, as in Fig. 319, we have

$$
\begin{aligned}
l &= O_1Q+QR-r_1 \\
&= O_1O_3 \operatorname{Cos}(\theta_1-\theta)+r_3-r_1 \\
&= d \operatorname{Cos}(\theta_1-\theta)+r_3-r_1 \qquad (21)
\end{aligned}
$$

$$\therefore \frac{dl}{dt}=d \,.\operatorname{Sin}(\theta_1-\theta)\left(\frac{d\theta}{dt}\right) \qquad (22)$$

and

$$\frac{d^2l}{dt^2}=-d \operatorname{Cos}(\theta_1-\theta)\left(\frac{d\theta}{dt}\right)^2 \qquad (23)$$

These expressions hold while the contact is on the nose of the cam, i.e. for values of θ from $\theta=\alpha$ to $\theta=\theta_1$.

It will be seen that the follower has simple harmonic motion whenever it is moving.

257. Comparison of the Various Types.—It is beyond the scope of this book to go at all fully into the relative merits of the above types of cam for use in engines, and for a full comparison the reader is referred to an article by B. B. Low, M.A., in *Engineering*, May, 25, 1923. Briefly it may be said that the convex cam with flat-footed follower gives the greatest value for the average valve lift, the concave cam with roller follower is only slightly inferior, the straight cam comes next, and the convex cam is the worst. The cam with flat-footed follower also gives by far the lowest value for the maximum negative acceleration, i.e. it requires the weakest valve springs, the straight cam coming next in order, the concave next, while the convex cam with roller follower is worst in this respect. On the other hand, the cam with flat-footed follower gives a much greater maximum positive acceleration than the others, the concave cam coming next, then the straight cam, while the convex cam with roller follower is the best in this respect. The cam with flat-footed follower is also worst as regards noise, if the latter is assumed proportional to the velocity of the follower at the moment that the necessary tappet clearance is taken up, this velocity being more than twice that with the other cams.

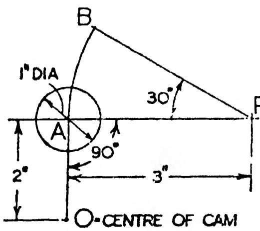
Mr. Low, in his article, has also shown that any type of cam will give smaller values of the maximum negative acceleration if there is no dwell than if a dwell is used, even though the maximum lift is increased by the elimination of the dwell; this is because the elimination of any dwell enables a greater nose radius to be used.

EXERCISES XIX

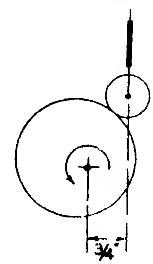
1. A reciprocating cam has a stroke of 6 in., which is performed at constant speed. It actuates a follower with a roller foot 1 in. dia. which moves in a straight line perpendicular to the line of stroke of the cam. During the first third of the stroke the cam lifts the follower, with S.H.M., a distance of 1 in. The middle third is a dwell and the last third brings the follower to the initial position with constant and equal acceleration and retardation. Design the cam.

2. Design a disc cam to give the following motion to a roller follower. Lift of 2 in. with S.H.M. followed by dwell for $\frac{1}{4}$ revolution of cam, return to initial position with S.H.M. followed by dwell for $\frac{1}{4}$ revolution. Minimum radius of cam $1\frac{1}{2}$ in. Central follower, roller 1 in. dia.

3. Determine the shape of the cam that will give the pivoted follower, shown in the figure, S.H.M. from A to B and back to A during each revolution.

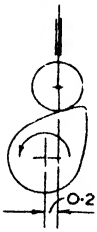


4. Repeat Question 2, but taking the line of stroke of the follower to be offset as shown in the figure.



5. The base circle dia. of a symmetrical, convex, I.C. engine cam is 1·2 in. It gives a lift of 0·4 in. to a central roller follower whose dia. is 1·0 in. The angle of opening is to be 120° and the nose radius is 0·1 in. There is to be no dwell. Determine the dimensions of the cam and the angular limits for contact to be on (*a*) the flank and (*b*) the nose.

6. If the angular speed of the cam of Question 5 is $\dot{\theta}$ rads./sec., find the acceleration of the follower when $\theta = 39°$. Check your answer by drawing an acceleration diagram.



7. Repeat Question 5, but assuming an offset as shown in the figure.

8. If the angular speed of the cam of Question 7 is $\dot{\theta}$ rads./sec., find the acceleration of the follower at the moment that contact (*a*) first occurs on the nose of the cam, (*b*) last.

9. A tangential cam works with a central roller follower. The lift $= 0·4$ in., base circle dia. $= 1·2$ in., angle of opening $= 120°$, and there is no dwell. Find the remaining dimensions of the cam and the limiting angles for contact on (*a*) flank and (*b*) nose.

10. If the angular speed of the cam of Question 9 is $\dot{\theta}$ rads./sec., find the acceleration when contact first occurs on the nose of the cam.

11. If the cam of Question 9 is offset as shown in the figure given in Question 7, find (1) the nose radius and the limiting angles for contact on flank and nose and (2) the acceleration of the follower when contact on the nose (*a*) commences and (*b*) finishes. Speed of rotation $= \dot{\theta}$ rads./sec.

12. A convex cam with a flat-footed follower has a base circle dia. of 1·2 in. The lift is 0·4 in., the nose radius $= 0·1$, and the angle of opening is 120°, there being no dwell. Determine the flank radius and the acceleration of the follower when contact first occurs on the nose.

CHAPTER XX

SPHERIC MECHANISMS ; UNIVERSAL JOINTS

258. Spheric mechanisms are those whose points have spheric motion as described in Art. 48. With a few exceptions they are of little practical importance. One of the exceptions is bevel gearing, which has already been dealt with, and another is the universal joint, which will be considered shortly. However, most of the link mechanisms described in Chapter VI have a spherical counterpart; in fact, the plane mechanisms of Chapter VI may be considered as special cases of spherical mechanisms, the radius of the sphere being infinite

259. The Spheric Four-bar Chain.—Such a chain is shown in Fig. 320, from which it will be seen that the only difference between it and an ordinary plain four-bar chain is that the axes of the turning pairs between the links converge on the point O. The links are shown curved so as to lie on a sphere having O as centre, but clearly this is not essential; they could equally well be straight.

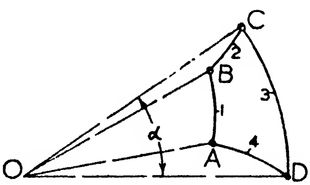


FIG. 320

It has been proved in Art. 48 that the instantaneous motion of any body having spheric motion can be produced by a rotation about an instantaneous axis which passes through the centre of the sphere; hence the instantaneous axis of the link *3* of the mechanism of Fig. 320 relative to link *1* is some line (the intersection of the planes OBC and OAD) passing through O. Consequently the axode of *3* relative to *1* is a conical surface in the general sense of the word "conical," i.e. it is a surface that can be swept out by a straight line which always passes through one fixed point. This axode can be considered to be fixed to the link *1*. Similarly the axode of *1* relative to *3* is also a conical surface which may be considered to be fixed to *3*. The motion of *3* relative to *1* due to the connexion provided by the links *2* and *4* could then equally well be produced by the rolling without slip of the one axode on the other.

Because the axodes are conical surfaces spheric mechanisms are sometimes referred to as *conic* mechanisms. Thus corresponding to the lever-crank, double-crank and double-lever mechanisms of Chapter VI there are conic lever-crank, conic double-crank and conic double-lever mechanisms.

The effective size of a link of a spheric mechanism is no longer measurable by the length of the link, but now must be measured by the angle between the axes of the joints at its ends. Thus the "length" of CD in Fig. 320 is defined by the angle α. It should be clear that two links subtending angles α and $180-\alpha$ are identical so far as the kinematics of any mechanism of which they are part are concerned, but, of course, their appearance may be quite different.

260. Disc Engines.—The spheric four-bar chain can be, and has been, constructed as an engine using a fluid such as water or steam as a working medium. Such engines have not been successful commercially for numerous reasons and so they will not be described here. Any reader who is interested in such engines is referred to *The Mechanics of Machinery* by Alex. B. W. Kennedy, published by Macmillan & Co., Ltd., and to Reuleaux's *Kinematics of Machinery*. A disc engine designed by Beauchamp Tower (famous for his experiments on the lubrication of bearings) is described in a paper by R. H. Heenan, in *Proc. Inst. Mech. E.*, 1885.

261. Universal Joints.—These are mechanisms that enable motion to be transmitted between two shafts whose axes intersect, and they are extensively used in many different kinds of machinery. The commonest form of universal joint is frequently referred to as a "Hooke's" or "Cardan" joint, after the supposed inventor of it, but according to Prof. Willis (who gives an account of the history of the joint in his book *The Principles of Mechanism*) neither Hooke nor Cardan was the actual inventor.

As will be seen subsequently, the velocity ratio between two shafts connected by a Hooke's joint is not constant, and the irregularity in the motion is a great disadvantage in many applications; consequently many people have sought after a universal joint which would have a constant velocity ratio, and at the present time two or three forms of such joint are being manufactured. Such joints will be referred to as "constant-velocity universal joints," although a more correct appellation would be "constant-velocity-ratio universal joints."

262. Hooke's or Cardan's Universal Joint.—This joint is made in several different constructional forms, two of the most important of which are shown in Figs. 321 and 322. In both those figures the shafts A and B are free to turn in bearings in the

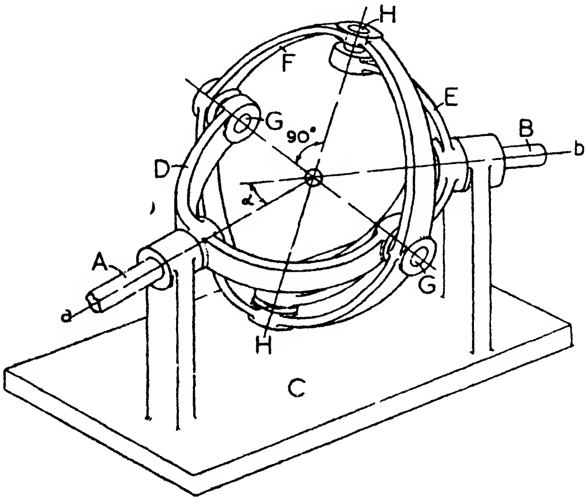


FIG. 321

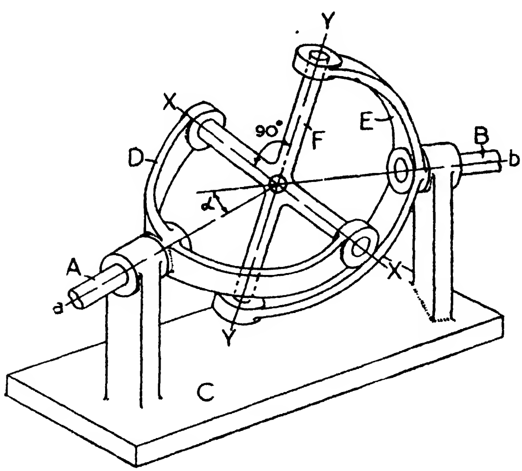


FIG. 322

frame C, and each has a " fork " (D and E respectively) fixed to it. In Fig. 321 the forks are connected by the ring F, which is provided with suitable bearings in which the pins GG and HH (integral with the forks D and E respectively) project and are free to turn. In Fig. 322 the forks are connected by the cross-member F, whose arms are pivoted in holes formed in the prongs of the forks. Kinematically the two forms are identical and are merely a special form of spheric four-bar chain, as will be seen on reference to Fig. 323, which shows the chain in skeleton form, with the corresponding links similarly lettered. The constructional forms of Figs. 321 and 322 are merely duplications of the mechanism of Fig. 323. Three of the links, namely, D, E and F, subtend angles of 90° at the centre O, the remaining link C subtending the angle α, which, as previously mentioned, may have any value up to about 60°.

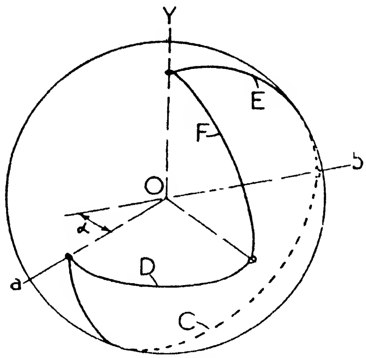


FIG. 323

263. The Velocity Ratio of the Hooke's Joint.—Fig. 324 shows two views of the joint. In that on the left both the shaft axes

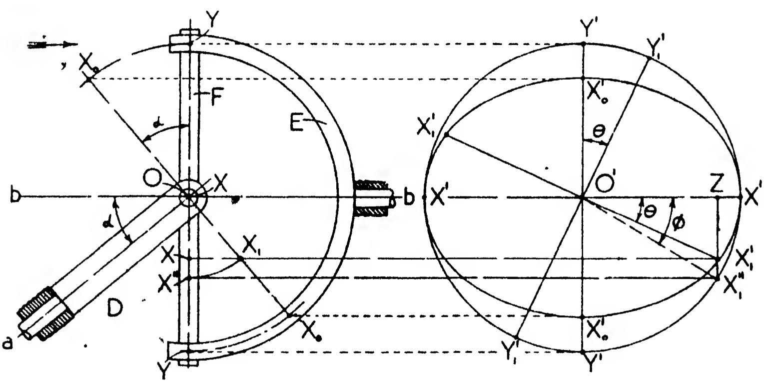


FIG. 324

Oa and Ob lie in the plane of the paper so that aOb is the true value of the angle α between the axes. The right-hand view is taken along the axis Ob in the direction of the arrow shown. Clearly in the latter view the path of the points Y,Y of the fork E, when that fork rotates about its axis, will be seen as a circle Y′,Y′. The corresponding points X,X of the other fork D will also describe a circle, but as the plane of that circle (X_0OX_0 in the left-hand view) is inclined at the angle α to the line of sight of the right-hand view, the circular path of X,X will appear in the latter view as the ellipse $X_0'X'X_0'X'$. In the initial position of the joint shown the cross-member F is seen in the right-hand view as the lines $Y'Y'X'X'$.

Suppose now that the fork E is turned through an angle θ as shown in the right-hand view. Then the new position of the cross-member is $Y_1'Y_1'X_1'X_1'$, and a little consideration will show that $X_1'X_1'$ is still at right-angles to $Y_1'Y_1'$, so that the angle $X_1'O'X'$, the angle through which the fork D has *apparently* moved, is also equal to θ. But since X_1' does not really lie in the plane of the paper, this apparent angle $X_1'O'X'$ is not the *true* value of the angle that D has turned through. To find that true value it is necessary to swing the plane of the ellipse $X_0'X'X_0'X'$ into the plane of the paper. When this is done, by swinging it about $X'X'$, the point X_1' will come to X_1'', and $X_1''O'X'$ is then the true value of the angle (ϕ) through which the fork D has turned as the result of turning the fork E through the angle θ.

Now $$\text{Tan}\,\theta = \text{Tan}\,\widehat{X_1'O'X'} = \frac{X_1'Z}{O'Z} = \frac{XO}{O'Z} = \frac{X_1O\,\text{Cos}\,\alpha}{O'Z}$$

and $$\text{Tan}\,\phi = \text{Tan}\,\widehat{X_1''O'X'} = \frac{X_1''Z}{O'Z} = \frac{X''O}{O'Z} = \frac{X_1O}{O'Z},$$

hence $$\frac{\text{Tan}\,\theta}{\text{Tan}\,\phi} = \text{Cos}\,\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and $$\text{Tan}\,\theta = \text{Tan}\,\phi\,\text{Cos}\,\alpha.$$

Differentiating with respect to time, we get

$$\text{Sec}^2\,\theta\,\frac{d\theta}{dt} = \text{Cos}\,\alpha\,.\,\text{Sec}^2\,\phi\,\frac{d\phi}{dt},$$

or, writing ω_b for $\dfrac{d\theta}{dt}$ and ω_a for $\dfrac{d\phi}{dt}$, we have

$$\frac{\omega_a}{\omega_b} = \frac{\text{Sec}^2\,\theta}{\text{Cos}\,\alpha\,.\,\text{Sec}^2\,\phi}$$

$$= \frac{1+\text{Tan}^2\,\theta}{\text{Cos}\,\alpha\,\text{Sec}^2\,\phi}$$

$$= \frac{1+\text{Tan}^2\,\phi\,\text{Cos}^2\,\alpha}{\text{Cos}\,\alpha\,.\,\text{Sec}^2\,\phi} \quad \text{since Tan}\,\theta = \text{Tan}\,\phi\,\text{Cos}\,\alpha$$

$$= \frac{\text{Cos}^2\,\phi+\text{Sin}^2\,\phi\,\text{Cos}^2\,\alpha}{\text{Cos}\,\alpha}$$

$$= \frac{1-\text{Sin}^2\,\phi\,\text{Sin}^2\,\alpha}{\text{Cos}\,\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

giving the velocity ratio in terms of ϕ and a. Similarly by substituting for ϕ instead of θ it can be shown that

$$\frac{\omega_a}{\omega_b}=\frac{\text{Cos } a}{1-\text{Cos}^2\,\theta\,\text{Sin}^2\,a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

giving the velocity ratio in terms of θ and a.

The ratio will be a maximum when Sin $\phi=0$, i.e. when $\phi=0$, π, etc., and a minimum when Sin $\phi=1$ or -1, i.e. when $\phi=\frac{\pi}{2}$, $\frac{3\pi}{2}$, etc., the values then being $\frac{1}{\text{Cos } a}$ and Cos a respectively.

The ratio will be unity when Cos $a=1-\text{Sin}^2\,\phi\,\text{Sin}^2\,a$; that is, when $\text{Sin}^2\,\phi=\frac{1-\text{Cos } a}{\text{Sin}^2\,a}$. Thus there will be four values of ϕ during each revolution when the angular velocities of the shafts will be equal. A polar diagram of the velocity of the shaft B, assuming that of A to be constant, and for a value of $a=30°$, is given in Fig. 325, the circle, radius unity, representing the con-

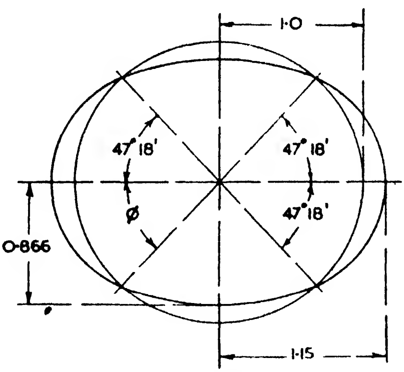


FIG. 325

stant velocity of A. The ratio is unity for $\phi=\pm 47°\ 18'$, and $\pm(180-47°\ 18')$.

264. The Acceleration of the Driven Shaft.—Since

$$\omega_a=\frac{\text{Cos } a}{1-\text{Cos}^2\,\theta\,\text{Sin}^2\,a}\cdot\omega_b=\frac{d\phi}{dt}\,.$$

on differentiating with respect to time we get, *assuming ω_b to be constant,*

$$\frac{d^2\phi}{dt^2}=-\frac{\omega_b{}^2\,\text{Cos } a(2\,\text{Cos}\,\theta\,\text{Sin}\,\theta\,\text{Sin}^2\,a)}{(1-\text{Cos}^2\,\theta\,\text{Sin}^2\,a)^2}$$

$$=-\frac{\omega_b{}^2\,\text{Sin } a\,\text{Sin } 2a\,\text{Sin } 2\theta}{2(1-\text{Cos}^2\,\theta\,\text{Sin}^2\,a)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

This acceleration will be a maximum for a certain value of θ which will now be determined. Since ω_b and a are both constants, the acceleration will be a maximum when $\dfrac{\text{Sin } 2\theta}{(1-\text{Cos}^2\,\theta\,\text{Sin}^2\,a)^2}$ is a maximum, and to find the value of θ for this the differential of the expression with respect to θ must be equated to zero.

Hence for maximum acceleration

$$\frac{d}{d\theta}\left\{\frac{\text{Sin } 2\theta}{(1-\text{Cos}^2\,\theta\,\text{Sin}^2\,a)^2}\right\}=0$$

$$\therefore\quad \frac{2\,\text{Cos}2\theta}{(1-\text{Cos}^2\,\theta\,\text{Sin}^2\,a)^2}-\frac{2\,\text{Sin}^2\,2\theta\,.\,\text{Sin}^2\,a}{(1-\text{Cos}^2\,\theta\,\text{Sin}^2\,a)^3}=0$$

$$\therefore\quad \text{Cos}2\theta(1-\text{Cos}^2\,\theta\,\text{Sin}^2\,a)-\text{Sin}^2\,2\theta\,\text{Sin}^2\,a=0$$

which reduces to

$$\text{Cos}^4\,\theta-\text{B}\,\text{Cos}^2\,\theta-\text{C}=0 \quad \text{where } \text{B}=\frac{3\,\text{Sin}^2\,a-2}{2\,\text{Sin}^2\,a}$$

$$\text{and } \text{C}=\frac{1}{2\,\text{Sin}^2\,a}$$

and this gives

$$\text{Cos}^2\,\theta=\frac{\text{B}+\sqrt{\text{B}^2+4\text{C}}}{2} \quad \ldots\ldots \quad (5)$$

from which Cos θ and hence θ may be found.

When $a=30°$ it will be found that the value of θ for maximum acceleration is 37° and the corresponding value of ϕ is 29° 30′. It will be noticed that this value is not the same as that for which the angular velocities of the shafts are equal. The value of the maximum acceleration in the example taken ($a=30°$) is $0{\cdot}2945\,\omega_b^2$, so that if $\omega_b=1000$ r.p.m. the maximum acceleration is 3229 rads./sec.^{2}.

265. Other Forms of Universal Joint.—Professor Reuleaux has proposed a variation of Hooke's joint on the lines of Fig. 326. The shafts A and B are carried in bearings in the frame (not shown) and B is provided with a fork in the prongs of which the " cross-member " F is pivoted, the axis of the pivot being perpendicular to that of the shaft. The axis is seen in the left-hand view as the point M. The other arm of cross-member is perpendicular to the first and is free to turn (about the axis MN) in the arm LN, which corresponds to the fork of a Hooke's joint so far as the shaft A is concerned. It will be seen that the construction differs from that of Hooke's joint only in that the axis MN is inclined at $90°-\beta$ to the axis of A instead of at 90°.

The relation between the angles turned through by the shafts may be found as follows. Starting from the position shown, let the shaft A be turned through an angle θ about its own axis. Then the projection OV of the arm will turn to OU through an

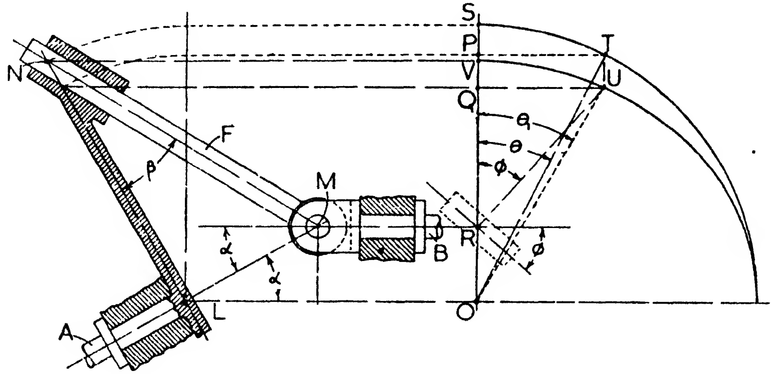


Fig. 326

apparent angle θ, as shown. As in Hooke's joint, the relation between θ and θ_1 is $\text{Tan}\,\theta = \text{Tan}\,\theta_1 \text{Cos}\,a$, a being the angle between the shaft axes. The new position of the arm MN will now be RU, and since the arms of the cross are perpendicular, it will be seen that the angle $\text{URV} = \phi$ is the true angle through which the shaft B has turned.

Now $$\text{Tan}\,\phi = \frac{\text{QU}}{\text{RQ}} = \frac{\text{QU}}{\text{OQ}-\text{OR}} = \frac{\text{QU}}{\text{OQ}-\text{LN Tan}\,\beta\,\text{Sin}\,a}$$

but $$\text{LN} = \text{OS} = \text{OT} = \frac{\text{PT}}{\text{Sin}\,\theta} = \frac{\text{QU}}{\text{Sin}\,\theta}$$

and $$\text{OQ} = \frac{\text{QU}}{\text{Tan}\,\theta_1} = \frac{\text{QU}\,.\,\text{Cos}\,a}{\text{Tan}\,\theta}$$

hence $$\text{Tan}\,\phi = \frac{\text{QU}}{\dfrac{\text{QU}\,.\,\text{Cos}\,a}{\text{Tan}\,\theta} - \dfrac{\text{QU}}{\text{Sin}\,\theta}\,\text{Tan}\,\beta\,\text{Sin}\,a}$$

$$= \frac{\text{Sin}\,\theta}{\text{Cos}\,\theta\,\text{Cos}\,a - \text{Tan}\,\beta\,\text{Sin}\,a} \qquad \ldots\ldots \quad (6)$$

If β is put equal to zero, i.e. if $\widehat{\text{LMN}} = 90°$, then the above expression reduces to that obtained for Hooke's joint.

The irregularity in the motion transmitted by this joint is greater than that produced by a Hooke's joint for the same shaft angle, and the joint is consequently of little practical importance except that by duplication it can be turned into a constant-velocity joint. Thus the joint shown in Fig. 327 is in reality two

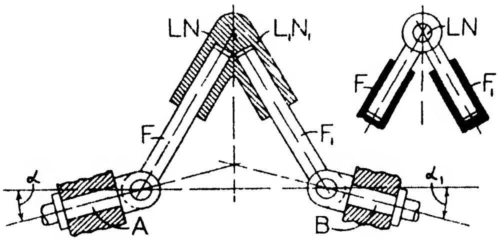


FIG. 327

joints, each similar to that of Fig. 326, but having the member LN in common. Provided the angles a and a_1 are equal, the motion of the shafts A and B will be identical. In an actual joint of this type the members F and F_1 were made hollow and the common member LN had two solid arms to fit into F and F_1. These two arms were actually pivoted to each other as indicated on the right in Fig. 327, but they might, kinematically, have been integral.

A variation of this joint was patented in the U.S.A. by Clemens in 1869. The member LN was dispensed with altogether, F being joined to F_1 by a ball-and-socket joint.

266. Constant-Velocity-Ratio Drives Using Hooke's Joints.—Although the Hooke's joint transmits an irregular motion, the irregularity can be eliminated by the use of two joints properly arranged. Thus in Fig. 328, provided the angles, a and β,

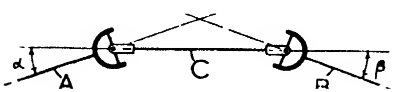


FIG. 328

between the intermediate shaft C and the shafts A and B are equal, and provided also that the axes of the turning joints between the shaft C and the cross—or ring—members connecting it to A and B are parallel, then the velocity ratio between A and B will be unity at every instant, the irregularity introduced by one joint being cancelled out by the other joint. If the axes at the ends of the intermediate shaft are perpendicular instead of parallel, then the irregularity of the motion will be doubled instead of being eliminated. This arrangement has been used successfully as a drive to the front (steering) wheels of motor cars and lorries.

A constant velocity ratio will also be obtained if in the above arrangement the shaft B is maintained parallel to A.

267. Constant-Velocity-Ratio Universal Joints.—Many efforts have been made to produce a single universal joint having a constant velocity ratio, and at the present time there are two or three such joints available. One such joint is derived from the arrangement of Fig. 328, by making the intermediate shaft C of zero length and using an old and imperfect form of Hooke's joint shown in Fig. 329. In this imperfect joint the axes, XX and Y, of the turning joints between the forks and the cross-member are perpendicular, as in a true Hooke's joint, but they do not intersect, being separated by a distance a. If two such joints are arranged as in Fig. 327, and then the intermediate shaft is made of zero length, the joint shown in Fig. 330 is derived, in which the irregularity is cancelled out and a constant velocity ratio is obtained. The only drawback of this joint, other than its complication, is that as the shafts turn they receive a slight axial motion which must be allowed for. This axial motion is due to the distance between the points L and M of the joint in the position shown in Fig. 330 being less than the corresponding

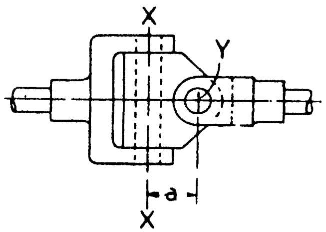


Fig. 329

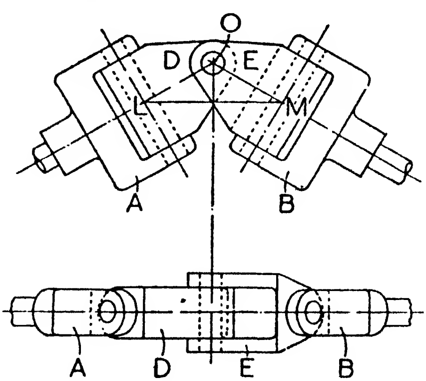


Fig. 330

distance after the joint has been turned through a quarter of a turn when the new distance between L and M will be equal to

LO + OM. This joint has also been used successfully in a front-wheel-driven motor car, and the axial motions of the shafts are eliminated by leaving out the pin connecting D and E. The frame supporting the shafts (not shown in the figure) is used to constrain the shafts not only as regards the angular relation between their axes, but also as regards their axial positions.

268. The Weiss and Rzeppa Joints.—These joints, which are in extensive production, are based on the same principle, which is illustrated by the arrangement shown in Fig. 331. The shafts A and B are provided with arms C and D having grooves formed in their faces; these grooves accommodate a ball E which consequently must always lie at the intersection of the grooves. If the latter are similarly arranged with respect to the shafts A and B, *the ball will have its centre in the plane* KK *which bisects the angle between the shaft axes*. Provided this latter condition is always fulfilled, then any motion of the shaft A about its axis will produce an exactly equal motion of the shaft B, and thus motion with a constant velocity ratio can be transmitted. It can be shown that the grooves in the arm A (and that in B) must be such that if the ball is rolled along it the centre of the ball will move in a plane containing the axis of the shaft, otherwise an irregularity in the motion will occur.

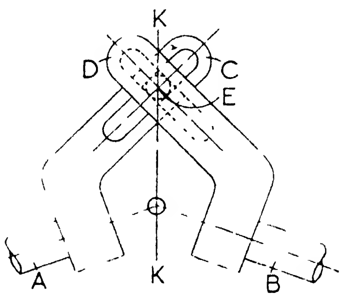


FIG. 331

To enable motion to be transmitted from either shaft to the other in either direction the arrangement must be duplicated.

The Weiss joint is a practical adaptation of the above arrangement, while the Rzeppa joint differs only in that the grooves C and D are made concentric with the intersection, O, of the axes of the shafts, and consequently the ball has to be controlled by additional mechanism so that its centre always lies in the plane KK. This additional mechanism is shown in the drawing of the joint in Fig. 332. The balls E are controlled by a cage F whose

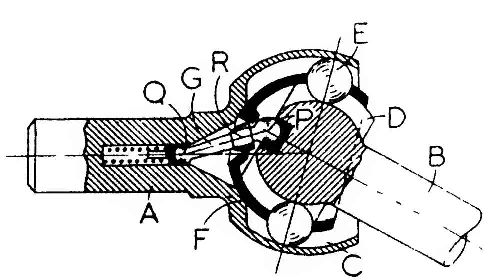


FIG. 332

position is determined by the ball-ended rod G. The end P of this rod engages a socket in the end of the shaft B, while the other end Q works in a cylindrical hole in the shaft A. The rod G has an enlargement at R which engages a hole in the cage F. The mechanism is shown in diagrammatic form in Fig. 333, from which it is clear that OP and PQ are the crank and connecting-rod of a slider-crank chain. Provided the point R is suitably chosen, the angle α that the member OR (the cage F) turns through can be made very nearly equal to half the angle θ that the crank OP (the shaft B) turns through, for values of θ up to about 50°. Thus the control in the Rzeppa joint is only approximate, but the error is quite small and the joint has proved satisfactory in practice.

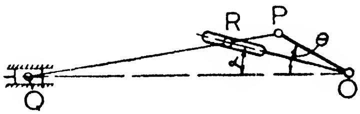


FIG. 333

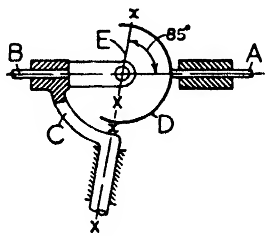
EXERCISES XX

1. A Hooke's joint connects a shaft running at 1000 r.p.m. to a second shaft, the angle between the axes being 15°. Find the angular velocity and acceleration of the driven shaft at the instant when the fork of the driving shaft has turned through an angle of 10° from the plane containing the shaft axes.

2. Derive expressions for the angular velocity and acceleration of a shaft connected by a Hooke's joint to a shaft that rotates at constant speed.

3. Find the maximum angular acceleration of a shaft that is connected by a Hooke's joint to a shaft rotating at a constant speed of 1000 r.p.m. Angle between shaft axes 10°.

4. The figure shows part of a linkage used for operating the front-wheel brakes of a motor car. The shaft A is carried in bearings in the axle beam, while B is carried by the stub axle (part of which is indicated at C), which is turned about the axis XX to steer the car. A is connected to B by a modified Hooke's joint, the



modification being that the axis xx of the connexion between the fork D and the cross E is at an angle of 85° to the axis of A instead of being at right-angles. When the brakes are off the axis xx coincides with XX, and in the straight-ahead position the axis of B coincides with that of A. In this position let A be turned through an angle of 10°, and then, keeping A fixed, let C be turned about XX through an angle of 20°. Find the angle the shaft B turns through about its axis.

CHAPTER XXI

RATCHETS, ESCAPEMENTS, ETC.

269. Ratchet mechanisms are kinematic chains in which, for some positions of one of the links, relative motion of the links is either impossible or is possible in one direction only. Simple examples of the two kinds are shown in Fig. 334 (*a*) and (*b*);

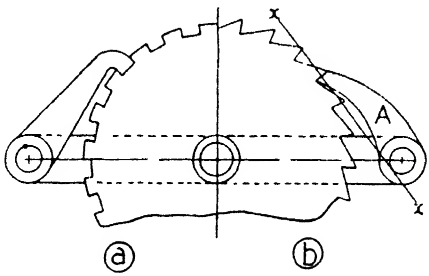


Fig. 334

these types have been called respectively *stationary ratchets* and *running ratchets*. In order that the *pawl, detent* or *click* A shall not be forced out of engagement with the *ratchet wheel* the line of thrust xx must be arranged to fall on the proper side of the axis of rotation of the pawl.

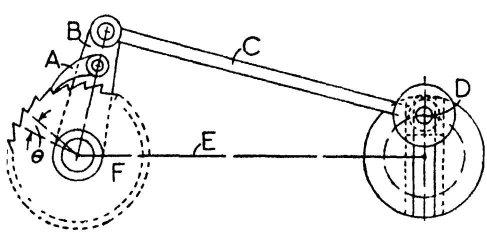


Fig. 335

Such ratchets are very widely used in all kinds of machinery. In machine tools they are used to give an intermittent motion to the work or cutter and are arranged as indicated in Fig. 335, where the pawl A is carried by the arm B which forms one link of a four-bar chain B, C, D, E, of which the crank D rotates continuously, thereby giving an oscillating motion to B. For anti-clockwise motion of the latter the pawl A imparts the motion to the ratchet wheel F, which is fixed to the screw controlling the work-table or cutter-head. If the crank D is made adjustable, as shown, then the amount of the feed per revolution of the crank can be varied. A variable feed can also be obtained using a fixed crank as shown in Fig. 336 and arranging an adjustable shield G to keep the pawl out of engagement with the ratchet wheel for variable portions of its stroke.

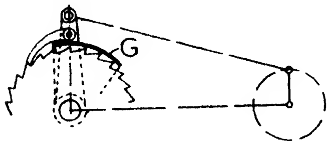


FIG. 336

If the motion imparted to the lever B in Fig. 335 is less than a certain amount, approximately equal to the angle θ subtended by the ratchet-wheel teeth, the pawl will not come into engagement with a fresh tooth at the end of its stroke; thus θ is the smallest angular movement which can be imparted to the ratchet wheel. This minimum movement could, of course, be made as small as desired by reducing the pitch of the ratchet-wheel teeth; this, however, makes those teeth weak. The difficulty can be circumvented by using multiple pawls as shown in Fig. 337, where three pawls, A, B and C, are used. Clearly the minimum motion of the ratchet wheel is now $\theta/_3$ and the pawls will act in turn.

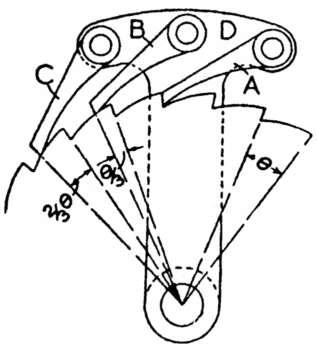


FIG. 337

Alternatively a pawl with multiple teeth may be used as shown in Fig. 338.

FIG. 338

It is sometimes required that the motion of the ratchet wheel shall be in either direction at will, and this can be done by using symmetrical ratchet-wheel teeth and a reversible pawl as shown in Fig. 339. The common normal at the point of contact P must be arranged to pass between the axis of the shaft and the centre A as shown, in order that the force acting may keep the pawl in engagement.

The pawl can also be connected to the arm by a sliding joint instead of a turning joint. Such a pawl is shown in Fig. 340. When the ratchet wheel overruns the pawl the latter falls off the

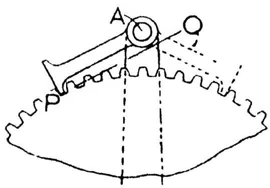


FIG. 339

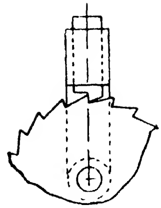


FIG. 340

tip of each successive tooth and thus makes a chattering noise which is sometimes objectionable. This can be obviated by using a *silent ratchet*, the principle of which will be clear from Fig. 341. The ring A is free on the boss of the ratchet wheel, but the fit is such that considerable friction exists and the ring tends to turn with the boss, thereby moving the pawl away from the ratchet teeth. As soon as the ratchet wheel reverses its motion the ring A brings the pawl back into contact again. The engagement and disengagement of the pawl can be made positive by using the arrangement shown in Fig. 342. The member A

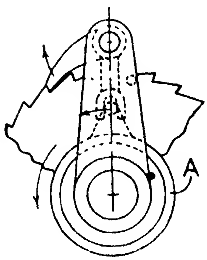


FIG. 341

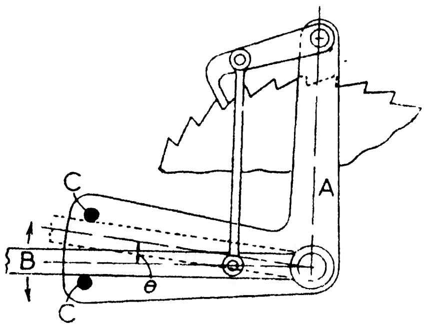


FIG. 342

carrying the pawl is free on the shaft of the ratchet wheel, as is also the actuating lever B; the latter can move independently of A through an angle θ determined by the stops CC, and this angle is made sufficient to allow of engagement and disengagement of the pawl. Motion of B after the angle θ has been described is either communicated to the ratchet wheel or is free return motion of the pawl.

270. Friction Ratchets.—It is not essential that the pawl should make positive engagement with the ratchet wheel; frictional contact can be equally as effective, and friction ratchets of the type shown in Fig. 343 are frequently used. The pawl A, pivoted at B to the arm C, has a cam face *aa* which is in frictional contact with the circular face of the wheel D. Supposing the arm to be

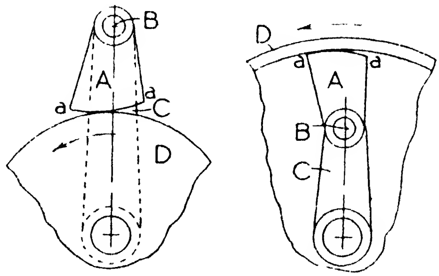


FIG. 343

fixed, then anticlockwise motion of D will simply cause the pawl to swing clear, but clockwise motion will cause it to jam against the wheel, thus locking it.

Friction ratchets besides being silent can be made so that the minimum motion possible to the wheel is less than with toothed ratchets, unless very fine pitches are used for the teeth, with the attendant disadvantages of weakness and high cost of manufacture.

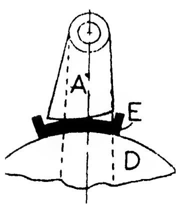


FIG. 344

An improved form of friction ratchet is shown in Fig. 344, where a pad E is interposed between the cam face of the pawl A and the wheel D.

Friction ratchets in which the pawl takes the form of a ball or roller are commonly used in the " free wheels " of bicycles and motor cars, four different arrangements being shown in Fig. 345. For anticlockwise rotation of the member A relative to B the balls or rollers C roll towards the converging end of their pockets and jam, thus locking A and B together ; for clockwise rotation of A relative to B the balls or rollers run towards the large end of their pockets, where they are free of the member B. Springs of various forms may be used as at D to keep the balls or rollers in a position such that they can act immediately A begins to turn anti-

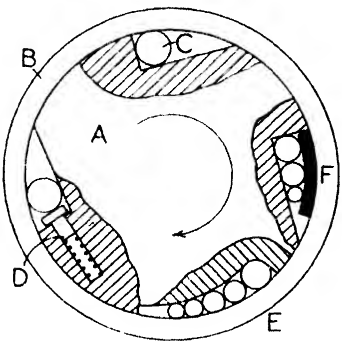


FIG. 345

clockwise relative to B. A number of balls or rollers of varying sizes is sometimes used, as shown at E, and a pad may also be placed between the balls or rollers and the outer drum as shown at F. Sometimes also the inner member is made circular and the outer member is shaped so that the balls or rollers jam for one direction of relative rotation. Also, by making the pockets in the shaped member symmetrical, the balls or rollers may be made to jam for both directions of relative rotation, but the rollers must then be controlled in some way so that the direction of rotation for which they will jam may be selected at will. In Fig. 346, for example, the rollers are prevented by the cage A, which is fixed relative to the outer member, from moving towards the jamming position when the inner member turns in the clockwise direction relative to the outer member, but they can jam for anti-clockwise motion. If the cage were released from the outer member and turned through an angle α, and again locked, then the inner member would jam for clockwise motion relative to the outer member.

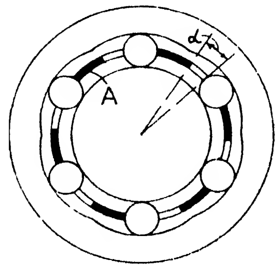


FIG. 346

In the Humphrey-Sandeberg roller ratchet the inner and outer members (A and B) are parts of hyperboloids of revolution, and parallel rollers, with their axes making a suitable angle with the common axis of A and B, are arranged between the latter. Then for one direction of rotation of A relative to B the rollers will tend to cause axial motion of A relative to B, so that jamming ensues and causes the two members to revolve as one. For the opposite direction of rotation the rollers will tend to cause axial motion such as to free the members. By controlling the axial position of the inner member relative to the outer the mechanism can be made to function as a friction clutch and to slip when the torque acting exceeds a predetermined figure.

271. Spring Ratchets.—A coil spring can be used as a ratchet as is shown in Fig. 347. The plate A is quite free to turn on the member B, whereas the plate C is mounted so that there is a certain amount of friction tending to keep it from turning relative to B. The spring D is coiled round B and at its ends is anchored to A and C respectively. If now A is turned in a clockwise direction when looked on in plan view, then since C will tend to remain at rest relative

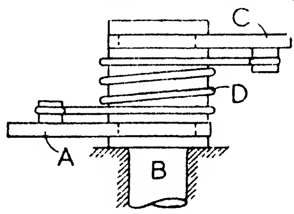


FIG. 347

to B, the spring will be coiled up round B and will grip it so that all three members A, B and C turn as one. But if A is turned in the opposite direction, then the spring uncoils and does not grip B. Then only the friction between C and B will be tending to make B turn, and provided this friction is less than the torque tending to hold B at rest, the members A and C, and the spring D, will be able to turn independently of B.

272. Lock Mechanisms.—An example of a rather different kind of ratchet is afforded by the ordinary lever lock, a simple form of which is shown diagrammatically in Fig. 348. The bolt A is free to slide in the frame B, but in the position shown is locked by the lever C, which is engaging the projecting tongue D, which is part of A. When the key is inserted and turned in a clockwise direction the land E first comes into contact with the face *lm* of the lever and raises the latter so that subsequently when the land F of the key engages the slot H in the bolt the latter can be slid back, the tongue D passing through the gap K of the lever.

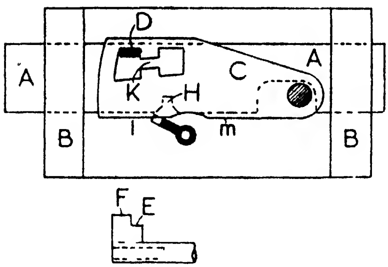


Fig. 348

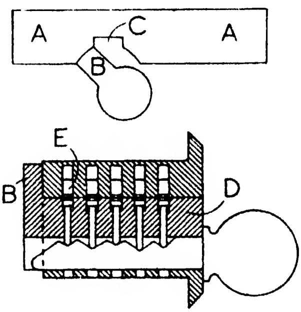


Fig. 349

Clearly unless the land E is just big enough then the tongue D will catch on the lever and the key will be prevented from sliding the bolt. By using a number of levers whose faces *lm* are differently shaped the lock can be made more difficult to pick.

The Yale lock consists of two separate mechanisms; one is shown on the top in Fig. 349, where the bolt A is locked by the cam B in the position shown. When B is turned it first of all unlocks the bolt and then, on engaging the slot C, it slides the bolt back. The second mechanism serves to lock the cam B and is shown at the bottom of the figure. The cam B is part of the cylinder D, which can only turn when all the plungers E (which are of different lengths) are brought into the positions shown, by the insertion of the proper-shaped key. When the key is removed the plungers E enter holes drilled in the cylinder and thus lock the latter, and the cam B, against rotation.

The Bramah lock is the simplest of all and is shown in Fig. 350. The bolt A carries a lug in which is formed a slot whose shape is as shown. Projecting into this slot is a pin which is part of a shaft that can turn about the axis O. In the position shown this pin is locking the bolt. When the pin is turned it engages the portion B of the slot and slides the bolt. When the bolt has been slid back the pin can go on turning, moving the while along the semicircular portion C of the slot.

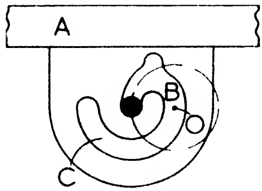


FIG. 350

273. The "Geneva Stop" Mechanism.—The principle of this is shown by Fig. 351. The shaft A turns about its axis and

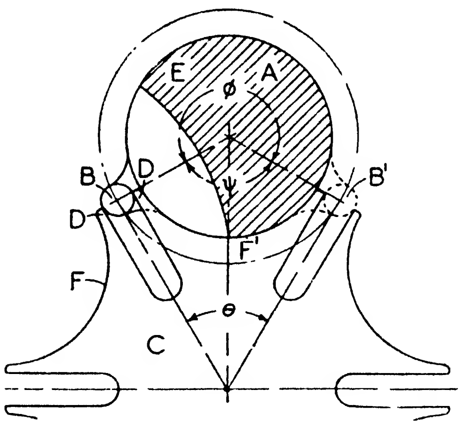


FIG. 351

has a projecting arm which carries a pin B, which is shown just entering a slot in the wheel C, so as to rotate the latter through the angle θ. During this action the wheel C is free to rotate, because the shaft A is cut away as indicated by the unshaded portion so as to allow the corners D of the wheel to clear. As the pin B leaves the slot in the position B′ the portion E of the shaft A comes round and engages the circular portion F of the wheel C, which has now arrived in the position F′. The shaft then locks the wheel against rotation until the pin arrives at the position B once again. Thus the wheel C gets an intermittent motion through the angle θ for each revolution of the shaft A, and if the latter revolves at a constant speed, then the time during which the wheel C is moving is proportional to the angle ψ and the time during which it is stationary is proportional to ϕ. This mechanism is fairly widely used in machine tools.

During the engagement of the pin B with the slot in the wheel C the mechanism is a reduced form of slider-crank chain. If the pin B were reduced in diameter and provided with a block that fitted the slot, the mechanism would be identical with the first inversion of the slider-crank chain as described in Art. 122, Chapter IX.

Expressions for the angular velocity and acceleration of the wheel C in terms of the (constant) angular velocity of A are given in Question 7 of Exercises IX.

274. The Mauser Revolver Mechanism.—This also gives an intermittent motion to the driven member and is shown diagrammatically in Fig. 352. The piece A can slide up and down the rod B and carries the pin lever C. Supposing the action to commence in the position shown, then as A moves upward the pin slides in the slot D and causes the member E to rotate about its axis XX. The pin C eventually falls over the ledge F into the slot G, so that when A goes down again the member E is held stationary. Towards the bottom of the stroke of A the pin C falls over another ledge into the slot H and the cycle of operations may be repeated. This mechanism is used also in machine tools.

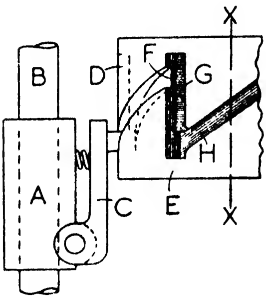


Fig. 352

Escapements.—The number of different forms of escapement is very large, and it is possible to give here only a few of them. For details of the types not described the reader is referred to books on clockwork and to Reuleaux's *Der Constructor*.

275. Graham's Escapement.—This is shown in Fig. 353 and is actuated by a pendulum which in swinging to and fro swings the anchor A. The pendulum is supposed to be swinging to the left, and the tip B of a tooth of the escape wheel has just escaped from the pallet C of the anchor ; the escape wheel now turns under the influence of the clock spring or weights until it is checked by another tooth coming into contact with the surface DE of the other pallet. When the pendulum and anchor swing back to the right the escape-wheel will again be freed and will turn until the tooth F is checked against the surface GH of the pallet C. As the tips of the escape-wheel teeth move across the surfaces HC and DK of the pallets they give an impulse to the anchor and thus to the pendulum, and this keeps the latter swinging. This impulse occurs when the pendulum is near to the middle of its swing and is an essential feature of all such escapements.

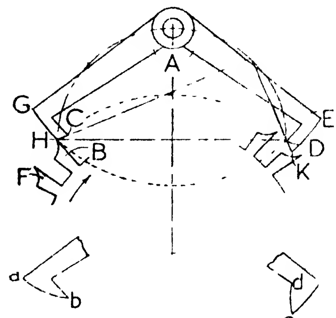


Fig. 353

Graham's escapement is an improvement of an older form known as the Anchor or Recoil escapement, in which the pallets were shaped as shown at the bottom of Fig. 353. The tips of the escape-wheel teeth on being released by the tips of the pallets were checked by the surfaces *ab* or *cd*. The pressure of the teeth on these surfaces tended to bring the pendulum to rest, but as this could not be done instantaneously, the anchor forced the escape wheel to revolve backwards slightly, i.e. to recoil. After the pendulum had come to rest and started on its return it received the necessary impulse from the pressure of the teeth on the surfaces *ab* and *cd*. The recoil of the escape wheel and its action against the motion of the pendulum have been proved to be bad. One important difference between the recoil escapement and Graham's is that, supposing the pendulum to be removed, then in the former the escape wheel can force the anchor to vibrate, whereas in the latter it cannot.

276. The Chronometer Escapement.—This is used in conjunction with a balance wheel, controlled by a hair spring, which oscillates and, being fixed to the same shaft or stem as the members P and Q (Fig. 354), which carry projecting pieces L and

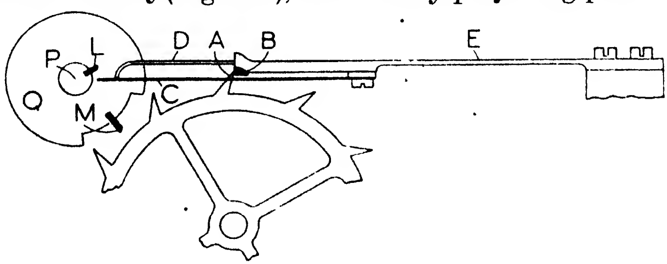


Fig. 354

M as shown, oscillates them also. In the position shown the escape-wheel tooth A is held against the locking stone B and the balance wheel is turning in a clockwise direction, so that eventually L strikes the light spring C, which deflects and allows it to pass and the balance wheel to travel on to the end of its swing. On its return L again strikes C, but the latter is now backed up by the stiff arm D, and so the whole assembly has to deflect by bending the thin portion E. This allows the tooth A to escape and the wheel begins to turn; this brings a tooth up against the projection M, and the latter, and thus the balance wheel, receive the necessary impulse to keep them vibrating. The projection L having passed the spring C, the latter and the arm D return in time for the locking stone B to check the next tooth of the escape wheel.

277. The Lever Escapement.—This is sketched in Fig. 355, where A is a disc fixed to the stem of the balance wheel and C is

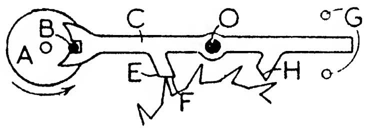


FIG. 355

the lever, pivoted at O. When the balance wheel is nearing the centre of its swing the pin B, carried by A, enters a slot in the lever, turns the latter, and thus releases a tooth of the escape wheel which has hitherto been locked against the corner E of the pallet F, or against the corresponding corner of H. In the figure the balance wheel is shown moving anticlockwise and the above action has been completed. When the escape wheel, having been unlocked, commences to move, its tooth presses against the face of the pallet F (or the corresponding face of H) and imparts an impulse to the lever and, through the pin B, to the balance wheel. Thus, when the unlocking action has been completed by the pin bearing against one side of the slot and turning the lever, the lever moves ahead of the pin and the other side of the slot comes against the pin so as to impart the impulse. Stops G limit the motion of the lever. The angle turned through by the balance wheel during the unlocking action is about one-third of that during which the impulse is given and the sum of these angles is about one-twentieth of the total angle of swing of the balance wheel.

278. The Geneva or Cylinder Escapement.—This is also known as Graham's cylinder escapement and is shown in Fig. 356. A is a portion of a thin cylinder which is attached, coaxially, to the balance wheel. The tooth B having just escaped past the edge of the cylinder, which is rotating in a clockwise direction, gives the cylinder and balance wheel the necessary impulse through the action of the face C of the tooth against the edge of the

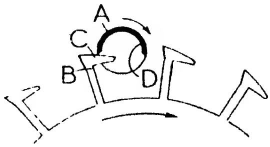


FIG. 356

cylinder. The tooth is subsequently checked against the inside of the cylinder at D and the escape wheel remains at rest until on the return swing of the balance wheel the tooth is released by the other edge of the cylinder.

CHAPTER XXII

MISCELLANEOUS MECHANISMS

279. Mechanisms Using Only Sliding Pairs.—What is probably the simplest of all mechanisms consists of only three links joined by three sliding joints, and is shown in Fig. 357. The links B and

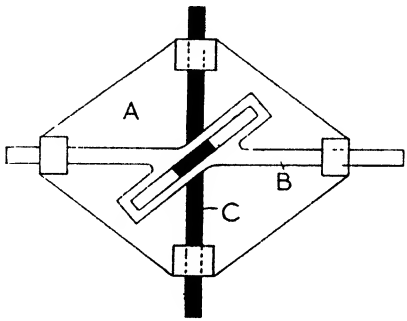


FIG. 357

C are free to slide in the frame A, and clearly motion of B produces motion of C and vice versa. A somewhat similar mechanism has four links joined by four sliding pairs. Such mechanisms seem to be of very little practical value.

280. Mechanisms Using Screw Pairs.—Mechanisms in which one or more screw pairs are used are in quite common use, but as these mechanisms are relatively simple, little need be said about them. The simplest consists of three links joined by three screw joints as shown in Fig. 358. Link A is connected to B by the screw thread a and to C by the thread c, while B is connected to C by the thread b. The threads must be

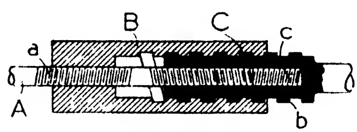


FIG. 358

of different pitches or, if two equal pitches be used, they must be of opposite hand. Let the pitches of the three threads be represented by a, b and c respectively, all being of the same hand, and let θ and ϕ be the angles turned through by A and C relative to B.

Then the axial motion of A relative to B is $a\theta$
and „ „ „ C „ „ B „ $b\phi$
$\therefore$ „ „ „ C „ „ A „ $b\phi - a\theta$

but this is equal to $c(\phi-\theta)$. Hence $\phi = \frac{(a-c)}{b-c} \cdot \theta$, and if a and c are small and b is large, then a large rotation of A will produce a small rotation of C. The writer has never come across this mechanism in his practical experience, but a mechanism in which A is connected to B by a turning joint (i.e. in which the pitch a is zero) has been used as a reduction gear for the steering of motor lorries, the steering wheel being fixed to A and C being connected to the drag link of the steering linkage (see *The Motor Vehicle*, by Newton and Steeds, page 321).

Another simple mechanism, and one that is widely used, consists of three links joined by a turning joint, a sliding joint and a screw joint; an example is an ordinary vice.

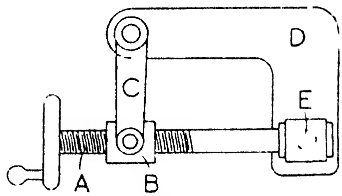


FIG. 359

A common application of a screw pair in a mechanism is shown in Fig. 359. It consists of five links A, B, C, D and E joined by one screw joint and four turning joints. The mechanism can be reduced to one of four links by joining A to D by a ball-and socket joint, the link E being eliminated. In both of these forms this mechanism has been used as a steering reduction gear for motor cars. The lever C, which is usually forked so as to straddle the nut B, is generally used to keep the nut from turning.

Two more mechanisms using screw pairs are shown in Figs. 360 and 361. In the former the nut is prevented from turning by the

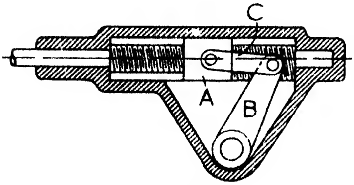


FIG. 360

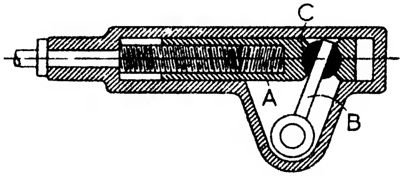


FIG. 361

casing and is coupled to the lever B by a connecting link (or links) C. In the latter the casing again prevents the nut from turning, but the nut is connected to the lever B by the link C, which is free to turn in the nut A and to slide along the lever B. Both of these mechanisms are commonly used as steering reduction gears in motor cars. Another mechanism used for this purpose is similar to that of Fig. 361, but the lever B carries a ball-ended pin, the ball of which fits in a parallel hole drilled in the nut A at right-angles to the axis of the screw ; this construction eliminates the link C.

Mechanisms using two screws of the same pitch, but of different hands, are also used in steering gears, and two examples are shown in Figs. 362 and 363. The former has been used in ships. The

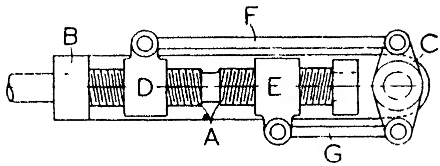


FIG. 362

double-threaded screw A has to be allowed a slight axial freedom in the bearings by which it is carried in the frame B, because otherwise the mechanism would be locked. This will be clear when it is realised that, because of the differing angularities of the connecting links F and G, a given angular movement of the link C will move the nuts D and E unequal amounts relative to the frame B, and if the screw were fixed axially this would be impossible. This difficulty is avoided in the mechanism shown in Fig. 363, where the nuts A and B (which are guided and prevented

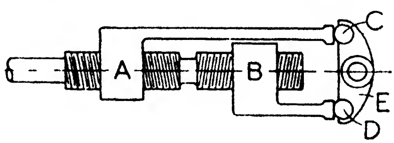


FIG. 363

from rotating by the frame, which is not shown in the diagram) are provided with extensions which bear directly on rollers C and D carried by the lever E. This form of the mechanism is used in motor cars ; it has the advantage that backlash in the screw joints can be eliminated by an axial adjustment of the screw, which, of course, must be fixed axially when the mechanism is working.

281. Skew or Crossed Kinematic Chains.—In the ordinary four-bar chain the axes of the four turning joints are parallel, and in the spherical four-bar chain they converge to a point, these being the only possible arrangements; but if more than four links are used it becomes possible to have mechanisms in which the axes of the turning joints are neither parallel nor intersecting. Also by the use of joints that allow more than one degree of freedom the number of links may be reduced to four. Such mechanisms are termed skew or crossed mechanisms, and for a fuller discussion of them than is possible in this book the reader is referred to Reuleaux's *Kinematics of Machines* (Macmillan, 1876), page 549. A few of these mechanisms only can be here considered.

In Fig. 364 is shown the mechanism of Robertson's steam engine, which was invented some time prior to 1870. The crank A is

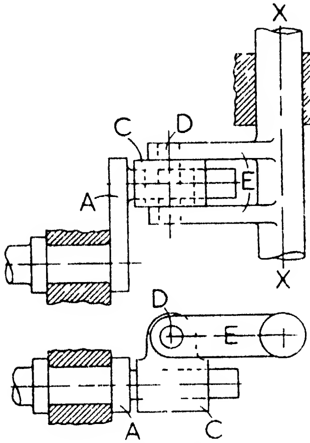


Fig. 364

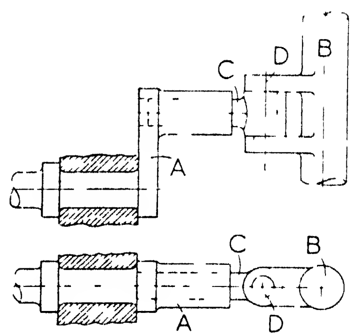


Fig. 365

connected to the member B (the piston rod) by the piece C, which is free both to slide and to turn on the crank-pin A and is free to turn about the pin D carried by the lugs E integral with B. As the crank A revolves the member B is moved with a screw motion, a combination of an axial motion up and down and a rotary oscillation about the axis XX. This does not seem a very sound arrangement mechanically, but it has worked satisfactorily as a mechanism for driving the sleeves of single-sleeve-valve internal combustion engines running at fairly high speeds. A sounder mechanical arrangement results from making the axes of the pin D and the crank-pin A intersect as shown in Fig. 365, where also, in the joint between C and A, C has been made the solid member and A the hollow member instead of the opposite arrangement of Fig. 364. This also has been successful in high-speed internal combustion engines.

A much sounder mechanical design is obtained, however, by duplicating the cranks A of the above mechanism (Fig. 364) and making the link C double-ended, as shown in Fig. 366. This also has been successfully used for the purpose mentioned above.

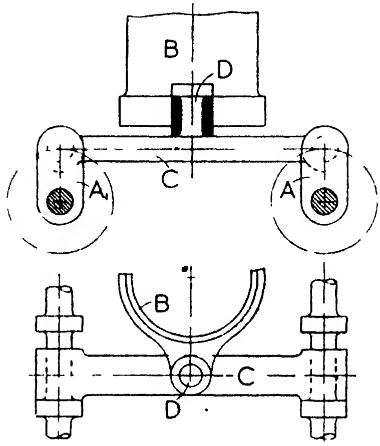


FIG. 366

Robertson also proposed an alternative mechanism on the lines of Fig. 367. The link C is now prevented from sliding along the crank-pin A and instead of being pivoted directly to B is connected by the link D, to which it is pivoted. The link D is free to slide and to turn in the member B.

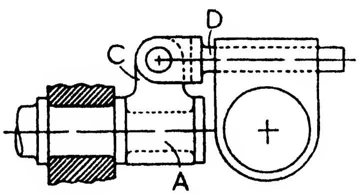


FIG. 367

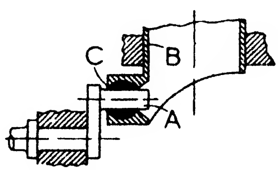


FIG. 368

What is probably the best mechanism for driving the sleeve of single-sleeve-valve engines is a reduction of the mechanism shown in Fig. 365. It is shown in Fig. 368, where the crank A is coupled to the sleeve B by the ball C, which is free to turn and to slide on the crank-pin A, and which fits in a socket in the sleeve.

Another good arrangement is shown in Fig. 369. A single crank A is used and the connecting-rod C couples this crank to a lever E. The connexion between the rod C and the sleeve B is by the ball D, which works in a socket formed in C and which is free to slide along the pin of the sleeve. In an alternative construction the end P of the rod C is guided along a straight line by a sliding block ; this, of course, is equivalent to making the lever E infinitely long.

A variation of the mechanism shown in Fig. 365 is given in Fig. 370. The members A and B are free to turn, about the axes

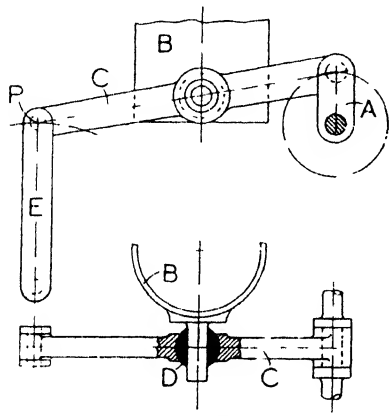


FIG. 369

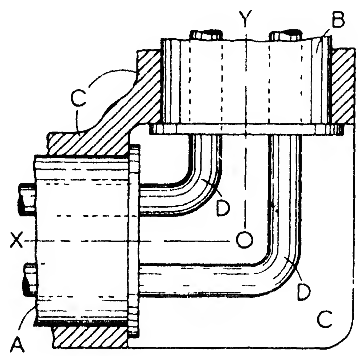


FIG. 370

OX and OY respectively, in the frame C, and are connected by the L-shaped pistons D which slide and turn in holes bored in A and B. To avoid dead-centres a number of pistons are used. The radii of the pitch circles of the cylinders in both members must be exactly equal or the members will not both be able to make complete revolutions. This mechanism is now used as a pump (the cylinder blocks A and B being positively driven) for operating the retractable undercarriages of aeroplanes. It has also been used as an engine, but not with any great commercial success.

282. The Swash-Plate Mechanism.—The term "swash-plate" is loosely applied to several mechanisms, but that to which it should, in the writer's opinion, be restricted is shown in Fig. 371. The actual swash-plate A is a slice of a circular cylinder, the faces *a* and *b* being parallel planes inclined at some angle a to the axis of the cylinder. The swash-plate is integral with its shaft and a turning joint connects it with the frame B. The member C is connected to the frame B by a sliding joint and to the pads D by ball-and-socket joints. The pads D slide on, and the centres O of their spherical portions are usually arranged to lie in, the faces of the swash-plate. If the latter condition holds, then it is easily seen that the motion of the plunger C is simple harmonic and its stroke is $\frac{2R}{\operatorname{Tan} \alpha}$, R being the distance between the axes of the swash-plate and plunger. Usually several plungers are actuated by the single swash-plate.

In this form the swash-plate has been successfully used as an engine mechanism and for air compressors, but only since the invention of the Michel thrust bearing and the application of its

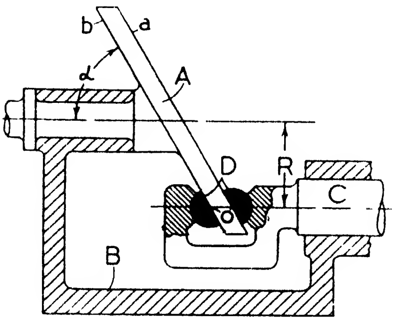


FIG. 371

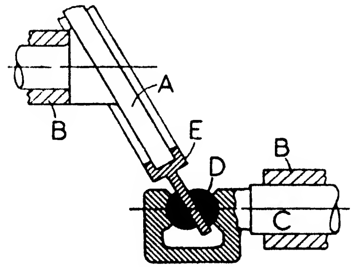


FIG. 372

principle to the lubrication of the contacts between the pads D and the swash-plate. For detailed descriptions of these applications the reader is referred to the technical Press.

If force closure is permissible, then contact need be made on one side of the swash-plate only. The speed and amount of sliding between the swash-plate and the pads D are reduced in the construction shown in Fig. 372, where the portion (E) of the swash-plate, against which the pads D bear, is made separate from and left free to turn on the body of the swash-plate A. Most of the sliding now occurs at this bearing (between E and A), but this can now be made a ball or roller bearing. The mechanism is, however, no longer a swash-plate, but has become a Z-crank.

283. The Z-Crank Mechanisms.—One form of this mechanism is shown in Fig. 373. The axis of the pin of the crankshaft A is inclined to the axis of rotation at some angle α, and the resulting shape of the shaft gives the name to the mechanism. The connecting link B, which is free to turn on the crank-pin, is connected by a ball-and-socket joint to the slider C, which is free to slide in the plunger D. The latter forms a sliding pair with the frame E. As the crankshaft rotates in the frame the plunger D receives a reciprocating motion. The link B is kept from rotating by its connexion with the slider C, which keeps the point P in the plane containing the axes of the crankshaft and plunger. In this form the mechanism has only rarely been used. Since the

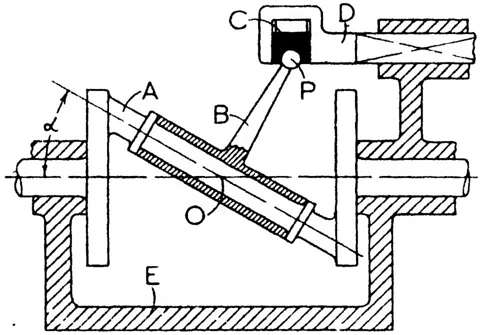


Fig. 373

axis of the crank-pin describes a cone, this kinematic chain is sometimes referred to as a conic-crank chain.

If this mechanism is used in a multi-cylinder form, i.e. with a number of plungers, D, D_1, D_2, etc., all operated by the single Z-crank, then the members B operating the plungers must be independent of each other, because during a revolution of the Z-crank the angles between the lines OP, OP_1, etc., will not be constant, but will vary slightly, and this would not be possible if a rigid member B, common to all the plungers, was used. As it is not very convenient to allow each of the members B to bear directly on the crank-pin, one of these members can be made a master member, embracing the crank-pin, while the others can embrace auxiliary pins, carried by the master member, and whose axes are parallel to that of the crank-pin.

284. The Motion of the Piston.—In Fig. 374 the front elevation shows the mechanism when the crank has turned through an

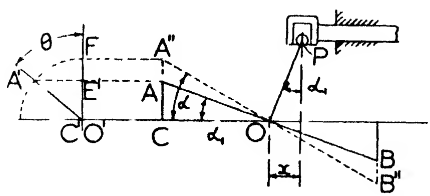


Fig. 374

angle θ, as seen in the end view, from the position in which it lies in the plane of the paper. The angle a_1 of the crank is thus not the true angle between the crank and the axis of rotation. If the crank is swung back into the plane of the paper, it will be seen as CA″OB″ and the angle A″OC will be the true angle a between the crank and the axis of rotation. Since the rod OP lies in the plane of the paper and is perpendicular to the crank, the angle POA is a right angle.

Then $\quad \text{Tan}\, a_1 = \dfrac{\text{AC}}{\text{CO}}$

and $\quad \text{AC} = \text{E}'\text{C}' = \text{C}'\text{A}' \,\text{Cos}\,\theta = \text{OF}\,\text{Cos}\,\theta = \text{A}''\text{C}\,\text{Cos}\,\theta.$

$$\therefore \quad \text{Tan}\, a_1 = \frac{\text{A}''\text{C}}{\text{CO}}\,\text{Cos}\,\theta$$

$$= \text{Tan}\, a\, \text{Cos}\,\theta.$$

Then $\quad x = \text{OP}\,\text{Sin}\, a_1$

$$= \frac{\text{OP Tan}\, a\, \text{Cos}\,\theta}{\sqrt{1 + \text{Tan}^2 a\, \text{Cos}^2 \theta}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and by differentiation and simplification it will be found that

$$\dot{x} = \frac{-\text{OP Tan}\, a\, \text{Sin}\,\theta}{\{1 + \text{Tan}^2 a\, \text{Cos}^2 \theta\}^{3/2}} \cdot \frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad (2)$$

and, if $\dfrac{d\theta}{dt}$ is constant,

$$\ddot{x} = \frac{-\text{OP Tan}\, a\, \text{Cos}\,\theta(1 + 3\,\text{Tan}^2 a - 2\,\text{Tan}^2 a\, \text{Cos}^2 \theta)\left(\dfrac{d\theta}{dt}\right)^2}{\{1 + \text{Tan}^2 a\, \text{Cos}^2 \theta\}^{5/2}} \quad . \quad (3)$$

285. A simplified form of this mechanism is shown in Fig. 375. In this form it has been and still is used occasionally. A some-

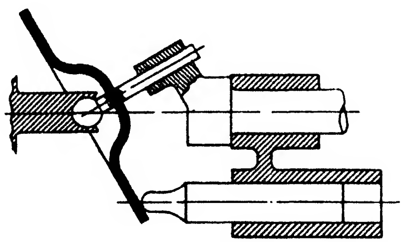


FIG. 375

what different and more widely used form of Z-crank mechanism is shown in Fig. 376, where the connexion between the link B and the plunger D is by a connecting-rod C with ball-and-socket joints at both ends. It is now necessary to anchor the link B against rotation, since clearly the rod C cannot supply the necessary constraint. This may be done as shown in the figure by providing the link B with an arm F, the end of which engages a block G that works in circular guides H fixed to the frame E. The axis OP of the link B is thereby constrained to lie in the plane containing the axis of the plunger D and the axis of rotation of the crankshaft. An alternative method is to connect the end of the arm F

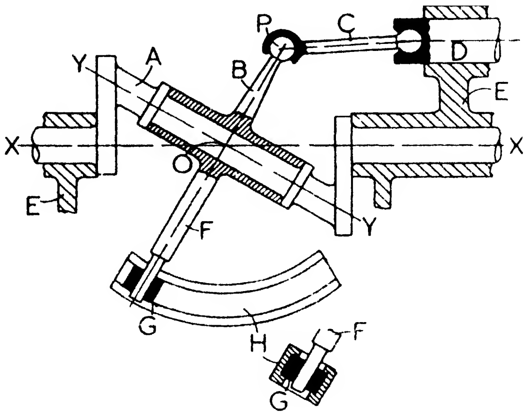


FIG. 376

to the frame by a rod (using a ball-and-socket joint at each end) whose axis, when the plunger D is in mid-stroke, is approximately perpendicular to the plane of the paper. If this is done the point P will no longer move in the plane of the paper, but will describe a figure of eight on a spherical surface, centre O.

The member B can also be controlled by making it integral with a bevel gear (axis YY) which is meshed with a similar bevel gear (axis XX) that is fixed to the frame. The pitch-cone angles of the two bevel gears must thus be each equal to half the obtuse angle between the axes XX and YY.

Whichever method of anchorage is adopted it is not essential to use a ball-and-socket joint between C and D—an ordinary turning pair is kinematically sufficient; but with the second form of anchorage (and also with the first form if the point P does not lie in the plane of the guide H) the plunger would receive a screw motion, and it was found that in the Bristol axial engine, * in which this construction was used, the torsional stresses set up in the rod C by the rotational component of the motion of the plunger brought about fracture of that rod. The trouble was obviated by using a ball-and-socket joint.

Usually in practical applications of this mechanism several plungers are operated by the single-Z-crank, and the link B takes

* See *The Engineer*, May 24th, 1935.

the form of a disc or star-shaped member connected at a number of points to the various plungers by the connecting-rods. Only the point P of the link B that lies in the plane of the guide H will move in a plane ; the other points corresponding to P will describe figures of eight as described above. The motions of the various plungers will not be identical, but the variations will be, in most practical applications, extremely small and may be neglected.

The motion of the plunger D will be very closely the same as the horizontal component of the motion of P, because the alteration in the angularity of the connecting-rod C will be quite small. The displacement, velocity and acceleration of the plunger will thus be given approximately by the equations (1), (2) and (3), page 300. As those expressions refer only to the point P, which moves in the plane of the guide H, expressions for the horizontal motion of a point P_1, not so situated, will now be derived.

286. In Fig. 377 the plane of the guide H of Fig. 376 is the plane of the paper in the elevation on the left and is seen as the line VOV

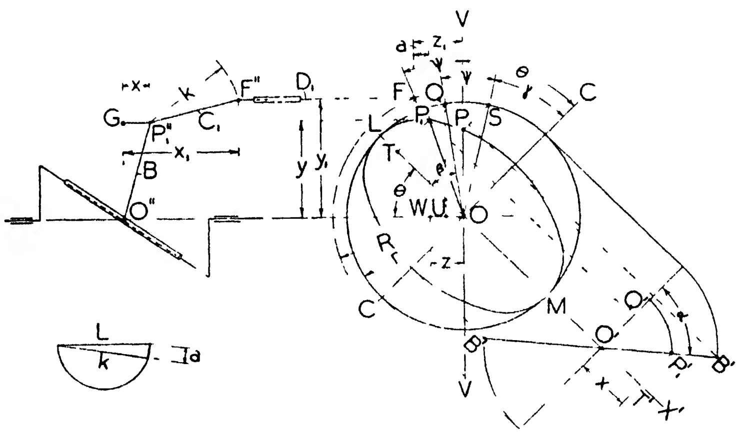


FIG. 377

in the end view. The point of the member B that lies in the plane of the guide is thus constrained to lie always in the line VOV and is seen at P. The member B, a circular plate on the circumference of which lie the points P, P_1, P_2, etc., is seen in the end view as the ellipse LP_1PM, the major axis LM of which lies in the plane of the end view and is perpendicular to the plane of the crank-shaft. The latter is shown as COC, making an angle θ with the vertical plane. In a view along the line LM the member B will be seen as a line B′B′ inclined to the vertical at the angle a, the true angle between the crank-pin and the axis of rotation of the crankshaft. The point P_1 on the circumference of B is the point whose motion parallel to the axis of rotation is being investigated. It does not lie in the plane of the elevation, nor does the axis of the plunger D_1, which is actuated by the point P_1 through the connecting-rod C_1. Since the points P, P_1, P_2, etc., will generally be situated at equal distances round the circumference of the member B, the true angle between P and P_1 will be $\frac{360°}{n}=\psi$, where n = the number of cylinders. In the end view this angle appears as POP_1, while the true value is obtained by swinging the member B about the line LM until it lies in the plane of the paper. It will then appear as the circle LQSM and the angle QOS will be the true angle ψ. If P_1 is projected to P_1' and $P_1'T'$ is drawn perpendicular to $O'X'$, then $O'T'=x$ is the distance, measured parallel to the axis of rotation, of the point P_1 from the origin O, the intersection of the axis of the crank-pin and the axis of rotation. Also $P_1W=y$ is the height of the point P_1 above the horizontal plane through the axis of rotation, and $WO=z$ is its distance out from the vertical plane through the axis of rotation.

Then

$$\begin{aligned} x &= O'T' = P_1'T' \operatorname{Tan} \alpha \\ &= P_1T \operatorname{Tan} \alpha \\ &= TQ \operatorname{Cos} \alpha \operatorname{Tan} \alpha \\ &= OQ \operatorname{Sin} \beta \operatorname{Sin} \alpha \\ &= r \operatorname{Sin} \alpha \operatorname{Cos} (\psi+\gamma) \qquad \ldots\ldots \quad (4) \end{aligned}$$

where r is the radius of the circle on which the points P lie, and $\operatorname{Tan} \gamma = \operatorname{Tan} \theta \operatorname{Cos} \alpha$.

If ψ is put equal to zero, then the expression will refer to the point P and reduces to

$$\begin{aligned} x &= r \operatorname{Sin} \alpha \operatorname{Cos} \alpha \\ &= \frac{r \operatorname{Sin} \alpha}{\sqrt{1+\operatorname{Tan}^2 \theta \operatorname{Cos}^2 \alpha}} \\ &= \frac{r \operatorname{Tan} \alpha \operatorname{Cos} \theta}{\sqrt{1+\operatorname{Tan}^2 \alpha \operatorname{Cos}^2 \theta}} \end{aligned}$$

which agrees with equation (1), page 300, r being equal to the OP of that equation.

It can easily be shown that

$$y = P_1W = r \operatorname{Sin} (\psi+\gamma) \operatorname{Sin} \theta + r \operatorname{Cos} (\psi+\gamma) \operatorname{Cos} \theta \operatorname{Cos} \alpha \quad . \quad (5)$$

and that

$$z = OW = r \operatorname{Sin} (\psi+\gamma) \operatorname{Cos} \theta - r \operatorname{Cos} (\psi+\gamma) \operatorname{Sin} \theta \operatorname{Cos} \alpha \quad . \quad (6)$$

Since the angle between the planes of the cylinders will be ψ, it will be seen that the co-ordinates of the end F″ of the connecting-rod C are $y_1 = \mathrm{R} \operatorname{Cos} \psi$, $z_1 = \mathrm{R} \operatorname{Sin} \psi$ and x_1, which is the unknown quantity, R being the distance between the axes of the cylinders and the axis of rotation. Then if L is the actual length of the connecting-rod C, we have

$$\mathrm{L}^2 = (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2$$

from which $$x_1 = x \pm \sqrt{\mathrm{L}^2 - (y_1 - y)^2 - (z_1 - z)^2}$$

If the expressions for x, y, z, y_1, z_1 and γ derived above are substituted in this equation, it will then give x_1 in terms of θ, but will be so complicated as to be practically unmanageable.

As stated previously, however, the motions of the plungers D will be very closely the same as the motions, parallel to the axis of rotation, of the points P, and thus may be taken to be given by the equation

$$x = r \operatorname{Sin} a \operatorname{Cos} (\psi + \gamma).$$

Then $$\dot{x} = -r \operatorname{Sin} a \operatorname{Sin} (\psi + \gamma) \frac{d\gamma}{d\theta} \cdot \frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad (7)$$

and, if $\frac{d\theta}{dt}$ is constant,

$$\ddot{x} = -r \operatorname{Sin} a \left\{ \operatorname{Cos} (\psi + \gamma) \left(\frac{d\gamma}{d\theta} \right)^2 + \operatorname{Sin} (\psi + \gamma) \frac{d^2\gamma}{d\theta^2} \right\} \left(\frac{d\theta}{dt} \right)^2 \quad . \quad . \quad (8)$$

where $$\operatorname{Tan} \gamma = \operatorname{Tan} \theta \operatorname{Cos} a$$

$$\frac{d\gamma}{d\theta} = \frac{\operatorname{Sec}^2 \theta \operatorname{Cos} a}{1 + \operatorname{Tan}^2 \theta \operatorname{Cos}^2 a}$$

and $$\frac{d^2\gamma}{d\theta^2} = \frac{2 \operatorname{Sec}^2 \theta \operatorname{Tan} \theta \operatorname{Cos} a \operatorname{Sin}^2 a}{(1 + \operatorname{Tan}^2 \theta \operatorname{Cos}^2 a)^2}$$

287. The position of the plunger D_1 when the crank is in the position shown, i.e. having turned through an angle θ from the plane of the guide, may be found exactly by graphical means as follows. The views on the right of Fig. 377 having been drawn, the point P_1'' in the elevation may be found by setting out $\mathrm{GP}_1'' = x = \mathrm{O'T'}$ along the projection line through P_1. Then with centre P_1'' and radius $k = \sqrt{\mathrm{L}^2 - a^2}$ an arc is struck to intersect the line of stroke of D_1 (projected from F, which is the end view of it) in F″, L being the actual length of the connecting-rod C and a (equal to $z_1 - z$) being the dimension shown in the end view, i.e. the difference between the distances of P_1 and F in front of the plane VOV. Clearly k may be obtained graphically by the construction shown in the left-hand corner of the figure.

288. The " Janney " Mechanism.—Another mechanism that is commonly referred to as a swash-plate mechanism, but which is actually an inversion of the Z-crank mechanism shown in Fig. 376, is shown in Fig. 378. The Z-crank is now the fixed member and

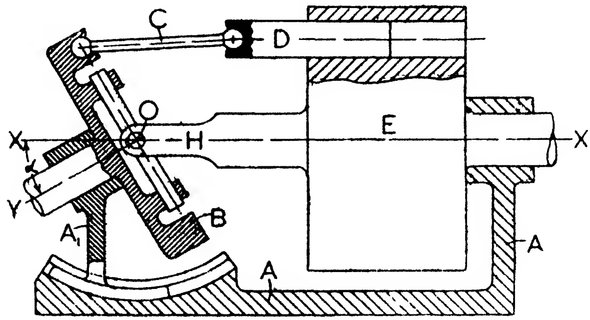


Fig. 378

is the frame AAA_1. The member B now revolves about the inclined axis OY of the Z-crank. This member is commonly referred to as the swash-plate of the mechanism. The member E now rotates about the axis XX of the Z-crank and is connected to the member B by a universal joint of which O is the cross and which corresponds to the guide FGH of Fig. 376. Since the mechanism is an inversion of that of Fig. 376, the motions of the pistons D parallel to the axis of rotation XX are the same as in that mechanism and are given approximately by the equations of Art. 284.

By varying the angle α the stroke of the pistons may be varied. To do this the member AAA_1 is made in two parts, AA and A_1, and A_1 is arranged to be able to turn about an axis, passing through the point O, relative to AA. Except during the actual alteration of the stroke the part A_1 is fixed relative to the part AA.

This mechanism has been used for many years as a variable-speed hydraulic drive, one of its first applications being in war-ships, where it was used to rotate the gun turrets. It was particularly suitable for this purpose, because, due to its ability to exert high torques at very low speeds, it gave a very precise control of the motion. It has since been used for many other purposes, such as actuating the rudders of ships, driving machine tools, in mine haulage gear, etc. It is known under several names, e.g. Williams-Janney, Wateringbury, Vickers-Janney, and is manufactured in England by Variable-Speed-Gears, Ltd.

289. The Wobble-Crank.—This is similar to the Z-crank, but whereas in the latter the axis of the crank-pin intersects the axis of rotation, in the wobble-crank it does not do so, the crank-pin axis and the axis of rotation being in a skew relationship. One application of this is as a drive to the sleeve valve of single-sleeve-valve internal combustion engines and is shown diagrammatically in Fig. 379. The member B, which is free to turn on the crank-pin, now receives an up-and-down motion in addition to a sideways

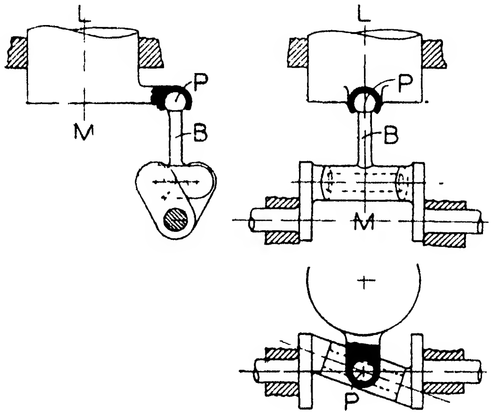
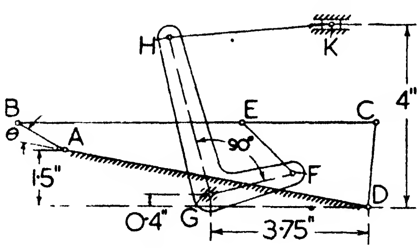


FIG. 379

rocking motion, and the ball-end P is thus able to impart an up-and-down motion along, combined with an angular oscillation about, the axis LM, which is the motion required with this valve. For a description of this application the reader is referred to *The Motor Vehicle*, 1st Edition, page 67.

MISCELLANEOUS EXERCISES

1. The figure shows a mechanism used in a machine for wrapping caramels. It consists of a four-bar chain ABCD of which AD is fixed and AB rotates at a constant speed. The point E of the link BC is connected as shown to one arm of a bell crank FGH pivoted on a fixed pivot at G. The other arm H is connected

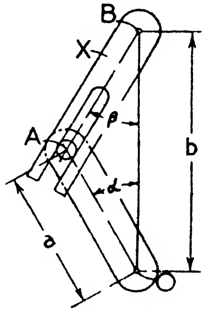


by a link to a slider K. Plot a diagram showing the displacement of the slider K for any position of the crank AB. Take the displacement to be zero when the angle θ is zero. AB = 1 in., BC = 6·5 in., CD = 2·0 in., AD = 6·5 in., BE = 4·25 in., EF = 1·25 in., FG = 1·75 in., GH = 3·25 in., HK = 6·0 in.

2. The figure shows a simplified version of the Geneva-stop mechanism. If OA is rotating about O with a constant angular velocity ω, prove that

$$\frac{d\beta}{dt}=\frac{a(b \operatorname{Cos} \alpha - a)}{a^2+b^2-2ab \operatorname{Cos} \alpha} \cdot \omega$$

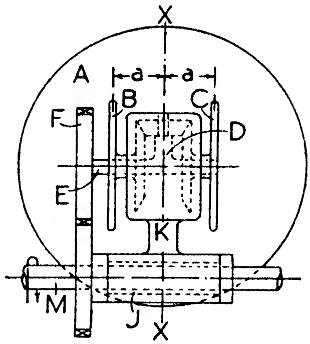
$$\frac{d^2\beta}{dt^2}=\frac{(a^3b-ab^3) \operatorname{Sin} \alpha}{(a^2+b^2-2ab \operatorname{Cos} \alpha)^2} \cdot \omega^2$$



and that the maximum acceleration of X occurs when

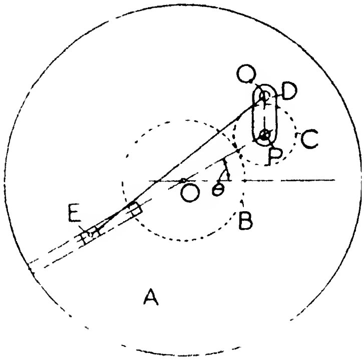
$$\operatorname{Cos} \alpha = -\frac{(a^2+b^2)}{4ab} \pm \sqrt{\left(\frac{a^2+b^2}{4ab}\right)^2+2}.$$

3. The figure shows an element of a mechanism used in an automatic steering gear for ships. A is a disc driven at a constant speed Ω. B and C are friction discs running on the face of A and coupled to the wheels of a differential D. The



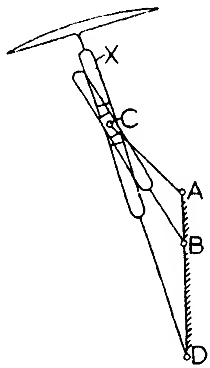
arm of the differential is fixed to the shaft E carrying the wheel F, which meshes with an equal wheel H. The latter is fixed to a nut J free to turn in the casing K and engaging a thread on the shaft M. At time t let the displacement of the casing K from the central position be x, let the angular displacement of M from the zero position be ϕ and the angular displacement of the nut J from the initial position be ψ. Prove that, when a state of equilibrium obtains, $\frac{d\theta}{dt}=k(\phi-\psi)$, where k is a constant.

4. The figure shows a mechanism used for turning approximately elliptical holes in boiler plates. The head A rotates about the centre O, concentric with which is a fixed gear B, which meshes with a gear C half the size of B. The gear C is fixed to the shaft of a crankshaft D carried by the head A, and whose crank-pin is coupled by a connecting-rod to a slider E working in a radial slot in A.



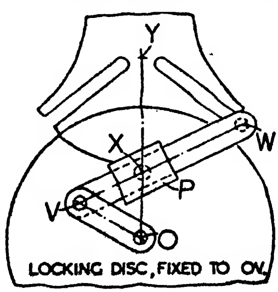
The slider E carries the cutting tool. If the speed of rotation of A is 30 r.p.m., find the acceleration of the slider E for the position shown. Note.—When $\theta = 0$ the crank D is on a dead-centre.

5. The mechanism shown in the figure is used in a knitting machine. ABC is a Whitworth quick-return motion. A second sliding block is pivoted on the pin



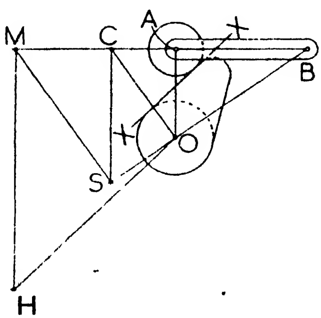
at C and engages the slotted link X pivoted at D. The *slotted link* BC rotates at a uniform speed. Find the necessary condition to make the times of the forward and return strokes of X equal.

6. The figure shows a modified form of Geneva-stop motion designed to reduce the acceleration of the slotted wheel at the commencement of its motion. OV is



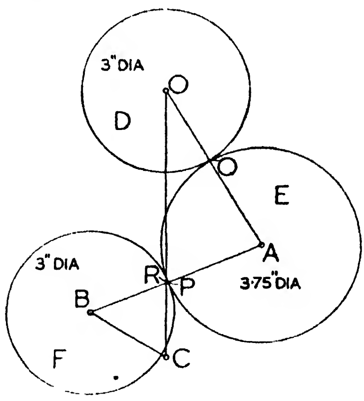
a crank that rotates at a constant speed; it is pivoted at V to the link VW, which carries the actuating pin at W. The link VW slides through the block P, which is pivoted on a fixed pivot at X. What is the necessary condition for the pin W to engage the slotted member without shock? Plot the complete path of the pin W.

7. A construction, due to Prof. G. L. Guillet, for determining the velocity and tangential acceleration of the centre of the roller of a pivoted follower actuated by a tangential cam is shown in the figure. OC is drawn perpendicular to OB,



CS perpendicular to BA, SM perpendicular to OB and HM perpendicular to BA, OH being parallel to XX. Prove that $_av_b = \omega_c \cdot AC$ in./sec. and that the tangential acceleration of A is given by $_aa^t_{b} = \omega_c^2 \cdot HM$ in./sec.^{2}, where ω_c = angular velocity of cam in radians/sec., and AC and HM are measured in inches on a full-size diagram.

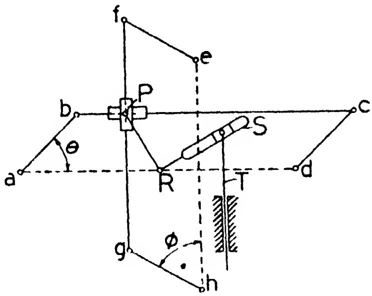
8. The figure shows a mechanism that has been used as an alternative to a Geneva motion for giving an intermittent motion to the wheel D. D, E and F



are toothed wheels and OABC is a four-bar chain, OC being the fixed link. The wheel F is *fixed to the link* CB, which rotates at 30 r.p.m. Draw the velocity diagram and find the angular velocity of the wheel D at the instant that the angle BCO equals 45°.

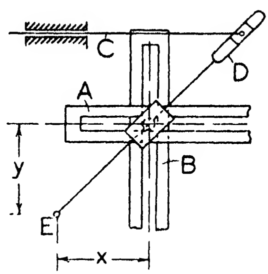
9. Referring to the mechanism of Question 8, if P and Q are the pitch points of E and F and E and D respectively, and if X is the intersection of CP, produced, with OA, prove that when the gear D has zero speed, X coincides with Q. Also if R is the intersection of CO and AB, and I is the instantaneous centre of AB relative to OC, prove that when D has zero speed, IR is perpendicular to CPQ.

10. The figure shows a mechanism that is used in a gunnery instrument. The equal four-bar chains *abcd*, and *efgh*, are coupled by the double-slider P, which can slide along *bc* and *fg*. The bell-crank PRS is pivoted on a fixed pivot at R, and



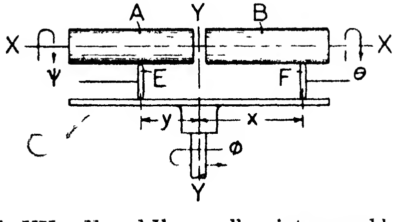
the arm RP engages a slider that is pivoted to the double-slider at P. The slotted arm RS carries a block that actuates the rod T. If the movement of the rod T is measured from the position when RS coincides with *ad*, prove that it is proportional to $\frac{\text{Sin } \theta}{\text{Sin } \phi}$.

11. The figure shows another mechanism used in the same instrument (see



Question 10). The sliders A and B move perpendicular to their respective slots and actuate the rod C through the slotted link D, which is pivoted at E. Prove that the movement of C is proportional to $\frac{x}{y}$.

12. In the mechanism shown in the figure A and B are cylinders of equal diameters free to rotate about their common axis XX, while C is a disc free to



rotate about its axis YY. E and F are rollers interposed between the cylinders and the disc. Prove that the rotation of A is equal to $\theta \cdot \frac{y}{x}$.

ANSWERS TO EXERCISES

Exercises I

1. Two. **2.** Two. **3.** Two. **4.** One. A helix. **5.** Three. **6.** One. **7.** One **8.** One. **9.** 2·225 yards at angle 37° 22′ W. of N. **10.** 1·118 miles at angle 26° 34′ E. of N. **11.** 0·078 ft. due E. and 0·293 ft. vertically upwards. **12.** 20 ft. per sec. **13.** 1·2 f.s.; 0; 1·26 f.s. **14.** 46·3 m.p.h.; 27° 14′ E. of N. **15.** 14·14 f.s. from W. to E. **16.** 81·82 m.p.h.; 1·36 miles. **17.** 58·91 m.p.h.; 0·214 miles. **18.** 5·7 ft./sec.^{2}; 56·2 ft., 14·0 ft./sec. **19.** 1·56 ft./sec.^{2} in direction 3° 18′ W. of N.

Exercises II

1. 104·72 ft./sec. **2.** 11·11 rads./sec. **3.** (*a*) 0; (*b*) 52·36 ft./sec.; (*c*) 26·18 ft./sec. **4.** 4·19 rads./sec.^{2}. **5.** 245·5 r.p.m.; 16·48 revs. **6.** 1·28 rads./sec.^{2}; 10·04 rads./sec.; 64 radians. **7.** 10,967 ft./sec.^{2}. **8.** 12·5 ft./sec.^{2}; 0·5 ft./sec.^{2} **9.** 28·68 ft./sec.^{2}.

Exercises III

1. $\dot{y} = 0{\cdot}144$; $\ddot{y} = 0{\cdot}192$; $\dot{x} = 0{\cdot}75$; $\ddot{x} = 0$. **3.** $\dot{x} = -2{\cdot}425$; $\dot{y} = 9{\cdot}701$; $\ddot{x} = -2{\cdot}768$; $\ddot{y} = 0{\cdot}693$. **4.** $\dot{x} = -1{\cdot}575$; $\dot{y} = 11{\cdot}701$; $\dot{x} = -24{\cdot}171$; $\ddot{y} = 1{\cdot}543$. **5.** $\dot{x} = 2{\cdot}5$; $\dot{y} = -2{\cdot}829$; $\dot{z} = -7{\cdot}5$. **6.** $\dot{x} = -0{\cdot}6025$; $\dot{y} = -3{\cdot}464$; $\dot{z} = 2{\cdot}764$; $\ddot{x} = 5{\cdot}828$; $\ddot{y} = -1{\cdot}7321$; $\ddot{z} = 0{\cdot}177$. **7.** Radial velocity $\dot{x} = r\dot{\alpha}\,\mathrm{Cos}\,\alpha$; axial velocity $\dot{y} = -r\dot{\alpha}\,\mathrm{Sin}\,\alpha$; transverse velocity $\dot{z} = r\omega\,\mathrm{Sin}\,\alpha$; radial acceleration $\ddot{x} = r\ddot{\alpha}\,\mathrm{Cos}\,\alpha - r\,\mathrm{Sin}\,\alpha\,(\dot{\alpha}^2 + \omega^2)$; axial acceleration $\ddot{y} = -r\dot{\alpha}^2\,\mathrm{Cos}\,\alpha - r\ddot{\alpha}\,\mathrm{Sin}\,\alpha$; transverse acceleration $\ddot{z} = r\dot{\omega}\,\mathrm{Sin}\,\alpha + 2r\omega\dot{\alpha}\,\mathrm{Cos}\,\alpha$. **8.** Radial acceleration = 55·933 towards axis; axial acceleration = 0·611 upwards; transverse acceleration = 18·32. **9.** Radial acceleration = −0·57; transverse acceleration = −0·297; axial acceleration = 1·963. **10.** Along OD_1 = 7850; along OE = 14,800.

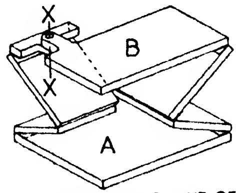
Exercises IV

1. (*a*) 4, 2 translations and 2 rotations; (*b*) 3 translations; (*c*) 3, 1 rotation and 2 compound motions, consisting of translations and rotations, in perpendicular planes; (*d*) 2, 1 combined translation and rotation and 1 rotation. **2.** The I.C. lies at the intersection of the perpendiculars to OX and OY drawn through A and B respectively. **4.** 1·72 f.s. **5.** The I.C. lies at the intersection of a perpendicular to OY through P and a perpendicular to the rod through Q. Centrode is a parabola, axis OX vertex at Q. 8·66 rads./sec. **6.** 8·89 in./sec. **7.** Zero. **8.** 94·25 in./sec. perpendicular to OA. **9.** 3·70 f.s., 45°. **10.** 48·37 r.p.m. anticlockwise looking from B to E; 59·77 r.p.m. about BF, clockwise looking from B to F. **11.** 145·466 r.p.m.; axis at 9° 54′ to OA, 0·969 in. from OA; 323·9 in./sec. **12.** 3·60 rads./sec.; 3·07 rads./sec.^{2}.

Exercises V

1. (*a*) 5, 3 translations and 2 rotations, one about a normal to the plane and the other about the line of the body; (*b*) 2, 1 translation along AB and 1 rotation about a line through P parallel to CD; (*c*) 4, 1 rotation about PQ and 3 compound motions; (*d*) 2, 1 rotation about PQ and 1 compound motion; (*e*) 1 rotation about PQ. **2.** 2, P. **3.** 3, 1. **4.** 90°. **6.** No. 4, translation in vertical plane. **7.** Ball and socket for one spring, ball-and-socket slide for the other.

Shackles at both ends of both springs. **8.** One. Freedom of rotation about an axis parallel to the line of translation of B, see figure.



XX PARALLEL TO LINE OF TRANSLATION OF B

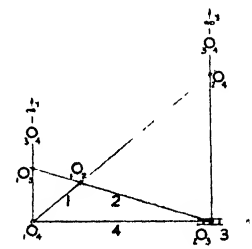
Exercises VI

6. $-6{\cdot}98$ f.s.^{2}; 0; $+6{\cdot}98$ f.s.^{2}.

Exercises VII

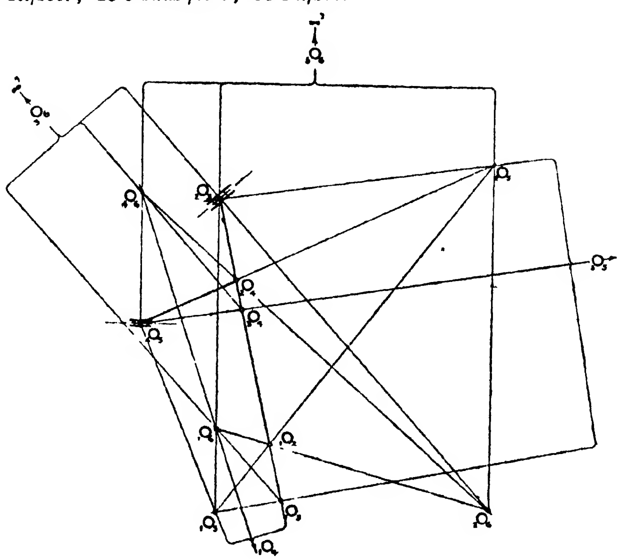
1. 753 in./min. by I.C.; 747 by calculation. **2.** 751 in./min. by velocity diagram.

3.



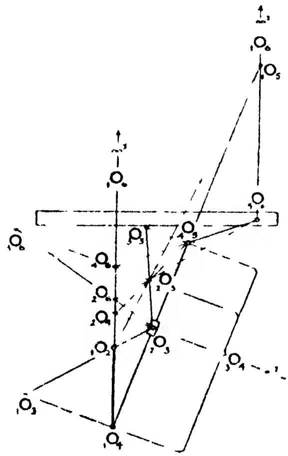
4. 25·5 ft./sec.; 13·9 rads./sec.; 91 in./sec.

5.



6. 152 ft./min.; 1·18 rads./sec.

7.



8. 509 in./min.; 36·1 rads./min; 136 in./min. **9.** 25·7 in./sec. **10.** 1·45 rads./sec.; 9·9 in./sec. **12.** 36·7 in./sec. **13.** 4·29 ft./sec. **14.** 3·1 f.s. **15.** Vel. E = 1·39 f.s.; Vel. of sliding of C = 0·694 f.s.; of D = 0·575 f.s. **16.** 9·74 f.s. **17.** 120 ft./min

Exercises VIII

1. 1050 ft./sec.^{2} by velocity diagram; 1095 ft./sec.^{2} by Klein's construction. **2.** 88·4 in./sec.^{2}; 8·6 rads./sec.^{2}, clockwise. **3.** 315 in./sec.^{2}. **5.** 4·85 in./sec.^{2}; 8·8 in./sec.^{2}; 0·21 rads./sec.^{2}, anticlockwise. **6.** 18·3 ft./sec.^{2}. **7.** 1·97 ft./sec.^{2}. **8.** 230 in./sec.^{2}.

Exercises IX

5. $\ddot{X} = a\ddot{x} - \omega^2 rb \operatorname{Cos}\theta$; $\ddot{Y} = -\omega^2 br \operatorname{Sin}\theta$. **6.** 8 in.; $\dfrac{\text{Max. return speed}}{\text{Max. cutting speed}} = \dfrac{3}{1}$.

9. $\dfrac{\text{Max. primary}}{\text{Max. secondary}} = \dfrac{3}{1}$.

Exercises XI

1. (*a*) 6 in. and 12 in.; (*b*) 18 in. and 36 in. (annulus). **4.** The arbitrary tooth shape when revolved through 180° about the point of intersection with the auxiliary pitch line must coincide with its original outline. **7.** 1·026 in.

Exercises XII

1. The base circle pitches of the teeth of the two gears must be equal. **2.** 4 in.; 4·4 in. **3.** 20 and 80; 4 in. and 16 in.; 4·4 in. and 16·4 in. **4.** Gear ratio exact, teeth are 34 and 85; C.D. = 7·4375 in.; C.D. exact, teeth are 34 and 86; gear ratio 2·529 : 1. **7.** New pressure angle = 18° 40′. **8.** (*a*) 0·5665 in.; (*b*) 0·3042; (*c*) Max. = 2, Min. 1. **9.** 27. **10.** 12. **11.** 4·533 in./sec. **12.** 10·783 in./sec. **13.** 0·0793 in.; 2·9919 in.; 10·1747 in. **14.** C.D. = 4·257 in.; 1·9028 in.; and 7·0112 in.; 6. **15.** 33° 33′.

Exercises XIV

1. 71° 33′ and 18° 27′. **2.** 54° 17′ and 15° 43′. **4.** 4 in. and 12 in., 21 and 189. **5.** 17° 1′ and 55° 35′.

Exercises XV

1. 2·550 in. and 6·450 in. ; 38° 19·4′ and 51° 40·6′. **2.** 1076 ft./min. **3.** 16 and 32 ; 1·789 in. and 7·184 in. ; 26° 33′ and 63° 27′ ; 4·486 in. **5.** 11·550 in. **7.** 0·7 r.p.m. ; 0·3 r.p.m.

Exercises XVI

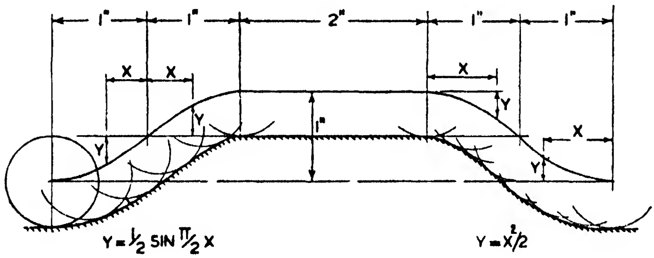
1. 50 r.p.m., clockwise. **2.** $\frac{1}{3}$ r.p.m., anticlockwise. **3.** Centre distance exact. Teeth are 32 × 48 and 27 × 53. Ratio = 2·844 : 1. Ratio exact. Teeth are 30 × 45 and 25 × 50. Centre distance = 3·75 in. **4.** +5000 r.p.m. **5.** +400 r.p.m. **6.** $\frac{\text{Arm}}{S_2} = +\frac{525}{203}$. **7.** +1250 r.p.m. **8.** +70 r.p.m. **9.** Ratio is $-\frac{11}{1}$. **10.** −615·4 r.p.m. **11.** $\frac{A+C}{n}$ = integer. **13.** 1·145 : 1. **14.** $-\frac{4}{5}$.

Exercises XVII

2. Steps are 12, 10·16, 8·06 and 6 in. **3.** Steps are 12, 10·04, 7·96 and 6 in.

Exercises XIX

1.



5. Flank radius = 6·2 in. ; (*a*) $\theta = 0$ to $\theta = 39°\ 19'$; (*b*) 39° 19′ to 80° 41′. **6.** $+1{\cdot}937\ \dot{\theta}^2$, by acceleration diagram $= +1{\cdot}97\ \dot{\theta}^2$. **7.** Dimension $d = 0{\cdot}895$ in. Flank rad. = 5·86 in. (*a*) $\theta = -10°\ 29'$ to $\theta = 30°\ 52'$; (*b*) $\theta = 30°\ 52'$ to $\theta = 72°\ 4'$. **8.** (*a*) $-1{\cdot}91\ \dot{\theta}^2$ (acceleration diagram $-1{\cdot}93\ \dot{\theta}^2$) ; (*b*) $-3{\cdot}58\ \dot{\theta}^2$ (acceleration diagram $-3{\cdot}6\ \dot{\theta}^2$). **9.** $d = 0{\cdot}8$. Nose rad. = 0·2. (*a*) $\theta = 0$ to $\theta = 32°\ 12'$; (*b*) $\theta = 32°\ 12'$ to $\theta = 87°\ 48'$. **10.** $-1{\cdot}65\ \dot{\theta}^2$. **11.** $d = 0{\cdot}7902$ in. Nose rad. = 0·2049 in. $\theta = -10°\ 29'$ to $\theta = 23°\ 0'$; $\theta = 23°\ 0'$ to $\theta = 79°\ 14'$. (*a*) $= 1{\cdot}15\ \dot{\theta}^2$; (*b*) $-1{\cdot}37\ \dot{\theta}^2$. **12.** 6·2 in. ; $-0{\cdot}546\ \dot{\theta}^2$.

Exercises XX

1. 1033·1 r.p.m. ; −277·6 rads./sec.². **3.** −335·8 rads./sec.². **4.** 41° 51′.

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