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ELEMENTARY ALGEBRA

PART II

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PREFACE

The general lines of procedure adopted in the first part of this book have been followed in this volume. General discussions in the text have been avoided wherever possible, methods have been indicated by illustrative examples, the exercises have been arranged to lead the pupil to construct his own formulae, and summaries of results, established in the exercises, have been added to consolidate the progress made and assist revision ; but all explanations which seem unsuitable for the pupil have been placed in an Introduction, which is included only in the Teacher's Edition.

In selecting their material, the authors have chiefly kept in mind the needs of the non-specialist, and have derived much assistance from the recommendations made in the *Reports of the Mathematical Association.* Apart from the concluding chapter, there is no section of the book which calls for any high degree of manipulative skill.

The order in which the chapters can be read may differ materially from that in which they are printed, without causing any inconvenience. Chapters XVI.-XVIII. on Calculus Ideas, Chapter XIX. on Series, Chapters XX.-XXI. on the Binomial Theorem, Chapters XXII.-XXIII. on Empirical Formulae and Nomography are practically independent sections which can be taken in any order or entirely omitted without prejudice to the others. Chapter XXIV. is intended solely for the specialist. The Revision Papers, 110 in number, are arranged to suit any variation of the order in which these sections may be taken.

The inclusion of a chapter on Nomography may invite criticism ; it is justified partly by its growing utility in practical mathematics, but also by the educational value which all such general methods possess. The authors have felt some hesitation in admitting a chapter on Series, a subject which often loses its value in an elementary course by being reduced to mere rule-of-thumb. It is hoped that this danger is guarded against in the treatment adopted.

Every effort has been made to introduce as much variety as possible into the examples, and to infuse them with practical interest. Repetitions of special types may eventually induce some kind of facility in handling them, but this method dulls the mind ; it is variety alone which makes the pupil "keep thinking."

Acknowledgment is due to The Controller of H.M. Stationery Office and to the Oxford and Cambridge Joint Board for kind permission to include questions set in recent examinations, and to Messrs. Macmillan & Co., who have sanctioned the use of their method of arrangement of logarithm tables. Special mention must also be made of Molesworth's *Pocket Hand-book for Engineers*, which contains in a concise form an admirable collection of formulae which has provided excellent material for examples. The authors also wish to take this opportunity of expressing their indebtedness to Mr. R. C. Fawdry, Head of the Military and Engineering Side, Clifton College ; to Mr. D. B. M'Quistan, Head of the Mathematical Department, Allan Glen's School ; to Mr. A. E. Broomfield of Winchester College ; and to Mr. H. K. Marsden of Eton College, whose criticisms have been most helpful.

C. V. D.
R. M. W.

January, 1921.

CONTENTS

INTRODUCTION

CHAPTER XII.

Indices. The laws for positive integral indices are as follows :

If m, n are positive integers,

(i) $a^m \times a^n = a^{m+n}$; (ii) $a^m \div a^n = a^{m-n}$, if $m > n$,

$= \dfrac{1}{a^{n-m}}$, if $m < n$;

(iii) $(a^m)^n = a^{mn}$.

The proofs of these laws follow immediately from the definition of the symbol a^m,

$$a^m = a \times a \times a \times a \times \ldots \text{ to } m \text{ factors.}$$

Exercise XII. a. illustrates the use of these laws, and is intended to show the reader how they are proved. Many numerical examples are usually required to clear away the haziness of idea most pupils have about indices, and in particular about (iii) ; and it is essential this should be done before proceeding to the interpretation of fractional and negative indices. It is hopeless to attempt to tackle these, until the pupil uses his reason and does not merely work by rule in his handling of positive integral indices. But when this stage is reached, there are few who, although they may never have seen or heard of fractional or negative indices, will not answer successfully such series of questions as :

1. What is the square root of (i) x^6 ; (ii) x^8 ; (iii) x^7 ?
2. If $x^5 \div x^8 = x^n$, what is n ?
3. If $x^7 \div x^7 = x^n$, what is n ? and so on.

The examples in Exercise XII. b., which to a large extent should be taken orally, are intended to lead the pupil to make his own interpretations. The conclusions are summarised at the end of the exercise, but some teachers will no doubt prefer to ask for formal interpretations while the working of the exercise is in progress.

It is worth noting that most boys are slow to realise that $\sqrt[n]{a^m}$ and $(\sqrt[n]{a})^m$ are identical, or to see the reason for it, or to make use of it when evaluating such expressions as $\sqrt{4^5}$.

Exercise XII. c. may well be postponed to a second reading.

Irrational numbers. The *theory* of irrational number is outside the scope of this book: for that, the reader is referred to Nunn's *Algebra*, Part II., page 13, or Hardy's *Pure Mathematics*, Chapter I. The present section is purely practical, and aims at giving facility in handling irrational expressions by examples on the statements

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}, \quad (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b.$$

The name "surd" is not used.

The graphical section which concludes the chapter is the natural link between indices and logarithms.

CHAPTER XIII.—LOGARITHMS AND SLIDE RULE.

Far more examples are given in the text than will be required by any single student, the object being to enable each teacher to make the selection he prefers. The graphical section should be taken orally.

No explanation is given in the text of the use of the slide rule, as oral instruction is almost essential. It is suggested that this should start as follows:

Draw a line 5 inches long, preferably on a strip of cardboard, and call it A_1A_{10}. Mark points $A_2, A_3, A_4, \dots A_9$ on it such that

(i) $A_1A_2 = 5(\log 2 - \log 1)$ inches $= 5 \log 2$ inches
$= 5 \times 0{\cdot}301'' = 1{\cdot}505''$.

(ii) $A_1A_3 = 5(\log 3 - \log 1)$ inches $= 5 \times 0{\cdot}477'' = 2{\cdot}385''$.

(iii) $A_1A_4 = 5(\log 4 - \log 1) = 5 \times 0{\cdot}602 = 3{\cdot}010''$,

and so on.

Thus $A_1A_7 = 5 \log 7 = 5 \times 0{\cdot}845 = 4{\cdot}225''$.

A part of this line is shown in Fig. 1, where the graduations are represented by 1, 2, 3, ... instead of $A_1, A_2, A_3, \dots$.

The line can now be subdivided by the same method.
To obtain the graduation 1·5 we have

$$5(\log 1{\cdot}5 - \log 1) = 5 \times 0{\cdot}176 = 0{\cdot}880'',$$

and so on.

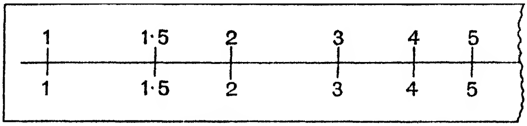


Fig. 1.

Now cut the cardboard along the straight line and you have the elements of a rough slide rule.

It is so graduated that the distance of any graduation n from the graduation 1 is proportional to $\log n$. Slide the lower half along until " 1 " on it comes below " 2 " on the upper half, then you will see that " 3 " on the lower half comes below 2×3 or 6 on the upper half. This is because

$$\log 2 + \log 3 = \log 6.$$

By this method any two numbers can be multiplied together: and by reversing the process we can find the quotient of any two numbers.

CHAPTER XIV.—RATIO AND PROPORTION.

Definition. If a, b are two numbers such that, if b is divided into an integral number q equal parts, a contains exactly p of these parts, a and b are said to be in the ratio $p : q$.

Or, If integers p, q exist such that $\frac{a}{b} = \frac{p}{q}$, then the ratio of a to b is said to be equal to the ratio of p to q.

For example, if $a = 2\frac{1}{3} = \frac{7}{3}$ and $b = 2\frac{1}{4} = \frac{9}{4}$, b can be divided into 27 equal parts, each equal to $\frac{1}{12}$, since $b = \frac{9}{4} = \frac{27}{12}$.

a can be divided into 28 equal parts, each equal to $\frac{1}{12}$, since $a = \frac{7}{3} = \frac{28}{12}$;

$$\therefore\ a : b = 28 : 27.$$

This process amounts to finding a common sub-multiple of the two numbers.

Next consider a square of side 3 inches; the length of its diagonal is $\sqrt{18}$ inches.

In this case it is impossible to find an integer q such that if the diagonal is divided into q equal parts, the side will contain an integral number of these parts.

Such related numbers as 3 and $\sqrt{18}$ are said to be *incommensurable.* The term 'ratio' can only be applied to such numbers if the definition given above is extended in scope. The necessary discussion is regarded as beyond the limits of this volume.

In addition to ratios of numbers, we may have ratios of *similar quantities*, *e.g.* the ratio of the populations of two towns or of the rents of two houses, etc. But it is meaningless to compare different kinds of magnitudes with each other, *e.g.* 5 inches with 2 hours.

The statement that the ratio of the populations of two towns X, Y is $a : b$ may be expressed in any of the following ways:

(i) The number of people in $X = \frac{a}{b} \times$ the number in Y.

(ii) $\dfrac{\text{The number of people in } X}{\text{The number of people in } Y} = \dfrac{a}{b}$.

(iii) If the people in X are arranged in groups of a and the people in Y are arranged in groups of b, there will be the same number of such groups in X as in Y.

(iv) If the population of X is az, the population of Y is bz.

It has been said that the uneducated mind compares two numbers or quantities by noting their difference, the educated mind by noting their ratio. Ratio provides the best test of comparison in general, because the absolute value of the difference shows nothing of its importance relative to the quantities concerned. A difference between the heights of two men of $\frac{1}{8}$ inch is considered trifling in our world; in Lilliput it would be noteworthy. Again, a rise of income of £1 a week is a big rise to a labourer earning £2 a week, but is insignificant to the millionaire.

Ratios appear in Mathematics in a variety of forms. Percentage is a special form of ratio; specific gravity, sines and cosines, coefficients of friction, of expansion, of elasticity, etc., are all examples of the use of ratio and furnish material for oral instruction.

Variation. The idea of variation is best explained by taking orally a number of simple examples, such as

(i) In uniform motion, distance $\propto$ time.

(ii) The number of Kms. between any two places $\propto$ the number of miles between those places.

(iii) The length of a shadow at any given moment $\propto$ the height of the object.

(iv) For substances of equal weight, the specific gravity varies inversely as the volume.

(v) For spheres of the same material, the weight varies as the cube of the diameter.

Oral questions should be taken of such types as the following (expressed as concrete examples):

(i) If $A \propto B$, what is the effect on A of doubling B or on B of doubling A?

(ii) If $A \propto \frac{1}{B}$, what happens to A if B is halved?

(iii) If $A \propto B^2$, in what ratio is A increased if B is increased in the ratio 3 : 2?

A boy cannot answer such questions unless he understands what he is talking about.

Graphical work should be taken side by side with each form of variation, and attention should be directed to the graphical tests for variation. Thus, to test whether A varies inversely as B, the values of A should be plotted against the corresponding values of $\frac{1}{B}$.

It helps also to clear up difficulties if pupils are made to examine or construct tables which follow a variation law, and some examples of functionality which are not variation laws should be taken to establish the distinction.

The obvious danger is unintelligent working. A boy easily acquires the habit of saying that if $y \propto x^3$, then $y = kx^3$, without thinking what it means. It is a much harder matter to make him realise that this variation-relation really implies that if x is altered in any ratio, then y is altered in the cube of that ratio. But his power to do this is the real measure of his understanding. Mere juggling with k is of little value.

It is left to the teacher to discuss and expand the statements given in the summary of results: this will naturally be done during the course of the exercises which precede it.

The discussion in the text on joint variation is rather fuller than usual, in order to indicate one way of introducing this idea. It is useful to point out to the student the difference in value of λ, according as $V = \lambda . h$ or $V = \lambda . r^2$. The construction of a double-entry table of values drives home the meaning of the double variation.

CHAPTER XV.—FUNCTIONS OF ONE VARIABLE.

THE idea of functionality is both difficult and important. It is fairly easy to learn the mechanical process of computing special values of a function and plotting them on squared paper. It is much harder to grasp the meaning of the "equation of a curve" and to form a mental picture of the graph of a simple function from observation of the manner in which the function changes. In this chapter we are mainly concerned with the general characteristics of simple functions; *e.g.* $(x-3)(x-2)(x-1)$ is large when x is large, decreases to zero as $x \to 3$, is negative for $3 > x > 2$, increases to zero as $x \to 2$, is positive for $2 > x > 1$, decreases to zero as $x \to 1$, is negative for $1 > x$ and tends to $-\infty$ as $x \to -\infty$. This is the *life history* of the function, and from it the reader should be able to picture its graph. Conversely, given the graph of a function, its life history can be outlined. The plotting of special values is irrelevant and should be discouraged in this connection, except for verification, and squared paper should not be used. Whether it is advisable to include the harder examples in XV. a. at a first reading depends on the capacity of the class, but many of the difficulties can be lightened by oral treatment.

Exercise XV. b. gives some examples on the construction of functions corresponding to various geometrical properties.

The use of accurate plotting of functions is illustrated in XV. c., applications being made to maxima and minima problems and the solution of equations—a type of question which appears to be a favourite with many examining bodies. Exercise XV. d. provides practice in functional notation, which is essential for the Calculus; but the second half of the exercise should only be attempted by students of special ability.

CHAPTER XVI.—LIMITS AND GRADIENTS.

No attempt has been made in the text to define and discuss exhaustively the meaning of a limit; the analytical idea is too subtle and delicate for most students at this stage ; when desired, reference should be made to the excellent expositions in Nunn's *Algebra*, Part II., Section VIII., and Hardy's *Pure Mathematics*.

The subject-matter of this chapter is arranged on the assumption that the work will be accompanied by oral instruction and general discussion, the extent of which must depend on the ability of the class. Example I. and Exercise XVI. a. are designed to bring out two fundamental ideas :

(i) That the statement $\underset{x \to a}{\mathrm{Lt}} f(x) = L$ implies that given any positive quantity ϵ, a positive quantity δ exists such that, for *all* values of x included in

$$0 < |x - a| < \delta, \quad |f(x) - L| < \epsilon.$$

(ii) That the statement $\underset{x \to a}{\mathrm{Lt}} f(x) = L$ does not necessarily require that a value of x exists for which $f(x) = L$.

The notation $\underset{x = a}{\mathrm{Lt}} f(x)$ should be avoided ; it is misleading, and may easily produce inaccurate ideas on the subject.

The explanation of the phrases $n \to \infty$ or $\underset{x \to a}{\mathrm{Lt}} f(x) \to \infty$ is left for oral treatment. As regards the first, the ordinary boy does not find much difficulty in forming a reasonably correct idea of what is represented : for example, suppose n takes in succession the values of the prime numbers 1, 2, 3, 5, 7, 11, 13, 17, ... arranged in a steadily increasing sequence. However large a number your opponent cares to select, you can always find a stage in the sequence beyond which *all* the numbers of the sequence exceed the number chosen by your opponent. In such a case you say n tends to infinity. The second expression may be treated similarly. The important idea for the pupil to grasp at this stage is that *infinity is not a number* and that expressions such as $\frac{1}{0}$ or $\frac{0}{0}$ are meaningless.

Kinematics probably furnishes the best illustration of the use of limits, and the exact meaning of the sentence " a car was travelling at 20 miles an hour and its speed was then gradually reduced to 15 miles an hour " should be discussed in detail and illustrated graphically. This leads on to the idea of the gradient of the graph, regarding the slope of the curve measured by the tangent at a special point as the limit of the average slope represented by a chord.

Although it is easier to grasp the meaning of rate of change referred to the time, it is important that some examples should be taken to illustrate the use of the word 'rate' referred to any independent variable. The rate at which a piece of ground on the side of a hill rises provides a simple and familiar example; but it should be pointed out that the measure of the slope or gradient is at variance with the ordinary usage of surveyors, who measure the gradient of a road by the sine of the angle of slope and not by the tangent of that angle.

The work on calculation of gradients is divided into two stages: the first deals with examples in which rates of change are estimated from observed data or statistics, and the second mainly with examples on formulae which permit of exact calculation.

CHAPTER XVII.—DIFFERENTIATION.

THE notation of the Calculus is best explained orally, and might follow some such line as the following:

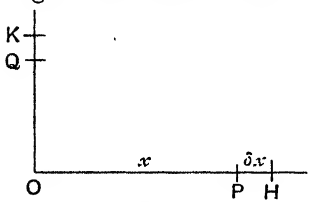


FIG. 2.

FIG. 3.

If x is the distance of a point P from O, and H is a point close to P on the line OP, the length of PH is represented by the symbol δx, which means "a small increment of the variable x," or colloquially "a little bit of x," and is positive if $OH > OP$ and negative if $OH < OP$.

Similarly, if $OQ = y$ and K is near Q on the line OQ, QK may be denoted by δy. Unless some condition is imposed, δx and δy may be any small quantities at all. But if x and y are connected by an equation, although δx can still be any small quantity whatever, δy can be calculated in terms of δx.

For example, suppose $xy = 36$.

Then δy is the change of y caused by a change δx in x, and is computed from the equation

$$(x + \delta x)(y + \delta y) = 36.$$

Again, suppose a train starts at O (Fig. 3), and arrives at N, where $ON = s$ feet, after t secs. and passes M after $t + \delta t$ secs., then the extra distance NM travelled in δt secs. is denoted by δs feet. Here δt might be anything at all. But if we know δt and know the equation connecting s and t, *i.e.* how the train travels, we can calculate δs in terms of δt.

It must be emphasised that in the symbol δs, the 'δ' and the 's' cannot be separated, that it is simply a shorthand notation for the phrase "a small increase in s," and does not mean δ multiplied by s.

Exercises XVII. a., b. are dull work ; but the reader must acquire facility in differentiating x^n (at any rate in simple cases) and handling easy polynomial expressions before proceeding to applications. A formal proof of the general formula $\frac{d}{dx}x^n = nx^{n-1}$ is excluded as unsuitable at this stage, but sufficient special cases are given as exercises to make its assumption plausible.

Exercise XVII. c. deals with the interpretation of the sign of $\frac{dy}{dx}$: this is not really difficult, but when the work is extended to cover the sign of $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, there is liable to be at first some confusion of idea. This line of thought is, however, so instructive that it is worth while devoting time to it.

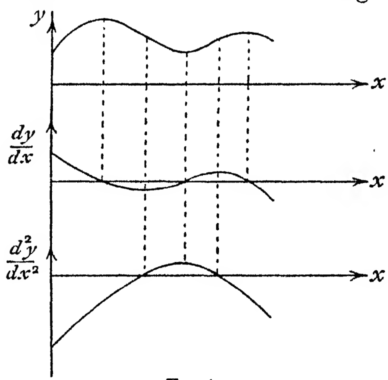


FIG. 4

The reader has already had a good deal of practice in the graphic representation of functions. This exercise carries the method a stage further forward, as it involves examining a figure to see whether the rate of increase is itself an increasing or decreasing function. The examples are intended to make the progress slow, and at first should be taken orally. It is useful and illuminating to exhibit, one below the other, the graphs of y and $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for some simple functions, as, for example, in Figure 4, basing the work on an appeal to the eye.

Exercise XVII. e. is introduced partly because methods of approximation have an importance of their own, but mainly as leading up to the integration work in the next chapter; the last six questions in it introduce the method for the differentiation of $F[\phi(x)]$. Anyone can be taught the process as a mere rule of thumb; but the understanding of the method is difficult and should at this stage be reserved for the clever boy who gets ahead of the class.

The work on calculation of rates of change in Exercise XVII. f. always presents difficulty, and may well be deferred to a second reading.

CHAPTER XVIII.—INTEGRATION.

Integration is treated as the reverse process of differentiation the idea of summation is more difficult and may well be deferred. The investigation of the analytical basis upon which Dufton's and Simpson's approximate rules depend is omitted as beyond the scope of this volume. For an account of the former, the reader is referred to *Nature*, Vol. CV., pp. 354, 455.

The temptation to introduce more physical applications into the examples was great, but was resisted on the ground that it would disturb the proportions of the book, which aims merely at a general introduction. Those who intend to go further will naturally turn to a special text-book. Problems on centres of gravity, pressure, moments of inertia, etc., are therefore omitted, except in cases where the formulae are supplied in the question itself.

CHAPTER XIX.—SIMPLE SERIES.

The inclusion of a chapter on this subject is open to criticism. Certainly there is little educational value in memorising the formulae for the sums of arithmetical and geometrical progressions, and in getting up the stereotyped questions which some examination papers still contain. But, examinations apart, there is a definite value in the appreciation of any kind of algebraic form and in the cultivation of the power to generalise ideas and to utilise formulae. If the reader can be led to construct formulae for himself and to understand the idea of order, he will gain definite advantage from the work of this chapter. It is therefore suggested that formulae should not be allowed to be used in Exercises XIX. a., b., c. In order that the pupil may not attach too much importance to the ordinary progressions, as much variety as possible has been introduced into Exercise XIX. a., which is intended to give training in the general idea of a law.

CHAPTER XX.—PERMUTATIONS AND COMBINATIONS.

A SHORT section on this subject is introduced as a necessary preliminary to a treatment of the Binomial Theorem. The illustrative examples have been made as brief as possible, and the main idea in each exercise has been to lead up through a large number of special cases to the general formula; the general formula should not be given until the pupil begins to see that there must be one, and that, if he knows it, it will simplify his work. A brief mention has been made of probability because of the intrinsic interest of the subject even, or perhaps especially, to the boy whose main interests are non-mathematical. The real importance of the subject lies in its application to Insurance and the Theory of Statistics; an excellent introductory account is given in Nunn's *Algebra*, Part II., where the idea of relative frequencies is clearly developed. A discussion of the Probability Curve is reluctantly omitted from this chapter as being disproportionate to the scheme of the book.

CHAPTER XXI.—BINOMIAL THEOREM.

IT was considered that an elementary book on Algebra would not be complete unless the Binomial Theorem, at any rate for a positive integral index, was included. Probably few teachers will take this chapter before doing the chapters on Functions and Differentiation, but there is nothing in the scheme of the book to prevent their doing so if they wish. The educational value of the chapter lies primarily in its giving the pupil a sense of algebraical form. If he sees that he can write down the expansion of $(x+a)(x+b)(x+c)(x+d)$ or of $(x+a)^4$ without any laborious multiplication, he obtains a sense of mastery of his subject and a better appreciation of the value of algebraic method. On the practical side the chief value of the Binomial is its application to approximate calculations, and a short section on this subject has therefore been included. The proof of the Binomial Theorem for any index is obviously unsuitable at this stage. It seems the best plan to follow the historical method, obtaining, as Newton did, the expansion for $(1+x)^{-1}$ by long division and for $(1+x)^{\frac{1}{2}}$ by the square-root rule. And it is of interest to point out to the reader Newton's brilliant induction of the general law from the special cases he considered. In the same way here, the general fact that, if x is small, $1+nx$ is an approximation for $(1+x)^n$ is an inference the student should be encouraged to draw from the examples he works out in Exercise XXI. c.

CHAPTER XXII.—EMPIRICAL FORMULAE.

GRAPHICAL work so far has consisted of :

(i) The representation of statistics or experimental results for which no functional relation is known.

(ii) The representation of functions of one variable.

We now proceed to consider a complementary problem: given a series of observations, is it possible to find an analytical formula which governs them to a fair degree of approximation ?

When this can be done, the result is called an *empirical* formula. Before investigating the problem, two cautionary remarks may be made : (i) however closely the formula may fit the data within the observed range, it is impossible to rely upon deductions from the formula *outside* this range. In other words, inferences based on *interpolation* are trustworthy, but those based on *extrapolation* are of little value. (ii) Unless the *form* of the expression has been determined from theoretical considerations, there is no reason to suppose that the functional character of the formula corresponds to any fundamental physical property.

We shall deal in the present section with results that either conform to or are reducible by various methods to a linear law. For practical work, a student should have either a fine black thread or a piece of transparent celluloid on which is cut a fine line running along its length down the middle from end to end.

The first example in the text (page 416) shows the results of an experiment plotted in the ordinary way. The points do not lie precisely on a straight line; their divergence may be due to errors of observation, imperfect conditions or faulty apparatus; but the divergence is sufficiently small to suggest that a straight line can be drawn which will represent the data within a reasonable degree of accuracy. The black thread, or better the celluloid, should now be used to determine that straight line which *best fits* the observations. Mark two points on this "best fit line" at *opposite* ends of it and read off their coordinates. Substitute them in the formula $\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}$, and reduce this to the form $y=ax+b$, where a, b are expressed correct to as many significant figures as the methods and data employed justify. The formula so obtained is only approximate, and therefore a, b should not be given in fractional form, and in no case to more figures than the data warrant.

In general, when experimental results are plotted, the graph is a curve instead of a straight line. In such cases it may be possible by some method to transform the curve into a straight line. The following list gives a few of the means which may be tried.

If y is a function of x, plot

(i) $\log y$ against x or $\log x$ against y;
(ii) $\log y$ against $\log x$;
(iii) xy against x or xy against y;
(iv) y against $\frac{1}{x}$ or x against $\frac{1}{y}$;
(v) $\frac{x}{y}$ against x.

Unless theoretical considerations furnish some kind of a clue, nothing but experience can teach the student how best to proceed.

When trying (i) and (ii), much time is saved by using specially ruled logarithmic or semi-logarithmic paper. [This can be obtained from scientific instrument makers, *e.g.* C. Baker, 244 High Holborn.]

For further information, the reader is advised to consult an excellent monograph by Professor Running. [*Empirical Formulae*: T. R. Running. Chapman & Hall.]

CHAPTER XXIII.—NOMOGRAPHY.

The object of this section is to give the reader some idea of the use now being made of nomograms by engineers. There is great variety of method, special problems requiring distinctive treatment, and only a brief outline is given here. Those who are interested in the subject, which is still in its infancy, will naturally turn to books specially devoted to it. The classics on the subject are d'Ocagne's *Calcul Graphique et Nomographie* and *Traité de Nomographie*, and it is d'Ocagne who is the real inventor of the idea.

The following English works provide an easier introduction:

- S. Brodetsky, *First Course in Nomography.* (Bell. 10s.)
- E. S. Andrews, *Alignment Charts.* (Chapman & Hall. 1s. 3d.)
- Capt. R. K. Hezlet, R.A., *Nomography.* (R.A. Institution. 3s.)
- E. W. Tipple, *Line Charts.* (By application to the author, 5 Cross Flatts Row, Beeston, Leeds. 2s.)

The reader may be sceptical of the practical value of a method which at first sight appears exceedingly laborious and leads, like all graphical methods, only to approximate results. But once the nomogram has been constructed in a durable form by a skilled draughtsman, practically no skill is required in reading from it any number of numerical results that may be required for engineering or commercial purposes. And that astonishing accuracy can be attained by skilled draughtsmen using graphical methods is shown by the results obtained by sound-ranging sections in France in locating the position of hostile guns from the intersections of hyperbolas.

The theoretical side of the subject, which is given here, is of educational value. The idea of non-uniform graduation gives an opportunity for discussing or revising the principle of the slide rule. In practice it will be found that the examples occupy a great deal of time; logarithmic graduation is a slow process until experience gives facility. How much of this work should be done by rule is open to question, but in any event there is abundant intellectual exercise to be obtained in such matters as the choice of units, the best positions for the axes and suitable methods for checking the work.

The illustrative examples in this section are worked out more fully than has been customary in earlier chapters owing to the comparative novelty of the subject; and a great deal of assistance is given in some of the exercises. This makes them appear longer than they really are, due to the fact that the statement of the problem is often almost a skeleton solution. Nothing has been said in the text of the use made in Nomography of the graduation of curves; but the outline of d'Ocagne's solution of the quadratic, which is given in the last example of XXI. c., indicates the use that may be made of them. The skeleton method has in fact made it possible to introduce into the examples a greater variety of method than could be included in the text.

The main point to be grasped is that the simple operation of laying a straight-edge between two points on graduated lines and noting the value of the graduation at which the straight-edge cuts a third line or curve also suitably graduated leads to approximate numerical solutions of an almost inexhaustible variety of formulae and equations.

CHAPTER XXIV.—FURTHER DEVELOPMENTS FOR THE SPECIALIST.

Although one of the sections in this chapter is explicitly described as Algebraic Form, the subject-matter of the whole chapter and the general characteristics of the exercises have been chosen with the idea of developing that analytical sense which is the true hall-mark of the specialist. There is not a great deal of bookwork, and the variety of methods that must be employed diminishes the usefulness of illustrative examples and increases proportionately the value of work of this type. The importance of insisting on the use of symmetrical methods, wherever possible, can hardly be exaggerated.

PART II.

CHAPTER XII.

INDICES AND IRRATIONALS.

INDICES.

Definition. If n is a positive integer (not zero), a^n is called the nth *power* of a and represents $a \times a \times a \times \ldots$ to n factors; the number n is called the *index* of a.

Note that the index of a itself is 1, for a really stands for a^1.

Example I. Prove that $a^7 \div a^3 = a^4$.

$$\frac{a^7}{a^3} = \frac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a}$$
$$= a \times a \times a \times a = a^4.$$

Example II. Prove that $(a^3)^4 = a^{12}$.

$$(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3$$
$$= (a \times a \times a) \times (a \times a \times a) \times (a \times a \times a) \times (a \times a \times a)$$
$$= a^{12}.$$

EXERCISE XII. a.

1. Prove *in full* that

(i) $a^2 \times a^3 = a^5$; (ii) $a^3 \times a^4 = a^7$;
(iii) $a^8 \div a^4 = a^4$; (iv) $a^{12} \div a^3 = a^9$.

2. Write the following in their simplest forms:

(i) $a^3 \times a^5$; (ii) $a^6 \div a^3$; (iii) $a \times a^5$; (iv) $a^{10} \div a^2$;
(v) $a^7 \times a^9$; (vi) $a^3 \times a^4$; (vii) $a^{12} \div a$; (viii) $x^{20} \div x^5$.

3. In what general statement or formula are the following special cases included?

$$a^2 \times a^8 = a^{10}; \quad a^{20} \times a^{41} = a^{61}; \quad a \times a^{80} = a^{81}.$$

4. In what general statement or formula are the following special cases included ?

$$a^7 \div a^4 = a^3; \quad a^{100} \div a^{10} = a^{90}; \quad a^{13} \div a = a^{12}.$$

5. Prove *in full* that

(i) $(a^2)^5 = a^{10}$; (ii) $(2x^2)^3 = 8x^6$.

6. Simplify

(i) $(a^4)^5$; (ii) $(a^{10})^{10}$; (iii) $(a^8)^6$; (iv) $(3a^9)^3$.

7. In what general statement or formula are the following special cases included ?

$$(a^5)^2 = a^{10}; \quad (a^{20})^9 = a^{180}; \quad (a^{100})^7 = a^{700}.$$

8. By what must x^7 be multiplied to give x^{21} ?

9. To what power must x^7 be raised to give x^{21} ?

10. By what must x^{50} be divided to give x^{10} ?

11. What is the square of x^4 ?

12. Is $(x^3)^4$ the same as $(x^4)^3$?

13. Is $(x^6)^{10}$ the same as $(x^3)^{20}$?

14. Divide $x^{15} \times x^{10}$ by x^5.

15. Divide $(x^2)^5$ by x^2.

16. Divide $(a^6)^4$ by $(a^4)^6$.

17. Multiply $(a^3)^4$ by $(a^3)^5$.

18. Divide $(a^7)^5$ by a.

19. What is the cube of $4x^4$?

20. Divide the cube of $2x^2$ by the square of $2x^3$.

21. How would the following facts be expressed without using the index notation ?

(i) the velocity of light is $3{\cdot}002 \times 10^{10}$ cm. per sec. ;

(ii) the population of London in 1920 was about $4{\cdot}5 \times 10^6$;

(iii) the value of steel imports in 1913 was about £$12{\cdot}2 \times 10^6$;

(iv) the distance of α Centauri is $2{\cdot}6 \times 10^{13}$ miles ;

(v) $2 \times 10^3 \times 4{\cdot}5 \times 10^6 = 9 \times 10^9$.

22. Write in a shorter form :

(i) 62,800,000 ; (ii) $67{\cdot}1 \times 67{\cdot}1 \times 67{\cdot}1 \times 1.000,000$.

SUMMARY OF RESULTS.

The examples of the above exercise illustrate the following facts:

If m, n are positive integers,

$$\text{(i)}\quad a^m \times a^n = a^{m+n};$$
$$\text{(ii)}\quad a^m \div a^n = a^{m-n}, \quad \text{for } m > n;$$
$$\text{(iii)}\quad (a^m)^n = a^{mn}.$$

The proofs of these general results follow the methods given in the worked-out examples on page 253 for special cases.

If m is not a positive integer, *e.g.* if $m = \frac{3}{4}$ or $m = -5$ or $m = -\frac{2}{3}$, the symbol a^m has as yet no meaning. We are now going to find out what meaning must be given to it, in order that the relations (i), (ii), (iii) may always hold good.

Note.—The symbol 0^0 is excluded from consideration and has no meaning.

Definition. If n is a positive integer (not zero), a is called *an* nth *root* of a^n.

Note the expression, *an* nth root; a number has more than one nth root; for example, 25 has *two* square roots, viz. $+5$ and -5.

If there is a positive nth root of x it is written $\sqrt[n]{x}$.

Example III. Assign a meaning to x^{-3}.

$$x^5 \div x^8 = x^{5-8} = x^{-3}, \quad \text{from (ii).}$$

But $$x^5 \div x^8 = \frac{x^5}{x^8} = \frac{1}{x^3}; \qquad \therefore\ x^{-3} = \frac{1}{x^3}.$$

Example IV. Assign a meaning to $x^{\frac{3}{4}}$.

$$x^{\frac{3}{4}} \times x^{\frac{3}{4}} \times x^{\frac{3}{4}} \times x^{\frac{3}{4}} = x^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = x^{\frac{12}{4}}, \quad \text{from (i),}$$
$$= x^3;$$
$$\therefore\ x^{\frac{3}{4}} \text{ is a fourth root of } x^3.$$

This may be written, $\quad x^{\frac{3}{4}} = \sqrt[4]{x^3}.$

Or, we may say $\quad (x^{\frac{3}{4}})^4 = x^{\frac{3}{4}\times 4}$, from (iii),

$$= x^3;$$
$$\therefore\ x^{\frac{3}{4}} \text{ is a fourth root of } x^3.$$

Example V. Assign a meaning to x^0.

$$x^4 \div x^4 = x^{4-4} = x^0, \qquad \text{from (ii).}$$

But
$$x^4 \div x^4 = \frac{x^4}{x^4} = 1;$$
$$\therefore\ x^0 = 1.$$

Example VI. Prove that $16^{\frac{3}{4}} = \sqrt[4]{16^3} = (\sqrt[4]{16})^3$, and find its value.

Exactly as in Example IV., we see that

$$16^{\frac{3}{4}} \times 16^{\frac{3}{4}} \times 16^{\frac{3}{4}} \times 16^{\frac{3}{4}} = 16^{\frac{12}{4}} = 16^3;$$
$$\therefore\ 16^{\frac{3}{4}} = \sqrt[4]{16^3}.$$

Again,
$$16^{\frac{3}{4}} = 16^{\frac{1}{4}} \times 16^{\frac{1}{4}} \times 16^{\frac{1}{4}} = (16^{\frac{1}{4}})^3.$$

But
$$16^{\frac{1}{4}} = \sqrt[4]{16};$$
$$\therefore\ 16^{\frac{3}{4}} = (\sqrt[4]{16})^3.$$

It is easier to compute $(\sqrt[4]{16})^3$ than $\sqrt[4]{16^3}$, as it involves smaller numbers.

$$16^{\frac{1}{4}} = \sqrt[4]{16} = 2;$$
$$\therefore\ 16^{\frac{3}{4}} = 2^3 = 8.$$

EXERCISE XII. b.

1. Express as powers of 10:

(i) $10^3 \div 10$; (ii) $10^3 \div 10^2$; (iii) $10^3 \div 10^3$;

(iv) $10^3 \div 10^4$; (v) $10^3 \div 10^5$.

What are the values of these expressions?

2. Express as powers of 2:

(i) $2^4 \div 2$; (ii) $2^4 \div 2^2$; (iii) $2^4 \div 2^3$;

(iv) $2^4 \div 2^4$; (v) $2^4 \div 2^5$; (vi) $2^4 \div 2^6$.

What are the values of these expressions?

3. Write down 7^5; divide it by 7, and write the quotient in the index form. Repeat this process 8 times. What are the successive quotients expressed in the index form? What are their values?

4. Find n if

(i) $x^{10} \div x^3 = x^n$; (ii) $x^{10} \div x^{15} = x^n$;

(iii) $x^{10} \div x^n = x^{-7}$; (iv) $x^{10} \div x^{10} = x^n$.

5. Find n if

(i) $x^4 \times x^7 = x^n$; (ii) $x^5 \times x^{-2} = x^n$; (iii) $x^3 \div x^5 = x^n$;
(iv) $x^n \div x^4 = x^6$; (v) $x^n \times x^3 = x^3$.

6. Express as powers of x, *i.e.* in the form x^n,

(i) $\dfrac{x^6}{x^2}$; (ii) $\dfrac{x^6}{x^9}$; (iii) $\dfrac{x^6}{x^6}$.

7. Express as powers of x

(i) $\dfrac{x^3}{x^8}$; (ii) $\dfrac{1}{x^5}$; (iii) $\dfrac{x^3}{x^3}$; (iv) 1.

8. Use the method of Example III. to assign a meaning to x^{-12}.

9. Simplify

(i) $a^7 \times a^0$; (ii) $a^7 \times 1$; (iii) $a^7 \times a^1$; (iv) $a^7 \times a$;
(v) $a^7 \times 0$; (vi) $a^0 \times 1$; (vii) $a^0 \times a$.

10. Express with positive indices

(i) a^{-2}; (ii) a^{-1}; (iii) $\left(\dfrac{1}{a}\right)^{-3}$; (iv) $a^{-1} \times a^{-1}$.

11. What are the values of

(i) 3^{-2}; (ii) 2^{-3}; (iii) 4^{-1}; (iv) 5^0;
(v) $(\frac{1}{4})^{-2}$; (vi) $(\frac{2}{3})^{-4}$; (vii) $(0{\cdot}1)^{-2}$; (viii) 1^{-3};
(ix) 1^0; (x) $(\frac{2}{3})^{-1}$.

12. If $a = 2$, $b = 3$, write down the values of

(i) $a^{-1} + b^{-1}$; (ii) $(a+b)^{-1}$; (iii) ab^{-1}; (iv) $(a-b)^{-1}$.

13. Find n if

(i) $x^n \times x^n = x^{16}$; (ii) $x^n \times x^n = x^{14}$; (iii) $x^n \times x^n = x^{15}$.

14. Find n if

(i) $x^6 \times x^6 = x^n$; (ii) $x^7 \times x^7 = x^n$; (iii) $x^{6\frac{1}{2}} \times x^{6\frac{1}{2}} = x^n$.

15. Find n if

(i) $x^n \times x^n = x$; (ii) $x^n \times x^n = 1$.

16. Express as powers of x the square roots of

(i) x^8; (ii) x^7; (iii) x.

17. Simplify

(i) $x^{\frac{2}{3}} \times x^{\frac{2}{3}} \times x^{\frac{2}{3}}$; (ii) $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}}$.

What is the meaning of $x^{\frac{2}{3}}$ and of $x^{\frac{1}{3}}$?

18. Assign a meaning, with reasons, to (i) $x^{\frac{1}{4}}$; (ii) $x^{\frac{3}{4}}$.

19. Find n if

(i) $(x^5)^3 = x^n$; (ii) $(x^{\frac{1}{2}})^3 = x^n$; (iii) $(x^{\frac{1}{2}})^2 = x^n$.

20. Find n if

(i) $(x^n)^3 = x^{27}$; (ii) $(x^n)^5 = x^3$; (iii) $(x^n)^8 = x^6$.

21. What are the values of

(i) $16^{\frac{1}{2}}$; (ii) $9^{\frac{3}{2}}$; (iii) $8^{\frac{1}{3}}$; (iv) $27^{\frac{2}{3}}$.

22. What are the values of

(i) $64^{\frac{4}{3}}$; (ii) $16^{1\cdot 25}$; (iii) $81^{\frac{3}{4}}$; (iv) $8^{-\frac{1}{3}}$; (v) $16^{\frac{9}{4}}$.

23. If $a = 16$, $b = 9$, find the values of

(i) $a^{\frac{1}{2}} + b^{\frac{1}{2}}$; (ii) $(a+b)^{\frac{1}{2}}$; (iii) $a^{-\frac{1}{2}} + b^{-\frac{1}{2}}$; (iv) $(a+b)^{-\frac{1}{2}}$.

24. What are the values of

(i) $8^{-\frac{1}{3}}$; (ii) $4^{-\frac{3}{2}}$; (iii) $16^{-\frac{3}{4}}$; (iv) $100^{-1\cdot 5}$; (v) $1^{-\frac{1}{4}}$.

25. Express as powers of x

(i) $\sqrt{x^3}$; (ii) $\sqrt[4]{x^{11}}$; (iii) $\sqrt[6]{x^3}$.

26. Simplify (i) $2x^{\frac{1}{2}} \times 3x^{\frac{1}{3}}$; (ii) $x \times x^{-\frac{1}{2}}$

27. Give reasons for the statement $x^{-\frac{1}{2}} = \dfrac{1}{\sqrt{x}}$.

28. Assign a meaning, with reasons, to

(i) $x^{-\frac{3}{4}}$; (ii) $x^{-2\frac{1}{2}}$.

29. By using tables, write down the values of

(i) $2^{0\cdot 5}$; (ii) $2^{0\cdot 25}$; (iii) $2^{0\cdot 125}$; (iv) $2^{0\cdot 0625}$; (v) $2^{0\cdot 03125}$.

What is the effect of repeating this process indefinitely ?

30. By using tables, write down the values of

(i) $10^{0\cdot 5}$; (ii) $10^{0\cdot 25}$; (iii) $10^{0\cdot 125}$; (iv) $10^{0\cdot 0625}$.

What is the effect of repeating this process indefinitely ?

31. Express the following facts without using negative indices :

(i) The charge of an electron is $4\cdot 7 \times 10^{-10}$ units.

(ii) The diameter of an electron is $0\cdot 3 \times 10^{-12}$ cm.

(iii) The mass of an atom of hydrogen is $1\cdot 6 \times 10^{-24}$ gr.

(iv) The diameter of an atom of oxygen is 3×10^{-8} cm.

(v) The time between two collisions of molecules of hydrogen is $1\cdot 06 \times 10^{-10}$ seconds.

(vi) The force of attraction between two grams at a distance 1 cm. apart is $0\cdot 648 \times 10^{-7}$ dynes.

(vii) The wave length of yellow light is 27×10^{-6} inches.

32. Express the following without using fractional or negative indices:

(i) lt^{-2}; mt^{-3}; ml^2t^{-2}.

(ii) $d=kt^{-3}$; $R=\mu^{-\frac{1}{2}}a^{\frac{3}{2}}$; $d=kH^{\frac{1}{3}}N^{-\frac{1}{3}}$.

(iii) $\frac{l}{L}\left(\frac{T}{t}\right)^{-2}$; $\left(\frac{l_1}{l_2}\right)^{-\frac{1}{2}}$; $pv^{0\cdot 25}$.

(iv) $a^{\frac{1}{2}}+b^{\frac{1}{2}}=c^{\frac{1}{2}}$; $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$; $t=2\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}$.

(v) $y=e^{-kt}$ when $k=8$, $t=0\cdot 5$.

33. If x is small (compared with 1), $(1+x)^n$ is approximately equal to $1+nx$.

Use this fact to find approximate values for

(i) $(1+\cdot 01)^{\frac{1}{2}}$; (ii) $\sqrt{1\cdot 02}$; (iii) $(1\cdot 01)^{-1}$;

(iv) $\frac{1}{\sqrt{1\cdot 02}}$; (v) $(0\cdot 99)^{\frac{1}{2}}$; (vi) $\frac{1}{0\cdot 99}$;

(vii) $\frac{1}{(1\cdot 001)^2}$; (viii) $\frac{1}{\sqrt{104}}$;

(ix) $\sqrt[3]{1003}$; (x) $\frac{1}{(1003)^{\frac{1}{3}}}$.

34. The "derived function" of x^n with respect to x is nx^{n-1} for all values of n. Use this fact to find the "derived functions" with respect to x of

(i) x^3; (ii) $x^{\frac{3}{2}}$; (iii) x^{-2}; (iv) $\sqrt{x}$;

(v) $\frac{1}{x^5}$; (vi) $\frac{1}{\sqrt{x}}$; (vii) x^0; (viii) $\frac{1}{\sqrt[3]{x^5}}$.

35. The "integral" of x^n with respect to x is $\frac{1}{n+1}\cdot x^{n+1}$ for all values of n, except $n=-1$. Use this fact to find the integrals with respect to x of

(i) x^2; (ii) $x^{\frac{1}{2}}$; (iii) x^{-2}; (iv) $\sqrt[3]{x}$; (v) $\frac{1}{x^3}$;

(vi) $\sqrt[2]{x^3}$; (vii) x; (viii) x^0; (ix) $\frac{1}{\sqrt[3]{x}}$.

Summary of Results.

The examples of the above exercise show what meanings must be given to a^n, when n is zero, fractional or negative, if the laws (i), (ii), (iii) on page 255 hold good.

The necessary interpretations are:

$$\text{(i)}\quad x^0 = 1\,;$$

$$\text{(ii)}\quad x^{-n} = \frac{1}{x^n}\,;$$

$$\text{(iii)}\quad x^{\frac{p}{q}} = \sqrt[q]{x^p} = (\sqrt[q]{x})^p.$$

In particular,

$$x^{\frac{1}{q}} = \sqrt[q]{x}\,;$$

$$x^{-\frac{p}{q}} = \frac{1}{x^{\frac{p}{q}}} = \frac{1}{\sqrt[q]{x^p}}.$$

The proofs of the general results follow the methods given in the worked-out examples on page 255.

EXERCISE XII. c.

1. Write with fractional indices

(i) $\sqrt[3]{x}$; (ii) $(\sqrt[3]{a})^2$; (iii) $(\sqrt[6]{a})^4$; (iv) $\sqrt{ab}$;
(v) $2\sqrt{ab^2}$; (vi) $3\sqrt{ab^3}$; (vii) $\sqrt{ab^2}$.

2. Write the following so that there is no denominator:

(i) $\dfrac{3}{x^2}$; (ii) $\dfrac{1}{x}$; (iii) $\dfrac{1}{xy}$; (iv) $\dfrac{5}{x\sqrt{y}}$.

3. Find the value of $p^{\frac{r-1}{r}}$ when (i) $r = 1{\cdot}5$, $p = 64$;
(ii) $r = 1{\cdot}25$, $p = 32$.

4. Find n, if (i) $2^n = 8$; (ii) $8^n = 2$; (iii) $3^n = \frac{1}{9}$;
(iv) $5^n = \dfrac{1}{5\sqrt{5}}$; (v) $(0{\cdot}01)^n = 10$.

5. Simplify (i) $2a^{\frac{1}{2}} \times a$; (ii) $a^{\frac{1}{4}} \times a^{\frac{1}{6}}$; (iii) $ab^{\frac{1}{2}} \times a^{\frac{1}{2}}b$.

6. Simplify (i) $a^{\frac{3}{2}} \div a$; (ii) $4a^{\frac{5}{2}} \div 2a$; (iii) $a \div a^{\frac{1}{3}}$.

7. Simplify (i) $a^2 \div \sqrt{a}$; (ii) $\sqrt[4]{a^3} \div (\sqrt[4]{a})^3$.

8. Multiply $x + x^{-1}$ by x.

9. Divide $\sqrt{x}$ by $\sqrt[3]{x}$.

10. Simplify $\dfrac{x - x^{-1}}{x - 1}$.

11. Find the value of $\sqrt[5]{32^4}$.

12. Simplify $\dfrac{1}{x^{-\frac{1}{2}}} \times \sqrt{x^3}$. **13.** If $16^n = 1$, find n.

14. Simplify

(i) $100^{2n} \div 10^n$; (ii) $\left(\dfrac{25x^4}{2\frac{1}{4}y^6}\right)^{-\frac{1}{2}}$; (iii) $(x^2)^{-2}$;

(iv) $(x^{-3})^{-3}$; (v) $\dfrac{1}{(x+y)^{-2}}$.

15. Simplify (i) $9^{\frac{1}{2}} + 16^{\frac{1}{2}}$; (ii) $(25+144)^{\frac{1}{2}}$;

(iii) $\dfrac{1}{8^{-\frac{1}{3}}} - \dfrac{1}{27^{-\frac{1}{3}}}$; (iv) $(4^n + 8^{2n}) \div 2^n$.

16. Solve $9^n = \dfrac{9}{3^n}$.

IRRATIONAL NUMBERS.

$\sqrt{2}$ is defined as a number whose square is 2. This number cannot be expressed as a fraction in the form $\dfrac{p}{q}$, where p and q are integers. It is, in fact, a new type of number, and is called an "*Irrational*" number.

Irrational numbers obey the ordinary laws of algebra, *e.g.*

$$\sqrt{a} \times \sqrt{b} = \sqrt{b} \times \sqrt{a} = \sqrt{ab};$$
$$\sqrt{a} + \sqrt{b} = \sqrt{b} + \sqrt{a};$$
$$\sqrt{a}(\sqrt{x} + \sqrt{y}) = \sqrt{ax} + \sqrt{ay}.$$

Example VII. Given that $\sqrt{2} = 1{\cdot}414$ approximately, find $\sqrt{98}$.

$$\begin{aligned}\sqrt{98} &= \sqrt{49} \times \sqrt{2}\\ &= 7\sqrt{2}\\ &= 9{\cdot}898 \text{ approx. or } 9{\cdot}90 \text{ correct to 3 figures.}\end{aligned}$$

Example VIII. Find $\dfrac{10}{\sqrt{2}}$ correct to 3 figures.

$$\frac{10}{\sqrt{2}} = \frac{10\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2} = 7{\cdot}07 \text{ approx.}$$

It is simpler to "rationalise" the denominator in this way than to divide 10 by 1·414, the approximate value of $\sqrt{2}$.

Example IX. Evaluate $\frac{\sqrt{3}+2}{\sqrt{3}-1}$, given $\sqrt{3}=1{\cdot}732$.

Rationalise the denominator by multiplying both numerator and denominator by $\sqrt{3}+1$.

Then
$$\frac{\sqrt{3}+2}{\sqrt{3}-1}=\frac{(\sqrt{3}+2)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$
$$=\frac{(\sqrt{3})^2+3\sqrt{3}+2}{(\sqrt{3})^2-1}$$
$$=\frac{5+3\sqrt{3}}{2}=\frac{5+5{\cdot}196}{2}\text{ approx.}$$
$$=\frac{10{\cdot}196}{2}$$
$$=5{\cdot}10 \text{ correct to 3 figures.}$$

EXERCISE XII. d.

1. Simplify

(i) $\sqrt{9}+\sqrt{16}$; (ii) $\sqrt{9+16}$; (iii) $\sqrt{9}\times\sqrt{16}$;

(iv) $\sqrt{9\times 16}$; (v) $\sqrt{36}\div\sqrt{9}$; (vi) $\sqrt{36\div 9}$.

2. Simplify

(i) $\sqrt{6}\times\sqrt{24}$; (ii) $\sqrt{32}\times\sqrt{18}$; (iii) $\sqrt{18}\div\sqrt{200}$.

3. Prove that $\sqrt{8}=2\sqrt{2}$; express in a similar way

(i) $\sqrt{18}$; (ii) $\sqrt{72}$; (iii) $\sqrt{50}$;

(iv) $\sqrt{2000}$; (v) $\sqrt{ab^2}$; (vi) $\sqrt{\left(\frac{a}{x^2}\right)}$.

4. Given $\sqrt{3}=1{\cdot}732$, write down to 3 significant figures the values of

(i) $\sqrt{300}$; (ii) $\sqrt{27}$; (iii) $\sqrt{0{\cdot}75}$.

5. Prove that $\frac{6}{\sqrt{3}}=2\sqrt{3}$; use this idea to obtain to 3 significant figures the values of (i) $\frac{1}{\sqrt{3}}$; (ii) $\frac{5}{\sqrt{12}}$.

6. Express without radical signs in the denominator

(i) $\frac{18}{\sqrt{2}}$; (ii) $\frac{5}{\sqrt{2}}$; (iii) $\frac{100}{\sqrt{10}}$; (iv) $\frac{6}{\sqrt{12}}$; (v) $\frac{b^2}{\sqrt{b}}$.

7. Evaluate to 3 significant figures, given

$$\sqrt{2}=1{\cdot}414,\quad \sqrt{5}=2\,236.$$

(i) $\sqrt{50}$; (ii) $\sqrt{500}$; (iii) $\sqrt{5000}$; (iv) $\frac{2}{\sqrt{5}}$.

8. (i) Simplify $(\sqrt{2}+1)(\sqrt{2}-1)$.

(ii) Evaluate to 3 figures $\dfrac{1}{\sqrt{2}-1}$.

9. (i) Simplify $(5\sqrt{2}-2\sqrt{3})(5\sqrt{2}+2\sqrt{3})$.

(ii) Express with a rational denominator $\dfrac{6}{5\sqrt{2}+2\sqrt{3}}$.

10. Evaluate to 3 figures

(i) $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$; (ii) $\dfrac{1}{\sqrt{3}-\sqrt{2}}$; (iii) $\dfrac{6-2\sqrt{5}}{\sqrt{5}-1}$.

11. Given $\sqrt[3]{10} \simeq 2 \cdot 154$, find to 3 figures $\sqrt[3]{80}$.

12. Evaluate x^2+2x when $x=\sqrt{2}-1$.

13. Simplify

(i) $\dfrac{1}{1+\sqrt{2}}+\dfrac{1}{\sqrt{2}-1}$; (ii) $\dfrac{\sqrt{6}}{(\sqrt{3}+1)(\sqrt{6}-\sqrt{2})}$.

14. Solve $x\sqrt{2}+x=7$: answer to 3 figures.

15. Evaluate $\left(x+\dfrac{1}{x}\right)^2$ if $x=\sqrt{6}-\sqrt{5}$.

16. The perimeter of an isosceles right-angled triangle is 5 inches; find, correct to $\frac{1}{100}$ inch, the length of each side.

17. The area of an equilateral triangle is 6 sq. inches; find correct to $\frac{1}{100}$ inch, the length of a side.

18. Draw a triangle ABC such that $AB=x$ in., $BC=\frac{1}{2}x$ in., $\angle ABC=90°$; from CA cut off CY equal to CB; from AB cut off AX equal to AY; prove that $AB \,.\, BX=AX^2$.

19. AB is the diameter of a circle, centre O; ABC is an equilateral triangle; CO cuts the circle at P; prove that the tangent from C to the circle equals AP.

20. OA, OB are two perpendicular radii of a circle; another circle is drawn touching OA, OB and the arc AB internally; prove that its radius $=OA\,(\sqrt{2}-1)$.

21. (1) What is the square of (i) $\sqrt{a}+\sqrt{b}$, (ii) $\sqrt{a+b}$?

(2) Is it possible for $x+y$ and $\sqrt{x^2+y^2}$ to be equal ?

(3) Interpret this result geometrically with reference to a right-angled triangle.

22. (1) What is the square of (i) $\sqrt{x}\times\sqrt{y}$, (ii) $\sqrt{xy}$?

(2) Is $\sqrt{x}\times\sqrt{y}$ equal to $\sqrt{xy}$?

23. Simplify (i) $\dfrac{1}{\sqrt{x}+\sqrt{y}}+\dfrac{1}{\sqrt{x}-\sqrt{y}}$;

(ii) $\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}$.

SUMMARY OF RESULTS.

(i) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

(ii) $\sqrt{a} + \sqrt{b}$ is *not* equal to $\sqrt{a+b}$, unless a or b is zero.

(iii) $\dfrac{a}{\sqrt{b}} = \dfrac{a\sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \dfrac{a\sqrt{b}}{b}$.

(iv) $\dfrac{a}{\sqrt{b} + \sqrt{c}} = \dfrac{a(\sqrt{b} - \sqrt{c})}{(\sqrt{b} + \sqrt{c})(\sqrt{b} - \sqrt{c})} = \dfrac{a(\sqrt{b} - \sqrt{c})}{b - c}$.

Example X. Draw the graph of 2^x from $x = -3$ to $x = +3$.

When $x = 3$, $2^x = 2^3 = 8$. When $x = 0$, $2^x = 2^0 = 1$.

When $x = -3$, $2^x = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8} = 0{\cdot}125$.

When $x = 0{\cdot}5$, $2^x = 2^{\frac{1}{2}} = \sqrt{2} = 1{\cdot}414$ approx.

When $x = 1{\cdot}5$, $2^x = 2^{1\frac{1}{2}} = 2 \times 2^{\frac{1}{2}} = 2 \times \sqrt{2} = 2{\cdot}828$ approx.

When $x = 2{\cdot}5$, $2^x = 2^{2\frac{1}{2}} = 2^2 \times 2^{\frac{1}{2}} = 4 \times \sqrt{2} = 5{\cdot}657$ approx.

We can now write down the following table of values :

x	−3	−2	−1	0	0·5	1	1·5	2	2·5	3
2^x	0·125	0·25	0·5	1	1·41	2	2·83	4	5·66	8

Plotting these values, we obtain the following graph

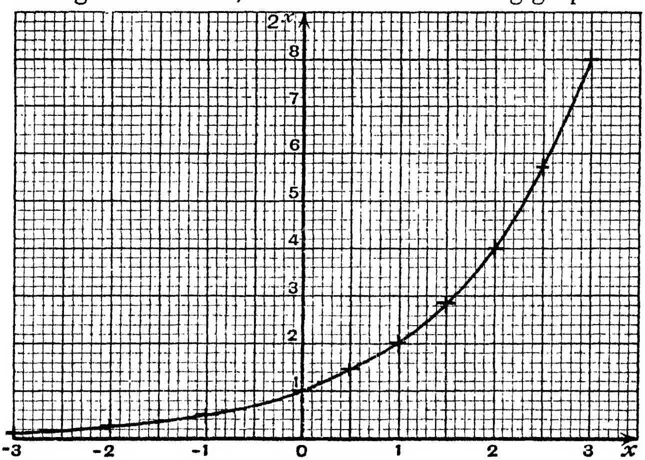


FIG. 5.*

* Figs. 1-4 appear in the Introduction.

We note that

(i) 2^x increases without limit as x increases;

(ii) 2^x is positive for all values of x;

(iii) for x positive $2^x > 1$,
for $x = 0$, $2^x = 1$,
for x negative $2^x < 1$;

(iv) as x is given increasingly large *negative* values, 2^x tends continually closer to the value 0.

EXERCISE XII. e.

1. Read off from the graph of 2^x approximate values of

(i) $2^{0 \cdot 2}$; (ii) $2^{-0 \cdot 5}$; (iii) $4^{1 \cdot 1}$.

2. Use the graph of 2^x to solve $2^x = 5$.

3. What is the greatest integral value of x for which

(i) $2^x < 0 \cdot 01$; (ii) $2^x < 0 \cdot 001$?

4. Show that $10^{1 \cdot 6} < 100$.

5. Show that $1 > 10^{-0 \cdot 8} > 0 \cdot 1$.

6. Show that $4^{2 \cdot 9} < 64$.

7. Show that $8^{1 \cdot 3} < 16$.

8. Make a rough estimate of the values of

(i) $8^{0 \cdot 33}$; (ii) $32^{-0 \cdot 39}$; (iii) $10^{0 \cdot 00002}$;

(iv) $10^{0 \cdot 99}$; (v) $10^{-2 \cdot 001}$; (vi) $4^{0 \cdot 497}$.

9. (i) By using such facts as $10^{0 \cdot 5} = \sqrt{10}$, $10^{0 \cdot 25} = \sqrt{(\sqrt{10})}$, $10^{0 \cdot 75} = 10^{\frac{3}{4}} = 10^{\frac{1}{2}} \times 10^{\frac{1}{4}}$, etc., compute the values needed for filling in the following table:

$x = 0$	0·125	0·25	0·375	0·5	0·625	0·75	0·875	1
$10^x =$								

(ii) Hence draw the graph of 10^x from $x = 0$ to $x = 1$.

(iii) From your graph, read off the values of $10^{0 \cdot 2}$, $10^{0 \cdot 4}$, $10^{0 \cdot 7}$, $10^{0 \cdot 9}$.

(iv) From your graph, read off the values of x for which

(*a*) $10^x = 3$; (*b*) $10^x = 5$; (*c*) $10^x = 8 \cdot 1$.

(v) From your graph, read off the values of

(*a*) $\sqrt[5]{10}$; (*b*) $\sqrt[3]{100}$; (*c*) $\sqrt[5]{1000}$.

10. Given that $2 = 10^{0.301}$ approximately, solve the equations

(i) $10^x = 8$; (ii) $5^x = 10$; (iii) $10^x = 200$; (iv) $10^x = 50$.

11. Without making a table of values, sketch roughly the graph of 5^x from $x = 2$ to $x = -1$.

12. Given that $9 = 10^{0.9542}$, express as powers of 10

(i) 3; (ii) 0·9; (iii) 2·7.

CHAPTER XIII.

LOGARITHMS.

We saw in the last chapter (Ex. XII. e. No. 9 (i)) how to construct the graph of 10^x.

Here is the necessary table of values :

$x=0$	0·125	0·25	0·375	0·5	0·625	0·75	0·875	1
$10^x=1$	1·33	1·78	2·37	3·16	4·22	5·62	7·50	10

Plotting them, we have the following graph :

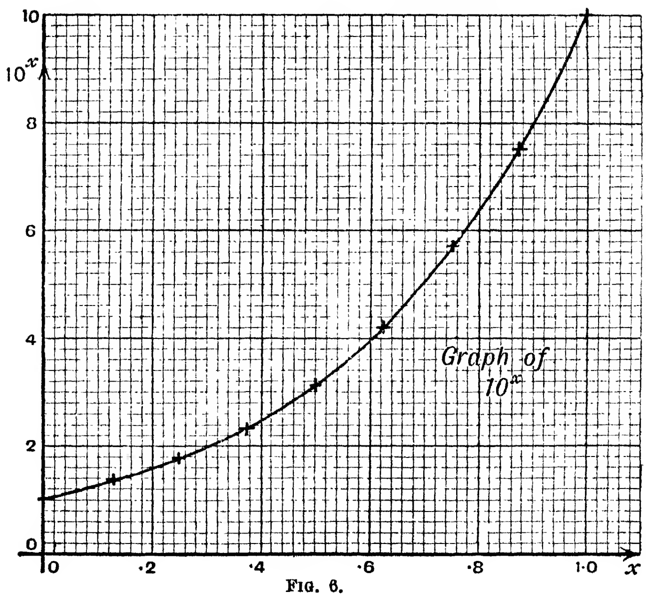


Fig. 6.

From this graph we can express any number between 1 and 10 as a power of 10,

$$e.g.\ 4 = 10^{0 \cdot 602} \text{ approximately.}$$

Definition. If $a = 10^x$, x is called the logarithm of a to base 10.

For example, (i) since $4 = 10^{0 \cdot 602}$,
the logarithm of 4 is 0·602 ;

(ii) since $1000 = 10^3$,
the logarithm of 1000 is 3 ;

(iii) since $1 = 10^0$,
the logarithm of 1 is 0.

EXERCISE XIII. a.

1. (i) Find from the graph the value of x for which $10^x = 7$.
(ii) What is the logarithm of 7 ?

2. (i) Use the graph to express 3·5 as a power of 10.
(ii) What is the logarithm of 3·5 ?

3. (i) Given that $2 = 10^{0 \cdot 301}$, express as powers of 10
(*a*) 20 ; (*b*) 200 ; (*c*) 2000 ; (*d*) 2,000,000.
(ii) What are the logarithms of
(*a*) 20 ; (*b*) 200 ; (*c*) 2000 ; (*d*) 2,000,000 ?

4. What are the logarithms of (i) 100 ; (ii) 1,000,000 ?

5. Given that $4 \cdot 342 = 10^{0 \cdot 6377}$, find the logarithms of
(i) 43·42 ; (ii) 434·2 ; (iii) 43,420 ; (iv) 4,342,000.

6. Given that $7 \cdot 286 = 10^{0 \cdot 8625}$, find the values of
(i) $10^{1 \cdot 8625}$; (ii) $10^{2 \cdot 8625}$; (iii) $10^{4 \cdot 8625}$; (iv) $10^{6 \cdot 8625}$.

7. What is x if (i) $27 \cdot 2 = 10^x \times 2 \cdot 72$;
(ii) $270 \cdot 2 = 10^x \times 2 \cdot 702$;
(iii) $71{,}000 = 10^x \times 7 \cdot 1$;
(iv) $568{,}123 = 10^x \times 5 \cdot 68$ approximately ?

8. What is the whole number in the index when each of the following numbers is expressed as a power of 10 ?
(i) 73 ; (ii) 7142·3 ; (iii) 8·61 ; (iv) $5 \cdot 6 \times 10^5$;
(v) 100,000 ; (vi) 1234 ; (vii) 12,345 ; (viii) 123,456 ;
(ix) 12,345,678 ; (x) 123,456,789.

9. (i) Is $10^{0.78}$ greater or less than 10?
(ii) Is $10^{0.78}$ greater or less than 1?
(iii) Is $10^{5.634}$ greater or less than a million?
(iv) Is $10^{4.81}$ greater or less than 10,000?

10. (i) Is 103 greater or less than $10^{1.9}$?
(ii) Is 863·84 greater or less than $10^{3.2}$?
(iii) Is 7·53 greater or less than $10^{1.1}$?

11. Use the graph to write down approximate values for
(i) $10^{2.35}$; (ii) $10^{4.71}$; (iii) $10^{7.15}$.

12. (i) Use the graph to express (approximately) as powers of 10
(*a*) 29; (*b*) 6100; (*c*) 8,400,000.
(ii) Use the graph to obtain approximate values for the logarithms of
(*a*) 33; (*b*) 54,000,000; (*c*) 7400.

Summary of Results.

The examples in Exercise XIII. a. lead us to the following conclusions:

(1) The logarithm of any number between 1 and 10 has a value between 0 and 1,

e.g. the logarithm of 2 is 0·301 approx.

(2) Moving the decimal point in a number affects the integral part of its logarithm, but has no effect on the figures in the logarithm after the decimal point;

e.g. the logarithm of 5·64 is 0·7513;
∴ the logarithm of 56·4 is 1·7513,
and the logarithm of 56,400 is 4·7513.

(3) The integral part of the logarithm can always be found by inspection;

e.g. $738{,}100 = 10^5 \times 7{\cdot}381$;

∴ the integral part of the logarithm of 738,100 is 5.

Note.—The integral part of a logarithm is called its ***characteristic.***

The figures in a logarithm after the decimal point are called its *mantissa.*

e.g. the logarithm of 738,100 is 5·8682.

The integer 5 is the characteristic.

The decimal fraction ·8682 is the mantissa.

Since the characteristic can be found by inspection, it is only the mantissa that is printed in logarithm tables.

For a set of logarithm 4-figure tables, see page 552. This table gives logarithms of numbers correct to 4 significant figures. It should be noted that if 4-figure tables are employed for computation, the answer is not necessarily *correct* to 4 figures, because the figures in the fifth place which have been neglected may, as the result of a series of additions or multiplications, affect the figure in the fourth place in the answer.

Example I. Find correct to 3 significant figures the value of

$$3{\cdot}142 \times 41{\cdot}27.$$

From the tables

$$\begin{aligned} 3{\cdot}142 \times 41{\cdot}27 &= 10^{0{\cdot}4972} \times 10^{1{\cdot}6156} \\ &= 10^{0{\cdot}4972+1{\cdot}6156} \\ &= 10^{2{\cdot}1128} \\ &= 129{\cdot}6 \\ &= 130 \text{ to 3 sig. fig.} \end{aligned}$$

$$\begin{array}{r} 1{\cdot}6156 \\ {\cdot}4972 \\ \hline 2{\cdot}1128 \end{array}$$

Rough estimate, $3 \times 40 = 120$.

Example II. Find correct to 3 significant figures the value of

$$407{\cdot}9 \div 14{\cdot}56.$$

From the tables

$$\begin{aligned} 407{\cdot}9 \div 14{\cdot}56 &= 10^{2{\cdot}6106} \div 10^{1{\cdot}1632} \\ &= 10^{2{\cdot}6106-1{\cdot}1632} \\ &= 10^{1{\cdot}4474} \\ &= 28{\cdot}02 \\ &= 28{\cdot}0 \text{ to 3 sig. fig.} \end{aligned}$$

$$\begin{array}{r} 2{\cdot}6106 \\ 1{\cdot}1632 \\ \hline 1{\cdot}4474 \end{array}$$

Rough estimate, $400 \div 14 =$ about 30.

Example III. Find correct to 3 significant figures the value of

$$\frac{31{\cdot}7 \times 920000}{415{\cdot}1 \times 1{\cdot}6}.$$

From the tables, we find the expression

$$
\begin{aligned}
&= \frac{10^{1\cdot5011} \times 10^{5\cdot9638}}{10^{2\cdot6181} \times 10^{0\cdot2041}} \\
&= \frac{10^{1\cdot5011+5\cdot9638}}{10^{2\cdot6181+0\cdot2041}} \\
&= \frac{10^{7\cdot4649}}{10^{2\cdot8222}} = 10^{7\cdot4649-2\cdot8222} \\
&= 10^{4\cdot6427} \\
&= 43{,}920 \\
&= 43{,}900 \text{ to 3 sig. fig.}
\end{aligned}
$$

$$
\begin{array}{r|r}
1\cdot5011 & 2\cdot6181 \\
5\cdot9638 & 0\cdot2041 \\
\hline
7\cdot4649 & 2\cdot8222 \\
2\cdot8222 & \\
\hline
4\cdot6427 &
\end{array}
$$

Rough estimate $= \dfrac{30 \times 900000}{400 \times 2} = \dfrac{27 \times 10^6}{800} \simeq 3 \times 10^4$
$\simeq 30{,}000.$

Example IV. Find correct to 3 significant figures the value of

$$(4\cdot172)^5.$$

From the tables

$$
\begin{aligned}
(4\cdot172)^5 &= (10^{0\cdot6203})^5 \\
&= 10^{0\cdot6203\times5} \\
&= 10^{3\cdot1015} \\
&= 1263 \\
&= 1260 \text{ to 3 sig. fig.}
\end{aligned}
$$

$$
\begin{array}{r}
0\cdot6203 \\
5 \\
\hline
3\cdot1015
\end{array}
$$

Rough estimate $= 4^5 = 1024.$

Example V. Find correct to 3 significant figures the cube root of 152·7.

From the tables

$$
\begin{aligned}
\sqrt[3]{152\cdot7} &= (152\cdot7)^{\frac{1}{3}} \\
&= 10^{2\cdot1838\times\frac{1}{3}} \\
&= 10^{0\cdot7279} \\
&= 5\cdot344 \\
&= 5\cdot34 \text{ to 3 sig. fig.}
\end{aligned}
$$

$$
3\,\underline{|\,2\cdot1838} \qquad 0\cdot7279
$$

Rough estimate, $\sqrt[3]{125} = 5.$

Note on Method.

Until the reader is fully experienced, he should follow the methods given above, in which all numbers are written down clearly as powers of ten, in the middle of the page. Any simplification that has to be done should be written in special columns at the side of the page.

In evaluating a complicated expression, such as in Example III., clearness is gained by having two such columns, one for logarithms of factors of the numerator and the other for logarithms of factors of the denominator.

When sufficient skill and experience has been acquired, it will be found that the middle of the page contains little more than the question and the answer, all the subsidiary calculations being done in the side columns; but the work in these side columns must be kept neat, and logarithms must not be allowed to straggle unattended into the middle of the page.

EXERCISE XIII. b.

Evaluate, correct to 3 significant figures,

1. $4{\cdot}83 \times 7{\cdot}24$.

2. $19{\cdot}1 \div 6{\cdot}87$.

3. $(8{\cdot}73)^2$.

4. 407×103.

5. $4{\cdot}723 \times 1{\cdot}246$.

6. $401{\cdot}6 \times 5{\cdot}073$.

7. $26{\cdot}7 \times 81{\cdot}31$.

8. $8{\cdot}138 \times 41{,}900$.

9. $412{\cdot}6 \div 37{\cdot}19$.

10. $46{\cdot}59 \div 5{\cdot}123$.

11. $91{,}923 \div 718{\cdot}6$.

12. $10 \div 4{\cdot}965$.

13. $(1{\cdot}738)^2$.

14. $(2{\cdot}716)^3$.

15. $(114{\cdot}9)^2$.

16. $(94{\cdot}56)^3$.

17. $\dfrac{31{\cdot}2 \times 7{\cdot}84}{9{\cdot}91}$.

18. $\dfrac{863}{4{\cdot}09 \times 11{\cdot}8}$.

19. $\dfrac{639 \times 41{\cdot}3}{83{\cdot}2 \times 55{\cdot}4}$.

20. $\dfrac{(8{\cdot}63)^2}{10{\cdot}08}$.

21. $4{\cdot}16 \times 11{\cdot}3 \times 7{\cdot}28$.

22. $\dfrac{(13{\cdot}01)^2}{(4{\cdot}82)^3}$.

23. $6{\cdot}14 \times (8{\cdot}231)^2$.

24. $\dfrac{6{\cdot}172 \times 19{\cdot}41}{(9{\cdot}835)^2}$.

25. $\dfrac{36{\cdot}32 \times 10000}{(21{\cdot}23)^3}$.

26. $\dfrac{100}{(3{\cdot}71)^3}$.

27. $\left(\dfrac{3{\cdot}18 \times 6{\cdot}05}{4{\cdot}45}\right)^2$.

28. $5{\cdot}018 \times (2{\cdot}63)^3$.

29. $2\frac{1}{3} \times 8{\cdot}61 \div 3{\cdot}4$.

30. $(13\frac{1}{4})^2 \div 4{\cdot}931$.

31. $\sqrt{4.256}$. **32.** $\sqrt{42.56}$.

33. $\sqrt[3]{1.523}$. **34.** $\sqrt[3]{15.23}$.

35. $\sqrt[3]{152.3}$. **36.** $\sqrt[3]{(16.35)^2}$.

37. (i) $\sqrt[3]{9}$; (ii) $\sqrt[3]{90}$; (iii) $\sqrt[3]{900}$; (iv) $\sqrt[3]{9000}$.

38. $6.14 \times \sqrt{8.231}$. **39.** $\sqrt{\dfrac{172.8}{9.43}}$.

40. $\dfrac{\sqrt{48.63}}{(2.59)^2}$. **41.** $\dfrac{4.13 \times 10^6}{517.2} \times \sqrt{39.41}$.

42. $(108.8)^{\frac{1}{4}}$. **43.** $\sqrt[3]{\dfrac{1.752 \times 753.6}{1111}}$.

44. $\dfrac{\sqrt{10}}{\sqrt{7}}$. **45.** $\sqrt{2\frac{1}{7} \times 17\frac{3}{11}}$.

EXERCISE XIII. c.

[In the following examples, take $\pi = 3.1416 = 10^{0.4971}$. Give all results correct to 3 significant figures.]

1. (i) Find the area of a circle of radius 4·37 cm.
(ii) Find the radius of a circle of area 150 sq. cm.

2. The volume of a sphere of radius r cm. is $\frac{4}{3}\pi r^3$ cu. cm.
(i) Find the volume of a sphere of radius 1·083 cm.
(ii) Find the radius of a sphere of volume 5 cu. cm.

3. The distance a body falls in a vacuum in t seconds is $\frac{1}{2} \times 32.2 \times t^2$ feet.
(i) How far does it fall in 1·63 seconds?
(ii) How long does it take to fall 83·2 feet?

4. A pendulum of length l feet makes one complete oscillation in $2\pi\sqrt{(l \div 32.2)}$ seconds.
(i) Find the time of oscillation for a pendulum of length 50 feet.
(ii) Find the length of a pendulum which takes 2 seconds for a complete oscillation.

5. Find the horse-power of a two-cylinder locomotive from the formula, horse-power $= \dfrac{p \,.\, l \,.\, a \,.\, n}{99000}$, where p = pressure in lb. wt. per sq. in. = 15, l = length of stroke in inches = 17·5, a = area of piston in sq. inches = 254, n = number of revolutions per minute = 250. Answer correct to nearest integer.

6. The following formula is given for the speed of a paddle steamer, V (in knots) $=\sqrt[3]{\dfrac{620P}{S}}$; find V, given that $P=$ horse-power $=1850$, $S=$ sectional area $=714$ sq. feet. Find also the horse-power necessary to secure a speed of $15\frac{1}{2}$ knots for the same vessel.

7. Use the formula for the volume of a cone, $V=\frac{1}{3}\pi r^2 h$, to find r if $V=182$, $h=11{\cdot}4$.

8. The horse-power H of a steam turbine is given by $H=\dfrac{6D^2U^3}{10^8 V}$; find H, if $D=$ mean diameter in feet $=2{\cdot}4$, $U=$ mean blade speed in ft. per sec. $=1530$, $V=$ specific volume of steam in cu. ft. $=285$.

9. The bursting pressure P lb. per sq. inch for a flat circular plate of thickness T inches and radius R inches is given by $P=\dfrac{72000T^2}{R^2}$; find P if $T=1{\cdot}125$, $R=8{\cdot}37$.

10. The diameter, D feet, for a 3-bladed propeller is found from Doig's formula $D=\dfrac{52H^{0\cdot2}}{R^{0\cdot6}}$; find D if the horse-power H is 9000 and the number of revolutions per minute R is 225.

11. The diameter d mm. of a tin wire which is just fused by a current of C amperes is given by the formula $d=\sqrt[3]{\left\{\left(\dfrac{C}{12{\cdot}8}\right)^2\right\}}$; find the value of d if $C=55$.

12. The force necessary to stop a train of weight 150 tons travelling v miles per hour in d feet is $\dfrac{2{\cdot}23v^2}{d}$ tons; what force will stop it within 87 yards when it is travelling 23 miles per hour?

13. If $M=\dfrac{\pi f d^3}{32{\cdot}2}$, find M when $f=6{\cdot}15$, $d=2{\cdot}73$.

14. If $d=k\sqrt[3]{\dfrac{H}{N}}$, find d when $k=173$, $H=46$, $N=3$.

15. If $P=\dfrac{KND^2}{5{\cdot}13}$, find P when $K=1{\cdot}016$, $N=6$, $D=4{\cdot}25$.

16. If $W=\dfrac{Hm}{D^2N^3}$, find W when $N=160$, $D=5{\cdot}75$, $H=15$, $m=1{\cdot}43\times10^7$.

17. If $W=1880\,\dfrac{p\,.\,A}{\sqrt{T}}$, find W when $p=14$, $A=7{\cdot}675$, $T=306$.

18. Find approximately the values of (i) 2^{30}; (ii) 3^{100}.

Example VI. Express as a power of 10, (i) 0·8634; (ii) 0·08634; (iii) 0·0008634.

$$0{\cdot}8634 = \frac{8{\cdot}634}{10} = 8{\cdot}634 \times 10^{-1}$$
$$= 10^{0{\cdot}9362} \times 10^{-1}$$
$$= 10^{-1+0{\cdot}9362};$$
$$0{\cdot}08634 = \frac{8{\cdot}634}{10^2} = 8{\cdot}634 \times 10^{-2}$$
$$= 10^{-2+0{\cdot}9362};$$
$$0{\cdot}0008634 = \frac{8{\cdot}634}{10^4} = 8{\cdot}634 \times 10^{-4}$$
$$= 10^{-4+0{\cdot}9362}.$$

The logarithm of 1 is 0, because $1 = 10^0$.

Therefore the logarithm of any positive number less than 1 is less than 0, *i.e.* is negative: this is illustrated in Example VI.

When the logarithm of a number is negative, the logarithm is always written so that the mantissa (*i.e.* the figures after the decimal point) is positive.

Thus the logarithm of 0·08634 is written as $-2+0{\cdot}9362$, or more shortly as $\bar{2}{\cdot}9362$, the $-$ being placed above the 2 to show that it refers only to the integer 2.

Example VII. Simplify $\dfrac{18{\cdot}61}{0{\cdot}0451}$.

$$\frac{18{\cdot}61}{0{\cdot}0451} = \frac{10^{1{\cdot}2697}}{10^{\bar{2}{\cdot}6542}}$$
$$= 10^{2{\cdot}6155}$$
$$= 412{\cdot}6.$$

$$\begin{array}{r} 1+{\cdot}2697 \\ -2+{\cdot}6542 \\ \hline 2+{\cdot}6155 \end{array}$$

Notes.—(i) At first it is best to write the work out in full as shown in the columns at the side.

(ii) If there is any difficulty in subtracting the integers, use the rule for subtraction: change the sign of the lower line and add.

Example VIII. Simplify $(0{\cdot}0832)^3$.

$$(0{\cdot}0832)^3 = 10^{\bar{2}{\cdot}9201 \times 3}$$
$$= 10^{\bar{4}{\cdot}7603}$$
$$= 0{\cdot}0005758.$$

$$\begin{array}{r} -2+\ {\cdot}9201 \\ 3 \\ \hline -6+2{\cdot}7603 \\ -4+\ {\cdot}7603 \end{array}$$

Example IX. Simplify $\sqrt[5]{0\cdot0736}$.

$$\sqrt[5]{0\cdot0736} = 10^{\bar{2}\cdot8669\times\frac{1}{5}}$$
$$= 10^{\bar{1}\cdot7734}$$
$$= 0\cdot5934.$$

$$\begin{array}{r} -2+\ \cdot8669 \\ 5)\overline{-5+3\cdot8669} \\ \hline -1+\ \cdot7734 \end{array}$$

Example X. Simplify $(0\cdot0862)^{0\cdot24}$.

$$(0\cdot0862)^{0\cdot24} = 10^{\bar{2}\cdot9355\times0\cdot24}$$
$$= 10^{-0\cdot48+0\cdot2245}$$
$$= 10^{-1+0\cdot52+0\cdot2245}$$
$$= 10^{\bar{1}\cdot7445}$$
$$= 0\cdot5552.$$

$$\begin{array}{r} 0\cdot9355 \\ 0\cdot24 \\ \hline 18710 \\ 37420 \\ \hline 0\cdot22452 \end{array}$$

EXERCISE XIII. d.

In Examples 1-30, write the answer so that the fractional part of it is positive; *e.g.* you would write $-1\cdot7$ as $\bar{2}\cdot3$.

Simplify

1. $2\cdot3+\bar{1}\cdot5$.
2. $2\cdot3+\bar{1}\cdot8$.
3. $2\cdot5+\bar{2}\cdot4$.
4. $\bar{2}\cdot5+\bar{3}\cdot1$.
5. $\bar{2}\cdot5+\bar{4}\cdot7$.
6. $\bar{4}\cdot6+2\cdot7$.
7. $\bar{2}\cdot4+1\cdot9+\bar{1}\cdot8$.
8. $2\cdot4-\bar{1}\cdot3$..
9. $\bar{3}\cdot7-2\cdot4$.
10. $3\cdot6-\bar{1}\cdot8$.
11. $\bar{3}\cdot4-1\cdot7$.
12. $3\cdot7-3\cdot9$.
13. $2\cdot6-3\cdot3$.
14. $2\cdot6-3\cdot9$.
15. $\bar{2}\cdot6-\bar{3}\cdot8$.
16. $\bar{1}\cdot4\times2$.
17. $\bar{1}\cdot6\times2$.
18. $\bar{2}\cdot7\times3$.
19. $\bar{3}\cdot9\times5\cdot8$.
20. $2\cdot4\times(-2)$.
21. $\bar{2}\cdot7\times(-3)$.
22. $\bar{2}\cdot6\div2$.
23. $\bar{1}\cdot6\div2$.
24. $\bar{5}\cdot7\div2$.
25. $\bar{3}\cdot9\div3$.
26. $\bar{4}\cdot4\div3$.
27. $\bar{2}\cdot8\div5$.
28. $\bar{1}\cdot6\div4$.
29. $\overline{10}\cdot1\div9$.
30. $\bar{3}\cdot1\div(-2)$.

EXERCISE XIII. e.

Find correct to 3 significant figures the values of

1. $0\cdot456\times7\cdot12$.
2. $0\cdot078\times3\cdot153$.
3. $0\cdot00461\times0\cdot912$.
4. $0\cdot000512\times712\cdot8$.
5. $47\cdot5\div89\cdot1$.
6. $0\cdot4523\div9\cdot165$.
7. $0\cdot2103\div0\cdot0172$.
8. $0\cdot3167\div0\cdot0715$.

9. $(0{\cdot}156)^2$. **10.** $(0{\cdot}0137)^3$.

11. $\sqrt{0{\cdot}783}$. **12.** $\sqrt{0{\cdot}0783}$.

13. $\sqrt[3]{0{\cdot}07825}$; $\sqrt[3]{0{\cdot}007825}$; $\sqrt[3]{0{\cdot}0007825}$.

14. $\dfrac{1}{7{\cdot}93}$. **15.** $\dfrac{1}{37{\cdot}4}$.

16. $\dfrac{0{\cdot}862 \times 0{\cdot}437}{3{\cdot}14}$. **17.** $\dfrac{415{\cdot}2 \times 0{\cdot}0302}{2468}$.

18. $\dfrac{0{\cdot}00531 \times 416{\cdot}2}{0{\cdot}1018}$. **19.** $\sqrt[3]{\dfrac{1}{915{\cdot}2}}$.

20. $\sqrt[4]{\dfrac{1}{0{\cdot}873}}$. **21.** $\sqrt[3]{\dfrac{4{\cdot}156 \times 0{\cdot}00612}{0{\cdot}0891}}$.

22. $0{\cdot}713 \times \sqrt[5]{0{\cdot}416}$. **23.** $(5{\cdot}16)^{1{\cdot}2}$; $(0{\cdot}516)^{1{\cdot}2}$.

24. $(1{\cdot}472)^{-0{\cdot}8}$. **25.** $(0{\cdot}628)^{-3{\cdot}4}$.

26. $(0{\cdot}372)^{0{\cdot}41}$. **27.** $(0{\cdot}0682)^{0{\cdot}08}$.

28. $(0{\cdot}0072)^{2{\cdot}34}$. **29.** $(0{\cdot}1)^{0{\cdot}1}$.

30. Find approximately the value of (i) $(0{\cdot}2)^{20}$; (ii) $(0{\cdot}9)^{100}$.

EXERCISE XIII. f.

Evaluate to 3 significant figures

1. $(415{\cdot}7)^2 - (112{\cdot}6)^2$. **2.** $(0{\cdot}718)^2 + (0{\cdot}513)^2$.

3. $\dfrac{0{\cdot}273 + (0{\cdot}864)^2}{10\frac{1}{4}}$. **4.** $\dfrac{5 + \sqrt{8{\cdot}67}}{10 - \sqrt{86{\cdot}7}}$.

5. From the formula $T_1 = T_2\left(\dfrac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}}$, find T_1 when $T_2 = 315$, $p_1 = 7{\cdot}16$, $p_2 = 5{\cdot}95$, $\gamma = 1{\cdot}5$.

6. Find the value of $\pi(R^2 - r^2)h$ when $R = 7{\cdot}16$, $r = 3{\cdot}12$, $h = 2{\cdot}59$.

7. The area of a triangle whose sides are of lengths a, b, c is $\sqrt{\{s(s-a)(s-b)(s-c)\}}$ where $s = \frac{1}{2}(a+b+c)$; find the area of a triangle whose sides are 14·5, 16·2, 18·1 inches.

8. The total surface of a closed circular cylinder is $2\pi r^2 + 2\pi rl$; find the area if $r = 7{\cdot}52$, $l = 5{\cdot}83$. [Use factors.]

9. Find the value of $A \,.\, e^{kt}$ when $A = 4{\cdot}7$, $e = 2{\cdot}718$, $k = 2{\cdot}4$, $t = \frac{1}{2}$.

10. (i) Find the area of a circle of radius 0·47 inch.
(ii) Find the radius of a circle of area 0·76 sq. inch.

11. In the formula $t=2\pi\sqrt{\frac{l}{32\cdot2}}$, find t if $l=1\cdot7$.

12. Find R from the formula $R=\frac{4l\rho}{\pi d^2}$ where $\rho=1\cdot7\times10^{-6}$, $l=94{,}300$, $d=0\cdot25$.

13. Find n from the formula $n=\frac{aH^{\frac{5}{4}}}{P^{\frac{1}{2}}}$ when $a=22\cdot25$, $H=6$, $P=180$.

14. The time t seconds for pneumatic transmission through a tube l feet long, d feet in diameter under a pressure of P lb. per sq. inch, is $t=0\cdot000482\sqrt{\frac{l^3}{Pd}}$; find t if $l=42\cdot5$, $d=0\cdot24$, $P=8\cdot57$.

15. A circular drain of fall 1 in 100 and diameter d feet can carry off $\frac{454\cdot3d^3}{1\cdot005+\sqrt{d}}$ cu. ft. of water per minute. Find in tons the weight of water removed in an hour by a drain 1·4 feet in diameter. [1 cu. ft. of water weighs 62·3 lb.]

16. Find f from the formula $f=1\cdot925\times10^{-8}q^2d^4$ when $q=500$, $d=0\cdot3162$.

17. Find the value of $71\cdot5e^{-kt}$ where $e=2\cdot718$, if (i) $k=0\cdot25$, $t=3$; (ii) $k=0\cdot25$, $t=100$.

18. From the formula for drilling mild steel, $P=35{,}500D^{0\cdot7}T^{0\cdot6}$, where P is the thrust in lb. required for a drill of diameter D inches to secure a feed of T inches per revolution, calculate P if $D=\frac{7}{8}$, $T=\frac{1}{80}$.

19. If $e=1-r\gamma^{-1}$, find e when $\gamma=1\cdot37$, $r=0\cdot4$.

20. Find, to three significant figures, the value of x if

$$x^3=r^3-\frac{14\cdot64}{r}\text{ and }\pi r^2=43\cdot7.$$

21. If $\frac{p_2}{p_1}=\sqrt{\left\{1-\frac{V^2l}{74{,}300{,}000d}\right\}}$, find $\frac{p_2}{p_1}$ if $V=88$, $l=5280$, $d=0\cdot75$.

22. Find the value of (i) $\left(1+\frac{1}{10^2}\right)^{10^2}$; (ii) $\left(1+\frac{1}{10^6}\right)^{10^6}$, given that $1\cdot000001=10^{0\cdot000000434294}$.

23. The formula for compound interest states that £P amounts to £$P\left(1+\frac{r}{100}\right)^n$ in n years at r per cent. Find, as accurately as your tables will allow, the amount for

(i) £1000 at 5% for 15 years;
(ii) 15s. 6d. at 6% for 5 years;
(iii) one penny at 4½% for 800 years.

24. What sum invested for a child at birth will amount to £1000 when he comes of age if it is invested at 6% compound interest ?

25. Simplify $$\frac{1+(1{\cdot}023)^{-10}}{1+(1{\cdot}023)^{-1}}.$$

26. The present value £P of an annuity of £A per annum to last for n years is given by $P=\frac{100A}{r}\left\{1-\left(1+\frac{r}{100}\right)^{-n}\right\}$, allowing r per cent. per annum compound interest.

(i) Find the present value of an annuity of £400 to last for 10 years, interest 6%.

(ii) A man is offered a choice between a single payment of £6000 down or a life annuity of £600 ; his expectation of life is 15 years ; which should he choose, reckoning interest at 6 per cent. ?

(iii) What life annuity can a man purchase with £5000 when his expectation of life is 20 years, reckoning interest at 4 per cent. ?

THE SLIDE RULE.

For remarks on the use of the slide rule, the reader is referred to the Introduction.

EXERCISE XIII. g.

With the aid of a slide rule, find to three significant figures the values of

1. $4{\cdot}62\times1{\cdot}6$. **2.** $4{\cdot}62\times1{\cdot}67$. **3.** $9{\cdot}71\times8{\cdot}5$.

4. $97{\cdot}1\times0{\cdot}85$. **5.** $312\div20{\cdot}7$. **6.** $3759\div177$.

7. $4156\div112$. **8.** $0{\cdot}0793\div7{\cdot}12$. **9.** $\sqrt{13{\cdot}9}$.

10. $\sqrt{1438}$. **11.** $\frac{1{\cdot}46\times7{\cdot}52}{7{\cdot}14}$. **12.** $\frac{4{\cdot}56\times6{\cdot}52}{9{\cdot}13\times1{\cdot}25}$.

13. $\frac{8{\cdot}35}{6{\cdot}91\times5{\cdot}13}$. **14.** $\frac{6{\cdot}31\times5{\cdot}64\times7{\cdot}12}{8{\cdot}15\times3{\cdot}14}$. **15.** $\sqrt{11{\cdot}7\times1{\cdot}56}$.

16. $\sqrt{\frac{62{\cdot}7}{21{\cdot}9}}$. **17.** $\left(\frac{8{\cdot}63}{5{\cdot}78}\right)^2$. **18.** $\sqrt{\frac{147\times1{\cdot}19}{27{\cdot}8}}$.

For additional examples, see Exercise XIII. b.

* LOGARITHMIC THEORY AND NOTATION.

If $x = a^p$, p is defined as the logarithm of x to base a. This may be written in the form

$$p = \log_a x.$$

So far we have only considered logarithms to base 10, which are the ordinary kind of logarithm for numerical work. If there is no doubt as to what base is being used, the symbol is abbreviated and simply written as $\log x$. In numerical work $\log x$ would be used to stand for $\log_{10} x$.

We shall now prove some of the results which have been already used.

I. $\log_a(xy) = \log_a x + \log_a y$.

Let $\qquad \log_a x = p$ and $\log_a y = q$;

$$\therefore\ x = a^p \text{ and } y = a^q;$$

$$\therefore\ xy = a^p \times a^q = a^{p+q};$$

$$\therefore\ \log_a(xy) = p + q = \log_a x + \log_a y.$$

II. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$.

Proceed as in I.

Then $\qquad \dfrac{x}{y} = \dfrac{a^p}{a^q} = a^{p-q}$;

$$\therefore\ \log_a\left(\frac{x}{y}\right) = p - q = \log_a x - \log_a y.$$

III. $\log_a(x^n) = n \log_a x$.

Proceed as in I.

Then $\qquad x^n = (a^p)^n = a^{pn}$;

$$\therefore\ \log_a(x^n) = pn = n \log_a x.$$

Note.—The result, given in III., holds with the same proof for any fractional or negative value of n.

Thus $\log \sqrt[n]{x} = \log\left(x^{\frac{1}{n}}\right) = \dfrac{1}{n}\log x$.

* This section may be omitted at a first reading: it is however advisable to take it before the chapters on Empirical Formulae and Nomography.

Example XI. If $W = \frac{Kbd^2}{l}$, find W when $K=16$, $b=5\cdot3$ $d=7\cdot14$, $l=90$.

$$\log W = \log \frac{Kbd^2}{l}$$
$$= \log K + \log b + 2\log d - \log l$$
$$= 1\cdot2041 + 0\cdot7243 + 2(0\cdot8537) - 1\cdot9542$$
$$= 1\cdot6816;$$
$$\therefore\ W = 48\cdot04.$$

1·2041
0·7243
1·7074
3·6358
1·9542
1·6816

Example XII. If $H = p\,.\,v^{1\cdot13}$, find $\log H$.

$$\log H = \log(p\,.\,v^{1\cdot13})$$
$$= \log p + \log(v^{1\cdot13})$$
$$= \log p + 1\cdot13\log v.$$

Example XIII. If $\log 5 = 0\cdot699$, find $\log 25$.

$$\log 25 = \log(5^2)$$
$$= 2\log 5 = 2\times0\cdot699$$
$$= 1\cdot398.$$

Example XIV. Find x, given that $4\cdot7^x = 0\cdot83$.

$$\log(4\cdot7)^x = \log 0\cdot83;$$
$$\therefore\ x\log 4\cdot7 = \log 0\cdot83;$$
$$\therefore\ x = \frac{\log 0\cdot83}{\log 4\cdot7} = \frac{\bar{1}\cdot9191}{0\cdot6721}$$
$$= \frac{-1+\cdot9191}{0\cdot6721} = -\frac{0\cdot0809}{0\cdot6721}$$
$$= -\frac{10^{\bar{2}\cdot9079}}{10^{\bar{1}\cdot8275}}$$
$$= -10^{\bar{1}\cdot0804}$$
$$= -0\cdot1203;$$
$$\therefore\ x = -0\cdot120.\quad \textit{Answer.}$$

$\bar{2}\cdot9079$
$\bar{1}\cdot8275$
$\bar{1}\cdot0804$

EXERCISE XIII. h.

[Unless otherwise stated, $\log x$ means $\log_{10} x$.]

1. Simplify

(i) $\log 8 - \log 2$; (ii) $\log 8 \div \log 2$;

(iii) $\log 1 \div \log 2$; (iv) $\log(\sqrt{3}) \div \log 3$.

2. Simplify

(i) $\dfrac{\log 25}{\log 5}$; (ii) $\dfrac{\log 100}{\log 10}$; (iii) $\log 2\frac{1}{4} \div \log 1\frac{1}{2}$.

3. Without using tables, state which is the greater:

(i) $\log 1 + \log 3$ or $\log (1+3)$;

(ii) $\log 2 + \log 4$ or $\log (2+4)$.

4. Without using tables, state which is the greater:

(i) $\log 2 - \log 1$ or $\log (2-1)$;

(ii) $\log 4 - \log 2$ or $\log (4-2)$;

(iii) $\log 6 - \log 3$ or $\log (6-3)$.

5. Find x if $$\frac{\log x}{\log 2} = \frac{\log 4}{\log 16}.$$

6. Simplify

(i) $\dfrac{\log (x^2)}{\log x}$; (ii) $\dfrac{\log \sqrt{x}}{\log x}$; (iii) $\log xy^2 - \log x$.

7. Simplify

(i) $\log x + \log \dfrac{1}{x}$; (ii) $\log \dfrac{x}{y} + \log y$.

8. If $x^4 = 1000$, find $\log x$.

9. Express the following in a form which does not involve the logarithmic notation:

(i) $\log x + \log y = \log 5$; (ii) $\log x + 3 \log y = 2$;

(iii) $x \log 3 = 3$; (iv) $3 \log x - 2 \log y = 1$;

(v) $x \log 5 = y \log 6$; (vi) $x \log 5 = y \log 6 + 2$.

10. Solve the equations

(i) $2^x = 8192$; (ii) $14{\cdot}5^n = 210{\cdot}3$; (iii) $7{\cdot}81^n = 0{\cdot}128$.

11. If $pv^n = 475$, find n when $p = 3{\cdot}62$, $v = 98{\cdot}5$.

12. If $h = 391Q^n$, find n when $h = 7{\cdot}95$, $Q = 0{\cdot}125$.

13. If $2^x = 3^y$, find the ratio of x to y.

14. (i) What is the error per cent. in taking $\log (1+x)$ equal to $0{\cdot}434x$ when $x = 0{\cdot}1$?

(ii) Hence find an approximate solution of the equation

$$\log (1+x) = \frac{\sqrt{x}}{10}.$$

15. What are the values of

$\dfrac{\log 4 - \log 1}{\log 2 - \log 1}$; (ii) $\dfrac{\log 8 - \log 1}{\log 2 - \log 1}$; (iii) $\dfrac{\log 9 - \log 1}{\log 3 - \log 1}$?

16. Find ϕ from the formula $\phi = \dfrac{\log t - \log 273}{\log 2{\cdot}718}$ when $t = 373$.

17. £P invested at r per cent. compound interest for n years amounts to £$P\left(1 + \dfrac{r}{100}\right)^n$.

In how many years will a sum of money invested at (i) 3%, (ii) 5%, compound interest, double itself ?

Show that the rough formula $n = \dfrac{70}{r}$ gives a good approximation in each case.

Prove that the correct formula is $n = \dfrac{\log 2}{\log\left(1 + \dfrac{r}{100}\right)}$.

18. Assuming that the population of a country increases according to a compound interest law, find the annual percentage increase in a country where the population increases by 50 per cent. in ten years.

19. If the barometer reading is h_1 inches at the foot and h_2 inches at the summit of a mountain, the height of the mountain is 60,360 $(\log h_1 - \log h_2)$ feet ; find the height of a mountain for which $h_1 = 30$, $h_2 = 25$.

20. Evaluate $\log_7 9$.

CHAPTER XIV.

RATIO AND PROPORTION. VARIATION.

THE *ratio* of two numbers a, b is measured by the fraction $\frac{a}{b}$ and is often written $a : b$.

The ratio $na : nb$ equals the ratio $a : b$, because $\frac{na}{nb} = \frac{a}{b}$.

The ratio-relation $x : y = a : b$ may be written either as

$$\frac{x}{y} = \frac{a}{b} \quad \text{or} \quad \frac{x}{a} = \frac{y}{b}.$$

Such ratio-relations as $x : y : z : u : v \ldots = a : b : c : d : e \ldots$ may be written either as

$$\frac{x}{y} = \frac{a}{b} \quad \text{and} \quad \frac{y}{z} = \frac{b}{c} \quad \text{and} \quad \frac{z}{u} = \frac{c}{d}, \text{ etc.,}$$

or as

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{u}{d} = \ldots .$$

Example I. Simplify the ratio $4\frac{1}{5} : 3\frac{1}{2}$.

$$\frac{4\frac{1}{5}}{3\frac{1}{2}} = \frac{4\frac{1}{5} \times 10}{3\frac{1}{2} \times 10} = \frac{42}{35} = \frac{6}{5};$$

$\therefore$ the ratio equals $6 : 5$.

Example II. If $x : y = 4 : 7$, find the value of the ratio $(x + 2y) : (4x - y)$.

First Method.

Suppose $x = 4a$, then $y = 7a$;

$$\therefore \frac{x + 2y}{4x - y} = \frac{4a + 14a}{16a - 7a} = \frac{18a}{9a} = \frac{2}{1}.$$

$\therefore$ **the ratio equals 2 : 1 or 2.**

Second Method.

$$\frac{x+2y}{4x-y}=\frac{\frac{x}{y}+2}{\frac{4x}{y}-1}=\frac{\frac{4}{7}+2}{\frac{16}{7}-1}=\frac{18}{7}\div\frac{9}{7}=2.$$

Example III. Given

$$x:y=2:3 \quad\text{and}\quad y:z=5:7 \quad\text{and}\quad z:w=3:2,$$

express in its simplest form $x:y:z:w$.

$$\frac{x}{2}=\frac{y}{3} \quad\text{and}\quad \frac{y}{5}=\frac{z}{7} \quad\text{and}\quad \frac{z}{3}=\frac{w}{2};$$

$$\therefore\ \frac{x}{10}=\frac{y}{15} \quad\text{and}\quad \frac{y}{15}=\frac{z}{21} \quad\text{and}\quad \frac{z}{21}=\frac{w}{14};$$

$$\therefore\ \frac{x}{10}=\frac{y}{15}=\frac{z}{21}=\frac{w}{14};$$

$$\therefore\ x:y:z:w=10:15:21:14.$$

EXERCISE XIV. a.

1. Simplify the following ratios :

(i) $3\frac{1}{3}:1\frac{1}{4}$; (ii) $a:(a+\frac{1}{2}a)$;

(iii) $\frac{a}{b}:ab$; (iv) $\left(1+\frac{1}{x}\right):(1+x)$;

(v) $15x^2y:12xy^2$; (vi) $10^2:20^2$;

(vii) $10^6:10^3$; (viii) $(a^2-b^2):(a^2-ab)$;

(ix) a yards : b feet; (x) x halfcrowns : y shillings.

2. If $3x+5y=7x-13y$, find $x:y$.

3. If $a:b=3:5$, evaluate $(a+b):(a-b)$.

4. If $\frac{x+11}{y+7}=\frac{11}{7}$, prove $\frac{x}{y}=\frac{11}{7}$.

5. Express in ratio form the following equations :

(i) $ab=cd$; (ii) $p^2=qr$; (iii) $(a+b)(c+d)=(p+q)(e+f)$;
(iv) $xyz=pqr$; (v) $abc=de$.

6. Two numbers are in the ratio $a:b$; the first is x; what is the second ?

7. y is the result of increasing x by 10 per cent.; find $x:y$.

8. By deducting discount at the rate of 3d. in the shilling a bill of £x is reduced to £y; find $x:y$.

9. A man's salary was £x before the war and has been increased since the war by £y. In what ratio has it been altered ?

10. $ABCD$ is a straight line; $AB=x''$, $BC=2x''$, $CD=4x''$; what is the ratio of AC to BD ?

11. The rents of two houses used to be in the ratio $a:b$; but the first has been raised 20 per cent. and the second 50 per cent.; what is their present ratio ?

12. Two partners share their profits in the ratio $a:b$; the profits are £p; how much does each receive ?

13. The ratio of the areas of two triangles is $a:b$ and the ratio of their heights is $p:q$; what is the ratio of their bases ?

14. (i) Find two numbers in the ratio $7:5$, whose difference is 8.

(ii) Find two numbers in the ratio $p:q$, whose difference is c.

15. The line AB is divided externally at P in the ratio $5:2$; $AB=x''$; find AP.

16. The ratio of the weights of two circular cylinders of the same material is $3:2$ and the ratio of their diameters is $3:4$; find the ratio of their heights.

17. AB is divided externally at P in the ratio $3:5$ and externally at Q in the ratio $9:7$; find $PQ:AB$.

18. A measuring rod contracts in the ratio $a:b$ where $a<b$; it is used to measure a line whose real length is x inches; what is the length obtained and what is the error per cent. ?

19. If x is positive, is the ratio $(3+x):(5+x)$ greater or less than the ratio $3:5$?

20. If x is positive, is the ratio $(8+x):(7+x)$ greater or less than the ratio $8:7$?

21. If $\frac{1}{a}+\frac{1}{b}=3\left(\frac{1}{a}-\frac{1}{b}\right)$, find $a:b$.

22. If $3x-y=2z$ and $y=6(z-x)$, find $x:y:z$.

23. A chain of length l feet hangs over a small rough peg; the portions on each side are in the ratio $a:b$; if $a>b$, find the depth of the mid-point of the chain below the peg.

24. $ABCD$ is a straight line; B divides AC in the ratio $p:q$; C divides BD in the ratio $x:y$; find (i) $AB:CD$, (ii) $AB:BD$.

25. If $(8x-2y)^2=(3x+4y)^2$, find $x:y$.

26. If $4x^2+9y^2=12xy$, find $x:y$.

27. The perimeter of a right-angled triangle is four times the shortest side; find the ratio of the other two sides.

28. The ratio of the radii of two circles is $a:b$ and the ratio of their areas is $(a-x):(b-x)$; express x in terms of a, b.

29. If $\frac{x+y}{x-y}=\frac{7}{4}$, evaluate $\frac{x^2+y^2}{x^2-y^2}$.

30. If x and y are each increased in the ratio $a:b$, in what ratio is $\frac{x+y}{x-y}$ altered?

31. At a height of h feet it is possible to see a distance of $\sqrt{\frac{3h}{2}}$ miles; an aeroplane climbs from 4000 feet to 9000 feet; in what ratio does the pilot increase his range of view?

32. The horse-power H required to drive W lb. of water per second through a small circular aperture of area A sq. inches is given by $H=0\cdot0015\frac{W^3}{A^2}$. In what ratio must the horse-power be altered if the radius of the aperture is doubled and if twice as much water per second is required?

33. The indicated horse-power H required to produce a speed of V knots in a paddle steamer is given by the approximate formula $H=\frac{A\,.\,V^3}{70}$, where A is its greatest section in sq. yards. (i) In what ratio must the horse-power be increased to raise the speed from 8 to 12 knots? (ii) The values of A for two steamers are in the ratio 6 : 5 and their speeds are in the ratio 5 : 2; what is the ratio of their indicated horse-powers?

34. If a carrier is driven through a tube l feet long, d feet in diameter under a pressure of P lb. per sq. inch, the time of transmission is approximately $0\cdot0048\sqrt{\frac{l^3}{Pd}}$ seconds; what is the effect on the time of transmission if the length is increased in the ratio 3 : 2, the pressure in the ratio 5 : 2, and the diameter decreased in the ratio 3 : 5?

35. If $a > b > x > 0$, arrange the ratios $(a-x):(b-x)$ and $(a+x):(b+x)$ and $a:b$ in ascending order of magnitude.

36. The ratio of a man's expenditure to his income is $a:b$; what is the ratio of his savings to his expenditure?

37. The ratio of the male to the female voters at an election is $a:b$. If c fewer men and d fewer women had voted, the ratio would have been $p:q$. What was the total number of votes cast in terms of these letters?

38. If h is small compared with 1000, find an approximate value of the ratio $[(1000+h)^2-1000^2]:h$.

39. If h is small compared with x and y, find an approximate value of the ratio $[(x+h)^2-x^2]:[(y+h)^2-y^2]$.

40. The incomes of A and B are in the ratio $5:4$; their expenditures are in the ratio $6:5$ and their savings are in the ratio $10:7$. Find the ratio of A's income to B's expenditure.

If $\frac{a}{b}=\frac{c}{d}$, the numbers a, b, c, d are said to be *in proportion*, and d is called the *fourth proportional* to a, b, c.

If $\frac{a}{b}=\frac{b}{c}$, c is called the *third proportional* to a, b and b is called the *mean proportional* between a, c.

If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=$ etc., $a, b, c, d, e, \ldots$ are said to be in *continued proportion*.

The ratio-equality $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{w}{d}=\ldots$ is often written in the form $x:y:z:w:\ldots=a:b:c:d:\ldots$.

Example IV. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a}{b}=\frac{a+c}{b+d}=\frac{a-c}{b-d}$.

Let $$\frac{a}{b}=\frac{c}{d}=k;$$

$$\therefore\ a=bk;\quad c=dk;$$

$$\therefore\ \frac{a+c}{b+d}=\frac{bk+dk}{b+d}=\frac{k(b+d)}{b+d}=k=\frac{a}{b}$$

and $$\frac{a-c}{b-d}=\frac{bk-dk}{b-d}=k=\frac{a}{b}.$$

EXERCISE XIV. b.

1. Find the fourth proportional to

(i) 2, 3, 4; (ii) ab, bc, cd; (iii) $\frac{1}{x}, y, \frac{1}{y}$.

2. Find the third proportional to

(i) 2, 3; (ii) ab, bc; (iii) $\frac{1}{x}, y$.

3. Find a mean proportional between

(i) 3, 12; (ii) a^3b, ab^3; (iii) a, b.

4. Fill up the gaps in

$$\frac{x}{8}=\frac{y}{3}=\frac{x+y}{\quad}=\frac{x-y}{\quad}=\frac{5x-12y}{\quad}.$$

5. If $x : y : z = 3 : 2 : 7$, find $\frac{x+3y}{y+z}$ and the ratios $\frac{1}{x} : \frac{1}{y} : \frac{1}{z}$.

6. If $\frac{x+y+z}{11} = \frac{x+y-z}{8} = \frac{x-y}{5}$, find $x : y : z$.

7. If $\frac{x}{5} = \frac{y+z}{11} = \frac{x-y+z}{2}$, find $x : y : z$.

8. If $\frac{a}{b} = \frac{c}{d}$, prove $\frac{a}{b} = \frac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}$.

9. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove $\frac{3a-5c}{3b-5d} = \frac{2c+7e}{2d+7f}$.

10. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, fill up the gaps in

$$\frac{a}{b} = \frac{a+c+e}{\dots} = \frac{\dots}{b-d+f} = \frac{10a-7c+2e}{\dots} = \frac{\frac{a}{2}+\frac{e}{3}}{\dots}.$$

11. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{ac}{bd} = \frac{a^2-3c^2}{b^2-3d^2}$.

12. If $\frac{x+y}{11} = \frac{x-y}{5}$, find $\frac{x^2-y^2}{xy}$.

13. If $6(x^2-y^2) = 5xy$, find $\frac{x+y}{x-y}$.

14. Solve $\frac{x-2y+3}{5} = \frac{3x+2y+1}{7} = \frac{2x+y}{8}$.

15. Solve $\frac{x^2-2x+3}{x^2+2x-3} = \frac{x+5}{x-5}$.

16. X, Y are points on the sides AB, AC of the triangle ABC such that $\frac{AX}{XB} = \frac{AY}{YC}$; prove that (i) $\frac{AX}{AB} = \frac{AY}{AC}$; (ii) $\frac{BX}{AB} = \frac{CY}{AC}$.

17. O is a point inside the triangle ABC; AO is produced to cut BC at D; prove that (i) $\frac{\triangle BAD}{\triangle CAD} = \frac{BD}{CD}$; (ii) $\frac{\triangle BAO}{\triangle CAO} = \frac{BD}{CD}$.

18. If $\frac{a}{b} = \frac{c}{d}$, prove that

(i) $\frac{a}{b} = \sqrt{\left(\frac{a^2+c^2}{b^2+d^2}\right)}$; (ii) $\frac{a-c}{b-d} = \sqrt[5]{\left(\frac{a^5+c^5}{b^5+d^5}\right)}$.

19. If b is the mean proportional between a, c, prove that $\frac{a^2}{b^2} = \frac{a}{c}$. Interpret this geometrically.

20. z is a function of x, y, and its values are shown in the following double-entry table:

y \ x	0	1	2	3
0	0	2	4	6
1	1	4	9	16
2	2	6	14	26
3	3	8	19	36

Find the probable values of z when

(i) $x=1,\ y=1{\cdot}5$; (ii) $x=2{\cdot}5,\ y=2$;
(iii) $x=1{\cdot}5,\ y=1{\cdot}5$; (iv) $x=2{\cdot}2,\ y=2{\cdot}6$.

21. The values of a function z of two variables x, y are shown in the following table:

y \ x	1	2	3
1	0·070	0·087	0·104
2	0·122	0·139	0·156
3	0·174	0·191	0·208

Find the probable values of z when

(i) $x=1{\cdot}5,\ y=1{\cdot}5$; (ii) $x=2{\cdot}6,\ y=2{\cdot}4$.

Summary of Results.

If
$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots,$$
then
$$\text{each fraction}\equiv\frac{pa+qc+re+\dots}{pb+qd+rf+\dots}$$
$$\equiv\sqrt[n]{\left\{\frac{pa^n+qc^n+\dots}{pb^n+qd^n+\dots}\right\}}.$$

VARIATION.

If a man is walking at a uniform rate of 4 miles an hour, he walks 1 mile in 15 minutes,

2 miles in 30 minutes,

3 miles in 45 minutes, and so on.

The distance he walks is directly proportional to the time taken.

If he walks x miles in t minutes, we have

$$x = \frac{t}{15} \quad \text{or} \quad \frac{x}{t} = \frac{1}{15}.$$

In this case, x is said to *vary directly* as t.

This is written $x \propto t$.

The result of plotting x against t is a straight line through the origin.

Suppose next there are a number of rectangles each of area 60 sq. in., but of different shapes.

If the breadth is 2″, the length is 30″.

If the breadth is 3″, the length is 20″.

If the breadth is 4″, the length is 15″, and so on.

The breadth is inversely proportional to the length.

If the breadth is b in. and the length is l in.,

$$b = \frac{30}{l} \quad \text{or} \quad \frac{b}{\frac{1}{l}} = 30.$$

In this case, b is said to *vary inversely as* l.

This is the same as saying that b varies directly as $\frac{1}{l}$ or

$$b \propto \frac{1}{l}.$$

And the result of plotting b against $\frac{1}{l}$ is a straight line through the origin.

Conversely, if we are given a table of values for x and y, and if the graph is a straight line through the origin when y is plotted against x, then $y \propto x$.

If the graph is a straight line through the origin when y is plotted against $\frac{1}{x}$, then $y \propto \frac{1}{x}$.

Example V. The weights of a number of circular discs of the same material and thickness but different diameters are given in the following table:

Diameter in cm. $= d =$ 2	3	4	5	6	7	8	9	10
Weight in gr. $= w =$ 20	45	80	125	180	245	320	405	500

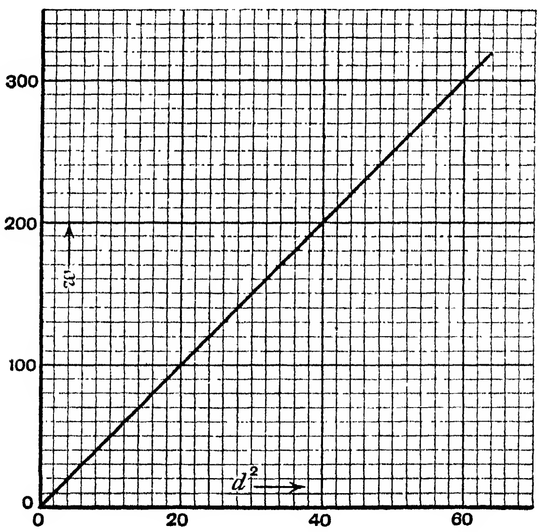
Show that the weight varies as the square of the diameter, and plot w against d^2.

The table connecting d^2 and w is

d^2	4	9	16	25	36	49	64	81	100
w	20	45	80	125	180	245	320	405	500

from which we see that $w = 5d^2$.

$$\therefore\ w \propto d^2.$$



The result of plotting w against d^2 gives a straight-line graph passing through the origin, which is the graphical test that w is proportional to d^2.

Example VI. It is stated that if the velocity of a stream of water is determined by a water-pressure gauge, the velocity varies as the square root of the height in the gauge.

(i) If a speed of 8 feet a second causes a height of 12 inches, what is the speed when the recorded height is 4 inches ?

(ii) If the velocity is v feet per second when the height recorded is h inches, find the equation connecting v and h.

(i) We have $$v \propto \sqrt{h}.$$

Now $v=8$, $h=12$ is one pair of corresponding values ; and it is required to find the value of v which corresponds to $h=4$.

$\therefore$ when $h=4$, the value of v is given by

$$\frac{v}{8}=\frac{\sqrt{4}}{\sqrt{12}};$$

$$\therefore\ v=\frac{8\sqrt{4}}{\sqrt{12}};$$

$$\therefore\ v=4{\cdot}62 \text{ feet per sec.}$$

(ii) Also we have $$\frac{v}{8}=\frac{\sqrt{h}}{\sqrt{12}};$$

$$\therefore\ v=\frac{8}{\sqrt{12}}\cdot\sqrt{h};$$

$$\therefore\ v=2{\cdot}31\,.\,\sqrt{h} \text{ approximately.}$$

EXERCISE XIV. c.

1. A man walking 3 miles an hour travels y yards in x minutes :

(i) Does y vary directly as x ?
(ii) What is the effect on y of doubling x ?
(iii) What is the effect on x of doubling y ?
(iv) What equation connects x and y, and sketch its graph ?

2. x men take y days to level a large field which it would take 12 men 40 days to level :

(i) Does y vary directly as x ?
(ii) What is the effect on y of doubling x ?
(iii) What is the effect on x of halving y ?
(iv) Express y in terms of x, and sketch its graph.

3. If y varies as x, (i) what is the effect on y of trebling x? (ii) what is the effect on x of halving y? (iii) what is the shape of the graph of y?

4. If y varies inversely as x, (i) what is the effect on y of doubling x? (ii) what is the effect on x of multiplying y by 10? (iii) what is the shape of the graph of y? (iv) if x is the length and y the height of a rectangle, what can you say about the area of the rectangle?

5. A rectangle of area 60 sq. cm. is x cm. long and y cm. wide.

(i) What is the variation-relation between y and x?
(ii) Complete the table:

y	10	20	30	40	50
x					
$\frac{1}{x}$					

(iii) Plot $\frac{1}{x}$ against y, taking 3″ as the unit on the $\frac{1}{x}$-axis.

6. If y varies as x^2, what is the effect (i) on x of dividing y by 4; (ii) on y of dividing x by 4? What is the shape of the graph of y?

7. Construct a table of values of x and y to illustrate the cases where

(i) y varies as x;
(ii) y varies inversely as x;
(iii) y varies as x^3.

Sketch the graph of y in each case.

8. To produce a fixed intensity of illumination, the candle-power c of a lamp must be increased four-fold when the distance x feet from the light is doubled; what is the variation-relation between c and x?

9. If multiplying x by 10 always produces the effect of dividing y by 100, what is the variation-relation between x and y?

10. If $y = x + x^2$, in what ratio is y increased when x alters from (i) 1 to 2, (ii) 2 to 4, (iii) 4 to 8? Does y vary as any power of x?

11. In the following tables y varies directly or inversely as some power of x; find the relation between y and x in each case.

What method of plotting must be adopted to give straight-line graphs in the various cases ? Carry it out in each case.

(i)	$x=$	-3	-1	0	2	5	
	$y=$	-12	-4	0	8	20	
(ii)	$x=$	1	2	3	4	5	
	$y=$	2	8	18	32	50	
(iii)	$x=$	1	2	5	8	10	12
	$y=$	10	5	2	$1\frac{1}{4}$	1	$\frac{5}{6}$
(iv)	$x=$	1	4	9	25	36	
	$y=$	1	2	3	5	6	

12. A number of triangles each have a base 6 cm. long ; if the height of any one of them is h cm. and its area is A sq. cm., complete the following table; what is the variation between A and h ?

$h=3$	4	5	8	10
$A=$				

13. If y varies as x^2, complete the following table :

$x=0$	1	2	3	5	10
$y=$		12			

14. If y varies as $\frac{1}{x}$, complete the table :

$x=1$	4	10	20
$y=$		0·8	

15. A cube of aluminium whose edge is x cm. weighs y gr.; complete the table :

$x=1$	2	4	5
$y=$			325

16. If y varies as $\sqrt{x}$, complete the table :

$x=0$	1	2	4	25	100
$y=$			4		

17. What is the variation between the variables in the following statements ?

(i) A circle of radius r cm. is of area A sq. cm.

(ii) The side of an equilateral triangle is x inches and its area is A sq. in.

(iii) A train travelling at a uniform rate of x miles an hour takes t hours to go 100 miles.

(iv) The simple interest on £x at 4 per cent. for 3 years is £y.

(v) If 5 pints of water are poured into a cylindrical vessel of diameter d inches, the depth is h inches.

(vi) In a circle of radius 3 cm., an angle of $x°$ at the centre stands on an arc of length y inches.

18. If $y \propto x^2$ and if $y=3$ when $x=2$, find the relation between x and y.

19. If $y \propto \frac{1}{x}$ and if $y=3$ when $x=5$, find the value of y when $x=2$.

20. If $y \propto x^2$, what is the effect on y of doubling x ?

21. If $y \propto \sqrt{x}$, what is the effect on y of increasing x in the ratio 9 : 4 ?

22. If $x \propto \frac{1}{y}$ and if $x \propto z^2$, what is the effect on y of doubling z ?

23. In the given graph, Fig. 8, $y \propto x^2$; find the relation between x and y.

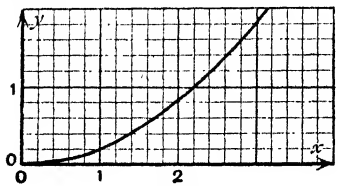


FIG. 8.

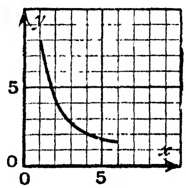


FIG. 9.

24. In the given graph, Fig. 9, y varies inversely as x; find the relation between x and y.

25. If y varies as x and if $y=3$ when $x=2$, represent the relation between y and x by a graph.

26. The area of the surface S sq. in. of a sphere varies as the square of its radius r in. ; its volume V cu. in. varies as the cube of its radius : (i) what is the effect on V of increasing S in the ratio 64 : 1 ? (ii) what is the variation-relation between S and V ?

27. The pressure on the wind-screen of a car travelling v miles an hour on a still day is P lb. per sq. foot. It is found that $P \propto v^2$. Is this consistent with the following measurements ?

$v = 5$	20	30
$P \doteq 0{\cdot}123$	1·97	4·43

Express P in terms of v as accurately as the data permit.

28. The range of view at sea varies as the square root of the height of the observer above sea-level. At a height of 20 feet the look-out can see 5·5 nautical miles; what is the range of vision at a height of 40 feet ? If a height of h feet commands N nautical miles, express N in terms of h.

29. Two hot-water cans are the same shape: one is 8″ high and holds a quart, the other is 2′ high. What does it hold ?

30. Two candlesticks are the same shape; it costs 15 shillings to gild the smaller which is 6″ high; what is the cost of gilding the other which is 15″ high ?

31. In what ratio must the radius of a sphere be increased to double (i) its surface, (ii) its volume ?

32. A lodger pays ninepence for a scuttle of coal 9 inches wide; what would he expect to pay for a scuttle of the same shape 12 inches wide ?

33. It takes x lb. of paint to paint the hull of a battleship 400 ft. long; how much will be required for the hull, similar in shape, of a battleship 530 feet long ?

34. The weights of two spheres are in the ratio 2 : 1 and the densities of the materials of which they are made are in the ratio 1 : 2. Compare the radii.

35. The time of a complete oscillation of a pendulum varies as the square root of its length. If a pendulum of length 100 cm. makes one complete oscillation in 2 seconds, find the time of a complete oscillation of a pendulum of length 25 cm. If the time is t sec. when the length is l cm., express t in terms of l.

36. The danger distance from the muzzle of a gun, within which the hearing may be injured by the firing, varies as the fifth root of the weight of the charge. For a 5-lb. charge of cordite the distance is 10 feet. What is the distance for a charge of 187 lb. of cordite ?

37. A beam supported at its ends carries a fixed load at its mid-point; for beams of the same cross-section and elasticity the sag in the middle varies as the cube of the length. What is the effect on the sag of trebling the length ?

38. (*a*) The weight of a body above ground varies inversely as the square of its distance from the centre of the earth; taking the radius of the earth as 4000 miles, make a table showing the weights of a body which weighs 100 lb. when on the ground at the following heights above the ground: (i) 100 miles, (ii) 1000 miles, (iii) 10,000 miles.

(*b*) Below ground the weight varies directly as its distance from the centre of the earth: make a table showing the weights of the same body at the following depths under ground: (i) 50 miles, (ii) 100 miles, (iii) 1000 miles.

39. If a wire of diameter d mm. fuses when the current in it reaches C ampères, C varies as $d^{\frac{3}{2}}$. The following results were obtained by experiment:

Diameter of copper wire in mm. - - -	0·10	0·51	1·02	1·84
Current fusing wire in ampères - - -	2·55	29·2	82	200

(i) Find the relation between C and d for copper wires as accurately as these data permit.

(ii) What current will fuse a copper wire of diameter 1·5 mm.?

(iii) Fill up the gap in the following table:

Diameter in mm.	Fusing current for copper wire in amp.	Fusing current for tin wire in amp.
0·10	2·55	0·41
1·02	82	

40. If a strong arc light is placed at A at the top of a vertical lamp-post AB, and if P is a point on the level ground, the illumination of the ground at P varies inversely as AP^3.

If $AB=15'$, $BP=20'$, the illumination of the ground at P is one unit, *i.e.* is equivalent to the illumination of a surface by a lamp of one candle-power distant one foot from it. What is the illumination of a point Q on the ground such that $BQ=30'$?

It is said that ordinary small print can be read if the illumination is 0·05 unit. What is the greatest distance from B at which an evening paper on the ground would be legible?

Summary of Results.

y is said to be a "function" of x, if y can be determined when x is known.

Suppose that y is a function of x, and that the values $y_1, y_2, y_3, y_4, \ldots$ of y correspond to the values $x_1, x_2, x_3, x_4, \ldots$ respectively of x. If the ratios $y_1 : y_2 : y_3 : y_4$, etc., are respectively equal to the ratios $x_1 : x_2 : x_3 : x_4$, etc., then y is said to *vary directly* as x, or simply *vary* as x.

This is sometimes written $y \propto x$.

If the ratios $y_1 : y_2 : y_3 : y_4$, etc., are equal to the ratios $x_1^n : x_2^n : x_3^n : x_4^n$, etc., then y is said to *vary directly as the nth power of* x, or $y \propto x^n$.

If the ratios $y_1 : y_2 : y_3 : y_4$, etc., are equal to the ratios $\frac{1}{x_1^n} : \frac{1}{x_2^n} : \frac{1}{x_3^n} : \frac{1}{x_4^n}$, etc., then y is said to *vary inversely* as the nth power of x, or $y \propto \frac{1}{x^n}$.

The statement that y varies as x^n may be expressed in any of the following ways:

(i) $y \propto x^n$.

(ii) $\frac{y_p}{y_q} = \frac{x_p^n}{x_q^n}$, where x_p, y_p are any pair of corresponding values of x and y, and x_q, y_q are any other pair.

(iii) $\frac{y_1}{x_1^n} = \frac{y_2}{x_2^n} = \frac{y_3}{x_3^n} = \ldots = k$ (say).

(iv) $y = k \cdot x^n$, where k is a *constant* number independent of the values of x and y.

(v) If x is altered in any ratio $\lambda : 1$, then y is altered in the ratio $\lambda^n : 1$.

(vi) The result of plotting y against x^n gives a straight line through the origin.

JOINT VARIATION.

First suppose we have a number of circular cylinders each of radius 5 in., but of different heights.

If the volume of one of these cylinders of height h in. is V cu. in., then

$$V = 25\pi h.$$

For this set of cylinders, $V \propto h$.

Next suppose we have a number of circular cylinders each of height 7 in., but of different radii.

If the volume of one of these cylinders of radius r in. is V cu. in., then

$$V = 7\pi r^2.$$

For this set of cylinders, $V \propto r^2$.

These facts can be expressed by saying that

$V \propto h$ when r is constant

and $V \propto r^2$ when h is constant.

Lastly, if both r and h vary, so that the cylinder may be of any height and radius,

$V = \pi r^2 h$

or $V \propto r^2 h$ when both r and h vary.

This example illustrates the general statement that:

if x varies as y when z is constant

and x varies as z when y is constant,

then x varies as yz when y and z both vary,

or x is said to vary jointly as y and z.

Example VII. The force necessary to stop a train in a given distance varies directly as the weight of the train and the square of its velocity and inversely as the distance. A force of 10 tons will stop a train weighing 200 tons and travelling 30 miles an hour in 200 yards. Find the formula which expresses the force F tons in terms of the velocity, v m.p.h., the distance, d feet, for a train weighing W tons.

$$F \propto \frac{Wv^2}{d}.$$

When $W = 200$, $v = 30$, $d = 600$, we have $F = 10$;

$$\therefore\ F : \frac{Wv^2}{d} = 10 : \frac{200 \times 30^2}{600} = 10 : 300 = 1 : 30$$

$$\therefore\ F = \frac{Wv^2}{30d}.$$

Note.—The statement $F \propto \frac{Wv^2}{d}$ can be expressed as follows :

$$F = k \,.\, \frac{Wv^2}{d}, \text{ where } k \text{ is a constant.}$$

In the above example, $k = \frac{1}{30}$.

EXERCISE XIV. d.

1. The volume, V cu. in., of a circular cone of height 6 in. and base-radius r in. is given by $V = 2\pi r^2$. The volume, V cu. in., of a circular cone of base-radius 6 in. and height h in. is given by $V = 12\pi h$. For a circular cone base-radius r in., height h, volume V cu. in.,

 (i) state how V varies if r is constant ;
 (ii) state how V varies if h is constant ;
 (iii) state how V varies if r and h both vary ;
 (iv) find the formula for V in terms of r and h.

2. One end of a cast-iron beam of rectangular section is built into a wall. For a beam 4″ deep, 10′ long, the greatest weight W tons that can be suspended from the other end is given by $W = 0{\cdot}3b$ where b in. is its breadth. For a beam d'' deep, 10′ long, 8″ broad, the greatest weight W tons is given by $W = 0{\cdot}15d^2$.

For a beam 2″ deep, l feet long, 4″ broad, the greatest weight W tons is given by $W = \frac{3}{l}$.

State how W varies

 (i) if d, l are constant ;
 (ii) if b, l are constant ;
 (iii) if b, d are constant ;
 (iv) if b, d, l all vary.

Find the formula for W in terms of b, d, l.

3. A $\frac{1}{4}$-lb. weight is fastened to the end of a string, and is whirled round in a circle. If the speed of the weight is 8 feet a second and the length of the string is l in., the strain in the string is T lb. where $T = \frac{6}{l}$. If the speed of the weight is v feet a second and the length of the string is 6 in., the strain is T lb. where $T = \frac{v^2}{64}$.

State how T varies (i) if v is constant ;
 (ii) if l is constant ;
 (iii) if v and l both vary.

Find the formula for T in terms of v and l. [P.T.O.]

If the string breaks under a strain of 5 lb., what is the maximum speed for a string of length 15 in.? How does v vary if T is constant?

4. A mass W lb. is fastened to the end of a string l feet long and is whirled round at the rate of n revolutions per sec.; the strain in the string is T lb.

If $W=1$, $n=4$, then $T=2\pi^2 l$.
If $W=2$, $l=4$, then $T=\pi^2 n^2$.
If $l=2$, $n=6$, then $T=9\pi^2 W$.

(i) How does T vary if l, n, W all vary?
(ii) Find the formula for T in terms of l, n, W.
(iii) How does n vary if T and W are constant?
(iv) How does l vary if W alone is constant?
(v) What is the effect on T of doubling n, halving l and leaving W unchanged?

5. p varies as t if v is constant, and varies inversely as v if t is constant. Complete the given double-entry table, giving the values of p.

		Values of t.		
		300	350	400
Values of v	40			
	50			
	60			240

6. V varies as x if y is constant and varies as y^2 if x is constant; complete the given double-entry table, giving values of V.

		Values of x.		
		5	10	15
Values of y	1			
	2		24	
	3			

7. $S=\frac{6}{x}$ if $y=2$, and $S=\frac{8}{y^2}$ if $x=3$; what is the simplest expression for S in terms of x and y?

8. Write down relations expressing the following facts :

(i) The area of the curved surface of a cone varies as the length of the slant side if the base-radius is constant, and varies as the base-radius if the length of the slant side is constant. [s, l, r.]

(ii) The pressure of a gas varies directly as the absolute temperature if the volume remains constant, and varies inversely as the volume if the temperature remains constant. [p, T, v.]

(iii) The time taken over a journey varies directly as the distance and inversely as the speed (uniform). [t, d, v.]

(iv) The pressure required to drive x cu. ft. of gas per hour through a pipe varies directly as the length of the pipe and as the square of x, and inversely as the fifth power of the diameter. [p, l, d.]

(v) The square of the time for pneumatic transmission through a tube varies directly as the cube of the length of the tube, and inversely as its diameter and the pressure. [t, l, d, p.]

9. On a railway curve the outer rail is raised above the inner by an amount which varies directly as the gauge and the square of the maximum velocity permitted, and inversely as the radius of the curve. If the gauge is 5 feet, the speed 15 m.p.h. and the radius 200 yards, the elevation is 1·5 inches. Express the elevation d inches in terms of the gauge W feet, the speed v m.p.h. and the radius R feet.

10. Lord Kelvin states that the most economical diameter for a copper wire in an electric circuit varies directly as the square root of the normal current and the fourth root of the cost per horse-power per annum, and inversely as the fourth root of the price of copper per lb. Express this by an equation. In what ratio should the diameter be altered if the cost of horse-power rises 50% and the price of copper rises 200%, and the current is reduced by 25% ?

11. In steamships of a certain type, if the displacement is d tons and if the indicated horse-power is H for a speed of v knots, the cube of H varies directly as the square of d and the ninth power of v. For a displacement of 1600 tons, the I.H.P. is 690 when the speed is 10 knots. Obtain a general formula.

12. The cost of the coal consumed by a steamer travelling at a steady speed varies directly as the distance and as the square of the speed. What relation connects the cost, the speed and the time taken for the journey ? If the coal consumed for a journey of 200 sea miles at a speed of 10 knots cost £80, find the equation connecting the cost, £C, the distance, d sea miles, and the time, T hours.

13. If a mine explodes near a ship, the pressure produced at a point D feet from the charge and at the same level is the sum of two parts, one of which varies directly as the weight of the charge and inversely as the distance, and the other varies directly as the weight of the charge and inversely as the cube of the distance. A charge of 100 lb. No. 1 dynamite produces a pressure of 11,250 lb. per sq. in. at a distance of 10 feet, and a charge of 180 lb. produces a pressure of 12,000 lb. per sq. in. at a distance of 15 feet at the same level. What charge must be used to produce a pressure of 18,000 lb. per sq. in. at a distance of 5 feet at the same level?

14. If x varies directly as y and inversely as the square root of z, and if $x=8$ when $y=4$ and $z=9$, find the equation connecting x, y, z.

15. If $A \propto BC^2$ and $B \propto xy^2$ and $C \propto \frac{y}{x}$, find the relation between A, x, y.

16. If a stone falls in a vacuum, its speed at any moment varies directly as the time since it started; and the distance it has fallen varies as the square of the time; what variation-relation gives the speed in terms of the distance?

17. The velocity of water issuing in a jet varies directly as the weight of water delivered per sec. and inversely as the area of the jet; the horse-power necessary to produce it varies directly as the cube of the weight of water delivered per sec. and inversely as the square of the area. What variation-relation gives the speed in terms of (i) the horse-power and the weight, (ii) the horse-power and the area?

If a 3 H.P. engine can just deliver 20 lb. of water per sec. in a jet of 2 sq. in., what weight of water per sec. is delivered in a jet of area 1 sq. in. by a 6 H.P. engine? Find also what weight of water per sec. is delivered at a speed of 15 feet per sec. by a 5 H.P. engine. [Take 1 cu. ft. of water to weigh 1000 ounces.]

18. The horse-power of a windmill varies directly as the total sail area and the cube of the velocity of the wind. If the sail area is 1000 sq. ft. and the velocity of the wind 15 m.p.h., the horse-power is 9·7. Find the horse-power if the sail area is 1200 sq. ft. and the velocity of the wind 20 m.p.h. Find also a general formula.

19. The time of pneumatic transmission through a tube varies directly as the square root of the cube of its length, and inversely as the square root of the pressure and the square root of the diameter. If a pressure of 9 lb. per sq. inch drives the carrier through a tube 400 feet long and 3 inches in diameter in 3 secs., find the pressure required to drive it through a tube 100 feet long, $2\frac{1}{2}$ inches in diameter, in 1 sec.

20. x varies inversely as y and z; y varies directly as the square root of x and the square root of w. If x increases in the ratio 3 : 2 and z in the ratio 4 : 3, how does w alter?

Summary of Results.

If x varies directly as the pth power of y and inversely as the qth power of z, then

$$x = k \cdot \frac{y^p}{z^q},$$

where k is a constant, independent of x, y, z.

If $x = a$, when $y = b$ and $z = c$, k is found from the equation

$$a = k \cdot \frac{b^p}{c^q}$$

or

$$\frac{x}{a} = \frac{y^p}{z^q} \div \frac{b^p}{c^q}.$$

If y is altered in the ratio $\lambda : 1$, and if z is altered in the ratio $\mu : 1$, then

$$x \text{ is altered in the ratio } \frac{\lambda^p}{\mu^q} : 1.$$

CHAPTER XV.

FUNCTIONS OF ONE VARIABLE.

A. REPRESENTATION OF FUNCTIONS.

If the area of a rectangle is fixed, the lengths of its sides cannot be chosen independently of one another; the choice of a length for one side depends upon the length chosen for the other. In other words, "the length of one side is a *known function* of the length of the other, when the area is given."

For example, if the area is 36 sq. inches, and if one side is of length x inches, then the other side must be of length $\frac{36}{x}$ inches.

Example I. *Sketch* the graph of the function $\frac{36}{x}$.

An accurate drawing is not required and it is not necessary to make a table of values. All that is necessary is to note a few important features of the function as follows:

(i) If x is positive, $\frac{36}{x}$ is positive;

if x is negative, $\frac{36}{x}$ is negative.

(ii) If x is large and positive, $\frac{36}{x}$ is small, and by making x sufficiently large, we can make $\frac{36}{x}$ as small as we please.

In symbols, when $x \to \infty$, $\frac{36}{x} \to 0$, which reads in words "when x tends to or approaches infinity, $\frac{36}{x}$ tends to or approaches zero."

(iii) If x is small and positive, $\frac{36}{x}$ is large and positive $\left[e.g. \text{ if } x=0{\cdot}1, \frac{36}{x}=360\right]$.

In symbols, when $x \to +0$, $\frac{36}{x} \to +\infty$.

(iv) When x actually $=0$, $\frac{36}{x}$ has no meaning.

(v) For a negative value of x, the value of $\frac{36}{x}$ is equal in magnitude but opposite in sign to its value for the corresponding positive value of x.

In particular, when $x \to -0$, $\frac{36}{x} \to -\infty$,

and when $x \to -\infty$, $\frac{36}{x} \to -0$.

The rough shape of the graph can now be sketched (see Fig. 10).

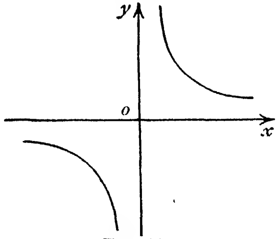


FIG. 10.

The graph is also said to correspond to the equation

$$y=\frac{36}{x} \quad \text{or} \quad xy=36.$$

EXERCISE XV. a.

[In the following examples, squared paper should not be used.]

1. Explain in words how the value of the function $(x-1)(x-3)$ varies as x varies from 1 to 3.

For what values of x does the function $=0$?

For what range of values of x is the function positive ?

2. Can you find a value of x for which the function x^2-4x+4 is negative ?

Construct another function of x which has a similar property.

3. For what values of x is the function x^2+x-6 zero ?

For what range of values of x is the function x^2+x-6 negative ?

4. (i) What is the greatest value of the function $9-(x-1)^2$?

(ii) For what values of x is this function zero?

(iii) To what value does this function tend when $x \to \infty$, and when $x \to -\infty$?

(iv) Sketch the graph of this function.

5. Sketch the graphs of (i) x^2, (ii) $(x-3)^2$, (iii) $(x+2)^2$.

6. Sketch the graphs of

(i) $+\sqrt{x}$, (ii) $-\sqrt{x}$, (iii) $\pm\sqrt{x-2}$, (iv) $\pm\sqrt{x+1}$.

7. (i) Can you find (*a*) a positive value of x, (*b*) a negative value of x, such that $\frac{1}{x^2} < 0{\cdot}01$?

(ii) When is this function $\frac{1}{x^2} > 100$?

(iii) Sketch the graph of $\frac{1}{x^2}$.

8. What is the value of the function $\frac{1}{x-3}$

(i) when x is large and positive, *e.g.* $+1003$?

(ii) when x is large and negative, *e.g.* -997?

(iii) when x is nearly equal to 3, (*a*) if $x > 3$, *e.g.* $3{\cdot}001$, (*b*) if $x < 3$, *e.g.* $2{\cdot}999$?

(iv) Sketch the graph of $\frac{1}{x-3}$.

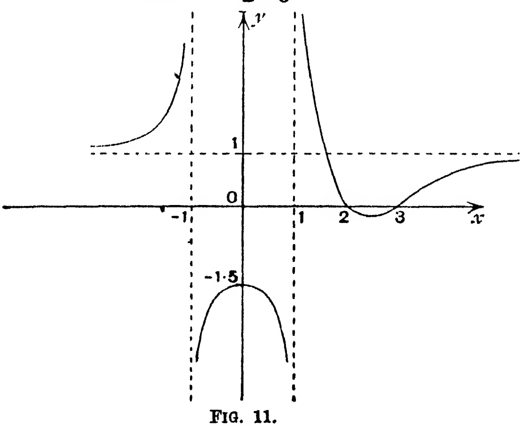


FIG. 11.

9. The graph of a certain function of x is shown in Fig. 11. Describe in words the variation in value of this function of x as x varies from $-\infty$ to $+\infty$.

10. ONP is a variable triangle with a right angle at N; ON lies along the axis Ox; the length of the hypotenuse OP is 5 and $ON = x$. What is the length of NP?

As x varies, what is the locus of P?

Of what function of x is this locus the graph?

11. The graph of a certain function of x is shown in Fig. 12. Describe in words how the value of the function varies as x varies from -2 to $+4$. If the same series of values of the function recurs again and again every time x is increased by 7, what can you say about the value of the function as $x \to \infty$?

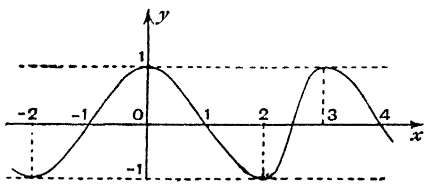


FIG. 12.

12. The values of a certain function of x can be calculated for all values of x, positive and negative. The function never has values greater than 1 or less than -2, and is zero when x equals -1 or 3 or 4. The function is positive if $-1 < x < 3$ or if $4 < x < \infty$, and for other values of x is negative.

Sketch the simplest graph of this function.

13. Sketch the simplest graph of a function which has the following characteristics:

(i) $\to +1$, when $x \to +\infty$.

(ii) $\to +\infty$, when $x \to +1$, provided $x > 1$.

(iii) Is never equal to 1.

(iv) Is not defined if $-1 < x < +1$.

(v) $\to -1$, when $x \to -\infty$.

(vi) $\to -\infty$, when $x \to -1$, provided $x < -1$.

14. (i) Can you find a value of x for which $\frac{x-1}{x-2}$ is firstly > 100 and secondly < -100?

(ii) For what value of x is this function zero?

(iii) For what range of values of x is this function negative?

(iv) Can you find a value of x for which the function equals 1?

(v) For what value of x is the function equal to 1·001?

(vi) For what value of x is the function equal to 0·999?

(vii) Sketch the graph of this function.

15. (i) For what values of x is the function $(x-1)(x-3)(x-4)$ zero ?

(ii) For what range of values of x is this function positive ?

(iii) Find the values of the function when $x=10$, 0, -10.

(iv) Can you find a value of x for which the function is $>1,000,000$?

(v) Sketch the graph of this function.

16. (i) Can you find a value of x between 0 and 10 for which the function

$$\frac{(x-1)(x-5)}{x-3}$$

is firstly >1000 and secondly <-1000 ?

(ii) Can you find the value of the function when $x=3$?

(iii) Describe the changes of sign in the function as x increases from -1 to $+8$, stating also where it vanishes.

(iv) What is the approximate error per cent. in taking the function as equal to x when $x=1000$ and when $x=-1000$?

(v) Sketch the graph of this function.

17. Sketch the graph of

(i) $(x-1)(x-2)(x-3)(x-4)(x-5)$;

(ii) $(x-1)(x-2)^2(x-4)(x-5)$;

(iii) $(x-1)(x-2)^3(x-5)$.

18. (i) Find approximately the value of the function

$$\frac{x-1}{(x-2)(x-3)}$$

if $x=1\cdot99$, $2\cdot01$, $2\frac{1}{2}$, $2\cdot99$, $3\cdot01$.

(ii) What can you say about the value of this function when x is large; about how much, for example, is it if x is a million ?

(iii) What is its approximate value if $x=-1,000,000$?

iv) For what range of values of x is this function negative ?

(v) Sketch the graph of the function.

19. Sketch the graph of $\dfrac{x-2}{(x-1)(x-3)}$.

20. Sketch the graph of $\dfrac{x-1}{(x-2)^2}$.

21. Sketch the graph of $\dfrac{(x+1)(x-6)}{(x-1)(x-4)}$.

22. Sketch the graph of $\dfrac{x(x-3)}{(x+2)(x-1)}$.

23. (i) For what range of values of x can the value of the function $\pm\sqrt{(1-x)(x+2)}$ be computed ?

(ii) For what values of x is this function zero ?

(iii) Sketch the graph of this function.

24. Sketch the graph of the function $\pm(x-3)\sqrt{x-2}$.

25. Sketch the graph of the function $+(x-4)\sqrt{(x-1)(x-3)}$.

Summary of Results.

The function $f(x)\equiv\dfrac{(x-a)(x-b)(x-c)\ldots}{(x-p)(x-q)(x-r)\ldots}$

vanishes when $x=a$ or b or c, etc., and tends to infinity when $x\to p$ or $\to q$ or $\to r$, etc.

The statement that $f(x)\to+\infty$ when $x\to p$ means that $f(x)$ can be made to exceed any number however large, if x is given a value sufficiently close to p or any value closer than this to p.

In order to find the range of values for which a function is positive (or negative), it is usually best to factorise it, when possible.

B. CONSTRUCTION OF FUNCTIONS.

EXERCISE XV. b.

1. In Fig. 13 the coordinates of A, B, P are respectively (2, 3) ; (6, 5) ; (x, y). Express y as a function of x.

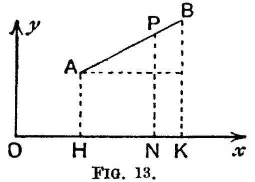


Fig. 13.

2. AB is the diameter of a circle APB ; PN is the perpendicular from P to AB ; if $AB=8$, $AN=x$, $PN=y$, express y as a function of x.

3. $ABCD$ is a rectangle ; P is a point such that the perpendicular PN from P to AB is equal to PC. If $BC=4$, $PN=x$, $NB=y$, express y as a function of x.

4. PN is an altitude of the triangle APB ; O is the mid-point of AB. If $PN=y$, $ON=x$, $AB=6$ and $PA^2+PB^2=30$, express y as a function of x.

5. AB is a diameter of a circle; a line AQR cuts the circle at Q and the tangent at B in R; P is a point on AQ such that $AP=QR$; PN, QK are the perpendiculars from P, Q to AB; $AB=4$, $AN=x$, $PN=y$, $QK=z$; express (i) z as a function of x (use the fact that $BK=AN$), (ii) y as a function of x.

Sketch the graph of y.

6. AOB is a triangle right-angled at O; PN is the perpendicular from a point P on AB to OB; if $AP=3$, $PB=4$, $ON=x$, $PN=y$, express y as a function of x.

7. A point R is taken on the side AB of a triangle ABC of area z sq. inches such that $AR=x \,.\, AB$, where $x > \frac{1}{2}$. RQ, RH are drawn parallel to BC, AC to meet AC, BC at Q, H; QK is drawn parallel to AB to meet BC at K. Express the area of $QRHK$ as a function of x and z. (C.S.C.)

8. AB is a diameter of a circle; CD is a chord parallel to AB and at distance b inches from it; any chord AQ cuts CD at R; RN is drawn perpendicular to AB; QP is drawn parallel to AB and cutting RN at P; if $AB=a$, $AN=x$, $NP=y$ inches, express y as a function of x.

9. AB is a fixed diameter of a given circle; the tangent at B meets a variable chord AP at Q; $AB=d$, $AP=x$, $BQ=y$; express y as a function of x.

10. $ABCD$ is a straight line and AH, BK, CL are three fixed lines perpendicular to it; $AB=a$, $BC=b$, $CD=c$; a variable line cuts AH, BK, CL at P, Q, R and DQ cuts CL at S; if $AP=y$, $CR=z$, $CS=x$, express x as a function of y, z.

11. Assuming the length, breadth and depth of an ordinary match-box are in the ratio $10:7:3$, express the volume of the box as a function of the area of match-board used in making it, *i.e.* box and drawer.

12. Taking Ox, Oy as perpendicular axes, sketch the graphs of x^2 and $\frac{1}{x}$ for positive values of x; any line perpendicular to Ox cuts the graphs at P, Q and Ox at N; if $ON=x$, express the area of the triangle OPQ as a function of x.

13. A sector of a circle of unit radius is folded to form a circular cone. If the angle of the sector is x right angles, express the volume of the cone as a function of x, assuming $x < 4$.

14. Take a rectangular sheet of paper of unit area and fold it in half; then fold it again in half with the second crease perpendicular to the first, so that it is now reduced to a quarter of the original size. Repeat this process, taking each time the new crease perpendicular to the last. Suppose the process of folding to be repeated x times in all.

(i) Express the area of the top of the packet as a function of x.

(ii) Supposing a triangular wedge is cut out of the middle of the last crease obtained and the paper is then unfolded, express the number of holes in the original sheet as a function of x.

15. Two metre rules AH, BK, whose zero graduations are at A, B, lie parallel to each other, and in the same sense, on a table; a third metre rule CL, zero graduation at C, touches them at A, B where the graduations on CL are a, b; a straight edge laid across AH, BK, CL meets them where the graduations are x, y, z respectively; express z in terms of x, y.

16. The triangle AOQ is right-angled at O; the bisector of $\angle AOQ$ cuts AQ at P; $OA = a$, $OP = y$, $OQ = x$. Draw PN perpendicular to OQ: (i) express PN and ON in terms of y; (ii) express y as a function of x.

17. AH, BK, CL are three lines, graduated in the same way, the graduations being that of a 10″ slide rule; A, B, C are the unit graduations in each case [*e.g.* the graduation for "7" is $10 \log 7$ inches from the unit graduation]. They are placed parallel to each other with C at the mid-point of AB. A straight-edge lies across them and meets AH, BK, CL at the graduations x, y, z respectively; (i) express z in terms of x, y; (ii) how is the expression altered if BK is in the opposite sense both to AH and CL?

18. With the data of Ex. 17, but supposing that C divides AB in the ratio $2 : 3$, express z in terms of x, y. How is this expression altered if C is the "2" graduation instead of the unit graduation on CL?

19. What is the simplest function of x which is zero when $x = 2$ and $x = 3$?

20. What is the simplest function of x which is zero when $x = 1$ and tends to infinity when $x \to 2$?

21. Find a function of x which is zero for $x = 1$ and $x = 3$ and is positive if $1 < x < 3$.

22. Find a function of x which is zero for $x = 1$ and $x = -1$ and is never positive.

23. Find a function of x which can be computed for any value of x between 2 and -2, but for no other values.

24. Find a function of x which is equal to 2 when $x=1$, but is never less than 2.

C. GRAPHICAL SOLUTIONS.

Example II. $ABCD$ is a rectangle such that $AB=4''$, $BC=8''$; P, Q are points on BC, CD, such that

$$CQ=\tfrac{1}{2}BP=x \text{ inches.}$$

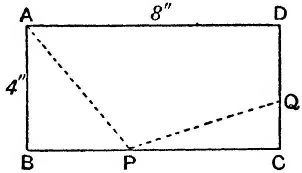


FIG. 14.

Express the area of $APQD$ in terms of x. Represent this function graphically: and find from the graph the length of CQ when the area of $APQD$ is 24 sq. inches.

Check the result by algebra.

(i) The area of $ABCD=4\times 8=32$ sq. inches.

$$CQ=x,\quad BP=2x;\quad \therefore\ PC=8-2x;$$

$\therefore$ area of triangle $ABP=\frac{1}{2}\times 4\times 2x=4x$ sq. inches,
area of triangle $PCQ=\frac{1}{2}(8-2x)\times x=x(4-x)$ sq. inches;

$$\begin{aligned}\therefore \text{ area of } APQD &= 32-4x-x(4-x)\\ &=32-4x-4x+x^2\\ &=32-8x+x^2 \text{ sq. inches.}\quad \textit{Answer.}\end{aligned}$$

(ii) To represent $x^2-8x+32$ by a graph.
Since CQ does not exceed CD, x is not greater than 4.
Construct a table of values for x from 0 to 4.

x	0	1	2	3	4
x^2 $-8x$ 32	0 0 32	1 -8 32	4 -16 32	9 -24 32	16 -32 32
$x^2-8x+32$	32	25	20	17	16

Plotting these, we obtain the required graph.

(iii) From the graph we see that the function equals 24 if $x = 1{\cdot}17$;

$\therefore$ $CQ = 1{\cdot}17''$ when the area of $APQD$ is 24 sq. in.

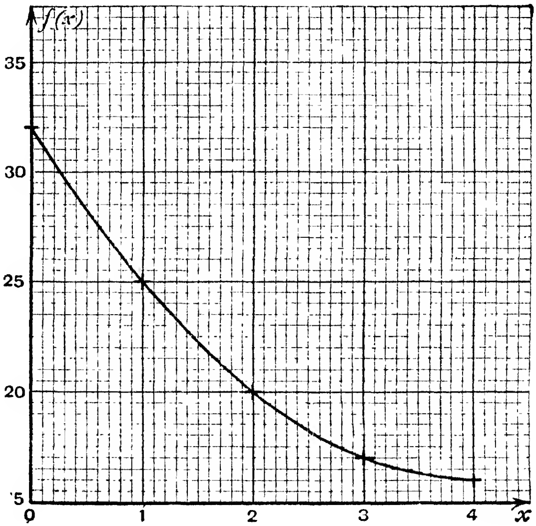


FIG. 15.

(iv) To solve by algebra, we take the equation

$$x^2 - 8x + 32 = 24;$$

$$\therefore\ x^2 - 8x + 8 = 0;$$

$$\therefore\ x = \frac{8 \pm \sqrt{64 - 32}}{2} = \frac{8 \pm \sqrt{32}}{2}$$

$$= \frac{8 \pm 5{\cdot}657}{2} = \frac{13{\cdot}657}{2} \quad \text{or} \quad \frac{2{\cdot}343}{2}$$

$$= 6{\cdot}828 \quad \text{or} \quad 1{\cdot}171.$$

The value $6{\cdot}828$ is excluded by geometrical considerations;

$$\therefore\ x = 1{\cdot}171.$$

EXERCISE XV. c.

[Examples 1-4 require the use of geometrical instruments.]

1. AB is the diameter of a circle APB; PN is the perpendicular from P to AB; $AB = 10$ cm. By taking different positions for P and making the necessary measurements, construct a table of values connecting the lengths of AP and AN. Represent it graphically, and find the length of AN when $AP = 2AN$.

2. Draw a quadrilateral $AKLB$ having $\angle AKL = 90° = \angle KLB$, $AK = 6$ cm., $KL = 12$ cm., $LB = 3$ cm.; P is any point on KL. Make the necessary measurements for a table of values connecting $AP + PB$ with KP; represent it graphically, and find the length of KP for which $AP + PB$ is least.

3. B is the foot of the perpendicular from A to a line CBD; $AB = 12$ cm., $BC = 4$ cm., $BD = 6$ cm.; P is any point on AB. Make the necessary measurements for a table of values connecting AP with $AP + CP + DP$; represent it graphically, and find the length of AP for which $AP + CP + DP$ is least.

4. With the notation and data of Ex. 1, make a table of values for the relation between AN and the area of the triangle APN; construct the graph, and find the length of AN for which the triangle APN is of maximum area.

If $AN = x$ cm., express the area of the triangle APN as a function of x.

5. A funnel is made in the shape of a pyramid of height a inches standing on a square base of side $2x$ inches. The funnel holds 48 cu. inches of liquid. Prove that (i) $a = \frac{36}{x^2}$; (ii) if the area of the four slant faces is y sq. inches, $y = 4x\sqrt{x^2 + a^2}$.

Tabulate the values of y for x equal to 1, 2, 3, 4, 5, and from the graph find the value of x for which y is least. (C.S.C.)

6. The perimeter of a right-angled triangle is 20 cm., and the shortest side is x cm.; if the area is A sq. cm., express A as a function of x. Find from a graph the value of x for which the area is a maximum. (C.S.C.)

7. The base of an open cistern is a square of side x feet; its volume is 200 cu. ft. The total area of the base and sides is A sq. ft. Express A as a function of x. Represent A graphically for values of x from 4 to 12. Find the depth of the cistern (i) when A is least, (ii) when the total area of the base and sides is 200 sq. ft. (C.S.C.)

8. A sheet of tin is 24 inches square; equal squares are cut out at the four corners, and the sides are then turned up to make a rectangular box; if the side of each square is x inches and the volume of the box is V cu. inches, express V as a function of x, and find graphically the maximum capacity of the box. (C.S.C.)

9. $CDEF$ is a rectangular sheet of paper; $CD = 30$ cm., $DE = 5$ cm.; A, B are the mid-points of CD, EF; P is a point on CA such that $PA = x$ cm. The paper is folded over with PB as crease, and the new position of PD cuts FB at Q; $QB = y$ cm.

Express y as a function of x. Find graphically (i) the least value of y, (ii) the length of AP if QB is 7 cm. Check the answer to (ii) by algebra. [Note that $PQ=QB$, and draw PN perpendicular to FB.] (C.S.C.)

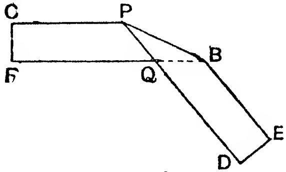


FIG. 16.

10. In the triangle ABC, $AB=AC=12$ feet, $\angle BAC=90°$; DHE, FHG are parallels to AC, AB; $AD=AF=x$ feet; express the area of $DHGB$ as a function of x, and find graphically the value of x for which this is a maximum. (C.S.C.)

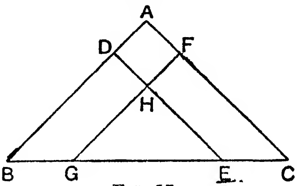


FIG. 17.

11. The base of a box is a square of side x feet, and the length of cord (exclusive of knots) needed for binding it once round each way is 30 feet. Express the volume V cu. ft. of the box as a function of x, and find from a graph the maximum volume of the box.

12. From a circular cone, height 12 cm., base diameter 12 cm., the greatest circular cylinder of radius r cm. is cut. Express its volume V cu. cm. as a function of r, and find from a graph the volume of the greatest cylinder.

13. Fill in the gaps in the following table, and so construct a table of values of the function $4+2x-x^2$ for values of x from -2 to $+4$:

x	-2	-1	0	1	2	3	4
4 $+2x$ $-x^2$	4 -4 -4					4 6 -9	
$4+2x-x^2$	-4					1	

Draw accurately the graph of this function, and from your graph answer the following questions :

(i) For what values of x does $4+2x-x^2=0$?

(ii) Between what values of x is $4+2x-x^2$ positive ?

(iii) What is the maximum value attained by the function $4+2x-x^2$, and for what value of x does the function have its maximum value ?

(iv) For what values of x does $4+2x-x^2=1\frac{1}{2}$? Solve the equation $2(4+2x-x^2)=3$.

(v) If $2x-x^2=-2$, what is the value of $4+2x-x^2$? For what values of x does $4+2x-x^2$ have this value ? Solve the equation $2x-x^2=-2$.

(vi) Solve the equation $4+2x-x^2=-1$ from the graph, and solve the equation $x^2-2x-5=0$ by formula. Compare the results.

14. Construct a table of values of the function $\frac{1}{5}(2x^2+2x-3)$ for values of x from -4 to $+4$ as follows :

x	-4	-3	-2	-1	0	1	2	3	4
$2x^2$ $2x$ -3	32 -8 -3						8 4 -3		
$2x^2+2x-3$	21						9		
$\frac{1}{5}(2x^2+2x-3)$	4·2						1·8		

Draw accurately the graph of this function, and from your graph answer the following questions :

(i) For what values of x is $2x^2+2x-3=0$?

(ii) Between what values of x is $2x^2+2x-3$ negative ?

(iii) What is the minimum value attained by the function $\frac{1}{5}(2x^2+2x-3)$, and for what value of x does the function have its minimum value ?

(iv) For what values of x does $\frac{1}{5}(2x^2+2x-3)=1$? Solve the equation $2x^2+2x-3=5$.

(v) If $2x^2+2x=7$, what is the value of $2x^2+2x-3$ and of $\frac{1}{5}(2x^2+2x-3)$? For what values of x does $\frac{1}{5}(2x^2+2x-3)$ have this value ? Solve the equation $2x^2+2x=7$.

(vi) Solve the equation $x^2+x=1$ from the graph. Check your result by solving the equation by the formula.

15. Construct a table of values of the function

$$\tfrac{1}{5}(x^3 - 3x^2 - 4x - 6)$$

for values of x from -3 to $+5$, and draw the graph of this function.

From your graph answer the following questions :

(i) For what values of x does $x^3 - 3x^2 - 4x - 6 = 0$?

(ii) For what range of values of x is $x^3 - 3x^2 - 4x - 6$ positive ?

(iii) What is the smallest value of the function

$$\tfrac{1}{5}(x^3 - 3x^2 - 4x - 6)$$

for positive values of x ? For what positive value of x does it have the smallest value ?

(iv) What is the largest value of the function for negative values of x, and for what negative value of x does it have its largest value ?

(v) Solve the equation $x^3 - 3x^2 - 4x - 6 = 7$ from the graph.

(vi) Solve the equations : (*a*) $x^3 - 3x^2 - 4x + 4 = 0$;
(*b*) $x^3 - 3x^2 - 4x + 19 = 0$;
(*c*) $x^3 - 3x^2 - 4x - 16 = 0$.

(vii) Solve the equation $x^3 - 3x^2 - 4x + 12 = 0$ from the graph, and also by factorising $x^3 - 3x^2 - 4x + 12$.

16. Draw a graph of the function $\dfrac{1}{x^2}$ for values of x from -3 to $+3$.

(i) With the same axes, and to the same scale, draw the graph of the function $2 - x$. For what values of x are the functions $\dfrac{1}{x^2}$ and $2 - x$ equal ? Hence solve the equation $x^3 - 2x^2 + 1 = 0$.

(ii) Draw the graph of the function $x - 2$ with the same axes, and solve the equation $\dfrac{1}{x^2} = x - 2$.

(iii) Draw the graph of the function $2x + 1$, and solve the equation $x^2(2x + 1) = 1$.

17. Draw graphs of the functions $\dfrac{1}{x}$ and $x^2 - 4$ with the same axes and to the same scale for values of x from -3 to $+3$.

For what values of x are these functions equal ?

Hence solve the equation $x^3 - 4x = 1$.

18. Draw graphs of the functions x^2 and $x - 1$ with the same axes and to the same scale. When are these functions equal ?

What does this tell you about the solutions of the equation $x^2 - x + 1 = 0$?

D. FUNCTIONAL NOTATION.

Any expression whose value can be determined when the value of x is known, can be represented by the symbol $f(x)$.

$f(x)$ is shorthand for the words "a function of x."

In a particular question $f(x)$ might be used to represent the function $5x^4-2x+8$.

Then $f(2)$ would mean the value of this function when $x=2$.

$$\therefore\ f(2) \text{ would mean } 5(2^4)-2(2)+8$$
$$=80-4+8$$
$$=84,$$

and
$$f(-1) \text{ would mean } 5(-1)^4-2(-1)+8$$
$$=5+2+8$$
$$=15,$$

and
$$f(0) \text{ would mean } 5(0)^4-2(0)+8$$

Example III. If $f(x)\equiv x^2-3+\frac{1}{x}$, find the value of $f(5)$ and $f(2a)$.

$$f(5)=5^2-3+\tfrac{1}{5}=25-3+\tfrac{1}{5}=22\tfrac{1}{5},$$
$$f(2a)=(2a)^2-3+\frac{1}{2a}=4a^2-3+\frac{1}{2a}.$$

Example IV. If $f(x)\equiv x^2+2x$, find the value of $f(x+h)-f(x)$.

Here
$$f(x+h)=(x+h)^2+2(x+h)$$
$$=x^2+2xh+h^2+2x+2h;$$
$$\therefore\ f(x+h)-f(x)=x^2+2xh+h^2+2x+2h-x^2-2x$$
$$=2xh+h^2+2h$$
$$=h(2x+2+h).$$

EXERCISE XV. d.

1. If $f(x)\equiv x^2+2$, find the values of $f(1)$; $f(0)$; $f(-1)$; $f(2a)$; $f(b^3)$.

2. If $f(x)\equiv 10^x$, find the values of $f(1)$; $f(2)$; $f(0)$; $f(-1)$; $f(2x)$.

3. If $f(x)\equiv \log x$, find the values of $f(1000)$; $f(2)$; $f(20)$; $f(1)$; $f(x^3)$.

4. If $f(x) \equiv x^2 - 3x + 5$, find the values of $f(2)$; $f(x+h)$; $f\left(\frac{1}{x}\right)$.

5. If $f(x) \equiv x^2 + 3x$, find the value of $\frac{f(x+h) - f(x)}{h}$. What is the approximate value of this when h is small compared with x?

6. If $f(x) \equiv \frac{1}{x}$, find the values of $f(1)$ and $\frac{f(1+h) - f(1)}{h}$. What is the approximate value of this last expression when h is small?

7. If $f(x) \equiv x^2 + 5$, simplify

(i) $f(3x)$;

(ii) $f(x+1) + f(x-1) - 2f(x)$.

8. If $f(x) \equiv x^2$, simplify

(i) $f(x^3)$;

(ii) $f(x+h) - f(x) - \{f(x) - f(x-h)\}$.

9. If $f(x) \equiv (x+7)(7-x)$, show that $f(-x)$ equals $f(x)$. Construct another function which has this property.

10. If $f(x) \equiv \frac{1+x^2}{x}$, show that $f\left(\frac{1}{x}\right)$ is equal to $f(x)$. Construct another function which has this property.

11. For what values of x does $f(x) = 0$ if $f(x)$ is $(x-1)(x-2)$?

12. If $f(x) \equiv 2x^2 - x$, solve the equation $f(x) = f(x-1)$.

13. If $f(x) \equiv \frac{1}{1-x}$ simplify $f(f(x))$.

Find simple functions which have the following properties:

14. $f(3) = 0, f(4) = 0$.

15. $f(3) = 0, f(4) = 8$.

16. $f(3) = 0$, $f(x) \to \infty$ when $x \to 1$, and also $f(1{,}000{,}000)$ is approximately equal to 3.

17. $f(x) = f(-x), f(1) = 3$.

18. $f(x) + f(y) = f(xy)$.

19. $f(x) \times f(y) = f(x+y)$.

20. $f(x) = -f(-x), f(1) = 0$.

21. $f(1) = f(3) = f(5) = -1, f(2) = f(4) = f(6) = +1$.

CHAPTER XVI.

LIMITS AND GRADIENTS.

It has already been noticed, in connection with the graphical representation of functions, that a function may have no meaning for one or more special values of the variable (*e.g.* $\frac{36}{x}$ has no meaning when $x=0$, see p. 307). We shall now examine more closely the behaviour of a function in the neighbourhood of such values. Examples I. and II. illustrate the meaning of a "Limit" of a function: a more formal discussion will be found in the Introduction.

Example I. Plot the values of the functions:

$$\text{(i) } 1-\frac{1}{n}; \qquad \text{(ii) } 1+\frac{1}{n}; \qquad \text{(iii) } 1+(-1)^n \cdot \frac{1}{n}$$

for positive *integral* values of n.

(i) $1-\frac{1}{n}$.

We have the following table of values:

$n=1$	2	3	4	5	6	7	8
$1-\frac{1}{n}=0$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$

which are represented in Fig. 18.

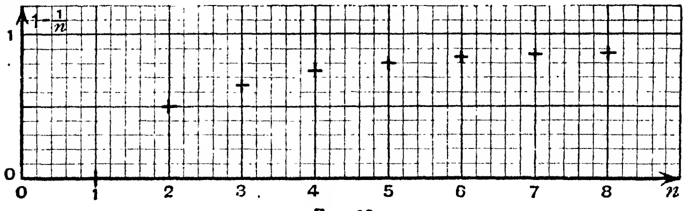


Fig. 18.

(ii) $1+\frac{1}{n}$.

We have the following table of values :

$n=1$	2	3	4	5	6	7	8
$1+\frac{1}{n}=2$	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{6}{5}$	$\frac{7}{6}$	$\frac{8}{7}$	$\frac{9}{8}$

which are represented in Fig. 19.

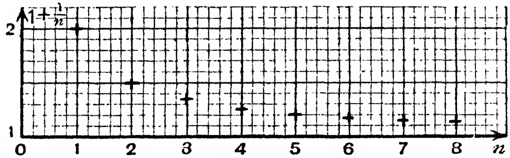


FIG. 19

(iii) $1+(-1)^n \cdot \frac{1}{n}$.

We have the following table of values :

$n=$ 1	2	3	4	5	6	7	8
$1+(-1)^n \cdot \frac{1}{n}=0$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{5}{4}$	$\frac{4}{5}$	$\frac{7}{6}$	$\frac{6}{7}$	$\frac{9}{8}$

which are represented in Fig. 20.

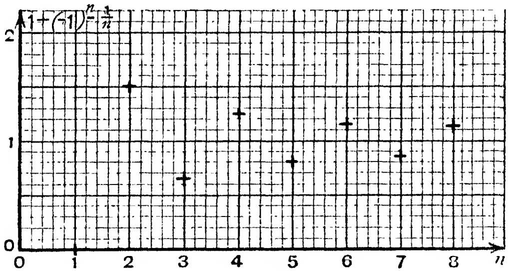


FIG. 20.

Either from the graphs or from examining the functions direct we note that

(i) $1-\frac{1}{n}$ is always less than 1, and that, as n increases, the function steadily increases and approaches the value 1.

(ii) $1+\frac{1}{n}$ is always greater than 1, and that as n increases the function steadily decreases and approaches the value 1.

(iii) $1+(-1)^n\cdot\frac{1}{n}$ is alternately less and greater than 1, and that as n increases the function increases and decreases alternately, but approaches the value 1.

In each case it is possible to find a value of n for which and for all greater values the function differs from 1 by less than a given amount, however small.

E.g. each function differs from 1 by less than 0·001 if $n > 1000$.

But it is impossible to find a value of n for which any of the functions actually equals 1.

Under these conditions we say that:

The Limit of $1+\frac{1}{n}$ as n tends to infinity is 1 [but the limit is not attained in this case]. And we write it as follows:

$$\underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)=1.$$

Similarly $\underset{n\to\infty}{\text{Lt}}\left(1-\frac{1}{n}\right)=1$ and $\underset{n\to\infty}{\text{Lt}}\left[1+(-1)^n\cdot\frac{1}{n}\right]=1.$

Example II. What is the limit of $\frac{(1+h)^2-1}{h}$ as h tends to 0?

$$\frac{(1+h)^2-1}{h}=\frac{1+2h+h^2-1}{h}=\frac{2h+h^2}{h}.$$

If $h=0$, $\frac{2h+h^2}{h}$ becomes $\frac{0}{0}$, which is a meaningless expression.

If $h\neq 0$, $\frac{2h+h^2}{h}=2+h$.

The smaller h becomes, the closer $2+h$ approaches to 2. It never attains the value 2, for h is never actually 0. But we can choose h so that $2+h$ differs from 2 for that and all smaller values of h by less than any given amount, however small;

$\therefore$ its limit is 2;

$$\therefore\ \underset{h\to 0}{\text{Lt}}\frac{(1+h)^2-1}{h}=2 \text{ (limit not attained).}$$

Note.—The x-coordinate of a point is sometimes called the *abscissa*, and the y-coordinate is called the *ordinate.*

For example, in Figure 21, ON is the *abscissa* of P and PN is the *ordinate* of P.

EXERCISE XVI. a.

1. (i) Plot the values of $\frac{2n+1}{n+1}$ for integral values of n from 3 to 10.

(ii) Can you find a value of n for which $\frac{2n+1}{n+1}$ differs from 2 by less than 0·001 ?

(iii) Can you find a value of n for which $\frac{2n+1}{n+1}$ equals 2 ?

(iv) What is the limit of $\frac{2n+1}{n+1}$ as n tends to infinity?

2. (i) If $s_1=1$; $s_2=1+\frac{1}{2}$; $s_3=1+\frac{1}{2}+\frac{1}{4}$; $s_4=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, etc., plot the values of s_n for values of n from 1 to 5.

(ii) Can you find a value of n for which s_n differs from 2 by less than 0·001 ?

(iii) Can you find a value of n for which s_n equals 2 ?

(iv) What is the value of $\underset{n\to\infty}{\text{Lt}}\ s_n$?

3. (i) Express as a decimal to 3 figures the value of $\frac{n^2}{2n^2+1}$ when $n=5$ and $n=10$.

(ii) What is the limit of $\frac{n^2}{2n^2+1}$ when n tends to infinity ?

(iii) Is this limit attained ?

4. (i) Find the values of $\frac{(0{\cdot}1)^n}{1+(0{\cdot}1)^n}$ for integral values of n from 1 to 4.

(ii) What is the value of $\underset{n\to\infty}{\text{Lt}}\ \frac{(0{\cdot}1)^n}{1+(0{\cdot}1)^n}$?

5. (i) Plot the values of $\frac{2^n}{1+2^n}$ for integral values of n from 1 to 4.

(ii) What is the value of $\underset{n\to\infty}{\text{Lt}}\ \frac{2^n}{1+2^n}$?

(iii) For what integral value of n does $\frac{2^n}{1+2^n}$ differ from 1 by less than $\frac{1}{100}$?

6. (i) Find the values of $\frac{x^2-1}{x^2-x}$ for $x=2$, 1·5, 1·1, 1·01.

(ii) What is the value of $\underset{x \to 1}{\text{Lt}} \frac{x^2-1}{x^2-x}$?

(iii) Is this limit attained?

(iv) Find a value of x for which $\frac{x^2-1}{x^2-x}$ differs from 2 by less than 0·001.

7. Fig. 21 shows part of the graph of $y=\frac{1}{2}x^3$.

$$ON=1, \quad NM=h.$$

PN, QM are ordinates and PR is perpendicular to QM.

(i) Express $\frac{QR}{PR}$ in terms of h.

(ii) What is the limit of $\frac{QR}{PR}$ as $h \to 0$? Interpret this result geometrically.

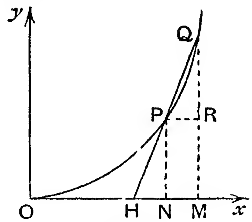


FIG. 21.

8. If a stone is dropped in a vacuum, it falls s feet in t seconds, where $s=16t^2$; the graph of this is represented in Fig. 22.

$$ON=\frac{1}{2}, \quad NM=h.$$

PN, QM are ordinates and PR is perpendicular to QM.

(i) Express $\frac{QR}{PR}$ in terms of h.

(ii) What is the limit of $\frac{QR}{PR}$ as $h \to 0$? Interpret this result.

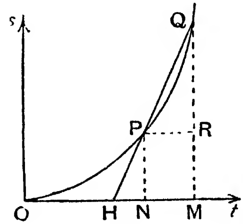


FIG. 22.

9. (i) Draw the graph of $5x - x^2$ from $x = 0$ to 5.

(ii) Draw the ordinate corresponding to $x = 1{\cdot}5$. What is its length ?

(iii) Take any point A on the x-axis and call $OA = a$. Give a geometrical meaning to the ratio

$$\frac{[5(a+h) - (a+h)^2] - [5a - a^2]}{h}.$$

(iv) What is the limit of this function when $h \to 0$?

(v) For what value of a is this limit equal to 0 ? Interpret this result geometrically.

10. (i) Draw the graph of $\frac{1}{x}$ from $x = 4$ to $\frac{1}{4}$.

(ii) Give a geometrical meaning to the ratio $\dfrac{\frac{1}{2+h} - \frac{1}{2}}{h}$.

(iii) What is the limit of this ratio when $h \to 0$? Interpret this result.

(iv) Give a geometrical meaning to the ratio $\dfrac{\frac{1}{a+h} - \frac{1}{a}}{h}$; evaluate its limit when $h \to 0$; and find the value of a for which this limit equals $-\frac{1}{4}$.

11. (i) What is the average of the numbers 1, 2, 3, 4, ... , $(n-1)$?

(ii) Prove that their sum is $\frac{n(n-1)}{2}$.

(iii) Find the value of $\underset{n \to \infty}{\text{Lt}} \frac{1}{n^2}[1 + 2 + 3 + \ldots + (n-1)]$.

12. ABC is an isosceles right-angled triangle ; $BC = 10$ cm. ; BC is divided into n equal parts, and through each point of division a line is drawn parallel to BA to meet CA, and rectangles are completed as in the figure. By using Ex. 11,

(i) find the sum of the areas of all the rectangles in terms of n :

(ii) find the limit of this sum when n tends to infinity ;

(iii) what is the area of the triangle ABC ?

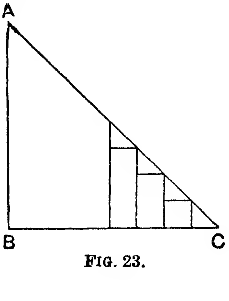


FIG. 23.

13. It can be proved that the sum of the series

$$1^2+2^2+3^2+4^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

(i) Find the value of $\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n^3}(1^2+2^2+3^2+\ldots+n^2)$.

(ii) Fig. 24 gives the graph of $y=x^2$; $OA=1$; OA is divided into n equal parts, and through each point of division lines are drawn parallel to Oy, and rectangles are completed as in the figure. Express the sum of the areas of all these rectangles in terms of n, and find the limit of this sum when n tends to infinity.

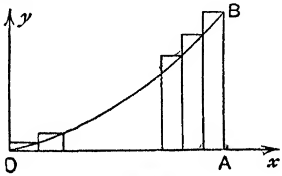


FIG. 24.

14. (i) Use the result given in No. 13 to find the value of

$$\frac{1}{n^3}[1^2+2^2+3^2+\ldots+(n-1)^2].$$

(ii) If the rectangles are drawn so that each ends below the graph of $y=x^2$ (the arrangement in Fig. 23), find the sum of the areas of the rectangles in terms of n; and find the limit of this sum when $n\to\infty$.

(iii) By comparing this with the result in Ex. 13, what can you say about the area of the figure bounded by OA, AB and the curve OB in Fig. 24?

CALCULATION OF RATES OF CHANGE FROM STATISTICS.

Example III. The following table gives the height of the mercury barometer at intervals of two hours during a day:

Time -	8 a.m.	10 a.m.	Noon	2 p.m.	4 p.m.	6 p.m.	8 p.m.
Height in inches -	28·57	28·65	28·85	29·10	29·22	29·12	29·0

Find the average rate at which the barometer was rising
(i) between 10 a.m. and noon;
(ii) between noon and 2 p.m.;
(iii) between 4 p.m. and 6 p.m.

Draw the barograph for the day, and find the rate at which the barometer was probably rising at noon.

(i) Between 10 a.m. and noon the barometer rose 0·2″ in 2 hours;

$\therefore$ the average rate of rise was 0·1″ per hour.

(ii) Between noon and 2 p.m. the barometer rose 0·25″ in 2 hours;

$\therefore$ the average rate of rise was 0·125″ per hour.

(iii) Between 4 p.m. and 6 p.m. the barometer fell 0·1″ in 2 hours;

$\therefore$ the average rate of rise was $-0·05''$ per hour.

Figure 25 gives the required barograph.

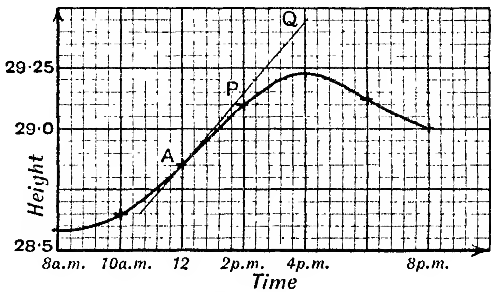


FIG. 25.

It is a curve and not a straight line, because the rate is altering throughout the day. If it continued to rise at the same rate after 12 o'clock as it is rising at 12 o'clock, the barograph would be the straight line formed by drawing the tangent APQ at the point A, which corresponds to 12 o'clock on the graph. In the figure, this tangent is drawn by eye, and cuts the 2 o'clock and 4 o'clock lines (or any other convenient lines) at P and Q. Now the difference in heights registered by P, Q is 0·30″.

$\therefore$ the rate of rise at A is 0·30″ in 2 hours,
or 0·15″ per hour.

EXERCISE XVI. b.

1. The following table gives the population of the United Kingdom for various years:

Year	1840	1850	1860	1870	1880	1890	1900
Population (in millions)	27·0	27·7	29·3	31·8	35·2	38·1	42·0

[Give answers in thousands per year.]

(*a*) What was the average rate of increase of population in the period 1840 to 1850?

(*b*) What was the average rate of increase of population for the whole period 1840 to 1900?

(*c*) During what period of ten years was the population increasing most rapidly, and what was the rate per year then?

2. At the end of a minutes a car has travelled x miles, and at the end of b minutes it has travelled y miles. What was its average speed

(*a*) for the first a minutes?

(*b*) for the first b minutes?

(*c*) during the period a minutes to b minutes?

If in the above question $a=3$ and $b=4$, what was the average speed of the car during the 4th minute?

3. The following table gives the distances a car travels, starting from rest:

Time in minutes	5	10	15	20	25	30	35	40	45
Distance in miles	0·7	2·6	5·2	8·3	11·4	14·2	16·7	19	20·5

[Give answers in the form miles per hour.]

(*a*) What was the average speed of the car for the first 5 minutes?

(*b*) What was the average speed for the first 20 minutes?

(*c*) What was the average speed for the whole period, 45 minutes?

(*d*) Draw a graph to illustrate the motion of the car, and by drawing a tangent to the graph at the point determined by $t=35$, find the probable speed of the car 35 minutes after it started.

(*e*) By drawing a tangent to the graph where it appears to be steepest, find the greatest speed attained by the car.

4. A lift ascends 90 ft. in 40 seconds, and its height h feet at intervals of 5 seconds is given by the following table :

t	5	10	15	20	25	30	35	40
h	4	15	30	45	62	77	86	90

Draw a graph to illustrate the motion, and draw tangents to the graph at the points determined by $t=10$, $t=20$, $t=30$.

(i) What is the average speed of the lift for the first 10 seconds ?

(ii) Estimate from your graph the speed of the lift after 10 seconds ?

(iii) What is the average speed of the lift for the first 20 seconds ?

(iv) Estimate its speed after 20 seconds.

(v) What is the average speed of the lift for the last 10 seconds ?

(vi) Estimate its speed after 30 seconds.

5. The weight that can be carried by a certain type of bridge varies with the diameters of the spars used in accordance with the following table :

Diameter of spar in inches -	3	5	7	9	11
Load carried in tons - -	0·1	0·7	2·4	6	12

[Give answers in the form tons per inch.]

What is the average rate at which the load carried increases when the diameter is increased from

(*a*) 3 inches to 5 inches ?

(*b*) 7 inches to 9 inches ?

(*c*) 9 inches to 11 inches ?

(*d*) 7 inches to 11 inches ?

Estimate the 'rate' of increase of the load when the diameter is 7 inches by drawing a graph.

6. Water runs out of a bath, and the volume that runs out is given by the following table :

Time in seconds - -	5	10	15	20	25
Volume in cubic feet - -	6	9	10·5	11·2	11·6

Draw a graph and find the rate of flow at the end of each period of 5 seconds. Show from your results that the rate of flow is proportional to the volume of water remaining in the bath, there being 12 cubic feet of water in the bath originally.

SUMMARY OF RESULTS.

(i) It is possible to have a function of x to which, for one or more special values of x, no meaning can be given.

E.g. $\dfrac{x^2-1}{x-1}$ has no meaning when $x=1$.

(ii) In such cases, the function may tend towards a definite limit as x tends to that value; but the limit is not attained.

E.g. $\underset{x\to 1}{\text{Lt}} \dfrac{x^2-1}{x-1}=2$, but no value of x exists for which $\dfrac{x^2-1}{x-1}$ equals 2.

(iii) If a function of x is represented by a graph, the rate at which the function is increasing for any value of x is represented by the slope of the tangent to the graph at the point corresponding to that value of x, provided that the units used in drawing the graph are taken into account.

CALCULATION OF RATES OF CHANGE FROM FORMULAE.

Example IV. Draw the graph of $y=2+3x+x^2$, and find the change of y per unit increase of x, when (i) $x=2$, (ii) $x=a$.

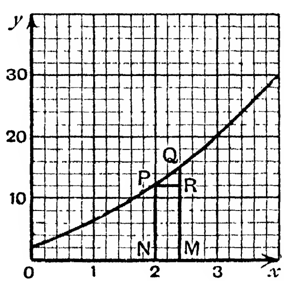


FIG. 26.

We have, by calculation, the table:

$x=0$	1	2	3	4
$y=2$	6	12	20	30

which is represented in Figure 26.

(i) When $ON=2$, $PN=2+3\times 2+2^2=2+6+4=12$.
When $OM=2+h$, $QM=2+3(2+h)+(2+h)^2=12+7h+h^2$;

$\therefore$ when x increases by h, y increases by $QM-PN$ or $QR=7h+h^2$;

$\therefore$ the average increase of y per unit increase of $x=\dfrac{7h+h^2}{h}=7+h$

or
$$\frac{QR}{PR}=7+h;$$
$$\therefore \operatorname*{Lt}_{h\to 0}\frac{QR}{PR}=7;$$

$\therefore$ at $x=2$, the rate of increase of y with respect to x is 7.

(ii) When $ON=a$, $PN=2+3a+a^2$.
When $OM=a+h$,
$$QM=2+3(a+h)+(a+h)^2=2+3a+3h+a^2+2ah+h^2;$$
$$\therefore QR=QM-PN=3h+2ah+h^2;$$
$$\therefore \frac{QR}{PR}=\frac{3h+2ah+h^2}{h}=3+2a+h;$$

$\therefore$ at $x=a$, the rate of increase of y with respect to x,
$$=\operatorname*{Lt}_{h\to 0}(3+2a+h)$$
$$=3+2a.$$

Definition. In Figure 26, $\dfrac{QR}{PR}$ is called the *average gradient* of the graph over the interval NM.

The limit of $\dfrac{QR}{PR}$ when h tends to 0 or $\operatorname*{Lt}_{h\to 0}\dfrac{QR}{PR}$ is called the *gradient* of the graph at P.

Figure 27 illustrates the way in which the "average gradient" changes as the point Q is taken nearer and nearer to P.

Q_1, Q_2, Q_3 are successive positions of Q, whilst P remains fixed.

The average gradient over the interval MN is $\dfrac{QR}{PR}$.

The limiting position to which the chord PQ tends as arc $PQ \to 0$ is indicated by the tangent TPT'.

$$\operatorname*{Lt}_{h \to 0} \frac{QR}{PR} = \text{gradient at } P = \text{gradient of the tangent } PT = \frac{PM}{MT}.$$

The gradient measures the "rate of change of y with respect to x," or the change in y per unit increase in x.

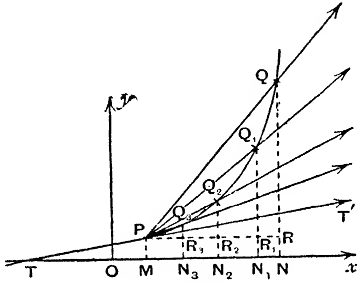


FIG. 27.

EXERCISE XVI. c.

1 What is the gradient of the slopes shown in Figs. 28-30 ?

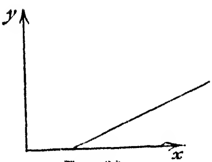


FIG. 28.

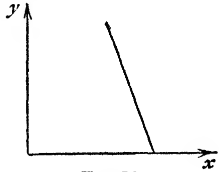


FIG. 29.

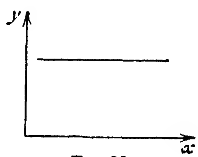


FIG. 30.

2. Figure 31 represents a hill; horizontal scale x-axis is 1 inch : 100 yards; vertical scale y-axis is 1 inch : 10 feet. Find (i) the average gradient from A to B, (ii) the average gradient from P to Q, (iii) the gradients at C, D, Q.

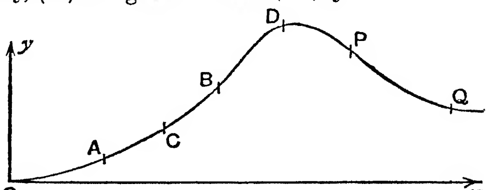


FIG. 31.

3. Draw lines of gradients (i) $\frac{3}{4}$, (ii) $-\frac{1}{2}$.

4. Find the coordinates of a point on the curve in Fig. 31, for the scale given in Ex. 2, where the slope is $\frac{1}{40}$.

5. (i) Draw the graph of $y=2x+3$.

(ii) Show that the points (1, 5) and (4, 11) lie on it.

(iii) What is the average gradient of the graph between these two points ?

(iv) What is the average gradient of the graph over the interval $x=2$ to $x=6$?

(v) What is the value of y when $x=a$ and when $x=a+h$, and what is the average gradient of the graph over this interval ?

(vi) Why does the average gradient of the graph not depend on the values of either a or h ?

6. (i) Draw the graph of $y=7-5x$.

(ii) What is the average gradient of the graph over the interval $x=3$ to $x=7$?

(iii) What is the gradient of the graph when $x=a$?

7. What is the gradient of the straight line joining the points (2, 5) and (7, 8) ?

8. A marble rolling down an inclined plane travels s feet in t seconds, where $s=3t^2$.

(i) How far has the marble travelled in 1 second ? What is its average speed for the 1st second ?

(ii) How far has the marble travelled in 2 seconds ? What is its average speed for the 1st two seconds ? What is its average speed during the 2nd second ?

(iii) How far has the marble travelled in 2·1 seconds ? What is its average speed in the interval 2 seconds to 2·1 seconds ?

(iv) How far has it travelled in $(2+h)$ seconds ? What is its average speed in the h seconds between 2 seconds and $(2+h)$ seconds ?

(v) What does your answer to Question (iv) become when $h=0·1$ second, when $h=0·01$ second and when $h=0·000001$ second ?

What is its speed exactly 2 seconds after it begins to move ?

Draw a graph to illustrate the motion from $t=0$ to $t=3$, and find its speed after 2 seconds by drawing a tangent.

9. The distance d ft. that a stone has fallen after t seconds is given by the formula $d=16t^2$.

(i) How far has the stone fallen after 3 seconds ? What is its average speed for the first 3 seconds ?

(ii) How far has the stone fallen after 2 seconds ? What is its average speed during the third second ?

(iii) How far has the stone fallen after 2·9 seconds ? What is its average speed during the interval from 2·9 seconds to 3 seconds ?

(iv) How far does it fall in $(3-h)$ seconds ? What is its average speed during the h seconds from $(3-h)$ seconds to 3 seconds ?

(v) What does your answer to Question (iv) become when $h=0{\cdot}1$ second, when $h=0{\cdot}01$ second and when $h=0{\cdot}000001$ second ?

What is the velocity of the stone 3 seconds after it is dropped ?

10. In the same way as in Ex. 9 work out the speed of the stone 1 second after it is dropped.

11. In the same way as in Ex. 9 work out the speed of the stone a seconds after it is dropped. Evaluate your result for $a=1$, 2 and 3, and compare with previous results.

12. The distance in feet travelled in t seconds by a body moving in a straight line from a fixed point A is given by the formula $AP=3t^2+5t+1$. Find AP when $t=2$ and when $t=2+h$. Find the average speed of the body P for the h seconds commencing with the end of the 2nd second. What is this speed when $h=0{\cdot}1$ sec., when $h=0{\cdot}01$ sec., when $h=0{\cdot}0001$ sec. and when $h=-0{\cdot}0000001$ sec. What is the speed of P 2 seconds after the motion starts ?

13. Draw a line AB of length 10 cm. and describe a semicircle with AB as diameter ; P is any point on the semicircle and PN is the perpendicular to AB. Let $AN=x$ and $PN=y$. Find by measurement or calculation the average gradient of the semicircle over the intervals (i) $x=0$ to 1 ; (ii) $x=0$ to 2 ; (iii) $x=3$ to 7 ; (iv) $x=8$ to 10. Interpret your results geometrically.

14. The area of a circular blot of ink is increasing at a steady rate of 2 sq. cm. per sec. ; if the radius is x cm. after t sec., find the average rate of increase of x over the interval (i) $t=5$ to $t=10$; (ii) $t=5$ to $t=5{\cdot}1$, assuming that $x=0$ when $t=0$.

15. The perimeter of a rectangle is 24 inches: if its area is y sq. in. when one side is of length x in., find

(i) the average rate of change of y over the interval $x=3$ to 5;

(ii) the average rate of change of y over the interval $x=3$ to $3+h$;

(iii) the average rate of change of y over the interval $x=a$ to $a+h$;

(iv) the rate of change of y when $x=a$;

(v) the value of x when the rate of change of y is zero; what does this mean?

(vi) Express y as a function of x and draw its graph.

16. If $y=3x+x^2$, find (i) the average rate of change of y over the interval $x=1$ to $1{\cdot}1$; (ii) the gradient at $x=1$; (iii) the value of x when the gradient is zero.

17. Calculate the gradient of the graph of $\frac{1}{x}$ when (i) $x=2$, (ii) $x=a$.

18. Calculate the gradient of the graph of $2x^3$ when $x=c$.

19. A marble rolling down a groove travels s feet in t seconds where $s=\frac{1}{4}t^2$; find (i) the average rate of change of s over the interval $t=1$ to $t=2$, (ii) the rate of change of s when $t=1$. What does this mean?

20. The graph of $y=mx+c$ is a straight line, m and c being any constant numbers. Find its gradient.

21. Find the gradient of $y=ax^2+bx+c$ when $x=2$.

22. Find the gradient of $y=ax^2+bx+c$ when $x=x_1$. For what value of x is this gradient equal to zero? What is the geometrical significance of a zero gradient?

23. Give geometrical meanings to the following, taking x to be the x-coordinate of a graph (do not simplify the expressions):

(i) $\dfrac{(x+h)^2-x^2}{h}$; (ii) $\underset{h\to 0}{\text{Lt}}\dfrac{(x+h)^2-x^2}{h}$;

(iii) $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$; (iv) $\underset{h\to 0}{\text{Lt}}\dfrac{\sqrt{x-h}-\sqrt{x}}{h}$;

(v) $\underset{h\to 0}{\text{Lt}}\dfrac{\pi(r+h)^2-\pi r^2}{h}$; (vi) $\dfrac{\frac{4}{3}\pi(r+h)^3-\frac{4}{3}\pi r^3}{h}$.

24. If $f(x)\equiv x^2$, evaluate $\underset{h\to 0}{\text{Lt}}\dfrac{f(x+h)-f(x)}{h}$ and interpret the result.

25. If $f(x)\equiv 5$, what is the gradient of the graph of $f(x)$?

Summary of Results.

Draw the graph of any function $y=f(x)$; see Figure 32.

Suppose $ON=x$ and $OM=x+h$, so that $PR=NM=h$.

Then $PN=f(x)$ and $QM=f(x+h)$.

$$\therefore \quad QR=f(x+h)-f(x).$$

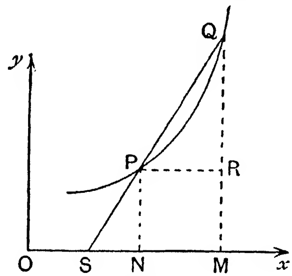


FIG. 32.

The average gradient of PQ is $\dfrac{f(x+h)-f(x)}{h}$.

And the gradient at P is $\underset{h\to 0}{\text{Lt}}\dfrac{f(x+h)-f(x)}{h}$.

The gradient at P is the slope of the tangent at P to the curve.

CHAPTER XVII.

DIFFERENTIATION.

NOTATION.

FIGURE 33 represents the graph of any function $y=f(x)$.

If $ON=x$, then $PN=y=f(x)$.

Q is any point close to P on the curve and QM its ordinate.

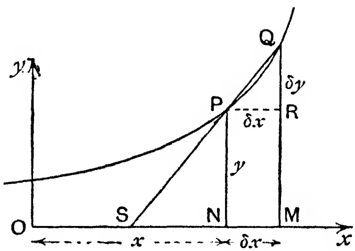


FIG. 33.

The length of NM is represented by the symbol δx, which means "a small increment of the variable x," or colloquially "a little bit of x."

And δy represents the *consequent* change in y; so that $\delta y = QM - PN = QR$.

Thus
$$ON=x; \quad OM=x+\delta x;$$
$$PN=y=f(x); \quad QM=y+\delta y=f(x+\delta x);$$
$$\therefore \ \delta y=f(x+\delta x)-f(x),$$
and
$$\frac{QR}{PR}=\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}.$$

Now $\frac{QR}{PR}$ = the gradient of the chord PQ.

$\therefore$ the gradient of the tangent at P

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x}$$

= rate of change of $f(x)$ with respect to x.

The expression $\underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x}$ is written $\frac{dy}{dx}$ or $\frac{d}{dx}y$, and is called the *differential coefficient* of y with respect to x.

The expression $\underset{\delta x \to 0}{\mathrm{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x}$ is written $\frac{df(x)}{dx}$ or $\frac{d}{dx}f(x)$, and is called the *differential coefficient* of $f(x)$ with respect to x.

The process of finding this limit is called "*differentiating with respect to x*," and the result is sometimes called the "*derived function*" of $f(x)$ and written $f'(x)$.

SUMMARY.

(i) $\frac{d}{dx}f(x) = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x}$.

(ii) If $y = f(x)$, $\frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x}$.

Example I. Differentiate $3x^2+7$ with respect to x.

$$\frac{d}{dx}(3x^2+7) = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{[3(x+\delta x)^2+7]-[3x^2+7]}{\delta x}$$

$$= \mathrm{Lt} \frac{[3\{x^2+2x\,\delta x+(\delta x)^2\}+7]-3x^2-7}{\delta x}$$

$$= \mathrm{Lt} \frac{6x\,\delta x+3(\delta x)^2}{\delta x}$$

$$= \mathrm{Lt}\,[6x+3\,\delta x], \text{ provided } \delta x \neq 0,$$

$$= 6x.$$

Example II. Find $\frac{d}{dx}\left(\frac{1}{x}\right)$.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \underset{\delta x \to 0}{\text{Lt}} \left(\frac{\frac{1}{x+\delta x} - \frac{1}{x}}{\delta x}\right)$$

$$= \text{Lt}\,\frac{x-(x+\delta x)}{x(x+\delta x)\,.\,\delta x}$$

$$= \text{Lt} - \frac{\delta x}{(x^2+x\,.\,\delta x)\,.\,\delta x}$$

$$= \text{Lt} - \frac{1}{x^2+x\,.\,\delta x}, \text{ provided } \delta x \neq 0,$$

$$= -\frac{1}{x^2}.$$

EXERCISE XVII. a.

1. (i) Simplify $5(x+\delta x)-5x$. (ii) Find $\frac{d}{dx}(5x)$.

2. (i) Simplify $(x+\delta x)^2-x^2$. (ii) Find $\frac{d}{dx}(x^2)$.

3. (i) Simplify $[5(x+\delta x)^2-3(x+\delta x)+7]-[5x^2-3x+7]$.

(ii) Find $\frac{d}{dx}[5x^2-3x+7]$.

4. (i) If $f(x) \equiv (1+x)^2$, what is $f(x+\delta x)$?

(ii) Find $\frac{d}{dx}(1+x)^2$.

5. Differentiate $x(1+x)$.

6. Differentiate (i) x^2; (ii) $3x^2$; (iii) $7x^2$, and write down the value of $\frac{d}{dx}(100x^2)$.

7. Differentiate (i) x^3; (ii) $4x^3$; (iii) $5x^3$; and write down the value of $\frac{d}{dx}(29x^3)$.

8. Given that $\underset{\delta x \to 0}{\text{Lt}}\,\frac{(x+\delta x)^4-x^4}{\delta x} = 4x^3$, write down the values of

(i) $\frac{d}{dx}(3x^4)$; (ii) $\frac{d}{dx}(10x^4)$.

9. Given that $\frac{d}{dx}x^5 = 5x^4$, write down the values of

(i) $\frac{d}{dx}(7x^5)$; (ii) $\frac{d}{dx}(x^5+7)$.

10. Given that $\frac{d}{dx}x^6 = 6x^5$, write down the values of

(i) $\frac{d}{dx}(\frac{3}{4}x^6)$; (ii) $\frac{d}{dx}(5 - x^6)$; (iii) $\frac{d}{dx}(x - 2x^6)$.

11. Express in the limit form the fact that $\frac{d}{dx}x^n = nx^{n-1}$.

12. Use the facts that

$$\underset{\delta x \to 0}{\text{Lt}} \frac{(x+\delta x)^2 - x^2}{\delta x} = 2x \quad \text{and} \quad \underset{\delta x \to 0}{\text{Lt}} \frac{(x+\delta x)^3 - x^3}{\delta x} = 3x^2$$

to write down the values of

(i) $\underset{\delta x \to 0}{\text{Lt}} \frac{[(x+\delta x)^3 + 2(x+\delta x)^2] - [x^3 + 2x^2]}{\delta x}$;

(ii) $\frac{d}{dx}(3x^2 + x^3)$; (iii) $\frac{d}{dx}(\frac{2}{3}x^3 + \frac{1}{2}x^2 + 5)$.

13. Express as limits

(i) $\frac{d}{dx}f(x)$; (ii) $\frac{d}{dx}\phi(x)$; (iii) $\frac{d}{dx}[3f(x) - 5\phi(x)]$.

If $\frac{d}{dx}f(x) = u$ and $\frac{d}{dx}\phi(x) = v$, what is $\frac{d}{dx}[3f(x) + 5\phi(x)]$?

14. Find

(i) $\frac{d}{dx}(3x)$; (ii) $\frac{d}{dx}(5x)$; (iii) $\frac{d}{dx}(3x \times 5x)$; (iv) $\frac{d}{dx}(3x + 5x)$.

Is $\frac{d}{dx}(3x \times 5x)$ equal to $\frac{d}{dx}(3x) \times \frac{d}{dx}(5x)$?

Is $\frac{d}{dx}(3x + 5x)$ equal to $\frac{d}{dx}(3x) + \frac{d}{dx}(5x)$?

15. (i) Express as limits $\frac{d}{dx}(3x^2)$; $\frac{d}{dx}(5x^3)$; $\frac{d}{dx}(3x^2 + 5x^3)$; $\frac{d}{dx}(3x^2 - 5x^3)$; $\frac{d}{dx}(3x^2 \times 5x^3)$; $\frac{d}{dx}(5x^3 \div 3x^2)$.

(ii) Is $\frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2)$?

(iii) Is $\frac{d}{dx}(3x^2 \pm 5x^3) = 3\frac{d}{dx}(x^2) \pm 5\frac{d}{dx}(x^3)$?

(iv) Is $\frac{d}{dx}(3x^2 \times 5x^3) = 3\frac{d}{dx}(x^2) \times 5\frac{d}{dx}(x^3)$?

(v) Is $\frac{d}{dx}(5x^3 \div 3x^2) = 5\frac{d}{dx}(x^3) \div 3\frac{d}{dx}(x^2)$?

16. What general formula covers the following facts?

$$\frac{d}{dx}x^2=2x; \quad \frac{d}{dx}x^3=3x^2; \quad \frac{d}{dx}x^4=4x^3; \quad \frac{d}{dx}x^5=5x^4.$$

17. What general formula covers the following facts?

$$\frac{d}{dx}(7x^2)=7\frac{d}{dx}(x^2); \quad \frac{d}{dx}(11x^4)=11\frac{d}{dx}(x^4); \quad \frac{d}{dx}\left(\frac{93}{x}\right)=93\frac{d}{dx}\left(\frac{1}{x}\right).$$

18. What general formula covers the following facts?

$$\frac{d}{dx}(x^2+x^3)=\frac{d}{dx}(x^2)+\frac{d}{dx}(x^3); \quad \frac{d}{dx}(7x^2-11x^5)=\frac{d}{dx}(7x^2)-\frac{d}{dx}(11x^5).$$

19. Take simple functions for $f(x)$ and $\phi(x)$ to show that $\frac{d}{dx}[f(x)\times\phi(x)]$ is not equal to $\frac{d}{dx}f(x)\times\frac{d}{dx}\phi(x)$.

20. Write down special cases of the general formula

$$\frac{d}{dx}(x^n)=nx^{n-1}$$

when n equals (i) 8; (ii) 50; (iii) -1; (iv) -3; (v) 1; (vi) 0; (vii) $\frac{3}{2}$; (viii) $\frac{1}{2}$; (ix) $-\frac{1}{2}$; (x) $-m$.

21. If $y=x^2$, find δy when $x=2$, $\delta x=0{\cdot}1$.

22. If $y=\frac{1}{x}$, find δy when $x=3$, $\delta x=\frac{1}{3}$.

23. If $y=x^3$, find δy when (i) $x=1$, $\delta x=0{\cdot}1$; (ii) $x=1$, $\delta x=0{\cdot}01$.

24. If $y=3x^2$, find δx when $x=2$, $\delta y=0{\cdot}1$.

25. If $y=x^2+x$, find δy when $x=10$, $\delta x=-1$.

Summary of Results.

(i) $\frac{d}{dx}(x^n)=nx^{n-1}$, where n is integral or fractional, positive or negative.

(ii) $\frac{d}{dx}(C)=0$, if C is any constant, *i.e.* a number independent of x.

(iii) $\frac{d}{dx}(Cx^n)=C\frac{d}{dx}(x^n)$
$=Cnx^{n-1}$, where C is any constant.

(iv) $\frac{d}{dx}[f(x)\pm\phi(x)]=\frac{d}{dx}[f(x)]\pm\frac{d}{dx}[\phi(x)]$.

Example III. Find $\frac{d}{dx}(x)$; $\frac{d}{dx}\left(\frac{1}{x^2}\right)$; $\frac{d}{dx}(\sqrt{x})$.

Since $$\frac{d}{dx}(x^n) = nx^{n-1}$$

(i) Put $n=1$, $\therefore \frac{d}{dx}x = 1x^{1-1} = x^0 = 1$,

or, more simply, $\frac{d}{dx}(x) = \underset{\delta x \to 0}{\text{Lt}} \frac{(x+\delta x)-x}{\delta x} = \text{Lt}\,\frac{\delta x}{\delta x} = 1.$

(ii) Put $n = -2$, $\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = -2 \times x^{-3}$

$$= -2 \times \frac{1}{x^3} = -\frac{2}{x^3}.$$

iii) Put $n = \frac{1}{2}$, $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

$$= \frac{1}{2} \times \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2} \times \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Example IV. Find $\frac{d}{dx}\left(3x^4 - 7x + 5 - \frac{2}{x}\right)$.

$$\begin{aligned}\text{The expression} &= \frac{d}{dx}(3x^4) - \frac{d}{dx}(7x) + \frac{d}{dx}(5) - \frac{d}{dx}\left(\frac{2}{x}\right)\\ &= 3\frac{d}{dx}(x^4) - 7\frac{d}{dx}(x) + 0 - 2\frac{d}{dx}(x^{-1})\\ &= 3(4x^3) - 7(1) + 0 - 2(-1)(x^{-1-1})\\ &= 12x^3 - 7 + 2x^{-2}\\ &= 12x^3 - 7 + \frac{2}{x^2}.\end{aligned}$$

After a little practice, most of the intermediate steps in the working can be omitted.

SUCCESSIVE DIFFERENTIATION.

If $y = x^4$, $$\frac{dy}{dx} = 4x^3;$$

$$\therefore \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(4x^3) = 4\frac{d}{dx}(x^3) = 4 \times 3x^2 = 12x^2.$$

For the sake of brevity, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is written $\frac{d^2y}{dx^2}$: this symbol is called the second differential coefficient of y with respect to x.

Similarly $\dfrac{d^2s}{dt^2}$ is short for $\dfrac{d}{dt}\left(\dfrac{ds}{dt}\right)$.

Example V. Find $\dfrac{d^2}{dx^2}(3x^2+4x-7)$.

$$\frac{d}{dx}(3x^2+4x-7)=\frac{d}{dx}(3x^2)+\frac{d}{dx}(4x)-\frac{d}{dx}(7)=6x+4\,;$$

$$\therefore\ \frac{d^2}{dx^2}(3x^2+4x-7)=\frac{d}{dx}(6x+4)=6.$$

EXERCISE XVII. b.

Differentiate with respect to x the expressions in Examples 1-30.

1. x^7. **2.** $10x^3$. **3.** $x+\dfrac{1}{x}$. **4.** $\dfrac{1}{x^5}$.

5. $\dfrac{3}{x}$. **6.** $\dfrac{2}{x^2}$. **7.** $3x^2-2x$. **8.** $\frac{1}{4}x^4-2$.

9. $x^2\times x^3$. **10.** $(5x)^3$. **11.** $(1-x)^2$. **12.** $\dfrac{1}{x^n}$.

13. ax. **14.** x^n. **15.** $\dfrac{c}{x^3}$. **16.** $b\sqrt{x}$.

17. $\sqrt{x^3}$. **18.** $\dfrac{1}{\sqrt{x}}$. **19.** x^0. **20.** $\sqrt[3]{x}$.

21. $\dfrac{2}{3}x^3-\dfrac{x}{5}$. **22.** $\dfrac{x}{10}+\dfrac{10}{x}$. **23.** $7x^3-2x-5$. **24.** $\dfrac{4}{3}\pi x^3$.

25. $3x^4-\dfrac{1}{2}x+7-\dfrac{6}{x}$. **26.** $5x^4+3x^2+\dfrac{1}{x}-\dfrac{1}{x^2}$.

27. $(x^2+1)(x+2)$. **28.** $\left(x+\dfrac{1}{x}\right)^2$. **29.** x^{2n}. **30.** x^{-k}.

31. If $y=3x^2+5$, find $\dfrac{dy}{dx}$ when $x=1$.

32. If $y=1+2x-x^2$, find $\dfrac{dy}{dx}$ when $x=-2$.

33. If $y=2x^3-9x^2+12x$, find the values of x for which $\dfrac{dy}{dx}=0$.

34. If $y=7x^3-9x$, find $\dfrac{d^2y}{dx^2}$. **35.** If $y^2=x$, find $\dfrac{d^2y}{dx^2}$.

36. If $y=6x^2$, prove that $x\dfrac{dy}{dx}=2y$.

37. If $y=x(1-x)$, prove that $1+\dfrac{dy}{dx}=\dfrac{2y}{x}$.

38. If $y=x^3$, prove that $\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}=18y$.

39. If $s=100t-16t^2$, find $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ when $t=1$ and $t=0$.

40. If $y=7x^4$, express (i) $2y\cdot\frac{dy}{dx}$; (ii) $\frac{d}{dx}(y^2)$ in terms of x.

41. If $y=2z^2$, $z=3+5x$, express in terms of x, (i) $\frac{dy}{dz}$; (ii) $\frac{dz}{dx}$; (iii) y; (iv) $\frac{dy}{dx}$; hence show that $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$.

42. If $y=z+z^3$, $z=1+x^2$, express in terms of x, (i) $\frac{dy}{dz}$; (ii) $\frac{dz}{dx}$; (iii) y; (iv) $\frac{dy}{dx}$; hence show that $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$.

43. The cubical elasticity of a fluid is equal to $-v\cdot\frac{dp}{dv}$ where the volume v and the pressure p are connected by the equation $pv=c$ (a constant). Simplify this expression.

44. The radius of the circle which approximates most closely to the shape of the curve $y=\frac{2x^2}{3}$ at any point is

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}\div\frac{d^2y}{dx^2};$$

find its value when $x=1$.

TURNING POINTS.

A function $f(x)$ is called an "*increasing function*" of x for any range of values of x in which, as x increases, the value of $f(x)$ also increases.

It is called a "*decreasing function*" of x if, as x increases, the value of $f(x)$ decreases.

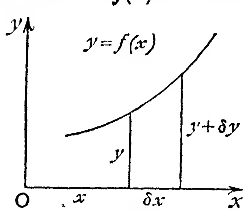


FIG. 34.

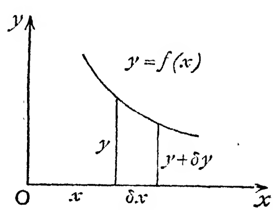


FIG. 35.

Figure 34 represents an increasing function.
Figure 35 represents a decreasing function.

If δx is positive, then δy is positive for an increasing function (Figure 34), and δy is negative for a decreasing function (Figure 35).

Since $\dfrac{dy}{dx} = \underset{x \to 0}{\text{Lt}} \dfrac{\delta y}{\delta x}$, we see that $\dfrac{dy}{dx}$ is positive for an increasing function, and is negative for a decreasing function.

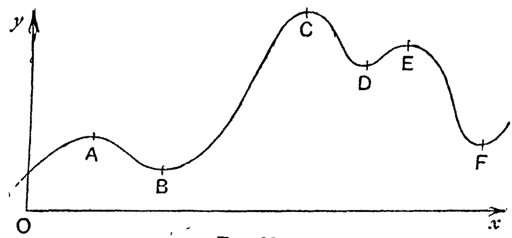


FIG. 36.

Figure 36 represents a function which is an increasing function for some ranges of values of x, and a decreasing function for others.

Thus $f(x)$ is a decreasing function from A to B, and an increasing function from B to C.

The separating point B is called a *turning point*, and the value of $f(x)$ corresponding to the point B is called a *turning value* of the function.

There are two kinds of turning points.

$A, C, E, \ldots$ correspond to values of x for which the function is greater than at any other point near it: at these points the function is said to be a *maximum*.

$B, D, F, \ldots$ correspond to values of x for which the function is less than at any other point near it: at these points the function is said to be a *minimum*.

At a turning point, $\dfrac{dy}{dx} = 0$, for the function is neither increasing $\left(i.e.\ \dfrac{dy}{dx}\text{ positive}\right)$ nor decreasing $\left(i.e.\ \dfrac{dy}{dx}\text{ negative}\right)$.

EXERCISE XVII. c.

1. Fig. 37 is the graph of $y=x^2$. Is y an increasing or decreasing function (i) from A to O, (ii) from O to B ? Is $\frac{dy}{dx}$ positive or negative (α) from A to O, (β) from O to B ? Has y a maximum or minimum value anywhere ?

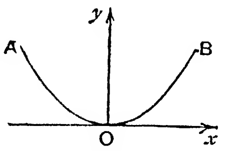


FIG. 37.

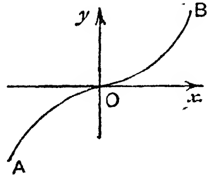


FIG. 38.

2. Fig. 38 is the graph of $y=x^3$. Answer the same questions as in Ex. 1.

3. Fig. 39 is the graph of $y=4+3x-x^2$. Is y an increasing or decreasing function from (i) A to B, (ii) B to C, (iii) C to D, (iv) D to E, (v) E to F ? What are the signs of $\frac{dy}{dx}$ for these five portions ? Has y a maximum or minimum value anywhere ?

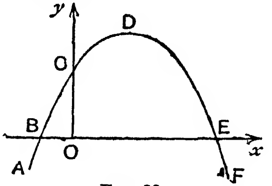


FIG. 39.

4. Draw freehand the graph of a function $y=f(x)$ which starts from the origin O, and is such that

(i) from O to A, y is a decreasing function ;

(ii) from A to B, $\frac{dy}{dx}$ is positive and y is negative ;

(iii) from B to C, y is a positive increasing function ;

(iv) from C to D, $\frac{dy}{dx}$ is negative and y is positive ;

(v) from D to E, $\frac{dy}{dx}$ is negative and y is negative ;

(vi) from E to F, $\frac{dy}{dx}$ is positive.

Has the function any maximum or minimum values ?

5. Draw freehand the graph of a function $y=f(x)$ for which

(i) x is negative, y is negative, $\frac{dy}{dx}$ is positive ;

(ii) x is negative, y is negative, $\frac{dy}{dx}$ is negative ;

(iii) x is negative, y is positive, $\frac{dy}{dx}$ is positive ;

(iv) x is negative, y is positive, $\frac{dy}{dx}$ is negative.

6. (i) Draw freehand the graph $ABCDE$ of a function $y=f(x)$ such that the values of $\frac{dy}{dx}$ (or the gradients of the graph) at A, B, C, D, E are respectively 1, $\frac{1}{2}$, 0, $-\frac{1}{2}$, -1.

(ii) What kind of a point is C ?

(iii) Is $\frac{dy}{dx}$ an increasing or decreasing function for the arc AC and the arc CE ?

(iv) What is the sign of $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$ for points on the arc AC and the arc CE ?

7. (i) Draw freehand the graph $ABCDE$ of a function $y=f(x)$ such that the values of $\frac{dy}{dx}$ (or the gradients of the graph) at A, B, C, D, E are respectively -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1.

(ii) What kind of a point is C ?

(iii) Is $\frac{dy}{dx}$ an increasing or decreasing function for the arc AC and the arc CE ?

(iv) What is the sign of $\frac{d^2y}{dx^2}$ for points on the arc AC and the arc CE ?

8. (i) Draw freehand the graph $ABCDE$ of a function $y=f(x)$ such that the values of $\frac{dy}{dx}$ (or the gradients of the graph) at A, B, C, D, E are respectively 1, $\frac{1}{2}$, 0, $\frac{1}{2}$, 1.

(ii) Is C a turning point ?

(iii) Is $\frac{dy}{dx}$ an increasing or decreasing function for the arc AC and the arc CE ?

(iv) What is the sign of $\frac{d^2y}{dx^2}$ for points on the arc AC and the arc CE ?

9. Answer the various questions in Ex. 8, taking the values of $\frac{dy}{dx}$ at A, B, C, D, E to be respectively -1, $-\frac{1}{2}$, 0, $-\frac{1}{2}$, -1.

10. AB, CD, EF, GH are portions of the graph of $y=f(x)$.

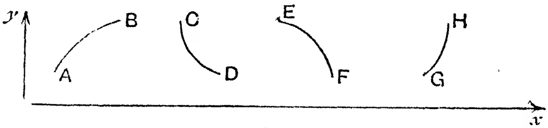


FIG. 40.

What can you say about the signs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the arcs

(i) AB; (ii) CD; (iii) EF; (iv) GH ?

11. Can one of the minimum values of a function be greater than one of its maximum values ? Illustrate by a figure.

12. Can a function have (i) exactly one minimum and two maximum values, (ii) exactly one minimum and three maximum values ? Illustrate by a figure.

13. Part of the graph of the function $y=f(x)$ is represented by the curve $ACBDPQ$ in Fig. 31, p. 334; copy this freehand, and underneath it draw roughly the corresponding portions of the graphs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

FIG. 41.

14. Fig. 41 represents part of the graph of $\frac{dy}{dx}$; draw roughly the corresponding portions of the graph of y and of $\frac{d^2y}{dx^2}$.

15. Part of the graph of the function $y=f(x)$ is represented by the curve $ABCDEF$ in Fig. 36, p. 347 ; show in tabular form the signs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the various portions of the graph, and state at which points either $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ is zero.

16. Fig. 42 shows the graph of $y=x^2$; the tangent Ox at T; PN is perpendicular to Ox; $ON=x$, $NP=$ that (i) $\frac{PN}{NT}=$ the slope at $P=2x$; (ii) $OT=TN$.

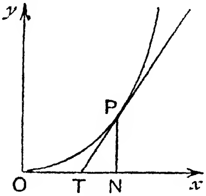


FIG. 42.

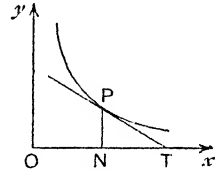


FIG. 43.

17. Fig. 43 shows the graph of $y=\frac{1}{x}$; the tangent at P meets Ox at T; PN is perpendicular to Ox; $ON=x$, $NP=y$; prove that (i) $\frac{PN}{NT}=-$ the slope at $P=\frac{1}{x^2}$; (ii) $ON=NT$.

18. If Fig. 42 represents the graph of $y=3x^2$ and if P is the point (2, 12), find the length of OT.

19. If Fig. 43 represents the graph of $y=\frac{8}{x^2}$ and if P is the point (2, 2), find the length of OT.

20. If Fig. 42 represents the graph of $y=x^3$, prove that

$$OT=\tfrac{2}{3}ON.$$

21. If Fig. 43 represents the graph of $y^2=\frac{1}{x^3}$, prove that

$$ON=\tfrac{2}{5}\,.\,OT.$$

SUMMARY OF RESULTS.

For the function $y=f(x)$,

(i) $\frac{dy}{dx}=0$, both at a maximum and a minimum.

(ii) $\frac{d^2y}{dx^2}$ is negative at a maximum;

$\frac{d^2y}{dx^2}$ is positive at a minimum.

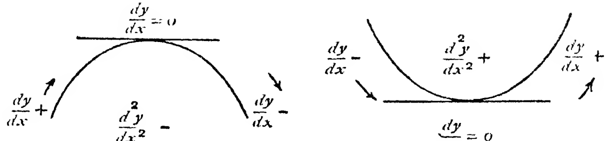


FIG. 44.

$\frac{dy}{dx}$ changes from + to − in passing through a maximum;

$\frac{dy}{dx}$ changes from − to + in passing through a minimum;

or $\frac{dy}{dx}$ is a decreasing function in passing through a maximum;

$\frac{dy}{dx}$ is an increasing function in passing through a minimum;

(iv) If $\frac{dy}{dx}=0$ but does not change sign, there is neither a maximum nor a minimum.

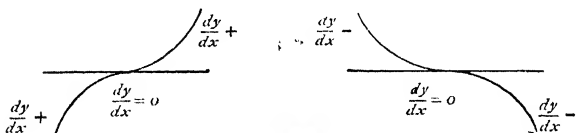


FIG. 45.

MAXIMA AND MINIMA PROBLEMS.

Example VI. The strength of a beam of uniform rectangular section varies as the breadth and the square of the depth. Find the breadth of the strongest rectangular beam that can be cut from a cylindrical tree-trunk of diameter 20 inches.

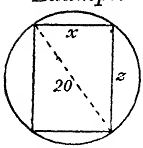


FIG. 46.

Let the breadth be x inches and the depth z inches;

$$\therefore\ x^2+z^2=20^2 \quad \text{(Pythagoras)}$$
$$=400.$$

Now the strength varies as xz^2

$$=kxz^2=kx(400-x^2), \text{ where } k \text{ is a constant,}$$
$$=k(400x-x^3);$$

$\therefore$ the function $y=400x-x^3$ is to be a maximum.

$$\frac{dy}{dx}=400-3x^2;$$

$$\therefore\ \frac{dy}{dx}=0 \text{ if } 400-3x^2=0 \quad \text{or} \quad x^2=\frac{400}{3}=133\cdot\dot{3};$$

$$\therefore\ x=\pm 11\cdot54.$$

Also $$\frac{d^2y}{dx^2}=\frac{d}{dx}(400-3x^2)=-6x;$$

$$\therefore\ \frac{d^2y}{dx^2} \text{ is negative if } x \text{ is positive;}$$

$$\therefore\ x=11{\cdot}54 \text{ gives a } maximum \text{ value for } y;$$

$$\therefore \text{ the required breadth is } 11{\cdot}5 \text{ inches.}$$

Note.—Sometimes it is necessary to find the maximum or minimum values of expressions which are not in a form that can be differentiated by the rules already considered. It may be possible however, either to change the form by substituting or to avoid the difficulty in some other way.

Example VII. Find (i) the value of x for which $\sqrt{x^2-8x+21}$ is a minimum, (ii) the value of x for which $\frac{x-1}{(x+1)^2}$ is a maximum.

(i) $\sqrt{x^2-8x+21}$ has its least value if $x^2-8x+21$ is a minimum;

$$\therefore\ \frac{d}{dx}(x^2-8x+21)=0;$$

$$\therefore\ 2x-8=0 \quad \text{or} \quad x=4.$$

It is a minimum because $\frac{d^2}{dx^2}(x^2-8x+21)=2$ and is therefore positive.

(ii) Let $y=\frac{x-1}{(x+1)^2}$ and put $x+1=z$;

$$\therefore\ y=\frac{z-2}{z^2}=\frac{1}{z}-\frac{2}{z^2};$$

$$\therefore\ \frac{dy}{dz}=-\frac{1}{z^2}+\frac{4}{z^3}=\frac{(-z+4)}{z^3};$$

$$\therefore\ \frac{dy}{dz}=0 \text{ if } z=4 \quad \text{or} \quad x+1=4 \quad \text{or} \quad x=3.$$

Also $$\frac{d^2y}{dz^2}=\frac{2}{z^3}-\frac{12}{z^4}=\frac{2z-12}{z^4}=\frac{-4}{4^4} \text{ if } z=4;$$

$$\therefore \text{ for } z=4,\ y \text{ is a maximum, since } \frac{d^2y}{dz^2} \text{ is negative;}$$

$$\therefore \text{ for } x=3,\ \frac{x-1}{(x+1)^2} \text{ is a maximum.}$$

EXERCISE XVII. d.

1. Find the values of x which correspond to turning values of the following functions; determine whether they are maxima or minima; and sketch roughly the graphs of the functions:

(i) $x^2 - 2x$; (ii) $x^2 + 4x + 3$; (iii) $3 + 8x - 10x^2$;

(iv) $x + \frac{1}{x}$; (v) $x - \frac{1}{x}$; (vi) $x^4 - 4x$;

(vii) $x^4 - x^3$; (viii) $x^3 - x^2 - x + 1$: (ix) $x^3 - 3x^2 - 9x + 7$;

(x) $x^3 - 3x^2 + 3x - 1$.

2. Find the area of the largest rectangular piece of ground that can be enclosed by 200 hurdles each 4 feet long.

3. The parcel post regulations require that the sum of the length and girth of a parcel shall not exceed 6 feet. Find the volume of the largest box with a square base that can be sent by post.

4. A closed rectangular cistern is to be constructed to contain 80 cu. feet. It is to be 5 feet long. Find the breadth when the *total* area of its surface is a minimum.

5. The strength of a rectangular beam varies as the breadth and the square of the depth. Find the breadth of the strongest rectangular beam which has a perimeter of 4 feet.

6. Find the area of the largest rectangular piece of ground that can be enclosed by 200 hurdles each 4 feet long, if an existing fence is utilised to form one side.

7. A box without a lid is to be made from a sheet of metal of negligible thickness and is to have square ends; it is to hold $4\frac{1}{2}$ cu. feet. What is the least area of metal required?

8. For a steamer travelling v knots, the cost of the coal is £$\frac{v^2}{18}$ per hour, and other expenses amount to £8 per hour. What is the most economical speed for a journey of 200 nautical miles?

9. An open gutter of rectangular section is formed out of a long rectangular strip of sheet iron 9 feet wide. Find the maximum area of the cross-section.

10. A is 8 miles north and B is 6 miles east of a point O. Two men, starting at the same time from A and B, walk towards O at 4 miles an hour. If their distance apart is x miles after t hours, prove that $x^2 = 32t^2 - 112t + 100$, and find when they are nearest together.

11. A rectangular sheet of cardboard is 8″ long and 5″ wide. Equal squares are cut out at each of the corners and the remainder is folded so as to form an open box. Find the maximum volume of the box.

12. Given $\pi r^2 h = 5$, find the value of r for which $2\pi r^2 + 2\pi rh$ is a minimum. What geometrical problem corresponds to this question?

13. A particle projected in a resisting medium is finally brought to rest: it travels s feet in t seconds, where $s = 6t - \frac{1}{2}t^3$. How far does it go?

14. Using the data of Ex. 3, find the volume of the largest circular cylinder that can be sent by parcel post.

15. If a very thin rod one foot long swings like a pendulum, the expression $x(1-x)^2$ measures the tendency to break at a place x feet from the point of suspension. Find where the rod is most likely to break.

16. The perimeter of a sheet of metal in the form of a circular sector is 1 foot. For what radius is the area a maximum?

17. From a circular sheet of paper a sector is removed and the remainder is folded to form a circular cone. What fraction must be removed in order to give the cone of maximum volume?

18. A rod one foot long is cut into two pieces to form the hypotenuse and one side of a right-angled triangle. How must it be cut to give the triangle of greatest area?

19. The efficiency of a screw of certain material is $\dfrac{4h - 3h^2}{3 + 4h}$, where h is the tangent of its pitch. What is the greatest efficiency? [Put $3 + 4h = x$.]

20. Assuming that the rigidity of a rectangular beam varies as its breadth and the cube of its depth, find the breadth of the most rigid beam that can be cut from a cylindrical trunk of diameter 3 feet.

21. A cylindrical vessel is open at one end and closed at the other; for a given surface, prove that the volume is greatest if its height equals the radius of the base.

22. A skeleton box with two square ends is formed with 12 pieces of wire, and four other pieces of wire form an equal square round the middle of it. The total amount of wire available is one yard. What is the maximum volume of the box?

23. A piece of wire 2 feet long is cut into two parts, one of which is bent to form a square and the other a circle. If the sum of the areas is a minimum, find the radius of the circle.

24. If $2x + y = 5$, what is the greatest value of $x^2 + 3xy + y^2$?

25. What is the greatest value of $\dfrac{x}{x^2 + 4}$?

26. What is the greatest value of $\dfrac{x}{(x^2 + 1)^2}$?

27. Find the minimum value of $\sqrt{\frac{x}{a}+\frac{a}{x}}$.

28. Find the radius of a circular cylinder which is cut from a sphere of radius 10 inches so that

(i) the volume of the cylinder is a maximum ;

(ii) the curved surface of the cylinder is a maximum.

29. P is a variable point on the line ABC ; $AB=2''$, $BC=3''$. Find the position of P for which $PA^2+PB^2-PC^2$ is a minimum.

30. ABC is a triangular field right-angled at A ; P, Q are points on AB, AC such that a fence from P to Q bisects the field. If $AB=a$, $AC=b$, $AP=x$, $PQ=y$ yards, express y as a function of x. If $a=625$, $b=800$, find the length of AP in order that the fence may be as short as possible; find also the length of the fence. (C.S.C.)

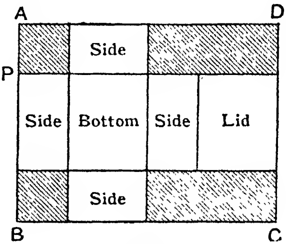


FIG. 47.

31. $ABCD$ is a rectangular sheet of cardboard (see Fig. 47) from which the shaded portions are removed. The remainder is used to make a closed box, as shown in the figure. If $AB=a$, $BC=b$, $AP=x$ inches, find the volume of the box in terms of a, b, x ; if $a=6$, $b=12$, find the value of x for which the volume is a maximum. (C.S.C.)

APPROXIMATION.

Figure 48 represents the graph of $y=f(x)$.

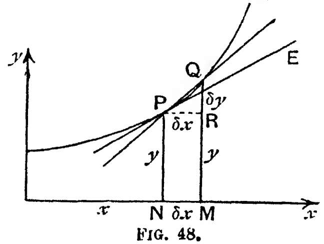


FIG. 48.

With the usual notation,

the slope of the chord PQ is $\frac{\delta y}{\delta x}$,

the slope of the tangent PE is $\frac{dy}{dx}$.

These are obviously not equal to each other.

But $\frac{dy}{dx}$ does equal the limit of $\frac{\delta y}{\delta x}$, when $\delta x \to 0$.

And we may say that

$$\frac{dy}{dx} \simeq \frac{\delta y}{\delta x}, \text{ when } \delta x \text{ is small,}$$

or
$$\delta y \simeq \frac{dy}{dx} \times \delta x, \text{ when } \delta x \text{ is small.}$$

The smaller δx is, the nearer the slope of the chord PQ is to the slope of the tangent PE.

Example VIII. If $y = x^2$, what is the error in the approximation $\delta y \simeq \frac{dy}{dx} \times \delta x$?

Now
$$y + \delta y = (x + \delta x)^2;$$
$$\therefore \; y + \delta y = x^2 + 2x \,.\, \delta x + (\delta x)^2,$$
but
$$y = x^2;$$
$$\therefore \; \delta y = 2x \,.\, \delta x + (\delta x)^2.$$
But
$$\frac{dy}{dx} = \frac{d}{dx}(x^2) = 2x;$$
$$\therefore \; \delta y = \frac{dy}{dx} \times \delta x + (\delta x)^2;$$
$$\therefore \; \delta y \simeq \frac{dy}{dx} \times \delta x \text{ with error } (\delta x)^2.$$

The relative size of the error may be seen in a figure.

If $y = x^2$, y sq. inches is the area of a square of side x inches, and $(y + \delta y)$ sq. inches is the area of a square of side $(x + \delta x)$ inches.

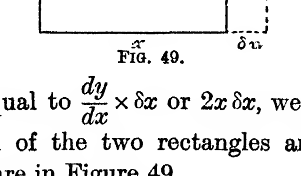


FIG. 49.

In taking δy equal to $\frac{dy}{dx} \times \delta x$ or $2x\,\delta x$, we are supposing its value is the sum of the two rectangles and neglecting the small shaded square in Figure 49.

Suppose x is 1″ and δx is 0·1″, the error is 0·01 sq. in.

EXERCISE XVII. e.

1. Find the error in the relation $\delta y \simeq \frac{dy}{dx} \times \delta x$ if

(i) $y=5x^2$; (ii) $y=x^3$.

2. The area of an isosceles right-angled triangle of side x inches is A sq. ins.

(i) Find geometrically the value of δA in terms of δx.

(ii) Express A in terms of x, and evaluate $\delta A - \frac{dA}{dx} \times \delta x$.

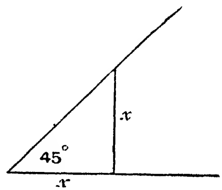


FIG. 50.

3. (i) Find an approximate expression for the increase in area of a circle when the radius increases from r to $r+\delta r$.

(ii) Illustrate the result geometrically.

(iii) Find approximately the difference in area of two circles of radii 10″ and 10·01″.

4. The radius of a spherical soap bubble increases from 1″ to 1·01″; find approximately its change of volume.

5. A body travels s feet in t sec., where $s=10t-\frac{1}{6}t^3$; find an approximate expression (i) for δs in terms of δt, (ii) for the distance it moves in the interval of time $t=2$ to $t=2{\cdot}1$.

6. A gas at constant temperature under a pressure of p lb. per sq. inch occupies v cu. inches, where $pv=c$ (a constant); if the pressure is increased from p to $p+\delta p$, find an approximate expression for the change of volume.

7. Two telegraph posts of equal height are at a distance

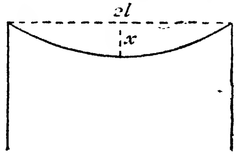


FIG. 51.

$2l$ feet apart. When the sag at the mid-point is x feet, the length of wire exceeds the distance between the poles by y feet, where

$y=\frac{4x^2}{3l}$. Find an approximate value for δy in terms of δx. What does this mean? If the distance between the poles is 30 yards, what is the effect of increasing the sag from 10 inches to 11 inches?

8. At sea-level, water boils at 212° F. At a height h feet above sea-level, the boiling point is lowered t degrees, where $h=520t+t^2$; find approximately the difference of heights of two places where the boiling points are 200° and 201° F. respectively.

9. The perimeter of a circular pond is measured as 1321 yards; assuming this is correct to the nearest yard, what error may be expected in the area of the pond when calculated from this result?

10. A beam AB 30 feet long supported at each end is just strong enough to carry a load of M tons placed at a point P on the beam such that $M=\frac{1}{10}\left(k+\frac{1}{k}+\frac{1}{2}\right)$, where $\frac{AP}{PB}=k$. Express δM in terms of k and δk. What does this mean? Find approximately what change in the load may be necessary when it is shifted from a point 5 feet from A to a point 6 feet from A?

11. Fig. 52 represents the graph of $y=x^2$, unit 1″ on each axis. If $ON=x$, it can be proved that the area bounded by ON, NP and the arc OP is $A=\frac{1}{3}x^3$ sq. in. Find approximately δA in terms of δx, and interpret the result geometrically.

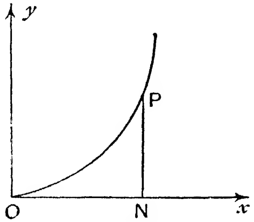


FIG. 52.

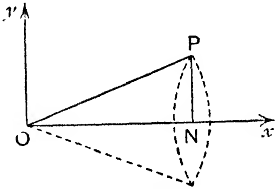


FIG. 53.

12. In Fig. 53 OP is the graph of $y=\frac{1}{2}x$; a circular cone is obtained by revolving OP about Ox. If $ON=x$ and the volume of the cone is V, prove that $V=\frac{1}{12}\pi x^3$. Find approximately δV in terms of δx, and interpret the result geometrically.

13. The bowl of a wine-glass is formed by revolving the graph of $y=x^2$ (see Fig. 52) about Oy. It can be proved that when the depth of wine in the glass is y inches, the volume is $V=\frac{1}{2}\pi y^2$ cu. in. Find approximately δV in terms of δy, and interpret the result geometrically.

14. If the depth of water in a hemispherical bowl of radius a in. is x in., the volume of the water is $V = \pi x^2\left(a - \frac{x}{3}\right)$ cu. in. Find approximately δV in terms of δx, and interpret the result geometrically.

15. If $y = z^5$ and $z = 1 + 3x$, (i) prove that $\delta y \simeq 5z^4 . \delta z$; (ii) express δz in terms of δx; (iii) express δy in terms of x and δx; (iv) hence find $\frac{dy}{dx}$ and $\frac{d}{dx}(1+3x)^5$.

16. If $y = \sqrt{z}$ and $z = 1 + x^2$, (i) express δy in terms of δz and δz in terms of δx, and so find a relation between δy, δx, x; (ii) hence find $\frac{d}{dx}(\sqrt{1+x^2})$.

17. If $y = z^3$ and $z = x^3 - x + 7$, (i) find a relation between δy, δx, x; (ii) hence find $\frac{d}{dx}[(x^3 - x + 7)^3]$.

18. If $y = \frac{1}{z^2}$ and $z = 3x - 5$, (i) find a relation between δy, δx, x; (ii) hence find $\frac{d}{dx}\left[\frac{1}{(3x-5)^2}\right]$.

19. If $u = y^2$, and if y is a function of x, (i) find a relation between δu, δy, y; (ii) hence express $\frac{du}{dx}$ in terms of $\frac{dy}{dx}$; (iii) prove that $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$.

20. If V cu. in. is the volume of a sphere of radius r in., then $V = \frac{4}{3}\pi r^3$. The radius of a sphere, which is expanding, is r in. after t sec.; (i) express δV in terms of r and δr; (ii) find a relation between $\frac{dV}{dt}$, $\frac{dr}{dt}$ and r; (iii) interpret this result.

RATE OF CHANGE.

Example IX. A vessel is in the shape of a circular cone of semi-vertical angle 45° Water is poured into it at the rate

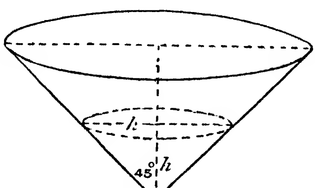


FIG. 54.

of 10 cu. in. per sec. At what rate is the level rising after 4 seconds?

When the depth is h in., the surface of the water is a circle of radius h in.;

$\therefore$ the volume of water in the vessel is

$$V = \tfrac{1}{3}\pi h^2 \times h = \tfrac{1}{3}\pi h^3 \text{ cu. in.};$$

$$\therefore \frac{dV}{dh} = \tfrac{1}{3}\pi \times 3h^2 = \pi h^2;$$

$$\therefore \delta V \simeq \pi h^2 . \delta h.$$

Suppose the volume increases from V to $V + \delta V$ in δt secs.

Then $$\delta V = 10 . \delta t;$$

$$\therefore \pi h^2 . \delta h \simeq 10 . \delta t;$$

$$\therefore \frac{\delta h}{\delta t} \simeq \frac{10}{\pi h^2}.$$

Now, when $t = 4$, $V = 40$;

$$\therefore \tfrac{1}{3}\pi h^3 = 40;$$

$$\therefore h^3 = \frac{120}{\pi};$$

$$\therefore h = \sqrt[3]{\frac{120}{\pi}} = 3{\cdot}368;$$

$$\begin{array}{r} 2{\cdot}0792 \\ 0{\cdot}4971 \\ 3\,|\,\overline{1{\cdot}5821} \\ \overline{0{\cdot}5274} \end{array}$$

$$\therefore \frac{\delta h}{\delta t} \simeq \frac{10}{\pi(3{\cdot}368)^2}$$

$$= 0{\cdot}2806;$$

$$\begin{array}{r} 1{\cdot}0000 \\ 1{\cdot}5519 \\ \overline{\bar{1}{\cdot}4481} \end{array} \qquad \begin{array}{r} 0{\cdot}5274 \\ 2 \\ \overline{1{\cdot}0548} \\ 0{\cdot}4971 \\ \overline{1{\cdot}5519} \end{array}$$

$\therefore$ the level is rising at $0{\cdot}28''$ per sec. approximately.

EXERCISE XVII. f.

1. The temperature of a metal cube is being raised steadily so that each edge expands at the rate of 0·01 inch per hour. At what rate is the volume increasing when the edge is 2 inches ?

2. The cross-section of a trough is an isosceles right-angled triangle : the trough is 10 feet long. Water is poured into it at the rate of 5 cu. feet per sec. Find the rate at which the level is rising after 8 seconds.

3. The area of a circular ink-blot starts from zero and grows at the rate of 4 sq. inches per sec. ; at what rate is the radius increasing (i) when the radius is 1 in., (ii) after 3 seconds ?

4. A man 6 ft. high walks towards an electric arc street light at $3\frac{3}{4}$ miles an hour ; the light is 24 feet above the level of the street. Find the rate in feet per sec. at which his shadow diminishes in length.

5. A certain quantity of gas is contained in a spherical envelope of volume v cu. in.: the pressure is p lb. per sq. inch, where $pv = 25$. The radius of the envelope increases at the rate of 2 in. per minute; find the rate at which the pressure is altering when the radius of the envelope is 5 inches.

6. A wine-glass is shaped so that when the depth of wine in it is y inches, its volume is $\frac{1}{8}y^4$ cu. in.; wine is poured in at the rate of 2 cu. in. per sec.; at what rate is the level rising when the depth is 2 in.?

7. The candle-power C of an incandescent lamp and its voltage V are connected by the equation $C = \frac{5V^6}{10^{11}}$. Find an expression for the rate of change of candle-power per unit increase of voltage, and evaluate it when $V = 100$.

8. ABC is a triangle right-angled at C; P is a point on AB; $PNCM$ is a rectangle with its corners N, M on CA, CB; $CA = 3$, $CB = 4$, $AP = x$ inches; P moves from A to B at 2 in. per minute; at what rate is the area of the rectangle increasing after (i) 60 sec., (ii) 75 sec., (iii) 90 sec.?

9. A cube is expanding so that its volume after t minutes is $1000 + 0{\cdot}2t + 0{\cdot}01t^2$ cu. inches; at what rate is its edge increasing in length after 10 minutes?

10. Sand is dropped on the ground at a steady rate of 10 cu. in. per sec. and forms a conical pile whose height remains equal to the radius of its base; at what rate is the height increasing after 5 seconds?

11. A bowl 5 inches deep is shaped so that when the depth of water in it is x inches, the amount of water is $8x + x^2$ cu. inches. Water is poured into it at the rate of 4 cu. in. per sec.; at what rate is the level rising when the depth is 3 inches?

12. The volume of a spherical envelope is increasing at the rate of 2 cu. cm. per sec.; at what rate is the area of the surface of the envelope stretching when the volume is 100 cu. cm.?

13. The volume of a circular cylinder is constant and equal to 150 cu. cm.; its height increases at the rate of 1 mm. a sec.; at what rate is its radius altering when its height is 3 cm.?

14. The heat required to raise the temperature of 1 gram of water from 0° C. to t° C. is Q units, where $Q = t + \frac{2}{10^5}t^2 + \frac{3}{10^7}t^3$. The specific heat is the rate of increase of the quantity of heat per degree rise in temperature. Find the specific heat at 60° C.

15. Water runs out of a bath at a rate proportional to the amount in the bath at any moment. Originally the bath contains

30 cu. feet of water; if t sec. later Q cu. feet have run out, express the data of the question by an equation.

Summary of Results.

(i) An approximate value for

$$f(x+\delta x)-f(x) \quad \text{is} \quad \frac{d}{dx}f(x)\times\delta x.$$

(ii) If in calculating the value of $f(x)$ there is a small error δx in the value substituted for x, the error in the result is approximately $\frac{d}{dx}f(x)\times\delta x$.

(iii) If a distance x feet is described in t seconds, the speed is $\frac{dx}{dt}$ feet per second.

If y is a function of x and x is a function of t, the rate at which y increases is given by

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt},$$

for $$\delta y \simeq \frac{dy}{dx}\times\delta x \quad \text{and} \quad \delta x \simeq \frac{dx}{dt}\times\delta t.$$

(iv) The formula $\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$ is used to differentiate complicated functions;

$$e.g. \text{ if } y=\sqrt{x} \quad \text{and} \quad x=3t^2-5,$$

then $$y=\sqrt{3t^2-5},$$

and $$\frac{d}{dt}(\sqrt{3t^2-5})=\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$$

$$=\frac{d}{dx}(\sqrt{x})\times\frac{d}{dt}(3t^2-5)$$

$$=\frac{1}{2\sqrt{x}}\times 6t$$

$$=\frac{3t}{\sqrt{3t^2-5}}.$$

CHAPTER XVIII.

INTEGRATION.

Example I. If $\frac{dy}{dx}=x^2+7$, express y in terms of x.

We know that $$\frac{d}{dx}x^3=3x^2;$$

$$\therefore \frac{d}{dx}(\tfrac{1}{3}x^3)=x^2.$$

Also $$\frac{d}{dx}(7x)=7;$$

$$\therefore \frac{d}{dx}(\tfrac{1}{3}x^3+7x)=x^2+7;$$

$\therefore$ *one* value of y is given by $y=\frac{1}{3}x^3+7x$.

But $$\frac{d}{dx}(\tfrac{1}{3}x^3+7x-99) \text{ is also } x^2+7,$$

or $$\frac{d}{dx}(\tfrac{1}{3}x^3+7x+2\tfrac{1}{4}) \text{ is also } x^2+7,$$

because $\frac{d}{dx}$ (any constant) is 0.

$\therefore$ $y=\frac{1}{3}x^3+7x+c$, where c is any constant, satisfies the given equation, and is called its *general solution.*

Note.—An equation involving $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$, etc., is called a *differential equation*: and the process of expressing y as a function of x is called *solving* or *integrating* the differential equation.

Whenever a differential equation is integrated, an arbitrary constant enters into the result.

EXERCISE XVIII. a.

1. Prove that $y = x^4 - 5$ is *a* solution of the differential equation $\frac{dy}{dx} = 4x^3$. What is the general solution ?

2. Prove that $y = \frac{1}{5}x^5 + \frac{1}{2}x^2 + 3$ is *a* solution of the differential equation $\frac{dy}{dx} = x^4 + x$. What is the general solution ?

3. The graph of a curve $y = f(x)$ is such that its gradient is everywhere equal to $2x$. (i) What is the differential equation for the curve ? (ii) Prove that $y = x^2$, $y = x^2 + 2$, $y = x^2 - 1$ are each possible solutions, and interpret this result geometrically by drawing the graphs.

4. Prove that $y = \frac{1}{12}x^4 + 97x - 13$ is *a* solution of the differential equation $\frac{d^2y}{dx^2} = x^2$. What is the general solution ?

5. Solve $\frac{dy}{dx} = x$.

6. Solve $\frac{dy}{dx} = 3x^2 - 2x$.

7. Solve $\frac{dy}{dx} = 4$.

8. Solve $\frac{d^2y}{dx^2} = 0$.

9. (i) What is $\frac{d}{dx} x^5$? (ii) Solve $\frac{dy}{dx} = x^4$.

(iii) Solve $\frac{dy}{dx} = 3x^4$.

10. (i) What is $\frac{d}{dx} x^8$? (ii) Solve $\frac{dy}{dx} = x^7$.

(iii) Solve $\frac{dy}{dx} = 5x^7 - 2$.

11. (i) What is $\frac{d}{dx}\left(\frac{1}{x}\right)$? (ii) Solve $\frac{dy}{dx} = \frac{1}{x^2}$.

(iii) Solve $\frac{dy}{dx} = \frac{4}{x^2}$.

12. (i) What is $\frac{d}{dx}\left(\frac{1}{x^4}\right)$? (ii) Solve $\frac{dy}{dx} = \frac{1}{x^5}$.

(iii) Solve $\frac{dy}{dx} = 1 - \frac{3}{x^5}$.

13. (i) What is $\frac{d}{dx}(\sqrt{x})$? (ii) Solve $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$.

(iii) Solve $\frac{dy}{dx} = x + \frac{3}{\sqrt{x}}$.

14. Solve $\frac{dy}{dx}=5x^2-3x+9$.

15. Solve $\frac{dy}{dx}=x^5-x^2+3$.

16. Solve $\frac{dy}{dx}=x^3+\frac{1}{x^3}$.

17. Solve $\frac{dy}{dx}=5\sqrt{x}$.

18. Solve $x^3\frac{dy}{dx}=3x+5$.

19. Solve $\frac{dy}{dx}=(x-2)(x-3)$.

20. Solve $\frac{d^2y}{dx^2}=x^2-x$.

21. Solve $\frac{dy}{dx}=3x^{100}+\frac{5}{x^{100}}$.

22. If $\frac{dy}{dx}=x-2$, prove that $y=\frac{1}{2}x^2-2x+c$, and find c if $y=5$ when $x=6$.

23. If $\frac{ds}{dt}=50-32t$, and if $s=48$ when $t=2$, find s in terms of t.

24. If $\frac{d^2s}{dt^2}=4t$, and if, when $t=0$, $\frac{ds}{dt}=5$ and $s=8$, express s in terms of t.

25. If a body travels s feet in t seconds, its velocity after t seconds is equal to $\frac{ds}{dt}$. Given that $\frac{ds}{dt}=3+4t$, and that it passed a point A after 5 seconds from the start, find its distance from A after another 5 seconds.

26. The gradient of the graph of the function in Fig. 55 is given by $\frac{dy}{dx}=3x^2-1$, and the graph cuts Oy at a distance 3 units from O. Find the function.

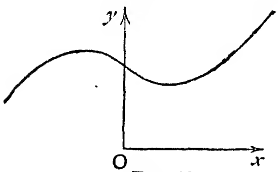


FIG. 55.

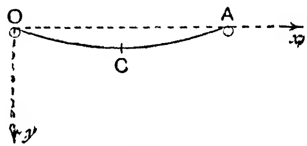


FIG. 56.

27. Fig. 56 represents a uniform lathe OA 20 feet long supported at its ends, which are at the same level. Bending slightly under its own weight, its form obeys the law $\frac{d^2y}{dx^2}=0\cdot0001\,(x^2-20x)$; unit 1 foot on each axis.

(i) State from first principles the value of $\frac{dy}{dx}$ at the mid-point C where $x=10$.
(ii) What is the value of y when $x=0$?
(iii) Find y in terms of x, and then put $x=20$ in the result; what should be the value of y ?
(iv) What is the slope of the lathe at each of its ends ?
(v) What is the sag of the lathe at its mid-point ?

28. A chain OA, 100 feet long, hangs vertically from O. Its weight per unit length steadily diminishes from O to A, and at any point x feet from O is $(4-0\cdot03x)$ lb. per foot. If the work done in winding it up on a drum at O is W foot-lb., then $\frac{dW}{dx}=4x-0\cdot03x^2$; find the work done in winding it up.

29. A stone is thrown horizontally with a velocity of 12 feet per sec. from the top of a tower O, 100 feet high; Ox is horizontal and Oy vertical (see Fig. 57). If P is its position t seconds after starting, and if $ON=x$ feet, $PN=y$ feet, then x, y obey the laws $\frac{d^2x}{dt^2}=0$; $\frac{d^2y}{dt^2}=32$, neglecting air resistance.

(i) Express x and y each in terms of t.

(ii) Express y in terms of x.

(iii) How long does it take to reach the ground?

(iv) How far from the foot of the tower does it strike the ground?

(v) What is the gradient of the path of the stone where it strikes the ground?

(vi) A man is standing on the ground 50 feet from the foot of the tower; where is the stone when it appears to him to be coming straight at him?

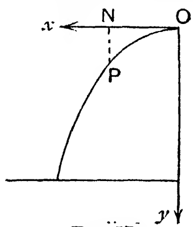


FIG. 57.

30. An elastic spiral spring attached to A, and hanging vertically, supports a body at O. The body is pulled down 3″ and let go. It now oscillates backwards and forwards. If its velocity is v inches per sec. when its distance from O is x in., v and x obey the law $\frac{d}{dx}(\frac{1}{2}v^2)=-\frac{1}{16}x$.

(i) Find its velocity when it passes O.

(ii) How far must it be pulled down in order to have a velocity of 2 in. per sec. when passing O?

31. The rate at which a body cools is proportional to the excess of its temperature above that of the atmosphere. The temperature of a body is $T°$ C. at x minutes past one in a room of temperature $K°$ C. Express the law of cooling by a differential equation.

32. In climbing a mountain, the rate at which the atmospheric pressure p lb. per sq. in. decreases per unit increase of the height x feet is proportional to the pressure. Express this law by a differential equation.

Notation. The relation $\frac{dy}{dx}=f(x)$ is usually expressed in the form $y=\int f(x)\,dx$.

In other words, $\int f(x)dx$ represents the function which when differentiated with respect to x gives $f(x)$.

For example $$\int x^3 dx \equiv \frac{x^4}{4}+c,$$

where c is a constant, because

$$\frac{d}{dx}\left(\frac{x^4}{4}+c\right)\equiv x^3.$$

AREAS AND VOLUMES.

Example II. Figure 58 represents the graph of $y=\frac{x^2}{5}+1$, unit $\frac{1}{2}''$ on each axis.

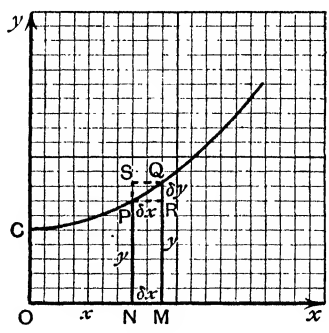


FIG. 58.

If $ON=x$ in., and if the area bounded by CO, ON, NP and arc CP is A sq. in., express A in terms of x.

With the usual notation

δA represents the area bounded by PN, NM, MQ and arc PQ;

$$\therefore \quad \delta A > \text{rectangle } PNMR;$$

$$\therefore \quad \delta A > y\delta x \quad \text{or} \quad \frac{\delta A}{\delta x} > y,$$

and δA < rectangle $SNMQ$;

$$\therefore \quad \delta A < (y+\delta y)\,\delta x \quad \text{or} \quad \frac{\delta A}{\delta x} < y+\delta y;$$

$$\therefore \quad y+\delta y > \frac{\delta A}{\delta x} > y;$$

$$\therefore \text{ when } \delta y \to 0, \ \frac{dA}{dx} = y = \frac{x^2}{5}+1;$$

$$\therefore \quad A = \frac{x^3}{15}+x+c, \text{ where } c \text{ is constant.}$$

But when $x=0$, the area $A=0$;

$$\therefore \quad c=0;$$

$$\therefore \quad A = \frac{x^3}{15}+x.$$

Would there be any difference in the above argument if the graph sloped downwards instead of upwards ?

Example III. Figure 59 represents the graph of $y=\frac{x^2}{5}+1$, not drawn to scale; if $OH=2$, $OK=7$, find the area bounded by AH, HK, KB and the arc AB.

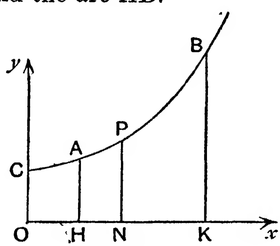


FIG. 59.

It is proved above in Example II. that if $ON=x$, the area of $CONP$ is A, where $\frac{dA}{dx}=\frac{x^2}{5}+1$;

$$\therefore \quad A = \int\left(\frac{x^2}{5}+1\right)dx$$

$$= \frac{x^3}{15}+x+c, \text{ where } c \text{ is a constant};$$

$$\therefore \text{ the area of } COKB = \frac{7^3}{15} + 7 + c,$$

$$\text{and the area of } COHA = \frac{2^3}{15} + 2 + c\,;$$

$$\therefore \text{ the area of } AHKB = \left[\frac{7^3}{15} + 7\right] - \left[\frac{2^3}{15} + 2\right]$$

$$= \frac{343 - 8}{15} + 5 = \frac{335}{15} + 5 = \frac{67}{3} + 5 = 27\tfrac{1}{3}.$$

Note the disappearance of the arbitrary constant c, owing to the subtraction.

Notation. The symbol $\int_a^b f(x)\,dx$ is used to express the value of $\int f(x)\,dx$ when $x = b$ *minus* the value of $\int f(x)\,dx$ when $x = a$.

Thus

$$\int_2^7 x^2\,dx \equiv \left[\frac{x^3}{3}\right]_{x=7} - \left[\frac{x^3}{3}\right]_{x=2} = \frac{7^3}{3} - \frac{2^3}{3} = \frac{343-8}{3} = \frac{335}{3},$$

and we often write $\quad \int_2^7 x^2\,dx \equiv \left[\frac{x^3}{3}\right]_2^7.$

Looking back at Example III. we see that the area of $AHKB$ can be written in any of the following forms:

$$\int_2^7 y\,dx \equiv \int_2^7 \left(\frac{x^2}{5} + 1\right) dx \equiv \left[\frac{x^3}{15} + x\right]_2^7.$$

The expression $\int f(x)\,dx$ is called an *indefinite* integral because its value contains an arbitrary constant.

The expression $\int_a^b f(x)\,dx$ is called a *definite* integral because the arbitrary constant of integration disappears automatically.

Examples II., III. show that $\int_a^b f(x)\,dx$ is the area bounded by two ordinates $x = a$, $x = b$, the x-axis and an arc of the graph of $y = f(x)$.

EXERCISE XVIII. b.

1. Fig. 60 represents the graph of $y=x^2$; $ON=x$; (i) find in terms of x the area bounded by ON, NP, arc OP; (ii) find the area $AHKB$, where $OH=1$, $OK=2$; (iii) express this area as a definite integral.

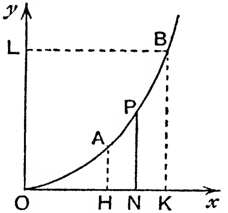


FIG. 60.

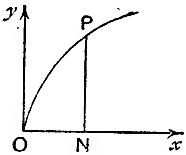


FIG. 61.

2. Fig. 61 represents the graph of $y^2=x$; $ON=x$; find the area ONP.

3. Interpret geometrically $\int_1^3 x\,dx$ and evaluate it (i) by direct calculation, (ii) geometrically.

4. Fig. 62 represents part of the graph of $y=(x-1)(4-x)$; (i) find OA and OB; (ii) find the area between Ox and the portion of the curve above Ox.

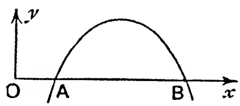


FIG. 62.

5. Draw the graph of $\dfrac{1}{x^2}$ and find the area (i) between it and the ordinates $x=1$, $x=2$ and the x-axis; (ii) between it and the lines $y=1$, $y=2$ and the y-axis.

6. If Fig. 60 represents the graph of $y=x^3$; prove that the area BLO is three times the area BKO.

7. Find the values of

(i) $\int_1^2 (x^2+5)\,dx$; (ii) $\int_2^4 \dfrac{dx}{x^3}$;

(iii) $\int_{-1}^{+1} (x-x^2)\,dx$; (iv) $\int^{\frac{1}{2}} (1+x)^2\,dx$.

8. Fig. 63 represents a solid formed by rotating part of the graph of $y=x^2$ about Ox.

$ON=x$, $NP=y$, $OM=x+\delta x$, $QM=y+\delta y$, $OH=\frac{1}{4}$, $OK=\frac{3}{4}$.

The figure shows cross-sections perpendicular to Ox.

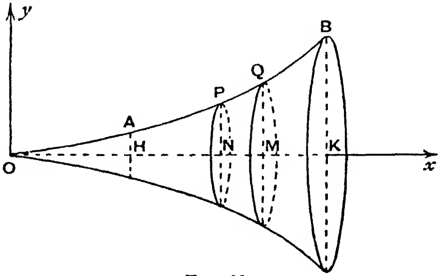


FIG. 63.

V is the volume of the solid between O and the cross-section through P.

(i) What is the area of the cross-section through P ?

(ii) What is the area of the cross-section through Q ?

(iii) Interpret $\pi y^2\,\delta x$ geometrically.

(iv) Prove that $\pi(y+\delta y)^2 . \delta x > \delta V > \pi y^2 . \delta x$.

(v) Prove that $V=\pi\int_0^x y^2\,dx$.

(vi) Express V in terms of x.

(vii) Find the volume between the cross-sections at A and B, and express it also as a definite integral.

(viii) If the graph of $y=f(x)$ is drawn, interpret geometrically $\int_0^x \pi y^2\,dx$.

(ix) If the graph of $y=f(x)$ is drawn, interpret geometrically $\int_0^y \pi x^2\,dy$.

9. (i) Draw the graph of $y=\frac{1}{3}x$; take a point P on it and draw PN perpendicular to Ox.

(ii) What solid is formed by revolving OP about Ox ?

(iii) Find, by the method of Ex. 8, the volume of the solid between O and the section through P perpendicular to Ox, in terms of x.

10. The graph of $y=\dfrac{1}{x}$ is rotated about Oy; find the volume of the solid between two planes perpendicular to Oy and at distances 1 and 2 from O.

11. Fig. 64 represents a section of a hemisphere of radius of 10 in., centre O; $ON = x$ in.

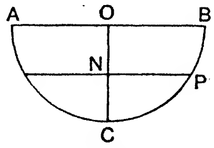


FIG. 64.

(i) Find PN in terms of x.

(ii) Find the area of the cross-section of the hemisphere at N perpendicular to ON.

(iii) Find an approximate expression for the volume of the hemisphere contained between two parallel planes at distances x and $x + \delta x$ from O.

(iv) If the volume between the sections through O and through N, perpendicular to ON, is V cu. in., express V in terms of x.

(v) Express as a definite integral the volume of the spherical segment between C and the section through N; and simplify it.

(vi) Find the volume of the hemisphere.

12. A basin is formed by revolving the graph of $y = \frac{1}{8}x^4$ about the vertical axis Oy; how much water is in the basin when the greatest depth is 4 inches? [Unit for each axis, one inch.]

13. (i) Sketch the graph of $y = \frac{1}{27}(2x+3)(9-2x)$ from $x = 0$ to $x = 3$.

(ii) A beer barrel is formed by rotating this portion of the graph about Ox. [Unit for each axis, one foot.] Find the capacity of the barrel in cu. feet.

14. (i) Sketch the graphs of $y^2 = x$ and $x^2 = 8y$ for positive values of x.

(ii) Show that they cross each other at the point (4, 2).

(iii) Show that the area of the portion common to both is

$\int_0^4 \left(\sqrt{x} - \frac{x^2}{8}\right) dx$, and evaluate it.

15. If $f(x) = \int_0^x \left(1 - \frac{y^2}{4}\right) dy$, find the value of $\int_0^1 f(x)\, dx$. [This may be stated more shortly as follows: find the value of

$$\int_0^1 \int_0^x \left(1 - \frac{y^2}{4}\right) dy\, dx.]$$

16. Using the notation explained in Ex. 15, find the value of

$$\int_{-1}^{1}\int_{1}^{x} y(1-y)\,dy\,dx.$$

17. Show that $\int_0^1 (x+2)(x+3)\,dx = \int_1^2 (x+1)(x+2)\,dx.$ Interpret this result geometrically.

18. A lathe of length 10 feet is built into a wall horizontally at one end and is deflected by a weight fastened at the other end. The deflection at x feet from the wall is $\int_0^x f(v)\,.\,dv$ feet, where $f(v) = \int_0^v 0{\cdot}0005\,(10-u)\,du$; [or more shortly,

$$\int_0^x\int_0^v 0{\cdot}0005(10-u)\,du\,dv].$$

Find the deflection (i) when $x=10$, (ii) when $x=5$.

19. A lathe of length 10 feet is supported at its ends and loaded at the middle. The deflection at x feet from one end is $\int_0^x f(v)\,.\,dv$ feet, where $f(v) = \int_v^5 0{\cdot}0005u(10-u)\,du$. Find the deflection at the mid-point.

20. The moment of inertia of a cylinder, radius r in., height h in., about its axis is $2\pi\rho h\int_0^r x^3\,.\,dx$, where ρ lb. is the mass per cu. inch. If the mass of the cylinder is W lb., express the moment of inertia in terms of r, W.

21. The moment of inertia of a sphere, radius r in., about a diameter is $\pi\rho\int_0^r (r^2-x^2)^2 dx$, where ρ lb. is the mass per cu. inch. If the mass of the sphere is W lb., express the moment of inertia in terms of r, W.

22. A plane area is enclosed between the curve $y^2 = 2x^3$ and the line $x=3$. The distance of the centre of gravity of this area measured along the axis of x from the origin is given by the formula $S = \int_0^3 yx\,dx \div \int_0^3 y\,.\,dx$. Evaluate this expression.

23. A solid is formed by the rotation of the portion of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ lying between the lines $x=0$ and $x=5$ about the axis of x. The distance of the centre of gravity of this solid measured from the origin along its central axis is given by the formula $S = \int_0^5 \pi y^2 x\,dx \div \int_0^5 \pi y^2\,.\,dx$. Evaluate this expression.

24. A gas is compressed in a cylinder by a piston. If the face of the piston is initially a inches from the inside end of the cylinder, and if the piston travels forward until this distance is b inches, the work done is $-\int_a^b \frac{660}{x^{1\cdot4}}dx$ ft.-lb. If $a=10$, find the work done in compressing the gas to half its volume.

25. A new cure for chilblains was tried on 8 people. It succeeded in 5 cases and failed in 3 cases. The chance that it succeeds in the next two cases is

$$\int_0^1 x^7(1-x)^3dx \div \int_0^1 x^5(1-x)^3dx.$$

Find what the chance is.

26. Two points are taken at random on a line 12 inches long. The chance that the distance between them does not exceed 3 inches is $\int_3^{12}(x-3)\,dx \div \int_0^{12} x\,dx$. Evaluate this chance.

27. The portion of the parabola $y=\frac{1}{4}x^2$ (the y-axis being drawn vertically downwards) above $y=5$ is taken and submersed in a liquid in a vertical plane with its vertex uppermost and 1 unit distance below the surface. Then the depth of the centre of pressure below the surface is

$$\int_0^5 (1+x)^2\sqrt{x}\,dx \div \int_0^5 (1+x)\sqrt{x}\,dx.$$

Evaluate this expression.

28. The density of a spherical body of radius a feet varies directly as its distance from the centre. If ρ is the density (*i.e.* the mass in lb. per cu. ft.) at the surface, (i) prove that the mass of the spherical shell whose inner and outer radii are x and $x+\delta x$ feet, is approximately $\frac{4\pi\rho x^3}{a}\,.\,\delta x$ lb.; (ii) find the total mass of the body.

29. The density of a chain AB varies as its distance from A (*i.e.* at a point x feet from A, the material weighs kx lb. per foot length); its length is 20 feet; the lighter half weighs 30 lb. Find the weight of the heavier half.

30. The velocity of a body after t sec. from its start is v feet per sec., where v is a given function of t, viz. $v=f(t)$; (i) what meanings can you give to $\int v\,dt$? (ii) Given the time-velocity graph of a moving particle, how could you find the distance travelled in a given time?

31. Fig. 65 represents a mound with the rectangle $ABCD$ as base and with its sides sloping up to the ridge EF, which is situated symmetrically. $AB=30'$, $AD=18'$, $EF=10'$; and the height of EF above the base is 12 feet.

(i) Prove that the area of the section of the mound parallel to the base and h feet below EF is

$$\tfrac{5}{2}h\,(h+6) \text{ sq. ft.}$$

(ii) Find the volume of the mound.

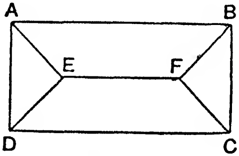


FIG. 65.

32. (i) Sketch the half of the ellipse $\frac{x^2}{4}+y^2=81$ which lies between $x=0$ and $x=18$.

(ii) This portion is rotated about Ox to form the explosive head of a torpedo; unit on each axis, 1 inch. Find the volume of the explosive in the head.

DUFTON'S RULE FOR APPROXIMATION OF AREAS.

PQ is an arc of a curve; PA, QB are the perpendiculars to a line AB.

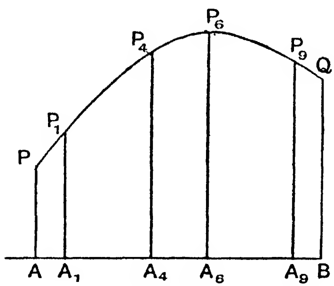


FIG. 66.

AB is divided into ten equal parts AA_1, A_1A_2, A_2A_3, ... A_9B.

Through the points A_1, A_4, A_6, A_9 perpendiculars A_1P_1, A_4P_4, A_6P_6, A_9P_9 are drawn to meet the curve.

Then the area contained by PA, AB, BQ, and the arc PQ is

$$\simeq \tfrac{1}{4} \,.\, AB \,.\, [A_1P_1+A_4P_4+A_6P_6+A_9P_9].$$

SIMPSON'S RULE FOR APPROXIMATION OF AREAS.

PA, *QB* are the perpendiculars from *P*, *Q* to a line *AB*.

AB is divided into any even number, say $2n$, equal parts, and the ordinates $y_1, y_2, y_3, \dots y_{2n+1}$ are drawn from each point of division and from the beginning and end of the line to the curve.

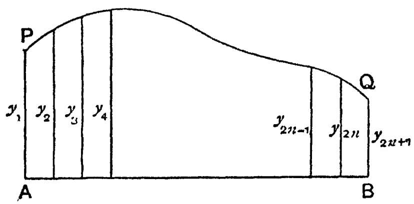


FIG. 67.

Let $$h=\frac{AB}{2n}.$$

Then the area contained by *PA*, *AB*, *BQ* and the arc *PQ* is

$$\simeq \tfrac{1}{3}h\,[y_1+y_{2n+1}+2(y_3+y_5+y_7+\dots+y_{2n-1}) \\ +4(y_2+y_4+y_6+\dots+y_{2n})].$$

Dufton's rule is obviously quicker to apply than Simpson's rule, as the numerical computation is less. The relative accuracy of the two rules depends on the nature of the curve, but in general Dufton's rule gives as high a degree of accuracy as is required.

EXERCISE XVIII. c.

1. (i) Write down the values of x^2 for $x=0{\cdot}1,\ 0{\cdot}4,\ 0{\cdot}6,\ 0{\cdot}9$, and then use Dufton's rule to find the area bounded by the x-axis, the graph of $y=x^2$ and the ordinate through $x=1$.

(ii) Evaluate this area by calculation, and find the error per cent. in the result obtained from Dufton's rule.

(iii) Divide the area into 4 strips and apply Simpson's rule to find the area.

2. (i) Draw the graph corresponding to the following table of values :

$x=0$	1	2	3	4	5	6	7	8	9	10
$y=0$	7	8	11	15	18	21	18	14	6	0

(ii) Find its area by "counting squares."
(iii) Find its area by "Dufton's rule."
(iv) Find its area by "Simpson's rule."

3. Draw a semicircle, diameter 10 cm., and find its area

(i) by Dufton's rule,
(ii) by Simpson's rule (ten strips),
(iii) by formula.

Give the approximate error per cent. of (i) and (ii).

4. Find the area bounded by the graph of $y=\frac{60}{x}$, the x-axis, and the ordinates $x=5$ and $x=10$

(i) by Dufton's rule,
(ii) by Simpson's rule.

Assuming that the area is 41·6, find the approximate error per cent. in each case.

5. The velocity of train after leaving a station is shown in the following table :

t, time in minutes	0	2	4	6	8	10	12	14	16	18	20
v, speed in yards per minute -	0	50	110	160	230	290	360	410	470	530	570

Find in miles the distance travelled in the first 20 minutes

(i) by Dufton's rule,
(ii) by Simpson's rule.

6. In finding the volume of earth in a cutting between two sections A and B, the following measurements were obtained :

	Distance from A in yards.	Area of section of cutting in sq. feet.
At A - -	0	2000
	8	3000
	12	3420
	19	3900
	24	3900
	30	3700

(i) Represent these results graphically.

(ii) Find the volume of earth in the cutting in cu. ft. (*a*) by Dufton's rule, (*b*) by Simpson's rule. (C.S.C.)

7. The speed of a car, v miles per hour, was noted at intervals of 12 seconds over a period of 2 minutes.

Time in seconds =	0	12	24	36	48	60	72	84	96	108	120
$v=$	5	3	2·5	3	5·7	10·3	14	10·5	5·6	2	0

Find the distance travelled in feet by two methods. (C.S.C.)

8. Use Dufton's rule to find an approximate result for the volume of a hemisphere, radius 10 cm., taking sections parallel to the base of the hemisphere. Compare this with the result given by Simpson's rule and by the formula $\frac{2}{3}\pi r^3$ for a hemisphere.

9. (i) Sketch the graph of $y=\log x$ from $x=1$ to $x=10$.

(ii) Interpret geometrically $\int_1^2 \log x \,.\, dx$.

(iii) Use Dufton's rule and Simpson's rule to find an approximate value of this definite integral.

10. By using Dufton's and Simpson's rules, find an approximate value of $\int_0^{10} \sqrt{1+x^3}\, dx$.

CHAPTER XIX.

SIMPLE SERIES.

Example I. The following sequence of numbers obeys a simple law; find (i) the 15th term, (ii) the nth term in the sequence

$$11, 15, 19, 23, 27, 31, \ldots.$$

We see that each term exceeds the term before it by 4.

The 2nd term $= 15 = 11 + 4$.

The 3rd term $= 19 = 11 + 4 + 4 = 11 + 2 \times 4$.

The 4th term $= 23 = 11 + 4 + 4 + 4 = 11 + 3 \times 4$,

and so on;

$\therefore$ the 15th term $= 11 + 14 \times 4 = 11 + 56 = 67$,

and the nth term $= 11 + (n-1) \times 4 = 11 + 4n - 4 = 4n + 7$.

Example II. Find the nth term in each of the following sequences:

(i) $7, 21, 63, 189, \ldots$:

(ii) $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \ldots$.

(i) We see that each term is three times the term

The 2nd term $= 21 = 7 \times 3$.

The 3rd term $= 63 = 7 \times 3 \times 3 = 7 \times 3^2$.

The 4th term $= 189 = 7 \times 3 \times 3 \times 3 = 7 \times 3^3$

and so on;

$\therefore$ the nth term $= 7 \times 3^{n-1}$.

(ii) We notice that the numerators are 3.

$\therefore$ the numerator of the nth

The denominators are 4, $4 + 3$, $4 + 2 \times$

$\therefore$ the denominator of the nth

$4 + (n-1) \times 3 = 4 + 3n$

$\therefore$ the nth t

EXERCISE XIX. a.

1. The following sequences of numbers obey simple laws; write down in each case the 6th and the nth number in the sequence:

(i) 2, 4, 6, 8, ... ; (ii) 1, 4, 9, 16, 25, ... ;

(iii) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$; (iv) 10, 200, 3000, 40000, ... :

(v) 3, 6, 12, 24, ... ; (vi) $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \frac{9}{32}, \ldots$;

(vii) 5, 12, 19, 26, 33, ... ; (viii) 10, 7, 4, 1, -2, ... ;

(ix) 0·1, 0·02, 0·003, 0·0004, ... ; (x) -1, 1, -1, 1, -1, ... ;

(xi) 8, 27, 64, 125, ... ; (xii) 14, 108, 1012, 10016, ...

2. Write down the first three terms of the sequence in which the nth term is

(i) $2n - 1$; (ii) $\frac{3^n}{18}$; (iii) $\frac{n+1}{n+2}$;

(iv) $n^2 + n$; (v) $3 \,.\, 10^{n-1} + 5n$; (vi) $n^3 - 1$.

3. The sum of the first n terms of a sequence is $\frac{n(n+1)}{2}$; (i) what is the first term? (ii) what is the sum of the first two terms? (iii) what is the second term? (iv) what is the 10th term? (v) what is the nth term?

4. The sum of the first n terms of a sequence is $\frac{n(n+1)(2n+1)}{6}$; find (i) the first term, (ii) the 4th term, (iii) the nth term.

5. What is the average of the following 14 numbers?

3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55.

Is it necessary to add them all up?
When you know their average, how can you find their sum? What is it?

6. Find (i) the average, (ii) the sum of the following sequences:

(i) 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, 50.

(ii) 60, 55, 50, 45, 40, 35, 30, 25, 20, 15, 10.

(iii) 20, 13, 6, -1, -8, -15, -22, -29, -36, -43, -50.

(iv) $3\frac{1}{2}$, $4\frac{3}{4}$, 6, $7\frac{1}{4}$, $8\frac{1}{2}$, $9\frac{3}{4}$, 11, $12\frac{1}{4}$, $13\frac{1}{2}$, $14\frac{3}{4}$, 16.

7. How many terms are there in the following sequences?

(i) 7, 9, 11, 13, 15, ... , 53, 55, 57, 59 ;

(ii) 22, 29, 36, 43, 50, 57, ... , 155, 162, 169;

(iii) 100, 97, 94, 91, 88, ... , 46, 43, 40 ;

(iv) $3\frac{1}{6}$, $4\frac{1}{8}$, $5\frac{1}{10}$, $6\frac{1}{12}$, ... , $20\frac{1}{40}$, $21\frac{1}{42}$.

8. Find the least number of three digits which belongs to the sequence:

(i) 6, 13, 20, 27, ... ;
(ii) 1, 2, 4, 8, 16, ... ;
(iii) 3, 14, 25, 36, ...

Find also the nth term in each case.

9. In a sequence the terms increase by a constant amount (positive or negative) [as, for example, in 6 (i), (iii)]; find the second term if

(i) the first term is 8 and the 10th term is 71 ;
(ii) the first term is 3 and the 15th term is 4 ;
(iii) the first term is 14 and the 8th term is -7 ;
(iv) the fourth term is 25 and the 12th term is 97.

10. Find the sum of the following :

(i) $25+28+31+34+\ldots+94+97+100+97+94+91+\ldots+52+49$;
(ii) $4-5+6-7+8-9+\ldots+40-41+42$.

ARITHMETICAL PROGRESSIONS.

Definition. A sequence of numbers in which each term exceeds the preceding term by the same amount, positive or negative, is called an *Arithmetical Progression* or, for short, an A.P.

E.g. (i) 5, 12, 19, 26, 33, ...,
or (ii) 10, 2, -6, -14, ...,
or (iii) $6\frac{1}{4}$, $7\frac{3}{4}$, $9\frac{1}{4}$, $10\frac{3}{4}$, ...

In (i) each term exceeds the term before it by 7.

This amount 7 is called the *common difference.*

In (ii) the common difference is $2-10=-8$,
or $-6-2=-8$.

In (iii) the common difference is $7\frac{3}{4}-6\frac{1}{4}=1\frac{1}{2}$.

If three numbers are in arithmetical progression, the middle number is called the *arithmetic mean* of the other two.

E.g. 21, 29, 37 are in A.P. ;

$\therefore$ 29 is the arithmetic mean of 21 and 37.

The arithmetic mean of two numbers is simply their average.

Example III. Find the sum of 24 terms of the sequence

$$-41,\ -33,\ -25,\ -17,\ \ldots.$$

The common difference $= -33-(-41) = -33+41 = 8$;

$\therefore$ the 24th term $= -41+23\times 8 = -41+184 = 143$;

$\therefore$ the sum $s = -41-33-25 \ldots +127+135+143.$

The average of this sequence is $\dfrac{-41+143}{2} = \dfrac{102}{2} = 51$;

$$\therefore\ s = 24\times 51$$
$$= 1224.$$

Another method.

$$s = -41-33-25-\ldots+127+135+143.$$

Write the sequence backwards;

$$\therefore\ s = 143+135+127+\ldots-25-33-41;$$

$\therefore$ adding $\quad 2s = 102+102+102+\ldots+102.$

But there are 24 terms;

$$\therefore\ 2s = 102\times 24;$$

$$\therefore\ s = \frac{102\times 24}{2} = 51\times 24 = 1224.$$

Example IV. The 10th term of an A.P. is 17 and the 25th term is 41; find the second term.

Let the first term be a and the common difference d;

$$\therefore\ \text{the 10th term is } a+9d;$$

$$\therefore\ a+9d = 17.$$

Similarly, $\quad a+24d = 41$;

$$\therefore\ -15d = -24;$$

$$\therefore\ d = \tfrac{24}{15} = \tfrac{8}{5} = 1\tfrac{3}{5};$$

$$\therefore\ a = 17-9d = 17-\tfrac{72}{5} = 17-14\tfrac{2}{5} = 2\tfrac{3}{5};$$

$$\therefore\ \text{the second term} = a+d = 2\tfrac{3}{5}+1\tfrac{3}{5}$$
$$= 4\tfrac{1}{5}.$$

EXERCISE XIX. b.

1. [illegible] of an A.P. is 10 and the third is 11; find t[illegible]

2. [illegible] of an A.P. is 11 and the 10th term is -4; find the 3rd [illegible]

3. The first ter[illegible] A.P. is 21, the last term is 193, and the common difference [illegible]nd the number of terms.

4. Find the arit[illegible] mean of 23 and 90.

5. If 12, x, y, z, 41 are in A.P., find x, y, z. This question might be worded as follows: "insert three arithmetic means between 12 and 41."

Sum the series in Ex. 6-15:

6. $7+8+9+\ldots+100$.

7. $22+19+16+\ldots$ 20 terms.

8. $5{\cdot}3+7{\cdot}2+9{\cdot}1+\ldots$ 25 terms.

9. $8\frac{1}{3}+6\frac{5}{6}+5\frac{1}{3}+\ldots$ 18 terms.

10. $3+7+11+15+\ldots$ n terms.

11. $a+(a+d)+(a+2d)+(a+3d)+\ldots$ 12 terms.

12. $a+(a+d)+(a+2d)+\ldots$ n terms.

13. $11+13\frac{1}{2}+\ldots+36$.

14. $23+\ldots+67$ (12 terms).

15. $a+\ldots+l$ (n terms).

16. In the sequence

[1]; [2, 3]; [4, 5, 6]; [7, 8, 9, 10]; [11, 12, 13, 14, 15], ...,

what is the first number in (i) the 8th bracket, (ii) the nth bracket?

17. Find the sum of all numbers less than 100 which are not divisible by 7.

18. The first two terms of an A.P. are a, x; what is (i) the third term, (ii) the nth term?

19. In the sequence

[1]; [3, 5]; [7, 9, 11]; [13, 15, 17, 19]; ...,

what is the first number in (i) the 10th bracket, (ii) the nth bracket?

20. The nth term of a sequence is $5n-11$; what is the common difference and the sum of the first n terms?

21. (i) Each term of an A.P. is multiplied ... is the result an A.P.?

(ii) Each term of an A.P. is multiplie... the result an A.P.?

22. What is the sum of all numbers divi... 1000 and 5000?

23. In a sequence of n terms, the last te... l and the common difference is d; what is the first term?

24. If a, b are integers, find the sum ...ntegers from a to b inclusive, assuming $a < b$.

SUMMARY OF RESULTS.

Any arithmetical progression can be written in the form

$$a, \quad a+d, \quad a+2d, \quad a+3d, \ \ldots;$$

(i) the nth term $=a+(n-1)d$;

or, if $l=$ the last term and $n=$ the number of terms,

$$l=a+(n-1)d;$$

(ii) the sum of n terms $=s=n\times$ average term;

or
$$s=\frac{n}{2}(a+l)$$

and
$$s=\frac{n}{2}[2a+(n-1)d];$$

(iii) the arithmetic mean of a, b is $\frac{a+b}{2}$.

GEOMETRICAL PROGRESSIONS.

Definition. A sequence of numbers in which each term bears a fixed ratio to the term preceding it is called a *Geometrical Progression* or, for short, a G.P.

E.g. (i) 4, 12, 36, 108, 324, ...,

or (ii) 7, -14, 28, -56, 112, ...,

or (iii) 12, 4, $1\frac{1}{3}$, $\frac{4}{9}$, $\frac{4}{27}$, ...

In (i), the ratio is $\frac{12}{4}=3=\frac{36}{12}=\frac{108}{36}=\ldots$.

This amount 3 is called the *common ratio.*

In (ii) the common ratio $=\frac{-14}{7}=-2$.

In (iii) the common ratio $=\frac{4}{12}=\frac{1}{3}$.

If three numbers are in geometrical progression, the middle number is called the *geometric mean* of the other two.

E.g. 4, 12, 36 are in G.P.;

$\therefore$ 12 is the geometric mean of 4 and 36.

If a, x, b are in G.P., $\frac{x}{a}=\frac{b}{x}$, by definition;

$$\therefore\ x^2=ab;$$

$$\therefore\ x=\pm\sqrt{ab}.$$

Therefore the geometric mean of a, b is $\pm\sqrt{ab}$.

Example V. Find the sum of 10 terms of the sequence

$$18,\ 12,\ 8,\ 5\tfrac{1}{3},\ \ldots.$$

The common ratio $= \dfrac{12}{18} = \dfrac{2}{3}$; $\left[\dfrac{8}{12} = \dfrac{2}{3} \text{ and } \dfrac{5\frac{1}{3}}{8} = \dfrac{2}{3}\right]$;

$$\therefore \text{ the 10th term} = 18 \times (\tfrac{2}{3})^9.$$

$$s = 18 + 12 + 8 + \ldots + 18 \times (\tfrac{2}{3})^7 + 18 \times (\tfrac{2}{3})^8 + 18 \times (\tfrac{2}{3})^9;$$

$$\therefore \ \tfrac{2}{3}s = 12 + 8 + 5\tfrac{1}{3} + \ldots + 18 \times (\tfrac{2}{3})^8 + 18 \times (\tfrac{2}{3})^9 + 18 \times (\tfrac{2}{3})^{10}.$$

The terms from 12 up to $18 \times (\frac{2}{3})^9$ are the same in both lines.

$$\therefore \ s - \tfrac{2}{3}s = 18 - 18 \times (\tfrac{2}{3})^{10} = 18[1 - (\tfrac{2}{3})^{10}];$$

$$\therefore \ 3s - 2s = 54[1 - (\tfrac{2}{3})^{10}];$$

$$\therefore \ s = 54[1 - (\tfrac{2}{3})^{10}].$$

An approximate value can be found by logarithms.

$$\tfrac{2}{3} = 0{\cdot}6667 = 10^{\bar{1}\cdot8240};$$

$$\therefore \ (\tfrac{2}{3})^{10} = 10^{\bar{2}\cdot240} = 0{\cdot}0174;$$

$$\therefore \ s = 54 \times 0{\cdot}9826$$

$$= 53{\cdot}1.$$

EXERCISE XIX. c.

1. Write down, using the index notation, the nth terms in geometrical progressions which start:

(i) $7,\ 35$; ii) $1,\ -2$; (iii) $8,\ 4$;
(iv) $-10,\ 6$; (v) $6,\ 7$; (vi) $2\frac{1}{3},\ 3$;
(vii) $-x,\ x^3$; (viii) $a,\ b$; (ix) $a,\ 1$.

2. Simplify (i) $8(-\frac{1}{2})^7$; (ii) $(-1)^{2n} - (-1)^{2n-1}$.

3. What is the first term in the sequence $1,\ \frac{3}{4},\ \frac{9}{16},\ \frac{27}{64},\ \ldots$, which is less than $0{\cdot}001$?

4. If $s = 3 + 6 + 12 + 24 + 48 + 96 + 192 + 384 + 768$, write down $2s$ in full; then subtract, and so find s.

5. If $s = 54 + 18 + 6 + 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \frac{2}{243}$, write down $\frac{1}{3}s$ in full; then subtract, and so find s.

6. If $s = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6$, write down rs in full; what is $rs - s$? Hence find s.

7. Sum the following sequences; do *not* multiply out the result:

(i) $4,\ 4 \times 3,\ 4 \times 3^2,\ 4 \times 3^3,\ \ldots$ to 10 terms;
(ii) $5,\ 5 \times \frac{2}{5},\ 5 \times (\frac{2}{5})^2,\ 5 \times (\frac{2}{5})^3,\ \ldots$ to 12 terms;
(iii) $\frac{3}{4},\ \frac{1}{2},\ \frac{1}{3},\ \ldots$ to 8 terms;
(iv) $7,\ -14,\ 28,\ -56,\ \ldots$ to 11 terms;
(v) $27,\ -12,\ 5\frac{1}{3},\ \ldots$ to 20 terms.

8. (i) Divide $1-r^6$ by $1-r$.

(ii) What is the quotient when $1-r^{10}$ is divided by $1-r$?

(iii) Express $1+r+r^2+r^3+\ldots+r^{18}+r^{19}+r^{20}$ as a simple fraction.

(iv) Express $a+ar+ar^2+ar^3+\ldots+ar^{20}$ as a simple fraction.

9. If $s=a+ar+ar^2+\ldots$ to n terms,

(i) what is the nth term?

(ii) what is the last-but-one term?

(iii) write down the value of rs, putting in the first three and the last three terms;

(iv) what is the value of $rs-s$? Hence find s.

10. (i) Prove that $1+\frac{1}{10}+\frac{1}{100}+\ldots$ to 8 terms $=\frac{10}{9}\left(1-\frac{1}{10^8}\right)$.

(ii) What is the value of this answer to 3 places of decimals?

(iii) What is the approximate error per cent. in taking the sum of these 8 terms as $\frac{10}{9}$?

(iv) What is the sum of 50 terms of the same sequence, and what is the approximate error per cent. in taking the sum as $\frac{10}{9}$?

(v) What is the meaning of $1\cdot\dot{1}$, and what is its value?

11. AB is a straight line, one inch long. Bisect AB at C_1; bisect C_1B at C_2; bisect C_2B at C_3; bisect C_3B at C_4; and so on.

(i) What are the lengths of AC_1, C_1C_2, C_2C_3, C_3C_4, C_4C_5, ...?

(ii) What are the lengths of AC_2, AC_3, AC_4, AC_5, AC_{10}, AC_{100}?

(iii) What is the limiting value of the length of AC_n when $n\to\infty$, and what series is this associated with?

(iv) What is the difference between the sum of the first 10 terms and the sum of the first 50 terms of the sequence 8, 4, 2, 1, $\frac{1}{2}$, ...?

(v) What is a fair approximation for the sum of the first 10 terms of this sequence?

(vi) What meaning would you give to the expression "the sum to infinity" of this sequence? Why is it better to call it the "limiting sum"? [This name is due to Prof. T. P. Nunn.]

12. Which of the following geometrical progressions can be "summed to infinity" or in other words has a "limiting sum"?

(i) $1, \frac{10}{9}, (\frac{10}{9})^2, (\frac{10}{9})^3, \ldots$; (ii) 100, 90, 81, ...;

(iii) $4, -6, +9, \ldots$; (iv) $1, -\frac{1}{2}, \frac{1}{4}, \ldots$.

13. Sketch freehand a graph (it will consist of a set of disconnected points) representing the sum of the first 1, 2, 3, 4, ... terms of the sequence as a function of the number of terms

(i) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$; (ii) $1, 2, 4, 8, \ldots$.

14. Express $0{\cdot}\dot{4}\dot{3}$ as a geometrical progression, and then find its "limiting sum."

15. Find the "limiting sum" of the following:

(i) $500 + 400 + 320 + 256 + \ldots$;

(ii) $10, \ -2, \ 0{\cdot}4, \ -0{\cdot}08, \ldots$;

(iii) $a + ar + ar^2 + ar^3 + \ldots$ if $1 > r > -1$.

16. Find, using logarithms, an approximate value of

(i) $50 + 50\,(1{\cdot}05) + 50\,(1{\cdot}05)^2 + \ldots$ to 10 terms;

(ii) $8 + 12 + 18 + \ldots$ to 18 terms.

Summary of Results.

Any geometrical progression can be written in the form

$$a, \quad ar, \quad ar^2, \quad ar^3, \ \ldots;$$

(i) the nth term $= ar^{n-1}$;

(ii) the sum of n terms $= \dfrac{a(r^n - 1)}{r-1} = \dfrac{a(1-r^n)}{1-r}$;

(iii) if $1 > r > -1$, the "limiting sum" or "sum to infinity"

$$= \frac{a}{1-r};$$

(iv) the geometric mean of a, b is $\pm\sqrt{ab}$.

MISCELLANEOUS EXAMPLES ON SERIES.

EXERCISE XIX. d.

1. A train travelling 12 m.p.h. gathers speed uniformly till five minutes later its speed is 40 m.p.h.; what was its speed at the end of the first minute, and how far did it go in the five minutes?

2. Thirty concentric circles are engraved at equal distances apart on a metal plate; the radii of the inner and the outer circles are 4·5″ and 7·3″; find the distance between two consecutive circles correct to $\frac{1}{1000}$ inch.

3. A ball thrown vertically upwards from the floor in a vacuum returns to the floor in 2 seconds; the times of successive rebounds are $1\frac{1}{2}$ sec., $1\frac{1}{8}$ sec., $\frac{27}{32}$ sec., ... ; how long elapses before the sixth bounce? Find also the time from the start till the ball comes to rest.

4. A man starting business loses £240 the first year, £160 the second year, £80 the third year; if the same improvement continues, what is his total gain or loss after 12 years?

5. Potatoes are planted in parallel rows in a field, each row containing 3 more than the row before; the first row holds 32 and the last 149. How many potatoes are there in the field?

6. The height of a tree is 20 feet and one year later is 24 feet; its growth each year is $\frac{3}{4}$ of its growth the previous year. What is the limiting height of the tree, however long it lives?

7. Find the sum of $0{\cdot}7 + 0{\cdot}71 + 0{\cdot}72 + 0{\cdot}73 + \ldots$ to 100 terms.

8. Find the sum of $2 + 22 + 222 + \ldots$ to 8 terms.

9. A square is divided into 25 equal squares, and the numbers 1, 2, 3, ... up to 25 are arranged in these squares so that the sum of the numbers in each row and each column is the same; what is the sum of the numbers in each row? [This arrangement is called a magic square.]

10. In a potato race, ten potatoes are placed in a line with the starting point at intervals of two yards, and the nearest is 20 yards from the starting point to which each potato must be brought separately. How many yards must a competitor run if he can reach one yard both picking up and putting down the potato?

11. A pendulum is set swinging; its first vibration is through 20°, and each succeeding vibration is $\frac{9}{10}$ of the one before it. What is the total angle through which it swings before coming to rest?

12. What is the sum of the first n odd integers?

13. Postal orders are issued for all multiples of sixpence from 6d. to 21s. inclusive, except for 20s. 6d.; the commissions are: 6d. to 2s. 6d., one penny; 3s. to 15s., three halfpence; 15s. 6d. to 21s., twopence. Find the cost of one complete set of postal orders.

14. In a group of n people every one writes once to every one else; how many letters are written altogether? [Count them as follows: how many letters does A write and receive? how many letters does B, not counting correspondence with A? how many letters does C, ignoring A, B? and so on.]

15. A starts with a salary of £500 a year and receives a yearly increase of £20 a year; B starts at the same time at £300 a year and receives a yearly increase of £50 a year. After how many years will the total sum received by B exceed that received by A, neglecting interest?

16. A starts at a salary of £500 a year and receives a yearly increase of £100 a year; B starts at £500 a year and receives a half-yearly increase of £50 per half-year. How much has each received altogether after 10 years, neglecting interest?

17. A gramophone record has a spiral line cut on it winding from the outer edge, which is a circle of 6 inches radius, to the inner edge, which is a circle of radius 2 inches. The record revolves 80 times a minute and takes 4 minutes to play. Assuming that the spiral path is equivalent to a number of equally spaced concentric circles, find the total length of the path in yards.

18. (i) To how much will £100 amount after 10 years at 5 per cent. compound interest, reckoned yearly?

(ii) On January 1st, 1900, a man invested £100 at 5 per cent. compound interest, reckoned yearly. He invested in the same way £100 on January 1st of each year up to 1910 inclusive. What was the value of his investments on January 1st, 1911?

19. (i) How much must be paid now in order to receive £100 at the end of 10 years, allowing 5 per cent. compound interest, reckoned yearly?

(ii) What lump sum was paid on January 1st, 1900, in order to secure a payment of £100 a year for ten years, starting on January 1st, 1915, allowing 5 per cent. compound interest?

20. What single payment was required on January 1st, 1900, in order to secure a payment of £100 a year on every succeeding 1st of January for ever, allowing 4 per cent. compound interest?

21. If $s = 1 + 2x + 3x^2 + \dots + nx^{n-1}$, what is $s - xs$ equal to? Hence find s.

22. The series $1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \dots$ is obviously greater than

$$1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}) + \dots \quad \text{or} \quad 1 + \tfrac{1}{2} + (\tfrac{2}{4}) + (\tfrac{4}{8}) + \dots .$$

How many terms of the first series will give a sum greater than 100?

23. Show that the sum of any number of terms of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ always lies between 1 and $\frac{1}{2}$.

24. Prove that the sum of any number of terms of the series $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$ is less than a sufficient number of terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$, and therefore is < 2.

[Group the terms of the first series as in Example 22 above.]

25. Find the sum of 20 terms of the series

$$\frac{1}{2\times3}+\frac{1}{3\times4}+\frac{1}{4\times5}+\frac{1}{5\times6}+\ldots.$$

[Note that $\frac{1}{2\times3}=\frac{1}{2}-\frac{1}{3}$, etc.]

26. Find the sum of 30 terms of the series

$$\frac{1}{3\times5}+\frac{1}{5\times7}+\frac{1}{7\times9}+\ldots.$$

[See hint for No. 25.]

27. The sum of n terms of the series $1^2+2^2+3^2+4^2+5^2+\ldots$ is of the form an^3+bn^2+cn, where a, b, c are constants; find a, b, c.

28. The sum of n terms of a series is n^3; find the first four terms.

29. Verify that $1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2$ for the case $n=5$.

30. Find the sum of 10 terms of the series

$$1+1\times2^2+1\times2\times3^2+1\times2\times3\times4^2+\ldots.$$

[Note that

$$1\times2\times3\times4^2=1\times2\times3\times4\,(5-1)$$
$$=1\times2\times3\times4\times5-1\times2\times3\times4.]$$

31. If all rational numbers are written down as follows:

$$\tfrac{1}{1},\ \tfrac{2}{1},\ \tfrac{1}{2},\ \tfrac{3}{1},\ \tfrac{2}{2},\ \tfrac{1}{3},\ \tfrac{4}{1},\ \tfrac{3}{2},\ \tfrac{2}{3},\ \tfrac{1}{4},\ \tfrac{5}{1},\ \tfrac{4}{2},\ \tfrac{3}{3},\ \tfrac{2}{4},\ \tfrac{1}{5},\ \ldots,$$

(i) how many terms in the sequence are there before $\frac{1}{15}$?

(ii) what will be the 100th term?

32. Find the sum of 198 terms of the series

$$\log 1\tfrac{1}{2}+\log 1\tfrac{1}{3}+\log 1\tfrac{1}{4}+\log 1\tfrac{1}{5}+\ldots.$$

GROWTH FUNCTIONS.

Example VI. If £1 is invested at 3 per cent. compound interest, find its amount at the end of x years.

The interest on £100 is £3;

∴ the interest on £1 is £0·03;

∴ £1 amounts to £1·03 at the end of 1 year.

We therefore start the second year with £1·03.

This amounts to £(1·03)(1·03) at the end of the second year;

∴ the amount at the end of the second year is £$(1\cdot03)^2$.

Similarly, the amount at the end of the third year is £$(1\cdot03)^3$, and so on;

∴ the amount at the end of x years is £$(1\cdot03)^x$.

Such a function as $(1\cdot03)^x$ is called a *Growth function*, because it represents the magnitude of a quantity which increases in such a way that its growth over a definite period (in the above case one year) is proportional to its magnitude at the beginning of that period.

EXERCISE XIX. e.

1. (i) Draw the graph of the growth function $(1\cdot5)^x$ as x varies from 0 to 4.

(ii) A plant grows so that its height is $(1\cdot5)^x$ inches after $10x$ days, as long as $x < 4$; what is its height (1) to start with, (2) at the end of 4 days, 8 days, 12 days, 16 days?

(iii) In what ratio does its height increase (1) from the end of the 4th to the end of the 8th day, (2) from the end of the 8th day to the end of the 12th day, (3) from the end of the 12th day to the end of the 16th day?

2. Assuming that the population of a country is a growth function of the time, find the population in 1900, given that the population in 1880 was 23·6 million and in 1890 was 27·4 million.

3. The excess of the temperature of a liquid cooling in a room of fixed temperature above that of the room is a growth function of the time. If the temperature of the room is 30° C., and if at 1 p.m. the temperature of the liquid is 70° C. and at 1.10 p.m. it is 60° C., what is its temperature at 1.30 p.m.?

4. A merchant finds that his capital is a growth function of the time. After 10 years' work his capital is £50,000, and after 20 years' work it is £80,000; what will it probably be after 25 years' work?

5. If $r^x = 10$ and $r^{x+4} = 20$, what is r^{x+8} and what is r?

6. In an influenza epidemic it was found that the number of cases was a growth function of the time. In ten days the number had increased from 320 to 480; what was the daily increase per cent. ?

7. (i) If $f(x)$ is a growth function of x, prove that

$$\frac{f(x)}{f(x+a)}=\frac{f(x+a)}{f(x+2a)}=\frac{f(x+2a)}{f(x+3a)}, \text{ etc.}$$

(ii) Given a graph of a function, how would you test whether the function is or is not a growth function ?

CHAPTER XX.

PERMUTATIONS AND COMBINATIONS.

PERMUTATIONS.

Example I. Given five different letters a, b, c, d, e, how many different arrangements are there if each arrangement consists of two different letters ?

One way of answering this is to write down all the possible arrangements in a tabular form as follows :

Initial letter -	a	b	c	d	e
Arrangements -	ab ac ad ae	ba bc bd be	ca cb cd ce	da db dc de	ea eb ec ed

This shows that there are 5 groups each containing 4 arrangements ;

$\therefore$ the total number of arrangements is $5 \times 4 = 20$.

Note.—This number is called the number of *permutations* of 5 things taken 2 at a time, and is written ${}^5P_2$.

The argument can be put more concisely as follows, without writing down all the possible arrangements.

Imagine two compartments thus :

Initial Letter.	Second Letter.

Into the first compartment any one of the 5 letters can be put ; *i.e.* it can be filled in 5 ways.

And then, whatever letter has been put first, there are always 4 letters left, any one of which can be put in the second compartment.

Therefore the total number of possible arrangements is

$$5 \times 4 = 20.$$

Example II. How many different arrangements can be made with the 26 letters of the alphabet, if each arrangement is to consist of three different letters ?

Imagine three compartments :

First Letter.	Second Letter.	Third Letter.

The first can be filled in 26 ways.

The second can then be filled in 25 ways, however the first has been filled.

∴ there are 26×25 ways of filling the first two compartments. The third compartment can then be filled in 24 ways, however the first two have been filled.

∴ there are $26 \times 25 \times 24$ ways of filling the three compartments, *i.e.* the number of permutations of 26 things taken 3 at a time, or ${}^{26}P_3$ is $26 \times 25 \times 24$.

Similarly, it could be shown that the number of permutations of 5 things taken 5 at a time is $5 \times 4 \times 3 \times 2 \times 1$.

This product is usually written $\underline{|5}$ or $5!$, and called "factorial five," so that $${}^5P_5 = \underline{|5},$$

and $${}^nP_n = n(n-1)(n-2) \ldots \times 3 \times 2 \times 1 = \underline{|n},$$

where n is any positive integer.

EXERCISE XX. a.

1. Write down in tabular form all the different arrangements that can be made with the six letters a, b, c, d, e, f, if each arrangement is to consist of two different letters. How many arrangements are there ? What is ${}^6P_2$?

2. How many different arrangements can be made with six letters, if each arrangement is to consist of three different letters ? What is the number of permutations of six things taken three at a time ?

3. What is the value of (i) ${}^4P_2$, (ii) ${}^{10}P_3$?

4. In how many different ways can six books be arranged on a shelf ?

5. How many different numbers between 100 and 1000 can be formed out of the digits 2, 3, 4, 5, 6, 7, 8, if no digit may occur twice in the same number ?

6. What is the value of (i) ${}^7P_4$, (ii) $\dfrac{\lfloor 7}{\lfloor 3}$?

7. (i) There are ten people, each of whom sends a Christmas card to each of the others ; how many cards are required altogether ?

(ii) Ten houses play in a league competition, each house playing every other house ; how many games are played altogether ?

8. In how many different ways can the algebraical expression *abcd* be written ?

9. There are 8 stations on a local line ; how many different kinds of single third-class tickets must be printed in order that it may be possible to book from any one station to any other ?

10. One pound notes are numbered as follows : $\frac{\text{N}}{27}$ 803407. How many differently numbered pound notes can be issued if any one of the 26 letters of the alphabet may be used, if the number under the letter may be any number from 1 to 99 inclusive and the number following may be any number from 1 to 999999 inclusive ?

11. In how many ways can the order of batting of a cricket XI. be arranged if a particular batsman is to go in last and it is immaterial which of the first two batsmen is put down first ?

12. How many terms are there in the expansion of the products

(i) $(a+b)(x+y+z)$;

(ii) $(a+b)(c+d+e)(f+g+h+k)$;

(iii) $(1+a)(1+b)(1+c)(1+d)(1+e)$?

13. There are 12 houses on an open plain, no three of which are in a straight line. A man starts from one particular house and ends at another particular house ; what is the number of possible routes followed if he calls at 2 other houses on the way ?

14. A railway carriage has 8 seats ; in how many ways can 8 people arrange themselves (i) if there is no restriction, (ii) if three people have to sit on the side facing the engine, (iii) if two people have to sit in corner seats ?

15. How many numbers greater than 5000 can be formed out of the digits (i) 3, 4, 5, 6, 7, (ii) 0, 3, 4, 5, 6, if no digit is repeated ?

16. In how many ways can five people sit at a round table (i) if there is no restriction, (ii) if two particular people insist on sitting together, (iii) if two particular people refuse to sit together ?

17. In how many ways can 8 people be arranged in a row of 8 chairs if two particular people occupy the end seats ?

18. In how many ways can 6 men and 3 ladies be arranged at a round table so that no two ladies sit together ?

19. Six letters are written and the six corresponding envelopes are addressed ; in how many different ways can the letters be put one into each envelope so that (i) exactly two go wrong, (ii) at least two go wrong ?

20. In how many ways can five Latin books, four Greek books and four French books be arranged on a shelf so that the books in each language are together ?

21. How many even numbers between 100 and 1000 can be formed of the digits 2, 3, 4, 5, 6 ?

22. The guests at a dinner party consist of six ladies and six gentlemen : the host and hostess sit at opposite ends of an oblong table, the guests sit 6 on each side. In how many ways can the table be arranged if ladies and gentlemen sit in alternate places ?

23. (i) Show that $\dfrac{\underline{|11}}{\underline{|6}} = 11 \times 10 \times 9 \times 8 \times 7.$

(ii) Show that $14 \times 13 \times 12 = \dfrac{\underline{|14}}{\underline{|11}}.$

24. Express in factorial form as in Ex. 23 (ii) :

(i) $20 \times 19 \times 18 \times 17 \times 16$; (ii

(iii) ${}^{30}P_{12}$; (iv) ${}^{n}P_{12}$; (v

25. Simplify (i) $\underline{|n} \div \underline{|n-1}$;

(ii) $\underline{|n} - \underline{|n-1}$.

26. Simplify $\dfrac{\underline{|2n}}{1 . 3 . 5 . 7 . 9 \ldots (2n-3)(2n-1)}$.

27. Use logarithms to find the value of ${}^{200}P_{10}$ tc figures.

28. If $\underline{|20}$ is expressed in prime factors, to (i) 2, (ii) 3 occur ?

Summary of Results.

The number of permutations of n things taken r at a time is written ${}^nP_r$ and is equal to $\dfrac{\underline{|n}}{\underline{|n-r}}$ or

$$n(n-1)(n-2)\ldots(n-r+1).$$

COMBINATIONS.

Example III. Find the number of ways in which from the seven letters a, b, c, d, e, f, g, a selection of three letters can be made, no regard being paid to the order.

Any selection of 3 letters can be rearranged in $\underline{|3}=6$ ways; *e.g.* the selection b, c, f yields the 6 distinct arrangements bcf, bfc, cbf, cfb, fbc, fcb.

The total number of arrangements of 7 things, taken 3 at a time, is $7\times6\times5$.

But each selection yields $\underline{|3}$ arrangements;

$$\therefore \text{ the number of selections} = \frac{7\times6\times5}{\underline{|3}} = 35.$$

Note.—This is called the number of *combinations* of 7 things taken 3 at a time, and is written ${}^7C_3$.

A difference in order makes a new *arrangement*, but does not he *selection*.

umber of different ways of selecting r things from things is called the number of *combinations* of r at a time, and is denoted by ${}^nC_r$.

nt in Example III. shows that

$${}^nC_r = {}^nP_r \div \underline{|r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}.$$

EXERCISE XX. b.

any ways can 2 men be chosen from 5 men? e of ${}^5C_2$?

any ways can a football eleven be chosen from is ${}^{14}C_{11}$?

3. In how many ways can 3 cards be chosen from a pack of 52 cards ?

4. How many different hands is it possible for a player to have at bridge if his hand consists of 13 cards chosen from a pack of 52 ? Evaluate this number by logarithms to two significant figures.

5. There are 12 points in a plane, no 3 of which are collinear ; (i) how many lines can be drawn joining pairs of points ? (ii) how many triangles can be drawn each having 3 of these points as vertices ?

6. What is the greatest number of points of intersection of

(i) 12 lines with each other ;
(ii) 12 circles with each other ?

7. How many different sums of money can be made up if one oι each of the following coins is available : a halfpenny, a penny, a sixpence, a sl.illing. ρ florin ?

8. In how many ways can a committee of 4 men and 2 ladies be appointed from a society of 12 men and 8 ladies ?

9. (i) £12 is distributed among six people so that two receive £3 each, two receive £2 each, and two receive £1 each ; in ow many ways can this be done ?

(ii) In how many ways can six people be divided into three pairs ?

Why are the answers to (i) and (ii) different ?

10. (i) In how many ways can two sides of six players each be chosen from 12 men ?

(ii) In how many ways can three sides of four players be chosen from 12 men ?

11. In how many ways can the partners at a da
if there are 6 ladies and 6 gentlemen ?

12. (i) Four people play a game in which 13 card
each of them. How many different d
the cards are possible ?

(ii) In how many ways can a pack of 52 ca
in 4 hands of 13 cards each ?

13. A cricketing party on tour consists of 19
are primarily batsmen, 5 primarily bowlers, 2
keepers : in how many ways can an eleven co
men, 3 bowlers, 1 wicket-keeper be selected ?

14. In how many ways can a committee c
10 candidates so as to include the young

15. In how many ways can 5 dots and 3 dashes be arranged in a row ?

16. In how many ways can 6 dots and 4 dashes be arranged in a row if no two dashes are together ?

17. Prove that (i) ${}^{7}C_{5} = {}^{7}C_{2}$;
(ii) ${}^{n}C_{r} = {}^{n}C_{n-r}$

18. Prove that (i) ${}^{10}C_{4} + {}^{10}C_{5} = {}^{11}C_{5}$;
(ii) ${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$.

19. Simplify (i) ${}^{n}C_{r} \div {}^{n}C_{r-1}$;
(ii) ${}^{n}C_{r} \div {}^{n-1}C_{r-1}$.

Prove that ${}^{n}C_{r} + 2 \, . \, {}^{n}C_{r-1} + {}^{n}C_{r-2} = {}^{n+2}C_{r}$.

20. (i) If n is a positive integer > 2, prove that $n(n-1)(n-2)$ is divisible by 6.

(ii) If n is a positive integer $> r$, prove that
$$n(n-1)(n-2) \ldots (n-r+1)$$
is divisible by $\underline{|r}$.

21. If $n = a + b + c$, and if n, a, b, c are positive integers, prove that $\underline{|n}$ is divisible by $\underline{|a} \times \underline{|b} \times \underline{|c}$.

22. A selection of r letters is made from the n letters, a_1, a_2, a_3, ... a_n ; (i) how many of these include a_1 ? (ii) how many of these do not include a_1 ? (iii) hence deduce that
$${}^{n-1}C_{r} + {}^{n-1}C_{r-1} = {}^{n}C_{r}.$$

MISCELLANEOUS EXAMPLES

(Including Simple Probability).

7V. There are 10 tickets in a bag which are
3, ... up to 10. Two are drawn out at random.
ance that at least one of these is greater than 5 ?

of ways in which 2 tickets can be selected from
$\frac{9}{-} = 45$.

f ways in which 2 tickets can be selected from
, 2, 3, 4, 5 is ${}^{5}C_{2} = 10$;
hat *both* tickets are not greater than 5 is $\frac{10}{45} = \frac{2}{9}$;
ıat *at least* one ticket is greater than 5 is
$$1 - \tfrac{2}{9} = \tfrac{7}{9}.$$
that at least one ticket is greater than 5 are

When it is said that the probability of an event is, say, $\frac{3}{4}$, this does not mean that in every four trials the event will occur exactly 3 times ; nor does it follow that in four million trials, it will occur exactly three million times. But it does mean that if it occurs r times in n trials, it will usually be found that the fraction $\frac{r}{n}$ approaches closer to $\frac{3}{4}$, the larger n becomes.

If the probability of an event occurring is $\frac{r}{n}$, then the probability against it occurring is $1-\frac{r}{n}$ or $\frac{n-r}{n}$,

and the odds on it occurring are $r:(n-r)$ on it,

or $(n-r):r$ against it.

EXERCISE XX. c.

1. Two cards are drawn from a pack of 52 cards.

(i) In how many ways can this be done ?

(ii) What is the number of ways if none of the four aces are chosen ?

(iii) What is the number of ways if at least one of the two cards is an ace ?

(iv) What is the chance that at least one of the two cards is an ace ?

(v) What are the odds against one of the two cards being an ace ?

2. Two Americans and four Englishmen toss up to decide which four shall play in a set of tennis.

(i) In how many ways can the four players be chosen ?

(ii) In how many of these ways will both Americans be chosen ?

(iii) What is the chance that both Americans play in the first set ?

3. Twelve boys are chosen at random from a division of twenty boys, which includes three boys from a particular house. What is the chance that these three boys are all among the chosen twelve ?

4. You are one of 15 boys from whom an eleven is picked at random.

(i) In how many different ways can the eleven be picked ?

(ii) In how many of these ways will you yourself be picked ?

(iii) What is your chance of being picked ?

5. Work out the general case of which Question 4 above is a particular example, and show that the method of Question 4 gives your chance of being one of m boys picked at random from a total of n boys as ${}^{n-1}C_{m-1} \div {}^{n}C_{m}$, and that this simplifies to the result $\frac{m}{n}$ which ordinary common sense would lead you to expect.

6. (i) Two coins are put down on a table. What is the chance that they both show 'heads'?

(ii) Three coins are put down on a table. What is the chance that they all three show 'heads'?

(iii) n coins are put down on a table. What is the chance that all the coins show 'heads'?

7. Three coins are put down on a table.

(i) What is the chance that they are either all three heads or all three tails?

(ii) What is the chance that they show either two heads and a tail or two tails and a head?

(iii) What is the chance that they show two heads and a tail?

8. Two people each draw a card from an ordinary pack of 52 cards; what is the chance that their cards form a pair?

9. The six faces of two dice are numbered from 1 to 6. The dice are thrown on the table.

(i) What is the chance that 1 is the upper face on both dice?

(ii) What is the chance that the sum of the numbers on the two upper faces is 9?

10. There are ten entries for a mile race, and numbers are drawn at random for the starting positions. What is the chance that the slowest runner is one of the inside three? What is the chance that the slowest runner is one of the inside three and also the fastest runner one of the outside three?

11. Six boys, including yourself and a friend, are arranged at random in a line. What is the chance that you are next to your friend?

12. Two sides of eleven are picked at random from twenty-two people. What is the chance that A and B are on opposite sides?

13. How many different arrangements, using all five letters, can be made of the letters

(i) a_1, a_2, a_3, b_1, b_2;

(ii) a_1, a_2, a_3, b, b;

(iii) a, a, a, b, b?

14. (i) In how many different ways can fifteen differently coloured beads be arranged on a straight thread ?

(ii) What is the number of ways if five of the beads are all red and exactly alike ?

(iii) What is the number of ways if also the other ten are all blue and exactly alike ?

(iv) What is the number of ways if the thread is circular instead of being straight and the beads all of different colour ?

15. Three prizes are awarded to a division of twelve boys, and any boy may win all the prizes. In how many ways can each of the three prizes be awarded separately ? What is the total number of ways in which the three prizes can be awarded ?

16. How many numbers of three digits can be formed with the digits 1, 3, 5, 7, (i) if no repetitions are allowed, (ii) if the digits may be repeated ?

17. How many numbers of not more than four digits can be formed with the digits 1, 3, 5, 7, if repetitions are allowed ?

18. A certain periodical announces a competition, in which a prize is awarded to the reader sending in correct results to sixteen different football matches. If the result may be either a win, a loss or a draw in each match, what is the total number of different forecasts that may be sent in ? Use logarithms to evaluate this number to two significant figures, and state the chance of a reader hitting upon the correct solution if he fills up the results at random.

19. A cricket XI. is being photographed. Five of the eleven sit on chairs, four stand and two sit on the ground. How many different arrangements can be made, disregarding difference of order in standing or sitting, if the captain is to sit on a chair ?

20. In how many ways can 6 different books be distributed between A, B and C, so that A has one book, B has two and C has three ?

21. In how many ways can 6 people be divided into three groups, if there need not be more than 1 person in a group ?

22. In how many ways can $2n$ people be divided into n couples ?

23. A golfer wants to buy three golf balls at the professional's shop, and says that he wants either "Silver Kings," "Dunlops," "Colonels" or "Why Nots." The professional has only 3 Silver Kings, 2 Dunlops, 1 Colonel and 1 Why Not left in stock. In how many ways can the golfer choose 3 balls from these ?

24. Compute the chance that a player at bridge who has 13 cards dealt to him from a pack of 52 in the ordinary way holds all four aces in his hand.

25. A man has 3 pennies and 4 half-crowns in his pocket, and he pulls out 3 coins at random. How many different sums can he pull out, and which of these is the most likely? What would you be prepared to give him for the three coins before learning what they were?

26. A pessimist went during the war to France as one of a draft of 120 men. He calculated that in 3 months' time $\frac{1}{10}$th of the draft would be dead, $\frac{1}{5}$th would be in hospital and $\frac{1}{12}$th would be captured. At what amount should he have fixed the probability of his being still on active service in 3 months' time?

CHAPTER XXI.

THE BINOMIAL THEOREM.

If we multiply out $(x+a)(x+b)$ we obtain various terms, each of which has two factors, one from the first bracket and the other from the second bracket.

Thus
$$(x+a)(x+b) \equiv x^2+ax+bx+ab.$$

In the same way, if we multiply out $(x+a)(x+b)(x+c)$ we get various terms, each of which has three factors, one from each bracket. These terms can be conveniently grouped according to the number of x factors chosen in each term.

If the x is chosen from each bracket, we get the term x^3.

If the x is chosen from two brackets and the remaining letter from the third bracket, we get ${}^3C_1$ terms, namely the terms $ax^2+bx^2+cx^2$.

If the x is chosen from one bracket and the remaining letter from the other two brackets, we get ${}^3C_2$ terms (the number of ways in which the other two letters can be chosen), and these terms are $abx+bcx+cax$.

Finally, if no x term is chosen, we get one term abc.

Thus $(x+a)(x+b)(x+c)$ can be written down by inspection as
$$x^3+x^2(a+b+c)+x(ab+bc+ca)+abc.$$

Note that if we now make b and c equal to a, we find that
$$(x+a)(x+a)(x+a) \equiv x^3+x(a+a+a)+x(a^2+a^2+a^2)+a^3$$
$$\equiv x^3+3ax+3a^2x+a^3.$$

EXERCISE XXI. a.

1. *Write down* the expansions of

(i) $(x+a)(x+b)(x+c)(x+d)$;

(ii) $(x+a)(x+a)(x+a)(x+a)$;

(iii) $(x+1)(x+1)(x+1)(x+1)$;

(iv) $(x-1)^4$.

2. (i) How many terms are there in the expansion of

$(x+a)(x+b)(x+c)(x+d)(x+e)(x+f)$?

(ii) How many of these terms have a factor x and not x^2 ?

(iii) How many of these terms have x^2 as a factor and not x^3 ?

3. What is the coefficient of x^2 in the expansions of

(i) $(x+a)(x+b)(x+c)(x+d)(x+e)$;

(ii) $(x+a)^5$;

(iii) $(x-1)^5$?

4. How many terms in the following expansions contain x^3 but not x^4 as a factor:

(i) $(x+a)(x+b)(x+c)(x+d)$;

(ii) $(x+a_1)(x+a_2)(x+a_3)(x+a_4)(x+a_5)(x+a_6)(x+a_7)$?

What is the coefficient of x^3 in $(x+1)^7$?

5. Expand in descending powers of x:

(i) $(x+1)(x+2)(x+3)$;

(ii) $(x+1)(x+2)(x+3)(x+4)$.

6. Expand in descending powers of x:

(i) $(x+1)(2x+1)(3x+1)$;

(ii) $(x+1)(2x+1)(3x+1)(4x+1)$.

The examples of the last exercise show how to write down the expansion of any expression of the form

$$(x+a_1)(x+a_2)(x+a_3)(x+a_4)\dots(x+a_n).$$

It is, in fact,

$$x^n + x^{n-1}(a_1+a_2+a_3+\dots) + x^{n-2}(a_1a_2+a_1a_3+a_2a_3+\dots)$$
$$+ x^{n-3}(a_1a_2a_3+a_1a_2a_4+\dots)+\dots.$$

The coefficient of x^{n-2} is the sum of the products, taken *two* at a time, of the n letters $a_1, a_2, \dots, a_n$.

The coefficient of x^{n-3} is the sum of these products taken *three* at a time.

The coefficient of x^{n-4} is the sum of these products taken *four* at a time, and so on.

Therefore, there are n terms in the coefficient of x^{n-1},

and $^nC_2$ .... x^{n-2},

and $^nC_3$ .... x^{n-3},

and so on.

Suppose now $a_1 = a_2 = a_3 = a_4 = \ldots = a_n = a$.

Then each term in the coefficient of x^{n-1} equals a,

and .... x^{n-2} .... a^2,

and .... x^{n-3} .... a^3,

and so on.

$\therefore$ the coefficient of x^{n-1} is $n \cdot a$,

.... x^{n-2} is $^nC_2 \cdot a^2$,

.... x^{n-3} is $^nC_3 \cdot a^3$,

and so on.

$\therefore$ if n is any positive integer,

$$(x+a)^n \equiv x^n + n \cdot x^{n-1}a + {}^nC_2 \cdot x^{n-2}a^2 + {}^nC_3 \cdot x^{n-3}a^3 + \ldots + {}^nC_r x^{n-r}a^r + \ldots + a^n,$$

or

$$(x+a)^n \equiv x^n + n \cdot x^{n-1}a + \frac{n(n-1)}{1 \cdot 2}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}a^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \ldots r}x^{n-r}a^r + \ldots + a^n.$$

This result is called the *Binomial Theorem.*

EXERCISE XXI. b.

Write down the expansions of Nos. 1-8.

1. $(1+x)^3$. **2.** $(1+x)^4$. **3.** $(1-x)^4$. **4.** $(x-2)^3$.

5. $(x-y)^6$. **6.** $\left(x+\frac{1}{x}\right)^5$. **7.** $(2x-3y)^3$. **8.** $(a^2-2b)^4$.

9. What is the coefficient of x^3 in (i) $(x-1)^7$; (ii) $(1+2x)^{\text{..}}$

10. What is the 4th term in the expansion of $(2-x)^{10}$?

11. What is the term independent of x in the exp $\left(x+\frac{2}{x}\right)^4$?

12. How many terms are there in the expansion of $(2x - 3y)^{10}$?

13. What is the middle term in the expansion of $(x - y)^6$?

14. What is (i) the 4th term from the beginning; (ii) the 4th term from the end in $(x + y)^{10}$?

15. What is the rth term in the expansion of (i) $(1 + 2x)^n$; (ii) $(1 - 2x)^n$?

16. What is (i) the rth term from the beginning; (ii) the rth term from the end in $(x + y)^n$? Are the coefficients equal ?

17. Simplify $(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$.

18. Simplify $1 - 4(x+1) + 6(x+1)^2 - 4(x+1)^3 + (x+1)^4$.

19. In the identity

$$(1+x)^n \equiv 1 + {}^nC_1 \,.\, x + {}^nC_2 \,.\, x^2 + \ldots + {}^nC_r x^r + \ldots + x^n,$$

what results are obtained by putting (i) $x = 1$; (ii) $x = -1$?

20. What is the coefficient of x^4 in $(1-x)(1+x)^7$?

21. What is the coefficient of x^r in $(1-x)(1+x)^n$?

22. Expand $(1 + x + x^2)^n$ as far as terms involving x^3.

23. Simplify $(\sqrt{2}+1)^4 + (\sqrt{2}-1)^4$.

24. Use the binomial theorem to evaluate 101^4 and 99^5.

25. Use the binomial theorem to evaluate to 5 significant figures $1{\cdot}02^5$ and $0{\cdot}97^6$.

APPROXIMATION.

The binomial expansion

$$(1+x)^n \equiv 1 + nx + \frac{n(n-1)}{1\,.\,2}x^2 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\,.\,2\,.\,3\ldots r}x^r + \ldots$$

has been proved, if n is a positive integer.

If n is fractional or negative, this result still holds good, provided that x lies between $+1$ and -1. The general proof is, however, beyond the scope of this book.

˜ʳ n is a positive integer, the expansion contains $n + 1$ terms.

ʻf n is fractional or negative, the expansion in this form terminate : in this case, if x is small compared with 1, w terms of the expansion may give a useful approxi-

EXERCISE XXI. c.

1. Prove by long division that

(i) $\dfrac{1}{1-x} \equiv 1 + x + \dfrac{x^2}{1-x}$;

(ii) $\dfrac{1}{1-x} \equiv 1 + x + x^2 + \dfrac{x^3}{1-x}$;

(iii) $\dfrac{1}{1-x} \equiv 1 + x + x^2 + x^3 + \dfrac{x^4}{1-x}$.

2. Find R if $\dfrac{1}{1-x} \equiv 1 + x + x^2 + x^3 + x^4 + R$.

3. Assuming the binomial theorem, if x is a positive fraction less than 1, write down the first five terms of the expansion of

$$\frac{1}{1-x} \equiv (1-x)^{-1}.$$

4. If $x = 0{\cdot}1$, what is approximately the error in taking $\dfrac{1}{1-x}$ equal to (i) $1 + x$; (ii) $1 + x + x^2$?

5. Find R by long division if

(i) $\dfrac{1}{1+x} \equiv 1 - x + R$;

(ii) $\dfrac{1}{1+x} = 1 - x + x^2 + R$;

(iii) $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + R$.

6. Assuming the binomial theorem, if x is a positive fraction less than 1, write down the first five terms of the expansion of

$$\frac{1}{1+x} \equiv (1+x)^{-1}.$$

If $x = 0{\cdot}01$, what is approximately the error in taking $\dfrac{1}{1+x}$ equal to (i) $1 - x$; (ii) $1 - x + x^2$?

7. Find R if (i) $\dfrac{1}{(1-x)^2} \equiv 1 + 2x + R$;

(ii) $\dfrac{1}{(1+x)^2} = 1 - 2x + R$.

8. Assuming the binomial theorem, if x is a positive fraction less than 1, write down the first four terms of the expansions of

$$\frac{1}{(1-x)^2} \equiv (1-x)^{-2} \text{ and } \frac{1}{(1+x)^2} \equiv (1+x)^{-2}.$$

What is approximately the error in taking $(1-x)^{-2} = 1 + 2x$ and $(1+x)^{-2} = 1 - 2x$ if $x = 0{\cdot}01$?

9. If x is a positive fraction less than 1, prove that

(i) $(1-\frac{1}{2}x)^2 > 1-x$; (ii) $(1-\frac{1}{2}x-\frac{1}{2}x^2)^2 < 1-x$;

(iii) $1-\frac{1}{2}x-\frac{1}{2}x^2 < \sqrt{1-x} < 1-\frac{1}{2}x$;

(iv) the error in taking $\sqrt{1-x}=1-\frac{1}{2}x$ is less than $\frac{1}{2}x^2$.

10. (i) Find by the square root rule the value of $\sqrt{1-x}$ as far as terms involving x^3.

(ii) What are the first four terms of the binomial expansion for $\sqrt{1-x} \equiv (1-x)^{\frac{1}{2}}$, assuming the Binomial Theorem?

(iii) If $x=0{\cdot}01$, what is the approximate error in taking $\sqrt{1-x}=1-\frac{1}{2}x$?

11. (i) Find by the square root rule the value of $\sqrt{1+x}$ as far as terms involving x^3.

(ii) If x is a positive fraction less than 1, what are the first four terms of the binomial expansion for

$$\sqrt{1+x} \equiv (1+x)^{\frac{1}{2}},$$

assuming the Binomial Theorem ?

(iii) If $x=0{\cdot}1$, what is the approximate error in taking $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x$?

12. (i) Prove that $\dfrac{1}{x+h} \equiv \dfrac{1}{x}\left(1+\dfrac{h}{x}\right)^{-1}$.

(ii) If h is small compared to x, find an approximation for $\dfrac{1}{x+h}$.

(iii) Express approximately $\dfrac{1}{1{\cdot}02}$ as a decimal.

13. (i) Prove that $\dfrac{1}{\sqrt{x^2+h}} \equiv \dfrac{1}{x}\left(1+\dfrac{h}{x^2}\right)^{-\frac{1}{2}}$.

(ii) If h is small compared to x, find an approximation for $\dfrac{1}{\sqrt{x^2+h}}$.

(iii) Express approximately $\dfrac{1}{\sqrt{4{\cdot}01}}$ as a decimal.

(iv) Express approximately $\dfrac{1}{\sqrt{9{\cdot}05}}$ as a decimal.

Summary of Results.

I. If x is small compared with 1,

$$(1+x)^n \simeq 1+nx,$$
$$(1-x)^n \simeq 1-nx.$$

In particular,
$$\frac{1}{1+x} \equiv (1+x)^{-1} \simeq 1-x,$$
$$\frac{1}{1-x} \equiv (1-x)^{-1} \simeq 1+x,$$
$$\sqrt{1+x} \equiv (1+x)^{\frac{1}{2}} \simeq 1+\tfrac{1}{2}x,$$
$$\sqrt{1-x} \equiv (1-x)^{\frac{1}{2}} \simeq 1-\tfrac{1}{2}x.$$

II. If h is small compared with x,
$$(x+h)^n \equiv x^n\left(1+\frac{h}{x}\right)^n \simeq x^n\left(1+\frac{nh}{x}\right).$$

Example I. If h is small, find an approximation for $\frac{(1-h)^2}{\sqrt{1+h}}$, and deduce an approximate value of $\frac{(0\cdot98)^2}{\sqrt{1\cdot02}}$.

$$\frac{(1-h)^2}{\sqrt{1+h}} = (1-h)^2 \,.\, (1+h)^{-\frac{1}{2}}$$
$$\simeq (1-2h)(1-\tfrac{1}{2}h) \simeq 1-2h-\tfrac{1}{2}h$$
$$\simeq 1-\frac{5h}{2}.$$

If $h=0\cdot02$, we have
$$\frac{(0\cdot98)^2}{\sqrt{1\cdot02}} \simeq 1-\frac{5\times0\cdot02}{2} \quad \text{or} \quad 1-0\cdot05$$
$$\simeq 0\cdot95.$$

Example II. The cost per hour of an electric lamp varies as $\frac{V^2}{R}$, where V is the voltage and R is the resistance. Find the approximate percentage change in the cost if the voltage is increased by 2 per cent. and the resistance is decreased by 3 per cent.

If the original cost per hour is $\frac{kV^2}{R}$ pence, the new cost is
$$\frac{k(V\times1\cdot02)^2}{R\times0\cdot97} \text{ pence} = \frac{kV^2}{R}(1+0\cdot02)^2(1-0\cdot03)^{-1}$$
$$\simeq \frac{kV^2}{R}(1+0\cdot04)(1+0\cdot03)$$
$$\simeq \frac{kV^2}{R}(1+0\cdot04+0\cdot03) \quad \text{or} \quad \frac{1\cdot07kV^2}{R} \text{ pence};$$

$\therefore$ the increase of cost is $\frac{0\cdot07kV^2}{R}$ pence;

$\therefore$ **the increase is 7 per cent.**

EXERCISE XXI. d.

1. (i) Simplify $(1+a)(1+b)-(1+a+b)$.

(ii) What is the error in taking $1{\cdot}01 \times 1{\cdot}02$ equal to

$$1 + {\cdot}01 + {\cdot}02 \quad \text{or} \quad 1{\cdot}03\ ?$$

(iii) Write down the approximate value of $1{\cdot}03 \times 1{\cdot}05$.

2. (i) If h is small, what is an approximate value of $(1+h)^3$?

(ii) Write down an approximate value of $(1{\cdot}02)^3$.

3. Each linear dimension of a rectangular block of steel expands $0{\cdot}000{,}011$ of itself for every 1° C. rise of temperature. In what ratio does the volume increase if the temperature rises (i) 1° C., (ii) 3° C. ?

4. If an error of 1 per cent. is made in measuring the length and breadth of a field, what may be the approximate error per cent. in the calculated area ?

5. A man estimates the dimensions of a room, and makes an error of 1 per cent. in the length, 2 per cent. in the breadth and 3 per cent. in the height ; what may be the approximate error per cent. in the calculated volume ?

6. (i) If the diameter of a sphere increases by 1 per cent., what is the approximate increase per cent. (*a*) in its volume, (*b*) in its surface ?

(ii) If the superficial area of a spherical soap-bubble increases by 1 per cent., what is the approximate increase per cent. in its volume ?

7. If h is small,

(i) show that $\dfrac{1-h}{1+h} \simeq 1-2h$;

(ii) find an approximation for $\dfrac{1+h}{1-h}$, and hence for $\dfrac{1{\cdot}01}{0{\cdot}99}$.

8. Find approximations for

(i) $(1{\cdot}002)^5$; (ii) $(0{\cdot}98)^3$; (iii) $\dfrac{1}{0{\cdot}997}$;

(iv) $\sqrt{1{\cdot}008}$; (v) $\sqrt[3]{1{\cdot}015}$; (vi) $\dfrac{1}{5{\cdot}003}$.

9. If h is small, find approximate values of

(i) $\dfrac{(1+3h)(1+4h)}{1+5h}$; (ii) $\dfrac{(1+3h)(1+4h)}{1+12h}$.

What is the approximate value of $\dfrac{1{\cdot}03 \times 1{\cdot}04}{1{\cdot}12}$?

10. The time of oscillation of a pendulum of length l feet is approximately $1{\cdot}11\sqrt{l}$ seconds. Find approximately the increase in the time when the length is altered from (i) 4 feet to 4·1 feet, (ii) l feet to $(l+h)$ feet, where h is small compared with l.

11. A given mass of gas at constant temperature obeys the law $pv=$ constant, where p is the pressure in lb. per sq. in. and v is its volume in cu. in.

(i) If the pressure is increased by 2 per cent., what is the approximate change in the volume ?

(ii) If the pressure alters from p to $p+p_1$, what is approximately the change of volume if p_1 is small compared with p ?

12. (i) What are the roots of $x^5-x^3=0$?

(ii) One root of $x^5-x^3=0{\cdot}01$ is nearly equal to 1; by putting $x=1+h$ and treating h as small, find a closer approximation to this root.

13. If h is small compared with x, find approximate values for

(i) $\dfrac{(x+h)^3-x^3}{h}$; (ii) $\dfrac{(x+h)^{10}-x^{10}}{h}$;

(iii) $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$; (iv) $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$.

14. The maximum deflection of a beam of length l feet and depth d inches, supported at its ends, is $\dfrac{kl^4}{d^2}$ in., where k is a constant; in what ratio is the deflection altered if l increases from 20 to 20·1 and d increases from 4 to 4·1 ?

15. A given mass of gas, absolute temperature T°, volume v cu. in. at pressure p lb. per sq. in., obeys the law $\dfrac{pv}{T}=$ constant. Find the percentage change in p if v increases by 2 per cent. and T decreases by 3 per cent.

16. The sides containing the right angle of a right-angled triangle are of lengths x in. and h in.; if h is small compared with x, find approximately by how much the hypotenuse exceeds the longer side.

17. If θ is small, find an approximation for $\dfrac{\theta-\frac{\theta^3}{6}}{1-\frac{\theta^2}{2}}$.

18. The time of pneumatic transmission through a pipe l yards long of diameter d inches under a pressure of p lb. per sq. inch varies as $\dfrac{l^{\frac{3}{2}}}{d^{\frac{1}{2}}p^{\frac{1}{2}}}$. What is the approximate percentage change in the time if l increases by 1 per cent., d increases by 2 per cent. and p decreases by 1 per cent. ?

19. In the Michelson-Morley experiment for the determination of the velocity of the earth relative to the ether, the effect to be observed was measured by $2(l_1+l_2)\left\{\frac{c}{c^2-v^2}-\frac{1}{\sqrt{c^2-v^2}}\right\}$, where v is the velocity required and c the velocity of light. Prove that if $\frac{v}{c}$ is small, this expression $\doteq (l_1+l_2)\cdot\frac{v^2}{c^3}$.

CHAPTER XXII.

EMPIRICAL FORMULAE.

AN experiment was made to determine the extension produced in a s[illegible] spring by weights of various amounts attached to its ext[illegible] and the results were tabulated as follows :

Length of spring in cm., l	28·3	29·5	31·5	35·6	39·7
Weight attached in gr., W	10	25	50	100	150

These observed results can be shown to obey *approximately* the law $l = 0{\cdot}081\,W + 27{\cdot}5$.

A law obtained in this way is called an *empirical formula.*

If the result of plotting y against x gives a straight line, y and x are connected by an equation of the form $y = ax + b$ where a, b are constants. The following example shows how these constants are found, if the straight-line graph is given.

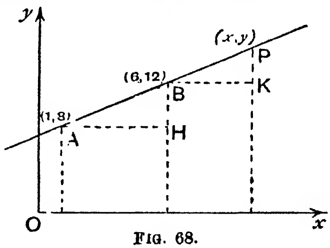


FIG. 68.

Example I. Two points on a straight-line graph which represents the relation between two variables x and y, are found to be $x = 1$, $y = 8$, and $x = 6$, $y = 12$.

Find the equation connecting x and y in the form $y = ax + b$.

A, B are the points (1, 8) and (6, 12).

Denote the coordinates of *any other* point P on the line by (x, y).

Through A, B, P draw lines parallel to Ox, Oy forming the triangles AHB, BKP, which are similar.

$$\therefore \frac{KB}{HA} = \frac{PK}{BH};$$

$$\therefore \frac{x-6}{6-1} = \frac{y-12}{12-8}; \qquad \therefore \frac{x-6}{5} = \frac{y-12}{4};$$

$$\therefore 4x - 24 = 5y - 60; \quad \therefore 5y = 4x + 36; \quad \therefore y = 0{\cdot}8x + 7{\cdot}2,$$

which is the functional relation required.

By exactly the same method, it may be shown that if (x_1, y_1) and (x_2, y_2) are the coordinates of any two points on a straight line graph, the relation which the graph represents is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}.$$

Example II. The observations made in an experiment were tabulated as follows :

$x=1$	1·5	2	2·5	3	[illegible]	4
$y=2{\cdot}50$	2·20	1·92	1·65	1·32	0·94	0·60

Find the most probable relation which expresses y in terms of x.

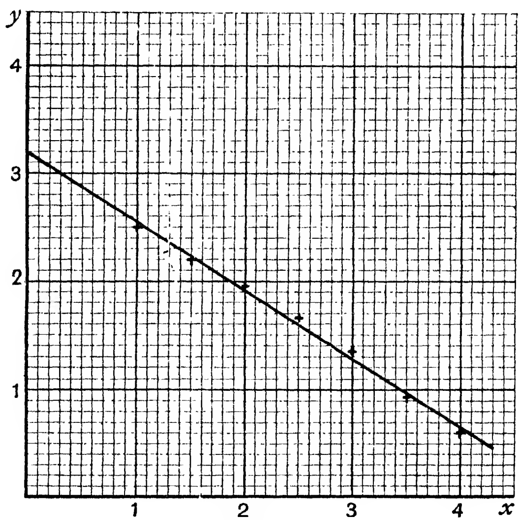


FIG. 69.

First plot the observations. We see that the points obtained, although not on a straight line, do not differ much from the straight. Now use a fine black thread to determine the "best-fit line" and read off the coordinates of two points at opposite ends: *e.g.* $x=0\cdot3$, $y=3$ and $x=3\cdot9$, $y=0\cdot7$.

The relation is therefore

$$\frac{x-3\cdot9}{3\cdot9-0\cdot3}=\frac{y-0\cdot7}{0\cdot7-3}$$

or

$$\frac{x-3\cdot9}{3\cdot6}=\frac{y-0\cdot7}{-2\cdot3};$$

$$\therefore\ y-0\cdot7=-\frac{2\cdot3}{3\cdot6}(x-3\cdot9);$$

$$\therefore\ y=-\frac{2\cdot3}{3\cdot6}x+\frac{2\cdot3\times3\cdot9}{3\cdot6}+0\cdot7;$$

$$\therefore\ y=-0\cdot64x+3\cdot2,$$

where each constant is calculated correct to two significant figures.

Example III. A load of 100 lb. is fastened to one end of a rope which passes round a rough circular peg and measurements are made of the least pull which must be applied at the other end of the rope in order to raise the load slowly. The magnitude of the pull required depends on the amount of the rope in contact with the peg. Suppose the least pull is P lb. when the portion of the rope in contact with the peg is θ right angles. The following observations were made:

θ = right angles	1	2	4	6	8
P lbs. - -	140	181	359	672	1190

Find the most probable relation expressing P in terms of θ.

If we plot these results as they stand, the graph will be a curve. If, however, we plot $\log P$ against θ we see that the points do not diverge far from a straight line. The table is

$\theta=$ 1	2	4	6	8
$\log P=$ 2·1461	2·2577	2·5551	2·8274	3·0755

These results are plotted in Fig. 70: and a fine thread is then used to discover the "best-fit line." We see that two points on this line are

$$\left.\begin{array}{r}\theta=7\cdot6\\ \log P=3\cdot038\end{array}\right\}\ \text{and}\ \left.\begin{array}{r}\theta=1\cdot3\\ \log P=2\cdot179\end{array}\right\};$$

$\therefore$ the relation is $\dfrac{\log P - 3{\cdot}038}{3{\cdot}038 - 2{\cdot}179} = \dfrac{\theta - 7{\cdot}6}{7{\cdot}6 - 1{\cdot}3}$;

$\therefore \dfrac{\log P - 3{\cdot}038}{0{\cdot}859} = \dfrac{\theta - 7{\cdot}6}{6{\cdot}3}$ or $\log P = \dfrac{0{\cdot}859}{6{\cdot}3}\theta - \dfrac{0{\cdot}859 \times 7{\cdot}6}{6{\cdot}3} + 3{\cdot}038$;

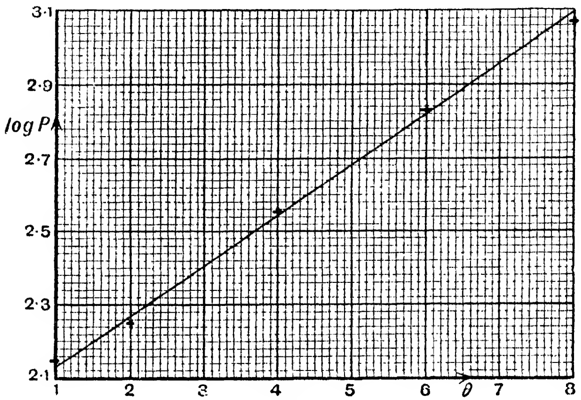


FIG. 70.

$\therefore \log P = 0{\cdot}136\theta + 2{\cdot}002$; $\quad \therefore P = 10^{0{\cdot}136\theta + 2{\cdot}002}$;

$\therefore P = 10^{2{\cdot}002} \times 10^{0{\cdot}136\theta}$; $\quad \therefore P = 100 \times 10^{0{\cdot}136\theta}$.

This approximate formula is the required relation.

EXERCISE XXII. a.

1. The following observations were made:

$x = 1$	2	4	5	7
$y = 0{\cdot}95$	2·8	5·8	7·7	10·7

Find the best fit equation for y in terms of x.

2. The length of a spiral spring is measured when different weights are suspended from its end; the results, when tabulated, are as follows:

Weight in gr., W	10	15	30	50	75
Length in cm., l -	22·1	24·1	30·2	38·4	48·7

Find the best fit equation for l in terms of W.

3. The following results were obtained with a screw jack:

Load in lb., W -	10	20	30	50	75
Effort in lb., P -	3·28	4·51	5·72	8·11	11·12

Find the best fit equation for P in terms of W.

4. If $y=\dfrac{100}{\sqrt{100+x}}$, make a table showing corresponding values of x and y between $x=10$ and $x=50$, and find a relation of the form $y=a+bx$, which best fits this function in this range. What is the error when $x=44$?

5. The following observations were made:

x	1	2	3	4	5
y	8	13	18	28·5	34

It is believed that they obey a linear law. In which observation is the error probably greatest? Ignoring this, what is the best fit equation for y in terms of x? And how would the table then run?

6. The pressure P lb. per sq. foot exerted by a wind of v miles an hour is measured in the following cases:

v	5	10	15	20	25	30
P	0·12	0·51	1·15	1·88	2·95	4·58

Plot P against v^2, and hence find the probable relation expressing P in terms of v.

7. In Giffard's Injector, the following values are given for the delivery of Q gallons of water per hour under a pressure of 60 lb. per sq. inch for a throat diameter D inches.

D	0·1	0·15	0·2	0·25	0·3
Q	300	730	1200	2000	2700

Plot D against $\sqrt{Q}$, and hence find the probable relation expressing Q in terms of D.

8. The following table gives the resistance R lb. to a train of weight 100 tons running at V miles an hour:

V	10	20	30	40	50	60
R	680	980	1380	2050	2880	3800

Plot R against V^2, and then express R in terms of V.

9. A body of density d, weighing 1 gr. in air (when brass weights are used), weighs 1 gr. $+ k$ mgr. in a vacuum. The following observations were recorded:

$d = 0{\cdot}5$	1	3	5	6
$k = 2{\cdot}26$	1·06	0·26	0·10	0·06

Plot k against $\frac{1}{d}$, and express k in terms of d.

10. A marble rolls down a groove, at the side of which are markers which are so adjusted as to point the position of the marble at intervals of one second. The following observations were made:

Time in seconds, t	1	2	3	4	5
Distance rolled in inches, s	4·1	16·3	36·8	65·4	102·1

Find the probable relation expressing s in terms of t.

11. The following results were recorded in an optical experiment:

x	10	11	12	15	20	30
y	39·9	33·1	28·9	22·4	18·5	15·7

Plot $\frac{1}{x}$ against $\frac{1}{y}$, and find the best fit equation expressing y in terms of x.

12. The following list of British amateur running records is taken from *Whitaker's Almanack*:

Distance in yd., d	150	200	220	440	880	1000
Time in sec., t	14·6	19·4	21·8	48·4	112·2	132·4

Plot $\log t$ against $\log d$, and hence find a relation expressing t in terms of d.

Professor Perry has pointed out that all forms of race records (men and animals) conform to a law of the same kind.

13. The following observations were made for the pressure and volume of saturated steam:

Volume, cu. ft. per lb. water, v -	297	173	82·4	55·1	21·3
Pressure, lb. per sq. in., p - -	1·13	2·02	4·42	6·77	18·7

Plot $\log p$ against $\log v$, and hence express p in terms of v.

14. The following census returns for England and Wales, correct to 4 significant figures, are taken from *Whitaker's Almanack*:

Year - -	1851	1861	1871	1881	1891	1901	1911
Population in millions	17·93	20·07	22·71	25·97	29·00	32·53	36·07

If the population in the year $(1851 + 10t)$ is P millions, show that an approximate formula of the type $P = a \times 10^{bt}$ exists, and find it. What, according to your formula, was the population in 1841, and what will it be in 1921?

15. In an experiment to determine the relation between the emissive power at a given temperature of a black body for the wave length for which the power is a maximum and that temperature, the following results were recorded by *Lummer* and *Pringsheim*: (See Preston's *Theory of Heat.*)

Absolute temp., T -	621·2	723	908·5	1094·5	1259	1460·4
Emissive power, E -	2·026	4·28	13·66	34·0	68·8	145·0
Wave-length, λ -	4·73	4·08	3·28	2·71	2·35	2·04

Plot $\log E$ against $\log T$, and express E in terms of T.

16. With the data of Ex. 15, plot $\log \lambda$ against $\log T$, and express λ in terms of T.

17. In a radiation experiment, the following records were made:

Absolute temp., T	373·1	492·5	723	810	1378	1497
Deviation of galvanometer, δ	156	638	3320	5150	44700	61600

Plot δ against T^4, and express δ in terms of T. [Stefan's Law.]

18. The following table gives the age to which a woman expects to live at various ages:

Present age, x - - -	10	15	20	30	35
Age probably reached, y -	62·0	62·6	63·4	65·4	66·5
Present age, x - - -	40	50	55	60	
Age probably reached, y -	67·7	70·6	72·1	73·9	

Show that y can be expressed in terms of x by a relation of the form $y=61+ax+bx^2$ approximately; find a and b. To what age would a woman 25 years old expect to live?

$$\left[\text{Plot } \frac{y-61}{x} \text{ against } x.\right]$$

19. The following table for the specific heat of water at various temperatures is due to *Prof. Callendar*:

Temperature centigrade, $60+t$	60°	70°	80°	90°
Specific heat, S - - -	1·0	1·0016	1·0033	1·0053
Temperature centigrade, $60+t$	100°	140°	180°	
Specific heat, S - - -	1·0074	1·0176	1·0308	

Show that S can be expressed in terms of t by a relation of the form $S=1+at+bt^2$ approximately. Find a, b. What is the probable value of S at 160° C.?

20. For a slope of 1 in 100, the velocity of flow in a full pipe, diameter d inches, is v feet per sec., where v, d are connected as follows:

d	4	10	15	30	48	60
v	0·56	1·2	1·6	2·7	3·7	4·3

It is thought that a relation of the form $v=\frac{ad}{1+b\sqrt{d}}$ exists, where a, b are constants. What method of plotting would you adopt to try this? Find a, b.

SUMMARY OF RESULTS.

(i) If the graph obtained by plotting y against x is, or approximates to, a straight line, and if (x_1, y_1), (x_2, y_2) are two points on it, preferably at opposite ends, then y can be expressed as a function of x by the equation

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}.$$

(ii) If the graph obtained by plotting y against x is a curve, it may be possible to obtain a straight line by plotting

(*a*) $\log y$ against $\log x$,
(*b*) $\log y$ against x,
(*c*) y against $\log x$,
(*d*) y against $\frac{1}{x}$, etc.

(iii) Where the data are experimental, coefficients in the functional relation should be expressed as decimals, and must not be given to more significant figures than the data or the method justify.

CHAPTER XXIII.

NOMOGRAPHY.

LINE CHARTS.

Example I. Construct a line chart connecting inches and centimetres, and read off from it, (i) the number of centimetres corresponding to 2·5 inches, (ii) the number of inches corresponding to 7 cm.

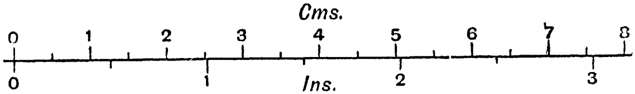


FIG. 71.

Draw a straight line; take an initial point O near the left end of the line; graduate the upper side of the line in cm. and the lower side in inches, each scale beginning at O.

The result is the required line chart.

Opposite the 2·5 graduation on the lower (inch) scale can be read off the number 6·3 on the upper (cm.) scale, so that $2{\cdot}5'' \equiv 6{\cdot}3$ cm. approx.

Opposite the 7 cm. graduation on the upper (cm.) scale can be read off the number 2·75 on the lower (inch) scale, so that 7 cm. $= 2{\cdot}75''$ approx.

Example II. Construct a line chart connecting numbers with their logarithms.

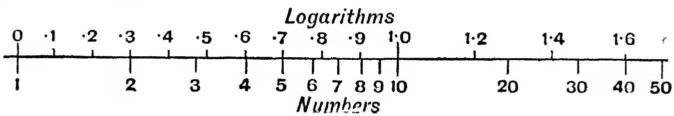


FIG. 72.

This chart embodies the principle of the slide rule. It is simply constructed by graduating the upper side of the line uniformly on a scale of (say) 5 cm. to a unit and marking the graduations 1, 2, 3, 10, 20, ... etc. on the lower side at distances from the O graduation on the upper side equal to 5 log 1, 5 log 2, 5 log 3, 5 log 10, 5 log 20 cm., etc., the values of the logarithms being taken from the tables.

EXERCISE XXIII. a.

1. Construct a simple line chart connecting numbers and their squares from 1 to 9.

Read off from your chart (i) the square of 7·2;
(ii) $\sqrt{72}$;
(iii) $(2{\cdot}6)^2$.

2. In the figure of Example I. above, read off the number of inches in 4 cm., and the number of cm. in 1·5 inches.

3. Construct a simple line chart connecting numbers and their cube roots from 1000 to 2000.

Read off from your chart (i) $\sqrt[3]{1420}$;
(ii) the cube of 112.

4. Construct a simple line chart connecting the circumferences of circles and their diameters for diameters from 1″ to 10″.

Read off from your chart (i) the circumference of a circle whose diameter is 7·5″; (ii) the diameter of a circle whose circumference is 25″.

5. Construct, using sine tables, a simple line chart connecting the length of a chord, and the angle it subtends at the centre of a circle of radius 10 cm., for angles from 0 to 60°.

6. Construct a simple line chart connecting x and y, where $y = 3{\cdot}2x + 2{\cdot}6$ for $0 < x < 5$. Read off the value of x when $y = 10$.

7. Construct a simple line chart for reducing marks which run from 250 to 500 to the scale 0 to 100. To what does 340 reduce?

We will now illustrate another method of obtaining, graphically, a large number of approximate numerical results from a simple formula. Under the ordinary system of uniform graduation of the axes, a graph could be drawn to illustrate any formula involving one dependent variable. The graph would usually be a smooth curve. By adopting a system of non-uniform graduation, it is often possible to obtain a straight-line graph, and this method is particularly useful when approximate numerical results are required for a large number of closely-related formulae. Under the ordinary system a "family" of curves would be required; by adopting the method explained below, a "family" of straight lines can be drawn with very little more trouble than is required to draw one.

In cases of non-uniform graduation, the axes must be graduated at sufficiently small intervals to allow proportional differences to be read without introducing undue error.

Example III. If a current of C ampères fuses a wire of diameter d mm., $C^2 = k \cdot d^3$, where k is a constant depending on the material. For copper wire, $k = 6400$; for lead, $k = 1250$, etc.

Represent the functional relation by a straight-line chart for the range $d = 0$ to $d = 1$.

Read off from the chart (i) the current which will fuse a copper wire of diameter 0·8 mm., (ii) the current which will fuse a lead wire of diameter 0·95 mm.

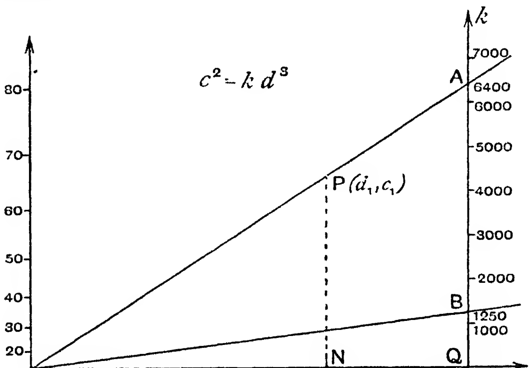


FIG. 73.

The d-axis (horizontal) must be graduated so that distances from O vary as the cube of the values of d: the C-axis (vertical) must be graduated so that distances from O vary as the square of the values of C. [Notice that both axes are thus simple line charts as in Examples I. and II., with the uniform set of graduations omitted.]

Choose any convenient unit for the d-axis, say 10 cm. Measure 10 cm. along the d-axis from O and mark the point Q, so obtained, 1. [*N.B.*—Fig. 73 is printed on a reduced scale.]

To obtain the graduation 0·9 on the d-axis, measure off

$$10 \times (0\cdot9)^3 = 7\cdot29 \text{ cm.}$$

from O, and so on.

For the C-axis, take any convenient unit, say $\frac{1}{1000}$ cm. To obtain 80 on the C-axis, measure off $\frac{1}{1000} \times (80)^2 = 6\cdot4$ cm. from O, and so on.

For copper, $k = 6400$;

$$\therefore \text{ when } d = 1, \ C^2 = 6400 \text{ or } C = 80.$$

Mark the point A, $d = 1$, $C = 80$, and join it to O.

This is the required line chart for copper wire. For, if P be any point on this line with coordinates (d_1, C_1), as taken from the graduations marked, the actual distances of the point P from these axes are not d_1 and C_1, but $10 \times d_1^3$ and $\frac{1}{1000} \times C_1^2$ cm.

Draw PN perpendicular to OQ, then, by the similar triangles PNO, AQO,

$$\frac{ON}{OQ} = \frac{PN}{AQ},$$

$$\frac{10 \, . \, d_1^3}{10} = \frac{\frac{1}{1000} C_1^2}{6\cdot4};$$

$$\therefore \ C_1^2 = 6400 d_1^3,$$

so that the reading (d_1, C_1) from the scales marked does give a solution of the formula $C^2 = 6400d^3$.

We can now give the first numerical result required, for the point on the line OA with coordinate 0·8 on the d-scale has coordinate 57 approximately on the C-scale.

Therefore a current of 57 ampères will fuse a copper wire of diameter 0·8 mm.

To obtain the line chart for lead, we have $k = 1250$;

$$\therefore \text{ when } d = 1, \quad C = \sqrt{1250} = 35\cdot4.$$

Mark the point $d = 1$, $C = 35\cdot4$, and join it to O.

This is the required line chart for lead, and the reader can easily see how this would be proved.

[To obtain the point whose coordinates are $d = 1$, $C = \sqrt{1250}$, it is only necessary, with the scale chosen for C, to measure upwards from Q a distance

$$QB = \tfrac{1}{1000}(\sqrt{1250})^2 = \tfrac{1}{1000} \times 1250 = 1\cdot25 \text{ cm.};$$

so there is no need to find the square root.]

Any member of the family $C^2 = k \, . \, d^3$ can be drawn as follows:

Draw a line, parallel to the C-axis, through Q and call it the k-axis.

Graduate this k-axis by measuring off distances 1 cm., 2 cm., 3 cm., ... and marking the points obtained 1000, 2000, 3000, ...

To obtain the line chart for any value of k, say 4000, join the point graduated 4000 on the k-axis to O and this is the line required.

Example IV. Represent the relation $C^2=6400d^3$ by a straight-line chart, using logarithmic graduation. Read off the value of C when $d=0{\cdot}8$.

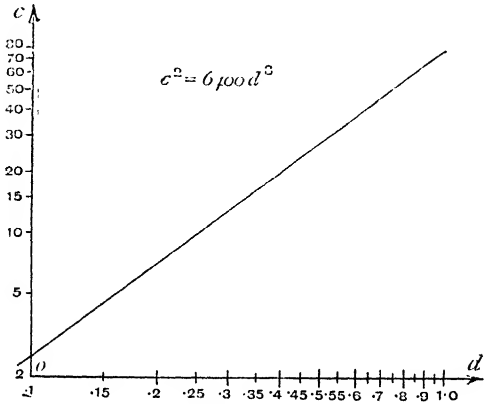


FIG. 74.

Since
$$\log C^2=\log(6400d^3)$$
$$=\log 6400+\log d^3,$$
$$\therefore\ 2\log C=3{\cdot}806+3\log d.$$

This is of the first degree in $\log C$ and $\log d$.

Therefore if the C-axis and the d-axis are graduated logarithmically, the graph of this relation will be a straight line.

We will represent d for the range 0·1 to 1, so that the greatest value of C is 80.

On the d-axis mark $d=0{\cdot}1$ and $d=1$ at a convenient distance apart, say 10 cm. [*N.B.*—Fig. 74 is printed on a reduced scale.] Call the point O at which $d=0{\cdot}1$.

Then 10 cm. on this scale represents $\log 1-\log(0{\cdot}1)=0-(\bar{1})=1$.

To find the position of any other graduation, say $d=0{\cdot}6$, we have $\log 0{\cdot}6-\log 0{\cdot}1=\bar{1}{\cdot}7782-(\bar{1})=0{\cdot}7782$.

$\therefore$ distance of "0·6" graduation from 0 is $10\times 0{\cdot}7782=7{\cdot}78$ cm.

Similarly, on the C-axis, put $C=2$ (say) at 0 and $C=20$ at a convenient distance, 5 cm., from it.

Then on this scale 5 cm. represents $\log 20-\log 2=\log 10=1$.

For any other value of C, say $C=80$,
$$\log 80-\log 2=\log 40=1{\cdot}6021\,;$$

∴ distance of the "80" graduation from 0 is

$$5 \times 1{\cdot}6021 = 8{\cdot}01 \text{ cm., and so on.}$$

Both scales have now been graduated in the same way as the simple line chart in Example II., with the uniform graduations omitted.

Now the relation $C^2 = 6400d^3$ is satisfied by the pairs of values

$$\left.\begin{matrix} d = 1, \\ C = 80 \end{matrix}\right\} \quad \text{and} \quad \left.\begin{matrix} d = 0{\cdot}25, \\ C = 10. \end{matrix}\right\}$$

Mark these points; the line joining them is the required chart. Any numerical result required can then be read from the chart. For example, when $d = 0{\cdot}8$, $c = 57$.

It is easy to see that, with this system of graduation, the family

$$C^2 = k \, . \, d^3, \quad \text{or} \quad 2 \log C = \log k + 3 \log d,$$

is represented by a system of parallel lines. And it is easy to draw a k-axis, as in Example III., so that any member of the system can be drawn at once.

Note.—The labour of graduation can be lightened by using specially prepared logarithmically scaled paper (which can be bought) or by taking the distances from a slide rule.

It may also save time to remember that the distance between two graduations depends only on their ratio; *e.g.* the distance from 2 to 6 must be the same as from 7 to 21.

Thus if 0·1 and 0·2 are marked on the scale, the positions of 0·4, 0·8, 1·6, etc., can be marked without any further calculation.

EXERCISE XXIII. b.

1. From the figure of Example III. above, read off

(i) the current which will fuse a lead wire of diameter 0·95 mm., and of 0·85 mm.;

(ii) the current which will fuse a copper wire of diameter 0·95 mm., and of 0·85 mm.

[Only a rough result can be expected, as not many secondary graduations are marked.]

2. From the figure of Example IV. above, read off

(i) the current which will fuse a copper wire of diameter 0·25 mm., and of 0·45 mm;

(ii) the thickness of copper wire required if a current of 30 ampères does not fuse the wire.

3. The distance between the graduations 1 and 2 on a slide rule is 2·5 inches; find the distances between the graduations, (i) 2 and 4; (ii) 2 and 8; (iii) 4 and 16; (iv) 2 and 5; (v) 2 and 10.

4. The distance between the graduations 1 and 10 on a slide rule is 10 inches; what is the distance between the graduations (i) 2 and 3; (ii) 1 and 5?

5. A scale Ox is graduated so that the distances from O vary as the squares of the values of x. The distance of the graduation "5" from O (the zero graduation) is 10 cms.; what is the distance between the graduations "1" and "2"?

6. A scale Ox is graduated so that, for a uniform graduation of Oy, the graph of $y = \sqrt{x}$ is a straight line. The graduation $x = 10$ is 5 inches from O; find the distances of the graduations $x = 1$ and $x = 5$ from O.

7. Take two axes Ox, Oy graduated uniformly, unit for each 1 cm., and draw the graphs of $y = 2x$ and $y = 3x$. Now graduate the x-axis so that the distances from O of the graduations 1, 2, 3, 4, ... are 1, 4, 9, 16, ... cm. What functional relations do these graphs now represent?

8. The time, t seconds, taken by a marble to roll s feet down a certain plane satisfies the equation $s = 3{\cdot}4t^2$. Represent this by a straight-line chart for $0 < t < 4$. Show how you would represent the family $s = kt^2$. Read off the value of s if $t = 2{\cdot}9$.

9. The diameter d inches of Cornish boilers is connected with the indicated horse-power H by the relation $d = 11{\cdot}4\sqrt{H}$.
Represent this by a straight-line chart for $9 < H < 49$.
What is d when $H = 30$?
Show how you would represent the family $d = k\sqrt{H}$.

10. Represent by straight-line charts on the same diagram:

(i) $y = 3\sqrt{x} + 1$; (ii) $y = 4\sqrt{x} + 1$; (iii) $y = 3\sqrt{x} + 5$;
(iv) $y = 4\sqrt{x} + 5$; for $0 < x < 25$.

11. Represent by a straight-line chart the formula

$$y = \frac{1{\cdot}2}{x} - 1{\cdot}4 \text{ for } 1 < x < 5.$$

What is y when $x = 3{\cdot}5$?

In Ex. 12-16, use logarithmic graduation.

12. Represent by a straight-line chart the formula

$$p \,.\, v^{1\cdot 06} = 475 \text{ for } 10 < p < 50.$$

What is v when $p = 38$?

13. Represent by a straight-line chart the formula

$$t = 0{\cdot}28d^{1\cdot 6} \text{ for } 200 < d < 500.$$

What is t when $d = 320$?

14. Thomson's formula for the discharge of water through a notch is $D^2 = 3{\cdot}91H^5$; draw a straight-line chart for

$$0{\cdot}1 \leqslant H \leqslant 0{\cdot}6.$$

What is D when $H = 0{\cdot}44$?

15. Represent by a straight-line chart the relation

$$y = (2{\cdot}7)^x \text{ for } 0 < x < 2.$$

What is y when $x = 1{\cdot}2$?

16. Represent by a straight-line chart the relation

$$xy^2 = 25 \text{ for } 1 < y < 5.$$

What is y when $x = 12$?

NOMOGRAMS—THREE OR MORE VARIABLES.

The methods already illustrated are unsuitable for formulae containing more than two variables.

Different types of methods have been evolved for such formulae, and are illustrated in the examples which follow; the diagrams constructed for this object are called *Nomograms.*

Example V. To construct a nomogram for the function $ax + by$, where x, y are variable and a, b are given constants.

Let $$z = ax + by.$$

The problem is to construct a nomogram from which we can read off the values of z corresponding to any values of x and y.

We shall measure values of x, y and z along three *parallel* lines, which we shall call the x-axis, y-axis and z-axis.

Suppose $1''$ on x-axis $= \xi$ units,
$1''$ on y-axis $= \eta$ units.

Draw a line EGF making

$$EG = \lambda . b . \eta \text{ in.},$$
$$GF = \lambda . a . \xi \text{ in.},$$

where λ is any convenient number.

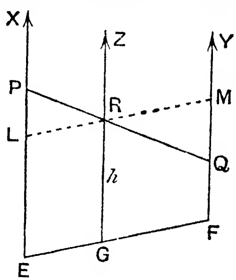


FIG. 75.

Through E, G, F draw any three parallel lines, EX, GZ, FY; these are to be the three axes.

Along EX, FY, measure off lengths EP, FQ to represent any definite values (x_1, y_1) of x and y respectively. Then

$$EP = \frac{x_1}{\xi} \text{ inches,}$$

$$FQ = \frac{y_1}{\eta} \text{ inches.}$$

Join PQ, cutting GZ in R; let $GR = h$ inches.
Draw LRM parallel to EGF cutting EX, FY at L, M.
By similar triangles and parallels,

$$\frac{PL}{QM} = \frac{LR}{RM} = \frac{EG}{GF},$$

$$\therefore \frac{\frac{x_1}{\xi} - h}{h - \frac{y_1}{\eta}} = \frac{\lambda \, . \, b\eta}{\lambda \, . \, a\xi} = \frac{b\eta}{a\xi};$$

$$\therefore a\xi\left(\frac{x_1}{\xi} - h\right) = b\eta\left(h - \frac{y_1}{\eta}\right);$$

$$\therefore ax_1 - a\xi h = b\eta h - by_1;$$

$$\therefore ax_1 + by_1 = h(a\xi + b\eta);$$

$$\therefore z_1 = h(a\xi + b\eta).$$

Choose the unit on the z-axis so that $1''$ on z-axis $= a\xi + b\eta$ units.

Then z equals the number represented by GR on this scale. And this is true, however x_1 and y_1 vary, for the position of GZ and the scale on GZ do not depend on x_1 and y_1.

This result may be expressed in the following rule:

To chart $z = ax + by$.

(1) Choose convenient scales for the x-axis and y-axis.
Suppose $1''$ on x-axis $= \xi$ units,
$1''$ on y-axis $= \eta$ units.

(2) Take three points E, G, F on a line so that

$$EG : GF = b\eta : a\xi.$$

(3) Draw three parallel lines EX, GZ, FY.

(4) GZ is the z-axis, and its scale is $1'' = a\xi + b\eta$ units.

(5) If a straight edge crosses the parallel lines cutting off lengths representing x and y from EX, FY, then it cuts off a length representing $ax + by$ or z from GZ.

The z-axis is called the *support* line or *Reference* line.

Example VI. For screw propellers, if the horse-power is increased by x per cent. and the number of revolutions per minute by y per cent., the diameter of the propeller should be increased by z per cent., where $z = 0{\cdot}2x - 0{\cdot}6y$ (adapted from Doig's formula): chart z for variations of x from 0 to 10 per cent. and of y from 0 to 15 per cent.

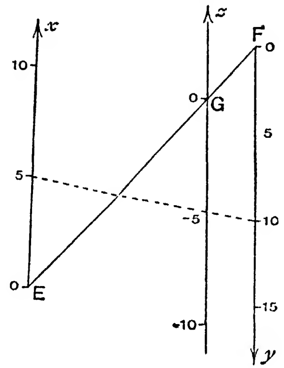


FIG. 76.

Take for unit on x-axis $1'' = 4$ units, and on y-axis $1'' = 5$ units. [*N.B.*—Figure 76 is printed on a reduced scale.]

$$\frac{EG}{GF} = \frac{5 \times 0{\cdot}6}{4 \times 0{\cdot}2} = \frac{15}{4} = \frac{3}{0{\cdot}8};\ \text{draw } EG = 3'',\ GF = 0{\cdot}8''.$$

Unit on z-axis is $1'' = 0{\cdot}2 \times 4 + 0{\cdot}6 \times 5$ units $= 3{\cdot}8$ units.

Owing to the minus sign in $0{\cdot}2x - 0{\cdot}6y$, the y-axis must be graduated in the opposite sense to that of the x-axis.

With these data, the chart can now be constructed.

The dotted line on the chart gives the reading $z = -5$ corresponding to $x = 5$, $y = 10$, and any number of numerical results can be similarly read from the chart.

NOMOGRAMS CAN BE CONSTRUCTED FOR ANY NUMBER OF VARIABLES.

Example VII. Chart $z = \frac{1}{2}u + 3v + 9x + 5y + 4$ for the ranges $10 < u < 40$, $10 < v < 15$, $0 < x < 5$, $0 < y < 10$.

Let $p = \frac{1}{2}u + 3v$ and $q = 9x + 5y$;

$$\therefore\ z = p + q + 4.$$

We first construct the "p" and "q" supports, and from them construct the "z" support.

Take as units :

1″ on u-axis	v-axis	x-axis	y-axis
10 units	2 units	2 units	4 units

[*N.B.*—Figure 77 is printed on a reduced scale.]

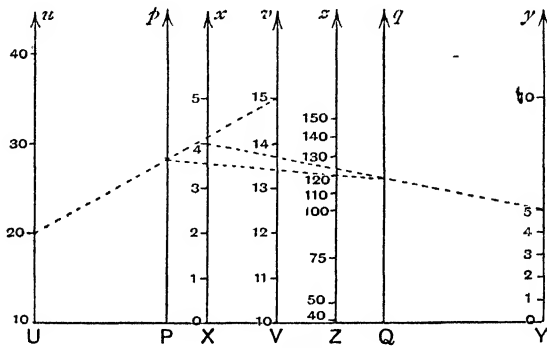


FIG. 77.

(1) For "p" support, $\frac{UP}{PV}=\frac{3\times 2}{\frac{1}{2}\times 10}=\frac{6}{5}=\frac{1\cdot 5}{1\cdot 25}$; take $UP=1\cdot 5''$, $PV=1\cdot 25''$.
Scale on p-axis is $1''=3\times 2+\frac{1}{2}\times 10=11$ units.

(2) For "q" support, $\frac{XQ}{QY}=\frac{5\times 4}{9\times 2}=\frac{20}{18}=\frac{2}{1\cdot 8}$; take $XQ=2''$, $QY=1\cdot 8''$.
Scale on q-axis is $1''=5\times 4+9\times 2=38$ units.

(3) For "z" support, $\frac{PZ}{ZQ}=\frac{1\times 38}{1\times 11}=\frac{38}{11}$; divide PQ at Z in ratio 38 : 11.
Scale on z-axis is $1''=1\times 38+1\times 11=49$ units.

(4) By substitution, when $u=10$, $v=10$, $x=0$, $y=0$, we have $z=5+30+4=39$.

$\therefore$ the graduation at the point Z in the figure is 39.

$\therefore$ we can now graduate the z-axis.

(5) The dotted lines in the figure show how z is obtained for $u=20$, $v=15$, $x=4$, $y=5$; giving $z=120$.

Note.—(*a*) In actually drawing the figure, it is best to try to arrange so that axes are not too near together.

Take the points in the following order :

(i) U, P, V at the proper distances apart.

(ii) Q. The line PQ has to be divided in ratio 38 : 11 ; so choose PQ a convenient length, say $\frac{1}{2}$ of $\frac{(38+11)}{10}=2{\cdot}45$ inches. Then Z is half of $3{\cdot}8=1{\cdot}9''$ from P.

(iii) X, Y, Z.

(*b*) There is no need to graduate the support lines p, q.

(*c*) It is usually best to graduate the z-axis by taking special values, and so obtaining two points of known graduations on it.

EXERCISE XXIII. c.

1. It has been suggested that the Chancellor of the Exchequer should regulate the tax on motor-cars by the following formula. For a car of H horse-power and weight W cwt., the tax should be £T, where $T=0{\cdot}4H+0{\cdot}8W$. Construct a nomogram to show the amount of the tax for cars of 10 to 40 H.P. and of weights 10 cwt. to 30 cwt. Proceed as follows: (i) Draw two parallel lines AH, BW 6 cm. apart and 15 cm. long and a line perpendicular to them cutting them at A, B ; (ii) on AB, take a point C, so that $AC=4$ cm., and through C draw CT parallel to AH ; (iii) the lines AH, BW, CT drawn in the same sense form the H, W, T axes ; take as scale for the H-axis, 1 cm. $=2$ H.P., and as scale for the W-axis, 1 cm. $=2$ cwt. ; then the scale for the T-axis is

$$1 \text{ cm.} = 2\times 0{\cdot}4+2\times 0{\cdot}8=\pounds 2{\cdot}4\ ;$$

(iv) when $H=10$ and $W=10$, we have $T=12$; mark at A, B, C the graduations 10, 10, 12 respectively. Graduate the H-axis and W-axis according to the units given in (iii). Measure off 5 cm. along CT, and mark the point K so obtained $12+5\times 2{\cdot}4=\pounds 24$. Divide CK into 12 equal parts, so that each graduation now represents £1.

This is the required nomogram. Use it to read off the tax on a car of

(*a*) 20 H.P. weighing 18 cwt.,
(*b*) 25 H.P. weighing 30 cwt.,
(*c*) 30 H.P. weighing 30 cwt.

2. Draw a line ACB so that $AC=4$ cm., $CB=3$ cm. ; draw three parallel lines AX, CZ, BY (in the same sense) ; regarding this as a nomogram, take A, B, C as zero graduations on the X, Y, Z axis ; as units take 1 cm. $=5$ units on X-axis, 1 cm. $=4$ units on Y-axis, 1 cm. $=10$ units on Z-axis. Graduate the axes, and read off the values of Z for (i) $X=20$, $Y=20$; (ii) $X=15$, $Y=30$; (iii) $X=25$, $Y=12$. Calculate the function Z in terms of X and Y.

Next suppose that the graduations at A, B, C are respectively 10, 20, 30, but that the units remain the same, read off the new answers to (i), (ii), (iii), and calculate the new function Z in terms of X and Y.

3. Work out all the results asked for in Question 2, taking BY in the opposite sense to that of AX and CZ.

4. Draw the nomogram for the relation $z=4{\cdot}3x+8{\cdot}6y$ for $0<x<5$, $0<y<5$, and read off the values of z for (i) $x=2$, $y=4$; (ii) $x=3$, $y=1$.

5. For snapped rivets, $l=d+\frac{5t}{4}+\frac{1}{32}$, where l = length of rivet before closing, d = diameter of hole, t = thickness of plate, all in inches. Construct a nomogram for $0<d<1\frac{1}{4}$, graduated to $\frac{1}{8}$ inch, and $0<t<4$ graduated to $\frac{1}{2}$ inch. Read off the values of l for (i) $d=\frac{3}{4}$, $t=1\frac{1}{2}$; (ii) $d=1\frac{1}{8}$, $t=2\frac{1}{2}$.

6. Draw three lines OU, OV, OF, so that $\angle UOF=60°=\angle FOV$. Call these the u-axis, the v-axis, the f-axis, and graduate each on a scale 1 cm. equals 1 unit. Regarding this as a nomogram, read off the values of f corresponding to (i) $u=4$, $v=4$; (ii) $u=10$, $v=6$.

If any straight line cuts OU, OV, OF at A, B, C, it can be proved that $\frac{1}{OA}+\frac{1}{OB}=\frac{1}{OC}$. Use this to write down the relation represented by the above nomogram.

By producing VO to T and taking OT as the t-axis, construct a nomogram for $\frac{1}{z}=\frac{1}{u}+\frac{1}{v}+\frac{1}{t}$.

7. Draw the nomogram for the relation $z=5{\cdot}1x-3{\cdot}4y+20$ for $10<x<20$, $0<y<10$, and read off the values of z if (i) $x=12$, $y=6$; (ii) $x=18$, $y=9$.

8. Draw the nomogram for the relation $z=3x-2y+5t$ for $0<x$ or y or $t<10$, and read off the values of z if (i) $x=1$, $y=3$, $t=2$; (ii) $x=3{\cdot}5$, $y=7{\cdot}5$, $t=4{\cdot}5$.

9. Draw the nomogram for the relation $z=2x+y-3t-4w$ for $50<x$ or $y<100$, $10<t$ or $w<30$, and read off the values of z if (i) $x=60$, $y=80$, $t=20$, $w=25$; (ii) $x=75$, $y=65$, $t=25$, $w=28$.

10. Draw a line $CABZ$ such that $CA=3$ cm., $AB=5$ cm.; draw two lines AX, BY *perpendicular* to AB in *opposite* senses. Regarding this as a nomogram, take C, A, B as the zero graduations for the Z, X, Y axes respectively; as units take 1 cm. = 2 units on X-axis, 1 cm. = 3 units on Y-axis, 1 cm. = 10 units on Z-axis. Graduate the axes and read off the values of Z for (i) $X=8$, $Y=12$; (ii) $X=6$, $Y=20$; (iii) $X=-5$, $Y=15$. Calculate the function Z in terms of X and Y.

Suppose next that the graduations at A, B for AX, BY are 10, 40 respectively, calculate the new function Z in terms of X and Y.

11. Draw two parallel lines XAW, ZBY in the same sense 6 cm. apart and a line ABT perpendicular to them cutting them at A, B. Regarding this as a nomogram and AX, BY BZ, AW as the x-axis, y-axis, z-axis, w-axis with zero graduations at A, B and ABT as a support line, graduate each axis on a scale 1 cm. equals 1 unit. Suppose the straight edge joining any two graduations x, y cuts the support line AT at K and the straight edge joining any graduation z to K cuts AW at w, then w depends on x, y, z. (i) What function is w of x, y, z? (ii) Read off from the nomogram the values of w corresponding to $x=4$, $y=3$, $z=5$. (iii) Use the nomogram to read off the values of $\frac{3\cdot5\times2\cdot7}{4\cdot2}$ and $\frac{6\cdot3\times5\cdot4}{7\cdot1}$.

USE OF NON-UNIFORM GRADUATION.

In the nomograms so far considered, the graduations of the axes have been uniform. By using non-uniform systems of graduation, the method can be extended to more complicated formulae. For example, the formula $z=3x^2+2y^2$ could be charted by graduating the axes so that the distance of any graduation marked "x" is at a distance up the scale proportional to x^2, etc., and then proceeding as in Example VI. We add one further example to illustrate this idea, the data of which are due to Professor Perry.

Example VIII. For the Thomson turbine, if P is the total horse-power of the waterfall, H the height of the fall in feet, n the number of revolutions per minute, $n=22\cdot75H^{1\cdot25}\times P^{-0\cdot5}$. Construct a nomogram for n, for $50<H<200$ and $50<P<100$.

We have $\log n=1\cdot357+1\cdot25\log H-0\cdot5\log P$.

[The number 1·357 in the formula merely affects the position of the origin on the n scale: and this we shall fix by taking a special case.]

$\log H$ varies from $\log 50$ to $\log 200$ or $1\cdot699$ to $2\cdot301$,

$\log P$ varies from $\log 50$ to $\log 100$ or $1\cdot699$ to 2.

For $\log H$ axis, take $1''=0\cdot15$ unit.

For $\log P$ axis, take $1''=0\cdot1$ unit.

If any line cuts the $\log H$, $\log P$, $\log n$ axes at E, F, G,

$$\frac{EG}{GF}=\frac{0\cdot5\times0\cdot1}{1\cdot25\times0\cdot15}=\frac{0\cdot1}{0\cdot375}=\frac{1}{3\cdot75}.$$

Scale for $\log n$ axis, $1''=0\cdot5\times0\cdot1+1\cdot25\times0\cdot15=0\cdot2375$ units or 1 unit $=4\cdot210''$.

The log P axis must be graduated in the opposite sense to that of the log H axis.

On the log H axis, the distance between (say) the 50 and 100 graduation is $(\log 100 - \log 50) \div 0\cdot15$ in. $= \dfrac{0\cdot301}{0\cdot15} = 2\cdot007''$: in this way we can graduate the log H axis, and similarly the log P axis. [*N.B.*—Figure 78 is printed on a reduced scale.]

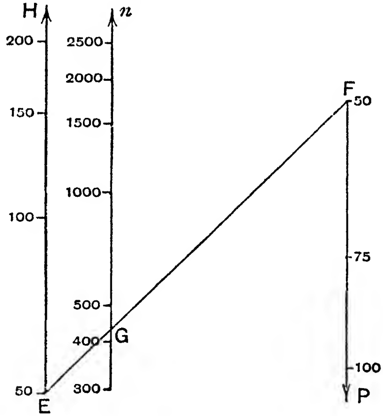


FIG. 78.

By calculation, when $H=50$ and $P=50$, we have $n=427\cdot6$. Join these two points on the H and P scale ; where the join cuts the "n" support, we mark the graduation $427\cdot6$, and graduate the scale upwards from this point.

EXERCISE XXIII. d.

1. If a car of H horse-power and weight W cwt. is taxed £T, a possible way of choosing T would be to use the formula $T = \sqrt[4]{HW^3}$. Construct a nomogram to show the amount of the tax for cars of 10 to 40 H.P. and of weights 10 cwt. to 30 cwt. Proceed as follows : (i) Write down $\log T$ in terms of $\log H$ and $\log W$. (ii) Draw a line ACB so that $AC=3''$, $CB=1''$, and draw three lines AH, CT, BW perpendicular to AB and on the same side of it ; these lines form the H, T, W axes, which will be graduated logarithmically. (iii) Take as unit both for $\log H$ and $\log W$, 1 inch $=0\cdot1$ unit ; then the unit for $\log T$ is

$$1 \text{ inch} = \tfrac{1}{4} \times 0\cdot1 + \tfrac{3}{4} \times 0\cdot1 = 0\cdot1 \text{ unit}.$$

(iv) Mark at A the graduation 10 H.P. and at B the graduation 10 cwt. What is T when $H=10$, $W=10$? What is the graduation at C? (v) $\log 20 - \log 10 = 0{\cdot}301$; but 1 inch $=0{\cdot}1$ unit; therefore the 20 graduation is $\frac{0{\cdot}301}{0{\cdot}1}$ or 3·01 inches from A. Graduate in this way the three axes. (vi) From the nomogram read off the tax on a car of

(a) 20 H.P. weighing 20 cwt.,
(b) 20 H.P. weighing 30 cwt.,
(c) 15 H.P. weighing 25 cwt.

2. The horse-power H transmitted by each inch width of a belt $\frac{1}{4}$ inch thick is given by $H=0{\cdot}0036dn$, where d = diameter of pulley in feet, n = number of revolutions per minute. Construct a nomogram for $100 < n < 300$, $1 < d < 10$, and read off from it (i) the value of H if $n=250$, $d=6$; (ii) the value of n if $H=5{\cdot}5$, $d=8$.

3. Represent by a nomogram Edwards' torsional resistance formula $d^3 = \frac{m}{0{\cdot}196f}$ for $10 < m < 50$, $4 < f < 7$, and read off (i) the value of d if $m=35$, $f=5{\cdot}5$, (ii) the value of f if $m=20$, $d=2{\cdot}7$.

4. Represent by a nomogram the formula $z=2xyt^2$ for $1 < x$ or y or $t < 3$, and read off the values of z if (i) $x=1{\cdot}5$, $y=2{\cdot}5$, $t=2{\cdot}1$; (ii) $x=2{\cdot}5$, $y=1{\cdot}8$, $t=1{\cdot}4$.

5. The discharge of gas in pipes is given by $Q=20000\sqrt{\frac{D^5H}{L}}$, where Q cu. ft. of gas are discharged per hour through a pipe of diameter D inches and length L yards under a pressure equivalent to a head H inches of water. Construct a nomogram for

$$80 < L < 100, \quad 1 < D < 5, \quad 0{\cdot}5 < H < 1$$

Read off the values of Q if (i) $L=85$, $D=2$, $H=0{\cdot}8$; (ii) $L=92$, $D=3{\cdot}5$, $H=0{\cdot}65$.

6. [*D'Ocagne's nomogram for any quadratic equation.*]

To construct a nomogram for solving the quadratic $t^2-at-b=0$, proceed as follows:

(i) Make an accurate drawing of the graph of $y=\frac{x^2}{1-x}$ from $x=0$ to 0·9 (see Fig. 79), taking the unit on the x-axis as 5″ and the unit on the y-axis as $\frac{1}{2}$″.

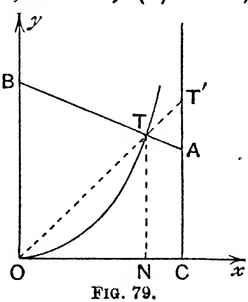


FIG. 79.

(ii) Take C on Ox at unit distance from O (*i.e.* 5″), and draw CA parallel to Oy; draw any line cutting Oy, CA and the curve at B, A, T, and let OT produced meet CA at T'. Draw TN perpendicular to OC.

(iii) Let $ON = x$ units, $NT = y$ units, $OB = b$ units, $CA = a$ units, and let $\frac{y}{x} = t$.

Prove that (1) $t = \frac{x}{1-x}$, $x = \frac{t}{1+t}$, $y = \frac{t^2}{1+t}$,

(2) $\frac{y-a}{1-x} = \frac{b-a}{1}$,

(3) $t^2 - at - b = 0$,

(4) $CT' = \frac{y}{x} = t$.

(iv) Regarding the figure as a nomogram, take OB, CA as the b-axis and a-axis, and graduate them in the same way as Oy, viz. $\frac{1}{2}$ inch equals 1 unit.

To graduate the curve, join O to any point on CA, *e.g.* to $a = 2$, and mark the point where this line meets the curve 2. This means that for this point on the curve

$t = \frac{y}{x} = CT' = 2$.

In this way, the different values of t are marked all along the curve.

(v) Use the nomogram to solve $t^2 - 2t - 3 = 0$. Join the points $a = 2$, $b = 3$, and read off the value of t where this line cuts the curve.

(vi) Use the nomogram to find one root of

(1) $t^2 - 2t - 5 = 0$,
(2) $t^2 + 2t - 3 = 0$,
(3) $t^2 + 2t - 5 = 0$.

What inference about the roots of the equation

$$t^2 + 2t + 2 = 0$$

can be drawn from the nomogram?

(vii) The sum of the roots of $t^2 - at - b = 0$ is a; use this fact to write down the second root in the equations given in (v) and (vi).

(viii) How could you use the nomogram to solve the equation

$$x^2 - 23x - 485 = 0 ?$$

In the preceding pages, it has been impossible to do more than give a brief introductory account: those who wish to pursue the subject further should consult either Brodetsky's *First Course in Nomography*, or d'Ocagne's *Traité de Nomographie*, or one of the other books mentioned in the Introduction.

CHAPTER XXIV.

FURTHER DEVELOPMENTS FOR THE SPECIALIST.

THEORY OF QUADRATICS.

[*Note.*—It is assumed in this section that the letters a, b, c, p, q, r, stand for real numbers, unless otherwise stated.]

If α, β are the roots of $ax^2+bx+c=0$, then [see Part I. p. 153]

$$ax^2+bx+c \equiv a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) \equiv a(x-\alpha)(x-\beta)$$
$$\equiv a[x^2-x(\alpha+\beta)+\alpha\beta];$$

$\therefore$ the sum of the roots $=\alpha+\beta=-\dfrac{b}{a}$,

the product of the roots $=\alpha\beta=\dfrac{c}{a}$.

[These two results should be learnt by heart.]

By formula, the solution of the quadratic is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Hence (i) if b^2-4ac is a perfect square, the roots are rational and unequal;

(ii) if $b^2-4ac>0$, but not a perfect square, the roots are real, irrational and unequal;

(iii) if $b^2-4ac=0$, the roots are real and equal;

(iv) if $b^2-4ac<0$, the roots are imaginary.

The expression b^2-4ac is called the *discriminant* of the quadratic function ax^2+bx+c. Its sign determines the nature of the factors of this function; and the factors are rational if the discriminant is a perfect square.

Example I. For what range of values of x is the function $5+7x-6x^2$ positive? Find also its greatest value.

(i) $5+7x-6x^2 \equiv (1+2x)(5-3x) \equiv 2(\frac{1}{2}+x)(-3)(x-\frac{5}{3})$

$\equiv -6(x-\frac{5}{3})(x+\frac{1}{2}).$

The function is positive if $(x-\frac{5}{3})(x+\frac{1}{2})$ is negative.
If $x > \frac{5}{3}$, each factor is positive;

$\therefore$ the product is positive.

If $\frac{5}{3} > x > -\frac{1}{2}$, the first factor is negative, the second is positive;

$\therefore$ the product is negative.

If $-\frac{1}{2} > x$, each factor is negative;

$\therefore$ the product is positive;

$\therefore$ the range of values is $\frac{5}{3} > x > -\frac{1}{2}$.

(ii) $$5+7x-6x^2 \equiv 5-6\left(-\frac{7x}{6}+x^2\right)$$

$$\equiv 5-6\left(\frac{7}{12}-x\right)^2+\frac{49}{24}$$

$$=7\tfrac{1}{24}-6\left(\frac{7}{12}-x\right)^2.$$

Now the least value of $\left(\frac{7}{12}-x\right)^2$ is zero.

$\therefore$ the greatest value of $5+7x-6x^2$ is $7\frac{1}{24}$, and the function has this value when $x=\frac{7}{12}$.

Example II. Prove that the roots of

$$(1+a^2)x^2-2(1+ab)x+1+b^2=0$$

are imaginary.

The discriminant $=4(1+ab)^2-4(1+a^2)(1+b^2)$
$=4[1+2ab+a^2b^2-1-a^2-b^2-a^2b^2]$
$=-4(a^2-2ab+b^2)=-4(a-b)^2,$

which is negative;

$\therefore$ the roots are imaginary.

Example III. If α, β are the roots of $ax^2+bx+c=0$, find (i) the value of $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$, (ii) the equation whose roots are 2α, 2β.

(i) $\alpha+\beta=-\frac{b}{a}$; $\alpha\beta=\frac{c}{a}$.

$$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]}{\alpha\beta}$$

$$=\frac{\frac{-b}{a}\left[\frac{b^2}{a^2}-\frac{3c}{a}\right]}{\frac{c}{a}}=\frac{-b(b^2-3ac)}{a^2c.}.$$

(ii) Put $y=2x$, so that when $x=\alpha$, $y=2\alpha$;

$$\therefore\ x=\frac{y}{2};\quad \therefore\ a\left(\frac{y}{2}\right)^2+b\left(\frac{y}{2}\right)+c=0;$$

$$\therefore\ ay^2+2by+4c=0$$

is the required equation.

EXERCISE XXIV. a.

1. State whether the roots of the follow
coincident or imaginary; and if real, what
are *not* required.

(i) $x^2-3x-5=0$; (ii)

(iii) $x^2-2x+5=0$; (iv

(v) $3x^2+105x+1=0$; (v

2. Form th quations whose roots

(i) 2, -1; (ii) 0, $\frac{3}{4}$; (iii)

(v) $\pm\sqrt{2}$; (vi) $\sqrt{2}+1$, $\sqrt{2}-1$;

3. Express as simply as possible the roots of $ax^2+2bx+c=0$.

4. Find the greatest value of a for which $x^2-5x+a=0$ has real roots.

5. What must be added to $4x^2-5x$ to make the result a perfect square? Find the least value of this function.

6. If x is real, find (i) the greatest value of $1+2x-x^2$, (ii) the least value of $5x^2-x$.

7. If x is real, prove that x^2-3x+3 is positive.

8. For what range of values of x is (i) $3-2x-8x^2$ positive? (ii) $2x^2-x-3$ negative?

9. How is x limited if x^2+3x-4 is positive?

10. For what range of values are the following functions positive?

(i) $(x+1)(x-1)(x-3)$; (ii) $(2x-3)(x+2)(3x-1)$;
(iii) $(1-x)(x+4)(2x+1)$; (iv) $(x-1)(x-3)(x+1)(x+3)$.

11. State the values of x for which $\frac{(x-2)(x+3)}{x+1}$ is (i) positive, (ii) negative.

12. State the values of x for which $\frac{(1-2x)(x+4)}{(x+3)(1-3x)}$ is (i) positive, (ii) negative.

13. (i) One root of $5x^2-17x+k=0$ is $1{\cdot}4$; what is the other?
(ii) One root of $5x^2-kx-39=0$ is $2{\cdot}6$; what is the other?

14. One root of $5x^2+9x+c=0$ is double the other; find them: also find c.

15. Find b if $3x^2+bx+4=0$ has equal roots.

16. e root of $x^2-px+12=0$ is three times the other; find p.

17. I re roots of $x^2+qx+r=0$, find the condition that

(ii) $\alpha=-\beta$; (iii) $\frac{\alpha}{\beta}=\frac{2}{3}$;
(v) $\alpha=0$; (vi) $\alpha-\beta=1$.

ts of $x^2-2ax-b^2=0$ are real.

$b-c)^2+4c(a+b)$ is a perfect square.
$+b)x^2+x(a+b-c)-c=0$ has rational

, + oxy has real factors.

the roots of $(a^2+b^2)x^2-2(a+b)x+2=0$ are imaginary.

22. Prove that the roots of $(a+4b)x-2(a+b)x+a-2b=0$ are rational.

23. Prove that the roots of $(x-a)(x-b)=c^2$ are real.

24. Prove that the roots of $ax^2+bx+c=0$ are rational if $a+b+c=0$.

25. Prove that $(a^2+bc)x^2+(a-b)(a-c)x-a(b+c)=0$ has rational roots.

26. Prove that $\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}=0$ has real roots.

27. Find k if $(3x-2y)^2+k(x-y)(x-2y)$ is a perfect square.

28. Find the sum of the squares of the roots of $3x^2-7x-5=0$.

29. If α, β are the roots of $2x^2 - 5x = 8$, find the values of

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$; (ii) $\alpha^2 - \alpha\beta + \beta^2$; (iii) $(1 + \alpha^2)(1 + \beta^2)$.

30. Find the conditions that the roots of $x^2 + bx + c = 0$ are respectively $\frac{2}{3}$ of the roots of $x^2 + qx + r = 0$.

31. Find the condition that the roots of $x^2 + bx + c = 0$ differ by $5b$.

32. (i) Form the equation whose roots are ten times the roots of $x^2 - 4x = 9$.

(ii) Form the equation whose roots are 1 less than the roots of $2x^2 - x = 4$.

33. (i) Form the equation whose roots are the squares of the roots of $2x^2 - 4x = 3$.

(ii) Form an equation whose roots exceed by 2 the roots of $x^2 - 4x = 7$.

34. What is the connection between the roots of $ax^2 + bx + c = 0$ and those of $ax^2 + bpx + cp^2 = 0$?

35. Find a condition, independent of p, that the roots of $ax^2 + bx + c = 0$ are (i) p and $p + 1$; (ii) $1 + \frac{1}{p}$ and $1 + \frac{1}{p+1}$.

36. What can you say about the nature of the roots of

$$x^2 + 2(a + b)x + 2a^2 + 2b^2 = 0 \text{ ?}$$

37. Prove that $3x^2 + 7xy + 2y^2 - 2x + y - 1$ has rational factors.

38. The sum of the roots of $x^2 - (a + 7)x + 2(2a + 1) = 0$ is half their product. Find a.

39. Find a if the equation $x^2 - 4(1 - 2a)x + 4 - 5a = 0$ has equal roots.

40. What is the condition that $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have one and only one common root ?

41. What is the condition that $x^2 + xy - 2y^2$ and $ax^2 + 2hxy + by^2$ have a common factor ?

42. Given that $3x^2 - 29x - 44 = 0$ and $3x^2 + 73x + 92 = 0$ have a common root, find it.

43. What is the condition that $x^2 + bx + c = 0$ and $x^2 - cx - b = 0$ have a common root ?

44. If $a + b + c = 0$, prove that $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root.

45. What is the condition that $ax^2 + ax + c$ is positive for all real values of x ?

46. If x, y, z are real, prove that $x^2+y^2+z^2-xy-yz-zx$ cannot be negative. What follows if the expression is zero ?

47. If x is real, what limits are there to the value of $x+\frac{1}{x}$?

48. (i) Solve the equation $\frac{x}{x^2+1}=k$.

(ii) What is the condition that the roots of this equation are real ?

(iii) If x is real, what is the least value of $\frac{x}{x^2+1}$?

49. If x is real, find the limits within which $\frac{x+4}{(x+1)(x-8)}$ cannot lie.

50. If x is real, find the limits within which $\frac{3x^2+2}{2x^2-2x+1}$ must lie.

51. If x is real, prove that $\frac{x+1}{x^2-4}$ is capable of any real value.

52. If α, β are the roots of $x^2+qx+r=0$, form the equation whose roots are (i) $\frac{1}{\alpha}$, $\frac{1}{\beta}$; (ii) $\alpha^2-\alpha\beta$, $\beta^2-\alpha\beta$.

53. Find the condition that the roots of $x^2+qx+r=0$ differ by the same amount as those of $x^2+bx+c=0$.

54. Can real values of x and y be found such that

(i) $x^2+y^2-4x+2y+5=0$;

(ii) $x^2+y^2-6y+10=0$?

55. (i) Write down the equation whose roots are α, β, γ, and express it in the form $x^3+px^2+qx+r=0$.

(ii) If α, β, γ are the roots of $ax^3+bx^2+cx+d=0$, express $\alpha+\beta+\gamma$ and $\alpha\beta+\beta\gamma+\gamma\alpha$ and $\alpha\beta\gamma$ in terms of a, b, c, d.

56. If α, β, γ, δ are the roots of $x^4+px^3+qx^2+rx+s=0$, express p, q, r, s in terms of α, β, γ, δ.

57. What can you say about the roots of the equations

(i) $x^3+qx+r=0$;

(ii) $x^3+px^2+qx+p=0$;

(iii) $x^4+qx^2+s=0$;

(iv) $x^3+px^2+qx=0$.

58. The roots of $x^3+px^2+qx+r=0$ are 3, 1, -2 ; what is the value of q ?

59. Two of the roots of $x^3+qx+r=0$ are 4, 3 ; what is the other root ?

ALGEBRAIC FORM.

The ideas included under this heading are of too general a nature for illustrative examples.

EXERCISE XXIV. b.

1. Use the identity $x^3+y^3 \equiv (x+y)(x^2-xy+y^2)$ to factorise

(i) x^3-y^3; (ii) $(a+b)^3+(b+c)^3$; (iii) $(a+b)^3-(b+c)^3$.

2. Use the identity $a^4+a^2b^2+b^4 \equiv (a^2+ab+b^2)(a^2-ab+b^2)$ to factorise

(i) a^4+4a^2+16; (ii) $(x+y)^4+(x^2-y^2)^2+(x-y)^4$.

3. Given $(x+y)^4 \equiv x^4+4x^3y+6x^2y^2+4xy^3+y^4$, write down the expansion of $(2x-3y)^4$.

4. Given that tho square root of $1+x(x+1)(x+2)(x+3)$ is $1+3x+x^2$, write down the square root of

(i) $1+(x+1)(x+2)(x+3)(x+4)$;

(ii) $y^4+(x+y)(x+2y)(x+3y)(x+4y)$.

5. Prove that

$$(x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3 \equiv (y+z)^3+3(y+z)^2x+3(y+z)x^2+x^3.$$

6. If $x+y+z=0$, prove that

(i) $x^2+y^2+z^2=2(x^2-yz)$;

(ii) $y^2-zx=z^2-xy$.

7. Given that the area of the triangle whose sides are of lengths x, y, z is

$$\tfrac{1}{4}\sqrt{\{(x+y+z)(x+y-z)(y+z-x)(z+x-y)\}},$$

find the area of the triangle whose sides are $x+y$, $y+z$, $z+x$.

8. Given that $x=1$ is a root of $x^6+2x^4+2x^2=5$, write down a second root.

9. Given that $x=\frac{2}{3}$ is a root of $6x^3-7x^2-7x+6=0$, write down a second root.

10. Given $x+2y$ is a factor of $4x^4-17x^2y^2+4y^4$, write down the other three factors.

11. Given $\dfrac{3}{(x-4)(x-1)} \equiv \dfrac{1}{x-4}-\dfrac{1}{x-1}$, express $\dfrac{3}{(x+4)(x+1)}$ as the difference of two fractions.

12. Given that $(x+y+z)^3-x^3-y^3-z^3=3(x+y)(y+z)(z+x)$, factorise $x^3y^3+y^3z^3+z^3x^3-(xy+yz+zx)^3$.

13. Factorise $(a+3b)^2-5(a+3b)(7a-11b)-14(7a-11b)^2$.

14. Given that $a+b+c$ is a factor of

$$a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2,$$

write down the other three factors.

15. What is the effect of writing px for x and py for y in the expressions

$$\text{(i)}\ \frac{ax^n-by^n}{xy(cx^{n-2}-dy^{n-2})};\quad \text{(ii)}\ \frac{\sqrt[3]{x^3+y^3}}{xy}?$$

16. Simplify $\dfrac{2\,.\,4\,.\,6\,.\,8\ldots(2n-2)}{1\,.\,2\,.\,3\,.\,4\ldots n}$.

17. Generalise

$$(x-y)(x+y)(x^2+y^2)(x^4+y^4)(x^8+y^8)\equiv x^{16}-y^{16}.$$

18. Given that $x+y=a$ and $x^2+y^2=b^2$, state with reasons which of the following statements are obviously untrue:

(i) $xy=\frac{1}{2}(a^2-b^2)$; (ii) $\frac{1}{x}+\frac{1}{y}=\frac{2a^2}{a^2-b^2}$;

(iii) $\frac{x}{y}+\frac{y}{x}=\frac{2b^2}{a^2-b^2}$; (iv) $x^3+y^3=\frac{ab}{2}(3b^2-a^2)$.

19. Given that $(ab+xy)^2+(ay-bx)^2\equiv(a^2+x^2)(b^2+y^2)$,

(i) write down the factors of $(ab-xy)^2+(ay+bx)^2$;

(ii) express $(a^2+x^2)(b^2+y^2)(c^2+z^2)$ as the sum of two squares.

20. How many terms are there in the expansion of

$$(a+b)(c+d)(e+f)(g+h)(k+l)\ ?$$

Which of the following terms belong to this expansion (i) *acfhk*, (ii) *aceg*, (iii) *bdefk*, (iv) *bdfgk*, (v) *lgfcb* ?

21. Find the coefficient of x^7 in

$$(1-2x+3x^3-x^5+x^9)(1+x^2+2x^3-4x^6).$$

22. Find the coefficient of x^{20} in

$$(1-x^3+x^6-x^9+x^{12}-\ldots)(1+x^5+x^{10}+x^{15}+\ldots).$$

23. Find the coefficient of x^n in

(i) $(1+x^{n-1})(1+2x+3x^2+4x^3+\ldots)$;

(ii) $(1+x)(1+x^2+x^4+x^6+\ldots)$;

(iii) $(1+x^2)(1+x^2+x^4+x^6+\ldots)$;

(iv) $(1-x)(1+x+x^2+x^3+\ldots)$.

24. Find the coefficient of x^{100} in

$$(1+2x^3+3x^6+4x^9+5x^{12}+\ldots)^3.$$

25. Find the coefficient of (i) x^6, (ii) x^n in

$$(1+x+x^2+x^3+\ldots)(1+x+x^2+x^3+\ldots).$$

26. What is the coefficient of x^{2n} in

(i) $1+2x+3x^2+4x^3+\ldots$;

(ii) $(1-x)(1+2x+3x^2+4x^3+\ldots)$;

(iii) $(1-2x+x^2)(1+2x+3x^2+4x^3+\ldots)$?

27. What is the coefficient of x^n (n odd or n even) in

(i) $1-x+x^2-x^3+x^4-\ldots$;

(ii) $1+x-x^2-x^3+x^4+x^5-x^6-x^7+\ldots$?

28. Given that, if n is a positive integer,

$$(1+x)^n \equiv 1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots$$
$$+\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3.4\ldots r}x^r+\ldots,$$

write down the expansion of (i) $(1+x)^5$; (ii) $(a+b)^4$.

29. Assuming the identity in Ex. 28, write down the coefficient of a^4b^3 in $(a+b)^7$.

30. Assuming that the identity in Ex. 28 is true for *all* values of n, provided that $1 > x > -1$, find approximate values for (i) $\sqrt{1+x}$, (ii) $\frac{1}{1+x}$, if x is so small that x^3 and higher powers of x can be neglected.

31. If h is so small compared with x that $\left(\frac{h}{x}\right)^3$ can be neglected, use the statement in Ex. 30 to find an approximate value for $\sqrt{x+h}$.

32. If $1 > x > -1$, use Ex. 30 to expand (i) $(1+x)^{-2}$; (ii) $(1-x)^{-n}$.

33. It can be proved that, if x, y, z are any positive quantities,

$$\frac{x+y}{2} \not< \sqrt{xy} \quad \text{and} \quad \frac{x+y+z}{3} \not< \sqrt[3]{xyz}.$$

Assuming these results, prove that

(i) $(x+y)(y+z)(z+x) \not< 8xyz$;

(ii) $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \not< 9$.

34. What is the connection between the graphs of

$$\frac{x(x-3)}{(x-4)(x-5)} \quad \text{and} \quad \frac{x(x+3)}{(x+4)(x+5)}\ ?$$

35. Prove that the graphs of

$$\frac{x(x-1)}{(x+1)(x-2)} \quad \text{and} \quad \frac{2}{(x+1)(x-2)}$$

are of the same size and shape.

36. Given that

$$(a^2-pb^2)(c^2-pd^2) \equiv (ac+pbd)^2 - p(ad+bc)^2,$$

express $(a^2-pb^2)(c^2-pd^2)(e^2-pf^2)$ in the form $Y^2 - p \,.\, Z^2$.

37. Given that $x=\dfrac{br-cq}{aq-bp}$ satisfies the equations $ax+by+c=0$, $px+qy+r=0$, write down the value of y given by the equations.

38. Given $ax^2+2hxy+by^2=0$, $lx+my=1$, it can be proved that $x^2(am^2-2hlm+bl^2)+2x(mh-bl)+b=0$; write down an equation in y independent of x.

39. Given that $x=\dfrac{k}{(a-b)(a-c)}$ is the value of x satisfying $x+y+z=0$, $ax+by+cz=0$, $a^2x+b^2y+c^2z=k$, write down the values of y, z.

40. Given that $x=\dfrac{(a+\lambda)(a+\mu)(a+\nu)}{(a-b)(a-c)}$ is the value of x arising from $\dfrac{x}{a+\lambda}+\dfrac{y}{b+\lambda}+\dfrac{z}{c+\lambda}=\dfrac{x}{a+\mu}+\dfrac{y}{b+\mu}+\dfrac{z}{c+\mu}=\dfrac{x}{a+\nu}+\dfrac{y}{b+\nu}+\dfrac{z}{c+\nu}=1$, write down the values of y, z.

PROPORTION.

Example IV. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that each fraction is equal to $\dfrac{pa+qc+re}{pb+qd+rf}$ and to $\sqrt[3]{\left(\dfrac{pa^3+qc^2e}{pb^3+qd^2f}\right)}$.

Let $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=k$; $\therefore$ $a=bk$, $c=dk$, $e=fk$;

$$\therefore \frac{pa+qc+re}{pb+qd+rf}=\frac{pbk+qdk+rfk}{pb+qd+rf}=k=\frac{a}{b},$$

and

$$\sqrt[3]{\left(\frac{pa^3+qc^2e}{pb^3+qd^2f}\right)}=\sqrt[3]{\left(\frac{pb^3k^3+qd^2fk^3}{pb^3+qd^2f}\right)}=\sqrt[3]{k^3}=k=\frac{a}{b}.$$

Example V. If $\dfrac{x+y}{4}=\dfrac{y+z}{5}=\dfrac{z+x}{6}$, find $x:y:z$.

$$\text{Each fraction}=\frac{(x+y)+(y+z)-(z+x)}{4+5-6}=\frac{2y}{3}.$$

Similarly, each $=\dfrac{2x}{4+6-5}=\dfrac{2x}{5}$ and $=\dfrac{2z}{5+6-4}=\dfrac{2z}{7}$;

$$\therefore\ \frac{x}{5}=\frac{y}{3}=\frac{z}{7} \quad \text{or} \quad x:y:z=5:3:7.$$

Example VI. If a, b, c, d are in continued proportion, prove that

(i) $b=dk^2$, $a=dk^3$, where $\dfrac{c}{d}=k$;

(ii) $\dfrac{a-b}{b-c}=\dfrac{b-c}{c-d}$.

(i) $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$ and $\dfrac{c}{d}=k$;

$$\therefore\ c=dk;\ b=ck=dk\,.\,k=dk^2;$$

$$a=bk=dk^2\,.\,k=dk^3.$$

(ii) $\dfrac{a-b}{b-c}=\dfrac{dk^3-dk^2}{dk^2-dk}=k$ and $\dfrac{b-c}{c-d}=\dfrac{dk^2-dk}{dk-d}=k$;

$$\therefore\ \frac{a-b}{b-c}=\frac{b-c}{c-d}.$$

Or, as follows:

$\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$; $\therefore\ \dfrac{a}{b}=\dfrac{b-c}{c-d}$, by Example IV., p. 288, and $\dfrac{a}{b}=\dfrac{a-b}{b-c}$;

$$\therefore\ \frac{a-b}{b-c}=\frac{b-c}{c-d}.$$

CROSS-MULTIPLICATION.

Example VII.

Given $\begin{cases} a_1x+b_1y+c_1z=0, \\ a_2x+b_2y+c_2z=0, \end{cases}$ find $x:y:z$.

We have
$$a_1c_2x+b_1c_2y+c_1c_2z=0,$$
$$a_2c_1x+b_2c_1y+c_2c_1z=0;$$
$$\therefore\ x(a_1c_2-a_2c_1)+y(b_1c_2-b_2c_1)=0;$$
$$\therefore\ x(c_1a_2-c_2a_1)=y(b_1c_2-b_2c_1);$$
$$\therefore\ \frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\text{similarly }\frac{z}{a_1b_2-a_2b_1}.$$

Determinant Notation.

The symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is used to represent $ad-bc$, and is called a *determinant*.

The relation found above for $x:y:z$ can therefore be written

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

EXERCISE XXIV. c.

1. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $\frac{a}{d}=\frac{b^3}{c^3}$.

2. If $\frac{x^2}{p}=\frac{x}{q}=\frac{1}{r}$, prove that $q^2=pr$.

3. If $\frac{x}{y}=\frac{y}{z}$, prove (i) $\frac{x-y}{y-z}=\frac{y}{z}$; (ii) $\frac{x^2+y^2}{y^2+z^2}=\frac{x}{z}$.

4. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{a}$, prove that $\frac{c^2}{d^2}=\frac{e}{b}$.

5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, state which of the following equations are identities:

(i) $\frac{a+c}{b+d}=\frac{c-e}{d-f}$; (ii) $\frac{a-c}{c-e}=\frac{b+d}{d+f}$; (iii) $\frac{ac}{bd}=\frac{c^2}{f^2}$;

(iv) $\frac{2a+c}{2b+d}=\frac{e^2}{f^2}$; (v) $\frac{3a+c}{3b+e}=\frac{2a-d}{2b-f}$; (vi) $\frac{a^2+7c}{b^2+7d}=\frac{e^2}{f^2}$;

(vii) $\frac{a^3-a^2c}{b^3-b^2d}=\frac{c^3+e^3}{d^3+f^3}$; (viii) $\sqrt{\frac{a^2-c^2}{b^2-d^2}}=\frac{c}{d}$;

(ix) $\frac{b-5f}{a-5e}=\frac{d-b}{c-a}$; (x) $\frac{ac-e^2}{a^2+c^2}=\frac{bd-f^2}{b^2+d^2}$.

6. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $b+c$ is a mean proportional between $a+b$ and $c+d$.

7. If v, w, x, y are in continued proportion, prove that

$$(v+w+x+y)^2=(v+w)^2+2(w+x)^2+(x+y)^2.$$

8. Solve $\frac{x+y}{14}=\frac{y+z}{17}=\frac{z-t}{8}=\frac{x+t}{11}$, $x+y+z+t=7$.

9. If $\frac{a}{b}>\frac{c}{d}$, and if a, b, c, d are positive, prove that

$$\frac{a}{b}>\frac{a+c}{b+d}>\frac{c}{d}.$$

10. Solve $\frac{x^2-2x+5}{3x^2+4x-1}=\frac{x^2+2x-5}{3x^2-4x+1}$.

11. Given $2x+y+5z=0$, $3x+2y+z=0$, write down $x:y:z$.

12. (i) Given $3x-y-2z=0$, $5x+y-3z=0$, write down $x:y:z$.
(ii) Solve the equations $3x-y-2=0$, $5x+y-3=0$.

13. Given $x+by-cz=0$, $ax-y+z=0$, write down $x:y:z$.

14. Write down the solution of the equations $ax+by+c=0$, $px+qy+r=0$.

15. Write the equations

$$a_1x^2+2h_1xy+b_1y^2=0, \quad a_2x^2+2h_2xy+b_2y^2=0$$

in the form $$\frac{x^2}{\;} = \frac{2xy}{\;} = \frac{y^2}{\;},$$

and hence find a relation independent of x and y.

16. Solve the equations

$$x+2y+5z=0, \quad 5x-4y+4z=0, \quad x^2-y^2-z^2=3.$$

17. If $a_1x+b_1y+c_1z=0$, $a_2x+b_2y+c_2z=0$, $a_3x+b_3y+c_3z=0$,

prove that $$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0.$$

This is usually written $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

18. If $5x-2y+z=3x+y-4z=0$, find the value of $\dfrac{xy-z^2}{x^2+z^2}$.

19. If $\dfrac{a}{x-y}=\dfrac{b}{y-z}=\dfrac{c}{z-x}$, prove that $a+b+c=0$.

20. What number must be added to each of the numbers 27, 60, 3, 18 so as to make them in proportion?

21. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, fill up the gaps in $\dfrac{a^4}{b^4}=\dfrac{a^4+3c^3e}{\;}=\dfrac{a^4+3ac^3e-5ace^2}{\;}$.

[Notice the different dimensions.]

22. If $\dfrac{a}{b}=\dfrac{c}{d}$, fill up the gap in $\dfrac{a+c}{b+d}=\sqrt[4]{\left(\dfrac{(a^5-c^6)}{\;}\right)}$.

23. Complete $\dfrac{x^2+y^2}{a}=\dfrac{xy}{b}=\dfrac{x^2-y^2}{\;}$.

24 If $\dfrac{x}{y}+\dfrac{y}{x}=\dfrac{a}{b}$, express $\dfrac{x+y}{x-y}$ in terms of a, b.

25. If $p^2+q^2=1$, complete $\dfrac{p}{a}=\dfrac{q}{b}=\dfrac{1}{\;}$.

26. If $\dfrac{x}{a}=\dfrac{y}{b}$, prove that $\dfrac{x^2+y^2}{ax+by}=\dfrac{xy(y-x)}{ay^2-bx^2}$.

Σ NOTATION.

$f(x, y)$ is called a *homogeneous* function of x and y of *degree* n if $f(kx, ky) \equiv k^n f(x, y)$.

E.g. $x^3 - 7x^2y + 5y^3$ is a homogeneous function of x and y of degree 3, because

$$(kx)^3 - 7(kx)^2(ky) + 5(ky)^3 \equiv k^3(x^3 - 7x^2y + 5y^3).$$

$f(x, y)$ is called a *symmetrical* function of x and y if

$$f(x, y) \equiv f(y, x).$$

E.g. $3x^2 - 7xy + 3y^2 + \frac{2}{5}x + \frac{2}{5}y - 3\frac{1}{4}$ is a symmetrical function of x and y, because the expression remains unaltered when x and y are interchanged.

The Σ notation is a short-hand method of representing the sum of a number of things of the same type.

E.g. if we are using 3 letters a, b, c,

$$\Sigma\, 5a \equiv 5a + 5b + 5c,$$
$$\Sigma\,(ab) \equiv ab + bc + ca,$$
$$\Sigma\,(a^2b) \equiv a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2.$$

In an analogous fashion,

$\sum_1^n r^2$ is used to represent $1^2 + 2^2 + 3^2 + \ldots + n^2$,

and $\sum_k^n r^2$ is used to represent $k^2 + (k+1)^2 + (k+2)^2 + \ldots + n^2$,

where n, k are positive integers and $n > k$.

Sometimes it is necessary to show what letters are involved in the expression.

Thus $$\sum_{a, b, c, d} (ab) \equiv ab + ac + ad + bc + bd + cd.$$

Example VIII. Prove that

(i) $\sum_{a, b, c} (a^3) - 3abc \equiv \Sigma\,(a) \,.\, \Sigma(a^2 - bc)$;

(ii) $\Sigma\,(y^3z^3) + 3xyz \,.\, \Sigma\,(x) \,.\, \Sigma\,(yz) - [\Sigma\,(yz)]^3 \equiv 3x^2y^2z^2$, where Σ extends to x, y, z.

(i) $\Sigma(a) \equiv a + b + c$; $\Sigma(a^2 - bc) \equiv a^2 - bc + b^2 - ca + c^2 - ab$
$\equiv a^2 + b^2 + c^2 - bc - ca - ab.$

By direct multiplication,

$$(a+b+c)(a^2+b^2+c^2-bc-ca-ab)\equiv a^3+b^3+c^3-3abc\ ;$$

$$\therefore\ \sum(a)\,.\sum(a^2-bc)\equiv\sum(a^3)-3abc.$$

(ii) We have $\sum(a^2-bc)\equiv\sum(a^2+2bc)-3\sum(bc)$

$$\equiv[\sum(a)]^2-3\sum(bc)\ ;$$

$\therefore$ from (i), $\sum(a^3)-3abc\equiv\sum(a)\,.\,\{[\sum(a)]^2-3\sum(bc)\}$

$$\equiv[\sum(a)]^3-3\sum(a)\,.\sum(bc).$$

Put $a=yz$, $b=zx$, $c=xy$.

Then $\sum(y^3z^3)-3x^2y^2z^2\equiv[\sum(yz)]^3-3\sum(yz)\,.\sum(x^2yz)$

$$\equiv[\sum(yz)]^3-3xyz\sum(yz)\,.\sum(x),$$

which is equivalent to the required result.

EXERCISE XXIV. d.

1. Simplify (i) $\sum\limits_{x,y,z}(x-y)$; (ii) $\sum\limits_{x,y,z}(x^2-y^2)$;

(iii) $\sum\limits_{a,b,c}a(b-c)$; (iv) $\sum\limits_{1}^{n}[(r+1)^2-r^2]$.

Prove that $\sum\limits_{a,b,c}[bc(b-c)]+\sum[a(b^2-c^2)]=0$.

2. How many terms are there in the expansion of

(i) $\sum\limits_{a,b,c,d}ab$; (ii) $\sum\limits_{a,b,c}(a)\,.\sum(ab)$; (iii) $\sum\limits_{a,b,c,d}(a^2b)$; (iv) $\sum\limits_{a,b,c,d}\left(\frac{a}{b}\right)$,

$\left[\sum\left(\frac{a}{b}\right)\right.$ and $\sum\left(\frac{b}{a}\right)$ represent the same thing.$\left.\right]$

3. Express as simply as possible in the $\sum$ notation :

(i) $[\sum(a)]^2-\sum(a^2)$; (ii) $\sum\limits_{a,b,c,d}(a+b)^2$;

(iii) $\sum\limits_{x,y,z}(x)\,.\sum\left(\frac{1}{x}\right)-\sum\left(\frac{x}{y}\right)$; (iv) $\sum\limits_{a,b,c,d}(a)\,.\sum(ab)-\sum(a^2b)$;

and simplify (v) $\sum\limits_{1}^{n}\log\left(1+\frac{1}{r}\right)$; (vi) $\sum\limits_{k}^{n}\log\left(1+\frac{1}{r}\right)$.

4. The identity $(r+1)^2-r^2\equiv 2r+1$ leads at once to

$$\sum_1^n[(r+1)^2-r^2]\equiv\sum_1^n(2r+1)=2\sum_1^n(r)+\sum_1^n(1).$$

(i) What is the value of $\sum\limits_1^n(1)$?

(ii) What does $\sum\limits_1^n(r)$ represent ?

(iii) Use the identity to prove that $\sum\limits_1^n(r)=\dfrac{n(n+1)}{2}$.

5. (i) Prove that $\sum_1^n [(r+1)^3 - r^3] \equiv \sum_1^n (3r^2+3r+1)$.

(ii) Use this identity and the result of Ex. 4 (iii) to prove that

$$\sum_1^n r^2 = \frac{n(n+1)(2n+1)}{6}.$$

6. (i) Prove that $\sum_1^n [(r+1)^4 - r^4] \equiv \sum_1^n (4r^3+6r^2+4r+1)$;

(ii) Use Ex. 4, 5 to prove that $\sum_1^n r^3 = \left[\frac{n(n+1)}{2}\right]^2$.

7. Given that $\sum_1^n r = \frac{n(n+1)}{2}$, find the value of $\sum_1^{2n} (r-1)$.

8. Given that $\sum_1^n 2^{r-1} = 2^n - 1$, find the value of $\sum_1^{2n} (2^{r-1} - 2)$.

9. (i) Find the value of $\sum_1^n \left(\frac{1}{r} - \frac{1}{r+1}\right)$;

(ii) What series does $\sum_1^n \frac{1}{r(r+1)}$ represent, and what is its sum?

10. Given that $\sum_1^n r^2 \equiv \frac{n(n+1)(2n+1)}{6}$, find the value of $\sum_p^q r^2$, where p, q are positive integers, $p < q$.

11. Given that $\sum_1^n r(r+1) \equiv \frac{n(n+1)(n+2)}{3}$, find the value of $\sum_p^q r(r+1)$ where p, q are positive integers, $p < q$.

12. Given that $\sum_1^n \frac{1}{r(r+1)} = \frac{n}{n+1}$, find the value of $\sum_n^{2n} \frac{1}{r(r+1)}$.

13. (i) Use the Σ notation to write down the most general homogeneous symmetrical function of x, y, z of the second degree.

(ii) Find the function if its value is 3 when $x=y=z=1$, and also 3 when $x=y=0$, $z=1$.

14. If $f(x, y, z) \equiv a\Sigma(x^3) + b\Sigma(x^2y)$, find a, b, given that

$$f(1, 1, 1) = 12 \quad \text{and} \quad f(0, 0, 1) = 2.$$

15. (i) Express $f(x, y, z)$ in the Σ notation, if $f(x, y, z)$ is the general symmetrical homogeneous function of the third degree.

(ii) Find the constants if $f(0, 0, 2) = 8$; $f(0, 1, 1) = 6$; $f(1, 1, 1) = 3$.

16. Given $\sum_{x, y, z} x = a$ and $\Sigma(x^2) = b^2$, express in terms of a, b

(i) $\Sigma(xy)$; (ii) $\Sigma(x^3) - 3xyz$.

17. Given $\sum_{x,y,z}(x)=a$ and $\sum(xy)=b^2$, express $\sum(x-y)^2$ in terms of a, b.

18. If $\sum_{x,y,z} x=0$, prove that

$$\text{(i)}\quad (x+y)(y+z)(z+x)+xyz=0;$$

$$\text{(ii)}\quad \sum(x^2y)=-3xyz.$$

19. Prove that $\sum(y-z)^3=3(x-y)(y-z)(z-x)$.

20. If $f(x, y, z)\equiv\sum(x^2)-\sum(yz)$, prove that

$$f(x+h, y+h, z+h)\equiv f(x, y, z).$$

21. If $\sum_{x,y,z}(x)=a$, $\sum x^2=3b^2$, $\sum x^3=3c^3$, express $(x-a)(y-a)(z-a)$ in terms of a, b, c.

22. (i) Express $2\sum_{x,y,z}(x^2-yz)$ as the sum of three squares.

(ii) If x, y, z are real, and if $\sum(x^2)=\sum(yz)$, prove $x=y=z$.

23. If $\sum_{x,y,z}(x)\,.\,\sum\left(\frac{1}{x}\right)=1$, prove that $\sum x=x$ or y or z.

24. If $\sum_{x,y,z}(x)=0$, express $xy+2yz$ in the form $ax^2+by^2+cz^2$.

REMAINDER THEOREM—FACTORS.

If the function $f(x)\equiv a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ is divided by $x-k$, it is required to prove that the remainder is $f(k)$.

Suppose the quotient is $Q(x)$ and the remainder is R, where R is independent of x.

Then $$f(x)\equiv(x-k)\,.\,Q(x)+R.$$

In this identity, put $x=k$;

$$\therefore\ f(k)=0+R;$$

$$\therefore\ \text{the remainder}=f(k).$$

In particular, if k is such that $f(k)=0$, the remainder is 0, and therefore $x-k$ is a factor of $f(x)$.

Example IX. Find the value of a if $x+2$ is a factor of $x^3-ax^2+7x+10$.

If $x+2$ is a factor, the expression vanishes if $x=-2$;

$$\therefore\ (-2)^3-a(-2)^2+7(-2)+10=0;$$

$$\therefore\ -8-4a-14+10=0;$$

$$\therefore\ -4a=12;$$

$$\therefore\ a=-3.$$

Example X. Factorise $\sum_{x,y,z} x(y-z)(y+z-x)^4$.

The given expression $=f(x, y, z)$

$\equiv x(y-z)(y+z-x)^4+y(z-x)(z+x-y)^4+z(x-y)(x+y-z)^4$.

If $x=0$, the function $=yz(z-y)^4+z(-y)(y-z)^4$

$=yz(y-z)^4-yz(y-z)^4=0$;

$\therefore$ x is a factor of $f(x, y, z)$.

By symmetry, y and z are also factors.

Again, if $y=z$, the function $=z(z-x)x^4+z(x-z)x^4$

$=x^4z(z-x)+x^4z(x-z)=0$;

$\therefore$ $y-z$ is a factor of $f(x, y, z)$.

By symmetry, $z-x$ and $x-y$ are also factors;

$\therefore$ $f(x, y, z)\equiv A\,.\,xyz(y-z)(z-x)(x-y)$.

But $f(x, y, z)$ is a homogeneous function of degree 6, and $xyz(y-z)(z-x)(x-y)$ is a homogeneous function of degree 6.

$\therefore$ A must be purely numerical, independent of x, y, z;

$\therefore$ putting $x=1$, $y=2$, $z=3$, we have

$$(1)(-1)(4)^4+2(2)(2)^4+3(-1)(0)^4$$
$$=A(1)(2)(3)(-1)(2)(-1);$$
$$\therefore\ -256+64=12A;$$
$$\therefore\ 12A=-192 \quad \text{or} \quad A=-16;$$
$$\therefore\ \Sigma x(y-z)(y+z-x)^4\equiv -16xyz(y-z)(z-x)(x-y).$$

Note.—Sometimes the value of A is best found by equating coefficients of a particular term.

EXERCISE XXIV. e.

1. Prove that $x+1$ is a factor of x^3-2x-1.
2. Prove that $x+2y$ is a factor of $x^4+10xy^3+4y^4$.
3. Find a if $x-2$ is a factor of x^3+ax-4.
4. Find the factors of $2a^3+a^2b+b^3$.
5. Find a, b if x^2+x-6 is a factor of x^3-ax^2-bx-6.
6. If n is a positive integer, find the condition that

(i) $x-1$ is a factor of x^n-1;

(ii) $x+1$ is a factor of x^n+1;

(iii) x^2+1 is a factor of x^n+1.

And prove that if 2^n+1 is a prime number, n must be a power of 2.

7. Find by inspection one factor of

$$(x+b+c)(x+c+a)(x+a+b)+abc.$$

8. Factorise $\sum_{a,b,c} bc(b-c)$.

9. Factorise $[\sum_{x,y,z}(x)]^3 - \sum(x^3)$.

10. Factorise $\sum_{x,y,z}(yz) \,.\, \sum(x) - xyz$.

11. Factorise $\sum_{x,y,z}(y-z)^3$.

12. Factorise $\sum_{x,y,z}(x^3+y^3)(x-y)$.

13. Factorise $[\sum_{a,b,c}(a)]^4 + \sum(a^4 - [b+c]^4)$.

14. Prove that $x^2 - yz$ is a factor of $[\sum(x)]^3 \,.\, xyz - [\sum(yz)]^3$: hence factorise it.

15. Prove that $xy - zw$ is a factor of $[\sum(xyz)]^2 - xyzw \,.\, [\sum(x)]^2$: hence factorise it.

16. Prove that

$$\sum_{a,b,c}(a[b+c-a]^2) + (b+c-a)(c+a-b)(a+b-c) \equiv 4abc.$$

17. Prove that $(x-1)^2$ is a factor of $x^n - nx + n - 1$.

18. If $x+y+z$ is a factor of

$$a(x^3+y^3+z^3) + b(y+z)(z+x)(x+y) + cxyz,$$

prove that $3a - b + c = 0$.

MISCELLANEOUS EQUATIONS.

No general rule can be given. Suitable transformations and substitutions can be learnt only by experience. Whenever possible, symmetrical methods of working should be adopted.

If expressions are squared, or if any other step is taken which is not reversible, it is necessary to verify by substitution that the final solutions satisfy the original equation.

The symbol $\sqrt{x}$ is used for the *positive* square root of x.

EXERCISE XXIV. f.

Solve the following equations:

1. $x + y = 8xy,\ x^2 + y^2 = 40x^2y^2$.

2. $xy = a^2,\ yz = b^2,\ zx = c^2$. 3. $ax = by + c = xy$.

4. $xy - x^2 = 2,\ 2y^2 - 3x^2 = 6$. 5. $xy + x - y = 7 = x^2 - xy + y^2$.

6. $\frac{1}{x^2}+\frac{1}{xy}=10,\ \frac{1}{y^2}+\frac{1}{xy}=15.$ **7.** $\frac{x}{a^2}=\frac{y}{b^2}=\frac{z}{c^2}=xyz.$

8. $x(y+z)=4,\ y(z+x)=3,\ z(x+y)=-6.$

9. $xy=1+\frac{30}{xy},\ \frac{3}{x}+\frac{4}{y}=3.$ **10.** $x^2+cy=y^2+cx=7c^2.$

11. $x^3+y^3=9=3x+3y.$

12. $2xy=x+2y=\frac{x}{y}+\frac{2y}{x}.$

13. $x+\frac{1}{y}=y+\frac{1}{x},\ x+\frac{2}{y}=3.$

14. $(x-2)(y-2)=2\frac{1}{2}=\frac{1}{2x}+\frac{1}{2y}.$ [Put $x+y=u,\ xy=v.$]

15. $xy+y=6,\ xz+z=8,\ yz=3.$

16. $\frac{x^3}{y}=10-xy,\ \frac{y^3}{x}=40-xy.$ [Form an equation in xy.]

17. Solve $x+\frac{1}{y}=\frac{8}{x},\ y+\frac{1}{x}=\frac{x}{2}.$ [Multiply the equations together.]

18. Solve $\frac{x}{3}+\frac{y}{4}=6,\ \frac{3}{x}+\frac{4}{y}=\frac{2}{3}.$

19. $(x-1)(y+2)=10,\ (y+2)(z+5)=15,\ (z+5)(x-1)=6.$

20. $x-6=y+4=z-3=\sqrt[3]{(xyz)}.$

21. $xy-\frac{x}{y}=9,\ xy-\frac{y}{x}=11\frac{2}{3}.$

22. $x^3-y^3=19,\ x^2y-xy^2=6.$ [Divide one equation by the other.]

23. $x(1-y^2)=3y,\ y(1-x^2)=3x.$ [Subtract and factorise.]

24. $x^3=5x-y,\ y^3=5y-x.$

25. $x^2+y^2=10,\ (x-y)(x^3-y^3)=52.$

26. $xy+x+y=-5,\ yz+y+z=7,\ zx+z+x=-3.$
[Note $xy+x+y\equiv(x+1)(y+1)-1.$]

27. $2x-\sqrt{x}=6.$ [Put $x=y^2$.]

28. $\sqrt{3x+1}-\sqrt{x+8}+1=0.$ [Test your answers.]

29. (i) $\sqrt{x+1}+\sqrt{x-2}=3$; (ii) $\sqrt{x+1}-\sqrt{x-2}=3.$

30. $5+\sqrt{x-2}=2x.$ **31.** $\sqrt{12+x}-\sqrt{4+x}=\sqrt{1-x}.$

32. $\sqrt{3x+1}+\sqrt{2x+2}=\sqrt{2x+3}+\sqrt{3x}.$

33. $\sqrt{3x+1} - \sqrt{2x-1} = \sqrt{x}$.

34. $x^2 + 3x + 3 = \sqrt{2x^2 + 6x + 5}$.
(Put $x^2 + 3x = y$ *or* $2x^2 + 6x + 5 = z^2$.)

35. $\dfrac{2}{1+\sqrt{x}} + \dfrac{3}{1-\sqrt{x}} = 5$. **36.** $(x^2 - 2x)(x^2 - 2x + 5) = 24$.

37. $(x^2 + 4x)^2 - 4x^2 - 16x - 5 = 0$.

38. $\left(x - 1 + \dfrac{2}{x}\right)\left(x - 2 + \dfrac{2}{x}\right) = 2$.

39. $xy = x + y + 2,\ yz = y + z - 4,\ zx = z + x - 2$.

40. $xy + x + y = -1,\ yz + 2y + 2z = 24,\ zx + 2z + 2x = 3$.

41. $x^2 + y^2 + z^2 = 133,\ x + y + z = 19,\ y^2 = xz$.

42. $(x+1)(x+3)(x+7)(x+9) = 45$. [Put $x + 5 = y$.]

43. $x^4 - 5x^3 - 12x^2 - 5x + 1 = 0$. $\left[\text{Divide by } x^2, \text{ put } x + \dfrac{1}{x} = y.\right]$

44. $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$.

45. $x^2 + y^2 + z^2 = 14,\ x + y + z = 6,\ xyz = 6$.
[Eliminate x, y, z from $(t-x)(t-y)(t-z) = 0$.]

46. $(4 - 6x)^{\frac{1}{3}} + (2 - x)^{\frac{1}{3}} + (7x - 6)^{\frac{1}{3}} = 0$.
[Note that if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.]

47. $\sqrt{x+a-b} + \sqrt{x-a} = \sqrt{2x-b}$.

48. $x^2 + xy + xz = p^2,\ xy + y^2 + yz = q^2,\ xz + yz + z^2 = r^2$.

49. $\dfrac{x+y}{2a} = \dfrac{y+z}{2b} = \dfrac{z+x}{2c} = yz$.

50. $x = \dfrac{y + 2a - 6}{ay - 3y + a - 2},\quad y = \dfrac{x + 2a - 6}{ax - 3x + a - 2}$.

Consider the cases $a = 1$, $a = 3$, $a = 4$, $a = 2\frac{1}{2}$.

51. $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} = \dfrac{xyz}{x+y+z}$.

52. $(y+z)^2 - x^2 = a^2,\ (z+x)^2 - y^2 = b^2,\ (x+y)^2 - z^2 = c^2$.

ELIMINATION.

Example XI. If $\quad a + \dfrac{1}{x} = b + \dfrac{1}{y} = c + \dfrac{1}{z},$

$$x + y + z = 0, \quad x^2 + y^2 + z^2 = 0,$$

prove that a relation exists between a, b, c, and find it.

Let $a+\frac{1}{x}=\lambda$; $\therefore\ b+\frac{1}{y}=\lambda=c+\frac{1}{z}$;

$$\therefore\ \frac{1}{x}=\lambda-a,\quad \frac{1}{y}=\lambda-b,\quad \frac{1}{z}=\lambda-c.$$

Now $\Sigma(x)=0$ and $\Sigma(x^2)=0$;

$$\therefore\ \Sigma(xy)=\tfrac{1}{2}[\{\Sigma(x)\}^2-\Sigma(x^2)]=0;$$

$$\therefore\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{\Sigma(yz)}{xyz}=0;$$

$$\therefore\ \lambda-a+\lambda-b+\lambda-c=0;$$

$$\therefore\ \lambda=\tfrac{1}{3}(a+b+c);$$

$$\therefore\ x=\frac{1}{\lambda-a}=\frac{3}{3\lambda-3a}=\frac{3}{b+c-2a}.$$

But $\Sigma(x)=0$; $\therefore\ \Sigma\left(\frac{1}{b+c-2a}\right)=0,$

which is the required relation.

This process is called *eliminating* x, y, z from the given equations.

EXERCISE XXIV. g.

1. If $x+ay=1$, $x+by=2$, $x+cy=3$, show that there must be a relation between a, b, c, and find it.

2. If $xy+x=a$, $xy+y=b$, $x+y=c$, show that a relation exists between a, b, c, and find it.

3. Eliminate m, given $x=am^2$, $y=2am$.

4. Eliminate t, given $x=t+\frac{1}{t}$, $y^2=t^2+\frac{1}{t^2}$.

5. Eliminate t, given $x=t+\frac{1}{t}$, $y=3t-\frac{1}{t}$.

6. Eliminate x, y, given $\frac{x+2y}{a}=\frac{2x+y}{b}=\frac{x+y}{c}$.

7. Eliminate x, given $\frac{x^2}{a+b}=\frac{x}{b+c}=\frac{1}{c+a}$.

8. Eliminate a, b, c, given $\frac{2x-y}{a}=\frac{2y+z}{b}=\frac{3x+y}{c}$, $a+b+c=0$.

9. Eliminate p, q, given $\frac{p+q}{x}=\frac{p-q}{y}=\frac{a}{b}$, $p^2+q^2=1$.

10. Eliminate t, given $x=\dfrac{a(1-t^2)}{1+t^2}$, $y=\dfrac{2at}{1+t^2}$.

11. Given $\dfrac{x}{y}+\dfrac{y}{x}=a$, $\dfrac{1}{x}+\dfrac{1}{y}=b$, $x+y=c$, show that a relation exists between a, b, c, and find it.

12. Given $\dfrac{x}{y}+\dfrac{y}{x}=c$ and $ax^2+bxy+ay^2=0$, must there be any relation between a, b, c? If so, find it.

13. Given $x+y=axy$, $y+z=byz$, $z+x=czx$, $xyz=1$, find the relation which must connect a, b, c.

14. Eliminate t, given $x=a+bt+ct^2$, $y=b+ct$.

15. Find the relation between a, b, c, d if

$$\sum_{x,y,z}(x)=a, \quad \sum(xy)=b^2, \quad \sum(x^3)=c^3, \quad xyz=d^3.$$

16. Eliminate x, given $\lambda=ax^2+bx+c=cx^2+bx+a$ and $c\neq a$.

17. Eliminate x, y, z, given $xy=a$, $yz=b$, $zx=c$, $x^2+y^2+z^2=d$.

18. If $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$, prove that

$$\frac{x^2}{b_1c_2-b_2c_1}=\frac{x}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1};$$

hence eliminate x.

19. Eliminate x, y, z, given $x=p(y+z)$, $y=q(z+x)$, $z=r(x+y)$.

20. Eliminate b, c, given $b=\dfrac{ax}{x-a}$, $c=\dfrac{bx}{x-b}$, $d=\dfrac{cx}{x-c}$.

REVISION PAPERS.

Papers.	
E. 1–10.	(Part I.)
F. 1–16.	(Chap. XII., XIII.)
G. 1–10.	(„ XII.–XIV.)
H. 1–10.	(„ XII.–XVI.)
K. 1–16.	(„ XII.–XVII.)
L. 1–10.	(„ XII.–XVIII.)
M. 1–10.	(„ XII.–XIV. and XIX.)
N. 1–10.	(„ XII.–XIV. and XX., XXI.)
P. 1–8.	(„ XII.–XVIII. and XXII., XXIII.)
Q. 1–10.	(„ XXIV.)

REVISION PAPERS.

E.

(PART I.)

E. 1.

1. If an oak beam l feet long, b inches wide and d inches thick is fixed at one end, and a weight W cwt. is placed at the other end, it will break if W is larger than $\frac{5bd^2}{4l}$. Find whether such a beam 10 feet long, 1 foot wide and 9 inches thick will break when a load of 6 tons is placed at the free end.

2. Simplify the following expressions :

(i) $3(x+y)-(3x+y)$. (ii) $(3x)^2-3x^2$.

(iii) $\left(\frac{x}{2}\right)^3-\frac{x^3}{2}$. (iv) $\left(a+\frac{b}{2}\right)-\left(\frac{a+b}{2}\right)$.

3. Make w the subject of the formula $P=\frac{W+nw}{8}$.

4. If the square of $(3x+7)$ is equal to the square of $(3x+8)$, find the value of x.

5. Which of the numbers 0, 1, 2, 3, 4 (if any), are roots of the equation $(2-x)^5+(4-x)^5=(6-2x)^5$?

6. A farmer has 60 hurdles, each 2 yards long, with which to make a rectangular enclosure of which one side is already formed by the fence of the field. Complete the following table :

Number of hurdles in side parallel to fence - - -	10	20	30	40	50	60
Number of hurdles in each of the other sides - - -	25					0
Area of enclosure in square yards - - - - -	1000					0

Draw a graph showing how the area depends on the number of hurdles placed in the side parallel to the fence, and find the number which gives the largest area possible.

E. 2.

1. Find the value of a, if $x=2$ is a solution of the equation

$$3x^2+2ax-20=0.$$

2. A flexible rod AB is built in at A, and supported at B; its length is l in. and the sag x in. from A is y in., where

$$y=\frac{1}{10l^3}x^2(l-x)(3l-2x).$$

(i) Find the sag halfway between A, B; (ii) compare the sags at $\frac{1}{5}$ of the distance from each end.

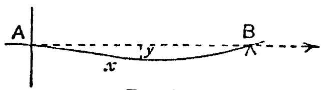


FIG. 80.

3. In a certain town eggs are advertised for sale at $3x$ shillings a dozen, and in another town they are being sold at x eggs for a shilling. What does one egg cost in each town? Find x if these two prices are the same.

4. Simplify the following expressions:

(i) $(x^3)^2\div(x^2)^3$. (ii) $(x^3)^2\div(x^3\times x^2)$.

(iii) $(a^2b^3)^2\div(ab^2)^3$. (iv) $6^{2x}\div3^x$.

5. For what values of x is $(x+1)^3=x^3+1$?

6. The rule for finding the tax on the earned income of a bachelor is as follows: Deduct one-tenth from the income, then deduct £135 from the result; of the remainder £225 is taxed at 3s. in the £ and the rest at 6s. in the £. Find a formula for the tax on an income of £x, where $x>1000$. What is the least value of x for which this formula holds good?

E. 3.

1. A working man gives a shillings a week to his wife for house-keeping expenses, spends b shillings a week on beer and tobacco, c shillings a month on tram fares, £d a year on boots and clothes, and £r a year on rent. How much will he have left from a wage of £4 a week at the end of the year, allowing exactly 52 weeks in a year? Give the answer in £ in as simple a form as possible.

2. Solve the equations: $\begin{cases}2{\cdot}6x+0{\cdot}16y=4{\cdot}7,\\3{\cdot}8x-0{\cdot}48y=3{\cdot}3.\end{cases}$

3. Make V the subject of the formula $\frac{1}{R}=\frac{CL}{t}\cdot\frac{V}{V-V_1}$.

4. Draw a graph to illustrate the relation between P and W, if $P=\frac{1}{5}W+4\cdot6$, for values of W from 0 to 40. Read off the value of W when P is 10.

5. $\frac{x-1}{2}$ and $\frac{2x-1}{5}$ are consecutive integers, $\frac{x-1}{2}$ being the greater of the two. What integers are they? Could they be consecutive integers if $\frac{2x-1}{5}$ were the greater?

6. The formula $h=\frac{12dV^2}{5R}$ gives the height, h inches, which the outer rail of a curved railway should be raised, where R yards is the radius of the curve, d feet the distance between the rails and V miles per hr. the maximum velocity of trains on the curve. Rewrite the formula with V as subject, and find the safe speed on a curve of radius 2160 yards when the outer rail is raised 2 inches and the distance between the rails is 4 feet 6 inches.

E. 4.

1. At a school of 600 boys, 300 boys subscribe on the average a shillings each, 200 boys subscribe on the average b pence each, and 100 boys subscribe nothing to a certain charity. Find in pence the average subscription for the whole school? Evaluate your answer when $a=2$ and $b=9$.

2. Prove that a triangle whose sides are (m^2+n^2) inches, (m^2-n^2) inches and $2mn$ inches is a right-angled triangle with (m^2+n^2) as its longest side whatever numbers m and n are. Give a numerical result when $m=5$ and $n=3$.

3. Factorize: (i) $16x^2-25$. (ii) $3x^2+7x-6$. (iii) $3x^2+x$.

4. An effort of P lb. is required to lift a weight of W lb. by a certain machine, where P and W are known to be connected by the equation $P=aW+b$, where a and b are constants, *i.e.* independent of the values of P and W. It is found that when $W=10$, $P=15$, and when $W=14$, $P=16\cdot2$. Find the values of a and b.
What is the value of W when $P=9$ and what does this mean?

5. If $x=y^3$ and $y^2=z^5$, express x^2 in terms of z only.

6. A spring 4 inches long is lengthened $\frac{W}{4}$ inches when a weight W lb. is hung on it. Another spring $3\frac{1}{2}$ inches long is lengthened $\frac{W}{3}$ inches when a weight W lb. is hung on it. For what value of W will the two springs have the same length?

E. 5.

1. A farmer keeps pigs and poultry only, and boasts that he has in his farmyard 200 heads and 464 legs, excluding those of himself and his household. How many pigs has he got in his farmyard ?

2. A rectangular piece of paper $ABCD$ is 5″ by 3″. The corner A is folded over the line BG so that it falls on the point F in the side DC, the dotted line BG being the position of the crease formed.
Calling AG x″ long, find the lengths of GF and GD in terms of x, and show that DF is 1 inch. Hence obtain an equation to find x and determine the position of the point G.

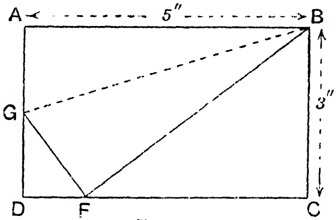


FIG. 81.

3. (i) Write down the L.C.M. of $2xy(x-y)^2$, $3y^2(x^2-y^2)$ and $4x^2(x+y)^2$.
(ii) Find the factors and H.C.F. of x^3-2x^2-3x and $x^4-5x^3-6x^2$.

4. Solve the equation $\dfrac{1}{x-1}+\dfrac{1}{x-2}=\dfrac{2}{x}$.

5. The weight of a foot of iron piping is $2{\cdot}45(D^2-d^2)$ lb., where D is the external and d is the internal diameter of the pipe in inches.
(i) Find the internal diameter of a pipe of external diameter 5·2″, which weighs 9·8 lb. per foot length ; (ii) Express the weight in terms of the mean radius r in. and the thickness t in.

6. A man walks a certain distance, x miles, up hill at 3 miles an hour, and walks back at 6 miles an hour. How long does he take going uphill ? How long does he take coming back ? How far has he been altogether, and how long has he taken ? Show that his average speed for the whole journey is 4 miles an hour.

[Lewis Carroll.]

E. 6.

1. A rectangle l″ long and b″ broad is such that its area would be increased by 44 square inches, if its length and breadth were both increased by 1 inch. Express this fact by an equation

connecting l and b, and prove that the perimeter of the rectangle is 86 inches.

2. Find the value of $\frac{x^{n+1}}{y^{n+1}} - \frac{x^n}{y^n}$ when $x=3$, $y=2$ and $n=2$.

3. A weight of 100 lb. falls freely 20 inches and is then brought to rest by stretching a spring a further distance of x inches. The work done by gravity is $100(20+x)$ inch-lb., and this equals the work done in stretching the spring which is $50x^2$ inch-lb. How much is the spring stretched ?

4. Express the following statements in a single formula :

The tax on £200 is £$\frac{200-160}{8}$.

The tax on £380 is £$\frac{380-160}{8}$.

The tax on £460 is £$\frac{460-160}{8}$.

Use your formula to calculate the tax on the same system on £240.

5. Simplify the following expressions :

$$\text{(i) } \frac{1}{h}\left[\frac{1}{x+h} - \frac{1}{x}\right], \qquad \text{(ii) } \frac{1}{h}\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right],$$

and find approximately their values when $x=1$, $h=0{\cdot}001$.

6. Find the value of the constants a and b in the formula

$$\frac{1}{w} = \frac{a}{u} + \frac{b}{v}$$

if $w=5$ when $u=4$ and $v=10$; and $w=6$, when $u=3$ and $v=9$.

E. 7.

1. $2n$ is the middle one of five consecutive even numbers. What are the others in terms of n ?

If the sum of all five is 60, find the value of the middle one.

2. Find the value of $\frac{2a+3b}{2a-5b}$, when $a=5b$.

Construct any similar expression in a and b, whose value will be independent of the values of a and b, provided $a=5b$; and find its value if (i) $a=10$, $b=2$; (ii) $a=15$, $b=3$.

3. Solve the equations : $\begin{cases} 4P = 5Q, \\ 14P - 13Q = 1{\cdot}8. \end{cases}$

4. a and b are two positive numbers $(a > b)$ such that

$$a^2+ab+b^2=79 \quad \text{and} \quad a^2+b^2=58.$$

Calculate in turn the values of (i) ab, (ii) $a^2+2ab+b^2$, (iii) $a+b$, (iv) $a^2-2ab+b^2$, (v) $a-b$. Hence find the values of a and b.

5. If $x=\sqrt{yz^3}$, and $y=2t^4$, $z=2s^2$, express x in terms of s and t.

6. In Fig. 82, $ABCD$ represents a rectangle with $AB=10''$, $BC=8''$. The lengths AF, AK, CG, CH are all equal to x''. What is the length of (i) FB, (ii) BG? Calculate the areas of the four triangles in the figure, and so express the area of the parallelogram $FGHK$ in terms of x. Give x the values 0, 1, 2 ... 8 in turn and calculate the corresponding values of the area $FGHK$. Show these results in a graph and from your graph find the value of x, which makes the area $FGHK$ a maximum.

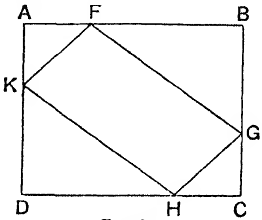


FIG. 82.

E. 8.

1. Fig. 83 represents some smoothed statistics for a seven year period. Curve AA shows the percentage of unemployment, curve BB shows number charged for drunkenness, curve CC shows felonies committed, drawn on separate scales. What general conclusions, if any, could be deduced from these graphs?

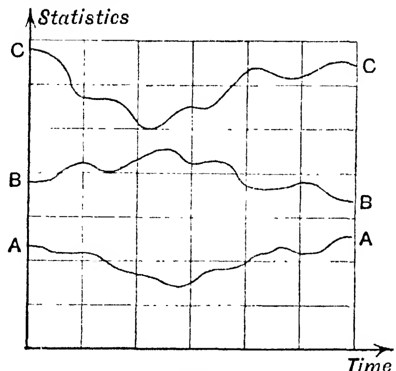


FIG. 83.

2. A boy is told to think of a number, subtract 7 from it and divide the remainder by 6. By mistake he subtracts 6 and divides by 7, but gets the same answer as he otherwise would have done. Find the number he thought of, and the answer he obtained.

3. Generalise the result of Question 2 by considering a boy who subtracts a and divides by b, instead of subtracting b and dividing by a. Prove that if he gets the same answer as he otherwise would have done, then he must have thought of the number $(a+b)$, and the answer he gets is 1.

4. (i) Simplify $\frac{2}{x^2-1}+\frac{3}{x-1}+\frac{5}{x+1}$.

(ii) Solve the equation $\frac{2}{x^2-1}+\frac{3}{x-1}+\frac{5}{x+1}=0$.

5. Solve the equations: (i) $6x^2-x-15=0$.

(ii) $2x+\frac{2}{x}=5$.

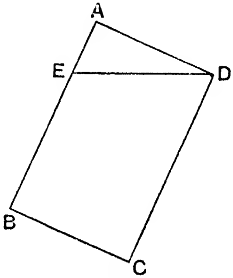


FIG. 84.

6. A rectangular tank of which $ABCD$ is a vertical section, is full of water and is tilted about the edge through C so that the water runs out. AB is 3 feet, BC is 2 feet and the length of the tank (not shown in the figure) is 5 feet.

DE represents the level of the water in the tank and AE is x ft.; find an expression in terms of x for the volume, V cubic feet, of water remaining in the tank, and also find the value of x in terms of V. What are the limits in the values of x and V, for which these formulae will hold?

E. 9.

1. Make W the subject of the formula $t=\frac{2aW}{W+w}\left(1+\frac{V}{v}\right)$.

2. A ship is ordered to make a run of 240 sea-miles at 16 knots. After steaming for m hours, she is x miles short of the distance she ought to have traversed in m hours. Find a formula for the number of hours she will be late if she cannot increase her average speed.

3. ABC is an acute-angled triangle having $AB=7$ cm., $BC=8$ cm., $CA=9$ cm., and AD is drawn perpendicular to BC meeting it at D. Draw a rough figure and take BD to be x cm. long. Express CD in terms of x and show that $AD=\sqrt{(49-x^2)}$. Obtain an equation for x from the right-angled triangle ACD, and hence find the value of x and the area of the triangle.

4. Solve the equation $\frac{1}{2x-1}=\frac{3x-4}{3x}$.

5. It is known that $\frac{6a-7}{2}$ is greater than $\frac{5a+4}{3}$. If a is an integer, find the smallest possible value of a.

6. Sealed cylindrical tins of volume 8 cubic inches are to be made to hold two ounces of tobacco.

If the volume of a cylinder is $\pi r^2 h$ cubic inches, and the surface of the tin used is $2\pi r(r+h)$ square inches, where r'' is the radius and h'' is the height of the cylinder, find h in terms of r when the volume is 8 cubic inches and hence show that the area of the surface is $\left(2\pi r^2 + \frac{16}{r}\right)$ square inches.

Tabulate the values of this expression, when r has the values $\frac{1}{2}$, 1, 2, 3, taking $\pi = 3{\cdot}14$, and represent the result graphically. Hence find the value of r, so that the quantity of tin used may be a minimum.

E. 10.

1. A bath is filled by the cold-water tap in a minutes, what fraction of it is filled in 1 minute ?

It is filled by the hot-water tap in b minutes: what fraction of it is filled in 1 minute ?

What fraction of the bath is filled, if both taps are turned on, in 1 minute, and how long will it be before the bath is full ?

The waste-pipe would empty the bath in c minutes.

Find an equation connecting a, b and c, if the level of the water remains unchanged when both taps are on, and the waste-pipe is open.

2. The volume of a sphere is $\frac{4}{3}\pi r^3$ cubic inches, if the radius is r''. Find the volume of metal in a spherical shell whose inner radius is $4''$ and outer radius $5''$, taking $\pi = \frac{22}{7}$.

3. If $t = 2x + 1$, write x in terms of t. Hence write $4x^2 + 4x - 3$ in terms of t, and find the value of t if $4x^2 + 4x - 3 = 0$.

4. Factorize (i) $5x^3 - 20x$, (ii) $7a^2 - 14a - 21$, (iii) $b(b-4) - 12$.

5. Solve the equations: $\begin{cases} y = x + 2. \\ x^2 + 2y = 7. \end{cases}$

6. In Fig. 85 ABC is a right-angled triangle in which $AB = 6$ cm., $BC = 4$ cm. A rectangle $PQRB$ is drawn as shown. If PQ is x cm., what is the length of RC ? It is known that

$$\frac{RC}{BC} = \frac{QR}{AB}.$$

Use this fact to show that $y = 6 - \frac{3x}{2}$, where y cm. is the length of QR. Hence express the area of the rectangle $PQRB$ in terms of x, and tabulate the values of the area of the rectangle when x has integral values from 0 to 4.

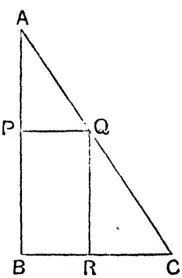


Fig. 85.

F.

(CHAPTERS XII.-XIII.)

F. 1.

1. A railway fare is increased by half; it is then increased again by $\frac{1}{6}$th of its new value. Prove that the final result is the same as if the original fare had been increased by three-quarters of its value.

2. Using the data of Question 1, find the original cost of a ticket which after the two increases costs 15s. 2d.

3. The following observed values of x and y will give a straight-line graph, subject to the errors of experiment:

x	1	2	3	4	5	6	7	8
y	7·2	12·1	16·7	21·8	26·6	31·5	36·3	41·2

Plot these values and draw the straight line which fits them best. Read off the most probable value of y when $x=4{\cdot}7$, and of x, when $y=30$.

4. If $x=\sqrt{y}$, $y^3=4z^2$, $z=4t^2$, (i) express y in terms of t, (ii) express x in terms of t.

5. If $2x-x^2=\frac{1}{2}$, find the value of (i) $(x-1)^2$, (ii) $(x-1)^6$.

6. Write the following numbers without using the index notation:

(i) $1{\cdot}8\times10^6$. (ii) $1{\cdot}8\times10^{-2}$.

(iii) $1{\cdot}8\times10^{-4}$. (iv) $1{\cdot}8\times100^{\frac{3}{2}}$.

and express 73800 and 0·0000738 in this form.

F. 2.

1. The equation $s=a+bt+t^2$ is satisfied by the two pairs of values $t=3$, $s=5$ and $t=5$, $s=29$.

Find the values of a and b, if they are constants.

2. The outer dimensions of a picture frame are 16 inches by 12 inches. The width of the frame is the same all round, and the surface area of the frame alone is 75 square inches. Find the width of the frame.

3. A piece of wire 1 foot long is bent into the shape of a rectangle. If one side of the rectangle is x'', find the area enclosed in terms of x. Find also the sides of the rectangle if the area enclosed is $6\frac{3}{4}$ square inches.

4. Rewrite the following expressions without using negative or fractional indices.

(i) $(1-x)^{-2}$. (ii) klt^{-2}. (iii) $(2x)^{\frac{1}{2}}$. (iv) $2x^{\frac{1}{2}}$.

(v) $2x^{-\frac{1}{2}}$.

5. Find the value of $P^{\frac{1}{2}}H^{-\frac{3}{4}}$, when $P=144$, $H=16$.

6. Simplify :

$$\frac{1}{a-1}-\frac{a}{1-a}+\frac{2a}{2a-1}-\frac{1}{1-2a}.$$

Check your answer by putting $a=2$.

F. 3.

1. A string 14 inches long has its ends fastened to two points, A and B, which are 10 inches apart. A pencil P is pressed against the string and is moved round so that the two portions of the string AP and PB are kept taut. Find the lengths of these two portions of the string in a position in which the angle APB is a right angle.

2. For what values of x, if any, are the following statements true ?

(i) $(x-1)^2=(1-x)^2$. (ii) $(x-1)=(1-x)$.

(iii) $(x-1)^2=(x-2)^2$. (iv) $(x-1)=(x-2)$.

3. The efficiency, e, of a certain machine is given by the formula $e=\frac{W(a-b)}{2P.a}$. Express a in terms of e, P, W and b.

4. Use logarithm tables to find x and y, if $x=10^{3\cdot1462}$ and $10^y=47$.

5. Write the following expressions without using fractional or negative indices.

(i) $(a^2+b^2)^{\frac{1}{2}}$. (ii) $a^{\frac{2}{3}}+b^{\frac{2}{3}}$.

(iii) $W.l.b^{-1}.d^{-2}$. (iv) $K.H^{1\cdot25}P^{-0\cdot5}$.

6. If $a=K.H^{1\cdot25}.P^{-0\cdot5}$ express P in terms of a, K and H.

F. 4.

1. Find the value when $a=2$, $b=-1$, and $n=3$ of the following expressions :

(i) $(a+b)^n$. (ii) a^n+b^n. (iii) $(a+b)^{n-1}$.

(iv) $a^{n-1}+b^{n-1}$. (v) $\frac{a^n}{b^n}\div\frac{a^{n-1}}{b^{n-1}}$.

What will be the value of (v) when $n=10$?

2. A man intends to travel by car from A to B. He has the choice of two routes, one x miles long on which he can travel y miles an hour, the other 6 miles longer on which he will be able to travel on the average 5 miles an hour faster. Prove that the extra time that he would take travelling by the longer route is given by the expression $\frac{x+6}{y+5}-\frac{x}{y}$ hours.

Simplify this expression and prove that he will save time in going by the longer route, provided $x>\frac{6y}{5}$, and that he can only save time if the journey is going to take at least 72 minutes anyhow.

3. If $x=2y+3$, express x^2-3y^2 in terms of y only.
Hence solve the equations: $\begin{cases} x=2y+3, \\ x^2-3y^2=22. \end{cases}$

4. Write down the logarithms of 1000, $\sqrt{1000}$, 1 and $(\sqrt[3]{100})^2$.

5. Use logarithms to find the value, correct to 3 significant figures, of
(i) $9,724,185 \div 45,620$;
(ii) $(1{\cdot}479)^5$.

6. Find the value of $\sqrt{\frac{H \cdot D^5}{L}}$, correct to 3 figures if $H=25{\cdot}7$, $D=2{\cdot}32$, $L=880$.

F. 5.

1. An extension ladder is formed of x separate ladders, each of length l feet, and the overlap at each junction is n feet. What is the total length of the ladder?

2. Find the value when $n=3$ of

(i) $8^{\frac{1}{n}}$; (ii) 8^{-n}; (iii) $8^{1-\frac{1}{n}}$; (iv) 16^{1-n}; (v) 16^{n-3}.

3. The perimeter of a rectangle is 34 inches, and the length of its diagonal is 13 inches. Find the lengths of its sides.

4. Two quantities s and t are connected by the formula

$$s=6t-4t^2.$$

Tabulate the values of s, when t has the values -2, -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, 2. Draw a graph to show the relation between s and t, and from it read off the largest value of s, and also the value of t when $s=1$.

5. Find, correct to 3 figures, the value of:

(i) $\frac{4{\cdot}712 \times 17{\cdot}95}{6{\cdot}834}$; (ii) $\sqrt{\frac{129{\cdot}1}{7{\cdot}32}}$.

6. The volume of a sphere is given by the formula $V=\frac{4}{3}\pi r^3$, where $\pi=3{\cdot}142$ and r is the radius.

Write the formula with r as the subject, and find the radius of a sphere whose volume is 16 cubic inches.

F. 6.

1. Prove that $\frac{n^2(n+1)^2}{4}+(n+1)^3=\frac{(n+1)^2(n+2)^2}{4}$.

[Note that $(n+1)^2$ is a factor of the left-hand side of this identity.]

Write down (without multiplying out) the special cases of this identity for $n=0$, 1, 2, 3, 4, 5, 6. By adding these results together, prove that $1^3+2^3+3^3+4^3+5^3+6^3+7^3=\frac{7^2\times 8^2}{4}$. By generalising this method, find an expression for the sum of the cubes of the first n integers.

2. Make μ the subject of the optical formula $\frac{1}{f}=(\mu-1)\left(\frac{1}{r}-\frac{1}{R}\right)$.

3. Solve the equations : (i) $3x^2-2x-8=0$;

(ii) $3x^2-2x-7=0$,

giving the result of the latter correct to 3 figures.

4. Simplify the following expressions :

(i) $2x^6\div x^2$. (ii) $3x^{\frac{1}{2}}\div x^{\frac{1}{3}}$. (iii) $4x^2\times 3x^4$.

(iv) $6x^{\frac{2}{3}}\times x^{\frac{4}{5}}$. (v) $(2x^2)^3$. (vi) $(2x^{\frac{1}{6}})^3$.

5. Find the value of pv^n when $p=14{\cdot}21$, $v=7{\cdot}7$ and $n=4$, correct to 3 figures.

6. In finding the value of $\frac{1124\times 79{\cdot}3}{74{\cdot}25}$ correct to 3 significant figures, a boy obtained the answer 11,900.

Show as shortly as possible why this answer is clearly wrong, and obtain the correct answer.

F. 7.

1.

$3\times 4=12$; $\therefore\ 35^2=1225$.

$4\times 5=20$; $\therefore\ 45^2=2025$.

$7\times 8=56$; $\therefore\ 75^2=5625$.

$12\times 13=156$; $\therefore\ 125^2=15625$.

What general formula covers these facts ? Prove it.

2. A bank clerk, in entering a sum of xs. yd. in a column of figures, writes it down as £x. ys. 0d. by mistake, with the result that he finds the column adds up £16. 12s. 2d. too much.

Find the sum of money which was entered wrongly.

3. A stone is thrown upwards with a velocity of 64 feet per second. Its height in feet after t seconds is given by the formula $h = 64t - 16t^2$.

Find when it is at a height of 48 feet, and explain the double answer.

4. Simplify

(i) $3a^3 \div 6a^{-2}$. (ii) $8a^{3\cdot5} \div 2a^{5\cdot5}$.

(iii) $\dfrac{2a^{\frac{2}{3}} \times 3a^2}{6a^{\frac{1}{6}}}$. (iv) $\dfrac{4a^{-2} \times a^3}{6a^{-4}}$.

5. Find the value, correct to 3 figures, of

(i) $\dfrac{1\cdot462}{7\cdot136}$; (ii) $\dfrac{10}{\sqrt{107\cdot1}}$.

6. Find the value of $e^x + e^{-x}$, when $e = 2\cdot718$ and $x = 2$, correct to 3 figures.

F. 8.

1. Find the value of (i) $(10^3)^{\frac{1}{3}}$; (ii) $(10^7)^{\frac{1}{2}}$; (iii) $(10^{7\cdot2})^{\frac{1}{4}}$.

2. Given that $x^2 - y^2 = 10$ and $x - y = 2$.
Find the value of $(x + y)$, and hence find x and y.

3. Solve the equations : (i) $6x^2 - 7x - 3 = 2$;
(ii) $6x^2 - 7x - 3 = 1$,

giving the solution of the latter correct to 3 figures.

4. In the figure, O is the centre of a circle whose radius

$$OA = OR = 5''.$$

AB is a chord at right angles to OR, and $NR = x''$. What is the length of ON in terms of x? What is the length of AN in terms of x?

Tabulate the values of AN when x has the values 0, 1, 2, 3, 4, 5. Show these results as a graph. What can you say about the shape the graph will take?

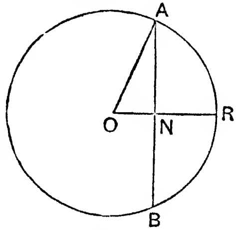


FIG. 86.

5. Factorise and find the value of $\pi(R^2 - r^2)l$, if $\pi = 3\cdot142$, $R = 15\cdot42$, $r = 14\cdot42$ and $l = 250$.

6. The length of the tangent from a point on the outer of two concentric circles to the inner circle is 5 inches. The difference of their radii is 2 inches. Find their radii.

F. 9.

1. (i) Simplify $\frac{a}{a+1} - \frac{a+1}{a+2}$. Test your result when $a=1$.

(ii) Simplify $\frac{(x^2-4)(x^2-4x+3)}{(x^2-1)(x^2-5x+6)}$.

2. Express in a single formula the following facts:

$$\sqrt{6} \times \sqrt{7} = \sqrt{6 \times 7} = \sqrt{42}.$$
$$\sqrt{7} \times \sqrt{10} = \sqrt{70}.$$
$$\sqrt{3} \times \sqrt{5} = \sqrt{15}.$$

Use this formula to find the value as simply as possible of

(i) $\sqrt{32} \times \sqrt{2}$;
(ii) $\sqrt{6} \times \sqrt{24}$;
(iii) $\sqrt{ab^3} \times \sqrt{a^3b}$.

3. The area of a triangle is $\sqrt{s(s-a)(s-b)(s-c)}$, where a, b, c are the sides and $s=\frac{a+b+c}{2}$. What does this formula become for an equilateral triangle, side a? Verify your result by finding a formula for the area of an equilateral triangle independently.

4. In a right-angled triangle the hypotenuse is 15″, and the difference between the other two sides is 3″. Find the other two sides.

5. The bore, d inches, of a pipe through which a pump of horse-power H can deliver G gallons per second is given by

$$d = 1\cdot25 \,.\, G^{\frac{3}{4}}H^{-\frac{1}{4}}.$$

(i) Find the bore necessary for an engine of 15 H.P. to deliver 300 gallons per minute.

(ii) Rewrite the formula, making G the subject. (C.S.C.)

6. Find the value, correct to 3 figures, of

(i) $\sqrt[3]{0\cdot012}$. (ii) $\sqrt[4]{0\cdot08}$.

F. 10.

1. In Fig. 87, AB represents a line 10″ long, and P is a point such that $AP=x''$. Squares are described on AB, AP and PB as shown.

Write down the areas of these squares and express the area of the shaded portion of the figure in terms of x.

Tabulate the values of this area as x varies from 0 to 10, and show the results on a graph. Find from your graph the position of P which makes the shaded area greatest.

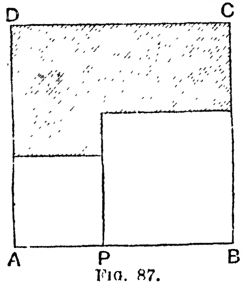


FIG. 87.

2. Find a general formula to include the following facts:

$$\frac{\sqrt{30}}{\sqrt{5}}=\sqrt{\frac{30}{5}}=\sqrt{6}.\quad \frac{\sqrt{24}}{\sqrt{12}}=\sqrt{\frac{24}{12}}=\sqrt{2}.\quad \frac{\sqrt{90}}{\sqrt{18}}=\sqrt{\frac{90}{18}}=\sqrt{5}.$$

Use this formula to find the value of

(i) $\frac{\sqrt{8}}{\sqrt{2}}$; (ii) $\frac{\sqrt{72}}{\sqrt{8}}$; (iii) $\frac{\sqrt{ab^3}}{\sqrt{a^3b}}$.

3. Solve the equations: $\begin{cases} x+y=1, \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{6}=0. \end{cases}$

4. Find, without using tables, the logarithms of

$$\sqrt[3]{10},\quad \sqrt[4]{0{\cdot}1},\quad \sqrt[5]{0{\cdot}001},\quad \sqrt[3]{0{\cdot}1}.$$

5. Find the value of $\frac{l^2\omega}{8s}$ when $l=80$, $\omega=0{\cdot}072$, $s=276$.

6. It is stated by Professor Perry that the record times for races of all kinds by men or animals obey the law $t=k\,.\,m^{1{\cdot}175}$, where t sec. is the record for a race of m miles and k depends on the conditions of the race, but is independent of the distance. The record for the "half-mile" is 1 min. $52\frac{1}{2}$ sec. (*Whitaker's Almanack*). What should be the record for the "mile"? [Up to the present it is reported to be 4 min. $12\frac{3}{4}$ sec.]

F. 11.

1. Prove that the equations:

$$\left.\begin{aligned} 3x+y-2z&=7, \\ x-y-2z&=5, \\ 2x+y-\ \ z&=3, \end{aligned}\right\}$$

are inconsistent by obtaining two equations independent of y and comparing them.

2. Simplify $$\frac{x-y}{(x+y)^2}+\frac{1}{x+y}+\frac{x^2+y^2}{(x+y)^3}.$$

3. $(1+x)^n \simeq 1+nx$, when x is small. Use logarithms to find the error per cent. in assuming that $(1+x)^n=1+nx$, when $x=0{\cdot}025$ and $n=4$.

4. Find, correct to 3 figures the roots of the equations :

$$\text{(i)}\ 4x^2+5x-2=0;$$
$$\text{(ii)}\ 4x^2+5x-2=4.$$

5. A small object 3 mm. long when seen through a magnifying glass held d mm. from it, appears to have a length l mm. given by $l=\frac{3f}{f-d}$, where f is a constant depending on the kind of magnifying glass used.

(i) Find the value of f for a certain magnifying glass in which the apparent length l is 12 mm., when the glass is held 15 mm. away.

(ii) Find also the apparent length of the same object seen through the same glass, when the glass is held 12 mm. away.

6. The safe width of a dam at a depth of x feet below water level is $\sqrt{\left\{\frac{0{\cdot}05\,.\,x^3}{9+0{\cdot}03x}\right\}}$ feet.

How wide should a dam be at a depth of 28 feet below water level ? (Certificate.)

F. 12.

1. Two boys bicycle a distance d miles together ; the wheels of one bicycle are a'' in diameter, the wheels of the other are b'' $(a > b)$. Show that the number of revolutions made by the smaller wheels is $\frac{1760\times 36d}{\pi}\,.\left(\frac{1}{b}-\frac{1}{a}\right)$ more than the number made by the larger. Write this in a form suitable for calculation by logarithms, and evaluate when $a=28$, $b=26$ and $d=10$.

$$[\pi=3{\cdot}142=10^{0{\cdot}4972}.]$$

2. The following equations are found to connect a force of P lb., and the distance x'' at which it acts from a fulcrum :

$$\begin{cases} P\,.\,x=88,\\ P(x+2)=120.\end{cases}$$

Find the values of P and x.

3. Find x and y if $x=\frac{1}{y}=\frac{2}{x-1}$.

4. Subtract $\frac{x^n}{(x+y)^m}$ from $\frac{x^{n-1}}{(x+y)^{m-1}}$.

5. Evaluate the following as shortly as possible, given that $\sqrt{2}=1{\cdot}414...$, $\sqrt{3}=1{\cdot}732...$:

(i) $\frac{4}{\sqrt{2}}$; (ii) $\frac{6}{\sqrt{2}}$; (iii) $\frac{\sqrt{6}}{\sqrt{3}}$; (iv) $\sqrt{12}$.

6. When water flows at speed v through a pipe of diameter d, the loss of pressure in a length l is equal to a head of h feet of water, where $h=\frac{0{\cdot}045}{d^{1{\cdot}16}}\times\frac{v^2 l}{2g}$.

Find h when $l=75$, $v=7{\cdot}4$, $d=2{\cdot}3$, $g=32{\cdot}2$. (C.S.C.)

F. 13.

1. Fig. 88 represents a white flag, 20 ft. by 30 ft., with a red cross; the red stripes forming the cross are of equal width, and the area of the cross is $\frac{1}{3}$ of the area of the whole flag. Find the width of the strips to the nearest $\frac{1}{10}$th of an inch. (C.S.C.)

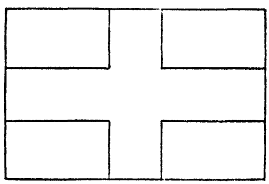


FIG. 88.

2. In a sailing-boat the direction of the wind makes an angle α with the direction of the boat's keel, and the sail makes an angle β with the boat's keel. The following table shows the best angle to choose for β when sailing into a wind:

α in degrees	60°	50°	40°	30°	20°	10°
β in degrees	$34\frac{1}{2}$°	28°	22°	16°	10°	5°

Exhibit these results on a graph, and read off from the graph the best angle at which to set the sail when the course required is due North and the wind is blowing from the North-East.

3. Find the value of $W\cdot\frac{\sqrt{2ah-h^2}}{b-h}$, when $W=112$, $b=10{\cdot}5$, $h=8{\cdot}25$ and $a=5$.

4. Find x if (i) $3^x=81$; (ii) $81^x=3$; (iii) $3^x=\frac{1}{9}$; (iv) $81^{-x}=3$.

5. Find the value of (i) 6^{20}; (ii) 2^{60}, correct to one significant figure, in the approximate form $a\times10^n$, a being an integer less than 10.

6. Write in as simple a form as possible:

(i) $\frac{\sqrt{60}}{\sqrt{3}}$; (ii) $\frac{\sqrt{20}}{\sqrt{5}}$; (iii) $\frac{3\sqrt{5}}{\sqrt{15}}$; (iv) $\frac{4^{\frac{2}{3}}}{2^{\frac{1}{3}}}$.

F. 14.

1. A line AB, $2a$ inches long, is bisected at M. P is a point in the line between M and B, such that $MP = x''$. Prove that

$$AP^2 - PB^2 = 4ax.$$

2. Write 4^n, 8^{n-1} as powers of 2, and simplify $\frac{2^n \times 4^n}{8^{n-1}}$.

3. Fig. 89 represents a circle inscribed in the triangle ABC. $AR = x''$, $BP = y''$, $CQ = z''$.

Also $AB = 5''$, $AC = 6''$, $BC = 7''$.

Write down three equations connecting x, y and z.

By adding these up, show that $x + y + z = 9$, and hence find the values of x, y and z.

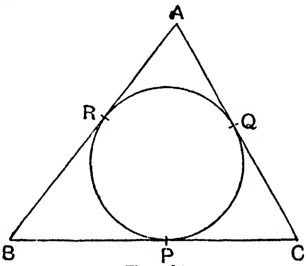


FIG. 89.

4. Find the value of $e^{\mu\theta}$ when $e = 2{\cdot}718$, $\mu = 0{\cdot}56$, $\theta = 2$.

5. Given that a sum of £P invested at r per cent. per annum, interest being reckoned half-yearly, will amount at Compound Interest in n years to a sum £A = £$P\left(1 + \frac{r}{200}\right)^{2n}$, find the amount on £1750 invested for 15 years at 6 per cent. per annum paid half-yearly, giving the answer in pounds, correct to 3 figures.

6. Q cubic feet of water flow over a rectangular weir per second given by $Q = 6{\cdot}78\left(b - \frac{h}{5}\right)h^{\frac{3}{2}}$, where b = width of weir in feet = 60 feet, h = height of water over weir in feet = 1·25 feet. Find how many tons of water flow over the weir in an hour, if a cubic foot of water weighs $62\frac{1}{2}$ lb.

F. 15.

1. Write 27, $\frac{1}{9}$, $3\sqrt{3}$ and $\sqrt[4]{3}$ as powers of 3, and write down their logarithms to the base 3.

2. The tendency of a plank bridge to break when a man is standing on it, is measured by the quantity $\frac{w}{l}(l - x)\,x$, where l feet is the length of the bridge, w lb. is the weight of the man and x feet is his distance from one end. Tabulate the values of this expression for a man, 150 lb. weight, standing on a bridge 20 feet long, when his distance from one end is 0, 4, 8, 12, 16, 20 feet respectively.

Draw a graph to show how the tendency to break varies, and find where the man is when the bridge is most likely to break.

3. Two quantities R and v are connected by the equation $R = a + b\sqrt{v}$, where a and b are constants.

If $R = 10$ when $v = 64$, and $R = 11{\cdot}2$ when $v = 81$, find the values of a and b.

4. There is a day-and-night motor service from A to B, the cars leaving A at each hour; also a service from B to A, the cars leaving B at the half-hour. The journey occupies x hours, where x is an integer.

How many cars will a man pass on the road when he travels from A to B in one of these cars ?

5. Evaluate

$$\text{(i)}\ \frac{1{\cdot}56 \times 10^6}{412{\cdot}7 \times 1{\cdot}175}; \qquad \text{(ii)}\ \sqrt[3]{\frac{0{\cdot}03167}{12{\cdot}58}}.$$

6. If a sum of money £P amounts to £A when invested for n years at r per cent., then $A = P\left\{1 + \frac{r}{100}\right\}^n$. Write this formula with r as the subject, and hence find the rate per cent. paid on a "War Savings Certificate," which costs 15s. 6d. and amounts to £1 in 5 years.

F. 16.

1. A man buys a number of motor-cars at £400 each. He keeps one for his own use and sells the remainder at 10 per cent. profit. As the result he finds he has got his own car for nothing. How many cars did he buy ?

2. Simplify $$\left(\frac{1}{x} - \frac{1}{1+x}\right) \div \left(\frac{1}{x} + \frac{1}{1-x}\right).$$

3. Given that the volume of a sphere, radius r'', is $\frac{4}{3}\pi r^3$, find the diameter of a hemispherical bowl which will hold a gallon of water. [$\pi = 3{\cdot}142$ and 1 cubic foot $= 6\frac{1}{4}$ gallons.]

4. If $(h-a)(1-a) = (h-b)(1-b) = k$, find k in terms of a and b only. (C.S.C.)

5. A certain sheet of transparent material absorbs $\frac{1}{10}$th of the light of a certain colour falling on it. What fraction of the light of that colour gets through two such sheets ?

How many sheets must be placed one over another to reduce the light of that colour by 60 per cent. ?

6. A spherical bomb, when dropped from a height of s feet, reaches the ground with a velocity v feet per second after t seconds. Taking air-resistance into account,

$$s = \frac{L^2}{2kg} \log\left(\frac{L^2}{L^2 - v^2}\right),$$

$$t = \frac{L}{2kg} \log\left(\frac{L+v}{L-v}\right),$$

where $L = 400$, $k = 0{\cdot}4343$, $g = 32{\cdot}2$.

Find the time taken to reach the ground from a height of 1000 feet. (C.S.C.)

G.

(CHAPTERS XII.-XIV.)

G. 1.

1. Write the following expressions without using fractional or negative indices:

(i) $5x^{\frac{1}{2}}$; (ii) $(9x)^{\frac{1}{2}}$; (iii) $1{\cdot}4 \times 10^{-6}$; (iv) $2\left(\frac{a}{3b}\right)^{-1}$.

2. In an examination some candidates take 4 papers and some take 6 papers. There are 250 candidates, and they do 1078 papers in all. Find how many took 6 papers.

3. The following formulae connect the velocity v of a body which starts with initial velocity u, and moves with uniform acceleration f, with the time t it has been moving, and the distance s it has gone:

(i) $v = u + ft$;

(ii) $s = ut + \frac{1}{2}ft^2$.

Rewrite the first formula so as to express t in terms of u, v and f.

Substitute this expression for t in the second formula, and so obtain the formula $v^2 = u^2 + 2fs$.

Find also a formula connecting s, v, f, t.

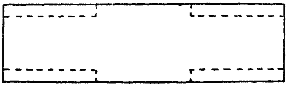


FIG. 90.

4. From a rectangular strip of paper, a'' by b'' $(a > b)$, rectangles x'' by y'' $(x > y)$ are cut out at each corner, the sides of length x'' being parallel to those of length a''; the remainder is formed into a box by folding the projections upwards, and the longer projections are folded again, and so just form the lid of the box.

Find (i) the relation between x, y, a; (ii) the volume of the box in terms of a, b, y.

5. In a certain Parliament the ratio of the majority to the minority is 3 : 2. Eighteen members desert the majority and join the minority, thus reducing the ratio to 5 : 4. Find the number of members. (Certificate.)

6. ABC is a triangle with a right angle at A, and AD is perpendicular to BC.

It is known that the ratio $BD : DA$ is equal to the ratio $DA : DC$. If BD is 5·4″ and DC is 9·6″, find the length of AD. Find also the length of BA

(i) from the fact that $BD : BA = BA : BC$;

(ii) by applying Pythagoras' theorem to the triangle ADB.

G. 2.

1. Find the values of $(1-x)(x+2)$ when x has integral values from -3 to $+3$. Plot a graph to illustrate these figures, and read off from it the greatest value of $(1-x)(x+2)$ for all values of x in this interval.

2. Solve the equations: $\begin{cases} 2{\cdot}7x + 0{\cdot}13y = 5{\cdot}6, \\ 3{\cdot}9x + 0{\cdot}7y = 6{\cdot}4, \end{cases}$

giving the results correct to two significant figures.

3. A man buys two horses, paying £250 for the two. He sells one at a loss of 10 per cent. and the other at a profit of 50 per cent., receiving £291 in all. What did he pay for each horse?

4. Einstein's theory of relativity gives the formula $V = \dfrac{7 \times 10^{41}}{\sqrt{\rho^3}}$, connecting V the volume of world space and ρ the mean density of matter throughout space.

(i) Write this formula with ρ as subject.

(ii) Assuming space to be spherical and the volume of a sphere to be $\frac{4}{3}\pi r^3$, find a formula for the radius r of space in the form $r = \dfrac{a \, . \, 10^b}{\sqrt{\rho}}$, with b integral and a correct to one figure.

5. Two glasses are not quite full; one contains n spoonsful of water and the other n spoonsful of wine. A spoonful of wine is taken from the second glass and mixed in the first. What is the proportion of wine in the mixture? A spoonful of the mixture is now taken out of the first glass and put in the second. What is the proportion of wine to water in the second glass? Is this greater or less than the proportion of water to wine in the first glass?

6. AD bisects the angle A of a triangle BAC internally, and meets BC at D. It is known that the ratio $BD : DC = BA : AC$.

If the sides of the triangle are $AB = 5''$, $BC = 6''$, $AC = 7''$ and BD is x'', write down the equation to find x, and obtain the lengths of BD and DC.

G. 3.

1. A train in which there is seating accommodation for 600 has 84 compartments, some of which seat 6 and some seat 10. Find how many compartments there are of each kind.

2. Write the following expressions without using fractional or negative indices: (i) $a^{\frac{1}{2}}b^{\frac{3}{2}}$, (ii) $H^{\frac{3}{4}}G^{-\frac{1}{4}}$, (iii) $1{\cdot}4 \times 10^{-11}$, and write down the value of 7^{-2}, $9^{-\frac{1}{2}}$, 11^{0}.

3. Write the formula $W = 17{\cdot}6 \dfrac{PR^2}{r}$ with R as the subject, and find the value of R, correct to 3 figures, when $W = 4480$, $r = 0{\cdot}29$, $P = 9{\cdot}6$.

4. (i) Solve the equations: $\begin{cases} x + y = 2a + 3b, \\ 4x + 3y = 7a + 5b. \end{cases}$

(ii) If $\dfrac{x+y}{x-y} = \dfrac{8}{5}$, find the ratio $x : y$.

5. The "displacements" of similar ships vary as the cubes of their lengths. Find the displacement of a ship which is half as long again as a similar ship whose displacement is 2000 tons.

6. (i) The perimeter of a triangle is 2 feet, and one side is three times another. Show that the length of the shortest side lies between 3 and 4 inches.

(ii) Show that in any triangle in which one side is three times another, the ratio of the shortest side to the perimeter lies between $\frac{1}{6}$ and $\frac{1}{8}$.

G. 4.

1. The triangle ABC is right-angled at B, and has $AB = 12$ cm., $BC = 10$ cm. P is a point in AB such that $AP = x$ cm., and PQ is drawn parallel to BC to meet AC at Q. It is known that the ratio $AP : AB$ equals the ratio $PQ : BC$. Hence express PQ in terms of x, and show that the area of the $\triangle APQ$ is $\dfrac{5x^2}{12}$ square cm. Tabulate the values of this area as x takes values from 0 to 12; show the results on a graph, and estimate from your graph the value of x when the area of the triangle APQ is exactly half the area of the triangle ABC.

2. Find the value of e^{-kt} when $e=2\cdot718$, $k=5\cdot6$ and $t=5$.

3. A pendulum is set swinging so that the angle it swings through is 10°. The angle of each succeeding swing is $\frac{9}{10}$th of the angle of the preceding swing. Show that after 22 swings the angle it swings through is less than 1°.

[One swing is considered to be the movement from one extreme position to another extreme position on the opposite side of the vertical.]

4. Express $x+\dfrac{(1+x)x}{1-x}+\dfrac{1-x}{1+x}$ as a single fraction.

5. Solve, correct to 3 figures, the equations:

(i) $12x^2-43x+45=10$;

(ii) $12x^2-43x+44=10$.

6. The time of oscillation of a simple pendulum varies as the square root of its length. If the time for a pendulum 2 feet long is $1\frac{1}{2}$ seconds, find the time of oscillation of a pendulum one yard long (correct to two figures).

G. 5.

1. A triangle ABC is inscribed in a rectangle as shown in Fig. 91. Find the area of the triangle ABC in the simplest possible form in terms of the lengths p, q, r, s shown in the figure.

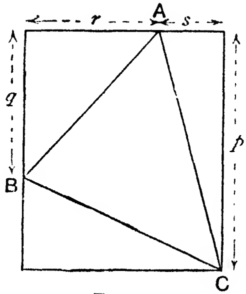


FIG. 91.

2. Simplify

(i) $a^{\frac{5}{6}}\times a^{\frac{1}{6}}$; (ii) $(a^{\frac{2}{3}})^6$; (iii) $\sqrt[3]{a^{\frac{9}{2}}}$; (iv) $\sqrt[3]{ab^2}\times\sqrt{a^2b}$.

3. Find the value of pv^n, when $p=17\cdot23$, $v=4\cdot56$ and $n=1\cdot05$.

4. Show that a triangle whose sides are $\sqrt{5}-1$, $\sqrt{5}+1$ and $2\sqrt{3}$ inches is right-angled, and find its area.

5. A man can scull at 6 miles an hour in still water for an indefinite time. He sculls a distance d miles upstream against a current of x miles an hour, and then sculls the same distance down with the current. Show that the ratio of the time he takes to the time he would have taken if there had been no current is $\frac{36}{36-x^2}$.

6. If a light beam is supported at each end and carries a given load at its middle point, the sag at the middle varies as the cube of the length of the beam. For a beam 6 feet long the sag is $1\frac{1}{2}$ inches. What will it be for a similar beam equally loaded 8 feet long? (Certificate.)

G. 6.

1. If a bookmaker lays odds of m to n against a horse, he will, if the horse wins, pay a man who backs it a sum which bears a ratio $m:n$ to the stake for which it was backed (in addition to returning the stake if already deposited). A man wants to back two horses for which the odds are 2 to 1 against and 7 to 4 against, so as to win exactly £5 on balance if either of the horses wins. Find how much he should put on each horse. How much does he lose if neither horse wins?

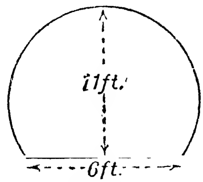


FIG. 92.

2 The section of a tube railway is a segment of a circle as shown in Fig. 92. The greatest height of the tube is 11 feet, and the width at the bottom is 6 feet. Find the radius of the tube.

3. If $\frac{a+b}{a-b}=\frac{c}{d}$, prove that $\frac{c+d}{c-d}=\frac{a}{b}$.

4. Simplify (i) $\frac{4^m \times 3^{n+2}}{2^m \times 2^{m-1} \times 9}$; (ii) $\frac{\sqrt[3]{a^4}}{\sqrt[3]{a^2}}$.

5. Find the value of $\pi a^2 - \pi b^2$ to 3 significant figures, when $\pi = 3{\cdot}142$, $a = 7{\cdot}528$, $b = 1{\cdot}461$.

6. It is found that the current required to fuse a wire of given material varies as $D^{\frac{3}{2}}$, where D is the diameter of the wire. A current of 154·1 ampères fuses a leaden wire of diameter 0·232″; find in ampères, correct to 3 figures, the current which will fuse a leaden wire of diameter 0·316″. (Certificate.)

G. 7.

1. In Fig. 93 ABC is any triangle with $BC=a''$; AN is perpendicular to BC and $AN=p''$; $PQRS$ is a square of side x''. The ratio $AS:AB=SR:BC=x:a$. Hence write down the value of the ratio $BS:BA$; use the fact that $BS:BA=SP:AN$, to obtain an equation connecting a, x and p, and show that

$$\frac{1}{x}=\frac{1}{a}+\frac{1}{p}.$$

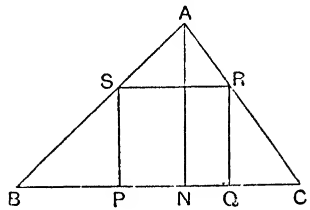


FIG. 93.

2. In the figure of Question 1, express in terms of a, x and p

(i) the area of the triangle ASR ;

(ii) the sum of the areas of the triangles BPS, CQR ;

(iii) the area of the square ;

(iv) the area of the whole triangle.

By equating the sum of the first three of these to the fourth, obtain the same equation connecting a, x, p as in Question 1.

3. (i) If $p\,.\,v^{1\cdot4}=C$, express v in terms of C and p.

(ii) If $p\,.\,v^r=C$, express v in terms of C, p and r.

4. Simplify $\left\{\frac{a+b}{a-b}+\frac{a-b}{a+b}\right\}\div\left\{\frac{a+2b}{a+b}+\frac{a-2b}{a-b}\right\}$.

5. Fill in the gaps in the following:

$$\text{(i)}\ \frac{2x}{y}=\frac{6x}{\cdots}=\frac{\cdots}{5y}=\frac{14x^2}{\cdots}=\frac{\cdots}{6y^2};$$

$$\text{(ii)}\ \frac{a}{b}=\frac{c}{d}=\frac{a+c}{\cdots}=\frac{a-c}{\cdots}=\frac{\sqrt{ac}}{\cdots}.$$

6. The weight of the shell of a d-inch gun (*i.e.* a gun whose internal diameter is d inches) is proportional to d^3. Find the internal diameter of a gun whose shell is twice as heavy as the shell of a 4·7 inch gun.

G. 8.

1. A car will travel n miles on a gallon of petrol, which costs half-a-crown a gallon. Find the smallest value of n for which it will be more economical to make a journey by car, allowing for cost of petrol only, than to travel by train third-class at $1\frac{3}{4}$d. per mile.

2. A golf-ball is dropped from a height of 6 feet on a stone floor, and each time it bounces it rises to a height two-thirds of the previous height. Find how many times it will have bounced before the bounces are less than 2 inches.

3. Find the value of $214\sqrt{H^3 + 0{\cdot}035V^2H^2}$, when $H = 0{\cdot}42$, $V = 0{\cdot}785$.

4. Find x if (i) $10^x = 4{\cdot}619$; (ii) $5^x = 125$; (iii) $\log_{10} x = 1{\cdot}6125$; (iv) $\log_7 x = -2$.

5. Solve the equations: $\begin{cases} x - y = 1{\cdot}6, \\ \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1. \end{cases}$ (Certificate.)

6. It is known that the horse-power required to propel a ship of given pattern varies as $(\text{speed})^3 \times (\text{displacement})^{\frac{2}{3}}$.
A ship of 1000 tons displacement is found to require a certain horse-power at 10 knots; by what factor must this horse-power be multiplied in order to propel a similar ship of 8000 tons at 12 knots? (Certificate.)

G. 9.

1. Two trains make a journey of 200 miles. One train goes 10 miles an hour faster on the average than the other and takes an hour less on the journey. Find the speed of each train.

2. Simplify (i) $\dfrac{2^{n+1} + 2^n}{2^n - 2^{n-1}}$; (ii) $\left(\dfrac{125}{8}\right)^{-\frac{2}{3}}$.

3. If $x : y = 3 : 5$, find the value of

(i) $\dfrac{x+y}{x-y}$; (ii) $\dfrac{x^2+y^2}{x^2-y^2}$.

4. The following table gives values for the pressure and volume of a mass of gas kept at constant temperature:

Volume in c.c. -	47·1	44·0	42·4	40·4	38·4	36·3
Pressure in atmospheres - -	1·47	1·57	1·63	1·71	1·80	1·90

Illustrate these figures by a graph, and read off the volume when the pressure is 1·85 atmospheres.

5. If d is the mean distance of a planet from the sun, then the time of its revolution round the sun or its "year" varies as $d^{\frac{3}{2}}$. If the mean distances of Jupiter and the Earth from the sun are roughly in the ratio 16 to 3, find the length of Jupiter's "year" in days.

6. The central load which breaks a wooden beam l feet long, b inches wide and h inches thick, supported at its ends, varies as $\frac{bh^2}{l}$. A specimen of pitch pine for which $b=1$, $h=1$, $l=2$ just broke under a load of 200 lb.

What load can be carried by a pitch-pine beam 8 feet long, 6 inches wide and 1 foot thick, if the load is not to exceed $\frac{1}{5}$ of the breaking load? (C.S.C.)

G. 10.

1. If y is known to vary inversely as x, complete the following table:

x	4	2	1	$\frac{1}{2}$			0·1	
y	3	6			15	20		100

2. The resistance of a wire varies as the length of the wire and inversely as the square of the diameter. Compare the resistances of two copper wires, one 100 feet long and of diameter $\frac{1}{16}$ inch, the other 50 feet long and diameter 0·1 inch.

3. Given $P=aW+b$. If $P=14$ when $W=6$, and $P=17{\cdot}2$ when $W=8$, find the values of a and b.

4. For what values of x, if any, are the following equations true?

(i) $(2x+1)x=(x+1)x+(x-1)x$;

(ii) $(2x+1)x=(x+1)x+1$;

(iii) $(2x+1)x=(x+2)x+(x-1)x$;

(iv) $(2x+1)(x-1)=x^2+x(x-1)$.

5. OAB are three points in order on a straight line. The distances of O from A and B are a inches and b inches respectively. P is a point in AB dividing AB so that the ratio $AP:PB=k:1$. Find the distance of O from P in terms of a, b and k. (Certificate.)

6. A clear electric lamp is found to illuminate a horizontal desk satisfactorily when it is 6 feet above it. When the glass of the lamp is frosted, its illuminating power is diminished by

25 per cent. If the illumination from a light varies inversely as the square of the distance, how much lower must the frosted lamp be hung to illuminate the desk as much as before ?

(Certificate.)

H.

(CHAPTERS XII.-XVI.)

H. 1.

1. (i) For what values of x is the function $(x-5)(x-6)$ equal to 0 ?

(ii) What sort of value does this function have (*a*) when x is large and positive, say $+100$? (*b*) when x is large and negative, say -100 ?

(iii) Between what values of x is this function negative, and what can you say about its value then ?

(iv) Sketch roughly without squared paper the graph of this function.

2. (i) Find a value of x for which the function $\frac{1}{x-6}$ is

(*a*) positive and $< 0{\cdot}001$,

(*b*) negative but $> -0{\cdot}001$.

(ii) Find a value of x for which this function is

(*a*) positive and > 1000,

(*b*) negative and < -1000.

3. Solve the equation $\frac{x}{x+2}+\frac{2x}{2x+1}=1$.

4. Find the value of $a^{\frac{1}{2}}b^{-\frac{3}{4}}$, when $a=0{\cdot}523$, $b=1{\cdot}467$.

5. The weight of a column of mercury in a tube varies as the length and the square of the radius of the tube. If a column 10 cm. long in a tube, diameter 1 cm., weighs about 107 grams, find the weight of a column 45 cm. long and diameter 5 mm.

6. The ratio of a man's income to his expenditure for one year is 5 to 4. Next year his income remains unaltered, but his expenditure is increased by £100, so that the ratio then is 6 : 5. Find his income.

H. 2.

1. Make a table of values of the function $3+4x-x^2$ for values of x from -1 to 5. Draw an accurate graph of the function between these values of x, and from it read off the answers to the following questions :

(i) What is the maximum value of the function ?

(ii) For what value of x is the function a maximum ?

(iii) Between what values of x is the function positive ?

(iv) For what values of x is the function equal to 4 ?

(v) Solve the equation $3+4x-x^2=6$.

(vi) Solve the equation $x^2-4x-4=0$.

2. Fig. 94 shows a circular arch, height h and base $2a$. If r is the radius of the complete circle, obtain an expression for r in terms of a and h, and find the value of r when $a=50$ ft. and $h=10$ ft.

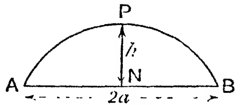


FIG. 94.

3. Find the value of x when

(i) $2^x=4^3$; (ii) $9^x=(3^3)^{\frac{2}{5}}$; (iii) $10^x=\frac{1}{\sqrt{10}}$.

4. Find correct to 2 significant figures two numbers whose sum is 2, such that the sum of their squares is 3.

5. If $\frac{x}{y}=\frac{7}{10}$, find the value of

(i) $\frac{x+y}{x-y}$; (ii) $\frac{x+2y}{2x-y}$.

6. The force of attraction between two magnetic poles varies inversely as the square of the distance between them. If two magnetic poles attract each other with a force of 5·6 dynes, when 5 cm. apart, find the force of attraction between them when they are 7·5 cm. apart.

H. 3.

1. For what values of x is the function $\frac{x^2}{100}=0$? Can the function $\frac{x^2}{100}$ ever be negative ? What is its value when x is large, *e.g.* when $x=\pm 100$? Sketch roughly the graph of this function.

2. The volume of a gas varies as the absolute temperature and inversely as the pressure. When the pressure is 15 lb. per square inch and the absolute temperature is 280°, the volume of a gas is 200 cubic inches. Find its volume when the pressure is 10 lb. per square inch and the absolute temperature 300°.

3. In Fig. 95, semi-circles are drawn on AN, NB and AB as diameters. NP is perpendicular to AB, and it is known that NP is a mean proportional between AN and NB. Prove that the area shaded in the figure is equal in area to a circle drawn on PN as diameter.

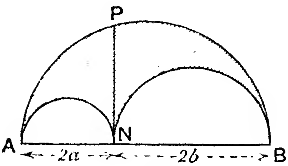


FIG. 95.

4. The ratio of the incomes of two men is 5 to 4. The income of each is raised by £150 a year and the ratio of their incomes is then 11 to 9. Find their incomes before they were raised.

5. Plot the graph of the function $\frac{x^2}{2}+\frac{1}{2x}$ from $x=3$ to $x=-3$, and from it read off the roots of the equation $\frac{x^2}{2}+\frac{1}{2x}=3$.

(Certificate.)

6. (i) Find the error in assuming $\frac{1-x}{1+x}$ to be equal to $1-2x$ if $x=0\cdot1$.

(ii) Find the ratio of the error to the correct value.

(iii) Find the "percentage error."

Give all answers correct to two significant figures.

H. 4.

1. Find a general statement to include the following:

the ratio 3 : 4 is greater than the ratio 2 : 3;
the ratio 6 : 7 is greater than the ratio 5 : 6;
the ratio 100 : 101 is greater than the ratio 99 : 100.

Find a value of x for which the ratio $\frac{x}{x+1}$ is greater than $0\cdot999$.

2. What is the value of the function $\frac{x}{x+1}$, (*a*) when $x \to +\infty$, (*b*) when $x \to -\infty$, (*c*) when $x \to -1$, but is < -1, (*d*) when $x \to -1$, but is > -1? For what value of x is the function zero? Draw a rough graph of this function without using squared paper.

3. Programmes are sold at a charity concert for "a shilling and anything more you like to give." A programme seller finds that everyone gives him either a shilling or a halfcrown for a programme. He sells 64 programmes and takes in all £5. 13s. 6d. Find how many people paid half a crown for their programme and how many paid a shilling.

4. For what values of x, if any, are the following true ?

(i) $(x+1)^2+(x+2)^2=2(x+1)(x-2)+1$;

(ii) $(x+1)^2+(x+2)^2=2(x+1)(x+2)+2$.

5. Find the value of $4{\cdot}75H^3d^{-4}$, when $H=1{\cdot}723$, $d=0{\cdot}1614$.

6. Sketch the time-velocity graph of a train which starts from rest and gathers speed at a uniform rate for five minutes, after which it travels steadily at 40 miles an hour for ten minutes and then slows down uniformly so as to come to rest in five minutes.

Unit for time-axis, 1″ : 5 minutes;
unit for velocity axis, 1″ : 20 miles an hour.

How far has it travelled altogether ? Suppose the graph has been accurately drawn with these units. How would a person describe the motion who thought

the time-axis unit was 1″ : 2 minutes
and the velocity-axis unit was 1″ : 5 miles an hour ;

and what would he give as the total distance travelled ?

H. 5.

1. Sketch roughly the graph of the function $x(x-2)(x-4)$.

2. Draw accurate graphs of the functions $\frac{x^2}{10}$ and $\frac{1}{x}$ on the same piece of paper and with the same scales, for values of x between 6 and -6.

For what values of x are these two functions equal ? Read off this value, (i) from the graph, (ii) using log tables.

3. If $f(x)=x^2+7x-6$, find the values of $f(1)$, $f(0)$, $f(2a)$ and $f(x+1)$.

4. Lidless rectangular tins with a square base are to be made to hold 12 cubic inches.

(i) Express the height, h'', as a function of the side, x'', of the base.

(ii) Express the surface, S square inches, as a function of h and x, and then of x only, showing that $S=x^2+\frac{48}{x}$.

Plot the values of S when x has values from 1 to 6, and hence find the value of x for which S is a minimum.

5. Simplify $\frac{3(x+h)^3 - 3x^3}{h}$, and find its value when $x = 1$, $h = 0{\cdot}001$, correct to 3 figures.

6. Find the error in assuming $(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x$, when $x = 0{\cdot}1$. Find also the percentage error. Give both answers correct to one significant figure.

H. 6.

1. If $f(x) = x^2 - \frac{10}{x}$, what is the value of $f(1), f(10), f(0{\cdot}1)$? Has $f(0)$ any meaning? What are $f(-0{\cdot}1)$, $f(-0{\cdot}01)$?

2. (i) When is the function $\frac{x-3}{x+3}$ equal to zero?

(ii) Find values of x for which this function is (a) > 1000 (b) < -1000.

(iii) To what value does this function tend

(a) when $x \to +\infty$?

(b) when $x \to -\infty$?

(iv) Sketch the graph of this function.

3. The following table gives the heights in feet of certain points on a road above sea-level, and their horizontal distances in yards from the starting point:

Height h	100	112	127	139	140	138	130
Distance d	0	50	100	150	200	250	300

Show these results on a graph. Calculate the average gradient

(a) for the first 50 yards horizontally,

(b) for the first 100 yards horizontally.

By drawing a tangent, estimate the gradient at the point 100 yards distant from the starting point.

4. The strength of a rectangular beam of given material varies as the product of the breadth and the square of the depth of the cross-section. Find the breadth of a beam 4 inches deep which is twice as strong as a beam of the same material which is 4 inches broad and 3 inches deep. (Certificate.)

5. The area of a right-angled triangle is 7·5 square inches and its hypotenuse is 6·5 inches. Find the lengths of its sides.

6. Simplify and find the value, correct to 3 figures, of

(i) $\frac{\sqrt{32}}{2\sqrt{2}}$; (ii) $\frac{\sqrt[3]{81}}{\sqrt[3]{3}}$; (iii) $\frac{10}{\sqrt{5}}$; (iv) $\sqrt{6} \times \sqrt{30}$.

[Take $\sqrt{5} = 2{\cdot}236$.]

H. 7.

1. An underground train travels from one station to the next, a distance of 1000 yards, in 80 seconds in accordance with the following table :

Time in seconds	10	20	30	40	50	60	70	80
Distance travelled in yards	50	135	260	435	610	780	900	1000

Draw a graph to illustrate these numbers.

Calculate (i) the average speed for the whole journey;
(ii) the average speed for the first 20 seconds;
(iii) the average speed for the period 20 sec. to 40 sec.

By drawing a tangent, estimate the actual speed 20 seconds after the train started. Give all results in the form feet per second.

2. Sketch roughly without squared paper the graph of the functions (i) $\pm\sqrt{x^3}$, (ii) $\pm x^{\frac{5}{2}}$.

3. If $f(x)=\frac{x^2}{2}+1+\frac{1}{2x^2}$, (i) prove that $f(x)=f\left(\frac{1}{x}\right)$, and write down another function which has this property. (ii) Also show that $f(x)=f(-x)$. What does this tell you about the graph of the function ?

4. If $x^3=175t^2$ and $t^5=y$, express x in terms of y only, and find the value of x, when $y=0{\cdot}0162$, correct to 3 figures.

5. The sides of a rectangle are in the ratio 3 : 2, and its area is 54 square inches. Find its sides.

6. Find a general statement to include the following facts :

the ratio 36 : 25 is greater than the ratio 6 : 5,
the ratio 81 : 49 is greater than the ratio 9 : 7,
but the ratio 16 : 49 is less than the ratio 4 : 7,
and the ratio 9 : 25 is less than the ratio 3 : 5.

H. 8.

1. Plot values of the function $6x-x^2$ as x varies from -1 to $+7$, and draw an accurate graph.

What is the change in the function per unit change in x
(i) as x changes from 1 to 3 ?
(ii) as x changes from 1 to $1{\cdot}1$?
(iii) as x changes from 1 to $1+h$?

What is the limit of the last expression as $h\to 0$?

Draw a tangent to the graph at the point given by $x=1$; estimate its gradient, and compare with the result obtained by calculation.

2. If $f(x)=x^2+6$, find the value of $f(1)$ and $f(-1)$.

Write down the value of $\frac{f(x+h)-f(x)}{h}$, and simplify this expression. To what limit does it tend, when $h \to 0$?

3. The expenses of a hotel-keeper are partly constant and partly vary directly as the number of visitors in the hotel. When there are 10 visitors, the weekly expenses are £55, and when there are 30 visitors, the weekly expenses are £135. Express the weekly expenses as a function of the number (n) of visitors in the hotel.

4. Sketch free-hand the graph of a function, the *average* gradient of which is given in the following table:

x	0	0 to 1	1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6
Average gradient	0	1	2	$\frac{1}{2}$	-1	-2	$-\frac{1}{2}$	0

The graph is to be a smooth curve.

5. A line AB, 4″ long, is produced to Q so that $AQ:BQ=5:4$; find the length of AQ.

6. If $x=\frac{a-1}{a+1}$ and $y=\frac{2a-1}{2a+1}$, find a in terms of x only, and then in terms of y only, and so obtain an equation connecting x and y and not involving a.

H. 9.

1. Make a table of values of the function $y=x^2+2x-3$ from $x=-4$ to $x=+2$; draw an accurate graph, and read off answers to the following questions:

(i) What is the minimum value of the function ?

(ii) For what value of x is the function a minimum ?

(iii) Between what values of x is the function negative ?

(iv) For what values of x does the function equal 4 ?

(v) Solve the equations : (a) $x^2+2x-3=1$,
(b) $x^2+2x \quad =2$.

2. With the function and graph of Question 1, what is the average change in x per unit change in y

(i) When x changes from -4 to -2?

(ii) When x changes from -3 to -2?

(iii) When x changes from $-(2+h)$ to -2?

(iv) What is the limit of this last expression as $h \to 0$?

(v) Draw a tangent to the graph at the point given by $x = -2$, and estimate its gradient. Compare with the answer to part (iv).

3. If $y = f(x) = \dfrac{ax+b}{cx-a}$, prove that $x = f(y)$.

4. Solve the equation $4{\cdot}5x^2 + 3x - 6{\cdot}7 = 0$, giving results correct to 3 significant figures.

5. Simplify
(i) $(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})$;
(ii) $(\sqrt{5}+\sqrt{2})^2$;
(iii) $(\sqrt{5}+\sqrt{2})^{-1}$,

and give the values of these expressions correct to 3 significant figures.

6. If $f(x)$ denotes the greatest integer in x, sketch the graph of (i) $f(x)$, (ii) $x - f(x)$, when x is positive.

H. 10.

1. Make a table of values of the function $\frac{1}{10}(x^3 - 4x)$ for values of x from -4 to $+4$; draw an accurate graph of the function, and read off answers to the following questions:

(i) For what positive value of x is the function a minimum?

(ii) What is the maximum value of the function for negative values of x?

(iii) For what values of x is the function negative?

(iv) For what values of x does the function equal $0{\cdot}2$?

(v) Solve the equations (*a*) $\frac{1}{10}(x^3 - 4x) = 0{\cdot}1$,
(*b*) $x^3 - 4x = 1$,
(*c*) $x^3 - 4x + 1 = 0$.

2. Find the gradient of the graph of Ex. 1 at the point given by $x = 2$, (*a*) by drawing a tangent to the graph,
(*b*) by calculation.

3. Sketch roughly the graph of the function $(x-4)^2(x-3)$.

4. $f(x)$ is a function of x such that $f(x) \times f(y) = f(x+y)$, for all values of x and y; and $f(1) = 2$. Prove that $f(2) = 4$, $f(3) = 8$, and find the values of $f(0)$ and $f(\frac{1}{2})$.
What is the simplest form of this function of x?

5. Prove that if a, b, c are three consecutive integers, then $b^2 = ac + 1$.

6. A sum of a guinea can be made up of either a shillings and b florins, or of a sixpences and b half-crowns. Find the values of a and b.

K.

(CHAPTERS XII.-XVII.)

K. 1.

1. Soundings are made on a sea-shore at low tide to discover the shape of the sea-bottom, and the following measurements are recorded:

Distance from low water mark in yards - - - - - -	50	100	150	200	250	300
Depth of water in fathoms - -	0·7	2	4	4·5	5	5·1

Draw a graph to illustrate these figures.
Estimate the average gradient of the sea-bottom
(i) For the first 300 yards out.
(ii) Between points 100 yards and 250 yards out.
(iii) Draw a tangent and find the gradient at a point 150 yards out. [1 fathom = 6 feet.]

2. Find the value of (i) $\underset{h \to 0}{\text{Lt}} \frac{1}{h}\{(x+h)^2+4(x+h)-x^2-4x\}$.

(ii) $\underset{h \to 0}{\text{Lt}} \frac{1}{h}\left\{\frac{1}{x+h-1}-\frac{1}{x-1}\right\}$.

3. Find, by any method, the gradient of the graph of the function x^3-5x.
(i) At the point where $x=1$.
(ii) At the point where $x=a$.

4. Sketch roughly a graph of the function $(x-2)^3(x-4)$.

5. If $x=t+1$ and $y=t^2+t$, express (i) t as a function of x, (ii) y as a function of x only. For what values of t are x and y equal?

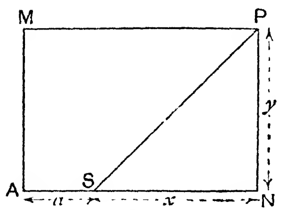


FIG. 96.

6. In Fig. 96 AS is a constant length a. P is a variable point, $PNAM$ is a rectangle such that $PS=PM$. If $PN=y$ and $NS=x$, find an equation connecting x and y, and express y as a function of x.

K. 2.

1. Draw roughly with the same axes, the graphs of

(i) $y = 3 - x$. (ii) $y = 5 - x$. (iii) $y = 5$.

What are the gradients of these lines ?

2. Differentiate the following functions with respect to x.

(i) $4x - 3x^2$. (ii) $3 + 4x - 3x^2$.

(iii) $6 + 4x - 3x^2$. (iv) $\frac{3}{2}x + \frac{2}{5}x^2$.

(v) $6x^{-1}$. (vi) $8\sqrt{x}$.

3. (i) Find the gradient of the graph $y = 7 + 6x + 3x^2$ at the point $x = 2$.

(ii) At what point is the gradient equal to 1 ?

4. y is known to be a function of x of the type $mx + c$, m and c being constants. If $y = 43$ when $x = 17$, and $y = 34$ when $x = 14$, express y as a function of x.

5. Prove that, if a, b and n are all positive then the ratio $(a+n) : (b+n)$ is greater or less than the ratio $a : b$ according as a is less or greater than b.

[Consider the value of $\frac{a+n}{b+n} - \frac{a}{b}$, and simplify it.]

6. With the figure and data of paper E. 2, Question 2, find the maximum sag of the rod. Find also the inclination of the rod to the horizontal at the support B.

K. 3.

1. Find $\frac{dy}{dx}$ (i) when $y = 4x^2 - 3 + \frac{1}{x}$,

(ii) when $y = 16x^3 - 4x^2 + 2x - 5$,

(iii) when $y = 5x^{\frac{1}{3}} + x^{-\frac{1}{3}}$.

2. Find the gradient at $x = a$ of the graph

$$y = x^3 - 6x^2 - 15x + 5.$$

Find for what values of x the gradient of this graph is zero. Sketch the graph roughly.

3. Sketch roughly the graph of the function $(x-3)\sqrt{x-2}$, where the positive value of $\sqrt{x-2}$ is always taken.

4. The distance, s feet, of a particle moving in a straight line from a fixed origin is given by $s = 4 + 6t + 3t^2$, where t is the number of seconds for which it has been moving. Find its velocity, (i) 2 seconds, (ii) 10 seconds after it started.

5. A particle moves in a straight line in such a way that its velocity after any time t varies inversely as its distance x from the origin. Write down a differential equation connecting $\frac{dx}{dt}$ and x.

6. If $f(x) \equiv x^3 - 5x^2 + x + 7$ and if $f(y+a)$ when expanded contains no term in y^2, find a.

K. 4.

1. Find the value of (i) $\underset{h \to 0}{\text{Lt}} \left\{ \dfrac{\dfrac{1}{(x+h)^2 - 2} - \dfrac{1}{x^2 - 2}}{h} \right\}$.

(ii) $\underset{n \to \infty}{\text{Lt}} \dfrac{n^2+1}{2n^2+1}$.

2. Differentiate the following with respect to x:

(i) $\sqrt{6x}$. (ii) $6\sqrt{x}$. (iii) $\frac{4}{x^3}$. (iv) $x^2 + \frac{1}{x^2}$.

3. For what values of x is the function $x^3 - 12x$ equal to zero? Find its derived function and find for what values of x the derived function is zero. From these data sketch a rough graph of the function.

4. There are 15 cubic feet of water in a bath. The water is allowed to run out and after t seconds θ cubic feet of water have run out. If the rate at which the water runs out is proportional to the quantity of water left in the bath, find a differential equation connecting $\frac{d\theta}{dt}$ and θ.

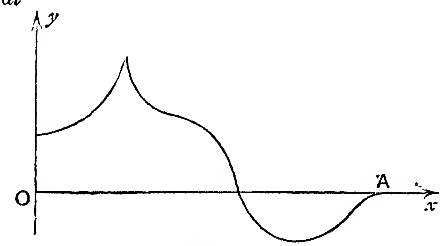


FIG. 97.

5. Fig. 97 represents the graph of $y = f(x)$, unit 1 cm. on each axis; describe in general terms the variation in the values of (i) y; (ii) $\frac{dy}{dx}$; (iii) $\frac{d^2y}{dx^2}$ as x varies from 0 to OA.

6. The force of attraction between two planets varies directly as their masses and inversely as the square of the distance between them. Compare the force with which the planets Jupiter and Mars attract the earth, at a time when Jupiter is 500 million miles away, and Mars is 150 million miles away, if the mass of Jupiter is about 340 times that of the Earth, and the mass of Mars is about one-seventh that of the Earth.

K. 5.

1. Differentiate the following functions of t with respect to t:

(i) $4t + \frac{4}{t}$. (ii) $3t^3 - 4t^{-1} + 5t^{-3}$. (iii) $5t^{\frac{5}{2}}$.

2. Differentiate the function $x^3 - 27x$. For what values of x is its rate of change zero? What are the maximum and minimum values of this function. Sketch its graph.

3. The weight of a certain solid is given by the formula

$$10x^2(15 - 2x),$$

where x is variable. Show that as x increases from 0 to 5 the solid gets steadily heavier. What happens after that?

4. If $R = 0{\cdot}0075(1 - e^{-0{\cdot}2t})$, find R when $t = 10$, if $e = 2{\cdot}718$. Find also the value of R when $t = 20$.
What is the average rate of change of R per unit change in t, as t changes from 10 to 20.

5. If $f(x) = x^2 + 3x$, find the value of

(i) $\dfrac{f(x+h) - f(x)}{h}$.

(ii) $\dfrac{f(x+h) - f(x) - \{f(x) - f(x-h)\}}{h^2}$.

Find the limits to which these two expressions tend, as $h \to 0$.

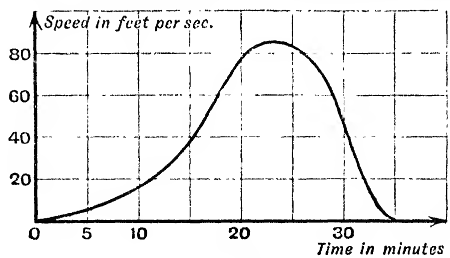


FIG. 98.

6. Fig. 98 represents the time-velocity graph of a train, units as shown on diagram.
Construct the distance-time graph and write down the distance travelled, (i) at the end of 15 minutes, (ii) in the whole journey.

K. 6.

1. Find the turning points of the function x^3+3x^2-9x+6 and determine whether they are maxima or minima.

2. Show that the function x^3+5x-6 steadily increases, as x increases from $-\infty$ to $+\infty$.

One root of the equation $x^3+5x-6=0$ is $x=1$. Show why it cannot have another real root.

3. A piece of wire 8 inches long is bent so as to form a rectangle. If one side of the rectangle is x'', express the area as a function of x, and find the value of x for which the area is a maximum.

4. (i) Express $5(p^2+q^2)$ as the sum of two squares.

(ii) Express 5^4 as the sum of two squares.

5. AB is a straight line 6 inches long. M is a point on the line between A and B, and N is a point in AB produced, such that the ratio $AM : MB = AN : BN = 6 : 4$. Find the length of MN.

6. The distance, s feet, that a stone falls is proportional to the square of the time, t seconds, for which the stone has been falling. Express this fact as an equation connecting s and t, and prove that the velocity of the stone varies directly as the time for which it has been falling.

K. 7.

1. Differentiate the function $5-3x-2x^3$ and show that the function steadily decreases as x varies from $-\infty$ to $+\infty$, and that it can only be zero, when $x=1$. Draw roughly the graph of this function.

2. A particle moves along the straight line OA. B is a point on OA such that $OB=5$ feet, and the particle starts from B and moves away from O in such a way that its velocity is proportional to its distance from O. If its initial velocity is 10 feet per second, find an equation connecting $\frac{dx}{dt}$ and x, where x feet is its distance from the starting point B, and t seconds is the time for which it has been moving.

3. The volume of a cone is $\frac{1}{3}\pi r^2 h$ and its curved surface is πrl, where r is the radius, h the vertical height and l $(=\sqrt{h^2+r^2})$ the slant height of the cone. A conical tent is to be made to hold 400 cubic feet of air. Express its height as a function of r, and then express its surface S, as a function of r. Find the value of S, if $r=7$. For what value of r is s least ?

4. Draw a rough sketch of the graph of the function $18x^2 - x^4$, and find its maximum and minimum values. (Certificate.)

5. If $x : y = 7 : 6$, find the value of

$$\text{(i) } \frac{3x+5y}{6x+4y}. \qquad \text{(ii) } \frac{x^2+y^2}{x^2-y^2}.$$

6. The tension of a wire of given material varies as the square of its length if it is to produce a given musical note. What must be the tension in a wire one foot long to produce the same note as a wire 9″ long at a tension of 11 lb. ?

K. 8.

1. Tabulate the values of the function $x + \frac{4}{x}$, as x varies from -4 to $+4$. Draw an accurate graph of this function and read off the values of x for which this function is a maximum or minimum.

Verify your answers by differentiating the function, and calculating the values of x required.

2. Draw a tangent to your graph (Question 1) at the point $x = 3$, and estimate its gradient. Check your answer by calculating the value of $\frac{dy}{dx}$ when $x = 3$.

3. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$,

(i) when $y = 3x^3 - 7x$; (ii) when $y = 5x^2 - 4$;

(iii) when $y = 7x + 6$; (iv) when $y = 5$;

(v) when $y = \frac{6}{x}$.

4. Draw a rough sketch of the graph of the function $75x - 4x^3$ and find its maximum and minimum values. (Certificate.)

5. A line AB, 4 inches long, is divided at the point P, in such a way that the square on AP is 1 square inch larger than the square on PB. Find the length of AP.

6. A rectangular plate 20 inches long and $3x$ inches wide has a circular hole, radius x inches, cut out of it. Express the area of the remaining portion as a function of x, and find the value of x for which the area remaining is a maximum.

K. 9.

1. (i) Is $\dfrac{x^2-1}{x-1}$ always equal to $x+1$?

(ii) Can you find a value of x for which $\dfrac{x^2-1}{x-1}$ equals 2 ?

(iii) Find the value of $\underset{x \to 1}{\text{Lt}} \dfrac{x^2-1}{x-1}$.

2. A bullet fired vertically upwards rises s feet in t seconds, where $s = 500t - 16t^2$. (i) Find its height after 10 sec., 11 sec., 10·1 sec. ; (ii) What is its average speed in the intervals 10 to 11 sec. and 10 to 10·1 sec.? (iii) What is its average speed in the interval 10 to $10+h$ sec.? (iv) What is its velocity after 10 sec. ?

3. If $y = 2x^2$ and $x = 3z + 1$, express (i) δy in terms of δx ; (ii) δx in terms of δz ; (iii) δy in terms of δz. Hence find $\dfrac{dy}{dz}$.

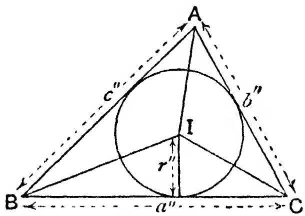


FIG. 99.

4. I is the centre and r the radius of a circle inscribed in the triangle ABC. Show by considering the areas of the triangles AIB, BIC, CIA, that the area of the whole triangle is equal to rs, where $s = \dfrac{a+b+c}{2}$. Hence find the radius of the circle inscribed in the right-angled triangle whose sides are 3″, 4″ and 5″.

5. If y varies as x^3, complete the table :

$x=2$	4	30
$y=5$		

6. Express $\left(\dfrac{1}{x}-\dfrac{b}{a}\right) \div \left(\dfrac{1}{x}+\dfrac{a}{b}\right)$ in a form not containing x, $a(1-x) = b(1+x)$.

K. 10.

1. If $\frac{a}{xy}=\frac{b}{x^2}=\frac{c}{yz}$, express the ratio $x:y:z$ in terms of a, b, c.

2. The sides of a triangle are 5, 5, $2x$ cm.; its area is A sq. cm. Express A as a function of x. Represent this function by a graph and find from the graph the maximum area of the triangle. (C.S.C.)

3. A hoop of radius b feet is bowled along with velocity v feet per sec. and comes to a step of height h feet. The hoop will climb the step if $v^2 > \frac{128bh}{(2b-h)^2}$. If the velocity of the hoop is 8 feet per sec., and if its radius is 18 inches, find to the nearest inch the height of the tallest step it can climb. (Certificate.)

4. Ox, Oy are horizontal and vertical lines: the graph of $y=x^2$ represents a hill side on the scale, unit for x-axis $=100$ yards, unit for y-axis $=1$ foot. What is the average slope of the hill, (i) from $x=1$ to $x=2$; (ii) from $x=1$ to $x=1\cdot1$; (iii) from $x=1$ to $x=1+h$. What is the gradient of the hill at $x=1$?

5. A, B, C are three points on Ox; APB is a semicircle above Ox and BQC a semicircle below OX; the two semicircles form part of the graph of $y=f(x)$. Describe the changes of sign of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ between A and C.

6. An open cardboard box with square ends is fitted with an overlapping lid which covers the open top, the whole of two square ends and one other side. The total area of the cardboard is 8 sq. feet. What is the maximum volume of the box?

K. 11.

1. The highest and lowest marks in an examination were 72 and 17. These were scaled so that the highest and lowest were 200 and 100. What was the mark corresponding to an original mark of 53?

2. Prove that $2^{\sqrt{3}}-2^{-\sqrt{3}} \simeq 3$.

3. A road of constant width curves through a right angle, the walls being concentric circular arcs AB, CD. The radius of the inner curve is 60 yards and it is just possible to see D from C. How wide is the road, to the nearest yard? (Certificate.)

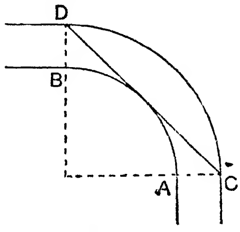


FIG. 100.

4. The electromotive force of a certain type of cell has been found to vary with the temperature as follows:

Temperature (Centigrade)	15	20	25	30
E.M.F. in volts - - -	1·4340	1·4284	1·4233	1·4188

What is the average fall in E.M.F. per degree of temperature, when the temperature rises from (i) 15° C. to 30° C.; (ii) 15° C. to 25° C.; (iii) 15° C. to 20° C.? Illustrate the table by a graph and by drawing a tangent estimate the rate of fall of E.M.F. at 20° C.

5. Find $\dfrac{dy}{dx}$ if (i) $y=\dfrac{x^3-1}{x}$; (ii) $y^3=2x^2$.

6. If a steamer travels at v knots, the cost of a certain journey is $8\left(\dfrac{1500}{v}+0{\cdot}4v^2\right)$£. What is the most economical speed?

K. 12.

1. (i) If y varies inversely as the square of x, what is the effect on y of increasing x in the ratio 4 : 1?

(ii) The illuminating power of a light varies inversely as the square of the distance. Which gives the better light, a 10 candle power lamp, 3 feet away, or an 18 candle power lamp, 4 feet away?

2. Simplify the result of substituting $x=\dfrac{3ab}{b-a}$ in

$$\frac{1}{x-2a}+\frac{2}{x+b}+\frac{1}{b}.$$

3. From the formula $p=475v^{-1{\cdot}06}$, find p when $v=0{\cdot}874$.

4. (i) Plot the function $x+(-1)^x$ for positive integral values of x.

(ii) What is the least integral value of x, for which and for all greater values of x, $x+(-1)^x>100$?

(iii) What is $\underset{x\to\infty}{\text{Lt}}\,[x+(-1)^x]$?

5. If $y=ax^2+bx$, prove that $x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=0$.

6. ABP is a triangle, right-angled at B, having $AB>BP$; $AB=12''$; the perpendicular bisector of AP cuts AB at Q. Find the length of AQ if triangle PBQ is of maximum area.

K. 13.

1. A regular pyramid, height h in., stands on a square base of side $2x$ in. and is used as a metal funnel to hold 48 cu. in. Assume the formulae: volume $= \frac{4}{3}hx^2$; surface of slant sides $= y = 4x\sqrt{x^2+h^2}$; and find the values of h and y when $x = 1, 2, 3, 4, 5$. Plot y against x and find from the graph the value of x which makes y as small as possible. (C.S.C.)

2. Use calculus methods to solve Question 1.

3. The following table gives the best wheel bases for trucks which have to travel on a curved track.

Radius of curve in feet $= r$	20	30	40	50	60
Wheel base in inches $= b$	66	69		75	78

What simple relation connects b and r? Complete the table.

4. (i) Find the circumference of a circle whose area is 3·85 sq. in.

(ii) The area of a circle is x sq. in., the circumference is y in.; express y as a function of x.

5. For what values of x is the function $108x - x^4$, (i) positive, (ii) an increasing function, (iii) a maximum? Sketch its graph.

6. A closed vessel which tapers to a point at its top and bottom is such that when the depth of liquid in it is x feet, the volume of the liquid is $12x^2 - x^3$ cu. feet. Find (i) its internal height; (ii) the area of its cross-section halfway up; (iii) the total volume.

K. 14.

1. If $\frac{x}{3} = \frac{y}{5}$, prove that $\frac{x^2 - y}{x - y^2} = \frac{3x - 5}{3 - 5y}$.

2. Kepler's Third Law states that the square of the time a planet takes for one revolution round the sun varies as the cube of its mean distance from the sun. In the following table, the mean distance of the earth from the sun (*i.e.* 93,000,000 miles) is taken as the unit of distance.

	Earth	Mercury	Venus	Mars	Jupiter	Uranus
Time in years	1	0·24	0·62	1·88		
Mean distance	1	0·38		1·52	5·20	19·18

Verify Kepler's Law and complete the table.

3. (i) Solve $x\sqrt{3}-x=4$.

(ii) Find the value of $x-\frac{1}{x}$ if $x=\sqrt{2}-1$.

4. Use logarithm tables to evaluate $x \times \delta(\log x) \div \delta x$, when (i) $x=5$, $\delta x=0 \cdot 1$; (ii) $x=7$, $\delta x=0 \cdot 1$; (iii) $x=9$, $\delta x=0 \cdot 1$. What do you infer from these results ?

5. Draw freehand the graph of a function $y=f(x)$ for which

(i) over a portion AB $\frac{dy}{dx}$ is positive and $\frac{d^2y}{dx^2}$ is negative ;

(ii) over a portion CD $\frac{dy}{dx}$ is negative and $\frac{d^2y}{dx^2}$ is positive.

6. A circular electric current of radius 1 foot exerts a force P on a small magnet placed with its axis perpendicular to the plane of the wire and at a distance x feet from the centre where P varies as $\frac{x}{(1+x^2)^{\frac{5}{2}}}$. For what value of x is P greatest ?

K. 15.

1. The weight of rails, W lb. per yard, necessary when the maximum velocity is v miles per hour and the greatest load on the driving wheel is L tons, is given by

$$W=17\sqrt[3]{\{L+0 \cdot 0001Lv^2\}^2}. \quad \text{Find } W \text{ if } L=8 \cdot 3,\ v=47.$$

2. If V varies directly as the square of x and inversely as y, complete the given table.

Values of x.

		1	2	3
	1			
Values of y	2			270
	3		80	

3. If $\frac{1}{x}+\frac{1}{y}=\frac{2}{r}$, express $\frac{1}{r-x}+\frac{1}{r-y}$ in terms of x, y.

4. From the given table evaluate $\frac{1}{10^x} \times \delta(10^x) \div \delta x$ for $10^x=5$ and $10^x=6$ and $10^x=8$:

10^x	5	5·0001	6	6·0001	8	8·0001
x	0·6989700	0·6989787	0·7781513	0·7781585	0·9030900	0·9030954

What do you infer from these results ?

5. If $xy=5$, find

(i) $\frac{dy}{dx}$ in terms of x; (ii) $\frac{dx}{dy}$ in terms of x; (iii) $\frac{dy}{dx}\times\frac{dx}{dy}$.

6. What is the greatest value of $3x-5x^2y$ if $xy^2=4$ and y is positive?

K. 16.

1. (i) Express with positive indices $\frac{a^2b^{-3}}{2}$.

(ii) Evaluate $8^{1\frac{1}{3}}$; $16^{1\frac{1}{4}}$; $(0\cdot25)^{-\frac{1}{2}}$; $10^{-2}\times10^{-1}$.

2. A man pays income tax only on part of his income; the tax is y per cent. of the portion subject to tax, which works out as equivalent to x per cent. of his total income. Find the ratio of the portion subject to tax to the portion free of tax.

3. The perimeter of a right-angled triangle is 20 cm.; the lengths of the two shorter sides are a cm., b cm., and the area is A sq. cm. (i) Prove that $(20-a)(20-b)=200$; (ii) express A in terms of a, and draw a graph showing the relation between A and a as a varies from 0 to 10 cm.; (iii) find from the graph the maximum area of the triangle. (C.S.C.)

4. If $f(x)=x^2-3x+5$, (i) express $\frac{f(x+h)-f(x)}{h}$ in terms of x, h; (ii) find the limit of this expression when $h\to0$; (iii) for what value of x is this limit zero; (iv) interpret the results of (i), (ii), (iii) geometrically.

5. If $y=\frac{1}{z}$ and $z=1-x^2$, (i) express δy in terms of z, δz; (ii) express δz in terms of x, δx; (iii) express δy in terms of x, δx; (iv) find $\frac{d}{dx}\left(\frac{1}{1-x^2}\right)$.

6. The function $y=3x+\frac{a}{x}$ decreases as x increases from 0 to 5, and increases as x increases from 5 to 10. (i) What is the value of a? (ii) what is the value of $\frac{d^2y}{dx^2}$ when $x=5$ and $x=-5$? (iii) find maximum and minimum values of the function, distinguishing between them; (iv) explain your results by a sketch of the function.

L.

(CHAPTERS XII.-XVIII.)

L. 1.

1. A beam AB supported at its ends A, B carries a load at P. The greatest load the beam can stand varies directly as its breadth, its length and the square of its depth, and inversely as AP and PB. The breaking load for a chestnut beam 20 feet long, 8 in. wide, 5 in. deep at 2 feet from its mid-point is $2\frac{3}{4}$ tons. Obtain a general formula for any chestnut beam.

2. In what ratio is the expression $\frac{0 \cdot 043x^3y}{x+y}$ altered if x and y are each altered in the ratio 2 : 3 ?

3. (i) Describe the changes of sign in the function $\frac{3(x-2)(x-4)}{(x-1)(x-5)}$ when x increases from -1 to 6.

(ii) Find correct to one significant figure its value when $x = 1 \cdot 0001$.

(iii) Can you find a value of x for which the function equals 3 ?

(iv) Find by logarithms the value of the function when $x = 1000$.

(v) Simplify $\frac{3(x-2)(x-4)}{(x-1)(x-5)} - 3$, and evaluate

$$\underset{x \to \infty}{\mathrm{Lt}} \frac{3(x-2)(x-4)}{(x-1)(x-5)}.$$

(vi) Sketch the graph of the function for positive values of x.

4. Find correct to two significant figures the value of x for which $\sqrt{x} - x^{1 \cdot 06}$ is a maximum.

5. (i) Solve $\frac{dy}{dx} = x - \sqrt{x}$, given $y = 1$ when $x = 1$.

(ii) Solve $\frac{d^2y}{dx^2} = \frac{1}{x^3}$, given $y = 4$ when $x = \frac{1}{2}$ and $y = -2$ when $x = -\frac{1}{2}$.

6. Find the values of (i) $\int \left(5x^2 + \frac{4}{x^2}\right) dx$; (ii) $\int \left(x^{1 \cdot 2} + \frac{1}{x^{1 \cdot 2}}\right) dx$.

L. 2.

1. Fig. 101 represents the end-view of a circular cylinder on a table between two rectangular blocks, distant d in. apart ; the smaller is of height h in. and the radius of the cylinder is r in. Find r in terms of d, h. (C.S.C.)

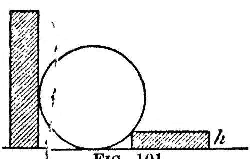


FIG. 101.

2. In what ratio must x be altered to become

$$\text{(i) } y\,; \qquad \text{(ii) } x+xy\,; \qquad \text{(iii) } \frac{x}{y}\,?$$

3. Find a and b, if x^3 is a factor of $(1+x)(1+bx)^2-(1+ax)^2$.

4. Two points A, B at the same level are 200 feet apart. A wire fixed at A, B carries a heavy weight at its mid-point O; the portions AO, OB may be taken as straight, but the length of the wire varies with the temperature. When the wire is $2s$ feet long, the depth of O below AB is y feet; when the length is $2(s+h)$ feet, the depth of O is $y+k$ feet. (i) Find a relation between s, y; (ii) find a relation between h, k, s, y; (iii) find a simple form of this relation when h, k are small; (iv) find an expression for $\frac{dy}{ds}$ in terms of s; (v) if the wire stretches from 210 to 210·1 feet, find approximately the increase in depth of O. (C.S.C.)

5. Interpret geometrically $\int_1^1 (2x+3)dx$; evaluate it (i) by direct calculation and (ii) geometrically.

6. The volume of a bowl of depth x cut from a sphere of radius a is $\pi\int_{a-x}^{a}(a^2-x^2)dx$. Compare the volumes of two bowls of depths $\frac{a}{2}$ and $\frac{a}{3}$.

L. 3.

1. (i) If $x\sqrt{x}=x^n$, find n; (ii) what is the value of $(-1)^{-\frac{1}{3}}$? (iii) simplify $(1000)^x \div 10^{2x}$.

2. Two cylindrical glasses contain equal amounts of liquid; the depths of liquid in the glasses are in the ratio $a:b$. What is the ratio of their diameters?

3. (i) One solution of the equations $x^2+y^2=25$, $\frac{x^2}{9}+\frac{y^2}{16}=2$ is $x=3$, $y=4$; write down the other three pairs of solutions.

(ii) If $x^2+y^2=25$, what are the greatest and least values of x, if y is real?

(iii) If $\frac{x^2}{9}+\frac{y^2}{16}=2$, what are the greatest and least values of y if x is real?

(iv) *Sketch* the graphs of $x^2+y^2=25$ and of $\frac{x^2}{9}+\frac{y^2}{16}=2$.

4. Four rods each of length 10 cm. are jointed together at their ends to form a rhombus ; if the length of one diagonal is x cm., express the area of the rhombus as a function of x, and find (i) the value of x for which the area is greatest, and (ii) the maximum area.

5. The distance of the centre of gravity of a circular cone of height h in. and base-radius r in. from the vertex is

$$\frac{\int_0^h \pi x y^2 dx}{\int_0^h \pi y^2 dx} \text{ in.,}$$

where $y = \frac{r}{h}x$. Evaluate this expression.

6. A body moves along a line OA *towards* O in such a way that its velocity at any point is proportional to its distance from O. Its velocity at A, where OA is 30 feet, is 12 feet per sec., and t sec. after passing A, its distance from O is x feet. Find a differential equation connecting x and t, and express $\frac{d^2x}{dt^2}$ in terms of x.

L. 4.

1. (i) Simplify the inequality $\frac{x-1}{2} > 7$.

(ii) Within what limits must x lie if $x^2 < 3x$?

2. The air resistance to a shell varies as the square of its diameter and the cube of its velocity. The resistance to a shell 5 in. in diameter when travelling 1200 feet per sec. is 150 lb. ; find the resistance to a shell 4·5″ in diameter when travelling 900 feet per sec. Find also a general formula.

3. (i) Sketch the graph of $y=(1-x)(x-5)$. Calculate the gradient of the tangents at the points where it crosses the x-axis, and find where the gradient is zero.

(C.S.C.)

(ii) Find the area of the portion of the curve which lies on the positive side of the x-axis.

4. If a force of P lb. acts on a body for t sec., the impulse given to the body is $\int_0^t P\,.\,dt$ lb.-sec. units. Find the impulse given by a force which acts for t sec. and is such that $P=6-2t$.

5. If the graph of $x^2=5y$ is rotated about the x-axis, find the volume of the portion between $x=1$ and $x=4$.

6. In Fig. 102 the dotted line is a time-velocity graph and the continuous line is a time-distance graph. If the units are properly chosen, can the two graphs both refer to the same motion? Describe the motion in general terms, the time shown being 35 minutes in each case and the maximum velocity 30 miles per hour. Find the distance gone after 14, 28, 35 minutes.

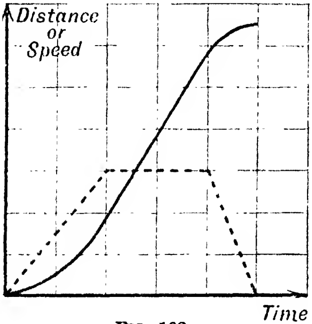


FIG. 102.

L. 5.

1. Express $(x-2)(x-3)$ in the form $a(x-1)^2+b(x-1)+c$.

2. $ABCD$ is a rectangle; P is a point on CD; AP meets BD at Q; $AB=12$, $AD=8$, $PD=x$ cm.; the area of the triangle DPQ is y sq. cm.; express y as a function of x, and find the length of DP when the area is 16 sq. cm.

3. Find the value of $a \,.\, e^{pt}$ when $a=32{\cdot}6$, $e=2{\cdot}718$, $p=-0{\cdot}37$, $t=4{\cdot}5$.

4. (i) For what range of values of x is $2x^3-9x^2$ a decreasing function?

(ii) Find a minimum value of this function.

5. If a triangle, base b in., height h in., is immersed in water and held in a vertical plane with its base in the surface, the depth of the centre of pressure is given by $\int_0^h by^2\left(\frac{h-y}{h}\right)dy \div \frac{1}{6}bh^2$. Simplify this expression.

6. Fig. 103 is a perspective view of a piece of steel formed by two cylinders of radius a, whose axes are horizontal and cut at right angles. A horizontal plane is drawn at height x above the axes; what is the shape of the portion of the plane which lies inside both cylinders ? Prove that its area is $4(a^2 - x^2)$. Hence prove that the volume common to the two cylinders is $\frac{16}{3}a^3$.

(Trin. Coll.)

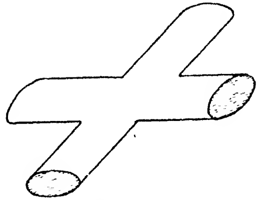


FIG. 103.

L. 6.

1. The rate at which a man can dig a trench is constant for the first 15 minutes, and afterwards, without any abrupt change, varies inversely as the square root of the total number of minutes he has been at work. At the end of 2 hours he is excavating at the rate of 22 cu. feet per hour. Find the rate of work (i) at the end of a 4-hour spell, (ii) at starting. (C.S.C.)

2. How is the value of $\frac{x^3 + 7x^2y}{3xy^2 - y^3}$ affected if x and y are each altered in the ratio $a : b$?

3. If $x = y^2 = z^3$ and $xy = t^2$, express z in terms of t.

4. Find the radius of a cylinder which is cut from a cone of height h in. and base-radius r in. if (i) the volume of the cylinder is a maximum, (ii) the *total* surface is a maximum, given $h > 2r$.

5. The portion of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ lying between $x = 1$ and $x = 2$ is rotated about the x-axis so as to form a surface. The area of this surface is $\int_1^2 2\pi y \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} \,.\, dx$. Evaluate this expression.

6. The differential equation of a graph is $\frac{d^2y}{dx^2} = 4x - 6x^2$; the graph cuts the x-axis at the origin and the point (2, 0) ; find the value of y when $x = 1$ and the slope at the origin.

L. 7.

1. (i) Draw the graph of $x + 2y = 3$.

(ii) How can you represent graphically all pairs of values of x, y such that $x + 2y > 3$?

(iii) Represent graphically the pairs of values of x and y for which simultaneously $x + 2y > 3$, $x - y < 1$, $5y - 2x < 10$.

(iv) Hence obtain all the integral values of x and y which satisfy the inequalities in (iii).

2. Two glasses contain equal amounts of spirits and water respectively: $\frac{1}{n}$th of the first glass is transferred to the second and stirred up, and then an equal amount of the mixture is transferred back again. Is the ratio of spirits to water in the first glass greater than, equal to or less than the ratio of water to spirits in the second glass ?

3. Given that $2=10^{0\cdot3010}$ and $3=10^{0\cdot4771}$, express as powers of 10 without using tables, (i) 1·2 ; (ii) 250 ; (iii) 0·015 ; (iv) 6·4.

4. (i) Sketch the graph of $y=x(x-1)(x-2)$ for values of x from -1 to $+3$.

(ii) Prove that $\int_0^2 x(x-1)(x-2)\,dx=0$, and interpret this result geometrically.

(iii) Find the minimum and maximum values of y. (C.S.C.)

5. If a force of P lb. acting on a body moves it s feet in the direction of the force, then the "work done" by the force is $\int_0^s P\,.\,ds$ ft.-lb. Find the work done by a variable force $P=3s+2$, which moves a body 10 feet.

6. A train runs from stop to stop in 10 minutes, and its speed is as follows :

Time in minutes -	0	1	2	3	4	5	6	7	8	9	10
Speed in miles per hr.	0	12	24	36	44	49·5	51	50	45	31	0

Use (i) Dufton's rule, (ii) Simpson's rule to find the total length of run. (C.S.C.)

L. 8.

1. Sketch the graph of the function $y=\frac{700x}{x+540}$ as x varies from 160 to 10,000. It has been suggested [see Edgeworth, *Levy on Capital*] that the following is a fair income-tax scheme. For an income of £x ($x>160$), an abatement of £y is allowed, where $y=\frac{700x}{x+540}$, and the tax is £T, where $T=\frac{x-y}{10}$: for x = or <160, there is no tax. Express T in terms of x, and show that the percentage of a man's income taken as tax increases steadily with x, but never exceeds 10 per cent. Find the percentage for the following incomes: (i) £160 ; (ii) £460 ; (iii) £10,000 ; (iv) £100,000.

2. The volume of a gas varies directly as its absolute temperature and inversely as its pressure. At an absolute temperature of 300°, the volume is 900 cu. cm. under a pressure of 420 mm. (mercury). Find the pressure if the volume becomes 800 cu. cm. and the absolute temperature 280°.

3. The cubical elasticity of a gas whose volume is v and pressure p is measured by $-v\dfrac{dp}{dv}$. Find the cubical elasticity of a gas for which $p \cdot v^{1.408}=c$, where c is constant.

4. The portion of the curve $y=x^2+1$ between $x=0$ and $x=2$ is rotated about the x-axis to form a solid of revolution. The x-coordinate of the centre of gravity of this solid is

$$\int_0^2 (x^2+1)^2 x\,dx \div \int_0^2 (x^2+1)^2\,dx.$$

Evaluate this expression.

5. If a horizontal circular disc of radius 4″ is under the influence of a unit charge of electricity at a point 5″ from and vertically above the centre, the density of the induced distribution at a point on the disc x in. from the centre is $\dfrac{3}{2\pi^2}\cdot\dfrac{1}{(x^2+25)\sqrt{16-x^2}}$ units per sq. in. For what value of x is this a minimum ?

6. Given $x^4\dfrac{d^2y}{dx^2}=1$ and that when $x=1$, $\dfrac{dy}{dx}=1$ and $y=2$, express y in terms of x.

L. 9.

1. (i) How is x limited if $\dfrac{3x-2}{2}$ is greater than $x+1$? Illustrate the result graphically.
   (ii) Is it possible to have $a>b$ and $ax<bx$ simultaneously ?

2. $ABCD$ is a rectangle; $AB=p$, $BC=q$ in. $(p>q)$; H, K are points on AB such that $AH=KB$, and $CDHK$ is a quadrilateral in which a circle can be inscribed. Find the length of AH in terms of p, q.

3. The rate of flow in a circular sewer of diameter D inches is $\dfrac{9{\cdot}4\sqrt[3]{D^2}}{\sqrt{i}}$ feet per sec., where i = length of sewer ÷ fall.
If the diameter is decreased in the ratio 5 : 6, and if the fall is increased from 5 feet in a mile to 6 feet in a mile, find the ratio in which the velocity is altered.

4. The distance of the centre of gravity of a uniform hemisphere of radius a from the centre of the hemisphere is

$$\int_0^a \pi(a^2-x^2)x\,dx \div \int_0^a \pi(a^2-x^2)\,dx.$$

Evaluate this expression.

5. Sketch the graph of $y=\dfrac{1}{x^2}$; mark the point $P(2, \frac{1}{4})$ on it, and draw the tangent at P cutting the x-axis, Ox, at T. Find (i) the gradient of PT, (ii) the length of OT.

6. Interpret geometrically the expressions,

$$\text{(i)} \int_1^2 \frac{1}{x^2}\,dx; \qquad \text{(ii)} \int_1^2 \pi \cdot \frac{1}{x^4}\,dx,$$

and evaluate them.

L. 10.

1. Fig. 104 represents a bridge whose span AB is a feet, supported on an arch in the form of a circular arc of radius r feet; the heights of the roadway above C, D and the top of the arch E are b, b, c feet; express r in terms of a, b, c. Find r if $a=80$, $b=16$, $c=1$. (C.S.C.)

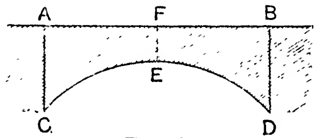


FIG. 104.

2. A_1, A_2, A_3, A_4, A_5, ... are points on a line such that the lengths of A_1A_2, A_1A_3, A_1A_4, A_1A_5, ... are log 2, log 3, log 4, log 5, ... feet. Find the lengths of

(i) $A_1A_2 - A_2A_4$;
(ii) $A_1A_8 - 3 . A_1A_2$;
(iii) $A_1A_{10} - A_1A_2 - A_1A_5$.

What facts do these results illustrate?

3. If z varies directly as x and inversely as y, complete the table:

Values of x.

		1	2	3	4	5
Values of y.	1					
	2			90		
	3					100

4. If a rectangle of height a in. is immersed in water with its plane vertical and its upper edge h in. below the surface, then the depth of the centre of pressure below the surface is

$$\int_0^a (x+h)^2\,dx \div \int_0^a (x+h)\,dx.$$

Simplify this expression.

5. A glass is so shaped that when the depth of liquid in it is x inches, the volume of the liquid is $\frac{1}{10}x^4$ cu. in.; water is poured into it at the rate of 2 cu. in. per sec.; at what rate is the level rising after 3 seconds ?

6. Given $\frac{d^2y}{dx^2} = 1 - x^3$ and $x=0$ when $y=0$, and $x=2$ when $y=2$, find $\frac{dy}{dx}$ when $x=0$, and express y in terms of x.

M.

(CHAPTERS XII.-XIV. and XIX.)

M. 1.

1. Write down the 1st and 10th terms of sequences whose nth term is

(i) $5n-4$; (ii) $(n+1)(n+2)$; (iii) n^2-1.

2. Fill in the gaps in the following sequence: 2, 5, 8, , 14, 17, , 23, , , 32 and write down the value (i) of the 15th term, (ii) of the nth term.

3. How many terms are there in the sequence 1, 5, 9, 13,... 101 ? Find the sum of these terms.

4. A line AB, 4 inches long, is divided at P so that the square on AP is twice the square on PB. Find the length of AP correct to 3 figures.

5. Show that a triangle with sides $\sqrt{2}-1$, $\sqrt{2}+1$ and $\sqrt{6}$ will be a right-angled triangle, and find its area.

6. Two ships are similar (in the mathematical sense), but are of tonnage 8000 and 27,000 respectively. Assuming that the tonnage of similar ships varies as the cube of the linear dimensions, and that it costs £A to paint the smaller ship, what will it cost to paint the larger ship ? (Certificate.)

M. 2.

1. Find the sum of all the numbers up to and including 100 which are divisible by 5.

2. A car starts from rest and is steadily accelerated in such a way that its speed after 1 second is 1 mile per hour, after 2 seconds, 3 miles per hour, after 3 seconds 5 miles an hour, and so on. Find what its speed will be according to the same law after 8 seconds, and find when it will attain a speed of 20 miles an hour.

3. Solve the equations: $\begin{cases} 4{\cdot}5x + 7{\cdot}2y = 1{\cdot}4, \\ 3{\cdot}6x + 2{\cdot}4y = 0{\cdot}49. \end{cases}$

4. If $x : y = 10 : 9$, find the value of (i) $\dfrac{x+y}{x-y}$, (ii) $\dfrac{x^2+y^2}{x^2-y^2}$.

5. The triangle ABC in Fig. 105 is right-angled at B, and has $AB = 8''$, $BC = 6''$.

The side AB is divided into 8 equal parts and rectangles are constructed as in the figure. If MN is the side of any one of these, then the ratio $MN : BC = AM : AB$. Hence find the length of MN and the area of the rectangle which has MN as its side. Find the sum of the areas of all these rectangles, and find the difference between their sum and the area of the $\triangle ABC$.

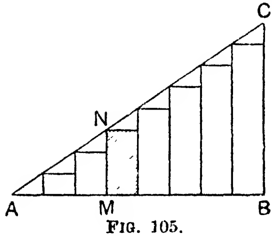


FIG. 105.

6. Show that if in Question 5 the side AB is divided into n equal parts, so that $(n-1)$ rectangles are formed, each with base $\dfrac{8}{n}$ inches, then the smallest rectangle formed has area $\dfrac{48}{n^2}$ square inches, and the largest $\dfrac{48(n-1)}{n^2}$ square inches, and that the sum of all of them differs from the area of the triangle by $\dfrac{24}{n}$ square inches.

M. 3.

1. A solid sphere of radius $4''$ weighs 32 lb. Find the weight of a shell of the same material, whose internal and external radii are $3''$ and $5''$. (Certificate.)

2. If $x = t + \dfrac{1}{t}$, $y = t - \dfrac{1}{t}$, express $x^2 + xy + y^2$ in terms of t only.

3. The nth term of a series is $5n - 7$. What is the 1st term? what is the $(n+1)$th term? Find the sum of $(n+1)$ terms.

4. Draw with the same axes and scales the graphs of x^2-7 and $\frac{4}{x}$ for values of x between the limits ± 3. Show that the points of intersection of the graphs will give the solution of the equation $x^3-7x-4=0$, and hence determine how many real roots this equation has, and their approximate values. (Certificate.)

5. An equilateral triangle ABC has all its sides divided into 12 equal parts. Corresponding points are joined as in Fig. 106 (6 parts only are shown), and a number of smaller equilateral triangles thus formed. How many small triangles are there (i) in the fifth row (PQ) ? (ii) in the 12th row ? How many small triangles are there in all ?

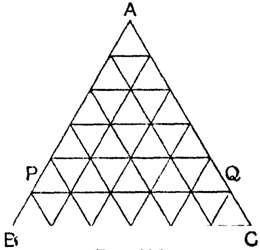


FIG. 106.

6. Generalise Question 5, taking any triangle ABC and dividing each side into n equal parts. By joining corresponding points, a number of similar triangles are formed as in Fig. 106. How many are there in the rth row ? how many are there in all ? What geometrical proposition does this result illustrate ?

M. 4.

1. The horse-power of a certain type of water-wheel is given by the formula H.P. $=0\cdot00113QH$, where Q is quantity of water flowing in cubic feet per min. and H is the head of water in feet. Find the horse-power of a wheel of this type driven by 850 cubic feet of water per minute from a height of $2\cdot6$ feet.

2. Convert the relation $a=2\cdot5c^{\frac{1}{2}}d^{-\frac{3}{4}}$ into a relation of similar type, giving c in terms of a and d. (Certificate.)

3. Solve the equation $x^2+3x-5\cdot23=0$, giving the values of x correct to 3 significant figures.

4. What is the nth term of the sequence $2+1$; 2^2+2 ; 2^3+3 ; 2^4+4 ; ... ?

If the nth term of this sequence is a, the $(n+1)$th term b and the $(n+2)$th term c, prove that $3b=2a+c+1$.

5. Find the nth term and the sum of n terms of the sequence 2, 6, 18, 54, ...

6. A clerk's commencing salary is £100 a year, and he is offered a choice between a yearly rise of £5 and a rise of £22 every 4 years. Calculate the total sum that he will receive in the course of 33 years under each arrangement. (Certificate.)

M. 5.

1. An engine exerting a pull of F tons draws a train weighing W tons (including the engine) up an incline of 1 in m on a road whose resistance is r lb. per ton. It can be proved that in t seconds the train will have moved from rest through s feet, where

$$s=\frac{1}{2}\left(\frac{F}{W}-\frac{1}{m}-\frac{r}{2240}\right)gt^2.$$

Express F in terms of the remaining letters. (Certificate.)

2. If $x=\frac{4y-5}{5y+1}$ and $y=\frac{4z-5}{5z+1}$, find the values of y and z, when $x=1$.

3. Find the value of $\frac{8132h^{2\cdot5}}{n^2}$, when $h=4\cdot65$ and $n=325$.

4. Find x from the following equations :

(i) $a^x\times a^5=a^9$;

(ii) $(a^x)^6=a^9$;

(iii) $10^x=417$ to 3 significant figures ;

(iv) $3^x=2$ to 3 significant figures.

5. Prove that $\frac{101+103+105+\ldots+199}{1+3+5+\ldots+99}=3.$

6. Take any positive number x_1 ; take its positive square root, add one and square the result, call the resulting number x_2. Repeat the process, starting with x_2 instead of x_1 and call the result x_3 ; repeat with x_3 and call the result x_4, and so on. Find x_{10} in terms of x_1. Also find x_n in terms of x_1.

M. 6.

1. A certain engineering formula is of the type

$$H=\frac{ml}{d^x}\cdot\frac{V^n}{2g}.$$

Rewrite this formula, expressing V in terms of the other letters.

2. If $U=0\cdot4343\,.\,P\,.\,V\,.\log\frac{p_1}{p_2}$, find U when $P=16$, $V=7$, $p_1=15\cdot8$ and $p_2=7\cdot4$.

3. The proprietor of a tea-shop finds that the average expenditure of each of ninety customers is 5d. ; but that while each man spends on the average 7d., each woman spends on the average 4d. How many of his customers are men ? (Certificate.)

4. A sequence is formed whose nth term is $6n-4$; write down the first three terms, and show that the difference between two successive terms is always 6.

5. What is the nth term of the sequence 27, 9, 3, 1, ... ? Find its value approximately when $n = 20$, using logs.

6. A long strip of paper, thickness 0·005 inch, is wound on to a cylindrical bobbin of diameter $3\frac{1}{2}$ inches, and the outer diameter of the roll thus made is 14 inches. How many layers of paper are there in the roll ? Assuming that the lengths of successive layers increase in Arithmetic Progression, and that each length is a complete circle, find in yards the total length of the strip.

(Certificate.)

M. 7.

1. According to a certain scheme for paying income tax a man paid at the rate of 2s. 6d. in the £ on "earned" income and 4s. in the £ on "unearned" income (*i.e.* on dividends from investments, etc.). His income was £800, and his total income tax was £115. How much of his income was "earned" ?

2. If $W = \sqrt{\dfrac{w(p_1 - p_2)d^5}{L}}$, find the value of W when $w = 0{\cdot}0012$, $p_1 = 18{\cdot}4$, $p_2 = 17{\cdot}8$, $d = 2{\cdot}5$ and $L = 1760$.

3. Find the 16th term and the sum of 16 terms of the sequence 1, $\frac{4}{3}$, $\frac{5}{3}$, 2, ...

4. What would appear to be the form of the nth term of a sequence whose first three terms are formed as follows ?

(i) 3×2; 3×2^2; 3×2^3;
(ii) $3 + 2$; $4 + 2^2$; $5 + 2^3$.

5. Find the sum of 9 terms of a G.P. whose first two terms are 1 and $\frac{3}{5}$, using logs to get the result correct to 3 significant figures.

6. A window is in the shape of a rectangle surmounted by the semi-circle drawn on the upper side of the rectangle, this side being one of the shorter sides.

If h feet is the height of the window, measured from the bottom side to the highest point of the semi-circle, and the width of the window is a feet, show that its area is $(ah - \frac{3}{28}a^2)$ square feet. [Take the area of a circle, radius r feet, to be $\frac{22}{7}r^2$ sq. feet.]

If the area is $9\frac{1}{2}$ square feet and the height is 5 feet, find the width of the window to the nearest half-inch. (Certificate.)

M. 8.

1. Solve, by a graph if possible, the following problem :

A party, climbing a mountain 4000 feet high by the regular track, climbs at the rate of 1000 feet per hour, measured vertically ; it starts at 11.5 a.m. and halts half-an-hour for lunch at 1.30 p.m. Another party, whose rate of descent is 1800 feet per hour, measured vertically, leaves the summit at 1.45 p.m., following the regular track. At what time will it meet the first party ?

(Certificate.)

2. Simplify $\frac{3x}{x-1} - \left(\frac{x}{x+1} \div \frac{1}{1-x}\right)$.

3. If $T = \frac{0 \cdot 196(D^4 - d^4)}{D}$, find T when $D = 3 \cdot 12$, $d = 1 \cdot 74$.

4. Show that the recurring decimal 0·7777... is equal to the "limiting sum" of a G.P., and find its value as a fraction.

5. If $x : y = 13 : 4$ and $2x + 3y = 19$, find the values of x and y.

6. The area cut off from the surface of a cone between two circular sections, the distances of whose edges from the vertex measured along a slant side of the cone are a'' and b'', is $c(b^2 - a^2)$ sq. inches, where c is a constant depending on the angle of the cone. A cone, for which the constant c is 0·8, and whose slant side is 30″, is painted in alternate strips of red and blue, the red starting at the vertex, the width of each strip being 1″. Show that the areas of the consecutive strips of each colour form two arithmetic progressions, and find the total areas painted in the two colours. (Certificate.)

M. 9.

1. The following formula connects the penetration t inches into steel-armour plates, with the velocity v ft. per sec. of the shell:

$$t = \frac{15v}{6200 - v}.$$

Find the velocity of the shell that penetrates 7″. (Certificate.)

2. OAB are three points in order on a straight line, such that $OA = a''$ and $OB = b''$. A point P divides AB so that the ratio $AP : PB = 5 : 3$. Find the length of OP in terms of a and b.

3. If $H = \frac{G^2 L}{(3d)^5}$, express d in terms of H, b, and L, using fractional and negative indices.

4. Find the value of $T\left(\frac{p_1}{p_2}\right)^{\frac{n-1}{n}}$ correct to 3 significant figures, when $T = 147$, $p_1 = 16 \cdot 5$, $p_2 = 13 \cdot 8$ and $n = 1 \cdot 6$.

5. An estimate for boring a well is 4s. for the first yard, 4s. 2d. for the second yard, and so on; each yard costing 2d. more than the preceding yard. If the sum to be spent on boring is £7. 2s., how deep is the well to be? (Certificate.)

6. After heavy rain the quantity of water passing a point on a small stream in hilly country is found to be 4000 cubic feet in one hour, 2000 cubic feet in the next, 1000 cubic feet in the next, and so on. Show that the stream will have practically dried up in twelve hours, if this rate of decrease continues, and find how much water will have passed the point then.

M. 10.

1. In any triangle ABC, AD is perpendicular to BC and meets BC at D. $AD = 3''$, $BC = 6''$.

Denoting the length of BD by x'', express AB^2 and AC^2 in terms of x. If $AC^2 = k \, . \, AB^2$, form an equation to find x. What is the meaning of the roots of this equation, (i) when $k = 1$, (ii) when $k = 5$?

2. If $L = 6{\cdot}6d^{\frac{2}{5}}$, express d in terms of L, and find the value of d, when $L = 118$.

3. Find the first term and the common difference of an arithmetic progression whose 3rd term is 15 and 10th term 43.

4. What is the smallest number larger than 100 which is divisible by 9 ? What is the largest number less than 200 and divisible by 9 ? Find the sum of all the numbers between 100 and 200 which are divisible by 9.

5. Find the sum to n terms of the series $1 - \frac{1}{4} + \frac{1}{16} - \dots$.
What is the "limiting sum" of this series ?

6. A War Savings Certificate costs 15s. 6d., and amounts to 20s. at compound interest in 5 years. Find the number of years in which a sum invested at this rate of compound interest will amount to double the original principal; in other words, solve the equation
$\left(\frac{20}{15{\cdot}5}\right)^{\frac{x}{5}} = 2$, calculating x correct to one place of decimals.

(Certificate.)

N.

(CHAPTERS XII.-XIV. and XX., XXI.)

N. 1.

1. The senior and junior marks obtained in an examination are 137 and 29. These are scaled from 150 to 0. If x marks, when scaled, becomes y marks, express y in terms of x. What mark is unaltered by the scaling ?

2. Give the general rule which includes the following :

(i) ${}^7P_5 = \frac{|\underline{7}}{|\underline{2}}$; (ii) ${}^{10}P_7 = \frac{|\underline{10}}{|\underline{3}}$; (iii) ${}^{11}P_6 = \frac{|\underline{11}}{|\underline{5}}$,

and evaluate ${}^{20}P_2$.

3. (i) What is the coefficient of x^2 in the expansion of
$$(x + a + b)(x + c)(x + d) \text{ ?}$$

(ii) What are the terms in the expansion of
$$(a + b + c)(a - b + d)(a + c - d)(b - c - d),$$
which have b^3 as a factor ?

4. If $p \cdot v^{1\cdot4} = 28\cdot7$, find v when $p = 17\cdot3$.

5. The distance a spring stretches (*i.e.* its extension) when a weight is suspended from it varies (within certain limits) directly as the weight. The following measurements were made in an experiment, state which (if any) of them are probably incorrect, and find the probable equation between the extension l cm. and the weight w grams which produces it:

Total length of spring in cm., l -	21·0	22·7	26	27·7	34·2	36	39·5	41
Weight attached in gr., w - -	0	5	15	20	35	45	50	60

6. (i) There are n circles in a plane, each lying outside the rest; what is the greatest number of lines that can be drawn, each of which touches two circles ?

(ii) Two sets of 3 concentric circles are drawn in a plane, and no two circles intersect. How many lines can be drawn, each of which touches two circles ?

N. 2.

1. The resistance R lb. to a parachute of area A sq. ft. moving v ft. per sec. varies as A if v is constant, and as v^2 if A is constant. Complete the table for R.

How does v vary if R is constant ?

Values of A.

		4	8	10
	5			
Values of v.	10		1	
	15			

2. The following formulae occur in the mensuration of a circular cone : $s = \pi rl$; $l^2 = r^2 + h^2$; $V = \frac{1}{3}\pi r^2 h$.

If $\frac{r}{3} = \frac{h}{4}$, evaluate $\frac{s^3}{V^2}$.

3. In the Morse code, letters are represented by dots or dashes or combinations of both ; how many letters can be represented by using not more than four symbols for any one letter ?

4. (i) Simplify $(x+y)^3 - 3y(x+y)^2 + 3y^2(x+y) - y^3$.
(ii) What is the coefficient of x^2 in $(1+2x+x^2)^5$?

5. (i) Find the roots of $x^3 - 100x = 0$.
(ii) One root of $x^3 - 100x = 1$ is obviously nearly equal to 10; by putting $x = 10 + h$, where h is small, and neglecting h^2 and h^3, find a closer approximation to this root.

6. If the diameter of a circular cylinder increases 2 per cent. and the height increases 1 per cent., find the approximate percentage increase in the volume.

N. 3.

1. Fig. 107 represents the graph of $y=(x-1)(x-2)$ on a certain scale. Sketch, with the same units, the graphs of

(i) $y = 1 + (x-1)(x-2)$;
(ii) $y=(x+1)(x+2)$;
(iii) $y=(x-2)(x-3)$;
(iv) $y=\frac{1}{(x-1)(x-2)}$.

Sketch also the graph of
$$y=(x-1)(x-2),$$
if the x-unit is halved and the y-unit is doubled.

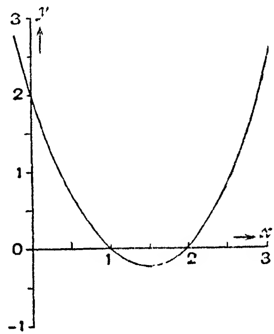


FIG. 107.

2. In an ordinary box of safety matches the ratio of the length, the breadth and the height is 4 : 3 : 1. If the area of the matchwood used in making the box is A sq. inches and the volume of the box is V cu. inches, find an equation between A and V.

3. A tea-pot with a spout 2″ long holds a pint; how much will a tea-pot of the same shape with a spout 3″ long hold ?

4. A carriage holds 4 people inside and 2 outside; in how many ways can 6 people be divided between the inside and outside places, the arrangement inside and outside being immaterial ?

5. A pendulum of length l feet vibrates $\frac{108 \cdot 4}{\sqrt{l}}$ times a minute. What is the reduction in the number of vibrations per hour if the length is increased, (i) from 9 to 9·1 feet, (ii) from l to $l+h$ feet, where h is small compared with l ?

If a grandfather-clock gains, would you screw the weight at the end of the pendulum up or down in regulating it ? And why ?

6. What is the coefficient of x^3 in

(i) $(1-x^2)^2(1+x)^8$; (ii) $(1-x)^2(1+x)^8$?

N. 4.

1. If the length of a pendulum is l feet and the time of vibration is t sec., it is found by observation that when $l=1$, $t=1{\cdot}11$, and when $l=2$, $t=1{\cdot}58$, and when $l=3$, $t=1{\cdot}93$. If these obey a law of the form $t=al^n$, find a and n.

2. The side BA of the triangle ABC is produced to E; if the bisector of $\angle EAC$ cuts BC produced at P, it can be proved that $\frac{BP}{CP}=\frac{BA}{AC}$. (i) If $AB=7''$, $AC=3''$, $BC=5''$, calculate CP; (ii) if $AB=z$, $BC=x$, $CA=y$ in., calculate CP.

3. In signalling by semaphore, a single arm can be shown in any one of 7 positions, exclusive of the position of rest. These positions denote the first seven letters of the alphabet. The remaining letters can be signalled by showing two arms in any pair of these seven positions. How many letters can be signalled altogether ?

4. If $a=b+h$, where h is small compared with b, show that $\sqrt{\frac{a}{b}}$ and $\frac{3a+b}{a+3b}$ have nearly the same values. What is approximately the value of each when $a=11$ and $b=10$?

5. (i) What is the result of putting $x=2y-1$ in the expression

$$x^4+4x^3-14x^2-36x+45 \text{ ?}$$

(ii) Solve the equation $x^4+4x^3-14x^2-36x+45=0$.

6. A police-court witness is asked to identify 2 suspected persons from a group of 12. If he really knows nothing about them, what is the chance that he picks out the two suspects ?

N. 5.

1. Draw the graph of $y=x^2-x$. With the same axes and without any further calculation, sketch the graphs of

(i) $x=y^2-y$; (ii) $y=x^2+x$; (iii) $y=x^2-x+1$.

2. When the horizontal course of an aeroplane is altered, there is a tendency for the axis of rotation of the propeller to turn up or down measured by $\frac{\pi R^2 MN}{32{\cdot}2T}$ foot-lb. Find the turning tendency for a "Gnome" motor, where R = radius of gyration $=0{\cdot}742$ feet, M = weight $=247$ lb., N = number of revolutions per sec. $=20$, T = number of seconds aeroplane takes to turn through a complete circle $=19{\cdot}5$.

3. Solve $$\frac{xy}{4}=\frac{yz}{9}=\frac{zx}{6}=\frac{y+z}{2\cdot5}.$$

4. A, B, C, D are 4 points in a plane, no three of which are collinear; how many different angles at A, B, C, D can be formed by lines joining these points, but not produced beyond them?

5. How many arrangements can be made of the letters of the word *referce*?

6. (i) If x is small, prove that $\sqrt{x^2+16}-\sqrt{x^2+9}\simeq 1-\frac{x^2}{24}$.

(ii) If x is large, prove that $\sqrt{x^2+16}-\sqrt{x^2+9}\simeq\frac{7}{2x}$.

(iii) Hence sketch the graph of $\sqrt{x^2+16}-\sqrt{x^2+9}$.

N. 6.

1. (i) If $\log_a x=p$, express x in terms of a, p.

(ii) If $\log_b x=q$, find an equation between a, b, q, p.

(iii) Hence prove that $\frac{\log_a x}{\log_b x}$ is independent of x.

(iv) What is the value of $\frac{\log_8 x}{\log_2 x}$?

(v) Given the graph of $\log_2 x$, how would you deduce the graph of $\log_8 x$?

2. If the lengths of the sides of a cyclic quadrilateral taken in order are a, b, c, d in., the area is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where s is its semi-perimeter.

(i) If the sides of the quadrilateral touch a circle, it can be proved that $a+c=b+d$; simplify the expression for the area.

(ii) Interpret geometrically the result of putting $d=0$ in the given formula.

(iii) What is the result of putting $a=b=c$ and $d=0$ in the given formula, and what is the geometrical interpretation?

3. In how many ways can 8 people be seated at two round tables, each of which holds four? [Arrangements in which at least one person has a different right-hand neighbour are to be counted as different arrangements.]

4. The area of the wing surface of an aeroplane with load W lb. for a speed of v miles an hour varies as $\frac{W}{v^2}$. How must the area be altered approximately if W is increased by 3 per cent. and v is reduced by 2 per cent.?

5. Find (without using logarithms) the approximate value of the cube root of (i) 125·5; (ii) 127.

6. (i) In the game of Crown and Anchor, two dice are used, the faces of each of which are marked with a crown, an anchor, a spade, a heart, a diamond, a club. The gambler chooses one of these on which to stake. The dice are then thrown. If both dice show the figure he has chosen, the Banker pays him 4 times his stake and returns his stake; if one of the dice shows his figure, the Banker pays him twice his stake and returns his stake. If his choice does not turn up at all, the Banker takes his stake. If during a session the total stakes amount to £600, how much profit will the Banker expect to make?

(ii) If three dice are used, the Banker pays, besides returning the stake, four times the stake for a triple success, twice the stake for a double success and a sum equal to the stake for a single success; in the event of failure he keeps the stake. What fraction of the total sum staked does the Banker expect to win?

N. 7.

1. The horse-power H required to drive an electric fan of diameter d feet to deliver V cu. feet of air per second is given by $V = 14d^2\sqrt{H}$.

(i) Express this as a formula in which the units are inches and cu. inches for the diameter and volume.

(ii) What is the effect on the horse-power if the volume and diameter are each doubled?

2. In a game of chess, the first player (white) can make any one of twenty moves. Black can then also make any one of twenty moves. White can play his second move in about twenty ways and Black has the same choice for his second move. Find approximately the number of ways in which the first two moves on each side can be played.

3. (i) If h is small compared with x, find an approximation for $\sqrt{x^2+h^2} - x$.

(ii) The height, h in., of an isosceles triangle ABC is small compared with the base BC; the excess of $AB+AC$ over BC is d in.; prove that the area of the triangle is approximately $\frac{h^3}{d}$.

4. A tank, fitted with a tap enabling $\frac{1}{N}$th of the contents to be drawn off, is full of a certain liquid. As much as possible is drawn off, and the tank is filled up with water. What fraction of the original liquid remains after the process has been repeated, (i) twice, (ii) p times?

If $N = 10$, how many times must the process be repeated in order that less than half the original liquid may remain? (C.S.C.)

5. A policeman on a motor bicycle, pursuing a thief who has stolen a car, comes to a cross-roads; he turns down a road at random, and then comes to a point at which the road branches into two; he again chooses at random. What is the chance he is still on the right road?

6. It can be proved that if water stands in a spherical vessel of radius a in. to a depth of d in., the volume of the water is

$$f(d) \equiv \pi d^2\left(a - \frac{d}{3}\right).$$

(i) Use this formula to obtain the volume of a sphere; (ii) evaluate $f(d) + f(2a - d)$.

N. 8.

1. The electrical resistance of a copper wire varies directly as its length and inversely as the square of the diameter of the cross-section. How is the resistance altered if the length is doubled and the diameter is halved?

2. What would the graph of $(-1)^x$ drawn for all positive values of x look like? What is its chief peculiarity?

3. How many different signals can be sent, using one or more of 4 flags of different colours, (i) if different orders count as distinct signals, (ii) if the order does not matter?

4. (i) If y is so small that y^3 can be neglected, express $\dfrac{1}{1+y}$ in the form $a + by + cy^2$.

(ii) If x is so small that x^3 can be neglected, express

$$\frac{2n + (n+1)x}{2n + (n-1)x}$$

in the form $a + bx + cx^2$. Express also $\sqrt[n]{1+x}$ in the same form.

(iii) What can you deduce from the results of (ii)?

5. (i) In how many ways can 12 competitors in a race be divided into two heats of six, if the order of the heats is immaterial?

(ii) What is the chance that the two best runners are in the same heat?

6. The volume of a circular cone, base-radius r in. and height h in., is $\frac{1}{3}\pi r^2 h$. A frustum of a cone is the portion between two planes perpendicular to the axis. If the areas of the two plane faces of the frustum are A, B sq. inches, and if the distance between them is h in., prove that the volume of the frustum is $\frac{1}{3}h(A + B + \sqrt{AB})$.

N. 9.

1. If a disc of diameter d feet revolves at the rate of n revolutions per minute, the power wastage due to the air is P ft.-lb. per sec., where $P^2=10^{-22}d^{11}n^7$.

Express this formula with P as its subject for a disc of diameter x inches revolving m times a second.

2. Motor-cars are numbered with either one or two letters (*e.g.* P; AA; LH) followed by any number from 1 to 9999. How many cars can be numbered differently if all letters in the alphabet are used ?

3. If water to a depth of d in. stands in a hemispherical bowl of radius a in., the volume of water is $\pi d^2\left(a-\frac{d}{3}\right)$ cu. in.

If the contents form $\frac{1}{n}$th of the bowl, where n is large, find an approximate value of d in terms of a, n.

4. Four people wish to settle how to play a foursome ; three of them spin coins, and the odd man out plays with the fourth. What is the chance that the first spin is decisive ?

5. (i) There are 4 teams competing against each other on the "knock-out" system. In how many ways can the draw be made ?

(ii) What is the number of ways if there are 8 teams ?

(iii) What is the number of ways if there are 16 teams ?

6. (i) A framework of rods has 5 corners, no four of which are coplanar; what is the greatest number of rods in the framework ?

(ii) If the rods are jointed freely at their extremities, how many can be removed without affecting the rigidity of the framework ?

(iii) A framework of rods has n corners, no four of which are coplanar; what is the greatest number of rods in the framework ?

(iv) If the rods are jointed freely at their extremities, what is the greatest number that can be removed without affecting the rigidity of the framework ?

N. 10.

1. The acceleration g feet per sec. of a falling body near the surface of a planet varies directly as the mass of the planet and inversely as the square of the radius. The time of oscillation of a pendulum varies directly as the square root of its length and inversely as the square root of g. Find how the mass of a planet varies in terms of the time of oscillation of a pendulum of given length and the radius of the planet ?

2. (i) How many terms are there in the expansion of

$$(1+2+2^2+2^3)(1+5+5^2+5^3+5^4)\ ?$$

(ii) In how many ways can 5000 (*i.e.* $2^3 \times 5^4$) be expressed as the product of two factors ?

(iii) In how many ways can 10,000 be expressed as the product of two factors ?

3. A domino consists of two halves, each of which may have any number of pips marked on it from 0 to 6 ; how many different kinds of dominoes are there ?

4. What is the coefficient of t^3 in the expansion of

(i) $(1+at+a^2t^2+a^3t^3+a^4t^4)(1+bt+b^2t^2+b^3t^3+b^4t^4) \times (1+ct+c^2t^2+c^3t^3+c^4t^4)$;

(ii) $(1+t+t^2+t^3+t^4)^3$?

5. $f(x)$ is a function of x such that

(i) $f(x) \equiv x-2n$ if $2n < x \leqslant 2n+1$,

and (ii) $f(x) \equiv 2n-x$ if $2n+1 < x \leqslant 2n+2$, where n is zero or any positive integer. Sketch the graph of $f(x)$ for positive values of x.

6. A, B play a match of the best three sets at tennis. Either is equally likely to win any set. A wins the first set ; what is his chance then of winning the match ?

P.

(CHAPTERS XII.-XVIII. and XXII., XXIII.)

P. 1.

1. In a certain formula ah is taken as an approximation for $\dfrac{a^2h}{a-h}$; find the error as a percentage of the true value: and evaluate it for the special case $h=2$, $a=4000$. (Certificate.)

2. Two men measure independently the length of a fence; the first makes it a yards, and his error is less than x per cent.; the second makes it b yards, and his error is less than y per cent. Find two inequalities connecting a, b, x and y.

3. An experiment to determine the force P lb. required to lift a weight of W lb. by a system of pulleys gave the following results:

W	5	10	15	20	30	40
P	1·23	2·15	3·06	3·98	5·82	7·64

Find the best fit equation for P in terms of W.

4. Describe in general terms the value of the function $\frac{x-4}{x-3}$, when x is (i) large and positive; (ii) slightly greater than 4; (iii) slightly less than 4; (iv) slightly greater than 3; (v) slightly less than 3; (vi) less than -1000.

For what range of values of x is this function negative? Sketch its graph.

5. Simplify (i) $\frac{d}{dx}(x+2)(x-3)$; (ii) $\frac{dy}{dx}$ if $xy=5$;

(iii) $\int\left(x^2-\frac{1}{x^2}\right)dx$; (iv) $\int_{-1}^{+1}(x+1)^2\,dx$.

6. (i) If a sphere of radius a in. contains water to a depth of x in., the volume is $\int_0^x \pi(2ax-x^2)\,dx$. Simplify this expression.

(ii) Water is poured into a spherical bowl of radius 5 in. at the rate of 4 cu. in. per sec. At what rate is the water-level rising when the depth is 3 inches?

P. 2.

1. Given $\log x = 3{\cdot}2 \log t + 7{\cdot}4$, express x in terms of t without using logarithmic notation.

2. I is an index-number representing the cost of living compared with a fixed pre-war standard. A workman receives a fixed wage of 40s. a week plus a bonus directly proportional to the cost of living. The purchasing power P of his wages is measured by $W \div I$, where W shillings is the weekly wage. When $I=250$, he receives 90s. a week. Express W and P as functions of I. Prove that, when the wages increase, their purchasing power decreases.

3. The following readings connect the candle-power and voltage of an incandescent lamp:

Candle-power -	20·68	23·24	26·00	28·96
Voltage - -	94	98	102	106

If the candle-power varies as the nth power of the voltage, find n.

4. The horizontal cross-sections of the crater of a volcano are circles with their centres on a vertical line, the radii of the circles at different heights are as follows (measurements in feet):

Height -	0	10	20	30	40	50	60	70	80	90	100
Radius -	0	17	24	27	30	35	41	52	65	78	94

Find its volume in cu. ft. to two significant figures, (i) by Dufton's rule, (ii) by Simpson's rule.

5. The work done in stretching a spring of natural length a inches to a length b inches is $\frac{1}{30}\int_a^b (x-a)\,dx$ foot-lb. Find the work done in stretching the spring from its natural length, 10 inches, to a length of 14 inches.

6. A particle projected in a resisting medium finally comes to rest. It travels s feet in t sec., where $s=50t-\frac{1}{6}t^3$. How far does it go before stopping?

P. 3.

1. Simplify (i) $\log(x^3)-\log x$; (ii) $\frac{\log(x^3)}{\log x}$;

(iii) $\log(10x)+\log\left(\frac{1}{x}\right)$; (iv) $\log x$ if $x^3=100$.

2. (i) If $xyz=a$, $xzw=b$, $xyw=c$, express the ratios $y:z:w$ in terms of a, b, c.

(ii) Divide a^2-b^2 into two parts in the ratio $a:b$.

3. The following table gives the distance, d yards, in which a train running V miles per hour can be stopped:

V	30	40	45	50	60
d	100	170	224	271	400

Plot d against V^2, and then express d in terms of V.

4. The moment of inertia of a spherical segment, radius r in., height h in., about its axis is $\frac{\pi d}{2}\int_{r-h}^{r}(r^2-x^2)^2.dx$, where d is its density. A sphere of radius r is cut into two segments by a plane dividing a diameter in the ratio 3 : 1. Compare the moments of inertia of the two segments about their axes.

5. A piece of wire two feet long is cut into two parts, one of which is bent to form a square and the other an equilateral triangle. If the sum of the areas is a minimum, find the side of the equilateral triangle.

6. A weight hangs at the end O of a spiral spring: it is pulled down a certain distance and released, and then oscillates backwards and forwards. If its velocity is v inches per sec. when its distance from O is x inches, the differential equation of the motion is $\frac{d}{dx}(v^2)=-\frac{x}{10}$. (i) If it is pulled down 5″ and released, find its velocity when passing O; (ii) how far must it be pulled down to pass O with a velocity of 1 inch per sec.? (iii) with the data of (i), find an equation expressing δt in terms of x, δx, and write down the integral which gives the time of a complete oscillation.

P. 4.

1. If $pv^n = 329$, find n when $p = 11{\cdot}6$, $v = 3{\cdot}82$.

2. The connection between the speed v feet per sec. of a column of water produced by a pressure of head h feet is shown in the following table:

h	4	16	25	36	64
v	15·7	32·7	39·3	49·1	63·3

Plot v against $\sqrt{h}$ and find the most probable relation expressing v in terms of h.

3. A circular cylinder of radius a in., height l in., is closed at one end with a flat cover, and at the other with a spherical cap of height h in. Find the total surface. [The area of a spherical cap of height h which is part of a sphere of radius r is $2\pi rh$.]

4. Given $y = (x-1)^3$: (i) find the value of $\frac{dy}{dx}$ when $x = 1$; (ii) find the sign of $\frac{dy}{dx}$ when $x = 0{\cdot}9$ and when $x = 1{\cdot}1$; (iii) is $x = 1$ a turning point of the function $(x-1)^3$? (iv) find the sign of $\frac{d^2y}{dx^2}$ when $x = 0{\cdot}9$ and when $x = 1{\cdot}1$; (v) sketch the graph of $(x-1)^3$ from $x = 0$ to $x = 2$, and explain the geometrical interpretation of results (i) to (iv).

5. The gradient of a curve which passes through the origin is given by the equation $\frac{dy}{dx} = x^2(1-x)$; find the area between the curve, the x-axis and the ordinate $x = 1$.

6. If a gas occupies v cu. feet under a pressure of p lb. per sq. inch, where $p \cdot v^{1{\cdot}42} = 300$, the work done in expanding from v_1 cu. ft. to v_2 cu. ft. is $\int_{v_1}^{v_2} p \cdot dv$ ft.-lb.; find the work done in expanding from 5 to 8 cu. ft.

P. 5.

1. If a exceeds b by x per cent., by how much per cent. does $2a$ exceed b? In what ratio must b be increased to give a?

2. The following figures are given for torsional resistance (steel):

Diameter of shaft in inches, d -	1	2	2·5	3	3·5
Torsional resistance in tons, m -	1·28	10·2	19·9	34·5	54·7

Plot $\log m$ against $\log d$; hence find the best fit equation expressing m in terms of d.

3. Construct a simple line chart connecting Centigrade and Fahrenheit temperatures from 0° to 50° C., given 0° C. = 32° F. and 100° C. = 212° F. Read off the equivalents of 18° C., 98° F.

4. If fuel contains C per cent. carbon, H per cent. hydrogen, N per cent. oxygen, the calorific value of 1 lb. of fuel in British thermal units is V, where $V = 145C + 620H - 77{\cdot}5N$. Construct a nomogram for $70 < C < 90$, $2 < H < 6$, $4 < N < 8$. Read off from it the value of V when $C = 75$, $H = 3$, $N = 6$.

5. If $z = \frac{1}{\sqrt{y}}$ and $y = 49 - x^2$, express δz in terms of y, δy, and express δy in terms of x, δx; hence find δz in terms of x, δx, and deduce the value of $\frac{d}{dx}\left(\frac{1}{\sqrt{49 - x^2}}\right)$.

6. Sketch the graph of $(x-2)(5-x)$, and find the area of the portion of it which lies above the x-axis.

P. 6.

1. If $t = 2\pi\sqrt{\frac{l}{g}}$ is the same formula as $l = kt^2$, find k correct to 3 significant figures, given $\pi = 3{\cdot}142$, $g = 32{\cdot}2$.

2. If a jet of water is projected from a $\frac{3}{4}$-inch nozzle under a pressure of P lb. per sq. in., the effective height, h feet, of the jet is connected with P as follows:

P	40	50	60	70	80	90	100
h	61	67	72	76	79	81	83

Plot $\frac{P}{h}$ against P, and hence find an approximate formula for h in terms of P.

3. Represent by a straight-line graph the formula $l = 39t^2$ for $0{\cdot}5 < t < 1{\cdot}5$.

4. For ordinary wrought-iron shafting, if D = diameter of shaft in inches, H = indicated horse-power transmitted, N = number of revolutions per minute, $D=\sqrt[3]{\left\{\frac{65H}{N}\right\}}$; construct a nomogram for $10 < H < 100$, $10 < N < 300$. Read off from it the value of D when $H=80$, $N=100$.

5. Sketch the graph of the function $y=f(x)$ from $x=0$ to $x=8$, from the data in the following table:

x	0	0 to 1	1	1 to 2	2 to 4	4	4 to 6	6	6 to 8
y	0			+	−			−	
$\frac{dy}{dx}$	+	+	0	−	−	0	+	0	−

6. Interpret graphically $\int_0^1 \sqrt{1-x^2}\,dx$.

By using (i) Dufton's rule, (ii) Simpson's rule, find an approximate value of this definite integral.

P. 7.

1. If a 1 candle-power lamp is at A, and if $AB=1$ foot, the amount of illumination at B of a plane perpendicular to AB is called the unit of illumination. The amount of illumination varies inversely as the square of the distance from the light and directly as the candle-power. (i) How many units of illumination are given by an x candle-power lamp at a distance of r feet? (ii) $AC=15$ feet, a 20 candle-power lamp is at A and a 40 candle-power lamp at C; a screen perpendicular to AC cuts AC at P; find the position of P if both sides of the screen are equally illuminated.

2. Given that 1 litre = 1·76 pints, construct a simple line chart connecting pints and litres, up to 1 gallon.

3. The formula given for staying flat surfaces in boilers is $d=t\sqrt{\left(\frac{16000}{P}\right)}$, where d = pitch of stays, t = thickness of plate in inches and P = pressure of steam in lb. per sq. inch. Construct a nomogram for $0 < t < 1$, $0 < P < 200$.

4. The following swimming records are taken from *Whitaker's Almanack*:

Distance in yards, d -	150	220	300	440	500
Time in seconds, t - -	92·4	145·4	210	323	367·2

Plot log t against log d, and express t in terms of d. The record for 300 metres is 230·2 sec. Is this what your formula would lead you to expect? [1 metre = 1·094 yards.]

5. (i) At what point on the graph of $y = x^2 - 2x$ is the gradient 1, if the unit for each axis is 1 inch?

(ii) How is the answer to (i) affected if the unit for the x-axis is 1 inch and the unit for the y-axis 2 inches?

6. If the graph of $y = f(x)$ passes through the origin and has a maximum at $x = 1$ and a minimum at $x = 2$, but no other turning points, and if the slope at the origin is 1, find a value of $f(x)$.

P. 8.

1. $ABCD$ is a rectangular sheet of paper; $AB = a$ in., $BC = b$ in. $(a > b)$: a rectangle $APQD$ is cut off the ends just large enough to allow two equal circles to be cut from it so that when the remainder $PBCQ$ is rolled into a circular cylinder, these two circles will form its ends. (i) Express the volume of this cylinder in terms of a, b. (ii) If the area of the paper is 20 sq. in., for what value of b is the volume of this cylinder greatest?

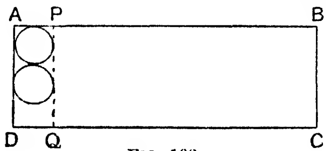


FIG. 108.

2. What equation can be solved from the graphs of $xy = 4$ and $y = x^2 - x$? By *sketching* their graphs state how many real roots this equation has. Check your answer by an algebraic solution, showing that $x = 2$ is one answer.

3. The following table gives the diameters of Cornish boilers in inches for various indicated horse-powers:

Horse-power, H - -	10	20	30	40	60
Diameter d, - - -	36	51	62·5	72·2	88·5

Plot log d against log H, and hence express d in terms of H.

4. Represent by a nomogram the formula (p. 300),

$$F = \frac{Wv^2}{30d} \text{ for } 150 < W < 200,\ 40 < v < 60,\ 150 < d < 200,$$

and read off the value of F for $W = 180$, $v = 50$, $d = 160$.

5. (i) Sketch the graph of $xy=4+x^2$.

(ii) At what points on this graph is the tangent parallel to the x-axis ?

6. The weight per inch length of a rod AB varies as the cube of its distance from A ; the rod is 20 inches long and the weight at B is 5 lb. per inch length. Find (i) the total weight of the rod, (ii) the length of AC if the portions AC, CB are of equal weight.

Q.

(CHAPTER XXIV.)

Q. 1.

1. If $a \neq b$, prove that $x(x+a)+b$ and $x(x+b)+a$ have a common factor if, and only if, $a+b=-1$.

2. If a number and its square root contain respectively p and q digits, prove that $2p-4q+1=(-1)^p$.

3. Write $5(a+2b)(3a-4b)$ as the difference of two squares.

4. What digit must the symbol x represent if the product of the two numbers $x5$ and $3x$ is the number $2xx5$?

5. In the sequence u_1, u_2, u_3, ... , if $u_n=2u_{n-1}+a^n$ and $u_1=a$, write down the value of u_2, u_3, and express u_n as a function of a and n.

6. Solve the equations: $\begin{cases} xy=72, \\ yz=60, \\ x+z=11. \end{cases}$

Q. 2.

1. What is the condition that the values of x and y found from the equations $ax+by=p+qx=p+qy$ may be, (i) inconsistent, (ii) indeterminate ?

2. Prove that the fifth power of any whole number has the same last digit as the whole number.

3. If $ab+bc+ca=0$. prove that $\sqrt{\dfrac{a+c}{b+c}}=\pm\dfrac{a}{b}$.

4. Prove that the relation $y=\dfrac{3(2x-5)}{x-2}$ can be written in the form

$$\frac{y-p}{y-q}=\frac{3(x-p)}{x-q},$$

and find the numerical values of p and q.

5. If a is the nth root of x and if y is nearly equal to x, then the nth root of y is approximately $\frac{y+(n-1)x}{nx}a$. Use this result to find the cube root of 1020, and compare your answer with that obtained by logarithms.

6. A company advances £95 to one of its employees for the purchase of £100 War Loan, and it is to be repaid by 10 equal annual instalments, the first to be paid in one year's time. Allowing compound interest at 5 per cent., find the amount of each instalment.

Q. 3.

1. If $a+\frac{1}{b}=c$, $b+\frac{1}{c}=d$, $c+\frac{1}{d}=a$, prove that $cd=\frac{1}{2}$, $bc=-\frac{1}{2}$, and find the numerical value of ab.

2. If $x(a-b)=y^2$ and $y(a+b)=x^2$, express a^2-b^2 and $\frac{a}{b}$ in terms of x and y.

3. Solve the equations : $\begin{cases} x+y=z=2xy, \\ \frac{1}{x}-\frac{1}{y}=1. \end{cases}$

4. If $(x+1)^2$ is greater than $5x-1$ and less than $7x-3$, find the only possible *integral* value of x.

5. Solve the equation $\frac{x^2-5}{x^2-1}+3+\frac{2}{x-1}=0$,

and explain the appearance of the root $x=1$, which does not solve the original equation.

6. If the result of substituting $\frac{X+Y}{\sqrt{2}}$ for x and $\frac{X-Y}{\sqrt{2}}$ for y in $ax^2+2hxy+by^2$ is $AX^2+2HXY+BY^2$, prove that

(i) $A+B=a+b$; (ii) $AB-H^2=ab-h^2$.

Q. 4.

1. Rewrite the formula $T_1=T_2\left(\frac{p_1}{p_2}\right)^{\frac{n-1}{n}}$ with n as the subject.

2. Solve the equations : $\begin{cases} x(1+y)=b(a-1), \\ y(1+x)=a(b-1). \end{cases}$

3. Prove that $x=\frac{1}{1-a}$ is a root of the equation :

$$x+\frac{1}{1-x}+\frac{x-1}{x}=a+\frac{1}{1-a}+\frac{a-1}{a}.$$

What are the other two roots ?

4. If
$$\frac{x^2-yz}{a}=\frac{y^2-zx}{b}=\frac{z^2-xy}{c},$$
prove that
$$\frac{a^2-bc}{x}=\frac{b^2-ca}{y}=\frac{c^2-ab}{z}.$$

5. A betting-man learns that only three horses are going to start in a certain race, and finds that the odds * quoted against them by a bookmaker are 3 to 2 against, 3 to 1 against and 2 to 1 against. Show that the man by adjusting his bets properly can make certain of winning £a, whichever horse wins, and find how much he must bet on each horse to do this.

6. Generalise Question 5 by considering the case in which three horses only are running and the odds quoted against them are p_1 to 1, p_2 to 1 and p_3 to 1; show that a man betting on them can be sure of winning £a, whichever horse wins, provided

$$p_1p_2p_3-p_1-p_2-p_3-2 \text{ is positive.}$$

Examine the case when $p_1=5$, $p_2=2$ and $p_3=1$.

Q. 5.

1. If b^2 is an arithmetic mean between a^2 and c^2, prove that $\frac{1}{a+c}$ is an arithmetic mean between $\frac{1}{a+b}$ and $\frac{1}{b+c}$.

2. If $x+y=1$, prove that $(x^2+y)^2=(y^2+x)(1-xy)$.

3. If $p=ny-mz$, $q=lz-nx$, $r=mx-ly$ and $x+y+z=0$, prove that $\frac{qrx+rpy+pqz}{xyz}$ can be expressed in terms of l, m, n only.

4. A circle of radius r is divided into 12 equal sections by 6 diameters. Through the extremity A of one of these diameters a perpendicular AN_1 is drawn to an adjacent diameter; then N_1N_2 is drawn perpendicular to the next diameter, and so on. Find an expression for the distance of N_n from the centre of the circle, and show that the total length of the spiral so formed cannot be greater than $r(\sqrt{3}+2)$.

[Use the fact that $\cos 30°=\frac{\sqrt{3}}{2}$, $\sin 30°=\frac{1}{2}$.]

5. Prove that $\sum_{a,b,c}(b-c)(x-b)(x-c)\equiv-(a-b)(b-c)(c-a)$.

6. (i) Find a symmetrical, homogeneous expression of the third degree in x and y, which has $3x-y$ as a factor, and is equal to 8 when $x=y=1$.

(ii) Find the most general rational integral algebraic function of x, of the fourth degree, which leaves remainder 1, when divided by $(x-1)$, $(x-2)$, $(x-3)$ or $(x-4)$.

* For the meaning of this, see paper G. 6, Question 1.

Q. 6.

1. If $\frac{1}{a+b}+\frac{1}{c}=\frac{1}{a}+\frac{1}{b+c}$, prove that $\frac{1}{(a+b)^3}+\frac{1}{c^3}=\frac{1}{a^3}+\frac{1}{(b+c)^3}$.

2. If $$f(x,\ y)\equiv 2xy+x+y+2=0,$$
and if $$X=\frac{x^3-3x-1}{3x(x+1)},\quad Y=\frac{y^3-3y-1}{3y(y+1)}$$
prove that $$f(X,\ Y)=0.$$

3. If $$\begin{cases} x^2=y^2+z^2-2lyz, \\ y^2=z^2+x^2-2mzx, \\ z^2=x^2+y^2-2nxy, \end{cases}$$
prove that $$\frac{x^2}{1-l^2}=\frac{y^2}{1-m^2}=\frac{z^2}{1-n^2}.$$

4. Find the conditions that $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ may be the square of an expression of the type $lx+my+nz$.

5. Find the factors of (i) $\sum_{x,y,z}(y-z)^3$;

(ii) $\sum_{a,b,c} a(b-c)^3$.

6. Prove that if x is real and a and b are positive quantities, such that $b>a$, then $\frac{x^2-ax+\frac{1}{4}b^2}{x^2+ax+\frac{1}{4}b^2}$ cannot be less than $\frac{b-a}{a+b}$ or greater than $\frac{a+b}{b-a}$.

Q. 7.

1. If $x^2-yz=a$, $y^2-zx=b$, $z^2-xy=c$ and $xy+yz+zx=0$, prove that $x^2y^2z^2=\frac{a^2b^2c^2}{(a+b+c)^3}$.

2. Prove that the values of x, y which satisfy the equations $(2x+y)^2=2x-3y+1$ and $x+2y=1$ must also satisfy the equation $x^2-xy+3y^2=0$.

3. If $x^{\frac{1}{2}}+y^{\frac{1}{3}}=1$, prove that $y^2-2y(1+3x)+(1-x)^3=0$.

4. Solve the equations: (i) $(x+1)(x+3)(x+5)(x+7)=384$;

(ii) $\sqrt[3]{1-x}+\sqrt[3]{2-x}=\sqrt[3]{3-2x}$.

5. Find the limits between which m must lie if both roots of the equation $x^2-(m-1)x+(m-1)(m-2)=0$ are real.

6. (*a*) What is the equation whose roots are α, β, γ?

(*b*) If α, β, γ are the roots of the equation $x^3+px+q=0$, prove that $\alpha+\beta+\gamma=0$; and express in terms of p, q the values of (i) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$;

(ii) $\Sigma(\alpha^2)$;

(iii) $\Sigma(\alpha^3)$.

Q. 8.

1. (i) If
$$\frac{ab}{b+x}-\frac{cd}{d+y}=\frac{bc}{x}-\frac{ad}{y}=z,$$
prove that either $\frac{x}{b}+\frac{y}{d}+1=0$ or $z=a-c$.

(ii) Given $ax^2=\frac{1}{y}+\frac{1}{z}$, $by^2=\frac{1}{z}-\frac{1}{x}$, $cz^2=\frac{1}{x}+\frac{1}{y}$,
prove that $$abcx^2y^2z^2=b+c-a.$$

2. If y is small, an approximate value for $\sqrt{(1+4y)}$ is $\frac{1+3y}{1+y}$; if $y=z+z^2$, find in terms of z the error in the approximation.

3. If $\Sigma x=0$, prove that
$$(x+y-z)(y+z-x)(z+x-y)=-8xyz.$$

4. Determine the values of a, b, c and d so that
$$a+b(x-1)+c(x-1)(x-2)+d(x-1)(x-2)(x-3)$$
may be identically equal to $4x^3-5x^2+6x-7$.

5. The following transformation is used in the theory of Relativity, $x_1=\beta(x-vt)$; $t_1=\beta\left(t-\frac{vx}{c^2}\right)$, where $\beta=\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$.
Express x and t in terms of x_1, t_1 etc., in as simple a form as possible. What do you notice about the forms of x and t so obtained ?

6. Solve the equations : (i) $\sqrt{a+x}+\sqrt{b+x}=\sqrt{a+b+2x}$;
(ii) $2x^4-7x^3+9x^2-7x+2=0$.

Q. 9.

1. If $x=m^2-n^2$ and $y=(2m+n)n$, prove that x^2+xy+y^2 is a perfect square.

2. If $x^3=2$, prove that $(l+mx+nx^2)(3-x+x^2)$
$$\equiv(3l+2m-2n)+(-l+3m+2n)x+(l-m+3n)x^2.$$

3. Find an equation whose roots are those of $2x^2-x-5=0$, each diminished by c, and find the value of c in order that in the new equation the sum of the roots may be zero.

4. Find the factors of : (i) $(a+b)(b+c)(c+a)+abc$;
(ii) $(a+b+c)^3-\Sigma(b+c-a)^3$.

5. Solve the equations : $\begin{cases} x^2+2yz=y^2+2zx=3, \\ z^2+2xy=19. \end{cases}$

6. (i) Write $\dfrac{7-3x}{(2-x)^2(3-x)}$ in the form

$$\frac{A}{(2-x)^2}+\frac{B}{(2-x)}-\frac{C}{3-x}.$$

(ii) When $f(x)$ is divided by $x-1$, the quotient is $\phi(x)$, and the remainder is 2 ; when $\phi(x)$ is divided by $x-3$, the quotient is $\psi(x)$ and the remainder is 4. What is the remainder when $f(x)$ is divided by $(x-1)(x-3)$?

Q. 10.

1. If $\dfrac{a^2-d^2}{b^2c^2-d^4}=\dfrac{a}{bcd}$, prove that either $\dfrac{a}{b}=\dfrac{c}{d}$ or $d^3=-abc$.

2. Prove that $a+b(1-a)+c(1-a)(1-b)+d(1-a)(1-b)(1-c)$

$$\equiv 1-(1-a)(1-b)(1-c)(1-d).$$

3. If $\dfrac{ax-b}{1}=\dfrac{bx-c}{y}=\dfrac{cx+a-1}{y^2}$ and $b^2 \neq ac$, prove that x, y are the roots of a certain quadratic equation.

4. The following is a method in use amongst Russian peasants for multiplying by numbers larger than 2.

To multiply 34×43, halve the left-hand number and double the right-hand number, ignoring fractions, and continue the process until the left-hand number is 1, thus : writing each line of working immediately under the preceding figures.

Cross out all the lines in which the numbers in left-hand column are even, and add up remaining numbers in right-hand column.

~~34 ×~~	~~43~~
17 ×	86
~~8 ×~~	~~172~~
~~4 ×~~	~~344~~
~~2 ×~~	~~688~~
1 ×	1376
	1462

Then $\qquad 34\times 43=1462.$

Show that any number can be written in the form

$$a\,.\,2^x+b\,.\,2^{x-1}+c\,.\,2^{x-2}+\ldots+k,$$

where $a, b, c, \ldots k$ are either 1 or 0, and hence justify this method of multiplication.

5. A number of cannon-balls are arranged in the form of a triangular pyramid, so that the sides of the successive layers are made up of 1, 2, 3, 4, ... balls, while the layers contain 1, 3, 6, 10, ... balls. It is known that when there are n layers, the number of balls in the pyramid is given by an expression of the form

$$an+bn^2+cn^3,$$

where a, b and c are independent of n. Find their values.

(Army C.)

6. Prove that $(\sum\limits_{x,y,z,} x)^3 - 2(\Sigma x)(\Sigma x^2) - 8xyz$ has a factor $x+y-z$, and hence find all its factors.

7. If $x=\phi(a)$ is one root of the equation

$$f(x)=f(a),$$

prove that $x=\phi[\phi(a)]$ is also a root.

Use this result to find the remaining roots of the equation in Paper Q. 4, No. 3.

SYMBOLS

$\simeq$ is approximately equal to.

$a : b$ equals $\frac{a}{b}$. See p. 284.

$a : b : c : d : e : \ldots$. See p. 288.

$\propto$ varies as.

$\leqq$ is less than or equal to.

$\geqq$ is greater than or equal to.

$\sqrt[n]{x}$. See p. 255.

$\log_a x$. See p. 280.

$f(x)$ a function of x.

Σ. See p. 454.

$\rightarrow$ tends to.

$\rightarrow +0$ tends to zero through positive values.

$\rightarrow -0$ tends to zero through negative values.

∞ infinity. See p. xv, 311, 324.

$\left.\begin{matrix} |a-b| \\ a \sim b \end{matrix}\right\}$ the positive value of $\pm(a-b)$ if a, b are real.

$\underset{x \to a}{\text{Lt}} f(x)$ the limit of $f(x)$ when x tends to a. See p. xv, 324.

δx a small change in x. See p. xvi, 339.

$\frac{d}{dx}$. See p. 340.

$\int f(x)dx$, $\int_a^b f(x)dx$. See pp. 368, 370.

${}^nP_r$, ${}^nC_r$. See p. 398.

$\underline{|n}$. See p. 395.

GLOSSARY AND INDEX

References in Roman numerals are to pages of the Introduction in the Teacher's Edition.

Factorial. Factorial n, where n is an integer is written $\underline{|n}$ or $n!$ and means the product of the numbers $n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1$.

Function. A number, or quantity, is said to be a function of one or more variables, if its value can be determined when the values of these variables are known. p. 306, p. xiv.

Functional notation. p. 320.

Geometrical mean (G.M.). p. 385.

Geometrical progression (G.P.). p. 385.

Gradient. A measure of the steepness of a graph. pp. 333, 338, p. xv.

Graphical solutions. p. 314.

Growth function. p. 392.

Homogeneous function. p. 454.

Incommensurable. Two numbers whose ratio cannot be expressed in the form $p:q$, where p and q are integers, are said to be incommensurable. p. xii.

Increasing function. p. 346.

Indefinite integral. p. 370.

Independent variable. A number which varies unconditionally and independently of other variable numbers.

Indices. Fractional. p. 260, p. ix.
Integral. p. 253.
Negative. p. 260.

Infinity. pp. 311, 324, p. xv.

Integer. A whole number.

Integral. p. 370.

Integration. p. 364.

Interpolation. When the value of a function is known for some values of the variables within a certain range, approximate values of the function may be determined from these for other values of the variables within this range ; this process is known as interpolation. p. xx. See Exx. 20, 21, p. 290, and Exx. 18,19, p. 422.

Irrational number. A number which cannot be expressed in the form $\frac{p}{q}$, where p and q are integers. p. 261, p. x.

Limit. p. 324, p. xv.

Limiting sum. The number to which the sum of a series tends as the number of terms of the series tends to infinity. p. 388.

Line chart. p. 424.

Logarithms. pp. 267, 269, 275, p. x.

Logarithmic graduation. p. 428.

Logarithmic notation. p. 280.

Logarithm paper. p. 429, p. xxii.

Mantissa. The decimal fraction of a logarithm. p. 270.

Maximum. The value of a function which is greater than all other values of the function in the immediate neighbourhood. p. 347.

Mean. The average of two or more quantities. See also Arithmetic mean and Geometric mean.

Minimum. The value of a function which is less than all other values of the function in the immediate neighbourhood. p. 347.

Nomogram.
Nomography. The graphical representation of equations and formulae containing more than two variables. pp. 424, 431 *et seq.*, p. xxii.

Ordinate. The y-coordinate of a point. p. 325.

Sequence. A succession of numbers which may or may not obey some definite law. p. 380.

Series. A succession of positive or negative numbers obeying some definite law and regarded as a single group. p. 380.

Variable. A number whose value changes.
See Dependent variable and Independent variable.

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

π may be taken to be 3·1416.

log π may be taken to be 0·4971.

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

ANSWERS.

PART II.

Note.—Where only one kind of unit occurs in the question, the unit is not specified in the Answer.

EXERCISE XII. a. (p. 253.)

2. $a^8, a^3, a^6, a^8, a^{16}, a^{12}, a^{11}, x^{15}$. **3.** $a^m \times a^n = a^{m+n}$.

4. $a^m \div a^n = a^{m-n}$. **6.** $a^{20}, a^{100}, a^{48}, 27a^{27}$.

7. $(a^m)^n = a^{mn}$. **8.** x^{14}. **9.** 3. **10.** x^{40}.

11. x^8. **12.** Yes. **13.** Yes. **14.** x^{20}.

15. x^8. **16.** 1. **17.** a^{27}. **18.** a^{34}. **19.** $64x^{12}$. **20.** 2.

21. 30,020,000,000 cm. per sec.; 4,500,000; £12,200,000.
26,000,000,000,000 miles; $2000 \times 4{,}500{,}000 = 9{,}000{,}000{,}000$.

22. $6{\cdot}28 \times 10^7$, $(6{\cdot}71)^3 \times 10^9$.

XII. b. (p. 256.)

1. Indices 2, 1, 0, −1, −2; values 100, 10, 1, 0·1, 0·01.

2. Indices 3, 2, 1, 0, −1, −2; values 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$.

3. Indices 5, 4, 3, 2, 1, 0, −1, −2, −3; values 16807, 2401, 343, 49, 7, 1, $\frac{1}{7}$, $\frac{1}{49}$, $\frac{1}{343}$.

4. 7, −5, 17, 0. **5.** 11, 3, −2, 10, 0. **6.** x^4, x^{-3}, x^0.

7. x^{-5}, x^{-5}, x^0, x^0. **8.** $\frac{1}{x^{12}}$. **9.** $a^7, a^7, a^8, a^8, 0, 1, a$.

10. $\frac{1}{a^2}, \frac{1}{a}, a^3, \frac{1}{a^2}$. **11.** $\frac{1}{9}, \frac{1}{8}, \frac{1}{4}$, 1, 16, $\frac{81}{16}$, 100, 1, 1, $\frac{3}{2}$.

12. $\frac{5}{6}, \frac{1}{6}, \frac{2}{3}$, −1. **13.** 8, 7, $7\frac{1}{2}$. **14.** 12, 14, 13.

15. $\frac{1}{2}$, 0. **16.** $x^4, x^{\frac{7}{3}}, x^{\frac{1}{3}}$. **17.** $x^2, x, \sqrt[3]{x^2}, \sqrt[3]{x}$.

18. $\sqrt[4]{x}, \sqrt[4]{x^3}$. **19.** 15, 6, 1. **20.** 9, $\frac{3}{5}$, $\frac{3}{4}$.

21. 4, 27, 2, 9. **22.** 256, 32, 27, 128, 64. **23.** 7, 5, $\frac{7}{12}$, $\frac{1}{5}$.

24. $\frac{1}{2}, \frac{1}{3}, \frac{1}{8}$, 0·001, 1. **25.** $x^{\frac{3}{2}}, x^{\frac{11}{4}}, x^{\frac{1}{2}}$. **26.** $6x^{\frac{5}{6}}, x^{\frac{1}{2}}$.

28. $1/\sqrt[4]{x^3}$, $1/\sqrt{x^5}$. **29.** 1·414, 1·189, 1·090, 1·044, 1·022, $2^0=1$.

30. 3·162, 1·778, 1·334, 1·154, $10^0=1$.

31. Number of zeros after decimal point is 9, 12, 23, 7, 9, 7, 4.

32. $\frac{l}{t^2}$, $\frac{m}{t^3}$, $\frac{ml^2}{t^2}$; $dt^3=k$, $\mu R^2=a^3$, $Nd^3=Hk^3$; $\frac{lt^2}{LT^2}$, $\sqrt{\frac{l_2}{l_1}}$, $p\sqrt[4]{v}$;
$\sqrt{a}+\sqrt{b}=\sqrt{c}$, $\sqrt[3]{x^2}+\sqrt[3]{y^2}=\sqrt[3]{a^2}$, $gt^2=4\pi^2 l$; $ye^4=1$.

33. 1·005, 1·01, 0·99, 0·99, 0·995, 1·01, 0·998, 0·098, 10·01, $10^{-10}\times 9{\cdot}91$.

34. $3x^2$, $\frac{3}{2}\sqrt{x}$, $-\frac{2}{x^3}$, $\frac{1}{2\sqrt{x}}$, $-\frac{5}{x^6}$, $-\frac{1}{2}x^{-\frac{3}{2}}$, 0, $-\frac{5}{3}x^{-\frac{8}{3}}$.

35. $\frac{1}{3}x^3$, $\frac{2}{3}x^{\frac{3}{2}}$, $-\frac{1}{x}$, $\frac{3}{4}x^{\frac{4}{3}}$, $-\frac{1}{2x^2}$, $\frac{2}{5}x^{\frac{5}{2}}$, $\frac{1}{2}x^2$, x, $\frac{3}{2}x^{\frac{2}{3}}$.

XII. c. (p. 260.)

1. $x^{\frac{1}{3}}$, $a^{\frac{2}{3}}$, $a^{\frac{2}{3}}$, $a^{\frac{1}{2}}b^{\frac{1}{2}}$, $2a^{\frac{1}{2}}b$, $3a^{\frac{1}{2}}b^{\frac{3}{2}}$, $a^{\frac{1}{2}}b$. **2.** $3x^{-2}$, x^{-1}, $x^{-1}y^{-1}$, $5x^{-1}y^{-\frac{1}{2}}$.

3. 4, 2. **4.** 3, $\frac{1}{8}$, -2, $-\frac{3}{2}$, $-\frac{1}{2}$.

5. $2a^{\frac{3}{2}}$, $a^{\frac{5}{12}}$, $a^{\frac{3}{2}}b^{\frac{3}{2}}$. **6.** $a^{\frac{1}{2}}$, $2a^{\frac{3}{2}}$, $a^{\frac{2}{3}}$. **7.** $a^{\frac{3}{2}}$, 1. **8.** x^2+1.

9. $x^{\frac{1}{6}}$. **10.** $\frac{x+1}{x}$. **11.** 16. **12.** x^2.

13. 0. **14.** 10^{3n}, $\frac{3y^3}{10x^2}$, x^{-4}, x^9, $(x+y)^2$.

15. 7, 13, -1, 2^n+2^{5n}. **16.** $\frac{2}{3}$.

XII. d. (p. 262.)

1. 7, 5, 12, 12, 2, 2. **2.** 12, 24, 0·3.

3. $3\sqrt{2}$, $6\sqrt{2}$, $5\sqrt{2}$, $20\sqrt{5}$, $b\sqrt{a}$, $\frac{1}{x}\sqrt{a}$. **4.** 17·3, 5·20, 0·866.

5. 0·577, 1·44. **6.** $9\sqrt{2}$, $\frac{5}{2}\sqrt{2}$, $10\sqrt{10}$, $\sqrt{3}$, $b\sqrt{b}$.

7. 7·07, 22·36, 70·7, 0·894. **8.** 1, 2·41.

9. 38, $\frac{3}{10}(5\sqrt{2}-2\sqrt{3})$. **10.** 0·268, 3·15, 1·24.

11. 4·31. **12.** 1. **13.** $2\sqrt{2}$, $\frac{1}{2}\sqrt{3}$.

14. 2·90. **15.** 24. **16.** 1·465, 2·07. **17.** 3·72.

21. $a+b+2\sqrt{(ab)}$; $a+b$; x or y must be zero. **22.** xy, xy, yes.

23. $\frac{2\sqrt{x}}{x-y}$, $\frac{x+y}{\sqrt{(xy)}}$.

XII. e. (p. 265.)

1. 1·15, 0·707, 4·59. **2.** 2·32. **3.** -7, -10.

8. 2, $\frac{1}{4}$, 1, 10, 0·01, 2.

9. (i) See p. 267. (iii) 1·58, 2·51, 5·01, 7·94. (iv) 0·48, 0·70, 0·91. (v) 1·58, 4·64, 3·98.

10. 0·903, 1·43, 2·301, 1·699.

12. Indices are 0·4771, −1 +0·9542, 0·4313.

XIII. a. (p. 268.)

Note.—Answers are given correct to three figures but, if according to four-figure tables the fourth figure is 5, this is shown.

1. 0·845. **2.** 0·544. **3.** Indices are 1·301, 2·301, 3·301, 6·301.

4. 2, 6. **5.** 1·6377, 2·6377, 4·6377, 6·6377.

6. 72·86, 728·6, 72860, 7,286,000. **7.** 1, 2, 4, 5.

8. 1, 3, 0, 5, 5, 3, 4, 5, 7, 8. **9.** Less, greater, less, greater.

10. Greater, less, less. **11.** 224, 51300, 14100000.

12. (i) Indices are 1·46, 3·785, 6·92. (ii) 1·52, 7·73, 3·87.

XIII. b. (p. 272.)

1. 35·0. **2.** 2·78. **3.** 76·2. **4.** 41900. **5.** 5·88.

6. 2040. **7.** 2170. **8.** 341000. **9.** 11·1. **10.** 9·09.

11. 128. **12.** 2·01. **13.** 3·02. **14.** 20·0. **15.** 13,200.

16. 845500. **17.** 24·7. **18.** 17·9. **19.** 5·73. **20.** 7·39.

21. 342. **22.** 1·51. **23.** 416. **24.** 1·24. **25.** 38·0.

26. 1·96. **27.** 18·7. **28.** 91·3. **29.** 5·91. **30.** 35·6.

31. 2·06. **32.** 6·52. **33.** 1·15. **34.** 2·48. **35.** 5·34.

36. 6·44. **37.** 2·08, 4·48, 9·65, 20·8. **38.** 17·6. **39.** 4·28.

40. 1·04. **41.** 50100. **42.** 3·23. **43.** 1·06. **44.** 1·195.

45. 6·08.

XIII. c. (p. 273.)

1. 60·0 sq. cm., 6·91 cm. **2.** 5·32 cu. cm., 1·06 cm.

3. 42·8 ft., 2·27 sec. **4.** 7·83 sec., 3·26 ft.

5. 168. **6.** 11·7, 4290. **7.** 3·90. **8.** 4·34.

9. 1300. **10.** 12·5. **11.** 2·64. **12.** 4·52 tons.

13. 12·2. **14.** 430. **15.** 21·5. **16.** 1·58.

17. 11550. **18.** $1{\cdot}1 \times 10^9$, 5×10^{47}.

XIII. d. (p. 276.)

1. 1·8. **2.** 2·1. **3.** 0·9. **4.** $\bar{5}$·6. **5.** $\bar{5}$·2.
6. $\bar{1}$·3. **7.** 0·1. **8.** 3·1. **9.** $\bar{5}$·3. **10.** 3·8.
11. $\bar{5}$·7. **12.** $\bar{1}$·8. **13.** $\bar{1}$·3. **14.** 2·7. **15.** 0·8.
16. 2·8. **17.** $\bar{1}$·2. **18.** $\bar{4}$·1. **19.** $\overline{13}$·82. **20.** $\bar{5}$·2.
21. 3·9. **22.** $\bar{1}$·3. **23.** $\bar{1}$·8. **24.** $\bar{3}$·85. **25.** $\bar{1}$·3.
26. $\bar{2}$·8. **27.** $\bar{1}$·76. **28.** $\bar{1}$·9. **29.** $\bar{2}$·9. **30.** 1·45.

XIII. e. (p. 276.)

1. 3·25. **2.** 0·246. **3.** 0·004205. **4.** 0·365. **5.** 0·533.
6. 0·04935. **7.** 12·2. **8.** 4·43. **9.** 0·0243. **10.** $10^{-6} \times 2{\cdot}57$.
11. 0·885. **12.** 0·280. **13.** 0·428, 0·199, 0·09215. **14.** 0·126.
15. 0·0267. **16.** 0·120. **17.** 0·00508. **18.** 21·7. **19.** 0·103.
20. 1·035. **21.** 0·6585. **22.** 0·598. **23.** 7·16, 0·452.
24. 0·734. **25.** 4·86. **26.** 0·667. **27.** 0·807.
28. $10^{-6} \times 9{\cdot}685$. **29.** 0·794. **30.** $10^{-14} \times 1{\cdot}05$, $10^{-5} \times 2{\cdot}6$.

XIII. f. (p. 277.)

1. $10^5 \times 1{\cdot}60$. **2.** 0·779. **3.** 0·0994. **4.** 11·5. **5.** 335.
6. 338. **7.** 112 sq. in. **8.** 631. **9.** 15 6. **10.** 0·694, 0·492.
11. 1·44. **12.** 3·27. **13.** 15·6. **14.** 0·0931. **15.** 951 tons.
16. $10^{-5} \times 4{\cdot}81$. **17.** 33·8, $10^{-10} \times 9{\cdot}93$. **18.** 2330. **19.** 0·708.
20. 3·63. **21.** 0·516. **22.** 2·69, 2·72.
23. £2080; 20s. 9d.; £$10^{12} \times 8$. **24.** £294. **25.** 0·909.
26. £2940; annuity is worth £5830; £368.

XIII. g. (p. 279.)

1. 7·39. **2.** 7·71. **3.** 82·5. **4.** 82·5. **5.** 15·1.
6. 21·2. **7.** 37·1. **8.** 0·0111. **9.** 3·73. **10.** 37·9.
11. 1·54. **12.** 2·605. **13.** 0·236. **14.** 9·90. **15.** 4·27.
16. 1·69. **17.** 2·23. **18.** 2·51.

XIII. h. (p. 281.)

1. 2 log 2, 3, 0, $\frac{1}{2}$. **2.** 2, 2, 2. **3.** log (1 + 3), log 2 + log 4.
4. log 2 − log 1, equal, log (6 − 3). **5.** $\sqrt{2}$.

6. 2, $\frac{1}{2}$, 2 $\log y$. **7.** 0, $\log x$. **8.** $\frac{3}{4}$.

9. $xy=5$, $xy^3=2$, $3^x=1000$, $x^3=10y^2$, $5^x=6^y$, $5^x=100\times 6^y$.

10. 13, 2, -1. **11.** 1·06. **12.** 1·87. **13.** 1·585.

14. 4·8, 0·053. **15.** 2, 3, 2. **16.** 0·312. **17.** 23·5, 14·2.

18. 4·1. **19.** 4780 ft. **20.** 1·13.

XIV. a. (p. 285.)

1. 8 : 3, 2 : 3, 1 : b^2, 1 : x, $5x : 4y$, 1 : 4, 1000 : 1, $(a+b) : a$, $3a : b$, $5x : 2y$.

2. 9 : 2. **3.** -4.

5. $\frac{a}{c}=\frac{d}{b}$, $\frac{p}{q}=\frac{r}{p}$, $\frac{a+b}{p+q}=\frac{e+f}{c+d}$, $\frac{x}{p}\cdot\frac{y}{q}\cdot\frac{z}{r}=1$, $\frac{a}{d}\cdot\frac{b}{e}\cdot\frac{c}{1}=1$.

6. $\frac{bx}{a}$. **7.** $\frac{10}{11}$. **8.** $\frac{4}{3}$.

9. $\frac{x+y}{x}$. **10.** $\frac{1}{2}$. **11.** $\frac{4a}{5b}$.

12. £$\frac{ap}{a+b}$, $\frac{bp}{a+b}$. **13.** $aq : bp$. **14.** 28, 20 ; $\frac{pc}{p-q}$, $\frac{qc}{p-q}$.

15. $\frac{5x}{3}$. **16.** 8 : 3 **17.** 6 : 1.

18. $\frac{bx}{a}$, $\frac{100(b-a)}{a}$. **19.** Greater. **20.** Less.

21. $\frac{1}{2}$. **22.** 8 : 6 : 9. **23.** $\frac{l(a-b)}{2(a+b)}$ ft.

24. $\frac{px}{qy}$, $\frac{px}{q(x+y)}$. **25.** $\frac{6}{5}$ or $-\frac{2}{11}$ **26.** 3 : 2.

27. 5 : 4. **28.** $\frac{ab}{a+b}$. **29.** $\frac{65}{56}$.

30. 1. **31.** 3 : 2. **32.** 1 : 2.

33. 27 : 8, 75 : 4. **34.** Increased in ratio 3 : 2.

35. $\frac{a+x}{b+x}<\frac{a}{b}<\frac{a-x}{b-x}$. **36.** $\frac{b-a}{a}$.

37. $\frac{(a+b)(qc-pd)}{qa-pb}$. **38.** 2000. **39.** $\frac{x}{y}$. **40.** 8 : 5.

XIV. b. (p. 288.)

1. 6, $\frac{c^2d}{a}$, x. **2.** $4\frac{1}{2}$, $\frac{bc^2}{a}$, xy^2. **3.** 6, a^2b^2, $\sqrt{(ab)}$.

4. 11, 5, 4. **5.** 1, 14 : 21 : 6. **6.** 29 : 9 : 6.

7. 5 : 7 : 4. **10.** $b+d+f$, $a-c+e$, $10b-7d+2f$, $\frac{b}{2}+\frac{f}{3}$.

12. $\frac{55}{24}$. **13.** 5 or $-\frac{1}{5}$. **14.** -2, $\frac{1}{3}$. **15.** $\frac{3}{7}$ or 0.

20. 5, 20 (19), 8, 20 ; ($z=x^2y+2x+y$). **21.** 0·104, 0·170.

XIV. c. (p. 293.)

1. Yes, doubles y, doubles x, $y=88x$.

2. No, halves y, doubles x, $y=\frac{480}{x}$.

3. Trebles y, halves x, line through origin.

4. Halves y, divides x by 10, curve, area is constant.

5. $y \propto \frac{1}{x}$; $x=6, 3, 2, 1·5, 1·2$; $\frac{1}{x}=\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}$.

6. Halves x, divides y by 16, curve. **8.** $C \propto x^2$.

9. $y \propto \frac{1}{x^2}$. **10.** 3 : 1, 10 : 3, 18 : 5, no.

11. $y=4x$, $y=2x^2$, $xy=10$, $y=\sqrt{x}$. **12.** $A \propto h$; 9, 12, 15, 24, 30.

13. 0, 3, 12, 27, 75, 300. **14.** 8, 2, 0·8, 0·4.

15. 2·6, 20·8, 166·4. **16.** 0, 2, 2·83, 4, 10, 20.

17. $A \propto r^2$, $A \propto x^2$, $t \propto \frac{1}{x}$, $y \propto x$, $h \propto \frac{1}{d^2}$, $y \propto x$. **18.** $y=\frac{3}{4}x^2$.

19. 7·5. **20.** Quadruples y.

21. Increases y in ratio 3 : 2. **22.** Divides y by 4.

23. $x^2=5y$. **24.** $xy=8$.

26. Increases in ratio 512 : 1, $V^2 \propto S^3$. **27.** Yes, $P=0·00492v^2$.

28. 7·78 mi., $N=1·23\sqrt{h}$. **29.** $6\frac{3}{4}$ gallons.

30. £4. 13s. 9d. **31.** 1·41 : 1, 1·26 : 1.

32. 1s. 9d. **33.** $1·755x$ lbs. **34.** 1·59 : 1.

35. 1 sec., $t=0·2\sqrt{l}$. **36.** 20·6 ft. **37.** Multiplies sag by 27.

38. 95·2, 64, 8·16 ; 98·75, 97·5, 75 lbs.

39. $C=80·2d^{\frac{3}{2}}$, 147 amp., 13·4 amp. **40.** 0·414 units, 66·2 ft.

XIV. d. (p. 301.)

1. $V \propto h$, $V \propto r^2$, $V \propto r^2h$, $V=\frac{1}{3}\pi r^2h$.

2. $W \propto b$, $W \propto d^2$, $W \propto \frac{1}{l}$, $W \propto \frac{bd^2}{l}$, $W=\frac{3bd^2}{16l}$.

3. $T \propto \frac{1}{l}$, $T \propto v^2$, $T \propto \frac{v^2}{l}$, $T=\frac{3v^2}{32l}$, 28·3 ft. per sec., $v \propto \sqrt{l}$.

4. $T \propto ln^2W$, $T = \frac{\pi^2 ln^2 W}{8}$, $n \propto \frac{1}{\sqrt{l}}$, $l \propto \frac{T}{n^2}$, doubles T.

5. Top row 270, 315, 360 ; second, 216, 252, 288 ; third, 180, 210, 240.

6. Top row 3, 6, 9 ; second 12, 24, 36 ; third 27, 54, 81.

7. $\frac{24}{xy^2}$.

8. $S \propto lr$, $p \propto \frac{T}{V}$, $t \propto \frac{d}{v}$, $p \propto \frac{lx^2}{d^5}$, $t \propto \sqrt{\left(\frac{l^3}{dp}\right)}$.

9. $d = 0{\cdot}8 \frac{Wv^2}{R}$.

10. $d = \lambda . \sqrt[4]{\left(\frac{i^2H}{P}\right)}$, reduced 27 per cent.

11. $H = 0{\cdot}00504 d^{\frac{2}{3}} v^3$.

12. $C \propto Tv^3$, $C = \frac{d^3}{250T^2}$.

13. $P = \frac{900W}{D} + \frac{22500W}{D^3}$, 50 lbs.

14. $xz^2 = 162y$.

15. $A \propto \frac{y^4}{x}$.

16. $v \propto \sqrt{d}$.

17. $v \propto \sqrt{\frac{H}{W}}$, $v \propto \sqrt[3]{\frac{H}{A}}$, 15·9 lbs., 79 lbs.

18. 27·6 H.P., $H = 10^{-6} \times 2{\cdot}87 A v^3$.

19. 1·52 lbs. per sq. in.

20. Decreases in ratio 1 : 6.

XV. a. (p. 307.)

1. 1, 3 ; $x > 3$ and $x < 1$.

2. No.

3. 2, -3 ; $2 > x > -3$.

4. 9 ; 4, -2 ; $-\infty$, $-\infty$.

7. $x > 10$ or $x < -10$; $0{\cdot}1 > x > -0{\cdot}1$.

8. Small, positive ; small, negative ; large, positive ; large, negative.

10. $\sqrt{(25-x^2)}$; circle, centre O ; $\pm\sqrt{(25-x^2)}$.

14. Yes, 2·01, 1·999, 1 ; $2 > x > 1$; no, 1002, -998.

15. 1, 3, 4 ; $x > 4$ and $3 > x > 1$; 378, -12, -2002 ; yes, *e.g.* 104.

16. Yes, *e.g.* 2·999 and 3·001 ; no ; 1, 5 ; 0·3, 0·3 per cent.

18. 98, -102, -6, -201, 199 ; small, 0·000001 ; $-0{\cdot}000001$; $3 > x > 2$ and $1 > x$.

23. $1 > x > -2$; 1, -2.

XV. b. (p. 311.)

1. $y = \frac{1}{2}(x+4)$.

2. $y = \sqrt{(8x - x^2)}$.

3. $y = \sqrt{(8x - 16)}$.

4. $y = \sqrt{(6 - x^2)}$.

5. $z = \sqrt{(4x - x^2)}$; $y = \sqrt{\frac{x^3}{4-x}}$.

6. $y = \frac{4}{3}\sqrt{(9 - x^2)}$.

7. $z(1-x)(3x-1)$.

8. $y = \frac{abx}{x^2 + b^2}$.

9. $y = \frac{d}{x}\sqrt{(d^2 - x^2)}$.

10. $x = \frac{c(by + az)}{(a+b)(b+c)}$.

11. $V = 0{\cdot}0293\, A^{1{\cdot}5}$.

12. $\frac{1}{2}(x^3 \sim 1)$. **13.** $\frac{\pi x^2}{192}\sqrt{(16-x^2)}$. **14.** 2^{-x}, 2^{x-1}.

15. $z=\frac{bx-ay}{x-y}$. **16.** $\frac{y}{\sqrt{2}}$, $\frac{y}{\sqrt{2}}$; $y=\frac{ax\sqrt{2}}{x+a}$.

17. $z=\sqrt{(xy)}$; $z=\sqrt{\frac{x}{y}}$. **18.** $z=\sqrt[5]{(x^3y^2)}$; $z=2\sqrt[5]{(x^3y^2)}$.

19. $(x-2)(x-3)$. **20.** $\frac{x-1}{x-2}$. **21.** $(3-x)(x-1)$

22. $-(x^2-1)^2$. **23.** $\sqrt{(4-x^2)}$. **24.** $(x-1)^2+2$.

XV. c. (p. 315.)

1. 2·5. **2.** 8. **3.** 9·2. **4.** 7·5, $\frac{1}{2}x\sqrt{(10x-x^2)}$

5. 2·94. **6.** $A=\frac{10x(x-10)}{x-20}$, 5·86.

7. $A=x^2+\frac{800}{x}$, 3·68, 10·1 or 1·54. **8.** $V=4x(12-x)^2$, 1024 cu. in.

9. $y=\frac{x^2+25}{2x}$, 5, 2·10 or 11·90. **10.** $\frac{3}{2}x(8-x)$, 4.

11. $V=\frac{1}{2}x^2(15-4x)$, 15·6. **12.** $V=2\pi r^2(6-r)$, 201.

13. −4, 1, 4, 5, 4, 1, −4; (i) 3·24 or −1·24; (ii) $3{\cdot}24 > x > -1{\cdot}24$; (iii) 5, $x=1$; (iv) 2·87 or −0·87; (v) 2, 2·73 or −0·73; (vi) 3·45 or −1·45.

14. 4·2, 1·8, 0·2, −0·6, −0·6, 0·2, 1·8, 4·2, 7·4; (i) 0·82 or −1·82; (ii) $0{\cdot}82 > x > -1{\cdot}82$; (iii) −0·7, $x=-0{\cdot}5$; (iv) 1·56 or −2·56; (v) 4, 0·8, 1·44 or −2·44; (vi) 0·62 or −1·62.

15. −9·6, −3·6, −1·2, −1·2, −2·4, −3·6, −3·6, −1·2, 4·8; (i) 4·27; (ii) $x>4{\cdot}27$; (iii) −3·83, $x=2{\cdot}53$; (iv) −0·974, $x=-0{\cdot}528$; (v) 4·52; (vi) (*a*) 3·78, 0·71, −1·49; (*b*) −2·31; (*c*) 4·62; (vii) 3, 2, −2.

16. 1, 1·62, −0·62; 2·20; 0·66. **17.** 2·11, −0·25, −1·86.

18. No real solutions.

XV. d. (p. 320.)

1. 3, 2, 3, $4a^2+2$, b^6+2. **2.** 10, 100, 1, 0·1, 10^{2x}.

3. 3, 0·301, 1·301, 0, 3 log x. **4.** 3, $(x+h)^2-3(x+h)+5$, $\frac{1}{x^2}-\frac{3}{x}+5$.

5. $2x+h+3$; $2x+3$. **6.** 1, $-\frac{1}{1+h}$, −1. **7.** $9x^2+5$, 2.

8. x^6, $2h^2$. **11.** 1, 2. **12.** $\frac{3}{4}$. **18.** $\frac{x-1}{x}$.

14. $(x-3)(x-4)$. **15.** $8(x-3)$. **16.** $\frac{3(x-3)}{x-1}$.

17. $3x^2$ or x^2+2. **18.** $\log x$. **19.** 10^x. **20.** x^3-x. **21.** $(-1)^x$.

XVI. a. (p. 325.)

1. Yes, *e.g.* 3000; no; 2. **2.** 11, no, 2.

3. 0·490, 0·4975; 0·5; no. **4.** Approx. 0·09, 0·01, 0·001, 0·0001; 0.

5. -1, 7. **6.** 1·5, 1·67, 1·91, 1·99; 2; no; $0{\cdot}9991 < x < 1{\cdot}001$.

7. $\frac{1}{2}(h^2+3h+3)$, 1·5. **8.** $16(h+1)$, 16. **9.** 5·25, $5-2a$, 2·5.

10. $-\frac{1}{4}, -\frac{1}{a^2}, \pm 2$. **11.** $\frac{n}{2}, \frac{1}{2}$. **12.** $\frac{50(n-1)}{n}$, 50, 50.

13. $\frac{1}{3}, \frac{(n+1)(2n+1)}{6n^2}, \frac{1}{3}$. **14.** $\frac{1}{3}; \frac{(n-1)(2n-1)}{6n^2}, \frac{1}{3}; \frac{1}{3}$.

XVI. b. (p. 330.)

1. 70; 250; 1890 to 1900, 390.

2. $\frac{x}{a}, \frac{y}{b}, \frac{y-x}{b-a}$, $y-x$ miles per min.

3. 8·4, 24·9, 27·3, 29, 37·2 m.p.h.

4. 1·5, 2·6, 2·25, 3·2, 1·3, 2·4 ft. per sec.

5. 0·3; 1·8; 3; 2·4, 1·2 tons per in.

6. For $t=0$, 5, 10, 15, 20, 25, rate is 1·76, 0·88, 0·43, 0 21, 0·11, 0·05 cu. ft. per sec.

XVI. c. (p. 334.)

1. 0·47, $-2{\cdot}9$, 0. **2.** 0·021, $-0{\cdot}020$, 0·021, 0·009, $-0{\cdot}0077$.

4. 0·93, 0·38 ins. **5.** 2, 2, 2. **6.** -5. **7.** 0·6.

8. 3 ft., 3 ft. per sec.; 12 ft., 6 ft. per sec., 9 ft. per sec.; 13·23 ft., 12·3 ft. per sec.; $3(2+h)^2$ ft., $12+3h$ ft. per sec.; 12·3, 12·03, 12·000003, 12 ft. per sec.

9. 144 ft., 48 ft. per sec.; 64 ft., 80 ft. per sec.; 134·56 ft., 94·4 ft. per sec.; $16(3-h)^2$ ft., $96-16h$ ft. per sec.; 94·4, 95·84, 95·999984, 96 ft. per sec.

10. 32 ft. per sec. **11.** $32a$ ft. per sec.

12. 23; $3h^2+17h+23$; $3h+17$ ft. per sec.; 17 ft. per sec.

13. 3, 2, 0, -2. **14.** 0·15, 0·18 cm. per sec.

15. 4, $6-h$, $12-2a-h$, $12-2a$, 6, $y=12x-x^2$. **16.** 5·1, 5, $-1{\cdot}5$.

17. $-\frac{1}{4}, -\frac{1}{a^2}$. **18.** $6c^2$. **19.** $\frac{3}{4}, \frac{1}{2}$. **20.** m.

21. $4a+b$. **22.** $2ax_1+b, -\frac{b}{2a}$. **24.** $2x$. **25.** 0.

XVII. a. (p. 341.)

1. $5 . \delta x$, 5.

2. $2x . \delta x + (\delta x)^2$, $2x$.

3. $10x . \delta x - 3\delta x + 5(\delta x)^2$, $10x - 3$.

4. $(1 + x + \delta x)^2$, $2x + 2$.

5. $1 + 2x$.

6. $2x$, $6x$, $14x$, $200x$.

7. $3x^2$, $12x^2$, $15x^2$, $87x^2$.

8. $12x^3$, $40x^3$.

9. $35x^4$, $5x^4$.

10. $4\frac{1}{2}x^5$, $-6x^5$, $1 - 12x^5$.

11. $\underset{\delta x \to 0}{Lt} \dfrac{(x + \delta x)^n - x^n}{\delta x} = nx^{n-1}$.

12. $3x^2 + 4x$, $6x + 3x^2$, $2x^2 + x$.

13. $3u + 5v$.

14. 3, 5, $30x$, 8, no, yes.

15. Yes, yes, no, no.

16. $\dfrac{d}{dx} x^n = nx^{n-1}$.

17. $\dfrac{d}{dx}(cx^n) = c \dfrac{d}{dx}(x^n)$.

18. $\dfrac{d}{dx}(Ax^m + Bx^n) = \dfrac{d}{dx}(Ax^m) + \dfrac{d}{dx}(Bx^n)$.

20. The differentials of x^8, x^{50}, x^{-1}, x^{-3}, x, x^0, $x^{\frac{3}{2}}$, $x^{\frac{1}{2}}$, $x^{-\frac{1}{2}}$, x^{-m} are $8x^7$, $50x^{49}$, $-x^{-2}$, $-3x^{-4}$, 1, 0, $\frac{3}{2}x^{\frac{1}{2}}$, $\frac{1}{2}x^{-\frac{1}{2}}$, $-\frac{1}{2}x^{-\frac{3}{2}}$, $-mx^{-m-1}$.

21. 0·41. **22.** $-\frac{1}{30}$. **23.** 0·331, 0·030301. **24.** 0·008. **25.** -20.

XVII. b. (p. 345.)

1. $7x^6$.

2. $30x^2$.

3. $1 - \dfrac{1}{x^2}$.

4. $-\dfrac{5}{x^6}$.

5. $-\dfrac{3}{x^2}$.

6. $-\dfrac{4}{x^3}$.

7. $6x - 2$.

8. x^3.

9. $5x^4$.

10. $375x^2$.

11. $2x - 2$.

12. $-\dfrac{n}{x^{n+1}}$.

13. a.

14. ax^{a-1}.

15. $-\dfrac{3c}{x^4}$.

16. $\dfrac{b}{2\sqrt{x}}$.

17. $\frac{3}{2}\sqrt{x}$.

18. $-\dfrac{1}{2x^{\frac{3}{2}}}$.

19. 0.

20. $\frac{1}{3}x^{-\frac{2}{3}}$.

21. $2x^2 - \frac{1}{5}$.

22. $\frac{1}{10} - \dfrac{10}{x^2}$.

23. $21x^2 - 2$.

24. $4\pi x^2$.

25. $12x^3 - \frac{1}{2} + \dfrac{6}{x^2}$.

26. $20x^3 + 6x - \dfrac{1}{x^2} + \dfrac{2}{x^2}$.

27. $3x^2 + 4x + 1$.

28. $2x - \dfrac{2}{x^3}$.

29. $2nx^{2n-1}$.

30. $-kx^{-k-1}$.

31. 6.

32. 6.

33. 1, 2.

34. $42x$.

35. $\mp\frac{1}{4}x^{-\frac{3}{2}}$

39. 68, 100; -32, -32.

40. $392x^7$, $392x^7$.

41. $12 + 20x$, 5, $2(3 + 5x)^2$, $60 + 100x$.

42. $4 + 6x^2 + 3x^4$, $2x$, $2 + 4x^2 + 3x^4 + x^6$, $8x + 12x^3 + 6x^5$

43. p.

44. $\frac{125}{36}$.

XVII. c. (p. 348.)

1. Decr., incr., $-$, $+$, yes. **2.** Incr., incr., $+$, $+$, no.

3. Incr., incr., incr., decr., decr., $+$, $+$, $+$, $-$, $-$, yes.

4. Max. C; min. A, E. **6.** Max.; decr., decr.; $-$, $-$.

7. Min.; incr., incr.; $+$, $+$. **8.** No; decr., incr.; $-$, $+$.

9. No; incr., decr.; $+$, $-$. **10.** $+$, $-$; $-$, $+$; $-$, $-$; $+$, $+$.

11. Yes.

15. AB $-$, $-$ $+$; BC $+$, $+$ $-$; CD $-$, $-$ $+$; DE $+$, $+$ $-$; EF $-$, $-$ $+$. **18.** 1. **19.** 3.

XVII. d. (p. 354.)

1. $x=1$ min.; $x=-2$ min.; $x=0{\cdot}4$ max.; $x=1$ min., $x=-1$ max.; none; $x=1$ min.; $x=\frac{3}{4}$ min., not $x=0$; $x=1$ min., $x=-\frac{1}{3}$ max.; $x=3$ min., $x=-1$ max.; none.

2. 40,000 sq. ft. **3.** 2 cu. ft. **4.** 4 ft. **5.** 8 in.

6. 80,000 sq. ft. **7.** $13\frac{1}{2}$ sq. ft. **8.** 12 knots. **9.** $10\frac{1}{8}$ sq. ft.

10. $1\frac{3}{4}$ hrs. **11.** 18 cu. ins. **12.** 0·927. **13.** 8 ft.

14. 2·55 cu. ft. **15.** 4 ins. from top. **16.** 3 ins.

17. 0·184. **18.** 8, 4 ins. **19.** $\frac{1}{4}$. **20.** 1·5 ft.

22. 12 cu. in. **23.** 1·68 in. **24.** $31\frac{1}{4}$. **25.** $\frac{1}{4}$.

26. $\frac{3\sqrt{3}}{16}$. **27.** $\sqrt{2}$. **28.** 8·16, 7·07 in.

29. On BA produced, $PA=3$ in. **30.** $y=\frac{\sqrt{(4x^4+a^2b^2)}}{2x}$, 500, 707 yds.

31. $\frac{1}{2}x(a-2x)(b-2x)$, 1·27 ins.

XVII. e. (p. 358.)

1. $5(\delta x)^2$, $3x(\delta x)^2+(\delta x)^3$. **2.** $x.\delta x+\frac{1}{2}(\delta x)^2$, $A=\frac{1}{2}x^2$, $\frac{1}{2}(\delta x)^2$.

3. $2\pi r.\delta r$, 0·63 sq. in. **4.** 0·126 cu. in.

5. $\delta t(10-0{\cdot}6t^2)$, 0·76 ft. **6.** $-\frac{c}{p^2}.\delta p$.

7. $\delta y=\frac{8x}{3l}.\delta x$; length of wire increases 0·05 in. **8.** 540 ft.

9. 210 sq. yd. **10.** $\delta M=\frac{\delta k}{10}\left(1-\frac{1}{k^2}\right)$, $-0{\cdot}12$ tons.

11. $\delta A=x^2.\delta x$. **12.** $\delta V=\frac{1}{4}\pi x^2.\delta x$.

13. $\delta V = \pi y \, . \, \delta y.$ **14.** $\delta V = \pi(2ax - x^2) \, . \, \delta x.$

15. $3 \, . \, \delta x$; $15(1+3x)^4 \, . \, \delta x$; $15(1+3x)^4$.

16. $\dfrac{1}{2\sqrt{z}} \, . \, \delta z$; $2x \, . \, \delta x$; $\delta y \simeq \dfrac{x \, . \, \delta x}{\sqrt{(1+x^2)}}$; $\dfrac{x}{\sqrt{1+x^2)}}$.

17. $\delta y \simeq 3(x^3 - x + 7)^2(3x^2 - 1) \, . \, \delta x$; $3(x^3 - x + 7)^2(3x^2 - 1)$.

18. $\delta y \simeq \dfrac{-6 \, . \, \delta x}{(3x-5)^3}$; $-\dfrac{6}{(3x-5)^3}$. **19.** $\delta u = 2y \, . \, \delta y$; $\dfrac{du}{dx} = 2y \dfrac{dy}{dx}$.

20. $\delta V \simeq 4\pi r^2 \, . \, \delta r$; $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$.

XVII. f. (p. 361.)

1. 0·12 cu. in. per hr. **2.** 1·5 in. per sec.

3. 0·637, 0·326 in. per sec. **4.** $\frac{11}{6}$ ft. per sec.

5. 0·057 lbs. per sq. in. per min. **6.** 0·31 in. per sec.

7. $10^{-10} \times 3V^5$, 3. **8.** 0·96, 0, −0·96 sq. in. per min.

9. 0·00133 in. per min. **10.** 0·242 in. per sec.

11. 0·286 in. per sec. **12.** 1·39 sq. cm. per sec.

13. −0·665 mm. per sec. **14.** 1·00564.

15. $\dfrac{dQ}{dt} = k(30 - Q)$, where k is a constant.

XVIII. a. (p. 365.)

1. $y = x^4 + c.$ **2.** $y = \frac{1}{5}x^5 + \frac{1}{2}x^2 + c.$

3. $\dfrac{dy}{dx} = 2x$, parallel curves. **4.** $y = \frac{1}{12}x^4 + ax + b.$

5. $y = \frac{1}{2}x^2 + c.$ **6.** $y = x^3 - x^2 + c.$

7. $y = 4x + c.$ **8.** $y = ax + b.$

9. $5x^4$, $y = \frac{1}{5}x^5 + c$, $y = \frac{3}{5}x^5 + c.$ **10.** $8x^7$, $y = \frac{1}{8}x^8 + c$, $y = \frac{5}{8}x^8 - 2x + c.$

11. $-\dfrac{1}{x^2}$, $y = -\dfrac{1}{x} + c$, $y = -\dfrac{4}{x} + c.$ **12.** $-\dfrac{4}{x^5}$, $y = -\dfrac{1}{4x^4} + c$, $y = x + \dfrac{3}{4x^4} + c.$

13. $\dfrac{1}{2\sqrt{x}}$, $y = 2\sqrt{x} + c$, $y = \frac{1}{2}x^2 + 6\sqrt{x} + c.$ **14.** $y = \frac{5}{3}x^3 - \frac{3}{2}x^2 + 9x + c.$

15. $y = \frac{1}{6}x^6 - \frac{1}{3}x^3 + 3x + c.$ **16.** $y = \frac{1}{4}x^4 - \dfrac{1}{2x^2} + c.$

17. $y = \frac{10}{3}x^{\frac{3}{2}} + c.$ **18.** $y = -\dfrac{3}{x} - \dfrac{5}{2x^2} + c.$

19. $y=\frac{1}{3}x^3-\frac{5}{2}x^2+6x+c.$ 20. $y=\frac{1}{12}x^4-\frac{1}{6}x^3+ax+b.$

21. $y=\frac{3}{101}x^{101}-\frac{5}{99x^{99}}+c.$ 22. $c=-1.$

23. $s=12+50t-16t^2.$ 24. $s=\frac{2}{3}t^3+5t+8.$

25. 165 ft. 26. $y=x^3-x+3.$

27. 0 ; 0 ; $y=\frac{0\cdot0001}{12}x(20-x)(400+20x-x^2)$, 0 ; $\frac{1}{15}$, $-\frac{1}{15}$; 5 in.

28. 10,000 ft. lb.

29. $x=12t$, $y=16t^2$; $y=\frac{1}{9}x^2$; $2\frac{1}{2}$ sec. ; 30 ft. ; $\frac{20}{3}$; $x=10$, $y=\frac{100}{9}$.

30. $\frac{3}{4}$ in. per sec., 8 in. 31. $\frac{dT}{dx}=a(K-T).$ 32. $\frac{dp}{dx}=-ap.$

XVIII. b. (p. 371.)

1. $\frac{1}{3}x^3$, $2\frac{1}{3}$, $\int_1^2 x^2dx$. 2. $\frac{2}{3}x^{\frac{3}{2}}$. 3. 4.

4. 1, 4 ; $4\frac{1}{2}$. 5. $\frac{1}{2}$, 0·83. 7. $7\frac{1}{3}$, $\frac{3}{32}$, $-\frac{2}{3}$, $\frac{19}{24}$.

8. πy^2, $\pi(y+\delta y)^2$, $\frac{1}{5}\pi x^5$, 0·148. 9. $\frac{1}{27}\pi x^3$. 10. 1·57.

11. $\sqrt{(100-x^2)}$, $\pi(100-x^2)$, $\pi(100-x^2).\delta x$, $\pi(100x-\frac{1}{3}x^3)$,
$\frac{\pi}{3}(2000-300x+x^3)$, 2090 cu. in.

12. 47·4 cu. in. 13. 14·2 cu. ft. 14. $2\frac{2}{3}$. 15. $\frac{23}{48}$.

16. 0. 17. Each $=8\frac{5}{8}$. 18. 2, $\frac{5}{8}$ in. 19. 1·56 in.

20. $\frac{1}{2}Wr^2$. 21. $\frac{2}{5}Wr^2$. 22. $2\frac{1}{7}$. 23. $1\frac{7}{8}$.

24. 210 ft. lb. 25. $\frac{21}{55}$. 26. $\frac{9}{16}$. 27. $4\frac{3}{7}$.

28. $\pi\rho a^3$. 29. 90 lb. 31. 2520 cu. ft. 32. 3050 cu. in.

XVIII. c. (p. 377.)

1. 0·335 ; $\frac{1}{3}$, 0·6 per cent. ; 0·33. 2. 120·2, 122·5, 118·7.

3. 39·495, 38·687, 39·270 ; 0·5, 1·4 per cent.

4. 41·63 ; 41·59 ; 0·07, 0·02 per cent. 5. 3·32, 3·28 mi.

6. $10^5\times3\cdot040$ cu. ft., each method. 7. Dufton 1090, Simpson 1030 ft.

8. 2088 ; 2094, 2094 cu. cm. 9. 0·1676, 0·1678.

10. 128·1, 127·5.

XIX. a. (p. 381.)

1. 12, $2n$; 36, n^2 ; $\frac{1}{13}$, $\dfrac{1}{2n+1}$; 6×10^6, $n\times 10^n$; 96, $3\times 2^{n-1}$; $\frac{11}{64}$, $\dfrac{2n-1}{2^n}$; 40, $7n-2$; -5, $13-3n$; 6×10^{-6}, $n\times 10^{-n}$; 1, $(-1)^n$; 343, $(n+1)^3$; 10^6+24, 10^n+4n.

2. 1, 3, 5 ; $\frac{1}{6}$, $\frac{1}{2}$, $\frac{3}{2}$; $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$; 2, 6, 12 ; 8, 40, 315 ; 0, 7, 26.

3. 1, 3, 2, 10, n.

4. 1, 16, n^2.

5. 29, no, $14\times 29=406$.

6. 29, 435 ; 35, 385 ; -15, -165 ; $9\frac{1}{4}$, $101\frac{3}{4}$.

7. 27, 22, 21, 19.

8. 104, $7n-1$; 128, 2^{n-1} ; 102, $11n-8$.

9. 15, $3\frac{1}{14}$, 11, 7.

10. 2866, 23.

XIX. b. (p. 383.)

1. $19\frac{1}{2}$.
2. $7\frac{2}{3}$.
3. 87.
4. $56\frac{1}{2}$.
5. $19\frac{1}{4}$, $26\frac{1}{2}$, $33\frac{3}{4}$.
6. 5029.
7. -130.
8. 702·5.
9. $-79\frac{1}{2}$.
10. $2n^2+n$.
11. $12a+66d$.
12. $\frac{1}{2}n[2a+d(n-1)]$.
13. $258\frac{1}{2}$.
14. 540.
15. $\frac{1}{2}n(a+l)$.
16. 29 ; $\frac{1}{2}(n^2-n+2)$.
17. 4215.
18. $2x-a$; $x(n-1)+a(2-n)$.
19. 91, n^2-n+1.
20. 5, $\dfrac{n}{2}(5n-17)$.
21. Yes, no.
22. 999000.
23. $l-(n-1)d$.
24. $\frac{1}{2}(b+a)(b-a+1)$.

XIX. c. (p. 386.)

1. $7\times 5^{n-1}$, $(-2)^{n-1}$, 2^{4-n}, $-10(-0{\cdot}6)^{n-1}$, $\dfrac{7^{n-1}}{6^{n-2}}$, $\dfrac{3^{2n-3}}{7^{n-2}}$, $(-1)^n x^{2n-1}$, $\dfrac{b^{n-1}}{a^{n-2}}$, $\dfrac{1}{a^{n-2}}$.

2. $-\frac{1}{16}$, 2.

3. $(\frac{3}{4})^{25}$.

4. 1533.

5. $80\frac{242}{243}$.

6. ar^7-a ; $\dfrac{a(r^7-1)}{r-1}$.

7. $2(3^{10}-1)$, $\frac{26}{3}[1-(0{\cdot}4)^{12}]$, $\frac{9}{4}-(\frac{2}{3})^6$, $\frac{7}{3}(2^{11}+1)$, $\frac{243}{13}[1-(\frac{4}{9})^{20}]$.

8. $1+r+r^2+r^3+r^4+r^5$; $1+r+r^2+\ldots+r^9$; $\dfrac{1-r^{21}}{1-r}$, $\dfrac{a(1-r^{21})}{1-r}$.

9. ar^{n-1}, ar^{n-2}, $ar+ar^2+ar^3+\ldots+ar^{n-2}+ar^{n-1}+ar^n$, ar^n-a, $\dfrac{a(r^n-1)}{r-1}$.

10. 1·111 ; 0·000001 per cent., $\frac{10}{9}(1-10^{-50})$, 10^{-48} per cent. ; $\frac{10}{9}$.

11. $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ... ins. ; $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$, $\frac{31}{32}$, $1-2^{-10}$, $1-2^{-100}$ ins. ; 1, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$; $2^{-6}-2^{-46}$; 16.

12. (ii), (iv). **14.** $\frac{43}{36}$. **15.** 2500, $8\frac{1}{3}$, $\frac{a}{1-r}$.

16. 630, 23600.

XIX. d. (p. 388.)

1. 17·6 m.p.h., $2\frac{1}{5}$ m. **2.** 0·0966. **3.** 6·58, 8 sec.

4. £2400. **5.** 3620. **6.** 36 ft.

7. 119·5. **8.** $\frac{2}{81}(10^9-82)$. **9.** 65.

10. 541 yd. **11.** 200°. **12.** n^2.

13. £21. 16*s*. $4\frac{1}{2}d$. **14.** n^2-n. **15.** 15.

16. £9500, £14500. **17.** 223 yd. **18.** £163, 1490.

19. £61·4, 390. **20.** £2500.

21. $\frac{1-(n+1)x^n+nx^{n+1}}{1-x}$; $\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$.

22. Less than 2^{200}. **25.** $\frac{5}{11}$. **26.** $\frac{10}{63}$.

27. $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{6}$. **28.** 1, 7, 19, 37. **29.** Each = 225.

30. 2.3.4.5.6.7.8.9.10.11 − 1.

31. 265 ; $\frac{6}{5}$, $\left(\text{before } \frac{n}{m}, \text{ the number is } \frac{(n+m-1)(n+m-2)}{2}+m-1\right)$.

32. 2.

XIX. e. (p. 392.)

1. 1, 1·18, 1·38, 1·63, 1·91 ; 1·18 : 1 in each case. **2.** 31·8 millions.

3. 46·9° C. **4.** £101,000. **5.** 40, 1·19. **6.** 4·1 per cent.

XX. a. (p. 395.)

1. 30. **2.** 120. **3.** 12 ; 720.

4. 720. **5.** 210. **6.** 840 ; 840.

7. 90, 45. **8.** 24. **9.** 56.

10. $26\times 99\times 999999$. **11.** $\frac{1}{2}|\underline{10}$. **12.** 6, 24, 32.

13. 90. **14.** $|\underline{8}$, 2880, 8640. **15.** 192, 144.

16. 24, 12, 12. **17.** 1440. **18.** 14400.

19. 15 ; 719. **20.** $|\underline{6}\,(|\underline{4})^2=414720$. **21.** 36.

22. $(|\underline{6})^2=518400$. **24.** $\frac{|\underline{20}}{|\underline{15}}$, $\frac{|\underline{20}}{|\underline{14}}$, $\frac{|\underline{30}}{|\underline{18}}$, $\frac{|\underline{n}}{|\underline{n-12}}$, $\frac{|\underline{n}}{|\underline{n-r}}$.

25. n, $(n-1)|\underline{n-1}$. **26.** $2^n\,.\,|\underline{n}$. **27.** $10^{22}\times 8{\cdot}1$. **28.** 18, 8.

XX. b. (p. 398.)

1. 10. **2.** 364. **3.** 22100.

4. $\frac{\lfloor 52}{\lfloor 39 \lfloor 13}$, $10^{11} \times 6{\cdot}4$. **5.** 66; 220. **6.** 66; 132.

7. 31. **8.** 13860. **9.** 90, 15.

10. 462, 5775. **11.** 720. **12.** $\frac{\lfloor 52}{(\lfloor 13)^4}$, $\frac{\lfloor 52}{24(\lfloor 13)^4}$.

13. 15840. **14.** 56. **15.** 56.

16. 35. **19.** $\frac{n-r+1}{r}$, $\frac{n}{r}$. **22.** ${}^{n-1}C_{r-1}$, ${}^{n-1}C_r$.

XX. c. (p. 401.)

1. 1326, 1128, 198, $\frac{33}{221}$; 188 : 33 against. **2.** 15, 6, $\frac{2}{5}$.

3. $\frac{11}{57}$. **4.** 1365, 1001, $\frac{11}{15}$. **6.** $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{2^n}$.

7. $\frac{1}{4}$, $\frac{3}{4}$, $\frac{3}{8}$. **8.** $\frac{1}{17}$. **9.** $\frac{1}{36}$, $\frac{1}{6}$.

10. $\frac{3}{10}$, $\frac{1}{10}$. **11.** $\frac{1}{3}$. **12.** $\frac{11}{21}$.

13. 120, 60, 10. **14.** $\lfloor 15$, $\frac{\lfloor 15}{\lfloor 5}$, $\frac{\lfloor 15}{\lfloor 10 \lfloor 5}$, $\frac{1}{2}\lfloor 14$. **15.** 12, 1728.

16. 24, 64. **17.** 340. **18.** $3^{16} = 10^7 \times 4{\cdot}3$, $10^{-8} \times 2{\cdot}3$.

19. 3150. **20.** 60. **21.** 90.

22. $\frac{\lfloor 2n}{2^n \cdot \lfloor n}$. **23.** 11. **24.** $\frac{711}{4165}$.

25. 4; one penny, two half-crowns; $52\frac{4}{7}$ pence, say 4s. $4\frac{1}{2}d$. **26.** $\frac{37}{60}$.

XXI. a. (p. 406.)

1. $x^4 + x^3(a+b+c+d) + x^2(ab+ac+ad+bc+bd+cd) + x(abc+bcd+cda+dab) + abcd$;
$x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$; $x^4 + 4x^3 + 6x^2 + 4x + 1$;
$x^4 - 4x^3 + 6x^2 - 4x + 1$.

2. 64, 6, 15.

3. $abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde$; $10a^3$; -10.

4. 4, 35, 35.

5. $x^3 + 6x^2 + 11x + 6$; $x^4 + 10x^3 + 35x^2 + 50x + 24$.

6. $6x^3 + 11x^2 + 6x + 1$; $24x^4 + 50x^3 + 35x^2 + 10x + 1$.

XXI. b. (p. 407.)

1. $1+3x+3x^2+x^3$. **2.** $1+4x+6x^2+4x^3+x^4$.

3. $1-4x+6x^2-4x^3+x^4$. **4.** $x^3-6x^2+12x-8$.

5. $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$.

6. $x^5+5x^3+10x+\frac{10}{x}+\frac{5}{x^3}+\frac{1}{x^5}$. **7.** $8x^3-36x^2y+54xy^2-27y^3$.

8. $a^8-8a^6b+24a^4b^2-32a^2b^3+16b^4$. **9.** 35, $\frac{1}{3}n(n-1)(n-2)$.

10. $-15360x^3$. **11.** 24. **12.** 11.

13. $-20x^3y^3$. **14.** $120x^7y^3$, $120x^3y^7$.

15. ${}^nC_{r-1}\,.\,2^{r-1}x^{r-1}$; $(-1)^{r-1}\,.\,{}^nC_{r-1}\,.\,2^{r-1}x^{r-1}$.

16. ${}^nC_{r-1}x^{n-r+1}y^{r-1}$; ${}^nC_{n-r+1}x^{r-1}y^{n-r+1}$; Yes. **17.** x^3. **18.** x^4.

19. ${}^nC_1+{}^nC_2+\dots+{}^nC_n=2^n-1$; $1-{}^nC_1+{}^nC_2-{}^nC_3+\dots=0$.

20. 0. **21.** ${}^nC_r-{}^nC_{r-1}=\frac{(n-2r+1)\,\lfloor n}{\lfloor r\,\lfloor n-r+1}$.

22. $1+nx+\frac{n(n+1)}{2}\,.\,x^2+\frac{n(n-1)(n+4)}{6}x^3$. **23.** 34.

24. 104060401, 9509900499. **25.** 1·1041, 0·83297.

XXI. c. (p. 409.)

2. $\frac{x^5}{1-x}$. **3.** $1+x+x^2+x^3+x^4$.

4. 0·01, 0·001. **5.** $\frac{x^2}{1+x}$, $-\frac{x^3}{1+x}$, $\frac{x^4}{1+x}$.

6. $1-x+x^2-x^3+x^4$; 0·0001, 0·000001. **7.** $\frac{3x^2-2x^3}{(1-x)^2}$, $\frac{3x^2+2x^3}{(1+x)^2}$.

8. $1+2x+3x^2+4x^3$, $1-2x+3x^2-4x^3$; 0·0003, 0·0003.

10. $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3$, 0·00001. **11.** $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$, 0·001.

12. $\frac{1}{x}-\frac{h}{x^2}$, 0·98. **13.** $\frac{1}{x}-\frac{h}{2x^3}$, 0·499, 0·332.

XXI. d. (p. 412.)

1. ab, 0·0002, 1·08. **2.** 1·06, $1+3h$. **3.** 1·000033 : 1, 1·000099 : 1.

4. 2 per cent. **5.** 6 per cent. **6.** 3, 2; 1·5 per cent.

7. $1+2h$, 1·02. **8.** 1·01, 0·94, 1·003, 1·004, 1·005, 0·1999.

9. $1+2h$, $1-5h$, 0·95. **10.** 0·028, $0·555\frac{h}{\sqrt{l}}$.

11. 2 per cent. decrease, $-\frac{vp_1}{p}$.

12. 0, ± 1; 1·005.

13. $3x^2$, $10x^9$, $-\frac{1}{x^2}$, $\frac{1}{2\sqrt{x}}$.

14. Decreased about 3 per cent.

15. 5 per cent. decrease.

16. $\frac{h^2}{2x}$ in.

17. $\theta + \frac{1}{3}\theta^3$.

18. 1 per cent. increase.

XXII. a. (p. 418.)

Note.—Some of the formulae below are given to a greater degree of accuracy than can be attained by drawing alone.

1. $y = 1{\cdot}62x - 0{\cdot}54$.

2. $l = 17{\cdot}95 + 0{\cdot}41W$.

3. $l = 0{\cdot}12W + 2{\cdot}1$.

4. $y = 9{\cdot}79 - 0{\cdot}033x$; 0·005.

5. $y = 6{\cdot}9x + 0{\cdot}1$; when $x = 3$, $y = 21$.

6. $P = 0{\cdot}0049v^2$.

7. $Q = 31000D^2$.

8. $R = 600 + 0{\cdot}9V^2$.

9. $k = \frac{1{\cdot}20}{d} - 0{\cdot}14$.

10. $S = 4{\cdot}09t^2$.

11. $y = \frac{12x}{x-7}$.

12. $t = 0{\cdot}0419d^{1{\cdot}163}$.

13. $p = 475v^{-1{\cdot}06}$.

14. $P = 18{\cdot}0 \times 10^{0{\cdot}051t}$, 16·0 (actually 15·91), 41·0.

15. $E = 10^{-14} \times 2{\cdot}185T^5$.

16. $\lambda = \frac{2960}{T}$.

17. $\delta = 10^{-8} \times 1{\cdot}24T^4 - 90$.

18. $y = 61 + 0{\cdot}068x + 0{\cdot}0024x^2$ (but better to write 61·1 for 61); 64·2 (actually 64·37).

19. $a = 10^{-4} \times 1{\cdot}5$; $b = 10^{-7} \times 9{\cdot}0$; 1·024.

20. Plot $\frac{d}{v}$ against $\sqrt{d}$; $a = 0{\cdot}21$, $b = 0{\cdot}25$.

XXIII. a. (p. 425.)

1. 51·8, 8·49, 6·76.

2. 1·58 in., 3·81 cm.

3. 11·2, $1{\cdot}405 \times 10^6$.

4. 23·6, 7·96 in.

6. 2·31.

7. 36.

XXIII. b. (p. 429.)

1. 32·7, 27·7 amp.; 74·1, 62·7 amp.

2. 10, 24·2 amp.; 0·52 mm.

3. 2·5, 5, 5, 3·3, 5·8 in.

4. 1·76, 6·99 in.

5. 1·2 cm.

6. 1·58, 3·54 in.

7. $y = 2x^2$, $y = 3x^2$.

8 28·6.

9. 62·4.

11. $-1{\cdot}06$.

12. 10·8.

13. 2850.

14. 0·254.

15. 3·29.

16. 1·44.

XXIII. c. (p. 435.)

1. £22. 8*s*., £34, £36.

2. 45·7, 55·7, 38·6 ; $Z=\frac{1}{7}(6X+10Y)$; 38·6, 48·6, 31·4 ; $Z=\frac{1}{7}(6X+10Y-50)$.

3. − 11·4, − 30, 4·3 ; $Z=\frac{1}{7}(6X-10Y)$; 38·6, 20, 54·3 ; $Z=\frac{1}{7}(6X-10Y+350)$.

4. 43, 21·5. **5.** 2·66, 4·28. **6.** 2, 3·75, $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$.

7. 60·8, 81·2. **8.** 7, 18. **9.** 40, 28.

10. 55, 45·5, − 20 ; $Z=\frac{60(4X+Y)}{3X+2Y}$; $Z=\frac{60(4X+Y-80)}{3X+2Y-110}$.

11. $w=\frac{xz}{y}$, 6·67, 2·25, 4·79.

XXIII. d. (p. 438.)

1. $\log T=\frac{1}{4}\log H+\frac{3}{4}\log W$; 10, £10 ; £20 ; £27·1 ; £22.

2. 5·4, 191. **3.** 3·19, 5·18. **4.** 33·1, 17·6.

5. 11000, 38500.

6. (v) $t=3$. (vi) 3·45, 1, 1·45, imaginary. (vii) − 1, − 1·45, − 3, − 3·45. (viii) put $x=10z$ and solve $z^2-2{\cdot}3z-4{\cdot}85=0$.

XXIV. a. (p. 443.)

1. Real +, − ; real +, − ; imaginary ; coincident + ; real −, − ; imaginary.

2. $x^2-x-2=0$, $4x^2-3x=0$, $x^2-2ax+a^2-b^2=0$; $x^3-x^2(a+b+c)+x(ab+bc+ca)-abc=0$, $x^2-2=0$, $x^2-2x\sqrt{2}+1=0$; $bdx^2+x(bc-ad)-ac=0$.

3. $\frac{-b\pm\sqrt{(b^2-ac)}}{a}$. **4.** $6\frac{1}{4}$. **5.** $\frac{25}{16}$, $-\frac{25}{16}$.

6. 2, − 0·05. **8.** $\frac{1}{2}>x>-\frac{3}{4}$, $1\frac{1}{2}>x>-1$.

9. $x>1$ or $x<-4$.

10. $x>3$ or $1>x>-1$; $x>\frac{3}{2}$ or $\frac{1}{3}>x>-2$; $1>x>-\frac{1}{2}$ or $x<-4$; $x>3$ or $1>x>-1$ or $x<-3$.

11. (i) $x>2$ or $-1>x>-3$. (ii) $2>x>-1$ or $x<-3$.

12. (i) $x>\frac{1}{2}$ or $\frac{1}{3}>x>-3$ or $x<-4$. (ii) $\frac{1}{2}>x>\frac{1}{3}$ or $-3>x>-4$.

13. 2, − 3. **14.** − 0·6, − 1·2 ; 3·6. **15.** $\pm 4\sqrt{3}$.

16. ± 8. **17.** $2q^2 = 9r$, $q = 0$, $6q^2 = 25r$, $r = 1$, $r = 0$, $q^2 = 4r + 1$.

27. 16 or 0. **28.** $8\frac{7}{9}$. **29.** $-\frac{2}{3}$, $18\frac{1}{4}$, $31\frac{1}{4}$.

30. $3b = 2q$, $9c = 4r$. **31.** $6b^2 + c = 0$. **32.** $y^2 - 40y = 900$, $2y^2 + 3y = 3$.

33. $4y^2 - 28y + 9 = 0$, $y^2 - 8y + 5 = 0$.

34. Roots of second equation are p times roots of first.

35. $b^2 = a^2 + 4ac$, $(a + b + c)^2 = b^2 - 4ac$. **36.** Imaginary.

38. 6. **39.** $\frac{11}{16}$ or 0. **40.** $a = -2$.

41. $a + b + 2h = 0$ or $4a + b - 4h = 0$. **42.** $-\frac{4}{3}$.

43. $b + c = 0$ or $b - c = 1$. **45.** $4c > a > 0$. **46.** $x = y = z$.

47. Not between 2 and -2. **48.** $\dfrac{1 \pm \sqrt{(1 - 4k^2)}}{2k}$, $\frac{1}{2} > k > -\frac{1}{2}$; $-\frac{1}{2}$.

49. Between $-\frac{1}{3}$ and $-\frac{1}{27}$. **50.** 6 and 1.

52. $ry^2 + qy + 1 = 0$; $y^2 + y(4r - q^2) - r(q^2 - 4r) = 0$. **53.** $q^2 - 4r = b^2 - 4c$.

54. $x = 2$, $y = -1$, no.

55. $(x - \alpha)(x - \beta)(x - \gamma) = 0$, $x^3 - x^2(\alpha + \beta + \gamma) + x(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma = 0$; $-\dfrac{b}{a}$, $\dfrac{c}{a}$, $-\dfrac{d}{a}$.

56. $p = -\Sigma\alpha$, $q = \Sigma\alpha\beta$, $r = -\Sigma\alpha\beta\gamma$, $s = \alpha\beta\gamma\delta$.

57. Sum is zero ; sum = product ; roots are of form $\pm a$, $\pm b$; one is zero.

58. -5. **59.** -7.

XXIV. b. (p. 447.)

1. $(x - y)(x^2 + xy + y^2)$, $(a + 2b + c)(a^2 + b^2 + c^2 + bc - ca + ab)$ $(a - c)(a^2 + 3b^2 + c^2 + 3bc + ca + 3ab)$.

2. $(a^2 + 2a + 4)(a^2 - 2a + 4)$, $(3x^2 + y^2)(x^2 + 3y^2)$.

3. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$. **4.** $x^2 + 5x + 5$, $x^2 + 5xy + 5y^2$.

7. $\sqrt{[xyz(x + y + z)]}$. **8.** -1. **9.** $\frac{3}{2}$.

10. $(x - 2y)(2x + y)(2x - y)$. **11.** $\dfrac{1}{x + 1} - \dfrac{1}{x + 4}$.

12. $-3xyz(x + y)(y + z)(z + x)$. **13.** $16(5b - 3a)(15a - 19b)$.

14. $(a - b + c)(a + b - c)(a - b - c)$. **15.** Unchanged ; divided by p.

16. $\dfrac{1}{n} \cdot 2^{n-1}$. **17.** $(x - y)(x + y)(x^2 + y^2) \ldots (x^{2^n} + y^{2^n}) = x^{2^{n+1}} - y^{2^{n+1}}$.

18. (ii), (iv).

19. $(a^2 + x^2)(b^2 + y^2)$, $[c(ab + xy) + z(ay - bx)]^2 + [z(ab + xy) - c(ay - bx)]^2$.

20. 32 ; *acfhk*, *bdfgk*, *lgfcb*. **21.** 7. **22.** 0.

23. $n+3$; 1 ; 2 if n is even, 0 if n is odd ; 0 if $n \neq 0$. **24.** 0.

25. 7, $n+1$. **26.** $2n+1$; 1 ; 0 if $n \neq 0$. **27.** $(-1)^n, (-1)^{\frac{n(n-1)}{2}}$.

28. $1+5x+10x^2+10x^3+5x^4+x^5$, $a^4+4a^3b+6a^2b^2+4ab^3+b^4$. **29.** 35.

30. $1+\frac{1}{2}x-\frac{1}{8}x^2$, $1-x+x^2$. **31.** $\sqrt{x}\left(1+\frac{h}{2x}-\frac{h^2}{8x^2}\right)$.

32. $1-2x+3x^2-4x^3+\ldots+(-1)^n(n+1)x^n+\ldots$,

$1+nx+\frac{n(n+1)}{1.2}x^2+\frac{n(n+1)(n+2)}{1.2.3}x^3+\ldots$.

34. Reflections in the y-axis. **35.** $\frac{x(x-1)}{(x+1)(x-2)} \equiv 1+\frac{2}{(x+1)(x-2)}$.

36. $[e(ac+pbd)+pf(ad+bc)]^2-p[f(ac+pdb)+e(ad+bc)]^2$.

37. $\frac{ar-cp}{bp-aq}$. **38.** $y^2(bl^2-2hlm+am^2)+2y(lh-am)+a=0$.

39. $\frac{k}{(b-a)(b-c)}, \frac{k}{(c-a)(c-b)}$.

40. $\frac{(b+\lambda)(b+\mu)(b+\nu)}{(b-a)(b-c)}, \frac{(c+\lambda)(c+\mu)(c+\nu)}{(c-a)(c-b)}$.

XXIV. c. (p. 452.)

5. (i), (iii), (vii), (viii), (ix), (x). **8.** $x=2$, $y=1\frac{1}{2}$, $z=2\frac{3}{4}$, $t=\frac{3}{4}$.

10. 1·6 or 0. **11.** $-9:13:1$. **12.** $5:-1:8$; $\frac{5}{8}, -\frac{1}{8}$.

13. $(c-b):(1+ac):(1+ab)$. **14.** $\frac{br-cq}{aq-bp}, \frac{cp-ar}{aq-bp}$.

15. $\frac{x^2}{h_1b_2-h_2b_1}=\frac{2xy}{b_1a_2-b_2a_1}=\frac{y^2}{a_1h_2-a_2h_1}$,

$(a_1b_2-a_2b_1)^2=4(a_1h_2-a_2h_1)(b_2h_1-b_1h_2)$.

16. 4, 3, -2 or -4, -3, 2. **18.** $\frac{4}{17}$. **20.** 17.

21. b^4+3d^3f ; $b^4+3ad^3f-5bdf^2$. **22.** $ab^4-c^2d^4$.

23. $\sqrt{(a^2-4b^2)}$. **24.** $\sqrt{\left(\frac{a+2b}{a-2b}\right)}$. **25.** $\sqrt{(a^2+b^2)}$.

XXIV. d. (p. 455.)

1. 0, 0, 0, n^2+2n. **2.** 6, 7, 12, 12.

3. $2\Sigma ab$, $3\Sigma a^2+2\Sigma ab$, 3, $3\Sigma abc$, $\log(n+1)$, $\log\left(\frac{n+1}{k}\right)$.

4. n, $1+2+3+\ldots+n$. **7.** $n(2n-1)$. **8.** $2^{2n}-4n-1$.

9. $\frac{n}{n+1}$; $\frac{1}{1.2}+\frac{1}{2.3}+\ldots+\frac{1}{n(n+1)}$; $\frac{n}{n+1}$.

10. $\frac{1}{6}[q(q+1)(2q+1)-p(p-1)(2p-1)]$.

11. $\frac{1}{3}[q(q+1)(q+2)-p(p-1)(p+1)]$.

12. $\dfrac{n+1}{n(2n+1)}$.

13. $a\Sigma x^2+b\Sigma xy,\ 3\Sigma x^2-2\Sigma xy$.

14. 2, 1.

15. $a\Sigma x^3+b\Sigma x^2y+cxyz$; $a=1,\ b=2,\ c=-12$.

16. $\frac{1}{2}(a^2-b^2),\ \frac{1}{2}a(3b^2-a^2)$.

17. $2a^2-6b^2$.

21. $c^3-\dfrac{a^3}{3}$.

22. $(x-y)^2+(y-z)^2+(z-x)^2$.

24. $\frac{1}{2}(x^2-3y^2-z^2)$.

XXIV. e. (p. 458.)

3. -2.

4. $(a+b)(2a^2-ab+b^2)$.

5. $-2,\ 5$.

6. Always, n odd, if $\frac{n}{2}$ is an odd integer.

7. $x+a+b+c$.

8. $-(a-b)(b-c)(c-a)$.

9. $3(x+y)(y+z)(z+x)$.

10. $(x+y)(y+z)(z+x)$.

11. $3(x-y)(y-z)(z-x)$.

12. $(x-y)(y-z)(z-x)(x+y+z)$.

13. $12abc(a+b+c)$.

14. $(x^2-yz)(y^2-zx)(z^2-xy)$.

15. $(xy-zw)(yz-wx)(zx-wy)$.

XXIV. f. (p. 459.)

Note.—Imaginary roots are omitted.

1. 0 or $\frac{1}{2}$ or $\frac{1}{3}$, 0 or $\frac{1}{3}$ or $\frac{1}{2}$.

2. $\pm\dfrac{ac}{b},\ \pm\dfrac{ab}{c},\ \pm\dfrac{bc}{a}$.

3. 0 or $\dfrac{ab+c}{a}$, $-\dfrac{c}{b}$ or a.

4. $\pm 2,\ \pm 3$.

5. 3 or -2 or $\pm\sqrt{7}$, 2 or -3 or $\pm\sqrt{7}$.

6. $\pm\frac{1}{2},\ \pm\frac{1}{3}$.

7. $\pm\dfrac{a}{bc}$ or 0, $\pm\dfrac{b}{ca}$ or 0, $\pm\dfrac{c}{ab}$ or 0.

8. $\pm\sqrt{\frac{37}{14}},\ \pm\sqrt{\frac{21}{10}},\ \mp\sqrt{\frac{35}{26}}$.

9. $1\frac{1}{2}$ or 3 or $\dfrac{-15\pm\sqrt{465}}{8}$, 4 or 2 or $\dfrac{-15\mp\sqrt{465}}{6}$.

10. $3c$ or $-2c$ or $-\dfrac{c}{2}(1\pm\sqrt{29})$, $-2c$ or $3c$ or $-\dfrac{c}{2}(1\pm\sqrt{29})$.

11. 2 or 1, 1 or 2.

12. 3, $\frac{3}{4}$.

13. 1 or 2 or -3, 1 or 2 or $\frac{1}{3}$.

14. $\frac{1}{2}$ or $\frac{1}{3}$, $\frac{1}{3}$ or $\frac{1}{2}$.

15. 3 or -5, $\pm\frac{3}{2}$, ± 2.

16. $\pm 2,\ \pm 4$.

17. $\pm 2,\ \pm\frac{1}{2}$.

18. 9, 12.

19. 3 or -1, 3 or -7, -2 or -8.

20. 12 or $\frac{18}{5}$, 2 or $-\frac{32}{5}$, 9 or $\frac{3}{5}$.

21. $\pm 6,\ \pm 2$.

22. 3 or -2, 2 or -3.

23. ± 2 or 0, ∓ 2 or 0.

24. 0 or ± 2 or $\pm\sqrt{6}$ or $\frac{1}{2}(\pm\sqrt{7}\pm\sqrt{3})$, 0 or ± 2 or $\mp\sqrt{6}$ or $\frac{1}{2}(\pm\sqrt{7}\mp\sqrt{3})$.

25. $+3$ or ± 1, ± 1 or ± 3. **26.** 0 or -2, -5 or 3, -3 or 1.

27. 4, not $2\frac{1}{4}$. **28.** 1, not 8. **29.** None, 3. **30.** 3.

31. -3 or $-2{\cdot}2$. **32.** 2. **33.** 1, not $-\frac{1}{2}$. **34.** -1 or -2.

35. 0. **36.** 3 or -1. **37.** $1, -5, -2\pm\sqrt{3}$.

38. 1 or 2. **39.** 0 or 2, -2 or 4, 2 or 0.

40. -1 or $-\frac{7}{4}$, 2 or -1, 5 or 26. **41.** 9 or 4, 6 or 6, 4 or 9.

42. $-4, -6, \pm\sqrt{19}-5$. **43.** $-1, \dfrac{7\pm 3\sqrt{5}}{2}$.

44. $3, 2, \frac{1}{2}, \frac{1}{3}$. **45.** $x, y, z = 1, 2, 3$ any order.

46. $2, \frac{2}{3}, \frac{9}{7}$. **47.** $a, b-a$.

48. $\dfrac{x}{p^2}=\dfrac{y}{q^2}=\dfrac{z}{r^2}=\pm\dfrac{1}{\sqrt{(p^2+q^2+r^2)}}$.

49. $\dfrac{x}{c+a-b}=\dfrac{y}{a+b-c}=\dfrac{z}{b+c-a}=\dfrac{1}{(b+c-a)(a+b-c)}$ or 0.

50. If $a=3$, $x=y=$ any number; if $a=1$, $(x+\frac{1}{2})(y+\frac{1}{2})=2\frac{1}{4}$; if $a=4$, $x=y=1$; if $a=2\frac{1}{2}$, $x=y=-2$; in all other cases $x=y=1$ or -2.

51. $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}=\dfrac{a+b+c}{abc}$ or 0.

52. $\dfrac{x}{b^2+c^2}=\dfrac{y}{c^2+a^2}=\dfrac{z}{a^2+b^2}=\pm\dfrac{1}{2\sqrt{(a^2+b^2+c^2)}}$.

XXIV. g. (p. 462.)

1. $a+c=2b$. **2.** $(a-b+c)(a-b-c)+2a+2b-2c=0$.

3. $y^2=4ax$. **4.** $x^2=y^2+2$. **5.** $(x+y)(3x-y)=16$.

6. $a+b=3c$ unless $x=y=0$. **7.** $(b+c)^2=(a+b)(a+c)$.

8. $5x+2y+z=0$. **9.** $x^2+y^2=\dfrac{2b^2}{a^2}$. **10.** $x^2+y^2=a^2$.

11. $a+2=bc$. **12.** $b+ac=0$ unless $x=y=0$.

13. $(a+c-b)(a+b-c)(b+c-a)=8$. **14.** $y^2-by-cx+ac=0$.

15. $a^3-3ab^2-c^3+3d^3=0$. **16.** $\lambda=a\pm b+c$.

17. $a^2b^2+b^2c^2+c^2a^2=abcd$. **18.** $(c_1a_2-c_2a_1)^2=(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)$.

19. $2pqr+pq+qr+rp=1$, unless $x=y=z=0$. **20.** $ax-dx+3ad=0$.

REVISION PAPERS.

E. 1. (p. 465.)

1. No. **2.** $2y$, $6x^2$, $-\frac{3x^3}{8}$, $\frac{a}{2}$. **3.** $w=\frac{8P-W}{n}$.

4. $-2\frac{1}{2}$. **5.** 2, 3, 4.

6. Hurdles 25, 20, 15, 10, 5 ; area 1000, 1600, 1800, 1600, 1000 ; 30.

E. 2. (p. 466.)

1. 2. **2.** $\frac{l}{40}$; 13 : 28. **3.** $3x$, $\frac{12}{x}$ pence ; 2.

4. 1, x, a, 12^x. **5.** 0, -1. **6.** £$(0{\cdot}27x-74\frac{1}{4})$, £400.

E. 3. (p. 466.)

1. £$(208-2{\cdot}6a-2{\cdot}6b-0{\cdot}6c-d-r)$. **2.** 1·5, 5.

3. $V=\frac{tV_1}{t-RCL}$. **4.** 27. **5.** 6, 5 ; yes, -4, -3.

6. $V=\sqrt{\left(\frac{5Rh}{12d}\right)}$, 20 m.p.h.

E. 4. (p. 467.)

1. $6a+\frac{1}{3}b$., 1*s*. 3*d*. **2.** 34, 30, 16.

3. $(4x+5)(4x-5)$, $(3x-2)(x+3)$, $x(3x+1)$.

4. 0·3, 12 ; -10 ; effort insufficient to start machine.

5. z^{15}. **6.** 6.

E. 5. (p. 468.)

1. 32. **2.** x, $3-x$ in. ; $x=1\frac{2}{3}$.

3. $12x^2y^2(x^2-y^2)^2$, $x(x-3)(x+1)$, $x^2(x-6)(x+1)$, $x(x+1)$.

4. $1\frac{1}{3}$. **5.** 4·8 in., 19·6 *ft.* lb. **6.** $\frac{x}{3}$; $\frac{x}{6}$ hrs. ; $2x$ mi, $\frac{x}{2}$ hrs.

E. 6. (p. 468.)

1. $l+b=43$. 2. $\frac{9}{8}$. 3. 7·40 in.

4. The tax on £x is £$\frac{x-160}{8}$, £10.

5. $-\frac{1}{x(x+h)}$, $-\frac{2x+h}{x^2(x+h)^2}$, -1, -2. 6. -1, $4\frac{1}{2}$.

E. 7. (p. 469.)

1. $2n-4$, $2n-2$, $2n+2$, $2n+4$; 12. 2. 2·6.

3. 0·5, 0·4. 4. 21, 100, 10, 16, 4; $a=7$, $b=3$.

5. $x=\pm 4s^3t^2$. 6. $10-x$; $8-x$; $18x-2x^2$; 4·5 in.

E. 8. (p. 470.)

1. The fewer there are out of work, the fewer felonies are committed, but the more cases of drunkenness occur.

2. 13, 1. 4. $\frac{8x}{x^2-1}$, 0. 5. $\frac{5}{3}$ or $-\frac{3}{2}$, 2 or $\frac{1}{2}$.

6. $V=30-5x$, $x=6-0{\cdot}2V$, $0\leqq x\leqq 3$, $30\geqq V\geqq 15$.

E. 9. (p. 471.)

1. $W=\frac{wvt}{2av+2aV-vt}$. 2. $\frac{15x}{16m-x}$ hrs.

3. 2, 26·8 sq. cm. 4. 2 or $\frac{1}{3}$. 5. 4. 6. $\frac{8}{\pi r^2}$, 1·08.

E. 10. (p. 472.)

1. $\frac{ab}{a+b}$ min., $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$. 2. 256 cu. in. 3. t^2-4, ± 2.

4. $5x(x+2)(x-2)$, $7(a-3)(a+1)$, $(b-6)(b+2)$.

5. 1 or -3, 3 or -1. 6. $4-x$; $6x-1{\cdot}5x^2$; 0, 4·5, 6, 4·5, 0.

F. 1. (p. 473.)

2. 8*s*. 8*d*. 3. 25·2, 5·69. 4. $y=4t^{\frac{4}{3}}$, $x=2t^{\frac{2}{3}}$.

5. $\frac{1}{2}$, $\frac{1}{8}$. 6. 1,800,000, 0·018, 0·00018, 1800, $10^4\times 7{\cdot}38$, $10^{-5}\times 7{\cdot}38$.

F. 2. (p. 473.)

1. -16, 4. 2. $1\frac{1}{2}$ in. 3. $6x-x^2$; 4·5, 1·5 in.

4. $\frac{1}{(1-x)^2}$, $\frac{kl}{t^2}$, $\sqrt{(2x)}$, $2\sqrt{x}$, $\frac{2}{\sqrt{x}}$. 5. 1·5.

6. $\frac{4a^2-2}{(a-1)(2a-1)}$, each $=4\frac{2}{3}$.

F. 3. (p. 474.)

1. 6, 8 in. **2.** All, 1, 1·5, none. **3.** $\frac{Wb}{W-2Pe}$. **4.** 1401, 1·6721.

5. $\sqrt{(a^2+b^2)}$, $\sqrt[3]{a^2}+\sqrt[3]{b^2}$, $\frac{Wl}{bd^2}$, $K \cdot \frac{\sqrt[4]{H^5}}{\sqrt{P}}$. **6.** $P=\frac{K^2}{a^2} \cdot \sqrt{H^5}$.

F. 4. (p. 474.)

1. 1, 7, 1, 5, -2, -2. **3.** $y^2+12y+9$, 5 or -23, 1 or -13.

4. 3, 1·5, 0, $\frac{4}{3}$. **5.** 213, 7·08. **6.** 1·40.

F. 5. (p. 475.)

1. $x(l-n)+n$ ft. **2.** 2, $\frac{1}{512}$, 4, $\frac{1}{256}$, 1. **3.** 12, 5 in.

4. $2\frac{1}{4}$, 1·31 or 0·19. **5.** 12·4, 4·20. **6.** $r=\sqrt[3]{\frac{3V}{4\pi}}$, 1·56 in.

F. 6. (p. 476.)

1. $\frac{1}{4}n^2(n+1)^2$. **2.** $\mu=1+\frac{Rr}{f(R-r)}$. **3.** 2 or $-\frac{4}{3}$, 1·90 or $-1·23$.

4. $2x^4$, $3x^{\frac{1}{3}}$, $12x^6$, $6x^2$, $8x^6$, $8x^{\frac{1}{2}}$. **5.** 50,000. **6.** 1200.

F. 7. (p. 476.)

1. $(10x+5)^2=100x(x+1)+25$. **2.** 17*s.* 10*d.*

3. After 1 or 3 sec. **4.** $\frac{1}{2}a^5$, $\frac{4}{a^2}$, $\sqrt{a^5}$, $\frac{2}{3}a^5$.

5. 0·205, 0·966. **6.** 7·52.

F. 8. (p. 477.)

1. 0·1, 0·178, 0·1995. **2.** 5, $3\frac{1}{2}$, $1\frac{1}{2}$.

3. $-\frac{1}{2}$ or $\frac{5}{3}$, 1·59 or $-0·420$. **4.** $\sqrt{(10x-x^2)}$, a circle.

5. 23400. **6.** $5\frac{1}{4}$, $7\frac{1}{4}$ in.

F. 9. (p. 478.)

1. $-\frac{1}{(a+1)(a+2)}$, each $=-\frac{1}{6}$, $\frac{x+2}{x+1}$.

2. $\sqrt{a}\times\sqrt{b}=\sqrt{(ab)}$, 8, 12, a^2b^2. **3.** $\frac{a^2\sqrt{3}}{4}$.

4. 12, 9 in. **5.** 2·12 in., $G=0·846d^{\frac{4}{3}}H^{\frac{1}{3}}$. **6.** 0·229, 0·532.

F. 10. (p. 478.)

1. $(20x-2x^2)$ sq. in., $AP=5$ in.

2. $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$, 2, 3, $\frac{b}{a}$.

3. 3 or -2, -2 or 3.

4. 0·25, $\bar{1}$·75, $\bar{1}$·4, 1·6667.

5. 0·209.

6. 4 min. 14 sec.

F. 11. (p. 479.)

2. $\frac{3x^2+2xy+y^2}{(x+y)^3}$.

3. 0·36 per cent.

4. 0·319 or -1·57, $\frac{3}{4}$ or -2.

5. 20, 7·5 mm.

6. 10·6 ft.

F. 12. (p. 480.)

1. 554.

2. 16, $5\frac{1}{2}$.

3. 2 or -1, $\frac{1}{2}$ or -1.

4. $\frac{x^{n-1}y}{(x+y)^m}$.

5. 2·83, 4·24, 1·41, 3·46.

6. 1·09.

F. 13. (p. 481.)

1. 52·6 in.

2. 25° with keel.

3. 189.

4. 4, $\frac{1}{4}$, -2, $-\frac{1}{4}$.

5. 4×10^{15}, 10^{18}.

6. $2\sqrt{5}$, 2, $\sqrt{3}$, 2.

F. 14. (p. 482.)

2. 2^{2n}, 2^{3n-3}, 8.

3. 2, 3, 4 in.

4. 3·065.

5. £4240.

6. 56900 tons.

F. 15. (p. 482.)

1. Logarithms 3, -2, 1·5, 0·25.

2. At the middle.

3. 0·4, 1·2.

4. $2x$ cars.

5. 3220, 0·136.

6. $100\left[\sqrt[n]{\frac{A}{P}}-1\right]$, 5·2 per cent.

F. 16. (p. 483.)

1. 11.

2. $\frac{1-x}{1+x}$.

3. 10·2 in.

4. $k=(a-1)(1-b)$.

5. $\frac{81}{100}$, 9(8·7).

6. $v=230$, 8·15 sec.

G. 1. (p. 484.)

1. $5\sqrt{x}$, $3\sqrt{x}$, 0·0000014, $\frac{6b}{a}$.

2. 39.

3. $t=\frac{v-u}{f}$, $s=vt-\frac{1}{2}ft^2$.

4. $4x-2y=a$, $\frac{1}{2}(a-2y)(b-2y)y$.

5. 405.

6. 7·2, 9 in.

G. 2. (p. 485.)

1. 2·25. 2. 2·2, −3·3. 3. £140, £110.

4. $\rho = 8 \times 10^{27} V^{-\frac{2}{3}}$, $a = 5{\cdot}5$, $b = 13$. 5. $\frac{1}{n+1}$, $n : 1$, equal.

6. 2·5, 3·5 in.

G. 3. (p. 486.)

1. 60, 24. 2. $\sqrt{(ab^3)} \cdot \sqrt[4]{\frac{H^3}{G}}$, 0·000,000,000,014 ; $\frac{1}{45}$, $\frac{1}{3}$, 1.

3. $R = \sqrt{\left(\frac{Wr}{17{\cdot}6P}\right)}$, 2·77. 4. $a - 4b$, $a + 7b$; 13 : 3. 5. 6750 tons.

G. 4. (p. 486.)

1. 8·485. 2. $6{\cdot}91 \times 10^{-13}$. 4. $\frac{1+3x^2}{1-x^2}$.

5. $2\frac{1}{3}$ or $1\frac{1}{4}$, 2·41 or 1·18. 6. 1·8 sec.

G. 5. (p. 487.)

1. $\frac{1}{2}(pr + qs)$. 2. a, a^4, $a^{\frac{3}{2}}$, $a^{\frac{4}{3}} b^{\frac{7}{6}}$. 3. 84·75.

4. 2. 6. 3·56 in.

G. 6. (p. 488.)

1. £5½, £6 ; £11½. 2. $5\frac{10}{11}$ ft. 4. 2×3^n, $a^{\frac{2}{3}}$.

5. 171. 6. 245 amp.

G. 7. (p. 489.)

1. $\frac{a-x}{a} = \frac{x}{p}$. 2. $\frac{1}{2}x(p-x)$, $\frac{1}{2}x(a-x)$, x^2, $\frac{1}{2}pa$.

3. $v = \left(\frac{c}{p}\right)^{\frac{5}{7}}$, $v = \left(\frac{c}{p}\right)^{\frac{1}{r}}$. 4. $\frac{a^2+b^2}{a^2-2b^2}$.

5. $3y$, $10x$, $7xy$, $12xy$; $b+d$, $b-d$, $\sqrt{(bd)}$. 6. 5·9 in.

G. 8. (p. 490.)

1. $n > 17\frac{1}{7}$. 2. 9 times. 3. 59·7.

4. 0·6645, 3, 40·98, $\frac{1}{49} = 0{\cdot}0204$. 5. 4 or $-\frac{84}{41}$, 2·4 or $-\frac{748}{205}$.

6. 6·912.

G. 9. (p. 490.)

1. 40, 50 m.p.h. 2. 6, $\frac{14}{25}$. 3. -4, $-\frac{17}{8}$.

4. 37·3 c.c. 5. 4500 days. 6. 8640 lb.

G. 10. (p. 491.)

1. $x=0{\cdot}8,\ 0{\cdot}6,\ 0{\cdot}12$; $y=12,\ 24,\ 120$. **2.** 5·12 : 1. **3.** 1·6, 4·4.

4. 0, ± 1, all, none. **5.** $\dfrac{a+bk}{1+k}$. **6.** 9·65 in.

H. 1. (p. 492.)

1. 5, 6; (*a*) and (*b*) large and positive; $6>x>5$; >-1 (actually $\geq -\frac{1}{4}$).

2. 1007, -995; 6·0009, 5·9991. **3.** ± 1.

4. 0·542. **5.** 120 gr. **6.** £3000.

H. 2. (p. 492.)

1. $-2,\ 3,\ 6,\ 7,\ 6,\ 3,\ -2$; 7; 2; $4{\cdot}65>x>-0{\cdot}65$; 3·73 or 0·27 ; 1 or 3 ; 4·83 or $-0{\cdot}83$.

2. $\dfrac{a^2}{2h}+\dfrac{h}{2}$, 130 ft. **3.** 6, 0·6, $-0{\cdot}5$. **4.** 1·7, 0·29.

5. $-\frac{17}{3}$, $\frac{27}{4}$. **6.** 2·5 dynes.

H. 3. (p. 493.)

1. 0, no, large. **2.** 321 cu. in. **4.** £1500, £1200

5. 2·36, 0·17, $-2{\cdot}53$. **6.** 0·018, 0·022, 2·2.

H. 4. (p. 494.)

1. $\dfrac{x}{x+1}>\dfrac{x-1}{x}$ if $x>0$, 1000. **2.** 1, 1, $\rightarrow+\infty$, $\rightarrow-\infty$, $x=0$.

3. 33, 31. **4.** -1, no value. **5.** 35800. **6.** 10 mi., 1 mi.

H. 5. (p. 495.)

2. 2·154. **3.** 2, -6, $4a^2+14a-6$, x^2+9x+2.

4. $h=\dfrac{12}{x^2}$, $S=x^2+4xh$, 2·885 in. **5.** $9x^2+9xh+3h^2$, 9·01.

6. 0·003, 0·3 per cent.

H. 6. (p. 496.)

1. -9, 99, $-99{\cdot}99$; no ; 100·01, 1000·0001.

2. 3, $-3>x>-3{\cdot}006$, $-3<x<-2{\cdot}995$, 1, 1.

3. $\frac{2}{25}$, $\frac{9}{100}$, $\frac{3}{25}$. **4.** $4\frac{1}{2}$. **5.** 6, 2·5 in.

6. 2, 3, $2\sqrt{5}=4{\cdot}47$, $6\sqrt{5}=13{\cdot}4$.

H. 7. (p. 497.)

1. 37·5, 20·25, 45, 30 ft. per sec.
4. $x = \sqrt[3]{175} \,.\, y^{\frac{2}{15}}$, 3·23.
5. 9, 6 in.
6. $\frac{x^2}{y^2} >$ or $< \frac{x}{y}$ as $x >$ or $< y$.

H. 8. (p. 497.)

1. 2, 3·9, $4-h$, 4.
2. 7, 7 ; $2x+h$ if $h \neq 0$; $2x$.
3. £$(15+4n)$.
5. 20 in.
6. $a = \frac{1+x}{1-x} = \frac{1+y}{2(1-y)}$, $xy - 3x + 3y = 1$.

H. 9. (p. 498.)

1. -4 ; -1 ; $1 > x > -3$; 1·83 or -3·83 ; 1·24 or -3·24 ; 0·73 or -2·73.
2. -4 ; -3 ; $-h-2$ if $h \neq 0$; -2.
4. 0·932 or -1·60.
5. 3, $7+2\sqrt{10} = 13{\cdot}3$, $\frac{1}{3}(\sqrt{5}-\sqrt{2}) = 0{\cdot}274$.

H. 10. (p. 499.)

1. 1·15 ; 0·308 ; $2 > x > 0$ and $x < -2$; 2·214, -0·539, -1·675 ; (*a*) and (*b*) 2·115, -0·254, -1·861 ; (*c*) 1·861, 0·254, -2·115.
2. 0·8.
4. 1, $\sqrt{2}$, 2^r.
6. 7, 7.

K. 1. (p. 500.)

1. $\frac{17}{500}$, $\frac{1}{25}$, about 0·03.
2. $2x+4$, $-\frac{1}{(x-1)^2}$.
3. -2, $3a^2-5$.
5. $t = x-1$, $y = x^2 - x$, $t = \pm 1$.
6. $y^2 = 2ax + a^2$, $y = \sqrt{(2ax+a^2)}$.

K. 2. (p. 501.)

1. -1, -1, 0.
2. $4-6x$, $4-6x$, $4-6x$, $\frac{3}{2} + \frac{4}{3}x$, $-\frac{6}{x^2}$, $\frac{4}{\sqrt{x}}$.
3. 18 ; $-\frac{5}{6}$, $4\frac{1}{12}$.
4. $y = 3x - 8$.
6. 0·026l at distance 0·578l from A ; gradient $\frac{1}{10}$.

K. 3. (p. 501.)

1. $8x - \frac{1}{x^2}$, $48x^2 - 8x + 2$, $\frac{5}{3}x^{-\frac{2}{3}} - \frac{1}{3}x^{-\frac{4}{3}}$.
2. $3a^2 - 12a - 15$; 5, -1.
4. 18, 66 ft. per sec.
5. $\frac{dx}{dt} = \frac{k}{x}$.
6. $1\frac{2}{3}$.

K. 4. (p. 502.)

1. $-\dfrac{2x}{(x^2-2)^2}, \tfrac{1}{2}$. 2. $\sqrt{\dfrac{3}{2x}}, \dfrac{3}{\sqrt{x}}, -\dfrac{12}{x^4}, 2x-\dfrac{2}{x^3}$.

3. $0, \pm 2\sqrt{3}$; ± 2. 4. $\dfrac{d\theta}{dt}=k(15-\theta)$. 6. 214 : 1

K. 5. (p. 503.)

1. $4-\dfrac{4}{t^2}, 9t^2+\dfrac{4}{t^2}-\dfrac{15}{t^4}, 12{\cdot}5t^{\frac{3}{2}}$. 2. $3x^2-27$; ± 3; 54, -54.

3. Grows lighter and weight vanishes when $x=7{\cdot}5$.

4. 0·00649, 0·00736, 0·000087. 5. $2x+3+h$, 2, $2x+3$, 2.

6. 4800, 29500 yds.

K. 6. (p. 504.)

1. $x=1$ (min.), $x=-3$ (max.). 3. $4x-x^2$, $x=2$.

4. $(2p+q)^2+(p-2q)^2$, 20^2+15^2 or 24^2+7^2. 5. 14·4. 6. $s=k\,.\,t^2$.

K. 7. (p. 504.)

1. $-3-6x^2$. 2. $\dfrac{dx}{dt}=2(x+5)$.

3. $h=\dfrac{1200}{\pi r^2}$, $S=\sqrt{\left(\dfrac{144\times 10^4}{r^2}+\pi^2 r^4\right)}$, 230 sq. ft., 6·46 ft.

4. 81, 0. 5. $\frac{17}{22}, \frac{85}{13}$. 6. $19\frac{5}{9}$ lb.

K. 8. (p. 505.)

1. 2 (min.), -2 (max.). 2. $\frac{5}{6}$.

3. $9x^2-7, 18x$; $10x, 10$; $7, 0$; $0, 0$; $-\dfrac{6}{x^2}, \dfrac{12}{x^3}$.

4. ± 125. 5. $2\frac{1}{8}$ in. 6. $60x-\pi x^2$, 9·55 in.

K. 9. (p. 506.)

1. Not if $x=1$, no, 2.

2. 3400, 3564, 3417·84 ; 164, 178·4 ; $180-16h$; 180 ft. per sec.

3. $\delta y=4x\,.\,\delta x+2\,.\,(\delta x)^2$; $\delta x=3\,.\,\delta z$; $\delta y=12(3z+1)\,.\,\delta z+18\,.\,(\delta z)^2$; $12(3z+1)$.

4. 1 in. 5. 40, 16875. 6. $\dfrac{b}{a}$.

K. 10. (p. 507.)

1. $ab : a^2 : bc$.
2. $x\sqrt{(25-x^2)}$, $12\frac{1}{2}$ sq. cm.
3. Less than 13·75 in.
4. $\frac{1}{100}$, $\frac{7}{1000}$, $\frac{2+h}{300}$, $\frac{1}{150}$.
5. $\frac{dy}{dx} +, -, -, +$; $\frac{d^2y}{dx^2} -, -, +, +$.
6. 0·87 cu. ft.

K. 11. (p. 507.)

1. 165.
3. 25 yd.
4. 0·00101, 0·00107, 0·00112, 0·00107.
5. $2x + \frac{1}{x^2}$, $\frac{2}{3}\sqrt[3]{\frac{2}{x}}$.
6. 12·3.

K. 12. (p. 508.)

1. Decreased in ratio 1 : 16, 18 c.p.
2. $\frac{1}{a}$.
3. 548.
4. 102, $\to\infty$.
6. 8 in.

K. 13. (p. 509.)

1. $x = 2·94$.
3. $b = 60 + 0·3r$, 72.
4. 6·955 in., $y = 2\sqrt{(\pi x)}$.
5. Positive if $0 < x < 4·76$, increasing if $x < 3$; $x = 3$.
6. 8 ft., 48 sq. ft., 256 cu. ft.

K. 14. (p. 509.)

2. 0·73, 11·9, 84·0.
3. $2(1+\sqrt{3}) = 5·46$, -2.
4. Each $\simeq 0·43$, $\frac{d}{dx}\log x \simeq \frac{0·43}{x}$.
6. $\frac{1}{2}$.

K. 15. (p. 510.)

1. 79·6.
2. 60, 240, 540; 30, 120, 270; 20, 80, 180.
3. $\frac{x+y}{xy}$.
4. Each $\simeq 2·3$, $\frac{d}{dx}(10^x) \simeq 2·3 \times 10^x$.
5. $-\frac{5}{x^2}$, $-\frac{x^2}{5}$, 1.
6. $\frac{1}{25}$ when $y = +10$.

K. 16. (p. 511.)

1. $\frac{a^2}{2b^3}$; 16, 64, 2, 0·001.
2. $x : (y - x)$.
3. $A = \frac{10a(a-10)}{a-20}$, 17·2 sq. cm.
4. $2x + h - 3$, $2x - 3$, 1·5.
5. $\frac{2x}{(1-x^2)^2}$.
6. 75; 1·2, $-1·2$; 30 (min.), -30 (max.).

L. 1. (p. 512.)

1. $W = \dfrac{114bld^2}{x(l-x)}$ (in tons), where $AP = x$ feet and all measurements are in feet.

2. 27 : 8. **3.** (ii) -20000. (iii) No. (iv) 3·000. (v) $\dfrac{9}{(x-1)(x-5)}$, 3.

4. 0·26. **5.** $y = \frac{1}{3}(3x^2 - 4x^{\frac{3}{2}} + 7)$, $y = 4x + 1 + \dfrac{1}{2x}$.

6. $\dfrac{5}{3}x^3 - \dfrac{4}{x} + c$; $\dfrac{5}{11}x^{2\cdot2} - \dfrac{5}{x^{0\cdot2}} + c$.

L. 2. (p. 512.)

1. $d + h - \sqrt{(2dh)}$. **2.** $y : x$, $(1+y) : 1$, $1 : y$.

3. $\frac{3}{4}$, $\frac{1}{4}$. **4.** $ky = hs$, $\dfrac{s}{\sqrt{(s^2 - 10000)}}$, 2 in.

5. 24. **6.** 135 : 64.

L. 3. (p. 513.)

1. 1·5, -1, 10^3. **2.** $\sqrt{b} : \sqrt{a}$. **3.** $+3$, ±4; 5, -5; $4\sqrt{2}$, $-4\sqrt{2}$.

4. $\dfrac{x}{2}\sqrt{(400 - x^2)}$, $10\sqrt{2}$, 100. **5.** $\frac{3}{4}h$. **6.** $\dfrac{dx}{dt} = -0\cdot4x$, $0\cdot16x$.

L. 4. (p. 514.)

1. $x > 15$, $0 < x < 3$. **2.** 51 lb., $R = \dfrac{d^2v^3}{10^6 \times 288}$.

3. 4, -4; $x = 3$; $10\frac{2}{3}$. **4.** $6t - t^2$ lb.-sec.

5. $\dfrac{1024\pi}{125} = 25\cdot7$. **6.** Yes; 6160, 18480, 21560 yd.

L. 5. (p. 515.)

1. $(x-1)^2 - 3(x-1) + 2$. **2.** $y = \dfrac{4x^2}{x+12}$, 9·21. **3.** 6·17.

4. $0 < x < 3$, -27. **5.** $\frac{1}{2}h$. **6.** A square.

L. 6. (p. 516.)

1. 15·6, 62·2 cu. ft. per hr. **2.** No change. **3.** $z = t^{\frac{4}{3}}$.

4. $\dfrac{2r}{3}$, $\dfrac{rh}{2(h-r)}$ in. **5.** $\dfrac{47\pi}{16} = 9\cdot23$. **6.** 1·5, $\frac{4}{3}$.

L. 7. (p. 516.)

1. 1, 2; 2, 2; 3, 3. **2.** Equal.

3. Indices are 0·0791, 2·3980, $\bar{2}\cdot1761$, 0·8060. **4.** $\pm\dfrac{2\sqrt{3}}{9} = \pm0\cdot385$.

5. 170 ft.-lb. **6.** 5·75, 5·79 mi.

L. 8. (p. 517.)

1. $T = \frac{x^2 - 160x}{10x + 5400}$; 0, 3, 9·335, 9·93 per cent.

2. 441 mm. **3.** $1{\cdot}408p$. **4.** $\frac{155}{103} = 1{\cdot}50$.

5. $\frac{1}{3}\sqrt{21} = 1{\cdot}53$. **6.** $y = \frac{1}{8}\left(\frac{1}{x^2} + 8x + 3\right)$.

L. 9. (p. 518.)

1. $x > 4$, yes. **2.** $\frac{p^2 - q^2}{2p}$. **3.** (Decreased) 0·97 : 1.

4. $\frac{3a}{8}$. **5.** $-\frac{1}{4}$, 3. **6.** $\frac{1}{2}$, $\frac{7\pi}{24} - 0{\cdot}916$.

L. 10. (p. 519.)

1. $\frac{b-c}{2} + \frac{a^2}{8(b-c)}$, $60\frac{5}{6}$ ft. **2.** 0, 0, 0; $\log x + \log y = \log (xy)$

3. $z = \frac{60x}{y}$. **4.** $\frac{2(a^2 + 3ah + 3h^2)}{3(a + 2h)}$.

5. 0·232 in. per sec. **6.** 0·8, $y = \frac{1}{20}(16x + 10x^2 - x^5)$.

M. 1. (p. 520.)

1. 1, 46; 6, 132; 0, 99. **2.** 11, 20, 26, 29; 44; $3n - 1$

3. 26, 1326. **4.** 2·34 in. **5.** $\frac{1}{2}$. **6.** £$\frac{9A}{4}$.

M. 2. (p. 520.)

1. 1050. **2.** 15 m.p.h., $10\frac{1}{2}$ sec. **3.** $\frac{1}{60}$, $\frac{3}{16}$.

4. 19, $\frac{181}{19}$. **5.** 21, 3 sq. in.

M. 3. (p. 521.)

1. 49 lb. **2.** $3t^2 + \frac{1}{t^2}$ **3.** -2, $5n - 2$, $\frac{(n+1)(5n-4)}{2}$.

4. 2·90, $-0{\cdot}60$, $-2{\cdot}30$. **5.** 7, 23, 144. **6.** $2r - 1$, n^2.

M. 4. (p. 522.)

1. 2·50. **2.** $c = 0{\cdot}16a^2d^{1\cdot5}$. **3.** 1·23 or $-4{\cdot}23$.

4. $2^n + n$. **5.** $2 \times 3^{n-1}$, $3^n - 1$. **6.** £5940, £5940.

M. 5. (p. 523.)

1. $F = \frac{W}{m} + \frac{Wr}{2240} + \frac{2Ws}{gt^2}$. **2.** -6, $-\frac{1}{34}$. **3.** 3·59

4. 4, 3, 2·62, 0·631. **6.** $(9 + \sqrt{x_1})^2$, $(n - 1 + \sqrt{x_1})^2$.

M. 6. (p. 523.)

1. $V=\left(\frac{2gH}{ml}\right)^{\frac{1}{n}}\times d^{\frac{x}{n}}$. **2.** 16·0. **3.** 30.

4. 2, 8, 14. **5.** 3^{4-n}, $10^{-8}\times 2{\cdot}3$. **6.** 1050, 802 yd.

M. 7. (p. 524.)

1. £600. **2.** 0·00632. **3.** 6, 56.

4. 3×2^n, $n+2+2^n$. **5.** 2·47. **6.** 24 in.

M. 8. (p. 524.)

1. 2·24 p.m. **2.** $\frac{x^3+x^2+4x}{x^2-1}$. **3.** 5·38. **4.** $\frac{7}{6}$.

5. $6\frac{1}{2}$, 2. **6.** Red 348, blue 372 sq. in.

M. 9. (p. 525.)

1. 1973 ft. per sec. **2.** $\frac{3a+5b}{8}$. **3.** $d=\frac{1}{3}G^{0{\cdot}4}\,.\,L^{0{\cdot}2}\,.\,H^{-0{\cdot}2}$.

4. 157. **5.** 24 yd.

6. Rate of flow about $\frac{1}{2}$ cu. in. per sec., 7998 cu. ft.

M. 10. (p. 526.)

1. $(k-1)x^2+12x+9(k-5)=0$. **2.** $d=0{\cdot}059L^{1{\cdot}5}$, 75·6.

3. 7, 4. **4.** 108, 198, 1683. **5.** $\frac{4}{5}[1-(-\frac{1}{4})^n]$, 0·8. **6.** 13·6 yrs.

N. 1. (p. 526.)

1. $y=\frac{25}{18}(x-29)$, 104. **2.** ${}^nP_r=\frac{\lfloor n}{\lfloor n-r}$, 380.

3. $a+b+c+d$, $b^3(d-a-c)$. **4.** 1·435.

5. Readings for $w=35$, 50; $l=21+\frac{w}{3}$. **6.** $2n(n-1)$, 36.

N. 2. (p. 527.)

1. $R=\frac{Av^2}{800}$, $v\propto\frac{1}{\sqrt{A}}$. **2.** $\frac{375\pi}{16}=73{\cdot}6$. **3.** 30.

4. x^3, 45. **5.** 0, ± 10; 10·005. **6.** 5 per cent.

N. 3. (p. 528.)

2. $18A^3 = 24389V^2$. **3.** $3\frac{3}{8}$ pints. **4.** 15.

5. 12, $\frac{3252h}{l\sqrt{l}}$, down. **6.** 40, 8.

N. 4. (p. 529.)

1. $a = 1{\cdot}115$, $n = 0{\cdot}5$. **2.** $3\frac{3}{4}$, $\frac{xy}{z-y}$. **3.** 28.

4. 1·049. **5.** $16y^4 - 80y^2 + 64$; 1, ± 3, -5. **6.** $\frac{1}{60}$.

N. 5. (p. 529.)

2. 13·6 ft.-lb. **3.** 0, 0, 0 or 4, 6, 9. **4.** 12. **5.** 105.

N. 6. (p. 530.)

1. $\log_a b$, $\frac{1}{5}$. **2.** $\sqrt{(abcd)}$, $\frac{1}{4}a^2\sqrt{3}$. **3.** 1260.

4. Increased 7 per cent. **5.** 5·0067, 5·027. **6.** £16. 13*s*. 4*d*., $\frac{2}{27}$.

N. 7. (p. 531.)

1. $V = 168d^2\sqrt{H}$, reduced in ratio 1 : 4. **2.** 160,000.

3. $\frac{h^2}{2x}$. **4.** $\left(1-\frac{1}{N}\right)^2$, $\left(1-\frac{1}{N}\right)^p$, 7.

5. $\frac{1}{6}$. **6.** $\frac{4}{3}\pi a^3$, $\frac{4}{3}\pi a^3$.

N. 8. (p. 532.)

1. Increased in ratio 8 : 1.

2. The lines $y = \pm 1$; everywhere discontinuous. **3.** 64, 15.

4. $1 - y + y^2$; each is $1 + \frac{x}{n} - \frac{(n-1)x^2}{2n^2}$. **5.** 462, $\frac{5}{11}$.

N. 9. (p. 533.)

1. $P = 1{\cdot}94 \times 10^{-11} \times d^{5\cdot5} \,.\, n^{3\cdot5}$. **2.** 7019298. **3.** $a\sqrt{\frac{2}{3n}}$.

4. $\frac{3}{4}$. **5.** 3, 315, $\frac{\lfloor 16}{2^{15}} = 638512875$. **6.** 10, 1, $\frac{n(n-1)}{2}$, $\frac{1}{2}(n-3)(n-4)$.

N. 10. (p. 533.)

1. $M \propto \frac{lr^2}{t^2}$. **2.** 20, 10, 13. **3.** 28.

4. $c^3 + c^2(a+b) + c(a^2+ab+b^2) + a^3 + a^2b + ab^2 + b^3$, 10. **6.** $\frac{3}{4}$.

P. 1. (p. 534.)

1. $\dfrac{100h}{a}$, 0·05 per cent.

2. $\dfrac{a}{100-x} > \dfrac{b}{100+y}$, $\dfrac{b}{100-y} > \dfrac{a}{100+x}$.

3. $P = 0{\cdot}183W + 0{\cdot}32$.

5. $2x-1$, $-\dfrac{5}{x^2}$, $\dfrac{x^3}{3}+\dfrac{1}{x}+c$, $2\frac{2}{3}$.

6. $\pi x^2\left(a-\dfrac{x}{3}\right)$, $\dfrac{4}{21\pi} = 0{\cdot}0606$ in. per sec.

P. 2. (p. 535.)

1. $x = 10^7 \times 2{\cdot}51t^{3\cdot 2}$.

2. $W = 40 + \frac{1}{5}I$, $P = \dfrac{40}{I}+\dfrac{1}{5}$.

3. 2·80.

4. $10^5 \times 7{\cdot}03$, $10^5 \times 7{\cdot}09$.

5. $\frac{1}{10}$.

6. $333\frac{1}{3}$ ft.

P. 3. (p. 536.)

1. $2 \log x$, 3, 1, $\frac{2}{3}$.

2. $ac : ab : bc$; $a^2 - ab$, $ab - b^2$.

3. $d = 0{\cdot}11V^2$.

4. 459 : 53.

5. 4·52 in.

6. 1·12 in. per sec., 4·47 in., $2\displaystyle\int_{-5}^{5}\sqrt{\frac{20}{25-x^2}}\,dx$.

P. 4. (p. 537.)

1. 2·50.

2. $v = 8\sqrt{h}$.

3. $\pi(2a^2 + 2al + h^2)$.

4. 0 ; +, + ; no ; −, +.

5. $\frac{1}{30}$.

6. 65·1 ft.-lb.

P. 5. (p. 537.)

1. $(2x + 100)$ per cent., $(100 + x) : 100$.

2. $m = 1{\cdot}276d^3$.

3. 64·4° F., 36·7° C.

4. 12300.

5. $\delta z \simeq -\dfrac{\delta y}{2y^{\frac{3}{2}}}$, $\delta y \simeq -2x\,\delta x$, $-\dfrac{x}{(49-x^2)^{\frac{3}{2}}}$.

6. $4\frac{1}{2}$.

P. 6. (p. 538.)

1. 0·8155.

2. $h = \dfrac{P}{0{\cdot}0092P + 0{\cdot}29} = \dfrac{3{\cdot}4P}{1 + 0{\cdot}031P}$.

4. 3·73.

6. Area of quadrant of circle, 0·787, 0·783.

P. 7. (p. 539.)

1 1. $\dfrac{x}{r^2}$, $AP = 6{\cdot}21$ ft.

4. $t = 0{\cdot}28d^{1\cdot 16}$, 232 sec.

4 5. $\frac{3}{2}$, $-\frac{3}{4}$; $\frac{5}{4}$, $-\frac{5}{16}$.

6. $\dfrac{x}{12}(2x^2 - 9x + 12)$.

P. 8. (p. 540.)

1. $\frac{b^2}{4\pi^2}(\pi a - b)$, 4·58 in. **2.** $x^3 - x^2 - 4 = 0$, one. **3.** $d = 11{\cdot}4\sqrt{H}$.

4. 93·7. **5.** $(2, 4)$; $(-2, -4)$. **6.** 25 lb., 16·8 in.

Q. 1. (p. 541.)

3. $(4a + 3b)^2 - (a + 7b)^2$. **4.** 7.

5. $2a + a^2$, $4a + 2a^2 + a^3$, $\frac{a(a^n - 2^n)}{a - 2}$. **6.** 6, 12, 5.

Q. 2. (p. 541.)

1. $a + b - q \neq 0$ and $p \neq 0$; $q = 0$ or if $a + b = q$ and $p = 0$. **4.** 3, 5.

5. 10·067, 10·07. **6.** £12·3.

Q. 3. (p. 542.)

1. $-1\frac{1}{2}$. **2.** xy, $\frac{x^3 + y^3}{x^3 - y^3}$. **3.** $\frac{2}{3}$, 2, $\frac{8}{3}$. **4.** 3. **5.** −

Q. 4. (p. 542.)

1. $n = \log\left(\frac{p_1}{p_2}\right) \div \log\left(\frac{p_1 T_2}{p_2 T_1}\right)$. **2.** $-b$ or $a - 1$, $-a$ or $b - 1$.

3. a, $\frac{a-1}{a}$. **5.** $24a$, $15a$, $20a$.

Q. 5. (p. 543.)

3. $-(l + m + n)^2$. **4.** $r\left(\frac{\sqrt{3}}{2}\right)^n$.

6. $(3x - y)(3y - x)(x + y)$; $1 + A(x-1)(x-2)(x-3)(x-4)$.

Q. 6. (p. 544.)

4. $f^2 = bc$, $g^2 = ca$, $h^2 = ab$.

5. $3(y - z)(z - x)(x - y)$; $(a - b)(b - c)(c - a)(a + b + c)$.

Q. 7. (p. 544.)

4. 1 or −9 ; 1, $1\frac{1}{2}$, 2. **5.** $2\frac{1}{3} \geq m \geq 1$.

6. $(x - \alpha)(x - \beta)(x - \gamma) = 0$, q, $-2p$, $-3q$.

Q. 8. (p. 545.)

2. $\frac{2z^3}{1 + z + z^2}$. **4.** −2, 19, 19, 4.

5. $x = \beta(x_1 + vt_1)$, $t = \beta\left(t_1 + \frac{vx_1}{c^2}\right)$. **6.** $-a$, $-b$; 2 or $\frac{1}{2}$.

Q. 9. (p. 545.)

1. $(m^2+mn+n^2)^2$. **3.** $2y^2+y(4c-1)+2c^2-c-5=0$, $\frac{1}{4}$.

4. $(a+b+c)(bc+ca+ab)$, $24abc$. **5.** ± 3 or $\pm\frac{1}{3}$, ± 3 or $\pm\frac{1}{3}$, ∓ 1 or $\pm 4\frac{1}{3}$.

6. (i) $\frac{1}{(2-x)^2}+\frac{2}{2-x}-\frac{2}{3-x}$; (ii) $4x-2$.

Q. 10. (p. 546.)

3. $t^2(b^2-ac)-t(a^2+bc-a)+c^2+ab-b=0$.

5. $\frac{1}{3}, \frac{1}{2}, \frac{1}{6}$. **6.** $(x+y-z)(y+z-x)(x+z-y)$

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