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ELEMENTARY ALGEBRA

PART I

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ELEMENTARY ALGEBRA

PART I

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*** *This volume (Part I.) may be had with or without answers.*

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PREFACE

THE object has been to provide a text book of practical interest and utility, fulfilling the latest requirements of the various examining bodies, and following, to a great extent, the recommendations of the Mathematical Association.

Part I. is intended for beginners and therefore includes a large number of examples which may be taken orally.

Multiplication and Division by polynomials are deferred until after simultaneous equations of the first degree have been treated.

Algebraic processes are identified with those of Arithmetic.

Methods are referred to first principles ; *e.g.* in the solution of equations each step is shown to be a logical application of some axiom and not a matter of arbitrary rules.

A great part of the mere gymnastics of the subject, such as the reduction of complicated specimens of fractions, is made subordinate to useful and suggestive work.

It has been recognised that many learners acquire some facility in manipulation of algebraic expressions without getting any power of dealing with the most important part, the solution of problems. Much practice is therefore given in translating questions into a symbolical form, in order to lead the student easily to the solution of problems.

A very large number of examples are introduced at every stage.

Stress is laid on the importance of testing solutions and checking results, and of using approximations.

Graphical work, involving largely the use of squared paper, is freely employed and interwoven throughout the book. It is used in connection with solution of equations, square and cube roots, statistics, height and distance problems, rate problems of various kinds, indeterminate equations, logarithms, ratio and variation.

Facility in finding factors and in the use of labour-saving methods is aimed at, and the Remainder Theorem is freely employed.

Students are introduced at a fairly early stage to the idea of a function and to the use of functional notation.

The bookwork is expressed in the manner suggested by much experience with learners as the one most readily grasped and retained.

Sets of revision papers are inserted at various stages, usually at the end of what may be considered a term's work.

With a view to practical utility and as a stimulus to interest, logarithms are introduced as early as possible, viz., immediately after Proportion.

Thanks are due to various bodies, from whose examination papers many examples have been taken, especially to the Oxford and Cambridge Local Examination Delegates, and the Controller of His Majesty's Stationery Office.

Some teachers will prefer to leave Chapters, Articles and Examples marked with an asterisk (*) until the student is firmly grounded in the rest of this volume.

A number of easy " Problems involving Quadratics " (XXIX. a.) have been added.

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PART I

ELEMENTARY ALGEBRA

CHAPTER I

DEFINITIONS, ETC.

1. It is assumed that the beginner is already acquainted with the meanings and use of the ordinary symbols of operation, $+$, $-$, $\times$, $\div$, ( ), as employed in Arithmetic. The symbol / is sometimes used to denote the operation of division.

Thus $$10/7 = 10 \div 7 = \tfrac{10}{7}.$$

2. In Arithmetic we denote quantities by *numbers*, each number having a fixed value. In Algebra we denote quantities by *symbols*, generally letters, to which we may assign any value we please.

Thus, in Arithmetic, 2×3 is always equal to 6, whereas $2 \times a$, or more shortly, $2a$, will have different values according to the numerical value we assign to the symbol a.

When $a = 3$, $2a = 2 \times 3 = 6$. If $a = 8$, then $2a = 2 \times 8 = 16$, and so on.

In Arithmetic,
$$2 \times 6 + 3 \times 6 + 5 \times 6 = (2 + 3 + 5) \times 6 = 10 \times 6 = 60.$$

So in Algebra, $2a + 3a + 5a = 10 \times a$, or $10a$.

In the same way, $6b - 2b = 4b$.

We must also remember that since the symbols stand for numerical quantities, we may apply the ordinary Arithmetical laws in using them. Algebraic proofs of the various Arithmetical laws will be given at a later stage.

As in Arithmetic $2 \times 7 = 7 \times 2$, so in Algebra $a \times b = b \times a$, or $ab = ba$.

In the same way, just as 2 and 7 are the factors of the product 2×7, so a and b are the factors of the product ab, remembering that by ab we mean $a \times b$.

Also $a \times b \times c = a \times c \times b = b \times a \times c$, or $abc = acb = bac$,

just as $2 \times 7 \times 8 = 7 \times 2 \times 8 = 7 \times 8 \times 2$.

Thus
$$\begin{aligned} &3abc + 2acb + 7cab \\ &= 3abc + 2abc + 7abc \\ &= 12abc. \end{aligned}$$

In performing the above addition we look upon abc as a single quantity.

Examples. I. a.

Write down, or read off, the values of the following:

1. $3x + 4x$. 2. $a + a$. 3. $2a - a$. 4. $7x - 3x$.
5. $11x - 4x$. 6. $x - x$. 7. $3ab + 5ab$. 8. $2ab + 3ba$.
9. $ab - ba$. 10. $11xy - 7xy$. 11. $9xy - 3yx$. 12. $6ab - ba$.
13. $8abc - 3cab$. 14. $3x + 4x + 5x$. 15. $3ab + 4ab + 2ab$.
16. $5ab + 6ba + 11ab$. 17. $a + 6a + 7a + 2a$. 18. $3abc + 4cab + 7acb$.
19. $a + a + a + a + a$. 20. $3x + 4x + x + 2x + 5x$.

What is the value of $8x$

21. when $x = 2$, 22. when $x = 4$, 23. when $x = \frac{1}{2}$,
24. ...... $x = \cdot 4$, 25. ...... $x = \frac{3}{4}$, 26. ...... $x = 2\frac{1}{2}$?

What is the value of $\frac{x}{2}$

27. when $x = 4$, 28. when $x = 16$, 29. when $x = 3$,
30. ...... $x = \frac{1}{4}$, 31. ...... $x = \cdot 5$, 32. ...... $x = 2\cdot 5$?

Find the value of $3x$

33. when $x = 1$, 34. when $x = 3$, 35. when $x = \frac{5}{3}$,
36. ...... $x = 2\frac{1}{6}$, 37. ...... $x = 2\cdot 4$, 38. ...... $x = 1\cdot 6$.

Find the value of $\frac{x}{3}$

39. when $x = 6$, 40. when $x = 12$, 41. when $x = 7\cdot 5$,
42. ...... $x = 2\cdot 4$, 43. ...... $x = \cdot 6$, 44. ...... $x = \cdot 024$.

3. Symbolical Expression.

$5£ = (20 \times 5)$ shillings,

$\therefore a£ = 20a$ shillings.

In the same way, $a£ = 240a$ pence.

Again, $\quad 360 \text{ shillings} = (360 \div 20)£,$

$\therefore \quad a \text{ shillings} = (a \div 20)£$

$$= \frac{a}{20}£.$$

x half-crowns $= 30x$ pence,

just as $\quad$ 7 half-crowns $= (30 \times 7)$ pence.

$£x + y$ shillings $= (20x + y)$ shillings.

If I give 6 pence to each of 4 boys, I give away (6×4) pence altogether.

.... 6 .... a .... $6a$ ....

.... x .... 4 .... $4x$ ....

.... x .... a .... ax ....

Examples. I. b.

1. What is the number which is 2 greater than x?

2. What is the number which is 3 less than x?

3. If each article costs x pence,
(i) what is the cost of 3 articles? (ii) what is the cost of 7 articles?
(iii) ....11 .... ? (iv) .... a .... ?

4. Express $x£$ (i) in shillings, (ii) in half-sovereigns, (iii) in half-crowns, (iv) in florins, (v) in pence.

5. If I walk x miles an hour, how far do I walk
(i) in 2 hours? (ii) in 7 hours?
(iii) in half-an-hour? (iv) in a hours?

6. Express x yards (i) in feet, (ii) in inches.

7. Express x inches (i) in feet, (ii) in yards.

8. If I give 2 shillings to each boy, how many shillings do I give to x boys? How many pence do I give them?

9. If I divide x shillings equally amongst 7 boys, how many shillings does each boy get? How many pence does each boy get?

10. If there are x forms in a school, how many boys are there in the school
(i) when each form contains 16 boys?
(ii) .... y boys?

11. What is the total number of pence in $£x$, and y shillings?

12. What is the cost in pence of x articles at y pence each? How many shillings do they cost?

13. Express x square feet in square inches.

14. Express x square inches in square feet.

15. Express x metres
(i) in decimetres, (ii) in centimetres,
(iii) in millimetres, (iv) in kilometres.

16. Express x millimetres
(i) in centimetres, (ii) in decimetres,
(iii) in metres, (iv) in kilometres.

17. What is the double

(i) of x? (ii) of $3x$? (iii) of $7x$? (iv) of ax?

(v) of $\frac{x}{2}$? (vi) of $\frac{3x}{2}$? (vii) of $\frac{7x}{4}$?

18. If I buy a horse for £x and sell it for £y, how much do I gain?

19. If I buy a horse for £x and sell it at a loss of £y, how much do I sell it for?

20. If I buy a horse for £x and gain £y by selling it, how much do I sell it for?

4. An Algebraic Expression. Any collection of symbols, figures, and signs involving only arithmetical operations, is called **an algebraic expression.**

Term. The different parts of the expression connected by the signs plus ($+$) and minus ($-$) are called **terms.**

Thus, $5x+7y-4z$ is an algebraic expression, and $5x$, $7y$, and $-4z$ are its *terms*.

When no sign is prefixed to a term, the positive sign ($+$) is always understood.

A *simple expression* consists of one term only; a *compound expression* of two or more terms.

An expression of one term is sometimes called a **monomial.**

Coefficient. In the case of a product, such as 3×7, each of the factors 3 and 7 is said to be the **coefficient** of the other. In the same way, a is the *coefficient* of bc in the product abc, or b is the coefficient of ac, or c of ab.

When one of the factors is expressed in figures, it is called the **numerical coefficient** of the product of the other factors.

Thus in the expression $12xyz$, 12 is the numerical coefficient of xyz.

Power. The **power** of any number or quantity is the result obtained when the number or quantity is multiplied by itself once or any other number of times.

Thus aa is called the *second power* of a, aaa the *third power*, and so on.

Instead of writing aa, we write it thus a^2, and call it 'a squared.' In the same way we write a^3 instead of aaa, a^5 instead of $aaaaa$, and so on.

Hence a^4 denotes the fourth power of a.

Index. The number written above, called the **index** or *exponent*, indicates the number of factors.

$$a \times a \times a \times a \times a \ldots \text{ to } n \text{ factors} = a^n.$$

Square; Cube. The second power of a quantity is called its **square**, the third power its **cube**.

N.B.—a^1 is the same as a.

Square root. The **square root** of a number is that number which, multiplied by itself, gives the original number.

The symbol $\sqrt{}$ is used to denote a square root.

Thus $\sqrt{25} = 5$, for $5 \times 5 = 25$.

$\sqrt{16a^2} = 4a$, for $4a \times 4a = 16a^2$.

Cube root. The **cube root** of a quantity is that quantity whose third power is equal to the original quantity.

Thus, since $2^3 = 8$, 2 is the cube root of 8.

The cube root of a is written thus, $\sqrt[3]{a}$.

In the same way the **fourth**, **fifth**, etc., root of any quantity is that quantity whose *fourth*, *fifth*, etc., power is equal to the original quantity.

The n^{th} root of a is written thus, $\sqrt[n]{a}$.

Like and Unlike Terms. In any algebraic expression, those terms which differ only in their numerical coefficients are said to be *like* terms.

In the expression

$$6ax^2 - 7a^2x - 9abcx - 11a^2x - bcd - 3ax^2$$

$6ax^2$ and $-3ax^2$ are *like* terms, also $-7a^2x$ and $-11a^2x$; $-9abcx$ and $-bcd$ are *unlike* to one another and to all the other terms.

5. Examples. $a^2 \times a = a \times a \times a = a^3$.

$a^2 \times a^3 = a \times a \times a \times a \times a = a^5$.

$a^4 \times a^7 = \text{eleven } a\text{'s multiplied together} = a^{11}$.

N.B.—a^3 is not a multiplied by itself three times, but is the product of three factors, a, a, a.

$$a^2b \times b = a \times a \times b \times b = a^2b^2.$$

$$a^3b^2 \times a^2b^4 = a^3 \times a^2 \times b^2 \times b^4 = a^5b^6.$$

$$a^2 \times a^3x = a^5x.$$

$$3ab \times 3a = 9 \times a \times a \times b = 9a^2b.$$

$$12abc \times 2a^2bc = 24 \times a \times a^2 \times b \times b \times c \times c = 24a^3b^2c^2.$$

The square of $a^2 = a^2 \times a^2 = a^4$.

.... $a^5 = a^5 \times a^5 = a^{10}$.

.... $4a^2 = 4 \times 4 \times a^2 \times a^2 = 16a^4$.

The square root of a^4 is a^2, for $a^2 \times a^2 = a^4$.

.... a^6 is a^3, for $a^3 \times a^3 = a^6$.

Examples. I. c.

1. Give three examples of
   (1) a simple algebraic expression,
   (2) a compound algebraic expression,
   (3) a simple algebraic expression with a numerical coefficient.

2. Express the product abx^2 in different forms.

3. Do the same with $3x^2y^3$, $6a^2b^3c^4$, $12ab^3x$.

What is the

4. second power of 3, 5. third power of 4, 6. fifth power of 2,

7. product of x and x^2, 8. product of a^2 and a^3,

9. .... a^3 and x^2, 10. .... a^2b and b^2c,

11. .... $4a$ and $3b$, 12. .... $4a^2$ and $5a^3$,

13. .... $12abc$ and $3abc$, 14. .... $12a^3y^2$ and $7ayz$,

15. square root of x^2, 16. square root of x^6,

17. .... $16a^2$, 18. .... x^{12},

19. square of 5, 20. square of x^3,

21. .... a^4b, 22. .... $4x^3y^4$,

23. cube of x^2, 24. cube of ay^3, 25. cube of $2a^2y^4$,

26. cube root of x^6, 27. cube root of $8a^3$, 28. cube root of $27a^6$?

29. What is the coefficient of a in the expression $6a$,

30. .... a^2 .... $3a^2b$,

31. .... y .... x^2y,

32. .... y^2 .... y^2x,

33. .... a^4 .... $3a^4b^2c$,

34. .... x .... $\frac{3}{4}abx$?

Find the values of

35. 2^2+3^2, 36. $(2+3)^2$, 37. 3^2+4^2, 38. $(3+4)^2$,

39. 7^2-5^2, 40. $(7-5)^2$, 41. $\sqrt{25}-\sqrt{16}$, 42. $\sqrt{25-16}$,

43. 13^2-5^2, 44. $(13-5)^2$, 45 $\sqrt{25}-\sqrt{9}$, 46. $\sqrt{25-9}$.

6. Substitution.

(1) If $a=3$, $2a = 2 \times 3 = 6$.

$a^2 = a \times a = 3 \times 3 = 9$.

$4a^3 = 4 \times a \times a \times a = 4 \times 3 \times 3 \times 3 = 12 \times 9 = 108$.

(2) If $x=5$, $4x = 4 \times 5 = 20$.

$4x^2 = 4 \times 5 \times 5 = 100$.

$\frac{6}{5}x^3 = \frac{6}{5} \times 5 \times 5 \times 5 = 6 \times 5 \times 5 = 150$.

(3) If $a=2$, $b=3$, $c=4$,

$$abc=2\times3\times4=24.$$
$$a^2b=2\times2\times3=12.$$
$$ab^2c=2\times3\times3\times4=6\times12=72.$$

(4) If $a=0$, $b=1$, $c=3$, $x=3$,

$$a^2=0. \quad a^3=0 \quad a^4=0.$$
$$abc=0\times1\times3=0.$$
$$a^2bc=0\times0\times1\times3=0.$$
$$b^2c^2=1\times1\times3\times3=9.$$
$$b^3c^4=1\times1\times1\times3\times3\times3\times3=81.$$
$$x^x=3^3=3\times3\times3=27.$$
$$x^b=3^1=3.$$
$$\sqrt[x]{27}=\sqrt[3]{27}=3.$$

Examples. I. d.

If $a=5$, $b=3$, $c=1$, $x=7$, find the value of

1. $3a$. 2. $3b$. 3. c^3. 4. x^2. 5. $3b^2$. 6. $4a^2$.
7. $9c^2$. 8. cx. 9. b^4. 10. $4a^3$. 11. $2x^2$. 12. $11c^4$.

If $a=1$, $b=2$, $c=3$, $x=4$, $y=5$, evaluate the following:

13. $7a^2b$. 14. $6abc$. 15. $9x^2y$. 16. a^4bc.
17. $\frac{3}{4}b^2c$. 18. $\frac{16}{5}acy$. 19. $8a^5b$. 20. $8ax$.
21. $\frac{3}{16}b^4$. 22. a^b. 23. c^a. 24. b^c.
25. a^{2b}. 26. b^{ac}. 27. $\frac{4}{15}a^a$. 28. $\frac{2}{3}c^b$.
29. $\frac{x^4}{16}$. 30. $\frac{7}{8}b^2cx^2$. 31. $\frac{4}{27}a^3c^3x$. 32. $\frac{6}{125}xy^2$.

If $a=0$, $b=1$, $c=2$, $x=\frac{1}{2}$, evaluate the following:

33. $7a^2$. 34. $6ab$. 35. $3ax$. 36. $4cx^2$.
37. $abcx$. 38. a^3c^4x. 39. $\frac{1}{2}b^3c^3x^2$. 40. $\frac{3}{4}b^2cx^3$.
41. $a^7b^7c^7$. 42. $\sqrt[c]{b^2c^2}$. 43. $\sqrt[c]{\frac{1}{4}b^4c^4}$. 44. $\sqrt[3]{\frac{8}{27}b^3c^3}$.

CHAPTER II

NEGATIVE QUANTITIES

7. Any quantity with the sign $+$ prefixed, or understood, is called a positive quantity, and any quantity with the sign $-$ prefixed is called a negative quantity.

Negative Quantities. Arithmetically we cannot subtract 6 from 3, *i.e.* the expression $3-6$ has no arithmetical meaning.

In Algebra however such an expression has an intelligible interpretation.

This is best seen by considering a few examples.

If a farmer buys 7 cows, and sells 4 cows, he has 3 *more* than he had at the start. On the other hand, if he buys 4 cows, and sells 7, he has 3 *less* than at first.

We express this algebraically thus,

$$7 \text{ cows} - 4 \text{ cows} = +3 \text{ cows}.$$

$$4 \text{ cows} - 7 \text{ cows} = -3 \text{ cows}.$$

Again, if a man gains £10 and loses £6, he has £10 − £6, *i.e.* £4, *more* than at first. If, on the other hand, he gains £6 and loses £10, he has £4 *less* than at first,

i.e. $\text{£}10 - \text{£}6 = +\text{£}4,$

and $\text{£}6 - \text{£}10 = -\text{£}4.$

Moreover, if he loses £10 and then gains £6, he will then have £4 *less* than at first,

i.e. $-\text{£}10 + \text{£}6 = -\text{£}4.$

If a man runs 120 yds. along a road, and then runs 90 yds. towards his starting point, he will be 30 yds. from his starting place. But if he first runs 90 yds. and then 120 yds. backwards, he will still be 30 yds. from his starting place, but *on the opposite side of it.*

$$120 - 90 = 30, \qquad 90 - 120 = -30.$$

Thus we see that $+4$ and -4 are the exact *opposite* of one another. If we consider a man's income, +£4 will represent an *increase*, whilst −£4 will represent an equal *decrease*. +4 yds. and −4 yds. represent 4 yds. *in opposite directions*, and so on.

Suppose a man loses first £10 and then again loses £4, he is £14 poorer than at first.

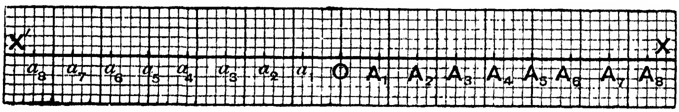
That is, $-\text{£}10 - \text{£}4 = -\text{£}14.$

Thus $-3-2=-5$, and $-5-6=-11$.

Now instead of using £, or cows, or yards, let us use a symbol a.

We then have,
$$10a - 6a = +4a.$$
$$6a - 10a = -4a.$$
$$-6a - 10a = -16a.$$
$$-10a - 6a = -16a.$$
$$-10a + 6a = -4a.$$

8. Graphical Illustrations. Take a str. line XOX′ of unlimited length, and let all distances measured *to the right* be considered positive, whilst all distances measured in the opposite direction, from right to left, are taken as negative.



Take $\quad OA_1 = A_1A_2 = A_2A_3 = \ldots = b$ along OX,

and $\quad Oa_1 = a_1a_2 = a_2a_3 = \ldots = b$ along OX′.

Taking O as the starting point in each case,

OA_6 denotes $+6b$, whilst Oa_6 denotes $-6b$, and so on.

Also A_3A_7 denotes $+4b$, whilst A_7A_3 denotes $-4b$.

Thus $6b$ is denoted by OA_6 (6 spaces *to the right*), and A_6A_4 denotes $-2b$ (2 spaces *to the left*);
$$\therefore\ 6b - 2b = OA_4 = 4b.$$

Again, still starting from O, $-2b$ is denoted by Oa_2 (2 spaces to *the left*) and $+5b$ by a_2A_3 (5 spaces *to the right*)
$$\therefore\ -2b + 5b = OA_3 = 3b.$$

Again, $-3b$ is denoted by Oa_3, and $-4b$ by a_3a_7, both distances being measured *to the left*,
$$\therefore\ -3b - 4b = Oa_7 = -7b.$$

Once more,

$-7b$ is denoted by Oa_7 (7 spaces in the negative direction)

$+4b$ .... a_7a_3 (4 .... positive ....),
$$\therefore\ -7b + 4b \text{ is denoted by } Oa_3,$$

i.e.
$$-7b + 4b = -3b.$$

Examples. II. a.

What is the value of

1. $5-3$.	**2.** $3-5$.	**3.** $11-7$.	**4.** $-3-2$.
5. $-7-11$.	**6.** $7-11$.	**7.** $4a-2a$.	**8.** $2a-4a$.

What is the value of

9. $-2a-4a$.
10. $-4a+6a$.
11. $3x-9x$.
12. $9x-3x$.
13. $7a^2-3a^2$.
14. $-3x^2-11x^2$.
15. $-11x^2+8x^2$.
16. $2a^2-9a^2$.
17. a^2-4a^2.
18. $8ab-4ab$.
19. $-8ab-4ab$.
20. $-ab-ab$.
21. $4ab-11ab$.
22. $3xy-8xy$.
23. $3a^2b-12a^2b$.
24. $ab-ab$.
25. $ab-5ab$.
26. $-4-5$.
27. $-4x+7x$.
28. $-5ab+2ab$.
29. $-abc-11abc$.
30. $3abc-5cab$.
31. $-2xy-5yx$.
32. $-3abc+7acb$.
33. $-3abc-7bca$.
34. $14x-11x$.
35. $11x-14x$.
36. $-12x+15x$.
37. $-x^3-x^3$.
38. $12x-17x$.
39. $-12x-17x$.
40. $-13x+17x$.
41. $-15x^2+6x^2$.

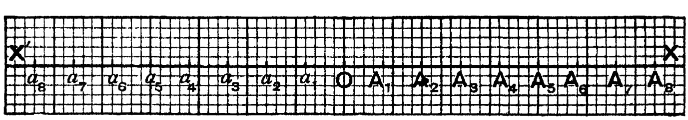
Graphical Examples.

Use graphical illustrations to prove the following (squared paper will be found useful):

42. $4-3=1$.
43. $7-4=3$.
44. $6-2=4$.
45. $-8+5=-3$.
46. $2-5=-3$.
47. $-7+2=-5$.
48. $-2-3=-5$.
49. $-4-5=-9$.
50. $5x-3x=2x$.
51. $-3x+8x=5x$.
52. $-2x-4x=-6x$.
53. $-5x+x=-4x$.
54. $-2x-3x=-5x$.
55. $-7x+4x=-3x$.

9. The order in which additions and subtractions are performed is immaterial. If you take 4 from 6 and then add 3 the result is the same as if you first add the 3 to the 6 and then subtract the 4. The same principle holds good with regard to algebraical expression, thus $6a-4b+3c$ is equal to $6a+3c-4b$.

This is generally accepted as axiomatic, but may with advantage be illustrated graphically.



With the above diagram, using the same hypotheses with regard to signs, etc., as in Art. 8,

$4b+3b-5b$ takes us from O to A_4 (4 spaces), then from A_4 to A_7 (3 spaces), then from A_7 to A_2 (5 spaces in the negative direction);

$$\therefore\ 4b+3b-5b=OA_2=2b.$$

In the same way $4b-5b+3b$ takes us first from O to A_4, then from A_4 to a_1 (5 spaces in the negative direction), then from a_1 to A_2 (3 spaces in a positive direction), *i.e.* to the same point as in the first case;

$$\therefore\ 4b+3b-5b \text{ is the same as } 4b-5b+3b.$$

Again, $6b-4b-3b$ takes us first from O to A_6 (6 spaces), then from A_6 to A_2 (4 spaces in the negative direction), then from A_2 to a_1 (3 spaces in the negative direction);

$$\therefore\ 6b-4b-3b=Oa_1=-b.$$

In the same way $-4b-3b+6b$ takes us first from O to a_4 (4 spaces in the negative direction), then from a_4 to a_7 (3 spaces in the negative direction), and then from a_7 to a_1 (6 spaces in the positive direction);

$$\therefore\ -4b-3b+6b=Oa_1=-b,$$

i.e. $$6b-4b-3b=-4b-3b+6b.$$

Graphical Examples. II. b.

Prove the following graphically, using squared paper:

1. $6+5-3=8$.
2. $3-4+2=1$.
3. $-5+4-2=-3$.
4. $-1-2-3=-6$.
5. $7-7+2=2$.
6. $-6+3+4=1$.
7. $8-5-3=0$.
8. $1-2+3-4+5=3$.
9. $-2+1-3+2-4+3=-3$.
10. $-2+5-7+4=0$.
11. $6a-7a+4a=3a$.
12. $3a-4a-5a=-6a$.
13. $3a+4a-9a=-2a$.
14. $-4a-3a+7a=0$.
15. $-6x+4x+5x=3x$.
16. $-7x+4x+x=-2x$.
17. $3a-5a+4a-2a=0$.
18. $-9a+8a+3a-5a=-3a$.
19. $-a-3a-6a=-10a$.
20. $-7a+4a-3a+6a=0$.

10. Substitutions.

Example 1. When $a=2$, $b=3$, $c=1$, $d=0$, find the value of $\sqrt{\dfrac{a^2b^2}{c}}$.

$$\sqrt{\frac{a^2b^2}{c}}=\frac{ab}{\sqrt{c}}=\frac{2\times 3}{1}=6.$$

Example 2. With the same values of a, b, c and d, find the value of $a^2-b^2+c^2-qd$.

$$\begin{aligned} a^2-b^2+c^2-qd &= 2\times 2-3\times 3+1\times 1-q\times 0 \\ &= 4-9+1 \quad q\times 0=0) \\ &= -4. \end{aligned}$$

Example 3. With the same values of a, b, c and d, evaluate the expression $\frac{3}{4}\sqrt[3]{\frac{4a}{b^3}}-\frac{1}{8}\sqrt{\frac{bc^4}{3}}+\sqrt[3]{a^3b^3c^3}$.

$$\text{The given expression } =\tfrac{3}{4}\sqrt[3]{\frac{4\times2}{3\times3\times3}}-\tfrac{1}{8}\sqrt{\frac{3\times1}{3}}+abc$$
$$=\tfrac{3}{4}\times\tfrac{2}{3}-\tfrac{1}{8}+6$$
$$=\tfrac{1}{2}-\tfrac{1}{8}+6$$
$$=6\tfrac{3}{8}.$$

Example 4. Find the values of x^2-5x+4 for the following values of x:—0, 1, 2, 3, 4, 5.

When

x =	0	1	2	3	4	5
x^2 =	0	1	4	9	16	25
$-5x$ =	0	-5	-10	-15	-20	-25
4 =	4	4	4	4	4	4
x^2-5x+4 =	4	0	-2	-2	0	4

$\therefore$ 4, 0, -2, -2, 0, 4 are the required values.

Examples. II. c.

If $a=3$, find the value of

1. a^3. 2. $-a^2$. 3. $a-4$. 4. a^2-2. 5. $3a^2-2a$. 6. $a-2a^3$.

If $x=1$, $y=2$, find the value of

7. $2x^2+y$. 8. $x-2y$. 9. x^2y. 10. xy^2. 11. x^2-y^2. 12. $4x^2-y^2$.

If $a=-3$, find the value of

13. $a+2$. 14. $a+3$. 15. $2a-7$. 16. $5a+15$. 17. $\frac{a}{2}+1$. 18. $\frac{3a}{2}+4\frac{1}{2}$.

If $x=0$, $y=4$, $a=7$, $b=3$, $c=8$, find the value of

19. $\sqrt{\frac{c^2}{y}}$. 20. $\sqrt[3]{\frac{y^3}{c}}$. 21. $4\sqrt{a^3b^2x}$. 22. $\frac{\sqrt{b^4c^2}}{y}$. 23. $\sqrt{\frac{1}{a^2y}}$. 24. $\sqrt[3]{\frac{1}{b^3c}}$.

25. $a^2+b^2+c^2$. 26. x^3. 27. x^3y. 28. px^5. 29. $qx^2+bc-20y$.

30. $3ab-4bc-2ay$. 31. $a^2+b^2+c^2-x^2-y^2$. 32. $\frac{1}{7}ab-\frac{1}{4}cy-\frac{3}{8}y^2$.

33. $abx^2-7acy^2+9a^2cy$.

If $a=0$, $b=4$, $c=9$, $d=25$, find the value of

34. $\sqrt{ab}-\sqrt{bc}+\sqrt{cd}$. 35. $\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{d}}$. 36. $\frac{d^2}{25}-\frac{c^2}{81}-\frac{bc}{9}+\frac{bcd}{36}$.

37. $\sqrt{bcd}-\sqrt{acd}-\sqrt[3]{2b}+\sqrt[3]{5d}$. 38. $b\sqrt{cd}+a\sqrt{bd}-4\sqrt{bc}-\sqrt[3]{6bc}$

39. Find the values of x^2-6x+9, when x has the values 0, 1, 2, 3, 4, 5. Tabulate the work.

40. Find the values of $2x^2-3x-10$, when x has the values 0, 2, 4, 6, 8. Tabulate the work.

41. Find the values of $4x^2 - 5x + 4$ when x has the values 0, ·5, 1, 1·5, 2. Tabulate the work.

42. Prove that $2x^2 - 23x + 63 = 0$, when $x = 7$.

43. Prove that $x^2 - \frac{8x}{5} - \frac{21}{5} = 0$, when $x = 3$.

11. An algebraic expression consisting entirely of unlike terms cannot be simplified unless the values of the symbols are given.

If a man has 7 pigs, 3 cows, and 3 geese, he does not know the value of 7 pigs + 3 cows + 5 geese, unless he knows the value of a pig, the value of a cow, and the value of a goose.

In the same way we cannot simplify the expression $7a + 3b + 5c$, unless we are given the values of a, b, and c.

On the other hand, **if an algebraical expression consists entirely of like terms, we can collect these terms into one.**

Just as 2 cows + 3 cows + 5 cows = 10 cows,

so $$2a + 3a + 5a = 10a.$$

7 pigs − 3 pigs = 4 pigs.

In the same way $$7a - 3a = 4a.$$

11 geese − 4 geese = 7 geese ;

$$\therefore\ 11x - 4x = 7x.$$

12 horses − 7 horses + 2 horses = 7 horses.

In the same way $$12y - 7y + 2y = 7y.$$

12. In Arithmetic we know that

$$2(3+4) = 2 \times 3 + 2 \times 4 = 6 + 8 = 14.$$

Or otherwise, $$2(3+4) = 2 \times 7 = 14.$$

In Algebra $$2(3a + 4a) = 2 \times 3a + 2 \times 4a = 6a + 8a = 14a.$$

Or otherwise, $$2(3a + 4a) = 2 \times 7a = 14a.$$

Let us now consider the expression $2(3a + 4b)$, noticing that the terms $3a$ and $4b$ are *unlike*.

$2(3a + 4b) = 2 \times 3a + 2 \times 4b = 6a + 8b$, and this expression cannot be further simplified unless the values of a and b are given, for the terms $6a$ and $8b$ are unlike.

Thus we see that the second method used in the above arithmetical examples cannot be used in Algebra when the terms are unlike.

13. Example 1. Express $4a+2b-3c-2a+b-c$ in its simplest form.

$$4a+2b-3c-2a+b-c$$
$$=4a-2a+2b+b-3c-c$$
(collecting like terms)
$$=2a+3b-4c.$$

Example 2. Find the simplest form of

$$3x^2y-4x^3-4xy^2-6x^2+2xy^2-3x^2y-5x^2-3x^3+6.$$

The given expression

$$=3x^2y-3x^2y-4x^3-3x^3-4xy^2+2xy^2-6x^2-5x^2+6$$
(collecting like terms)
$$=-7x^3-2xy^2-11x^2+6.$$

Examples. II. d.

Find simple forms of the following expressions:

1. $11-7+4-3+2.$
2. $-6+9-11+2.$
3. $3a-6a+4a-a.$
4. $-11a-4a+2a.$
5. $3bc-7bc-9bc+18bc.$
6. $-3x^2y-7x^2y+4xy^2-3xy^2.$
7. $9x^2-14xy+2y^2+6xy-6x^2-5y^2.$
8. $2(6a-4a+2a).$
9. $\frac{1}{2}(9a-3a-2a).$
10. $\dfrac{16a^2-a^2-7a^2}{4}.$

Prove that the following statements are true when $x=1$, $y=2$ and $z=4$.

11. $x^2+y^2+z^2=21.$
12. $x^2y+y^2z=18.$
13. $yz^2-2y^2z-5x^3=-5.$
14. $\frac{y}{x}-\frac{z}{y}=0.$
15. $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=5.$
16. $\frac{z^2}{y}-\frac{y^2}{x}+\frac{x^2}{z}=4\frac{1}{4}.$
17. $\frac{yz}{x}-\frac{xz}{y}+\frac{xy}{z}=6\frac{1}{2}.$
18. $x^2-y^2-z^2=-19.$
19. $\sqrt[3]{yz}-\sqrt[3]{16xz}+\sqrt[3]{x^6y^2z^2}=2.$
20. $y^x+x^z+z^y=19.$

CHAPTER III

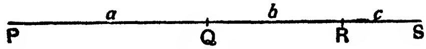
SIMPLE BRACKETS

14. In Arithmetic when a number of terms are included within brackets ( ) it is understood that the terms within the brackets should be considered *as a whole.*

Thus $8+(7+5)$ means that we first add 7 and 5, and then add the result to 8.

When a group of terms within brackets has the positive sign (+) prefixed, the brackets may be removed without changing any of the signs within the brackets.

I. To prove that $a+(b+c)=a+b+c$.

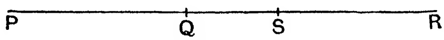


Let the straight lines PQ, QR, RS represent a, b, c respectively.

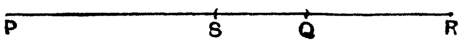
Then $\quad a+(b+c)=\mathsf{PQ}+(\mathsf{QR}+\mathsf{RS})=\mathsf{PQ}+\mathsf{QS}$

$\qquad\qquad =\mathsf{PQ}+\mathsf{QR}+\mathsf{RS}=a+b+c.$

II. *To prove that* $a+(b-c)=a+b-c$.



(Fig 1)



(Fig 2)

Representing a, b, c by straight lines as before, remembering that we must draw RS in the opposite direction to PQ and QR, (see Art. 9) $a+(b-c)=\mathsf{PQ}+(\mathsf{QR}-\mathsf{SR})$

$=\mathsf{PQ}+\mathsf{QS}$ in fig. (1) and $\mathsf{PQ}-\mathsf{SQ}$ in fig. (2)

$=\mathsf{PS}$ in each case

$=\mathsf{PQ}+\mathsf{QR}-\mathsf{SR}$ in each case

$=a+b-c.$

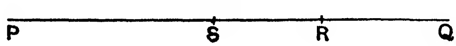
Also, since we may write algebraic terms in any order,

$$-c+b=b-c;$$

$$\therefore\ a+(-c+b)=a+(b-c)=a+b-c=a-c+b.$$

We have thus proved the rule.

When a group of terms within brackets has the negative sign ($-$) prefixed, the brackets may be removed on changing the signs of all the terms within the brackets.



As above $a-(b+c)=\mathsf{PQ}-(\mathsf{RQ}+\mathsf{SR})=\mathsf{PQ}-\mathsf{SQ}=\mathsf{PS}$

$\qquad =\mathsf{PQ}-\mathsf{RQ}-\mathsf{SR}=a-b-c.$

Also $\quad a-(b-c)=\mathsf{PQ}-(\mathsf{RQ}-\mathsf{RS})=\mathsf{PQ}-\mathsf{SQ}=\mathsf{PS}$

$\qquad =\mathsf{PQ}-\mathsf{RQ}+\mathsf{RS}=a-b+c.$

Again, since terms may be written in any order,

$$a-(-c+b)=a-(b-c)=a-b+c=a+c-b.$$

The rule is therefore established.

15. In addition to the ordinary brackets, we sometimes use a line, called a "vinculum," drawn over the terms to be connected.

Thus $a-\overline{2b+3c}$ is the same as $a-(2b+3c)$.

In Arithmetic we know that $\frac{3+5}{2}$ is the same as $\frac{3}{2}+\frac{5}{2}$.

So in Algebra $\frac{3x+4a}{5}$ is the same as $\frac{3x}{5}+\frac{4a}{5}$.

Here the "vinculum" ________, drawn underneath, has the same value as a pair of brackets.

For instance $3+\frac{2x-4}{3}=3+\frac{1}{3}(2x-4)=3+\frac{2x}{3}-\frac{4}{3}$.

Also $3-\frac{2x-4}{3}=3-\frac{1}{3}(2x-4)=3-\frac{2x}{3}+\frac{4}{3}$.

As in Arithmetic $\quad 3(2+5)=3\times 2+3\times 5$,
so in Algebra $\quad 4(a+b)=4a+4b$.

16. **Example 1.** Prove, by removing the brackets, that

$$7-(x+2)+(3-2x)-(-6x+3)=5+3x.$$

$$\begin{aligned}\text{The given expression}&=7-x-2+3-2x+6x-3\\&=7+3-2-3+6x-x-2x\\&=10-5+6x-3x\\&=5+3x.\end{aligned}$$ Q.E.D.

Example 2. Prove that $4a-2(a+b)+3(a-b)=5a-5b$.

$$\begin{aligned}4a-2(a+b)+3(a-b)&=4a-2a-2b+3a-3b\\&=4a+3a-2a-2b-3b\\&=7a-2a-5b\\&=5a-5b.\end{aligned}$$ Q.E.D.

Example 3. Simplify the expression

$$\frac{5x-15}{5}-\frac{12-42x}{6}+\frac{27x-54}{9}.$$

$$\begin{aligned}\text{The given expression}&=\frac{5x}{5}-\frac{15}{5}-\frac{12}{6}+\frac{42x}{6}+\frac{27x}{9}-\frac{54}{9}\\&=x-3-2+7x+3x-6\\&=11x-11.\end{aligned}$$

Examples. III. a.

What are the values of

1. $6+(4-2)$. 2. $6-(3+1)$. 3. $9+(3-4)$.
4. $9-(3-4)$. 5. $11-(8+4)$. 6. $10+(5-10)$.
7. $14-(3+11)$. 8. $11-(-2-3)$. 9. $17+(5-6)$.
10. $-2-(3+4)$. 11. $-2-(2-4)$. 12. $-7+(-4+11)$.

13. $21-(25-23)$.
14. $-(4+7)+15$.
15. $6a+(4a-2a)$.
16. $6a-(4a-2a)$.
17. $6a-(4a+2a)$.
18. $6a-(-4a-2a)$.
19. $a-(a+a)$.
20. $a+(a-a)$.
21. $-a-(a+a)$.
22. $-(a+a)+5a$.
23. $3a^2-(5a^2-7a^2)$.
24. $6ab-(2ab+4ab)$.
25. $-x^2-(-3x^2)+(-5x^2)$.
26. $-x^2+(7x^2-6x^2)$.

Prove the following by removing brackets :

27. $6+(x-2)-(3+4x)+(6x+1)=3x+2$.
28. $(3x-2)-(4x-5)+(x+7)=10$.
29. $(9a-b)+(-2a+3b)-(6a+5b)=a-3b$.
30. $x-6a-(2x-3a)-(a-6x)=5x-4a$.
31. $(a+b-c)-(a-b-c)+(a-b+c)=a+b+c$.
32. $3a-2b+3c-(2a-5b-3c)+(3a-3b-2c)=4a+4c$.
33. $\overline{a-b}+\overline{b-c}-\overline{a-c}=0$.
34. $4a-2b+5c-\overline{2a-3b+7c}+\overline{3b+9c-2a}=4b+7c$.
35. $2(x-1)+3(1-x)-2(2-3x)=5x-3$.
36. $3(2-a)-7(a+6)+6(2a+7)=2a+6$.
37. $2(a+b)-(2a-b)=3b$.
38. $3(2a-c)-7(c-3a)-4(5a-2c)=7a-2c$.
39. $3(a-b+c)-4(b+a-c)-2(c-a-b)=a-5b+5c$.
40. $2(3x+12)+3(x-4)-4(2x+3)=x$.
41. $\dfrac{2x+4}{2}+\dfrac{3x-6}{3}=2x$.
42. $\dfrac{3x-9}{3}+\dfrac{4x-12}{2}-\dfrac{8x+12}{4}=x-12$.
43. $\dfrac{3x+12}{3}-\dfrac{2x-4}{2}-\dfrac{22-33x}{11}=3x+4$.
44. $\dfrac{6x-8}{2}+\dfrac{10x-5}{5}-\dfrac{14x-21}{7}=3x-2$.
45. $\dfrac{8-9x}{3}-\dfrac{7-21x}{7}+\dfrac{20+25x}{5}=5x+5\tfrac{2}{3}$.

ADDITION

17. In Arithmetic the sum of 2 and 3 may be written $2+3$.

So in Algebra the sum of a and b is $a+b$.

Using the rules for removing brackets, the sum of a and $-b$ is

$$a+(-b)=a-b.$$

When like terms are to be added together, they may (Art. 9) be collected into one term.

Unlike terms cannot thus be collected.

The sum of $2a$, $-3a$, and $5a$ is

$$2a+(-3a)+5a=2a-3a+5a=4a.$$

The sum of x^2, $-3x$, and -6 is $x^2+(-3x)+(-6)$ which is equal to x^2-3x-6; and this cannot be shortened, since the terms are all unlike.

When a number of like terms are collected into one term, the result is called their **algebraic sum**, even though some of the terms may be connected by the negative or minus sign.

18. Example 1. Add together $\frac{5x}{6}$ and $\frac{x}{5}$.

$$\text{The sum required} = \frac{5x}{6}+\frac{x}{5}$$
$$= \frac{5\times 5x}{5\times 6}+\frac{6x}{5\times 6}, \text{ (as in Arithmetic)}$$
$$= \frac{25x+6x}{30} = \frac{31x}{30}.$$

Example 2. Find the sum of $\frac{x^2}{3}$ and $-\frac{2x^2}{7}$.

$$\text{The sum required} = \frac{x^2}{3}-\frac{2x^2}{7}$$
$$= \frac{7x^2}{7\times 3}-\frac{3\times 2x^2}{7\times 3}$$
$$= \frac{7x^2-6x^2}{21} = \frac{x^2}{21}.$$

Examples. III. b.

Add together the following quantities:

1. 4 and -7.
2. 5 and -3.
3. -4 and -2.
4. -7 and 6.
5. -4 and 4.
6. 9 and -9.
7. $3x$ and $-2x$.
8. $-2x$ and $-4x$.
9. $-7x$ and $9x$.
10. $-7x$ and $3x$.
11. $3a$ and $4a$.
12. $3a$ and $-4a$.
13. $-3a$ and $-6a$.
14. $6a$ and $-2a$.
15. $-2a$ and $7a$.
16. x^2 and $-3x^2$.
17. abc and acb.
18. bca and $-cab$.
19. x and $\frac{x}{2}$.
20. x and $-\frac{x}{2}$.
21. $-2x$ and $-\frac{1}{2}x$.
22. $-\frac{x}{2}$ and $3x$.
23. $2a^2$ and $2a$.
24. $3a^2$ and $-3a$.
25. $-6x^2$ and $-2x$.
26. $-2x^3$ and x.
27. $\frac{x}{2}$ and $\frac{x}{4}$.
28. $\frac{x}{2}$ and $-\frac{x}{4}$.
29. $\frac{x}{4}$ and $-\frac{x}{2}$.
30. $-\frac{x}{2}$ and $-\frac{x}{4}$.
31. $\frac{3x}{8}$ and $\frac{x}{4}$.
32. $-\frac{x}{4}$ and $\frac{3x}{8}$.
33. $\frac{3}{4}xyz$ and $-\frac{1}{2}xyz$.
34. $\frac{x}{6}$ and $-\frac{x}{3}$.
35. $\frac{5x^2}{8}$ and $-\frac{3x^2}{4}$.
36. $3x^2$ and $-2y^2$.

19. **Example 1.** The sum of $3x-4a$ and $2x+3a$

$$=3x-4a+2x+3a$$
$$=3x+2x-4a+3a$$
$$=5x-a.$$

Example 2. The sum of $4(x-y)$ and $5(x-y)$

$$=9(x-y).$$

Here we look upon $x-y$ as a single quantity, and just as

$$4a+5a=9a,$$

or
$$4\text{ cats}+5\text{ cats}=9\text{ cats},$$

so
$$4(x-y)+5(x-y)=9(x-y).$$

Example 3. Find the sum of $\frac{5}{9}(2a-b)$ and $\frac{4}{9}(2a-b)$.

Here we may look upon $\frac{1}{9}(2a-b)$ as a single quantity, and therefore the sum required

$$=9\text{ times }\tfrac{1}{9}(2a-b)$$
$$=\tfrac{9}{9}(2a-b)$$
$$=2a-b.$$

Examples. III. c.

Find the sum of

1. $a+b$ and $a-b$.
2. $2x-a$ and $3x+a$.
3. $-x+a$ and $x+a$.
4. $2x+a$ and $3x+a$.
5. $a-3b$ and $a+2b$.
6. $2a-b$ and $3a-b$.
7. x^2+y^2 and x^2-y^2.
8. $2x^2-y^2$ and $3x^2-2y^2$.
9. $\frac{a}{2}+\frac{b}{2}$ and $\frac{a}{2}-\frac{b}{2}$.
10. $\frac{a}{2}+\frac{b}{2}$ and $\frac{a}{2}+\frac{b}{2}$.
11. $\frac{1}{3}a+\frac{2}{3}b$ and $\frac{2}{3}a+\frac{1}{3}b$.
12. $\frac{3}{4}a-\frac{1}{3}b$ and $\frac{1}{4}a+\frac{2}{3}b$.
13. $a-b$ and $b-c$.
14. $a-c$ and $b-c$.
15. $2a-3b$ and $a-3c$.
16. $2x^2+5x$ and $x+4$.
17. $3x^2-5x$ and $2x-3$.
18. x^3-3x^2 and $2x^2-x$.
19. $x^2-\frac{x}{2}$ and $\frac{x}{2}+2$.
20. $3x^2+\frac{x}{2}$ and $\frac{x}{2}-5$.
21. $a+b-c$ and $a-b+c$.
22. $3a-2b-2c$ and $3a+2b-c$.
23. $x+y-z$ and $3x-2y+4z$.
24. $a^2-b^2-c^2$ and $-a^2+2b^2+c^2$.
25. $3x^2+4x+1$ and $2x^2-x-1$.
26. $x^2-2xy+y^2$ and $x^2+2xy+y^2$.
27. $3(a-b)$ and $2(a-b)$.
28. $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a+b)$.
29. $\frac{3}{4}(x^2-y^2)$ and $\frac{1}{4}(x^2-y^2)$.
30. $\frac{7}{8}(x+5)$ and $\frac{1}{8}(x+5)$.
31. $\frac{9}{5}(a-b)$ and $-\frac{4}{5}(a-b)$.
32. $-\frac{8}{9}(x-3)$ and $-\frac{1}{9}(x-3)$.
33. 9 times $8\frac{1}{2}$ and -8 times $8\frac{1}{2}$.
34. 5 times $3\frac{3}{4}$ and -4 times $3\frac{3}{4}$.
35. 3 times $1\frac{1}{5}$ and twice $1\frac{1}{5}$.
36. 8 times $1\frac{2}{5}$ and -3 times $1\frac{2}{5}$.
37. $4(a-b)$ and $2(a+b)$.
38. $3(x+y)$ and $-2(x-y)$.
39. $5(x-1)$ and $5(x-2)$.
40. $7(1-x)$ and $2(1+x)$.
41. $3(1+2x)$ and $2(3-2x)$.
42. $x(a-b)$ and $x(a+b)$.

20. **Example 1.** The sum of $3a$, $-4a$, $6a$, $-2a$, $7a$

$$\begin{aligned}&=3a-4a+6a-2a+7a\\&=3a+6a+7a-4a-2a\\&=16a-6a=10a.\end{aligned}$$

Example 2. The sum of $9x^2$, $-6x^2$, $3x^2$, $-2x^2$, $6x^2$, $-3x^2$

$$\begin{aligned}&=9x^2-6x^2+6x^2+3x^2-3x^2-2x^2\\&=9x^2-2x^2=7x^2.\end{aligned}$$

Examples. III. d.

Find the sum of

1. $2a$, $3a$, $4a$, $5a$.
2. $2a$, $-a$, $3a$, $-2a$.
3. $-x$, $-2x$, $-3x$, $-4x$.
4. $5x^2$, $-3x^2$, $-2x^2$, $9x^2$.
5. $7y$, $-3y$, $-2y$, $-5y$.
6. $6p$, $-4p$, $3p$, $-2p$, $-3p$.
7. $-3ab$, $-7ab$, $10ab$, $5ab$.
8. $7a$, $-3a$, $9a$, $-7a$, $3a$, $-9a$.
9. $2x^3$, $7x^3$, $-3x^3$, $-2x^3$, $-7x^3$.
10. $\frac{3}{4}x$, $2x$, $\frac{1}{4}x$, $-x$.
11. $\frac{7}{8}a$, $-\frac{3}{8}a$, $-\frac{4}{8}a$, $6a$, $-2a$.
12. $2\frac{x}{y}$, $-7\frac{x}{y}$, $9\frac{x}{y}$.
13. $\frac{5}{8}x$, $\frac{3}{4}x$, $\frac{1}{4}x$, $-\frac{3}{8}x$.
14. $2x$, $-\frac{5}{9}x$, $-\frac{1}{9}x$, $\frac{2}{3}x$.

Collect the terms in the following:

15. $3a-2a+4a-a$.
16. $7x^2-3x^2-x^2+2x^2$.
17. $3ab-7ab+ab-2ab+9ab$.
18. $11x^2y-8x^2y-2x^2y+4x^2y-x^2y$.
19. $4abc-9abc+6abc-7abc$.
20. $-3x^4-4x^4-7x^4-x^4$.
21. $-9x^3-6x^3+8x^3-2x^3+9x^3$.
22. $\frac{2x}{3}-\frac{x}{3}+x-\frac{2x}{3}$.
23. $\frac{5}{9}x+\frac{2}{9}x-\frac{8}{9}x$.
24. $-\frac{5}{3}a^2+\frac{2}{3}a^2-a^2-2a^2$.

21. **Example 1.** Find the sum of $3a-4b-2c$, $4a+2b-c$ and $2a-b-3c$.

First Method. The required sum

$$\begin{aligned}&=3a-4b-2c+(4a+2b-c)+(2a-b-3c)\\&=3a-4b-2c+4a+2b-c+2a-b-3c\\&=3a+4a+2a-4b+2b-b-2c-c-3c\\&\quad\text{(collecting like terms)}\\&=9a-3b-6c.\end{aligned}$$

Second Method. Arrange the given expressions in lines so that the like terms appear in the same vertical columns: then add each column.

$$\begin{array}{r}3a-4b-2c\\4a+2b-c\\2a-b-3c\\\hline 9a-3b-6c.\end{array}$$

Example 2. Find the sum of $4x^3-1-3x^2$, $5x^2-3x+2x^3$, and $7-2x+2x^2$.

Arranging the expressions so that like terms appear in the same vertical column,

$$\begin{array}{rrrr}
4x^3 & -3x^2 & & -1 \\
2x^3 & +5x^2 & -3x & \\
 & 2x^2 & -2x & +7 \\
\hline
6x^3 & +4x^2 & -5x & +6,
\end{array}$$
the required sum.

Example 3. Find the sum of $\frac{2}{3}(x-y+3z)$, $\frac{3}{4}(4x-8y-z)$, $\frac{1}{2}(2x+2y-2z)$.

$$\begin{aligned}
\text{The reqd. sum} &= \frac{2x}{3}-\frac{2y}{3}+2z+3x-6y-\frac{3z}{4}+x+y-z \\
&= \frac{2x}{3}+3x+x-\frac{2}{3}y-6y+y+2z-\frac{3z}{4}-z \\
&\quad \text{(collecting like terms)} \\
&= x\left(\tfrac{2}{3}+3+1\right)+y\left(1-6-\tfrac{2}{3}\right)+z\left(2-1-\tfrac{3}{4}\right) \\
&= \tfrac{14}{3}x-\tfrac{17}{3}y+\tfrac{1}{4}z.
\end{aligned}$$

Examples. III. e.

Find the sum of

1. $a^2-b^2+c^2$, $-a^2-b^2-c^2$, $a^2+b^2+c^2$.
2. $2a+3b-4c$, $3a-2b+4c$, $a+5b+6c$.
3. $3x-4y+4z$, $-2x+6y-5z$, $x-3y-8z$.
4. $-a-b-c$, $-2a-2b-2c$, $-3a-3b-3c$.
5. $4ax-3by+5cz$, $7ax+8by-2cz$, $2ax-2by+cz$.
6. $a+b$, $b+c$, $c+a$. 7. $2(a-b)$, $2(a+b)$.
8. $a+b-c$, $3(a-b+c)$, $4(a-b-c)$.
9. $x^2+2xy+y^2$, x^2-y^2, $2xy+y^2$.
10. $x^3+3x^2y-3xy^2+y^3$, $x^3-3x^2y+3xy^2-y^3$, x^3+y^3.
11. $4x-6x^2-1+2x^3$, $3x^2-4-x^3+5x$, $12-x$.
12. $3a^3-2c^3-d^3$, $b^3+c^3+4d^3$, $a^3-3b^3-4c^3$.
13. $x^3-3x^2y+3xy^2$, $-2x^2y-xy^2-y^3$, x^3+4y^3.
14. $4p^2-3q^2-4r-3$, q^2-2r-4, $6r-2-3p^2$, $9-q^2$.
15. $7x^2yz-5xyz^2$, $3xy^2z-4x^2yz$, $-5xy^2z-7xyz^2$, $2x^2yz-4xy^2z+6xyz^2$.
16. $a^2-bc-2ac$, b^2+ac-c^2, $c^2-3ac-4bc$, $ab+ac+bc$.
17. $a^3-b^3-3a^2c$, $b^3-3abc+3ac^2$, $6abc+7a^2c-2ac^2$.
18. $4(a+b+c)$, $3(2a-b-c)$, $8(b-a+2c)$.
19. $\frac{1}{3}(x+y-z)$, $\frac{2}{3}(x-y-z)$, $\frac{5}{3}(-x+y+z)$.
20. $\frac{2}{3}a+\frac{1}{3}b$, $\frac{1}{3}a-c$, $\frac{5}{3}b+6c$.
21. $\frac{3}{4}(8x-12y)$, $\frac{2}{3}(6x-9y)$, $\frac{1}{6}(12x+30y)$.

SUBTRACTION

22. $2a$ subtracted from $5a = 5a - 2a = 3a$.

$2a$ .... $-5a = -5a - 2a = -7a$.

$-3a$ .... $7a = 7a - (-3a) = 7a + 3a = 10a$.

$-4a$ .... $-2a = -2a - (-4a) = -2a + 4a = 2a$.

$x - y$ .... $x + y = x + y - (x - y) = x + y - x + y = 2y$.

$x - 2$ ....$x^2 - 5x = x^2 - 5x - (x - 2)$

$$= x^2 - 5x - x + 2$$

$$= x^2 - 6x + 2.$$

Examples. III. f.

Subtract

1. a from $4a$.
2. $-a$ from $4a$.
3. $2a$ from $-3a$.
4. $-b$ from $6b$.
5. $-b$ from $-6b$.
6. $-5b$ from $-5b$.
7. $-8b$ from $11b$.
8. x from $-x$.
9. $-2y$ from $2y$.
10. $3x^3$ from x^3.
11. $7ax^2$ from $11ax^2$.
12. $-7ax^2$ from $-11ax^2$.
13. $-7ax^2$ from $11ax^2$.
14. $7ax^2$ from $-13ax^2$.
15. a from 0.
16. $11a$ from 0.
17. $-3a$ from 0.
18. $3a + 2b$ from 0.
19. $a - b$ from 0.
20. $a - b$ from $a + b$.
21. $2a - b$ from $3a - 3b$
22. $\frac{1}{2}a - \frac{1}{2}b$ from $\frac{1}{2}a + \frac{1}{2}b$.
23. $\frac{1}{2}a + \frac{1}{2}b$ from $a + b$
24. $\frac{1}{2}a - \frac{1}{2}b$ from $a - b$.
25. c from $a + b$.
26. $a + b$ from c.
27. a from ax.
28. $-a$ from ax.
29. $-a$ from $-ax$.
30. x from x^2.

What must be added to

31. $2a - b$ to make $2a$?
32. $2a + 3b$ to make $2a$?
33. $a + b - c$ to make a ?
34. $3a - b - c$ to make $3a + b$?
35. $x^2 - y^2 - z^2$ to make $3y^2 + z^2$?
36. $x^2 - 5x - 6$ to make $5x + 6$?
37. $x^2 + px + q$ to make $3x^2 - px$?

23. Example 1. *Subtract* $3a - 2b + 2c$ *from* $5a + 3b - 4c$.

The reqd. result $= 5a + 3b - 4c - (3a - 2b + 2c)$

$= 5a + 3b - 4c - 3a + 2b - 2c$ ....(1)

$= 5a - 3a + 3b + 2b - 4c - 2c$ ....(2)

(collecting like terms) $= 2a + 5b - 6c$.

Example 2. *Subtract* $3x - 2x^2 - 6$ *from* $7x - 5 - 2x^2 + 4x^3$.

In cases such as this it is generally best to arrange the expressions in *ascending* or *descending* powers of x.

Arranging the expressions in *descending* powers of x,

$$\begin{aligned}\text{the reqd. result} &= 4x^3 - 2x^2 + 7x - 5 - (-2x^2 + 3x - 6)\\ &= 4x^3 - 2x^2 + 7x - 5 + 2x^2 - 3x + 6 \quad\ldots\ldots\ldots\ldots(1)\\ &= 4x^3 - 2x^2 + 2x^2 + 7x - 3x - 5 + 6 \quad\ldots\ldots\ldots\ldots(2)\\ &= 4x^3 + 4x + 1.\end{aligned}$$

When the student has had a little practice, he will be able to shorten the work by omitting lines marked (1) and (2) in the above.

24. The work of subtraction is often conveniently arranged as follows.

Subtract $5a - 3b + 4c$ *from* $6a - 5b - 3c$.

$$\begin{array}{r} 6a - 5b - 3c \\ 5a - 3b + 4c \\ \hline a - 2b - 7c. \end{array}$$

Explanation. We see from the examples previously worked out, that **we must change the signs of all terms in the expression to be subtracted and then take the algebraic sum of the two lines.**

$$6a - 5a = a, \qquad -5b + 3b = -2b, \qquad -3c - 4c = -7c.$$

The signs need not be actually changed; the change may be made *mentally*.

Subtract $3a^4 - 4a^3 + 2a^2 + 5a$ *from* $2a^5 + 3a^4 - 5a + 4$.

$$\begin{array}{rrrrrr} 2a^5 & +3a^4 & & & -5a & +4 \\ & 3a^4 & -4a^3 & +2a^2 & +5a & \\ \hline 2a^5 & & +4a^3 & -2a^2 & -10a & +4. \end{array}$$

Explanation. $2a^5 - 0 = 2a^5$, $3a^4 - 3a^4 = 0$, $0 + 4a^3 = 4a^3$, $0 - 2a^2 = -2a^2$, $-5a - 5a = -10a$, $4 - 0 = 4$.

Examples. III. g.

Subtract

1. $a^2 + 2ab - b^2$ from $a^2 + 2ab + b^2$.
2. $x + 3y + 3z$ from $5x + 7y - 2z$.
3. $5x^2 - 3x + 2$ from $7x^2 - 5x + 6$.
4. $3x^2 - 2xy - 3y^2$ from $x^2 + 2xy + 5y^2$
5. $2a - b - 4d$ from $a - 3b + c$.
6. $3x - 4a + 11$ from $5x - 8a - 2$.
7. $-3ab - 2b^2 + 11$ from $6b^2 + 5ab + 2$.
8. $5a - 3c + 4d$ from $6a - 2b - 3c - 2d$.
9. $x^3 - 6x^2y - 3xy^2$ from $x^3 - 9x^2y - 5xy^2 + y^3$.

From

10. $6a - b + c - 3d$ take $3a + b - c - d$.
11. $6x - 3y - 4z + 7$ take $5x + 2y - 3z + 9$.
12. $5a^2 - 7ab - 12$ take $-3ab + 2$.

From

13. $3x - 4x^3 + 7x^2 - 9$ take $8 - 2x - 8x^3 - 2x^2$.
14. $5a^3 - 9a^2 + 3$ take $4a^3 - 6a - 3$.
15. $ab - bc - cd - ad$ take $-ab + bc - 3cd$.
16. $a^2 - 1 - 2a^4 - 3a + 5a^3$ take $3a^3 - 4a^4 + 6a^2 - 2$.
17. $6x^4 - 36 + 8x^2 - 9x$ take $3x^3 - 7 + 8x^2 - 3x$.

By how much does

18. 7 exceed 4 ?
19. 7 exceed -4 ?
20. -7 exceed -9 ?
21. $3a$ exceed $-a$?
22. $2x^2 + 1$ exceed $x^2 + 1$?
23. $x^2 - 2x + 1$ exceed $2x + 1$?
24. $a - b$ exceed $a - 3b$?
25. $3a - 4x$ exceed $a + 7x$?

Find the excess of

26. $6a$ over $-2a$.
27. $7a$ over 5.
28. $3x^2$ over $-x$.
29. $6 - x^2$ over $-x^2$.
30. $3(a + b)$ over $2(a - b)$.
31. 8 times $3\frac{1}{2}$ over 6 times $3\frac{1}{2}$.
32. 9 times $3\frac{1}{3}$ over 5 times $3\frac{1}{3}$.
33. Subtract the sum of $3a - b$ and $a + 2b$ from $6a - 7b$.
34. Subtract $3x - y - z$ from the sum of $x + y - z$, and $3y - z$.
35. By how much does zero exceed $7x - 6$?
36. Subtract $3a^2 - b^2 + c^2$ from zero ?
37. Subtract the sum of $3a - b + 2c - 5d$ and $a + b - 2c + 3d$ from the excess of $6a - c - d$ over $a - b - c$.
38. Take 3 from $2x^2$ and the result from $x^2 - 3x - 3$.

CHAPTER IV

MULTIPLICATION

Rule of Signs.

25. We know that

$$+2 \times +3 = +6\,; \text{ also } +a \times +b \text{ is represented by } +ab. \quad \ldots(1)$$

Again, $-3 \times +2$ means -3 taken twice.

i.e. $$-3 \times +2 = -3 + (-3) = -3 - 3 = -6.$$

We therefore deduce that $-a \times +b = -ab$. $\quad \ldots\ldots\ldots\ldots(2)$

Next let us consider $+3 \times -2$.

This means $+3$ taken -2 times, and therefore has no arithmetical meaning.

It bears however an algebraic interpretation.

Remembering the convention of signs for direction (Art. 8), we see that $+3$ taken -2 times is the same as $+3$ taken $+2$ times, *but in the opposite direction.*

$$\therefore \; +3 \times -2 = +3 \times +2 \text{ with the opposite sign,}$$
$$= +6 \text{ with the opposite sign,}$$
$$= -6.$$

Algebraically therefore,

$$+a \times -b = -ab. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Lastly let us consider the product -3×-2.

This denotes -3 taken -2 times.

$\therefore$ remembering the convention of sign for direction, this is the same as -3 taken twice, *but in the opposite direction,*

$$= -6 \textit{ in the opposite direction,}$$
$$= +6.$$

$$\therefore \text{ in algebra we say that } -a \times -b = +ab. \quad \ldots\ldots\ldots(4)$$

Examining the results (1), (2), (3), (4), we have the following **rule of signs.**

Terms with like signs multiplied together give plus (+).

Terms with unlike signs multiplied together give minus (−).

Indices.

26. By definition, $a^3 = a \times a \times a,$

and $a^4 = a \times a \times a \times a.$

$$\therefore \; a^3 \times a^4 = a \times a \times a \times a \times a \times a \times a \qquad \text{(7 factors)}$$
$$= a^7 \text{ by definition.}$$

In the same way $a^2 \times a^3 = a \times a \times a \times a \times a$

$$= a^5.$$

In each case the index of the product is the **sum** of the indices of the factors.

We therefore deduce the following law.

To multiply two powers of the same quantity, add the indices of the factors.

The **continued product** of a number of quantities is the result when they are all multiplied together.

Thus the continued product of 2, 3, 4 is $2 \times 3 \times 4 = 24$.

.... a, b, c, is abc.

.... a^2, a^3, a^4 is a^9.

.... $-a, 2a, -3a$ is $6a^3$.

.... $-a, -2a, -3a$ is $-6a^3$.

27. Examples.

(1) $$a^2b^3 \times a^5b^2 = a^2 \times a^5 \times b^3 \times b^2 = a^7b^5.$$

(2) $$3a^2b \times -4b = -3 \times 4 \times a^2 \times b \times b \quad \text{(Unlike signs give minus)} = -12a^2b^2.$$

(3) $$-4x^2y \times -5x^3y = +4 \times 5 \times x^2 \times x^3 \times y \times y \quad \text{(Like signs give plus)} = 20x^5y^2.$$

(4) $$(3a - 4b) \times -2 = -6a + 8b.$$

(5) $$-4x^2y^3(x^2 - 3yz + 5z^2)$$
$$= -4x^2y^3 \times x^2 - 4x^2y^3 \times (-3yz) - 4x^2y^3 \times (5z^2)$$
$$= -4x^4y^3 + 12x^2y^4z - 20x^2y^3z^2.$$

(6) $$24a(\tfrac{2}{3}a^2 - \tfrac{1}{4}b^2 + \tfrac{3}{8}bc) = 24a \times \tfrac{2}{3}a^2 - 24a \times \tfrac{1}{4}b^2 + 24a \times \tfrac{3}{8}bc$$
$$= 16a^3 - 6ab^2 + 9abc.$$

(7) $$(\tfrac{1}{6}a - \tfrac{2}{3}b - c) \times -\tfrac{3}{5}ab^2c = -\tfrac{3}{5}ab^2c \times \tfrac{1}{6}a + \tfrac{3}{5}ab^2c \times \tfrac{2}{3}b + \tfrac{3}{5}ab^2c \times c$$
$$= -\tfrac{1}{10}a^2b^2c + \tfrac{2}{5}ab^3c + \tfrac{3}{5}ab^2c^2.$$

Examples. IV. a.

Multiply

1. $2a$ by 3.
2. $3a$ by -3.
3. $-2a$ by -4.
4. a by $2a^2$.
5. $-2a^2$ by a^2.
6. $-3ab$ by $2ab$.
7. $3x$ by $4y$.
8. $-3x$ by $-2y$.
9. $-5x$ by $3y$.
10. $7x^2$ by $-2x$.
11. abc by abc.
12. a^2b by $-b^2c$.
13. $-a^2$ by x^3.
14. $-2a^2$ by $-3ab$.
15. $4x^2$ by $-2x^3$.
16. p^{11} by $-p^3$.
17. $-p^7q$ by $-pq^7$.
18. $-3p^2q$ by $2pq^3$.
19. $a^2b^3c^4$ by ab^2c^3.
20. $\frac{1}{2}a$ by $\frac{1}{3}b$.
21. $\frac{3}{4}a^2$ by $-\frac{4}{3}b^2$.
22. $\frac{5}{8}x^3$ by $-\frac{8}{5}x$.
23. $-\frac{9}{4}x^2y$ by $-\frac{2}{9}y^2z$.
24. $-\frac{3}{11}a^2b$ by $\frac{33}{5}bc^2$.

Write down, or read off, the continued product of

25. $-2, -3, 4$.
26. $a, -b, c$.
27. $a^2, -b^2, c$.
28. $b^2, -c^2, -a$.
29. $2a, 3b, 5c$.
30. $3a, -2b, -4c$.
31. $a^2x, x, -y$.
32. $3a, x, -x^2$.
33. $-a, -a, -a$.
34. $-2a, -2a, -2a$.
35. $a^2, b^3, 2c^4$.
36. $3p^2, 2pq, 4qr$.

Write down, or read off, the values of

37. $(-a)^2$.
38. $(-a)^3$.
39. $(-a)^6$.
40. $(-2a)^3$.
41. $(x^2)^3$.
42. $(x^3)^2$.
43. $(-x^2)^3$.
44. $(-2xy)^3$.
45. $(-2xy)^4$.
46. $(-1)^7$.
47. $(-1)^8$.
48. $(-1)^{11}$.
49. $(-x^2)^7$.
50. $(-x^3)^5$.
51. $(-2x^2)^6$.
52. $(-2a^2b)^3$.
53. $(-3x^2y)^3$.
54. $(-3xy^2)^4$.

Examples. IV. b.

Multiply

1. $a+5b-3c$ by 5.
2. $2a-3b+2c$ by -4.
3. $a+b+c$ by $2a$.
4. $3a^2-2a+5$ by $-2a$.
5. $6a^3-4a^2-2a-5$ by $7a^2$.
6. $ab-bc+ca$ by bc.
7. $2ab-3bc-4ca$ by $-3abc$.
8. $x^2-2xy+y^2$ by x^3.
9. $x^3-3x^2y+3xy^2-y^3$ by $-3x^2$.
10. $a^2+ab+b^2-ac-bc$ by $-c$.
11. $3ab+2ac-bc$ by abc.
12. $1-3x-2x^2+x^3$ by $-2x$.
13. x^3-3x^2+3x+1 by $2x$.
14. $3x^4-2x^2+6$ by $-5x^2$.
15. $-3a^2-2ab+b^2$ by $-2b^2$.
16. $-5a^3-ab^4c^3+9b^5c^2$ by $-12a^6b^4c^8$.

Find the continued product of

17. $a-b$, a, b.
18. $a^2-2ab-b^2$, $2a$, and $3c$.
19. x^2-5x+3, $2x$, and $-3x$.
20. $x^4-3x^3+2x^2-3$, $-6x$, and $-2x$.

Following the law of indices, what is the product of

21. a^m and a^n.
22. a^m and $-a^n$.
23. a^m and a^m.
24. a^m and a^{2m}.
25. $-a^3$ and $-a^n$.
26. $-a^5$ and a^n.
27. a^{2m} and a^{3m}.
28. a^{2m} and a^{2n}.
29. $-2a^m$ and a^m.
30. $-3a^mb^n$ and $-5a^nb^m$.
31. a^x+a^{2x} and a^x.
32. $e^{2x}-e^x+1$ and e^{2x}.
33. a^{m-1} and a^{m+1}.
34. a^{m-6} and a^{m-2}.

When $a=-2$, what is the value of

35. a^2-2.
36. $2a^2-a+4$.
37. a^3+8.
38. $3a^2+2a-16$.
39. $2a^3+16$.
40. $a^4+3a^3+2a^2-a$.

When $a=-1$, $b=2$, find the value of

41. a^2+b.
42. a^3-3b.
43. a^2+b^2.
44. $8a^2-b^3$.
45. a^2+ab+b^2.
46. a^3+b^3.

When $x=0$, $y=-1$, $z=2$, find the value of

47. $x^2-2yz+y^2$.
48. $xy+yz+zx$.
49. $x^3+y^3+z^3$.
50. $x^2+y^2+z^2-xy-yz-zx$.
51. $x^4+y^4+z^4$.
52. $(x-y)^2+(y-z)^2+(z-x)^2$.

28. *To find the product of* $(x+3)$ *and* $(x+4)$.

First let us regard $(x+3)$ as a single quantity, a suppose.

$$\begin{aligned}(x+3)\times(x+4)&=a\times(x+4)\\&=ax+4a\\&=(x+3)\times x+4(x+3)\\&=x^2+3x+4x+12\\&=x^2+7x+12.\end{aligned}$$

Examining the above, we see that it is the same as multiplying $(x+3)$ by x and by 4 and adding the results.

To find the product of $(x-2)$ and $(x-5)$.

Regarding $(x-2)$ as a single quantity, a suppose,

$$\begin{aligned}(x-2)\times(x-5)&=a\times(x-5)\\&=ax-5a\\&=x(x-2)-5(x-2)\\&=x^2-2x-5x+10\\&=x^2-7x+10.\end{aligned}$$

Again, we see that this is the same as multiplying $(x-2)$ by x and by -5, and then taking their algebraic sum.

The work may conveniently be arranged thus:

$$\begin{array}{rl}x-2 & \\ \underline{x-5} & \\ x^2-2x\quad\;\; & \text{(multiplying } x-2 \text{ by } x) \\ \underline{-5x+10} & \text{(multiplying } x-2 \text{ by } -5\text{, and placing like} \\ x^2-7x+10. & \text{(adding)} \qquad \text{terms underneath one another)}\end{array}$$

N.B.—$(x+3)\times(x-2)$ is usually written thus, $(x+3)(x-2)$.

29. Example 1. Multiply $x+a$ by $x+b$.

$$\begin{array}{l}x+a\\ \underline{x+b}\\ x^2+ax\\ \underline{\qquad\;\; bx+ab}\\ x^2+ax+bx+ab.\end{array}$$

This may be written $\quad x^2+(a+b)x+ab.$

This result is true whatever values we give to a and b, positive or negative.

Hence $\quad (x+2)(x+5)=x^2+(5+2)x+5\times2=x^2+7x+10.$

$(x-3)(x-5)=x^2+(-3-5)x+(-5)(-3)=x^2-8x+15.$

$(x-3)(x+7)=x^2+(-3+7)x+(-3)(7)=x^2+4x-21.$

$(x+3)(x-9)=x^2+(3-9)x+(3)(-9)=x^2-6x-27.$

After a little practice the student will be able to write down such products at sight.

Example 2. Multiply $5+3x$ by $7-2x$.

$$\begin{array}{l} 5+3x \\ \underline{7-2x} \\ 35+21x \\ \underline{\quad\;\; -10x-6x^2} \\ 35+11x-6x^2 \end{array}$$

Example 3. Multiply $ay+b$ by $cy-d$.

$$\begin{array}{l} ay+b \\ \underline{cy-d} \\ acy^2+bcy \\ \underline{\qquad\qquad -ady-bd} \\ acy^2+bcy-ady-bd \end{array}$$

Example 4. Multiply $a+b$ by $a-b$.

$$\begin{array}{l} a+b \\ \underline{a-b} \\ a^2+ab \\ \underline{\quad -ab-b^2} \\ a^2-b^2 \end{array}$$

i.e. $(\mathbf{a}+\mathbf{b})(\mathbf{a}-\mathbf{b})=\mathbf{a}^2-\mathbf{b}^2$.

This result is very important. It is true for all values of a and b.

Hence $(a+2)(a-2)=a^2-2^2=a^2-4.$

$(a+1)(a-1)=a^2-1.$

$(x+a)(x-a)=x^2-a^2.$

$$(2x+3a)(2x-3a)=(2x)^2-(3a)^2$$
$$=4x^2-9a^2.$$

Examples. IV. c.

[After a little practice, the student will be able to write down the results in many of the following, without showing any work.]

Find the product of

1. $x+2$, $x+3$.	2. $x-2$, $x-3$.	3. $x+2$, $x-3$.
4. $x-2$, $x+3$.	5. $x+3$, $x+9$.	6. $x-3$, $x+6$.
7. $x-11$, $x-7$.	8. $x+11$, $x-7$.	9. $1+x$, $1+2x$.
10. $1+4x$, $1-3x$.	11. $1-x$, $1-2x$.	12. $2+x$, $3+x$.
13. $5+x$, $6+x$.	14. $3+x$, $7+x$.	15. $1-9x$, $1+7x$.
16. $1-7x$, $1+3x$.	17. $x+1$, $x-1$.	18. $x+2$, $x-2$.
19. $x-3$, $x+3$.	20. $x-7$, $x+7$.	21. $1-x$, $1+x$.
22. $2+x$, $2-x$.	23. $7-x$, $7+x$.	24. $9-x$, $9+x$.
25. $x+y$, $x+y$.	26. $x+2y$, $x+3y$.	27. $x-2y$, $x+2y$.
28. $x-3y$, $x-2y$.	29. $x-3y$, $x+2y$.	30. $x-5y$, $x+4y$.

Find the product of

31. $2x+y,\ 2x+y.$
32. $3x-y,\ 3x-y.$
33. $2x-3,\ 3x+4.$
34. $2x-1,\ 3x-4.$
35. $5x+6,\ 2x+3.$
36. $3x-7,\ 5x+2.$
37. $2-3x,\ 3-2x.$
38. $5-4x,\ 6+7x.$
39. $2-3x,\ 2+3x.$
40. $2x-5,\ 2x+5.$
41. $5x-7,\ 5x+7.$
42. $6x-5,\ 6x+5.$
43. $9x+8,\ 9x-8.$
44. $4x+7,\ 4x-7.$
45. $x-a,\ x+b.$
46. $x+a,\ x-b.$
47. $a+b,\ a+b.$
48. $ax+b,\ ax+b.$
49. $a-b,\ a-b.$
50. $ax-b,\ ax-b.$
51. $px-q,\ px-q.$
52. $p+qx,\ p+qx.$
53. $a+3x,\ a-5x.$
54. $3-x,\ 7+2x.$
55. $x+ay,\ x-ay.$
56. $px-q,\ px+q.$
57. $px+q,\ px+q.$
58. $cx-d,\ cx-d.$
59. $3x-4y,\ 4x-3y.$
60. $3x+4y,\ 4x-5y.$
61. $7x+8c,\ 6x-4c.$
62. $2ax+3,\ 3ax+2.$
63. $a^2-b^2,\ a^2+b^2.$
64. $a^2-4b,\ a^2+4b.$
65. $a^2+6b,\ a^2-4b.$
66. $a^2-3b,\ a^2-5b.$
67. $4a^2-3b,\ 4a^2+3b.$
68. $5a^2-2b^2,\ 5a^2+2b^2.$
69. $x^2-2a^2,\ x^2+2a^2.$
70. $x^2-p,\ x^2+p.$
71. $a-b^3,\ a+b^3.$
72. $a-b^3,\ a-b^3.$
73. $x^3+1,\ x^3-1.$
74. $x^3-2,\ x^3+2.$
75. $ax^2+1,\ ax^2-1.$
76. $bx^2+c,\ bx^2-c.$
77. $ax+1,\ bx+1.$
78. $ax+1,\ bx-1.$
79. $x+2y,\ 3x+1.$
80. $2x-a,\ 3x+b.$
81. $a+b,\ c+d.$
82. $a-b,\ c-d.$
83. $2a-b,\ 3c+4d.$
84. $a+3b,\ 2c-5d.$
85. $x^2+a,\ x^2-3b.$
86. $ax^2+bx,\ ax+b.$
87. $ax^2-bx,\ ax+b.$
88. $x^2+a^2,\ x+a.$
89. $x^2-a^2,\ x+a.$
90. $x^2-4y^2,\ x-2y.$

SQUARES

30.
$$(\mathbf{x}+\mathbf{a})^2=(x+a)(x+a)=x^2+ax+ax+a^2$$
$$=\mathbf{x^2+2ax+a^2}.$$

This is true for all values of a.

Hence
$$(x+2)^2=x^2+4x+4.$$
$$(x+7)^2=x^2+14x+49.$$
$$(\mathbf{x}-\mathbf{a})^2=(x-a)(x-a)=x^2-ax-ax+a^2$$
$$=\mathbf{x^2-2ax+a^2}.$$

This is also true for all values of a.

Hence
$$(x-3)^2=x^2-6x+9.$$
$$(x-8)^2=x^2-16x+64.$$

From the above we gather that:

The square of the sum of two quantities is equal to the sum of their squares plus twice their product.

The square of the difference of two quantities is equal to the sum of their squares minus twice their product.

Examples. IV. d.

Doing all the work mentally, write down the expanded values of the following:

1. $(a+b)^2$. 2. $(a+x)^2$. 3. $(c+d)^2$. 4. $(x+4)^2$.
5. $(x+7)^2$. 6. $(p+3)^2$. 7. $(a-b)^2$. 8. $(a-x)^2$.
9. $(c-d)^2$. 10. $(x-4)^2$. 11. $(x-9)^2$. 12. $(p-4)^2$.
13. $(2p+3)^2$. 14. $(3p+q)^2$. 15. $(2p-5)^2$. 16. $(4p-1)^2$.
17. $(x-1)^2$. 18. $(3x-1)^2$. 19. $(1-x)^2$. 20. $(1-2x)^2$.
21. $(1-5x)^2$. 22. $(1+p)^2$. 23. $(1+7p)^2$. 24. $(2a+3b)^2$.
25. $(4x-3y)^2$. 26. $(-a+b)^2$. 27. $(-2a+x)^2$. 28. $(2x-3a)^2$.
29. $(-2x+3a)^2$. 30. $(4p+5q)^2$. 31. $(5p-4q)^2$. 32. $(a^2+b^2)^2$.
33. $(a^2-b^2)^2$. 34. $(a^2+b)^2$. 35. $(a^2-p)^2$. 36. $(2a^2-3b^2)^2$.
37. $(4a^2+3b^2)^2$. 38. $(a^3+b)^2$. 39. $(x^3+y^3)^2$. 40. $(x^3-y^3)^2$.
41. $(2x^2+a)^2$. 42. $(3x^2-y^2)^2$. 43. $(1-2x^2)^2$. 44. $(-1-x)^2$.
45. $(-1-2x)^2$. 46. $(x^4+a^4)^2$. 47. $(x^4-y^4)^2$. 48. $(2x^4-3y^4)^2$.
49. $(2p^3+3q^2)^2$. 50. $(x^5-a^5)^2$.

31. Example 1. $(x+2)(x-2)=x^2-2^2=x^2-4$. (See Art. 29, Ex. 4.)
Example 2. $(2x-3)(2x+3)=(2x)^2-(3)^2=4x^2-9$.
Example 3. $(-a+x)(-a-x)=(-a)^2-x^2=a^2-x^2$.
Example 4. $(px-q)(px+q)=p^2x^2-q^2$.

Examples. IV. e.

Write down the following products:

1. $(x+1)(x-1)$. 2. $(x-2)(x+2)$. 3. $(1+x)(1-x)$.
4. $(x+5)(x-5)$. 5. $(3-y)(3+y)$. 6. $(7-x)(7+x)$.
7. $(b-a)(b+a)$. 8. $(2p+q)(2p-q)$. 9. $(3p+q)(3p-q)$.
10. $(a-3b)(a+3b)$. 11. $(3p+2q)(3p-2q)$. 12. $(5x-4a)(5x+4a)$.
13. $(-a-b)(-a+b)$. 14. $(-2a+x)(-2a-x)$. *15. $(a-7b)(a+7b)$.
16. $(-a-7b)(-a+7b)$. 17. $(x^2-y^2)(x^2+y^2)$. 18. $(a^2+2b^2)(a^2-2b^2)$.
19. $(px-q)(px+q)$. 20. $(a-bx)(a+bx)$. 21. $(x^3-a^3)(x^3+a^3)$.
22. $(-x^2-a)(-x^2+a)$. 23. $(2a^3+x)(2a^3-x)$. 24. $(2a^2-3x)(2a^2+3x)$.
25. $(1-x^3)(1+x^3)$. 26. $(1+ax^2)(1-ax^2)$. 27. $(3-a^3)(3+a^3)$.
28. $(11-7x)(11+7x)$. 29. $(9-8x)(9+8x)$. 30. $(7x-9)(7x+9)$.

***32.** The formulae

$$(a+b)^2=a^2+2ab+b^2 \quad \text{and} \quad (a-b)^2=a^2-2ab+b^2$$

may be used with great advantage in arithmetical work.

$$99^2=(100-1)^2=10{,}000-200+1=9{,}801.$$
$$101^2=(100+1)^2=10{,}000+200+1=10{,}201.$$
$$105^2=(100+5)^2=10{,}000+1000+25=11{,}025.$$
$$100{\cdot}5^2=(100+{\cdot}5)^2=10{,}000+100+{\cdot}25=10{,}100{\cdot}25.$$

These formulae may often be used in approximations.

$$\begin{aligned}(100{\cdot}03)^2 &= (100+{\cdot}03)^2\\ &= 10{,}000+200\times{\cdot}03+{\cdot}0009\\ &= 10{,}000+6+{\cdot}0009\\ &= 10{,}006{\cdot}00 \text{ correct to two dec. places.}\end{aligned}$$

In giving approximate values, **·5 or more counts as unity.** Thus 79·7, 79·5, 79·8 would count as 80, correct in whole numbers.

On the other hand, 79·3, 79·2 would be taken as 79.

In the same way, 6·035729 would be taken as

6·04 correct to two decimal places.
6·036 .... three ....
6·0357 .... four ....
6·03573 .... five ....

Using the formula $(a+b)(a-b)=a^2-b^2$.

$$\begin{aligned}99\times101 &= (100-1)(100+1)\\ &= 10{,}000-1=9999.\end{aligned}$$

Also
$$\begin{aligned}99{\cdot}6\times100{\cdot}4 &= (100-{\cdot}4)(100+{\cdot}4)\\ &= 10{,}000-{\cdot}16\\ &= 9999{\cdot}84.\end{aligned}$$

$$\begin{aligned}15{\cdot}6\times14{\cdot}4 &= (15+{\cdot}6)(15-{\cdot}6)\\ &= 225-{\cdot}36\\ &= 224{\cdot}64.\end{aligned}$$

*Examples. IV. f.

Without doing the actual multiplication, find the value of

1. 98^2.	**2.** 201^2.	**3.** 102^2.	**4.** 103^2.
5. 107^2.	**6.** 9999^2.	**7.** 1001^2.	**8.** 1002^2.
9. $9{\cdot}9^2$.	**10.** $10{,}003^2$.	**11.** $20{,}001^2$.	**12.** $999{\cdot}8^2$.
13. $20{,}010^2$.	**14.** $2{,}005^2$.	**15.** $100{\cdot}3^2$.	**16.** 1008^2.
17. 999^2.	**18.** $99{\cdot}97^2$.	**19.** $80{\cdot}2^2$.	**20.** $600{\cdot}5^2$.
21. $899{\cdot}6^2$.	**22.** $500{\cdot}3^2$.	**23.** $9{\cdot}006^2$.	**24.** $7{\cdot}996^2$.

25. $100{\cdot}02^2$, correct to three decimal places.
26. $1{\cdot}005^2$, ... four ....
27. $10{\cdot}08^2$, ... three ....
28. $999{\cdot}96^2$, ... two ....
29. $10{\cdot}005^2$, ... four ....

30. 1002×998.	**31.** 203×197.	**32.** 97×103.	**33.** 83×77.
34. $11{\cdot}5\times10{\cdot}5$.	**35.** $9{\cdot}3\times10{\cdot}7$	**36.** 82×78.	**37.** $20{\cdot}04\times19{\cdot}96$.
38. $1{\cdot}72\times1{\cdot}68$.	**39.** $1{\cdot}96\times2{\cdot}04$.	**40.** $9000{\cdot}4\times8999{\cdot}6$.	

33. Example 1. Multiply x^2-2x+5 by $x+2$.

$$\begin{array}{l} x^2-2x+5 \\ \quad\;\; x+2 \\ \hline x^3-2x^2+5x \\ \quad\;\; 2x^2-4x+10 \\ \hline x^3 \qquad\quad +x+10 \end{array}$$

Example 2. Multiply $a-b+c$ by $b-c$.

$$\begin{array}{l} a-b+c \\ \quad b-c \\ \hline ab \qquad -b^2+bc \\ \quad -ac \qquad +bc-c^2 \\ \hline ab-ac-b^2+2bc-c^2 \end{array}$$

Examples. IV. g.

Find the product of

1. $x^2-2x+1,\ x-1$.
2. $x^2+4x+4,\ x+1$.
3. $2x^2-3x+1,\ 2x-1$.
4. $x^2-2x+4,\ x+2$.
5. $9x^2+3x+1,\ 3x-1$.
6. $3x^2-2x+4,\ 2x+5$.
7. $x^2-ax+a^2,\ x-a$.
8. $25x^2+5x+1,\ 5x-1$.
9. $a^2+b^2,\ a+b$.
10. $x^2+ax+a^2,\ x-a$.
11. $a^2-b^2,\ a+b$.
12. $x^2-6x+9,\ x-3$.
13. $4x^2+2x+1,\ 2x-1$.
14. $4x^2-2x-5,\ 2x-7$.
15. $4x^2-3,\ x-2$.
16. $x^2+3x-4,\ x^2-2$.
17. $9x^2-3x+1,\ 3x+1$.
18. $x^3+3x^2+3x+1,\ x-1$.
19. $x-a,\ x-b,\ x-c$.
20. $x-2a,\ x+2a,\ x^2+4a^2$.
21. $x+3b,\ x-3b,\ x^2-9b^2$.
22. $2x+3,\ 2x-7,\ 3x+2$.
23. $a-b,\ a+b,\ a-c$.
24. $a+b-c,\ a-b$.
25. $2a+3b-c,\ 3a-4b$.

Examples. IV. h.

Find, *by inspection*, the coefficient of

1. x in the product $(x+2)(x+7)$.
2. x .... $(x-3)(x+7)$.
3. x .... $(2x-1)(3x-1)$.
4. x .... $(2x+3)(3x+4)$.
5. x .... $(3x-5)(x+2)$.
6. x .... $(5x-4)(2x-1)$.
7. a .... $(a+2)(x+3)$.
8. b .... $(x-2)(x+3b)$.
9. a .... $(x+2a)(3x-5)$.
10. a .... $(x+2a)(x-5a)$.
11. x^2 .... $(2x^2+x+1)(x+2)$.

Find, *by inspection*, the coefficient of

12. x^2 in the product $(3x^2-2x+4)(5x+7)$.
13. x^2 .... $(5x^2-3x-11)(5x+3)$.
14. x^2 .... $(ax^2+3x+4)(2x-1)$.
15. x^2 .... $(6x^2-ax+7)(6x+a)$.
16. x^2 .... $(3x^2-2x+4)(5x-7)$.
17. x^2 .... $(ax^2+bx+c)(x+d)$.
18. x^2 .... $(ax^2-bx+c)(ax+b)$.
19. x .... $(5x^2-2x+4)(5x+7)$.
20. x .... $(9x^2-8x+3)(5x-2)$.
21. x .... $(ax^2-bx+c)(cx-b)$.
22. x .... $(ax^2+bx+c)(bx-c)$.
23. Simplify $[a(3-b)+b(a+1)-2a]\times(a+b)$.
24. Find the product of $3x(x-3)+2(2x^2+1)$, and $4(x-1)-(x-9)$.
25. Simplify $(x+3)^2-(x-2)(x+2)+(x+1)(x-13)$.
26. Without doing the complete multiplication, determine the coefficient of x^2 in the product $(5x^3-9x^2-7x-13)(3x-7)$.
27. If $X=3x-2a$, and $Y=2x-3a$, find the value of $(2X-Y)(3X-2Y)$.
28. Find the value of $(X+Y)(X-Y)$ when $X=5x-2$ and $Y=3x-2$.
29. Simplify $(x+1)(x+9)-4(x-2)^2+3(x+1)(x-1)$.
 Check your result by using some particular value of x.
30. If $X=3px^2-px-4$, and $Y=16+qx-3qx^2$, find the value of $qX+pY$.
31. Multiply the sum of $2x(x-1)-(x-4)$, $2x-3$, and x^2+1 by the remainder when $(x+1)(x-1)-(x+6)$ is subtracted from

$$(x-2)(x+2)+2(x-2).$$

32. Simplify $\left(\frac{3a+3b}{2}-\frac{a-b}{2}\right)\left(\frac{3b+3a}{2}-\frac{b-a}{2}\right)$.
33. Find the value of $(3x-1)(4x+5)-2(2x-1)^2-4(x-1)(x+5)$, when $x=-2$.
34. Prove that $4(2x+1)^2-3(x-2)(2x-1)-2(5x-1)(x+2)=13x+2$.
35. Simplify $2(x+2)^2-(x-1)(x+1)-(x-3)^2$.

CHAPTER V

DIVISION

34. Rule of signs. $+ab=+a\times+b$;

$$\therefore\ +ab\div+a=+b,$$

or

$$\frac{+ab}{+a}=+b. \quad\text{........................(1)}$$

$$-ab = -a \times +b,$$

$$\therefore\ -ab \div -a = +b,$$

or $$\frac{-ab}{-a} = +b. \quad \text{.......................(2)}$$

$$+ab = -a \times -b;$$

$$\therefore\ +ab \div -a = -b,$$

or $$\frac{+ab}{-a} = -b. \quad \text{.......................(3)}$$

$$-ab = +a \times -b;$$

$$\therefore\ -ab \div +a = -b,$$

or $$\frac{-ab}{+a} = -b. \quad \text{.......................(4)}$$

Examining the results in (1), (2), (3), (4), we have the following **rule of signs for division.**

Terms with like signs divided by one another give plus (+). Terms with unlike signs divided by one another give minus (−).

N.B.—The rule of signs in division is the same as that in multiplication.

35. $a^5 = a \times a \times a \times a \times a$, by definition,

and $a^3 = a \times a \times a$;

$$\therefore\ a^5 \div a^3 = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a \times a$$
$$= a^2.$$

In the same way, $a^7 \div a^3 = \dfrac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a}$
$$= a^4.$$

In each case the index of the *quotient* is the index of the dividend *diminished by* the index of the divisor.

We therefore deduce the following law.

To divide one power of a quantity by another power of the same quantity, subtract the index of the divisor from the index of the dividend.

36. Examples.

(1) $$5x^2 \div 5 = \frac{5 \times x^2}{5} = x^2.$$

(2) $$5x^7 \div -5x^2 = -\frac{5x^7}{5x^2} \quad \text{(Unlike signs give minus.)}$$
$$= -x^5. \quad (7 - 2 = 5.)$$

(3) $$-35a^3b^2c \div -7abc$$
$$= +\frac{35a^3b^2c}{7abc} \quad \text{(Like signs give plus.)}$$
$$= 5a^2b.$$

(4) $$(6a-9b+3c) \div -3$$
$$= -\frac{6a}{3} + \frac{9b}{3} - \frac{3c}{3}$$
$$= -2a + 3b - c.$$

(5) $$(28a^7b^4 - 20a^5b^3 - 36a^4b^5) \div 4a^2b^2$$
$$= \frac{28a^7b^4}{4a^2b^2} - \frac{20a^5b^3}{4a^2b^2} - \frac{36a^4b^5}{4a^2b^2}$$
$$= 7a^5b^2 - 5a^3b - 9a^2b^3.$$

(6) $$\frac{4x^2y - 14xy^2 - 22xy}{2xy} = \frac{4x^2y}{2xy} - \frac{14xy^2}{2xy} - \frac{22xy}{2xy}$$
$$= 2x - 7y - 11.$$

After a little practice, the student will be able to write the answer down at once in examples like the above.

Examples. V. a. (*Oral.*)

Divide

1. $3x$ by 3.
2. $3x$ by x.
3. $-3x$ by -3.
4. $-3x$ by 3.
5. $7abc$ by $7a$.
6. $7abc$ by $-7a$.
7. a^2 by a.
8. a^2 by $-a$.
9. $-x^2$ by x.
10. $-x^2$ by $-x$.
11. a^5 by a^2.
12. $-a^4$ by a^3.
13. a^2 by a^2.
14. a^3 by $-a^3$.
15. $24x^4$ by $6x^2$.
16. $21x^3$ by $-7x$.
17. $8a^2$ by $-4a^2$.
18. $-6a^3$ by $-2a$.
19. $7a^3x^4$ by $-ax$.
20. $-a^4b^7$ by $-a^2b^2$.
21. $-54a^2bc$ by $6abc$.
22. $16a^2b^2c^2$ by $4abc$.
23. $-21a^3x^4$ by $7a^3x$.
24. $63a^2b^5c^7$ by $-7ab^3c^2$.

Simplify the following:

25. $\frac{12a}{4}$.
26. $\frac{6a}{a}$.
27. $\frac{-6a^3}{a}$.
28. $\frac{-8a^2b}{-ab}$.
29. $\frac{24a^2b^2}{-4a}$.
30. $\frac{x^2y^2z^2}{xy}$.
31. $\frac{96a^7b^6}{4a^2b^2}$.
32. $\frac{-27p^6q^7x^2}{-9p^3q^3x}$.
33. $\frac{-56a^9b^8c^6}{8a^6b^8c^2}$.
34. $\frac{49pq^2r}{-7pq}$.
35. $\frac{-32l^2mn}{4lm}$.
36. $\frac{-72a^3b^5c^7}{8abc}$.
37. $\frac{54a^2bx^4}{-3ab}$.
38. $\frac{132x^3y^7}{12x^2y^2}$.

Examples. V. b.

Divide

1. $3a-6b$ by 3.
2. $3a-9b$ by -3.
3. $4x^2-3x$ by x.
4. y^2-6y by $-y$.
5. a^2+ab by a.
6. $-b^2+ab$ by $-b$.
7. $3a^2-6ab$ by $3a$.
8. $4a^2b-12ab^2$ by $4ab$.

9. $9a^3b - 21ab^3$ by $-3ab$.

10. $ab + ac$ by a.

11. $ax + bx$ by $-x$.

12. $4x^3 - 5x^2$ by x^2.

13. $-7x^4 + 9x^3$ by $-x^3$.

14. $a^4b^3 - a^3b^4$ by a^2b^2.

15. $-3a^2bc + 7ab^2c$ by $-abc$.

16. $6x^7y^9z^3 - 5x^5y^7z^6$ by $x^3y^4z^2$.

17. $14a^2b - 7ab^2$ by $-7ab$.

18. $-33x^4y^2 - 18x^3y^3$ by $-3x^3y^2$.

19. $12a^4 - 24a^2b^2$ by $6a^2$.

20. $-5m^3n + 20m^2n^2$ by $-5mn$.

21. $12a - 9b - 18c$ by -3.

22. $ab + bc + bd$ by b.

23. $3ac - 4cd - 12cx$ by $-c$.

24. $-a^2x - ax^2 - a^2x^2$ by ax.

25. $2a^2 - 8ab + 16ac$ by $-2a$.

26. $x^3 + 3x^2 - 3x$ by x.

27. $ax^4 - a^2x^3 + a^3x^2$ by $-ax^2$.

28. $7a^4b^2 + 35a^3b^4 - 21a^3b^3$ by $7a^3b^2$.

29. $a^2bc - ab^2c + abc^2$ by $-abc$.

30. $4x^4 - 2x^3 + 8x^2 - 2x$ by $-2x$.

31. $15y^4 - 5y^3x - 30yx^3$ by $5y$.

32. $9x^2y^2 - 21xy^3 - 3x^3y$ by $-3xy$.

33. $4x^4y^8 - 8x^5y^6 - 28x^6y^4$ by $-4x^3y^3$.

34. $27x^4y^5z^6 - 45x^5y^4z^5 + 54x^6y^7z^4$ by $9x^3y^3z^2$.

Following the law of indices, what is the quotient when

35. a^m is divided by a^n.

36. a^n is divided by a^3.

37. x^4 .... x^p.

38. $6x^n$ .... $-2x^4$.

39. $27x^my^n$ .... $3x^ny^m$.

40. $-54x^3y^3$ .... $-6x^ny^n$.

37. We have already seen that $x(x+2) = x^2 + 2x$.
The converse therefore is true, viz.

$$x^2 + 2x = x(x+2).$$

Hence $$(x^2 + 2x) \div (x+2) = \frac{x(x+2)}{x+2} = x.$$

Divide $x^2 + 5x + 6$ by $x + 2$.

$$\begin{aligned}(x^2 + 5x + 6) \div (x+2) &= \frac{x^2 + 5x + 6}{x+2} \\ &= \frac{x^2 + 2x + 3x + 6}{x+2} \\ &= \frac{x^2 + 2x}{x+2} + \frac{3x+6}{x+2} \quad \text{(Just as } \tfrac{3+9}{5} = \tfrac{3}{5} + \tfrac{9}{5} \text{ in Arithmetic.)} \\ &= \frac{x(x+2)}{x+2} + \frac{3x+6}{x+2} \\ &= x + \frac{3x+6}{x+2} \\ &= x + \frac{3(x+2)}{x+2} \\ &= x + 3.\end{aligned}$$

The above is worked out in full detail and should be studied carefully.

The work however is more conveniently arranged as follows:

$$\begin{array}{r l}
x+2\,)\,x^2+5x+6\,(\,x+3 & \\
\underline{x^2+2x\quad\;\;} & \\
+3x+6 & \\
\underline{+3x+6} &
\end{array}$$

If the two methods are compared, it will be seen that they differ only in arrangement.

It should be observed that the second method is analogous to that used in Arithmetic.

38. Example 1. Divide $15x^2-26x+8$ by $5x-2$.

$$\begin{array}{r l}
5x-2\,)\,15x^2-26x+8\,(\,3x-4 & \\
\underline{15x^2-\;\;6x\qquad\;} & \ldots\ldots\ldots\ldots(1) \\
-20x+8 & \ldots\ldots\ldots\ldots(2) \\
\underline{-20x+8} & \ldots\ldots\ldots\ldots(3)
\end{array}$$

$15x^2 \div 5x = 3x$; $\therefore$ $3x$ is the first term of the quotient.

$3x(5x-2)=15x^2-6x$, and we thus obtain line (1).

Line (2) is obtained by subtraction, and by bringing down the term $+8$.

$-20x \div 5x = -4$; $\therefore$ -4 is the second term of the quotient.

$-4(5x-2) = -20x+8$, and we thus obtain line (3).

There is no remainder.

Example 2. Divide x^2-16 by $x+4$.

$$\begin{array}{r}
x+4\,)\,x^2-16\,(\,x-4 \\
\underline{x^2+4x\qquad\;\;} \\
-4x-16 \\
\underline{-4x-16}
\end{array}$$

Example 3. Divide $6-13a+6a^2$ by $2-3a$.

$$\begin{array}{r}
2-3a\,)\,6-13a+6a^2\,(\,3-2a \\
\underline{6-\;\;9a\qquad\qquad} \\
-\;\;4a+6a^2 \\
\underline{-\;\;4a+6a^2}
\end{array}$$

Examples. V. c.

Divide

1. $x^2+7x+12$ by $x+3$.
2. $x^2-7x+12$ by $x-3$.
3. a^2+3a+2 by $a+2$.
4. a^2-5a+4 by $a-4$.
5. $b^2+13b+42$ by $b+6$.
6. x^2+6x+9 by $x+3$.
7. $x^2-14x+49$ by $x-7$.
8. x^2-2x+1 by $x-1$.
9. $a^2-15a+54$ by $a-9$.
10. $y^2+13y+36$ by $y+4$.
11. $2x^2-3x-2$ by $2x+1$.
12. $10x^2-14x-12$ by $2x-4$.
13. $2x^2+3x-2$ by $x+2$.
14. $3x^2-x-14$ by $x+2$.
15. $9x^2-3x-2$ by $3x-2$.
16. $10x^2-14x-12$ by $5x+3$.

17. $4+4x+x^2$ by $2+x$.
18. $1-5x+6x^2$ by $1-3x$.
19. $9-6x+x^2$ by $3-x$.
20. $3a^2-8a+4$ by $3a-2$.
21. $25-30a+9a^2$ by $5-3a$.
22. $35y^2+32y-99$ by $7y-9$.
23. x^2-a^2 by $x+a$.
24. $25x^2-16$ by $5x-4$.
25. a^2-4x^2 by $a-2x$.
26. $25-x^2$ by $5+x$.
27. $1-4x^2$ by $1-2x$.
28. $x^2-xy+6y^2$ by $x-3y$.
29. $1-16pq+64p^2q^2$ by $1-8pq$.
30. $12a^2-7ab+b^2$ by $4a-b$.
31. $a^2-b^2c^2$ by $a+bc$.
32. $4x^4-49$ by $2x^2-7$.
33. $81x^6-1$ by $9x^3+1$.
34. $25x^4-16y^4$ by $5x^2-4y^2$.
35. $100-x^2$ by $10+x$.
36. $1-100b^4$ by $1-10b^2$.

Prove the following by division :

37. $\dfrac{x^2+7x+15}{x+3}=x+4+\dfrac{3}{x+3}$.
38. $\dfrac{x^2-14x+48}{x-7}=x-7-\dfrac{1}{x-7}$.
39. $\dfrac{a^2-15a+50}{a-9}=a-6-\dfrac{4}{a-9}$.
40. $\dfrac{10x^2+14x-16}{5x-3}=2x+4-\dfrac{4}{5x-3}$.
41. $\dfrac{35a^2+32ab-91b^2}{7a-9b}=5a+11b+\dfrac{8b^2}{7a-9b}$.
42. $\dfrac{1-5x^2}{1-2x}=1+2x-\dfrac{x^2}{1-2x}$.
43. $\dfrac{25-3x^2}{5-x}=5+x-\dfrac{2x^2}{5-x}$.
44. $\dfrac{1-19x^2}{1+4x}=1-4x-\dfrac{3x^2}{1+4x}$.

39. Example 1. Divide $x^3-ax^2+a^2x-a^3$ by $x-a$.

$$
\begin{array}{l}
x-a\;)\;x^3-ax^2+a^2x-a^3\;(\;x^2+a^2\\
\qquad\quad\underline{x^3-ax^2}\\
\qquad\qquad\qquad\;+a^2x-a^3\\
\qquad\qquad\qquad\;\underline{+a^2x-a^3}
\end{array}
$$

Example 2. Divide $35x^2+5acx+7pqx-acpq$ by $7x-ac$.

$$
\begin{array}{l}
7x-ac\;)\;35x^2-5acx+7pqx-acpq\;(\;5x+pq\\
\qquad\qquad\underline{35x^2-5acx}\\
\qquad\qquad\qquad\qquad\;+7pqx-acpq\\
\qquad\qquad\qquad\qquad\;\underline{+7pqx-acpq}
\end{array}
$$

Examples. V. d.

Find the quotient in the following cases :

1. $(x^3+ax^2+a^2x+a^3)\div(x+a)$.
2. $(x^2+ax+bx+ab)\div(x+a)$.
3. $(x^2-ax-bx+ab)\div(x-b)$.
4. $(3x^2+xy+3x+y)\div(3x+y)$.
5. $(x^3+ax^2+a^2x+a^3)\div(x^2+a^2)$.
6. $(3x^2+xy-6x-2y)\div(3x+y)$.
7. $(px^2+p^2x+x+p)\div(x+p)$.
8. $(3px^2+qx+3px+q)\div(3px+q)$.
9. $(x^3-ax^2+a^2x-a^3)\div(x^2+a^2)$.
10. $(px^2+2x-p^2x-2p)\div(x-p)$.
11. $(ax^2-7ax-5cx+35c)\div(x-7)$.
12. $(a^2x^2+abx+acx+bc)\div(ax+b)$.
13. $(ax^2-7ax+5cx-35c)\div(ax+5c)$.
14. $(5apx^2-3aqx+5bpx-3bq)\div(5px-3q)$.
15. $(21apx^2-3aqx+14bpx-2bq)\div(7px-q)$.

Find the quotient in the following cases :

16. $(a^2x^2 - abx - acx + bc) \div (ax - c)$.
17. $(27x^2 + 3bcx - 9ax - abc) \div (3x - a)$.
18. $(14x^2 - 2apx + 7bqx - abpq) \div (7x - ap)$.
19. $(abx^2 - 2bcx + acx - 2c^2) \div (ax - 2c)$.
20. $(5apx^2 - 5bpx + 3aqx - 3bq) \div (ax - b)$.
21. Divide the sum of $x(x-3)$ and $2(3-x)$ by $x-2$.
22. Divide the product of $3x - 6a$ and $5x - 15a$ by $x - 2a$.
23. Simplify $[6x(x-1) + 5(x-3)] \div (3x-5)$. Check your result by putting $x = 3$.
24. Divide the sum of $x^3 + 1$ and $3x(x+1)$ by $x+1$. Check your result.
25. Simplify $(3x+9)(7x-21) \div (x-3)$.
26. Find the product of $2x^2 - 9x - 5$ and $x - 1$, and divide it by $2x + 1$.
27. Simplify $[6x(x-1) + (x-6)] \div (3x+2)$. Check your result.
28. Find the expanded value of $(a+b)(a-b)^2$.
29. Without doing all the multiplication, determine the coefficient of x^2 in the product $(x^3 - 2x^2 + 6x - 9)(2x - 3)$.
30. Divide $2x^2 - 17x$ by $x - 3$, and hence determine what number must be added to the first expression to make it exactly divisible by the second.
31. Divide the sum of $2x - 7 - 3x^2$, $5x^2 + 1 - 3x$, and $7 - 4x + 2x^2$ by $4x - 1$.
32. Divide $5(x-1)(x+1) + 3x(3x+1)$ by $7x + 5$.
33. What must be added to the expression $3x^3 - 8x^2 + 10x$ to make it exactly divisible by $3x - 2$?
34. Divide $x(bx - c) + c(bx - c)$ by $x + c$.
35. Simplify $[a^2(x^2 - 1) + (a-b)(a+b)] \div (ax + b)$.
36. Divide $(a - 2b)(a + 2b) + 4b(a + b) + 4b^2$ by $a + 2b$.

CHAPTER VI

REVISION EXAMPLES

VI. a. (*Oral.*)

1. Read off the simplest form of

(i) $\frac{x}{2} + \frac{x}{2}$. (ii) $x + \frac{x}{2}$. (iii) $x - \frac{x}{2}$.

(iv) $4ab + \frac{ab}{2}$. (v) $3abc - \frac{1}{2}bca$. (vi) $2a - \frac{a}{2} + a$.

2. What is the value of $5x - 1$ when

(i) $x = 2$, (ii) $x = -2$, (iii) $x = \cdot 2$,

(iv) $x = \cdot 4$, (v) $x = -\cdot 8$, (vi) $x = \cdot 3$?

3. What is

(i) the second power of 5, (ii) the second power of -3,

(iii) .... $\frac{1}{3}$, (iv) .... $-\frac{1}{2}$,

(v) the square of -1, (vi) the cube of -1,

(vii) .... $-\dfrac{ab}{2}$, (viii) .... $-\dfrac{ab}{2}$?

4. What are the values of

(i) $(-2)^2+(-3)^2$, (ii) $(-2-3)^2$, (iii) $(-2)^2-(-3)^2$,

(iv) $(-2+3)^2$, (v) $1-(-2)^3$, (vi) $[1-(-2)]^3$?

5. Simplify

(i) $7-5+3$, (ii) $7a-a-7a$, (iii) $-a-5a+3a$,

(iv) $x^2-3x^2+9x^2$, (v) $3xy-7yx+4xy$, (vi) $5-4+3-2+2-1$

6. What is the value of x^2-1 when

(i) $x=-1$, (ii) $x=2$, (iii) $x=\frac{1}{2}$,

(iv) $x=-3$, (v) $x=-1\frac{1}{2}$, (vi) $x=2\frac{1}{2}$?

7. What is the value of x^2-5x+7 when

(i) $x=0$, (ii) $x=1$, (iii) $x=-1$,

(iv) $x=2$, (v) $x=3$, (vi) $x=-3$?

8. What is the value of x^3-2x^2+2x-1 when

(i) $x=0$, (ii) $x=1$, (iii) $x=-1$,

(iv) $x=2$, (v) $x=3$, (vi) $x=-3$?

9. Read off the simplest values of

(i) $5-5(1-x)$. (ii) $6a+(-3a+2a)$.

(iii) $2x^2-(3x^2-4x^2)$. (iv) $-2ab-(3ab-7ab)$.

(v) $2(x-1)+3(x-2)+4(x-3)$. (vi) $3(2x-1)-2(3x+1)+7$.

10. Simplify

(i) $\dfrac{3x-6}{3}-\dfrac{2x-8}{2}$. (ii) $\dfrac{9-3x}{3}-\dfrac{12-8x}{4}$.

(iii) $\dfrac{4-2x}{2}-\dfrac{5x-5}{5}+\dfrac{9x-3}{3}$. (iv) $\dfrac{3x-1}{4}+\dfrac{x-3}{4}$.

(v) $\dfrac{7x-9}{8}+\dfrac{x+1}{8}$. (vi) $\dfrac{7x-5}{4}-\dfrac{3x-13}{4}$.

(vii) $\dfrac{23x+7}{5}-\dfrac{3x-3}{5}$.

(viii) $(a+b-c)-(a-b-c)+(a-b+c)$.

11. In the expression $ax^3+bx^2y-2cxy^2+2y^3$, what is the coefficient of

(i) y, (ii) y^2, (iii) a?

12. In the expression $ax^2-bx-c-bx^2+cx+d$, what is the coefficient of

(i) x^2, (ii) x?

13. What is the sum of

(i) $3a$ and $-7a$.
(ii) $2a,\ -5a,\ 7a$.
(iii) $-\frac{x}{2},\ -\frac{x}{2},\ x$.
(iv) $-\frac{x}{4},\ \frac{x}{2},\ x$.
(v) $\frac{5x^2}{8},\ \frac{3x^2-8}{8}$.
(vi) $x^2-2x,\ 2x+1$.
(vii) $x^3-3x^2,\ 3x^2-4x,\ 4x+1$.
(viii) $x^2-3x,\ 1-2x$.
(ix) $3(x-1),\ 4(x-1)$.
(x) $\frac{1}{3}(x-3),\ \frac{2}{3}(x-3)$.
(xi) $\frac{5}{6}(a+bx),\ \frac{1}{6}(a+bx)$.
(xii) $\frac{1}{2}(a+b),\ \frac{1}{2}(a-b)$.
(xiii) $2x(b-c),\ 2x(b+c)$.
(xiv) $\frac{1}{a}(a+x),\ \frac{1}{a}(a-x)$.

14. Add together

(i) $x-2y+3z,\ 2x+y-3z,\ x-2y+z$.
(ii) $x^2-2x+1,\ 3x-1,\ 2x^2-x$.
(iii) $2(a-b+c),\ 3(a+b-c),\ 4(b+c-a)$.
(iv) $x^3-4x^2y+5xy^2,\ 3x^2y-2xy^2+y^3,\ -2xy^2-y^3$.
(v) $3x^3-7x^2+5x,\ x^3-7x+2,\ 3x^2+2x-7$.
(vi) $\frac{5a}{6}-\frac{3b}{4}+\frac{7c}{8},\ a+\frac{7b}{4}-\frac{c}{2},\ \frac{a}{6}-2b-\frac{3c}{8}$.

15. In each of the following cases, subtract the second expression from the first:

(i) $x,\ -3x$.
(ii) $x^2,\ -xy$.
(iii) $\frac{x}{2},\ \frac{x}{4}$.
(iv) $0,\ 2x-3y$.
(v) $-a^2x,\ -3a^2x$.
(vi) $a+3b,\ a-5b$.
(vii) $2(x^2-1),\ 2x^2-2$.
(viii) $a-b+c,\ b+c-a$.
(ix) $3(x-2),\ 7(x-2)$.
(x) $3a,\ 2a-b$.
(xi) $a,\ 3a-2b$.
(xii) $x^2-3x-2,\ x^2-5x+4$.
(xiii) $x^3-1,\ x^2-1$.
(xiv) $5x^3-6x^2+3,\ 2x^2-5x+2$.
(xv) $4(x-y),\ 2(x-y)$.
(xvi) $5(2a-b),\ 7(2a-b)$.
(xvii) $3(x^2-3x+2),\ 3(2-3x)$.
(xviii) $c(a+b),\ c(a-b)$.
(xix) $7(x-y)-z,\ 5(x-y)-3z$.

16. In each of the following cases find the excess of the first expression over the second:

(i) $2x,\ -2x$.
(ii) $7x^2,\ 4$.
(iii) $-3x^2,\ -2x^2$.
(iv) $-3a^2x,\ -5a^2x$.
(v) $6-x^2,\ x^2$.
(vi) $2(a-b),\ -2(a-b)$.
(vii) $x^3-7x^2,\ 7x^2-5$.
(viii) $-5(a^2-b^2),\ 2(a^2-b^2)$.
(ix) 3 times 141, twice 141.
(x) 5 times $2\frac{1}{2}$, 3 times $2\frac{1}{2}$.
(xi) 4 times the square of 9, 3 times the square of 9.
(xii) 5 times the cube of 2, twice the cube of 2.

17. Simplify the following:

(i) $-2a \times 3b$. (ii) $-2a \div 2a$.

(iii) $-\frac{3}{4}a \times \frac{4}{3}x$. (iv) $\frac{7}{8}a^2x \div \frac{7}{4}ax$.

(v) $\frac{2}{3}ab^2c \times \frac{9}{2}a^2bc^2$. (vi) $-\frac{3}{4}ab^2 \div -\frac{1}{4}ab$.

(vii) $-\frac{33}{10}x^5 \times \frac{5}{11}x^2$. (viii) $\frac{27}{4}x^3 \div \frac{3}{4}x$.

(ix) $\frac{2}{3}a \times \frac{a^2}{8} \times -\frac{4x}{3}$. (x) $\frac{9}{4}x^2y \div \frac{3}{2}xy$.

(xi) $-\frac{15}{20}x \times \frac{2a}{3} \times -2x$. (xii) $-x^2 \times a^2 \div ax$.

(xiii) $(-a)^3 \times (-a)^4$. (xiv) $(-a^5) \div (-a)^4$.

(xv) $(-a^7) \times a^3$. (xvi) $(-a)^2 \times (-a)^3 \div a^5$.

18. Read off the products of the following expressions:

(i) $\frac{ax}{3} - \frac{ay}{4}$, $12xy$. (ii) $\frac{x^2}{9} - \frac{x}{3} + \frac{1}{18}$, $-18x$.

(iii) $12x^2 + 16x - 8$, $\frac{1}{4}$. (iv) $12x^3 - 6x^2 + 9x$, $\frac{1}{3x}$.

(v) $\frac{x^4}{9} - \frac{2x^3}{27} - \frac{x^2}{3}$, $-\frac{27}{x^2}$. (vi) $3x^2 - 2x + 1$, $3x$, $-2x$.

19. Multiply out:

(i) $(1+x)(1-x)$. (ii) $(1+x)^2$. (iii) $(1-2x)^2$.

(iv) $(a+2b)^2$. (v) $(x+3)(x+5)$. (vi) $(x-3)(x+2)$.

(vii) $(x-2y)(x-3y)$. (viii) $(3x+1)(3x-1)$. (ix) $(5-p)(6-p)$.

(x) $(a^2-3)(a^2+3)$. (xi) $(3x-5)(3x+5)$. (xii) $(a^2x+1)^2$.

(xiii) $2(x-4)(x+4)$. (xiv) $(x^2+3y)(x^2+2y)$. (xv) $(1-2x)(1+4x)$.

(xvi) $\frac{1}{2}(2a+4b)(a-2b)$. (xvii) $\frac{1}{3}(3+6x)(1+2x)$. (xviii) $\frac{3}{4}(2a+2x)(2a-2x)$.

(xix) $4(a-\frac{1}{2})(a+\frac{1}{2})$. (xx) $9(x^2-\frac{1}{3})(x^2+\frac{1}{3})$.

20. Give the following expressions in their expanded form:

(i) $(3a-2b)^2$. (ii) $(2a-y)^2$. (iii) $(a^2-2)^2$.

(iv) $\left(x+\frac{a}{2}\right)^2$. (v) $4(x-\frac{1}{2})^2$. (vi) $9(x-\frac{1}{3})^2$.

(vii) $(7-x)(3+x)$. (viii) $3(5-x)(5+x)$. (ix) $2(x-y)^2$.

(x) $(x+c)(x-a)$. (xi) $6\left(\frac{x}{2}-1\right)\left(\frac{x}{3}-1\right)$. (xii) $(x-\frac{2}{3})(x+\frac{2}{3})$.

(xiii) $(a-2x)(a+4x)$. (xiv) $(ax-1)(bx-1)$. (xv) $(3a-\frac{1}{2})(3a+\frac{1}{2})$.

(xvi) $9(2x+\frac{1}{3})(2x-\frac{1}{3})$. (xvii) $(5x-3)(2x+3)$. (xviii) $(3x+7)(5x-2)$.

(xix) $(3x+2)(5x+1)$. (xx) $(7x-3y)(2x+y)$.

21. Read off the coefficient of x^2 in the products:

(i) $(x^2+2x+1)(x+1)$. (ii) $(x^2-3x+4)(2x-1)$.

(iii) $(6x^2-5x+2)(3x-2)$. (iv) $(x^3-2x)(x+4)$.

22. Read off the coefficients of x in the above products.

23. Read off the quotients in the following:

(i) $\frac{x^5}{-x^2}$. (ii) $\frac{-4a^3}{-2a}$. (iii) $\frac{7a^2bc}{abc}$.

(iv) $\frac{5a^2x}{3ax}$. (v) $\frac{24p^2qr^2}{6p^2qr}$. (vi) $\frac{-27p^6q^4}{4p^3q^3}$.

(vii) $(6ab-8a^2)\div 2a$. (viii) $(-9x^3-3x)\div -3x$. (ix) $\frac{3a^2x-4ax^2}{ax}$.

(x) $\frac{12ab^2c-16a^2bc}{4abc}$. (xi) $\frac{a^2b-b^2c+bc^2}{-b}$. (xii) $\frac{4x^2-9x^2}{5x}$.

(xiii) $\frac{(a-x)^3}{(a-x)^2}$. (xiv) $4(a-b)^2\div 2(a-b)$. (xv) $\frac{x^2-2x}{x-2}$.

(xvi) $\frac{5a^2-10ab}{a-2b}$. (xvii) $\frac{(a+x)^3}{a+x}$. (xviii) $\frac{27a^2x-5ax^2}{27a-5x}$.

(xix) $\frac{6a^2-4b^2}{3a^2-2b^2}$. (xx) $\frac{(a-x)^4}{(a-x)^2}$.

REVISION PAPERS

VI. b.

1. What is the value of x^2-2x+1,
(i) when $x=1$, (ii) when $x=2$, (iii) when $x=-2$?

2. Arrange the following expression in descending powers of x, and then collect like terms:

$$3x-4x^3+7x^2+7+2x-3x^3+2x^4-7x^2-10.$$

What is the coefficient of x^3, and what is the coefficient of x^2 in the result?

3. Prove that $4+2(6-3)=10$, by two different methods.

4. Find the sum of $6a-(2a-b)$ and $b-(3a-2b)$; and subtract $a-2b$ from the result.

5. Multiply $2x+5a$ by $3x-4a$, and find the continued product of a, $x-a$, $x+a$.

6. Write down the quotients in the following cases:
(i) $7x^3\div x^2$. (ii) $-9x^3\div 3x$. (iii) $(2a^3-3a^2b+4ab^2)\div a$.

7. Divide $6x^2-5xy+y^2$ by $2x-y$, and check your result by multiplication.

VI. c.

1. What is the value of x^2+2x+1,
(i) when $x=-1$, (ii) when $x=2$, (iii) when $x=-2$?

2. Arrange the following expression in ascending powers of a, and then collect like terms:

$$a^2b^2-7a^3b+5ab^3+4a^3b-3ab^3+a^4+b^4+4a^2b^2.$$

What is the coefficient of a^3 in the result?

3. Prove that $a-2(4a-a)=-5a$ by two different methods.

4. Subtract $4x^2-5$ from the sum of $3x^2-(x+1)$ and $x+2x^2-5$.

5. Find the product of $x-3a$ and $x+3a$; and the continued product of x^2, $x-2a$, $x+a$.
6. Write down the quotients in the following cases:
   (i) $-7x^2 \div -7x$. (ii) $(-3ax+x^2)\div x$. (iii) $a^4bc\div(-a)^2$.
7. Divide $6a^2-ab-12b^2$ by $2a-3b$, and check your result by multiplication.

VI. d.

1. What is the value of $a^2-5ab+6b^2$,
   (i) when $a=0, b=1$, (ii) when $a=-1, b=1$, (iii) when $a=2b$?
2. Arrange the following expression in descending powers of x; then collect like terms, and find the value of the expression when $x=1$:
$$x-7-8x^2+4x^3+2x-3x^3+5x^2+6.$$
3. Simplify the expressions:
   (i) $5(x-3)-3(x-2)-(2x-9)$. (ii) $\frac{5x-10}{5}-\frac{7x+21}{7}+\frac{3x-9}{3}$.
4. Take $4c-2b$ from the sum of $2a-3b-4c$, $a+2b-3c$, and $5b-2a-2c$.
5. State the results of the following multiplications:
   (i) $(-a)^3(-b)^2$. (ii) $(-a^2x)^2(ax)^3$. (iii) $(-a^2bc)(-ab^2c)(-abc^2)$.
6. Multiply $3x+12a$ by $2x-3a$, and divide the result by $x+4a$.
7. Multiply $7p-9q$ by $3p+4q$, and check your result by division.

VI. e.

1. What is the value of $(x+1)^3$,
   (i) when $x=0$, (ii) when $x=-2$, (iii) when $x=3$?
2. Use squared paper to illustrate the following:
   (i) $7-5=2$. (ii) $7-2-8=-3$.
3. Simplify the expressions:
   (i) $7a-2\left(x-\frac{a}{2}\right)+4\left(x+\frac{a}{2}\right)$. (ii) $x^3-(x-2)+3(x^2-2-5x)$.

   Find the value of the second expression when $x=-2$.
4. Subtract the sum of $2x^2-3(x-1)$ and $2x+3(x^2-2)$ from the sum of $5x^2-(x-2)$ and $x^2-2(x+1)$.
5. If X stands for $x-a$, and Y for $2x+a$, find the product of $X+Y$ and $X+2Y$.
6. Divide $ax^2-5ax+6a$ by $x-2$.
7. Find the remainder when $14x^2-27xy+3y^2$ is divided by $7x-3y$.

VI. f.

1. What is the value of $(2x-a)^3$,
   (i) when $x=0, a=1$, (ii) $x=-1, a=-2$, (iii) when $x=2, a=4$?
2. Use squared paper to illustrate the following:
   (i) $2a+5a-3a=4a$. (ii) $a-7a+3a=-3a$.
3. Simplify the expressions:
   (i) $(x^2-4x-21)\div(x+3)$. (ii) $4(x-1)-\frac{3}{2}(x-1)-\frac{1}{2}(x-1)$.

4. Find the value of the sum of $x^3-3x(x-1)$, $x^2+2(x-1)$, and $x-2x(x-x^2)$ when $x=2$.
5. If X stands for $2x-a$, and Y for $x+2a$, find the product of $2X+3Y$ and $X-Y$.
6. Multiply $5x^2-2(x^2-a)$ by $2a-3(a-2x^2)$.
7. Divide $10(x^2-2ax)-3(ax-4a^2)$ by $2x-3a$.

VI. g.

1. What is the value of a^2-3b^2-2ac,
   (i) when $a=0, b=-1, c=1$, (ii) when $a=-2, b=2, c=-3$?
2. A man walks 4 miles East, then 7 miles West, then again 5 miles East. How far is he then from his starting point? Illustrate with a diagram.
3. Simplify the expressions:
   $$\text{(i) } (x^3-3ax^2+3a^2x-a^3)\div(x-a).$$
   $$\text{(ii) } a(a-x)-\frac{a}{2}(2a-2x)+\frac{x}{3}(3a-6x).$$
4. If X stands for $ax^2+5bx+5c$, and Y for $ax^2-6bx-6c$, find the value of $6X+5Y$.
5. Find the expanded value of $ap-bp$ when p stands for $2a-3b$.
6. Write down the results of the following multiplications:
   (i) $(2x-a)(2x+a)$. (ii) $(x^2-3)(x^2+3)$. (iii) $(a-p^2)(a+p^2)$.
7. Prove that $[(x^2-6x+9)\div(x-3)]+[(y^2+y-6)\div(y-2)]=x+y$.

VI. h.

1. Find the value of $(a+b-c)^2+(b+c-a)^2+(a+c-b)^2$,
   (i) when $a=b=c=3$. (ii) when $a=-b=c=2$.
2. What must be added to $x^3-3x(x-1)-1$ to make it equal to $x^3+3x(x+1)+1$?
3. Find the sum of $3(x-a)+2(y-a)$ and $2(x+a)-3(y+a)$.
4. If X stands for $x+\frac{2}{x}$, and Y for $x-\frac{3}{x}$, find the product of $3X+2Y$ and X.
5. Find the values of $5x^2+x-3$ when $x=-2, -1, 0, 1, 2$. Tabulate your work.
6. Find the continued product of $(x-2y)$, $(x+2y)$, $(x-2y)$.
7. Divide $2a^2x^2+6apx+aqx+3pq$ by $2ax+q$.

VI. k.

1. Find the value of $(2x-y)^2-(3y-x)^2$,
   (i) when $x=-1, y=2$. (ii) when $x=-1, y=-2$.
2. By how much does $5x^2-2(x+3)$ exceed $3(x^2-2)+x$?
3. Subtract $a(b+c-a)$ from the sum of $b(c+a-b)$ and $c(a+b-c)$.

4. If X stands for $a(x+y)$, and Y for $b(x-y)$, find the values of
$$\frac{X}{a}+\frac{Y}{b} \text{ and } \frac{X}{a}-\frac{Y}{b}.$$

5. Find the values of $3x^2-5x+1$ when $x=-2, -1, 0, 1, 2$. Tabulate your work.

6. Find the continued product of $x-a$, $x+a$, $x+a$.

7. Divide $4bx^2-5bx-16cx+20c$ by $bx-4c$.

CHAPTER VII

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY

40. When we express algebraically the fact that two expressions are equal, that statement is called an **equation.**

Thus $2a-3b=-3b+2a$ is an equation.

Moreover, the above equation is true for all values of a and b, the symbols used.

On the other hand, the equation $x+3=5$, is evidently only true when x is equal to 2; $x-3=0$ is true only when x is equal to 3.

An equation which is only true when the symbols have certain particular values is called a **conditional equation**, or an **equation of condition.**

An equation which is true for all values of the symbols used is called an **identity.**

Simple Equations of Condition.

The two parts of an equation on either side of the sign of equality are called its **sides** or **members.**

We see that the equation $x-4=0$ is true when $x=4$.

The value 4 is said to **satisfy** the equation.

The process of finding that value of x which will satisfy an equation is called **solving the equation.**

An equation which, when simplified, involves one symbol in the first degree only is called a **simple equation** with regard to that symbol, and the symbol used is called the **unknown quantity.**

The value of the unknown quantity which satisfies an equation is called a **root of the equation, a solution of the equation.**

41. It will be seen later, that the solution of equations is a most important branch of Mathematics.

In the case of Simple Equations with one unknown quantity the process consists mainly in the use of four axioms.

(1) **If equals be added to equals the sums are equal.**

Thus if $x=a, \qquad x+2=a+2.$

(2) **If equals be taken from equals the remainders are equal.**

If $x=b, \qquad x-5=b-5.$

(3) **If equals be multiplied by equals the products are equal.**

If $x=a, \qquad 3x=3a.$

(4) **If equals be divided by equals the quotients are equal.**

If $5x=10, \qquad x=2.$

Examples. VII. a.

Find the values of x which satisfy the following equations :

1. $2x=6.$ 2. $3x=9.$ 3. $5x=20.$ 4. $4x=-20.$

5. $17x=51.$ 6. $11x=-33.$ 7. $-x=6.$ 8. $7x=0.$

9. $-3x=-15.$ 10. $\frac{x}{2}=1.$ 11. $\frac{x}{3}=4.$ 12. $-\frac{x}{2}=4.$

13. $\frac{x}{5}=-4.$ 14. $-4x=0.$ 15. $2x=5.$ 16. $3x=7.$

17. $\frac{2x}{3}=6.$ 18. $\frac{x}{6}=\frac{1}{3}.$ 19. $\frac{3x}{4}=\frac{6}{8}.$ 20. $15x=10.$

21. $\frac{x}{6}=\frac{1}{12}.$ 22. $-\frac{x}{3}=\frac{1}{12}.$ 23. $\frac{5x}{4}=10.$ 24. $\frac{6x}{5}=-18.$

25. $\frac{3x}{4}=0.$ 26. $\frac{5x}{7}=\frac{15}{14}.$ 27. $6x-2x=12.$ 28. $2x-5x=9.$

29. $-5x+7x=7-5.$ 30. $x+2x-6x=0.$ 31. $9x-3x=-36+30.$

32. $-11x+7x=-8+12.$ 33. $x-5x-4x=-16.$

34. $7x-2x-x=19-3.$ 35. $-3x-4x-7x=-48+20.$

36. $15x-3x+x=37-11.$ 37. $7x+x-5x=21-16+4.$

38. $-x-2x-3x=-7-4-10.$ 39. $11x-5x+6x=-35+11.$

40. $\cdot5x=1.$ 41. $\cdot2x=4.$ 42. $\cdot7x=2\cdot1.$ 43. $\cdot3x=\cdot9.$

44. $5x=\cdot05.$ 45. $7x=\cdot21.$ 46. $-8x=\cdot24.$

42. Example 1. Solve the equation $3x+2=22-7x$.

$$3x+2=22-7x.$$

Adding $7x$ to both sides, $3x+7x+2=22-7x+7x$, (*Ax.* 1.)

i.e. $10x+2=22$.

Taking 2 from each side, $10x+2-2=22-2$, (*Ax.* 2.)

i.e. $10x=20$.

Dividing both sides by 10, $x=2$; (*Ax.* 4.)

$\therefore$ 2 is the reqd. root of the equation.

To verify the fact that 2 *is a root of the equation* $3x+2=22-7x$.

When $x=2$, $3x+2=3\times 2+2=8$.

.... $22-7x=22-7\times 2=22-14=8$.

$\therefore$ .... $3x+2=22-7x$, *i.e.* the equation is then **satisfied**. Q.E.D.

Examples. VII. b.

Solve the following equations, giving reasons for each step, and verifying each solution :

1. $x=6-2x$.
2. $3x=12+2x$.
3. $4x=42-2x$.
4. $5=16-11x$.
5. $17-7x=-4$.
6. $-5x=-6x+12$.
7. $3x-4=0$.
8. $6x+18=0$.
9. $4x-6=3x-6$.
10. $5x-13=7x-13$.
11. $5x+6=2x+12$.
12. $8x-12=x+2$.
13. $2x+5=35-4x$.
14. $13x-21=12x-24$.
15. $-2x-4=-5x+11$.
16. $17x-35=13x-19$.
17. $6x+15=9x+13-5x$.
18. $5-6x-6=7x-1$.
19. $9-3x=6+2x-12$.
20. $3x+4+2x+6=0$.

[When denominators occur, **multiply both sides of the equation by the least common multiple of the various denominators.**

This operation will clear away the fractions.

Thus if $$\frac{3x-4}{10}=\frac{5}{12},$$

multiply both sides by 60,

$$\frac{60}{10}\times(3x-4)=\frac{5}{12}\times 60,$$

or $6(3x-4)=25$; $\therefore$ $18x-24=25$.]

21. $\frac{x}{3}=\frac{1}{2}$.
22. $\frac{2x}{3}=\frac{5}{6}$.
23. $\frac{7x}{9}=-21$.
24. $\frac{2x}{3}-\frac{1}{4}=\frac{3}{4}$.
25. $\frac{3}{4}x=-\frac{9}{2}$.
26. $\frac{5x}{7}=\frac{3}{4}$.
27. $\frac{3}{4}=-\frac{x}{12}$.
28. $\frac{x}{4}+\frac{17}{8}=0$.
29. $\frac{11x}{13}-\frac{19x}{31}=0$.
30. $\frac{x-3}{5}=0$.
31. $\frac{2x-5}{7}=0$.
32. $3(x-1)=3$.
33. $\frac{x-1}{4}=1$.
34. $\frac{2x-1}{3}=3$.
35. $\frac{3x+5}{7}=2$.
36. $\frac{2x}{3}-\frac{5}{6}=0$.
37. $6(x-3)=0$.
38. $3(x+5)=0$.
39. $\frac{2}{3}(x-10)=0$.
40. $5(2x-7)=0$.
41. $3(3x+7)=0$.
42. $\frac{4}{9}(6x-15)=0$.
43. $\frac{13}{11}(19x-27x)=0$.
44. $\frac{47}{135}\left(\frac{x}{2}-1\right)=0$.

43. Let us consider the equation $2x+5=10-4x$.

Adding $4x$ to both sides, $2x+4x+5=10$.

[*N.B.*—The result of this operation is that $-4x$ disappears from the right hand side, and appears on the left, *with its sign changed.*]

$$i.e.\quad 6x+5=10.$$

Taking 5 from each side, $6x=10-5$.

[*N.B.*—Again, the result is that 5 disappears from the left hand side, and appears on the right, *with its sign changed.*]

We therefore deduce the following most important rule.

Any term may be transposed from one side of an equation to the other by changing its sign.

Example 1. Solve the equation $3x-4+5x-4=3x-10+7x+16$.

Transposing so that we have all the terms containing x on the left, and the other terms on the right,

$$3x+5x-3x-7x=-10+16+4+4,$$
$$i.e.\quad 8x-10x=24-10,$$
$$-2x=14.$$

Dividing both sides by -2, $x=-7$, the required solution.

Verification. When $x=-7$, the left side

$$=-7\times3-4-7\times5-4=-21-4-35-4=-64.$$

When $x=-7$, the right hand side

$$=-7\times3-10-7\times7+16=-21-10-49+16=-64=\text{the left hand side.}$$

Q.E.D.

Example 2. Solve the equation $x^2-8x+23=x(x-3)-2(x-4)+3$.

Removing the brackets, $x^2-8x+23=x^2-3x-2x+8+3$.

Transposing all the terms containing x, or powers of x, to the left, and other terms to the right,

$$x^2-8x-x^2+3x+2x=8+3-23;$$
$$i.e.\quad -8x+5x=-23+11,$$
$$-3x=-12.$$

Dividing both sides by -3, $x=4$, the required solution.

Verification. When $x=4$,

$$\text{the left hand side}=4\times4-8\times4+23$$
$$=16-32+23=7.$$

$$\text{When } x=4,\text{ the right hand side}=4(4-3)-2(4-4)+3$$
$$=4+3=7$$
$$=\text{the left hand side.}$$

Q.E.D.

Example 3. Solve the equation

$$(x-1)(x+6)=(x-2)(x-3)+3.$$

Multiplying out, $x^2+5x-6=x^2-5x+6+3.$

Transposing, $x^2+5x-x^2+5x=6+6+3,$

$$10x=15,$$

$$x=1\tfrac{1}{2}.$$

Examples. VII. c.

[*The beginner is advised to verify each solution.*]

Solve the following equations:

1. $6x-18=4x-8-3x+5.$
2. $10x-10-6x-27=3.$
3. $24x+10-20x+100=5x+96.$
4. $6x-18-12x+60=3x+3-8x+17.$
5. $12x-18-3x+3-4x=0.$
6. $6x+18=4x-8+3x-2.$
7. $7x+15-3x+4=2x-3.$
8. $5(x-1)=4(x-2).$
9. $3x-(2x-5)=12.$
10. $3(3x+1)-(x-1)=6(x+10).$
11. $3(2x+5)-4(x-3)=5(3x+1)-4.$
12. $11(x-2)-2(4-3x)-4(1-2x)=17(x-1)+7.$
13. $x(x+4)=x^2+36.$
14. $(x+3)(x-2)=x^2-26.$
15. $x^2+8=(x+2)^2.$
16. $x(x-2)=x^2-4.$
17. $2x^2-7=x(2x-3).$
18. $3x^2-5-x(3x+1)=0.$
19. $(x+1)(x+4)=x(x+2).$
20. $2(x-1)(x+1)=2x^2-4x.$
21. $(x-3)^2=x^2+4x+29.$
22. $(x-4)^2=(x-1)^2-3.$
23. $(x-2)^2=(x-5)^2-15.$
24. $(x-3)(x+3)=(x+4)(x-7)+40.$
25. $x(x-9)-4=(x-7)(x+7).$
26. $2(x-6)(x+6)+12=(2x-1)(x-3).$

44. When the equations are in fractional form, the fractions should be cleared first.

Example 1. Solve the equation $\frac{x}{4}+\frac{3}{5}=\frac{1}{4}-\frac{x}{5}+\frac{7}{2}.$

Multiplying both sides by 20, the L.C.M. of 4, 5, and 2,

$$5x+12=5-4x+70.$$

Transposing, $5x+4x=5+70-12,$

$$9x=63,$$

$$x=7.$$

Example 2. Solve the equation $\frac{3}{5}+\frac{4}{10x}=\frac{23}{5x}+1.$

Multiplying both sides by $10x$,

$$3\times 2x+4=23\times 2+10x,$$

$$6x+4=46+10x,$$

$$6x-10x=46-4,$$

$$-4x=42.$$

Dividing both sides by -4, $x=-\frac{42}{4}=-\frac{21}{2}=-10\tfrac{1}{2}.$

Verification. When $x=-10\frac{1}{2}\left(=-\frac{21}{2}\right)$,

the left hand side $=\frac{3}{5}+\frac{4}{10}\div\left(-\frac{21}{2}\right)$

$=\frac{3}{5}-\frac{4}{10}\times\frac{2}{21}=\frac{3}{5}-\frac{4}{105}$

$=\frac{63-4}{105}=\frac{59}{105}$.

When $x=-10\frac{1}{2}$, the right hand side $=\frac{23}{5}\div\left(-\frac{21}{2}\right)+1$

$=-\frac{23}{5}\times\frac{2}{21}+1=-\frac{46}{105}+1$

$=\frac{-46+105}{105}=\frac{59}{105}$

$=$ the left hand side. Q.E.D.

Example 3. Solve the equation $\dfrac{x-3}{4}-\dfrac{x-5}{2}=\dfrac{x+1}{8}-\dfrac{x-4}{3}$.

Multiplying both sides by 24, the L.C.M. of 4, 2, 8, and 3,

$$6(x-3)-12(x-5)=3(x+1)-8(x-4),$$

i.e. $$6x-18-12x+60=3x+3-8x+32.$$

Transposing, $$6x-12x-3x+8x=3+32+18-60,$$

i.e. $$-x=-7,$$

$$x=7.$$

Verification. When $x=7$, the left hand side $=\frac{7-3}{4}-\frac{7-5}{2}=1-1=0$.

When $x=7$, the right hand side $=\frac{7+1}{8}-\frac{7-4}{3}$.

$=1-1=0$

$=$ the left hand side. Q.E.D.

Useful facts to note in connection with decimals.

$$4\times\cdot25=1,\quad\therefore\ \frac{1}{\cdot25}=\frac{4}{4\times\cdot25}=4.$$

$$\text{Thus } \frac{7}{\cdot25}=\frac{7\times4}{1}=28.\quad \text{Also } \frac{1}{\cdot125}=\frac{8}{8\times\cdot125}=8.$$

$$\frac{1}{\cdot025}=\frac{40}{40\times\cdot025}=40.\qquad \frac{7}{\cdot75}=\frac{7\times4}{4\times\cdot75}=\frac{28}{3}.$$

Example 4. Solve the equation $\dfrac{x+\cdot15}{\cdot125}-\dfrac{x-\cdot25}{\cdot25}=3\cdot3$.

$$\frac{8(x+\cdot15)}{1}-\frac{4(x-\cdot25)}{1}=3\cdot3,$$

$$8x+1\cdot2-4x+1=3\cdot3,$$

$$4x=3\cdot3-2\cdot2,$$

$$4x=1\cdot1,$$

$$x=\cdot275.$$

Examples. VII. d.

Solve the equations :

1. $\frac{x}{2}-\frac{x}{3}=3.$

2. $\frac{x}{3}=\frac{x}{4}+1.$

3. $\frac{x}{5}-\frac{1}{2}=\frac{x}{6}.$

4. $\frac{3x}{4}-\frac{2x}{3}=\frac{1}{3}.$

5. $\frac{x}{7}=\frac{x}{5}-4.$

6. $\frac{x}{3}+\frac{x}{4}=\frac{x}{8}+5\frac{1}{2}.$

7. $\frac{x}{2}-4\frac{5}{8}+3x=2x+1\frac{3}{8}.$

8. $\frac{x+1}{3}-5=0.$

9. $\frac{2x-3}{5}-7=0.$

10. $\frac{x-3}{4}=\frac{x-2}{5}.$

11. $\frac{x-1}{6}+\frac{2x-1}{7}=\frac{25}{42}.$

12. $\frac{2x-1}{4}-\frac{x-1}{5}=1.$

13. $2-\frac{5}{x}=\frac{10}{x}-1.$

14. $7+\frac{9}{2x}=9+\frac{1}{2x}.$

15. $\frac{14}{3}+\frac{4}{x}=1-\frac{x-1}{6x}.$

16. $12-\frac{5x-10}{7x}=\frac{35}{x}-22\frac{2}{7}.$

17. $\frac{6x+1}{5}-\frac{5x-6}{7}=\frac{2x+1}{3}.$

18. $\frac{x-1}{2}-\frac{x-2}{3}-\frac{x-3}{4}=0.$

19. $\frac{1}{2}(x-3)-\frac{1}{3}(x-4)=1.$

20. $\frac{x-3}{4}-6-\frac{x-1}{5}=\frac{x-5}{3}-8.$

21. $\frac{x-2}{4}+\frac{1}{3}=x-\frac{2x-1}{3}.$

22. $x-1-\frac{x-2}{2}+\frac{x+3}{3}=0.$

23. $\frac{x}{3}-\frac{x}{4}+\frac{x-2}{5}=3.$

24. $\frac{3x}{4}+x=\frac{7x}{8}+2x-9.$

25. $\frac{2}{3}(4x-1)-\frac{1}{7}(3x+2)=6+\frac{1}{9}(5x-2).$

26. $\frac{7x+8}{8}-\frac{9x-12}{16}=\frac{3x+1}{10}-\frac{29-8x}{20}.$

27. $\frac{x}{4}-\frac{x-2}{5}=5+\frac{14-x}{2}-\frac{5x}{12}.$

28. $\frac{x}{12}-\frac{8-x}{8}-\frac{1}{4}(5+x)+\frac{11}{4}=0.$

29. $\frac{3x-5}{4}-\frac{7x+9}{16}+\frac{8x+19}{8}+8\frac{5}{8}=0.$

30. $5x-\frac{2x-1}{3}+1=3x+\frac{x+2}{2}+7.$

31. $\frac{7x}{2}-\frac{x-8}{3}-\frac{4}{5}(4x+2)=\frac{5x-3}{7}.$

32. $\frac{1}{7}(3x+5)-\frac{1}{3}(2x+7)=\frac{3x}{5}-10.$

33. $\frac{7x-11}{8}-\frac{9x-17}{10}=\frac{7}{20}.$

34. $\frac{1}{5}(2x+11)-\frac{1}{2}(5-6x)=7x+1\frac{1}{2}.$

35. $\frac{3(x+2)}{11}-2(x-3)+\frac{3(2x+1)}{4}=5\frac{1}{3}+\frac{9x+4}{12}.$

36. $\frac{49}{4}-7(\frac{3}{4}-x)=10(x+3)-2.$

37. $\frac{5x-4}{7}-\frac{x-1}{1\frac{2}{3}}=\frac{x-3}{2\frac{9}{13}}-\frac{3x-8}{7}.$

38. $\frac{4x}{3}-5x+8(x+\frac{1}{2})=4x+3\frac{1}{3}.$

39. $1\frac{1}{3}-\frac{1}{3}(3x-2)=\frac{1}{2}(2-x).$

40. $\frac{x}{6}-\frac{5}{3}=\frac{6x-2}{5}-\frac{x+8}{3}.$

41. $\frac{1}{5}(x-5)+\frac{1}{3}(x-3)=\frac{1}{12}(5x-3).$

Solve the equations :

42. $\frac{x-7}{5}-\frac{x-11}{6}+\frac{x-10}{7}=2.$

43. $\frac{1}{3}(5x+1)+\frac{1}{7}(x+3)=x.$

44. $\frac{3x+5}{8}+5x-39=\frac{21+x}{3}.$

45. $\frac{5x-3}{7}-\frac{8-x}{3}=\frac{7x}{2}-\frac{4}{5}(4x+2).$

46. $\frac{x+4}{5}-\frac{x-3}{4}=2\frac{2}{5}-\frac{x+2}{5}.$

47. $\frac{1}{5}(3x-\frac{1}{2})-\frac{3}{4}\left(\frac{x}{5}-\frac{1}{3}\right)=\frac{3}{20}(2x+3).$

48. $\frac{1}{3}(x-\frac{5}{2})-\frac{3}{5}(x+\frac{4}{3})+\frac{7}{2}=0.$

49. $\frac{3x+1}{3}+\frac{2x+1}{5}=1.$

50. $19-3(14x-31)=4\left(5\frac{1}{4}-\frac{35x}{12}\right).$

51. $\frac{x+3}{4}-\frac{x+4}{5}=\frac{x+5}{6}-\frac{x+6}{7}.$

52. $\frac{x+7}{3}-\frac{3x}{5}=x-2-\frac{1}{2}(3x-11).$

53. $\frac{1}{7}(x+2)-\frac{1}{6}(x-6)=3\frac{1}{3}-\frac{5}{21}(x-4).$

54. $75-\frac{2}{3}(2x-7)=5x+\frac{x-4}{10}-\frac{3x-2}{4}.$

55. $\frac{2x+7}{7}-\frac{9x-8}{11}-\frac{x-11}{2}=0.$

56. $\frac{2}{5}(x-1)+\frac{2x}{7}-\frac{x-7}{14}=\frac{x-1}{5}+13.$

57. $\cdot7x+\cdot5=\cdot5x+1\cdot1.$

58. $1\cdot4+\cdot3x=\cdot5x-1\cdot7.$

59. $\cdot09x-\cdot01x=\cdot14-\cdot06x.$

60. $\cdot03x+\cdot02=\cdot17-\cdot07x.$

61. $\cdot004x+\cdot412=\cdot007x-\cdot008.$

62. $\frac{x}{\cdot5}-\frac{x}{\cdot75}=\cdot46.$

63. $\frac{x}{\cdot125}=\frac{x}{\cdot75}+20.$

64. $\frac{x-1}{\cdot25}-\frac{x-2}{\cdot125}=4\cdot2.$

65. $\frac{2x-3}{2\cdot5}=\frac{3x-4}{12\cdot5}+\cdot24.$

66. $\frac{\cdot25x-\cdot025}{\cdot125}=\frac{2x-\cdot45}{1\cdot25}+\cdot6.$

67. What value of x will make $(5-3x)(7-2x)$ equal to $(11-6x)(3-x)$?

68. What value of x will make $\frac{1}{x}+\frac{1}{2x}-\frac{3}{4x}-\frac{5}{12}$ equal to the fraction $\frac{7}{24}$?

69. Under what circumstances is

$$(x+3)(x+4) \text{ equal to } (x+5)(x+7)\ ?$$

70. Simplify the expression $(x-2)^2-(x-3)(x-1)$.
What do you deduce about the equation $(x-2)^2-(x-3)(x-1)=0$?

71. Go through the process of solving the equation
$(2x-1)(3x-4)=(6x-5)(x-1)$. What do you deduce?

Approximate Solutions.

45. In finding approximate values,
One half, or more than one half, counts as unity,

i.e. ·5 .... ·5 ....,
·05 .... ·05 and < ·1 ....·1,
·005 .... ·005 and < ·01 ....·01, and so on.

Thus if $x = 3{\cdot}74526$........................,

$x = 3{\cdot}7$ correct to one dec. place,

$= 3{\cdot}75$ .... two dec. places,

$= 3{\cdot}745$ .... three ....,

$= 3{\cdot}7453$ ...... four ....

In solving the equation $7x = 25$,

dividing both sides by 7, $x = 3{\cdot}571428$.........

$\therefore x = 4$, to the nearest integer,

$= 3{\cdot}6$ correct to one dec. place,

$= 3{\cdot}57$ .... two ......places,

$= 3{\cdot}571$ .... three ... .

Thus, in approximations, if the first figure neglected is 5 or more than 5, increase by one the last figure retained.

Examples. VII. e.

Find approximate values of x in the following equations:

1. $10(x-1) - 6x - 26 = 3$, correct to the nearest integer.
2. $5(x-1) = 11(x-3)$, correct to one dec. place.
3. $3x^2 - 7 - 3x(x+3) = 0$, correct to two dec. places.
4. $(x-2)^2 = (x-5)^2 + 5$, correct to two dec. places.
5. $(x-3)(x+3) = (x-7)(x+7) + 7x$, correct to two dec. places.
6. $\frac{x}{7} = \frac{x}{3} - 5$, correct to the nearest integer.
7. $\frac{x-1}{6} + \frac{2x-1}{7} = \frac{35}{42}$, correct to two dec. places.
8. $\frac{2x-1}{4} - \frac{x-1}{5} = 14$, correct to two dec. places.
9. $\frac{x-1}{2} + \frac{x-2}{3} - \frac{x-8}{4} = 0$, correct to two dec. places.
10. $\frac{1}{7}(3x+5) - \frac{1}{3}(2x+7) = \frac{3x}{5} - 2$, correct to two dec. places.
11. $4\frac{3}{4} - \frac{3}{4}(14x - 31) = 5 - \frac{35x}{12}$, correct to two dec. places.
12. $\frac{2x-3}{2{\cdot}5} = \frac{3x-4}{12{\cdot}5} + {\cdot}262$ correct to two dec. places.

CHAPTER VIII

SYMBOLICAL EXPRESSION

46. Algebra is largely used for solving problems of various kinds, but before attempting this the beginner must learn how to express given statements symbolically, *i.e.* in algebraic form.

Let us take a few simple cases.

There are (3×4) ft. in 4 yards.

Thus we see that there are $3x$ ft. in x yds.

There are (20×5) shillings in £5.

Hence there are $20x$ shillings in £x.

There are (12×7) pence in 7 shillings.

There are $12x$ pence in x shillings.

Just as 2×6 is a number which is double of 6, so $2a$ represents a number which is double the number represented by a.

The number which is 3 greater than 6 is $6+3$.

The number which is 3 greater than x is $x+3$.

The number which is a greater than x is $x+a$.

7 buns at 2 pence each, cost 7×2 pence.

Hence x buns at 2 pence each cost $(x \times 2)$ pence, *i.e.* $2x$ pence.

$$235 \text{ shillings} = (235 \div 20)\text{£};$$

$$\therefore\ x \text{ shillings} = (x \div 20)\text{£}$$

$$= \frac{x}{20}\text{£}.$$

14 pounds and 6 shillings are the same as $(14 \times 20 + 6)$ shillings.
In the same way x pounds $+ y$ shillings $= (20x + y)$ shillings.

6 pounds $+ 5$ shillings $+ 4$ pence $= (6 \times 240 + 5 \times 12 + 4)$ pence.
$\therefore$ x pounds $+ y$ shillings $+ z$ pence $= (240x + 12y + z)$ pence.

If 13 articles cost 54 shillings, each article costs $\frac{54}{13}$ shillings.

If x .... 54 .... $\frac{54}{x}$ ....

... x .... y .... $\frac{y}{x}$ ....

An even number is a number which has 2 for a factor.

$\therefore$ if x is any whole number,
$2x$ is an even number.
$\therefore$ if x is any whole number,
$2x + 1$ is an odd number.
$2x - 1$ is also an odd number.

47. Example 1. What is the cost of a articles at b shillings each? 12 articles at 3 shillings each cost 12×3 shillings.

$\therefore$ by analogy, a articles at b shillings each cost ab shillings.

Example 2. A man walks x miles an hour. How far does he walk in y hours? If he walks 4 miles an hour, he will walk 4×6 miles in 6 hours.
$\therefore$ by analogy, if he walks x miles an hour, he will walk xy miles in y hours.

Example 3. A man has x crowns and y florins, how many shillings has he? x crowns $= 5x$ shillings, and y florins $= 2y$ shillings,

$\therefore$ he has $(5x + 2y)$ shillings.

Example 4. If I spend x shillings out of £y, how many pence have I left? £$y = 240y$ pence, and x shillings $= 12x$ pence,

$\therefore$ I have $(240y - 12x)$ pence left.

Examples. VIII. a.

1. One part of x is 20: what is the other part?
2. One part of 35 is y: what is the other part?

3. What number is less than x by 20 ?

4. What number is less than 34 by x ?

5. What number multiplied by x will give 56 ?

6. What number divided by x will give 35 ?

7. If 16 is less than x by 5, what is the value of x ?

8. The sum of two numbers is x, and one of them is 23 : what is the other ?

9. The sum of two numbers is y, and one of them is x : what is the other ?

10. The difference of two numbers is 13, and x is the greater : what is the other ?

11. How many times is x contained in 78 ?

12. How many times is y contained in x ?

13. How many times is $3a$ contained in $5b$?

14. I have £x and give away y shillings : how many shillings have I left ?

15. The sum of three numbers is 96. One of them is x, another y : what is the third ?

16. The sum of two numbers is $a+5b$, and one of them is $3b$: what is the other ?

17. The difference of two numbers is $x-y$, and the greater is y : what is the other ?

18. If a book costs x pence, how many can be bought for y pence ?

19. If a penknife costs x pence, how many can be bought for y shillings ?

20. I gave x shillings for y pencils : how many pence did I give for each ?

21. If I spend x half-crowns out of a sum of £y, how many shillings have I left ?

22. What number exceeds x by 4 ?

23. What number exceeds 4 by x ?

24. By how much does 20 exceed x ?

25. What number is less than 40 by a ?

26. If 75 contains x three times, what is the value of x ?

27. If x oranges cost fourpence, what is the price of one ?

28. I am x years old now : how old shall I be in 7 years ?
How old shall I be in y years ?
How old was I 11 years ago ?

29. Find a number half as great again as x ?

30. If I walk x miles in 6 hours, how many do I walk in one hour ?
How many do I walk in y hours ?
How long do I take to walk one mile ?
How long do I take to walk y miles ?

31. The sum of two numbers is $a+b$; one of them is $a-b$; what is the other ?

32. I row x miles at the rate of y miles an hour : how many hours do I take to do it ?

33. What is the value of x eggs at 3 pence apiece ?

34. What is the value of x eggs at 3 pence a dozen ?

35. By how much does $x-5$ exceed $x-7$?

36. If eggs sell at x pence a dozen, how much does each egg cost ? How many will you get for a shilling ? How many will you get for y shillings ?

37. If 3 lbs. of sugar cost 8 pence, what will x lbs. cost ?

38. If x lbs. of sugar cost y pence, what will z lbs. cost ?

39. Write down three consecutive numbers of which n is the least.

40. Write down three consecutive numbers of which n is the greatest.

41. Write down three consecutive numbers of which n is the middle one.

42. The greatest of four consecutive numbers is $n+3$: what are the others ?

43. Write down five consecutive numbers of which the middle one is n.

44. What is the cost in pounds of x cakes at y shillings apiece ?

45. By how much does $3x-y$ exceed $x+y$?

46. What number added to $a-3b$ will make $a+b$?

47. A bill is made up of £a, b shillings, and c pence: what is the total number of pence in it ?

48. A train travels at the rate of x miles an hour: how many yards does it go in a minute ?

49. How far is it from A to B, if a man, bicycling at the rate of 10 miles an hour, does the journey in x hours ?

50. A horse eats x bushels a week. How many days will it take him to eat 76 bushels ? How many days will it take y horses to eat the same amount ?

51. What is the number which exceeds one-quarter of x by 25 ?

52. Write down five consecutive numbers of which $2n-3$ is the middle one.

53. Write down five consecutive odd numbers of which $2n-1$ is the middle one.

54. What is the area in square feet of a room a feet long and b feet wide ?

55. The area of a room is x square feet and its length is y feet: what is its width ?

56. A square has sides x feet long: what is its area ?

Express the following statements in the form of equations:

57. The excess of x over 20 is y.

58. Three times x exceeds y by 25.

59. The sixth part of $x-8$ is equal to the seventh part of $2x+3$.

60. Three times $x-4$ is equal to five times $x-1$.

61. There are x shillings in £y and z florins.

62. There are a pence in £b, c half-crowns, and d shillings.

63. The product of two consecutive numbers, of which x is the greater, is y.

64. The product of three consecutive numbers, of which x is the middle one, is a^2.

65. A is x years old, B is 5 years older. The sum of their ages is y.

66. A man is x years old, and his son y years younger. The sum of their ages is a years.

67. A has £x, and B £y. After B has given A £a, they have equal amounts.

68. When x is divided by y, the quotient is 15 and the remainder 7.

69. When a is divided by b, the quotient is x and the remainder y.

70. The area of a room x feet long, and y feet wide is a square feet.

71. The area of a courtyard, a feet by b feet, is x square yards.

72. The product of x and y is three times the excess of a over b.

73. The excess of x over y is five times the excess of a over b.

Substitution in formulae.

48. If r is the radius of a circle, and C its circumference, the two quantities r and C are connected by the formula

$$C = 2\pi r, \text{ where } \pi = \tfrac{22}{7}.$$

(This is only an approximate value of π.)

Thus if we know the radius of a circle, we can find its circumference.

Example 1. Find the circumference of a circle whose radius is 21 feet.

If C denote the circumference, substituting the given value of r in the formula $C = 2\pi r$,

$$\begin{aligned} C &= 2\pi \times 21 \text{ feet} \\ &= 2 \times \tfrac{22}{7} \times 21 \text{ feet, for } \pi = \tfrac{22}{7}, \\ &= 2 \times 22 \times 3 \text{ feet} \\ &= 6 \times 22 = 132 \text{ feet.} \end{aligned}$$

Example 2. Given that the circumference of a circle is 99 ft. in length, find its radius.

If r denote its radius, $\quad 2\pi r = 99$;

$$\begin{aligned} \therefore\ 2 \times \tfrac{22}{7} r &= 99, \\ r &= \frac{7 \times 99}{2 \times 22} = \frac{7 \times 9}{4} \text{ feet} \\ &= \tfrac{63}{4} = 15\tfrac{3}{4} \text{ feet} \\ &= 15 \text{ feet } 9 \text{ inches.} \end{aligned}$$

The area, A, of the floor of a room whose length is l, and breadth b, is given by the formula

$$A = l \times b.$$

Example 3. Find the area of a room $16\frac{1}{2}$ feet long and $10\frac{1}{2}$ feet wide.

If A denote the area, substituting in the above formula,

$$\begin{aligned} A &= 16\tfrac{1}{2} \times 10\tfrac{1}{2} \text{ sq. ft.} \\ &= \tfrac{33}{2} \times \tfrac{21}{2} = \tfrac{99 \times 7}{4} = \tfrac{693}{4} \text{ (multiplying by factors)} \\ &= 173\tfrac{1}{4} \text{ sq. ft.} \end{aligned}$$

Example 4. Find, **to the nearest inch,** the length of the circumference of a circle of radius 6 inches.

Let C denote the circumference in inches.

Substituting the values of π and r in the formula

$$C = 2\pi r,$$
$$C = 2 \times \tfrac{22}{7} \times 6 \text{ inches}$$
$$= \tfrac{44}{7} \times 6 \text{ inches}$$
$$= \tfrac{264}{7} \text{ inches}$$
$$= 37{\cdot}7\ldots \text{ inches}$$
$$= 38 \text{ in. (to the nearest inch).}$$

Example 5. Given that the area of a circle (A) and its radius (r) are connected by the formula $A = \pi r^2$ when $\pi = \frac{22}{7}$, find, **to the nearest tenth of a square inch,** the area of a circle of radius 3 inches.

If A sq. in. denote the reqd. area, substituting the values of π and r in the formula

$$A = \pi r^2,$$
$$A = \tfrac{22}{7} \times (3)^2 = \tfrac{22}{7} \times 9 = \tfrac{198}{7}$$
$$= 28{\cdot}28\ldots \text{ sq. inches}$$
$$= 28{\cdot}3 \text{ sq. in. (to the nearest tenth).}$$

Examples. VIII. b.

Given that the circumference (C) of a circle and its radius (r) are connected by the formula $C = 2\pi r$, where $\pi = \frac{22}{7}$, find:

1. The circumference of a circle of radius 7 inches.
2. .... 9 inches.
3. The radius of a circle whose circumference is 110 feet long.
4. .... 12 feet long.
5. The circumference (correct to a tenth of an inch) of a circle whose radius is 5 in. long.
6. The radius (correct to a tenth of an inch) of a circle whose circumference is 16 inches long.
7. The radius (correct to a tenth of an inch) of a circle whose circumference is 20 inches long.

The area (A) of a circle is connected with its radius (r) by the formula

$$A = \pi r^2, \text{ where } \pi = \tfrac{22}{7}.$$

8. Find the area (correct to a tenth of a square inch) of a circle whose radius is 4 inches.
9. Find the radius of a circle whose area is 154 sq. inches.

The area (A) of a room is connected with its length (l) and its breadth (b) by the formula

$$A = lb.$$

10. Find the area of a room $15\frac{1}{2}$ ft. long and 12 ft. wide.
11. Find, to the nearest foot, the length of a room whose area is 246 sq. ft. and width 11 ft.

12. Find, to the nearest inch, the length of a room whose area is 112 sq. feet and width 9 feet.

If A is the area of the walls of a room, l its length, b its breadth, h its height,

$$A = 2h(l + b).$$

13. Find the area of the walls of a room, 10 ft. high, 16 ft. long, and 12 ft. wide.

14. The area of the walls of a room is 750 sq. ft.; its length is 18 ft. and its breadth 12 feet: find its height.

15. The area of the walls of a room is 650 sq. ft.; its length is 18 ft. and its breadth 12 ft.: find its height.

The volume (V) of a cylinder on a circular base of radius r, and of height h, is given by the formula

$$V = \pi r^2 h, \text{ where } \pi = \tfrac{22}{7}.$$

16. Find the volume of a cylinder of height 7 feet on a circular base of radius 3 feet.

17. The volume of a cylinder on a circular base of radius 7 ft. is 693 cubic feet: find its height.

The area, A, of a triangle of height h, on a base b, is given by the formula

$$A = \tfrac{1}{2}hb.$$

18. Find the area of a triangle of height 3 feet and base 2 ft. 3 in.

19. A triangle of area 36 sq. ft. stands on a base of 10 ft.: find its height to the nearest inch.

If a body falls freely under the acceleration, g, of gravity for t seconds, the space (in feet) it falls through is given by the formula

$$S = \tfrac{1}{2}gt^2, \text{ where } g = 32.$$

20. Find the space a body under the acceleration of gravity falls through in 6 secs.

21. Find how long a body under the acceleration of gravity takes to fall through 144 feet.

If a body, starting with a velocity of u feet per second, and moving under an acceleration f, acquires a velocity of v ft. per second in t seconds, v is given by the formula

$$v = u + ft.$$

22. Find the velocity of a body in 7 seconds if it starts with a velocity of 3 ft. per second and moves under an acceleration 4.

a, $a+b$, $a+2b$, $a+3b$ ... being a series of numbers, the value, p, of the n^{th} is given by the formula

$$p = a + (n-1)b.$$

23. Find the twenty-first number of the following series:

$$1,\ 3,\ 5,\ 7 \ldots.$$

24. Find the twenty-fifth term of the series:

$$-4,\ -1,\ 2,\ 5,\ 8 \ldots.$$

If in a series of numbers the numbers increase by regular intervals, their sum is given by the formula

$$S = \frac{n}{2}(a + l),$$

where S denotes the sum, n the number of terms, a the first term, and l the last term of the series.

25. Find the sum of the first 25 natural numbers.

26. Find the sum of the consecutive numbers from 9 to 31 inclusive.

Find the sum of the series:

27. 9, 12, 15, 18 ... to 11 terms.

28. 6, 10, 14, 18 ... to 12 terms.

29. 97, 94, 91 ... 37, 34, 31, 28.

Find the sum of:

30. The first 43 even numbers.

31. The first 21 odd numbers.

32. All the even numbers between 5 and 51.

33. All the odd .... 40 and 90.

34. The first 17 numbers each of which is divisible by 4.

35. .... 21 .... 3.

The sum (S) of the squares of the first n natural numbers is given by the formula

$$S = \frac{n(n+1)(2n+1)}{6}.$$

Find the sum of:

36. The squares of the first 15 natural numbers.

37. The squares of all numbers from 7 to 21 inclusive.

38. The squares of all numbers between 12 and 35.

The volume (v) of a sphere of radius r, is given by the formula

$$v = \tfrac{4}{3}\pi r^3, \text{ where } \pi = \tfrac{22}{7}.$$

39. Find, correct to two decimal places, the volume in cubic feet of a sphere of radius 3 feet.

40. The volume of a sphere is 4851 cubic feet: find its radius.

41. A clerk starting with a salary of 100£, has a salary of 105£ in his second year, 110£ in his third year, 115£ in his fourth year, and so on. By means of the formula in Example 23, find his salary in his twenty-first year of service.

If when A is divided by B, Q is the quotient and R the remainder,

$$A = BQ + R.$$

42. A certain number when divided by 22 has a quotient 15 and a remainder 4: find the number.

If two sides of a triangle, of lengths a and b, contain a right angle, the third side c is obtained from the formula $c^2 = a^2 + b^2$.

[*N.B.*—The above may be written, $c^2 - a^2 = b^2$, or $c^2 - b^2 = a^2$.]

Which of the triangles whose sides are of the following lengths will be right-angled?

43. 3, 4, 5 feet. 44. 13, 12, 6 inches. 45. 25, 24, 7 centimetres.

46. 1·5, 2, 2·5 yards. 47. 1·3, 1·2, ·7 feet. 48. $30a$, $24a$, $18a$.

FUNCTIONAL NOTATION

When we speak of a **function of x** we mean an expression containing x or powers of x. It may also contain constants and various symbols of operation.

It is called an *algebraic* function if these symbols are only those of the algebraic operations, addition, subtraction, multiplication, division and extraction of a root.

A function of x may be denoted by $f(x)$, $F(x)$, $\phi(x)$ or a similar form. $2x^2+3x+7$ is a function of x: so we might write

$$f(x)=2x^2+3x+7;$$

and $f(4)$ would here mean the value of $2x^2+3x+7$ when 4 was substituted for x.

Thus $\quad f(4)=2\times 4^2+3\times 4+7=51.$

$$f(1)=2\times 1^2+3\times 1+7=12.$$

$$f(-1)=2\times(-1)^2+3\times(-1)+7=2-3+7=6.$$

$$f(0)=7, \text{ for } 2\times 0^2=0 \text{ and } 3\times 0=0.$$

Sometimes a function of x is denoted by another letter, usually the letter y.

Thus, in the above case, we might write $y=2x^2+3x+7$.

In such a case the student must be careful to express clearly what is meant when he uses different values of x.

$$y=2\times 4^2+3\times 4+7 \text{ would not be sufficient.}$$

Write $y=2\times 4^2+3\times 4+7$ **when $\mathbf{x}=4$, to make it quite clear.**

Examples. VIII. b_1.

1. If $f(x)=2x+3$, find the value of

(i) $f(5)$; (ii) $f(1)$; (iii) $f(-1)$; (iv) $f(0)$.

2. If $f(x)=5x+7$, find the value of $f(1)$ and $f(2)$.

3. If $f(x)=a+bx$, what does $f(1)\times f(-1)$ become ?

4. If $f(x)=x^2-4x+3$, find the value of $f(1)+f(2)+f(3)$.

5. If $f(x)=x+\frac{1}{x}$, prove that $f(2)=f\left(\frac{1}{2}\right)$.

6. If $f(x)=x^3-2x$, prove that $f(x)+f(-x)=0$.

7. If $f(x)=3x-4$ and $\phi(x)=5x+7$, find the value of

(i) $f(1)+\phi(1)$, (ii) $f(2)+\phi(3)$.

8. If two sides of a rectangle are $3x+5$ and $3x-5$ respectively, and $f(x)$ denotes its area, express $f(x)$ in its simplest form, and find the value of $f(10)$.

9. If $f(x)=12x-3$, for what value of x is $f(x)$ equal to 33 ?

10. If $f(x)=3x+9$, find the value of x which makes $f(x)$ equal to -2.

Examples. VIII. c.

1. If $f(x)=x^2+x+1$, find the value of

(i) $f(0)$, (ii) $f(1)$, (iii) $f(2)$.

2. If $f(n)=\frac{n\,.\,\overline{n+1}}{2}$, find the value of

(i) $f(5)$, (ii) $f(7)$, (iii) $f(-3)$, (iv) $f(n+1)$, (v) $f(n-3)$.

3. If $\phi(x)=(x-1)(x-2)(x-3)$, find the value of

(i) $\phi(0)$, (ii) $\phi(1)$, (iii) $\phi(3)$, (iv) $\phi(5)$, (v) $\phi(-2)$.

4. If $\phi(n)=(2n-1)(2n+1)-(n-1)$, find the value of

(i) $\phi(0)$, (ii) $\phi(3)$, (iii) $\phi(2n)$,
(iv) $\phi(2n+1)$, (v) $\phi(n+1)$, (vi) $\phi(\frac{1}{2})$.

5. If $f(x)=2x^2-5x+3$, prove that

(i) $f(x+1)+f(x-1)-2f(x)=4$.
(ii) $f(x+2)+f(x-2)-2f(x)=16$.

6. If $f(x)=2x^2-6x+5$ and $\phi(x)=2x^2-6x+7$, find the value of

(i) $\phi(0)-f(0)$, (ii) $\phi(2)-f(2)$, (iii) $\phi(4)-f(2)$.

7. If $f(x)=2x^2+x$ and $\phi(x)=x^2+2x$, find the value of

$$f(x+1)-\phi(x-1).$$

8. If $f(x)=ax^2+bx+c$, and $\phi(x)=ax^2-bx+c$, find the value of

$$f(x+1)-\phi(x+1).$$

9. If $f(x)=ax^2+bx+c$, and $\phi(x)=a-bx+cx^2$, find the value of

(i) $f(0)-\phi(0)$, (ii) $f(1)-\phi(1)$,

(iii) $f(2)-\phi(2)$, (iv) $f(3)-\phi(2)$.

10. If $\phi(x)=x^3+3x^2+3x+1$, find the value of $\phi(x-1)$ in its simplest form.

11. A man walked for $3x$ hours at the rate of x miles an hour; then he walked back towards his starting-point for 2 hours at $x+1$ miles per hour, and then for 1 hour at 4 miles an hour in his original direction. Express as a function of x (i) his final distance from the starting-point, (ii) the total distance travelled.

12. If $\phi(t)$ denote the distance in feet fallen by a body in the first t seconds of its fall, what will denote the distance fallen in the third second? Find also the numerical result if $\phi(t)=16t^2$.

CHAPTER IX

EASY PROBLEMS

49. We will now proceed to solve some easy problems:

Example 1. Three times a certain number diminished by 15 comes to 45: find the number.

Let x be the number required.

Three times the number diminished by 15 is $3x-15$,

$$\therefore\ 3x-15=45;$$

$$\therefore\ 3x=45+15=60;$$

$$\therefore\ x=20,$$

i.e. the required number is 20.

Verification. $3\times20-15=60-15=45.$

Example 2. A man is twice as old as his son, and ten years ago he was three times as old. Find the present ages of the father and son.

Let x be the present age of the son.

Then, by hypothesis, the present age of the father is $2x$ years.

10 years ago the son was $x-10$ years old.

Also 10 years ago the father was $2x-10$ years old.

$$\begin{aligned} \therefore\ 2x-10 &= 3(x-10), \\ 2x-10 &= 3x-30, \\ 2x-3x &= -30+10, \\ -x &= -20, \\ \therefore\ x &= 20. \end{aligned}$$

$\therefore$ the father is now 40, and the son 20 years old.

The student should verify the result.

Example 3. A man paid a bill of £6. 10*s*. in sovereigns and florins. If he used three times as many florins as sovereigns, find the number of sovereigns he paid away and the number of florins.

Let x be the number of sovereigns he used.

Then $3x$ is the number of florins he used.

x sovereigns $=20x$ shillings, and $3x$ florins $=6x$ shillings.

Also £6. 10*s*. $=130$ shillings,

$$\begin{aligned} \therefore\ 20x+6x &= 130, \\ 26x &= 130, \\ x &= 5, \end{aligned}$$

i.e. he used 5 sovereigns and 15 florins.

Example 4. The number 55 is divided into two parts such that one-third of one part, together with one-fifth of the other part, is equal to 17. Find the parts.

Let x be one part. Then $55-x$ is the other part.

$$\therefore\ \frac{x}{3}+\frac{55-x}{5}=17.$$

Multiply both sides by 15,

$$\begin{aligned} 5x+3(55-x) &= 17\times 15, \\ 5x+165-3x &= 255, \\ 5x-3x &= 255-165, \\ 2x &= 90; \\ \therefore\ x &= 45; \end{aligned}$$

and $55-x=55-45=10$.

$\therefore$ 45 and 10 are the reqd. parts.

Example 5. A and B travel in opposite directions from two places 54 miles apart, and meet in 6 hours. If A goes twice as fast as B, find their rates of travelling.

Suppose B travels x miles an hour, then A travels $2x$ miles an hour.

In 6 hours, B goes $6x$ miles.

.... A goes $12x$ miles.

But the total distance travelled by A and B in 6 hours is 54 miles.

$$\begin{aligned} \therefore\ 6x+12x &= 54, \\ x &= 3, \end{aligned}$$

i.e. A travels 6 miles an hour, and B 3 miles an hour.

Examples. IX. a.

1. One man has £x, another man £$2x$, and they together have £30. How much has each man ?

2. A boy has a certain number of apples, and when he is given 20 more he finds he has three times as many as at first: how many had he at first ?

3. A certain number when trebled is 54 more than before: what is the number ?

4. A has a certain sum of money, and B has £10 more than A. They together have £40: how much has each ?

5. To three times a certain number of apples I add 17, and then find I have 77. How many apples had I at first ?

6. From four times a certain number I take 23, and obtain 61 as the result: what was the original number ?

7. A man walked a certain number of miles, and then bicycled for three hours at 10 miles an hour. He finds he has altogether travelled four times as far as he walked: how many miles did he walk ?

8. A man has a certain number of shillings, and an equal number of sovereigns. His total sum of money is 63 shillings. How many sovereigns has he ?

9. A man has a certain number of half-crowns, and double that number of florins. If his total sum of money amounts to £3. 18*s*., how many half-crowns has he ?

10. A man is 28 years older than his son, and the sum of the ages of father and son is 48. Find their ages.

11. Find the number which exceeds its sixth part by 30.

12. A man has five children, each three years older than the next one, and their united ages amount to 70. Find the age of the eldest.

13. Three persons A, B, C together have £144. B has £10 more than A, and C £10 less than A. How much has each ?

14. Two numbers differ by 18, and their sum is 42. Find them.

15. Find the number which exceeds its fourth part by 15.

16. Find a number such that its third part exceeds 24 by as much as 24 exceeds its fifth part.

17. Out of a cask of wine $\frac{4}{5}$ full, 10 gallons are drawn, and the cask is then $\frac{2}{3}$ full. How much can it hold ?

18. Find the three consecutive numbers whose sum is 96.

19. Ten times a certain number exceeds 24 by as much as 102 exceeds four times the number: find the number.

20. A man has a certain number of pennies, one half that number of shillings, and one-third that number of florins, his total sum of money amounting to 22*s*. 6*d*. How many of each coin has he ?

21. Two men have £49 between them. If one has six times as much as the other, how much has each ?

22. A has £3 less than B, and they together have £41. Find the share of each.

23. £500 is divided between A and B, so that A receives £172 more than B. Find their shares.

24. The sixth and seventh parts of a certain sum amount to £2. 12*s.*: what is the whole ?

25. A is 25 years older than B, and in five years he will be twice as old as B. Find their present ages.

26. A is 23 years older than B, and A's age is as much below 90 as B's age is above 13. Find their ages.

27. A is three times as old as B, and 9 years ago their united ages amounted to 66. Find their ages.

28. A is 6 times as old as B, and A's age 32 years ago is equal to B's age 28 years hence: find their ages.

29. Three boys A, B, C divide the apples on a tree. A takes one-third of the apples, B takes 21 and C the rest. If A has 2 more apples than C, how many apples were there on the tree ?

30. Find a number such that, if you divide it by 2 and add 11, the result will be three times as great as that which you would obtain by multiplying it by 2 and adding 11.

31. The half of a certain integer exceeds the third of the next greater integer by three: find the integer.

32. A man bought a house, and gained five-sixths of what he gave for it by selling it for £770. How much did he give for it ?

33. The sum of three consecutive numbers is 105: find them.

34. The sum of three consecutive odd numbers is 135. Find them.

35. A sheep costs twice as much as a turkey, and I spend £18. 1*s.* in buying 6 sheep and 7 turkeys. Find the price of each sheep and each turkey.

36. A man walks a certain distance, bicycles twice that distance, swims half as far as he walked, and finds he has covered 14 miles. How far did he swim ?

37. A and B divide a sum of £40 between them, so that A has £6. 10*s.* more than B. What is the share of each ?

38. Two persons have £4320 between them: if the first has five times as much as the second, how much has each ?

39. Divide £36 into two shares so that one-third of the less is equal to one-fifth of the greater.

40. The number 57 is divided into two parts, so that one-third of the first and one-seventh of the second are together equal to 11: what are the parts ?

41. In a village consisting of 151 persons, there are 17 more women than men, and 30 more children than women: how many men, women, and children are there ?

42. A man makes 304 runs in 15 innings at cricket: how many must he make in the next three innings to have an average of 20 ?

43. A, travelling half as fast again as B, and starting 9 miles behind him, catches him up in 6 hours: find their rates of travelling.

44. Two trains, one of which travels half as fast again as the other, start at the same time from two places 300 miles apart, and meet in 5 hours. Find their rates of travelling.

45. A and B run round a circular course of 1000 yards, starting from the same point, at the same time, and in the same direction. A, after running $2\frac{1}{2}$ times round the course in 10 minutes, just overtakes B: find B's rate of travelling.

46. A travels from P to Q, a distance of 30 miles, and back again at the rate of 9 miles an hour. On his way back, he meets B, who travels at the rate of 6 miles an hour, and who started at the same time from P. Find the distance of their meeting point from P.

47. A starts at noon to travel from P to Q at the rate of 6 miles an hour, and B starts at 1 p.m. to travel from Q to P at the rate of 5 miles an hour. If they meet at 4.30 p.m., find the distance from P to Q.

48. A man does one-third of a journey at the rate of 4 miles an hour, one-third at 5 miles an hour, and the remaining third at 6 miles an hour, completing the journey in 6 hours and 10 minutes. Find the length of the journey.

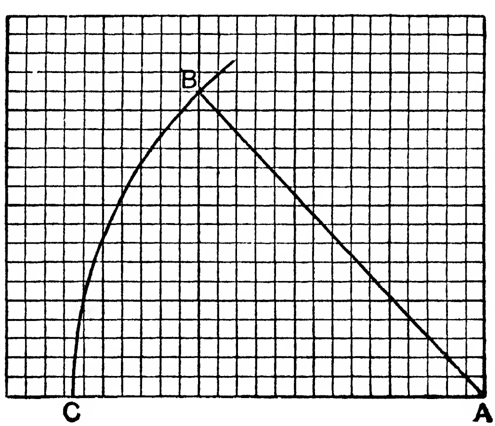
49. A man walks one-half of a journey at the rate of 4 miles an hour, bicycles one-third at 12 miles an hour, and rides the remainder on horseback at 9 miles an hour, completing the journey in 6 hours and 10 minutes. Find the length of the journey.

50. In a journey of 72 miles, a man does one-quarter of the distance at the rate of 6 miles an hour, one-third at the rate of 9 miles an hour, and does the whole journey in 7 hours and 40 minutes. What is his rate of travelling over the last part ?

USE OF SQUARED PAPER

[The most convenient paper for beginners is that ruled to show inches and tenths of an inch.]

50. *To find the length of a straight line joining the corners of any two squares, with the aid of a pair of compasses.*

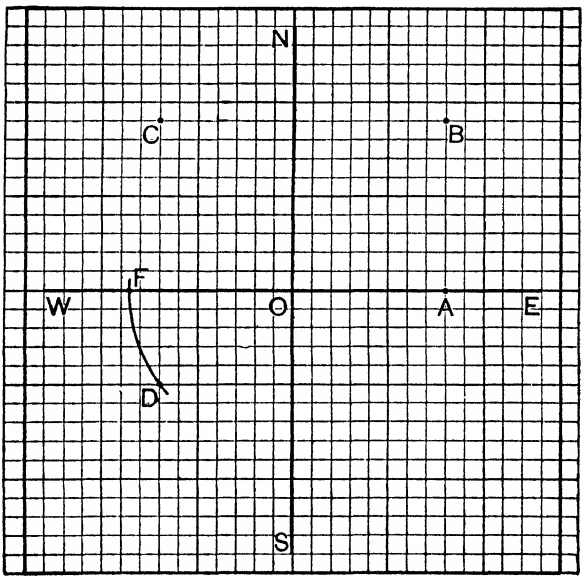


Take points A and B at corner of squares.

With centre A and radius AB describe an arc of a circle cutting the horizontal line through A at C. We see that the point C falls as nearly as possible at the middle point of a side of a small square.

Therefore, from the diagram AB = AC = 2·15 inches.

51. *A man travels* 8 *miles due east, then* 9 *miles north, then* 15 *miles west, and finally* 14 *miles south. Find to the nearest half-mile his distance at the finish from the starting point.*



Using a side of each square to represent one mile, with the accompanying diagrams, 8 m. east takes him from O to A,

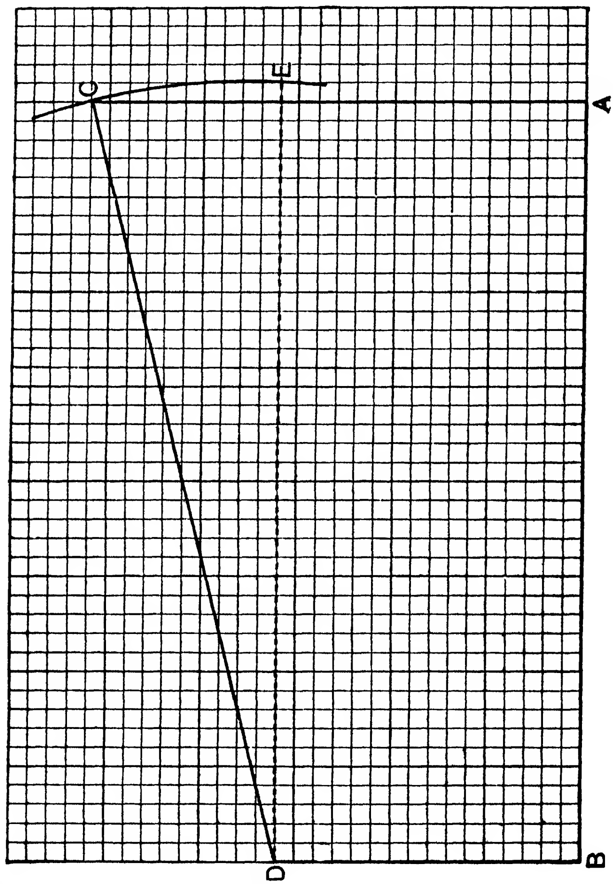
9 m. north .... A to B,

15 m. west .... B to C,

and 14 m. south .... C to D.

With centre O and radius OD describe a circle cutting the line OW at F. The reqd. distance = OD = OF = $8\frac{1}{2}$ miles to the nearest half-mile, from the diagram.

52. *Two vertical posts, 16 ft. and 26 ft. high, are 40 ft. apart. Find, to the nearest foot, the length of the straight wire joining their upper ends.*



Taking one-tenth of an inch to represent one foot, one inch will represent 10 feet.

Mark the points A and B 4 inches apart, also the point C 2·6 inches vertically above A, and the point D 1·6 inches vertically above B. Join CD.

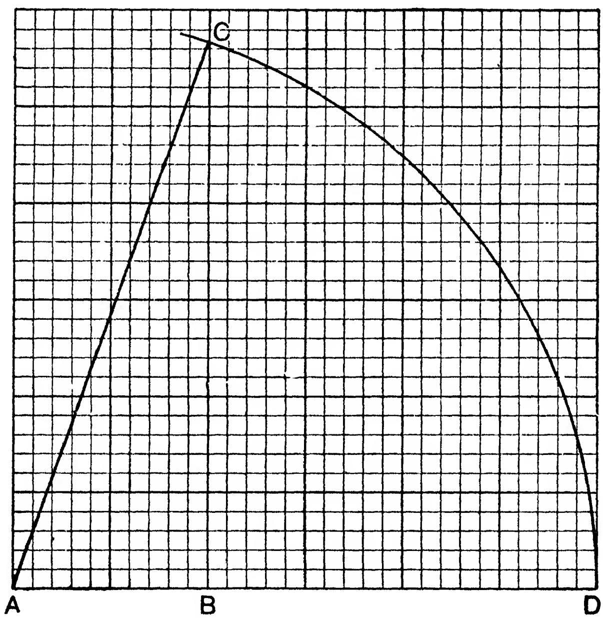
AB = 4 inches and therefore represents 40 feet.
AC = 2·6 inches .... 26 feet.
CD = 1·6 inches .... 16 feet.

Therefore CD represents the wire whose length is required. With centre D and radius DC, describe an arc of a circle to cut the horizontal line through D at E.

From the diagram we see that DE = 4·1 inches.

∴ DC = 4·1 inches, and the wire is $10 \times 4{\cdot}1$, *i.e.* 41 feet long.

53. *A ladder* 30 *ft. long has its foot at a distance of* 10 *feet from a vertical wall. How far up the wall does it reach?*



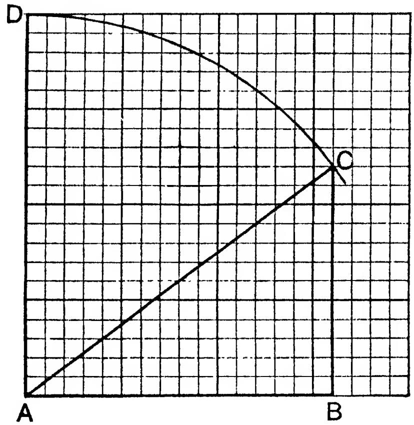
Let A be the foot of the ladder, and, taking a side of a square to represent one foot, take B 10 units in a horizontal line from A, so that B is the foot of the wall.

With centre A and radius 30 units describe a circle to cut the vertical through B at C.

AC represents 30 feet so that C is the point in the wall to which the ladder reaches.

From the diagram it is seen that BC the required distance $= 28 \cdot 3$ feet. Here we estimate the decimal of a foot by eye.

54. *Two sides of a triangle contain a right angle and are* 1·6, *and* 1·2 *feet long respectively: to find, by means of squared paper, the length of the third side.*



Taking an inch to represent a foot, AB 1·6 in. long represents the longer side, and BC at right angles to it and 1·2 in. long represents the shorter side. Join AC.

With centre A and radius AC, describe an arc of a circle cutting the vertical line through A at D.

$AC = AD = 2$ in. from the diagram.

$\therefore$ the side required is 2 feet long.

Those who are familiar with the proposition in geometry which proves that "the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on its sides" can readily verify the above as follows.

$$\begin{aligned} AC^2 - AB^2 = 2^2 - 1 \cdot 6^2 &= (2 + 1 \cdot 6)(2 - 1 \cdot 6) \\ &= 3 \cdot 6 \times \cdot 4 \\ &= 1 \cdot 44 = 1 \cdot 2^2 = BC^2 \\ \textit{i.e. } AC^2 &= AB^2 + BC^2. \end{aligned}$$

Examples. IX. b.

PROBLEMS INVOLVING THE USE OF SQUARED PAPER

1. A man travels 9 miles west, then 11 miles south, and finally 4 miles east: how far from the starting point, to the nearest mile, is he at the finish?

2. A man after travelling 7 miles due east, and a certain distance due north, finds himself 15 miles from his starting point. How far north did he travel?

3. A ship steaming at the rate of 8 miles an hour due east, drifts at the same time with a current at the rate of 3 miles an hour due north. Find its distance from its starting point in 2 hours.

4. A ship steaming at the rate of 10 miles an hour due west, and drifting due north with a current is found to be 32 miles from its starting point in 3 hours. Find the rate at which the current flows.

5. A balloon after sailing 5 miles horizontally from its starting point, is found to be at an altitude of 2 miles. Prove that it is approximately 5·4 miles from its starting point.

6. Two vertical posts, 6 ft. and 9 ft. high, are four feet apart: find the length of the straight line joining their upper ends.

7. A ladder with its foot at a horizontal distance of 20 ft. from a vertical wall, just reaches a point on the wall 30 ft. from the ground: find, to the nearest tenth of a foot, the length of the ladder.

8. A ball rolls 3 ft. east, then 5 ft. north, then 1 ft. west, and lastly 3 ft. in a direct line towards its starting point. How far is it then from its starting point?

9. A man walks 2 miles east, then 3 miles north-east: how far is he then from his starting point?

10. A man, having walked a certain distance in a north-westerly direction, finds that he is 25 miles west of his starting point: how far has he walked?

11. A boy bicycles 2·7 miles east, and then 3·4 miles north: how far is he then from his starting point, to the nearest half-mile?

12. A man swims in a north-easterly direction until he is 2 miles north of his original position, and then 3 miles to the north-west: how far is he then from his starting-point?

13. A room is 5·6 metres long, and 3·4 metres wide: find the distance between two opposite corners, as accurately as you can.

14. On a base of 3 inches, describe a triangle whose other sides are 4 inches and $4\frac{1}{2}$ inches long: find the altitude of the triangle to the nearest tenth of an inch.

15. Find, as accurately as you can, the length of the diagonal of a square whose sides are three inches long.

16. Find, as accurately as possible, the length of the diagonal of a rectangular board 2 ft. wide and 3 ft. long.

17. Find the altitude of an equilateral triangle whose sides are 3 inches long.

18. Draw two circles of $1\frac{1}{2}$ inches radius, with their centres 2 inches apart. Find the length of the line joining their points of intersection.

19. With centres 3 inches apart, draw two circles of radii 2 in. and $2\frac{1}{2}$ in. Find the length of the line joining their points of intersection.

20. A man walks due east from a town P which lies 4 miles due north of a town Q. How far from Q is he when he has walked 5 miles ?

21. A man walks south-east from a place P which lies 3 miles north of Q. How far from Q is he when he has walked 4 miles ?

22. Multiply 2·3 by 3·5 by means of squared paper.

23. Multiply 3·4 by 4·7 by means of squared paper.

24. The road from A to B is inclined upwards at 30° to the horizon for 2 miles, then at 20° for 2 miles, and then descends at an inclination of 27° to B, which is on the same level as A. Measure the length of the descent to B.

25. A travels east at 12 miles an hour, and B, starting at the same time from the same place, travels north-east at 20 miles an hour. Find, to the nearest mile, their distance apart at the end of 1, 2 and 3 hours. (Use one-tenth of an inch to represent one mile.)

26. A and B are two places 6 miles apart, B lying due east of A. One man walks at 2 miles an hour from A towards the north-east, another man, starting at the same time, walks north-west from B at 3 miles an hour. Find their distances apart to the nearest tenth of a mile in one hour. (Use one inch to represent one mile.)

27. A donkey tethered to a post can graze over a circle of 24 ft. radius. The shortest distance from the post to a straight hedge is 17 ft. Over what length of hedge can the donkey graze ?

28. A man walks 2·8 miles north, then 3·4 miles west, and then 1·6 miles south-east. How far is he then from his starting point ?

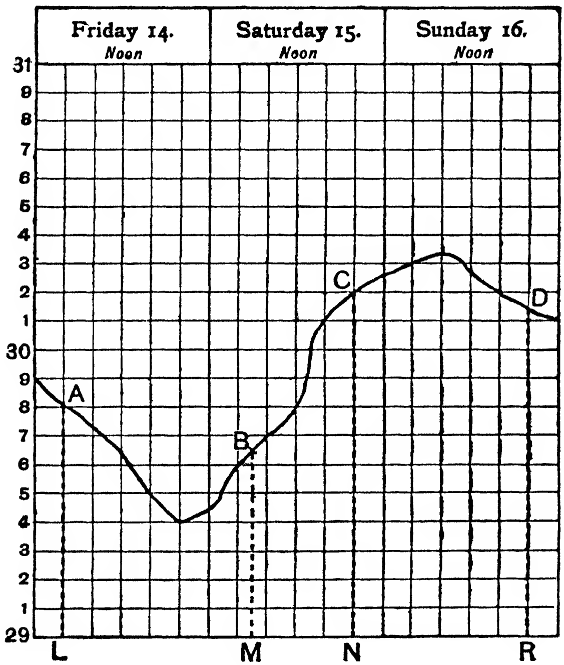
55. Exhibition of Statistics by means of Graphs. The accompanying diagram gives a portion of a barometric chart, from which we can read off the height of the barometer at any hour of the dates given.

We determine the height of the barometer from the vertical lines, and the date and hour from the horizontal lines.

Thus the height of the barometer at

4 a.m. on the	14th is given by	AL = 29·8 inches.	
6 a.m.	15th	BM = 29·65	......
8 p.m.	15th	CN = 30·2	......
8 p.m.	16th	DR = 30·15	......

August.

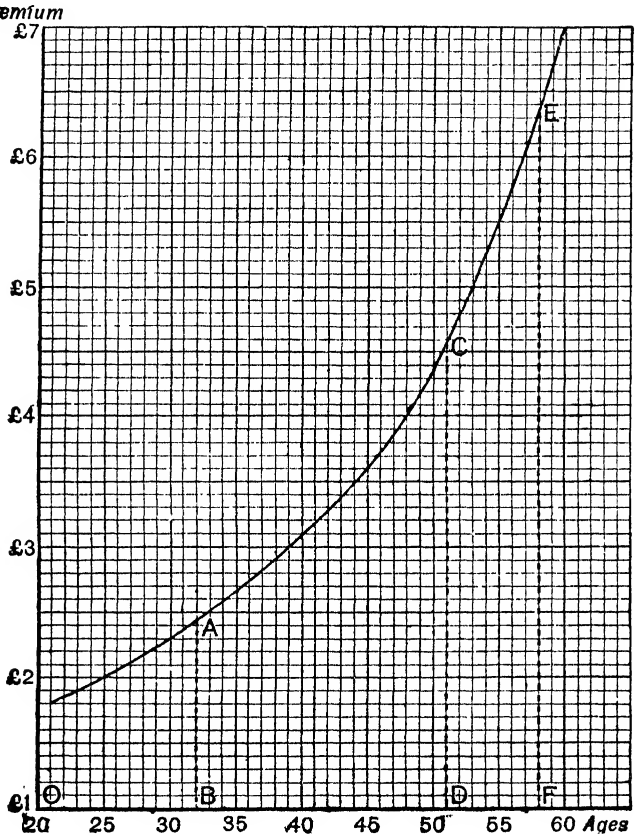


Also we see that the barometer was falling from midnight Thurs. 13th to 8 p.m. on Fri. 14th, and rising from 8 p.m. on the 14th to 8 a.m. on the 16th.

56. Construct a graph to exhibit the following:
Premiums of Life-insurance at various ages (for 100£).

Age in years.	21	25	30	35	40	45	50	55	60
Premium.	£1. 16*s*.	£2.	£2. 6*s*.	£2. 13*s*.	£3. 2*s*.	£3. 12*s*.	£4. 7*s*.	£5. 10*s*.	£7. 1*s*.

From the diagram estimate the premium at the ages of 32, 51, and 58.



Measuring the ages horizontally, the premiums vertically, we plot the given points as shown in the diagram, the point O denoting age 20, and premium 1£ (not premium 1£ at age 20).

The dotted lines AB, CD, EF give the premiums at the ages 32, 51, 58 respectively.

They are £2. 9*s.*, £4. 11*s.*, £6. 8*s.*

Examples. IX. c.

1. Construct a graph to show the following:
Premiums of Life-insurance at various ages (for 100£).

Age in years	20	25	30	35	40	45	50	55	60
Premium in £	2	2·2	2·5	2·8	3·2	3·8	4·6	5·5	6·9

Estimate the premium for £1000 insurance at ages 28 and 43 to the nearest £.

2. Population of England and Wales.

Year	1801	1811	1821	1831	1841	1851	1861	1871	1881	1891
Number in Millions	8·9	10·2	12·0	13·9	15·9	17·9	20·0	22·7	26·0	29·0

Draw a graph to exhibit the above. Estimate the population in 1837, and the year in which the population was 24 millions.

3. The temperature taken every two hours one day showed:

Midnight,	46·0°	2 p.m.,	66·7°
2 a.m.,	44·8°	4 p.m.,	67·5°
4 a.m.,	44·6°	6 p.m.,	58·5°
6 a.m.,	47·5°	8 p.m.,	54·6°
8 a.m.,	52·6°	10 p.m.,	51·4°
10 a.m.,	56·8°	Midnight,	50·6°
Noon,	61·0°		

Draw a curve to show the variation of temperature throughout the day and estimate the temperature at 3 p.m.

4. The following table shows a patient's temperature at the given times. Construct his temperature chart.

Mon.		Tues.		Wed.		Thurs.		Fri.		Sat.		Sun.	
a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.	a.m.	p.m.
99·4°	99·8°	100·6°	102·4°	101·1°	102·2°	100·4°	100·9°	100·2°	99·8°	98·7°	98·4°	98·2°	98·2°

5. Rainfall in 1903 at Greenwich.

	Inches.	Average of 50 years.		Inches.	Average of 50 years.
January,	2·12	1·99	July,	5·27	2·47
February,	1·36	1·48	August,	4·81	2·35
March,	2·22	1·46	September,	2·23	2·21
April,	1·84	1·66	October,	4·44	2·81
May,	1·95	2·00	November,	2·09	2·29
June,	6·07	2·02	December,	1·31	1·77

In the same figure and on the same scale construct a chart of the above, showing the actual rainfall in continuous lines, and the average rainfall in dotted lines.

6. If P ozs. is the weight required to stretch an elastic string until its length is x inches, show the following in a graph :

Length in inches	9	10	11	12	13	14
Weight in ozs.	0·9	1·2	1·5	1·8	2·1	2·4

Determine the weight necessary to stretch the string to a length of 16 inches.

7. The price on Jan. 1st (in pence) of silver per Troy ounce in London was as follows :

1890	1891	1892	1893	1894	1895	1896	1897	1898	1899
45	40	36	29	30	31	28	27	27	28

Exhibit the above in a graph.

8. Table giving the boiling-point of water in degrees Fahr. at different heights above sea-level.

Height above sea-level in feet.	0	1000	2000	3000	4000	5000	6000
Boiling-pt. deg. Fahr.	212°	210·1°	208·2°	206·3°	204·4°	202·5°	200·6°

Exhibit the above graphically and read off the height above sea-level where the boiling point is 203·5°, and the boiling point at a height of 3700 feet.

9. Table giving the height of the barometer at various heights above sea-level.

Height above sea-level in feet.	0	2000	4000	6000	8000	10000	12000
Height of barometer in inches.	30	27·8	25·7	23·8	22·1	20·5	19

Show the above in a graph, and from it read off the height of the barometer at an altitude of 3000 ft. and 6400 ft. Also the altitudes when the readings of the barometer are 20 in. and 24·4 in.

10.

Diameter of circle.	10	11	12	13	14	15
Corresponding area.	78·5	95·0	113·1	132·7	153·9	176·7

Show the above graphically, and deduce the areas of circles whose diameters are 11·7 in. and 14·4 ft.; also the diameter of the circle whose area is 136·8 sq. in.

CHAPTER X

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE IN TWO UNKNOWNS

57. Take the equation $3x-4y=12$.

$$3x=4y+12. \quad \therefore\ x=\frac{4y+12}{3}.$$

For every value we give to y, we get a corresponding value of x.

Thus, if $y=1$, $\quad x=\frac{4+12}{3}=\frac{16}{3}$,

if $y=2$, $\quad x=\frac{8+12}{3}=\frac{20}{3}$,

if $y=3$, $\quad x=\frac{12+12}{3}=8$,

if $y=-2$, $\quad x=\frac{-8+12}{3}=\frac{4}{3}$, and so on.

Hence we see that the equation $3x-4y=12$ has an infinite number of solutions, *i.e.* an infinite number of values of x and y can be found which will satisfy the equation.

But suppose we are given two equations,

$$3x-4y=12, \quad \text{.........(1)}$$
$$5x+2y=46. \quad \text{.........(2)}$$

We can now find values of x and y which will satisfy *both* equations.

From (1) $3x=4y+12$, $\therefore\ x=\frac{4y+12}{3}$.

" (2) $5x=46-2y$, $\therefore\ x=\frac{46-2y}{5}$.

Hence, if the value of x is the same in both equations,

$$\frac{4y+12}{3}=\frac{46-2y}{5}.$$

Multiplying both sides by 15, $5(4y+12)=3(46-2y)$,

$$20y+60=138-6y,$$
$$26y=78,$$
$$y=3.$$

Substituting this value of y in equation (1),

$$3x-4\times 3=12,$$
$$3x=24,$$
$$x=8.$$

Thus the values $x=8$, $y=3$, will satisfy both equations.

Verification. When $x=8$, and $y=3$,

$$3x-4y=3\times 8-4\times 3=12.$$

$\therefore$ equation (1) is satisfied.

Again, when $x=8$, and $y=3$, $5x+2y=5\times 8+2\times 3=46$.

$\therefore$ equation (2) is also satisfied. Q.E.D.

58. We notice in the above, that in order to find the value of y we first get rid of x.

This process of getting rid of an unknown quantity is called *elimination*.

We might have effected the above solution by eliminating y, and obtaining the value of x first. We should then obtain the value of y by substituting this value of x in one of the original equations.

Also we notice that having first found the value of y, we may substitute that value in either equation. It is advisable, of course, to choose the simpler equation for this substitution.

If we put $y=3$ in equation (2), we have

$$5x+2\times 3=46,$$
$$5x=46-6=40,$$
$$x=8, \text{ as before.}$$

Also we must observe that two simultaneous equations of the first degree have only one solution.

59. The following method of elimination is the most common.

Example 1. Solve the simultaneous equations,

$$3x+5y=29, \quad\ldots\ldots(1)$$
$$2x+7y=34. \quad\ldots\ldots(2)$$

Multiplying (1) by 7, $\quad 21x+35y=203,$

" (2) by 5, $\quad 10x+35y=170.$

(N.B.—*The coefficients of y in the two equations are now equal.*)

Subtracting, $\quad 11x=33,$

$$x=3.$$

Substituting this value of x in equation (1),

$$3\times 3+5y=29,$$
$$5y=29-9=20,$$
$$y=4.$$
$$\therefore \left.\begin{array}{l} x=3 \\ y=4 \end{array}\right\} \text{ is the reqd. solution.}$$

Verification. When $x=3$ and $y=4$, $3x+5y=3\times 3+5\times 4=29$.
.... $2x+7y=2\times 3+7\times 4=34$.

Example 2. Solve the simultaneous equations,

$$3x+2y=2,$$
$$5x-2y=-18.$$

(N.B.—The coefficients of y are equal *but of opposite sign.*)

Adding, $8x=-16,$

$$x=-2.$$

Substituting this value of x in either equation we obtain the value of y. This is left as an exercise for the student.

The work may often be shortened if the coefficients of x or y have common factors.

Example 3. Solve the simultaneous equations,

$$38x+17y=127, \quad \text{.................................(1)}$$
$$133x+71y=479. \quad \text{.................................(2)}$$

These equations may be written,

$$2\times 19x+17y=127,$$
$$7\times 19x+71y=479.$$

Multiplying (1) by 7, $266x+119y=889,$

Multiplying (2) by 2, $266x+142y=958.$

Subtracting, $-23y=-69,$

$$y=3.$$

Substituting this value of y in equation (1),

$$38x+51=127,$$
$$38x=76,$$
$$x=2.$$
$$\therefore \left.\begin{array}{l} x=2 \\ y=3 \end{array}\right\} \text{ is the reqd. solution.}$$

Examples. X. a.

Eliminate x from the following equations (1–6):

1. $x+y=4,\ x+3y=8.$
2. $3x-2y=14,\ 2x-5y=2.$
3. $y-x=5,\ 3y+x=7.$
4. $y=3-4x,\ 5x-4y=7.$
5. $y=3x+5,\ 2y+3x=9.$
6. $\frac{x}{3}+y=1,\ \frac{x}{5}+\frac{y}{2}=-4.$

Eliminate x from the following equations (7–10):

7. $2x+3y=7,\ 5x-y=9.$

8. $x-\dfrac{14y}{3}=\dfrac{2x+2y+1}{5},\ \dfrac{x-2y}{5}=2.$

9. $\dfrac{5}{x}-\dfrac{3}{y}=7,\ \dfrac{5}{x}+\dfrac{8}{y}=4.$

10. $\dfrac{3}{x}-\dfrac{2}{y}=9,\ \dfrac{4}{x}+\dfrac{3}{y}=11.$

11. If $x=3$ find the value of y when $3x+4y=17$.

12. If $x=5$ find the value of y when $7y-6x=5$.

13. If $y=-3$ find the value of x when $3x-7y=30$.

14. If $y=-2$ find the value of x when $\dfrac{x-2}{2}+\dfrac{y+10}{4}=3.$

15. If $x=\frac{1}{2}$ find the value of y when $6x-1+\dfrac{y-3}{4}=4.$

16. If $y=-\frac{1}{3}$ find the value of x when $\dfrac{6y+1}{3}+\dfrac{2x-3}{4}=\frac{1}{6}.$

Solve the equations:

17. $x+2y=12,\ x-3y=2.$

18. $3x-y=26,\ x-5y=4.$

19. $2x+y=5,\ x+3y=5.$

20. $3x+2y=7,\ 5x+y=7.$

21. $4x-y=10,\ 2x-y=4.$

22. $7x-3y=31,\ 9x-5y=41.$

23. $x+y+8=0,\ x-y=2.$

24. $x+y=3,\ x-y=1\frac{1}{2}.$

25. $x+y=4\frac{1}{2},\ x-y=4\frac{1}{2}.$

26. $x-10y=5,\ 2x+10y=40.$

27. $2x+3y=28,\ 3x+2y=27.$

28. $4x-3y=14,\ 3x-4y=0.$

29. $7x-3y=-6,\ x+5y=10.$

30. $5x-7y=20,\ 3x-2y=12.$

31. $15x+2y=27,\ 3x+7y=45.$

32. $7x-3y=41,\ 3x-y=17.$

33. $11x+13y=23,\ 13x+11y=25.$

34. $2x+3y=47,\ 4x-y=45.$

35. $5x+y=5,\ 7x-y=13.$

36. $5x-4y=8\frac{1}{3},\ 2x+3y=14.$

37. $4x-5y=2,\ x+10y=41.$

38. $4x+6y=11,\ 17x-5y=1.$

39. $4x+3=3y+2,\ 5x+4y=22.$

40. $2x-3y=5,\ 3x+2y=1.$

41. $4x+3y=43,\ 3x-2y=11.$

42. $5x-4y=x-y=-2.$

43. $8x-4y=9x-3y=6.$

44. $3x+2y=2x-y-56=0.$

45. $10y=7y-x=20.$

46. $5x-2y=7x+2y=x+y+11.$

60. If necessary first simplify the equations.

Example 1. Solve the equations.

$$\frac{x+y}{3}=2+2y, \quad\text{............(1)}\qquad \frac{2x-4y}{5}=4\tfrac{3}{5}-y \quad\text{..................(2)}$$

Multiplying (1) by 3, and simplifying,

$$x+y=6+6y,$$
$$x-5y=6. \quad\text{...(3)}$$

Multiplying (2) by 5, and simplifying,

$$2x-4y=23-5y,$$
$$2x+y=23. \quad\text{...(4)}$$

We now solve equations (3) and (4) in the usual manner.

Example 2. Solve the equations,

$$\frac{2}{x}-\frac{3}{y}=3, \quad \text{.......................................(1)}$$

$$\frac{5}{x}+\frac{6}{y}=48. \quad \text{.......................................(2)}$$

In such cases as this, it is advisable to solve first for $\frac{1}{x}$ and $\frac{1}{y}$.

Thus, multiplying (1) by 2, $\quad \frac{4}{x}-\frac{6}{y}=6.$

Adding this to (2), $\quad \frac{9}{x}=54.$

$$\therefore \frac{1}{x}=6,$$

$$x=\tfrac{1}{6}.$$

Substituting this value of x in (2), $5\times 6+\frac{6}{y}=48,$

$$\frac{6}{y}=48-30=18,$$

$$\frac{1}{y}=3,$$

$$y=\tfrac{1}{3}.$$

$$\therefore \left.\begin{array}{l} x=\tfrac{1}{6} \\ y=\tfrac{1}{3} \end{array}\right\} \text{ is the required solution.}$$

Examples. X. b.

Solve the equations:

1. $\frac{x}{3}-\frac{y}{4}=-1, \quad \frac{x}{2}+\frac{y}{5}=10.$

2. $\frac{x}{5}-\frac{y}{3}=0, \quad \frac{x}{4}-\frac{y}{2}=-1.$

3. $\frac{x}{6}+\frac{y}{16}=6, \quad \frac{y}{12}-\frac{x}{9}=2.$

4. $\frac{x}{8}+\frac{y}{5}=1, \quad \frac{x}{4}-\frac{y}{5}=14.$

5. $2y-\frac{x}{2}=22, \quad 3y+\frac{x}{5}=14.$

6. $\frac{x}{5}+\frac{y}{8}+9=0, \quad \frac{x}{4}+\frac{y}{10}+9=0.$

7. $3x-\frac{y-3}{5}=6, \quad 4y+\frac{x-2}{3}=12.$

8. $\frac{7x+2}{6}-(y-3)=4, \quad \frac{7y+3}{6}-(x+2)=-3.$

9. $\frac{x-y}{3}=\frac{2x+3y}{5}=-4.$

10. $\frac{x}{5}+\frac{y}{2}=14, \quad \frac{x}{9}-\frac{y}{5}=3.$

11. $\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4}, \quad \frac{2y+4}{3}-\frac{2x+y}{8}=\frac{x+13}{4}.$

12. $3x+\frac{7y}{2}=11y-\frac{2x}{5}+2=22.$

13. $\frac{x+y}{3}=2+2y, \quad \frac{2x-4y}{5}=\frac{23}{5}-y.$

14. $\frac{x+4}{7}-\frac{x-y-1}{4}=2x-4, \quad 2y-4-\frac{3x-2y}{3}=3x.$

15. $8(2x-3y)-(2x+3y)=1, \quad (2x-3y)+\tfrac{1}{7}(2x+3y)=2.$

Solve the equations :

16. $\frac{x+1}{y+2}=\frac{x+3}{2y+1}=2.$ 17. $\frac{5x+6}{10}-\frac{11y-5}{21}=11,\quad \frac{55y-12}{25}=\frac{7x}{5}-37.$

18. $\frac{1}{5}(3x-4y)=\frac{1}{2}(x-y-3),\quad \frac{1}{4}(x-y+7)=\frac{1}{9}(4x-3y).$

19. $\frac{x+1}{y}=7,\quad \frac{x}{1+y}=6.$ 20. $\frac{x-1}{3}-\frac{y+5}{12}=\frac{x+2}{60},\quad (x-1\frac{1}{2})(y-1\frac{1}{3})=xy-5.$

21. $\cdot 3x+4y=11,\quad \cdot 2x+3y=8.$ 22. $1\cdot 2x+\cdot 6y=\cdot 6,\quad \cdot 3x-\cdot 2y=\cdot 01.$

23. $\cdot 6x+\cdot 7y+3\cdot 95=0,\quad \frac{x}{\cdot 5}+\frac{y}{\cdot 7}+10=0.$

24. $\cdot 03x+\cdot 06y=\cdot 05,\quad \cdot 09y-\cdot 03x=\cdot 05.$

25. $2x+\cdot 4y=1\cdot 2,\quad 3\cdot 4x+\cdot 02y=\cdot 126.$

26. $\frac{x}{\cdot 2}+\frac{y}{\cdot 5}=12\cdot 3,\quad \frac{x}{\cdot 6}+\frac{y}{\cdot 8}=5\cdot 5.$

27. If $3x+5y=16$, and $2x-3y=17$, find the value of $x+y$.

28. If $3x+2y=8$, and $2x+3y=2$, find the values of $x+y$, and $x-y$.

29. If $7x+11y=2$, and $8x+13y=1$, find the value of $5x+8y$.

30. Given that $13x-11y=17$, and $11x-13y=7$, find the values of $x+y$ and $x-y$.

31. $\frac{1}{x}+\frac{1}{y}=5,\quad \frac{1}{x}-\frac{1}{y}=3.$ 32. $\frac{1}{x}+\frac{2}{y}=12,\quad \frac{1}{x}-\frac{2}{y}=4.$

33. $\frac{2}{x}+\frac{1}{y}=5,\quad \frac{1}{x}+\frac{3}{y}=5.$ 34. $\frac{2}{x}+\frac{3}{y}=28,\quad \frac{3}{x}+\frac{2}{y}=27.$

35. $\frac{7}{x}-\frac{3}{y}=41,\quad \frac{3}{x}-\frac{1}{y}=17.$ 36. $\frac{7}{x}-\frac{5}{y}=3,\quad \frac{2}{x}+\frac{25}{2y}=12.$

37. $\frac{12}{x}-\frac{8}{y}=2,\quad \frac{3}{x}+\frac{4}{y}=2.$ 38. $\frac{1}{x}+\frac{1}{y}=1,\quad \frac{1}{x}-\frac{1}{y}=9.$

39. $\frac{1}{4}\left(\frac{2}{x}-\frac{3}{y}\right)=3\frac{1}{4},\quad \frac{1}{3}\left(\frac{2}{x}+\frac{3}{y}\right)+1\frac{2}{3}=0.$

SIMULTANEOUS EQUATIONS WITH THREE UNKNOWN QUANTITIES

***61.** The method is similar to that for solving equations with two unknowns. Here however we shall need *three* equations.

Example 1. Solve the equations,

$$2x+3y-z=5, \qquad \text{.........(1)}$$
$$3x-4y+2z=1, \qquad \text{.........(2)}$$
$$4x-6y+5z=7. \qquad \text{.........(3)}$$

First let us eliminate z from equations (1) and (2).

Multiplying (1) by 2, $\quad 4x+6y-2z=10$

Adding (2), $\quad 3x-4y+2z=1$

$$7x+2y\qquad=11 \qquad \text{.........(4)}$$

Next eliminate z from equations (1) and (3).

$$\begin{array}{lr} \text{Multiplying (1) by 5,} & 10x+15y-5z=25 \\ \text{Adding (3),} & \underline{4x-\ \ 6y+5z=\ \ 7} \\ & 14x+\ 9y\qquad =32 \quad \ldots\ldots(5) \end{array}$$

Now let us solve equations (4) and (5).

$$\begin{array}{lr} \text{Multiplying (4) by 2,} & 14x+4y=22. \\ \text{Subtracting from (5),} & 5y=10, \\ & y=2. \end{array}$$

$$\begin{array}{lr} \text{Substituting this value of } y \text{ in (4),} & 7x+4=11, \\ & 7x=7, \\ & x=1. \end{array}$$

Substituting *for both x and y* in equation (1),

$$\begin{aligned} 2+6-z &= 5, \\ -z &= -3, \\ z &= 3. \end{aligned}$$

$$\therefore \left.\begin{aligned} x&=1 \\ y&=2 \\ z&=3 \end{aligned}\right\} \text{ is the reqd. solution.}$$

Example 2. Solve the equations $\frac{1}{x}+\frac{1}{y}=7$, ...(1)

$$\frac{2}{x}-\frac{3}{z}=-9, \quad \ldots\ldots(2)$$

$$\frac{3}{y}+\frac{4}{z}=32. \quad \ldots\ldots(3)$$

Here we shall first solve for $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$.

First eliminate $\frac{1}{z}$ from (2) and (3).

$$\begin{array}{lr} \text{Multiplying (2) by 4,} & \frac{8}{x}-\frac{12}{z}=-36 \\ \ldots\ldots\ldots\ldots\text{ (3) by 3,} & \underline{\frac{9}{y}+\frac{12}{z}=\ \ 96} \\ \text{Adding,} & \frac{8}{x}+\frac{9}{y}=\ \ 60 \\ \text{Multiplying (1) by 9,} & \underline{\frac{9}{x}+\frac{9}{y}=\ \ 63} \\ \text{By subtraction,} & \frac{1}{x}=3, \\ & x=\frac{1}{3}. \end{array}$$

Substituting for x in equation (1), $3+\frac{1}{y}=7$,

$$\frac{1}{y}=4,$$

$$y=\frac{1}{4}.$$

Substituting for y in equation (3),

$$3 \times 4 + \frac{4}{z} = 32,$$

$$3 + \frac{1}{z} = 8,$$

$$\frac{1}{z} = 5,$$

$$z = \tfrac{1}{5},$$

$$\therefore \left.\begin{aligned} x &= \tfrac{1}{3} \\ y &= \tfrac{1}{4} \\ z &= \tfrac{1}{5} \end{aligned}\right\} \text{ is the reqd. solution.}$$

Example 3. Solve the equations $\frac{x}{3} = \frac{y}{8} + 1 = \frac{z}{2} - 3,$

$$\frac{y}{2} - \frac{z}{5} = 2.$$

From the first equation $\frac{x}{3} = \frac{y}{8} + 1.$

Multiplying both sides by 24, $8x = 3y + 24,$

$$8x - 3y = 24. \quad \text{.................................(1)}$$

Also from the first equation $\frac{y}{8} + 1 = \frac{z}{2} - 3.$

Multiplying both sides by 8, $y + 8 = 4z - 24,$

$$y - 4z = -32. \quad \text{..............................(2)}$$

Multiplying both sides of $\frac{y}{2} - \frac{z}{5} = 2$ by 10,

$$5y - 2z = 20. \quad \text{..............................(3)}$$

Multiplying by 2, $10y - 4z = 40$

Subtracting (2), $y - 4z = -32$

$$\begin{aligned} 9y &= 72 \\ y &= 8. \end{aligned}$$

Substituting this value of y in equation (1),

$$\begin{aligned} 8x - 24 &= 24, \\ 8x &= 48, \\ x &= 6. \end{aligned}$$

Substituting for y in equation (2),

$$\begin{aligned} 8 - 4z &= -32, \\ -4z &= -40, \\ z &= 10. \end{aligned}$$

$$\therefore \left.\begin{aligned} x &= 6 \\ y &= 8 \\ z &= 10 \end{aligned}\right\} \text{ is the reqd. solution.}$$

*Examples. X. c.

Solve the following equations:

1. $3x+4y-z=19,$ $5x+2y+z=15,$ $2x+3y+2z=11.$

2. $x+2y+z=16,$ $x-2y+3z=12,$ $4x+2y+z=22.$

3. $5x-3y+4z=35,$ $x+3y-4z=-23,$ $2x-5y+6z=43.$

4. $x+y+z=12,$ $5x+6y-3z=2,$ $3x+4y-4z=-14.$

5. $3x-2y-z=1,$ $4x-3y+4z=-3,$ $2x+y-5z=-2.$

6. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1,$ $\frac{x}{3}+\frac{y}{4}-\frac{z}{2}=-8,$ $\frac{x}{4}-\frac{y}{2}+\frac{z}{3}=19.$

7. $x+y-z=2,$ $3x+y-z=8,$ $x-y+2z=-6.$

8. $x+y+z=18,$ $x-y+z=12,$ $x+y-z=6.$

9. $x-2y=10,$ $3y+4z=-26,$ $y-4z=18.$

10. $2x-y=12,$ $3x-4z=36,$ $x-z=11.$

11. $x+y+z=20,$ $8x+4y+2z=50,$ $27x+9y+3z=64.$

12. $\frac{x}{2}+\frac{y}{4}+\frac{z}{3}=24,$ $\frac{x}{4}+\frac{y}{3}+\frac{z}{2}=29,$ $\frac{x}{3}+\frac{y}{2}+\frac{z}{4}=25.$

13. $x-y=y-z=\frac{x+z}{6}=2.$

14. $x=\frac{3y-4z+26}{3}=\frac{34-2x-3y}{2}=2(z-y).$

15. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=9,$ $\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=3,$ $\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=1.$

16. $\frac{2}{x}+\frac{3}{y}=18,$ $\frac{2}{y}+\frac{3}{z}=23,$ $\frac{2}{z}+\frac{3}{x}=19.$

17. $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-5}{6},$ $x+y+z=33.$

18. $\frac{3x}{2}=\frac{4y}{3}=\frac{5z}{4},$ $x+2y-z=82.$

CHAPTER XI

BRACKETS

62. When two or more pairs of brackets occur within one another, the best plan is to remove the *outermost* first. After a little practice, several pairs may be removed in one step.

Example 1. Prove that $8a - \{3a + (2a - 5)\} = 3a + 5$.

(In removing the curly bracket we must look upon all the terms in the plain bracket as a single quantity.)

The given expression $= 8a - 3a - (2a - 5)$

$= 5a - 2a + 5$

$= 3a + 5$. Q.E.D.

Example 2. Simplify $3\{6x - 2(2x - 1)\}$.

[Every term inside the curly brackets must be multiplied by 3, and each term inside the plain brackets must be multiplied by 2 *as well.*]

The given expression $= 18x - 6(2x - 1)$

$= 18x - 12x + 6$

$= 6x + 6$.

Examples. XI. a.

Prove the following:

(Remove one pair of brackets at a time.)

1. $a - \{b - (c + d)\} = a - b + c + d$.
2. $6a - \{2a + (a - 5)\} = 3a + 5$.
3. $4a - \{3a - (2a - a)\} = 2a$.
4. $7x + \{2x - (3x - 4)\} = 6x + 4$.
5. $a - \{a - (a - a)\} = 0$.
6. $3 - \{4x - (2x + 4) + 1\} = 6 - 2x$.
7. $9x + \{3x - (4x - 2) + x\} = 9x + 2$.
8. $7 - \{4x + (2x - 3) + 7\} = 3 - 6x$.
9. $14 - \{12 - (2x - 6) - 9x\} = 11x - 4$.
10. $12x - \{3x - (7x - 9) + (2x - 3)\} = 14x - 6$.
11. $24 - \{5x - (2x + 5) - (3x - 7)\} = 22$.
12. $2\{x + 3(x - 2)\} = 8x - 12$.
13. $3\{7x - 2(3x - 4)\} = 3x + 24$.
14. $4\{3a - (a - 2a)\} = 16a$.
15. $2 - 3\{x - 2 - 5(x - 1)\} = 12x - 7$.
16. $6 - 2\{x - 3 - (x + 4) + 3(x - 2)\} = 32 - 6x$.
17. $7\{2 - 3(x - 4) + 4(x - 6)\} = 7x - 70$.
18. $6\{x - \frac{1}{2}(x - 1)\} = 3x + 3$.
19. $8\{2x - \frac{1}{4}(6x + 5)\} = 4x - 10$.
20. $6\{x - \frac{1}{3}(2x - 7) + \frac{1}{2}(x - 5)\} = 5x - 1$.

Simplify the following, removing both pairs of brackets in one step:

21. $3x + \{2x - (x + 2)\}$.
22. $6 - \{5 - (3 - x)\}$.
23. $2x - \{3x + (x - 2)\}$.
24. $6x + \{5 - (2x - 5)\}$.
25. $9 - \{-2 + (2x - 7)\}$.
26. $a - \{-b - (c - d)\}$.
27. $a + [2a - (7a - 1) - (9 - 8a)]$.
28. $6y - [3x - (2y - x) + (3y - 5x)]$.
29. $9a - [3b + (2a - 5b) - (3a + 5b)]$.
30. $11c + [-3d - (4c - 3d) + c]$.
31. $a - [-(a - b) + (a + b)]$.
32. $2\{3x + 3(x - 1)\}$.
33. $3\{2x - 5(2x - 3)\}$.
34. $7\{x - 2(3 - x)\}$.
35. $3\{6a - 5(a - 1)\}$.
36. $9\{2(a - 1) - 3(a - 7)\}$.
37. $4\{a - 2(a - 1) + 3(a - 2)\}$.
38. $5\{2a - 3(a - 1) - (1 - a)\}$.
39. $2x - 7\{3 - (2x - 1) - 2(x - 2)\}$.
40. $9x - 3\{y - 2(3x + y) + (3y - x)\}$.

63. Example 1. Prove that $a + [3b - \{4a - (a - b)\}] = -2a + 2b$.

The given expression $= a + 3b - \{4a - (a - b)\}$.

(In removing the square brackets [ ] we must look upon all the terms within the curly brackets as a single quantity.)

$= a + 3b - 4a + (a - b)$.

[Regarding $(a-b)$ as a single quantity as before.]

$$=a+3b-4a+a-b$$
$$=2a-4a+3b-b$$
$$=-2a+2b.$$

Or, more shortly, the given expression

$$=a+3b-4a+a-b$$
$$=2a-4a+3b-b$$
$$=-2a+2b.$$

This is easy to understand if we remember that the plus preceding the square bracket does not alter the minus preceding the curly bracket, whilst the minus preceding the curly bracket changes the minus preceding the plain bracket into plus.

Example 2. Simplify the expression

$$4[a-3\{a-2(b-c)+2c\}-4(a-b)].$$

Every term inside the square brackets must be multiplied by 4.
Every term inside the curly brackets must be multiplied by 3 *as well.*
Also $(b-c)$ must be multiplied by 2 as well as by 3 and 4.
$(a-b)$ must be multiplied by 4×4.

The given expression

$$=4a-12\{a-2b+2c+2c\}-16(a-b)$$
$$=4a-12a+24b-24c-24c-16a+16b$$
$$=-24a+40b-48c.$$

Example 3.

$$a-2b-[3a-5b-\{2a-3c+(5a-2c-\overline{3a-b+2c})\}]$$
$$=a-2b-3a+5b+2a-3c+5a-2c-3a+b-2c$$
$$=2a+4b-7c.$$

Explanation. The minus preceding the first square bracket ([) operating on the minus preceding the first curly bracket ({) makes it plus.

Thus the plus in front of the first plain bracket remains plus and the minus preceding the vinculum remains minus.

The work of the above might be given in greater detail thus:

The given expression

$$=a-2b-3a+5b+\{2a-3c+(5a-2c-\overline{3a-b+2c})\}$$
$$=a-2b-3a+5b+2a-3c+(5a-2c-\overline{3a-b+2c})$$
$$=a-2b-3a+5b+2a-3c+5a-2c-\overline{3a-b+2c}$$
$$=a-2b-3a+5b+2a-3c+5a-2c-3a+b-2c$$

as before.

Examples. XI. b.

Remove the brackets and collect the like terms in the following expressions:

1. $4a-\{3a-(2a-a)\}$.
2. $a-[a-(a-\overline{a-c})]$.
3. $a-\{a+(a-\overline{a+b})\}$.
4. $2x-[3x-\{5x-(5x-6x)+2x\}]$.
5. $7+[6-2(3+x)-4(x-2)]$.
6. $4a-3[a-4(1-a)]$.
7. $a^2+b^2-[a(a+b)-b(b-a)]$.
8. $1-\frac{1}{2}\{1-\frac{1}{4}(1-x)\}$.
9. $6[a-2\{b-4(c+d)\}]-4[a-2\{b-3(c-d)\}]$.
10. $\frac{4x-8}{2}-\frac{3x-9}{3}-\frac{15x+5}{5}$.
11. $\frac{1}{2}(x+y)+\frac{1}{2}(x-y)$.
12. $\frac{1}{2}(x+y)-\frac{1}{2}(x-y)$.
13. $a(b-c)+b(c-a)+c(a-b)$.
14. $-[-\{-(-x)\}]-[-\{-(-x-y)\}]$.

Prove the following:

15. $3b-\{5a-[6a+(12a-3b)]-a\}=14a$.
16. $9(b-c)-[-\{a-b-4(c-b+a)\}]=-3a+12b-13c$.
17. $5x^2-(3x-\overline{x^2-4})+2(x^2-\overline{x-5})=8x^2-5x+6$.
18. $4a-[2a-\{2b(x+y)-2b(x-y)\}]=2a+4by$.

When $a=1$, $b=2$, $c=0$, prove that

19. $a-2(b-c)+3(2a-4b)-6(c-2a-3b)=27$.
20. $3b-[5a-\{6a+(14a-3b)-2a\}]=13$.
21. $3bc-[4ab+\{3a-(12a-7b)-2abc\}]=-13$.
22. $4[a-2(b-c)-\{a-(b-2)\}]=-16$.

Express the following in their simplest forms:

23. $7a-[5b-\{4a-(3a-2b)\}]$.
24. $a-(b-c)-\{b-(a-c)\}-[a-\{2b-(a-c)\}]$.
25. $a-[3a+c-\{4a-(3b-c)\}+3b]$.
26. $5a-[2a-2\{a-(a-1)\}+2]$.
27. $6a-[3b-\{2a-(6a-3b)\}]$.
28. $a-[3b+\{3c-2a-(a-b)\}+2a-(b-3c)]$.
29. $3\{a-2[b-4(c-d)]\}-4\{a-3[b+4(c+d)]\}$.
30. $a-[2a-\{3a-(4a-\overline{5a-7})\}]$.
31. $4x^2-2x(x-2y)+2y(2y+x)-2x^2$.
32. $2[3ab-a\{-b+b(2+a)\}+3\{a(2-b)+a^2b\}]$.
33. $x^3-2x\{x^2-x(2-x)\}+3[x^3-x(x-1)]$.
34. $3a-2[3a-2\{3a-2(3a-\overline{2a+b})+b\}+b]$.
35. $5a-4[2a-3\{4a-\overline{3a-b}\}-4b]+24a$.
36. $4\{4-4(4-a)+a\}-3\{a-3(a-3)+3\}$.
37. $3[xy+x\{y-y(3+x)\}+2\{x(3-y)-x^2y\}]$.
38. $x-x[x+x(x-\overline{1-x})]$.

Prove the following:

39. $\frac{3x-1}{4}-\frac{2-x}{5}+\frac{1}{5}=\frac{1}{2}$, when $x=1$.

40. $\dfrac{6}{x-1}=\dfrac{5}{x-2}$, when $x=7$.

41. $\dfrac{5}{3x-2}-\dfrac{19}{7x-1}=0$, when $x=\frac{3}{2}$.

42. $\dfrac{7}{x-2}-\dfrac{4}{x+4}=0$, when $x=-12$.

43. $\dfrac{2(x+1)}{5}-8=\dfrac{2x}{16}-1$, when $x=24$.

44. $\dfrac{x-4}{5}-\dfrac{x-5}{6}=\dfrac{x-2}{24}$, when $x=14$.

45. $x-1-\dfrac{x^2+3}{x+2}=0$, when $x=5$.

46. $\dfrac{6x+1}{x+1}-\dfrac{3+6x^2}{x^2-1}=-4\frac{2}{3}$, when $x=2$.

Insertion of brackets.

***64.** In the preceding articles we have dealt with the removal of brackets. Sometimes it is necessary to insert brackets, and the rules for doing so will obviously be the converse of the rules for their removal.

Any number of terms may be placed within brackets with the positive sign (+) prefixed, without changing the signs of the terms included in the brackets.

Any number of terms may be placed within brackets with the negative sign (−) prefixed, provided that the sign of each term included in the brackets is changed.

Thus $\quad 2a+3b-4c-5d=2a+(3b-4c-5d)$.

Also the same expression $=2a+3b-(4c+5d)$.

$$ac-bd+bc-ad=ac-(bd-bc+ad)$$
$$=ac-(bd-bc)-ad.$$

When all the terms within a pair of brackets have a common factor, that common factor may be removed and placed outside the bracket as a multiplier.

$$4a-(5a-5d)=4a-5(a-d).$$
$$x^3-(2x^2-4x+6)=x^3-2(x^2-2x+3).$$

Example. Collect in brackets the like powers of x in the expression

$$ax^3-cx^2-dx-bx^3-dx^2+ax.$$

The given expression

$$=ax^3-bx^3-cx^2-dx^2-dx+ax$$
$$=x^3(a-b)-x^2(c+d)-x(d-a).$$

*Examples. XI. c.

Arrange the following expressions in descending powers of x, bracketing the coefficients of the different powers of x:

1. $2x^2 - 6x + a + x^3 + ax^2 - 2ax - 7$.
2. $x^2 - 2ax + a^2 + x^2 - 2bx + b^2 + x^2 - 2cx + c^2$.
3. $x^2y - y^2x + x^3 - y^3 - xz^2 + x^2z$.
4. $a^3 - 3a^2x + 3ax^2 - x^3 + b^3 - 3b^2x + 3bx^2 - x^3$.
5. $a - ax + bx^3 - bx^2 - bx + c + ax^2$.
6. $p^2x^2 + 2px + p^2 - q^2x^2 - 2qx - q^2$.

Bracket the powers of x in the following expressions in descending order and so that the signs preceding the brackets are all positive:

7. $ax^3 - bx^2 + cx + d - bx^3 + cx^2 - ax - e$.
8. $2x^4 - 3x^2 + 6x^3 - 7x + bx^2 - ax - ax^3 - ax^4$.
9. $x^3 + y^3 - 3xy^2 + 3x^2y + 3xz^2 - 3x^2z$.
10. $ax^2 - bx + c - cx^3 + cx - bx^2 + ax^2$.
11. $ax^4 - bx^3 - cx^2 - px^4 + qx^3 + rx^2$.
12. $3(m+n)x^2y - 2mxy^2 - 2(m-n)x^2y + 2nxy^2$.

Bracket the powers of x in the following expressions so that the signs preceding the brackets are all negative:

13. $ax^3 + px^2 - qx + c - bx^3 - cx^2 - dx - p$.
14. $ax^2 - bx - c - bx^3 - bx^2 + cx + d - ax^3$.
15. $ax^2 - (a-1)x + 2a + (3-2a)x - bx^2$.

65. Identities. **An equation which is true for all values of the symbols used is called an identity.**

The symbol $\equiv$ is often used to denote that two expressions are **identically equal**, *i.e.* that they are equal for all values of the symbols used.

Thus when we write $a - b \equiv -b + a$, we mean that $a - b$ and $-b + a$ are equal whatever values we assign to the symbols a and b.

Example 1. Prove the truth of the following identity

$$4a - \frac{2a-b}{3} + \frac{4a+4b}{6} \equiv 4a + b.$$

$$\begin{aligned} 4a - \frac{2a-b}{3} + \frac{4a+4b}{6} &\equiv 4a - \frac{2a}{3} + \frac{b}{3} + \frac{4a}{6} + \frac{4b}{6} \\ &\equiv 4a - \frac{2a}{3} + \frac{2a}{3} + \frac{b}{3} + \frac{2b}{3} \\ &\equiv 4a + b. \end{aligned}$$

Q.E.D.

To prove the truth of an identity when both sides of the equation are somewhat complicated, it is often advisable to simplify each side separately.

Example 2. Prove the truth of the identity

$$3x-y+4\left[x-\left(3y-x-\frac{2x-4}{2}\right)\right]\equiv 5(x-y-1)-4(y-x)-4y+6x-3.$$

$$\begin{aligned}3x-y+4\left[x-\left(3y-x-\frac{2x-4}{2}\right)\right]&\equiv 3x-y+4x-4(3y-x-x+2).\\&\equiv 3x-y+4x-12y+8x-8\\&\equiv 15x\quad 13y-8. \quad\ldots\ldots\ldots\ldots(1)\end{aligned}$$

Again, taking the right hand side,

$$\begin{aligned}5(x-y-1)-4(y-x)-4y+6x-3&\equiv 5x-5y-5-4y+4x-4y+6x-3\\&\equiv 15x-13y-8. \quad\ldots\ldots\ldots\ldots(2)\end{aligned}$$

$\therefore$ from (1) and (2),

$$3x-y+4\left[x-\left(3y-x-\frac{2x-4}{2}\right)\right]\equiv 5(x-y-1)-4(y-x)-4y+6x-3.$$

Q.E.D.

Example 3. Simplify the expression $x-5-[3+\{x-(3+x)\}]$, and hence determine what value of x will make it equal to zero.

The given expression $= x-5-3-x+3+x$

$= x-5$;

$\therefore$ it is equal to zero when $x=5$.

Example 4. Prove that $\frac{7x}{2}-\frac{x-8}{3}-\frac{4}{5}(4x+2)\equiv\frac{32-x}{30}$.

$\frac{7x}{2}-\frac{x-8}{3}-\frac{4}{5}(4x+2)$ (The L.C.M. of 2, 3 and 5 is 30.)

$$\begin{aligned}&\equiv\frac{15\times 7x}{15\times 2}-\frac{10(x-8)}{10\times 3}-\frac{6\times 4(4x+2)}{6\times 5}\\&\equiv\frac{105x-10x+80-96x-48}{30}\\&\equiv\frac{105x-10x-96x+80-48}{30}\\&\equiv\frac{32-x}{30}.\end{aligned}$$

Q.E.D.

Example 5. Find the simplest form of the expression

$$\frac{x-1}{5}-\frac{2x-3}{4}+\frac{3x-1}{2}.$$

The L.C.M. of 5, 4 and 2 is 20.

Therefore multiplying numerator and denominator

of the first fraction by 4,

.........second 5,

.........third 10,

$$\begin{aligned}\text{the given expression}&=\frac{4(x-1)}{4\times 5}-\frac{5(2x-3)}{5\times 4}+\frac{10(3x-1)}{10\times 2}\\&=\frac{4x-4-10x+15+30x-10}{20}\\&=\frac{24x+1}{20}.\end{aligned}$$

Examples. XI. d.

Prove the following identities:

1. $6a - \frac{9a-a}{2} \equiv 2a.$

2. $7a - \frac{21a-b}{3} \equiv \frac{b}{3}.$

3. $2a + 2[a - 2(b-c)] \equiv 4(a-b+c).$

4. $\frac{2x-3}{4} - \frac{6-3x+y}{2} \equiv \frac{4x-y}{2} - 3\frac{3}{4}.$

5. $\frac{4x-3}{2} - \frac{8x-6}{4} \equiv 0.$

6. $\frac{x-3}{4} - 2 - \frac{x-1}{5} \equiv \frac{x-51}{20}.$

7. $\frac{x-2}{4} + \frac{2x-1}{3} - \frac{x}{2} \equiv \frac{5x-10}{12}.$

8. $x - 1 - \frac{x-2}{2} + \frac{x+3}{3} \equiv \frac{5x+6}{6}.$

9. $\frac{3x}{4} + x - \frac{7x}{8} - 2x + 9 \equiv \frac{72-9x}{8}.$

10. $5x - \frac{2x-1}{3} + 1 - 3x - \frac{x+2}{2} \equiv \frac{5x+2}{6}.$

11. $\frac{7x-11}{8} - \frac{9x-17}{10} - \frac{7}{20} \equiv -\frac{x+1}{40}.$

12. $10(x+3) + 7(\frac{3}{4} - x) - \frac{49}{4} \equiv 3x + 23.$

13. $4x - 3\{5x - 8(x+\frac{1}{2})\} \equiv 13x + 12.$

14. $\frac{x+7}{3} - \frac{3x}{5} - (x-2) + \frac{1}{2}(3x-11) \equiv \frac{7x-35}{30}.$

15. $\frac{1}{7}(3x+5) - \frac{1}{3}(2x+7) - \frac{3x}{5} \equiv -\frac{88x+170}{105}.$

16. $\frac{3x-5}{4} - \frac{7x+9}{16} + \frac{8x+19}{8} \equiv \frac{21x+9}{16}.$

17. Simplify the expression $12 - [4x - 2(3-x) - 5(x-3)]$, and hence determine what value of x will make it equal to zero.

18. What value of x will make the expression $5(x-3) - 4(x-2)$ equal to zero?

19. What value of x will make the expression

$$5x - 10 - (3x-7) - \{4 - 2x - (6x-3)\} \text{ equal to zero?}$$

20. What value of x will make

$$\frac{2x-3}{5} - \frac{4x-6}{3} + \frac{6x+16}{10} \text{ equal to zero?}$$

Simplify the following expressions:

21. $\frac{x+1}{2} + \frac{2x+1}{3}.$

22. $\frac{x-3}{3} - \frac{x-4}{4}.$

23. $\frac{x}{5} + \frac{x-3}{2}.$

24. $\frac{x}{5} - \frac{x-1}{7}.$

25. $\frac{3x}{5} - \frac{x-3}{4}.$

26. $\frac{x-3}{4} + \frac{x-4}{3}.$

27. $\frac{4x-3}{6} - \frac{x-2}{4}.$

28. $\frac{3x+5}{6} - \frac{4x+5}{8}.$

29. $\frac{x-6}{5} - \frac{2x-1}{3} + \frac{x+5}{2}.$

30. $\frac{x-8}{4} - \frac{3x-7}{6} + \frac{2x+3}{2}.$

31. $\frac{2x-3}{6} - \frac{3x-5}{9} + \frac{x+2}{4}.$

32. $\frac{3x-8}{8} + \frac{2x+7}{10} - \frac{7x-6}{20}.$

CHAPTER XII

REVISION PAPERS

XII. a.

1. Prove that $\frac{2x-3}{3}-\frac{3x-5}{5}+\frac{5x+3}{6}-\frac{7x+5}{10}\equiv\frac{x}{5}$.

2. Multiply $3x-5y$ by $5x+7y$, and find the remainder when the result is divided by $5x-8y$.

3. Solve the equation $\frac{3x-7}{2}-\frac{2x-3}{5}=1\frac{1}{2}$. Check your result.

4. Find values of x and y which will satisfy both the equations,

$$\frac{3x}{2}-2y=7,\quad 2x-\frac{3y}{2}=7.$$

Check your result.

5. How many pence are there in £$a+b$ half-crowns $+c$ florins ?

How many pounds are there in a half-sovereigns $+b$ half-crowns $+c$ shillings ?

6. On squared paper take two lines AB, AC, at right angles, such that AB $=2{\cdot}4$ in., and AC $=3{\cdot}2$ in. Find, without actual measurement, the length of BC.

7. Three-quarters of a certain number exceeds two-thirds of it by 4. Find the number. Check your result.

XII. b.

1. Simplify the expression $\frac{3x-4}{4}-\frac{2x-5}{5}+\frac{7x-3}{6}$.

Check your result by putting $x=5$.

2. Divide $21a^2-ab-10b^2$ by $7a-5b$, and multiply the quotient by $3a-2b$.

3. Solve the equation $\frac{3}{2}(x-1)+\frac{5}{3}(1-2x)-2=0$. Check your result.

4. What values of x and y will make both $5x-3y$, and $3(y-x)$ equal to 3 ? Check your result.

5. A man walks a miles in b hours. How many miles does he walk in an hour ? How many minutes does he take to walk one mile ? How long does he take to walk x miles ?

6. Solve the following problem on squared paper, without actual measurement. A man walks $1\frac{1}{2}$ miles East, and then 3 miles North. How far is he then from his starting point ?

7. From a cask $\frac{7}{8}$ths. full 36 gallons are drawn, and the cask is then found to be half full. How many gallons does it contain when full ? Check your result.

XII. c.

1. Divide $22x^2 - 67x - 35$ by $2x - 7$. Check your result by using $x = 2$.

2. Simplify $(2x + 3)(3x - 1) + (2x - 5)(5x - 3) - (4x - 3)^2$.

3. Solve the equation $(x - 3)^2 - (x - 4)^2 = 3$.

4. What values of x and y will make both

$$\frac{x - 2y}{3} \text{ and } \frac{x + y}{5} \text{ equal to } x - 10\ ?$$

5. I was x years old 5 years ago. How old shall I be 7 years hence? How old was I 21 years ago? In how many years from now shall I be $x + 21$ years old? In how many years from now shall I be 45 years old?

6. A man walks 3·7 miles South, and then in a direction due West, until he is 5 miles in a straight line from his starting point. Find by means of squared paper, without actual measurement, the distance he walked in a westerly direction, to the nearest tenth of a mile.

7. A man sold half his oranges and half an orange more, and then found he had 25 left. How many had he at first? Check your result.

XII. d.

1. Simplify the expression $5[3x - 2(1 - 3x) + \frac{1}{5}\{3 - (4 - x)\} + 2]$.

2. Prove that $(3x - 1)(3x + 1) - (1 - x)(1 + x) + 3(1 - 2x)(1 + 2x) \equiv 1 - 2x^2$.

3. Solve the equation $(x - 3)(x + 1) - (x + 2)(x - 5) = 0$. Check your result.

4. Prove that if $\frac{x - 3}{4} - \frac{2(x - y)}{3} + \frac{x + 9}{12} = 0$, then $x = 2y$. Hence write down three positive integral solutions of the equation.

5. If a lbs. of cheese cost b pence, how much will 1 lb. cost? How much will x lbs. cost? How much cheese shall I get for a shilling?

6. A straight wire joins the top ends of two vertical posts, 17 ft. and 24 ft. high respectively, 35 feet apart. By means of squared paper, without actual measurement, find the length of the wire to the nearest foot.

7. A is 13 years older than B. Also A is as much above 57 as B is below 50. Find their ages. Check your result.

XII. e.

1. Divide $apx + qx - 5ap - 5q$ by $x - 5$. Check your result by multiplication.

2. Prove that $(x - a)^2 + (x + a)^2 - (2x - a)(x - 2a) \equiv 5ax$.

3. What value of x will make $\frac{5x - 3}{7} - \frac{3}{2}(x - 4) + 2(x - 3) - \frac{11}{14}$ equal to zero? Check your result.

4. Solve the equations $\frac{x}{2} - \frac{y - 3}{3} = 3$,

$$\frac{x - 3y}{4} = 1 - \frac{4y - x}{8}.$$

5. Write down the number which exceeds one-third of x by 14.
.... one-quarter of 52 by x.
.... $x+1$ by $x-1$.
.... $\frac{x-8}{4}$ by 2.

6. A man walks $2\frac{1}{2}$ miles East, then 3 miles North. He then walks due South-west until he is due North of his starting point. How far is he then from home? and how far has he walked? Solve the problem on squared paper without actual measurement.

7. A is 10 years older than B. In 8 years B's age will be $\frac{1}{2}$ of A's. Find their ages. Check your result.

XII. f.

1. Simplify the expression $\frac{8}{15}+\frac{2x-5}{2}-\frac{3x+7}{3}+\frac{5x-1}{5}+2\frac{1}{2}$, and hence determine what value of x will make it equal to zero.

2. Prove that $2(x+3a)^2+3(x-2a)^2-5(x^2+6a^2)=0$.

3. What value of x will make $6[3\frac{1}{2}-\frac{1}{3}\{2x-5(x-1)\}+2]$ equal to zero? Check your result.

4. Find the values of a and y if $\frac{a-x}{3}=\frac{y-4x}{2}=1$, when $x=2$.

5. Eggs sell at a pence a score. How much will 100 eggs cost? How much will a dozen cost? How many eggs sell for a shilling?

6. A man walks 4 miles West, 3·4 miles North, and then straight towards his starting point until he is one mile from it. How far has he walked?

7. If $f(x)=3x^2-2x+1$, and $\phi(x)=4x^2-3x-2$, find the value of

$$3f(3)-2\phi(2).$$

XII. g.

1. Find the value of $1+3x-4x^2$, when $x=-3, -2, -1, 0, 1, 2, 3$. Tabulate your work.

2. The weight (W lbs.) of a square-cut beam of ash is given by the formula $W=45a^2l$, where l feet is its length, and a feet the length of an edge of its square end. Find the weight of such a beam in lbs.

(1) 20 feet long and 6 in. square.
(2) 15 feet long and 8 in. square.

3. Solve the equation $(x+1)(x-2)(x+5)=(x-1)(x+2)(x+3)$.

4. Divide 224 into two parts which differ by 10.

5. What values of x and y will make both

$$\frac{3x-4y}{9} \text{ and } \frac{x-5y}{4}-2 \text{ equal to 3?}$$

6. Solve the equations

$$\frac{x}{2}-\frac{3y}{4}+z+1=0,$$
$$3(x-y)+5z+4=0,$$
$$x+6y-2z=9.$$

7. A donkey tethered to a post can graze over a circle of 40 feet radius. The shortest distance from the post to a straight hedge is 25 feet. Over what length of hedge can the donkey graze? Solve on squared paper.

XII. h.

1. Find the values of $3x^2-4x+7$ when $x=-3, -2, -1, 0, 1, 2, 3$. Tabulate your work.

2. If a room is l feet long, b feet wide, and h feet high, the area of its walls is $2h(l+b)$. Find the area of the walls of a room 10 feet high, 13 ft. 6 in. wide, and 15 feet long.

3. Solve the equation $4(x-1)^2-(2x-1)(2x-5)=5$.

4. If $5x-y=8$, and $5y-x=20$, find the values of $x+y$ and $x-y$.

5. The sum of five consecutive odd numbers is 275: find them.

6. A man walks 2·6 miles West, then 3·5 miles North, and then 2 miles South-east. How far is he then from his starting point?

7. Solve the equations

$$2(x-y+2z)=12+y-z,$$
$$3(x+y)=z-y-16,$$
$$5(x+y)=2(y-2z-2).$$

CHAPTER XIII

CO-ORDINATES, AND GRAPHS OF STRAIGHT LINES

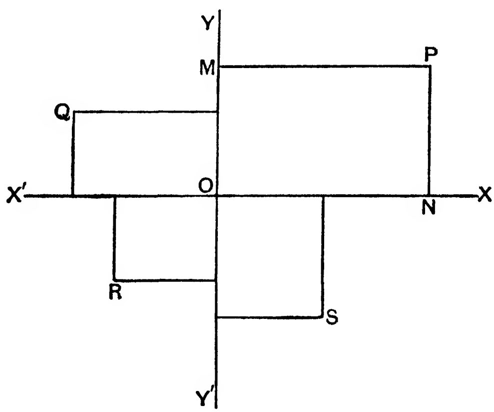
[All graphs should be drawn on **squared paper.** It should be ruled to show inches and tenths of an inch, or centimetres and millimetres.]

66. Take two straight lines, XOX′, YOY′, at right angles to one another. Let P be any point in their plane, and draw PN, PM perpendicular to XOX′ and YOY′ respectively.

Let $\text{PM}=x$, and $\text{PN}=y$.

These values, x and y, determine the position of the point P; *i.e.* if we know the values of x and y, we can draw the point P.

For instance, if $x=5$, and $y=3$; along OX measure $\text{ON}=5$, and along OY measure $\text{OM}=3$ units of length. Then $\text{PM}=\text{ON}=5$, and $\text{PN}=\text{OM}=3$, and therefore P is the point we required to find.



x and y are called the *co-ordinates* of the point P; XOX′, YOY′ the *axes of co-ordinates*, or, more shortly, the *axes*; O the *origin*.

P is often described as the point (x, y).

x is called the *abscissa*, and y the *ordinate* of the point P.

If lines drawn in one direction are taken as positive, then lines drawn in the opposite direction must be taken as negative.

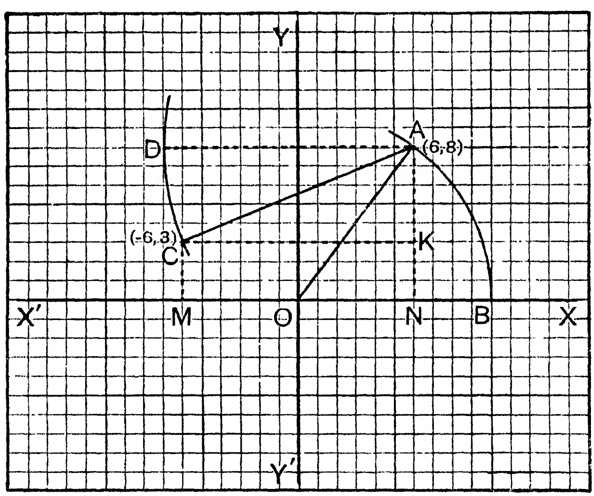
Lines drawn in the directions OX, OY are usually considered positive, and therefore lines drawn in the directions OX′, OY′ are taken as negative.

For example, in the accompanying diagram, at Q the abscissa is negative, and the ordinate positive. At R the abscissa is negative, and also the ordinate. At S the abscissa is positive and the ordinate negative.

In practice, it is simplest to draw the point (5, 3) in the following way.

Along OX measure ON = 5; and at N draw NP perpendicular to ON in the direction OY, the positive direction, and make NP = 3. We then have the same point as in the paragraph above.

Example 1. Plot the point (6, 8) and find its distance from the origin.



Draw axes XOX′, YOY′, and using a side of each square as unit, take ON = 6 units along OX.

Along the vertical line through N, and *in the positive direction,* take NA = 8 units.

A is the point (6, 8).

With centre O and radius OA describe a circle cutting OX at B.

The distance reqd. $= OA = OB = 10$ units, as we see from the diagram.

Example 2. Plot the points (6, 8) (− 6, 3), and find the length of the line joining them.

Plot the pt. (6, 8). (See diagram in above example.)

Along OX′ take $OM = 6$ units, and along the vertical line through M, and in the positive direction, take $MC = 3$ units.

C is the pt. (− 6, 3).

With centre A and radius AC, describe a circle cutting the horizontal line through A at the point D.

The length reqd. $= AC = AD = 13$ units, as we see from the diagram.

We might also find the length of AC in the following manner.

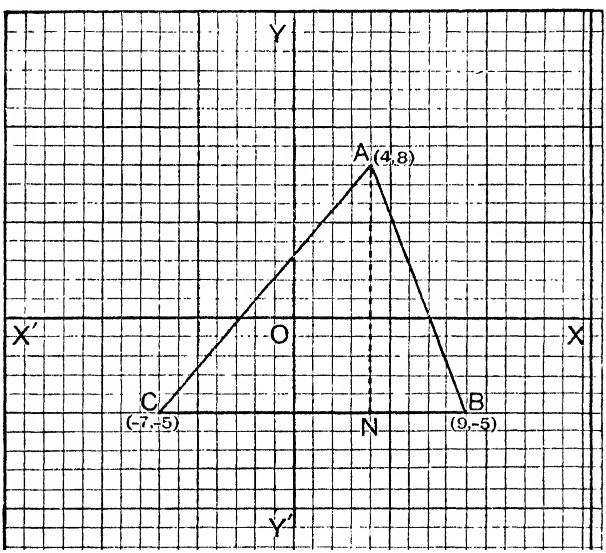
From the diagram, $AK = 5$ units, and $CK = 12$ units.

$$AC^2 = AK^2 + CK^2 = 5^2 + 12^2 = 169;$$

$$\therefore\ AC = 13 \text{ units.}$$

Example 3. To find the area of the triangle formed by joining the points (4, 8), (9, −5), (− 7, −5).

[The area of a triangle is equal to one-half the product of its base and altitude.]



Plot the points as shown in the diagram, and form the triangle ABC by joining them.

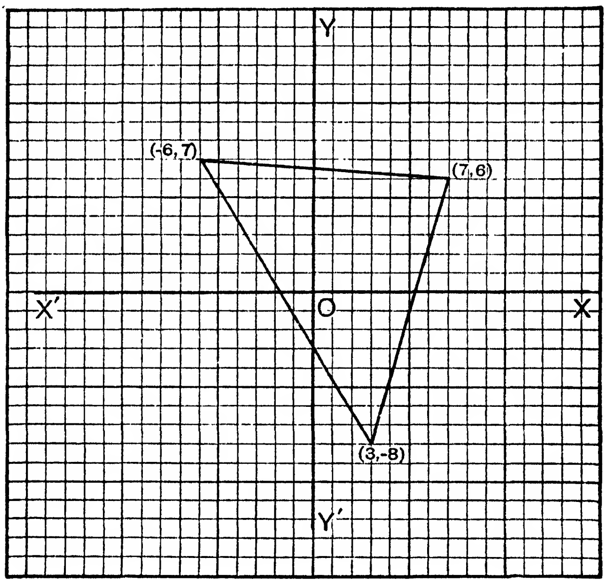
We see that the base $BC = 16$ **units.**

Also if the vertical line through A meets the base at N, AN is the altitude of the triangle, and is equal to 13 units.

$\therefore$ the area of the $\triangle = \frac{1}{2}\text{BC} \times \text{AN} = \frac{1}{2} \times 16 \times 13 = 8 \times 13 = 104$ square units.

Example 4. To find the area of a triangle by **counting squares.**

Find the area of the triangle joining the points (7, 6), (−6, 7), (3, −8).



Plot out the points as shown in the diagram, and form the triangle.

Now let us count up the number of squares in the triangle, **counting as whole squares those which are equal to or greater than half a square, and ignoring those which are less than half a square.**

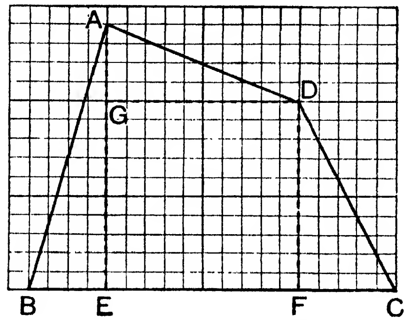
Beginning with the top horizontal row, the numbers in the different rows are 7, 12, 11, 10, 9, 9, 8, 6, 6, 5, 4, 3, 2, 1.

Adding these up, the total number of squares is 93.

$\therefore$ the area of the triangle is 93 square units.

When one side of a rectilineal figure is drawn along a line of squared paper, its area can easily be found by dividing the figure into rectangles and right-angled triangles.

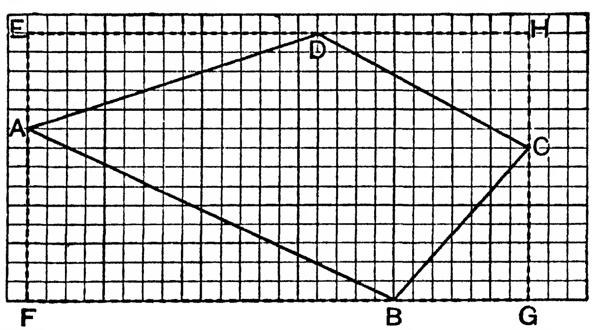
Example 5. Find the area of the figure ABCD in the diagram.



Draw AE and DF perpendicular to BC, and DG perpendicular to AE.

$$
\begin{aligned}
\triangle ABE &= \tfrac{1}{2}BE \times AE = \tfrac{1}{2} \times 4 \times 14 = 28 \text{ sq. units} \\
\triangle AGD &= \tfrac{1}{2}AG \times GD = \tfrac{1}{2} \times 4 \times 10 = 20 \ \ldots\ldots\ldots \\
\triangle DFC &= \tfrac{1}{2}DF \times FC = \tfrac{1}{2} \times 10 \times 5 = 25 \ \ldots\ldots\ldots \\
\text{Fig. } DFEG &= DF \times EF = 10 \times 10 = \underline{100} \ \ldots\ldots\ldots \\
\therefore \text{ the area of } ABCD &= 173 \text{ sq. units.}
\end{aligned}
$$

Example 6. To find the area of the figure ABCD in the diagram.



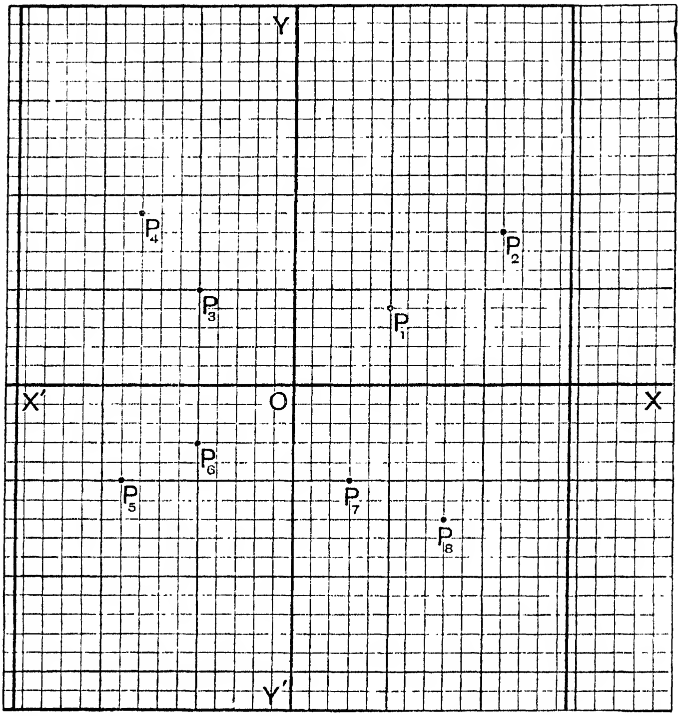
Through A, B, C, D, draw lines along the lines of the paper so as to form the rectangle EFGH.

$$
\begin{aligned}
\triangle AED &= \tfrac{1}{2}AE \times DE = \tfrac{1}{2} \times 5 \times 15 = 37\tfrac{1}{2} \text{ sq. units.} \\
\triangle AFB &= \tfrac{1}{2}AF \times BF = \tfrac{1}{2} \times 9 \times 19 = 85\tfrac{1}{2} \ldots\ldots\ldots\ldots \\
\triangle BGC &= \tfrac{1}{2}BG \times CG = \tfrac{1}{2} \times 7 \times 8 = 28 \ \ldots\ldots\ldots\ldots \\
\triangle DHC &= \tfrac{1}{2}DH \times CH = \tfrac{1}{2} \times 11 \times 6 = \underline{33} \ \ldots\ldots\ldots\ldots \\
& \qquad\qquad\qquad\qquad\qquad\quad 184
\end{aligned}
$$

$$
\begin{aligned}
\therefore ABCD &= EF \times FG - 184 \text{ sq. units.} \\
&= 14 \times 26 - 184 \ \ldots\ldots\ldots\ldots \\
&= 364 - 184 \ \ldots\ldots\ldots\ldots \\
&= 180. \ \ldots\ldots\ldots\ldots
\end{aligned}
$$

Examples. XIII. a.

1. Write down the co-ordinates of the points P_1, P_2, P_3,... shown in the diagram below.



2. Plot the following points on squared paper:

$$(2, 3),\ (2, -4),\ (-3, 3),\ (-2, -4).$$

3. Plot the following pairs of points, and determine the co-ordinates of the middle points of the lines joining them:

(i) $(2, 4), (-2, -4)$. (ii) $(3, 4), (3, -4)$.

(iii) $(6, 8), (-2, -4)$. (iv) $(-3, 5), (-5, 3)$.

4. Plot the points $(5, 2)$, $(5, 1)$, $(5, -2)$, $(5, -4)$, $(5, -3)$. Join them. What do you notice about them?

5. Plot the points $(0, 6)$, $(4, 0)$. Join them, and determine the area of the triangle this line forms with the axes of co-ordinates.

6. Plot the points $(3, 4)$, $(3, -4)$, $(-3, 4)$, $(-3, -4)$. Determine the number of square units in the area of the figure formed by joining them.

7. Plot the points (3, 4), (4, 8). Join them, and write down the ordinates of the points on this line whose abscissae are respectively 2 and 5. Write down also the abscissae of the points whose ordinates are respectively -2 and 6.

8. Plot the points (3, -2), (-3, -2), (0, 4). Join them, and, by counting squares, determine as accurately as you can the area of the triangle so formed. Verify your result by calculation.

9. Determine the perimeter of the triangle formed by joining the points (8, 0), (-8, 0), (0, 6).

10. Find the perimeter of the triangle formed by joining the points (7, 9), (-11, 20), (-17, -5).

11. Draw the triangle (10, 0), (-10, 0), (0, 18). Find its area by counting squares and verify your result by multiplying half the altitude by the base.

12. Draw a semi-circle of radius 1·5 in. and find its area by counting squares.

13. Find the area of the triangle joining the points (4, 2), (4, 7), (-2, 3), using half an inch as unit.

Find the lengths of the lines joining the following pairs of points:

14. (0, 0), (15, 20).
15. (9, 8), (-10, 19).

16. (7, 13), (-16, 3).
17. (15, -12), (-15, 4).

In the following, use an inch as unit, and when necessary estimate the value of the second decimal place.

Find, to the nearest hundredth of an inch, the lengths of the lines joining the following pairs of points:

18. (0, 0), (2·4, 1·3).
19. (3·2, 1·8), ($-0·4$, 2·7).

20. (2·3, 0·9), ($-1·1$, $-1·4$).
21. (0·5, $-0·9$), ($-0·9$, 2·3).

Find the area (in squares of your paper) of the figures formed by joining the following points:

22. (2, 6), (2, 1), (8, 6), (8, 1).
23. (0, 0), (0, 9), (8, 0), (8, 9).

24. (5, -6), (5, 5), (-4, -6), (-4, 5).

25. (0, 0), (10, 0), (14, 7), (4, 7).
26. (-9, 5), (7, 5), (16, 13), (0, 13).

27. (0, 0), (17, 0), (0, 12).
28. (13, 0), (0, 8), (13, 8).

29. (10, 5), (-6, 5), (6, 17).
30. (-9, 20), (-9, 5), (11, 24).

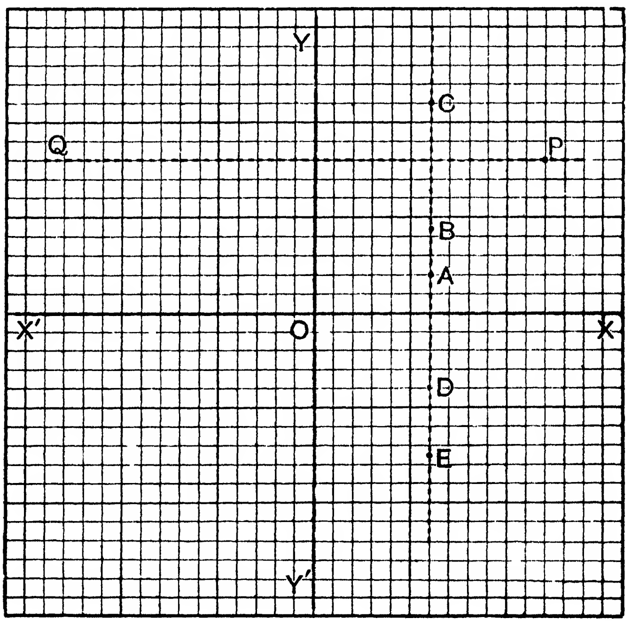
31. (5, 12), (-15, 8), (-4, 17).
32. (10, 7), (3, 16), (-8, 3).

67. Draw axes XOX′, YOY′, and mark a number of points whose abscissae are equal to 6 taking any convenient unit of length.

A, B, C, D, E, in the diagram, are such points.

We thus see that all points, whose abscissae are equal to 6, lie on the straight line parallel to OY and distant 6 units from it.

Moreover, if we look at any other point *not on this line,* we see that its abscissa is *not equal to* 6. In other words, $x=6$ for all points on the straight line CE, and for no other points.



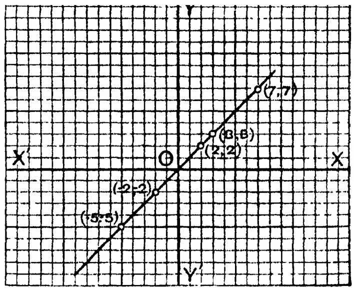
The line CE is therefore called the **graph** of $x=6$.

We notice too that the equation $x=6$ is true for all points on the line however far we produce it in either direction.

In the same way, if we mark a number of points whose ordinates are all equal to 8 and join them, we get a straight line PQ parallel to OX, and *it is the graph of* $y=8$.

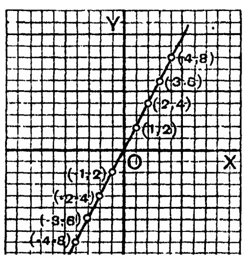
68. If in a diagram we mark the points (2, 2), (3, 3), (4, 4), (5, 5) and so on, and join them, we get a straight line. Also if

(x, y) be the co-ordinates of any point on this line, we see that $x=y$. *Hence this line is the graph of* $x=y$.



It will be seen that the points (0, 0), $(-1, -1)$, $(-2, -2)$, $(-3, -3)$, etc., all lie on this graph.

69. *Draw the graph of* $y=2x$.



When

$x=1$	2	3	4	...
$y=2$	4	6	8	...

When

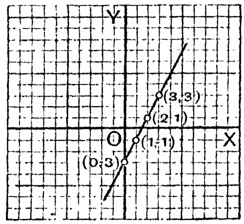
$x=0$	-1	-2	-3	-4	...
$y=0$	-2	-4	-6	-8	...

Joining the points thus found, we have the graph required. It will be seen to be *a straight line through* O *the origin*.

N.B.—The line is of unlimited length.

70. *Draw the graph of the expression* $2x-3$.

N.B.—This is the same as the graph of $y=2x-3$.



Let $y=2x-3$.

When

$x=0$	1	2	3	...
$y=-3$	-1	1	3	...

Marking in a diagram the points thus found, and joining them, we have the graph reqd.

It will be seen that the graph is a straight line of unlimited length.

71. *Draw the graph of the expression* $\dfrac{2x-3}{5}$.

Let $y=\dfrac{2x-3}{5}$.

When

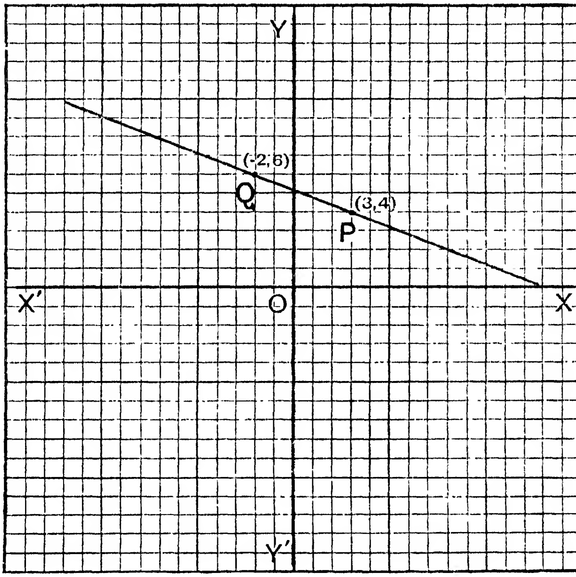
$x=$	0	1	2	3	4
$y=$	$-\cdot 6$	$-\cdot 2$	$\cdot 2$	$\cdot 6$	1

Marking these points in a diagram and joining them, we have the graph reqd.

N.B.—**It will be seen that all graphs of expressions of the first degree, *i.e.* graphs obtained from equations of the first degree, are straight lines.**

72. *To draw the graph of the expression* $\dfrac{26-2x}{5}$, *i.e. the graph of the equation* $y=\dfrac{26-2x}{5}$.

[The equation being of the first degree, its graph is a straight line. It will therefore be sufficient if we plot two points on the graph, for only one straight line can be drawn through two given points.]



Choose convenient points.

When $$x=3,\ y=\frac{26-6}{5}=4.$$

$\therefore$ the pt. (3, 4) is on the graph.

When $$x=-2,\ y=\frac{26+4}{5}=6.$$

$\therefore$ the pt. $(-2, 6)$ is also on the graph.

Joining these points, P and Q in the diagram, the line PQ is the graph reqd.

73. *Solve graphically, on squared paper, the following equations :*

$$2x-y=11. \quad x-2y=10.$$

In the first equation, when $y=1$, $x=6$. Mark this pt. on the squared paper.

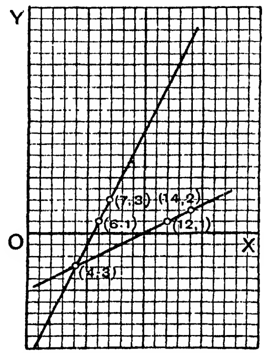
In the same equation, when $y=3$, $x=7$. Mark this pt. also.

The str. line joining these pts. is the graph of the first equation.

In the second equation, when $y=1$, $x=12$. Mark this pt. in the same diagram.

Also in the second equation, when $y=2$, $x=14$. Mark this pt.

The line joining these last two pts. gives the graph of the second equation.



From the diagram it will be seen that the str. lines meet at the pt. $(4, -3)$.

Hence $x=4$, $y=-3$, is the reqd. solution.

Verification. In the first equation, when

$$x=4,\quad 2\times 4-y=11,$$

$$-y=11-8=3,\quad y=-3.$$

$$\therefore\ x=4,\quad y=-3 \text{ satisfy the first equation.}$$

In the second equation, when $x=4$,

$$4-2y=10,\quad -2y=10-4,$$

$$y=-3.$$

$$\therefore\ x=4,\quad y=-3 \text{ satisfy this equation also.}$$

74. The following are very important :

(1) The co-ordinates of the origin are (0, 0).

(2) If a point lies on the axis of x, its ordinate is zero.

(3) If a point lies on the axis of y, its abscissa is zero.

Thus we see that the graph of $x=0$ is the axis of y ; and the graph of $y=0$ is the axis of x.

(4) The graph of $x=a$, where a is constant, is a str. line $\|$ to the axis of y.

The student should illustrate this by drawing graphs of $x=2$, $x=5$, $x=-7$, and so on.

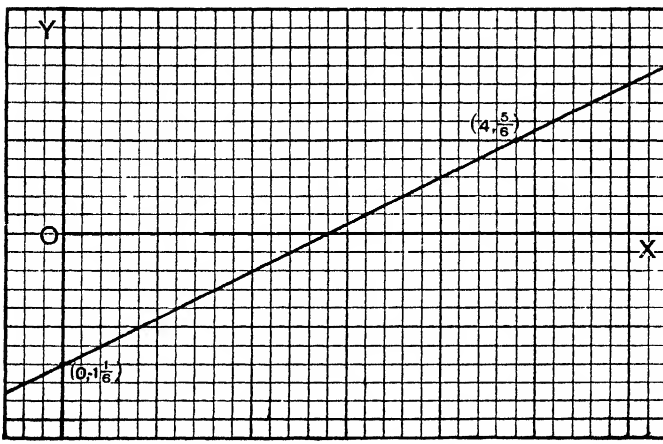
(5) The graph of $y=b$, where b is constant, is a str. line $\|$ to the axis of x.

Illustrate this by drawing the graphs of $y=3$, $y=4$, $y=-8$.

75. It is sometimes advisable to work with other units than an inch, or a tenth of an inch.

Draw the graph of $\dfrac{3x-7}{6}$.

Note that here we draw the graph of a function of x.



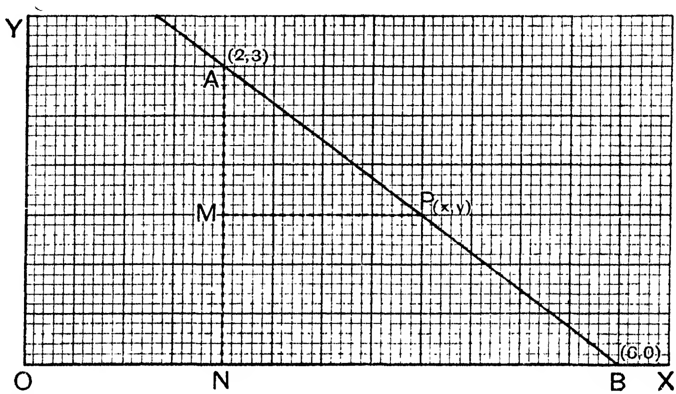
Let $y=\dfrac{3x-7}{6}$. The graph is a str. line since the equation is of the first degree. When

$x=$	0	4
$y=$	$-\frac{7}{6}$	$\frac{5}{6}$

Taking 6 tenths of an inch to represent unity, we have the graph as shown in the diagram.

76. *To find the equation of the graph which passes through the points*
$(2, 3)(4, 1\cdot5)(6, 0)(8, -1\cdot5)(10, -3)$.

[In the diagram 10 sides of a small square are taken to represent unity.]



When we plot these points we see that they lie in a str. line.

$\therefore$ the equation of the graph is of the first degree.

Let $ax+by=c$ be the equation reqd.

The pt. (2, 3) is on the graph,

$$\therefore \; x=2,\ y=3 \text{ satisfy the equation } ax+by=c,$$

$$i.e. \; 2a+3b=c. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The pt. (6, 0) is on the graph,

$$\therefore \; x=6,\ y=0 \text{ satisfy the equation } ax+by=c.$$

$$i.e. \; 6a=c;$$

$$\therefore \; a=\frac{c}{6};$$

$$\therefore \text{ from (1) } 3b=c-\frac{c}{3}=\frac{2c}{3},$$

$$b=\tfrac{2}{9}c.$$

$$\therefore \; \frac{cx}{6}+\frac{2cy}{9}=c,$$

i.e. $3x+4y=18$ is the equation reqd.

The equation might also be found as follows:

Let P (x, y) be any pt. on the line.

$\triangle$s AMP, ANB are equiangular, and therefore their sides are proportional.

$$\therefore \frac{\mathrm{AM}}{\mathrm{PM}} = \frac{\mathrm{AN}}{\mathrm{BN}};$$

$$i.e. \ \frac{3-y}{x-2} = \tfrac{3}{4} \text{ (see diagram).}$$

Whence $3x+4y=18$, as before.

Before drawing any graph, first tabulate the values of x and y, and then choose a convenient unit.

Make it a rule to state, in a prominent position on the squared paper, the unit employed.

Let your work be very neat, and do not use a pencil with a thick point.

Examples. XIII. b.

[*In each case state the unit employed. Small units are inadvisable.*]

1. In separate diagrams draw the graphs of the following:

(i) $x=4$. (ii) $y=5$. (iii) $x=-2$. (iv) $y=-3$.

2. In the same diagram draw graphs of the following:

(i) $y=3x$. (ii) $y=-2x$.

Distinguish the graphs by writing their equations on each.

3. In the same diagram draw graphs of:

(i) $y=\frac{1}{2}x$. (ii) $y=-\frac{1}{2}x$.

Distinguish them as in the previous example.

Trace on squared paper the graphs of the following:

4. $y+4=0$.	**5.** $x+2$.	**6.** $x-2$.	**7.** $y-x=5$.
8. $y=x+6$.	**9.** $y=2x+1$.	**10.** $2x+3$.	**11.** $4-3x$.
12. $5-6x$.	**13.** $y=6+2x$.	**14.** $3x+4y=12$.	**15.** $3x-4y=12$.
16. $\frac{3x-5}{6}$.	**17.** $\frac{5-3x}{6}$.	**18.** $\frac{y}{3}=\frac{x-1}{4}$.	**19.** $\frac{x}{2}-\frac{y}{3}=1$.
20. $15x=19y$.	**21.** $3x+4y=0$.	**22.** $7x-3y=0$.	**23.** $\frac{x}{5}-\frac{y}{9}=0$.
24. $2y=4x-1$.	**25.** $x-3y=6$.	**26.** $2y-x=6$.	**27.** $6x=3y-5$.
28. $6x=5-3y$.	**29.** $11x+11y=9$.		

Solve the following equations graphically, and verify your result by Algebra.

30. $x+2y=12$, $x-3y=2$. (Use half an inch, or a centimetre, as unit.)

31. $4x-y=10$, $2x-y=4$. (Use an inch as unit.)

32. $4x-3y=14$, $3x-4y=0$. (Half-inch unit.)

33. $5x-7y=20$, $3x-2y=12$. (Half-inch unit.)

34. $x=5$, $y-x=3$. (Half-inch unit.)

35. $y=3$, $\frac{x}{8}+\frac{y}{6}=1$. (Half-inch unit.)

Solve the following equations graphically, and verify your result by Algebra :

36. $x=2{\cdot}8,\ \dfrac{x}{2}=\dfrac{y}{3}$. (Half-inch unit.)

37. $y-2x=-3,\ 2y+x=14$.

38. $2x+7y=52,\ 3x-5y=16$.

39. $5x+9y=188,\ 13x-2y=57$.

40. $3y-4x=0,\ y+x=21$.

41. $x-\dfrac{y-2}{7}=5,\ 4y-\dfrac{x+10}{3}=3$.

42. $\dfrac{x+y}{3}+5=10,\ \dfrac{x-y}{2}+7=\dfrac{19}{2}$.

In the following, plot the points given, and find the equation of the graph in each case :

43.

$x=1$	3	5	7	9
$y=3$	9	15	21	27

44.

$x=\ 0$	1	3	7	9
$y=-4$	-3	-1	3	5

45.

$x=-2$	0	2	4	6
$y=\ 11$	7	3	-1	-5

46.

$x=-2$	-1	0	2
$y=\ 10$	5	0	-10

47.

$x=\ 0$	·5	1	3	3·2	3·6
$y=-5$	-4	-3	1	1·4	2·2

48.

$x=0$	-1	·3	2	·8
$y=4$	1	4·9	10	6·4

49.

$x=-4$	-3	-2	-1	0
$y=\ 0$	1·5	3	4·5	6

50.

$x=0$	1	2	3	4
$y=1\frac{2}{3}$	2	$2\frac{1}{3}$	$2\frac{2}{3}$	3

CHAPTER XIV

PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS

77. Example 1. Find two numbers such that twice the first added to three times the second is equal to 45, and also such that five times the first added to four times the second is equal to 74.

Let x be the first number, and y the second.

Twice the first + 3 times the second $=2x+3y$,

$\therefore\ 2x+3y=45$, (by hypothesis). ....(1)

5 times the first + 4 times the second $=5x+4y$. ....(2)

$\therefore\ 5x+4y=74$, (by hypothesis).

Multiplying (1) by 4, $8x+12y=180$, ...(3)

.... (2) by 3, $15x+12y=222$. ....(4)

Subtracting (3) from (4), $7x=42$,

$$x=6.$$

Substituting this value of x in (1),

$$2\times 6+3y=45,$$
$$3y=45-12=33,$$
$$y=11.$$

$\therefore$ 6 and 11 are the reqd. numbers.

Verification. $2\times 6+3\times 11=12+33=45$,

$5\times 6+4\times 11=30+44=74$.

Example 2. Five years ago A was twice as old as B, and 6 years hence their united ages will come to 82. Find their present ages.

Let x years be A's present age, and y years B's present age.

5 years ago, A's age was $x-5$, and B's age $y-5$.

$\therefore$ by hypothesis, $x-5=2(y-5)$,

$$x-5=2y-10,$$

$x-2y=-5$, ...(1)

6 years hence, A's age will be $x+6$ years, and B's age $y+6$,

$\therefore$ by hypothesis, $x+6+y+6=82$,

$x+y=70$, ...(2)

Subtracting (1) and (2) $-3y=-75$,

$$y=25.$$

Substituting in (1), $x-50=-5$,

$$x=45.$$

$\therefore$ A's present age is 45, and B's 25.

In representing numbers of more than one digit algebraically, we must remember that 23 means $2\times 10+3$, and not 2×3.

Thus the number, whose tens' digit is x and units' digit y, is $10x+y$, and not xy, for xy denotes $x\times y$.

Example 3. The sum of the digits of a certain number, less than 100, is 11, and if the digits are reversed, the number is diminished by 9. Find the number.

Since the number is less than 100, it has two digits.

Let x be the tens' digit, and y the units' digit.

By the first hypothesis, $x+y=11$. ....(1)

The number obtained by reversing the digits is $10y+x$.

$\therefore$ by the second hypothesis,

$$10x+y-(10y+x)=9,$$
$$10x+y-10y-x=9,$$
$$9x-9y=9,$$
$$x-y=1. \quad \text{......................(2)}$$

Adding (1) and (2) $\quad 2x=12,$

$$x=6.$$

Substituting this value in (1), $\quad y=5.$

$\therefore$ the reqd. number is $\quad 10\times 6+5=65.$

Verification. The sum of the digits $=6+5=11$,

$$65-56=9.$$

Example 4. A man walks two-thirds of a journey at 4 miles an hour, then bicycles back for one quarter of the whole journey at 8 miles an hour, and turning round, runs the rest of the way, taking 9 hours over the whole journey. If he had run the whole distance at the rate at which he did the last part, he would have taken $4\frac{4}{7}$ hours: find his rate of running.

Let a miles be the whole distance, and suppose he ran x miles per hour.

He walks 4 miles in 1 hour;

$\therefore$ .... 1 mile in $\frac{1}{4}$ hour;

$\therefore$ .... $\frac{2a}{3}$ miles in $\frac{2a}{3}\times\frac{1}{4}=\frac{a}{6}$ hours. ....(1)

He bicycles 8 miles in an hour;

$\therefore$ .... 1 mile in $\frac{1}{8}$ hour;

$\therefore$ .... $\frac{a}{4}$ miles in $\frac{a}{32}$ hours. ....(2)

His distance now from the end of his journey

$$=a-\frac{2a}{3}+\frac{a}{4}=\frac{7a}{12}.$$

He runs x miles an hour;

$\therefore$ .... 1 mile in $\frac{1}{x}$ hour;

$\therefore$ .... $\frac{7a}{12}$ miles in $\frac{7a}{12x}$ hours. ....(3)

From (1), (2), (3), $\quad \frac{a}{6}+\frac{a}{32}+\frac{7a}{12x}=9.$

Simplifying this, $\quad 19a+56\frac{a}{x}=864.$ ....(4)

He runs x miles in an hour;

$\therefore$ .... a miles in $\frac{a}{x}$ hours;

$$\therefore \frac{a}{x}=4\tfrac{4}{7}=\tfrac{32}{7}. \quad \text{.................................(5)}$$

Substituting this value of $\frac{a}{x}$ in (4),

$$19a+8\times 32=864,$$

whence $$a=32 \text{ miles.}$$

From (5), $$x=\frac{7a}{32}=7 \text{ miles an hour.}$$

Examples. XIV. a.

1. The sum of two numbers is 29, and their difference is 5: find them.

2. Three times the sum of two numbers is 51, and their difference is 7: find them.

3. Find two numbers such that three times the first and twice the second together make 34, and three times the first together with five times the second make 58.

4. Half the sum of two numbers is 11, and half their difference is 2: find the numbers.

5. Six pounds of sugar and three pounds of cheese cost 4*s*. 3*d*., and five pounds of sugar and six pounds of cheese cost 6*s*. 2*d*.: find the cost of sugar and cheese per pound.

6. I have 10 coins consisting of half-crowns and florins, together amounting to 23*s*. 6*d*. How many coins have I of each sort?

7. At a meeting of a cricket club to elect a captain, 75 members were present, and the captain was elected by a majority of 13, all voting. How many voted for and against?

8. Six years ago I was three times as old as my brother, and now I am twice as old: find our present ages.

9. The daily wages of 10 men and 7 boys amount to £2. 2*s*.: if a man earns in two days as much as a boy earns in seven days, find what each earns per day.

10. Four times A's age exceeds B's age by 16, and one-fifth of A's age is equal to one-sixteenth of B's age. Find their ages.

11. Ten years ago a father was seven times as old as his son, two years hence twice his age will be equal to five times his son's. What are their present ages?

12. When A and B begin to trade, B's capital is four-ninths of A's. Each of them gains £50 and then A's capital is twice B's. Find the original capitals.

13. A man's age is three times that of his son, in fifteen years it will be double that of his son. How old is each now?

14. A man receives 3*s*. 6*d*. for every day that he works, but is fined one shilling for every day that he is absent. After 20 days he receives the same wages that he would have earned by steadily working for 11 days. How many days was he absent from work?

15. A sum of £2. 15*s*. 6*d*. is paid in florins and half-crowns, there being 25 coins in all: how many are there of each ?

16. The sum of two digits of a number is 9; if the digits are reversed, the new number is four-sevenths of what it was before. Find the number.

17. A man travels the first half of a journey at a uniform speed, and the second half at double the speed, completing the journey in 10 hours 48 minutes. He travels the whole way back at a mile an hour faster than he originally started, and does the return journey in 12 hours. Find the length of the journey, and the man's starting pace.

18. Two men start from two places 48 miles apart. When they travel in opposite directions, they meet in 4 hrs. 48 minutes; when they travel in the same direction, one overtakes the other in 9 hrs. 36 minutes. Find their rates of travelling.

19. If A were to give B twelve shillings, A would have half the sum which B then has; but, if B were to give A thirteen shillings, B would have one-third of what A then has. How much money has each originally ?

20. A is three times as old as B; in eleven years he will be four times as old as B was the year before last. What are their ages ?

21. A bag contains £5 in shillings and sixpences. If there were twice as many shillings and half as many sixpences the amount would be increased by half-a-crown. How many coins are there in the bag ?

22. At an examination, A obtained 11 marks less than B; if he had gained half as many marks again as he did, he would have beaten B by 17. How many marks did each receive ?

23. If £2. 11*s*. 6*d*. is paid in florins and half-crowns, the number of coins being 24, how many are there of each ?

24. A number is composed of two digits of which one is three times the other, but if the digits were transposed, the number would be reduced by 54. Find the number.

25. Two persons starting at the same time from places 40 miles apart, ride towards one another, and meet at a distance of 18 miles from one end. If the faster one had gone 1 mile an hour slower, and the slower one 1 mile an hour faster, they would have met half-way. At what rate was each riding ?

26. A merchant has two sorts of wine worth respectively 6*s*. 8*d*. and 4*s*. a gallon; how much of each must he take to obtain a mixture of 40 gallons worth 4*s* 8*d*. a gallon.

27. At a certain election there were two rival candidates, and their supporters were conveyed to the polling-booths in carriages capable of accommodating 8 and 12 voters respectively. If the voters, 740 in all, just filled 75 carriages, find by what majority the election was won.

28. A traveller walks a certain distance. Had he gone half a mile an hour faster, he would have walked it in four-fifths of the time; had he gone half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance, and his rate of walking.

29. A's age is twice B's. Four years hence B's will be twice C's, and 12 years after that A's will be twice C's. Find their present ages.

30. Certain annual parish expenses were met by collections on alternate Sundays with an annual donation of £15. It was determined to have a collection on every Sunday, with the result that, though each collection

was one-fourth less than before, there was enough without the donation to meet the expenses and £3 to spare. Find the expenses.

31. Some smugglers discovered a cave, which would exactly hold the cargo of their boat, consisting of 13 bales of silk and 33 casks of rum. Whilst they were unloading, a Custom House cutter coming in sight, they sailed away with 9 casks and 5 bales, leaving the cave two-thirds full. How many bales or casks would the cave hold ?

32. On two successive days a man bought a shilling's worth of eggs and a shilling's worth of oranges. On the second day the number of eggs was 25 per cent. greater, and the number of oranges was 15 per cent. less than the numbers of those he got on the previous day. On both days the number of eggs and oranges united was 32. How many eggs did he receive on the first day ?

33. If the floor of a room were 9 feet longer and 6 feet narrower it would take 4 square yards less carpet; but if it were 6 feet shorter and 6 feet wider, it would not change its area. Find its dimensions.

34. At a school treat it was calculated that if each teacher gave *5s.* there would be *3d.* for each child and *3d.* over: but two more teachers arrived bringing a third as many children as there were before, and it was now found that each child would receive $3\frac{1}{2}d.$ if each teacher gave *5s. 6d.* How many children and teachers were there at first and at last ?

35. A certain dole was *25s.* more than would give the recipients a florin apiece, and there were fifteen too many to receive half-a-crown apiece. What was the amount of the dole ?

36. The difference of the perimeters of two square fields expressed in linear yards is one-fourth of the difference between their areas expressed in square yards, and the sum of the perimeters of the fields is eight times the difference of their perimeters. Find the areas of the fields.

37. A's age is equal to the combined ages of B and C. Ten years ago A was twice as old as B. Show that ten years hence A will be twice as old as C.

38. A bill of 25 guineas is paid with crowns and half-guineas, and twice the number of half-guineas exceeds three times that of the crowns by 17 : how many of each are used ?

39. The united ages of a man and his wife are at present six times those of their children; two years ago their united ages were 10 times, and six years hence they will be 3 times, the united ages of their children. How many children have they ?

40. A man does a journey at a certain rate, and finds that if he had travelled 6 miles an hour faster, he would have done the journey in one-third of the time. What was his slower rate of travelling ?

41. A man does a journey in a motor car at a uniform speed in 6 hours. On his return he is delayed at half-way for half-an-hour, but quickening his pace by 3 miles an hour does the journey in the same time. Find his original speed and the length of the journey.

42. In going the shortest way from A to B, a man had to go back one mile to pick up something he had dropped, and took $3\frac{1}{4}$ hours over the walk. He went back by a route which was half-a-mile longer, and took 3 hours over the return walk. Find his rate of walking, and the shortest distance from A to B.

43. In walking from A to B a man meets a friend and rides back with him in his motor-car for 3 miles at the rate of 12 miles an hour. Resuming his walk he arrives at B 7 hours after his start. If he had walked straight through, he would have taken 6 hours over the walk. Find his rate of walking, and the length of the walk.

44. Two men run a course of 4000 feet at uniform rates. One starts 30 seconds after the other and arrives 10 seconds before him. Where does he pass him ?

45. A man pays a certain tax on the whole of his income. If his income had been one-tenth more, and the tax 1*d.* in the £ lower, the tax paid by him would have been exactly £1 less ; but if his income had been one-fifteenth less, and the tax 1*d.* in the £ higher, the amount of his tax would have been exactly £1 more. Find his income and the rate per £ of the tax.

46. The road from A to B ascends five miles, is then level for four miles, and finally descends six miles. A man walks from B to A in four hours, the next day he walks half-way to B and back again in three hours fifty-five minutes, and returns on the third day to B in three hours fifty-two minutes. What are his rates of walking (*a*) uphill, (*b*) downhill, (*c*) on level ground, if these rates do not vary from day to day ?

47. Two ships (S_1, S_2) start at the same time in the same direction from two stations (A_1 and A_2 respectively) on the same route. After a certain time S_1 overtakes S_2, when it is found that they have sailed 1500 miles between them, that S_1 passed A_2 four days ago, and that S_2 is now nine days' sail from A_1. Find the distance between A_1 and A_2 and the average rates of sailing of the vessels.

EASY GRAPHICAL PROBLEMS

78. *A man, starting at noon, walks at the rate of* 6 *miles an hour. Draw a graph of his motion, and from the diagram, read off, as accurately as you can, the time when he is* 22 *miles from his starting point, and the distance he has travelled in* 2 *hours* 24 *minutes.*

Measure distance along OX, taking a side of each square to represent a mile. Measure times along OY, at right angles to OX, taking 10 sides to represent an hour, so that each side represents 6 minutes.

Taking OA along OX equal to 30 miles, (30 squares), and AB at right angles to OA equal to 5 hours, (50 squares), B represents the man's position in 5 hours, for he travels 30 miles in 5 hours.

Join OB. OB is the graph of his motion.

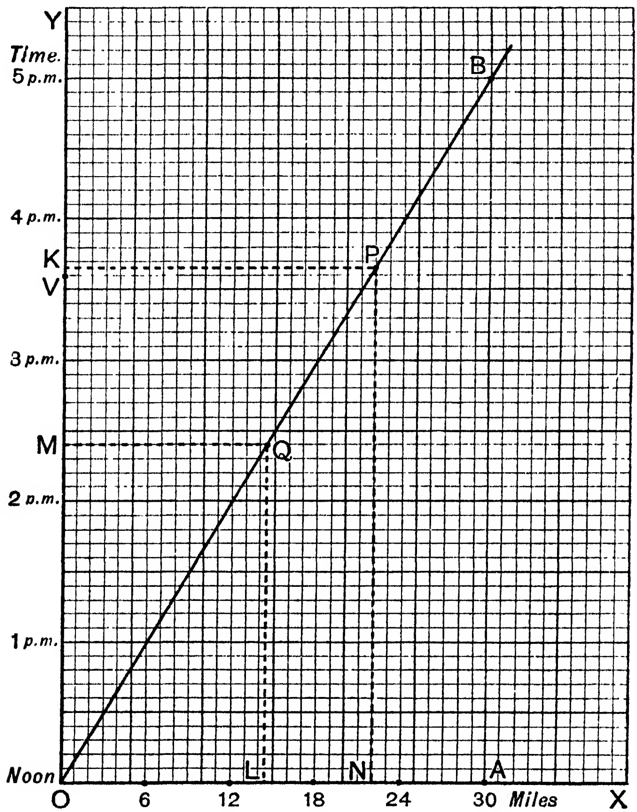
By this we mean that any ordinate PN represents the time taken to walk the distance represented by the abscissa ON.

To find the time when he is 22 miles from the start, take ON equal to 22 miles and draw the corresponding ordinate NP.

This ordinate represents the time reqd.

Drawing PK parallel to OX, and *estimating* the value of the portion KV of the side of a square, we see that the reqd. time is 3.40 p.m.

To find the distance travelled at 2.24 p.m., take OM along OY equal to 2 hours 24 minutes, and draw MQ parallel to OX. Draw



the ordinate QL at Q, OL represents the distance reqd., and is equal to $14\frac{1}{2}$ miles nearly.

The student should verify these results by calculation.

He should also verify the fact that OB is the graph of the man's motion by taking simple distances, and reading off the corresponding times; *e.g.* 6 miles (time 1 hour), 12 miles (2 hours) and so on.

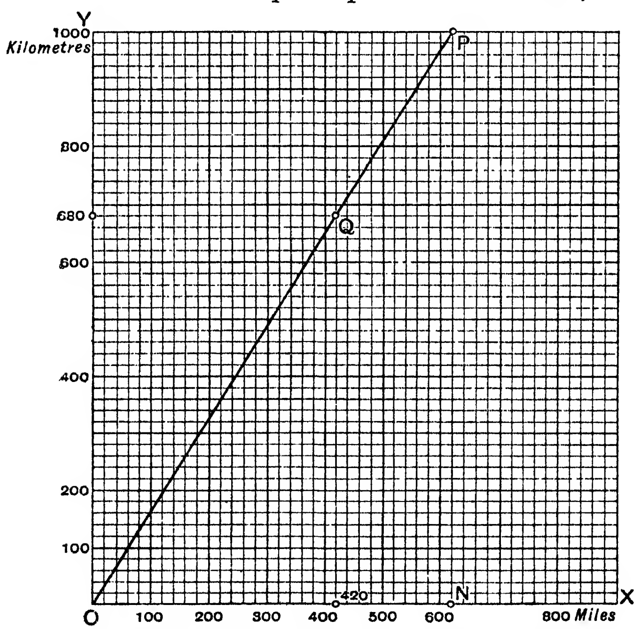
79. *Given that ·62 of an English mile = 1 kilometre, construct a graph from which you can read off any number of miles in kilometres and any number of kilometres in miles. From it write down the number of kilometres in 420 miles and the number of miles in 580 kilometres. Calculate the results to the nearest* 10 *kilometres or miles.*

If x miles $= y$ kilometres, $\frac{x}{62} = \frac{y}{100}$.

Take an abscissa ON = 62 units (31 sides of a sq.),
and an ordinate NP = 100 units (50 ,, ,,).

Join OP. OP is the graph of $\frac{x}{62} = \frac{y}{100}$.

∴ taking each horizontal side of a sq. to represent 20 miles, and each vertical side of a sq. to represent 20 kilometres,



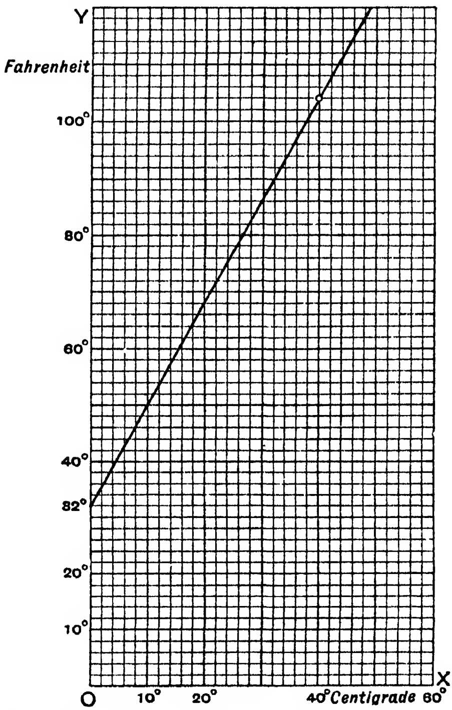
the abscissa of the pt. Q represents 420 miles ;
∴ its ordinate represents 420 miles in kilometres.

∴ from the diagram 420 miles = 680 kilometres nearly.

Also from the diagram 580 kilometres = 360 miles.

80. *Construct a graph which will enable you to convert, at sight, degrees Fahrenheit into degrees Centigrade, and vice versa.*

Let $x°$ in the Centigrade scale be the same temperature as $y°$ in the Fahrenheit scale.



In the Centigrade scale, freezing point stands at 0°; in the Fahrenheit at 32°.

In the Centigrade scale, boiling point is at 100°; in the Fahrenheit at 212°.

$$\therefore \frac{x}{100} = \frac{y-32}{212-32},$$

whence $$9x = 5y - 160.$$

Therefore if we draw the graph of this equation, the abscissae will give us temperatures in Centigrade scale, whilst the corresponding ordinates will give us the corresponding temperatures in Fahrenheit scale.

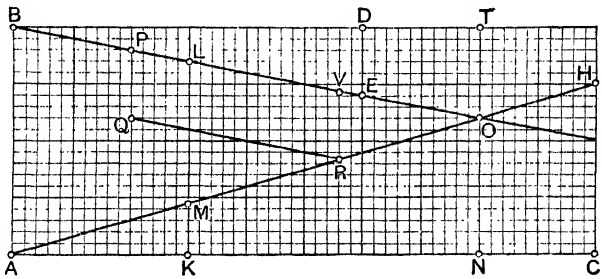
Thus from the graph,

$$80^\circ \text{ F.} = 26{\cdot}7^\circ \text{ C. and } 40^\circ \text{ C.} = 104^\circ \text{ F.}$$

A graph may often be drawn without the use of an equation, but the student must realize that every graph has its corresponding equation, and *vice versa*, every equation will have its corresponding graph.

81. *Two men start at noon to walk: the one from* A *to* B, *the other from* B *to* A. *If* A *and* B *are* 20 *miles apart, and the men walk at the rate of* 3 *miles an hour and* 2 *miles an hour respectively, construct a graph which will enable you to determine when and where they meet.*

Read off from the graph their distance apart at 1.30 *p.m. and also find at what time they are first at a distance of* 6 *miles from one another.*



On squared paper, take pts. A and B on a vertical line 20 units apart. Horizontally take AC = 50 units (10 units to an hour) and vertically CH = 15 units. Join AH. Then since the first man walks 15 miles in 5 hours (50 units), AH is the graph of the first man's motion; *i.e.* the ordinate of any pt. on AH denotes the distance he has walked in the time denoted by the abscissa of the pt.

Considering the second man, take BD horizontally 30 units in length, to denote 3 hours, and DE vertically downwards 6 units in length. Join BE.

Then BE is the graph of the second man's motion if we read his times along BD, and his distances walked at right angles to BD and downwards.

Hence if AH and BE meet at O, AN denotes the time when they meet, and ON, OT denote the distances walked by the two men in that time.

Thus from the diagram, we read off that they meet at 4 o'clock, that the first man has then walked 12 miles and the second 8 miles.

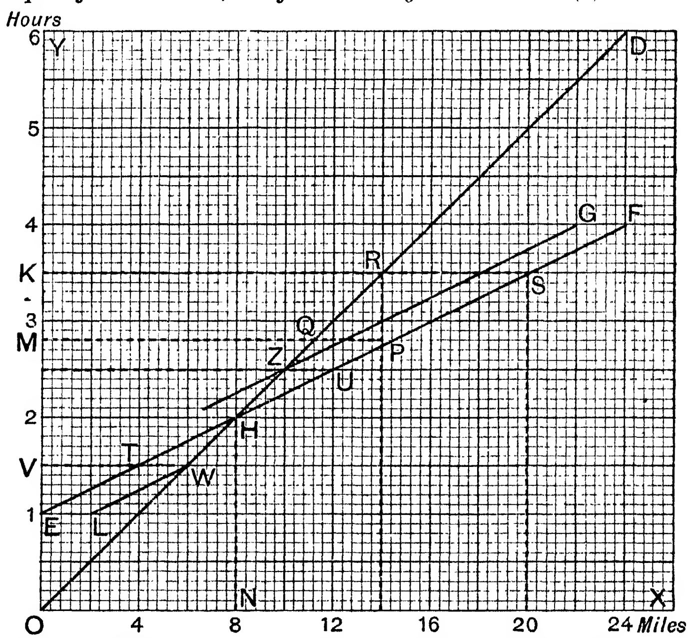
If AK denotes $1\frac{1}{2}$ hours, and KML is drawn vertically, LM is their distance apart at 1.30 p.m. From the diagram LM = 12·5 miles.

To find when the men are first 6 miles apart, take a pt. P on BE where it passes through a corner of a square, and take PQ vertically downwards equal to 6 units.

Draw QR ∥ to BE to meet AH at R. If the ordinate through R meet BE at V, VR = PQ = 6 units.

∴ the abscissa of R gives the time reqd. From the diagram we read this off as 2·8 hrs. after noon, *i.e.* at 48 minutes after 2 o'clock.

***82.** *A walks a distance of 24 miles at the rate of 4 miles an hour, and B, starting an hour later, does the distance in 3 hours less. Draw graphs of their motion, and from the diagram determine* (1) *when and*



where B overtakes A, (2) *their distance apart after B has been walking $2\frac{1}{2}$ hours,* (3) *the times when they are 2 miles apart.*

Measure distances horizontally from O along OX, taking 10 sides of a square to represent 4 miles.

Measure times vertically from O along OY, taking 10 sides of a square to represent one hour.

Take the point D whose abscissa is 24 miles and ordinate 6 hours.

Join OD. OD is the graph of A's motion, for he walks 24 miles in 6 hours.

Take the point E at the one hour point in OY. This is B's starting time.

Take the point F, whose abscissa is 24 m. and ordinate (reckoned from the level of E) 2 hrs. less than the time represented by the ordinate of D. Join EF.

EF is the graph of B's motion, for he walks the 24 miles in 2 hrs. less than A.

The co-ordinates ON, HN of the pt. H, where OD and EF intersect, give the place and time of meeting.

Thus we see that B overtakes A 8 miles from the start, and one hour after B's start.

Looking at the horizontal line PQM, we see that

PM represents the distance walked by B in time OM,

QM .... A ....

$\therefore$ PQ represents their distance apart at the time OM.

$\therefore$ taking K in OY so that $EK = 2\frac{1}{2}$ hrs. and drawing the horizontal line KRS, RS represents their distance apart when B has been walking $2\frac{1}{2}$ hours. From the figure we see that RS = 6 miles.

To determine when they are 2 miles apart, we have to find the point, or points, where the horizontal distance between the graphs represents 2 miles.

Taking EL horizontally equal to 2 miles, draw LW || to EF to meet OD at W. Draw WTV horizontally.

WT = EL = 2 miles. $\therefore$ EV represents the time after B's start when they are 2 m. apart.

From the figure EV = half an hour.

Taking the point G, 2 m. horizontally from F, draw GZ || to EF to meet OD at Z.

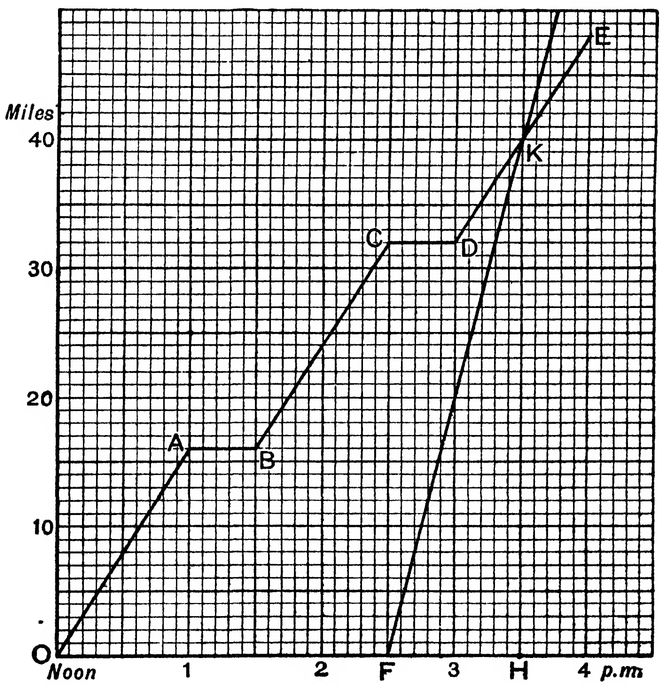
Draw the horizontal line UZ to meet EF at U.

UZ = GF = 2 miles and we see from the diagram that the corresponding time is $1\frac{1}{2}$ hours from B's start. ∴ they are again 2 m. apart in $1\frac{1}{2}$ hours after B's start.

This problem should be studied carefully.

The beginner must draw a figure for himself, using an inch to represent 4 miles, and one hour.

83. P *motors at* 16 *m. an hour, starting at noon and stopping for half an hour at the end of each hour ;* Q, *starting at* 2.30 *p.m. motors, without stoppages, at* 40 *m. an hour. Where, and at what time does he pass* P ?



Measuring time horizontally, and miles vertically, as shown in the figure, OA is P's graph for the first hour.

From 1 to 1.30 p.m. he stops, ∴ AB is his graph for that time.

In the same way BC is his graph from 1.30 to 2.30, CD from 2.30 to 3, DE from 3 to 4.

Q starting at 2.30, FK is his graph, where FH = 1 hour and HK = 40 miles.

From the figure we see that Q catches P up at 3.30 p.m. 40 miles from the start.

[*N.B.*—Remember that during a stoppage time advances, whilst the distance from the start, *i.e.* vertical distance on paper, remains the same.]

Examples. XIV. b.

1. If £1 is worth 25 francs, construct a graph from which you can read off the value of any number of shillings up to £3, in francs. Write down from the diagram the value of 35 shillings in francs, and 35 francs in shillings.

2. 60 oranges sell for six and eight pence. Make a graph to show the cost of any number up to 60, and from it write down the cost of 27 oranges, and the number of whole oranges you would get for 2*s.* 3*d.*

3. A train travels at a uniform rate for an hour and a half, and covers 40 miles in that time. Draw the graph of its motion and write down the time it takes to travel 17 miles, and how far it has travelled in 12 minutes. Give the results to the nearest mile and minute.

4. A body starts moving with a velocity of 4 ft. per second, and its velocity after t secs. is given by the formula $4+3t$. Draw a graph which gives its velocity at any time. Read off its velocity after 3 secs., and 4·5 secs., and the time when its velocity is 11·5 ft. per sec.

5. Given that 1 kilogramme = 2·2 lbs., draw a graph which will enable you to read off any number of lbs. in kilogrammes (up to 50 lbs.), and read off the values of 25 and 38 kilogrammes in lbs., and of 32·5 and 38 lbs. in kilogrammes.

6. Given that 1 cubic inch = 16·4 cubic centimetres, make a graph to convert c. cms. into c. ins., and read off the values of 80 and 40 c. cms. in c. ins., and of 2·5 c. ins. in c. cms.

7. In a Reaumur thermometer the freezing point stands at 0°, and the boiling point at 80°; in a Fahrenheit, freezing point at 32°, and boiling at 212°. Construct a graph to convert R. degrees into F. degrees, and *vice versa.* Read off 60° R. in F. degrees, and 43° F. in Reaumur degrees.

8. A man starts at noon at the rate of 4 miles an hour to walk from Cambridge to Clare, a distance of 29 miles; a second man bicycles from Clare to Cambridge, starting at 2 p.m., and riding at 10 miles an hour. Draw a graph to show where and when they meet, and determine also from it the times when they are 10 miles apart.

9. A starts running at the rate of 100 yds. in 30 secs. and B starts from the same spot 6 secs. later at the rate of 100 yds. in 12 secs. Draw a graph to find when and where B catches A up.

10. In the ten years from 1881 to 1890, the population of one town increases uniformly from 30,000 to 50,000, whilst that of another town decreases from 60,000 to 40,000. From a graph determine the year and month when the two populations were equal.

11. The top boy in a form gets 88 marks, and the last boy 33. These have to be scaled so that the top boy gets 100 and the last boy 0. Draw a graph which will effect this, and read off (to the nearest integer) the scaled marks of the boys who get 65, 54, 49.

12. Given that 1 inch $=2{\cdot}54$ centimetres, construct a graph to convert centimetres into inches. Read off the value of 5·6 cms. in inches, and the value of 4·9 inches in centimetres, as accurately as you can.

13. Given that 1 centimetre $=\cdot 39$ inches, draw a graph to convert inches into centimetres. Read off the value of 3·6 in. in centimetres, and the value of 8·6 cms. in inches, as accurately as you can.

14. On an examination paper of maximum 69 the marks gained by 10 candidates were: 60, 54, 46, 35, 32, 29, 27, 26, 25, 12. Draw a graph to raise the maximum to 100, and read off (to the nearest integer) the raised marks of the candidates.

15. 50 articles cost 4*s.* 10*d.* Construct a graph from which you can read off the cost (to the nearest halfpenny) of any number of articles up to 50. Write down the cost of 23 things, and the number you would get for 3*s.*

16. The first 100 copies of a pamphlet cost 27*s.* to print, but every 100 in excess of the first costs only 3*s.*; make a graph to show the cost of any number up to 800, and read off the cost of 370 copies. Write down the number of copies you would get for £2. 2*s.* 6*d.*

17. A clerk is paid at the rate of £120 a year: make a graph to determine (to the nearest pound) his wages for any given number of weeks. Write down his wages for 23 weeks.

18. I want a ready means of finding approximately 0·866 of any number up to 10. I select a point O at the corner of the squared paper where two thicker lines cross, and find a second point P by going 10 inches to the right and then 8·66 inches up (or 5 to the right and 4·33 up), and join O to P. The two thick lines passing through O are scaled off in inches, OX to the right, OY up. Explain clearly why the distance from OX of any point in OP is 0·866 of its distance from OY. Read off from the scales, and mark on the appropriate places on the paper, 0·866 of 3, 0·866 of 6·5, 0·866 of 4·8, and $\frac{1}{0{\cdot}866}$ of 5.

19. For a certain book it costs a publisher £100 to prepare the type and 2*s.* to print each copy. Find an expression for the total cost in pounds of x copies. Also make a diagram on the scale of 1 inch to 1000 copies and 1 inch to £100 to show the total cost of any number of copies up to 5000. Read off the cost of 2500 copies, and the number of copies costing £525.

20. A starts walking at the rate of 4 miles an hour, and 15 minutes later B starts at the rate of 8 miles an hour. Find, graphically, when and where B overtakes A.

21. Two ships 72 miles apart sail towards one another at the rates of 7 and 9 miles an hour. Find, graphically, when they meet.

22. A walks at 4 miles an hour, but takes a rest of half an hour at the end of every 4 miles. B starting at the same time and walking at a uniform rate, without any rests, catches A up just as he is starting after his third rest. Find, graphically, B's rate of travelling.

23. A travelling at 4 miles an hour, walks 4 miles, then rests for half an hour, then walks 8 miles further, and then walks straight back at the

same rate. He meets B, who walks uniformly and without resting, a mile and a half from home. Find B's rate of travelling, if he started at the same time as A.

24. A travels at 5 miles an hour, but takes a rest of half an hour at the end of each hour. B starting 2 hours after A, and travelling uniformly, without resting, overtakes A $17\frac{1}{2}$ miles from home. Find, graphically, B's rate of travelling.

25. A and B, travelling at 8 and 12 miles an hour respectively, bicycle towards one another from two places 50 miles apart, starting at the same time. Find, graphically, when and where they meet, and when they are 10 miles from one another.

26. Solve the above problem graphically, as accurately as you can, when B starts an hour after A.

27. A motorist starts to do a journey of 8 miles in half an hour, but after travelling for $22\frac{1}{2}$ minutes finds himself behind time. He quickens his pace to 24 miles an hour, and just completes his journey in time. Find his initial rate of travelling.

28. A motorist does a journey of 80 miles in 6 hours. During the first part of the journey he travels at 10 miles an hour, and during the latter part at 18 miles an hour. How far does he travel at each rate ?

*CHAPTER XV

LONG MULTIPLICATION

84. Further examples of the use of the formulae

$$(\mathbf{a} \pm \mathbf{b})^2 = \mathbf{a}^2 \pm 2\mathbf{ab} + \mathbf{b}^2.$$

Example 1. Find the expanded value of $\{x+(a+b)\}^2$.

Regarding $(a+b)$ as a single quantity,

$$\{x+(a+b)\}^2 = x^2 + 2(a+b)x + (a+b)^2$$
$$= x^2 + 2ax + 2bx + a^2 + 2ab + b^2$$

(if we wish to expand the expression fully).

Example 2. $\{a+b-c\}^2$

$$= \{(a+b)-c\}^2$$
$$= (a+b)^2 - 2(a+b)c + c^2$$
$$= a^2 + 2ab + b^2 - 2ac - 2bc + c^2 \quad \text{(expanded fully).}$$

Example 3. $(a+2b+2c+d)^2 = \{(a+2b)+(2c+d)\}^2$

$$= (a+2b)^2 + 2(a+2b)(2c+d) + (2c+d)^2$$
$$= a^2 + 4ab + 4b^2 + 2(2ac + ad + 4bc + 2bd) + 4c^2 + 4cd + d^2$$
$$= a^2 + 4ab + 4b^2 + 4ac + 2ad + 8bc + 4bd + 4c^2 + 4cd + d^2.$$

Examples. XV. a.

Find the fully expanded values of the following :

1. $\{x+(a-b)\}^2$.
2. $\{x-(a+b)\}^2$.
3. $\{(a+b)+2\}^2$.
4. $\{a+(b+c)\}^2$.
5. $\{a-(b+c)\}^2$.
6. $\{a-(b-c)\}^2$.
7. $\{(a-b)-2\}^2$.
8. $\{2x+(y+z)\}^2$.
9. $\{x-(2y+z)\}^2$.
10. $(a+2b+3c)^2$.
11. $(a-2b+3c)^2$.
12. $(3x+a-b)^2$.
13. $(2x+3a-b)^2$.
14. $(2x^2+x+1)^2$.
15. $(3x^2-x+1)^2$.
16. $(x^2+x-8)^2$.
17. $(x^2+2x+1)^2$.
18. $(x^2-x-4)^2$.
19. $(2x^2-x-5)^2$.
20. $(x+y-3)^2$.
21. $(2x-y+4)^2$.
22. $(1-x+x^2)^2$.
23. $(2+x-x^2)^2$.
24. $(3-x+2x^2)^2$.
25. $(5-2x+3x^2)^2$.
26. $(a+b+c+d)^2$.
27. $(a+b+c-d)^2$.
28. $(a-b+c-d)^2$.
29. $(a+b+2c+d)^2$.
30. $(a+b+2c-2d)^2$.
31. $(x+y+z-3)^2$.
32. $(x-y-z+3)^2$.
33. $(2x-y+2z-1)^2$.
34. $(3a-2b+2c-d)^2$.
35. $(x^3+x^2+x+1)^2$.
36. $(x^3+2x^2-2x+1)^2$.
37. $(x^3-x^2+x-1)^2$.
38. $(x^3-3x^2+3x-1)^2$.

85. Further examples of the use of the formula.

$$(a+b)(a-b)=a^2-b^2.$$

Example 1. $(a+b+c)(a+b-c)=(a+b)^2-c^2$

[Looking upon $a+b$ as a single quantity.]

$$=a^2+2ab+b^2-c^2.$$

Example 2.

$$\begin{aligned}&(x+a-2b)(x-a+2b)\\&=(x+\overline{a-2b})(x-\overline{a-2b})\\&=x^2-(a-2b)^2\\&=x^2-a^2+4ab-4b^2.\end{aligned}$$

Example 3.

$$\begin{aligned}&(a+b+c+d)(a+b-c-d)\\&=(\overline{a+b}+\overline{c+d})(\overline{a+b}-\overline{c+d})\\&=(a+b)^2-(c+d)^2\\&=a^2+2ab+b^2-c^2-2cd-d^2.\end{aligned}$$

Examples. XV. b.

1. $(a-b+c)(a-b-c)$.
2. $(a+b+2c)(a+b-2c)$.
3. $(x+y+1)(x+y-1)$.
4. $(x+2y+b)(x+2y-b)$.
5. $(a+b+x)(a-b-x)$.
6. $(a+2b-c)(a-2b+c)$.
7. $(2x+a+b)(2x+a-b)$.
8. $(3y-a-b)(3y+a+b)$.
9. $(a-4x+y)(a+4x-y)$.
10. $(1+a+b)(1-a-b)$.
11. $(4-a+b)(4+a-b)$.
12. $(a^2+ab+b^2)(a^2-ab+b^2)$.
13. $(1-a-b)(1-a+b)$.
14. $(x+2y+b)(x+2y-b)$.
15. $(p-2q+3r)(p+2q-3r)$.
16. $(1-2x+3y)(1+2x-3y)$.
17. $(x+3y-4)(x+3y+4)$.
18. $(x^3+x+1)(x^2-x+1)$.
19. $(1-2x+7y)(1-2x-7y)$.
20. $(2x+3y-5)(2x+3y+5)$.

21. $(3x^2+x-2)(3x^2-x+2)$. **22.** $(2x-4y-5)(2x+4y+5)$.
23. $(5a-2b+3)(5a+2b+3)$. **24.** $(a^2-2ab+b^2)(a^2+2ab+b^2)$.
25. $(1-2x+3x^2)(1+2x+3x^2)$. **26.** $(a-b+c-d)(a-b-c+d)$.
27. $(2x+y+a+b)(2x+y-a-b)$. **28.** $(x+a+y-b)(x+a-y+b)$.
29. $(2x-a-y+2b)(2x-a+y-2b)$.
30. $(3x-2a+2y-3b)(3x-2a-2y+3b)$.
31. $(1-x+y-z)(1-x-y+z)$. **32.** $(2-a-3b+c)(2-a+3b-c)$.

86. When we have more than two terms in the multiplier or multiplicand, the process is similar to that in simpler cases.

Example 1. Multiply a^2+ab+b^2 by $a-b$.

$$\begin{array}{llll}
a^2 & +ab & +b^2 & \\
a & -b & & \\
\hline
a^3 & +a^2b & +ab^2 & \\
 & -a^2b & -ab^2 & -b^3 \\
\hline
a^3 & & & -b^3
\end{array}$$

Example 2. Multiply $x^2-2xy+4y^2$ by $x^2+2xy+4y^2$.

$$\begin{array}{lllll}
x^2 & -2xy & +4y^2 & & \\
x^2 & +2xy & +4y^2 & & \\
\hline
x^4 & -2x^3y & +4x^2y^2 & & \\
 & 2x^3y & -4x^2y^2 & +8xy^3 & \\
 & & 4x^2y^2 & -8xy^3 & +16y^4 \\
\hline
x^4 & & +4x^2y^2 & & +16y^4
\end{array}$$

Example 3. Multiply $4yz-3xy-2xz+x^2+y^2-z^2$ by $-y+2x-z$.

Here we first arrange both multiplier and multiplicand in order of powers of x, and during the multiplication place like terms under one another.

$$\begin{array}{llllllllll}
x^2 & -3xy & -2xz & +y^2 & +4yz & -z^2 & & & & \\
2x & -y & -z & & & & & & & \\
\hline
2x^3 & -6x^2y & -4x^2z & +2xy^2 & +8xyz & -2xz^2 & & & & \\
 & -x^2y & & +3xy^2 & +2xyz & & -y^3 & -4y^2z & +yz^2 & \\
 & & -x^2z & & +3xyz & +2xz^2 & & -y^2z & -4yz^2 & +z^3 \\
\hline
2x^3 & -7x^2y & -5x^2z & +5xy^2 & +13xyz & & -y^3 & -5y^2z & -3yz^2 & +z^3
\end{array}$$

87. By multiplication it will be found that

$$(\mathbf{a}+\mathbf{b})^3=\mathbf{a}^3+3\mathbf{a}^2\mathbf{b}+3\mathbf{a}\mathbf{b}^2+\mathbf{b}^3.$$

$$(\mathbf{a}-\mathbf{b})^3=\mathbf{a}^3-3\mathbf{a}^2\mathbf{b}+3\mathbf{a}\mathbf{b}^2-\mathbf{b}^3.$$

These results are useful and should be committed to memory.

88. ***Analogy between Algebraical and Arithmetical methods of multiplication.***

Multiply 213 by 23.

$$\begin{array}{r} 213 \\ 23 \\ \hline 426 \\ 639 \\ \hline 4899 \end{array}$$

This is an abbreviated form of the following:

$$\begin{array}{llll} 2\,.\,10^2+1\,.\,10 & +3 & & \\ 2\,.\,10 & +3 & & \\ \hline 4\,.\,10^3+2\,.\,10^2 & +6\,.\,10 & & \\ & 6\,.\,10^2+3\,.\,10+9 & & \\ \hline 4\,.\,10^3+8\,.\,10^2+9\,.\,10+9=4899 & & & \end{array}$$

If we now multiply $2x^2+x+3$ by $2x+3$ we at once see the analogy between the two methods.

$$\begin{array}{rrrr} 2x^2 & +\;x & +3 & \\ & 2x & +3 & \\ \hline 4x^3 & +2x^2 & +6x & \\ & 6x^2 & +3x & +9 \\ \hline 4x^3 & +8x^2 & +9x & +9 \end{array}$$

89. Detached coefficients. The work in the above example is much shortened if we omit the powers of x, just as we omit powers of 10 in Arithmetic.

The multiplication then stands thus:

$$\begin{array}{rrrr} 2x^2 & +\;x & +3 & \\ & 2x & +3 & \\ \hline 4 & +2 & +6 & \\ & 6 & +3 & +9 \\ \hline 4x^3 & +8x^2 & +9x & +9 \end{array}$$

inserting the requisite powers of x in the last line.

Example 1. Multiply $4x^3-3x^2-11x+2$ by $2x^2-5x+9$.

$$\begin{array}{rrrrrr} 4x^3 & -\;3x^2 & -11x & +\;2 & & \\ 2x^2 & -\;5x & +\;9 & & & \\ \hline 8 & -\;6 & -22 & +\;4 & & \\ & -20 & +15 & +55 & -\;10 & \\ & & 36 & -27 & -\;99 & +18 \\ \hline 8x^5 & -26x^4 & +29x^3 & +32x^2 & -109x & +18 \end{array}$$

When powers of x are missing, 0 must be inserted as in Arithmetic.

Example 2. Multiply $3x^3-7x+9$ by $2x^2-3$.

$$\begin{array}{llllll}
3x^3+0x^2- & 7x & + & 9 & & \\
2x^2+0x & - & 3 & & & \\
\hline
6 \quad +0 & -14 & +18 & & & \\
 & -\ 9 & -\ 0 & +21 & -27 & \\
\hline
6x^5 & -23x^3 & +18x^2 & +21x & -27 &
\end{array}$$

Examples. XV. c.

[*Nearly all the following examples are best done by the method of detached coefficients.*]

Multiply

1. x^3+2x^2+x-4 by $x-2$.
2. $a^2+2ab+b^2$ by $a-b$.
3. x^2+xy+y^2 by $x-y$.
4. x^2-4y^2 by $x+3y$.
5. x^2+2x-5 by x^2-3x+6.
6. x^2+2x+3 by x^2-2x-5.
7. $a^3-3a^2b-3ab^2$ by $a^2-5ab+2b^2$.
8. x^2+xy+y^2 by $-x^2+xy-y^2$.
9. $a^2-5ab+6b^2$ by $3ab+2a^2-b^2$.
10. x^2+x+1 by $x-1$.
11. x^2-2x+4 by $x+2$.
12. $4x^2+2x+1$ by $2x-1$.
13. $x-2y$ by $x^2+2xy+4y^2$.
14. $9a^2-6ab+4b^2$ by $3a+2b$.

Find the product of the following:

15. x^2-x+1 and $x+1$.
16. a^2-ab+b^2 and $a+b$.
17. $x-2$ and x^2+2x+4.
18. x^2+3y^2 and $x-4y$.
19. x^3-2x^2+4x+5 and $x-3$.
20. $x+x^2-5$ and x^2-x-7.
21. $c^2-5cd-5d^2$ and $c^2+5cd+5d^2$.
22. x^2+xy+y^2 and x^2-xy+y^2.
23. $ab+cd-ac-bd$ and $ab+cd+ac+bd$.
24. $2a^2-3ab+4b^2$ and $-5a^2+3ab+4b^2$.
25. x^2+3x+1 and x^2-5x+2.
26. $3x^2-7x+5$ and $4x^2-2x+1$.
27. $4+3x-2x^2$ and $5-x-2x^2+x^3$.
28. $2-x+3x^2y$ and $3+2x-x^2y$.
29. $x^2+2xy+y^2+x+y+1$ and $x+y-1$.
30. x^4-5x^2+6 and x^2+3x+4.
31. $3x-1+4x^3-5x^2$ and $2x-4+x^3$.
32. $1+2a^2-3a^4-a$ and $3a-5+2a^2$.
33. $3x^3-2x^2y-xy^2$ and $7xy-5y^2$.
34. $2(x^2+2xy+y^2)$ and $3(x-y)^2$.
35. $3(a^2-ab+b^2)$ and $\frac{1}{3}(a+b)^2$.
36. $a^2+b^2+c^2-bc-ca-ab$ and $a+b+c$.

*CHAPTER XVI

LONG DIVISION

90. Example 1. Divide $8x^3-6x^2+3x-18$ by $2x-3$.

$$\begin{array}{r|l}
2x-3 & 8x^3-6x^2+3x-18 \ (\,4x^2+3x+6 \\
 & \underline{8x^3-12x^2} \\
 & \qquad 6x^2+3x \\
 & \qquad \underline{6x^2-9x} \\
 & \qquad\qquad 12x-18 \\
 & \qquad\qquad \underline{12x-18}
\end{array}$$

Before starting the work of division both divisor and dividend should be arranged in the same order (ascending or descending) of powers of one of the symbols used.

Example 2. Divide $5x-3+x^3+x^4-4x^2$ by $2x-3+x^2$.

Arranging the expression in descending powers of x,

$$\begin{array}{rll}
x^2+2x-3\,) & x^4+x^3-4x^2+5x-3\,(\,x^2-x+1 & \\
 & \underline{x^4+2x^3-3x^2} & \ldots\ldots(1) \\
 & \phantom{x^4+{}}-x^3-x^2+5x & \ldots\ldots(2) \\
 & \phantom{x^4+{}}\underline{-x^3-2x^2+3x} & \ldots\ldots(3) \\
 & \phantom{x^4+2x^3-{}}x^2+2x-3 & \ldots\ldots(4) \\
 & \phantom{x^4+2x^3-{}}\underline{x^2+2x-3} &
\end{array}$$

$\frac{x^4}{x^2}=x^2$; $\therefore$ x^2 is the first term of the quotient.

$$x^2(x^2+2x-3)=x^4+2x^3-3x^2,$$

and we thus obtain line (1) as in Arithmetic.

Line (2) is obtained by subtraction, and by bringing down the term $+5x$.

$\frac{-x^3}{x^3}=-x$; $\therefore$ $-x$ is the second term of the quotient.

$$-x(x^2+2x-3)=-x^3-2x^2+3x,$$

and we thus obtain line (3).

Line (4) is obtained in the same way as line (2).

$\frac{x^2}{x^2}=1$; $\therefore$ 1 is the last term of the quotient.

There is no remainder, as we see by subtracting the last line.

91. The analogy between the Algebraical and Arithmetical methods of division is at once seen if we compare the following:

Arithmetical method. *Algebraical method.*

$$\begin{array}{rl}
121\,) & 14883\,(\,123 \\
 & \underline{121} \\
 & 278 \\
 & \underline{242} \\
 & 363 \\
 & \underline{363}
\end{array}$$

$$\begin{array}{rl}
10^2+2.10+1\,) & 10^4+4.10^3+8.10^2+8.10+3\,(\,10^2+2.10+3 \\
 & \underline{10^4+2.10^3+10^2} \\
 & \phantom{10^4+{}}2.10^3+7.10^2+8.10 \\
 & \phantom{10^4+{}}\underline{2.10^3+4.10^2+2.10} \\
 & \phantom{10^4+2.10^3+{}}3.10^2+6.10+3 \\
 & \phantom{10^4+2.10^3+{}}\underline{3.10^2+6.10+3}
\end{array}$$

$$\begin{array}{rl}
x^2+2x+1\,) & x^4+4x^3+8x^2+8x+3\,(\,x^2+2x+3 \\
 & \underline{x^4+2x^3+x^2} \\
 & \phantom{x^4+{}}2x^3+7x^2+8x \\
 & \phantom{x^4+{}}\underline{2x^3+4x^2+2x} \\
 & \phantom{x^4+2x^3+{}}3x^2+6x+3 \\
 & \phantom{x^4+2x^3+{}}\underline{3x^2+6x+3}
\end{array}$$

Example 1. Divide $x^3 - y^3$ by $x - y$.

$$\begin{array}{l}
x - y\ )\ x^3 - y^3\ (\ x^2 + xy + y^2 \\
\qquad\quad \underline{x^3 - x^2y} \\
\qquad\qquad\quad x^2y - y^3 \\
\qquad\qquad\quad \underline{x^2y - xy^2} \\
\qquad\qquad\qquad\quad xy^2 - y^3 \\
\qquad\qquad\qquad\quad \underline{xy^2 - y^3}
\end{array}$$

Example 2. Divide $x^5 + 1$ by $x + 1$.

$$\begin{array}{l}
x + 1\ )\ x^5 + 1\ (\ x^4 - x^3 + x^2 - x + 1 \\
\qquad\quad \underline{x^5 + x^4} \\
\qquad\qquad - x^4 + 1 \\
\qquad\qquad \underline{- x^4 - x^3} \\
\qquad\qquad\qquad x^3 + 1 \\
\qquad\qquad\qquad \underline{x^3 + x^2} \\
\qquad\qquad\qquad\quad - x^2 + 1 \\
\qquad\qquad\qquad\quad \underline{- x^2 - x} \\
\qquad\qquad\qquad\qquad\quad x + 1 \\
\qquad\qquad\qquad\qquad\quad \underline{x + 1}
\end{array}$$

92. Detached Coefficients. From the preceding we see that in Division as in Multiplication we can shorten the work by using the method of *detached coefficients*.

Example 1. Divide $6x^4 - 7x^3 + 7x^2 + 18x - 24$ by $2x^2 - 3x + 6$.

$$\begin{array}{l}
2 - 3 + 6\ )\ 6 - 7 + \ 7 + 18 - 24\ (\ 3x^2 + x - 4 \\
\qquad\qquad\ \underline{6 - 9 + 18} \\
\qquad\qquad\qquad 2 - 11 + 18 \\
\qquad\qquad\qquad \underline{2 - \ 3 + \ 6} \\
\qquad\qquad\qquad\quad - \ 8 + 12 - 24 \\
\qquad\qquad\qquad\quad \underline{- \ 8 + 12 - 24}
\end{array}$$

Example 2. Divide $6x^5 - 23x^3 + 18x^2 + 21x - 27$ by $2x^2 - 3$.

$$\begin{array}{l}
2 + 0 - 3\ )\ 6 + 0 - 23 + 18 + 21 - 27\ (\ 3x^3 - 7x + 9 \\
\qquad\qquad\ \underline{6 + 0 - \ 9} \\
\qquad\qquad\qquad - 14 + 18 + 21 \\
\qquad\qquad\qquad \underline{- 14 + \ 0 + 21} \\
\qquad\qquad\qquad\qquad 18 + \ 0 - 27 \\
\qquad\qquad\qquad\qquad \underline{18 + \ 0 - 27}
\end{array}$$

Examples. XVI. a.

[*All the following divisions may be done by the method of Detached Coefficients.*]

Divide

1. $x^3 - 3x^2 + 4x + 28$ by $x + 2$.
2. $x^3 - 9x^2 + 13x + 15$ by $x - 3$.
3. $2x^3 - 3x^2 + 7x - 3$ by $2x - 1$.
4. $6x^3 + 2x^2 + 11x - 10$ by $3x - 2$.
5. $24x^3 - 35x^2 - 36x + 5$ by $8x - 1$.
6. $15 - 17x - 30x^2 - 28x^3$ by $3 - 7x$.
7. $x^3 + 3x^2 + 3x + 1$ by $x^2 + 2x + 1$.
8. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^2 - 2xy + y^2$.
9. $x^3 - 6x^2 + 12x - 8$ by $x^2 - 4x + 4$.
10. $8x^3 + 12x^2 + 6x + 1$ by $4x^2 + 4x + 1$.
11. $27a^3 - 54a^2b + 36ab^2 - 8b^3$ by $9a^2 - 12ab + 4b^2$.
12. $125x^3 - 27y^3 - 225x^2y + 135xy^2$ by $25x^2 + 9y^2 - 30xy$.
13. $9x^3 - 18x^2 + 26x - 24$ by $3x - 4$.
14. $x^3 - 4x^2 + 5x - 2$ by $x^2 - 3x + 2$.
15. $x^3 - y^3$ by $x - y$.
16. $x^3 - 27$ by $x^2 + 3x + 9$.
17. $27x^3 - 1$ by $3x - 1$.
18. $a^3 + b^3$ by $a + b$.
19. $x^4 - 1$ by $x - 1$.
20. $x^4 - 1$ by $x + 1$.

21. x^3+x^2+x+1 by $x+1$.
22. x^3-x^2+x-1 by $x-1$.
23. $81x^4-16$ by $3x+2$.
24. x^4+x^2+1 by x^2+x+1.
25. x^4+x^2+1 by x^2-x+1.
26. $x^4+4x^3+6x^2+4x+1$ by x^2+2x+1.
27. $x^3-6x^2+12x-8$ by $x-2$.
28. $12x^3-38x^2+38x-20$ by $6x^2-7x+5$.
29. $6a^2-2a-a^4-4a^3+a^5$ by a^3-4a+2.
30. $-141x^2-180x+5x^4-32-58x^5+24x^6+92x^3$ by $2x^2-4-3x$.
31. $6x^5-x^4+10x^3-14x^2-25$ by $3x^2+4x+5$.
32. $x^6-3x^5+x^3+358x-357$ by x^2+2x-3.

Harder Examples in Division.

93. Example 1. Divide $9a^2-4b^2-c^2+4bc$ by $3a-2b+c$.

$$\begin{array}{r l}
3a-2b+c\ ) & 9a^2-4b^2-c^2+4bc\ (\ 3a+2b-c \\
& \underline{9a^2-6ab+3ac} \\
& 6ab-3ac-4b^2+4bc-c^2 \\
& \underline{6ab \qquad\ \ -4b^2+2bc \quad\ \ } \\
& \qquad -3ac \qquad +2bc-c^2 \\
& \qquad \underline{-3ac \qquad +2bc-c^2}
\end{array}$$

Example 2. Divide $a^3-b^3+c^3+3abc$ by $a-b+c$.

Arranging divisor and dividend in descending powers of a,

$$\begin{array}{r l l}
a-b+c\ ) & a^3+3abc-b^3+c^3\ (\ a^2+ab-ac+b^2+bc+c^2 & \\
& \underline{a^3-a^2b+a^2c} & \\
& \quad a^2b-a^2c+3abc & \text{(rem. arranged in descending powers of } a\text{)} \\
& \quad \underline{a^2b-ab^2+\ abc} & \\
& \qquad -a^2c+ab^2+2abc & (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots) \\
& \qquad \underline{-a^2c \qquad +\ abc-ac^2} & \text{(placing like terms under one another)} \\
& \qquad\qquad ab^2+\ abc+ac^2-b^3 & \text{(bringing down } -b^3\text{)} \\
& \qquad\qquad \underline{ab^2 \qquad\qquad -b^3+b^2c} & \\
& \qquad\qquad\quad abc+ac^2-b^2c & \\
& \qquad\qquad\quad \underline{abc \qquad -b^2c+bc^2} & \\
& \qquad\qquad\qquad ac^2-bc^2+c^3 & \text{(bringing down } c^3\text{)} \\
& \qquad\qquad\qquad \underline{ac^2-bc^2+c^3} &
\end{array}$$

Example 3. Divide $\frac{9}{16}x^4-\frac{3}{4}x^3y-\frac{7}{4}x^2y^2+\frac{4}{3}xy^3+\frac{16}{9}y^4$ by $\frac{3}{2}x^2-xy-\frac{8}{3}y^2$.

$$\begin{array}{r l}
\frac{3}{2}x^2-xy-\frac{8}{3}y^2\ ) & \frac{9}{16}x^4-\frac{3}{4}x^3y-\frac{7}{4}x^2y^2+\frac{4}{3}xy^3+\frac{16}{9}y^4\ (\ \frac{3}{8}x^2-\frac{xy}{4}-\frac{2}{3}y^2 \\
\frac{9}{16}x^4\div\frac{3}{2}x^2=\frac{9x^4}{16}\times\frac{2}{3x^2}=\frac{3x^2}{8}. & \underline{\frac{9}{16}x^4-\frac{3}{8}x^3y-\ x^2y^2} \\
& \qquad -\frac{3}{8}x^3y-\frac{3}{4}x^2y^2+\frac{4}{3}xy^3 \\
-\frac{3x^3y}{8}\div\frac{3x^2}{2}=-\frac{3x^3y}{8}\times\frac{2}{3x^2}=-\frac{xy}{4}. & \qquad \underline{-\frac{3}{8}x^3y+\ \frac{x^2y^2}{4}+\frac{2}{3}xy^3} \\
-x^2y^2\div\frac{3x^2}{2}=-x^2y^2\times\frac{2}{3x^2}=-\frac{2}{3}y^2. & \qquad\qquad -\ x^2y^2+\frac{2}{3}xy^3+\frac{16}{9}y^4 \\
& \qquad\qquad \underline{-\ x^2y^2+\frac{2}{3}xy^3+\frac{16}{9}y^4}
\end{array}$$

Examples. XVI. b.

Divide

1. $a^2+4ab+4b^2-c^2$ by $a+2b-c$.
2. a^4+4b^4 by $a^2-2ab+2b^2$.
3. $a^2+b^2+c^2+2ab+2bc+2ac$ by $a+b+c$.
4. $9a^2-4b^2-c^2-4bc$ by $3a-2b-c$.
5. x^6-a^6 by x^2+ax+a^2.
6. $a^2-b^2-c^2+2bc$ by $a+b-c$.
7. $2x^4+x^5-31x+9x^2+15+4x^3$ by $2x+x^2-3$.
8. $x^3-y^3+6y^2-12y+8$ by $x-y+2$.
9. $1+a^5+a^{10}$ by a^2+a+1.
10. $6x^4+5y^4-13xy(x^2+y^2)+23x^2y^2$ by $3x^2+y^2-2xy$.
11. $a^3+b^3+c^3-3abc$ by $a+b+c$.
12. $a^3+b^3-c^3+3abc$ by $a+b-c$.
13. $x^3-y^3+8+6xy$ by $x-y+2$.
14. x^6-1 by $x+1$.
15. $x^4+x^2y^2+y^4$ by x^2-xy+y^2.
16. $a^3+b^3+c^3-3abc$ by $a^2+b^2+c^2-ab-bc-ac$.
17. $a^2(b+c)+b^2(c+a)-c^2(a+b)+abc$ by $a+b-c$.
18. x^8-y^8 by x^2-y^2.
19. $64a^6-1$ by $2a-1$.
20. $a^2(b-c)+b^2(c-a)+c^2(a-b)$ by $a-b$.
21. $x^3+\frac{2}{3}ax^2+\frac{8}{3}a^2x-a^3$ by $x-\frac{1}{3}a$.
22. $\frac{x^3}{2}+\frac{2}{3}ax^2-\frac{8}{3}a^2x+\frac{4}{3}a^3$ by $\frac{x}{2}-\frac{a}{3}$.
23. $\frac{x^3}{8}+\frac{x^2y}{2}-\frac{17xy^2}{12}+\frac{y^3}{2}$ by $\frac{x^2}{2}-xy+\frac{y^2}{3}$.
24. $\frac{a^3}{8}-\frac{b^3}{27}$ by $\frac{a}{2}-\frac{b}{3}$.
25. $\frac{x^3}{64}+\frac{y^3}{125}$ by $\frac{x}{4}+\frac{y}{5}$.
26. $\frac{a^4}{16}+\frac{a^2b^2}{36}+\frac{b^4}{81}$ by $\frac{a^2}{4}+\frac{ab}{6}+\frac{b^2}{9}$.
27. $\frac{a^3}{27}-\frac{a^2b}{21}+\frac{ab^2}{49}-\frac{b^3}{343}$ by $\frac{a}{3}-\frac{b}{7}$.
28. $\frac{a^3}{125}-\frac{3a^2b}{100}+\frac{3ab^2}{80}-\frac{b^3}{64}$ by $\frac{a^2}{25}-\frac{ab}{10}+\frac{b^2}{16}$.

Remainder Theorem.

94. *If ax^2+bx+c is divided by $x-p$ until the remainder is independent of x, that remainder will be ap^2+bp+c.*

$$
\begin{array}{r|l}
x-p\ ) & ax^2+bx+c\ (\ ax+(ap+b) \\
 & \underline{ax^2-apx} \\
 & (ap+b)x+c \\
 & \underline{(ap+b)x-(ap+b)p} \\
 & \qquad\qquad ap^2+bp+c
\end{array}
$$

This proves the theorem.

It should be observed that this remainder may be obtained by substituting p for x in the dividend.

The above is true for all values of the symbols used, and hence

$$\text{when } 3x^2-4x+5 \text{ is divided by } x-2,$$

$$\begin{aligned}\text{the remainder} &= 3\times 2^2-4\times 2+5\\ &= 12-8+5=9.\end{aligned}$$

This of course can be tested by actual division.

Again when $4x^2-7x+9$ is divided by $x+5$,

$$\begin{aligned}\text{the remainder} &= 4(-5)^2-7(-5)+9\\ &= 100+35+9\\ &= 144.\end{aligned}$$

95. *If ax^3+bx^2+cx+d is divided by $x-p$ until the remainder is independent of x, that remainder will be*

$$ap^3+bp^2+cp+d.$$

First method. Performing the actual division,

$$\begin{array}{l} x-p\;)\;ax^3+bx^2+cx+d\;(\;ax^2+(ap+b)x+(ap^2+bp+c)\\ \qquad\quad \underline{ax^3-apx^2}\\ \qquad\quad (ap+b)x^2+cx\\ \qquad\quad \underline{(ap+b)x^2-(ap+b)px}\\ \qquad\qquad\qquad (ap^2+bp+c)x+d\\ \qquad\qquad\qquad \underline{(ap^2+bp+c)x-(ap^2+bp+c)p}\\ \qquad\qquad\qquad\qquad\qquad\quad ap^3+bp^2+cp+d \end{array}$$

This proves the theorem.

As before, the remainder may be obtained by substituting p for x in the dividend.

When $4x^3-3x^2+7x-9$ is divided by $x-11$,

$$\begin{aligned}\text{the remainder} &= 4\times 11^3-3\times 11^2+7\times 11-9\\ &= 5324-363+77-9\\ &= 5029.\end{aligned}$$

When x^3-4x^2+6x-4 is divided by $x-2$,

$$\begin{aligned}\text{the remainder} &= 2^3-4\times 2^2+6\times 2-4\\ &= 8-16+12-4\\ &= 0.\end{aligned}$$

$\therefore$ x^3-4x^2+6x-4 is divisible by $x-2$ without remainder.

We thus have a ready means of testing whether any expression is exactly divisible by a given binomial expression.

Second method. When ax^3+bx^2+cx+d is divided by $x-p$ until the remainder is independent of x, let P denote the quotient, and R the remainder.

Then $ax^3+bx^2+cx+d=(x-p)\times \mathsf{P}+\mathsf{R}$. ....(1)

[Just as in Arithmetic when we divide 57 by 9, $57=9\times 6+3$.]

Considering the equation (1), R is independent of x, by hypothesis. Also the equation is true whatever value we assign to x.

Let $x=p$. Then the equation becomes

$$ap^3+bp^2+cp+d=\mathsf{R}, \quad \text{for} \quad (x-p)\mathsf{P}=(p-p)\mathsf{P}=0.$$

This proves the theorem.

96. *For what value of p is $x^2-(p+2)x+6$ divisible by $x-p$ without remainder?*

When the division is performed the remainder, by the preceding articles,

$$\begin{aligned} &=p^2-(p+2)p+6 \\ &=p^2-p^2-2p+6 \\ &=-2p+6. \end{aligned}$$

$\therefore$ the reqd. value of p is obtained by equating this remainder to zero, in which case

$$\begin{aligned} -2p+6&=0, \\ 2p&=6, \\ p&=3. \end{aligned}$$

97. *For what value of p is $x^3-(p+6)x^2+(6p+c)x+d$ divisible by $x-p$ without remainder?*

When the division is performed the remainder

$$\begin{aligned} &=p^3-(p+6)p^2+(6p+c)p+d \\ &=p^3-p^3-6p^2+6p^2+cp+d \\ &=cp+d. \end{aligned}$$

$\therefore$ the reqd. value of p is obtained from the equation

$$cp+d=0,$$

i.e.

$$\begin{aligned} cp&=-d, \\ p&=-\frac{d}{c}. \end{aligned}$$

Examples. XVI. c.

Without actual division, find the remainder when

1. $x^3-7x^2+11x-5$ is divided by $x-3$.
2. $2x^3+7x^2-9x+2$ is divided by $x-2$.

3. x^3-3x^2-4x+6 is divided by $x+2$.

4. $4x^3-5x^2+11x-7$ is divided by $x+9$.

5. $5x-6x^2-7+2x^3$ is divided by $2x-3$.

6. $4x^4-3x^2+8$ is divided by x^2-3.

7. For what value of p is $3x^2-px+10$ divisible by $3x-5$ without remainder ?

8. For what value of p is x^2-7x+p divisible by $x-2$ without remainder ?

9. For what value of p is $3x^3-7x^2-9x-p$ divisible by $x-3$ without remainder ?

Employ the second method of Art. 95 to find the remainder when the following divisions are performed :

10. $(x^3-7x^2-11x+16)\div(x-3)$.

11. $(4x^3-5x^2+7x-3)\div(2x+3)$.

12. $(9x^4-4x^2+16)\div(x^2-2)$.

13. $(4x^6+5x^4-4x^2-7)\div(2x^2-3)$.

Employ the second method of Art. 95 to prove that there is no remainder when the following divisions are performed :

14. $(x^4-y^4)\div(x-y)$.

15. $(x^{11}-y^{11})\div(x-y)$.

16. $(x^9+y^9)\div(x+y)$.

17. $(a^{12}-b^{12})\div(a^2-b^2)$.

CHAPTER XVII

REVISION PAPERS

XVII. a.

1. In the following expression, first remove the brackets, then rebracket the coefficients of the different powers of x, making the first term in each bracket positive:

$$(x-p)(x-q)-(x+q)(x+r)+(x-r)(x-p).$$

2. Plot the points (10, 5), (-5, 15), (10, 22) and find the area of the triangle formed by joining them.

3. Draw the graphs of $\frac{x}{10}+\frac{y}{12}=1$, and $5y=6x$. Hence solve these simultaneous equations, and verify your solution by algebra.

4. A bill of £1. 3*s*. 3*d*. is paid in half-crowns and three-penny pieces. If there were 12 coins altogether, how many were there of each kind.

5. Multiply x^2-x+2 by x^2+x+2. Check your answer by using $x=2$.

6. Divide $x^3-4x^2y+3xy^2-12y^3$ by $x-4y$.

7. Find the remainder when $2x^4-x^3+10x^2-2x+18$ is divided by $2x^2+x+5$.

XVII. b.

1. A is x years old, and B is y years younger.

(i) What is the sum of their ages ?
(ii) What will be the sum of their ages 10 years hence ?
(iii) What was the sum of their ages 10 years ago ?
(iv) What was the difference of their ages 10 years ago ?

2. Plot the points (10, 4), (-7, 4), (-7, 13), (10, 13) and find the area of the quadrilateral formed by joining them.

3. In the same diagram draw the graphs of

$$\frac{x}{12}+\frac{y}{16}=1, \quad 4x-3y=0, \quad y-x=2.$$

What do you deduce as to the three simultaneous equations ?

4. The sum of the two digits of a number is ten. By reversing the digits the number is increased by 36. Find the number.

5. Multiply $a^2+2ab-b^2$ by $a^2+2ab+b^2$. Check your result by putting $a=b=1$.

6. Find the continued product of $2a-b$, $2a+b$, $4a^2+b^2$.

7. Divide $6ax^3-x^4-9a^2x^2+4a^4$ by $2a^2+3ax-x^2$.

XVII. c.

1. I buy apples at the rate of x apples for threepence.

(i) How many do I get for half-a-crown ?
(ii) What will 100 apples cost me ?

2. Find the length of the line joining the points (1·6, 3·6), (-1·6, 1·2).

3. Make a table to show six pairs of corresponding values of x and y which satisfy the equation $3x+4y=13$. Choosing a suitable unit, plot the points accurately, and draw the graph.

4. Find the value of $(x^2+1)^4$. Check your result by using $x^2=1$.

5. Express the following in the form of an algebraic equation. The cost of x things at half-a-crown each, y things at $9d.$ each, and z things at $4\frac{1}{2}d.$ each is £a.

6. Find the continued product of x^2-3y^2, x^2+3y^2, x^4+9y^4.

7. Divide $6x^4-21-5x^2-x-19x^3$ by $2x^2-5x-7$.

XVII. d.

1. A man runs at the rate of x yards in y minutes.

(i) How many yards does he run in an hour ?
(ii) How long does he take to run a mile ?

2. Plot the points (0, 0), (8, 5), (12, 18), (0, 23) and find the area of the quadrilateral formed by joining them.

3. Draw the graphs of $3x-4y=10$, and $3x+5y=15$, and hence find approximate solutions of the simultaneous equations. Verify by substitution.

4. Multiply x^4-3x^2+1 by x^2-3x+2. Check your result by putting $x=10$.

5. Find two consecutive even numbers such that 73 times their difference is equal to their sum.

6. Simplify $(x^2+ax+b)^2-(x^2-ax+b)^2$.

7. Divide $a^2-5ab+6b^2-a+b-2$ by $a-2b+1$.

XVII. e.

1. How far does a train travel :

(i) In x hours at y miles an hour ?
(ii) In x hours at y miles a minute ?
(iii) In x minutes at y miles an hour ?

2. Plot the points (15, 0), (19, 6), (10, 14), (– 14, 8) and find the area of the quadrilateral formed by joining them.

3. Find the area of the triangle formed by the graphs of $y=8$, $x=18$, $x-y+8=0$.

4. If C is the circumference of a circle and D its diameter, $C=\frac{22}{7}D$. Draw a graph and from it read off the circumferences of circles whose diameters are 4 in., 11 in., 20 in., and the radii of circles whose circumferences are 47 in. and 31·4 in.

5. Find the value of $(x^2-x+1)^3$. Check your result by putting $x=1$.

6. The sum of any number which has an even number of digits and the number formed by reversing its digits is divisible by 11. Prove this in the case of a number of two digits.

7. Divide $6a^2+ab-b^2-a+7b-12$ by $2a+b-3$.

XVII. f.

1. Write down the cost of:
   (i) x things at y pence each.
   (ii) x things at 3 a penny.
   (iii) x things at y a penny.
   (iv) x things when y things cost 3 pence.

2. Solve the equation $(3x-1)^2+(4x-2)^2=(5x-3)^2$.

3. Plot the points given by the table below, and deduce the equation of the graph which passes through them.

$x=$	0	1	2	3	4
$y=$	·75	3·5	6·25	9	11·75

4. A walks at 4 m. an hour, and 4 hours after his start B bicycles after him at 10 m. an hour. Find, graphically, as accurately as you can, how far from the start B catches A up.

5. Multiply $2x^2-5x+3$ by x^2-3x+1, checking your result by putting $x=2$.

6. Simplify $(2x+a)(2x+b)-(2x+a)(2x+c)+(2x+c)(2x-b)$.

7. Divide $a^2-ab-6b^2+ac+17bc-12c^2$ by $a+2b-3c$.

XVII. g.

1. From the sum of $5b-3a-4c$, $4a-2b-\frac{c}{2}$, and $\frac{a}{2}-\frac{5b}{2}+5c$, subtract
$$\frac{a}{2}-\frac{b}{2}+\frac{c}{2}.$$

2. Simplify $[3(x+y)-2(y-z)-(2x+z)][2(x-z)-(x-y)+z]$.

3. Solve the equation $2(x-3)+(x-2)(x-4)=x(x+1)-33$. Test your result.

4. Two men bicycle a journey of 45 miles in opposite directions, one man doing the journey in 6 hours, the other in 4 hours. Where do they meet? Solve the problem graphically, and test your result in any way you please.

5. Solve the equations $5(x-1)+11(y-4)=97,$
$11(x-5)+5(y-11)=0.$

6. Divide the sum of $6x(x-1)^2$, $(3x+1)^2$, and $-2(8x^2+3)$ by $2x-5$.

7. Divide 104 into two parts, such that four times their difference may exceed by 2 the sum of one-fourth of the greater and one-third of the less.

XVII. h.

1. From the excess of 5 over $x-3$, subtract x^2-2x+8.

2. Find the product of $a(x-b)-a(1-b)$ and $(x+3)(x-1)-(x+2)(x-1)$.

3. Choosing a suitable unit, draw accurately the graph of $3y=2x+7$.

4. A does a journey at a uniform rate in 6 hours. B starting at the same time, but at twice A's rate, is delayed for $2\frac{1}{2}$ hours when he has gone half way. He, however, reached the end of the journey at the same time as A. Prove graphically that if B travelled at the pace at which he did the second half, he would do the complete journey in 4 hours.

5. What values of x and y will make both
$3(x-4)-2(y+3)$ and $2(x-15)+3(y-4)$ equal to unity?

6. Simplify $[27(x-y)(x+y)-8y(6x+y)]\div(9x+5y)$.

7. A certain number of shillings, and two-thirds of that number of half-crowns, are together less than four guineas by two-thirds of the same number of florins. What is the number?

XVII. k.

1. From the excess of $2x(x-5)$ over $5(1-2x)$
take the excess of $x(x-3)$ over $3(4-x)$.

2. Find the values of $4x-3x^2$ for integral values of x from -3 to 3. Tabulate your work.

3. Solve the equation $8(x+1)^3-10(x+2)(6x-7)=(2x-3)^3-150x$. Test your result.

4. A does a journey of 42 miles in $5\frac{1}{4}$ hours, and B starting an hour later does the reverse journey in four hours. Find, graphically, as accurately as you can, how far their meeting place is from A's starting point. Test your result.

In how many minutes after B's start were they first 20 miles apart?

5. Solve the equations $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$, $\frac{1}{x}-\frac{1}{y}=\frac{1}{3}$. Test your result.

6. Simplify $[2x(x-1)(x-2)-(x+9)^2+76]\div(x-5)$.

7. A debt, which might have been paid exactly with $5x$ half-sovereigns and x half-crowns, was paid out of a £10 note, and the change was found to be equal to $15x$ half-crowns and x half-sovereigns. Find x and the amount of the debt.

XVII. l.

1. Find the continued product of
$$x+y, \quad x-y, \quad x^2+y^2, \quad x^4+y^4.$$

2. A is x years old, B y years old, C z years old: what was the sum of their ages a years ago?

3. Solve the equation $(x+1)(x+3)(x+5)=(x+7)(x+9)(x-7)$. Test your result.

4. Taking 7 cms. $=2\cdot76$ inches, draw a graph which will enable you to convert centimetres to inches and *vice versâ*.

From the figure read off the value of

(i) $4\cdot3$ cms. in inches, (ii) $5\cdot7$ cms. in inches,
(iii) $1\cdot5$ in. in cms. (iv) $2\cdot2$ in. in cms.

5. Simplify $[6x(x-2)^2-5(x-2)(x+2)+2x+1]\div(3x-7)$.

6. A man buys a case of oranges at $8d.$ a dozen. He finds 54 spoiled, and selling the rest at 7 for $5d.$, he loses $2s.\ 6d.$ on the transaction. How many did he buy ?

7. Solve the equations $7y-2x=1,\quad 2w-x=15,$
$2y+z=7,\quad 10y+3x=19.$

CHAPTER XVIII

RESOLUTION INTO FACTORS

98. When an algebraic expression is expressed as a product of its factors, it is said to be **resolved into factors**, and the process of finding the factors is called **resolution into factors.**

We have already dealt with some of the simpler forms of factorization ; thus we have seen that $2x-6=2(x-3)$.

In other words the factors of $2x-6$ are 2 and $(x-3)$.

Example 1. Resolve $4a^2-3a$ into factors.

a is common to both terms,

$$\therefore\ 4a^2-3a=a(4a-3),$$

or, the factors of $4a^2-3a$ are a and $(4a-3)$.

Example 2. $6x^3-7x^2-2x=x(6x^2-7x-2)$.

Example 3. $3a^2bc-5ab^2c+4abc^2=abc(3a-5b+4c)$.

Example 4. $15x^2y^3-5xy^4-20x^4y^2=5xy^2(3xy-y^2-4x^3)$.

N.B.—The above results should be checked by removing the brackets.

Examples. XVIII. a.

[*Check results by removing brackets.*]

Resolve the following expression into factors :

1. $ax+ab$.
2. $ax-a^2$.
3. x^2-3ax.
4. x^3-5ax^2.
5. $ax^2-a^2x+a^6$.
6. $3a^2-3ab$.
7. $5x^3-15x^2y$.
8. x^2-xy.
9. $21-56x$.
10. $25x^2-20xy$.
11. $ax-bx+cx$.
12. $-2x^3+4x$.
13. $-ay+by+cy$.
14. $p^2x^2-apxy+pbxy$.
15. $76a^2x^3-57a^3x^2$.
16. $3p^2x^2-9px+12$.
17. $x^2yz+xy^2z-xyz^2$.
18. $7ab-7bc-21bx$.
19. $14x^3-7x^2y+56xy^2$.
20. $36x^2yz-54xy^2z+48xyz^2-18x^2y^2z^2$.

TRINOMIAL EXPRESSIONS

99. An algebraic expression of three terms is called **a trinomial.** Examine the four multiplications given below.

$$\begin{array}{l} x+2 \\ \underline{x+3} \\ x^2+2x \\ \underline{\quad\;+3x+6} \\ x^2+5x+6 \end{array}$$

$\therefore\ (x+2)(x+3)$
$\quad = x^2+5x+6\ldots$(i).

$$\begin{array}{l} x-2 \\ \underline{x-3} \\ x^2-2x \\ \underline{\quad\;-3x+6} \\ x^2-5x+6 \end{array}$$

$\therefore\ (x-2)(x-3)$
$\quad = x^2-5x+6\ldots$(ii).

$$\begin{array}{l} x+2 \\ \underline{x-3} \\ x^2+2x \\ \underline{\quad\;-3x-6} \\ x^2-\ \ x-6 \end{array}$$

$\therefore\ (x+2)(x-3)$
$\quad = x^2-x-6\ldots$(iii).

$$\begin{array}{l} x-2 \\ \underline{x+3} \\ x^2-2x \\ \underline{\quad\;+3x-6} \\ x^2+\ \ x-6 \end{array}$$

$\therefore\ (x-2)(x+3)$
$\quad = x^2+x-6\ldots$(iv).

The results are different forms of the expression

$$x^2+px+q.$$

In each case we notice in the product that

(1) the coefficient of x is the algebraic sum of the second terms of the factors.

(2) the third term is the product of the second terms of the factors.

In (i) $2+3=5,\quad 2\times 3=6.$

In (ii) $-2-3=-5,\quad (-2)(-3)=6.$

In (iii) $2-3=-1,\quad (2)(-3)=-6.$

In (iv) $-2+3=1,\quad (-2)(3)=-6.$

Reversing the process, in order to find the factors of an expression of the form x^2+px+q, we must seek two numbers whose algebraic sum is p and whose product is q.

Examples.

$x^2+7x+12=(x+4)(x+3)$, for $4+3=7$ and $4\times 3=12$.

$x^2-7x+12=(x-4)(x-3)$, for $-4-3=-7$ and $(-3)(-4)=12$.

$(x^2-4x-12)=(x-6)(x+2)$, for $-6+2=-4$ and $(-6)(2)=-12$.

$(x^2+4x-12)=(x+6)(x-2)$, for $6-2=4$ and $(6)(-2)=-12$.

100. In more general form the above results may be expressed thus:

$$x^2+(a+b)x+ab=(x+a)(x+b),$$
$$x^2-(a+b)x+ab=(x-a)(x-b),$$
$$x^2+(a-b)x-ab=(x+a)(x-b),$$
$$x^2-(a-b)x-ab=(x-a)(x+b).$$

All the above can of course be checked by multiplying the factors.

We also see that

$$abx^2+(a+b)x+1=(ax+1)(bx+1),$$
$$abx^2-(a+b)x+1=(ax-1)(bx-1),$$
$$abx^2+(a-b)x-1=(ax-1)(bx+1),$$
$$abx^2-(a-b)x-1=(ax+1)(bx-1).$$

Thus
$$3x^2+4x+1=(3x+1)(x+1),$$
$$10x^2-3x-1=(5x+1)(2x-1),$$
$$10x^2+3x-1=(5x-1)(2x+1).$$

Also
$$x^2-11xy+10y^2=(x-10y)(x-y),$$
$$x^2-4xy-21y^2=(x-7y)(x+3y).$$

Examples. XVIII. b.

Resolve into factors:

1. $x^2+9x+20$.
2. $x^2-10x+21$.
3. $x^2+10x+24$.
4. $x^2+10x+21$.
5. $x^2-10x+24$.
6. x^2-8x+7.
7. x^2+3x+2.
8. x^2-4x+4.
9. x^2-x-2.
10. x^2+x-2.
11. x^2+2x+1.
12. x^2+4x-5.
13. x^2-4x-5.
14. $x^2+12x+35$.
15. x^2-6x+9.
16. $x^2-11x+10$.
17. $x^2-12x+27$.
18. $x^2+20x+51$.
19. $x^2-18x+65$.
20. $x^2-10x+25$.
21. x^2+x-42.
22. x^2-x-42.
23. $x^2+4x-45$.
24. $x^2-2x-35$.
25. $x^2+14x+49$.
26. $x^2+2x-63$.
27. $x^2-22x+120$.
28. $x^2-3x-130$.
29. x^2+x-72.
30. $1-3x+2x^2$.
31. $21+10x+x^2$.
32. $x^2+(p+q)x+pq$.
33. $x^2-(m+n)x+mn$.
34. $x^2+(m-n)x-mn$.
35. $x^2-(m-n)x-mn$.
36. $x^2+(2a+b)x+2ab$.
37. $x^2-(a+3b)x+3ab$.
38. $x^2-(2a-3b)x-6ab$.
39. $x^2+(4a-5b)x-20ab$.
40. $x^2-(5a-3b)x-15ab$.
41. $x^2+7x-18$.
42. $x^2-x-110$.
43. $1-5x+6x^2$.
44. $5-4x-x^2$.
45. $x^2+16x-17$.
46. $40-13x+x^2$.
47. $1-3x-130x^2$.
48. $x^2-14x-15$.
49. $40-3x-x^2$.
50. $x^2+x-110$.
51. $42-x-x^2$.
52. $66+5x-x^2$.

Resolve into factors :

53. $1-7x+6x^2$.
54. $72+x-x^2$.
55. $x^2-35x+216$.
56. $x^2+9xy-10y^2$.
57. $a^2+16ab+15b^2$.
58. $x^2-23x+132$.
59. $5x^2-4xy-y^2$.
60. $a^2-2ab-24b^2$.
61. $x^2-22xy+121y^2$.
62. $x^2-30x+225$.
63. $x^2-73x+72$.
64. $x^2-26xy+169y^2$.
65. $x^2-103x+102$.
66. $73x^2-74x+1$.
67. $x^2-14ax+45a^2$.
68. $54x^2-3xy-y^2$.
69. $26x^2+11x-1$.
70. $240x^2+x-1$.
71. $43x^2-42x-1$.
72. $1-5ab+6a^2b^2$.
73. $x^2y^2-4xy-32$.
74. $156x^2-x-1$.
75. $1-10xy+25x^2y^2$.
76. $51x^2y^2-20xy+1$.
77. $42a^2b^2-ab-1$.
78. $17x^2+16xy-y^2$.
79. $54x^2+21xy+y^2$.
80. $54x^2-15xy-y^2$.
81. $57-22x+x^2$.
82. $x^2y^2-16xy+55$.
83. $x^2y^2-13xy-48$.
84. $x^2-93x+92$.
85. $167-166x-x^2$.
86. $x^2+34x+289$.
87. $1-30x+225x^2$.
88. $81x^2+82x+1$.
89. $x^2-10xy-39y^2$.

101. An expression of four terms can often be factorized by **grouping the terms in pairs.**

Examples.

$$\begin{aligned} &ax-bx+ay-by \\ &=(a-b)x+(a-b)y \\ &=(a-b)(x+y)\ [\text{just as } cx+cy=c(x+y)]. \end{aligned}$$

$$\begin{aligned} &3ax-2by-3bx+2ay \\ &=(3ax-3bx)+(2ay-2by) \\ &=3x(a-b)+2y(a-b) \\ &=(a-b)(3x+2y). \end{aligned}$$

We might deal with $x^2-(a+b)x+ab$ in this way.

$$\begin{aligned} x^2-(a+b)x+ab &= x^2-ax-bx+ab \\ &= x(x-a)-b(x-a) \\ &= (x-a)(x-b). \end{aligned}$$

$$\begin{aligned} x^3-ax^2+a^2x-a^3 &= (x^3-ax^2)+(a^2x-a^3) \\ &= x^2(x-a)+a^2(x-a) \\ &= (x-a)(x^2+a^2). \end{aligned}$$

$$\begin{aligned} 15a^2-6ab-5ax^2+2bx^2 &= 15a^2-5ax^2-6ab+2bx^2 \\ &= 5a(3a-x^2)-2b(3a-x^2) \\ &= (3a-x^2)(5a-2b). \end{aligned}$$

$$\begin{aligned} x^3-2x^2-3x+6 &= x^2(x-2)-3(x-2) \\ &= (x-2)(x^2-3). \end{aligned}$$

Examples. XVIII. c.

Factorize the expressions :

1. $ax+bx+ay+by$.
2. $ax-bx-ay+by$.
3. $ax-2x-ay+2y$.
4. $6x-ax-6y+ay$.
5. $x^2+xy+xz+yz$.
6. $x^2-xy+xz-yz$.

7. $a^2c^2 - acd + abc - bd.$	8. $x^2 - 2x + xy - 2y.$
9. $3x - 3y + ay - ax.$	10. $a^2 + bc - ab - ac.$
11. $bc - a^2 - ab + ac.$	12. $a^2c^2 + bd + abc + acd.$
13. $a^2c + b^2d + b^2c + a^2d.$	14. $a^2c - a^2d - b^2d + b^2c.$
15. $x^3 - 3x^2 + 2x - 6.$	16. $x^3 - xy - 2x^2 + 2y.$
17. $x^5 - 15 + 5x^4 - 3x.$	18. $x^2y^2 + x^2 + y^2 + 1.$
19. $xy^2 - 1 - y^2 + x.$	20. $ab(x^2 + 1) - x(a^2 + b^2).$
21. $x^2 - y^2 - 4x + 4y.$	22. $a^2 + m(m+1)a + m^3.$
23. $x^3 + x^2 + x + 1.$	24. $x^5 + x^4 + x + 1.$
25. $2x^3 - x^2 + 2x - 1.$	26. $ax^2 - bx^2 + a - b.$
27. $2x^3 - 3x^2 + 4x - 6.$	28. $3x^3 - x^2 + 12x - 4.$
29. $7x^3 - 3x^2 - 21x + 9.$	30. $2x^3 - x^2 - 10x + 5.$
31. $2x^3 + 14x^2 - 3x - 21.$	32. $11x^3 + 55x^2 + 7x + 35.$
33. $a^2 - bc - b + a^2c.$	34. $x^2 - a^2 + x - a^2x.$
35. $2a - x^3 - 2x^2 + ax.$	36. $2x^3 + 6x^2 - cx - 3c.$

102. Difference of two squares. We know by multiplication that $a^2 - b^2 = (a+b)(a-b)$. Hence we see that if an expression can be written as the difference of two squares, we can at once resolve it into factors.

Examples.

$$x^2 - 4 = x^2 - 2^2 = (x+2)(x-2).$$
$$x^2 - 1 = (x+1)(x-1).$$
$$25x^2 - 9y^2 = (5x)^2 - (3y)^2 = (5x+3y)(5x-3y).$$
$$10^2 - 7^2 = (10+7)(10-7) = 17 \times 3 = 51.$$
$$25^2 - 24^2 = (25+24)(25-24) = 49.$$

Examples. XVIII. d.

Resolve into factors :

1. $1 - x^2.$	2. $1 - 4x^2.$	3. $x^2 - 4a^2.$	4. $a^2 - 49.$
5. $9a^2 - x^2.$	6. $9x^2 - 1.$	7. $25x^3 - 16.$	8. $x^2 - 9.$
9. $25x^2 - 49.$	10. $a^2 - 25.$	11. $121 - b^2.$	12. $a^2 - 9.$
13. $x^2 - 169.$	14. $4 - a^2.$	15. $16 - 121x^2.$	16. $a^2b^2 - c^2d^2.$
17. $9x^2y^2 - 16a^2b^2.$	18. $101^2 - 1.$	19. $11^2 - 3^2.$	20. $x^2y^2 - 1.$
21. $64 - c^2d^2.$	22. $1 - 9k^2.$	23. $9 - 4a^2.$	24. $9a^2b^2 - 16.$
25. $153^2 - 152^2.$	26. $x^2 - 10{,}000.$	27. $10{,}000x^2 - 1.$	28. $x^2y^2 - 81a^4.$
29. $a^6 - b^4.$	30. $b^4 - 25.$	31. $x^8 - a^2.$	32. $36x^{12} - y^8.$
33. $a^2b^6c^4 - x^2.$	34. $1 - 100x^2.$	35. $a^2b^2c^2 - d^2.$	36. $1 - 121a^4.$
37. $49x^2 - 36y^2.$	38. $p^2q^2 - 4.$	39. $144x^4 - y^4z^6.$	40. $a^2 - 225b^2.$
41. $81x^2 - 64.$	42. $4m^2n^2 - 1.$	43. $9p^2 - 49q^2.$	44. $x^2 - 169y^2.$

45. $81a^2b^2 - 1.$	46. $x^{36} - y^{18}.$	47. $a^2 - 289b^2.$
48. $121a^2 - 144b^2.$	49. $25x^{16} - 169a^{10}.$	50. $x^4y^2 - 100.$
51. $x^2y^4 - 144p^2.$	52. $1 - 100x^6y^4z^8.$	53. $121x^6y^8 - 1.$

Find by factorization the values of :

54. $385^2 - 285^2$. 55. $95^2 - 85^2$. 56. $999^2 - 1$. 57. $37^2 - 27^2$.
58. $1001^2 - 1$. 59. $237^2 - 37^2$. 60. $8275^2 - 8273^2$. 61. $35^2 - 33^2$.
62. $825^2 - 175^2$. 63. $97^2 - 94^2$. 64. $673^2 - 373^2$. 65. $998^2 - 4$.
66. $1896^2 - 1892^2$. 67. $97^2 - 9$. 68. $2753^2 - 2745^2$. 69. $109^2 - 81$.
70. $99999^2 - 1$. 71. $116^2 - 16$. 72. $125^2 - 25^2$. 73. $125^2 - 25$.
74. $249^2 - 49^2$. 75. $364^2 - 64^2$.

103. When the terms have a common factor, this should first be taken out. The expression can often then be further factorized.

Examples.

$$a^3 - ax^2 = a(a^2 - x^2) = a(a + x)(a - x).$$

$$12x^2 - 75 = 3(4x^2 - 25) = 3(2x + 5)(2x - 5).$$

$$27a^2b^4x^2 - 147a^2b^2 = 3a^2b^2(9b^2x^2 - 49)$$
$$= 3a^2b^2(3bx + 7)(3bx - 7)$$

Examples. XVIII. e.

Resolve the following expressions into their simplest factors :

1. $3x^2 - 12a^2$. 2. $7 - 7x^2$. 3. $2x^2 - 288$.
4. $45x^2y^2 - 80x^2a^2$. 5. $3a^8 - 3x^2$. 6. $112a^2x^2y^3 - 175a^2y$.
7. $54a^2b^2 - 24c^2d^2$. 8. $141a^9b^7 - 564a^3b^3$. 9. $7a^2 - 343b^2$.
10. $75x^2 - 48$. 11. $11 - 99b^2$. 12. $45a^2b^2 - 80$.
13. $13a^6 - 13b^2$. 14. $7x^2 - 1575a^2$. 15. $3x^4 - 300$.
16. $27ap^2 - 147aq^2$. 17. $605x^2c - 720b^2c$. 18. $13abc^2 - 52abd^2$.
19. $17 - 68p^2q^2$. 20. $7x^2y^2 - 28x^2y^4$.

104. Expressions in the form of the difference of two squares.

Example.

$$(x + b)^2 - (c + d)^2.$$
$$= [\overline{a + b} + \overline{c + d}][\overline{a + b} - \overline{c + d}]$$
$$= (a + b + c + d)(a + b - c - d).$$

Examples. XVIII. f.

Resolve into their simplest factors :

1. $(a-b)^2-c^2$.
2. $a^2-(b+c)^2$.
3. $(x-y)^2-4a^2$.
4. $(x+2y)^2-16b^2$.
5. $x^2-(2a-b)^2$.
6. $(x+y)^2-(a+b)^2$.
7. $(2x+3y)^2-(x+y)^2$.
8. $a^2-(4x-y)^2$.
9. $25x^2-(a-b)^2$.
10. $16a^2-25(x+y)^2$.
11. $(x+1)^2-(x-1)^2$.
12. $(2x+a)^2-(2x-a)^2$.
13. $(a-2b)^2-(c+d)^2$.
14. $(a+b+c)^2-(x+y+z)^2$.
15. $(3x-y)^2-(x+2y)^2$.
16. $(2x+5)^2-(2x-3)^2$.
17. $(5p+q)^2-(5p-q)^2$.
18. $9x^2-(3x-y)^2$.
19. $4(x+a)^2-9(y+b)^2$.
20. $9(x+y)^2-4(x-y)^2$.
21. $3(a+b)^2-12(c+d)^2$.
22. $64p^2-(q-4)^2$.
23. $(a+b)^2-(a-b)^2$.
24. $(2x+3y+a)^2-(x-y+a)^2$.
25. $(3x+2y)^2-(2x+3y)^2$.
26. $(4x-3a)^2-(4x+3a)^2$.
27. $1-(3x-2y)^2$.
28. $1-4(x-y)^2$.
29. $100-(2a-3b)^2$.
30. $16a^2-(4a-b)^2$.
31. $(a^2+b^2)^2-4a^2b^2$.
32. $(a^2+2b^2)^2-4a^2b^2$.
33. $a^2b^2-(ab-1)^2$.
34. $(3a-2)^2-(2a-3)^2$.
35. $(x^2-2x+3)^2-(x^2+2x-2)^2$.

105. Harder Examples.

Find the factors of

$$x^2-a^2+4y^2-b^2+4xy+2ab.$$

The given expression may be written thus :

$$\begin{aligned} &x^2+4xy+4y^2-(a^2-2ab+b^2)\\ &=(x+2y)^2-(a-b)^2\\ &=[\overline{x+2y}+\overline{a-b}][\overline{x+2y}-\overline{a-b}]\\ &=(x+2y+a-b)(x+2y-a+b). \end{aligned}$$

Examples. XVIII. g.

Resolve into factors :

1. $a^2-2ab+b^2-c^2$.
2. $c^2-a^2-2ab-b^2$.
3. $x^2+2ax+a^2-b^2$.
4. $y^2-a^2+2ax-x^2$.
5. $a^2-b^2-c^2+2bc$.
6. $1-a^2+2ab-b^2$.
7. $x^2-y^2+a^2+2ax$.
8. $x^2-4xy+4y^2-9a^2b^2$.
9. $x^2-2xy+y^2-9$.
10. $16-a^2-b^2+2ab$.
11. $1-4a^2-b^2+4ab$.
12. $a^2+2ax+x^2-y^2-2by-b^2$.
13. $4a^2-4ab+b^2-x^2-2cx-c^2$.
14. $a^2-2ab+b^2-c^2+2cd-d^2$.
15. $a^2+c^2-b^2-d^2-2ac-2bd$.
16. x^4-x^2-2x-1.
17. $a^2-b^2+c^2+2ac$.
18. $9a^2-4c^2+b^2-x^2-6ab-4cx$.
19. $5a^2-10ab+5b^2-20c^2$.

106. Factorization of trinomial expressions when the coefficient of the highest term is not unity.

This can often be done by inspection, but if the factors are not readily seen, the method described in the next article should be employed.

$$10x^2+29x-21=(5x-3)(2x+7).$$

Arrange the factors thus: $\begin{array}{c}5x-3\\ \times\\ 2x+7\end{array}.$

Firstly. We see that the *first* term of the product is the product of the *first* terms of the factors, and the *last* term of the product is the product of the *second* terms of the factors.

Thus if $6x^2+11x-35$ has factors,

their first terms must be $6x$ and x or $3x$ and $2x$.

Also, their second terms must be 35 and 1, or 5 and 7, with proper signs prefixed.

Secondly. We see that the coefficient of x is formed by the products $5x\times 7$ and $2x\times(-3)$. [Notice the **crossed lines** ($\times$) above.]

We **also** notice that if the *last* term of the product is *positive*, the *second* terms of the factors have the same sign: if the *last* term of the product is *negative*, the *second* terms of the factors have different signs.

Let us take a few cases.

Example. Factorize $3x^2-17x+10$.

The *first* terms of the factors must be $3x$ and x.

The *second* .... 10 and 1, or 2 and 5.

....................................are of the same sign, and *negative*.

We therefore have to choose from the following.

$$\left.\begin{array}{r}x-10\\ \times\\ 3x-1\end{array}\right\}\text{ the coeff. of } x \text{ would be } -(1+3\times 10).$$

$$\left.\begin{array}{r}x-1\\ \times\\ 3x-10\end{array}\right\}\text{ .... } -(10+3).$$

$$\left.\begin{array}{r}3x-5\\ \times\\ x-2\end{array}\right\}\text{ .... } -(3\times 2+5).$$

$$\left.\begin{array}{r}3x-2\\ \times\\ x-5\end{array}\right\}\text{ .... } -(3\times 5+2).$$

The last case is therefore the only possible one, and we see that the factors are $3x-2$ and $x-5$.

After a little practice it will easily be seen which cases may be rejected.

Example. Factorize $7x^2+32x-15$.

The *first* terms of the factors must be $7x$ and x.

The *second* .... have different signs.

$$\left.\begin{array}{c}7x+15\\ \times\\ x-1\end{array}\right\}\text{ coeff. of } x \text{ would be } -7\times1+15\times1=8.$$

$$\left.\begin{array}{c}7x-15\\ \times\\ x+1\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 7\times1-15\times1=-8.$$

$$\left.\begin{array}{c}7x-1\\ \times\\ x+15\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 7\times15-1=104.$$

$$\left.\begin{array}{c}7x+1\\ \times\\ x-15\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ -7\times15+1=-104.$$

$$\left.\begin{array}{c}7x+5\\ \times\\ x-3\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ -7\times3+5=-16.$$

$$\left.\begin{array}{c}7x-5\\ \times\\ x+3\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 7\times3-5=16.$$

$$\left.\begin{array}{c}7x+3\\ \times\\ x-5\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ -7\times5+3=-32.$$

$$\left.\begin{array}{c}7x-3\\ \times\\ x+5\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 7\times5-3=32.$$

$\therefore$ $7x-3$ and $x+5$ are the reqd. factors.

Example. Factorize $3x^2-8x-3$.

3 is *not* a factor of each term. $\therefore$ $3x-3$ cannot be a factor.

$\therefore$ the factors must be

$3x-1$ and $x+3$, or $3x+1$ and $x-3$.

The second pair are the factors, for $-3\times3+1=-8$.

107. When the factors cannot readily be seen by inspection the following method is recommended.

Example 1. Find the factors of $2x^2-5x+2$.

$$2x^2-5x+2=\tfrac{1}{2}\{(2x)^2-5(2x)+4\}.$$

(This is the same as multiplying by $\tfrac{2}{2}$).

(Writing y instead of $2x$) $\quad =\tfrac{1}{2}[y^2-5y+4]$

$$=\tfrac{1}{2}(y-4)(y-1)$$

$$=\tfrac{1}{2}(2x-4)(2x-1)$$

$$=(x-2)(2x-1).$$

Example 2. Factorize $12x^2-x-20$.

$$12x^2-x-20=\tfrac{1}{12}[(12x)^2-(12x)-240].$$

(Writing y instead of $12x$) $=\tfrac{1}{12}(y^2-y-240)$

$$=\tfrac{1}{12}(y-16)(y+15)$$
$$=\tfrac{1}{12}(12x-16)(12x+15)$$
$$=\left(\frac{12x-16}{4}\right)\left(\frac{12x+15}{3}\right)$$
$$=(3x-4)(4x+5).$$

Example 3. Factorize $28x^2+xy-45y^2$.

$$28x^2+xy-45y^2=\tfrac{1}{28}[(28x)^2+(28x)y-28\times 45y^2].$$

(Writing a instead of $28x$) $=\tfrac{1}{28}(a^2+ay-28\times 45y^2)$.

We now have to find two numbers whose product is -28×45, and whose algebraic sum is 1. This can easily be done if we put the product -28×45 into its prime factors.

$$-28\times 45=-2\times 2\times 7\times 5\times 3\times 3.$$
$$-7\times 5+2\times 2\times 3\times 3=-35+36=1;$$

$\therefore$ the given expression $=\tfrac{1}{28}(a+36y)(a-35y)$

$$=\tfrac{1}{28}(28x+36y)(28x-35y)$$
$$=(7x+9y)(4x-5y).$$

Examples. XVIII. h.

[*Results should always be checked by multiplication.*]

Find the factors of :

1. $5x^2-12x+4$.
2. $3x^2+14x+15$.
3. $3x^2-7x+2$.
4. $2x^2+11x-21$.
5. $3x^2-13x-30$.
6. $5x^2+42x-27$.
7. $2x^2+19x+9$.
8. $3x^2-22x+7$.
9. $4x^2-16x+15$.
10. $9x^2-18x+8$.
11. $16x^2-8x-15$.
12. $49x^2+21x+2$.
13. $9x^2+6x-8$.
14. $4x^2+4x-63$.
15. $6x^2+11x+3$.
16. $6x^2-11x+3$.
17. $6x^2-x-2$.
18. $12x^2-25x+12$.
19. $20x^2+41x+20$.
20. $12x^2-7x-12$.
21. $18x^2-9x-2$.
22. $24x^2-50x+25$.
23. $3-8x+4x^2$.
24. $5+9x-2x^2$.
25. $2x^2+5xy+3y^2$.
26. $2x^2+3xy-2y^2$.
27. $12x^2+8xy-15y^2$.
28. $14x^2+29x-15$.
29. $9x^2-9x-28$.
30. $14x^2-29x+12$.
31. $10x^2-13xy-9y^2$.
32. $7x^2+4xy-3y^2$.
33. $12x^2+17xy+5y^2$.
34. $26x^2-41x+3$.
35. $13x^2+41x+6$.

108. By Multiplication $(\mathbf{a}+\mathbf{b})(\mathbf{a}^2-\mathbf{ab}+\mathbf{b}^2)=\mathbf{a}^3+\mathbf{b}^3$
and $(\mathbf{a}-\mathbf{b})(\mathbf{a}^2+\mathbf{ab}+\mathbf{b}^2)=\mathbf{a}^3-\mathbf{b}^3$.

Example 1. $x^3-1=x^3-1^3=(x-1)(x^2+x+1).$

Example 2.
$$\begin{aligned}27a^3+8b^3&=(3a)^3+(2b)^3\\&=[3a+2b][(3a)^2-(3a)(2b)+(2b)^2]\\&=(3a+2b)(9a^2-6ab+4b^2).\end{aligned}$$

Example 3.
$$\begin{aligned}1-27x^3&=1-(3x)^3\\&=(1-3x)[1+(3x)+(3x)^2]\\&=(1-3x)(1+3x+9x^2).\end{aligned}$$

Example 4.
$$\begin{aligned}8x^3+729y^6&=(2x)^3+(9y^2)^3\\&=(2x+9y^2)[(2x)^2-(2x)(9y^2)+(9y^2)^2]\\&=(2x+9y^2)(4x^2-18xy^2+81y^4).\end{aligned}$$

Examples. XVIII. k.

Resolve into factors :

1. x^3+y^3. **2.** x^3-y^3. **3.** $1-x^3$. **4.** $1+x^3$. **5.** x^6+y^3.
6. x^6-y^3. **7.** $8x^3-1$. **8.** $1+8y^3$. **9.** $8a^3+b^3$. **10.** $1+27x^3$.
11. x^3+27. **12.** y^3-27. **13.** a^3+125. **14.** $125a^3-1$.
15. $8x^3-27y^3$. **16.** $8a^3+27b^3$. **17.** a^3-216. **18.** $343x^3-1$.
19. y^3-64. **20.** $64+y^3$. **21.** $1000x^3+1$. **22.** a^3b^3-1.
23. $1+a^3b^3$. **24.** a^3b^6-64. **25.** $8x^3y^3-1$. **26.** x^6+1.
27. $64a^3-125b^3$. **28.** $27x^3+p^3q^3$. **29.** $216a^3-b^3$. **30.** $512x^3+1$.
31. $729a^3-8x^3$. **32.** $1+729x^3$. **33.** a^6-b^6. **34.** x^6-64.

Miscellaneous Factors (Easy). Examples. XVIII. l.

1. $-8x^3+16x$. **2.** $a^2-11ab+30b^2$. **3.** $-3+3x^2$.
4. $3a^5b^3c^2-21a^3b^4c^3+18a^4b^4c^2$. **5.** $3a^2-27$.
6. $5a^3-40$. **7.** $10a^2+9ab-b^2$. **8.** $3(a-1)^2-3(a-2)^2$.
9. x^5y-3xy^5. **10.** $7a^2-175$. **11.** $-x^3-x^2-x-1$.
12. $11ac^2-33a^2c$. **13.** $3-21x+18x^2$. **14.** $3a^2b^2-3a^2-3b^2+3$.
15. $12-3x^2$. **16.** $p^6q^7r^4-3p^4q^6r^8+2p^9q^4r^4$.
17. $3\times 11^2-3^3$. **18.** $15x^2-36x+12$. **19.** $x^2-px+qx-pq$.
20. $4x^2-36xy-40y^2$. **21.** $5-45y^2$. **22.** $20x^2+30xy-20y^2$.
23. $11x^2-253xy+1452y^2$. **24.** $3-81x^3$. **25.** $4-(3-x)^2$.
26. $(x-y)^2-5x+5y$. **27.** $15x^4-15y^4$. **28.** $3x^2-6x+3$.
29. $3ab-6b-3ac+6c$. **30.** $117x^2-13$. **31.** $2x^3-250$.
32. $pqx^2+px+qx+1$. **33.** $2x^2-16x+14$. **34.** $7x^2-14x+7xy-14y$
35. $2a^2-50$. **36.** $a^2+ab-42b^2$. **37.** $18x^2-8y^2$.
38. $15p^2q^3-12p^3q^2+18p^2q^2$. **39.** $363-3x^2$. **40.** $9x^2+36x-45$.
41. $24x^2-2x-1$. **42.** $2-x^3-2x^2+x$. **43.** $5x^3-5y^3$.
44. $3x^2+27x+60$. **45.** $3x^3y^3-3$. **46.** $20p^2q^2-5$.
47. $8ab^3c^3-a$. **48.** $17x^2+51x+34$. **49.** $9(a-b)^2-4(a-c)^2$.
50. $7x^2y^4-700$. **51.** $2(x-y)^2-2$. **52.** $3-3(x-y)^2$.
53. $1-5x+6x^2$. **54.** $x^2-9xy+20y^2$. **55.** $3a^2-3b^2$.

56. $1-4(x-y)^2$. 57. $39x^2-26x$. 58. $2x^2+24xy+70y^2$.
59. $3-3(2x-1)^2$. 60. $x^2-30x+225$. 61. $18x^3-9x^2-2x$.
62. $3x^2-12$. 63. $5x+9x^2-2x^3$. 64. $15a^2b-30ab^2$.
65. $6x^4-x^3-2x^2$. 66. $7x^2-8x+1$. 67. $200-15x-5x^2$.
68. $4a^2bc-6ab^2c+8abc^2$. 69. $7x^2-7$. 70. x^4-27x.
71. $x^2+xy-42y^2$. 72. $9x^2-18x-315$. 73. $a^3x-125x$.
74. $3x-8x^2+4x^3$. 75. $4a^2+4ab+b^2$. 76. $7a^2+7a-770$.
77. $13x^4+41x^3+6x^2$. 78. $x^2+px-qx-pq$.

***109.** The **Remainder Theorem** (Art. 95) is often useful for purposes of factorization.

Factorize the expression x^3+4x^2+x-6.

When this expression is divided by $x-1$, the remainder

$$=1+4+1-6=0, \quad \text{..........................(i)}$$

i.e. the expression is divisible by $x-1$ without remainder; in other words $x-1$ is a factor.

Knowing this we write the expression thus:

$$\begin{aligned} &x^3-1+4(x^2-1)+x-1\\ &\qquad=(x-1)(x^2+x+1)+4(x-1)(x+1)+x-1\\ &\qquad=(x-1)(x^2+x+1+4x+4+1)\\ &\qquad=(x-1)(x^2+5x+6)\\ &\qquad=(x-1)(x+2)(x+3). \end{aligned}$$

From the above [see (i)] we observe that in any algebraical expression where x is the only symbol used, **if the algebraical sum of the numerical coefficients is zero, x − 1 is a factor of the expression.**

Example. Factorize the expression $6x^3+13x^2+2x-5$.

When we divide by $x+1$, the remainder is

$$-6+13-2-5=0; \quad \text{..............................(i)}$$

$\therefore$ $x+1$ is a factor of the expression.

Knowing this we write the expression in the form

$$\begin{aligned} &6(x^3+1)+13(x^2-1)+2(x+1)\\ &\qquad=6(x+1)(x^2-x+1)+13(x+1)(x-1)+2(x+1)\\ &\qquad=(x+1)(6x^2-6x+6+13x-13+2)\\ &\qquad=(x+1)(6x^2+7x-5)\\ &\qquad=(x+1)(3x+5)(2x-1). \end{aligned}$$

Hence, comparing (i) with the given expression, we observe that in any algebraical expression where x is the only symbol used, **if the algebraical sum of the coefficients of the even powers of x**

is equal to that of the odd powers of x, x + 1 is a factor of the expression.

***110.** *Prove that $(a-b)$, $(b-c)$, $(c-a)$ are factors of the expression*

$$a^3(b-c)+b^3(c-a)+c^3(a-b).$$

When we arrange the given expression in descending powers of a and divide by $a-b$, the remainder is equal to the value of the expression obtained by putting $a=b$. (Remainder Theorem.)

This remainder $=b^3(b-c)+b^3(c-b)=0$;

$\therefore$ $a-b$ is a factor of the given expression.

In the same way we may prove that $b-c$ and $c-a$ are factors of the same expression.

*111. Miscellaneous factors.

Example 1. Factorize the expression x^4-a^4.

$$\begin{aligned} x^4-a^4 &= (x^2+a^2)(x^2-a^2) \\ &= (x^2+a^2)(x+a)(x-a). \end{aligned}$$

Example 2. Factorize the expression x^6-a^6.

$$\begin{aligned} x^6-a^6 &= (x^3+a^3)(x^3-a^3) \\ &= (x+a)(x^2-ax+a^2)(x-a)(x^2+ax+a^2). \end{aligned}$$

In a case of this kind it is advisable to consider the expression as the difference of two squares *first*, as above.

Example 3. Resolve into factors $3x^4-3x^3y-18x^2y^2$.

$$\begin{aligned} 3x^4-3x^3y-18x^2y^2 &= 3x^2(x^2-xy-6y^2) \\ &= 3x^2(x-3y)(x+2y). \end{aligned}$$

Example 4. Resolve $(a+b)^3-1$ into factors.

$$\begin{aligned} (a+b)^3-1 &= [(a+b)-1][(a+b)^2+(a+b)+1] \\ &= (a+b-1)(a^2+2ab+b^2+a+b+1). \end{aligned}$$

Example 5. Resolve $32(x+y)^3-2x-2y$ into factors.

$$\begin{aligned} 32(x+y)^3-2x-2y &= 32(x+y)^3-2(x+y) \\ &= 2(x+y)[16(x+y)^2-1] \\ &= 2(x+y)[4(x+y)+1][4(x+y)-1] \\ &= 2(x+y)(4x+4y+1)(4x+4y-1). \end{aligned}$$

Example 6. Resolve $9x^2-49y^2-9x+21y$ into factors.

$$\begin{aligned} 9x^2-49y^2-9x+21y &= (3x+7y)(3x-7y)-3(3x-7y) \\ &= (3x-7y)(3x+7y-3). \end{aligned}$$

* Examples. XVIII. m.

Resolve the following expressions into their simplest factors :

1. a^4-b^4. **2.** $16a^4-1$. **3.** $32x^4-2y^4$.

4. x^4-x^2+2x-1. **5.** $3ax^6-3a^7$. **6.** $7(a+b)^2-7(a-b)^2$.

Resolve the following expressions into their simplest factors :

7. $(a-b)^2-4(c-d)^2$. 8. $(a^2-b^2)^2-(a-b)^4$. 9. $(x-y)^3-x+y$.

10. $4x^3-12x^2-x+3$. 11. $2x^3+x^2-18x-9$.

12. $ab(x^2+y^2)-xy(a^2+b^2)$. 13. $a(b+c-d)-c(a-b+d)$.

14. $4x^4-2x^3y-3xy^3-9y^4$. 15. x^4-13x^2+36.

16. $a^2b^2+a^5b^5$. 17. $a(a-b)^2-ac^2$. 18. $x^3-3a^2x+2a^3$.

19. $84x^2-8x-1$. 20. $4(2x+3)^2-9(x-3)^2$. 21. $1+2x+x^2-x^4$.

22. $a^2b-b(b-c)^2$. 23. a^4-16b^4. 24. a^6-1.

25. x^4-5x^2+4. 26. $(x^2+xy)^2-(xy+y^2)^2$.

27. $x^2+(1-a)x-a$. 28. $x^2+(2a+b)x-ab-3a^2$.

29. $x^2+3ax-3ab-b^2$. 30. $(a^2-b^2)(x^2-y^2)-4abxy$.

31. x^7+x^6+x+1. 32. $200x^2+10x-21$.

33. $(x^2-y^2-z^2)^2-4y^2z^2$. 34. $(x-2y)^3+(2x-y)^3$.

35. $x^4+4x^3-7x^2-10x$. 36. $(x^2+a^2)b+(a^2+b^2)x$.

37. $2x^3-9x^2+4x+15$. 38. $(ax+by)^2+(ay-bx)^2+c^2(x^2+y^2)$.

39. $15x^2-4x-35$. 40. $(x^2-a^2)b+(a^2-b^2)x$.

41. $(1-ab)^2(a+b)^2-(1+ab)^2(a-b)^2$. 42. $a(a+1)x^2+x-a(a-1)$.

43. $x^4-3x^3-2x^2+12x-8$. 44. $5x^4-4x^3-6x^2+4x+1$.

45. $6x^3-13x^2y-9xy^2+10y^3$. 46. x^3-4x^2+4x-3.

47. $a(a+2)x^2+2x-a^2+1$. 48. $a^2(1+b)-b^2(1+a)$.

49. $16a^4-(b-c)^4$. 50. $\left(\frac{a}{2}+2b-c\right)^2-\left(\frac{a}{2}-b+2c\right)^2$.

51. $15x^3-4x^2y-13xy^2+6y^3$. 52. x^3-6x+4.

53. $(x^2-xy)^2-(xy-y^2)^2$. 54. $x^2+(a-b)xy-aby^2$.

55. $5p^2-19pq+12q^2$. 56. $x+8a^3xy^3$.

57. $27x^4-48y^2$. 58. x^3-x^2-4.

59. $2x^2+7x-30$. 60. $a^2x+a(1-x^2)-x$.

61. xy^5-yx^5. 62. $4x^2-12x-432$.

63. $b(b-2)-(a^2-1)$. 64. $(x^2+3)^2-16x^2$.

65. $(2x+5)^2-(3x-6)^2$. 66. $(x^2-x)^2-8(x^2-x)+12$.

CHAPTER XIX

HIGHEST COMMON FACTOR

112. When a term is the product of several letters, each of the letters is called a **dimension** of the product. Also the number of letters, when expressed without indices, denotes the **degree** of the product.

$a^3bc=a\,.\,a\,.\,a\,.\,b\,.\,c$, and is therefore of five dimensions. Numerical coefficients are considered as of no degree.

$9x^2yz$, and $13x^2yz$ are therefore of the same degree, the fourth.

The highest common factor (H.C.F.) **or highest common divisor** (H.C.D.) of two or more integral algebraic expressions is the integral expression of the highest degree which will exactly divide each of them.

Consider the expressions $27a^2b^3c$, $15a^3b^5c^4$. 3 is the H.C.F. of the numerical coefficients 27 and 15.

The highest power of a which will divide both expressions is a^2.
.... b .... b^3.
.... c .... c.

$\therefore$ the H.C.F. of the two expressions is $3a^2b^3c$.

Example. Find the H.C.F. of $15a^5b^4c^6$, $60a^3b^5$, $25a^4b^2c^2$.

The H.C.F. of 15, 60, 25 is 5.
The highest power of a which divides all the expressions is a^3.
.... b .... b^2.
No power of c divides all three expressions.
$\therefore$ the reqd. H.C.F. $=5a^3b^2$.

Examples. XIX. a.

Find the highest common factor of :

1. $5a^2b$, $10ab^2$. **2.** x^2y^3, x^3y^2. **3.** abc, $3a^2b$.

4. $6xy^2z$, $8x^3yz^2$. **5.** $9a^2b^2c^2$, $15a^3bc^4$. **6.** $9a^2x^4$, $21b^2x^3$.

7. $6x^2y$, $3xy^2$, $9x^2y^2$. **8.** x^2y, y^2z, xy^2. **9.** $3a^2c^6$, $27a^4c^4$, $18a^3c^8$.

10. $26x^3y^2$, $13x^2z^3$, $39x^2y^2z^2$. **11.** $35a^6b^4c^2d^3$, $20a^5c^3d^4$, $45a^3b^2d$, $10a^7b^4cd^7$

12. $3abc^2$, $5a^2bc$, $7abc^2$, $9abcd$.

113. In *compound expressions* the H.C.F. can be determined by inspection as soon as the expressions are resolved into their simplest factors.

Example 1. Find the H.C.F. of

$$a^2bx+ab^2x \text{ and } a^2b-b^3.$$
$$a^2bx+ab^2x=abx(a+b),$$
$$a^2b-b^3=b(a^2-b^2)=b(a+b)(a-b).$$

By inspection the reqd. H.C.F. is $b(a+b)$.

Example 2. Find the H.C.F. of $x^2-17x+60$ and $x^2+7x-60$.

$$x^2-17x+60=(x-12)(x-5),$$
$$x^2+7x-60=(x+12)(x-5).$$
$$\therefore \text{ the reqd. H.C.F. is } x-5.$$

Example 3. Find the H.C.F. of x^2-4, x^2+3x+2, x^2+x-2.

$$x^2-4=(x-2)(x+2),$$
$$x^2+3x+2=(x+1)(x+2),$$
$$x^2+x-2=(x-1)(x+2).$$
$$\therefore\ x+2 \text{ is the H.C.F. reqd.}$$

Example 4. Find the H.C.F. of $x^3-ax^2+a^2x-a^3$ and $x^3-ax^2-a^2x+a^3$.

$$x^3-ax^2+a^2x-a^3=x^2(x-a)+a^2(x-a)=(x-a)(x^2+a^2),$$
$$x^3-ax^2-a^2x+a^3=x^2(x-a)-a^2(x-a)=(x-a)(x^2-a^2)$$
$$=(x+a)(x-a)^2.$$
$$\therefore \text{ the reqd. H.C.F. is } x-a.$$

Examples. XIX. b.

Find the H.C.F. of .

1. a^2-ax, a^2+ax.
2. $5x-10$, $4x-8$.
3. x^2+xy, $xy+y^2$.
4. x^2-4, $3x-6$.
5. a^2+2ab, $ab+2b^2$.
6. x^2+xy, x^2-y^2.
7. x^2-2xy, x^2-4y^2.
8. $x^2+2xy+y^2$, x^2-y^2.
9. x^3-3ax^2, $2x^2-6ax$.
10. $15x-45$, $3x^2-27$.
11. $3x^2+12xy$, $4x^2-64y^2$.
12. $4x^2-8xy$, $3xy^2-6y^3$.
13. x^2+3x+2, x^2+6x+5.
14. $1-2x+x^2$, $1-x^2$.
15. $1+2x+x^2$, $4x-4x^3$.
16. $x^2-7x+12$, $x^2-8x+15$.
17. x^3+y^3, $5x^2-5y^2$.
18. x^2-x-20, x^2+3x-4.
19. x^2-121, $x^2+12x+11$.
20. $x^2+17x+60$, $x^2-7x-60$.
21. $3x^3+3a^3$, $2x^2+4ax+2a^2$.
22. a^3+b^3, $a^2b-ab^2+b^3$.
23. x^2+x-42, $x^2-9x+18$.
24. $4x^2+12x-72$, $3x^2-3x-18$.
25. $24a^2b^2(a+b)^2$, $21a^3b^4(a^3+b^3)$.
26. $12x^2-x-1$, $6x^2-5x+1$.
27. $2x^2+5x-3$, $7x^2-63$.
28. x^3-2x^2-x+2, x^3-x^2-4x+4.
29. $(b+c)^2-a^2$, $(c+a)^2-b^2$, $(a+b)^2-c^2$.
30. $10x^2+13x-3$, $5x^2-11x+2$, $5x^2-16x+3$.
31. $x^2-7x+10$, x^2+2x-8, $3x^2-3x-6$.
32. $(a-b)^2-c^2$, $(a+c)^2-b^2$, $(c-b)^2-a^2$.
33. $x^2-10x+25$, x^2-25, x^3-125.
34. $x^2-(a-c)x-ac$, $x^2-(a+c)x+ac$.
35. $2x^2+x-1$, $2x^2-5x+2$, $6x^2+x-2$.
36. $16x^4+36x^2+81$, $8x^3+27$.
37. x^3-x^2-3x+3, x^3-3x^2+2.
38. x^4-x^3-2x+2, $2x^3-x-1$.
39. $15x^3-19x^2+4$, $9x^3-9x^2-4x+4$.
40. $x^2-7x+10$, $4x^3-25x^2+20x+25$.

***114.** When compound expressions cannot readily be factorized we find their H.C.F. by a method analogous to the Arithmetical method.

Before attempting any such, the student must grasp the principle underlying the Arithmetical method.

Let us find the H.C.F. of 782 and 5451.

$$\begin{array}{l}
782\ )\ 5451\ (\ 6 \\
\qquad\quad \underline{4692} \\
\qquad\qquad 759\ )\ 782\ (\ 1 \\
\qquad\qquad\qquad\quad \underline{759} \\
\qquad\qquad\qquad\quad 23\ )\ 759\ (\ 33 \\
\qquad\qquad\qquad\qquad\qquad \underline{69} \\
\qquad\qquad\qquad\qquad\qquad\ \ 69 \\
\qquad\qquad\qquad\qquad\qquad\ \ \underline{69}
\end{array}$$

23 is the reqd. H.C.F.

This method depends upon the fact that if any two numbers have a common factor, the remainder, when one is divided by the other, has the same factor.

Thus in the above,

any factor common to 782 and 5451 is a factor of 759.

.... 759 and 782 .... 23.

This principle, a rigid proof of which will be given later, being true for Arithmetical numbers must also be true in Algebra, since the symbols stand for numbers.

Let us now apply it to some examples.

Example 1. Find the H.C.F. of x^3+6x^2-8x-7 and $x^3+8x^2+10x+21$.

$$\begin{array}{l}
x^3+6x^2-8x-7\ )\ x^3+8x^2+10x+21\ (\ 1 \\
\qquad\qquad\qquad\quad \underline{x^3+6x^2-\ 8x-\ 7} \\
(a)\qquad\qquad 2\ )\ 2x^2+18x+28\ )\ x^3+6x^2-\ 8x-\ 7\ (\ x-3 \\
\qquad\qquad\qquad\ \ x^2+\ 9x+14\quad \underline{x^3+9x^2+14x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad -3x^2-22x-\ 7 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-3x^2-27x-42} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad (b)\quad 5\ )\ 5x+35\ (\ x^2+9x+14\ )\ x+2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x+\ 7\quad \underline{x^2+7x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ 2x+14 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ \underline{2x+14}
\end{array}$$

$x+7$ is the reqd. H.C.F.

(a) Here we see that 2 is a factor of $2x^2+18x+28$, *but not a factor of* x^3-6x^2-8x-7: we therefore reject it.

(b) We see that 5 is a factor of $5x+35$, *but not a factor of* $x^2+9x+14$: we therefore reject it.

The work will be considerably simplified if factors not common to both divisor and dividend are rejected in this way.

Time will be saved if the work is arranged as below:

$$\begin{array}{r|r|r|l}
x & x^3+6x^2-\ \ 8x-\ \ 7 & x^3+8x^2+10x+21 & 1 \\
 & x^3+9x^2+14 & x^3+6x^2-\ \ 8x-\ \ 7 & \\
\hline
-3 & -3x^2-22x-\ \ 7 & 2\ )\ 2x^2+18x+28 & \\
 & -3x^2-27x-42 & x^2+\ \ 9x+14 & x\ \ldots\ldots\ldots\ldots\text{(c)} \\
 & 5\ )\ 5x+35 & x^2+\ \ 7x & \\
 & x+7 & 2x+14 & 2 \\
 & & 2x+14 &
\end{array}$$

At the stage (c) we might have shortened the work thus. The factors of $x^2+9x+14$ are $x+2$ and $x+7$. $x+2$ is evidently not a divisor of the given expressions.

Dividing x^3+6x^2-8x-7 by $x+7$ we find that $x+7$ is the H.C.F.

When the given expressions have factors common to every term, these should be removed first, remembering that they themselves may have a common factor.

Example 2. Find the H.C.F. of

$$36x^4-78x^3+18x^2+12x \text{ and } 90x^4-207x^3+63x^2+36x.$$

$$36x^4-\ \ 78x^3+18x^2+12x=6x(6x^3-13x^2+3x+2).$$

$$90x^4-207x^3+63x^2+36x=9x(10x^3-23x^2+7x+4).$$

$3x$ is the H.C.F. of $6x$ and $9x$.

We now proceed to find the H.C.F. of the remaining factors.

$$\begin{array}{r|r|r|l}
3x & 6x^3-13x^2+3x+2 & 10x^3-23x^2+7x+4 & 2 \\
 & 6x^3-\ \ 9x^2-3x & 12x^3-26x^2+6x+4 & \\
\hline
-2 & -\ \ 4x^2+6x+2 & -x\ )\ -2x^3+\ \ 3x^2+\ \ x & \\
 & -\ \ 4x^2+6x+2 & 2x^2-\ \ 3x\ -\ \ 1 &
\end{array}$$

$\therefore$ the reqd. H.C.F. is $3x(2x^2-3x-1)$.

Example 3. Find the H.C.F. of

$$6x^3-19x^2+11x^2+6 \text{ and } 10x^3-19x^2+2x+6.$$

$$\begin{array}{r|r|r|l}
 & 6x^3-19x^2+11x+\ \ 6 & 10x^3-19x^2+\ \ \ \ 2x+6 & 1 \\
(c)\ldots\ldots & 2 & 6x^3-19x^2+\ \ 11x+6 & \ldots\ldots(a) \\
3x & 12x^3-38x^2+22x+12 & x\ )\ 4x^3\qquad\quad -\quad\ 9x & \ldots\ldots(d) \\
 & 12x^3\qquad\ -27x & 4x^2\qquad -\quad\ 9 & \\
-9 & -38x^2+49x+12 & 4x^2-98x\ \ +138 & 2 \\
(b)\ldots\ldots & -36x^2\qquad\ +81 & 49\ )\ 98x-147 & \\
 & -1\ )\ -\ \ 2x^2+49x-69 & 2x-\ \ 3 & \\
 & 2x^2-49x+69 & & \\
x & 2x^2-\ \ 3x & & \\
-23 & -46x+69 & & \\
 & -46x+69 & &
\end{array}$$

The reqd. H.C.F. is $2x-3$.

N.B.—It is not necessary that the first term of the divisor should go an exact number of times into the first term of the dividend. See (*a*) and (*b*).

It is, however, sometimes convenient, as at (*c*), to introduce a factor.

At (*d*) we reject the factor x, which is not a factor of either of the given expressions.

***115.** If A and B represent any integral algebraical expression, then if A and B have a common factor, their sum or difference has the same factor.

Let p be the common factor of A and B, and C and D the quotients when we divide them by p.

$$\text{Then } A = pC, \quad \text{and } B = pD.$$

$$\therefore\ A + B = p(C + D), \text{ i.e. } p \text{ is a factor of } A + B.$$

In the same way $A - B = p(C - D)$, $\therefore\ p$ .... $A - B$.

Further if A and B have a common factor p, p is also a factor of $mA + nB$ and $mA - nB$, where mA and nB are any multiples of A and B.

Let C and D be the quotients when we divide A and B by p, so that $A = pC$, and $B = pD$.

$$\therefore\ mA + nB = mpC + npD$$
$$= p(mC + nD);$$
$$\therefore\ p \text{ is a factor of } mA + nB.$$

In the same way, $mA - nB = p(mC - nD)$;

$$\therefore\ p \text{ is a factor of } mA - nB.$$

This can often be employed to shorten the work of finding a H.C.F.

Find the H.C.F. of

$$5x^3 + 16x^2 + 23x - 5148 \text{ and } 3x^3 + 48x^2 - 103x - 5148.$$

The difference of the two expressions

$$= 2x^3 - 32x^2 + 126x$$
$$= 2x(x^2 - 16x + 63)$$
$$= 2x(x - 7)(x - 9).$$

Now $2x$ is not a common factor, nor is $x - 7$, for 7 will not divide exactly into 5148.

$\therefore\ x - 9$ must be the H.C.F. if there is one.

* Examples. XIX. c.

Find the highest common factor of

1. $30a^2x^4 - 5a^3x^3 + 5a^5x$, $9ax^3 - a^3x + 2a^4$.
2. $x^4 - 2x^3y - 2x^2y^2 - 3xy^3$, $3x^3y + 2x^2y^2 + 2xy^3 - y^4$.

Find the highest common factor of:

3. $2x^4 - x^3 - x^2 - x - 3$, $2x^4 - 5x^3 + x^2 + 5x - 3$.
4. $2x^3 - 7x^2 + 8x - 4$, $6x^3 - 6x^2 - 11x - 2$.
5. $2x^3 - 5x + 6$, $4x^3 + x^2 - 12x + 4$.
6. $3x^3 + 14x^2 + 12x + 16$, $2x^4 + 7x^3 - 4x^2 - x - 4$.
7. $2x^4 + 9x^3 + 14x + 3$, $3x^4 + 15x^3 + 5x^2 + 10x + 2$.
8. $12x^3 + 9x^2 - 4x - 3$, $16x^3 + 8x^2 + x + 3$.
9. $2x^3 + 9x^2 - 17x - 45$, $6x^3 - 29x^2 + 31x + 10$.
10. $x^4 - 6x^3 + 8x^2 - 11x + 2$, $2x^4 - 11x^3 + 8x^2 - 6x + 1$.
11. $6x^3 + 11x^2 - 31x + 14$, $4x^3 - 47x + 7$.
12. $5x^3 + 12x^2 + 3x - 2$, $x^5 + 3x^4 + x^3 - x^2 - 4$.
13. $4x^3 - 17x^2 + 3x + 4$, $x^3 - 17x + 4$.
14. $2x^3 - 7x^2 - 46x - 21$, $2x^4 + 11x^3 - 13x^2 - 99x - 45$.
15. $15x^3 + 6x^2 - 45x - 18$, $-49x^3 + 28x^2 + 147x - 84$.
16. $6x^4 - 25x^2y^2 - 9y^4$, $3x^3 - 15x^2y + xy^2 - 5y^3$.
17. $3x^4 + 3x^3y - 27x^2y^2 + 33xy^3 - 12y^4$, $5x^4 - 5x^3y - 15x^2y^2 + 25xy^3 - 10y^4$.
18. $25x^4 + 5x^3 - x - 1$, $20x^4 + x^2 - 1$.
19. $x^3 + 4x^2 + 5x + 6$, $x^4 + 2x^3 + 5x^2 + 4x + 4$.
20. $3x^3 + 17x^2 - 62x + 14$, $7x^3 + 52x^2 - 46x + 8$.

REDUCTION OF FRACTIONS TO LOWEST TERMS

116. We shall assume throughout that as the symbols stand for numerical quantities, the ordinary Arithmetical rules concerning Vulgar Fractions apply to Algebra, leaving the proofs of those rules to a later stage.

In Arithmetic $\dfrac{6}{8} = \dfrac{3 \times 2}{4 \times 2} = \dfrac{3}{4}$.

So in Algebra $\dfrac{ma}{mb} = \dfrac{a}{b}$.

$$\frac{abc^2}{b^2c} = \frac{ac \times bc}{b \times bc} = \frac{ac}{b}.$$

$$\frac{ax - bx}{abx} = \frac{(a-b) \times x}{ab \times x} = \frac{a-b}{ab}.$$

$$\frac{4a^2 - 6ab}{6a^2 - 4ab} = \frac{2a(2a - 3b)}{2a(3a - 2b)} = \frac{2a - 3b}{3a - 2b}.$$

$$\frac{x^2 - 5x + 6}{x^2 - 4x + 4} = \frac{(x-2)(x-3)}{(x-2)^2} = \frac{(x-2)(x-3)}{(x-2)(x-2)} = \frac{x-3}{x-2}.$$

117. A fraction is reduced to its lowest terms by dividing its numerator and denominator by their H.C.F.

The H.C.F. should always be found by factorization, when possible.

Reduce $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}$ *to its lowest terms.*

$$\text{The given expression} = \frac{(3x-1)(x+1)}{x^2(x+1)-(x+1)}$$
$$=\frac{(3x-1)(x+1)}{(x^2-1)(x+1)}$$
$$=\frac{3x-1}{x^2-1} \text{ in its lowest terms.}$$

Reduce $\dfrac{a^3-7a^2+16a-12}{3a^3-14a^2+16a}$ *to its lowest terms*

The denominator $=a(3a^2-14a+16)=a(3a-8)(a-2)$.

Hence it is evident that if the numerator and denominator have a common factor, it is $a-2$.

Acting on this knowledge, we write the numerator to show $a-2$ as a factor, thus :

$$(a^3-2a^2)-(5a^2-10a)+6a-12$$
$$=a^2(a-2)-5a(a-2)+6(a-2)$$
$$=(a^2-5a+6)(a-2)$$
$$=(a-2)(a-3)(a-2)\,;$$

$\therefore$ the given expression $=\dfrac{(a-2)(a-3)(a-2)}{a(3a-8)(a-2)}=\dfrac{(a-2)(a-3)}{a(3a-8)}$ in its lowest terms.

Examples. XIX. d.

Reduce the following to their lowest terms :

1. $\dfrac{4a^3}{8a}$.
2. $\dfrac{10x^3}{5ax}$.
3. $\dfrac{10a^2b^2c}{24ab^2c^2}$.
4. $\dfrac{18x^5y^2z^3}{24x^3y^4z}$.
5. $\dfrac{18ab^4c^3}{12a^3b^2c^3}$.
6. $\dfrac{105m^3n^4p^6}{42m^2n^6p^2}$.
7. $\dfrac{a^2}{a^2+ab}$.
8. $\dfrac{x^2}{x^2-xy}$.
9. $\dfrac{3ax}{4ax-3ay}$.
10. $\dfrac{3ax}{3ax^2-3axy}$.
11. $\dfrac{6a^2-9ab}{8ab-12b^2}$.
12. $\dfrac{8x^2-12xy}{6x^2-4xy}$.
13. $\dfrac{3x^4-3x^2y^2}{5x^4-5x^2y^2}$.
14. $\dfrac{abx-bx^2}{acx-cx^2}$.
15. $\dfrac{xy-xyz}{3bz-3bz^2}$.

Reduce the following to their lowest terms:

16. $\dfrac{x^2-2x}{x^2-5x+6}$.

17. $\dfrac{3x-x^2}{x^2-5x+6}$.

18. $\dfrac{x^2+4x+4}{x^2+5x+6}$.

19. $\dfrac{1+3x+2x^2}{1-2x-3x^2}$.

20. $\dfrac{x^2+(a+b)x+ab}{x^2+(a+c)x+ac}$.

21. $\dfrac{a^2-b^2}{a^3-b^3}$.

22. $\dfrac{x^2-2xy+y^2}{x^2-y^2}$.

23. $\dfrac{b^2-a^2}{a^2+2ab+b^2}$.

24. $\dfrac{1+(a+b)x+abx^2}{1+(a+c)x+acx^2}$.

25. $\dfrac{2x^2-18}{3x^2+3x-18}$.

26. $\dfrac{x^4-3x^2+2}{x^4-x^2-2}$.

27. $\dfrac{x^2-(a-b)x-ab}{x^2-(a+c)x+ac}$.

28. $\dfrac{x^6-2x^3y^3+y^6}{x^6-y^6}$.

29. $\dfrac{x^2-7x+10}{2x^2-x-6}$.

30. $\dfrac{a^2+2ab+b^2-c^2}{a^2-b^2-2bc-c^2}$.

*31. $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}$.

32. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$.

33. $\dfrac{x^2-x-20}{x^2+x-12}$.

34. $\dfrac{x^4+x^2+1}{x^2+x+1}$.

35. $\dfrac{x^3+4x^2-5^*}{x^3-3x+2}$.

36. $\dfrac{x^2-1}{3x^3+7x-10}$.

37. $\dfrac{x^4-9a^2}{x^4-6ax^2+9a^2}$.

38. $\dfrac{x^3+4x^2-5x}{x^3-6x+5^*}$.

39. $\dfrac{(x-y)^2-1}{(x+1)^2-y^2}$.

40. $\dfrac{a^3+a^2+a-3^*}{a^3+3a^2+5a+3^\dagger}$.

41. $\dfrac{3a^2-7ab+4b^2}{3a^2-ab-2b^2}$.

42. $\dfrac{4-(x+y)^2}{(x+2)^2-y^2}$.

43. $\dfrac{6a^2-13ab+6b^2}{6a^2-5ab-6b^2}$.

44. $\dfrac{(2a+b)^2-c^2}{4a^2-(b+c)^2}$.

45. $\dfrac{27+a^3}{9+3a}$.

46. $\dfrac{3x^2+5x+2}{3x^2+x-2}$.

MULTIPLICATION AND DIVISION OF FRACTIONS

118. **Example 1.** Simplify $\dfrac{ab-ac}{ab-bc}\times\dfrac{3abc}{12a^2}\times\dfrac{a^2-ac}{b^2-bc}$.

The given expression $=\dfrac{a(b-c)}{b(a-c)}\times\dfrac{bc}{4a}\times\dfrac{a(a-c)}{b(b-c)}$

(factorizing and dividing numerator and denominator by $3a$)

$$=\frac{a^2bc(b-c)(a-c)}{4ab^2(a-c)(b-c)}$$

$$=\frac{ac}{4b}$$

[for a, b, $(b-c)$, $(a-c)$ are all common factors of numerator and denominator].

Example 2. Simplify $\dfrac{x^2+x-2}{x^2-2x}\times\dfrac{x^2-x-2}{x^2-2x-8}\div\dfrac{x^2-1}{x^2-5x}$.

The given expression $=\dfrac{(x+2)(x-1)}{x(x-2)}\times\dfrac{(x-2)(x+1)}{(x-4)(x+2)}\times\dfrac{x(x-5)}{(x-1)(x+1)}$

$$=\frac{x-5}{x-4}.$$

---

* The sum of the numerical coefficients is zero. $\therefore$ $x-1$ is a factor. (Art. 95.)

† The sum of the coefficients of even powers = the sum of the coefficients of odd powers. (Art. 95.)

Examples. XIX. e.

Simplify the following:

1. $\dfrac{x^2-y^2}{x^2+2xy+y^2} \times \dfrac{xy+y^2}{x^2-xy}$.

2. $\dfrac{x^2-49}{x^2-9} \div \dfrac{x+7}{x+3}$.

3. $\dfrac{x^2-4}{2x-4} \times \dfrac{2}{x+2}$.

4. $\dfrac{4x^2-1}{4y^2-1} \div \dfrac{2x+1}{2y-1}$.

5. $\dfrac{x^2-5x+6}{x^2-16} \times \dfrac{x^2+5x+4}{x^2-4} \div \dfrac{x-3}{x-4}$.

6. $\dfrac{x^2+(a+b)x+ab}{x^2-c^2} \div \dfrac{x+a}{x-c}$.

7. $\dfrac{x^2+5x+6}{x^2-25} \div \dfrac{x+3}{x-5}$.

8. $\dfrac{x^4-a^4}{x^2-2ax+a^2} \div \dfrac{x^2+a^2}{a(x-a)}$.

9. $\dfrac{25a^2-1}{9x^2-4y^2} \times \dfrac{3x+2y}{5a+1} \div \dfrac{5a-1}{3x-2y}$.

10. $\dfrac{x^2-x-6}{x^2+x-2} \div \dfrac{x^2-3x}{x^2-x}$.

11. $\dfrac{6x^2+5x+1}{6x^2-x-1} \times \dfrac{2x^2-11x+5}{2x^2-11x-6}$.

12. $\dfrac{x^4-27x}{x^2-9} \div \dfrac{x^2+3x+9}{x+3}$.

13. $\dfrac{2x^2+x-1}{2x^2-x-1} \div \left(\dfrac{x^2+4x+3}{x^2+4x-5} \times \dfrac{2x-1}{2x+1}\right)$.

14. $\dfrac{x^2-5x+6}{x^2-10x+21} \times \dfrac{3(x^2-49)}{x^2+5x-14} \div \dfrac{x^2+5x-6}{x^2-x}$.

15. $\dfrac{(a+b)^2-c^2}{a^2-(b+c)^2} \times \dfrac{(a-b)^2-c^2}{(a+b)^2-c^2}$.

16. $\dfrac{x^3-64}{x^2-16} \times \dfrac{(x-3)^2}{(x+4)^2-4x} \div \dfrac{x^2+2x-15}{4x^2+16x}$.

17. $\dfrac{8x^2+14x+3}{8x^2-10x+3} \times \dfrac{12x^2-6x}{4x^2+5x+1} \div \dfrac{18x^2-6x}{4x^2+x-3}$.

18. $\dfrac{(a^2+ax)^2}{(a^2-ax)^2} \times \dfrac{a^3-x^3}{a^3+x^3} \div \dfrac{a+x}{a-x}$.

19. $\dfrac{6x^2+6}{(x+1)^2-x} \times \dfrac{x^3-1}{x^3-3x^2} \times \dfrac{x^3+x^2}{x^4-1}$.

20. $\dfrac{(a-b)^2-c^2}{ab-b^2-bc} \times \dfrac{c}{a^2+ab-ac} \div \dfrac{ac-bc+c^2}{a^2-(b-c)^2}$.

21. $\dfrac{3x-6x^2}{1-9x+18x^2} \times \dfrac{1-8x^3}{(1-2x)^2} \div \dfrac{3+6x+12x^2}{1+3x-18x^2}$.

22. $\dfrac{x^3+216}{x^2-x-42} \times \dfrac{x^3-3x^2}{x^4-6x^3+36x^2} \div \dfrac{x^2+2x-15}{2x^2-98}$.

23. $\dfrac{x^4+x^2+1}{x^2-1} \times \dfrac{(x-1)^3}{x^3-1} \div \dfrac{x^3+8x^2-9x}{x+1}$.

24. $\dfrac{8x^2-26x+15}{3x^2-x-4} \times \dfrac{3x^2-7x+4}{2x^2-7x+5} \div \dfrac{4x^2+x-3}{x^2-1}$.

25. $\dfrac{x^4-a^4}{x^6+a^6} \times \dfrac{x^3+a^3}{x^4-2a^2x^2+a^4} \div \dfrac{(x^3-a^3)}{(x^4-a^2x^2+a^4)(x^2-2ax+a^2)}$.

26. $\dfrac{15x^2-31xy+14y^2}{10x^2+xy-21y^2} \times \dfrac{21x^2-9xy}{3x^2-2xy+3x-2y} \div \dfrac{27x^2-63xy}{2x^2+3xy+2x+3y}$.

CHAPTER XX

LOWEST COMMON MULTIPLE

119. The **lowest common multiple** (L.C.M.) of two or more integral algebraic expressions is the integral expression of the lowest degree which is exactly divisible by each of them.

The L.C.M. of a^3b^2 and ab^3 is a^3b^3.

.... a^2, a^7, a^2, a is a^7.

.... $12a^3$ and $18a^2$ is $36a^3$, for 36 is the L.C.M. of 12 and 18, and a^3 is the L.C.M. of a^3 and a^2.

Example 1. Find the L.C.M. of $21a^6b^3c$, $7a^3b^2c^4$, and $2a^2b^5c^3$.

The L.C.M. of 21, 7, and 2 is 42.

The L.C.M. of a^6b^3c, $a^3b^2c^4$, $a^2b^5c^3$

must contain a^6 or it would not be divisible by the first expression,

it .... b^5 .... third ....

and .... c^4 .... second ....

$\therefore$ $42a^6b^5c^4$ is the reqd. L.C.M.

Examples. XX. a.

Find the lowest common multiple of :

1. a^2bc, ab^2c.
2. ax^2, $4a^2x$.
3. $4a^3$, $6a^5$.
4. $6xy^2$, $15x^2y$.
5. $42x^2y$, $49y^2z$.
6. a^2, $2ab$, b^2.
7. $10x^4$, $12x^2y^2$, $4xy^3$.
8. xy, yz, zx.
9. $8a^3b$, $12a^2b^2$, $3ab^3$, $4b^4$.
10. a^4, $4a^3b$, $6a^2b^2$, $4ab^3$, b^4.
11. $9x^4y$, $12x^3y^2$, $54x^2y^3$.
12. ay^2, az^2, a^2y, a^2z.
13. a, $2a$, $3a$, $4a$, $5a$.
14. a^3b^3, a^2b^2, ab.
15. $6a^3b^2c^4$, $4ab^3c^2$, $9a^2b^4c$.
16. $8x^3y^4z^5$, $5x^5y^2z^5$, $12x^2y^4z^6$, $16x^6y^4z^2$.

120. The L.C.M. of *compound expressions* can be determined by inspection when the expressions have been resolved into their simplest factors.

Example 1. Find the L.C.M. of a^3b-a^2bx and ab^2c-b^2cx.

$$a^3b-a^2bx=a^2b(a-x),$$
$$ab^2c-b^2cx=b^2c(a-x).$$

Thus we see that the reqd. L.C.M. is $a^2b^2c(a-x)$.

Example 2. Find the L.C.M. of x^2-5x+6 and x^2+2x-8.

$$x^2-5x+6=(x-2)(x-3),$$
$$x^2+2x-8=(x-2)(x+4);$$
$$\therefore\ (x-2)(x-3)(x+4) \text{ is the reqd. L.C.M.}$$

Example 3. $4a^4b^2c+4a^3b^2cx$, $6a^3bc^2-6a^2bc^2x$, and $3a^2b^3c-3b^3cx^2$.

$$4a^4b^2c+4a^3b^2cx=4a^3b^2c(a+x),$$
$$6a^3bc^2-6a^2bc^2x=6a^2bc^2(a-x),$$
$$3a^2b^3c-3b^3cx^2=3b^3c(a^2-x^2)=3b^3c(a-x)(a+x);$$
$$\therefore\ 12a^3b^3c^2(a-x)(a+x) \text{ is the reqd. L.C.M.}$$

Examples. XX. b.

Find the least common multiple of

1. $4x,\ 4(a-x)$.
2. $a^2,\ a(a-b)$.
3. $2(a-x),\ 3(a+x)$.
4. $3(a+b),\ 7(a+b)$.
5. $a^2b(a-b),\ ab^2(a-b)$.
6. $xyz(x-y),\ xy$.
7. $2x^2(x+y),\ 4xy$.
8. $6(x-1),\ 2(x+1),\ (x^2-1)$.
9. $a^2,\ a^2-ax$.
10. $2a^3+2a^2x,\ 4ax$.
11. $3a-3b,\ 5a-5b$.
12. $4(x-y),\ 3(x^2-y^2)$.
13. $x^2,\ (x^2+1)^2,\ 6(x^2+1)$.
14. $3(ax-by),\ 4(ax+by),\ 6(a^2x^2-b^2y^2)$.
15. $x(x^2-y^2),\ y(x+y),\ x(x-y)$.
16. $8(1-x),\ 8(1+x),\ (1+x^2)$.
17. $3(x^3-1),\ 4(x^2+x+1),\ 6(x-1)$.
18. $x^2+3x+2,\ x^2+5x+6$.
19. $x^2-2x+1,\ x^2+x-2$.
20. $x^2-9x+14,\ x^2-10x+21$.
21. $x^2-3x-4,\ x^2+2x-24$.
22. $(a+b)^2-c^2,\ (a+c)^2-b^2$.
23. $6(x+y)^2,\ 9(x+y)^3$.
24. $2x^2-7x+3,\ 2x^2+5x-3$.
25. $3x^2-7x+2,\ 3x^2+8x-3$.
26. $x^2-y^2,\ (x+y)^2,\ (x-y)^2$.
27. $x^2-36y^2,\ x^2+7xy+6y^2,\ x^2+5xy-6y^2$.
28. $7(a^2b+ab^2),\ 21(a^2+ab),\ 35(b^2-ab)$.
29. $3(x^2-y^2),\ 6(x^2+xy),\ 4(x^3-x^2y)$.
30. $12x^2y(x^2-3x+2),\ 18xy^2(x-1),\ 8y^3(x-2)^2$.
31. $a^3-b^3,\ 2a^2-3ab+b^2,\ a^3+a^2b+ab^2$.
32. $2x^2-7x+3,\ 3x^2-7x-6$.
33. $x^2-5x+6,\ x^2-2x-3,\ x^2-x-2$.
34. $x^2-4,\ x^2-x-2,\ x^3+2x^2-x-2$.
35. $6(a^4-a^2b^2),\ 18ab(a^3-b^3),\ 9b(a^3b+b^4)$.
36. $6x(x^3-y^3),\ 9(x^3-xy^2),\ 12(x^3+2xy^2-2x^2y-y^3)$.
37. $x^2-4a^2,\ x^3+2ax^2+4a^2x+8a^3,\ x^3-2ax^2+4a^2x-8a^3$.
38. $x^2-(a+b)x+ab,\ x^2+3ax-3ab-b^2,\ x^2+(2a+b)x-ab-3a^2$.
39. $4x^3-12x^2-x+3,\ 2x^3+x^2-18x-9$.
40. $ab-b^2-ca+bc,\ bc-c^2-ab+ca$.

CHAPTER XXI

ADDITION AND SUBTRACTION OF FRACTIONS

121. We have already seen that, just as in Arithmetic

$$\frac{3}{7}+\frac{5}{7}=\frac{3+5}{7},$$

so in Algebra

$$\frac{x}{a}+\frac{y}{a}=\frac{x+y}{a},$$

and

$$\frac{x}{a}-\frac{y}{a}=\frac{x-y}{a}.$$

When in Arithmetic we wish to add or subtract fractions which have different denominators, the plan is to reduce all the fractions to *equivalent fractions having the same denominator.*

We adopt the same plan in Algebra.

Example 1. $\dfrac{x-3}{4}-\dfrac{x-2}{6}=\dfrac{3(x-3)}{3\times 4}-\dfrac{2(x-2)}{2\times 6}.$

[12 is the L.C.M. of the denominators 4 and 6. We therefore multiply numerator and denominator in the first fraction by 3, and in the second by 2.]

$$=\frac{3(x-3)-2(x-2)}{12}=\frac{3x-9-2x+4}{12} \text{ (removing brackets)}$$

$$=\frac{x-5}{12} \text{ (collecting like terms).}$$

Example 2. Simplify $\dfrac{x+3}{3x}-\dfrac{4x-3}{4x^2}+\dfrac{5}{2x^3}.$

The given expression $=\dfrac{4x^2(x+3)}{4x^2\times 3x}-\dfrac{3x(4x-3)}{3x\times 4x^2}+\dfrac{6\times 5}{6\times 2x^3}$

(the L.C.M. of $3x$, $4x^2$, $2x^3$ is $12x^3$)

$$=\frac{4x^3+12x^2-12x^2+9x+30}{12x^3}$$

$$=\frac{4x^3+9x+30}{12x^3}.$$

Examples. XXI. a.

Simplify the following expressions:

1. $\dfrac{1}{x}+\dfrac{1}{3x}+\dfrac{1}{2x}.$
2. $\dfrac{a}{x}+\dfrac{a}{3x}-\dfrac{a}{2x}.$
3. $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}.$
4. $\dfrac{1}{ax}+\dfrac{1}{bx}-\dfrac{1}{cx}.$
5. $\dfrac{a}{bc}+\dfrac{b}{ac}+\dfrac{c}{ab}.$
6. $\dfrac{x-3}{3}-\dfrac{x-4}{4}.$
7. $\dfrac{x}{6}-\dfrac{x+1}{7}.$
8. $\dfrac{2x-1}{3}-\dfrac{4x-8}{6}.$
9. $\dfrac{x-a}{a}-\dfrac{x-b}{b}.$

10. $\frac{3x-y}{xy}-\frac{3z-2y}{yz}$.

11. $\frac{x-3}{3x}-\frac{x-5}{5x}$.

12. $\frac{q+3r}{3qr}-\frac{2p-q}{2pq}$.

13. $\frac{x+1}{2}+\frac{x+2}{3}+\frac{x-4}{4}$.

14. $\frac{x+y}{5}-\frac{2x-7y}{10}+\frac{x-3y}{3}$.

15. $\frac{a-b}{b}-\frac{a+b}{a}+\frac{a^2+4ab+2b^2}{2ab}$.

16. $\frac{x-y}{xy}+\frac{y-z}{yz}+\frac{z-x}{zx}$.

17. $\frac{2c-a}{3c}-\frac{a-b}{2a}+\frac{3b}{4a}$.

18. $\frac{x-y}{x}+\frac{x^3+y^3}{xy^2}-\frac{x^2+y^2}{x^2}$.

19. $\frac{3b+4a}{2ab}+\frac{b-6c}{3bc}-\frac{a+6c}{4ac}$.

20. $\frac{2x+1}{3x}-\frac{3x+2}{5x}+\frac{1}{7}$.

21. $\frac{a^2-b^2}{a^2b^2}-\frac{c^2-b^2}{b^2c^2}+\frac{c^2-a^2}{a^2c^2}$.

22. $\frac{3x-6y}{3}-\frac{21x-14y}{7}+\frac{38x-57y}{19}$.

122. Note carefully the truth of the following statements:

$$\frac{1}{2-x}=\frac{-1}{x-2}=-\frac{1}{x-2}.$$

This is obtained by multiplying numerator and denominator by -1.

In the same way $$\frac{a-b}{c-d}=\frac{b-a}{d-c},$$

and again $$\frac{4x-3y}{y-x}=-\frac{4x-3y}{x-y}.$$

Example 1. $$\frac{7a}{a-b}-\frac{3a-2b}{b-a}=\frac{7a}{a-b}+\frac{3a-2b}{a-b}$$

$$=\frac{7a+3a-2b}{a-b}=\frac{10a-2b}{a-b}.$$

Example 2. Simplify $\frac{x+3y}{x+y}-\frac{x-6y}{x+2y}$.

The L.C.M. of $x+y$ and $x+2y$ is $(x+y)(x+2y)$.

Multiply numerator and denominator in the first fraction by $x+2y$,

...... second .... $x+y$,

the given expression $=\frac{(x+2y)(x+3y)}{(x+2y)(x+y)}-\frac{(x+y)(x-6y)}{(x+2y)(x+y)}$ ....(a)

$$=\frac{(x+2y)(x+3y)-(x+y)(x-6y)}{(x+2y)(x+y)}$$

$$=\frac{x^2+5xy+6y^2-(x^2-5xy-6y^2)}{(x+2y)(x+y)}$$

$$=\frac{x^2+5xy+6y^2-x^2+5xy+6y^2}{(x+2y)(x+y)}$$ ....(b)

$$=\frac{10xy+12y^2}{(x+2y)(x+y)}=\frac{2y(5x+6y)}{(x+2y)(x+y)}.$$

The above example is worked out in full. After a little practice such steps as (a) and (b) may be omitted.

The common denominator should generally be left in factors, and the result reduced to its lowest terms.

Example 3. Simplify $\dfrac{a^2-b^2}{ab+b^2}-\dfrac{a-b}{a+b}$.

$$\text{The given expression}=\frac{(a-b)(a+b)}{b(a+b)}-\frac{a-b}{a+b}$$
$$=\frac{a-b}{b}-\frac{a-b}{a+b}$$
$$=(a-b)\left[\frac{1}{b}-\frac{1}{a+b}\right]$$
$$=(a-b)\frac{a+b-b}{b(a+b)}$$
$$=\frac{a(a-b)}{b(a+b)}.$$

Examples. XXI. b.

Express the following in their simplest forms:

1. $\dfrac{1}{x+1}+\dfrac{1}{x-1}$.
2. $\dfrac{3}{x-1}+\dfrac{1}{1-x}$.
3. $\dfrac{1}{x+3}+\dfrac{1}{x+4}$.
4. $\dfrac{1}{x+3}-\dfrac{1}{x+4}$.
5. $\dfrac{6}{2x-3y}-\dfrac{3}{3y-2x}$.
6. $\dfrac{4}{x+6}-\dfrac{2}{x+3}$.
7. $\dfrac{3}{3x-1}-\dfrac{2}{2x+3}$.
8. $\dfrac{x}{x+y}+\dfrac{y}{x-y}$.
9. $\dfrac{x+2}{x+4}-\dfrac{x+5}{x+10}$.
10. $\dfrac{x+5}{x-2}-\dfrac{x-5}{2-x}$.
11. $\dfrac{x+3}{x-3}-\dfrac{x-3}{x+3}$.
12. $\dfrac{3}{1-x}+\dfrac{4}{(1-x)^2}$.
13. $\dfrac{2x-1}{x+1}-\dfrac{2x-1}{x-1}$.
14. $\dfrac{1}{x-y}+\dfrac{2x-y}{x^2-y^2}$.
15. $\dfrac{4x}{(x+y)^2}-\dfrac{4}{x+y}$.
16. $\dfrac{1}{1-2x}-\dfrac{2x}{1-4x^2}$.
17. $\dfrac{3a}{9a^2-4b^2}-\dfrac{1}{3a+2b}$.
18. $\dfrac{2y}{(x-2y)^2}+\dfrac{1}{x-2y}$.
19. $\dfrac{x}{x^2-y^2}+\dfrac{y}{y^2-x^2}$.
20. $\dfrac{4}{x-4}-\dfrac{16-3x}{x^2-16}$.
21. $\dfrac{x-y}{x^2-y^2}+\dfrac{1}{2x+3y}$.

(In the first fraction, $x-y$ is a common factor of numerator and denominator.)

22. $\dfrac{1}{y-x}+\dfrac{x}{(x-y)^2}$.
23. $\dfrac{a-b}{c-d}-\dfrac{b-a}{d-c}$.
24. $\dfrac{2a-b}{c-d}-\dfrac{a-2b}{d-c}$.
25. $\dfrac{1}{a(a-b)}+\dfrac{1}{b(a+b)}$.
26. $\dfrac{2x}{a^2-4x^2}-\dfrac{1}{2x+a}$.
27. $\dfrac{5}{3(a-b)}+\dfrac{3}{2(b-a)}$.
28. $\dfrac{x+a}{x-a}+\dfrac{x^2-a^2}{ax-a^2}$.
29. $\dfrac{a}{a^2-9b^2}+\dfrac{1}{3b-a}$.
30. $\dfrac{a+3b}{a-2b}-\dfrac{2a+6b}{2a+5b}$.
31. $\dfrac{1}{a^3-1}+\dfrac{a+1}{a^2+a+1}$.
32. $\dfrac{1}{x-2y}-\dfrac{x^2+4y^2}{x^3-8y^3}$.
33. $\dfrac{1}{9a^2-3ab+b^2}-\dfrac{3a}{27a^3+b^3}$.

34. $\dfrac{a^2-4b^2}{a-2b}-\dfrac{a^2-9b^2}{a+3b}$.

35. $\dfrac{x^3+y^3}{x^2-xy+y^2}+\dfrac{x^3-y^3}{x^2+xy+y^2}$.

36. $\dfrac{x^2+5x+4}{x+4}-\dfrac{x^2-5x+6}{x-2}$.

37. $\left(\dfrac{x+y}{x-y}\right)^2-1$.

38. $\dfrac{x-2}{x^2-x-2}+\dfrac{x-4}{x^2-5x+4}$.

39. $\dfrac{x+4}{x^2-3x-28}-\dfrac{x-5}{x^2+2x-35}$.

40. $\dfrac{x^2-4y^2}{x^2+2xy}-\dfrac{x-2y}{x}$.

41. $\dfrac{6x+5y}{4}-\dfrac{9x^2-y^2}{6x+2y}$.

*** Example 1.** $\dfrac{a}{a-b}-\dfrac{b}{a+b}-\dfrac{b^2}{a^2-b^2}-\dfrac{a^2}{a^2+b^2}$

$$=\frac{a(a+b)-b(a-b)-b^2}{a^2-b^2}-\frac{a^2}{a^2+b^2}$$

(taking the first three fractions together)

$$=\frac{a^2+ab-ab+b^2-b^2}{a^2-b^2}-\frac{a^2}{a^2+b^2}$$

$$=\frac{a^2}{a^2-b^2}-\frac{a^2}{a^2+b^2}$$

$$=a^2\left(\frac{1}{a^2-b^2}-\frac{1}{a^2+b^2}\right)$$

$$=\frac{a^2(a^2+b^2-a^2+b^2)}{a^4-b^4}$$

$$=\frac{2a^2b^2}{a^4-b^4}.$$

*** Example 2.** Simplify $\dfrac{3}{x-a}-\dfrac{1}{x-3a}-\dfrac{3}{x+a}+\dfrac{1}{x+3a}$.

The given expression $\left(\dfrac{3}{x-a}-\dfrac{3}{x+a}\right)+\left(\dfrac{1}{x+3a}-\dfrac{1}{x-3a}\right)$ (rearranging the fractions)

$$=\frac{3x+3a-3x+3a}{x^2-a^2}+\frac{x-3a-x-3a}{x^2-9a^2}$$

$$=\frac{6a}{x^2-a^2}-\frac{6a}{x^2-9a^2}$$

$$=6a\left(\frac{1}{x^2-a^2}-\frac{1}{x^2-9a^2}\right)$$

$$=\frac{6a(x^2-9a^2-x^2+a^2)}{(x^2-a^2)(x^2-9a^2)}$$

$$=\frac{-48a^3}{(x^2-a^2)(x^2-9a^2)}.$$

* Examples. XXI. c.

Simplify :

1. $\dfrac{1}{a+b}+\dfrac{1}{a-b}+\dfrac{2a}{a^2-b^2}$.

2. $\dfrac{1}{a+b}+\dfrac{1}{b-a}+\dfrac{4b}{a^2-b^2}$.

3. $\dfrac{1}{1-3x}+\dfrac{1}{1+3x}+\dfrac{1}{1-9x^2}$.

4. $\dfrac{1}{x^2-5x+4}-\dfrac{1}{x^2-4x+3}$.

Simplify :

5. $\dfrac{a}{a^2-b^2}-\dfrac{1}{3(a-b)}-\dfrac{1}{3(a+b)}$.

6. $\dfrac{1}{3(x-3)}+\dfrac{1}{x^2-9}-\dfrac{1}{2(x+3)}$.

7. $\dfrac{1}{x-1}-\dfrac{2}{x-2}+\dfrac{1}{x-3}$.

8. $\dfrac{a^2}{a^3+b^3}+\dfrac{a-b}{a^2-ab+b^2}+\dfrac{1}{a+b}$.

9. $\dfrac{1}{x-3}-\dfrac{8x}{x^3-27}-\dfrac{x-3}{x^2+3x+9}$.

10. $\dfrac{ab}{(a-b)(b-c)}+\dfrac{ac}{(a-c)(c-b)}$.

11. $\dfrac{1}{x^2-4x+3}+\dfrac{1}{x^2-5x+6}+\dfrac{1}{x^2-3x+2}$.

12. $\dfrac{1}{x-2y}+\dfrac{2(x+1)}{2x-y}-\dfrac{1+2x}{2x-y}$.

13. $\dfrac{1}{6x-2}-\dfrac{1}{2x-\frac{2}{3}}+\dfrac{1}{3x-1}$.

14. $\dfrac{3x}{x^2-3x+2}+\dfrac{4}{1-x}+\dfrac{1}{x-2}$.

15. $\dfrac{1}{2(x-2)}+\dfrac{1}{(x-2)(x-3)}+\dfrac{1}{(x-2)(x-3)(x-4)}$.

16. $\dfrac{4y}{x^2+2xy}-\dfrac{3x}{xy+2y^2}+\dfrac{3x-2y}{xy}$.

17. $\dfrac{1}{(x-2)(x-3)}-\dfrac{1}{x^2+x-6}-\dfrac{3}{9-x^2}$.

18. $\dfrac{a^2-3ab+2b^2}{a-2b}+\dfrac{6a^2-5ab-6b^2}{2a-3b}-\dfrac{6a^2+ab-2b^2}{3a+2b}$.

19. $\dfrac{a-b}{a^2-ab+b^2}+\dfrac{1}{a+b}+\dfrac{ab}{a^3+b^3}$.

20. $\dfrac{2}{x^2-8x+15}+\dfrac{2}{x^2-4x+3}+\dfrac{4}{6x-x^2-5}$.

21. $\dfrac{1}{x^4+2x^3}+\dfrac{1}{x^4-2x^3}+\dfrac{2}{x^4+4x^2}$.

22. $\dfrac{1}{x-1}+\dfrac{2}{x+1}+\dfrac{3x-2}{1-x^2}-\dfrac{1}{(x+1)^2}$.

23. $\dfrac{x+1}{2x^3-4x^2}+\dfrac{x-1}{2x^3+4x^2}-\dfrac{1}{x^2-4}$.

24. $\dfrac{8}{x^2-5x+6}-\dfrac{5}{x^2-3x+2}-\dfrac{3}{x^2-4x+3}$.

25. $\dfrac{x^3y-xy^3}{x^6-y^6}+\dfrac{x}{x^3-y^3}-\dfrac{y}{x^3+y^3}$.

26. $\dfrac{1}{x^2-x-2}+\dfrac{2}{1-x^2}+\dfrac{1}{x^2+x-2}$.

27. $\dfrac{2y}{x^2+xy-6y^2}+\dfrac{x}{x^2-9y^2}-\dfrac{1}{x-2y}$.

28. $\dfrac{5}{x^2-3x-28}+\dfrac{3}{x^2+x-12}+\dfrac{9}{x^2-10x+21}$.

29. $\dfrac{4}{x+3}-\dfrac{7}{x+4}+\dfrac{3}{x+7}$.

30. $\dfrac{1}{4(3a-x^2)}-\dfrac{1}{5(3a+x^2)}-\dfrac{9x^2}{10(9a^2-x^4)}$.

31. $\dfrac{1}{2x^2-4x+2}-\dfrac{1}{3x^2-3}+\dfrac{1}{4x^2+8x+4}$.

32. $\dfrac{x-3y}{x+3y}-\dfrac{x-2y}{x+2y}+2$.

33. $\dfrac{1+x^2}{1-x^2}-\dfrac{4x^2}{1-x^4}-\dfrac{1-x^2}{1+x^2}$.

34. $\dfrac{5x}{3x-2}-\dfrac{21x^2+6x}{9x^2+4}+\dfrac{2x}{3x+2}$.

35. $\dfrac{b}{a-b}-\dfrac{8b}{a-2b}+\dfrac{9b}{a-3b}$.

36. $\dfrac{1}{x^2+2xy-3y^2}+\dfrac{1}{y^2+2xy-3x^2}$.

37. $\dfrac{1+x^2}{1-x^2}+\dfrac{4x^2}{1+x^4}-\dfrac{1-x^2}{1+x^2}$.

38. $\dfrac{1}{a^2-2}-\dfrac{2}{a^2-1}+\dfrac{2}{a^2+1}-\dfrac{1}{a^2+2}$.

39. $\frac{x^2-7xy+12y^2}{4x^2-11xy-3y^2}-\frac{2x^2+7xy-4y^2}{8x^2-6xy+y^2}$.

40. $\frac{x}{x-y}-\frac{y}{x+y}-\frac{x^2}{x^2+y^2}+\frac{y^2}{y^2-x^2}$.

41. $\frac{4a^2b^2}{a^4-b^4}+\frac{2a^2}{a^2+b^2}+\frac{a}{a+b}-\frac{a}{b-a}$.

42. $\frac{(2a-5b)^2-4a^2}{4a-5b}+\frac{(3a-2b)^2-4b^2}{3a-4b}$.

43. $\frac{6x^2-5xy-6y^2}{14x^2-23xy+3y^2}-\frac{15x^2+8xy-12y^2}{35x^2+47xy+6y^2}$.

44. $\frac{x}{x-y-z}+\frac{y}{y+z-x}-\frac{x+y}{x+y+z}$.

45. $\frac{1}{a-5}-\frac{1}{a-3}+\frac{1}{a+5}-\frac{1}{a+3}$.

46. $\frac{1}{x+1}+\frac{2}{x^2+1}+\frac{4}{x^4+1}+\frac{8}{x^8-1}$.

47. $\frac{1}{a-b}-\frac{1}{2(a+b)}-\frac{a+3b}{2(a^2+b^2)}-\frac{4b^3}{a^4-b^4}$.

48. $\frac{5}{3-2x}-\frac{15}{(3-2x)^2}+\frac{30x}{(3-2x)^3}$.

49. $\frac{1+a}{1-a}+\frac{4a}{1+a^2}+\frac{8a}{1-a^4}-\frac{1-a}{1+a}$.

50. $\frac{3x^2+2x+4}{x^3-1}-\frac{x+1}{x^2+x+1}-\frac{2}{x-1}$.

51. $\frac{4}{x(x-2)}+\frac{1}{x^2-5x+6}-\frac{3}{x(x-3)}$.

52. $\frac{1}{x-1}-\frac{3}{x+1}+\frac{2(x-2)}{x^2+1}$.

53. $\frac{1}{a^2-3b^2+2ab}+\frac{1}{b^2-3a^2+2ab}-\frac{2}{3a^2+10ab+3b^2}$.

54. $\frac{2x+1}{x^2+x+1}-\frac{3}{x}-\frac{1}{1-x}$.

55. $\frac{b}{a+b}-\frac{ab}{(a+b)^2}-\frac{ab^2}{(a+b)^3}$.

56. $\frac{8x^3}{8x^3-y^3}-\frac{2x^2}{4x^2+2xy+y^2}+\frac{x}{y-2x}$.

57. $\frac{1}{x+4}-\frac{3}{x+3}+\frac{3}{x+2}-\frac{1}{x+1}$.

58. $\frac{2}{x+1}+\frac{3}{(x+1)^2}+\frac{2x+5}{x^2-2x+3}$.

59. $\frac{1}{2(x-1)}-\frac{x-5}{x^2-7x+10}+\frac{x-6}{2(x^2-9x+18)}$.

60. $\frac{x}{x^2+y^2-xy}+\frac{1}{x+y}+\frac{2xy-y^2}{x^3+y^3}$.

61. $\frac{1}{x-3}+\frac{1}{x+3}-\frac{1}{x-1}-\frac{1}{x+1}$.

62. $\frac{10x-11}{3(x^2-1)}-\frac{10x-1}{3(x^2+x+1)}+\frac{x^2-2x+5}{(x^3-1)(x+1)}$.

63. $\frac{3(x^2+x-2)}{x^2-x-2}-\frac{3(x^2-x-2)}{x^2+x-2}-\frac{8x}{x^2-4}$.

64. $\frac{a+2}{a}-\frac{a}{a+2}-\frac{a^3-2a^2}{2a^2-8}$.

65. $\frac{2}{x^2+x}+\frac{2x-1}{x^2-x+1}-\frac{2x^3-1}{x^4+x}$.

66. $\frac{2x+9}{x^2+7x+12}-\frac{x}{x^2+5x+6}-\frac{x}{x^2+3x+2}$.

67. $\frac{a}{b}-\frac{(a^2-b^2)x}{b^2}+\frac{a(a^2-b^2)x^2}{b^2(b+ax)}$.

68. $\frac{3}{3x-2}-\frac{2}{2x-1}-\frac{3}{4-3x}$.

69. $\frac{a-2b}{2a^2-11ab+12b^2}+\frac{2(2a-b)}{4a^2-4ab-3b^2}-\frac{3(a-b)}{2a^2-7ab-4b^2}$.

70. $\dfrac{\frac{1}{1+x}}{1-\frac{1}{1+x}}$.

71. $\dfrac{x+y-\frac{x^2+y^2}{x+y}}{x+y-\frac{2xy}{x+y}}$.

72. $\dfrac{\frac{1}{a+b}+\frac{1}{a-b}}{\frac{1}{a+b}-\frac{1}{a-b}}$.

Simplify :

73. $\dfrac{x^2+3x+2}{(x-1)^2}\left\{1-\dfrac{3(3x+2)}{3x^2+8x+4}\right\}$.

74. $\dfrac{1-\left(\dfrac{x-y}{x+y}\right)^2}{1+\left(\dfrac{x-y}{x+y}\right)^2}$.

75. $\dfrac{x-2-\dfrac{x^2-5x}{x-3}}{x+\dfrac{3x}{x-3}}$.

76. $\left(\dfrac{x+3}{x^2-4}+\dfrac{x+5}{x^3+8}\right)\div\dfrac{x^2+1}{x^2-2x+4}$.

77. $\dfrac{a+x-\dfrac{a^3}{a^2-ax+x^2}}{a+x-\dfrac{x^3}{a^2-ax+x^2}}$.

78. $\dfrac{a}{b}\left(\dfrac{b}{c}-\dfrac{c}{a}\right)+\dfrac{b}{c}\left(\dfrac{c}{a}-\dfrac{a}{b}\right)+\dfrac{c}{a}\left(\dfrac{a}{b}-\dfrac{b}{c}\right)$.

79. $\dfrac{\dfrac{a^2-b^2}{b}+a}{\dfrac{1}{b}+\dfrac{1}{a}}\div\dfrac{b^3-a^3}{a^2-b^2}$.

80. $\dfrac{(a+3b)^2-(a-3b)^2}{(3a+b)^2-(3a-b)^2}$.

81. $\dfrac{a^3-b^3}{a-b}-\dfrac{a^3+b^3}{a+b}+(a-b)^2$.

82. $\dfrac{\dfrac{m^2+n^2}{n}-m}{\dfrac{1}{m}-\dfrac{1}{n}}\div\dfrac{m^3+n^3}{m^2-n^2}$.

83. $\dfrac{a-\dfrac{b^2}{a}}{\dfrac{a^3}{b^2}-\dfrac{b^2}{a}}\times\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)$.

84. $\left\{\dfrac{x+2a}{a-2x}-\dfrac{a+2x}{x-2a}\right\}\times\left\{\dfrac{3}{2a-x}-\dfrac{1}{a-x}\right\}$.

85. $\left\{\dfrac{5a}{a-6b}-\dfrac{2b}{3a-2b}\right\}\div\left\{\dfrac{2a}{a+2b}-\dfrac{2b-a}{2b-3a}\right\}$.

86. $\dfrac{1}{x-\dfrac{3}{x-2}}-\dfrac{1}{x+\dfrac{2}{x+3}}$.

87. $\dfrac{x(x+1)(x+2)}{3}-\dfrac{x(x+1)(2x+1)}{6}$.

88. $\left(\dfrac{x}{x-2}+\dfrac{5}{x-8}\right)\times\left(\dfrac{x-3}{3x-8}-\dfrac{2}{x+2}\right)$.

89. $\left(\dfrac{x}{x-y}-\dfrac{y}{x+y}\right)(x^2+2xy-y^2)\div\left(\dfrac{x}{x-y}+\dfrac{y}{x+y}\right)$.

90. $\left(1-\dfrac{2xy}{x^2+y^2}\right)\div\left(\dfrac{x^3-y^3}{x-y}-3xy\right)$.

91. $\left(\dfrac{x+1}{x-1}+\dfrac{5}{x-7}\right)\left(\dfrac{x-2}{3x-5}-\dfrac{2}{x+3}\right)$.

92. $\left(\dfrac{a^3+b^3}{a^3-b^3}+\dfrac{a^3-b^3}{a^3+b^3}\right)\div\left(\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}\right)$.

93. $\dfrac{(2a+3)(a^2+3a+2)-2(a+1)(a^2+2a)}{(2a+3)a^2-a(a^2-2)}$.

94. $\dfrac{3}{2x+3-\dfrac{3}{1-\dfrac{x}{x+6}}}$.

95. $\dfrac{1+\dfrac{4a^2}{6ab+9b^2}}{1+\dfrac{9b^2}{4a^2-6ab}}\div\left(\dfrac{16a^4}{81b^4}-\dfrac{2a}{3b}\right)$.

96. $1-\dfrac{1}{\dfrac{x}{a}+3+\dfrac{1}{x-2a}\left(4a+\dfrac{a^3-x^3}{x^2+ax+a^2}\right)}$.

97. $\dfrac{\dfrac{a-x}{b+x}-\dfrac{b-x}{a+x}}{\dfrac{a+x}{b+x}-\dfrac{b-x}{a-x}}-\dfrac{\dfrac{a+x}{b+x}-\dfrac{b-x}{a-x}}{\dfrac{a+x}{b-x}-\dfrac{b+x}{a-x}}.$ 98. $\left(1+\dfrac{45}{x-8}-\dfrac{26}{x-6}\right)\left(3-\dfrac{65}{x+7}+\dfrac{8}{x-2}\right).$

99. $\left\{\dfrac{x-a}{(x+a)^2}+\dfrac{x+a}{(x-a)^2}\right\}\div\left\{\dfrac{1}{(x+a)^2}-\dfrac{1}{x^2-a^2}+\dfrac{1}{(x-a)^2}\right\}.$

100. $\dfrac{1-\dfrac{1}{1+\dfrac{x}{1-x}}}{1-\dfrac{1}{1-\dfrac{x}{1+x}}}.$

101. $\left(\dfrac{1}{1-x^2}+\dfrac{1}{1-x}-1\right)\dfrac{(1-x)^2}{1-x^3}.$

102. $\dfrac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}\div\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{x}{y}+\dfrac{y}{x}}.$ 103. $(a+b+c)\left(\dfrac{1}{bc}+\dfrac{1}{ca}+\dfrac{1}{ab}\right)-\dfrac{1}{abc}(a^2+b^2+c^2).$

104. $\dfrac{(ac+bd)^2-(ad+bc)^2}{(a-b)(c-d)}.$ 105. $\dfrac{\dfrac{c}{a+b}-\dfrac{a}{b+c}}{\dfrac{a}{b+c}-\dfrac{b}{c+a}}.$

106. $\left\{\left(x+\dfrac{1}{x}\right)^2-2\left(1+\dfrac{1}{x^2}\right)\right\}\div\left(x-\dfrac{1}{x}\right)^2.$

107. $\dfrac{\{ax^2+(b-c)x-f\}^2-\{ax^2+(b+c)x-f\}^2}{\{ax^2+(b+e)x-f\}^2-\{ax^2+(b-e)x-f\}^2}.$

108. $(yz+zx+xy)\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)-xyz\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}\right).$

109. $\left(2-\dfrac{3n}{m}+\dfrac{9n^2-2m^2}{m^2+2mn}\right)\div\left(\dfrac{1}{m}-\dfrac{1}{m-2n-\dfrac{4n^2}{m+n}}\right).$

110. $\left(x^2-1-\dfrac{6}{x^2}\right)\div\left(x^2-2x+3-\dfrac{4}{x}+\dfrac{2}{x^2}\right).$

111. $\dfrac{a^2-(b-c)^2}{(c+a)^2-b^2}+\dfrac{b^2-(c-a)^2}{(a+b)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}.$

112. $\left\{1+\dfrac{x^2-xy-y^2}{x^2+xy+y^2}\right\}\times\left\{\dfrac{1}{x}-\dfrac{y^3-xy^2}{x^4-x^2y^2}\right\}.$

113. $\dfrac{x^2-ax-2a^2}{x^2-(2a+b)x+2ab}-\dfrac{x^2+ax-2a^2}{x^2+(2a+b)x+2ab}.$

114. $\dfrac{9x^2-(y-z)^2}{(3x+z)^2-y^2}+\dfrac{y^2-(z-3x)^2}{(3x+y)^2-z^2}+\dfrac{z^2-(3x-y)^2}{(y+z)^2-9x^2}.$

115. $\left\{1+\dfrac{2b^2}{a(a-3b)}\right\}\left\{1+\dfrac{b}{2b-a}\right\}-\left(\dfrac{a^2}{b^2}+\dfrac{b}{a}\right)\left(\dfrac{a^2-ab}{a^2-ab+b^2}-1\right).$

116. $\dfrac{\{(a+b)(a+b+c)+c^2\}\{(a+b)^2-c^2\}}{\{(a+b)^3-c^3\}\{a+b+c\}}.$

117. $\left(1+\dfrac{y^2+z^2-x^2}{2yz}\right)\div\left(1-\dfrac{x^2+y^2-z^2}{2xy}\right).$

Simplify:

118. $\left(x-y-\frac{4y^2}{x-y}\right)\left(x+y-\frac{4x^2}{x+y}\right)\div\left\{3(x+y)-\frac{8xy}{x-y}\right\}$.

119. $\left(\frac{x^2}{y^2}-1\right)\left(\frac{x}{x-y}-1\right)+\left(\frac{x^3}{y^3}-1\right)\left(\frac{x^2+xy}{x^2+xy+y^2}-1\right)$.

120. $\dfrac{1}{x+\frac{1}{x+2}}\times\dfrac{1}{x+\frac{1}{x-2}}\div\dfrac{x-\frac{4}{x}}{x^2+\frac{1}{x^2}-2}$.

121. $(1+a)^2\div\left\{1+\dfrac{a}{1-a+\frac{a}{1+a+a^2}}\right\}$.

Prove that

122. $\frac{a-2b}{a-b}+\frac{a-2b}{a+3b}-\frac{2(a+b)}{a+2b}=\frac{2b(a+b)(2a+b)}{(b-a)(a+3b)(a+2b)}$.

123. $\frac{a}{ax-x^2}+\frac{b}{bx-x^2}+\frac{c}{cx-x^2}=\frac{1}{a-x}+\frac{1}{b-x}+\frac{1}{c-x}+\frac{3}{x}$.

*CHAPTER XXII

HARDER SIMPLE EQUATIONS INVOLVING FRACTIONS

123. The usual method of solution is to clear away the fractions by multiplying both sides of the equation by the L.C.M. of the denominators.

The work can often be shortened by sundry methods illustrated in the following worked-out examples.

Example 1. Solve the equation $\frac{3}{4x-3}=\frac{2}{3x-5}$.

Multiplying both sides by $(4x-3)(3x-5)$, the L.C.M. of the denominators,

$$3(3x-5)=2(4x-3),$$
$$9x-15=8x-6,$$
$$x=9.$$

Example 2. Solve the equation $\frac{3x}{x-1}-\frac{2x}{x+1}=\frac{x^2+10}{x^2-1}$.

Multiplying both sides by $(x-1)(x+1)$,

$$3x(x+1)-2x(x-1)=x^2+10,$$
$$3x^2+3x-2x^2+2x=x^2+10,$$
$$5x=10,$$
$$x=2.$$

Example 3. Solve the equation $\dfrac{2}{2x-1}-\dfrac{3}{3x+1}=\dfrac{3}{3x-1}-\dfrac{2}{2x+1}$.

Simplifying each side of the equation separately,

$$\frac{2(3x+1)-3(2x-1)}{(2x-1)(3x+1)}=\frac{3(2x+1)-2(3x-1)}{(3x-1)(2x+1)},$$

$$\frac{6x+2-6x+3}{(2x-1)(3x+1)}=\frac{6x+3-6x+2}{(3x-1)(2x+1)},$$

$$\frac{5}{(2x-1)(3x+1)}=\frac{5}{(3x-1)(2x+1)}.$$

Dividing both sides by 5, and multiplying up,

$$(3x-1)(2x+1)=(2x-1)(3x+1),$$
$$6x^2+x-1=6x^2-x-1,$$
$$2x=0,$$
$$x=0.$$

Example 4. Solve the equation $\dfrac{10x-14}{2x-3}=\dfrac{15x-24}{3x-5}$.

The equation may be written $\dfrac{5(2x-3)+1}{2x-3}=\dfrac{5(3x-5)+1}{3x-5}$,

$$i.e.\quad \cancel{5}+\frac{1}{2x-3}=\cancel{5}+\frac{1}{3x-5},$$
$$3x-5=2x-3,$$
$$x=2.$$

Example 5. Solve the equation $\dfrac{x-3}{x-5}-\dfrac{x-1}{x-3}=\dfrac{x-7}{x-9}-\dfrac{x-5}{x-7}$.

The equation may be written

$$\frac{\overline{x-5}+2}{x-5}-\frac{\overline{x-3}+2}{x-3}=\frac{\overline{x-9}+2}{x-9}-\frac{\overline{x-7}+2}{x-7};$$

$$\therefore\ \cancel{1}+\frac{2}{x-5}-\cancel{1}-\frac{2}{x-3}=\cancel{1}+\frac{2}{x-9}-\cancel{1}-\frac{2}{x-7}.$$

Dividing both sides by 2

$$\frac{1}{x-5}-\frac{1}{x-3}=\frac{1}{x-9}-\frac{1}{x-7}.$$

Simplifying each side separately,

$$\frac{(x-3)-(x-5)}{(x-5)(x-3)}=\frac{(x-7)-(x-9)}{(x-7)(x-9)},$$

$$i.e.\quad \frac{2}{(x-5)(x-3)}=\frac{2}{(x-7)(x-9)}.$$

Dividing both sides by 2, and multiplying up,

$$(x-7)(x-9)=(x-5)(x-3),$$
$$x^2-16x+63=x^2-8x+15$$
$$-8x=-48,$$
$$x=6.$$

* Examples. XXII.

(In the case of a fractional solution, express the result in decimals correct to two decimal places.)

Solve the equations :

1. $\frac{x-3}{x-4}=\frac{x+12}{x+8}$.

2. $\frac{x+3}{2x-3}=\frac{2x}{4x-9}$.

3. $3-\frac{22}{x+5}=\frac{6x-1}{2x+7}$.

4. $\frac{x}{x-3}+\frac{2}{x-5}=1$.

5. $\frac{x+1}{3x-4}=\frac{1}{5}+\frac{8x-3}{15x-20}$.

6. $\frac{6x-5}{8x-12}=\frac{1}{12}-\frac{3x-4}{6x-9}$.

7. $\frac{3}{x-3}+\frac{4}{x-4}=\frac{25}{x^2-7x+12}$.

8. $\frac{5x-7}{10x-5}=\frac{1}{10}-\frac{4x-3}{4x-2}$.

9. $\frac{11x}{x+20}+\frac{24}{x}=11+\frac{88}{x(x+20)}$.

10. $\frac{x-\frac{1}{2}}{x-1}-\frac{3}{5}\left(\frac{1}{x-1}-\frac{1}{3}\right)=\frac{23}{10(x-1)}$.

11. $\frac{9(12-x)}{4(x+1)}+\frac{5}{4}=\frac{17-x}{x-8}$.

12. $\frac{6x-7}{2x-3}-\frac{9x-12}{3x-5}=\frac{12x-25}{3x-7}-\frac{8x-18}{2x-5}$.

13. $\frac{x-4}{x-5}-\frac{x-2}{x-3}=\frac{x-10}{x-11}-\frac{x-8}{x-9}$.

14. $\frac{30+6x}{x+1}+\frac{60+8x}{x+3}=14+\frac{48}{x+1}$.

15. $\frac{6x+2}{x+15}+\frac{2x-9}{x-6}=6+\frac{2x-13}{x-6}$.

16. $\frac{3x-14}{x-5}-\frac{3x-8}{x-3}=\frac{3x-32}{x-11}-\frac{3x-26}{x-9}$.

17. $\frac{7x+1}{x-1}=\frac{35}{9}\left(\frac{x+4}{x+2}\right)+\frac{28}{9}$.

18. $\frac{\frac{x}{6}}{5x-4}=\frac{\frac{2x}{5}-\frac{27}{14}}{12x+\frac{33}{5}}$.

19. $\frac{x+2}{x-3}+\frac{x-2}{x-6}=2$.

20. $2\left(\frac{2x+3}{x-1}\right)+3\left(\frac{x-2}{x+2}\right)=7$

21. $\frac{3x+2}{x-1}+\frac{2x-4}{x+2}=5$.

22. $\frac{8x}{2x-3}-\frac{5}{3x-2}=4$.

23. $\frac{1}{x+1}-\frac{2}{x+2}+\frac{1}{x+4}=0$.

24. $\frac{3x}{x-1}-\frac{2x}{2x-1}=2$.

25. $\frac{1}{x-1}+\frac{1}{x-4}=\frac{2}{x+2}$.

26. $\frac{1}{15-10x}-\frac{1}{15-6x}=\frac{1}{15x+120}$.

27. $\frac{4(2x-1)}{3(x-2)}-\frac{2(7x-1)}{6x-13}=\frac{1}{3}$.

28. $\frac{3x-2}{2x-3}-\frac{x+17}{x+10}=\frac{1}{2}$.

29. $\frac{6x+1}{3x-5}-\frac{2x-5}{3x-4}=\frac{4}{3}$.

30. $\frac{1}{x-1}-\frac{1}{x-3}=3\left\{\frac{1}{x-2}-\frac{1}{x-3}\right\}$.

31. $\frac{10x+17}{18}-\frac{12x+2}{11x-8}=\frac{5x-4}{9}$.

32. $\frac{x-5}{x^2-6x+6}-\frac{x-7}{x^2-8x+15}=0$.

33. $\frac{x^2-x+1}{x-1}+\frac{x^2+x+1}{x+1}=2x$.

34. $\frac{x-1}{x-2}-\frac{x}{x-1}=\frac{x-8}{x-9}-\frac{x-7}{x-8}$.

35. $\frac{1+x}{1-x}-\frac{2+3x}{2-3x}=1+\frac{1+3x}{1-3x}$.

36. $\frac{1}{x-4}-\frac{1}{x-3}=\frac{1}{4}\left(\frac{1}{x-5}-\frac{1}{x-1}\right)$.

37. $\frac{x-1}{x-5}+\frac{x-5}{x-9}+\frac{x-9}{x-1}=3.$

38. $\frac{1}{x-3}-\frac{1}{x-5}-\frac{1}{x-7}+\frac{1}{x-9}=0.$

39. $\frac{3x-4\frac{1}{2}}{2x-3\frac{1}{3}}-\frac{7x+4}{8x-7}=\frac{5}{8}.$

40. $\frac{1}{x-3}-\frac{1}{x-4}=\frac{1}{x-6}-\frac{1}{x-7}.$

41. $\frac{5x-34}{x-7}+\frac{3x-26}{x-9}=\frac{5x-24}{x-5}+\frac{3x-32}{x-11}.$

42. $\frac{x-1}{x+1}+\frac{x+1}{x-2}+\frac{x-2}{x-1}=3.$

CHAPTER XXIII

MISCELLANEOUS FACTORS FOR REVISION

XXIII. a.

[Grouped in batches of 10.]

Resolve into their simplest factors :

1. $ax^2-bx.$ 2. $x^2+11x+10.$ 3. $3x^2-3.$ 4. $2x^2-8x+6.$
5. $ax-bx+a^2-b^2.$ 6. $1-2x-3x^2.$ 7. $4a^3-4b^3.$
8. $18x^2+24x+6.$ 9. $8x^2+14x-15.$ 10. $x^3+2x^2-x-2.$

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11. $20xy-15y^2.$ 12. $ax^2-ab^2.$ 13. $x^2-52x+51.$ 14. $4(a^2-\frac{1}{4}).$
15. $x^3+ax^2+a^2x+a^3.$ 16. $72-x-x^2.$ 17. $(a+b)^2-a-b.$
18. $16x^2-50x-21.$ 19. $a^2-b^2-c^2+2bc.$ 20. $abx^2-4ax-3bx+12.$

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21. $3-6x+3x^2.$ 22. $27x^2-12x+1.$ 23. $20a^2-45.$
24. $3ax+2by-2bx-3ay.$ 25. $3c^3-81.$ 26. $6+3x-2x^2-x^3.$
27. $35x^2+12x-32.$ 28. $x^2y^2+1-x^2-y^2.$ 29. $6-5x-2x^2+x^3.$
30. $a^3x^2+b^3y^2-a^3y^2-b^3x^2.$

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31. $63ab-21bc-245b^2.$ 32. $54x^2+15xy-y^2.$ 33. $6x-ay-ax+6y.$
34. $3x^2-\frac{1}{3}.$ 35. $27x^2-6x-8.$ 36. $343x^2-7y^2.$
37. $x^2y^2-1-x^2+y^2.$ 38. $(a-b)^3-a+b.$ 39. $x^6-64y^6.$
40. $(a+b)^2-5a-5b+6.$

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41. $p^3x^2-2p^2x+p.$ 42. $x^2-25x+156.$ 43. $x(x+8)+8(x+6).$
44. $33x^2+20xy-32y^2.$ 45. $x^2+2ax-7bx-14ab.$ 46. $(a+b)^3-(a-b)^3.$
47. $15x^2-2ab-5ax+6bx.$ 48. $2x^6-128.$ 49. $4x^3-7x-3.$
50. $(bx+ay)^2+(by-ax)^2-c^2(x^2+y^2).$

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51. $x^2-16(x-4).$ 52. $(a+\frac{1}{2})^2-(b+\frac{1}{2})^2.$ 53. $x^2+14x-147.$
54. $3(a-b)^2-3a+3b.$ 55. $12x^2-14ab+8ax-21bx.$ 56. $x^3+3+2x^2-2x.$
57. $27x^2+210x-125.$ 58. $x^2-3ay+3xy-a^2.$ 59. $a^4-16(b-c)^4.$
60. $a(a-1)x^2+x-a(a+1).$

Resolve into their simplest factors:

61. $a^2+2a+b^2+2b+2ab$. 62. $35x^2-74xy-24y^2$. 63. $3(x^2-y^2)-4x+4y$.

64. $b^2x^4-b^6$. 65. $x^4+2x^2y^3+y^6$. 66. $16\left(x^2-\frac{a^2}{16}\right)$.

67. $32x^3+352x^2+320x$. 68. $(x+y)^2(x-y)-(x-y)^2(x+y)$.

69. $4b^2c^2-(a^2-b^2-c^2)^2$. 70. $(2a-b)^4-(a-2b)^4$.

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71. $5a^2-a-5b^2+b$. 72. $39x^2+14x-8$. 73. $16(x^4-\frac{1}{16})$.

74. $ax+by-ay-cx-bx+cy$. 75. $(x^2-2)^2-x^2$.

76. $(x+y)^2-13(x+y)a+42a^2$. 77. $(3a-b)^4-(a-3b)^4$.

78. $a^2x+ac-abx-b^2y-bc+aby$. 79. $8(2x+y)^3+(x-2y)^3$.

80. $16x^4+4x^2y^2+y^4$.

REVISION PAPERS

XXIII. b.

1. Resolve the following into their simplest factors:

(i) ax^2-a^3. (ii) $x^2-2xy-99y^2$.

(iii) $75x^2-76x+1$. (iv) $x^2+xy-5x-5y$.

2. Find the H.C.F. of $2x^2-5x-3$ and $3x^3-81$.

3. Simplify $\frac{3}{x-1}-\frac{4}{x-2}+\frac{1}{x-3}$, and find a value of x which will make the expression equal to zero.

4. Multiply $x^2-ax+bx-ab$ by $x^2+ax-bx-ab$.

5. Using half an inch as x unit, and one-tenth of an inch as y unit, plot the points given by the table below, and join them by an even curve.

$x=-5$	-4	-3	-2	-1	0	1	2	3	4	5
$y=25$	16	9	4	1	0	1	4	9	16	25

Read off from the figure, the values of x when $y=7$ and 13, and the values of y when $x=1\cdot8$ and $-2\cdot4$.

6. Solve the equation $\frac{x^2-2x+4}{x-1}=\frac{x^2-5}{x+1}$.

7. A bicycles at the rate of 12 m. an hour, stopping for 6 minutes at the end of each hour. B starts 2 hours 24 minutes later on his motor car, and, pursuing him, catches him up 42 miles from the start without any stops. At what rate did B travel? Solve the problem graphically and algebraically.

XXIII. c.

1. Resolve the following into factors:

(i) $2x^2-8$. (ii) $2x^2-5x+2$.

(iii) $a^2+2ab+b^2-c^2$. (iv) $x^2-y^2-3x+3y$.

2. Simplify $\dfrac{(x^2-1)(x^2-4)}{(x^2+x-2)(x^2-x-2)}$.

3. Find the L.C.M. of $3a^3b-3a^2b^2$, $4ab^3-4a^2b^2$, $2a^3b^3$.

4. Simplify $[(x-1)^2+2(x-1)(2x-1)+(2x-1)^2]\div(3x-2)$.

5. Plot the points (10, 10), (15, 18), (30, 22), (39, 10). If the quadrilateral joining them represents a field, each square unit representing one-tenth of an acre, find the area of the field.

6. Solve the equations $\dfrac{1}{3x}-\dfrac{1}{4y}=\dfrac{11}{72}$, $\dfrac{1}{x}-\dfrac{1}{3y}=\dfrac{7}{18}$. Check your result.

7. A train does a journey without stoppages in 8 hours; if it had travelled 5 m. an hour faster, it would have done the journey in 6 hours 40 minutes. Find its slower speed.

XXIII. d.

1. Resolve into factors:

(i) $2x^2+7x+3$. (ii) $a^2-b^2-2bx-x^2$.

(iii) $c^2+ab-ac-bc$. (iv) $3-3b^3$.

2. Find the H.C.F. of $x^2-ax-bx+ab$, $x^2+cx-ax-ac$, and bx^2-a^2b.

3. Simplify $\dfrac{1}{x-y}-\dfrac{2x+y}{x^2-y^2}+\dfrac{x(x^2+y^2)}{x^4-y^4}$.

4. Draw the graph of $x+2y=8$, and from it write down all the positive integral solutions of the equation, not counting zero values.

5. Divide a^6-b^6 by a^2-ab+b^2.

6. Solve the equation $\dfrac{x^2-x-2}{x-2}+\dfrac{2x^2-x-1}{x-1}=\dfrac{4x^2+x-3}{x+1}$.

7. In an innings of a cricket eleven the team were accounted for in the following manner. Some were stumped, half as many again were caught, and half the wickets that fell were bowled. How many were stumped, caught, and bowled respectively?

XXIII. e.

1. Resolve into factors:

(i) $x^2-28x-128$. (ii) $ax-2y-2x+ay$.

(iii) x^3-5x^2+7x-3. (iv) $4+108a^3$.

2. Simplify $\dfrac{(a+b)^2-c^2}{(a-b)^2-c^2}\times\dfrac{(b+c)^2-a^2}{(c-b)^2-a^2}\div\dfrac{(a+b+c)^2}{c^2-(a-b)^2}$.

3. Find the L.C.M. of x^2-5x+6, x^2-x-2, x^2-2x-3.

4. A bicycles a journey of 36 miles in $5\frac{1}{2}$ hours, and B, starting $1\frac{1}{2}$ hours after him, arrives at the end of the journey 36 minutes before him. If they ride at uniform speeds, find graphically where B passes A. Calculate your result to the nearest tenth of a mile.

5. Divide $6x^4-5x^3+6x^2+17x+6$ by $6x^2+7x+2$.

6. Simplify $\dfrac{2x^2-5x+3}{2x-3}-\dfrac{3x^2+x-4}{x-1}+\dfrac{2(3x^2-13x-10)}{3x+2}$.

7. What value of x will make

$(x+\frac{1}{2})^2-(x-\frac{3}{2})^2$ equal to $2x+3$.

XXIII. f.

1. Resolve into factors :

(i) $2x^2+9x-5$. (ii) $(2a+b)^2-(a+2b)^2$.

(iii) $a(b+c-d)+d(a-b-c)$. (iv) $x^3-x^2z-xy^2+y^2z$.

2. Find the H.C.F. of $c^2-(a-b)^2$, $(a+c)^2-b^2$, $(c-b)^2-a^2$.

3. Simplify $\frac{2}{1-x}-\frac{2}{2-x}+\frac{1}{(1-x)^2}-\frac{5}{(2-x)^2}$. Check your result by putting $x=3$.

4. Draw the graph of $2x+3y=21$, and from it write down all positive integral solutions, counting zero values as positive.

5. Solve the equations $\frac{5}{y}-\frac{2}{x}=1\frac{1}{6}$,

$$\frac{36}{x}-\frac{24}{y}=1.$$ Check your results.

6. By doing a journey at the rate of $12\frac{1}{2}$ miles an hour a bicyclist completes it in 3 minutes less time than if he had travelled at 12 miles an hour. Find the length of the journey.

7. Solve the equation $\frac{x+5}{x+4}-\frac{x+7}{x+6}=\frac{x+10}{x+9}-\frac{x+12}{x+11}$. Test your answer.

XXIII. g.

1. Resolve into factors :

(i) $12x^2+7x-12$. (ii) $4a^2+b^2-c^2-d^2+4ab+2cd$.

(iii) $x^3-2-x+2x^2$. (iv) $x^2y^2-x^2-y^2+1$.

2. Simplify $\frac{x^4+x^2+1}{x^4-4}\times\frac{x^2-2}{x^3-1}\div\frac{x^3+1}{x^2+2}$.

3. Find the L.C.M. of $3(x^4-x^2y^2)$, $6(x^2y^2+y^4)$, $9(x^3-x^2y+xy^2-y^3)$.

4. The majority against a certain motion is equal to $6\frac{2}{3}$ per cent. of the total number voting. If 12 of those who voted against the motion had voted for it, the motion would have been carried by a single vote. Find the numbers voting on each side.

5. Divide $x^3-b(4a+b)x+(a+2b)(a^2+3b^2)$ by $x+a+2b$.

6. Solve the equation $\frac{2x+3}{x+1}-\frac{2x+9}{x+4}=\frac{3x+7}{x+2}-\frac{3x+16}{x+5}$. Test your answer.

7. A man travels at the rate of x feet per minute.

How long does he take to do a mile ?

How many yards does he travel in an hour ?

How many miles does he travel in y hours ?

XXIII. h.

1. Simplify $\left(x+\frac{1}{x}\right)^3-\left(x-\frac{1}{x}\right)^3$.

2. Solve the equation $\frac{2x^2+5x+4}{x+2}=\frac{4x^2+8x+6}{2x+3}$. Test your solution.

3. Plot the points (0, 0), (1, 1), (4, 2), (9, 3), (16, 4), (25, 5), (1, −1), (4, −2), (9, −3), (16, −4), (25, −5), using one-tenth of an inch as x unit and half an inch as y unit. Join the points by an even curve. Estimate the corresponding y values on the curve when $x=11$, and when $x=23$.

4. Simplify $\frac{a^2-b^2}{a^2}\times\left(1+\frac{2b}{a-b}\right)^2\div\frac{(a+b)^3}{a^3-a^2b}$.

5. A fraction is such that its denominator exceeds its numerator by 2, also if the numerator is diminished by unity and the denominator increased by unity, the fraction becomes equal to $\frac{1}{2}$. Find the fraction.

6. Solve the equations $\frac{x}{y}-2x=2\frac{1}{2}$,

$\frac{x}{y}+2x+5\frac{1}{2}=0$. Test your solution.

7. What is the interest on

(i) £300 for 1 year at x per cent. per annum ?
(ii) .... 4 years ...., simple interest ?
(iii) £a for 1 year ...., ... ?
(iv) .... y years ...., ... ?

XXIII. k.

1. Divide $x^2+1+\frac{1}{x^2}$ by $x-1+\frac{1}{x}$.

2. Solve the equation $\frac{4}{5x-1}-\frac{17}{25x^2-1}=\frac{3}{5x+1}$. Test your solution.

3. From the equation $\frac{3}{y-5}+\frac{4}{2-x}=\frac{14}{(x-2)(y-5)}$, find the value of $\frac{x}{y}$.

4. Simplify $\left(1-2\frac{y}{x}+\frac{y^2}{x^2}\right)\times\frac{x+y}{\frac{x}{y}-\frac{y}{x}}\div\left(\frac{x}{y}-\frac{y^2}{x^2}\right)$.

5. At what time (to the nearest minute) do the hands of a clock point in the same direction between 4 and 5 o'clock ?

6. Solve the equations $xy+4x=7$,

$xy-3x=14$. Test your solution.

7. In the equation $y=2x-x^2$, find the corresponding values of y to all integral values of x from −3 to 5. Tabulate your work. Using half an inch as x unit, and one-tenth of an inch as y unit, plot the points, and join them by an even curve.

XXIII. l.

1. Divide $(x^2-y^2)^2-(x^2-3xy+2y^2)^2$ by $(x-y)^2$.

2. Solve the equation $\frac{3x^2+14x+7}{x+4}=\frac{9x^2-5}{3x-2}$. Test your solution.

3. Simplify $\frac{a^2+b^2-c^2+2ab}{a^2+b^2-c^2-2ab}\div\frac{a+b+c}{a-b+c}$.

4. Find two numbers whose difference is 27, such that the larger divided by the smaller gives a quotient 7 and a remainder 3.

5. Find values of a and b which will satisfy both the equations

$$\frac{a}{x}-\frac{b}{y}=7,\quad \frac{2a}{x}-\frac{3b}{y}=2,\ \text{when } x=\tfrac{1}{2} \text{ and } y=\tfrac{1}{3}.$$

6. Solve the equations

$$3x+4y+14=0,$$
$$5x-2y+\ 6=0.$$

Deduce the solution of the equations

$$\frac{3}{x}+\frac{4}{y}+14=0,$$
$$\frac{5}{x}-\frac{2}{y}+\ 6=0.$$

7. If $2x-3y-1=0$, and $xy-3x+2=0$, prove that $3y^2-8y+1=0$.

XXIII. m.

1. Divide $(a^2+2ab-3b^2)^2-(a^2-4ab+3b^2)^2$ by $(a-b)^2$.

2. Solve the equation $\dfrac{3}{2x+3}-\dfrac{1}{2-x}=\dfrac{19}{2(2x+3)(x-2)}$. Test your solution.

3. From the equation $\dfrac{7}{y-4}-\dfrac{3}{x-2}+\dfrac{2}{(x-2)(y-4)}=0$, find the value of $\dfrac{x}{y}$.

4. Simplify $\dfrac{4x^4+8x^2+4}{(x^2-x+1)^2}\times\dfrac{x^4+x^2+1}{x^4+1}\div\dfrac{(x^4+x^2)^2}{x^3+1}$.

5. At what time (to the nearest minute) do the hands of a clock point in opposite directions between 4 and 5 o'clock ?

6. Try to solve the equation $\dfrac{1}{x+4}=\dfrac{2}{2x-7}+\dfrac{15}{(7-2x)(4+x)}$. What conclusion do you draw ?

7. A horse is bought for £85, and sold at a gain of x per cent. What is the selling price ?

By selling a horse for £92, a profit of x per cent. is made : what was the original price of the horse ?

CHAPTER XXIV

SQUARE ROOT

124. Every quantity has two square roots, equal in value but opposite in sign.

E.g. the square root of 4 is $+2$ or -2,

$$\text{for } (+2)^2=4, \quad\text{and}\quad (-2)^2=4.$$
$$\therefore\ \sqrt{4}=2 \quad\text{or}\quad -2,$$

or, as it is written more shortly, $\sqrt{4}=\pm 2$.

At present we will only deal with the positive root.

A square is always positive, for by the rule of signs

$$a \times a = a^2,$$
$$(-a) \times (-a) = a^2;$$

i.e. whether a quantity is positive or negative, its square is positive.

Hence we see that a negative quantity has no square root. The square root of a negative quantity however has an interpretation, but this hardly comes into the province of Elementary Algebra.

The square roots of simple algebraical expressions can be seen by inspection.

$$\sqrt{(a^4b^2)} = a^2b.$$
$$\sqrt{x^2y^4z^6} = xy^2z^3.$$
$$\sqrt{16a^4} = 4a^2.$$
$$\sqrt{\frac{81b^4}{x^2}} = \frac{9b^2}{x}.$$

Examples. XXIV. a.

Write down, or read off, the positive square roots of the following

1. x^8.	2. a^{10}.	3. y^{16}.	4. x^6y^4.
5. a^2b^4.	6. x^8y^6.	7. $4a^2b^2$.	8. $16a^4b^2$.
9. $49x^4y^6z^8$.	10. $\frac{4a^2}{b^2}$.	11. $\frac{9x^4}{y^6}$.	12. $\frac{81a^4b^6}{c^8}$.
13. $\cdot 01$.	14. $\cdot 25$.	15. $\cdot 64$.	16. $\frac{1}{\cdot 0001}$.
17. $\frac{1}{\cdot 16}$.	18. $\frac{\cdot 49}{\cdot 36}$.	19. $\cdot 01b^4c^2$.	20. $\frac{\cdot 16a^2}{4b^4}$.
21. $1\cdot 21a^6c^{10}$.	22. $\frac{16}{49}x^{12}y^{16}$.	23. $\frac{a^4}{\cdot 81b^2}$.	24. $\frac{\cdot 0064x^4}{\cdot 0001y^{12}}$.
25. $9(a-b)^2$.	26. $\frac{121}{9}(2x+y)^2$.	27. $\cdot 01(10x+10y)^2$.	

125. The square of a simple expression is also a simple expression.

E.g. $$(4a^2b^2)^2 = 16a^4b^4.$$

We know also that the square of a binomial expression is a trinomial expression.

E.g. $$(x+2)^2 = x^2 + 4x + 4.$$
$$(2x+3)^2 = 4x^2 + 12x + 9.$$

Thus we see that a binomial expression has no square root.

126. The square root of a trinomial expression which is a square can usually be determined by inspection.

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

Hence all trinomials which are perfect squares must be of the form
$$a^2 + 2ab + b^2.$$

Thus $\quad 4x^2 + 12xy + 9y^2 = (2x)^2 + 2(2x)(3y) + (3y)^2.$

$$\therefore \sqrt{4x^2 + 12xy + 9y^2} = 2x + 3y.$$
$$\sqrt{4x^2 - 12xy + 9y^2} = 2x - 3y.$$

The form of the square of a binomial $(a^2 \pm 2ab + b^2)$ is of great importance.

Consider the expression
$$x^2 + pax + a^2.$$

By comparing this with the above we see that if it has a square root, that root must be $x + a$.

But $\quad (x + a)^2 = x^2 + 2ax + a^2$;

$\therefore$ if $x^2 + pax + a^2$ is a perfect square,

p must be equal to 2.

Examples. XXIV. b.

Determine the square roots of the following expressions:

1. $x^2 + 2xy + y^2$.
2. $x^2 - 2xy + y^2$.
3. $a^2 + 4ab + 4b^2$.
4. $4a^2 - 4ab + b^2$.
5. $x^2 - 6x + 9$.
6. $1 - 4x + 4x^2$.
7. $25a^2 - 30ab + 9b^2$.
8. $49x^2 - 14xy + y^2$.
9. $4a^2 - 28ab + 49b^2$.
10. $9x^2 + 24xy + 16y^2$.
11. $121a^2 - 44ab + 4b^2$.
12. $1 - 2x^3 + x^6$.
13. $169a^2 + 52ab + 4b^2$.
14. $81a^2 - 18ab + b^2$.
15. $25x^2 - 70xy + 49y^2$.
16. $a^4 - 2a^2b^2 + b^4$.
17. $4a^4 + 4a^2b^2 + b^4$.
18. $x^4y^2 - 2x^2y + 1$.
19. $\frac{x^2}{9} - \frac{2x}{3} + 1$.
20. $a^4 + 4a^2b^2 + 4b^4$.
21. $x^2 - x + \frac{1}{4}$.
22. $\frac{a^2}{4} - ab + b^2$.
23. $\frac{x^2}{y^2} - 2 + \frac{y^2}{x^2}$.
24. $x^2 - 3xy + \frac{9y^2}{4}$.
25. $x^4 + \frac{1}{x^4} + 2$.
26. $a^2 - 5a + \frac{25}{4}$.
27. $(x + y)^2 + 2(x + y) + 1$.
28. $(a + b)^2 - 2(a^2 - b^2) + (a - b)^2$.
29. $(x - y)^2 - 4(x - y) + 4$.
30. $9(a + b)^2 + 6(a + b) + 1$.
31. $(a + b)^2 + 2(a + b)(c + d) + (c + d)^2$.
32. $(a + b)^2 + 2a(a + b) + a^2$.
33. $\left(\frac{a}{b} + 1\right)^2 - 2\left(\frac{a}{b} + 1\right) + 1$.
34. $16(x - y)^2 - 8(x - y) + 1$.
35. $(a + 2b)^2 + (a + 2b) + \frac{1}{4}$.
36. $(a + b)^2 - 2a(a + b) + a^2$.
37. $\left(\frac{a}{b} - 1\right)^2 - 2\left(\frac{a}{b} - 1\right) + 1$.

38. $16(x+y)^2-24(x^2-y^2)+9(x-y)^2$.

39. $\frac{a^6}{x^6}-2+\frac{x^6}{a^6}$.

40. $\frac{4a^4}{x^4}-4+\frac{x^4}{a^4}$.

41. $\frac{x^8}{4a^8}+2+\frac{4a^8}{x^8}$.

42. $\frac{(a+b)^2}{9}-\frac{(a+b)(x+y)}{3}+\frac{(x+y)^2}{4}$.

What must be added to the following expressions to make them complete squares?

43. a^2+b^2.

44. x^2-4x.

45. $9+x^2$.

46. $4x^2+25y^2$.

47. $(a+b)^2+2(a+b)$.

48. Determine the value of p if $x^2-4px+16$ is a perfect square.

49. For what value of a will x^2-2x+a be a perfect square?

50. What value of p will make $x^2+6pxy+q^2y^2$ a perfect square?

127. *To find the square root of any compound expression.*

The method depends upon the fact that the square of $a+b$ is $a^2+2ab+b^2$, which may be written in the form

$$a^2+b(2a+b). \quad \text{(i)}$$

Let us take an easy example.

The first term in the square root of $36x^2-84xy+49y^2$ is evidently $6x$.

$$\begin{array}{l} 36x^2-84xy+49y^2\ (\ 6x \\ \underline{36x^2} \\ \quad\quad -84xy+49y^2 \end{array}$$

Subtracting its square, *i.e.* $36x^2$, from the given expression, the remainder is $-84xy+49y^2$, which may be written

$$-7y(2\times 6x-7y).$$

Comparing this with (i), we see that in this case a is $6x$, and therefore b is $-7y$.

Hence we have the following rule.

Having obtained the first term, (6x), double it, (12x), and divide the first term (−84xy) of the remainder by it. The quotient (−7y) is the second term of the square root.

The full work is best arranged as below:

$$\begin{array}{rl} & 36x^2-84xy+49y^2\ (\ 6x-7y \\ & \underline{36x^2} \\ & \quad\quad -84xy+49y^2 \\ (12x-7y)\times(-7y) = & \quad\quad \underline{-84xy+49y^2} \end{array}$$

Explanation. Having obtained the first term of the square root, $6x$, we double it, $12x$, and divide it into $-84xy$, the first term of the remainder when $(6x)^2$ is subtracted. The quotient $(-7y)$ is the second term of the answer.

Add $-7y$ to $12x$ and multiply the result by $-7y$, placing the result $-84xy+49y^2$ under the remainder.

If the student carefully compares the following with the expression $a^2+b(2a+b)$, he will see the reasons for the different steps.

$$\begin{array}{rl}
 & a^2+2ab+b^2 \;(\; a \\
 & \underline{a^2\qquad\qquad\quad} \\
 & \qquad 2ab+b^2 \\
(2a+b)\times b = & \qquad \underline{2ab+b^2}
\end{array}$$

128. *Find the square root of*

$$25x^4-30px^3+49p^2x^2-24p^3x+16p^4.$$

$$\begin{array}{rl}
 & 25x^4-30px^3+49p^2x^2-24p^3x+16p^4 \;(\; 5x^2-3px \\
 & \underline{25x^4\qquad\qquad\qquad\qquad} \\
 & \quad -30px^3+49p^2x^2 \\
(10x^2-3px)\times(-3px)= & \quad \underline{-30px^3+\;\;9p^2x^2} \\
 & \qquad\qquad\qquad 40p^2x^2-24p^3x+16p^4
\end{array}$$

Thus far the work is exactly similar to that in the previous examples, the reasons being the same.

Thinking once more of the expression $a^2+b(2a+b)$, we see that if the given expression has a square root, the remainder $40p^2x^2-24p^3x+16p^4$ must be of the form $b(2a+b)$, remembering that now a is $5x^2-3px$.

We therefore *repeat the process of the first step.*

Double $5x^2-3px$, obtaining $10x^2-6px$.

$40p^2x^2\div 10x^2=4p^2$ gives us the next term of the answer.

Add this to $10x^2-6px$, obtaining $10x^2-6px+4p^2$; multiply this by $4p^2$, and place the result under the remainder.

The example is worked out in full below:

$$\begin{array}{rl}
 & 25x^4-30px^3+49p^2x^2-24p^3x+16p^4 \;(\; 5x^2-3px \\
 & \underline{25x^4\qquad\qquad\qquad\qquad} \qquad\qquad\qquad\qquad +4p^2 \\
 & \quad -30px^3+49p^2x^2 \\
(10x^2-3px)\times(-3px)= & \quad \underline{-30px^3+\;\;9p^2x^2} \\
 & \qquad\qquad\qquad 40p^2x^2-24p^3x+16p^4 \\
(10x^2-6px+4p^2)\times 4p^2= & \qquad\qquad\qquad \underline{40p^2x^2-24p^3x+16p^4}
\end{array}$$

$\therefore\; 5x^2-3px+4p^2$ is the reqd. sq. root.

129. The square root of a compound expression can often be seen by re-arrangement and inspection.

$$\begin{aligned}
&x^4-2x^3-x^2+2x+1\\
&\qquad=x^4-2x^3-2x^2+(x^2+2x+1)\\
&\qquad=x^4-2x^2(x+1)+(x+1)^2 \quad [a^2-2ab+b^2]\\
&\qquad=[x^2-(x+1)]^2;\\
&\therefore\ \sqrt{x^4-2x^3-x^2+2x+1}=x^2-x-1.
\end{aligned}$$

$$\begin{aligned}
&a^2+b^2+c^2-2bc-2ac+2ab\\
&\qquad=a^2+2a(b-c)+b^2+c^2-2bc
\end{aligned}$$

(arranging in descending powers of a)

$$\begin{aligned}
&\qquad=a^2+2a(b-c)+(b-c)^2\\
&\qquad=(a+b-c)^2;\\
&\therefore\ \sqrt{a^2+b^2+c^2-2bc-2ac+2ab}=a+b-c.
\end{aligned}$$

Find the square root of

$$\frac{4x^4}{25}+\frac{1}{9x^4}-\frac{4x^2}{5}-\frac{2}{3x^2}+\frac{19}{15}.$$

Arrange the expression in *descending* powers of x.

$$\begin{array}{ll}
 & \frac{4x^4}{25}-\frac{4x^2}{5}+\frac{19}{15}-\frac{2}{3x^2}+\frac{1}{9x^4}\left(\frac{2x^2}{5}-1+\frac{1}{3x^2}\right. \\
 & \frac{4x^4}{25} \\
\hline
 & \quad -\frac{4x^2}{5}+\frac{19}{15} \\
\left(\frac{4x^2}{5}-1\right)\times(-1) & \quad -\frac{4x^2}{5}+1 \\
\hline
 & \qquad \frac{4}{15}-\frac{2}{3x^2}+\frac{1}{9x^4} \\
\left(\frac{4x^2}{5}-2+\frac{1}{3x^2}\right)\times\frac{1}{3x^2} & \qquad \frac{4}{15}-\frac{2}{3x^2}+\frac{1}{9x^4} \\
\hline
\end{array}$$

Examples. XXIV. c.

Find the square roots of the following expressions :

1. $x^4+2x^3+3x^2+2x+1$.
2. $4x^4+4x^3+5x^2+2x+1$.
3. $x^4-2x^3+5x^2-4x+4$.
4. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.
5. $9x^4-12x^3+34x^2-20x+25$.
6. $4x^2+25y^2+16z^2-20xy-40yz+16xz$.
7. $16x^6+6x^3+17x^4+x^2+24x^5$.
8. $12a^3x-26a^2x^2+25x^4+9a^4-20ax^3$.

Find the square roots of the following expressions :

9. $x^4 - 6x^2 + 11 - \frac{6}{x^2} + \frac{1}{x^4}$.

10. $a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$.

11. $x^6 - 6x^4 + 49 + 42x - 14x^3 + 9x^2$.

12. $9x^4 - 12x^3y + 34x^2y^2 - 20xy^3 + 25y^4$.

13. $a^2 + 4b^2 + 9c^2 - 4ab - 12bc + 6ca$.

14. $9a^4 + 49b^4 + 121c^4 - 42a^2b^2 + 154b^2c^2 - 66a^2c^2$.

15. $4a^2b^2 + 9b^2c^2 + c^2a^2 - 4a^2bc - 12ab^2c + 6abc^2$.

16. $4x^2 + 9y^2 + 25z^2 - 12xy + 20xz - 30yz$.

17. $49x^4 + 109x^2y^2 + 36y^4 - 70x^3y - 60xy^3$.

18. $x^6 - 4x^3 + 2 + \frac{4}{x^3} + \frac{1}{x^6}$.

19. $4x^4 + 9y^4 + 49z^4 - 12x^2y^2 - 42y^2z^2 + 28x^2z^2$.

20. $\frac{x^2}{y^2} + \frac{y^2}{x^2} + 3 - 2\left(\frac{x}{y} + \frac{y}{x}\right)$.

21. $\frac{a^4}{4} - a^3 + 2a + 1$.

22. $\frac{a^4}{9} + \frac{2a^3}{3} + \frac{4a^2}{3} + a + \frac{1}{4}$.

23. $\frac{9a^4}{25} + \frac{4a^3}{5} + \frac{74a^2}{45} + \frac{4a}{3} + 1$.

24. $\frac{a^4}{9} - \frac{a^3}{3} + \frac{11a^2}{12} - a + 1$.

25. $x^6 - x^5 + \frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{3} + \frac{1}{9}$.

26. $\frac{x^4}{4} - 3x^3 + \frac{28x^2}{3} - 2x + \frac{1}{9}$.

27. $\frac{x^4}{9} + \frac{a^2}{4} - \frac{4x^3}{3} + \frac{ax^2}{3} + 4x^2 - 2ax$.

28. $9x^4 + \frac{64}{x^4} + 24x^2 - \frac{64}{x^2} - 32$.

29. $\frac{4x^2}{y^2} + \frac{9y^2}{4x^2} - \frac{x}{y} + \frac{3y}{4x} - \frac{95}{16}$.

30. $\frac{16}{25} - \frac{3a^3}{2} + a^4 - \frac{6a}{5} + \frac{173a^2}{80}$.

SQUARE ROOT OF NUMERICAL QUANTITIES

130. First study carefully the following example worked according to the algebraic method.

Example. Find the square root of 99225.

$$
\begin{array}{rl}
 & 99225 = 9.10^4 + 9.10^3 + 2.10^2 + 2.10 + 5\,(\,3.10^2 + 1.10 + 5 = 315 \\
 & \underline{9.10^4} \\
 & 9.10^3 + 2.10^2 \\
(6.10^2 + 1.10) \times (1.10) = & \underline{6.10^3 + 1.10^2} \\
 & 3.10^3 + 1.10^2 + 2.10 + 5 \\
(6.10^2 + 2.10 + \frac{10}{2}) \times (\frac{10}{2}) = & \underline{3.10^3 + 1.10^2 + 2.10 + 5}
\end{array}
$$

Below we give the same example in arithmetical form, omitting superfluous powers of 10.

$$
\begin{array}{rl}
 & 9\,92\,25\,.\,(\,315 \\
 & \underline{9} \\
 & 92 \\
(60 + 1) \times 1 = 61 & \underline{61} \\
 & 3125 \\
(620 + 5) \times 5 = 625 \times 5 = & \underline{3125}
\end{array}
$$

131. The following are very useful and should be learnt by heart:

$$13^2=169,\quad 17^2=289,$$
$$14^2=196,\quad 18^2=4\times81=324,$$
$$15^2=9\times25=225,\quad 19^2=361,$$
$$16^2=4\times64=256,\quad 21^2=9\times49=441.$$

132. The square roots of numerical quantities can often be best found by using factors.

$$1764=4\times441=4\times9\times49;\quad \therefore\ \sqrt{1764}=2\times3\times7=42.$$
$$53361=9\times5929=9\times7\times847=9\times7\times7\times121=3^2\times7^2\times11^2;$$
$$\therefore\ \sqrt{53361}=3\times7\times11=231.$$

Examples. XXIV. d.

Find the square root of

1. 1,764.	2. 18,225.	3. 16,900.	4. 2,704.
5. 34,969.	6. 390,625.	7. 213,444.	8. 7,056.
9. 15,876.	10. 4,020,025.	11. 9,006,001.	12. 3,892,729.
13. 5,499,025.	14. 408,120,804.	15. 1,825,201.	16. 12,173,121.

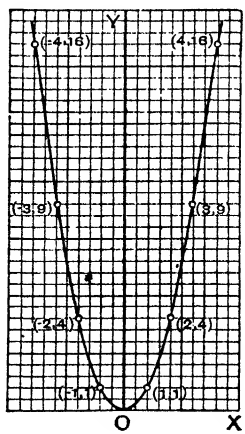
THE DETERMINATION OF THE SQUARE ROOTS OF NUMBERS BY GRAPHICAL METHODS

133. The student must first familiarize himself with the graph of the equation $y=x^2$.

Trace the graph of $y=x^2$.

When

$x=0$	± 1	± 2	± 3	± 4	± 5	...
$y=0$	1	4	9	16	25	...



Joining these points, we have the graph reqd., which we see is a curve.

For every value of y there are two equal and opposite values of x.

$\therefore$ the curve is symmetrical about the axis of y.

Moreover, as x increases indefinitely, y also increases indefinitely.

$\therefore$ the parts of the curve on either side of OY meet only at the origin.

Such a curve is called a **parabola**.

N.B.—In the above we have taken twice the length of the side of a square to denote unity.

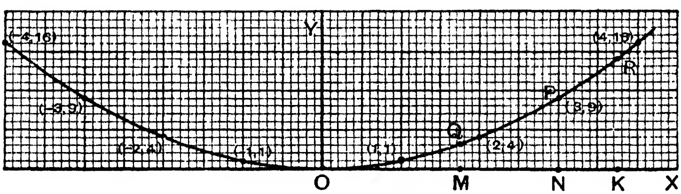
We observe that when x is greater than unity, the y value increases much more rapidly than the x value. This is well seen from the table of corresponding values of x and y below.

When

$x=$	5	6	7	8	9	10	11	...
$y=$	25	36	49	64	81	100	121	...

134. A better curve for working purposes will be obtained if we take 10 times the side of a square to denote unity for the abscissae, and one side of a square to denote unity for the ordinates.

Employing these units, we obtain the curve shown below.



Thus at P, the abscissa ON = 30 times the side of a sq. = 3 units,
and the ordinate PN = 9 times the side of a sq. = 9 units.

The effect of using different units for the x and y values in this way, is the same as uniformly stretching the paper in a direction parallel to the axis of x. If we took the larger unit for the y values, it would be the equivalent of stretching the paper parallel to the axis of y.

It will sometimes be found convenient to take the x unit still larger.

In connection with square roots, the important thing to observe is that since $\mathbf{y=x^2}$, or $\mathbf{x=\sqrt{y}}$, **for every point on the curve, the abscissa of any point on it is the square root of the corresponding ordinate.**

In the curve shown above take the point Q when the ordinate is 3 and draw the ordinate QM.

Now at every pt. on the curve $y=x^2$;

$$\therefore \text{ at Q } 3=\text{OM}^2, \text{ for there } y=3 \text{ and } x=\text{OM};$$

$$\therefore \text{ OM}=\sqrt{3}.$$

But from the figure we see that OM lies between 1·7 and 1·8 and somewhat nearer 1·7 than 1·8;

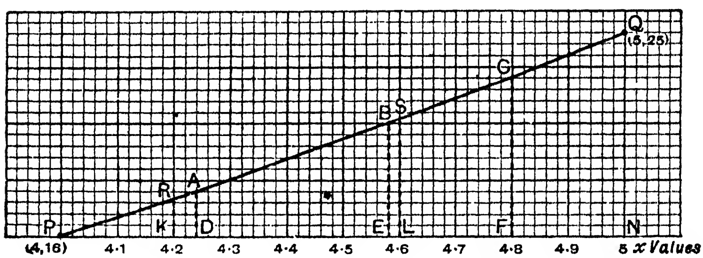
$$\therefore \sqrt{3}=1{\cdot}7 \text{ correct to one decimal place.}$$

Again take the pt. R where RK, the ordinate, $=14$.

$$14=\text{OK}^2;$$

$$\therefore \sqrt{14}=\text{OK}=3{\cdot}7 \text{ correct to one decimal place.}$$

135. *Construct a graph from which the square roots (correct to two decimal places) of numbers between* 16 *and* 25 *may be read off.*



We must draw the graph of $y=x^2$, and use a large unit for x values, for x has to be determined accurately to two decimal places.

We shall only need to draw that part of the curve where x lies between 4 and 5.

Take 50 sides of squares to represent unity in the x values, and 2 sides of squares to represent unity in the y values.

In the curve $y=x^2$, when $x=4$, $y=16$,

and when $x=5$, $y=25$.

Let P be the pt. (4, 16) and Q the pt. (5, 25) so that PN in the figure representing unity is equal to 50 sides of squares, and QN representing 9 is equal to 18 sides of squares.

(*N.B.*—QN is the difference of the ordinates of P and Q, and therefore $=25-16=9$ units.)

When $x=4{\cdot}2,\ y=x^2=(4{\cdot}2)^2=17{\cdot}64,$

$$17{\cdot}64-16=1{\cdot}64 \text{ units}=3{\cdot}28 \text{ sides of sqs.}$$

Hence estimating the value of ·28, R in the fig. is the pt. $(4{\cdot}2, (4{\cdot}2)^2)$.

(RK in the fig. = the diff. of the ordinates of R and P

$$=17{\cdot}64-16=1{\cdot}64 \text{ units}=3{\cdot}28 \text{ sides of sqs.)}$$

Again, when $x=4{\cdot}6,\ y=x^2=(4{\cdot}6)^2=21{\cdot}16$;

∴ estimating the value of ·16, S in the fig. is the pt. $(4{\cdot}6, (4{\cdot}6)^2)$.

(Here again, SL = the diff. of the ordinates of S and P

$$=21{\cdot}16-16=5{\cdot}16 \text{ units}=10{\cdot}32 \text{ sides of sqs.)}$$

The curve through the pts. P, R, S, Q is evidently so nearly a str. line that we need find no more pts. on the curve.

Join the pts. P, R, S, Q by the continuous curve as shown in the figure.

To find $\sqrt{18}$ from this graph we must take the pt. whose ordinate is 18, *i.e.* the pt. A. (*N.B.*—AD $=18-16=2$ units $=4$ sides of a sq.)

From the fig. we see that the abscissa of this pt. is $4+$PD, which is equal to 4·24;

$$\therefore\ \sqrt{18}=4{\cdot}24.$$

To find $\sqrt{21}$, we must take the pt. whose ordinate is 21, *i.e.* the pt. B. (*N.B.*—BE $=21-16=5$ units $=10$ sides of a sq.)

From the graph the abscissa of this point $=4+$PE $=4{\cdot}58$;

$$\therefore\ \sqrt{21}=4{\cdot}58.$$

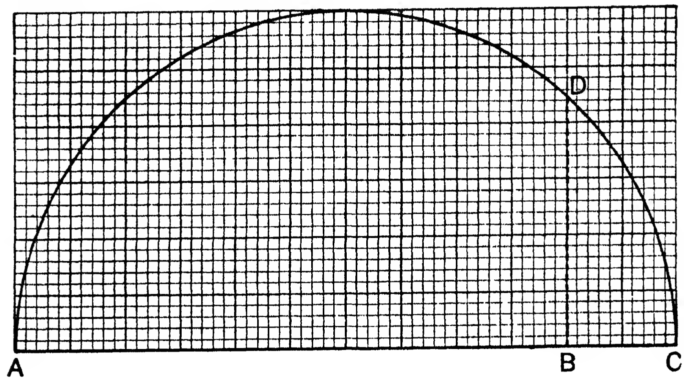
To find $\sqrt{23}$, we must take the pt. whose ordinate is 23, *i.e.* the pt. C;

$$\therefore\ \sqrt{23}=4+\text{PF}=4{\cdot}80.$$

The roots of other numbers between 16 and 25 can be read off in the same way.

136. The following geometrical methods may be used for determining the values of square roots in simple cases.

Example. To find the value of $\sqrt{5}$.



First Method. Take AB 5 units long, and produce it to C making BC equal to one unit. On AC as diameter describe the circle ADC. At B draw BD perp. to AC, meeting the circle at D.

From geometry we know that

$$DB^2 = AB \cdot BC = 5;$$

$$\therefore DB = \sqrt{5}.$$

From the diagram $\sqrt{5} = 2 \cdot 24$ approx.

(If squared paper is not used, DB must be measured.)

Second Method. On AB, 5 in. long, as diameter describe a circle.

In AB take a pt. D 1 in. from A, and draw DC perp. to AB to meet the circle at C. Join AC. With centre A and radius AC describe a circle cutting AB at E.

By geometry $AC^2 = AD \cdot AB = 5;$

$$\therefore AC = \sqrt{5};$$

$$\therefore AE = AC = \sqrt{5},$$

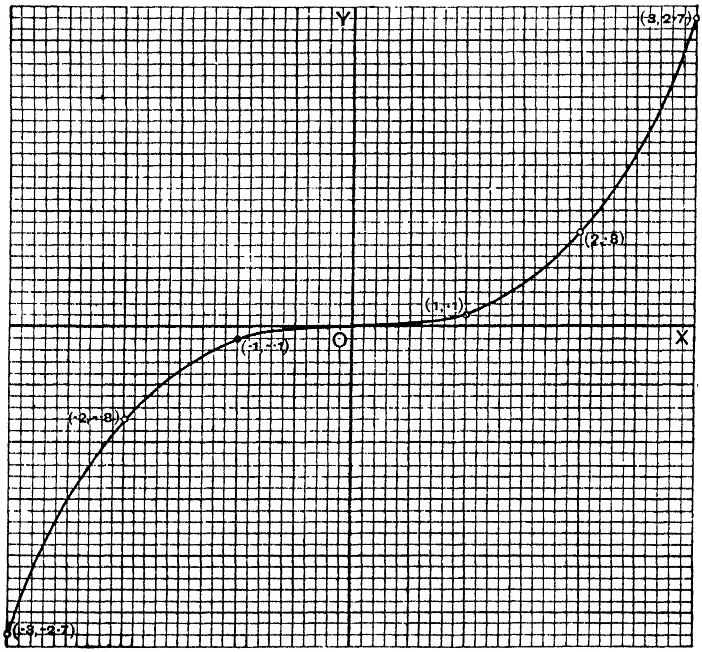
and if squared paper is used we can read off the value of $\sqrt{5}$ from the diagram.

Pythagoras' Theorem, which proves that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on its sides, may be sometimes used with advantage.

Thus to find $\sqrt{10}$, $10 = 1^2 + 3^2$, draw AB 3 units long, AC 1 unit long at rt. angles to AB. Join BC. $BC = \sqrt{10}$ units long.

CUBE ROOT BY GRAPHICAL METHOD

***137.** *Draw the graph of* $y = x^3$.



Use for the y values a unit one-tenth of that for the x values

When

$x = 1$	2	3	4	5	...	inches.
$y = 1$	8	27	64	125	...	tenths of an inch.

$x = -1$	-2	-3	...	inches.
$y = -1$	-8	-27	...	tenths of an inch.

Plot these points and we have the graph reqd.

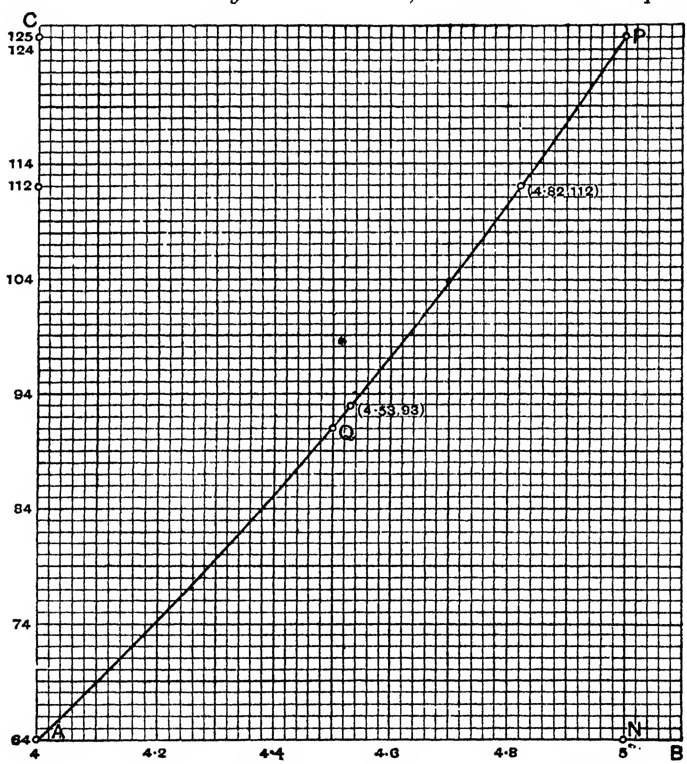
We see that the curve lies entirely in the first and third quadrants, and that the parts of the curve in those quadrants are similar.

For values of x greater than 1 or less than -1, as the numerical value of x increases, that of y increases much more rapidly; but for values of x between 1 and -1 the reverse happens. This shows that the axis of x is a tangent to the curve at the origin.

As x varies continuously from $-\infty$ through 0 to $+\infty$, y also varies continuously from $-\infty$ through 0 to $+\infty$.

From this graph we can read off cube roots and cubes of numbers.

***138.** *To construct a graph from which the cube root of any number between* 64 *and* 125 *may be written down, correct to two decimal places.*



Take a piece of squared paper ruled in inches and tenths of an inch.

Let the pt. A denote the pt. whose co-ors. are (4, 64).

In the horizontal line AB take 1 in. to represent ·2, so that AN (5 in. long) represents unity.

In the vertical line AC take an inch to represent 10.

On the paper plot the point (5, 125) P.

$(4{\cdot}5)^3 = 91{\cdot}125$. $\therefore$ plot the pt. (4·5, 91·125) Q, estimating the value of ·125.

Join the pts. A, Q, P by an even curve.

This curve will be seen to be part of the graph of $y = x^3$.

$\therefore$ we can read from it the values of the cube roots of numbers between 64 and 125.

E.g. $\sqrt[3]{112} = 4{\cdot}82$, $\sqrt[3]{93} = 4{\cdot}53$.

NOTE.—Great accuracy can be obtained in the above if a few more points are plotted; *e.g.* $[(4{\cdot}2), (4{\cdot}2)^3]$, $[(4{\cdot}8), (4{\cdot}8)^3]$.

Examples. XXIV. e.

[*Always state clearly, on the same sheet of paper as the graph, the units employed.*]

Plot the graphs of the following, using an x unit twice as large as the y unit.

1. $3x + 4y = 12$.
2. $3x - 4y = 12$.
3. $y = 2x$.
4. $y + 3x = 0$.
5. $5x - 2y = 1$.
6. $2x + 2y + 2 = 0$.

Plot graphs of the following using a y unit ten times as large as the x unit.

7. $x + y = 11$.
8. $x - 2y = 20$.
9. $10x = y$.
10. $20x + y = 0$.

Trace graphs of the equation $y = x^2$.

11. When the x unit is five times as large as the y unit.
12. .... four ....

Trace graphs of the equation $x^2 = y$.

13. When the x unit is equal to the y unit.
14. .... ten times as large as the y unit.
15. .... five ....

Trace graphs of the equation $y = 4x^2$.

16. When the x unit is equal to the y unit.
17. .... four times the y unit.

18. Construct a graph to show the square roots of numbers from 49 to 64. From it write down (correct to two decimal places) the square roots of 53·6, 57·8, 59·5, 61·6.

Verify one of your results by the Arithmetical method.

19. Construct a graph to show the square roots of numbers from 36 to 49. From it write down (correct to two decimal places) the square roots of 38·6, 39·7, 40, 42·6, 46·8.

[With the curve $y=x^2$, use 5 inches for the x unit, half an inch for the y unit.]

From the above graph read off approximate values of the squares of 6·44, 6·68, 6·82.

20. Plot the points $(7, 7^2)$, $(7·1, 7·1^2)$, $(7·2, 7·2^2)$, $(7·3, 7·3^2)$, $(7·4, 7·4^2)$. Join them and read off the square roots of 49·8, 50·7, 51·3, 53·9 correct to two decimal places.

[Use 10 inches for the x unit, one inch for the y unit.]

From the above graph write down approximate values of the squares of 7·05, 7·16, 7·28, 7·36.

21. Find from one graph, correct to two decimal places, the square roots of 54·6, 58·8, 62·4.

Verify one root by the Arithmetical method.

22. Plot the points $(8, 8^2)$, $(8·1, 8·1^2)$, $(8·2, 8·2^2)$. Join them and use the graph to determine, to one decimal place, the square roots of 6430, 6680.

23. Using 5 inches (or 10 centimetres) to denote ·1 in the x axis, and 5 inches (or 10 centimetres) to denote unity in the y axis, plot the points (8, 64), $(8·1, 8·1^2)$. Join them by a straight line. Assuming this straight line to be part of the graph of $y=x^2$, use it to determine the square roots (to two decimal places) of 6425, 6437, 6486.

Verify one of your results by the Arithmetical method.

In each of the following examples, use a single graph to determine the square roots of the given numbers (use large units).

In each case verify one answer by the Arithmetical method.

24. 81·96, 82·6, correct to three decimal places.

25. 8346, 8424, ... two ....

26. 101·68, 100·96,........... three ....

27. 152·8, 167·6, ... two ....

Use one of the methods of Art. 136 to find the approximate values of the following:

28. $\sqrt{3}$. 29. $\sqrt{6}$. 30. $\sqrt{7}$. 31. $\sqrt{11}$. 32. $\sqrt{5·6}$.

33. $\sqrt{4·8}$. 34. $\sqrt{6·6}$. 35. $\sqrt{4·5}$. 36. $\sqrt{5·7}$. 37. $\sqrt{4·3}$.

38. Draw a graph to find the cube root of any number between 125 and 216. Write down the cube roots of 144 and 198 correct to two decimal places.

39. Draw enough of the graph of $y=x^3$ to find the cube roots of numbers between 8 and 27.

Write down the cube roots of 15 and 21 correct to two decimal places.

40. Find the cube root of 8·25 correct to two decimal places. Test your result.

[Plot the points $(2, 2^3)$ $(2·1, 2·1^3)$, using a large x unit, say 5 inches, to denote ·1. Join the points by a *straight line*, and assume this straight line to be part of the graph of $y=x^3$.]

Find the cube roots of the following, correct to two decimal places:

41. 27·9. 42. 28·6. 43. 29·2. 44. 30. 45. 65·6.

46. 67 8. 47. 68·5. 48. 127. 49. 123·8. 50. 130.

CHAPTER XXV

QUADRATIC EQUATIONS

139. When an equation contains the square of the unknown quantity, and no higher power, it is called a **quadratic equation, or an equation of the second degree.**

$$\left.\begin{aligned} x^2-7x+12&=0,\\ 6x^2&=7x+3,\\ 12&=23x-5x^2,\\ x^2-4&=0 \end{aligned}\right\}\text{ are examples of such.}$$

140. Solution of quadratics by factorization.

Let us consider the equation $x^2-7x+12=0$.

It may be written $(x-3)(x-4)=0$.

We notice that when $x=3$,

$$\begin{aligned}\text{the left-hand side}&=(3-3)(4-4)\\&=0\times(-1)=0,\end{aligned}$$

i.e. the equation is satisfied, or 3 is a root of the equation.

Also when $x=4$,

$$\begin{aligned}\text{the left-hand side}&=(4-3)(4-4)\\&=1\times 0=0;\end{aligned}$$

$\therefore$ 4 also is a root of the equation.

It will be proved later on that every quadratic equation has two roots and only two.

N.B.—Every *multiple* of 0 is 0.

$$6\times 0=0,\qquad 1000\times 0=0,$$
$$0\times a=0,\qquad 0\times x^3=0.$$

Examples. XXV. a.

Write down the roots of the following equations:

1. $(x-1)(x-2)=0$.
2. $(x-1)(x+1)=0$.
3. $(x-a)(x-b)=0$.
4. $x(x-1)=0$.
5. $(x+2)(x+3)=0$.
6. $(x+a)(x-b)=0$.
7. $(x+2)x=0$.
8. $(x-2a)(x-b)=0$.
9. $(x+a)(x-2b)=0$.
10. $(x-\frac{1}{2})(x+\frac{3}{4})=0$.
11. $(x+\frac{1}{5})(x+\frac{3}{8})=0$.
12. $x(x+\frac{1}{3})=0$.
13. $\left(x-\frac{a}{2}\right)\left(x-\frac{b}{3}\right)=0$.
14. $(x-\overline{a+b})(x-\overline{a-b})=0$.

Write down the roots of the following equations:

15. $\left(x-\frac{a+b}{2}\right)\left(x+\frac{c+d}{2}\right)=0.$

16. $(x-\overline{p-2q})(x-\overline{2p-q})=0.$

17. $\{x-2(a+b)\}\{x+3(a-b)\}=0.$

18. $(x-a^2)(x+b^2)=0.$

19. $\{x+(a-b)^2\}\{x-(a+b)^2\}=0.$

20. $(x-3)^2=0.$

21. $x(x-a)=0.$

22. $x(x+4)=0.$

23. $(x+a)^2=0.$

24. $(x+2a)^2=0.$

141. *Solve the equation* $\quad x^2=x+20.$

Transposing all the terms to the left-hand side (or subtracting $x+20$ from both sides)

$$x^2-x-20=0,$$

factorizing,

$$(x-5)(x+4)=0;$$

$$\therefore\ x=5 \quad\text{or}\quad -4.$$

Verification. When $x=5$, $\quad x^2-x-20=25-5-20$

$$=0;$$

$\therefore$ 5 is a root of the equation.

When $\quad x=-4,\quad x^2-x-20=(-4)^2-(-4)-20$

$$=16+4-20=0;$$

$\therefore$ -4 is also a root.

Solve the equation $\quad 4x^2-16x=84.$

Transposing 84 to the left-hand side,

$$4x^2-16x-84=0.$$

Dividing both sides by 4, $x^2-4x-21=0,$

factorizing,

$$(x-7)(x+3)=0;$$

$$\therefore\ x=7 \quad\text{or}\quad -3.$$

Verification. When $\quad x=7$

$$\begin{aligned}4x^2-16x-84&=4\times 49-16\times 7-84\\&=196-112-84\\&=0;\end{aligned}$$

$\therefore$ 7 is a root of the equation.

When $\quad x=-3.\ 4x^2-16x-84=4\times 9-16(-3)-84$

$$\begin{aligned}&=36+48-84\\&=0;\end{aligned}$$

$\therefore$ -3 is also a root.

142. When an equation contains the square of the unknown quantity, and no first power of the unknown quantity, it is called

a pure quadratic. If it contains both the square and the first power of the unknown, it is called an **adfected quadratic.**

$x^2-4=0$ and $6x^2=54$ are examples of pure quadratics.

$x^2-7x+12=0$ is an adfected quadratic.

Pure quadratics are easily solved by factorization.

Solve the quadratic $\qquad 6x^2=54.$

Dividing both sides by 6, $\qquad x^2=9.$

Subtracting 9 from both sides, $\qquad x^2-9=0,$

$$i.e.\ (x-3)(x+3)=0,$$

$$\therefore\ x=3 \text{ or } -3.$$

Or we might proceed thus,

$$x^2=9 \text{ as before.}$$

Taking the square root of each side

$$x=\pm 3.$$

143. *Solve the equation* $x^2=12-x$.

Transposing all terms to the left-hand side (or subtracting $\mathbf{12-x}$ from both sides),

the equation becomes $\qquad x^2+x-12=0.$

Factorizing, $\qquad (x+4)(x-3)=0,$

from which we see that -4 and 3 are the roots reqd.

Verification. When $\qquad x=-4,$

the left-hand side $=(-4)^2=16,$

the right-hand side $=12-(-4)=16;$

$\therefore\ -4$ is a root.

When $x=3$, $\qquad$ the left-hand side $=(3)^2=9,$

the right-hand side $=12-3=9;$

$\therefore$ 3 is also a root.

Examples. XXV. b.

Solve the following equations, verifying the solutions in each case:

1. $x^2-7x+10=0.$	**2.** $x^2-5x+6=0.$	**3.** $x^2-4=0.$
4. $x^2-3x=0.$	**5.** $x^2+4x+3=0.$	**6.** $x^2+4x-5=0.$
7. $x^2=8x-7.$	**8.** $x^2-2=x.$	**9.** $x^2-3=1.$
10. $x^2+10=11x.$	**11.** $4x=45-x^2.$	**12.** $12x-27=x^2.$
13. $x^2=20-x.$	**14.** $x^2=7x.$	**15.** $2x^2-1=1.$
16. $x^2-4x+4=0.$	**17.** $x^2+3x=0.$	**18.** $21+10x+x^2=0.$

Solve the following equations, verifying the solutions in each case :

19. $14x+15=x^2$.
20. $40=3x+x^2$.
21. $x^2+225=30x$.
22. $2x^2-3=15$.
23. $4x^2=8x$.
24. $3x^2+21x=0$.
25. $103x=x^2+102$.
26. $x^2+16x+15=0$.

144. Let us take the equation $2x^2-11x+12=0$.

It may be written $(2x-3)(x-4)=0$.

We see that if $2x-3=0$, *i.e.* if $x=\frac{3}{2}$, the equation is satisfied, for $0\times(\frac{3}{2}-4)=0$.

Also if $x-4=0$, *i.e.* if $x=4$, the equation is again satisfied;

$\therefore$ $\frac{3}{2}$ and 4 are the roots of the equation.

Solve the equation $x^2=2(x+12)$.

Removing the brackets $x^2=2x+24$.

Transposing all terms to the left-hand side,

$$x^2-2x-24=0.$$

Factorizing, $(x-6)(x+4)=0$;

$\therefore$ 6 and -4 are the reqd. roots.

Solve the equation $x^2-4x+4=0$.

Factorizing, $(x-2)(x-2)=0$;

$\therefore$ in this case the roots are equal and each of them is 2.

145. If fractions or brackets occur in the given equation, they should first be cleared away.

Example 1. Solve the equation $3x-8=\frac{x^2}{4}$.

Multiplying both sides by 4, $12x-32=x^2$.

Transposing all terms to the left-hand side (or subtracting x^2 from both sides), $12x-32-x^2=0$.

Re-arranging and changing signs throughout [this is permissible, for if $a=b$, $-a=-b$; if $a=0$, $-a=0$],

$$x^2-12x+32=0.$$

Factorizing, $(x-4)(x-8)=0$;

$\therefore$ 4 and 8 are the reqd. roots, or $x=4$ or 8.

Verification. When $x=4$, the left-hand side $=3\times4-8=4$.

.... the right-hand side $=\frac{(4)^2}{4}=4$;

$\therefore$ 4 is a root.

When $x=8$, the left-hand side $=3\times8-8=16$.

.... the right-hand side $=\frac{(8)^2}{4}=\frac{64}{4}=16$;

$\therefore$ 8 is also a root.

Example 2. Solve the equation $\frac{7}{3x-1}-\frac{4}{x+1}=\frac{1}{4}$.

Multiplying both sides by $4(3x-1)(x+1)$, the L.C.M. of the denominators.

$$28(x+1)-16(3x-1)=(x+1)(3x-1),$$
$$28x+28-48x+16=3x^2+2x-1.$$

Transposing and arranging, $-3x^2-22x+45=0$,

$$3x^2+22x-45=0,$$
$$(3x-5)(x+9)=0;$$
$$\therefore\ \tfrac{5}{3} \text{ and } -9 \text{ are the reqd. roots.}$$

It is important to observe that **if $x-a$ is a factor of both sides of an equation, a is a root of the equation.**

This is at once seen by substitution.

Example 3. Solve the equation $2(2x-5)+7x(2x-5)=0$.

$2x-5$ is a factor throughout; $\therefore\ 2x-5=0$ gives a root

whence $x=\frac{5}{2}$.

Having divided by $2x-5$, we have left

$$2+7x=0$$
whence $x=-\frac{2}{7}$;

$\therefore$ the reqd. roots are $\frac{5}{2}$ and $-\frac{2}{7}$.

Examples. XXV. c.

Write down the roots of the following quadratic equations:

1. $(2x-3)(x-4)=0$.
2. $(3x+1)(2x-1)=0$.
3. $(3x+4)(5x+6)=0$.
4. $x(7x+9)=0$.
5. $(5x-7)(6x+1)=0$.
6. $(7x-8)^2=0$.
7. $(2x-a)(2x-b)=0$.
8. $(5x+a)(6x+b)=0$.
9. $(2x-\overline{a+b})(3x-\overline{c+d})=0$.
10. $3(4x+5)(2x-9)=0$.

Solve the following equations:

11. $x^2=2-x$.
12. $8x-x^2=15$.
13. $x^2=4(x+8)$.
14. $2(5x-12)=x^2$.
15. $x(x-4)=5$.
16. $4x^2=1$.
17. $x^2-4x=4(x-4)$.
18. $1+2x^2=3x$.
19. $x(x+4)=6(x+4)$.
20. $5x^2+17x=0$.
21. $x-10=x(x-10)$.
22. $4x(x+1)+1=0$.
23. $x^2+4{\cdot}8x+2{\cdot}87=0$.
24. $x+\frac{1}{x}=2$.
25. $x-\frac{9}{2}+\frac{2}{x}=0$.
26. $(2x-1)(3x+1)=11$.
27. $2x^2+\frac{13x}{2}=6$.
28. $5x(2x-3)+7(2x-3)=0$.
29. $x-1=\frac{2}{x}$.
30. $(2x+1)(x+8)=27$.
31. $\frac{x+10}{x-5}-\frac{10}{x}=\frac{11}{6}$.
32. $150x^2=299x+2$.
33. $(5x-3)(3x+1)=1$.
34. $6(4x+5)+\frac{7}{x}(4x+5)=0$.
35. $13x^2-6x-7=0$.
36. $x+35=70x^2$.
37. $9x^2=18x+16$.
38. $\frac{1}{x-1}-\frac{1}{x+3}=\frac{1}{35}$.

SOLUTION OF QUADRATICS BY COMPLETING SQUARES

146. Take the equation $a^2+2ab=0$.

Adding b^2 to both sides, $a^2+2ab+b^2=b^2$,

$$i.e.\ (a+b)^2=b^2.$$

The addition of b^2 to both sides **completed the square on the** left-hand side.

Take the equation $x^2-6x=0$.

Adding 9 to both sides, $x^2-6x+9=9$,

$$(x-3)^2=3^2.$$

Again the left-hand side becomes a complete square.

More generally, to complete the square on the left of the equation $x^2-2ax=0$ we must add a^2 to both sides.

The equation becomes $x^2-2ax+a^2=a^2$,

$$\text{or } (x-a)^2=a^2.$$

x^2+8x becomes $(x+4)^2$ by adding 16, *i.e.* 4^2............(1)

x^2-2cx .... $(x-c)^2$ .... c^2. ....(2)

x^2+10x .... $(x+5)^2$ .... 5^2. ....(3)

Thus we observe that any expression of the form $x^2\pm2px$ becomes a complete square when **we add the square of half the coefficient of x.**

In (1) we add $\left(\frac{8}{2}\right)^2$.

In (2) .... $\left(-\frac{2c}{2}\right)^2$.

In (3) .... $\left(\frac{10}{2}\right)^2$.

147. Let us now employ this to solve quadratic equations.

Example 1. Solve the quadratic $x^2+4x=32$.

Adding the sq. of half the coeff. of x to both sides,

$$x^2+4x+\left(\frac{4}{2}\right)^2=32+\left(\frac{4}{2}\right)^2,$$

$$i.e.\ x^2+4x+(2)^2=36,$$

$$(x+2)^2=36.$$

Taking the square root of both sides,

$$x+2=\pm6. \ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

With the positive sign $\quad x+2=6,$

$$x=4.$$

With the negative sign $x+2=-6$,

$$x=-8;$$

$\therefore$ 4 and -8 are the reqd. roots.

In connection with (i) we at first sight think we ought to say

$$\pm(x+2)=\pm 6,$$

for $\pm(x+2)$ is the sq. root of $(x+2)^2$ just as ± 6 is the sq. root of 36.

This however is unnecessary, as we see if we take the *four* different cases separately.

With positive signs on both sides, $x+2=6$, $x=4$ \
...... negative .... $-x-2=-6$, $x=4$ } the same result.

With the positive sign on the left and the negative sign on the right,

$$x+2=-6,\ x=-8.$$

With the negative sign on the left and the positive sign on the right,

$$-x-2=+6,$$

$x+2=-6$, $x=-8$, again the same result.

Thus it is sufficient if we attach the double sign ($\pm$) to one side.

We always attach it to the numerical square root.

148. Before completing squares the coefficient of x^2 should be reduced to unity.

Solve the equation $22-x=6x^2$.

Re-arranging by transposition, $6x^2+x=22$.

Dividing both sides by 6 to make the coefficient of x^2 equal to unity,

$$x^2+\frac{x}{6}=\frac{22}{6}.$$

Adding the sq. of half the coeff. of x, *i.e.* $\left(\frac{1}{12}\right)^2$, to both sides

$$x^2+\frac{x}{6}+\left(\frac{1}{12}\right)^2=\frac{22}{6}+\frac{1}{144},$$

$$\left(x+\frac{1}{12}\right)^2=\frac{528+1}{144}$$

$$=\frac{529}{144}$$

Taking the sq. root of both sides,

$$x+\frac{1}{12}=\pm\frac{23}{12}.$$

With the positive sign $\quad x+\frac{1}{12}=\frac{23}{12},$

$$x=\frac{23-1}{12}=\frac{11}{6}.$$

With the negative sign $\quad x+\frac{1}{12}=-\frac{23}{12},$

$$x=\frac{-23-1}{12}=-2;$$

$$\therefore \frac{11}{6} \text{ and } -2 \text{ are the reqd. roots.}$$

149. *To solve the general quadratic* $ax^2+bx+c=0$.

$$ax^2+bx=-c,$$

$$x^2+\frac{bx}{a}=-\frac{c}{a}.$$

Adding the square of half the coeff. of x to both sides.

$$x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$=\frac{b^2-4ac}{4a^2}.$$

Taking the sq. root of both sides,

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a};$$

$$\therefore\ \mathbf{x}=\frac{-\mathbf{b}\pm\sqrt{\mathbf{b}^2-\mathbf{4ac}}}{\mathbf{2a}}.$$

The above formula may be used for the solution of any quadratic equation.

There are therefore three methods of solving quadratics:

(1) by factorization, (2) by completing squares,

(3) by using the formula $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

The student should have considerable practice in all three methods.

When the factors cannot be seen *readily*, the second or third method should be employed.

Examples. XXV. d.

Solve the equations:

1. $6x^2 = 2 - x$.
2. $1 - 26x^2 = 11x$.
3. $x + 1 = 156x^2$.
4. $5x^2 = 4x + 1$.
5. $3x^2 + 10 = 17x$.
6. $7x^2 + 32x = 15$.
7. $2x^2 + 19x + 9 = 0$.
8. $(x - 1)^2 = 16$.
9. $2(x^2 + 1) - 5x = 0$.
10. $11x = 3(2x^2 + 1)$.
11. $3(x - 1)(x + 1) = 8x$.
12. $(x - 1)(x + 1) = \frac{7x}{12}$.
13. $15 = 4(3x^2 + 2x)$.
14. $(2x - 1)^2 = 25$.
15. $(3x - \frac{1}{2})^2 = 49$.
16. $3x(5x - 1) = 4(x + 9)$.
17. $25x^2 - 7x = 86$.
18. $5x - 11 = x(5x - 11)$.
19. $13x + 9 = 10x^2$.
20. $\left(\frac{x}{2} - 5\right)^2 - 36 = 0$.
21. $3(3x + 4) + 5x(3x + 4) = 0$.
22. $\frac{2x - 3}{2} = \frac{4x - 6}{x}$.
23. $x(x - 1) + \frac{1}{2}(x - 1) = 0$.
24. $\frac{2x - 3}{5} + \frac{2(2x - 3)}{3x} = 0$.
25. $7(3x - 6) + 11x(2x - 4) - 3x(5x - 10) = 0$.
26. $\frac{2}{3(x - 1)} - \frac{3}{2x + 1} = \frac{1}{15}$.
27. $\frac{6}{x - 2} = \frac{5}{x - 4} - \frac{6}{x - 3}$.
28. $\frac{x - 1}{x + 1} + \frac{x - 3}{x + 3} = \frac{2x + 1}{2x + 2}$.
29. $\frac{x}{5 + x} + \frac{7}{6 - 4x} = \frac{x - 7}{x - 6}$.
30. $\frac{3x + 4}{5} - \frac{30 - 2x}{x - 6} = \frac{7x - 14}{10}$.
31. $\frac{2}{x - 2} - \frac{3}{x - 3} = \frac{4}{x - 4} - \frac{5}{x - 5}$.
32. $\frac{2}{x + 3} + \frac{x + 3}{2} = \frac{10}{3}$.
33. $\frac{2x}{x - 1} + \frac{3x - 1}{x + 2} - \frac{5x - 11}{x - 2} = 0$.
34. $\frac{x - 3}{x + 3} - \frac{x + 3}{x - 3} + 6\frac{6}{7} = 0$.

When the quantity under the radical sign ($\sqrt{\ }$) is not a perfect square, the *approximate* values of the roots should be found by finding the square root to a few decimal places.

Thus if $$x=\frac{9\pm\sqrt{21}}{10},$$

$$x=\frac{9\pm 4{\cdot}583\ldots}{10} \text{ (for } \sqrt{21}=4{\cdot}583\ldots)$$

$$=1{\cdot}36, \text{ or } {\cdot}44, \text{ correct to two decimal places.}$$

Examples. XXV. e.

When the exact values of the roots of the following equations cannot be found, give results *correct to two decimal places, i.e.* to the nearest hundredth.

Solve

1. $x^2-2x=1$.
2. $x^2=2(1-x)$.
3. $x(x-3)=x-1$.
4. $x=\dfrac{x+4}{x-1}$.
5. $5x^2-9x-4=0$.
6. $\dfrac{x+1}{x+2}+\dfrac{x-3}{x-4}=0$.
7. $x^2=\sqrt{3}(2x-\sqrt{3})$.
8. $\dfrac{1}{x+3}+\dfrac{1}{x+6}+\dfrac{1}{x+9}=0$.
9. $\dfrac{2x-1}{3x+2}+\dfrac{x-3}{x+1}=0$.
10. $\dfrac{x-1}{x^2+3x+2}+\dfrac{x-3}{x^2+5x+6}=\dfrac{1}{x+2}$.
11. $2(x-1)=\dfrac{4-5x}{x+1}$.
12. $\dfrac{1}{x-2}+\dfrac{1}{x-3}+\dfrac{1}{x-4}=0$.
13. $\dfrac{3x+1}{3x-1}-\dfrac{3x-1}{3x+1}=2$.
14. $x^2-\sqrt{3}x-6=0$.

MISCELLANEOUS FORMS OF QUADRATIC EQUATIONS

*150 **Example 1.** Solve $\dfrac{x+2}{x-2}-\dfrac{x-3}{x+3}=\dfrac{x+4}{x-4}-\dfrac{x-1}{x+1}$.

Simplifying each side separately,

$$\frac{x^2+5x+6-(x^2-5x+6)}{(x-2)(x+3)}=\frac{x^2+5x+4-(x^2-5x+4)}{(x-4)(x+1)}$$

$$\frac{10x}{x^2+x-6}=\frac{10x}{x^2-3x-4};$$

$$\therefore\ x=0 \text{ or } \frac{1}{x^2+x-6}=\frac{1}{x^2-3x-4},$$

$$i.e.\quad x^2+x-6=x^2-3x-4.$$
$$4x=2,$$
$$x=\tfrac{1}{2};$$
$\therefore$ 0, $\frac{1}{2}$ are the reqd. solutions.

*Examples. XXV. f.

Solve the equations :

1. $x^4+100=29x^2$.

[Treat the equation as a quadratic for x^2.]

2. $x^2+\dfrac{324}{x^2}=45$.

3. $x^3+\dfrac{27}{x^3}=28$.

4. $\dfrac{x+2}{x-2}-\dfrac{x-5}{x+5}=\dfrac{x+3}{x-3}-\dfrac{x-4}{x+4}$.

5. $x^2-2x+\dfrac{36}{x^2-2x}=15$.

[Let $x^2-2x=v$, and first solve for v. Two values of v will be found, and we shall therefore have *four* values of x.]

6. $x^2-1+x^3-x=0$.

[Factorize the left-hand side.]

7. $5x^3-4x^2=5x-4$.

8. $x^2-4x-4=\dfrac{5}{x^2-4x}$.

9. $x-\dfrac{1}{x}=\dfrac{4}{21}\left(x^3-\dfrac{1}{x^3}\right)$.

10. $(x+1)(x+2)(x+3)(x+4)=24+34(x^2+5x)$.

11. $6x^3+(5-x)^3=5(5+x)(5+2x)$.

12. $(x+1)(x+2)(x+3)(x+4)=24$.

13. $\dfrac{x-1}{x+1}+\dfrac{x-4}{x+4}=\dfrac{x-2}{x+2}+\dfrac{x-3}{x+3}$.

14. $x(x+1)(x+2)(x+3)=120$.

15. $x^2+3x-\dfrac{9}{2}+\dfrac{2}{x^2+3x}=0$.

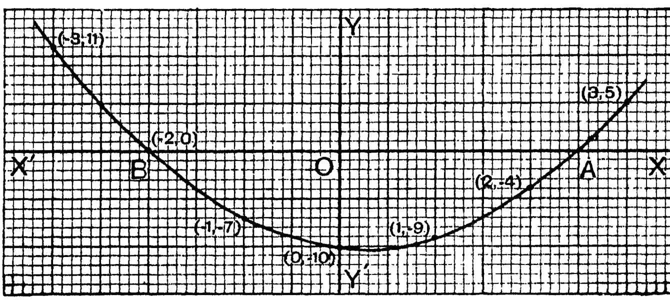
16. $16x(x+1)(x+2)(x+3)=9$.

17. $x^4+2x^3-11x^2+4x+4=0$.

CHAPTER XXVI

GRAPHS OF QUADRATIC FUNCTIONS OF x AND GRAPHIC SOLUTIONS OF QUADRATIC EQUATIONS

151. *Solve the equation* $2x^2 - x - 10 = 0$ *graphically.*



First Method. Let us trace the graph of $y = 2x^2 - x - 10$, using a unit for the x values 10 times as large as that for the y values, as in Art. 134.

When

$x = 0$	1	2	3	...	-1	-2	-3
$2x^2 = 0$	2	8	18	...	2	8	18
$-x - 10 = -10$	-11	-12	-13	...	-9	-8	-7
$y = 2x^2 - x - 10 = -10$	-9	-4	5	...	-7	0	11

$\therefore$ $(0, -10)$, $(1, -9)$, $(2, -4)$, $(3, 5)$, $(-1, -7)$, $(-2, 0)$, $(-3, 11)$ are points on the graph.

Marking these points as shown in the diagram, and drawing the curve carefully, we have the graph of $y = 2x^2 - x - 10$.

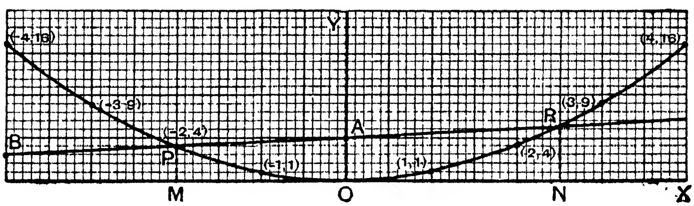
At the points A and B where this curve meets XOX′ the axis of x, $y=0$; $\therefore$ at those points $2x^2-x-10=0$.

But OA and OB are the values of x at these points;

$\therefore$ they are the roots of the given equation.

From the diagram we see that the roots are $2{\cdot}5$ and -2.

Second Method. First trace the graph of $y=x^2$, using a unit for the x values 10 times as large as that for the y values, as in Art. 134.



We thus obtain the curve POR as in the diagram.

Then trace in the same diagram, and *with the same units*, the graph of $$2y-x-10=0.$$

We know this to be a straight line. (Art. 71.)

When $x=0$, $y=5$; $\therefore$ (0, 5) is a point on the straight line.

Mark this point A.

When $x=-4$, $y=3$; $\therefore$ $(-4, 3)$ is also on the line.

Mark this point B, and join AB.

The straight line AB is the graph of $2y-x-10=0$.

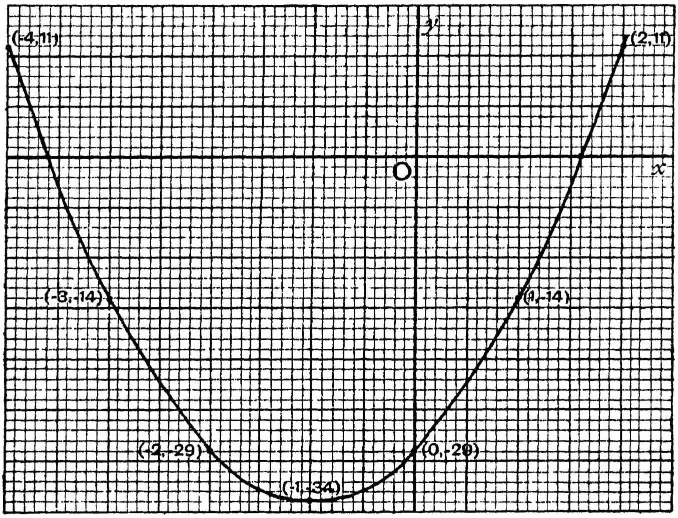
Mark the points P and R where this line meets the curve POR.

Now at the point P, the ordinate PM is the same for both graphs, *i.e.* y is the same in both the equations $y=x^2$ and $2y-x-10=0$; $\therefore$ at the point P, $2x^2-x-10=0$. OM is therefore a root of this equation. From the diagram OM $=-2$.

In precisely the same way, the ordinate at R is the same in both equations, $y=x^2$ and $2y-x-10=0$; $\therefore$ ON is another root of the equation $2x^2-x-10=0$. From the diagram ON $=2{\cdot}5$; $\therefore$ **the reqd. roots are -2 and $2{\cdot}5$.**

152. *Find graphically, correct to one decimal place, the roots of the equation* $5x^2+10x-29=0$.

Trace the graph of $y=5x^2+10x-29$.



When

$x=0$	1	2	3
$5x^2=0$	5	20	45
$10x-29=-29$	-19	-9	1
$y=-29$	-14	11	46

When

$x=-1$	-2	-3	-4
$5x^2=5$	20	45	80
$10x-29=-39$	-49	-59	-69
$y=-34$	-29	-14	11

Plotting the points (0, -29) (1, -14) (2, 11) (-1, -34) (-2, -29) (-3, -14) (-4, 11) and taking the x unit ten times as large as the y unit, we have the curve as shown in the diagram.

The equation is satisfied when $5x^2+10x-29=0$, *i.e.* when $y=0$, *i.e.* where the curve cuts the axis of x.

From the diagram, the roots required are

$$1{\cdot}6, \quad -3{\cdot}6.$$

Verification. When $x=1{\cdot}6$, $5x^2+10x-29=5(2{\cdot}56)+16-29$

$$=12{\cdot}8+16-29$$
$$=-{\cdot}2.$$

Thus when $x=1{\cdot}6$, $5x^2+10x-29$ is nearly zero.

$\therefore$ 1·6 is an approximate root. In the same way we can verify the fact that $-3{\cdot}6$ is an approximate root.

If we trace the graphs of $y=x^2$ and $y=x^2+bx+c$, where b and c have any assigned values, using the same units in each case, we shall obtain the same curve in different positions. This is easily seen by cutting out one curve and superimposing it on the other.

In general, it will be found that the graph of any equation in two variables, whose terms of the second degree form a perfect square, is a parabola.

For instance, if we plotted a number of points on the curve $(2x+3y)^2+3x-2y+5=0$ and joined them by an even curve we should obtain a parabola.

MAXIMUM AND MINIMUM VALUES OF QUADRATIC EXPRESSIONS OF ONE VARIABLE

153. These all hinge upon the fact that a perfect square is always positive, *i.e.* it cannot be less than zero.

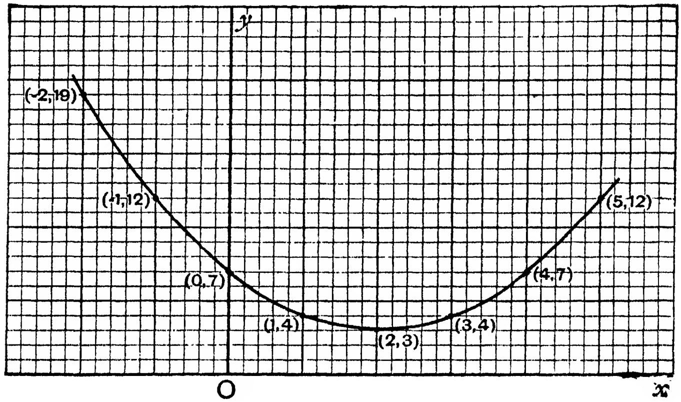
To find the minimum value of x^2-4x+7 for real values of x.

$$x^2-4x+7=(x-2)^2+3.$$

$\therefore$ the given expression is least when $(x-2)^2=0$.

The reqd. minimum value is therefore 3.

To find the minimum value of $x^2 - 4x + 7$ *graphically.*



Let us trace the graph of $y = x^2 - 4x + 7$.

When

$x = -2$	-1	0	1	2	3	4	5
$x^2 + 7 = 11$	8	7	8	11	16	23	32
$-4x = 8$	4	0	-4	-8	-12	-16	-20
$y = 19$	12	7	4	3	4	7	12

Plotting the pts. $(-2, 19)$ $(-1, 12)$ $(0, 7)$ $(1, 4)$ $(2, 3)$ $(3, 4)$ $(4, 7)$ $(5, 12)$ and joining them by an even curve, we have the curve shown in the diagram.

From it we see that the minimum value of y, *i.e.* of $x^2 - 4x + 7$, is 3.

[In the diagram the x unit is taken five times as large as the y unit.]

To find the maximum value of $3{\cdot}5 + 4x - 4x^2$ *for real values of* x.

$$3{\cdot}5 + 4x - 4x^2 = 4{\cdot}5 - (1 - 4x + 4x^2)$$
$$= 4{\cdot}5 - (1 - 2x)^2.$$

$\therefore$ the given expression is greatest when $(1-2x)^2$ is least, *i.e.* when $1-2x=0$.

Hence 4·5 is the maximum value reqd.

By plotting the graph of $y=3{\cdot}5+4x-4x^2$, we can find the maximum value graphically, as in the preceding example.

154. *Between what values of x is the expression* $19x-2x^2-35$ *positive?*

Let y denote the given expression.

$$y=-(2x^2-19x+35)=-(2x-5)(x-7)$$
$$=(2x-5)(7-x)=2(x-\tfrac{5}{2})(7-x).$$

When $x<2\tfrac{1}{2}$, $x-\tfrac{5}{2}$ is negative and $7-x$ is positive;

$$\therefore\ y \text{ is negative.}$$

When $x>2\tfrac{1}{2}$ but <7, $x-\tfrac{5}{2}$ is positive and $7-x$ is positive;

$$\therefore\ y \text{ is positive.}$$

When $x>7$, $x-\tfrac{5}{2}$ is positive and $7-x$ is negative;

$$\therefore\ y \text{ is negative.}$$

$\therefore$ the given expression is only positive as long as x is between $2\tfrac{1}{2}$ and 7.

This may be seen graphically by plotting the curve

$$y=19x-2x^2-35.$$

Examples. XXVI.

1. Draw the graph of $3x^2-5x-3$ for the following values of x, -2, -1, 0, 1, 2, 3,

   (i) Using an x unit ten times as large as the y unit.
   (ii) ....five ....

2. Draw the graph of $5x^2+4x-21$.

   (i) Using an x unit ten times as large as the y unit.
   (ii) ....five ....

3. Draw the graph of x^2-4x.

   (i) Using an x unit ten times as large as the y unit.
   (ii) ....five ....

4. Draw the graph of $4(x^2-1)$.

   (i) Using an x unit ten times as large as the y unit.
   (ii) ....five ....

   [*Tabulate values of x and y before choosing your units.*]

5. Prove graphically that the expression $x^2-6x+13$ is positive for all real values of x.

6. Show graphically that the expression $4x-6-x^2$ is never positive for real values of x.

Solve the following equations graphically:

7. $4x^2-4x-15=0$.
8. $4x^2-4x-35=0$.
9. $x^2+1\cdot 1x-\cdot 8=0$.
10. $x^2-3\cdot 3x+2=0$.
11. $6x^2-23x+21=0$, to the nearest tenth.
12. $10x^2+21x-13=0$.
13. $5x^2-3x-16=0$, to the nearest tenth.

14. Draw the graph of $4x^2-4x+1$. What do you deduce as to the roots of the equation $4x^2-4x+1=0$?

15. Plot the graph of $4x^2-3x+7$ using integral values of x from -2 to 3. What do you deduce as to the roots of the equation $4x^2-3x+7=0$?

16. Prove graphically that the expression $13-6x-x^2$ is never greater than 22 for real values of x.

17. Draw the graph of x^2-3x, and deduce approximate values of the roots of the equation $x^2-3x=3$.

18. Plot the graph of $5x^2-3x-24$, and from it deduce the roots of the equation $5x^2=3x+26$.

19. Draw the graphs of $y=x^2$, $2y=3x+14$ in the same diagram, and deduce the roots of the equation $2x^2-3x-14=0$.

20. Draw the graphs of $y=x^2$ and $5y-8x-69=0$ and deduce the roots of the equation $5x^2=8x+69$.

21. In the equation $y=5x^2-4x-10$, find the corresponding values of y to the values -2, -1, 0, 1, 2, 3 of x. Draw the portion of the curve thus given, and deduce approximate values of the roots of the equation $5x^2-4x-10=0$. Read off the minimum value of the expression $5x^2-4x-10$.

22. Find graphically the values of x for which the expression x^2-x-6 vanishes. Prove that for all values of x between these limits the expression is negative and for all other real values of x positive.

23. Draw the graphs of $y=x^2$ and $2y-3x-20=0$, and deduce the roots of the equation $2x^2=3x+20$.

24. Draw the graph of $y=(x-2)(x-3)$, and deduce approximate roots of the quadratic $(x-2)(x-3)=5$.

25. In the equation $y=3+3x-5x^2$, find the values of y corresponding to the values $-0\cdot 4$, $-0\cdot 2$, 0, $0\cdot 2$, $0\cdot 4$, $0\cdot 6$ of x. Plot the points thus obtained, using an inch to represent $0\cdot 2$ along the axis of x, and an inch to represent unity along the axis of y. Write down the maximum value of y.

26. Prove graphically that the line $y=6x-13$ meets the curve $y=x^2-4$ at one point only. Find its co-ordinates, and verify your result algebraically.

27. Find graphically, as accurately as you can, the minimum value of $4x^2-3x+2$ for real values of x. Verify your result algebraically.

28. Find graphically the maximum value of $6x-3-x^2$. Verify your result algebraically.

29. Find graphically the minimum value of $x^2-5x+\frac{35}{4}$. Verify your result algebraically and write down the corresponding value of x.

30. Find graphically the minimum value of $3x^2-6x+5\cdot 6$. Verify by algebra, and write down the corresponding value of x.

31. Find graphically the value of x which will give $2{\cdot}4+40x+5x^2$ a minimum value.

32. Find graphically between what limits the value of x must lie if $25x^2-30x-91$ is negative.

33. Between what limits must the value of x lie if the expression $20-2x^2-3x$ is positive? Find the limits graphically and by algebra.

CHAPTER XXVII

SIMULTANEOUS QUADRATIC EQUATIONS

155. In this chapter we shall consider simultaneous equations, where one at least is of a higher degree than the first.

The methods of solution are various, but the student should endeavour to reduce the equations to the forms

$$ax+by=c,$$
$$ax-by=c'.$$

Addition and subtraction will then effect the solution.

Example 1. Solve the equations $25x^2-y^2=84$, $5x-y=6$.

By division, $5x+y=14$.

Also $5x-y=6$.

Adding, $10x=20$, $\therefore\ x=2$.

Subtracting, $2y=8$, $\therefore\ y=4$.

$x=2$, $y=4$ is the reqd. solution.

Example 2. Solve the equations $3x+y=9$, ...(1)

$xy=6$. ....(2)

Squaring equation (1) $9x^2+6xy+y^2=81$.

From (2) $12xy=72$.

Subtracting, $9x^2-6xy+y^2=9$.

Taking the sq. root, $3x-y=\pm 3$.

We now have the two cases,

$$\left.\begin{aligned}3x+y&=9,\\3x-y&=3.\end{aligned}\right\}\qquad\qquad \begin{aligned}3x+y&=9,\\3x-y&=-3.\end{aligned}$$

Adding, $6x=12$, $x=2$. $\qquad$ $6x=6$, $x=1$.

Subtracting, $2y=6$, $y=3$. $\qquad$ $2y=12$, $y=6$.

$$\therefore\ \left.\begin{aligned}x&=2\\y&=3\end{aligned}\right\}\text{ and }\left.\begin{aligned}x&=1\\y&=6\end{aligned}\right\}\text{ are the reqd. solutions.}$$

Example 3. Solve the equations $9x^2+y^2=52$, ...(1)

$xy=8$. ....(2)

From (2) $6xy=48$. ....(3)

Adding to (1) *to complete the square,*

$$9x^2+6xy+y^2=100.$$

Taking the sq. root $3x+y=\pm 10.$

Also, in the same way, subtracting (3) from (1),

$$9x^2-6xy+y^2=4.$$

$$\therefore\ 3x-y=\pm 2.$$

There are now four cases,

$$\left.\begin{aligned}3x+y&=10,\\3x-y&=2.\end{aligned}\right\}\quad \left.\begin{aligned}3x+y&=\ \ 10,\\3x-y&=-2.\end{aligned}\right\}\quad \left.\begin{aligned}3x+y&=-10,\\3x-y&=\ \ 2.\end{aligned}\right\}\quad \begin{aligned}3y+y&=-10,\\3x-y&=-2.\end{aligned}$$

Adding, $6x=12$, $x=2$. $\quad 6x=8$, $x=\frac{4}{3}$. $\quad 6x=-8$, $x=-\frac{4}{3}$. $\quad 6x=-12$, $x=-2$.

Subtracting, $2y=8$, $y=4$. $\quad 2y=12$, $y=6$. $\quad 2y=-12$, $y=-6$. $\quad 2y=-8$, $y=-4$.

Hence the reqd. solutions are

$$\left.\begin{aligned}x&=2,\\y&=4.\end{aligned}\right\}\quad \left.\begin{aligned}x&=\tfrac{4}{3},\\y&=6.\end{aligned}\right\}\quad \left.\begin{aligned}x&=-\tfrac{4}{3},\\y&=-6.\end{aligned}\right\}\quad \left.\begin{aligned}x&=-2,\\y&=-4.\end{aligned}\right\}$$

Example 4. Solve the equations $4x^2+y^2=17$, ...(1)

$2x+y=5$. ....(2)

From (2) by squaring, $4x^2+4xy+y^2=25$. ....(3)

...... (1) by subtraction, $4xy=8$.

Subtracting this from (1) $4x^2-4xy+y^2=9$.

Taking the sq. root, $2x-y=\pm 3$.

Hence $\left.\begin{aligned}2x+y&=5,\\2x-y&=3.\end{aligned}\right\}$ or $\begin{aligned}2x+y&=5,\\2x-y&=-3.\end{aligned}$

Adding, $4x=8$, $x=2$. $\quad 4x=2$, $x=\frac{1}{2}$.

Subtracting, $2y=2$, $y=1$. $\quad 2y=8$, $y=4$.

$$\therefore\ \left.\begin{aligned}x&=2,\\y&=1.\end{aligned}\right\}\quad \left.\begin{aligned}x&=\tfrac{1}{2},\\y&=4.\end{aligned}\right\}$$ are the reqd. solutions.

The Examples in XXVII. a. can all be solved by substitution. The student must be careful to do the work methodically.

Example 1. $25x^2 - y^2 = 84, \quad \text{..........(1)}$

$5x - y = 6. \quad \text{..........(2)}$

From (2), $y = 5x - 6.$

$\therefore$ by substitution in (1),

$$25x^2 - (5x-6)^2 = 84,$$

whence $60x - 36 = 84, \quad \therefore x = 2.$

By substitution in (2), the simpler of the two given equations,

$$10 - y = 6, \quad \therefore y = 4.$$

$$\therefore \left.\begin{matrix} x=2 \\ y=4 \end{matrix}\right\} \text{ is the reqd. solution.}$$

Example 2. $3x + y = 9, \quad \text{..........(1)}$

$xy = 6. \quad \text{..........(2)}$

From (1), $y = 9 - 3x.$

$\therefore$ by substitution in (2), $x(9-3x) = 6,$

i.e. $3x^2 - 9x + 6 = 0,$

i.e. $x^2 - 3x + 2 = 0,$

i.e. $(x-1)(x-2) = 0;$

$\therefore x = 1$ or $2.$

When $x = 1$, from (1), $y = 9 - 3x = 9 - 3 = 6.$

.... $x = 2$, ... $= 9 - 6 = 3.$

$$\therefore \left.\begin{matrix} x=1 \\ y=6 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x=2 \\ y=3 \end{matrix}\right\} \text{ are the reqd. solutions.}$$

Example 3. $$9x^2+y^2=52, \quad\ldots\ldots(1)$$
$$xy=8. \quad\ldots\ldots(2)$$

From (2), $$y=\frac{8}{x}.$$

$\therefore$ from (1), by substitution,

$$9x^2+\frac{64}{x^2}=52,$$

$$i.e.\quad 9x^4-52x^2+64=0,$$

$$i.e.\quad (9x^2-16)(x^2-4)=0.$$

$$\therefore\quad x^2=\frac{16}{9} \text{ or } 4.$$

$$\therefore\quad x=\pm\frac{4}{3} \text{ or } \pm 2.$$

When $x=\pm\frac{4}{3}$, from (2), $y=\frac{8}{x}=\pm 8\times\frac{3}{4}=\pm 6$.

...... $x=\pm 2$, ... $=\pm\frac{8}{2}=\pm 4$.

Hence $\left.\begin{array}{l}x=\frac{4}{3}\\ y=6\end{array}\right\}$ $\left.\begin{array}{l}x=-\frac{4}{3}\\ y=-6\end{array}\right\}$ $\left.\begin{array}{l}x=2\\ y=4\end{array}\right\}$ $\left.\begin{array}{l}x=-2\\ y=-4\end{array}\right\}$ are the reqd. solutions.

Examples. XXVII. a.

Solve the equations:

1. $4x^2-y^2=35,$ $2x+y=7.$
2. $x^2-y^2=21,$ $x+y=3.$
3. $y^2-9x^2=28,$ $y-3x=2.$
4. $x^2-xy=35,$ $x-y=5.$
5. $4x^2+xy=51,$ $4x+y=17.$
6. $9x-3y=3,$ $9x^2-y^2=5.$
7. $5x-2y=12,$ $25x^2-4y^2=96.$
8. $4x^2-25y^2=-81,$ $4x-10y=54,$
9. $9x^2-49y^2=29,$ $6x-14y=2.$
10. $x+y=15,$ $xy=54.$
11. $x-y=2,$ $xy=15.$
12. $x-y=1,$ $xy=132.$

13. $x+y=4,$ $xy=-117.$

14. $x+y=6,$ $xy=-91.$

15. $xy=21,$ $x-y=4.$

16. $8xy=1,$ $4(x+y)=3.$

17. $4x+y=11,$ $xy=6.$

18. $5x-y=9,$ $xy=2.$

19. $3x-2y=14,$ $xy=12.$

20. $5x+4y=28,$ $xy=8.$

21. $x^2+y^2=53,$ $xy=14.$

22. $x^2+y^2=34,$ $xy=-15.$

23. $4x^2+y^2=17,$ $xy=2.$

24. $x^2+9y^2=18,$ $xy=3.$

25. $9x^2+4y^2=136,$ $xy=10.$

26. $16x^2+25y^2=544,$ $xy=12.$

27. $\frac{1}{x}+\frac{1}{y}=\frac{3}{4},$ $xy=8.$

28. $\frac{1}{x}-\frac{1}{y}=1,$ $xy=\frac{1}{6}.$

29. $\frac{1}{x}+\frac{1}{y}=\frac{14}{45},$ $x+y=14.$

30. $\frac{1}{x}-\frac{1}{y}=-\frac{2}{35},$ $x-y=2.$

31. $\frac{2}{x}+\frac{1}{y}=1,$ $xy=-1.$

32. $\frac{3}{x}+\frac{2}{y}+=12,$ $xy=\frac{1}{6}.$

33. $4x-3y=26,$ $\frac{4}{y}-\frac{3}{x}=-\frac{26}{10}.$

34. $5x+7y=17,$ $\frac{5}{y}+\frac{7}{x}=8\frac{1}{2}.$

35. $x^2+y^2=53,$ $x+y=5.$

36. $x^2+y^2=\frac{5}{16},$ $x-y=\frac{1}{4}.$

37. $4x^2+y^2=104,$ $2x+y=12.$

38. $9x^2+y^2=81,$ $3x-y=9.$

39. $x^2+xy+y^2=201,$ $x+y=16.$

40. $x^2-xy+y^2=157,$ $x-y=1.$

41. $x^2+2xy+4y^2=28,$ $x+2y=6.$

42. $9x^2+xy+4y^2=91,$ $3x-2y=13.$

***156.** In the following equations, the student's aim should be to reduce the equations to one of the forms exemplified earlier in this chapter.

Example. Solve the equations

$$x^3+y^3=91, \quad\quad (1)$$

$$x^2-xy+y^2=13. \quad\quad (2)$$

Dividing, $\quad x+y=7. \quad\quad (3)$

Squaring, $\quad x^2+2xy+y^2=49.$

$\therefore$ from (2), $\quad xy=12. \quad\quad (4)$

Now solve equations (3) and (4) as in Example 2, Art. 155.

*Examples. XXVII. b.

Solve the equations :

1. $x^3+y^3=9,$ $x+y=3.$

2. $x^3-y^3=37,$ $x-y=1.$

3. $8x^3+y^3=280,$ $2x+y=10.$

[Divide and then proceed as in the Example worked out.]

4. $x^3-8y^3=189,$ $x-2y=9.$

5. $27x^3+8y^3=35,$ $3x+2y=5.$

6. $8x^3-27y^3=485,$ $2x-3y=5.$

7. $$x^4+x^2y^2+y^4=21, \quad \text{.........................} \quad (1)$$
$$x^2+xy+y^2=3. \quad \text{.............................} \quad (2)$$

[Dividing (1) by (2), $$x^2-xy+y^2=7. \quad \text{.............................} \quad (3)$$

Now add and subtract equations (2) and (3), and proceed as in Example 3, Art. 155.]

8. $x^4+x^2y^2+y^4=1281,$ $x^2-xy+y^2=21.$

9. $x^4+x^2y^2+y^4=481,$ $x^2-xy+y^2=13.$

10. $x^4+x^2y^2+y^4=2613,$ $x^2+xy+y^2=67.$

11. $\frac{1}{x^2}+\frac{1}{y^2}=13,$ $\frac{1}{x}+\frac{1}{y}=5.$

12. $\frac{1}{x^2}+\frac{1}{y^2}=41,$ $\frac{1}{x}-\frac{1}{y}=-1.$

[See Note in Example 2, Art. 60.]

13. $\frac{4}{x^2}+\frac{1}{y^2}=109,$ $\frac{2}{x}+\frac{1}{y}=13.$

14. $\frac{9}{x^2}+\frac{1}{y^2}=\frac{26}{25},$ $\frac{3}{x}-\frac{1}{y}=\frac{4}{5}.$

15. $\frac{1}{x^2}+\frac{1}{y^2}=61,$ $30xy=1.$

16. $\frac{1}{x^2}+\frac{4}{y^2}=5,$ $xy=1.$

17. $15(x^2+y^2)=34xy,$ $\frac{1}{x}-\frac{1}{y}=2.$

18. $\frac{x}{y}+\frac{y}{x}=\frac{257}{16},$ $4(x+y)=17.$

19. $\frac{x}{y}+\frac{y}{x}=\frac{17}{4},$ $x-y=\frac{3}{2}.$

20. $\frac{4x}{y}+\frac{y}{x}=\frac{17}{2},$ $2x+y=20.$

21. $\frac{1}{x^3}+\frac{1}{y^3}=35,$ $\frac{1}{x}+\frac{1}{y}=5.$

22. $\frac{1}{x^3}-\frac{1}{y^3}=61,$ $\frac{1}{x}-\frac{1}{y}=1.$

23. $x^3+y^3=351,$ $x^2-xy+y^2=39.$

24. $x^3-y^3=702,$ $x^2+xy+y^2=117.$

25. $8x^3+y^3=2,$ $4x^2-2xy+y^2=1.$

26. $8x^3+27y^3=2,$ $4x^2-6xy+9y^2=1.$

***157.** Solve the equations $2x^2y^2 - 13xy + 18 = 0$, ...(1)

$$x + y = \tfrac{9}{2}. \qquad \text{..............................(2)}$$

Treating (1) as a quadratic for xy,

$$(2xy - 9)(xy - 2) = 0;$$

$$\therefore \; xy = \tfrac{9}{2} \text{ or } 2.$$

The complete solution is then obtained by first solving the equations

$$x + y = \tfrac{9}{2}, \quad xy = \tfrac{9}{2},$$

and then the equations $\quad x + y = \tfrac{9}{2}, \quad xy = 2$, as in Example 2, Art. 155.

***158.** When the variable terms in the equations are *homogeneous*, *i.e. of the same degree*, the following method may be used.

Solve the equations $12x^2 - 4xy + 11y^2 = 64$, ...(1)

$$16x^2 - 9xy + 11y^2 = 78. \qquad \text{........................(2)}$$

Eliminate the constant terms, by multiplying across (multiply the left-hand side of each equation by the right-hand side of the other).

$$78(12x^2 - 4xy + 11y^2) = 64(16x^2 - 9xy + 11y^2),$$

$$39(12x^2 - 4xy + 11y^2) = 32(16x^2 - 9xy + 11y^2).$$

Multiplying out, and re-arranging,

$$77y^2 + 132xy - 44x^2 = 0,$$

$$7y^2 + 12xy - 4x^2 = 0,$$

$$(7y - 2x)(y + 2x) = 0;$$

$$\therefore \; y = \frac{2x}{7} \text{ or } y = -2x.$$

$\left(\text{If the factors cannot be seen, solve as a quadratic for } \frac{y}{x}.\right)$

(1) When $y=\frac{2x}{7}$. Substituting this value of y in (1),

$$x^2\left(12-\frac{8}{7}+\frac{44}{49}\right)=64,$$

whence
$$x^2=\frac{49\times 64}{576}=\frac{49}{9},$$

$$x=\pm\frac{7}{3};$$

$$\therefore\ y=\frac{2x}{7}=\pm\frac{2}{3}.$$

(2) When $y=-2x$. Substituting this value in (1)

$$x^2(12+8+44)=64,$$

$$x^2=1,$$

$$x=\pm 1,$$

$$y=-2x=\mp 2;$$

$\therefore$ the reqd. solutions are
$$x=\pm\frac{7}{3},\ \pm 1,$$

$$y=\pm\frac{2}{3},\ \mp 2.$$

*159. When the above methods are inapplicable, substitution from one equation in the other may be employed.

Solve the equations $3x^2+4xy+5y^2=31,$ ....(1)

$x+2y=5.$ ....(2)

From (2) $x=5-2y$.

Substituting this value of x in (1),

$$3(5-2y)^2+4y(5-2y)+5y^2=31,$$

whence
$$9y^2-40y+44=0,$$

$$(9y-22)(y-2)=0;$$

$$\therefore\ y=\tfrac{22}{9} \text{ or } 2,$$

$$x=5-2y=5-\tfrac{44}{9} \text{ or } 5-4$$

$$=\tfrac{1}{9} \text{ or } 1.$$

*Examples. XXVII. c.

MISCELLANEOUS EXAMPLES IN SIMULTANEOUS QUADRATICS

Solve the following equations :

1. $x^2+xy=3,$
$y^2+xy=6.$

2. $2xy+y^2=16,$
$2x^2-xy=12.$

3. $x^2+y^2=xy+7,$
$x^2-y^2=xy-1.$

4. $3x^2 - 5xy = -2,$
$4xy - 3y^2 = 1.$

5. $x^2 - 2xy + 3 = 0,$
$2x + y = 4.$

6. $y^2 + xy = 4,$
$x^2 + 2y^2 - xy = 8.$

7. $x^2 + xy = 3,$
$y^2 + xy = 4.$

8. $6x^2 - 3xy + 11y^2 = 584,$
$x = 5y.$

9. $x^2 + 3xy + 2y^2 = 7,$
$x^2 - y^2 = 4.$

10. $x^2 + xy = 15,$
$xy - y^2 = 2.$

11. $3x^2 + 4xy + 5y^2 = 81,$
$3x = 2y.$

12. $2x^2 + 3xy = 26,$
$3y^2 + 2xy = 39.$

13. $x + y = 6,$
$(x^2 + y^2)(x^3 + y^3) = 1440.$

14. $6x^2 + 3xy - 18y^2 = 20,$
$3x^2 + 6xy = 8.$

15. $x^2 + y^2 = 5,$
$x^2 + xy = 6.$

16. $x^2 = 14 + xy,$
$y^2 = xy - 10.$

17. $x^3 - y^3 = 485,$
$x - y = 5.$

18. $x^2 - 4y = y^2 + 4x = 21.$

19. $\frac{1}{x} - \frac{1}{y} = \frac{1}{12},$
$\frac{4}{x^2} + \frac{6}{y^2} = \frac{5}{12}.$

20. $2x^2 + 3xy + 10 = 0,$
$x^2 + xy - y^2 + 11 = 0.$

21. $3xy + x^2 = 10,$
$5xy - 2x^2 = 2.$

22. $x^2 + xy + y^2 = 61,$
$x + y = 9.$

23. $(x+5)(y+7) = (x+27)(y+\frac{5}{7}),$
$xy = 1.$

24. $x^2 + 4y = 28,$
$3x = 4y.$

25. $9x^2 + 6xy - 4y^2 = 1,$
$3x - 2y = -1.$

26. $y^2 - xy = 15,$
$x^2 + xy = 14.$

27. $x^2 + 4y^2 - 3x + y = 67,$
$x - 2y = 1.$

28. $x^2 + xy + y^2 = 49,$
$x^4 + x^2y^2 + y^4 = 931.$

29. $x^2 + xy = 12,$
$xy - 2y^2 = 1.$

30. $2x + 3y = 1\frac{1}{2},$
$4x^2 + 9xy + 9y^2 = 11.$

31. $(x+y)^2 + 3(x-y) = 30,$
$xy + 3(x-y) = 11.$

32. $x^2 + 3xy + y^2 = 1,$
$x^2 - xy + y^2 = 13.$

33. $\frac{y}{x} - \frac{x}{y} = \frac{x+3}{x+4} = \frac{x+y}{xy}.$

34. $x^2 + xy = y^3 - 9x^2y + 64 = 0.$

35. $x^4 - x^2 + y^4 - y^2 = 84,$
$x^2 + x^2y^2 + y^2 = 49.$

GRAPHS. (CIRCLES.)

***160. The distance of the point (x, y) from the origin $= \sqrt{(\mathbf{x}^2 + \mathbf{y}^2)}$.**

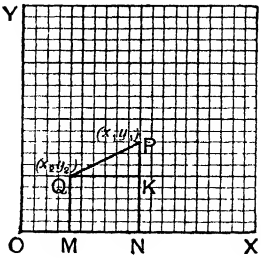
Using this, we may also determine the graph of $y = \sqrt{(25 - x^2)}$ as follows. The equation may be written, $x^2 + y^2 = 25$.

$$\therefore \sqrt{(x^2 + y^2)} = 5.$$

This shows us that the point (x, y) moves at a constant distance of 5 units from the origin.

The graph is therefore a circle, whose centre is at the origin, and whose radius $= 5$.

***161.** In the accompanying diagram, let P be the pt. (x_1, y_1) and Q the pt. (x_2, y_2).



Draw PN and QM perp. to the axis of x, and QK perp. to PN.

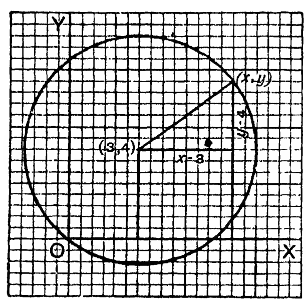
$$PK = y_1 - y_2, \text{ and } QK = x_1 - x_2.$$

$$\therefore \; PQ = \sqrt{(QK^2 + PK^2)} = \sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2]}.$$

Thus we see that the distance between the two pts. (x_1, y_1) and (x_2, y_2)

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

***162.** *Trace the graph* of $x^2 + y^2 - 6x - 8y = 0$.



This equation may be written $(x-3)^2 + (y-4)^2 = 25$.

$$\therefore \; \sqrt{(x-3)^2 + (y-4)^2} = 5.$$

It is important to notice that if *no constant term* occurs in an equation, the corresponding graph passes through the origin, for by substitution we see that when $x = 0$, one value of y is 0.

The graph of $x^2 + y^2 = 5$ is a circle whose radius is $\sqrt{5}$.

A line $\sqrt{5}$ units long may be drawn either by using Pythagoras' Theorem $(2^2 + 1^2 = 5)$ or by the method of Art. 136.

*Examples. XXVII. d.

Trace the graphs of the following:

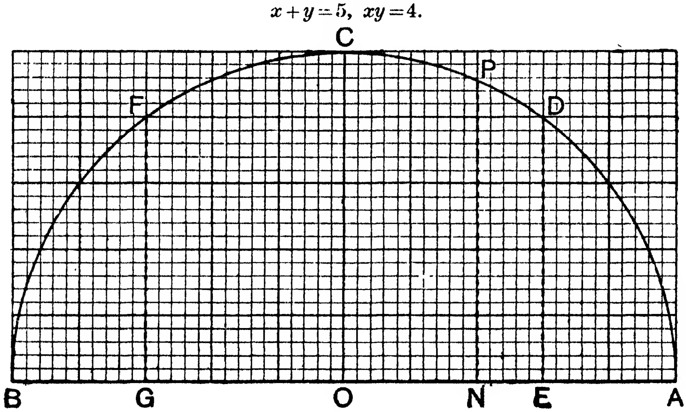
1. $x^2+y^2=36$.
2. $x^2+y^2=0$.
3. $x^2+y^2=49$.
4. $x^2+y^2=81$.
5. $x^2+y^2+8x-8y=0$.
6. $x^2+y^2-8x-6y=0$.
7. $(x-3)^2+(y-4)^2=36$.
8. $(x-1)^2+(y-2)^2=36$.
9. $(x+2)^2+(y-3)^2=25$.
10. $(x-3)^2+(y+3)^2=16$.
11. $\sqrt{(15-2x-x^2)}$.
12. $\sqrt{(21+4x-x^2)}$.
13. $\sqrt{(15+2x-x^2)}$.
14. $\sqrt{(14x-x^2-13)}$.
15. $x^2+y^2=2$.
16. $x^2+y^2=5$.
17. $x^2+y^2=13$.
18. $x^2+y^2=10$.
19. $x^2+y^2=20$.
20. $x^2+y^2=3$.
21. $x^2+y^2+2x+2y=0$.
22. $(x-1)^2+y^2=2$.
23. $(x+2)^2+(y-2)^2=5$.
24. $x^2+y^2+2x+2y=3$.
25. $x^2+y^2-6x+4y+3=0$.
26. $2x^2+2y^2=5$.
27. $2x^2+2y^2-4x+8y+3=0$.
28. $4x^2+4y^2-16x+8y+11=0$.
29. $4x^2+4y^2-24x+11=0$.

GRAPHICAL SOLUTION OF SIMULTANEOUS QUADRATIC EQUATIONS

***163.** Simultaneous quadratics can often be readily solved by graphical methods.

Example 1. Solve the following equations graphically:

$$x+y=5, \quad xy=4.$$



On AB, 5 in. long (the diagram is reduced in printing), describe the semi-circle ACB.

If P is any pt. on the curve and PN is drawn perp. to AB, we know, by Geometry, that

$$PN^2 = AN \cdot NB.$$

Mark the pts. D, F on the curve where the lengths of the perpendiculars DE, FG on AB are equal to 2 inches ($\sqrt{4}$).

Then $DE^2 = AE \cdot BE$, and $FG^2 = AG \cdot BG$.

$$\therefore \text{ if } AE = x \text{ and } BE = y,$$

$$x + y = AB = 5 \text{ and } xy = AE \cdot BE = DE^2 = 4.$$

$\therefore$ AE, BE are solutions of the given equation.

From the diagram $x = 1$, $y = 4$.

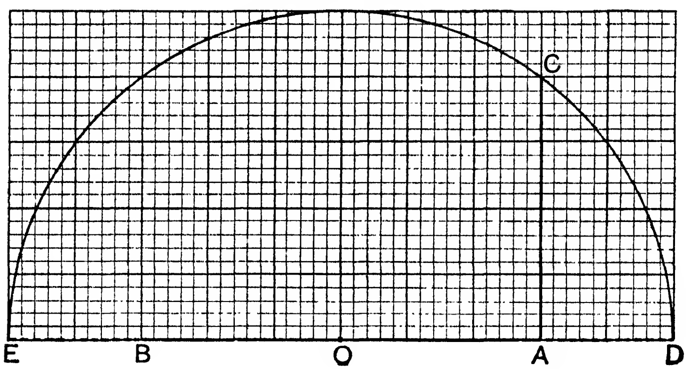
In the same way, AG and BG are solutions, and we have

$$x = 4, \quad y = 1.$$

$$\therefore \left.\begin{array}{l} x = 1 \text{ or } 4, \\ y = 4 \text{ or } 1, \end{array}\right\} \text{is the complete solution.}$$

Example 2. Solve the following equations by the graphical method:

$$x - y = 3, \quad xy = 4.$$



Take AB 3 in. long and AC at rt. $\angle$s to it 2 in. ($=\sqrt{4}$) long. With O, the mid. pt. of AB as centre, and OC radius, describe the semi-circle ECD, meeting AB produced at D and E.

As in the previous example, $CA^2 = DA \cdot AE$.

$$\therefore \text{ if } AE = x \text{ and } AD = y,$$

$$x - y = AE - AD = AE - BE = AB = 3.$$

Also $xy = EA \cdot AD = AC^2 = 4$.

$\therefore$ AE and AD give a solution of the given equations.

From the diagram see that $x = 4$, $y = 1$.

N.B. $x = -1$, $y = -4$ is also a solution of these equations. The above method does not give negative roots satisfactorily.

The methods of the two preceding examples may be employed to solve some quadratic equations.

Thus to solve $x^2 - 7x + 9 = 0$, we have to factorize the expression $x^2 - 7x + 9$, *i.e.* we have to find two numbers whose sum is 7 and product 9.

We can therefore use the method of Example 1.

In the same way, to solve $x^2 - 3x - 36 = 0$, we have to find two numbers whose difference is 3 and product 36.

We can therefore use the method of Example 2.

***164.** *Solve the following equations graphically:*

$$x^2 + y^2 - 4x - 2y + 1 = 0, \quad 2x - 3y = 3.$$

The first equation may be written

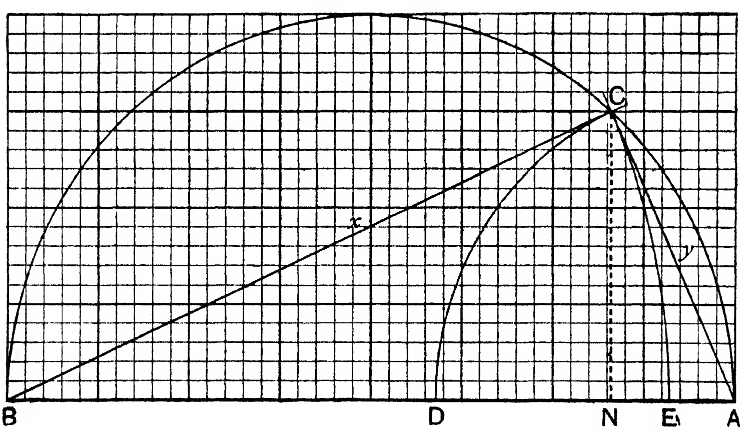
$$(x-2)^2 + (y-1)^2 = 4.$$

Hence its graph is a circle whose centre is at (2, 1) and whose radius is 2.

Draw the circle, and also draw, using the same axes and the same units, the graph of $2x - 3y = 3$, a str. line through the pts. (1·5, 0), (0, −1).

The pts. of intersection of the circle and str. line give the roots required.

***165.** *Find approximate solutions of the following equations by a graphical method:* $x^2 + y^2 = 16$, $xy = 6$.



The following method depends upon the fact that if ABC is a triangle, right-angled at C, and CN is drawn perp. to the hypotenuse AB, then AC . BC = 2△ABC = CN . AB. Now $\sqrt{16} = 4$, hence on AB, 4 in. long, describe a semi-circle ACB, and take the pt. C such that the perp. from C on $AB = \frac{6}{4} = 1\frac{1}{2}$ in. (Sqd. paper should be used.)

Then $$AC^2 + BC^2 = AB^2 = 16.$$

Also $$AC \cdot BC = CN \cdot AB = \tfrac{3}{2} \times 4 = 6;$$

$\therefore$ AC and BC are roots of the given equation.

With centre A and radius AC describe a circle cutting AB at D

$AC = AD = 1{\cdot}65$ approx. from the diagram.

In the same way $BC = 3{\cdot}65$ approx.;

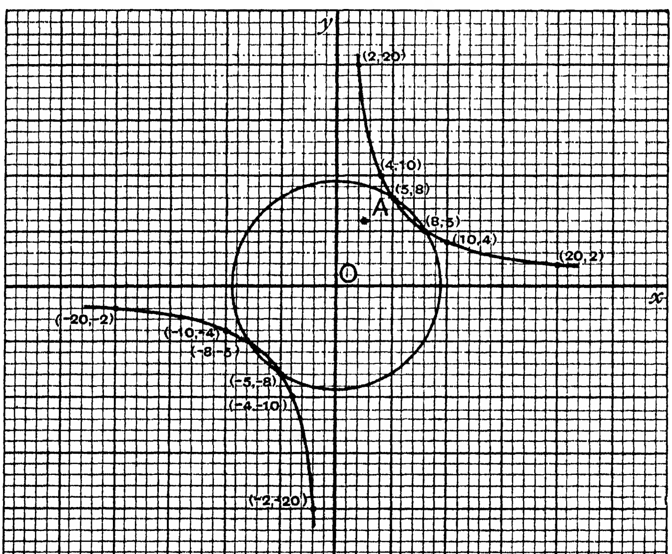
$\therefore$ 1·65, 3·65 are roots of the given equation.

***166.** *To trace the graph of* $xy = 40$.

When

$x = \pm 2$	± 4	± 5	± 8	± 10	± 20	...
$y = \pm 20$	± 10	± 8	± 5	± 4	± 2	...

the upper signs being taken together, and the lower signs together.



Plotting these pts. and joining them by an even curve, we have the figure shown in the diagram.

It is observed that the curve lies entirely in the first and third quadrants, and that the two branches are symmetrical in regard to both the bisectors of the angles between the axes of co-ordinates.

Hence we have another method of solution of equations of the following type :

$$x^2+y^2=89,$$
$$xy=40.$$

We first draw the graph of $xy=40$.

The graph of $x^2+y^2=89$ is a circle whose centre is at the origin, and radius $\sqrt{89}$. Since $89=25+64=5^2+8^2$, the length OA in the diagram is the radius. Describing the circle, and reading off the pts. of intersection of the two curves, we have the following solutions :

$$x=8,\ 5,\ -5,\ -8,$$
$$y=5,\ 8,\ -8,\ -5.$$

***167.** *Find approximate roots of the equations*

$$xy=80, \quad x-2y=10.$$

From the following table of values, draw the graph of $xy=80$.

$x=\pm 4$	± 5	± 8	± 10	± 20	...
$y=\pm 20$	± 16	± 10	± 8	± 4	...

Draw the graph of $x-2y=10$, a str. line through the pts. (10, 0), (0, -5).

The pts. of intersection of the two graphs give the reqd. roots. They will be found to be

$$\left.\begin{aligned} x&=18{\cdot}6,\ -8{\cdot}6 \\ y&=\ \ 4{\cdot}3,\ -9{\cdot}3 \end{aligned}\right\}\text{approx.}$$

Equations of the type of Examples 1 and 2 worked out in this chapter might also be solved by this method.

*Examples. XXVII. e.

Find, approximately, the values of the roots of the following equations by the use of graphical methods. Verify your results.

(In some cases the exact values of the roots can be obtained.)

1. $x+y=7,\ xy=9$.
2. $x+y=9,\ xy=16$.
3. $x-y=2,\ xy=16$.
4. $x-y=4,\ xy=9$.
5. $x+y=7,\ xy=5$.
6. $x-y=3,\ xy=8$.
7. $x^2-13x+36=0$.
8. $x^2-11x+25=0$.
9. $x^2-8x+13=0$.

Find, approximately, the values of the roots of the following equations, by the use of graphical methods. Verify your results.

10. $x^2-2x-16=0$.
11. $x^2+y^2=4$, $2x-y=1$.
12. $x^2+y^2=8$, $x+2y=2$.
13. $x^2+y^2-2x-4y+1=0$, $5y-5x=3$.
14. $4x^2+4y^2+8x-4y=11$, $x=2-2y$.
15. $x^2+y^2=9$, $4x+3y+6=0$.
16. $x^2+y^2=36$, $xy=15$.
17. $x^2+y^2=225$, $xy=80$.
18. $xy=80$, $2x-y=10$.

CHAPTER XXVIII

FURTHER EXAMPLES ON SYMBOLICAL REPRESENTATION

Examples. XXVIII.

1. A man rows x miles an hour in still water, and the current runs at the rate of y miles an hour:

(i) How many miles an hour does the man row with the current?
(ii) .... against .... ?
(iii) How long does he take to row a miles with the current?
(iv) ....against .... ?

2. Money is invested at simple interest at the rate of x per cent. per annum:

(i) What is the interest on 1£ for a year?
(ii) .... 1£ ... y years?
(iii) .... z£ ... . ?
(iv) What does z£ amount to in .... ?

3. Calculating simple interest at the rate of x per cent. per annum,

(i) What is the present value of 100£ due in one year?
(ii) .... a£ .... ?
(iii) .... 100£ .... y years?
(iv) .... a£ .... ?

4. A train runs at the rate of y miles an hour:

(i) How long does it take to do one mile?
(ii) .... z miles?
(iii) .... z miles at the above rate, and another z miles at double the rate?
(iv) How many miles does it run in a hours at the slower rate?

5. A can do a piece of work in x hours, B can do it in y hours:

(i) What fraction of the work do A and B do, working together, in one hour?
(ii) .... a hours? ...
(iii) How long do they take to do the work when working together?
(iv) .... three-quarters ....

6. One pipe, running alone, fills a cistern in x hours; a second, running alone, fills it in y hours; and a third, also running alone, empties it in z hours:

(i) What fraction of the cistern do they fill, all running together, in an hour?

(ii) How long do they take to fill the cistern, all running together?

7. x£ is the simple interest on y£ for z years:

(i) What is the simple interest on y£ for one year?
(ii) .... 1£ .... ?
(iii) .... 100£ .... ?
(iv) .... a£ ... b years?

8. In x years y£ amounts to z£ at simple interest:

(i) What is the interest on y£ for x years?
(ii) .... y£ ... one year?
(iii) .... 1£ .... ?
(iv) .... a£ ... b years?
(v) What is the rate of interest?

9. Apples cost x pence per dozen:

(i) What does a man give for one apple?
(ii) .... he .... y apples?
(iii) What does he give for one apple when the price is raised a penny per dozen?
(iv) What does he give for y apples at the higher price?
(v) How much do a apples cost at the cheaper price?
(vi) .... higher ...... ?

10. A man invests money at compound interest at the rate of x per cent. per annum:

(i) What is the interest on 1£ for one year?
(ii) .... amount of 1£ .... ?
(iii) .... a£ .... ?
(iv) .... interest on .... ?
(v) .... amount of 1£ ... 2 years?
(vi) .... 3 ...... ?
(vii) .... n ...... ?
(viii) ....P£ ... 2 years?
(ix) .... 3 ...... ?
(x) .... n ...... ?
(xi) .... interest on.............. ?

11. If simple interest is calculated at the rate of x per cent. per annum,

(i) What is the discount on 100£ due in one year?
(ii) .... a£ ....?
(iii) .... 100£ .... y years?
(iv) ....a£ ....?

12. A man can do a piece of work in x hours; a woman does half as much as a man, and a boy half as much as a woman. What fraction of the work will

(i) A man, a woman, and a boy together do in 1 hour?
(ii) 2 men, 3 women, and 4 boys .... ?

13. One man walks x miles an hour, and another y miles an hour starting at the same time, in the same direction.

(i) How much apart are they in an hour if the first man is the quicker walker?
(ii) How much apart are they in a hours?
(iii) How long does the first take to gain one mile on the other?
(iv) .... b miles .... ?

Express the following in the form of equations:

14. The product of two consecutive numbers of which x is the smaller is less than the product of the next higher two consecutive numbers by y.

15. A man bought a cows at x£ each, and b sheep at y£ each, and altogether spent z shillings.

16. Apples are sold at x pence a dozen, and pears at y pence for 10. a apples and b pears cost z shillings.

17. x men form a hollow square, four ranks deep, with y men on each outside face of the square.

18. A hollow square is formed by a men, y ranks deep, with z men on each outside face of the square.

19. A fraction whose numerator is x, and denominator y, is increased by a when the numerator is increased by b, and the denominator decreased by c.

20. x dozen of wine at a shillings a dozen, and y dozen at b shillings a dozen, cost c shillings a dozen on the average.

21. The area of a room x ft. long and y ft. wide is doubled when its length and breadth are each increased by a feet.

22. In travelling a yards, the fore wheel of a carriage makes n revolutions more than the hind wheel. Take x feet for the circumference of the fore wheel and y feet for that of the hind wheel.

23. One pipe will fill a cistern in x hours, a second will fill it in y hours; running together they fill it in z hours.

24. A starts off on a journey at x miles an hour; and n hours afterwards, B starts off at y miles an hour, and catches A up in a hours from A's start.

25. Two men start simultaneously to walk from A and B to B and A respectively, a distance of n miles. They walk at x miles an hour and y miles an hour, and meet in a hours.

26. Form the equation for the above problem when the second man starts b hours after the first, and they meet a hours after the first man started.

27. Between two places one mile apart there are x telegraph posts in a straight line, y yards apart.

28. Between two places a miles apart, there are x telegraph posts in a straight line, y yards apart.

29. A man spends one-third of his income of x£ in board and lodging, one-fifth in dress and one-tenth in sundries, and has y£ left at the end of the year.

30. A tradesman makes in a year a profit of x per cent. on his capital of y£ and has z£ at the end of the year.

31. A man gains x per cent. on a£ and loses y per cent. on b£, and altogether makes a profit of c£.

32. A man runs a miles at x miles an hour, b miles at y miles an hour, and c miles at z miles an hour, and takes d hours over the whole journey.

33. A man is hired for x days. He is paid y shillings a day for a days, and is fined z shillings a day for the rest of the time because he absents himself. He receives c£.

CHAPTER XXIX

PROBLEMS INVOLVING QUADRATIC EQUATIONS

168. Example 1. A number of two digits is less than four times the product of its digits by 11, and the digit in the tens' place exceeds the digit in the units' place by four. Find the number.

Let x be the digit in the units' place.

Then $x+4$ is the digit in the tens' place.

The number $=10(x+4)+x=11x+40$.

Four times the product of its digits $=4x(x+4)$;

$$\therefore\ 4x(x+4)-(11x+40)=11,$$
$$4x^2+16x-11x-40=11,$$
$$4x^2+5x-51=0,$$
$$(x-3)(4x+17)=0,$$
$$x=3 \text{ or } -\frac{17}{4}.$$

$\therefore$ 3 is the digit in the units' place, and $3+4(=7)$ the digit in the tens' place.

73 is therefore the reqd. number.

The solution $-\frac{17}{4}$ is inadmissible, because the digits of a number are *positive integers*.

Example 2. A reduction of 2 pence a dozen in the price of eggs will give 6 more for three shillings and sixpence: find the price per dozen.

Let x pence be the price of 12 eggs.

For 42 pence we obtain $\frac{12}{x}\times 42$ eggs.

When $x-2$ pence is the price of 12 eggs, we obtain $\frac{12}{x-2}\times 42$ for 3s. 6d.

$$\therefore\ \frac{12}{x-2}\times 42-\frac{12}{x}\times 42=6.$$

$$\frac{84}{x-2}-\frac{84}{x}=1,$$

$$84x-84(x-2)=x^2-2x,$$

$$x^2-2x-168=0,$$

$$(x-14)(x+12)=0,$$

$$x=14 \text{ or } -12.$$

$\therefore$ 14 pence a dozen is the reqd. price.

Example 3. A train does a journey of 240 miles at a uniform rate; if it had travelled 4 miles an hour slower, it would have taken 2 hours more over the journey: find its rate of travelling.

Let x miles an hour be the reqd. rate of travelling.

At the higher speed, the train took $\frac{240}{x}$ hours over the journey.

At the slower speed, $x-4$ miles an hour, it took $\frac{240}{x-4}$ hrs. over the journey.

$$\therefore \text{ by hypothesis, } \frac{240}{x}=\frac{240}{x-4}-2.$$

Multiplying up,

$$240(x-4)=240x-2x(x-4),$$

$$2x^2-8x-960=0,$$

$$x^2-4x-480=0,$$

$$(x-24)(x+20)=0;$$

$$\therefore x=24 \text{ or } -20.$$

$\therefore$ the train travels at the rate of 24 miles an hour, the negative solution being inadmissible.

It will be proved later on that every quadratic equation has two roots. As a consequence of this, inadmissible solutions of problems involving quadratic equations will often occur.* In this case, the negative solution would imply that the train travelled *backwards* at 20 miles an hour.

Example 4. A man invests his money at compound interest for two years at a certain rate per cent. and finds that he receives 5 shillings per cent. more than if he had invested it at simple interest. Find the rate per cent.

Let x be the rate per cent.

At compound interest, 100£ amounts to $(100+x)$£ in the first year.

The interest on $(100+x)$£ for the second year $=(100+x)\times\frac{x}{100}$.

$\therefore$ the interest on £100 for the two years $=x+\frac{(100+x)x}{100}$.

At simple interest, the interest on 100£ for the two years $=2x$.

$$\therefore x+\frac{(100+x)x}{100}=2x+\tfrac{1}{4},$$

whence

$$x^2=25,$$

and

$$x=\pm 5.$$

$\therefore$ 5 per cent. is the reqd. rate of interest.

Example 5. Two pipes running together will fill a cistern in $6\frac{2}{3}$ minutes. If one pipe, running alone, took a minute less to fill the cistern, and the other pipe, running alone, took 2 minutes more to do the same, then the two, running together, would fill the cistern in 7 minutes. Find in what time the cistern will be filled by each pipe running alone.

Let the first pipe, when running alone, fill the cistern in x minutes, and let the second pipe .... y ....

When running alone, the first pipe fills $\frac{1}{x}$ of the cistern in one minute

..................................second..........$\frac{1}{y}$ ....

But since by hypothesis they running together fill the cistern in $\frac{20}{3}$ min.

$\therefore$ in one minute .... $\frac{3}{20}$ of the cistern;

$$\therefore \frac{1}{x}+\frac{1}{y}=\frac{3}{20}. \qquad\qquad (1)$$

In the second case, the first pipe fills the cistern in $x-1$ min.

.... second .... $y+2$ ......

$$\therefore \frac{1}{x-1}+\frac{1}{y+2}=\frac{1}{7}. \qquad\qquad (2)$$

From (1),
$$\frac{1}{x}=\frac{3}{20}-\frac{1}{y}=\frac{3y-20}{20y}.$$

$$\therefore x=\frac{20y}{3y-20}. \qquad\qquad (3)$$

From (2),
$$\frac{1}{x-1}=\frac{1}{7}-\frac{1}{y+2}=\frac{y-5}{(7y+2)},$$

$$x-1=\frac{(7y+2)}{y-5}. \qquad\qquad (4)$$

From (3) and (4),
$$\frac{20y}{3y-20}-1=\frac{7(y+2)}{y-5}.$$

From this quadratic for y, $y=12$ will be found to be the only admissible solution.

Substituting in (3), $x=15$.

$\therefore$ the pipes would fill the cistern in 15 and 12 minutes respectively.

Examples. XXIX. a.

1. The difference of two numbers is 2, and the sum of their squares is 244 : find them.

2. A room is 4 feet more in length than in breadth, and its area is 192 sq. ft. : find its dimensions.

3. The product of two consecutive even numbers is 288. What are they ?

4. Find two consecutive numbers such that the sum of their squares is 481.

5. x yards of cloth at $x-3$ shillings per yard were bought for 13*s.* 9*d.* What was x?

6. What number when increased by 30 will be less by 12 than its square?

7. Find the number which, added to its square root, will make 182.

8. The length of a rectangular field is twice its breadth. If 20 yds. were added to its length and 30 to its breadth, its area would be 10,458 sq. yds. Find the dimensions of the field.

9. In a right-angled triangle one of the sides containing the right angle is 3 feet in length, and the square on the hypotenuse is 4 times the area of the triangle. Find the length of the remaining side.

10. A man bought x oxen for £120. Another bought 3 more for the same money. What was the cost of an ox to the first man, what to the second? If the difference was £2 per ox, what were the numbers bought?

11. A rectangular table 9 ft. by 6 ft. has a rectangular table-cloth which hangs down to the same depth at the ends and sides. What is that depth if the area of the cloth is twice that of the table?

12. The product of two numbers which differ by 3 is 40: find them.

13. When 13 times a certain number is subtracted from the square of the number, the result is 30. Find the number.

14. A motor-car does a journey of 192 miles at the average rate of x miles per hour, and a second car does the same journey at the average rate of $x+4$ miles per hour. How long does each car take over its journey?

If the difference of these times is 4 hours, find the value of x.

15. The difference of two numbers is 3, and the sum of their squares is 117. Find the numbers.

16. A man rents x acres of land for £54 per annum. How much does he pay per acre? If he sublets all except 8 acres at 5*s.* per acre more than this and receives £64 per annum, find the value of x.

17. A rectangular enclosure has an area of 2000 sq. yds., and its perimeter is 180 yds. in length. Find the lengths of its sides.

18. A man rows 6 miles down stream at x miles per hour, and the same distance up stream at $x-1$ miles per hour. How long does he take over each journey? If he takes $3\frac{1}{2}$ hours over the two journeys, find the value of x.

19. If the hind wheel of a carriage is x ft. in circumference, how many revolutions does it make in a mile? If the front wheel is 2 ft. smaller in circumference, and makes 24 more revolutions in a mile than the hind wheel, find the value of x.

20. A train travelling at x miles an hour for $x+12$ minutes goes 21 miles. Find x.

21. A bill of 80 shillings was shared equally between x persons. What did each pay? If two were excused, what would each pay? If this made a difference of 2 shillings to each, what was x?

22. 110 bushels of coals are equally divided among x poor persons. What number of bushels does each receive ? If this number is one less than the number of persons, how many are there ?

23. Two trains each run a distance of 330 miles, one at x miles per hour, the other at $x+5$. The faster takes half an hour less than the other for the whole distance. What are their speeds ?

24. A can do a piece of work in x days, B in $x+12$ days. What fraction of the work can they respectively do in a day ? If together they take 8 days, what times will they take separately ?

25. A cistern can be filled by two pipes in $1\frac{1}{3}$ hours. The larger pipe by itself will fill the cistern sooner than the smaller by 2 hours. Find what time each will take separately.

26. A car travels 15 miles an hour faster downhill than uphill, and takes $2\frac{1}{10}$ minutes to run up and down a hill one-quarter of a mile long, when the time taken in turning is deducted. Find its speed downhill.

27. A fraction, whose numerator is less than its denominator by 3, is doubled if 6 is added to the numerator and 5 to its denominator. Find its value.

28. The product of the two highest of five consecutive integers exceeds twice the product of the two smallest by 6. Find them.

29. The tens digit of a certain number is the square of a number which is 2 less than the units digit, and the sum of the two digits is 14. Find the number.

30. A rectangle whose area is 54 sq. ft. has its sides respectively diminished by 5 feet and 2 feet and so becomes a square. Find the length of a side of the square.

31. A train does a journey of 288 miles at a certain average speed and is one hour late. If it had travelled 4 miles per hour faster it would have been punctual. Find its speed.

32. A point travels for 8 secs. at the rate of x feet per sec., and then for $4x$ secs. at the same rate. If the total space described is 96 feet, find the value of x.

*Examples. XXIX. b.

1. Find two numbers whose difference is 2, such that twice the square of the less shall exceed the square of the greater by unity.

2. The plate of a looking-glass is 18 inches by 12 inches. It is to be framed with a frame of uniform width, the area of which is to be equal to that of the glass. Find the width of the frame.

3. Mr. Gladstone was born in the year A.D. 1809. In the year A.D. x^2 he was $x-3$ years old : find x.

4. When 17 times a certain number is subtracted from twice its square, the remainder is 84 : find the number.

5. The tens digit of a certain number is the square of the units digit, and the sum of its two digits is 12 : find the number.

6. A man runs 600 yards at a certain pace, and then doubling his pace, does another 600 yards. If he took $2\frac{1}{2}$ minutes over the 1200 yards, find the pace he started at, in yards per second.

7. Find two numbers whose difference is 3, and the sum of whose squares is 317.

8. A's rate of travelling is one mile an hour less than B's, and B can go 21 miles in 20 minutes less than it takes A to go 20 miles. How many miles an hour can A travel?

9. Find a number which together with its square amounts to 56.

10. Two trains each run a distance of 330 miles. One of them, whose average speed exceeds that of the other by 5 miles an hour, takes half-an-hour less to travel the whole distance. Find their average speeds.

11. A lady bought 28 yards of linen and a certain length of silk. The whole cost was 65*s*., the silk cost as many shillings per yard as there were yards of it, and 8 times as much as the same number of yards of linen. Find the price of the silk per yard.

12. P rides from A to B in one hour at a uniform speed. Q rides for one-third of the way 2 miles an hour faster than P, and for the rest of the journey 1 mile an hour slower than P, thus taking 40 seconds longer. Find the distance from A to B.

13. A person rents some land for £48. He cultivates 8 acres himself, and sub-letting the rest for 15*s*. per acre more than he pays, receives in rent £54 per annum. Find the number of acres.

14. One side of a room is 6 ft. longer than the other, and 924 square feet of paper are required to cover its walls. Now if the room were 3 feet higher, the same amount of paper would be required to cover three of its walls, one of the shorter walls being left uncovered. Find the dimensions of the room.

15. Of two square courtyards one contains as many square yards as it costs shillings to pave the other, and a side of the second contains as many linear yards as it costs pounds to pave the first, also the length of a side of the first exceeds that of the second by 3 yards, and the cost of paving the first exceeds that of paving the second by £2. Find the sizes of the courtyards, and the costs of paving.

16. Ten minutes after the departure of an express train a slow train is started, travelling on the average 20 miles less per hour, which reaches a station 250 miles distant $3\frac{1}{3}$ hours after the arrival of the express. Find the rate at which each train travels.

17. The length of a room is 2 feet more than its breadth, and its height is three-quarters of its breadth. If the area of the ceiling be 42 square feet more than that of the longer side, find the dimensions of the room.

18. A bicyclist, having ridden 72 miles and stopped an hour on the way, finds that, if he had ridden faster by one mile an hour and stopped two hours on the way, he would have accomplished the journey in the same time. At what pace did he ride?

19. In 100 minutes a boat's crew row $3\frac{1}{2}$ miles down a river and back again. If the river runs at 2 miles an hour, what is the pace of the boat in still water?

20. In going a quarter of a mile along a straight road the hind wheel of a bicycle turns 11 times more than the front wheel. Had the front wheel been 3 inches longer in circumference than it actually is, the hind wheel

would have turned 16 times more than the front wheel. Find the circumference of each wheel.

21. A battalion of soldiers when formed into a solid square present sixteen men fewer in the front than they do when formed into a hollow square four deep. Find the number of men.

22. A man buys pigs, geese, and ducks. If each of the geese had cost a shilling less, one pig would have been worth as many geese as each goose is actually worth shillings. A goose is worth as much as two ducks, and 14 ducks are worth seven shillings more than a pig. Find the price of a pig, a goose, and a duck respectively.

23. A sum of money is divided among A, B, and C, so that a third of the whole sum exceeds A's share as much as B's exceeds a quarter of the whole. What part does C get ?

24. A cyclist rides 3 miles an hour faster downhill than uphill ; and takes the same time to ride 22 miles downhill and 48 miles uphill that he takes to ride 50 miles downhill and 27 miles uphill. What is his speed uphill ?

25. A carrier charges 3*d.* each for all parcels not exceeding a certain weight ; and on heavier parcels he makes an additional charge for every 7 lbs. above that weight. The charge for half a cwt. is 1*s.* 3*d.*, and the charge for 9 stones is five times that for 1 qr. What is the scale of charges ?

26. A boat's crew row a certain distance against the stream in $8\frac{4}{7}$ minutes. If there were no current they would row the distance in 7 minutes less than it takes them to drift the distance down the stream. In what time would they row the course down the stream ?

27. A man being asked his age, answered, 'If you multiply my two digits together, the number formed will be my age 22 years ago, and if you add all the digits of the two ages you will have one-third of my present age.' How old is he ?

28. Three travellers A, B, C make the same journey. A's rate of travelling is 3 miles an hour greater than B's, and B's rate is 2 miles an hour greater than C's. A accomplishes the journey in 3 hours less time than B, and B in 4 hours less time than C. Find the rate of each, and the length of the journey.

29. A giant weighs 3 lbs. for every inch of his height, and the square of his height in feet exceeds his weight in stones by 31. Find his height and weight.

30. A labourer undertakes to carry a load a certain distance, agreeing to take one shilling for each cwt. moved one mile. He earns 4·05£, and the distance in miles exceeds the number of cwts. carried by 4·05. Find the load and the distance.

31. A rectangular enclosure is half an acre in area, and its perimeter is 201 yards. Find the lengths of its sides.

32. The sum of two numbers is six times their difference, and their product exceeds twice their sum by 11. Find the numbers.

33. If the longer side of a rectangle be increased by 3 yards, and the shorter by 2 yards, one side becomes double the other, and the area is doubled. Find the lengths of the sides.

34. A lawn, rectangular in shape, contains 864 square yards ; if it were 4 yards longer and 3 yards narrower its area would be the same. Find its dimensions.

35. The circumference of one wheel is 8 inches longer than that of another, and the first makes 72 fewer revolutions in a mile: find the circumference of each.

36. A slow train takes 5 hours longer in journeying between two given termini than an express, and the two trains when started at the same time, one from each terminus, meet 6 hours afterwards. Find how long each takes in travelling the whole journey.

37. The area of a rectangular room is 328 square feet, and its perimeter is 73 feet: find the lengths of its sides.

38. A boat's crew finds that the number of minutes which they just require to row 4 miles in a river against the stream exceeds by 31 the number of miles per hour they can row in still water; while it takes them 20 minutes to row the 4 miles with the stream. Find the rate at which the river flows.

39. In a mixed number the integer is 98 times the fraction. The numerator of the fraction being unity, and its denominator less by 7 than the integer, find the mixed number.

40. Two men start simultaneously from opposite ends of a road and meet at the end of 6 minutes. They pass one another, and each continuing to the end from which the other started, one ends his walk 5 minutes before the other. How long does each take?

41. A, B, and C walk from P to Q, a distance of 30 miles; A starts $2\frac{1}{2}$ hours before B, and B $1\frac{1}{2}$ hours before C, and they arrive at Q together. If B had started half-an-hour earlier, he would have passed A 2 hours before A reached Q. Find the rates at which A, B, and C walk.

42. A grocer has two weights, one as much over a lb. as the other is under a lb., and he finds that on selling 511 lbs. 14 ozs. of tea at 2*s.* 6*d.* a pound he gets £2 more by using the lighter weight than he would have done by using the heavier: what were the respective weights?

43. A gentleman arrives at the railway station nearest to his house an hour and a half before the time at which he had ordered his carriage to meet him. He sets out at once to walk at the rate of 4 miles an hour, and meeting his carriage when it had travelled 2 miles, reaches home exactly an hour earlier than he had originally expected. How far is his house from the station, and at what rate was his carriage driven?

44. The figures which express the pounds and the pence in a certain sum of money will change places if £2 19*s.* 9*d.* be added to it, and those which express the shillings and the pence would be interchanged by subtracting 2*s.* 9*d.* What alteration would be produced in the sum of money by interchanging the figures which express the pounds and shillings?

45. Two cyclists travel, one from A to B, the other from B to A, by the same road, and at uniform speeds. They start at the same moment. One reaches B $2\frac{1}{2}$ hours, the other reaches A 3 hours 36 minutes after they meet. How long was each on the journey?

46. A and B walk from one town to another. After walking 6 miles at a uniform speed A arrives at the top of a slope where he mends his pace by 1 mile an hour. B starts forty minutes later, and, after walking at a uniform speed, reaches the slope 10 minutes later than A: here increasing his speed by $\frac{1}{2}$ a mile an hour, he overtakes A just as the town is reached. A would have covered the distance in half an hour less, had he walked the whole distance with B's initial speed. Find the distance and the speeds.

47. Two towns A, B are connected by two roads, one of which is twice as long as the other. A man walked by the shorter road from A to B, and returning immediately by the longer road met one mile from B another man who started at the same time from A on a tricycle and travelled 3 miles an hour faster; and when he had walked 2 hours longer he again met the tricyclist who had passed through B and A without stopping. Find the lengths of the two roads, and the rate at which each man travelled.

48. What fraction will be increased by $\frac{1}{38}$ when unity is added to both numerator and denominator, and diminished by $\frac{8}{57}$ when 4 is subtracted from each of them?

49. A railway passenger observed the time of transit over three successive miles, and finds that the time for the first mile exceeds the time for the second by twice as much as the time for the second exceeds the time for the third. He also calculates that the average speed for the train in the first mile is 5 miles per hour less than in the second, and 8 miles per hour less than in the third. Find the time of traversing each of the three miles.

50. A cask A, of 20 gallons capacity, is filled with brandy, a certain quantity of which is afterwards drawn off into an equal cask B, which is then filled up with water. After this, A is filled up with some of the mixture in B; and when $6\frac{2}{3}$ gallons of the mixture now in A is poured back into B, the two casks contain equal quantities of brandy. How much was at first taken out of A?

CHAPTER XXX

EXAMPLES FOR REVISION

XXX. a. *(Oral.)*

Read off the square root of

1. $\cdot 25a^6b^2$. **2.** $\cdot 0001\dfrac{x^6}{y^2}$. **3.** $\dfrac{2\cdot 5}{10}x^4y^2$. **4.** $\dfrac{x^{10}}{\cdot 0064}$.

5. $4a^2-8ab+4b^2$. **6.** $\dfrac{1}{x^2-6x+9}$. **7.** $4x^2\pm 12xy+9y^2$.

8. $1\pm 4a^2b+4a^4b^2$. **9.** $x^2\pm 2+\dfrac{1}{x^2}$. **10.** $x^2\pm\dfrac{5ax}{2}+\dfrac{25a^2}{16}$.

11. $1\pm 2(a-b)+(a-b)^2$. **12.** $\left(\dfrac{a}{b}-2\right)^2+4\left(\dfrac{a}{b}-2\right)+4$.

13. $(x+5y)^2-10y(x+5y)+25y^2$. **14.** $(a+b)^2+2(a^2-b^2)+(a-b)^2$.

15. $4x^4\pm 2+\dfrac{1}{4x^4}$. **16.** $4x^4\pm 4+\dfrac{1}{x^4}$.

Read off the roots of the following quadratic equations:

17. $x^2-9x+20=0$. **18.** $x(x+3)=x+3$. **19.** $(x-4)(x-5)+2(x-5)=0$.

20. $(x^2-16)+(x-4)=0$. **21.** $x^2+5x=0$. **22.** $25x^2-16=0$.

23. $x(2x+1)-\frac{1}{3}(2x+1)=0$. **24.** $3x(4x-5)=7(4x-5)$.

Read off the roots of the following quadratic equations :

25. $3x(2x-3)+\frac{1}{3}(2x-3)=0.$

26. $3(x-a)+x(x-a)=0.$

27. $x-2+\dfrac{1}{x}=0.$

28. $7(5x-7)=\dfrac{3x}{2}(5x-7).$

29. $(x-1)^2=9.$

30. $x+2+\dfrac{1}{x}=0.$

31. $2x-2+x(x-1)=0.$

Find, by inspection, *one root* in each of the following equations :

32. $2x-2+(7x-3)(x-1)=0.$

33. $\dfrac{2x-3}{7}+\dfrac{27x}{17}(6x-9)=0.$

34. $\dfrac{13x}{11}(2x-1)-5(x-\frac{1}{2})=0.$

35. $7(3x-6)+11x(2x-4)-21x(5x-10)=0.$

36. $\dfrac{3x}{7}\left(3x-\dfrac{3}{2}\right)+(11x+14)\left(7x-\dfrac{7}{2}\right)=0.$

37. $\dfrac{5x-1}{x-7}+\dfrac{2x-\frac{2}{5}}{x+3}=0.$

XXX. b.

1. Simplify $\dfrac{a}{2x+3a}-\dfrac{a}{3a-2x}-\dfrac{4ax}{8x^2-18a^2}.$

Deduce the solution of the equation formed by equating the expression to zero. Test your result.

2. Write down (*a*) the square root of $(a+b)^2-2(a+b)+1,$
(*b*) the square of $a+b-c,$
(*c*) the cube of $a+b.$

3. Solve the equation $4x+\dfrac{3}{x-1}+4=0.$ Test your answer.

4. Draw enough of the graph of $y=x^2$ to determine $\sqrt{8}$ and $\sqrt{13}$. Use one inch as x unit and one-tenth of an inch as y unit.

5. Solve the equations $3x-7y=2,\ xy=3.$

6. Use the remainder theorem to prove that $x-a+b$ is a factor of
$$(x-a)^2+(2b-c)(x-a)+b^2-bc.$$

7. Find a fraction which becomes equal to $\frac{1}{2}$ if the numerator is increased by 2, and equal to $\frac{1}{3}$ if its denominator is increased by 3.

XXX. c.

1. Simplify $\dfrac{1}{x^2-ax+bx-ab}+\dfrac{1}{x^2-ax-bx+ab}.$ Check your result.

2. Determine values of a which will make $x^2-ax+25$ a complete square.

3. Solve the quadratic $x-4=1-\dfrac{14}{x+4}.$ Check your result.

4. Find the square root of $25x^4-70x^3+89x^2-56x+16.$

5. Draw the graph of $y=5x-x^2$. From your figure determine the value of x which gives $5x-x^2$ a maximum value. What is the value of y in this case ? Test your results algebraically.

6. Solve the equations $x^2+y^2=25,\ x+y=7$ graphically and by algebra.

7. Between one census and the next the native population of a town increased by 8 per cent., while the number of foreigners decreased from 200 to 150. The increase in the total population was 7 per cent. What was the total population at the second census ?

XXX. d.

1. Simplify $\dfrac{2a}{a+2b}+\dfrac{3a}{a-3b}+\dfrac{8a^2}{(6b-2a)(a+2b)}$.

2. Write down (i) the square root of $(x^2-x)^2-8(x^2-x)+16$.
(ii) the square of $a-2b+c$.
(iii) the cube of $a+2b$.

3. Using half an inch as x unit, and one-tenth of an inch as y unit, draw the graph of $y=x^2-3x+2$, for integral values of x, from -2 to 5. What do you deduce as to the equation $x^2-3x+2=0$? Give reasons.

4. Draw enough of the graph $y=x^2$ to determine the square roots of 54·8 and 58·5, correct to two decimal places. Use a large x unit.

5. Solve the equations $\dfrac{2}{x}-\dfrac{1}{y}=\dfrac{5}{12}$, $xy=12$.

6. Find the values of a which will make the expression

$$8x^3+a^2x^2-10ax-48$$

exactly divisible by $x-2$.

7. A clock is two minutes slow but is gaining. If it were three minutes slow, but were gaining half a minute a day more than it does, it would show correct time exactly 24 hours sooner. How much does the clock gain in a day ?

XXX. e.

1. Simplify $\dfrac{2-x}{3-2x-x^2}-\dfrac{x-3}{x^2+x-2}$.

2. What values of a will make $9x^2+axy+4y^2$ a complete square ?

3. Solve the quadratic $6(x^2-2)=x$, by completing squares, and verify your results by means of the formula for solving quadratic equations.

4. Determine graphically between what values of x the expression $35-4x-4x^2$ is positive. Verify your result by algebra.

5. Solve the equations $3x^2+4xy=11$,
$4y^2+3xy=22$.

6. Find the square root of $16x^4-16x^3+4x^2+8x-4+\dfrac{1}{x^2}$.

7. A sum of money is distributed among some children, each child receiving the same amount. If a shilling less had been given to each, 36 more children could have participated; and if a shilling more had been given to each, the number of children would have had to be reduced by 20. Find the sum distributed.

XXX. f.

1. Simplify $\dfrac{6x^2+x-1}{2x^2-5x-12}\times\dfrac{6x^2+11x+3}{9x^2-1}\div\dfrac{2x^2+9x+4}{x^2-16}$.

2. Prove that $x-a$ is a factor of $x^3-(a+b-c)x^2+(ab-bc-ca)x+abc$.

3. Solve, graphically, the equation $2x^2+x-13=0$. Get your results correct to one decimal place, and check your answer.

4. Find the maximum value of $7x-x^2$, and the minimum value of x^2-5x.

5. Solve the equations $x^2-5xy-14y^2=10$,
$x-7y=1$.

6. If $a^2=b^2+c^2$, prove that $(a+b+c)(b+c-a)(a+c-b)(a+b-c)=4b^2c^2$.

7. A fruiterer sold a certain quantity of oranges for £6. 10*s*. If he had given two more oranges for a shilling, the same quantity would only have realized £5. 17*s*. How many oranges did he sell ?

XXX. g.

1. Simplify $\dfrac{x^4+2x^2y^2+y^4}{x^4+x^2y^2+y^4}\times\dfrac{x^6-y^6}{x^4+x^2y^2}\div\left(1-\dfrac{y^4}{x^4}\right)$.

2. Prove that $(a-b)$, $(b-c)$, $(c-a)$ are factors of
$$a^4(b-c)+b^4(c-a)+c^4(a-b).$$

3. Solve the equation $4x^2-3x-12=0$ graphically and by algebra.

4. Use a geometrical method to find the value of $\sqrt{8}$.

5. Solve the equations $(x+2y)^2-3(x+2y)-28=0$,
$$x-2y=5.$$

6. Extract the square root of $x^4+1-12x(x^2+1)+38x^2$.

7. A man starts at 2 p.m. to walk to a place 13 miles off. He walks at a uniform speed till 4 p.m., when he increases his speed by one mile an hour, and reaches his destination at 5.30 p.m. At what speed did he walk during the first two hours ?

XXX. h.

1. Resolve into factors : (i) x^4-3x^2+9,
(ii) $512\left(x-\frac{1}{8}\right)^3-(8ax-a)^3$.

2. Simplify $$\frac{(a+b)x}{(x+a)(x-b)}+\frac{(b+c)x}{(x+c)(b-x)}.$$

3. Divide $(x^2-y^2)^3-z^6$ by $x^2-y^2-z^2$.

4. A certain port wine is worth 47*s*. a dozen now, and increases in value at the rate of 3*s*. a dozen per annum. Draw a graph to determine its worth in coming years, and read off its value per dozen in 7, 13, and 17 years.

5. Solve the equation $5x^2-5x-21=0$ graphically and by algebra, getting your results correct to one decimal place.

6. Solve the equations $x^2+y^2+1=3xy$,
$$2(xy+4)=3y^2.$$

7. One-fourth of the subscribers to a certain school gave a sovereign apiece, one-fourth of the remainder gave half-a-sovereign apiece, and the rest each gave a florin. If the three sets of subscribers raised their subscriptions to a guinea, half-a-guinea, and half-a-crown respectively, the total increase in the subscriptions would be £2. 10*s*. 0*d*. How many subscribers were there and what was the total amount subscribed ?

XXX. k.

1. Multiply $8a^5-12a^4b-54a^2b^3+243b^5$ by $2a+3b$, using the method of detached coefficients.

2. Express $\left(1-\dfrac{a^2+b^2-c^2}{2ab}\right)^2$ as a fraction with a numerator of four factors.

3. Solve the equation $\dfrac{4x-11}{x-3}-\dfrac{2x-17}{x-9}=\dfrac{3x-22}{x-7}-\dfrac{x-10}{x-9}$.

4. With the same axes draw the graphs of $y=x+4$ and $y=x^2$. Hence solve the equation $x^2-x-4+0$ as accurately as you can.

5. Two cyclists, riding 9 and 10 miles an hour respectively, start from two places 55 miles apart at noon towards one another. Find graphically, as accurately as you can, their time of meeting, and the times when they are 20 miles apart. Verify your results by algebra.

6. Solve the equations $(x+2y)^2+(2x-y)^2=85$, $xy=4$.

7. From two towns 445 miles apart, two cyclists start on Monday morning to meet each other. One travels at the rate of 48, the other at the rate of 57 miles a day. Find on what day they will meet.

XXX. l.

1. Multiply $2x^3-3x^2+4x-5$ by $3x^2+4x+5$.

2. Prove the identity $\dfrac{a}{ax-x^2}+\dfrac{b}{bx-x^2}+\dfrac{c}{cx-x^2}=\dfrac{1}{a-x}+\dfrac{1}{b-x}+\dfrac{1}{c-x}+\dfrac{3}{x}$.

3. Solve the equations $\dfrac{2}{x-3}+\dfrac{1}{y-2}=2$, $\dfrac{4}{x-3}+\dfrac{1}{y-2}=3$.

4. Solve the equations $x+y=7$, $xy=4$ by a geometrical method, as accurately as you can.

5. A cycles along a road starting at 15 miles an hour, but diminishing his pace by 3 m. an hour at the end of each hour. B starts at the same time, in the same direction, at 9 m. an hour, increasing his pace by one mile an hour at the end of each hour. Draw in one diagram a graph to give their positions at the end of each hour. Determine when and where they meet again, and how far apart they are in 5 hours.

6. Solve the equations
$$x^2-xy+y^2=21,$$
$$x^2-y^2=9.$$

7. A and B, who live p miles apart, start at the same time to visit each other. If A travel at the rate of q miles in an hour, and B at the rate of r miles in an hour, express in terms of p, q, and r the time which will elapse before they meet.

XXX. m.

1. Multiply $\dfrac{a^2-ab+b^2}{a^3-3ab(a-b)-b^3}$ by $\dfrac{a^2-b^2}{a^3+b^3}$.

2. Solve the equation $\dfrac{5x^2+x-3}{5x-4}=\dfrac{7x^2-3x-9}{7x-10}$.

3. Find the square root of $x^2+\dfrac{4a(x^2-3x+a)}{x^2-6x+9}$.

4. A man spends £75 in 64 days. Draw a graph to give his expenditure in any number of days. Write down his expenditure in 17, 35, and 49 days, to the nearest shilling.

5. Draw the graphs of $x^2+y^2-4x-8y=0$ and $2y-x=6$, in the same diagram, and hence solve the equations.

6. Solve the equations
$$(3x+y)^2-(3y+x)^2=24,$$
$$x^2+y^2=5.$$

7. A rectangular grass plot, 8 ft. longer than it is broad, is surrounded by a path 2 ft. 6 in. wide. The cost of making the path, at 1*s*. 6*d*. a square yard, is £3. 2*s*. 6*d*. Find the length and breadth of the plot of grass.

XXX. n.

1. Simplify $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^3-a^2b-ab^2+b^3}$.

2. Solve the equation $\dfrac{(1+x)^3}{1+x^3}=\dfrac{25}{13}$.

3. Resolve into factors (i) $(a^4-b^4)-(a+b)^2(a-b)^2+2b(a^3+b^3)$.
(ii) $x^3-10x^2+31x-30$.

4. Draw the graphs of $y=2x-x^2$, $2x+y=0$, and hence solve the equations.

5. Determine graphically the maximum value of $3-4x^2-12x$. Write down the value of x in that case, and verify your results by algebra.

6. Solve the equations $4x^2-6xy+y^2=11$,
$3y^2-2xy=14$.

7. A walks over a certain course and back again; B starting at the same time walks at half the pace of A over five-eighths of the course and back again. A passes B half a mile from the winning post: find the length of the course.

Solve the problem graphically or by algebra.

XXX. p.

1. Divide $ab(x^2+y^2)+(a^2+b^2)xy+(a-b)(x-y)-1$ by $ax+by-1$.

2. Solve the equation $6(x+4)^2+(x-4)^2=5(x^2-16)$.

3. Factorize (i) $a(a+b-c)(a-b+c)-b(b+c-a)(a+b-c)$.
(ii) $x^4-3x^2y^2+y^4$.

4. Draw the graph of $y=x^2-3x$, using a large x unit. Hence solve, as accurately as you can, the equation $x^2-3x=7$.

5. A, starting at noon, cycles 15 miles in the first hour, and diminishes his speed by 2 miles an hour at the end of each hour. B, starting at 2.30 p.m. in his motor car, catches him up at 4.30 p.m. How fast does B travel? Solve the problem graphically.

6. Solve the equations $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=13$,

$$\frac{1}{y}-\frac{1}{x}=1,$$

$$\frac{1}{xy}-\frac{2}{z}=0.$$

7. A woman has a fifth more apples than pears, but obtains a pound less for her apples when they sell at sixteen a shilling than for her pears, each of which is worth two apples. How many of each kind of fruit has she?

CHAPTER XXXI

LITERAL EQUATIONS

169. Instead of numerical coefficients, we sometimes have to deal with coefficients denoted by symbols whose values are supposed to be known. Such coefficients are called literal.

The methods of solution are the same as in dealing with numerical coefficients.

Simple Equations. (One unknown.)

Example 1. Solve the equation

$$\frac{x-a}{a-b}-\frac{x+a}{a+b}=\frac{2ax}{a^2-b^2}.$$

Multiplying both sides by a^2-b^2,

$$(x-a)(a+b)-(x+a)(a-b)=2ax.$$

Removing brackets, and transposing,

$$x(a+b-a+b-2a)=a^2-ab+a^2+ab,$$

$$2x(b-a)=2a^2.$$

Dividing both sides by $2(b-a)$,

$$x=\frac{a^2}{b-a}.$$

Examples. XXXI. a.

Solve the equations:

1. $\dfrac{x+a}{x-b}=1-\dfrac{x}{x-b}$.

2. $\dfrac{a}{bx}-\dfrac{b}{ax}=a^2-b^2$.

3. $\dfrac{a-b}{x-c}=\dfrac{a+b}{x+c}$.

4. $\dfrac{x}{a-2b}=2+\dfrac{x}{2a-b}$.

5. $\dfrac{acx}{b}+\dfrac{abx}{c}-\dfrac{1}{abc}=\dfrac{1}{abc}(1-b^2c^2x)$.

6. $\dfrac{x+a}{x-c}+\dfrac{x+c}{x-a}=2$.

7. $x-\dfrac{ax}{a+b}+a=\dfrac{a^2}{a-b}-\dfrac{b^2x}{a^2-b^2}$.

8. $\dfrac{x}{a+c}=\dfrac{x+1}{a+b+c}$.

9. $(x-a-b)^2=x^2-(a-b)^2$.

10. $\dfrac{3x}{a}+2b(a-c)+\dfrac{x}{b}=c(a+b)+\dfrac{2x}{c}$.

11. $\dfrac{1}{x-a}+\dfrac{1}{x-b}=\dfrac{2}{x}$.

12. $\dfrac{q-r}{x-p}+\dfrac{r-p}{x}+\dfrac{p-q}{x-r}=0$.

13. $\dfrac{x-2a}{x+2a}=\dfrac{x-a}{x+a}$.

14. $\dfrac{x-b}{x-a}-\dfrac{x-a}{x-b}=\dfrac{2(a-b)}{x-(a+b)}$.

15. $\dfrac{3\{ab-x(a+b)\}}{a+b}+\dfrac{(2a+b)b^2x}{a(a+b)^2}=\dfrac{bx}{a}-\dfrac{a^2b^2}{(a+b)^3}$.

Solve the equations :

16. $\dfrac{(a^2-1)(ax+1)}{a^2(x+a)}+\dfrac{(a^2+1)(x-a)}{ax+1}=\dfrac{ax+1}{x+a}+\dfrac{a(ax-1)}{ax+1}$.

17. $\dfrac{x}{ax+b}+\dfrac{x}{a+bx}=\dfrac{a+b}{ab}$. 18. $\dfrac{x-a}{x-b}+\dfrac{x-c}{x-d}=2$. 19. $\dfrac{x+2a}{x-2b}=\left(\dfrac{x+a}{x-b}\right)^2$.

20. $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{a+b}{x+a+b}$. 21. $\dfrac{x-2b}{a+b}+\dfrac{x-b}{a+2b}=\dfrac{2(x-a)}{3b}$.

22. $(x+a)(x+b)+(x+b)(x+c)=(x+c)(x+d)+(x+d)(x+a)$.

23. $\dfrac{1}{x-a}-\dfrac{1}{x-a+c}=\dfrac{1}{x-b-c}-\dfrac{1}{x-b}$.

24. $\dfrac{1}{x-a}-\dfrac{1}{x-b}=\dfrac{a-b}{x^2-ab}$. 25. $\dfrac{ax}{x-b}+\dfrac{bx}{x-a}=a+b$.

Simple Simultaneous Equations.

***170. Example 1.** Solve the equations $ax+by=p$ ....(1)

$bx-ay=q.$ ....(2)

Multiplying (1) by a and (2) by b,

$$a^2x+aby=ap,$$
$$b^2x-aby=bq.$$

Adding, $$x(a^2+b^2)=ap+bq,$$
$$x=\frac{ap+bq}{a^2+b^2}.$$

Instead of substituting for x to find the value of y, it will be simpler to eliminate x from the given equations.

Multiplying (i) by b and (2) by a,

$$abx+b^2y=bp,$$
$$abx-a^2y=aq.$$

Subtracting $$y(a^2+b^2)=bp-aq,$$
$$y=\frac{bp-aq}{a^2+b^2};$$

$$\therefore\ x=\frac{ap+bq}{a^2+b^2},\quad y=\frac{bp-aq}{a^2+b^2}\text{ is the reqd. solution.}$$

Example 2. Solve the equations

$$\frac{x}{a}+\frac{y}{b}=1, \text{ ....(1)}$$

$$\frac{x}{b}+\frac{y}{a}=1. \text{ ....(2)}$$

Subtracting, $$x\left(\frac{1}{a}-\frac{1}{b}\right)+y\left(\frac{1}{b}-\frac{1}{a}\right)=0,$$

i.e. $$x\left(\frac{1}{a}-\frac{1}{b}\right)-y\left(\frac{1}{a}-\frac{1}{b}\right)=0;$$

$$\therefore\ x=y.$$

Substituting in (1) or (2),

$$x\left(\frac{1}{a}+\frac{1}{b}\right)=1,$$

$$x=\frac{ab}{a+b}=y.$$

*Examples. XXXI. b.

Solve the equations:

1. $3(x-a)-2(y+a)=5-4a,$
 $2(x+a)+3(y-a)=4a-1.$
2. $(a+b)x+cy=bc,\ (b+c)y+ax=-ab.$
3. $ax+by=3(a^2+b^2),\ x+4b=y+2a.$
4. $ax+by=s,\ ax-by=t.$
5. $ax-by=a^2,\ bx-ay=b^2.$
6. $ax+by=a^2+2ab-b^2,\ bx+ay=a^2+b^2.$
7. $(a+b)x+(c+d)y=bc-ad,\ (a-b)x+(c-d)y=ad-bc.$
8. $\dfrac{x}{b-c}+\dfrac{y}{c-a}=\dfrac{1}{a-b},\ \dfrac{x}{c-a}+\dfrac{y}{a-b}=\dfrac{1}{b-c}.$
9. $a(x+y)-b(x-y)=2a,\ (a^2-b^2)(x-y)=4ab.$
10. $ax-by=2ab,\ 2bx+2ay=3b^2-a^2.$
11. $x(b-c)+by-c=0,\ y(c-a)-ax+c=0.$
12. $axy=c(bx+ay),\ bxy=c(ax-by).$
13. $c^2x+2a^2y=(c+a)(cx+2ay)=(c-a)^2.$
14. $axy+b=(a+c)y,\ bxy+a=(b+c)y.$
15. $\dfrac{x}{a+b}+\dfrac{y}{a-b}=\dfrac{a^2+b^2}{a^2-b^2},\ \dfrac{x}{b}+\dfrac{y}{a}=\dfrac{a^2+b^2}{ab}.$
16. $\dfrac{a}{x}+\dfrac{b}{y}=p,\ \dfrac{b}{x}+\dfrac{a}{y}=q.$
17. $(a-b)x+(a+b)y=2(a^2-b^2),\ ax-by=a^2+b^2.$
18. $ax+y=c,\ x+by=d.$
19. $ab(bx-ay)=c(a-b)(a^2+ab+b^2)=c(a^2x-b^2y).$
20. $\dfrac{2x-y}{10a+3b}=\dfrac{x-3y}{4b}=\dfrac{y+b}{2a}.$
21. $(a^2-1)x-2ay=a,\ 2ax+(a^2-1)y=1.$
22. $by+cz=a,\ cz+ax=b,\ ax+by=c.$
23. $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c},\ lx^2+my^2+nz^2=1.$
24. $a(y+z)=yz,\ b(z+x)=xz,\ c(x+y)=xy.$

QUADRATIC EQUATIONS

171. When the equation has been simplified, the factors can generally be seen by inspection.

Example 1. Solve the equation $x^2-3ax-18a^2=0.$

Factorizing,
$$(x-6a)(x+3a)=0;$$
$$\therefore\ x=6a \text{ or } -3a.$$

Example 2. Solve the equation $ax(x-1)+b(x+1)=2b$.

Removing brackets and re-arranging,

$$ax^2+x(b-a)-b=0.$$

Factorizing, $$(ax+b)(x-1)=0;$$

$$\therefore\ ax+b=0 \text{ or } x-1=0,$$

$$x=-\frac{b}{a} \text{ or } 1.$$

Examples. XXXI. c.

Solve the equations:

1. $x^2-2ax=15a^2$.
2. $x(5a-x)=6a^2$.
3. $bx\left(a-\frac{1}{x}\right)-c\left(a-\frac{1}{x}\right)=0$.
4. $x^2-(a+b)x+ab=0$.
5. $x^2-2ax+a^2=\frac{1}{a^2}$.
6. $px\left(x-\frac{1}{a}\right)+q\left(x-\frac{1}{a}\right)=0$.
7. $\frac{p-x}{p-a}=\frac{p+a}{p+x}$.
8. $\frac{a^2x^2}{b^2}+1=\frac{2ax}{b}$.
9. $abx^2+1=(a+b)x$.
10. $\frac{abx^2-1}{a-b}=x$.
11. $ax(x-3b)+2(x+2b)ab=16ab^2$.
12. $\frac{a^2x^2}{f^2}-\frac{2ax}{g}+\frac{f^2}{g^2}=0$.
13. $\frac{1}{2}(x+a)^2-\frac{1}{3}(2x-a)^2=\frac{19a^2}{24}$.
14. $\frac{1}{2x-5a}+\frac{5}{2x-a}=\frac{2}{a}$.
15. $x^2-2bx=4a^2+4ab$.
16. $4ax+b^2=4x^2+a^2$.
17. $(a^2-b^2)(x^2+1)=2(a^2+b^2)x$.
18. $\frac{a^2(x-b)}{a-b}+\frac{b^2(x-a)}{b-a}=x^2$.
19. $\frac{a}{x+a-1}+\frac{1}{x-a+1}=\frac{a}{x-1}$.
20. $\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=0$.
21. $4x^2-4ax+a^2=\frac{1}{b^2}$.
22. $\frac{b}{x-a}+\frac{a}{x-b}-2=0$.
23. $\frac{b-x}{a-x}+\frac{a-x}{b-x}=\frac{a}{b}+\frac{b}{a}$.
24. $\frac{ax^2-b}{ax+b}+\frac{a+bx^2}{a-bx}=\frac{2(a^2+b^2)}{a^2-b^2}$.
25. $bx^2+ay^2=a^3+b^3,\ x+y=a+b$.

EQUATIONS IN AN IRRATIONAL FORM

172. The square root of any quantity may always be regarded as having two values equal in magnitude but of opposite signs. For example, the square root of 49 is ± 7. When, however, such an expression as $\sqrt{2x+3}$ occurs in an equation it is usual to regard it as meaning the *positive* value of the square root of $2x+3$. It might be contended that $\sqrt{4x+7}-\sqrt{4x+3}=2$

was the same equation as $\sqrt{4x+7}+\sqrt{4x+3}=2$; but they are commonly regarded as being different, and instructions are given that after solving an equation of this sort, the answers obtained should be substituted in the original equation to see whether they satisfy it.

Example 1. Solve the equation $\sqrt{4x+7}+\sqrt{4x+3}=6$.

By transposition, $\sqrt{4x+3}=6-\sqrt{4x+7}$. ....(1)

Square; $\therefore \quad 4x+3=36-12\sqrt{4x+7}+4x+7$; ....(2)

$\therefore \quad 12\sqrt{4x+7}=36+7-3=40$;

$\therefore \quad \sqrt{4x+7}=\frac{10}{3}$.

Square; $\therefore \quad 4x+7=\frac{100}{9}$;

$\therefore \quad 4x=\frac{37}{9}$; $\quad \therefore \; x=\frac{37}{36}$.

This root will be found on substitution to satisfy the equation

$$\sqrt{4x+7}+\sqrt{4x+3}=6.$$

Example 2. Solve the equation $\sqrt{2x+3}+\sqrt{x-10}=6$. ....(1)

By transposing, $\sqrt{2x+3}=6-\sqrt{x-10}$.

Squaring, $2x+3=36-12\sqrt{x-10}+x-10$:

$\therefore \quad x-23=-12\sqrt{x-10}$. ....(2)

Squaring, $x^2-46x+529=144(x-10)$

$=144x-1440$;

$\therefore \quad x^2-190x=-1969$;

$\therefore \quad x=11$ or 179.

The result 11 satisfies the equation; 179 does not. The fact is that in solving equation (1) we have introduced an additional root through squaring. As we squared equation (2) it would have made no difference if we had written it $x-23=12\sqrt{x-10}$. Thus, in solving (1) we are also solving the equation $\sqrt{2x+3}-\sqrt{x-10}=6$; and this is the equation which is satisfied by the result 179.*

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* This may be expressed in general terms.

If we solve an equation $P=Q$ by squaring, we introduce generally an additional root.

The equation becomes $P^2=Q^2$,

i.e. $P^2-Q^2=0$,

i.e. $(P+Q)(P-Q)=0$.

Thus we have not only the original equation $P-Q=0$, but another one also, viz. $P+Q=0$, *i.e.* $P=-Q$.

Example 3. Solve $x^2 - x + 5\sqrt{2x^2 - 5x + 6} = \frac{1}{2}(3x + 33)$.

$$2x^2 - 2x + 10\sqrt{2x^2 - 5x + 6} = 3x + 33;$$

$$\therefore\ 2x^2 - 5x + 10\sqrt{2x^2 - 5x + 6} = 33.$$

Let $\sqrt{2x^2 - 5x + 6} = y$, *i.e.* $2x^2 - 5x + 6 = y^2$.

Then the equation becomes

$$y^2 - 6 + 10y = 33;$$

$$\therefore\ y^2 + 10y - 39 = 0;$$

$$\therefore\ (y-3)(y+13) = 0,$$

$$i.e.\ \sqrt{2x^2 - 5x + 6} = 3 \text{ or } -13,$$

$$\therefore\ 2x^2 - 5x + 6 = 9;$$

$$\therefore\ 2x^2 - 5x - 3 = 0.$$

By substitution it will be seen that the negative value (-13) of y will not satisfy the equation.

Thus the question has been reduced to the solution of a quadratic equation.

The following plan is sometimes useful.

Example 4. Solve $\sqrt{2x^2 + 9x - 1} + \sqrt{2x^2 - 7x + 7} = 6$. ....(1)

Now evidently $2x^2 + 9x - 1 - (2x^2 - 7x + 7) = 16x - 8$; ....(2)

$\therefore$ from (1) and (2) by division we obtain

$$\sqrt{2x^2 + 9x - 1} - \sqrt{2x^2 - 7x + 7} = \frac{8x - 4}{3}; \quad \text{......................(3)}$$

$\therefore$ by adding (1) and (3)

$$2\sqrt{2x^2 + 9x - 1} = \frac{8x-4}{3} + 6 = \frac{8x + 14}{3};$$

$$\therefore\ 6\sqrt{2x^2 + 9x - 1} = 8x + 14;$$

$$\therefore\ 3\sqrt{2x^2 + 9x - 1} = 4x + 7;$$

$\therefore$ by squaring, $18x^2 + 81x - 9 = 16x^2 + 56x + 49$:

$$\therefore\ 2x^2 + 25x - 58 = 0;$$

$$\therefore\ (2x + 29)(x - 2) = 0;$$

$$\therefore\ x = 2 \text{ or } -\tfrac{29}{2}.$$

Test, as before, to see whether the roots satisfy the equation.

Examples. XXXI. d.

Solve the following equations and verify the solutions by substitution:

1. $\sqrt{2x+3} = 5$.
2. $\sqrt{3x-5} = 1$.
3. $\sqrt[3]{4x-1} = 3$.
4. $5\sqrt{x-1} = \sqrt{x+1}$.
5. $\sqrt{x-1} = \sqrt{x} - 1$.
6. $\sqrt{x^2-9} = 4$.
7. $\sqrt{3x^2 - 4x + 9} = 3$.
8. $\sqrt{2x+3} + \sqrt{2x-2} = 5$.
9. $\sqrt{7x+1} - \sqrt{2x} = \sqrt{5x}$.
10. $\sqrt{5x+9} - \sqrt{3x+1} = \sqrt{2(x-6)}$.
11. $\sqrt{2x+10} + 2\sqrt{x+6} = 2$.
12. $\sqrt{2x+8} + 2\sqrt{x+5} = 2$.

13. $x+5=\sqrt{x+5}+6$.

14. $\sqrt{x+1}+\sqrt{x+8}=7$.

15. $\sqrt{x}-\sqrt{x-(a-b)^2}=a+b$.

16. $x^2=21+\sqrt{x^2-9}$.

17. $\sqrt{ax+b^2}+\sqrt{ax-2ab}=2a+b$.

18. $\sqrt{1+9x}+\sqrt{4x+1}=\sqrt{x+1}$.

19. $\dfrac{5}{\sqrt{x+2}}=\sqrt{x+2}+\sqrt{x-1}$.

20. $\sqrt{5ax+4b}+\sqrt{5ax-4b}=4\sqrt{b}$.

21. $\sqrt{x}+\sqrt{x-7}=\dfrac{21}{\sqrt{x-7}}$.

22. $\sqrt{x+1}+\sqrt{x+4}=\sqrt{x+9}$.

23. $\sqrt{x+2}+\sqrt{x}=\dfrac{4}{\sqrt{x+2}}$.

24. $\sqrt{x+a\sqrt{4x+2a^2}}=a+\sqrt{x}$.

25. $\sqrt{x-a^2}-\sqrt{x-b^2}=b-a$.

26. $x^2+\sqrt{x^2+3x+5}=7-3x$.

27. $x^2+\sqrt{x^2-5x+1}=5x+1$.

28. $x^2+2x+6\sqrt{x^2+2x+5}=11$.

29. $x^2+2x+4\sqrt{x^2+2x+8}=24$.

30. $3x^2-2\sqrt{3x^2-2x+1}=2(x+1)$.

31. $9x-3x^2+4\sqrt{x^2-3x+5}=11$.

32. $2x^2-\sqrt{(x-3)(2x-7)}=13x+9$.

33. $\sqrt{x^2+3x+6}-\sqrt{x^2+3x-1}=1$.

***173.** We now give some miscellaneous equations, of which the following are types.

Example 1. Solve the equations :

$$x+y+z=19, \quad\quad(1)$$
$$x^2+y^2+z^2=133, \quad\quad(2)$$
$$yz=x^2. \quad\quad(3)$$

Squaring (1), subtracting (2) from it, and dividing by 2,

$$xy+yz+zx=114, \quad\quad(4)$$

∴ from (3) $\quad x(y+x+z)=114,$

and from (1) $\quad x=6.$

Substituting this value of x and solving for y and z we obtain the following solutions

$$x=6,\ 6,$$
$$y=9,\ 4,$$
$$z=4,\ 9.$$

Example 2. Solve the equations :

$$x(y+z)=7, \quad\quad(1)$$
$$y(x+z)=4, \quad\quad(2)$$
$$z(x+y)=5. \quad\quad(3)$$

Adding (1), (2) and (3), and dividing by 2,

$$xy+yz+zx=8.$$

Subtracting (1), (2) and (3) from this, in succession,

$$yz=1,$$
$$xz=4,$$
$$xy=3.$$

Whence by multiplication, $\quad x^2y^2z^2=12$

$$xyz=\pm\sqrt{12};$$

$$\therefore\ x=\pm2\sqrt{3},\ y=\frac{\pm\sqrt{12}}{4},\ z=\frac{\pm\sqrt{12}}{3}.$$

Example 3. Solve the equations:

$$x^2+2yz=48, \quad \text{.........}(1)$$
$$y^2+2zx=48, \quad \text{.........}(2)$$
$$z^2+2xy=48. \quad \text{.........}(3)$$

Adding and taking the square root of both sides,

$$x+y+z=\pm 12. \quad \text{.........}(4)$$

Subtracting (2) from (1) and factorizing,

$$(x-y)(x+y-2z)=0.$$
$$\therefore\ x=y \quad \text{or} \quad x+y=2z.$$

(i) If $x=y$, from (1) $\qquad x^2+2xz=48,$

and from (3) $\qquad z^2+2x^2=48,$

whence $\qquad z=x.$

$\therefore\ x=y=z$, and from (1) or (2) or (3)

$$x=\pm 4=y=z.$$

(ii) If $\qquad x+y=2z,$

from (4) $\qquad z=\pm 4=x=y$ as before;

$\therefore\ x=y=z=\pm 4$ are the only solutions.

*Examples. XXXI. e.

Solve the equations:

1. $(x+y)^3+z^3=1125,$
$x+y+z=15,$
$xy=24.$

2. $xz=y^2,$
$x+y+z=13,$
$x^2+y^2+z^2=91.$

3. $x^3-2x=\frac{7}{8}.$

4. $\dfrac{x+y}{x-y}+10\dfrac{x-y}{x+y}=7,$
$xy^3=3.$

5. $xy+\dfrac{x}{y}=10,$
$xy^2-x=6y.$

6. $x+y=a+b,$
$\dfrac{a}{x}+\dfrac{b}{y}=2.$

7. $x^2+xy+y^2=2x^2+3xy+y^2=c^2.$

8. $x+y+z=7,$
$xy+xz+yz=14,$
$xyz=8.$

9. $\dfrac{x+a}{x+b}+\dfrac{x-a}{x-b}=\dfrac{a}{b}.$

10. $\dfrac{(x-a)(x-b)}{(x-c)(x-d)}=\dfrac{x-a-b}{x-c-d}.$

11. $(ax+by)^2+(ay-bx)^2=2\left(\dfrac{a}{b}+\dfrac{b}{a}\right)^2.$
$\dfrac{x}{y}+\dfrac{y}{x}=2\dfrac{a^2+b^2}{a^2-b^2}.$

12. $x(y+z)=5,$
$y(x+z)=4,$
$z(x+y)=3.$

13. $(x+y)(x+z)=1,$
$(y+z)(y+x)=4,$
$(z+x)(z+y)=9.$

14. Find the rational solutions of the equations,

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x}{y}-\frac{y}{x}=\frac{106}{9},\quad xy=3.$$

INDETERMINATE EQUATIONS

***174.** When we have but one equation involving *two* variables we can generally find any number of solutions. (Art. 57.)

Such equations, however, often admit of only a limited number of *positive integral* solutions.

Example. *Find the positive integral solutions of the equation*

$$5x + 12y = 193. \quad \text{..................................(1)}$$

By putting x or $y = 0$, 1, 2, and so on, one pair of roots can generally be found without difficulty.

Here we see *by trial* that one pair of roots is given by $x = 5$, $y = 14$.

$$\textit{i.e.}\ 5 \times 5 + 12 \times 14 = 193. \quad \text{..............................(2)}$$

Subtracting (2) from (1), $5(x-5) + 12(y-14) = 0$.

$$5(x-5) = 12(14-y),$$

$$\frac{x-5}{14-y} = \frac{12}{5}.$$

Now $\frac{12}{5}$ is in its lowest terms, and x and y must be positive integers,

$$\therefore\ x - 5 = 12p,$$

and $14 - y = 5p$, where p is an integer.

$$\textit{i.e.}\ x = 5 + 12p, \quad \text{..............................(3)}$$

$$y = 14 - 5p. \quad \text{..............................(4)}$$

From (3) p cannot be < 0, for then x would be negative.

...... (4) .... > 2 .... y ....

$$\therefore\ 0, 1, 2 \text{ are the only admissible values of } p.$$

Hence from (3) and (4) the only positive integral solutions of the given equations are

$$(p=0)\ \left.\begin{array}{l} x=5 \\ y=14 \end{array}\right\} \qquad (p=1)\ \left.\begin{array}{l} x=17 \\ y=9 \end{array}\right\} \qquad (p=2)\ \left.\begin{array}{l} x=29 \\ y=4 \end{array}\right\}.$$

***175.** *In how many ways can a bill of £2. 7s. 6d. be paid in half-crowns and half-sovereigns ?*

Let x be the number of half-crowns and y the number of half-sovereigns required to pay the bill.

Then $$\tfrac{5}{2}a + 10y = 47\tfrac{1}{2}$$

$$5x + 20y = 95$$

$$x + 4y = 19. \qquad \text{.............................(1)}$$

Now x and y must evidently be positive integers, and we see that the equation is satisfied by $x=3$, $y=4$,

$$\textit{i.e.} \;\; 3 + 4 \times 4 = 19. \qquad \text{..........................(2)}$$

Subtracting (2) from (1)

$$x - 3 + 4(y-4) = 0,$$

$$x - 3 = 4(4-y),$$

$$\frac{x-3}{4-y} = \frac{4}{1};$$

$$\therefore \;\; x - 3 = 4p,$$

$$4 - y = p, \text{ where } p \text{ is an integer.}$$

$$\textit{i.e.} \;\; x = 3 + 4p, \qquad \text{........................(3)}$$

$$y = 4 - p. \qquad \text{..........................(4)}$$

From (3) p cannot be < 0, for then x would be negative.

...... (4) .... > 4 .... y ....

$\therefore$ 0, 1, 2, 3, 4, are the only admissible values of p;

i.e. there are *five* ways of paying the bill.

This includes as a solution the case when no half-sovereigns are used, for when $p=4$, $y=0$.

GRAPHICAL SOLUTION OF INDETERMINATE EQUATIONS

***176. Example.** Find the positive integral solutions of the equation

$$3x+2y=30.$$

Use a half-inch unit.

When $x=0,\ y=15,$

$y=0,\ x=10.$

Joining the points (0, 15), (10, 0) by a str. line, we have the graph of the equation $3x+2y=30$.

The only points, whose co-ordinates are positive integers, through which the line passes, will be seen to be the points

(8, 3), (6, 6), (4, 9), (2, 12), not counting zero values.

$\therefore$ these are the reqd. solutions.

*Examples. XXXI. f.

Find the positive integral solutions of :

1. $2x+5y=35$. **2.** $2x+3y=15$. **3.** $5x+2y=27$.

4. $7x+3y=73$. **5.** $9x+5y=33$. **6.** $7x+13y=207$.

How many positive integral solutions are there of :

7. $2x+13y=185$. **8.** $2x+11y=165$.

9. $4x+9y=207$. **10.** $7x+3y=119$.

11. Prove that the equation $7x-5y=16$ has an infinite number of positive integral solutions.

Use graphical methods to find the positive integral solutions of :

12. $3x+2y=17$. **13.** $5x+y=18$. **14.** $3x+4y=48$.

15. $2x+7y=23$. **16.** $2x+3y=30$.

Find graphically, or by algebra, all integral solutions of the following equations which have positive values of x and negative values of y :

17. $x-2y=12$. **18.** $2x-3y=24$. **19.** $x-y=4$.

Find graphically, or by algebra, all integral solutions of the following equations which have negative values of x and y :

20. $2x+3y+24=0$. **21.** $4x+3y+24=0$. **22.** $x+2y+12=0$.

23. A man bought a number of books at 5*s*. each, and a number at 7*s*. each, and spent 38*s*. : how many of each did he buy ?

24. A man bought a number of geese at 7*s*. each, and a number of turkeys at 11*s*. each, and spent £4. 6*s*. : how many of each did he buy ?

25. In how many ways can I pay a bill of 31*s*. in sixpences and shillings, excluding zero solutions ?

26. Divide 59 into two parts so that one may be a multiple of 9 and the other of 4.

27. A has only four-shilling pieces, and B only half-crowns. What is the simplest way in which A can pay B the sum of 35*s*. ?

28. In how many ways can I pay a bill of 37*s*., if I have only florins and half-crowns in my pocket ?

29. The sum of two fractions is $2\frac{3}{28}$ and their denominators are 4 and 7. Find all the solutions of the problem.

30. Find general formulae to represent all the integral solutions of the equation $9x - 13y = 63$.

31. A has 25 four-shilling pieces, and B 25 half-crowns: in how many ways can A pay B the sum of 37*s.* ?

32. Find the positive integral solution of the equation $5x + 13y = 227$, for which the value of x is largest.

33. A man exchanges a number of geese at 7*s.* each, for a number of turkeys at 13*s.* each, and receives £4. 13*s.* in cash. Find the number of ways in which the exchange can be made, a condition being made that the man shall not take more than 20 turkeys.

CHAPTER XXXII

THEORY OF QUADRATIC EQUATIONS

177. *To prove that a quadratic equation cannot have more than two roots.*

If possible, let the general quadratic equation

$$ax^2 + bx + c = 0$$

have *three different* roots α, β, γ.

By hypothesis, each of these values of x satisfies the equation,
$\therefore$ by substitution

$$a\alpha^2 + b\alpha + c = 0, \qquad (1)$$

$$a\beta^2 + b\beta + c = 0, \qquad (2)$$

$$a\gamma^2 + b\gamma + c = 0. \qquad (3)$$

Subtracting (2) from (1), $a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$.

Dividing by $\alpha - \beta$, which by hypothesis is not equal to zero.

$$a(\alpha + \beta) + b = 0. \qquad (4)$$

In the same way, subtracting (3) from (1) and dividing by $\alpha - \gamma$,

$$a(\alpha + \gamma) + b = 0. \qquad (5)$$

Subtracting (5) from (4), $a(\beta - \gamma) = 0$;

$$\therefore\ a = 0 \text{ or } \beta - \gamma = 0,$$

which is impossible, for a is not equal to zero, nor is β equal to γ, by hypothesis.

$\therefore$ the quadratic cannot have more than two roots.

178. The square root of a negative quantity cannot be found. It is said to be '*imaginary*,' or '*unreal*,' or '*impossible*.'

The quadratic equation $ax^2+bx+c=0$, *will have*

(1) *real and different roots if* $b^2>4ac$,

(2) *real and equal roots if* $b^2=4ac$,

(3) *imaginary roots if* $b^2<4ac$.

We have seen (Art. 149), that the solution of this equation may be thus written:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

(1) If $b^2>4ac$, b^2-4ac is positive, and the value of $\sqrt{b^2-4ac}$ may be found;

$\therefore$ we then have two real and different roots,

viz. $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

(2) If $b^2=4ac$, $b^2-4ac=0$;

$\therefore$ $x=-\dfrac{b}{2a}$ is the only solution;

in other words the roots are equal, and each equal to

$$-\frac{b}{2a}.$$

(3) If $b^2<4ac$, b^2-4ac is negative, and the value of

$\sqrt{b^2-4ac}$ is imaginary.

Hence the equation in that case has no real roots.

By means of the above we can determine the *nature* of the roots of a quadratic, without actually effecting its solution.

The student must be careful to distinguish between *irrational* and *imaginary* roots.

If the roots of $ax^2+bx+c=0$ are rational, b^2-4ac must be a perfect square.

The roots of $x^2-2x-2=0$ are $1+\sqrt{3}$, and $1-\sqrt{3}$.

These are real but *irrational*.

The roots of x^2-2x+4 are $1+\sqrt{-3}$, and $1-\sqrt{-3}$.

These are *imaginary*.

179. *The roots of $ax^2+bx+c=0$ are equal, but of opposite sign if $b=0$.*

The roots are equal but of opposite sign,

if
$$\frac{-b+\sqrt{b^2-4ac}}{2a}=-\left[\frac{-b-\sqrt{b^2-4ac}}{2a}\right]$$
$$=\frac{b+\sqrt{b^2-4ac}}{2a};$$

i.e. if
$$\frac{2b}{2a}=0;$$

i.e. if
$$b=0.$$

Example 1. When we solve the equation $x^2+px-q^2=0$, the expression under the radical sign

$$=p^2+4q^2, \qquad (b^2-4ac)$$

which is positive.

$\therefore$ the roots of the equation are real and different for all values of p and q.

Example 2. When we solve the equation $5x^2-2x+4=0$, the quantity under the radical sign

$$=4-4\times 20, \text{ which is negative.}$$

$\therefore$ the equation has imaginary roots.

If we drew the graph of $y=5x^2-2x+4$, as in Art. 151, we should find that the curve does not meet the axis of x, *i.e.* no real value of x can be found which will make $5x^2-2x+4$ vanish.

Example 3. When we solve the equation $2x^2-px+8=0$, the expression under the radical sign

$$=p^2-4\times 16=p^2-64.$$

$$\therefore \text{ if } p^2=64, \text{ i.e. if } p=\pm 8,$$

the roots of $2x^2-px+8=0$ are equal.

180. *In the quadratic equation $ax^2+bx+c=0$,*

(1) *the sum of the roots* $=-\dfrac{b}{a}$,

(2) *the product of the roots* $=\dfrac{c}{a}$.

Let α and β be the roots.

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a},$$

$$\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Adding, $\alpha+\beta=-\dfrac{b}{a}$.

Multiplying, $\alpha\beta=\dfrac{b^2-(b^2-4ac)}{4a^2}$ $[(p+q)(p-q)=p^2-q^2]$

$$=\frac{4ac}{4a^2}$$

$$=\frac{c}{a}.$$

If we write the equation in the form $x^2+\dfrac{bx}{a}+\dfrac{c}{a}=0$, we may express these results as follows:

When the coefficient of x^2 in a quadratic equation is unity,

(1) the sum of the roots is equal to the coefficient of x with the sign changed;

(2) the product of the roots is equal to the constant term.

These results are of the greatest importance, and will be found most useful in solving problems concerned with the roots of quadratic equations.

181. *If* α *and* β *are the roots of* $ax^2+bx+c=0$,

$$ax^2+bx+c=a(x-\alpha)(x-\beta).$$

$$ax^2+bx+c=a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right)$$

$$=a[x^2-(\alpha+\beta)x+\alpha\beta]$$

$$=a(x-\alpha)(x-\beta).$$

In the same way, if α and β are the roots of $x^2+px+q=0$,

$$x^2+px+q=(x-\alpha)(x-\beta).$$

Example 1. The quadratic whose roots are -5 and 6 is

$$(x+5)(x-6)=0,$$

or $$x^2-x-30=0.$$

Example 2. If α and β are the roots of $x^2-px+q=0$, find the values of (1) $\alpha-\beta$, (2) $\alpha^2+\beta^2$, (3) $\alpha^3+\beta^3$.

(1) $\alpha+\beta=p$, ...(1)

$\alpha\beta=q$. ....(2)

Squaring (1) and subtracting four times (2),

$$(\alpha-\beta)^2=p^2-4q;$$

$$\therefore\ \alpha-\beta=\pm\sqrt{p^2-4q}.$$

(2) Squaring (1) and subtracting twice (2),

$$\alpha^2+\beta^2=p^2-2q.$$

(3) Squaring (1) and subtracting three times (2).

$$\alpha^2-\alpha\beta+\beta^2=p^2-3q.$$

Multiplying this with (1),

$$\alpha^3+\beta^3=p(p^2-3q).$$

Example 3. If α and β are the roots of $ax^2+bx+c=0$, form the equation whose roots are $\frac{1}{\alpha}$, and $\frac{1}{\beta}$.

The sum of the roots of the reqd. equation $=\frac{1}{\alpha}+\frac{1}{\beta}$

$$=\frac{\alpha+\beta}{\alpha\beta}=-\frac{b}{a}\div\frac{c}{a}=-\frac{b}{c}.$$

The product of the roots $=\frac{1}{\alpha\beta}=\frac{a}{c}$.

$\therefore$ the reqd. equation is

$$x^2+\frac{bx}{c}+\frac{a}{c}=0,$$

or

$$cx^2+bx+a=0.$$

182. *If a is positive and α, β are real roots of the equation $ax^2+bx+c=0$, the expression ax^2+bx+c vanishes when $x=\alpha$ or β, and is positive for all other values of x except for those lying between α and β.*

(1) The values α and β satisfy the equation ;

$\therefore$ the expression ax^2+bx+c is zero when $x=\alpha$ or β.

(2) $$\alpha+\beta=-\frac{b}{a},\quad \alpha\beta=\frac{c}{a}. \qquad \text{(Art. 180.)}$$

$$\therefore\ ax^2+bx+c=a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right)$$

$$=a[x^2-(\alpha+\beta)x+\alpha\beta]$$

$$=a(x-\alpha)(x-\beta). \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Let α be greater than β.

When $x>\alpha$, $x-\alpha$ is positive and $x-\beta$ is positive ;

$\therefore$ from (1) ax^2+bx+c is positive.

When $x<\alpha$ but $>\beta$, $x-\alpha$ is negative,

and $x-\beta$ is positive ;

$\therefore$ from (1) ax^2+bx+c is negative.

Lastly, when $x<\beta$, $x-\alpha$ is negative,

and $x-\beta$ is negative;

$\therefore$ from (1) ax^2+bx+c is positive.

$$\therefore\ ax^2+bx+c=0,\ \text{when } x=\alpha \text{ or } \beta,$$
is negative when x lies between α and β,
and is positive for all other values of x.

It follows that if a is negative and α and β are the roots of $ax^2+bx+c=0$, the expression ax^2+bx+c is zero when $x=\alpha$ or β, negative for all other values of x except for those lying between α and β.

Example 1. To prove *graphically* that the expression x^2+x-6

(i) vanishes when $x=2$ or -3,
(ii) is negative when x lies between 2 and -3,
(iii) is positive for all other values of x.

(i) If we draw the graph of $y=x^2+x-6$ as in Art. 151, we shall see that the curve cuts the axis of x where $x=2$ and $x=-3$.

(ii) When x lies between these values, y is negative.

(iii) For all other values y is positive.

Example 2. Show that $\dfrac{x^2-3x+4}{x^2+3x+4}$ can never be greater than 7 nor less than $\frac{1}{7}$ for real values of x.

Let $\dfrac{x^2-3x+4}{x^2+3x+4}=u$.

Multiplying up and rearranging as a quadratic for x,
$$x^2(1-u)-3x(1+u)+4(1-u)=0.$$
When we solve this quadratic for x, the expression under the radical sign
$$\begin{aligned} &=9(1+u)^2-16(1-u)^2 \qquad (b^2-4ac)\\ &=-7+50u-7u^2\\ &=(-7+u)(1-7u)\\ &=7(u-7)(\tfrac{1}{7}-u).\end{aligned}$$
Hence if $u>7$, $u-7$ is positive, and $\frac{1}{7}-u$ is negative.

$\therefore$ the expression under the radical sign is negative and x is imaginary.

If $u<7$ but $>\frac{1}{7}$, $u-7$ is negative and $\frac{1}{7}-u$ is negative.

$\therefore$ the expression under the radical sign is positive, and x is real.

If $u<\frac{1}{7}$, $u-7$ is negative, and $\frac{1}{7}-u$ is positive.

$\therefore$ the expression under the radical sign is negative and x is imaginary.

Thus for real values of x, u cannot be greater than 7 or less than $\frac{1}{7}$.

183. *Find the condition that the equations $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$ may have a common root.*

Let α be a common root of the equations.

Then by substitution $\quad a\alpha^2+b\alpha+c=0, \quad$ ....(1)

$$a'\alpha^2+b'\alpha+c'=0. \quad \text{........................(2)}$$

Multiplying (1) by b', and (2) by b, and subtracting,

$$a^2(ab' - a'b) + b'c - bc' = 0,$$

or
$$a^2 = \frac{bc' - b'c}{ab' - a'b}. \quad \text{.........................(3)}$$

Again multiplying (1) by a', and (2) by a, and subtracting,

$$a(a'b - ab') + a'c - ac' = 0,$$

or
$$a = \frac{a'c - ac'}{ab' - a'b}. \quad \text{.........................(4)}$$

$\therefore$ from (3) and (4) $\dfrac{bc' - b'c}{ab' - a'b} = \left(\dfrac{a'c - ac'}{ab' - a'b}\right)^2$,

or $(ab' - a'b)(bc' - b'c) = (a'c - ac')^2$; the reqd. condition.

Examples. XXXII.

Form the equations whose roots are:

1. $2, 5$.
2. $4, -5$.
3. $\frac{1}{2}, -\frac{1}{2}$.
4. $0, -3$.
5. $2a, -3a$.
6. $a+1, a-1$.
7. $1+\dfrac{1}{a}, 1-\dfrac{1}{a}$.
8. $m \pm \sqrt{m^2 - n}$.
9. $\dfrac{-m \pm \sqrt{m^2 - 4ln}}{2l}$.
10. $3+\sqrt{3}, 3-\sqrt{3}$.
11. $\dfrac{4-\sqrt{3}}{5}, \dfrac{4+\sqrt{3}}{5}$.

12. For what value of k will the roots of $x^2 - 10x = k$ be equal?

13. What is the condition that the roots of the equation $x^2 - px + q = 0$ may be rational?

14. Prove that the roots of $x^2 - 3x + k = 0$ will be imaginary if k is greater than $2\frac{1}{4}$.

15. Solve the equation $x^2 - px + q = 0$, and hence find (1) the sum of the roots, (2) the product of the roots.

16. If α and β are the roots of $ax^2 + bx + c = 0$, find the values of (1) $\alpha - \beta$, (2) $\alpha^2 + \beta^2$, (3) $\alpha^3 + \beta^3$, (4) $\alpha^4 + \beta^4$.

17. If α and β be the roots of the equation $x^2 - px + q = 0$, form the equation whose roots are $2\alpha, 2\beta$.

If α and β be the roots of the equation $ax^2 + bx + c = 0$, determine the equation whose roots are:

18. $-\alpha, -\beta$.
19. $3\alpha, 3\beta$.
20. $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$.
21. $\dfrac{2\alpha}{\beta}, \dfrac{2\beta}{\alpha}$.
22. $2\beta - \alpha, 2\alpha - \beta$.
23. $\dfrac{\alpha^2}{\beta}, \dfrac{\beta^2}{\alpha}$.

24. Find the numerical value of a in the equation $ax^2 + 2x + 3a = 0$, when the sum of its roots is equal to their product.

25. If one root of the equation $ax^2 + bx + c = 0$, is double the other, prove that $9ac = 2b^2$

26. Form an equation whose roots shall be $\frac{\alpha^2}{\beta}$, $\frac{\beta^2}{\alpha}$, where α, β are the roots of the equation $x^2 = px + p^2$.

27. If α, β be the roots of the equation $ax^2 + bx - a = 0$, determine the equation whose roots are $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$.

28. Find the sum of the cubes of the roots of $x^2 + px + q = 0$.

29. If α, β be the roots of the equation $px^2 + qx + r = 0$, find the equation whose roots are $\alpha + \beta$, $\alpha\beta$. Find also the value of $\alpha^4 + \beta^4$.

30. If α, β be the roots of the equation $ax^2 + bx + c = 0$, form the equation whose roots are α^2 and β^2.

31. Find the quadratic equation whose roots are the squares of the roots of the equation $x^2 = px + q$.

32. Prove that the equation $x^2 - 2(k+2)x - k^2 = 0$, cannot have equal roots for any real value of k. For what value of k will the roots be equal but of opposite sign?

33. If α, β be the roots of the equation $x^2 + px + q = 0$, prove that $x^2 + px + q$ will be a negative quantity, if x be put equal to $\frac{1}{3}\alpha + \frac{2}{3}\beta$.

34. Find the condition that the two quadratics $x^2 + px + q = 0$, $x^2 + p'x + q' = 0$, may have a common root.

35. If α, β be the roots of the equation $x^2 + px + q = 0$, prove that $\alpha^4 + \beta^4 = (p^2 - 2q)^2 - 2q^2$.

36. Show that one of the roots of the equation $px^2 + qx + r = 0$, will be double one of the roots of the equation $rx^2 + qx + p = 0$, if either $r = 2p$ or $2p + r = \pm q\sqrt{2}$.

37. If α, β be the roots of the equation $x^2 - px + q = 0$, prove that $\alpha^5 + \beta^5 = p^5 - 5p^3q + 5pq^2$.

38. Prove that, if one of the equations

$$x^2 - x(3c - b) + bc = 0, \quad x^2 - x(5c - b) + 4c^2 = 0,$$

has equal roots, so has the other.

39. If p, q be the roots of the equation $ax^2 + 2bx - c = 0$, find the equation whose roots are p^3, q^3.

40. One root of the equation $x^2 + ax + b = 0$ is double of the other; and one root of the equation $x^2 + ax + c = 0$ is equal to three times its other root. Find the value of $\frac{b}{c}$.

41. Prove that the roots of one of the two equations

$$8a^2x(2x - 1) + b^2 = 0, \quad 4a^2x^2 + b^2(4x + 1) = 0,$$

must be imaginary.

42. If $ax^2 + bx + c = 0$, $bcx^2 + cax + ab = 0$ have a common root, and if $a + b + c = 0$, prove that

$$b^4(a - c)^2 = a^2c^2(a - b)(b - c).$$

43. The roots of the quadratic $ax^2 + bx + c = 0$ are x_1, x_2; find in terms of a, b, c, the values of (1) $(ax_1 + b)(ax_2 + b)$, (2) $(bx_1 + c)(bx_2 + c)$.

44. If x_1, x_2 be the roots of the equation $ax^2 + bx + c = 0$, find, in terms of a, b, c, the value of

$$\frac{1}{(b + ax_1)^2} + \frac{1}{(b + ax_2)^2}.$$

45. Prove that, for real values of x, the expression $\frac{x^2+3x-15}{x-5}$ can have all numerical values except such as lie between 3 and 23.

46. Prove that $\frac{x^2+x+1}{x^2+1}$ cannot be greater than $\frac{3}{2}$, nor less than $\frac{1}{2}$, for real values of x.

47. Prove that $\frac{x^2-2x+4}{x^2+2x+4}$ cannot be greater than 3 or less than $\frac{1}{3}$, for real values of x.

48. For real values of x, prove that the expression $\frac{4x^2-5x+10}{3(x-2)}$ cannot lie between 9 and $-1\frac{2}{3}$.

49. Find the greatest value which the expression $x+\sqrt{6ax-7a^2-x^2}$ can have for real values of x.

50. Find the minimum value of $\frac{x^2-x+1}{x^2+x+1}$, for real values of x.

CHAPTER XXXIII

Examples. XXXIII. a.

1. Resolve into real elementary factors :

(i) $6x^2-23xy+20y^2$. (ii) $x^4-7x^2y^2+y^4$. (iii) x^6-1.

2. Simplify $\frac{9}{x^2-x-20}-\frac{7}{x^2+x-12}-\frac{2}{x^2-8x+15}$.

3. Find the squares of $x+y+2z-1$, and of $x+y-2z-1$. What is the value of the difference of these squares when $z=\frac{1}{2}(x+y)$?

4. Find the L.C.M. of x^5-xy^4, x^9+x^8y, $x^6+y^6+x^2y^2(x^2+y^2)$.

5. Solve the equations (i) $27x^2-57x=14$.

$$\text{(ii)}\quad x^2+y^2=5,\quad x^2-y^2=\frac{3xy}{2}.$$

6. A travels 42 miles in $5\frac{1}{4}$ hours. Find, graphically, how long he takes to travel 35 miles, and 29 miles. How far did he travel in 2 hrs. 36 min. ?

7. Solve the equations

$$x+2y-z+4=0,$$
$$3x+4y+z-1=0,$$
$$5x+6y-3z+18=0.$$

8. If α, β are the roots of the equation $x^2+px+q=0$, form the equation whose roots are $\alpha+2\beta$, $\beta+2\alpha$.

XXXIII. b.

1. Find the factors of (i) $x^2+16x+63$.

(ii) $y^3-43a^2y+42a^3$.

(iii) $x^7-14x^5+49x^3-36x$.

2. Find the square root of $9x^4-42x^3+37x^2+28x+4$.

3. Simplify $\dfrac{\dfrac{1}{x-a}-\dfrac{1}{x+a}-\dfrac{2a}{x^2+a^2}}{\dfrac{1}{x^3-a^3}-\dfrac{1}{x^3+a^3}}\left(\dfrac{1}{x^2+ax+a^2}+\dfrac{1}{x^2-ax+a^2}\right)$.

4. Solve the equations (i) $\dfrac{a}{bx}+\dfrac{b}{ax}=a^2+b^2$.

(ii) $(x-10)(x-7)+(2x-9)(x-8)=103$.

5. A person after paying income-tax of 6*d*. in the £ gave away one-thirteenth part of the remainder, and then had £540 left. What was his original income?

6. On an examination paper of maximum 58 the marks gained by six candidates were 52, 47, 41, 36, 24, 12. Draw a graph to raise the maximum to 100, and read off the raised marks of the candidates. Test one of your results.

7. Employ the Remainder Theorem to prove that $x^4-4x^3+2x^2+x+6$ is exactly divisible by x^2-5x+6.

XXXIII. c.

1. Remove the brackets in $7a+6[b-5\{c+4(b-3(a+2c))\}]$ and find its value when $a=2$, $b=3$, $c=1$.

2. Simplify $\dfrac{1}{x^2-4x+3}-\dfrac{4}{x^2+2x-15}+\dfrac{3}{x^2+4x-5}$.

3. Find the H.C.F. of $x^4-8x^3+13x^2-30x+8$
and $x^4-4x^3-11x^2-50x+16$.

4. Solve the equation $\dfrac{\dfrac{2x-1}{3}-\dfrac{4x^2-1}{x+3}}{x+1-\dfrac{5x^2-9}{3(x-3)}}=\dfrac{x-3}{x+3}\cdot\dfrac{10x+1}{2x+3}$.

5. Solve the equations:

(i) $(a+b)(c+x)+(b+c)(a+x)=(c+a)(b+x)$.

(ii) $x+y=3$, $\dfrac{2}{x}+\dfrac{1}{y}=2$.

6. I bought a horse and carriage for £80. I sold the horse at a profit of 20 per cent., and the carriage at a loss of 4 per cent., and found that on the whole transaction I had gained 5 per cent. What was the original cost of the horse?

7. Determine the values of k for which the equation

$$12(k+2)x^2-12(2k-1)x-38k-11=0$$

will have equal roots.

XXXIII. d.

1. Divide $x^5+x^4+4x^3+21x^2+23x-40$ by x^2+4x+5, using the method of detached coefficients.

2. Simplify $\left\{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}-3\left(\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}\right)+5\right\}\div\left(\dfrac{a}{b}-1-\dfrac{b}{a}\right)^2$.

3. Find the square root of $4x^4+12x^3-11x^2-30x+25$.

4. A man travels at the rate of x feet per second.

(i) How many yards does he travel per minute ?
(ii) .... miles .... hour ?
(iii) .... in y hours ?
(iv) How long does he take to travel y miles ?

5. Solve the equations :

$$\text{(i)} \quad \frac{7x}{1-\dfrac{2x-12}{3x-5}}=\frac{48}{1-\dfrac{1}{x}}.$$

$$\text{(ii)} \quad \frac{5}{x}-\frac{3}{y}=9, \quad 3y+2x=13xy.$$

6. A man on a bicycle, who travels at the rate of 10 miles an hour, and another walking at the rate of 4 miles an hour, start at the same time and from the same point to go round a field a quarter of a mile in circumference in the same direction. Find how soon the bicyclist is one-quarter of the whole circumference ahead of the walker.

7. Trace the graph of $y=3x-x^2$, and deduce the value of x when the expression $3x-x^2$ is a maximum. What is the maximum value of the expression ?

XXXIII. e.

1. Show that x^6+a^6 is divisible by $x^2+px+\dfrac{p^2}{3}$ if $p^6-27a^6=0$.

2. Find the product of $x-y$, $x+y$, x^2-xy+y^2, x^2+xy+y^2.

3. Find the square root of $n(n+1)(n+2)(n+3)+1$.

4. Express $\dfrac{\dfrac{1}{x}+\dfrac{1}{y-z}}{\dfrac{1}{x}-\dfrac{1}{y-z}}\left\{1-\dfrac{y^2+z^2-x^2}{2yz}\right\}$ in its simplest form.

5. Employ the Remainder Theorem to prove that

$$1-x^2-2x^3-2x^4-x^5+x^7$$

is exactly divisible by $x+1$ and by x^2+1.

6. Solve the equations

$$\text{(i)} \quad \frac{a(a-x)}{b}-\frac{b(b+x)}{a}=x.$$

$$\text{(ii)} \quad \frac{3}{3-x}=5-\frac{2}{2-x} \text{ (correct to two decimal places).}$$

7. Two travellers, one of whom travels 3 miles an hour faster than the other, set out to meet one another, starting simultaneously from two towns which are 216 miles apart. They meet after a lapse of 8 hours. Find the rate at which each of them travels.

8. Divide 1 into two fractions such that the sum of their cubes is $\frac{1}{3}$.

XXXIII. f.

1. Divide $(x+y)^4+(x^2-y^2)^2+(x-y)^4$ by $3x^2+y^2$.

2. Resolve each of the following into three real factors :

$$4x^3-23x^2+28x, \quad y^4+11y^2-180, \quad a^6+27b^6.$$

3. Solve the equations :

(i) $\dfrac{x+a}{x-a}-\dfrac{x-b}{x+b}=\dfrac{2(a+b)}{x}$.

(ii) $x^2+xy=28, \quad xy+y^2=21$.

4. Given that α, β are the roots of $x^2+px+q=0$, find the roots of $x^2+4px+16q=0$.

5. Prove that the difference of the squares of two consecutive numbers is equal to the sum of the numbers.

6. A, walking uniformly, but taking a rest of 20 minutes when he has gone half-way, does 5 miles in an hour. B, starting at the same time, and taking no rest, passes A $3\frac{1}{2}$ miles from the start. Find, by the graphical method, how long B takes to walk the $3\frac{1}{2}$ miles.

7. Show, by any method, that $a^3(b-c)+b^3(c-a)+c^3(a-b)$ contains $b-c$, $c-a$, $a-b$ as factors.

XXXIII. g.

1. Find the quotient and the remainder when $2x^4-3x^3-x^2+x-1$ is divided by $x-3$.

2. Find, to three places of decimals, a positive number such that if it is added to its square, the sum is unity.

3. Two workmen take the same time to earn £22 and £21 respectively. The former earns £15. 8*s*. in one day less time than the latter takes to earn the same sum. How much does each earn per day ?

4. Simplify the expressions

(i) $\left(\dfrac{a^3}{b}-\dfrac{b^3}{a}\right)\left(\dfrac{3a+b}{a+b}-\dfrac{3a-b}{a-b}\right)$.

(ii) $\dfrac{1}{(a^2-b^2)(a^2-c^2)}+\dfrac{1}{(b^2-c^2)(b^2-a^2)}+\dfrac{1}{(c^2-a^2)(c^2-b^2)}$.

5. Solve the equations

(i) $\dfrac{a}{x-a}+\dfrac{b}{x-b}=0$.

(ii) $\dfrac{a^2}{x}+\dfrac{b^2}{y}=\dfrac{(a+b)^2}{c}, \quad x+y=c$.

6. A man spends £70 in 45 days ; make a graph and read off from it his expenditure in 17, 32, and 41 days, to the nearest pound.

7. If α and β are the roots of the equation $ax^2-bx+c=0$, find the equation whose roots are 2α and 2β.

XXXIII. h.

1. Simplify $\dfrac{a^2x^m-b^2x^{m+4}}{a-bx^2}$.

2. If the coefficients of x^4 and of x in the product of $2x^3+3x^2+ax-10$ and $3x^3-ax^2-10x+4$ are equal to one another, find the value of a.

3. Find (i) the H.C.F., (ii) the L.C.M. of $a^4+a^2b^2+b^4$, $a^4-a^2b^2+2ab^3-b^4$.

4. In the same diagram draw the graphs of

$$y=x+3, \quad 2y-x=8, \quad \text{and} \quad 2y+5x=20.$$

What do you deduce as to the roots of the different pairs of equations ?

5. If α, β are the roots of $x^2-px-q=0$, form the equation whose roots are -3α, -3β.

6. Solve the equations (i) $(2x^2+3x-1)(2x^2+3x-2)=156$,
(ii) $2(x-1)(y-1)=6(x+y)=-3xy$.

7. The difference in the average rates of two trains is 13 miles per hour. The faster of the two takes 2 hours less time to travel 164 miles than the slower takes to travel 168 miles. Find their respective rates.

XXXIII. k.

1. If $\frac{x}{y}+\frac{y}{x}=a$, $\frac{y}{z}+\frac{z}{y}=b$, $\frac{z}{x}+\frac{x}{z}=c$, prove that $a^2+b^2+c^2-abc=4$.

2. Solve the equation $4x^2+2x-1=0$, giving results correct to two decimal places.

3. Simplify $\left(\frac{b-c}{a+b-c}-\frac{a-b+c}{c-b}\right)\left(\frac{1}{a}-\frac{c-b}{a^2}\right)$.

4. The denominator of a certain fraction exceeds its numerator by one. Two other fractions are formed, one of them by adding 9 to the denominator, and the other by subtracting 6 from the numerator, of the original fraction. These two fractions are equal. Find the original fraction.

5. An old clock increased uniformly in value from £4. 10*s.* in the year 1890, to £8. 10*s.* in 1899. Find graphically its value in 1893, 1894, and 1897, to the nearest shilling.

6. Solve the equations $x^2+y^2=2(a^2+b^2)$, $\frac{(x+y)^2}{a^2}+\frac{(x-y)^2}{b^2}=8$.

7. Construct an equation whose roots shall exceed by a quantity m the roots of the equation $ax^2+bx+c=0$.

XXXIII. l.

1. Resolve into factors (i) $a^4-8a^2b-48b^2$, (ii) $(a^2+b^2)c+(b^2+c^2)a$.

2. Multiply $a^3+4a^2b+8ab^2+8b^3$ by $a^3-4a^2b+8ab^2-8b^3$.

3. Show that if $a+b+c+d=0$, then
$$a^2-b^2+c^2-d^2=2(a+b)(a+d).$$

4. Find the area of the quadrilateral formed by joining the points (10, 20), (13, 9), (23, 8), (28, 20).

5. Solve the equations $x+y+z=6$, $4x+y=2z$, $x^2+y^2+z^2=14$.

6. If a, b, c are real quantities, determine the condition that the roots of the equation $ax^2+2bx+c=0$ may be imaginary.

7. The journey between two towns by one route consists of 233 miles by rail followed by 126 miles by sea; by another route it consists of 405 miles by rail, followed by 39 miles by sea. If the time occupied on the journey is 50 minutes longer by the first route than by the second, find the average speed by rail, assuming it to be the same by each route, and 25 miles an hour faster than the average speed by sea.

XXXIII. m.

1. Simplify $\frac{1}{a-b}\left\{\frac{(a-b)^3+(b-c)^3}{a-c}-(a+c-2b)^2\right\}$.

2. Resolve into factors (i) $18x^2+53x-35$.
(ii) $a^2+2bc-(c^2+2ab)$.
(iii) $(x-3b)^3-4b^2x+12b^3$.

3. Divide $x^6+6x^5-2x^4+37x^3-5x^2+13x-15$ by x^2-x+5, using the method of detached coefficients.

4. Find the value of $\sqrt{13}$ correct to two decimal places by any graphical or geometrical method.

5. Solve the equations (i) $\dfrac{x^2}{y}+\dfrac{y^2}{x}=\dfrac{3}{2}$, $x+y=1$.

(ii) $ab(x^2+1)=x(a^2+b^2)$.

6. Prove that if the roots of the equation $ax^2+2bx+c=0$ are imaginary, the roots of the equation $ax^2+2(a+b)x+a+2b+c=0$ are also imaginary.

7. The marks of a form ranged from 325 to 259. Draw a graph to scale them from 80 to 0, and read off the scaled marks corresponding to the following actual marks gained : 280, 295, 312. Verify one of your results.

XXXIII. n.

1. Find the relation between the constants when the three equations

$$ax+by=c,\quad bx+ay=d,\quad x^2+y^2=xy$$

are simultaneously true.

2. If $f(n)=\dfrac{n(n-1)}{2}$, and $\phi(n)=\dfrac{n(n+1)}{2}$, find the value of

(i) $f(n+1)-\phi(n)$, (ii) $[f(n+1)]^3-[\phi(n-1)]^3$.

3. Find the L.C.M. of $3x^2-4x-4$ and $4x^3-8x^2-x+2$.

4. Find graphically the maximum value of $6x-x^2-\frac{11}{2}$. Verify your result by algebra.

5. A merchant beginning business with a certain capital succeeded in doubling it, but afterwards lost £1000. He employed the remainder in a venture which brought him in a profit of 35 per cent., after which his capital was found to be £10 more than his original capital. Find the amount of that capital.

6. Solve the equations (i) $\dfrac{x^2-(a+b)x-bc}{x-b}=\dfrac{x^2-(a+c)x-bc}{x-c}$.

(ii) $ay^2+bxy=b$, $bx^2+axy=a$.

7. If α and β are the roots of the equation $ax^2+bx+c=0$, find the equation whose roots are $\dfrac{1+\alpha}{\beta}$, $\dfrac{1+\beta}{\alpha}$.

XXXIII. p.

1. Find the L.C.M. of x^4+x, x^4-x^2, x^5-x^2, and x^5+x^3+x.

2. Find the quotient when $x^3-y^3+z^3+3xyz$ is divided by $x-y+z$.

3. Multiply $4x^3+3x^2-7$ by $2x^3-x-5$, using the method of detached coefficients.

4. Draw the graph of $y=x^2+2x$, and hence solve the equation

$x^2+2x-7=0$. (Use a large x unit.)

5. Solve the equations (i) $\dfrac{1+2x-3x^2}{1-2x+3x^2}=\dfrac{3-2x+x^2}{3+2x-x^2}$.

(ii) $x^2-y=y^2-x=1\frac{5}{16}$.

6. A and B start in a long-distance race. For 15 minutes A goes at the rate of x yards per second, and B at the rate of $2x$ miles per hour, and then A is leading by 100 yards. Find the value of x.

7. If α, β are the roots of $x^2+px-q=0$, and γ, δ those of $x^2+px+r=0$, prove that $(\alpha-\gamma)(\alpha-\delta)=(\beta-\gamma)(\beta-\delta)=q+r$.

XXXIII. q.

1. Show that $\dfrac{(a+b)^3-c^3}{a+b-c}+\dfrac{(b+c)^3-a^3}{b+c-a}+\dfrac{(c+a)^3-b^3}{c+a-b}$ is equal to $2(a+b+c)^2+a^2+b^2+c^2$.

2. Solve the equations: (i) $ax+by=xy=cx+dy$.

(ii) $\left(\dfrac{x-a}{x+b}\right)^3=\dfrac{x-2a-b}{x+a+2b}$.

3. If $x=\dfrac{ab-cd}{(a-b)-(c-d)}$, show that $\dfrac{x+a}{x-b}=\dfrac{(a-c)(a+d)}{(b-d)(b+c)}$.

4. Find the L.C.M. of $8x^3+27$, $16x^4+36x^2+81$, $6x^2-5x-6$.

5. Draw enough of the graph of $y=x^2$ to enable you to find the square root of 95.

6. A dealer bought 200 sheep. He sold 80 of them so as to gain 4 per cent. on them, and the rest so as to gain $7\frac{1}{2}$ per cent. on them. His whole profit amounted to £21. 7*s*. What did he pay for each sheep?

7. Prove $x^3-px^2+qx-r=0$ to be the equation that results from the elimination of y and z from

$$x+y+z=p,$$
$$xy+yz+zx=q,$$
$$xyz=r.$$

XXXIII. r.

1. Find the factors of each of the following expressions:

$$x^2-1,\ x^2-6x-7,\ x^3-3x^2+2x,\ 3x^2-7x+2.$$

What is their L.C.M.?

2. Simplify (i) $(2x+3)(3x-1)+(2x-5)(5x-3)-(4x-3)^2$.

(ii) $\{(3a+2b)^2-(2a+b)^2\}\div\{7a-2b-(2a-5b)\}$.

3. Draw the graph of $y=x^2-3x$, and hence solve the quadratic $x^2-3x=14$. (Use a large x unit.)

4. Find the condition that $x^2+ax+b^2=0$, and $x^2-bx+a^2=0$ may have a common root.

5. In an election, if one-tenth of those who voted for A had refrained from voting, B would have been returned by a majority of 128, while if one-fifth of those who voted for B had transferred their votes to A, the latter would have been elected by a majority of 535. Which candidate was elected, and by what majority?

6. Solve the equations

$$x(x-y)=10,$$
$$y(x+y)=24.$$

7. If $x+y+z=a,\ x^2+y^2+z^2=b,\ x^3+y^3+z^3=c,$ find the product xyz in terms of a, b, c.

XXXIII. s.

1. Prove that $a+b+c$ is a factor of $a^3+b^3+c^3-3abc$.
Deduce the fact that $x+y+z$ is a factor of the expression

$$(x+y)^3+(y+z)^3+(z+x)^3-3(x+y)(y+z)(z+x).$$

2. Solve the equation $(a+b)(ax+b)(a-bx)=(a^2x-b^2)(a+bx)$.

3. If $f(n)=\dfrac{n(n+1)(2n+1)}{6}$, find the value of

(i) $f(n)-f(n-1)$.
(ii) $f(n)-f(n-2)$.

4. If α and β be the roots of the equation $x^2-px+q=0$, form the equation whose roots are $m\alpha^2+n\beta^2$, and $m\beta^2+n\alpha^2$.

5. Find the limits of value between which x must lie in order that $4x^2+4x-35$ may be positive.

6. Solve the equations

$$x+y+z=1,$$
$$x^2+y^2+z^2=9,$$
$$x^3+y^3+z^3=1.$$

7. A and B start from the same place at the same time. After an hour and a quarter A is found to be $7\frac{1}{2}$ miles ahead of B. If, however, A's rate of cycling had been greater by one-seventh, and B's by one-fifth, A would have been 8 miles ahead. Find their rates of cycling.

ANSWERS TO THE EXAMPLES.

PART I.

I. a. (p. 2).

1. $7x$.	**2.** $2a$.	**3.** a.	**4.** $4x$.	**5.** $7x$.	**6.** 0.	**7.** $8ab$.
8. $5ab$.	**9.** 0.	**10.** $4xy$.	**11.** $6xy$.	**12.** $5ab$.	**13.** $5abc$.	**14.** $12x$.
15. $9ab$.	**16.** $22ab$.	**17.** $16a$.	**18.** $14abc$.	**19.** $5a$.	**20.** $15x$.	
21. 16.	**22.** 32.	**23.** 4.	**24.** 3·2.	**25.** 6.	**26.** 20.	
27. 2.	**28.** 8.	**29.** $1\frac{1}{2}$.	**30.** $\frac{1}{8}$.	**31.** $\frac{1}{4}$.	**32.** 1·25.	
33. 3.	**34.** 9.	**35.** 5.	**36.** $6\frac{1}{2}$.	**37.** 7·2.	**38.** 4·8.	
39. 2.	**40.** 4.	**41.** 2·5.	**42.** ·8.	**43.** ·2.	**44.** ·008	

I. b. (p. 3).

1. $x+2$. **2.** $x-3$. **3.** $3x$ pence, $7x$ pence, $11x$ pence, ax pence.

4. $20x$, $2x$, $8x$, $10x$, $240x$. **5.** $2x$ miles, $7x$ miles, $\frac{x}{2}$ miles, ax miles.

6. $3x$, $36x$. **7.** $\frac{x}{12}$, $\frac{x}{36}$. **8.** $2x$, $24x$. **9.** $\frac{x}{7}$, $\frac{12x}{7}$. **10.** $16x$, xy.

11. $240x+12y$. **12.** xy pence, $\frac{xy}{12}$ shillings. **13.** $144x$.

14. $\frac{x}{144}$. **15.** $10x$, $100x$, $1000x$, $\frac{x}{1000}$.

16. $\frac{x}{10}$, $\frac{x}{100}$, $\frac{x}{1000}$, $\frac{x}{1000,000}$. **17.** $2x$, $6x$, $14x$, $2ax$, x, $3x$, $\frac{7x}{2}$.

18. $(y-x)$£. **19.** $(x-y)$£. **20.** $(x+y)$£.

I. c. (p. 6).

4. 9.	**5.** 64.	**6.** 32.	**7.** x^3.
8. a^5.	**9.** a^3x^2.	**10.** a^2b^3c.	**11.** $12ab$.
12. $20a^5$.	**13.** $36a^2b^2c^2$.	**14.** $84a^4y^3z$.	**15.** x.
16. x^3.	**17.** $4a$.	**18.** x^6.	**19.** 25.
20. x^6.	**21.** a^8b^2.	**22.** $16x^6y^8$.	**23.** x^6.
24. a^3y^9.	**25.** $8a^6y^{12}$.	**26.** x^2.	**27.** $2a$.
28. $3a^2$.	**29.** 6.	**30.** $3b$.	**31.** x^2.
32. x.	**33.** $3b^2c$.	**34.** $\frac{3}{4}ab$.	**35.** 13.
36. 25.	**37.** 25.	**38.** 49.	**39.** 24.
40. 4.	**41.** 1.	**42.** 3.	**43.** 144.
44. 64.	**45.** 2.	**46.** 4.	

I. d. (p. 7).

1. 15. **2.** 9. **3.** 1. **4.** 49. **5.** 27.
6. 100. **7.** 9. **8.** 7. **9.** 81. **10.** 500.
11. 98. **12.** 11. **13.** 14. **14.** 36. **15.** 720.
16. 6. **17.** 9. **18.** 48. **19.** 16. **20.** 32.
21. 3. **22.** 1. **23.** 3. **24.** 8. **25.** 1.
26. 8. **27.** $\frac{4}{15}$. **28.** 6. **29.** 16. **30.** 168.
31. 16. **32.** 24. **33.** 0. **34.** 0. **35.** 0.
36. 2. **37.** 0. **38.** 0. **39.** 1. **40.** $\frac{3}{16}$.
41. 0. **42.** 2. **43.** 2. **44.** $1\frac{1}{3}$.

II. a. (p. 9).

1. 2. **2.** -2. **3.** 4. **4.** -5. **5.** -18.
6. -4. **7.** $2a$. **8.** $-2a$. **9.** $-6a$.
10. $2a$. **11.** $-6x$. **12.** $6x$. **13.** $4a^2$.
14. $-14x^2$. **15.** $-3x^2$. **16.** $-7a^2$. **17.** $-3a^2$.
18. $4ab$. **19.** $-12ab$. **20.** $-2ab$. **21.** $-7ab$.
22. $-5xy$. **23.** $-9a^2b$. **24.** 0. **25.** $-4ab$.
26. -9. **27.** $3x$. **28.** $-3ab$. **29.** $-12abc$.
30. $-2abc$. **31.** $-7xy$. **32.** $4abc$. **33.** $-10abc$.
34. $3x$. **35.** $-3x$. **36.** $3x$. **37.** $-2x^3$.
38. $-5x$. **39.** $-29x$. **40.** $4x$. **41.** $-9x^2$.

II. c. (p. 12).

1. 27. **2.** -9. **3.** -1. **4.** 7. **5.** 21.
6. -15. **7.** 4. **8.** -3. **9.** 2. **10.** 4.
11. -3. **12.** 0. **13.** -1. **14.** 0. **15.** -13.
16. 0. **17.** $-\frac{1}{2}$. **18.** 0. **19.** 4. **20.** 2.
21. 0. **22.** 18. **23.** $\frac{1}{14}$. **24.** $\frac{1}{6}$. **25.** 122.
26. 0. **27.** 0. **28.** 0. **29.** -56. **30.** -89.
31. 106. **32.** -11. **33.** 7840. **34.** 9.
35. $1\frac{4}{15}$. **36.** 45. **37.** 33. **38.** 30.
39. 9, 4, 1, 0, 1, 4. **40.** -10, -8, 10, 44, 94. **41.** 4, $2\frac{1}{2}$. 3, $5\frac{1}{2}$, 10.

II. d. (p. 14).

1. 7. **2.** -6. **3.** 0. **4.** $-13a$. **5.** $5bc$.
6. $-10x^2y+xy^2$. **7.** $3x^2-8xy-3y^2$. **8.** $8a$. **9.** $2a$. **10.** $2a^2$.

III. a. (p. 16).

1. 8. **2.** 2. **3.** 8. **4.** 10. **5.** -1. **6.** 5. **7.** 0.
8. 16. **9.** 16. **10.** -9. **11.** 0. **12.** 0. **13.** 19. **14.** 4.
15. $8a$. **16.** $4a$. **17.** 0. **18.** $12a$. **19.** $-a$. **20.** a.
21. $-3a$. **22.** $3a$. **23.** $5a^2$. **24.** 0. **25.** $-3x^2$. **26.** 0.

III. b. (p. 18).

1. -3.	**2.** 2.	**3.** -6.	**4.** -1.	**5.** 0.	**6.** 0.
7. x.	**8.** $-6x$.	**9.** $2x$.	**10.** $-4x$.	**11.** $7a$.	**12.** $-a$.

13. $-9a$.	**14.** $4a$.	**15.** $5a$.	**16.** $-2x^2$.
17. $2abc$.	**18.** 0.	**19.** $\frac{3x}{2}$.	**20.** $\frac{x}{2}$.
21. $-\frac{5x}{2}$.	**22.** $\frac{5x}{2}$.	**23.** $2a^2+2a$.	**24.** $3a^2-3a$.
25. $-6x^2-2x$.	**26.** $-2x^3+x$.	**27.** $\frac{3x}{4}$.	**28** $\frac{x}{4}$.
29. $-\frac{x}{4}$.	**30.** $-\frac{3x}{4}$.	**31.** $\frac{5x}{8}$.	**32.** $\frac{x}{8}$.
33. $\frac{1}{4}xyz$.	**34.** $-\frac{x}{6}$.	**35.** $-\frac{x^2}{8}$.	**36.** $3x^2-2y^2$.

III. c. (p. 19).

1. $2a$.	**2.** $5x$.	**3.** $2a$.	**4.** $5x+2a$.	**5.** $2a-b$.
6. $5a-2b$.	**7.** $2x^2$.	**8.** $5x^2-3y^2$.	**9.** a.	**10.** $a+b$.

11. $a+b$.	**12.** $a+\frac{b}{3}$.	**13.** $a-c$.	**14.** $a+b-2c$.
15. $3a-3b-3c$.	**16.** $2x^2+6x+4$.	**17.** $3x^2-3x-3$.	**18.** x^3-x^2-x.
19. x^2+2.	**20.** $3x^2+x-5$.	**21.** $2a$.	**22.** $6a-3c$.
23. $4x-y+3z$.	**24.** b^2.	**25.** $5x^2+3x$.	**26.** $2x^2+2y^2$.
27. $5(a-b)$.	**28.** $a+b$.	**29.** x^2-y^2.	**30.** $x+5$.
31. $a-b$.	**32.** $-(x-3)$.	**33.** $8\frac{1}{2}$.	**34.** $3\frac{3}{4}$.
35. 6.	**36.** 7.	**37.** $6a-2b$.	**38.** $x+5y$.
39. $10x-15$.	**40.** $9-5x$.	**41.** $9+2x$.	**42.** $2ax$.

III. d. (p. 20).

1. $14a$.	**2.** $2a$.	**3.** $-10x$.	**4.** $9x^2$.
5. $-3y$.	**6.** 0.	**7.** $5ab$.	**8.** 0.
9. $-3x^3$.	**10.** $2x$.	**11.** $4a$.	**12.** $\frac{4x}{y}$.
13. $\frac{5x}{4}$.	**14.** $2x$.	**15.** $4a$.	**16.** $5x^2$.
17. $4ab$.	**18.** $4x^2y$.	**19.** $-6abc$.	**20.** $-15x^4$.
21. 0.	**22.** $\frac{2x}{3}$.	**23.** $-\frac{x}{9}$.	**24.** $-4a^2$.

III. e. (p. 21).

1. $a^2-b^2+c^2$.	**2.** $6a+6b+6c$.	**3.** $2x-y-9z$.
4. $-6a-6b-6c$.	**5.** $13ax+3by+4cz$.	**6.** $2a+2b+2c$.
7. $4a$.	**8.** $8a-6b-2c$.	**9.** $2x^2+4xy+y^2$.
10. $3x^3+y^3$.	**11.** x^3-3x^2+8x+7.	**12.** $4a^3-2b^3-5c^2+3d$.

13. $2x^3-5x^2y+2xy^2+3y^3$. **14.** p^2-3q^2.
15. $5x^2yz-6xy^2z-6xyz^2$. **16.** $a^2+b^2+ab-4bc-3ac$.
17. $a^3+4a^2c+3abc+ac^2$. **18.** $2a+9b+17c$.
19. $-\frac{2x}{3}+\frac{4y}{3}+\frac{2z}{3}$. **20.** $a+2b+5c$. **21.** $12x-10y$.

III. f. (p. 22).

1. $3a$. **2.** $5a$. **3.** $-5a$. **4.** $7b$. **5.** $-5b$.
6. 0. **7.** $19b$. **8.** $-2x$. **9.** $4y$.
10. $-2x^3$. **11.** $4ax^2$. **12.** $-4ax^2$. **13.** $18ax^2$.
14. $-20ax^2$. **15.** $-a$. **16.** $-11a$. **17.** $3a$.
18. $-3a-2b$. **19.** $-a+b$. **20.** $2b$. **21.** $a-2b$.
22. b. **23.** $\frac{a}{2}+\frac{b}{2}$. **24.** $\frac{a}{2}-\frac{b}{2}$. **25.** $a+b-c$.
26. $c-a-b$. **27.** $ax-a$. **28.** $ax+a$. **29.** $a-ax$.
30. x^2-x. **31.** b. **32.** $-3b$. **33.** $c-b$.
34. $2b+c$. **35.** $4y^2-x^2+2z^2$. **36.** $12+10x-x^2$. **37.** $2x^2-2px-q$.

III. g. (p. 23).

1. $2b^2$. **2.** $4x+4y-5z$. **3.** $2x^2-2x+4$.
4. $-2x^2+4xy+8y^2$. **5.** $-a-2b+c+4d$. **6.** $2x-4a-13$.
7. $8b^2+8ab-9$. **8.** $a-2b-6d$. **9.** $-3x^2y-2xy^2+y^3$.
10. $3a-2b+2c-2d$. **11.** $x-5y-z-2$. **12.** $5a^2-4ab-14$.
13. $4x^3+9x^2+5x-17$. **14.** a^3-9a^2+6a+6.
15. $2ab-2bc+2cd-ad$. **16.** $2a^4+2a^3-5a^2-3a+1$.
17. $6x^4-3x^3-6x-29$. **18.** 3. **19.** 11. **20.** 2.
21. $4a$. **22.** x^2. **23.** x^2-4x. **24.** $2b$. **25.** $2a-11x$. **26.** $8a$.
27. $7a-5$. **28.** $3x^2+x$. **29.** 6. **30.** $a+5b$. **31.** 7.
32. $13\frac{1}{3}$. **33.** $2a-8b$. **34.** $-2x+5y-z$. **35.** $6-7x$.
36. $-3a^2+b^2-c^2$. **37.** $a+b+d$. **38.** $-x^2-3x$.

IV. a. (p. 26).

1. $6a$. **2.** $-9a$. **3.** $8a$. **4.** $2a^3$.
5. $-2a^4$. **6.** $-6a^2b^2$. **7.** $12xy$. **8.** $6xy$.
9. $-15xy$. **10.** $-14x^3$. **11.** $a^2b^2c^2$. **12.** $-a^2b^3c$.
13. $-a^2x^3$. **14.** $6a^3b$. **15.** $-8x^5$. **16.** $-p^{14}$.
17. p^8q^8. **18.** $-6p^3q^4$. **19.** $a^3b^5c^7$. **20.** $\frac{ab}{6}$.
21. $-a^2b^2$. **22.** $-\frac{5x^4}{3}$. **23.** $\frac{x^2y^3z}{2}$. **24.** $-\frac{9a^2b^2c^2}{5}$.
25. 24. **26.** $-abc$. **27.** $-a^2b^2c$. **28.** ab^2c^2.
29. $30abc$. **30.** $24abc$. **31.** $-a^2x^2y$. **32.** $-3ax^3$.
33. $-a^3$. **34.** $-8a^3$. **35.** $2a^2b^3c^4$. **36.** $24p^3q^2r$.

37. a^2. **38.** $-a^3$. **39.** a^6. **40.** $-8a^3$. **41.** x^6.
42. x^6. **43.** $-x^6$. **44.** $-8x^3y^3$. **45.** $16x^4y^4$. **46.** -1.
47. 1. **48.** -1. **49.** $-x^{14}$. **50.** $-x^{15}$.
51. $64x^{12}$. **52.** $-8a^6b^3$. **53.** $-27x^6y^3$. **54.** $81x^4y^8$.

IV. b. (p. 27).

1. $5a+25b-15c$. **2.** $-8a+12b-8c$. **3.** $2a^2+2ab+2ac$.
4. $-6a^3+4a^2-10a$. **5.** $42a^5-28a^4-14a^3-35a^2$.
6. $ab^2c-b^2c^2+abc^2$. **7.** $-6a^2b^2c+9ab^2c^2+12a^2bc^2$.
8. $x^5-2x^4y+x^3y^2$. **9.** $-3x^5+9x^4y-9x^3y^2+3x^2y^3$.
10. $-a^2c-abc-b^2c+ac^2+bc^2$. **11.** $3a^2b^2c+2a^2bc^2-ab^2c^2$.
12. $-2x+6x^2+4x^3-2x^4$. **13.** $2x^4-6x^3+6x^2+2x$.
14. $-15x^6+10x^4-30x^2$. **15.** $6a^2b^2+4ab^3-2b^4$.
16. $60a^9b^4c^3+12a^7b^8c^6-108a^6b^9c^5$. **17.** a^2b-ab^2.
18. $6a^3c-12a^2bc-6ab^2c$. **19.** $-6x^4+30x^3-18x^2$.
20. $12x^6-36x^5+24x^4-36x^2$. **21.** a^{m+n}. **22.** $-a^{m+n}$.
23. a^{2m}. **24.** a^{3m}. **25.** a^{n+3}. **26.** $-a^{n+5}$. **27.** a^{5m}.
28. a^{2m+2n}. **29.** $-2a^{2m}$. **30.** $15a^{m+n}b^{m+n}$. **31.** $a^{2x}+a^{3x}$.
32. $e^{4x}-e^{3x}+e^{2x}$. **33.** a^{2m}. **34.** a^{2m-8}.
35. 2. **36.** 14. **37.** 0. **38.** -8. **39.** 0. **40.** 2.
41. 3. **42.** -7. **43.** 5. **44.** 0. **45.** 3. **46.** 7.
47. 5. **48.** -2. **49.** 7. **50.** 7. **51.** 17. **52.** 14.

IV. c. (p. 29).

1. x^2+5x+6. **2.** x^2-5x+6. **3.** x^2-x-6.
4. x^2+x-6. **5.** $x^2+12x+27$. **6.** $x^2+3x-18$.
7. $x^2-18x+77$. **8.** $x^2+4x-77$. **9.** $1+3x+2x^2$.
10. $1+x-12x^2$. **11.** $1-3x+2x^2$. **12.** $6+5x+x^2$.
13. $30+11x+x^2$. **14.** $21+10x+x^2$. **15.** $1-2x-63x^2$.
16. $1-4x-21x^2$. **17.** x^2-1. **18.** x^2-4.
19. x^2-9. **20.** x^2-49. **21.** $1-x^2$.
22. $4-x^2$. **23.** $49-x^2$. **24.** $81-x^2$.
25. $x^2+2xy+y^2$. **26.** $x^2+5xy+6y^2$. **27.** x^2-4y^2.
28. $x^2-5xy+6y^2$. **29.** $x^2-xy-6y^2$. **30.** $x^2-xy-20y^2$.
31. $4x^2+4xy+y^2$. **32.** $9x^2-6xy+y^2$. **33.** $6x^2-x-12$.
34. $6x^2-11x+4$. **35.** $10x^2+27x+18$. **36.** $15x^2-29x-14$.
37. $6-13x+6x^2$. **38.** $30+11x-28x^2$. **39.** $4-9x^2$.
40. $4x^2-25$. **41.** $25x^2-49$. **42.** $36x^2-25$.
43. $81x^2-64$. **44.** $16x^2-49$. **45.** $x^2-ax+bx-ab$.
46. $x^2+ax-bx-ab$. **47.** $a^2+2ab+b^2$. **48.** $a^2x^2+2abx+b^2$.
49. $a^2-2ab+b^2$. **50.** $a^2x^2-2abx+b^2$. **51.** $p^2x^2-2pqx+q^2$.
52. $p^2+2pqx+q^2x^2$. **53.** $a^2-2ax-15x^2$. **54.** $21-x-2x^2$

55. $x^2-a^2y^2$. 56. $p^2x^2-q^2$. 57. $p^2x^2+2pqx+q^2$.
58. $c^2x^2-2cdx+d^2$. 59. $12x^2-25xy+12y^2$. 60. $12x^2+xy-20y^2$.
61. $42x^2+20cx-32c^2$. 62. $6a^2x^2+13ax+6$. 63. a^4-b^4.
64. a^4-16b^2. 65. $a^4+2a^2b-24b^2$. 66. $a^4-8a^2b+15b^2$.
67. $16a^4-9b^2$. 68. $25a^4-4b^4$. 69. x^4-4a^4.
70. x^4-p^2. 71. a^2-b^6. 72. $a^2-2ab^3+b^6$.
73. x^6-1. 74. x^6-4. 75. a^2x^4-1. 76. $b^2x^4-c^2$.
77. $abx^2+ax+bx+1$. 78. $abx^2-ax+bx-1$.
79. $3x^2+6xy+x+2y$. 80. $6x^2-3ax+2bx-ab$.
81. $ac+bc+ad+bd$. 82. $ac-bc-ad+bd$.
83. $6ac-3bc+8ad-4bd$. 84. $2ac+6bc-5ad-15bd$.
85. $x^4+ax^2-3bx^2-3ab$. 86. $a^2x^3+2abx^2+b^2x$.
87. $a^2x^3-b^2x$. 88. $x^3+ax^2+a^2x+a^3$.
89. $x^3+ax^2-a^2x-a^3$. 90. $x^3-2x^2y-4xy^2+8y^3$.

IV. d. (p. 31).

1. $a^2+2ab+b^2$. 2. $a^2+2ax+x^2$. 3. $c^2+2cd+d^2$.
4. $x^2+8x+16$. 5. $x^2+14x+49$. 6. p^2+6p+9.
7. $a^2-2ab+b^2$. 8. $a^2-2ax+x^2$. 9. $c^2-2cd+d^2$.
10. $x^2-8x+16$. 11. $x^2-18x+81$. 12. $p^2-8p+16$.
13. $4p^2+12p+9$. 14. $9p^2+6pq+q^2$. 15. $4p^2-20p+25$.
16. $16p^2-8p+1$. 17. x^2-2x+1. 18. $9x^2-6x+1$.
19. $1-2x+x^2$. 20. $1-4x+4x^2$. 21. $1-10x+25x^2$.
22. $1+2p+p^2$. 23. $1+14p+49p^2$. 24. $4a^2+12ab+9b^2$.
25. $16x^2-24xy+9y^2$. 26. $a^2-2ab+b^2$. 27. $4a^2-4ax+x^2$.
28. $4x^2-12ax+9a^2$. 29. $4x^2-12ax+9a^2$. 30. $16p^2+40pq+25q^2$.
31. $25p^2-40pq+16q^2$. 32. $a^4+2a^2b^2+b^4$. 33. $a^4-2a^2b^2+b^4$.
34. $a^4+2a^2b+b^2$. 35. $a^4-2a^2p+p^2$. 36. $4a^4-12a^2b^2+9b^4$.
37. $16a^4+24a^2b^2+9b^4$. 38. $a^6+2a^3b+b^2$. 39. $x^6+2x^3y^3+y^6$.
40. $x^6-2x^3y^3+y^6$. 41. $4x^4+4ax^2+a^2$. 42. $9x^4-6x^2y^2+y^4$.
43. $1-4x^2+4x^4$. 44. $1+2x+x^2$. 45. $1+4x+4x^2$.
46. $x^8+2x^4a^4+a^8$. 47. $x^8-2x^4y^4+y^8$. 48. $4x^8-12x^4y^4+9y^8$.
49. $4p^6+12p^3q^2+9q^4$. 50. $x^{10}-2x^5a^5+a^{10}$.

IV. e. (p. 31).

1. x^2-1. 2. x^2-4. 3. $1-x^2$. 4. x^2-25.
5. $9-y^2$. 6. $49-x^2$. 7. b^2-a^2. 8. $4p^2-q^2$.
9. $9p^2-q^2$. 10. a^2-9b^2. 11. $9p^2-4q^2$. 12. $25x^2-16a^2$.
13. a^2-b^2. 14. $4a^2-x^2$. 15. a^2-49b^2. 16. a^2-49b^2.
17. x^4-y^4. 18. a^4-4b^4. 19. $p^2x^2-q^2$. 20. $a^2-b^2x^2$.
21. x^6-a^6. 22. x^4-a^2. 23. $4a^6-x^2$. 24. $4a^4-9x^2$.
25. $1-x^6$. 26. $1-a^2x^4$. 27. $9-a^6$. 28. $121-49x^2$.
29. $81-64x^2$. 30. $49x^2-81$.

IV. f. (p. 32).

1. 9604. **2.** 40401. **3.** 10404. **4.** 10609.
5. 11449. **6.** 99980001. **7.** 1002001. **8.** 1004004.
9. 98·01. **10.** 100060009. **11.** 400040001.
12. 999600·04. **13.** 400400100. **14.** 4020025.
15. 10060·09. **16.** 1016064. **17.** 998001.
18. 9994·0009. **19.** 6432·04. **20.** 360600·25.
21. 809280·16. **22.** 250300·09. **23.** 81·108036.
24. 63·936016. **25.** 10004·000. **26.** 1·0100.
27. 101·606. **28.** 999920·00. **29.** 100·1000.
30. 999996. **31.** 39991. **32.** 9991. **33.** 6391.
34. 120·75. **35.** 99·51. **36.** 6396. **37.** 399·9984.
38. 2·8896. **39.** 3·9984. **40.** 80999999·84.

IV. g. (p. 33).

1. x^3-3x^2+3x-1. **2.** x^3+5x^2+8x+4. **3.** $4x^3-8x^2+5x-1$.
4. x^3+8. **5.** $27x^3-1$. **6.** $6x^3+11x^2-2x+20$.
7. $x^3-2ax^2+2a^2x-a^3$. **8.** $125x^3-1$. **9.** $a^3+a^2b+ab^2+b^3$.
10. x^3-a^3. **11.** $a^3+a^2b-ab^2-b^3$. **12.** $x^3-9x^2+27x-27$.
13. $8x^3-1$. **14.** $8x^3-32x^2+4x+35$.
15. $4x^3-8x^2-3x+6$ **16.** $x^4+3x^3-6x^2-6x+8$.
17. $27x^3+1$. **18.** x^4+2x^3-2x-1.
19. $x^3-ax^2-bx^2-cx^2+abx+bcx+cax-abc$. **20.** x^4-16a^4.
21. $x^4-18b^2x^2+81b^4$. **22.** $12x^3-16x^2-79x-42$.
23. $a^3-a^2c-ab^2+b^2c$. **24.** $a^2-b^2-ac+bc$.
25. $6a^2+ab-3ac+4bc-12b^2$.

IV. h. (p. 33).

1. 9. **2.** 4. **3.** -5. **4.** 17. **5.** 1.
6. -13. **7.** $x+3$. **8.** $3x-6$. **9.** $6x-10$. **10.** $-3x$.
11. 5. **12.** 11. **13.** 0. **14.** $6-a$. **15.** 0.
16. -31. **17.** $ad+b$. **18.** 0. **19.** 6. **20.** 31.
21. c^2+b^2. **22.** 0. **23.** $a^2+2ab+b^2$.
24. $21x^3+8x^2-39x+10$. **25.** x^2-6x. **26.** 42.
27. $20x^2-5ax$. **28.** $16x^2-8x$. **29.** $26x-10$. **30.** $16p-4q$.
31. $9x^3-6x^2+7x-2$. **32.** $2a^2+5ab+2b^2$. **33.** 7. **35.** $14x$

V. a. (p. 36).

1. x. **2.** 3. **3.** x. **4.** $-x$. **5.** bc.
6. $-bc$. **7.** a. **8.** $-a$. **9.** $-x$. **10.** x.
11. a^2. **12.** $-a$. **13.** 1. **14.** -1.

15. $4x^2$. **16.** $-3x^2$. **17.** -2. **18.** $3a^2$.
19. $-7a^2x^3$. **20.** a^2b^5. **21.** $-9a$. **22.** $4abc$.
23. $-3x^3$. **24.** $-9ab^2c^5$. **25.** $3a$. **26.** 6.
27. $-6a$. **28.** $8a$. **29.** $-6ab^2$. **30.** xyz^2.
31. $24a^5b^4$. **32.** $3p^3q^4x$. **33.** $-7a^3c^4$. **34.** $-7qr$
35. $-8ln$. **36.** $-9a^2b^4c^6$. **37.** $-18ax^4$. **38.** $11xy^5$.

V. b. (p. 36).

1. $a-2b$. **2.** $-a+3b$. **3.** $4x-3$. **4.** $-y+6$.
5. $a+b$. **6.** $b-a$. **7.** $a-2b$. **8.** $a-3b$.
9. $-3a^2+7b^2$. **10.** $b+c$. **11.** $-a-b$. **12.** $4x-5$.
13. $7x-9$. **14.** a^2b-ab^2. **15.** $3a-7b$. **16.** $6x^4y^5z-5x^2y^3z$
17. $-2a+b$. **18.** $11x+6y$. **19.** $2a^2-4b^2$. **20.** m^2-4mn.
21. $-4a+3b+6c$. **22.** $a+c+d$. **23.** $-3a+4d+12x$.
24. $-a-x-ax$. **25.** $-a+4b-8c$. **26.** x^2+3x-3.
27. $-x^2+ax-a^2$. **28.** $a+5b^2-3b$. **29.** $-a+b-c$.
30. $-2x^3+x^2-4x+1$. **31.** $3y^3-xy^2-6x^3$. **32.** $-3xy+7y^2+x^2$.
33. $-xy^5+2x^2y^3+7x^3y$. **34.** $3xy^2z^4-5x^2yz^3+6x^3y^4z^2$.
35. a^{m-n}. **36.** a^{n-3}. **37.** x^{4-p}. **38.** $-3x^{n-4}$.
39. $9x^{m-n}y^{n-m}$. **40.** $9x^{3-n}y^{3-n}$.

V. c. (p. 38).

1. $x+4$. **2.** $x-4$. **3.** $a+1$. **4.** $a-1$.
5. $b+7$. **6.** $x+3$. **7.** $x-7$. **8.** $x-1$.
9. $a-6$. **10.** $y+9$. **11.** $x-2$. **12.** $5x+3$
13. $2x-1$. **14.** $3x-7$. **15.** $3x+1$. **16.** $2x-4$.
17. $2+x$. **18.** $1-2x$. **19.** $3-x$. **20.** $a-2$.
21. $5-3a$. **22.** $5y+11$. **23.** $x-a$. **24.** $5x+4$.
25. $a+2x$. **26.** $5-x$. **27.** $1+2x$. **28.** $x+2y$.
29. $1-8pq$. **30.** $3a-b$. **31.** $a-bc$. **32.** $2x^2+7$.
33. $9x^3-1$. **34.** $5x^2+4y^2$. **35.** $10-x$. **36.** $1+10b^2$

V. d. (p. 39).

1. x^2+a^2. **2.** $x+b$. **3.** $x-a$. **4.** $x+1$.
5. $x+a$. **6.** $x-2$. **7.** $px+1$. **8.** $x+1$.
9. $x-a$. **10.** $px+2$. **11.** $ax-5c$. **12.** $ax+c$.
13. $x-7$. **14.** $ax+b$. **15.** $3ax+2b$. **16.** $ax-b$.
17. $9x+bc$. **18.** $2x+bq$. **19.** $bx+c$. **20.** $5px+3q$.
21. $x-3$. **22.** $15(x-3a)$. **23.** $2x+3$. **24.** x^2+2x+1
25. $21(x+3)$. **26.** $2x^3-11x^2+4x+5$, x^2-6x+5. **27.** $2x-3$.
28. $a^3-a^2b-ab^2+b^3$. **29.** 18. **30.** 33. **31.** $x-1$.
32. $2x-1$. **33.** -4. **34.** $bx-c$. **35.** $ax-b$ **36.** $a+2b$.

VI. a. *Oral.* (p. 40).

1. (i) x. (ii) $\frac{3x}{2}$. (iii) $\frac{x}{2}$. (iv) $\frac{9ab}{2}$. (v) $\frac{5abc}{2}$. (vi) $\frac{5a}{2}$.

2. (i) 9. (ii) -11. (iii) 0. (iv) 1. (v) -5. (vi) ·5.

3. (i) 25. (ii) 9. (iii) $\frac{1}{9}$. (iv) $\frac{1}{4}$. (v) 1. (vi) -1. (vii) $\frac{a^2b^2}{4}$. (viii) $-\frac{a^3b^3}{8}$.

4. (i) 13. (ii) 25. (iii) -5. (iv) 1. (v) 9. (vi) 27.

5. (i) 5. (ii) $-a$. (iii) $-3a$. (iv) $7x^2$. (v) 0. (vi) 3.

6. (i) 0. (ii) 3. (iii) $-\frac{3}{4}$. (iv) 8. (v) $1\frac{1}{4}$. (vi) $5\frac{1}{4}$.

7. (i) 7. (ii) 3. (iii) 13. (iv) 1. (v) 1. (vi) 31.

8. (i) -1. (ii) 0. (iii) -6. (iv) 3. (v) 14. (vi) -52.

9. (i) $5x$. (ii) $5a$. (iii) $3x^2$. (iv) $2ab$. (v) $9x-20$. (vi) 2.

10. (i) 2. (ii) x. (iii) $x+2$. (iv) $x-1$. (v) $x-1$. (vi) $x+2$. (vii) $4x+2$. (viii) $a+b+c$.

11. (i) bx^2. (ii) $-2cx$. (iii) x^3. **12.** (i) $a-b$. (ii) $c-b$.

13. (i) $-4a$. (ii) $4a$. (iii) 0. (iv) $\frac{5x}{4}$. (v) x^2-1. (vi) x^2+1. (vii) x^3+1. (viii) x^2-5x+1. (ix) $7(x-1)$. (x) $x-3$. (xi) $a+bx$. (xii) a. (xiii) $4bx$. (xiv) 2.

14. (i) $4x-3y+z$. (ii) $3x^2$. (iii) $a+5b+3c$. (iv) $x^3-x^2y+xy^2$. (v) $4x^3-4x^2-5$. (vi) $2a-b$.

15. (i) $4x$. (ii) x^2+xy. (iii) $\frac{x}{4}$. (iv) $3y-2x$. (v) $2a^2x$. (vi) $8b$. (vii) 0. (viii) $2a-2b$. (ix) $4(2-x)$. (x) $a+b$. (xi) $2b-2a$. (xii) $2x-6$. (xiii) x^3-x^2. (xiv) $5x^3-8x^2+5x+1$. (xv) $2(x-y)$. (xvi) $2(b-2a)$. (xvii) $3x^2$. (xviii) $2bc$. (xix) $2(x-y+z)$.

16. (i) $4x$. (ii) $7x^2-4$. (iii) $-x^2$. (iv) $2a^2x$. (v) $6-2x^2$. (vi) $4(-b)$. (vii) x^3-14x^2+5. (viii) $-7(a^2-b^2)$. (ix) 141. (x) 5. (xi) 81. (xii) 24.

17. (i) $-6ab$. (ii) -1. (iii) $-ax$. (iv) $\frac{a}{2}$. (v) $3a^3b^3c^3$. (vi) $3b$. (vii) $-\frac{3x^7}{2}$. (viii) $9x^2$. (ix) $-\frac{a^3x}{9}$. (x) $\frac{3x}{2}$. (xi) ax^2. (xii) $-ax$. (xiii) $-a^7$. (xiv) $-a$. (xv) $-a^{10}$. (xvi) -1.

18. (i) $4ax^2y-3axy^2$. (ii) $-2x^3+6x^2-x$. (iii) $3x^2+4x-2$. (iv) $4x^2-2x+3$. (v) $-3x^2+2x+9$. (vi) $-18x^4+12x^3-6x^2$.

19. (i) $1-x^2$. (ii) $1+2x+x^2$. (iii) $1-4x+4x^2$. (iv) $a^2+4ab+4b^2$. (v) $x^2+8x+15$. (vi) x^2-x-6. (vii) $x^2-5xy+6y^2$. (viii) $9x^2-1$. (ix) $30-11p+p^2$. (x) a^4-9. (xi) $9x^2-25$. (xii) $a^4x^2+2a^2x+1$. (xiii) $2x^2-32$. (xiv) $x^4+5x^2y+6y^2$. (xv) $1+2x-8x^2$. (xvi) a^2-4b^2. (xvii) $1+4x+4x^2$. (xviii) $3a^2-3x^2$ (xix) $4a^2-1$. (xx) $9x^4-1$.

20. (i) $9a^2-12ab+4b^2$. (ii) $4a^2-4ay+y^2$. (iii) a^4-4a^2+4.
(iv) $x^2+ax+\frac{a^2}{4}$. (v) $4x^2-4x+1$. (vi) $9x^2-6x+1$.
(vii) $21+4x-x^2$. (viii) $75-3x^2$. (ix) $2x^2-4xy+2y^2$
(x) $x^2+cx-ax-ac$. (xi) x^2-5x+6. (xii) $x^2-\frac{4}{9}$.
(xiii) $a^2+2ax-8x^2$. (xiv) $abx^2-ax-bx+1$. (xv) $9a^2-\frac{1}{4}$.
(xvi) $36x^2-1$. (xvii) $10x^2+9x-9$. (xviii) $15x^2+29x-14$.
(xix) $15x^2+13x+2$. (xx) $14x^2+xy-3y^2$.

21. (i) 3. (ii) -7. (iii) -27. (iv) -2.

22. (i) 3. (ii) 11. (iii) 16. (iv) -8.

23. (i) $-x^3$. (ii) $2a^2$. (iii) $7a$. (iv) $\frac{5a}{3}$. (v) $4r$.
(vi) $-\frac{27p^3q}{4}$. (vii) $3b-4a$. (viii) $3x^2+1$. (ix) $3a-4x$. (x) $3b-4a$.
(xi) $-a^2+bc-c^2$. (xii) $-x$. (xiii) $a-x$. (xiv) $2(a-b)$. (xv) x.
(xvi) $5a$. (xvii) $(a+x)^2$. (xviii) ax. (xix) 2. (xx) $(a-x)^2$

VI. b. (p. 44).

1. 0, 1, 9. **2.** $2x^4-7x^3+5x-3$, -7, 0.
4. $a+4b$, $6b$. **5.** $6x^2+7ax-20a^2$, ax^2-a^3.
6. $7x$, $-3x^2$, $2a^2-3ab+4b^2$. **7.** $3x-y$.

VI. c. (p. 44).

1. 0, 9, 1. **2.** $b^4+2ab^3+5a^2b^2-3a^3b+a^4$, $-3b$.
4. x^2-1. **5.** x^2-9a^2, $x^4-ax^3-2a^2x^2$.
6. (1) x, (2) $x-3a$, (3) a^2bc. **7.** $3a+4b$.

VI. d. (p. 45).

1. 6, 12, 0. **2.** x^3-3x^2+3x-1, 0. **3.** 0, $x-8$.
4. $a-13c+6b$. **5.** $-a^3b^2$, a^7x^5, $-a^4b^4c^4$.
6. $6x-9a$. **7.** $21p^2+pq-36q^2$.

VI. e. (p. 45).

1. 1, -1, 64. **3.** $10a+2x$, $x^3+3x^2-16x-4$, 32
4. x^2-2x+3. **5.** $15x^2+3ax$.
6. $ax-3a$. **7.** $-6y^2$.

VI. f. (p. 45).

1. -1, 0, 0. **3.** $x-7$, $2(x-1)$.
4. $3x^3-4x^2+6x-2$, 18. **5.** $7x^2-17ax-12a^2$.
6. $18x^4+9ax^2-2a^2$. **7.** $5x-4a$.

VI. g. (p. 46).

1. $-3, \ -20.$ 2. 2 miles East.
3. $x^2-2ax+a^2, \ ax-2x^2.$ 4. $11ax^2.$
5. $2a^2-5ab+3b^2.$ 6. $4x^2-a^2, \ x^4-9, \ a^2-p^4.$

VI. h. (p. 46).

1. 27, 44. 2. $6x^2+2.$ 3. $5x-y-6a.$ 4. $5x^2+10.$
5. 15, 1, -3, 3, 19. 6. $x^3-2x^2y-4xy^2+8y^3.$ 7. $ax+3p.$

VI. k. (p. 46).

1. $-33, \ -25.$ 2. $2x^2-3x.$ 3. $a^2-b^2-c^2+2bc.$ 4. $2x, \ 2y.$
5. 23, 9, 1, -1, 3. 6. $x^3+ax^2-a^2x-a^3.$ 7. $4x-5.$

VII. a. (p. 48).

1. 3. 2. 3. 3. 4. 4. -5. 5. 3. 6. -3.
7. -6. 8. 0. 9. 5. 10. 2. 11. 12. 12. -8.
13. -20. 14. 0. 15. $2\frac{1}{2}$. 16. $2\frac{1}{3}$. 17. 9. 18. 2.
19. 1. 20. $\frac{2}{3}$. 21. $\frac{1}{2}$. 22 $-\frac{1}{4}$. 23. 8. 24. -15.
25. 0. 26. $1\frac{1}{2}$. 27. 3. 28. -3. 29. 1. 30. 0.
31. -1. 32. -1. 33. 2. 34. 4. 35. 2. 36. 2.
37. 3. 38. $3\frac{1}{2}$. 39. -2. 40. 2. 41. 20. 42. 3.
43. 3. 44. ·01. 45. ·03. 46. $-$·03.

VII. b. (p. 49).

1. 2. 2. 12. 3. 7. 4. 1. 5. 3.
6. 12. 7. $1\frac{1}{3}$. 8. -3. 9. 0. 10. 0.
11. 2. 12. 2. 13. 5. 14. -3. 15. 5.
16. 4. 17. -1. 18. 0. 19. 3. 20. -2.
21. $1\frac{1}{2}$. 22. $1\frac{1}{4}$. 23. -27. 24. $1\frac{1}{2}$. 25. -6.
26. $1\frac{1}{20}$. 27. -9. 28. $-8\frac{1}{2}$. 29. 0. 30. 3.
31. $2\frac{1}{2}$. 32. 2. 33. 5. 34. 5. 35. 3.
36. $1\frac{1}{4}$. 37. 3. 38. -5. 39. 10. 40. $3\frac{1}{2}$.
41. $-2\frac{1}{3}$. 42. $2\frac{1}{2}$. 43. 0. 44. 2.

VII. c. (p. 51).

1. 3. 2. 10. 3. 14. 4. 22. 5. 3. 6. 28.
7. -11. 8. -3. 9. 7. 10. 28. 11. 2. 12. 3.
13. 9. 14. -20. 15. 1. 16. 2. 17. $2\frac{1}{3}$. 18. -5.
19. $-1\frac{1}{3}$. 20. $\frac{1}{2}$. 21. -2. 22. 3.
23. 1. 24. 7. 25. 5 26. 9.

VII. d. (p. 53).

1. 18.	**2.** 12.	**3.** 15.	**4.** 4.	**5.** 70.	**6.** 12.
7. 4.	**8.** 14.	**9.** 19.	**10.** 7.	**11.** 2.	**12.** $3\frac{1}{2}$.
13. 5.	**14.** 2.	**15.** -1.	**16.** 1.	**17.** 4.	**18.** 11.
19. 7.	**20.** 11.	**21.** -6.	**22.** $-1\frac{1}{5}$.	**23.** 12.	**24.** 8.
25. 4.	**26.** 8.	**27.** 12.	**28.** 12.	**29.** -7.	**30.** 8.
31. 2.	**32.** 10.	**33.** -1.	**34.** $-\frac{1}{2}$.	**35.** $1\frac{2}{3}$.	**36.** -7.
37. 0.	**38.** -2.	**39.** 2.	**40.** 2.	**41.** 15.	**42.** 17.
43. $-\frac{16}{17}$.	**44.** 9.	**45.** 2.	**46.** 3.	**47.** 2.	**48.** 7.
49. $\frac{1}{3}$.	**50.** 3.	**51.** 1.	**52.** 5.	**53.** 14.	**54.** 14.
55. 7.	**56.** $30\frac{19}{29}$.	**57.** 3.	**58.** 15·5.	**59.** 1.	**60.** 1·5.

61. 140. **62.** ·69. **63.** 3. **64.** 1·95.

65. 2. **66.** 1·1. **67.** 1. **68.** $1\frac{1}{17}$.

69. When $x = -4\frac{3}{5}$. **70.** 1. The equation has no root. **71.** No root.

VII. e. (p. 55).

1. 10.	**2.** 4·7.	**3.** $-$·78.	**4.** 4·33.	**5.** 5·71.	**6.** 26.
7. 2·53.	**8.** 46·83.	**9.** -1·43.	**10.** ·45.	**11.** 3·03.	**12.** 2·04.

VIII. a. (p. 57).

1. $x-20$.	**2.** $35-y$.	**3.** $x-20$.	**4.** $34-x$.	**5.** $\frac{56}{x}$.
6. $35x$.	**7.** 21.	**8.** $x-23$.	**9.** $y-x$.	**10.** $x-13$.
11. $\frac{78}{x}$.	**12.** $\frac{x}{y}$.	**13.** $\frac{5b}{3a}$.	**14.** $20x-y$.	**15.** $96-x-y$.
16 $a+2b$.	**17.** $2y-x$.	**18.** $\frac{y}{x}$.	**19.** $\frac{12y}{x}$.	**20.** $\frac{12x}{y}$.
21. $20y-\frac{5x}{2}$.	**22.** $x+4$.	**23.** $4+x$.	**24.** $20-x$.	**25.** $40-a$.

26. 25. **27.** $\frac{4}{x}$ pence. **28.** $x+7$, $x+y$, $x-11$ years old.

29. $\frac{3x}{2}$. **30.** $\frac{x}{6}$ miles, $\frac{xy}{6}$ miles, $\frac{6}{x}$ hours, $\frac{6y}{x}$ hours. **31.** $2b$.

32. $\frac{x}{y}$. **33.** $3x$ pence. **34.** $\frac{x}{4}$ pence. **35.** 2.

36. $\frac{x}{12}$ pence, $\frac{144}{x}$ eggs, $\frac{144y}{x}$ eggs. **37.** $\frac{8x}{3}$ pence. **38.** $\frac{yz}{x}$ pence.

39. n, $n+1$, $n+2$. **40.** $n-2$, $n-1$, n. **41.** $n-1$, n, $n+1$.

42. n, $n+1$, $n+2$. **43.** $n-2$, $n-1$, n, $n+1$, $n+2$. **44.** $\frac{xy}{20}$.

45. $2x-2y$. **46.** $4b$. **47.** $240a+12b+c$.

48. $\frac{88x}{3}$. **49.** $10x$ miles. **50.** $\frac{532}{x}$ days, $\frac{532}{xy}$ days.

51. $\frac{x}{4}+25$. **52.** $2n-1$, $2n-2$, $2n-3$, $2n-4$, $2n-5$.
53. $2n-5$, $2n-3$, $2n-1$, $2n+1$, $2n+3$. **54.** ab sq. ft. **55.** $\frac{x}{y}$ feet.
56. x^2 sq. ft. **57.** $x-20=y$. **58.** $3x-y=25$.
59. $\frac{x-8}{6}=\frac{2x+3}{7}$. **60.** $3(x-4)=5(x-1)$. **61.** $20y+2z=x$.
62. $240b+30c+12d=a$. **63.** $x(x-1)=y$. **64.** $(x-1)x(x+1)=a^2$.
65. $2x+5=y$. **66.** $2x-y=a$. **67.** $x+a=y-a$.
68. $x=15y+7$. **69.** $a=bx+y$. **70.** $xy=a$.
71. $ab=9x$. **72.** $xy=3(a-b)$. **73.** $x-y=5(a-b)$.

VIII. b. (p. 61).

1. 3 ft. 8 in. **2.** 4 ft. $8\frac{4}{7}$ in. **3.** $17\frac{1}{2}$ ft. **4.** $1\frac{10}{11}$ ft.
5. 31·4 in. **6.** 2·5 in. **7.** 3·2 in. **8.** 50·3 sq. in.
9. 7 in. **10.** 186 sq. ft. **11.** 22 ft. **12.** 12 ft. 5 in.
13. 560 sq. ft. **14.** 12 ft. 6 in. **15.** 10 ft. 10 in. **16.** 198 cub. ft.
17. $4\frac{1}{2}$ ft. **18.** $3\frac{3}{8}$ sq. ft. **19.** 7 ft. 2 in. **20.** 576 ft.
21. 3 secs. **22.** 31 ft. per sec. **23.** 41. **24.** 68. **25.** 325.
26. 460. **27.** 264. **28.** 336. **29.** 1500. **30.** 1892. **31.** 441.
32. 644. **33.** 1625. **34.** 612. **35.** 693. **36.** 1240. **37.** 3220.
38. 13035. **39.** 113·14. **40.** $10\frac{1}{2}$ ft. **41.** £200. **42.** 334.
43. Right-angled. **44.** Not right-angled. **45.** and **46.** Right-angled.
47. Not right-angled. **48.** Right-angled.

VIII. b₁. (p. 63A).

1. (i) 13, (ii) 5, (iii) 1, (iv) 3. **2.** (i) 12, (ii) 17. **3.** a^2-b^2. **4.** -1.
7. (i) 11, (ii) 24. **8.** (i) $9x^2-25$, (ii) 875. **9.** When $x=3$. **10.** $-3\frac{2}{3}$.

VIII. c. (p. 64).

1. 1, 3, 7. **2.** 15, 28, 3, $\frac{(n+1)(n+2)}{2}$, $\frac{(n-3)(n-2)}{2}$.
3. -6, 0, 0, 24, -60.
4. 0, 33, $16n^2-2n$, $16n^2+14n+3$, $4n^2+7n+3$, $\frac{1}{2}$. **6.** 2, 2, 14.
7. x^2+5x+4. **8.** $2b(x+1)$. **9.** $c-a$, $2b$, $3a+4b-3c$, $8a+5b-3c$.
10. $\phi(x-1)=x^3$. **11.** (i) $(3x^2-2x+2)$ miles, (ii) $(3x^2+2x+6)$ miles.
12. (i) $\phi(3)-\phi(2)$, (ii) 80 feet.

IX. a. (p. 66).

1. £10, £20. **2.** 10. **3.** 27. **4.** £15, £25. **5.** 20. **6.** 21.
7. 10 miles. **8.** 3. **9.** 12. **10.** 38, 10 years old. **11.** 36. **12.** 20.
13. £48, £58, £38. **14.** 30, 12. **15.** 20. **16.** 90. **17.** 75 gallons.
18. 31, 32, 33. **19.** 9. **20.** 18 pennies, 9 shillings, 6 florins
21. £42, £7 **22.** £19, £22. **23.** £336, £164. **24.** £8. 8s.
25. 45, 20. **26.** 63, 40. **27.** 63, 21. **28.** 72, 12. **29.** 57.

30. -4. **31.** 20. **32.** £420. **33.** 34, 35, 36.
34. 43, 45, 47. **35.** 38 shillings, 19 shillings. **36.** 2 miles.
37. £23. 5s., £16. 15s. **38.** £3600, £720.
39. £13. 10s., £22. 10s. **40.** 15, 42.
41. 29 men, 46 women, 76 children. **42.** 56.
43. $4\frac{1}{2}$ miles an hour, 3 miles an hour.
44. 36 miles an hour, 24 miles an hour. **45.** 150 yards a minute.
46. 24 miles. **47.** $44\frac{1}{2}$ miles. **48.** 30 miles.
49. 36 miles. **50.** 15 miles an hour.

IX. b. (p. 73).

1. 12 miles, nearly. **2.** 13 miles, nearly. **3.** 17 miles, nearly.
4. 3.7 miles an hour. **6.** 5 feet. **7.** 36·1 feet.
8. 2·39 feet. **9.** 4·6 miles. **10.** 35·4 miles. **11.** 4·5 miles.
12. 4·1 miles. **13.** 6·55 metres. **14.** 3·9 in. **15.** 4·24 in.
16. 3·6 feet. **17.** 2·6 in. **18.** 2·2 in. **19.** 3·3 in.
20. 6·4 miles. **21.** 2·83 miles. **22.** 8·05.
23. 15·98. **24.** 3·7 miles. **25.** 14, 29, 43 miles.
26. 2·6 miles. **27.** 34 feet. **28.** 2·8 miles.

IX. c. (p. 77).

1. £24, £35. **2.** 15·1 millions, 1875. **3.** 67·1°.
6. 3 oz. **8.** 4475 feet nearly, 205°.
9. 26·8 in., 23·4 in., 10,600 ft., 5,300 ft.
10. 107·5 sq. in., 162·9 sq. ft., 13·2 in.

X. a. (p. 81).

1. $2y=4$. **2.** $11y=22$. **3.** $4y=12$. **4.** $21y=-13$.
5. $3y=14$. **6.** $y=46$. **7.** $17y=17$. **8.** $58y=87$.
9. $3y=-11$. **10.** $3y=-17$. **11.** 2. **12.** 5.
13. 3. **14.** 4. **15.** 11. **16.** $2\frac{1}{2}$.
17. $x=8,\ y=2$. **18.** $x=9,\ y=1$. **19.** $x=2,\ y=1$.
20. $x=1,\ y=2$. **21.** $x=3,\ y=2$. **22.** $x=4,\ y=-1$.
23. $x=-3,\ y=-5$. **24.** $x=2\frac{1}{4},\ y=\frac{3}{4}$. **25.** $x=4\frac{1}{2},\ y=0$.
26. $x=15,\ y=1$. **27.** $x=5,\ y=6$. **28.** $x=8,\ y=6$.
29. $x=0,\ y=2$. **30.** $x=4,\ y=0$. **31.** $x=1,\ y=6$.
32. $x=5,\ y=-2$. **33.** $x=1\frac{1}{2},\ y=\frac{1}{2}$. **34.** $x=13,\ y=7$.
35. $x=1\frac{1}{2},\ y=-2\frac{1}{2}$. **36.** $x=3\frac{1}{2},\ y=2\frac{1}{3}$. **37.** $x=5,\ y=3\frac{3}{5}$.
38. $x=\frac{1}{2},\ y=1\frac{1}{2}$. **39.** $x=2,\ y=3$. **40.** $x=1,\ y=-1$.
41. $x=7,\ y=5$. **42.** $x=6,\ y=8$. **43.** $x=\frac{1}{2},\ y=-\frac{1}{3}$.
44. $x=16,\ y=-24$. **45.** $x=-6,\ y=2$. **46.** $x=2,\ y=-1$.

X. b. (p. 83).

1. $x=12,\ y=20.$	**2.** $x=20,\ y=12.$	**3.** $x=18,\ y=48.$
4. $x=40,\ y=-20.$	**5.** $x=-20,\ y=6.$	**6.** $x=-20,\ y=-40.$
7. $x=2,\ y=3.$	**8.** $x=-2,\ y=-3.$	**9.** $x=-11\frac{1}{5},\ y=\frac{4}{5}.$
10. $x=45,\ y=10.$	**11.** $x=7,\ y=10.$	**12.** $x=5,\ y=2.$
13. $x=11,\ y=1.$	**14.** $x=3,\ y=6.$	**15.** $x=2,\ y=1.$
16. $x=7,\ y=2.$	**17.** $x=8\frac{4}{5},\ y=-11.$	**18.** $x=13,\ y=9\frac{1}{3}.$
19. $x=48,\ y=7.$	**20.** $x=3,\ y=2.$	**21.** $x=10,\ y=2.$
22. $x=\cdot 3,\ y=\cdot 4.$	**23.** $x=-2\cdot 5,\ y=-3\cdot 5.$	**24.** $x=\frac{1}{3},\ y=\frac{2}{3}.$
25. $x=\cdot 02,\ y=2\cdot 9.$	**26.** $x=1\cdot 5,\ y=2\cdot 4.$	**27.** 6.
28. 2, 6.	**29.** 1.	**30.** 5, 1.
31. $x=\frac{1}{4},\ y=1.$	**32.** $x=\frac{1}{8},\ y=\frac{1}{2}.$	**33.** $x=\frac{1}{2},\ y=1.$
34. $x=\frac{1}{5},\ y=\frac{1}{6}.$	**35.** $x=\frac{1}{5},\ y=-\frac{1}{2}.$	**36.** $x=1,\ y=1\frac{1}{4}.$
37. $x=3,\ y=4.$	**38.** $x=\frac{1}{5},\ y=-\frac{1}{4}.$	**39.** $x=\frac{1}{2},\ y=-\frac{1}{3}.$

X. c. (p. 87).

1. $x=2,\ y=3,\ z=-1.$	**2.** $x=2,\ y=4,\ z=6.$	**3.** $x=2,\ y=-3,\ z=4.$	**4.** $x=-2,\ y=6,\ z=8.$
5. $x=-2,\ y=-3,\ z=-1.$	**6.** $x=12,\ y=-24,\ z=12.$	**7.** $x=3,\ y=-11,\ z=-10.$	**8.** $x=9,\ y=3,\ z=6.$
9. $x=6,\ y=-2,\ z=-5.$	**10.** $x=8,\ y=4,\ z=-3.$	**11.** $x=-4\frac{1}{3},\ y=18,\ z=6\frac{1}{3}.$	**12.** $x=12,\ y=24,\ z=36.$
13. $x=8,\ y=6,\ z=4.$	**14.** $x=4,\ y=6,\ z=8.$	**15.** $x=\frac{1}{2},\ y=\frac{1}{3},\ z=\frac{1}{4}.$	**16.** $x=\frac{1}{3},\ y=\frac{1}{4},\ z=\frac{1}{5}.$
17. $x=5,\ y=11,\ z=17.$	**18.** $x=40,\ y=45,\ z=48.$		

XI. a. (p. 88).

21. $4x-2.$	**22.** $4-x.$	**23.** $2-2x.$	**24.** $4x+10.$
25. $18-2x.$	**26.** $a+b+c-d.$	**27.** $4a-8.$	**28.** $5y+x.$
29. $10a+7b.$	**30.** $8c.$	**31.** $a-2b.$	**32.** $12x-6$
33. $-24x+45.$	**34.** $21x-42.$	**35.** $3a+15.$	**36.** $171-9a.$
37. $8a-16.$	**38.** 10.	**39.** $30x-56.$	**40.** $30x-6y.$

XI. b. (p. 90).

1. $2a$. 2. c. 3. b. 4. $7x$. 5. $15-6x$.
6. $12-11a$. 7. $2b^2-2ab$. 8. $\frac{5}{8}-\frac{x}{8}$. 9. $2a-4b+24c+72d$.
10. $-2x-2$. 11. x. 12. y. 13. 0. 14. $2x+y$.
23. $8a-3b$. 24. c. 25. $2a-6b$. 26. $3a$. 27. $2a$.
28. $2a-3b-6c$. 29. $-a+6b+72c+24d$. 30. $3a-7$.
31. $6xy+4y^2$. 32. $12a-2ab+4a^2b$. 33. x^2+3x.
34. $a+10b$. 35. $33a+28b$. 36. $26a-84$.
37. $18x-9xy-9x^2y$. 38. $x-2x^3$.

XI. c. (p. 92).

1. $x^3+x^2(a+2)-x(6+2a)+a-7$. 2. $3x^2-2x(a+b+c)+a^2+b^2+c^2$.
3. $x^3+x^2(y+z)-x(y^2+z^2)-y^3$.
4. $-2x^3+3x^2(a+b)-3x(a^2+b^2)+a^3+b^3$.
5. $bx^3+x^2(a-b)-x(a+b)+a+c$. 6. $x^2(p^2-q^2)+2x(p-q)+p^2-q^2$.
7. $x^3(a-b)+x^2(c-b)+x(c-a)+d-e$.
8. $x^4(2-a)+x^3(6-a)+x^2(b-3)+x(-a-7)$.
9. $x^3+3x^2(y-z)+3x(z^2-y^2)+y^3$. 10. $x^3(a-c)+x^2(a-b)+x(c-b)+a$.
11. $x^4(a-p)+x^3(q-b)+x^2(r-c)$. 12. $x^2y(m+5n)+2xy^2(n-m)$.
13. $-x^3(b-a)-x^2(c-p)-x(d+q)-(p-c)$.
14. $-x^3(a+b)-x^2(b-a)-x(b-c)-(c-d)$.
15. $-x^2(b-a)-x(3a-4)+2a$.

XI. d. (p. 94).

17. 3. 18. 7. 19. 1. 20. 9.
21. $\frac{7x+5}{6}$. 22. $\frac{x}{12}$. 23. $\frac{7x-15}{10}$. 24. $\frac{2x+5}{35}$.
25. $\frac{7x+15}{20}$. 26. $\frac{7x-25}{12}$. 27. $\frac{5x}{12}$. 28. $\frac{5}{24}$.
29. $\frac{x+49}{30}$. 30. $\frac{9x+8}{12}$. 31. $\frac{9x+20}{36}$. 32. $\frac{9x}{40}$.

XII. a. (p. 95).

2. $15x^2-4xy-35y^2$, $-3y^2$. 3. 4.
4. $x=2$, $y=-2$. 5. $240a+30b+24c$, $\frac{a}{2}+\frac{b}{8}+\frac{c}{20}$.
6. 4 inches. 7. 48.

XII. b. (p. 95).

1. $\frac{91x-30}{60}$. 2. $3a+2b$, $9a^2-4b^2$.
3. -1. 4. $x=3$, $y=4$.
5. $\frac{a}{b}$ miles, $\frac{60b}{a}$ minutes, $\frac{bx}{a}$ hours. 6. 3·35 miles. 7. 96.

XII. c. (p. 96).

1. $11x+5$. **2.** 3. **3.** 5. **4.** $x=13$, $y=2$.
5. $x+12$, $x-16$, 16, $40-x$ years. **6.** 3·4 miles. **7.** 51.

XII. d. (p. 96).

1. $46x-1$. **3.** -7. **4.** $x=2, 4, 6$. $y=1, 2, 3$.
5. $\frac{b}{a}$ pence, $\frac{bx}{a}$ pence, $\frac{12a}{b}$ lbs. **6.** 36 feet. **7.** 60, 47.

XII. e. (p. 96).

1. $ap+q$. **3.** 1. **4.** $x=2$, $y=-3$.
5. $\frac{x}{3}+14$, $13+x$, $2x$, $\frac{x}{4}$. **6.** Half-a-mile, 9·04 miles **7.** 42, 32.

XII. f. (p. 97).

1. $x-2$, 2. **3.** $-3\frac{5}{8}$. **4.** $a=5$, $y=10$.
5. $5a$ pence, $\frac{3a}{5}$ pence, $\frac{240}{a}$ eggs. **6.** 11·65 miles. **7.** 50.

XII. g. (p. 97).

1. -44, -21, -6, 1, 0, -9, -26. **2.** 225 lbs., 300 lbs.
3. $-\frac{1}{2}$. **4.** 107, 117. **5.** $x=5$, $y=-3$.
6. $x=-1$, $y=2$, $z=1$. **7** 62·5 feet nearly.

XII. h. (p. 98).

1. 46, 27, 14, 7, 6, 11, 22. **2.** 570 sq. ft.
3. $1\frac{1}{2}$. **4.** 7, -2. **5.** 51, 53, 55, 57, 59.
6. 2·4 miles. **7.** $x=-4$, $y=0$, $z=4$.

XIII. a. (p. 104).

1. $P_1(5, 4)$, $P_2(11, 8)$, $P_3(-5, 5)$, $P_4(-8, 9)$, $P_5(-9, -5)$, $P_6(-5, -3)$, $P_7(3, -5)$, $P_8(8, -7)$.
3. (i) (0, 0), (ii) (3, 0), (iii) (2, 2), (iv) $(-4, 4)$.
4. They all lie on a straight line parallel to OY.

5. 12.	**6.** 48.	**7.** 0, 12, 1·5, 3·5.	**8** 18
9. 36.	**10.** 74·6.	**11.** 180.	**12.** 3·5 sq. in.
13. 15.	**14.** 25.	**15.** 22.	**16.** 25 nearly
17. 34.	**18.** 2·73 in.	**19.** 3·71 in.	**20.** 4·11 in
21. 3·49 in.	**22.** 30.	**23.** 72.	**24.** 99.
25. 70.	**26.** 128.	**27.** 102.	**28.** 52.
29. 96.	**30.** 150.	**31.** 68.	**32.** 95

XIII. b. (p. 113).

30. $x=8,\ y=2.$ 31. $x=3,\ y=2.$ 32. $x=8,\ y=6.$
33. $x=4,\ y=0.$ 34. $x=5,\ y=8.$ 35. $x=4,\ y=3.$
36. $x=2{\cdot}8,\ y=4{\cdot}2.$ 37. $x=4,\ y=5.$ 38. $x=12,\ y=4.$
39. $x=7,\ y=17.$ 40. $x=9,\ y=12.$ 41. $x=5,\ y=2.$
42. $x=10,\ y=5.$ 43. $y=3x.$ 44. $x-y=4.$
45. $2x+y=7.$ 46. $y+5x=0.$ 47. $y+5=2x.$
48. $y=3x+4.$ 49. $2y=3x+12.$ 50. $3y-x=5.$

XIV. a. (p. 117).

1. 17, 12. 2. 12, 5. 3. 6, 8. 4. 13, 9.
5. 4 pence, 9 pence. 6. 7 half-crowns, 3 florins.
7. 44 for, 31 against. 8. 24, 12. 9. 3*s.* 6*d.*, 1*s.* 10. 20, 64.
11. 14, 38. 12. £450, £200. 13. 45, 15. 14. 7.
15. 14 florins, 11 half-crowns. 16. 63.
17. 72 miles, 5 miles an hour. 18. $2\frac{1}{2}$, $7\frac{1}{2}$ miles an hour.
19. 32*s.*, 28*s.* 20. 57, 19. 21. 165.
22. 56, 67. 23. 17 florins, 7 half-crowns. 24. 93.
25. 9, 11 miles an hour. 26. 10, 30 gallons. 27. 100.
28. 15 miles, 2 miles an hour. 29. 24, 12, 4. 30. £51.
31. 24 bales, or 72 casks. 32. 12. 33. 24 feet long, 18 feet wide.
34. 5 teachers and 99 children at first, 7 teachers and 132 children at last.
35. £13. 15*s.* 36. 81, 49 sq. yds.
38. 21 crowns, 40 half-guineas. 39. 3. 40. 3 miles an hour
41. 15 miles an hour, 90 miles. 42. 3 miles an hour, $8\frac{1}{2}$ miles.
43. 4 miles an hour, 24 miles. 44. 3000 ft. from the starting point.
45. £400, 5 pence in the £. 46. 3, 4, 5 miles an hour.
47. 300 miles; 150, 100 miles a day.

XIV. b. (p. 128).

1. 44 francs, 28 shillings. 2. 3 shillings, 20. 3. 38 minutes, 5 miles.
4. 13 ft. per sec., 17·5 ft. per sec., 2·5 secs.
5. 55 lbs., 84 lbs., 14·8 kilogrammes, 17·3 kilogrammes.
6. 4·9 c. in., 2·45 c. in., 41 c. cms. 7. 167°, 5°.
8. They meet at 3.30 P.M. 14 miles from Cambridge; 10 miles apart at 2.48 P.M., and 4.12 P.M.
9. In 10 secs. from *A*'s start, 33·3 yds. from the starting point.
10. June, 1887. 11. 58, 38, 29.
12. 2·2 in., 12·45 cms. 13. 9·23 cms., 3·35 in.
14. 87, 78, 67, 51, 46, 42, 39, 38, 36, 17. 15. 2*s.* $2\frac{1}{2}$*d.*, 31 articles.
16. £1. 15*s.* 1*d.* approx.; 615 copies to the nearest 5. 17. £53.
18. 2·60, 5·63. 4·16, 5·77. 19. £350; 4250 copies.

20. In half an hour from A's start, A having travelled 2 miles.

21. In $4\frac{1}{2}$ hours.

22. 2·7 miles per hour.

23. ·25 of a mile per hour.

24. $5\frac{5}{6}$ miles per hour.

25. In $2\frac{1}{2}$ hours, 20 miles from A's starting point; 2 hours, 3 hours.

26. In 3·1 hours, 24·8 miles from A's starting point; 2·6 hours, 3·6 hours from A's start.

27. $13\frac{1}{3}$ miles an hour.

28. 35 miles, 45 miles.

XV. a. (p. 131).

1. $x^2+2ax-2bx+a^2-2ab+b^2$.
2. $x^2-2ax-2bx+a^2+2ab+b^2$.
3. $a^2+2ab+b^2+4a+4b+4$.
4. $a^2+b^2+c^2+2ab+2bc+2ca$.
5. $a^2+b^2+c^2-2ab+2bc-2ca$.
6. $a^2+b^2+c^2-2ab+2ac-2bc$.
7. $a^2-2ab+b^2-4a+4b+4$.
8. $4x^2+y^2+z^2+4xy+2yz+4xz$.
9. $x^2+4y^2+z^2-4xy+4yz-2zx$.
10. $a^2+4b^2+9c^2+4ab+12bc+6ca$.
11. $a^2+4b^2+9c^2-4ab-12bc+6ca$.
12. $9x^2+6ax-6bx+a^2-2ab+b^2$.
13. $4x^2+12ax-4bx+9a^2-6ab+b^2$.
14. $4x^4+4x^3+5x^2+2x+1$.
15. $9x^4-6x^3+7x^2-2x+1$.
16. $x^4+2x^3-15x^2-16x+64$.
17. $x^4+4x^3+6x^2+4x+1$.
18. $x^4-2x^3-7x^2+8x+16$.
19. $4x^4-4x^3-19x^2+10x+25$.
20. $x^2+2xy+y^2-6x-6y+9$.
21. $4x^2-4xy+y^2+16x-8y+16$.
22. $1-2x+3x^2-2x^3+x^4$.
23. $4+4x-3x^2-2x^3+x^4$.
24. $9-6x+13x^2-4x^3+4x^4$.
25. $25-20x+34x^2-12x^3+9x^4$.
26. $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$.
27. $a^2+b^2+c^2+d^2+2ab+2ac-2ad+2bc-2bd-2cd$.
28. $a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd$.
29. $a^2+b^2+4c^2+d^2+2ab+4ac+2ad+4bc+2bd+4cd$.
30. $a^2+b^2+4c^2+4d^2+2ab+4ac-4ad+4bc-4bd-8cd$.
31. $x^2+y^2+z^2+9+2xy+2yz+2zx-6x-6y-6z$.
32. $x^2+y^2+z^2+9-2xy+2yz-2zx+6x-6y-6z$.
33. $4x^2+y^2+4z^2+1-4xy-4yz+8xz-4x+2y-4z$.
34. $9a^2+4b^2+4c^2+d^2-12ab+12ac-6ad-8bc+4bd-4cd$.
35. $x^6+2x^5+3x^4+4x^3+3x^2+2x+1$.
36. $x^6+4x^5-6x^3+8x^2-4x+1$.
37. $x^6-2x^5+3x^4-4x^3+3x^2-2x+1$.
38. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

XV. b. (p. 131).

1. $a^2-2ab+b^2-c^2$.
2. $a^2+2ab+b^2-4c^2$.
3. $x^2+2xy+y^2-1$.
4. $x^2+4xy+4y^2-b^2$.
5. $a^2-b^2-2bx-x^2$.
6. $a^2-4b^2+4bc-c^2$.
7. $4x^2+4ax+a^2-b^2$.
8. $9y^2-a^2-2ab-b^2$
9. $a^2-16x^2+8xy-y^2$.
10. $1-a^2-2ab-b^2$.
11. $16-a^2+2ab-b^2$.
12. $a^4+a^2b^2+b^4$

13. $1-2a+a^2-b^2$. **14.** $x^2+4xy+4y^2-b^2$.
15. $p^2-4q^2+12qr-9r^2$. **16.** $1-4x^2+12xy-9y^2$.
17. $x^2+6xy+9y^2-16$. **18.** x^4+x^2+1.
19. $1-4x+4x^2-49y^2$. **20.** $4x^2+12xy+9y^2-25$.
21. $9x^4-x^2+4x-4$. **22.** $4x^2-16y^2-40y-25$.
23. $25a^2+30a+9-4b^2$. **24.** $a^4-2a^2b^2+b^4$.
25. $1+2x^2+9x^4$. **26.** $a^2-2ab+b^2-c^2+2cd-d^2$.
27. $4x^2+4xy+y^2-a^2-2ab-b^2$. **28.** $x^2+2ax+a^2-y^2+2by-b^2$.
29. $4x^2-4ax+a^2-y^2+4by-4b^2$.
30. $9x^2-12ax+4a^2-4y^2+12by-9b^2$.
31. $1-2x+x^2-y^2+2yz-z^2$. **32.** $4-4a+a^2-9b^2+6bc-c^2$.

XV. c. (p. 134).

1. x^4-3x^2-6x+8. **2.** $a^3+a^2b-ab^2-b^3$.
3. x^3-y^3. **4.** $x^3+3x^2y-4xy^2-12y^3$.
5. $x^4-x^3-5x^2+27x-30$. **6.** $x^4-6x^2-16x-15$.
7. $a^5-8a^4b+14a^3b^2+9a^2b^3-6ab^4$. **8.** $-x^4-x^2y^2-y^4$.
9. $2a^4-7a^3b-4a^2b^2+23ab^3-6b^4$. **10.** x^3-1.
11. x^3+8. **12.** $8x^3-1$. **13.** x^3-8y^3. **14.** $27a^3+8b^3$.
15. x^3+1. **16.** a^3+b^3.
17. x^3-8. **18.** $x^3-4x^2y+3xy^2-12y^3$.
19. $x^4-5x^3+10x^2-7x-15$. **20.** $x^4-13x^2-2x+35$.
21. $c^4-25c^2d^2-50cd^3-25d^4$. **22.** $x^4+x^2y^2+y^4$.
23. $a^2b^2+c^2d^2-a^2c^2-b^2d^2$. **24.** $-10a^4+21a^3b-21a^2b^2+16b^4$
25. $x^4-2x^3-12x^2+x+2$. **26.** $12x^4-34x^3+37x^2-17x+6$.
27. $20+11x-21x^2+7x^4-2x^5$. **28.** $6+x-2x^2+7x^2y+7x^3y-3x^4y^2$.
29. $x^3+3x^2y+3xy^2+y^3-1$. **30.** $x^6+3x^5-x^4-15x^3-14x^2+18x+24$.
31. $4x^5+3x^4-23x^3+25x^2-14x+4$.
32. $-5+8a-11a^2+4a^3+19a^4-9a^5-6a^6$.
33. $21x^4y-29x^3y^2+3x^2y^3+5xy^4$. **34.** $6x^4-12x^2y^2+6y^4$.
35. $a^4+a^3b+ab^3+b^4$. **36.** $a^3+b^3+c^3-3abc$.

XVI. a. (p. 136).

1. $x^2-5x+14$. **2.** x^2-6x-5. **3.** x^2-x+3.
4. $2x^2+2x+5$. **5.** $3x^2-4x-5$. **6.** $5+6x+4x^2$.
7. $x+1$. **8.** $x-y$. **9.** $x-2$.
10. $2x+1$. **11.** $3a-2b$. **12.** $5x-3y$.
13. $3x^2-2x+6$. **14.** $x-1$. **15.** x^2+xy+y^2.
16. $x-3$. **17.** $9x^2+3x+1$. **18.** a^2-ab+b^2.
19. x^3+x^2+x+1. **20** x^3-x^2+x-1. **21.** x^2+1.
22. x^2+1. **23.** $27x^3-18x^2+12x-8$. **24.** x^2-x+1.
25. x^2+x+1. **26.** x^2+2x+1 **27.** x^2-4x+4.
28. $2x-4$. **29.** a^2-a. **30.** $12x^4-11x^3+10x^2+39x+8$.
31. $2x^3-3x^2+4x-5$. **32.** $x^4-5x^3+13x^2-40x+119$.

XVI. b. (p. 138).

1. $a+2b+c$.
2. $a^2+2ab+2b^2$.
3. $a+b+c$.
4. $3a+2b+c$.
5. $x^4-ax^3+a^3x-a^4$.
6. $a-b+c$.
7. x^3+7x-5.
8. $x^2+xy-2x+y^2-4y+4$.
9. $a^8-a^7+a^5-a^4+a^3-a+1$.
10. $2x^2-3xy+5y^2$.
11. $a^2+b^2+c^2-ab-ac-bc$.
12. $a^2+b^2+c^2-ab+ac+bc$.
13. $x^2+y^2+4+xy+2y-2x$.
14. $x^5-x^4+x^3-x^2+x-1$.
15. x^2+xy+y^2.
16. $a+b+c$.
17. $ab+bc+ac$.
18. $x^6+x^4y^2+x^2y^4+y^6$.
19. $32a^5+16a^4+8a^3+4a^2+2a+1$.
20. $ab-ac-bc+c^2$.
21. $x^2+ax+3a^2$.
22. $x^2+2ax-4a^2$.
23. $\frac{x}{4}+\frac{3y}{2}$.
24. $\frac{a^2}{4}+\frac{ab}{6}+\frac{b^2}{9}$.
25. $\frac{x^2}{16}-\frac{xy}{20}+\frac{y^2}{25}$.
26. $\frac{a^2}{4}-\frac{ab}{6}+\frac{b^2}{9}$.
27. $\frac{a^2}{9}-\frac{2ab}{21}+\frac{b^2}{49}$.
28. $\frac{a}{5}-\frac{b}{4}$.

XVI. c. (p. 140).

1. -8.
2. 28.
3. -6.
4. -3427.
5. $-\frac{25}{4}$.
6. 35.
7. 11.
8. 10.
9. -9.
10. -53.
11. $-38\frac{1}{4}$.
12. 44.
13. $11\frac{3}{4}$.

XVII. a. (p. 141).

1. $x^2-2x(p+q+r)+(pq-qr+pr)$.
2. $127\frac{1}{2}$.
3. $x=5$, $y=6$.
4. 9 half-crowns, 3 threepenny pieces.
5. x^4+3x^2+4.
6. x^2+3y^2.
7. 3.

XVII. b. (p. 141).

1. $2x-y$, $2x-y+20$, $2x-y-20$, y.
2. 153.
3. Common roots, $x=6$, $y=8$.
4. 37.
5. $a^4+4a^3b+4a^2b^2-b^4$.
6. $16a^4-b^4$.
7. $2a^2-3ax+x^2$.

XVII. c. (p. 142).

1. $10x$ apples, $\frac{300}{x}$ pence.
2. 4.
3.

$x=-5$	-1	3	7	11	15
$y=7$	4	1	-2	-5	-8

4. $x^8+4x^6+6x^4+4x^2+1$.
5. $60x+18y+9z=480a$.
6. x^8-81y^8.
7. $3x^2-2x+3$.

XVII. d. (p. 142).

1. $\frac{60x}{y}$ yards, $\frac{1760y}{x}$ min.
2. 180.
3. $x=4{\cdot}07$, $y={\cdot}56$.
4. $x^6-3x^5-x^4+9x^3-5x^2-3x+2$.
5. 72, 74.
6. $4ax^3+4abx$.
7. $a-3b-2$.

XVII. e. (p. 142).

1. xy miles, $60xy$ miles, $\frac{xy}{60}$ miles. 2. 226.
3. 162. 4. 12·57, 34·57, 62·86, 7·5, 5 inches.
5. $x^6-3x^5+6x^4-7x^3+6x^2-3x+1$. 7. $3a-b+4$.

XVII. f. (p. 143).

1. xy pence, $\frac{x}{3}$ pence, $\frac{x}{y}$ pence, $\frac{3x}{y}$ pence. 2. $\frac{1}{2}$.
3. $4y-11x=3$. 4. 26·7 miles. 5. $2x^4-11x^3+20x^2-14x+3$.
6. $4x^2+ab-ac-bc$. 7. $a-3b+4c$.

XVII. g. (p. 143).

1. $a+b$. 2. $x^2+2xy+y^2-z^2$. 3. 7.
4. 27 m. from one end, 18 m. from the other. 5. $x=5$, $y=11$.
6. $3x^2-2x+1$. 7. 56, 48.

XVII. h. (p. 144).

1. $x-x^2$. 2. $ax^2-2ax+a$. 5. 11, 7.
6. $3x-7y$. 7. 21.

XVII. k. (p. 144).

1. x^2+7. 2. -39, -20, -7, 0, 1, -4, -15. 3. $-2\frac{1}{3}$.
4. 22 miles, 48 minutes. 5. $x=2\frac{2}{5}$, $y=12$.
6. $2x^2+3x+1$. 7. $x=2$, £5. 5*s*.

XVII. l. (p. 144).

1. x^8-y^8. 2. $x+y+z-3a$. 3. $-6\frac{1}{3}$.
4. 1·69 in., 2·25 in., 3·8 cms., 5·58 cms. 5. $2x^2-5x-3$.
6. 180. 7. $x=3$, $y=1$, $z=5$, $w=9$.

XVIII. a. (p. 145).

1. $a(x+b)$. 2. $a(x-a)$. 3. $x(x-3a)$.
4. $x^2(x-5a)$. 5. $a(x^2-ax+a^2)$. 6. $3a(a-b)$.
7. $5x^2(x-3y)$. 8. $x(x-y)$. 9. $7(3-8x)$.
10. $5x(5x-4y)$. 11. $x(a-b+c)$. 12. $-2x(x^2-2)$.
13. $-y(a-b-c)$. 14. $px(px-ay+by)$. 15. $19a^2x^2(4x-3a)$.
16. $3(p^2x^2-3px+4)$. 17. $xyz(x+y-z)$. 18. $7b(a-c-3x)$.
19. $7x(2x^2-xy+8y^2)$. 20. $6xyz(6x-9y+8z-3xyz)$.

XVIII. b. (p. 147).

1. $(x+4)(x+5)$. 2. $(x-3)(x-7)$. 3. $(x+4)(x+6)$.
4. $(x+3)(x+7)$. 5. $(x-4)(x-6)$. 6. $(x-1)(x-7)$.

7. $(x+1)(x+2)$. 8. $(x-2)^2$. 9. $(x-2)(x+1)$.
10. $(x+2)(x-1)$. 11. $(x+1)^2$. 12. $(x+5)(x-1)$.
13. $(x-5)(x+1)$. 14. $(x+5)(x+7)$. 15. $(x-3)^2$.
16. $(x-10)(x-1)$. 17. $(x-3)(x-9)$. 18. $(x+3)(x+17)$.
19. $(x-5)(x-13)$. 20. $(x-5)^2$. 21. $(x+7)(x-6)$.
22. $(x-7)(x+6)$. 23. $(x+9)(x-5)$. 24. $(x-7)(x+5)$.
25. $(x+7)^2$. 26. $(x+9)(x-7)$. 27. $(x-12)(x-10)$.
28. $(x-13)(x+10)$. 29. $(x+9)(x-8)$. 30. $(1-2x)(1-x)$.
31. $(7+x)(3+x)$. 32. $(x+p)(x+q)$. 33. $(x-m)(x-n)$.
34. $(x+m)(x-n)$. 35. $(x-m)(x+n)$. 36. $(x+2a)(x+b)$.
37. $(x-a)(x-3b)$. 38. $(x-2a)(x+3b)$. 39. $(x+4a)(x-5b)$.
40. $(x-5a)(x+3b)$. 41. $(x-2)(x+9)$. 42. $(x-11)(x+10)$.
43. $(1-3x)(1-2x)$. 44. $(5+x)(1-x)$. 45. $(x+17)(x-1)$.
46. $(8-x)(5-x)$. 47. $(1+10x)(1-13x)$. 48. $(x-15)(x+1)$.
49. $(8+x)(5-x)$. 50. $(x+11)(x-10)$. 51. $(7+x)(6-x)$.
52. $(6+x)(11-x)$. 53. $(1-6x)(1-x)$. 54. $(9-x)(8+x)$.
55. $(x-8)(x-27)$. 56. $(x+10y)(x-y)$. 57. $(a+15b)(a+b)$.
58. $(x-11)(x-12)$. 59. $(5x+y)(x-y)$. 60. $(a-6b)(a+4b)$.
61. $(x-11y)^2$. 62. $(x-15)^2$. 63. $(x-72)(x-1)$.
64. $(x-13y)^2$. 65. $(x-102)(x-1)$. 66. $(73x-1)(x-1)$.
67. $(x-9a)(x-5a)$. 68. $(9x+y)(6x-y)$. 69. $(13x-1)(2x+1)$.
70. $(16x-1)(15x+1)$. 71. $(43x+1)(x-1)$. 72. $(1-3ab)(1-2ab)$.
73. $(xy-8)(xy+4)$. 74. $(13x+1)(12x-1)$. 75. $(1-5xy)^2$.
76. $(17xy-1)(3xy-1)$. 77. $(7ab+1)(6ab-1)$. 78. $(17x-y)(x+y)$.
79. $(18x+y)(3x+y)$. 80. $(18x+y)(3x-y)$. 81. $(19-x)(3-x)$.
82. $(xy-5)(xy-11)$. 83. $(xy-16)(xy+3)$. 84. $(x-92)(x-1)$.
85. $(167+x)(1-x)$. 86. $(x+17)^2$. 87. $(1-15x)^2$.
88. $(81x+1)(x+1)$. 89. $(x-13y)(x+3y)$.

XVIII. c. (p. 148).

1. $(a+b)(x+y)$. 2. $(a-b)(x-y)$. 3. $(x-y)(a-2)$.
4. $(x-y)(6-a)$. 5. $(x+z)(x+y)$. 6. $(x-y)(x+z)$.
7. $(ac+b)(ac-d)$. 8. $(x+y)(x-2)$. 9. $(x-y)(3-a)$.
10. $(a-c)(a-b)$. 11. $(b+a)(c-a)$. 12. $(ac+d)(ac+b)$.
13. $(a^2+b^2)(c+d)$. 14. $(a^2+b^2)(c-d)$. 15. $(x-3)(x^2+2)$.
16. $(x-2)(x^2-y)$. 17. $(x+5)(x^4-3)$. 18. $(x^2+1)(y^2+1)$.
19. $(x-1)(y^2+1)$. 20. $(ax-b)(bx-a)$. 21. $(x-y)(x+y-4)$.
22. $(a+m)(a+m^2)$. 23. $(x+1)(x^2+1)$. 24. $(x+1)(x^4+1)$.
25. $(2x-1)(x^2+1)$. 26. $(a-b)(x^2+1)$. 27. $(2x-3)(x^2+2)$.
28. $(3x-1)(x^2+4)$. 29. $(7x-3)(x^2-3)$. 30. $(2x-1)(x^2-5)$.
31. $(x+7)(2x^2-3)$. 32. $(x+5)(11x^2+7)$. 33. $(a^2-b)(c+1)$.
34. $(x+1)(x-a^2)$. 35. $(2+x)(a-x^2)$. 36. $(x+3)(2x^2-c)$.

XVIII. d. (p. 149).

1. $(1-x)(1+x)$. **2.** $(1-2x)(1+2x)$.
3. $(x-2a)(x+2a)$. **4.** $(a-7)(a+7)$.
5. $(3a+x)(3a-x)$. **6.** $(3x+1)(3x-1)$.
7. $(5x-4)(5x+4)$. **8.** $(x+3)(x-3)$.
9. $(5x-7)(5x+7)$. **10.** $(a-5)(a+5)$.
11. $(11-b)(11+b)$. **12.** $(a-3)(a+3)$.
13. $(x-13)(x+13)$. **14.** $(2-a)(2+a)$.
15. $(4-11x)(4+11x)$. **16.** $(ab+cd)(ab-cd)$.
17. $(3xy+4ab)(3xy-4ab)$. **18.** 100×102.
19. 8×14. **20.** $(xy+1)(xy-1)$.
21. $(8-cd)(8+cd)$. **22.** $(1-3k)(1+3k)$.
23. $(3-2a)(3+2a)$. **24.** $(3ab-4)(3ab+4)$.
25. 1×305. **26.** $(x-100)(x+100)$
27. $(100x+1)(100x-1)$. **28.** $(xy-9a^2)(xy+9a^2)$
29. $(a^3-b^2)(a^3+b^2)$. **30.** $(b^2+5)(b^2-5)$.
31. $(x^4+a)(x^4-a)$. **32.** $(6x^6-y^4)(6x^6+y^4)$.
33. $(ab^3c^2-x)(ab^3c^2+x)$. **34.** $(1-10x)(1+10x)$.
35. $(abc+d)(abc-d)$. **36.** $(1-11a^2)(1+11a^2)$.
37. $(7x-6y)(7x+6y)$. **38.** $(pq-2)(pq+2)$.
39. $(12x^2+y^2z^3)(12x^2-y^2z^3)$ **40.** $(a-15b)(a+15b)$.
41. $(9x-8)(9x+8)$. **42.** $(2mn+1)(2mn-1)$.
43. $(3p-7q)(3p+7q)$. **44.** $(x-13y)(x+13y)$.
45. $(9ab+1)(9ab-1)$. **46.** $(x^{18}-y^9)(x^{18}+y^9)$.
47. $(a-17b)(a+17b)$. **48.** $(11a+12b)(11a-12b)$.
49. $(5x^8-13a^5)(5x^8+13a^5)$. **50.** $(x^2y-10)(x^2y+10)$.
51. $(xy^2-12p)(xy^2+12p)$. **52.** $(1-10x^3y^2z^4)(1+10x^3y^2z^4)$.
53. $(11x^3y^4-1)(11x^3y^4+1)$. **54.** 67,000.
55. 1800. **56.** 998,000. **57.** 640.
58. 1002,000. **59.** 54,800. **60.** 33,096.
61. 136. **62.** 650,000. **63.** 573.
64. 313,800. **65.** 996,000. **66.** 15,152.
67. 9,400. **68.** 43,984. **69.** 11,800.
70. 9,999,800,000. **71.** 13,440. **72.** 15,000.
73. 15,600. **74.** 59,600. **75.** 128,400.

XVIII. e. (p. 150).

1. $3(x-2a)(x+2a)$. **2.** $7(1-x)(1+x)$.
3. $2(x-12)(x+12)$. **4.** $5x^2(3y-4a)(3y+4a)$.
5. $3(a^4+x)(a^4-x)$. **6.** $7a^2y(4xy-5)(4xy+5)$.
7. $6(3ab+2cd)(3ab-2cd)$. **8.** $141a^3b^3(a^3b^2-2)(a^3b^2+2)$.
9. $7(a-7b)(a+7b)$. **10.** $3(5x-4)(5x+4)$.

11. $11(1-3b)(1+3b)$.
12. $5(3ab-4)(3ab+4)$.
13. $13(a^3-b)(a^3+b)$.
14. $7(x-15a)(x+15a)$.
15. $3(x^2-10)(x^2+10)$.
16. $3a(3p-7q)(3p+7q)$.
17. $5c(11x+12b)(11x-12b)$.
18. $13ab(c-2d)(c+2d)$.
19. $17(1-2pq)(1+2pq)$.
20. $7x^2y^2(1-2y)(1+2y)$.

XVIII. f. (p. 151).

1. $(a-b+c)(a-b-c)$.
2. $(a+b+c)(a-b-c)$.
3. $(x-y+2a)(x-y-2a)$.
4. $(x+2y+4b)(x+2y-4b)$.
5. $(x+2a-b)(x-2a+b)$.
6. $(x+y+a+b)(x+y-a-b)$.
7. $(3x+4y)(x+2y)$.
8. $(a+4x-y)(a-4x+y)$.
9. $(5x+a-b)(5x-a+b)$.
10. $(4a+5x+5y)(4a-5x-5y)$.
11. $4x$.
12. $8ax$.
13. $(a-2b+c+d)(a-2b-c-d)$.
14. $(a+b+c+x+y+z)(a+b+c-x-y-z)$.
15. $(4x+y)(2x-3y)$.
16. $16(2x+1)$.
17. $20pq$.
18. $y(6x-y)$.
19. $(2x+2a+3y+3b)(2x+2a-3y-3b)$.
20. $(5x+y)(x+5y)$.
21. $3(a+b+2c+2d)(a+b-2c-2d)$.
22. $(8p+q-4)(8p-q+4)$.
23. $4ab$.
24. $(3x+2y+2a)(x+4y)$.
25. $5(x+y)(x-y)$.
26. $-48ax$.
27. $(1+3x-2y)(1-3x+2y)$.
28. $(1+2x-2y)(1-2x+2y)$.
29. $(10+2a-3b)(10-2a+3b)$.
30. $b(8a-b)$.
31. $(a-b)^2(a+b)^2$.
32. $(a^2+2ab+2b^2)(a^2-2ab+2b^2)$.
33. $2ab-1$.
34. $5(a-1)(a+1)$.
35. $(2x^2+1)(5-4x)$.

XVIII. g. (p. 151).

1. $(a-b+c)(a-b-c)$.
2. $(c+a+b)(c-a-b)$.
3. $(x+a+b)(x+a-b)$.
4. $(y+a-x)(y-a+x)$.
5. $(a+b-c)(a-b+c)$.
6. $(1+a-b)(1-a+b)$.
7. $(x+a-y)(x+a+y)$.
8. $(x-2y+3ab)(x-2y-3ab)$.
9. $(x-y+3)(x-y-3)$.
10. $(4+a-b)(4-a+b)$.
11. $(1+2a-b)(1-2a+b)$.
12. $(a+x+b+y)(a+x-b-y)$.
13. $(2a-b+x+c)(2a-b-x-c)$.
14. $(a-b+c-d)(a-b-c+d)$.
15. $(a-c+b+d)(a-c-b-d)$.
16. $(x^2+x+1)(x^2-x-1)$.
17. $(a+c+b)(a+c-b)$.
18. $(3a-b+x+2c)(3a-b-x-2c)$.
19. $5(a-b+2c)(a-b-2c)$.

XVIII. h. (p. 154).

1. $(5x-2)(x-2)$.
2. $(x+3)(3x+5)$.
3. $(x-2)(3x-1)$.
4. $(x+7)(2x-3)$.
5. $(x-6)(3x+5)$.
6. $(x+9)(5x-3)$.
7. $(x+9)(2x+1)$.
8. $(x-7)(3x-1)$.
9. $(2x-5)(2x-3)$.
10. $(3x-2)(3x-4)$.
11. $(4x+3)(4x-5)$.
12. $(7x+1)(7x+2)$.

13. $(3x-2)(3x+4)$. 14. $(2x-7)(2x+9)$. 15. $(2x+3)(3x+1)$.
16. $(2x-3)(3x-1)$. 17. $(3x-2)(2x+1)$. 18. $(4x-3)(3x-4)$.
19. $(5x+4)(4x+5)$. 20. $(3x-4)(4x+3)$. 21. $(6x+1)(3x-2)$.
22. $(4x-5)(6x-5)$. 23. $(1-2x)(3-2x)$. 24. $(5-x)(1+2x)$.
25. $(2x+3y)(x+y)$. 26. $(2x-y)(x+2y)$. 27. $(6x-5y)(2x+3y)$
28. $(7x-3)(2x+5)$. 29. $(3x-7)(3x+4)$. 30. $(7x-4)(2x-3)$.
31. $(5x-9y)(2x+y)$. 32. $(7x-3y)(x+y)$. 33. $(12x+5y)(x+y)$
34. $(13x-1)(2x-3)$. 35. $(13x+2)(x+3)$.

XVIII. k. (p. 155).

1. $(x+y)(x^2-xy+y^2)$. 2. $(x-y)(x^2+xy+y^2)$.
3. $(1-x)(1+x+x^2)$. 4. $(1+x)(1-x+x^2)$.
5. $(x^2+y)(x^4-x^2y+y^2)$. 6. $(x^2-y)(x^4+x^2y+y^2)$.
7. $(2x-1)(4x^2+2x+1)$. 8. $(1+2y)(1-2y+4y^2)$.
9. $(2a+b)(4a^2-2ab+b^2)$. 10. $(1+3x)(1-3x+9x^2)$.
11. $(x+3)(x^2-3x+9)$. 12. $(y-3)(y^2+3y+9)$.
13. $(a+5)(a^2-5a+25)$. 14. $(5a-1)(25a^2+5a+1)$.
15. $(2x-3y)(4x^2+6xy+9y^2)$. 16. $(2a+3b)(4a^2-6ab+9b^2)$.
17. $(a-6)(a^2+6a+36)$. 18. $(7x-1)(49x^2+7x+1)$.
19. $(y-4)(y^2+4y+16)$. 20. $(4+y)(16-4y+y^2)$.
21. $(10x+1)(100x^2-10x+1)$. 22. $(ab-1)(a^2b^2+ab+1)$.
23. $(1+ab)(1-ab+a^2b^2)$. 24. $(ab^2-4)(a^2b^4+4ab^2+16)$.
25. $(2xy-1)(4x^2y^2+2xy+1)$. 26. $(x^2+1)(x^4-x^2+1)$.
27. $(4a-5b)(16a^2+20ab+25b^2)$. 28. $(3x+pq)(9x^2-3pqx+p^2q^2)$.
29. $(6a-b)(36a^2+6ab+b^2)$. 30. $(8x+1)(64x^2-8x+1)$.
31. $(9a-2x)(81a^2+18ax+4x^2)$. 32. $(1+9x)(1-9x+81x^2)$.
33. $(a-b)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$.
34. $(x-2)(x+2)(x^2+2x+4)(x^2-2x+4)$.

XVIII. l. (p. 155).

1. $-8x(x^2-2)$. 2. $(a-6b)(a-5b)$.
3. $3(x-1)(x+1)$. 4. $3a^3b^3c^2(a^2-7bc+6ab)$.
5. $3(a-3)(a+3)$. 6. $5(a-2)(a^2+2a+4)$.
7. $(10a-b)(a+b)$. 8. $3(2a-3)$.
9. $xy(x^4-3y^4)$. 10. $7(a-5)(a+5)$.
11. $-(1+x)(1+x^2)$. 12. $11ac(c-3a)$.
13. $3(1-6x)(1-x)$. 14. $3(a-1)(a+1)(b-1)(b+1)$.
15. $3(2+x)(2-x)$. 16. $p^4q^4r^4(p^2q^3-3qr^4+2p^5)$.
17. $3\times 14\times 8=3\times 7\times 2^4$. 18. $3(5x-2)(x-2)$.
19. $(x-p)(x+q)$. 20. $4(x-10y)(x+y)$.
21. $5(1-3y)(1+3y)$. 22. $10(2x-y)(x+2y)$.
23. $11(x-11y)(x-12y)$. 24. $3(1-3x)(1+3x+9x^2)$.
25. $(5-x)(x-1)$. 26. $(x-y)(x-y-5)$.

27. $15(x-y)(x+y)(x^2+y^2)$.
28. $3(x-1)^2$.
29. $3(a-2)(b-c)$.
30. $13(3x+1)(3x-1)$.
31. $2(x-5)(x^2+5x+25)$.
32. $(px+1)(qx+1)$.
33. $2(x-1)(x-7)$.
34. $7(x+y)(x-2)$.
35. $2(a-5)(a+5)$.
36. $(a+7b)(a-6b)$.
37. $2(3x+2y)(3x-2y)$.
38. $3p^2q^2(5q-4p+6)$.
39. $3(11+x)(11-x)$.
40. $9(x+5)(x-1)$.
41. $(4x-1)(6x+1)$.
42. $(1+x)(1-x)(2+x)$.
43. $5(x-y)(x^2+xy+y^2)$.
44. $3(x+5)(x+4)$.
45. $3(xy-1)(x^2y^2+xy+1)$.
46. $5(2pq+1)(2pq-1)$.
47. $a(2bc-1)(4b^2c^2+2bc+1)$.
48. $17(x+1)(x+2)$.
49. $(5a-3b-2c)(a-3b+2c)$.
50. $7(xy^2+10)(xy^2-10)$.
51. $2(x-y+1)(x-y-1)$.
52. $3(1+x-y)(1-x+y)$
53. $(1-2x)(1-3x)$.
54. $(x-5y)(x-4y)$.
55. $3(a-b)(a+b)$.
56. $(1+2x-2y)(1-2x+2y)$.
57. $13x(3x-2)$.
58. $2(x+5y)(x+7y)$.
59. $12x(1-x)$.
60. $(x-15)^2$.
61. $x(6x+1)(3x-2)$.
62. $3(x-2)(x+2)$.
63. $x(5-x)(1+2x)$.
64. $15ab(a-2b)$.
65. $x^2(3x-2)(2x+1)$.
66. $(7x-1)(x-1)$.
67. $5(8+x)(5-x)$.
68. $2abc(2a-3b+4c)$.
69. $7(x+1)(x-1)$.
70. $x(x-3)(x^2+3x+9)$
71. $(x+7y)(x-6y)$.
72. $9(x-7)(x+5)$.
73. $x(a-5)(a^2+5a+25)$.
74. $x(1-2x)(3-2x)$.
75. $(2a+b)^2$.
76. $7(a+11)(a-10)$.
77. $x^2(13x+2)(x+3)$.
78. $(x+p)(x-q)$.

XVIII. m. (p. 157).

1. $(a-b)(a+b)(a^2+b^2)$.
2. $(2a-1)(2a+1)(4a^2+1)$.
3. $2(2x-y)(2x+y)(4x^2+y^2)$.
4. $(x^2+x-1)(x^2-x+1)$.
5. $3a(x-a)(x+a)(x^2+ax+a^2)(x^2-ax+a^2)$.
6. $28ab$
7. $(a-b+2c-2d)(a-b-2c+2d)$.
8. $4ab(a-b)^2$.
9. $(x-y)(x-y+1)(x-y-1)$.
10. $(x-3)(2x-1)(2x+1)$.
11. $(x-3)(x+3)(2x+1)$.
12. $(ax-by)(bx-ay)$.
13. $(a+c)(b-d)$.
14. $(2x^2+3y^2)(2x-3y)(x+y)$.
15. $(x-2)(x+2)(x+3)(x-3)$.
16. $a^2b^2(1+ab)(1-ab+a^2b^2)$.
17. $a(a-b+c)(a-b-c)$.
18. $(x-a)^2(x+2a)$.
19. $(6x-1)(14x+1)$.
20. $(7x-3)(x+15)$.
21. $(1+x+x^2)(1+x-x^2)$.
22. $b(a+b-c)(a-b+c)$.
23. $(a^2+4b^2)(a-2b)(a+2b)$.
24. $(a-1)(a+1)(a^2+a+1)(a^2-a+1)$.
25. $(x-1)(x+1)(x-2)(x+2)$.
26. $(x+y)^3(x-y)$.
27. $(x+1)(x-a)$.
28. $(x+3a+b)(x-a)$.
29. $(x+3a+b)(x-b)$.
30. $[x(a+b)+y(a-b)][x(a-b)-y(a+b)]$.

31. $(x+1)(x^2+1)(x^4-x^2+1)$. **32.** $(20x+7)(10x-3)$.

33. $(x+y+z)(x-y-z)(x-y+z)(x+y-z)$.

34. $9(x-y)(x^2-xy+y^2)$. **35.** $x(x+1)(x-2)(x+5)$.

36. $(x+b)(bx+a^2)$. **37.** $(x+1)(2x-5)(x-3)$.

38. $(x^2+y^2)(a^2+b^2+c^2)$. **39.** $(3x-5)(5x+7)$.

40. $(x-b)(bx+a^2)$. **41.** $4ab(1+a)(1-a)(1+b)(1-b)$.

42. $[ax-(a-1)][(a+1)x+a]$. **43.** $(x-1)(x-2)^2(x+2)$.

44. $(x-1)^2(x+1)(5x+1)$. **45.** $(x+y)(3x-2y)(2x-5y)$.

46. $(x-3)(x^2-x+1)$. **47.** $[(a+2)x+a+1][ax-(a-1)]$.

48. $(a-b)(a+ab+b)$. **49.** $(2a+b-c)(2a-b+c)(4a^2+\overline{b-c}|^2)$.

50. $3(a+b+c)(b-c)$. **51.** $(x+y)(5x-3y)(3x-2y)$.

52. $(x-2)(x^2+2x-2)$. **53.** $(x-y)^3(x+y)$.

54. $(x+ay)(x-by)$. **55.** $(5p-4q)(p-3q)$.

56. $x(1+2ay)(1-2ay+4a^2y^2)$. **57.** $3(3x^2-4y)(3x^2+4y)$.

58. $(x-2)(x^2+x+2)$. **59.** $(2x-5)(x+6)$.

60. $(a-x)(1+ax)$. **61.** $xy(y+x)(y-x)(y^2+x^2)$.

62. $4(x-12)(x+9)$. **63.** $(b-1+a)(b-1-a)$.

64. $(x-1)(x+1)(x+3)(x-3)$. **65.** $(5x-1)(11-x)$.

66. $(x-3)(x+2)(x-2)(x+1)$.

XIX. a. (p. 159).

1. $5ab$.	**2.** x^4y^2.	**3.** ab.	**4.** $2xyz$.
5. $3a^2bc^2$.	**6.** $3x^3$.	**7.** $3xy$.	**8.** y.
9. $3a^2c^2$.	**10.** $13x^2$.	**11.** $5a^3d$.	**12.** abc.

XIX. b. (p. 160).

1. a.	**2.** $x-2$.	**3.** $x+y$.	**4.** $x-2$.
5. $a+2b$.	**6.** $x+y$.	**7.** $x-2y$.	**8.** $x+y$.
9. $x(x-3a)$.	**10.** $3(x-3)$.	**11.** $x+4y$.	**12.** $x-2y$.
13. $x+1$.	**14.** $1-x$.	**15.** $1+x$.	**16.** $x-3$.
17. $x+y$.	**18.** $x+4$.	**19.** $x+11$.	**20.** $x+5$.
21. $x+a$.	**22.** a^2-ab+b^2.	**23.** $x-6$.	**24.** $x-3$.
25. $3a^3b^2(a+b)$.	**26.** $3x-1$.	**27.** $x+3$.	**28.** $(x-1)(x-2)$.
29. $a+b+c$.	**30.** $5x-1$.	**31.** $x-2$.	**32.** $(a-b+c)$.
33. $x-5$.	**34.** $x-a$.	**35.** $2x-1$.	**36.** $4x^2-6x+9$.
37. $x-1$.	**38.** $x-1$.	**39.** $(x-1)(3x-2)$.	**40.** $x-5$.

XIX. c. (p. 163).

1. $a(3x^2-2ax+a^2)$.	**2.** x^2+xy+y^2.	**3.** $2x^2-x-3$.
4. $x-2$.	**5.** $x+2$.	**6.** $x+4$.
7. x^2+5x+1.	**8.** $4x+3$.	**9.** $2x-5$.
10. x^2-5x+1.	**11.** $2x+7$.	**12.** $x+2$.
13. $x-4$.	**14.** $2x^2+7x+3$.	**15.** x^2-3.
16. $3x^2+y^2$.	**17.** $x^3-3x^2y+3xy^2-y^3$.	
18 $5x^2-1$	**19.** x^2+x+2.	**20.** x^2+8x-2.

XIX. d. (p. 165).

1. $\frac{a^2}{2}$. 2. $\frac{2x^2}{a}$. 3. $\frac{5a}{12c}$. 4. $\frac{3x^2z^2}{4y^2}$. 5. $\frac{3b^2}{2a^2}$.

6. $\frac{5mp^4}{2n^2}$. 7. $\frac{a}{a+b}$. 8. $\frac{x}{x-y}$. 9. $\frac{3x}{4x-3y}$. 10. $\frac{1}{x-y}$.

11. $\frac{3a}{4b}$. 12. $\frac{2(2x-3y)}{3x-2y}$. 13. $\frac{3}{5}$. 14. $\frac{b}{c}$.

15. $\frac{xy}{3bz}$. 16. $\frac{x}{x-3}$. 17. $\frac{x}{2-x}$. 18. $\frac{x+2}{x+3}$.

19. $\frac{1+2x}{1-3x}$. 20. $\frac{x+b}{x+c}$. 21. $\frac{a+b}{a^2+ab+b^2}$. 22. $\frac{x-y}{x+y}$.

23. $\frac{b-a}{b+a}$. 24. $\frac{1+bx}{1+cx}$. 25. $\frac{2(x-3)}{3(x-2)}$. 26. $\frac{x^2-1}{x^2+1}$.

27. $\frac{x+b}{x-c}$. 28. $\frac{x^3-y^3}{x^3+y^3}$. 29. $\frac{x-5}{2x+3}$. 30. $\frac{a+b-c}{a-b-c}$.

31. $\frac{3x-1}{x^2-1}$. 32. $\frac{a+b-c-d}{a-b+c-d}$. 33. $\frac{x-5}{x-3}$. 34. x^2-x+1.

35. $\frac{x^2+5x+5}{x^2+x-2}$. 36. $\frac{x+1}{3x^2+3x+10}$. 37. $\frac{x^2+3a}{x^2-3a}$. 38. $\frac{x(x+5)}{x^2+x-5}$

39. $\frac{x+y-1}{x+y+1}$. 40. $\frac{a-1}{a+1}$. 41. $\frac{3a-4b}{3a+2b}$. 42. $\frac{2-x-y}{2+x-y}$.

43. $\frac{3a-2b}{3a+2b}$. 44. $\frac{2a+b-c}{2a-b-c}$. 45. $\frac{9-3a+a^2}{3}$. 46. $\frac{3x+2}{3x-2}$.

XIX. e. (p. 167).

1. $\frac{y}{x}$. 2. $\frac{x-7}{x-3}$. 3. 1. 4. $\frac{2x-1}{2y+1}$. 5. $\frac{x+1}{x+2}$.

6. $\frac{x+b}{x+c}$. 7. $\frac{x+2}{x+5}$. 8. $a(x+a)$. 9. 1. 10. 1.

11. $\frac{x-5}{x-6}$. 12. x. 13. $\frac{x+5}{x+3}$. 14. $\frac{3x}{x+6}$.

15. $\frac{a-b+c}{a+b+c}$. 16. $\frac{4x(x-3)}{x+5}$. 17. $\frac{2x+3}{3x-1}$. 18. $\frac{a^2+ax+x^2}{a^2-ax+x^2}$.

19. $\frac{6}{x-3}$. 20. $\frac{a-b+c}{ab}$. 21. $\frac{x(1+6x)}{1-6x}$. 22. $\frac{2(x+7)}{x+5}$.

23. $\frac{x^2-x+1}{x(x+9)}$. 24. $\frac{x-1}{x+1}$. 25. $\frac{x^2-ax+a^2}{x^2+ax+a^2}$. 26. $\frac{7x-3y}{3(3x-7y)}$.

XX. a. (p. 168).

1. a^2b^2c. 2. $4a^2x^2$. 3. $12a^5$. 4. $30x^2y^2$.

5. $294x^2y^2z$. 6. $2a^2b^2$. 7. $60x^4y^3$. 8. xyz.

9. $24a^3b^2$. 10. $12a^4b^4$. 11. $108x^4y^3$. 12. $a^2y^2z^2$.

13. $60a$. 14. a^3b^3. 15. $36a^3b^4c^4$. 16. $240x^6y^4z^6$.

XX. b. (p. 169).

1. $4x(a-x)$. 2. $a^2(a-b)$. 3. $6(a-x)(a+x)$.
4. $21(a+b)$. 5. $a^2b^2(a-b)$. 6. $xyz(x-y)$.
7. $4x^2y(x+y)$. 8. $6(x-1)(x+1)$. 9. $a^2(a-x)$.
10. $4a^2x(a+x)$. 11. $15(a-b)$. 12. $12(x-y)(x+y)$.
13. $6x^2(x^2+1)^2$. 14. $12(ax-by)(ax+by)$.
15. $xy(x+y)(x-y)$. 16. $8(1-x)(1+x)(1+x^2)=8(1-x^4)$.
17. $12(x-1)(x^2+x+1)=12(x^3-1)$. 18. $(x+1)(x+2)(x+3)$.
19. $(x-1)^2(x+2)$. 20. $(x-2)(x-3)(x-7)$.
21. $(x+1)(x-4)(x+6)$. 22. $(a+b+c)(a+b-c)(a-b+c)$.
23. $18(x+y)^3$. 24. $(2x-1)(x-3)(x+3)$.
25. $(3x-1)(x-2)(x+3)$. 26. $(x^2-y^2)^2$.
27. $(x+6y)(x-6y)(x+y)(x-y)$. 28. $105ab(a+b)(b-a)$.
29. $12x^2(x+y)(x-y)$. 30. $72x^2y^3(x-1)(x-2)^2$.
31. $a(a-b)(2a-b)(a^2+ab+b^2)$. 32. $(2x-1)(x-3)(3x+2)$.
33. $(x+1)(x-2)(x-3)$. 34. $(x-2)(x+2)(x-1)(x+1)$.
35. $18a^2b^2(a-b)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$.
36. $36x(x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2)$.
37. $(x-2a)(x+2a)(x^2+4a^2)$. 38. $(x-a)(x-b)(x+3a+b)$.
39. $(x-3)(x+3)(2x-1)(2x+1)$. 40. $(a-b)(b-c)(c-a)$.

XXI. a. (p. 170).

1. $\frac{11}{6x}$. 2. $\frac{5a}{6x}$. 3. $\frac{bc+ca+ab}{abc}$. 4. $\frac{bc+ca-ab}{abcx}$.
5. $\frac{a^2+b^2+c^2}{abc}$. 6. $\frac{x}{12}$. 7. $\frac{x-6}{42}$. 8. 1.
9. $\frac{bx-ax}{ab}$. 10. $\frac{-z+2x}{xz}$. 11. $\frac{2}{15}$. 12. $\frac{2p+3r}{6pr}$.
13. $\frac{13x+2}{12}$. 14. $\frac{10x-3y}{30}$. 15. $\frac{3a}{2b}$. 16. 0.
17. $\frac{2ac-4a^2+15bc}{12ac}$. 18. $\frac{x^4-y^4}{x^2y^2}$. 19. $\frac{1}{12c}$.
20. $\frac{22x-7}{105x}$. 21. 0. 22. $-3y$.

XXI. b. (p. 172).

1. $\frac{2x}{(x-1)(x+1)}$. 2. $\frac{2}{x-1}$. 3. $\frac{2x+7}{(x+3)(x+4)}$.
4. $\frac{1}{(x+3)(x+4)}$. 5. $\frac{9}{2x-3y}$. 6. $\frac{2x}{(x+6)(x+3)}$.
7. $\frac{11}{(3x-1)(2x+3)}$. 8. $\frac{x^2+y^2}{(x+y)(x-y)}$. 9. $\frac{3x}{(x+4)(x+10)}$.

10. $\frac{2x}{x-2}$. 11. $\frac{12x}{(x-3)(x+3)}$. 12. $\frac{7-3x}{(1-x)^2}$. 13. $\frac{2(1-2x)}{(x+1)(x-1)}$.

14. $\frac{3x}{(x^2-y^2)}$. 15. $\frac{-4y}{(x+y)^2}$. 16. $\frac{1}{1-4x^2}$. 17. $\frac{2b}{9a^2-4b^2}$.

18. $\frac{x}{(x-2y)^2}$. 19. $\frac{1}{x+y}$. 20. $\frac{7x}{x^2-16}$.

21. $\frac{3x+4y}{(x+y)(2x+3y)}$. 22. $\frac{y}{(x-y)^2}$. 23. 0.

24. $\frac{3a-3b}{c-d}$. 25. $\frac{a^2+b^2}{ab(a-b)(a+b)}$. 26. $\frac{4x-a}{a^2-4x^2}$.

27. $\frac{1}{6(a-b)}$. 28. $\frac{x}{a}\left(\frac{a+x}{x-a}\right)$. 29. $-\frac{3b}{a^2-9b^2}$.

30. $\frac{9b(a+3b)}{(a-2b)(2a+5b)}$. 31. $\frac{a^2}{(a-1)(a^2+a+1)}$. 32. $\frac{2xy}{x^3-8y^3}$.

33. $\frac{b}{27a^3+b^3}$. 34. $5b$. 35. $2x$.

36. 4. 37. $\frac{4xy}{(x-y)^2}$. 38. $\frac{2x}{(x-1)(x+1)}$.

39. $\frac{14}{(x-7)(x+7)}$. 40. 0. 41. $\frac{7y}{4}$.

XXI. c. (p. 173).

1. $\frac{4a}{a^2-b^2}$. 2. $\frac{2b}{a^2-b^2}$. 3. $\frac{3}{1-9x^2}$. 4. $\frac{1}{(x-1)(x-3)(x-4)}$.

5. $\frac{a}{3(a^2-b^2)}$. 6. $\frac{21-x}{6(x^2-9)}$. 7. $\frac{2}{(x-1)(x-2)(x-3)}$.

8. $\frac{3a^2-ab}{a^3+b^3}$. 9. $\frac{x}{x^3-27}$. 10. $\frac{a^2}{(a-b)(a-c)}$.

11. $\frac{3}{(x-1)(x-3)}$. 12. $\frac{3(x-y)}{(x-2y)(2x-y)}$. 13. 0.

14. $\frac{7}{(x-1)(x-2)}$. 15. $\frac{1}{2(x-4)}$. 16. $\frac{4}{x+2y}$.

17. $\frac{3x}{(x-2)(x-3)(x+3)}$. 18. $2(a+b)$. 19. $\frac{2a^2}{a^3+b^3}$. 20. 0.

21. $\frac{4}{(x^2-4)(x^2+4)}$. 22. $\frac{2}{(x-1)(x+1)^2}$. 23. $\frac{2}{x^2(x^2-4)}$.

24. $\frac{13}{(x-1)(x-2)(x-3)}$. 25. $\frac{x^4+y^4}{x^6-y^6}$. 26. $\frac{4}{(x^2-1)(x^2-4)}$.

27. $\frac{3y^2}{(x-2y)(x+3y)(x-3y)}$. 28. $\frac{17x}{(x-7)(x+4)(x-3)}$.

29. $\frac{1-5x}{(x+3)(x+4)(x+7)}$. 30. $\frac{3(a-3x^2)}{20(9a^2-x^4)}$. 31. $\frac{5x^2+6x+13}{12(x-1)^2(x+1)^2}$.

32. $\frac{2(x^2+4xy+6y^2)}{(x+3y)(x+2y)}$. 33. 0. 34. $\frac{24x(7x+2)}{(9x^2+4)(9x^2-4)}$.

35. $\dfrac{2a^2b}{(a-b)(a-2b)(a-3b)}$. 36. $\dfrac{2}{(x+3y)(3x+y)}$.

37. $\dfrac{8x^2}{1-x^8}$. 38. $\dfrac{12}{(a^4-4)(a^4-1)}$. 39. $\dfrac{34xy}{y^2-16x^2}$.

40. $\dfrac{2x^2y^2}{x^4-y^4}$. 41. $\dfrac{4a^2}{a^2-b^2}$. 42. $3a-5b$.

43. $\dfrac{34xy}{49x^2-y^2}$. 44. $\dfrac{2xz}{(x-y-z)(x+y+z)}$. 45. $\dfrac{32a}{(a^2-9)(a^2-25)}$.

46. $\dfrac{1}{x-1}$. 47. $\dfrac{b^2}{(a+b)(a^2+b^2)}$. 48. $\dfrac{20x^2}{(3-2x)^3}$.

49. $\dfrac{16a}{1-a^4}$. 50. $\dfrac{3}{x^3-1}$. 51. $\dfrac{2}{x(x-2)}$. 52. $\dfrac{-4(x-2)}{x^4-1}$.

53. 0. 54. $\dfrac{3}{x(x^3-1)}$. 55. $\left(\dfrac{b}{a+b}\right)^3$. 56. $\dfrac{xy^2}{y^3-8x^3}$.

57. $\dfrac{-6}{(x+4)(x+3)(x+2)(x+1)}$. 58. $\dfrac{2(2x+5)(x^2+2)}{(x+1)^2(x^2-2x+3)}$.

59. $\dfrac{1}{(x-1)(x-2)(x-3)}$. 60. $\dfrac{2x}{x^2-xy+y^2}$. 61. $\dfrac{16x}{(x^2-1)(x^2-9)}$.

62. $\dfrac{1}{x^2-1}$. 63. $\dfrac{4x}{x^2-1}$. 64. $\dfrac{4+2a-a^2}{2a}$. 65. $\dfrac{3}{x(x+1)}$.

66. $\dfrac{3}{(x+1)(x+4)}$. 67. $\dfrac{a+bx}{b+ax}$. 68. $\dfrac{18x^2-18x+2}{(3x-2)(2x-1)(3x-4)}$.

69. $-\dfrac{3b}{(2a-3b)(a-4b)}$. 70. $\dfrac{1}{x}$. 71. $\dfrac{2xy}{x^2+y^2}$.

72. $-\dfrac{a}{b}$. 73. $\dfrac{x+1}{x-1}$. 74. $\dfrac{2xy}{x^2+y^2}$.

75. $\dfrac{6}{x^2}$. 76. $\dfrac{1}{x-2}$. 77. $\dfrac{x^3}{a^3}$.

78. 0. 79. $-a$. 80. 1. 81. a^2+b^2. 82. $-m$.

83. $\dfrac{1}{a^2}$. 84. $\dfrac{5(a+x)}{(2a-x)^2}$. 85. $\dfrac{3(a+2b)}{a-6b}$.

86. $\dfrac{5}{(x+1)(x+2)(x-3)}$. 87. $\dfrac{x(x+1)}{2}$. 88. $\dfrac{(x-5)^2}{(x-8)(3x-8)}$.

89. x^2+y^2. 90. $\dfrac{1}{x^2+y^2}$. 91. $\dfrac{(x-4)^2}{(x-7)(3x-5)}$.

92. $\dfrac{a^4-a^2b^2+b^4}{a^4+a^2b^2+b^4}$. 93. $\dfrac{3}{a}$. 94. $\dfrac{2}{x}$.

95. $\dfrac{27b^3}{8a^3+27b^3}$. 96. $\dfrac{x^2-ax+a^2}{x^2-a^2}$. 97. $\dfrac{2(a-b)x}{(a+x)(b+x)}$.

98. 3. 99. $2x$. 100. -1. 101. $\dfrac{1}{1+x}$. 102. $\dfrac{2xy}{x^2-y^2}$.

103. $\dfrac{2(ab+bc+ca)}{abc}$. 104. $(a+b)(c+d)$. 105. $\dfrac{(c+a)(c-a)}{(a+b)(a-b)}$.

106. $\dfrac{x^2+1}{x^2-1}$. 107. $-\dfrac{c}{e}$. 108. $2(x+y+z)$.

109. $\dfrac{3n-m}{2}$. 110. $\dfrac{x^2-3}{(x-1)^2}$. 111. 1. 112. $\dfrac{2}{x+y}$.

113. $\dfrac{2x(a+b)}{x^2-b^2}$. 114. 1. 115. 2. 116. 1.

117. $\dfrac{x(x+y+z)}{z(x-y+z)}$. 118. $y-x$. 119. 2. 120. $\dfrac{1}{x}$. 121. $1+a-a^3$.

XXII. (p. 180).

1. 8. 2. 3. 3. -2. 4. $4\frac{1}{5}$. 5. 2. 6. 1.
7. 7. 8. 1. 9. 2. 10. 3. 11. 12. 12. 2.
13. 7. 14. 3. 15. 7. 16. 7. 17. -107. 18. $\frac{5}{8}=\cdot63$
19. $4\frac{2}{3}=4\cdot67$. 20. 16. 21. 6. 22. $\frac{9}{26}=\cdot35$. 23. 2. 24. $\frac{2}{5}=\cdot4$.
25. 2. 26. $\frac{5}{16}=\cdot31$ 27. $1\frac{1}{13}=1\cdot08$. 28. 4. 29. $1\frac{47}{120}=1\cdot39$.
30. -1. 31. 4. 32. $4\frac{5}{7}=4\cdot71$. 33. 0. 34. 5.
35. $\frac{1}{2}$. 36. 7. 37. $6\frac{1}{3}=6\cdot33$. 38. 6. 39. 2.
40. 5. 41. 8. 42. $\frac{5}{7}=\cdot71$

XXIII. a. (p. 181).

1. $x(ax-b)$. 2. $(x+1)(x+10)$.
3. $3(x-1)(x+1)$. 4. $2(x-1)(x-3)$.
5. $(a-b)(x+a+b)$. 6. $(1-3x)(1+x)$.
7. $4(a-b)(a^2+ab+b^2)$. 8. $6(3x+1)(x+1)$.
9. $(4x-3)(2x+5)$. 10. $(x-1)(x+1)(x+2)$.

11. $5y(4x-3y)$. 12. $a(x-b)(x+b)$.
13. $(x-1)(x-51)$. 14. $(2a+1)(2a-1)$.
15. $(x+a)(x^2+a^2)$. 16. $(9+x)(8-x)$.
17. $(a+b)(a+b-1)$. 18. $(2x-7)(8x+3)$.
19. $(a+b-c)(a-b+c)$. 20. $(ax-3)(bx-4)$.

21. $3(1-x)^2$. 22. $(3x-1)(9x-1)$.
23. $5(2a-3)(2a+3)$. 24. $(3a-2b)(x-y)$.
25. $3(a-3)(a^2+3a+9)$. 26. $(3-x^2)(2+x)$.
27. $(5x-4)(7x+8)$. 28. $(x-1)(x+1)(y-1)(y+1)$.
29. $(1-x)(2+x)(3-x)$. 30. $(x+y)(x-y)(a-b)(a^2+ab+b^2)$.

31. $7b(9a-3c-35b)$. 32. $(18x-y)(3x+y)$.
33. $(x+y)(6-a)$. 34. $\frac{1}{3}(3x-1)(3x+1)$.
35. $(3x-2)(9x+4)$. 36. $7(7x-y)(7x+y)$.
37. $(x^2+1)(y-1)(y+1)$. 38. $(a-b)(a-b+1)(a-b-1)$.
39. $(x-2y)(x+2y)(x^2+2xy+4y^2)(x^2-2xy+4y^2)$.
40. $(a+b-2)(a+b-3)$.

41. $p(px-1)^2$.
42. $(x-12)(x-13)$.
43. $(x+4)(x+12)$.
44. $(11x-8y)(3x+4y)$.
45. $(x+2a)(x-7b)$.
46. $2b(3a^2+b^2)$.
47. $(3x-a)(5x+2b)$
48. $2(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$.
49. $(x+1)(2x+1)(2x-3)$.
50. $(x^2+y^2)(a^2+b^2-c^2)$.

51. $(x-8)^2$.
52. $(a-b)(a+b+1)$.
53. $(x-7)(x+21)$.
54. $3(a-b)(a-b-1)$.
55. $(3x+2a)(4x-7b)$.
56. $(x+3)(x^2-x+1)$.
57. $(9x-5)(3x+25)$.
58. $(x-a)(x+a+3y)$.
59. $(a-2b+2c)(a+2b-2c)[a^2+4(b-c)^2]$.
60. $(\overline{a-1}\,.\,x+a)(ax-a+1)$

61. $(a+b)(a+b+2)$.
62. $(5x-12y)(7x+2y)$.
63. $(x-y)(3x+3y-4)$.
64. $b^2(x-b)(x+b)(x^2+b^2)$
65. $(x^2+y^3)^2$.
66. $(4x-a)(4x+a)$.
67. $32x(x+10)(x+1)$.
68. $2y(x+y)(x-y)$.
69. $(a+b-c)(a-b+c)(a+b+c)(b+c-a)$.
70. $3(a-b)(a+b)(5a^2-8ab+5b^2)$.

71. $(a-b)(5a+5b-1)$.
72. $(13x-4)(3x+2)$.
73. $(2x-1)(2x+1)(4x^2+1)$.
74. $(x-y)(a-b-c)$.
75. $(x-1)(x+1)(x-2)(x+2)$.
76. $(x+y-6a)(x+y-7a)$.
77. $16(a-b)(a+b)(5a^2-6ab+5b^2)$.
78. $(a-b)(ax+by+c)$.
79. $5x(13x^2+18xy+12y^2)$.
80. $(4x^2+2xy+y^2)(4x^2-2xy+y^2)$.

XXIII. b. (p. 182).

1. $a(x-a)(x+a)$, $(x+9y)(x-11y)$, $(75x-1)(x-1)$, $(x+y)(x-5)$.
2. $x-3$.
3. $\dfrac{2(4-x)}{(x-1)(x-2)(x-3)}$, 4.
4. $x^4-a^2x^2-b^2x^2+a^2b^2$.
5. $\pm 2{\cdot}6$, $\pm 3{\cdot}6$, $3{\cdot}2$, $5{\cdot}8$.
6. $\frac{1}{7}$
7. 30 miles an hour.

XXIII. c. (p. 182).

1. $2(x-2)(x+2)$, $(2x-1)(x-2)$, $(a+b-c)(a+b+c)$, $(x-y)(x+y-3)$.
2. 1.
3. $12a^3b^3(a-b)$.
4. $3x-2$.
5. $22{\cdot}4$ acres.
6. $x=3$, $y=-6$.
7. 25 miles an hour.

XXIII. d. (p. 183).

1. $(2x+1)(x+3)$, $(a+b+x)(a-b-x)$, $(b-c)(a-c)$, $3(1-b)(1+b+b^2)$.
2. $x-a$.
3. 0.
4. $\left.\begin{matrix} x=6, & 4, & 2, \\ y=1, & 2, & 3. \end{matrix}\right\}$
5. $a^4+a^3b-ab^3-b^4$.
6. 5.
7. 2 stumped, 3 caught, 5 bowled.

XXIII. e. (p. 183).

1. $(x-32)(x+4)$, $(x+y)(a-2)$, $(x-1)^2(x-3)$, $4(1+3a)(1-3a+9a^2)$.
2. $\dfrac{c-a+b}{c+a-b}$. 3. $(x+1)(x-2)(x-3)$. 4. 25·7 miles from the start.
5. x^2-2x+3. 6. -15. 7. $2\frac{1}{2}$.

XXIII. f. (p. 184).

1. $(2x-1)(x+5)$, $3(a-b)(a+b)$, $(b+c)(a-d)$, $(x-y)(x+y)(x-z)$.
2. $a-b+c$. 3. $\dfrac{3-2x^2}{(1-x)^2(2-x)^2}$. 4. $\left.\begin{matrix} x=9,\ 6,\ 3,\ 0, \\ y=1,\ 3,\ 5,\ 7. \end{matrix}\right\}$
5. $x=4$, $y=3$. 6. 15 miles. 7. $-7\frac{1}{2}$.

XXIII. g. (p. 184).

1. $(3x+4)(4x-3)$, $(2a+b+c-d)(2a+b-c+d)$, $(x-1)(x+1)(x+2)$, $(x-1)(x+1)(y-1)(y+1)$.
2. $\dfrac{1}{x^2-1}$. 3. $18x^2y^2(x^4-y^4)$. 4. 184 against, 161 for.
5. $x^2-x(a+2b)+3b^2+a^2$. 6. -3. 7. $\dfrac{5280}{x}$ min., $20x$ yds., $\dfrac{xy}{88}$ miles

XXIII. h. (p. 184).

1. $6x+\dfrac{2}{x^3}$. 2. 0. 3. 3·3, 4·8. 4. 1.
5. $\frac{5}{7}$. 6. $x=-2$, $y=1\frac{1}{3}$. 7. £$3x$, £$12x$, £$\dfrac{ax}{100}$, £$\dfrac{axy}{100}$.

XXIII. k. (p. 185).

1. $x+1+\dfrac{1}{x}$. 2. 2. 3. $\frac{4}{3}$.
4. $\dfrac{xy^3}{x^2+xy+y^2}$. 5. 22 min. past 4. 6. $x=-1$, $y=-11$.
7. -15, -8, -3, 0, 1, 0, -3, -8, -15.

XXIII. l. (p. 185).

1. $6xy-3y^2$. 2. 3. 3. $\dfrac{a+b-c}{a-b-c}$.
4. 31, 4. 5. $a=9\frac{1}{2}$, $b=4$. 6. $x=-2$, $y=-2$. $x=-\frac{1}{2}$, $y=-\frac{1}{2}$.

XXIII. m. (p. 186).

1. $12ab$. 2. $2\frac{1}{2}$. 3. $\frac{3}{7}$.
4. $\dfrac{4(x^2+x+1)(x+1)}{x^4(x^4+1)}$. 5. 55 min. past 4.
6. The equation is an identity. 7. £$\left(85+\dfrac{17x}{20}\right)$, £$\dfrac{9200}{100+x}$.

XXIV. a. (p. 187).

1. x^4. 2. a^5. 3. y^3. 4. x^3y^2. 5. ab^2. 6. x^4y^3. 7. $2ab$
8. $4a^2b$. 9. $7x^2y^3z^4$. 10. $\frac{2a}{b}$. 11. $\frac{3x^2}{y^3}$. 12. $\frac{9a^2b^3}{c^4}$.
13. $\cdot 1$. 14. $\cdot 5$. 15. $\cdot 8$. 16. 100. 17. $\frac{6}{2}$.
18. $\frac{7}{6}$. 19. $\frac{b^2c}{10}$. 20. $\frac{a}{5b^2}$. 21. $\frac{11a^3c^5}{10}$. 22. $\frac{4x^6y^5}{7}$.
23. $\frac{10a^2}{9b}$. 24. $\frac{8x^3}{y^6}$. 25. $3(a-b)$. 26. $\frac{11}{3}(2x+y)$. 27. $x+y$.

XXIV. b. (p. 188).

1. $x+y$. 2. $x-y$. 3. $a+2b$. 4. $2a-b$.
5. $x-3$. 6. $1-2x$. 7. $5a-3b$. 8. $7x-y$.
9. $2a-7b$. 10. $3x+4y$. 11. $11a-2b$. 12. $1-x^3$.
13. $13a+2b$. 14. $9a-b$. 15. $5x-7y$. 16. a^2-b^2.
17. $2a^2+b^2$. 18. x^2y-1. 19. $\frac{x}{3}-1$. 20. a^2+2b^2.
21. $x-\frac{1}{2}$. 22. $\frac{a}{2}-b$. 23. $\frac{x}{y}-\frac{y}{x}$. 24. $x-\frac{3y}{2}$.
25. $x^2+\frac{1}{x^2}$. 26. $a-\frac{5}{2}$. 27. $x+y+1$. 28. $2b$.
29. $x-y-2$. 30. $3(a+b)+1$. 31. $a+b+c+d$. 32. $2a+b$.
33. $\frac{a}{b}$. 34. $4(x-y)-1$. 35. $a+2b+\frac{1}{2}$. 36. b.
37. $\frac{a}{b}-2$. 38. $x+7y$. 39. $\frac{a^3}{x^3}-\frac{x^3}{a^3}$. 40. $\frac{2a^3}{x^2}-\frac{x^3}{a^2}$.
41. $\frac{x^4}{2a^4}+\frac{2a^4}{x^4}$. 42. $\frac{a+b}{3}-\frac{x+y}{2}$. 43. $\pm 2ab$. 44. 4.
45. $\pm 6x$. 46. $\pm 20xy$. 47. 1. 48. ± 2. 49. 1. 50. $\pm\frac{9}{2}$

XXIV. c. (p. 191).

1. x^2+x+1. 2. $2x^2+x+1$. 3. x^2-x+2.
4. $a^2-2ab+b^2$. 5. $3x^2-2x+5$. 6. $2x-5y+4z$.
7. $x(4x^2+3x+1)$. 8. $5x^2-2ax-3a^2$. 9. $x^2-3+\frac{1}{x^2}$.
10. $a-b-c$. 11. x^3-3x-7. 12. $3x^2-2xy+5y^2$.
13. $a-2b+3c$. 14. $3a^2-7b^2-11c^2$. 15. $2ab-3bc-ca$.
16. $2x-3y+5z$. 17. $7x^2-5xy+6y^2$. 18. $x^3-2-\frac{1}{x^3}$.
19. $2x^2-3y^2+7z^2$. 20. $\frac{x}{y}-1+\frac{y}{x}$. 21. $\frac{a^2}{2}-a-1$.
22. $\frac{a^2}{3}+a+\frac{1}{2}$. 23. $\frac{3a^2}{5}+\frac{2a}{2}+1$. 24. $\frac{a^2}{3}-\frac{a}{2}+1$.

25. $x^3-\frac{x^2}{2}+\frac{1}{3}$. 26. $\frac{x^2}{2}-3x+\frac{1}{3}$. 27. $\frac{x^2}{3}-2x+\frac{a}{2}$.
28. $3x^2+4-\frac{8}{x^2}$. 29. $\frac{2x}{y}-\frac{1}{4}-\frac{3y}{2x}$. 30. $a^2-\frac{3a}{4}+\frac{4}{5}$.

XXIV. d. (p. 193).

1. 42. 2. 135. 3. 130. 4. 52. 5. 187. 6. 625.
7. 462. 8. 84. 9. 126. 10. 2005. 11. 3001.
12. 1973. 13. 2345. 14. 20202. 15. 1351. 16. 3489.

XXIV. e. (p. 201).

18. 7·32, 7·60, 7·71, 7·85.
19. 6·21, 6·30, 6·33, 6·53, 6·84. 41·5, 44·6, 46·5.
20. 7·06, 7·12, 7·16, 7·34. 49·7, 51·3, 53, 54·2.
21. 7·39, 7·67, 7·90. 22. 80·2, 81·7. 23. 80·15, 80·23, 80·54
24. 9·053, 9·088. 25. 91·35, 91·78. 26. 10·084, 10·048.
27. 12·36, 12·94. 28. 1·73. 29. 2·45.
30. 2·65. 31. 3·32. 32. 2·37. 33. 2·19.
34. 2·57. 35. 2·12. 36. 2·39. 37. 2·07.
38. 5·24, 5·83. 39. 2·47, 2·76. 40. 2·02. 41. 3·03.
42. 3·06. 43. 3·08. 44. 3·11. 45. 4·03.
46. 4·08. 47. 4·09. 48. 5·03. 49. 5·05. 50. 5·07.

XXV. a. (p. 203).

1. 1, 2. 2. 1, -1. 3. a, b. 4. 0, 1.
5. -2, -3. 6. $-a$, b. 7. 0, -2. 8. $2a$, b.
9. $-a$, $2b$. 10. $\frac{1}{2}$, $-\frac{3}{4}$. 11. $-\frac{1}{5}$, $-\frac{3}{8}$. 12. 0, $-\frac{1}{8}$.
13. $\frac{a}{2}$, $\frac{b}{3}$. 14. $a+b$, $a-b$. 15. $\frac{a+b}{2}$, $-\frac{c+d}{2}$.
16. $p-2q$, $2p-q$. 17. $2(a+b)$, $-3(a-b)$.
18. a^2, $-b^2$. 19. $-(a-b)^2$, $(a+b)^2$. 20. 3.
21. 0, a. 22. 0, -4. 23. $-a$. 24. $-2a$.

XXV. b. (p. 205).

1. 5, 2. 2. 3, 2. 3. ± 2. 4. 0, 3. 5. -1, -3.
6. -5, $+1$. 7. 1, 7. 8. 2, -1. 9. ± 2. 10. 10, 1.
11. -9, 5. 12. 3, 9. 13. -5, 4. 14. 7, 0. 15. ± 1.
16. 2, 2. 17. -3, 0. 18. -7, -3. 19. 15, -1.
20. 5, -8. 21. 15, 15. 22. ± 3. 23. 0, 2.
24. 0, -7. 25. 102, 1. 26. -1, -15.

XXV. c. (p. 207).

1. $1\frac{1}{2}$, 4.	**2.** $-\frac{1}{3}, \frac{1}{2}$.	**3.** $-1\frac{1}{3}, -1\frac{1}{5}$.	**4.** 0, $-1\frac{3}{7}$.
5. $1\frac{2}{5}, -\frac{1}{6}$.	**6.** $1\frac{1}{7}$.	**7.** $\frac{a}{2}, \frac{b}{2}$.	**8.** $-\frac{a}{5}, -\frac{b}{6}$.
9. $\frac{a+b}{2}, \frac{c+d}{3}$.	**10.** $-1\frac{1}{4}, 4\frac{1}{2}$.	**11.** 1, -2.	**12.** 5, 3.
13. -4, 8.	**14.** 4, 6.	**15.** 5, -1.	**16.** $\pm\frac{1}{2}$.
17. 4, 4.	**18.** 1, $\frac{1}{2}$.	**19.** -4, 6.	**20.** 0, $-3\frac{2}{5}$.
21. 10, 1.	**22.** $-\frac{1}{2}, -\frac{1}{2}$.	**23.** $-4{\cdot}1$, $-{\cdot}7$.	**24.** 1, 1.
25. 4, $\frac{1}{2}$.	**26.** $\frac{3}{2}, -\frac{4}{3}$.	**27.** $\frac{3}{4}$, -4.	**28.** $\frac{3}{2}, -\frac{7}{5}$.
29. 2, -1.	**30.** $-9\frac{1}{2}$, 1.	**31.** 15, -4.	**32.** 2, $-\frac{1}{150}$.
33. $\frac{2}{3}, -\frac{2}{5}$.	**34.** $-\frac{5}{4}, -\frac{7}{6}$.	**35.** 1, $-\frac{7}{13}$.	**36.** $\frac{5}{7}, -\frac{7}{10}$.
37. $-\frac{2}{3}, \frac{8}{3}$.	**38.** 11, -13.		

XXV. d. (p. 211).

1. $\frac{1}{2}, -\frac{2}{3}$.	**2.** $\frac{1}{13}, -\frac{1}{2}$.	**3.** $\frac{1}{12}, -\frac{1}{13}$.	**4.** 1, $-\frac{1}{5}$.
5. $\frac{2}{3}$, 5.	**6.** $-5, \frac{3}{7}$.	**7.** $-9, -\frac{1}{2}$.	**8.** 5, -3.
9. 2, $\frac{1}{2}$.	**10.** $\frac{3}{2}, \frac{1}{3}$.	**11.** $-\frac{1}{3}$, 3.	**12.** $\frac{4}{3}, -\frac{3}{4}$.
13. $\frac{5}{6}, -\frac{3}{2}$.	**14.** 3, -2.	**15.** $\frac{5}{2}, -\frac{13}{6}$.	**16.** $\frac{9}{5}, -\frac{4}{3}$.
17. 2, $-\frac{43}{25}$.	**18.** $\frac{11}{5}$, 1.	**19.** $\frac{9}{5}, -\frac{1}{2}$.	**20.** 22, -2.
21. $-\frac{4}{3}, -\frac{3}{5}$.	**22.** $\frac{3}{2}$, 4.	**23.** 1, $-\frac{1}{2}$.	**24.** $\frac{3}{2}, -\frac{10}{3}$.
25. 2, -3.	**26.** 2, -14.	**27.** 5, $\frac{18}{7}$.	**28.** 5, $-\frac{3}{2}$.
29. 0, $7\frac{10}{23}$.	**30.** 12, 36.	**31.** 0, $3\frac{1}{2}$.	**32.** 3, $-2\frac{1}{3}$
33 $\frac{3}{2}$, 4.	**34.** 4, $-\frac{9}{4}$.		

XXV. e. (p. 212).

1. $1\pm\sqrt{2}=2{\cdot}41$ or $-{\cdot}41$.	**2.** $-1\pm\sqrt{3}={\cdot}73$ or $-2{\cdot}73$.
3. $2\pm\sqrt{3}=3{\cdot}73$ or ${\cdot}27$.	**4.** $1\pm\sqrt{5}=3{\cdot}24$ or $-1{\cdot}24$.
5. $\frac{9\pm\sqrt{161}}{10}=2{\cdot}17$ or $-{\cdot}37$	**6.** $1\pm\sqrt{6}=3{\cdot}45$ or $-1{\cdot}45$.
7. $\sqrt{3}=1{\cdot}73$.	**8.** $-6\pm\sqrt{3}=-7{\cdot}73$ or $-4{\cdot}27$
9. $\frac{6\pm\sqrt{176}}{10}=1{\cdot}93$ or $-{\cdot}73$.	**10.** $2\pm\sqrt{13}=5{\cdot}61$ or $-1{\cdot}61$.
11. $\frac{-5\pm\sqrt{73}}{4}=-3{\cdot}39$ or ${\cdot}89$.	**12.** $\frac{9\pm\sqrt{3}}{3}=3{\cdot}58$ or $2{\cdot}42$.
13. $\frac{1\pm\sqrt{2}}{3}={\cdot}80$ or $-{\cdot}14$.	**14.** $2\sqrt{3}=3{\cdot}46$ or $-\sqrt{3}=-1{\cdot}73$

XXV. f. (p. 214).

1. $\pm 5, \pm 2$. 2. $\pm 3, \pm 6$. 3. $1, 3$. 4. $0, -1$.
5. $3, -1, 1 \pm \sqrt{13} = 4{\cdot}61$ or $-2{\cdot}61$. 6. $1, -1, -1$. 7. $\pm 1, \frac{4}{5}$.
8. $5, -1, 2 \pm \sqrt{3} = 3{\cdot}73$ or ${\cdot}27$. 9. $\pm 2, \pm \frac{1}{2}, \pm 1$.
10. $-8, 3, 0, -5$. 11. $0, 5, -6$.
12. $0, -5$, (other roots imaginary). 13. $0, -\frac{5}{2}$.
14. $-5, 2$, (other roots imaginary).
15. $1, -4, \frac{-3 \pm \sqrt{11}}{2} = {\cdot}16$ or $-3{\cdot}16$.
16. $-\frac{3}{2}, \frac{\pm\sqrt{10} - 3}{2} = -3{\cdot}08, {\cdot}08$. 17. $1, 2, \frac{-5 \pm \sqrt{17}}{2} = -4{\cdot}56, -{\cdot}44$.

XXVI. (p. 219).

7. $2{\cdot}5, -1{\cdot}5$. 8. $-2{\cdot}5, 3{\cdot}5$. 9. ${\cdot}5, -1{\cdot}6$.
10. ${\cdot}8, 2{\cdot}5$. 11. $1{\cdot}5, 2{\cdot}3$. 12. ${\cdot}5, -2{\cdot}6$.
13. $2{\cdot}1, -1{\cdot}5$. 14. The roots are equal, ${\cdot}5$.
15. The roots are imaginary. 17. $3{\cdot}8, -{\cdot}8$. 18. $-2, 2{\cdot}6$.
19. $-2, 3{\cdot}5$. 20. $-3, 4{\cdot}6$.
21. $1{\cdot}87, -1{\cdot}07$. Minimum value $-10{\cdot}8$. 22. $-2, 3$.
23. $4, -2{\cdot}5$. 24. $4{\cdot}8, {\cdot}2$.
25. $-1, 2{\cdot}2, 3, 3{\cdot}4, 3{\cdot}4, 3$. Maximum value $3{\cdot}45$.
26. $(3, 5)$. 27. $1{\cdot}44$. 28. 6. 29. $2{\cdot}5$ $2{\cdot}5$.
30. $2{\cdot}6, 1$. 31. -4. 32. $-1{\cdot}4, 2{\cdot}6$. 33. $2{\cdot}5, -4$.

XXVII. a. (p. 222).

1. $x=3, y=1$. 2. $x=5, y=-2$. 3. $x=2, y=8$.
4. $x=7, y=2$. 5. $x=3, y=5$. 6. $x=1, y=2$.
7. $x=2, y=-1$. 8. $x=6, y=-3$. 9. $x=5. y=2$.
10. $x=6, 9.$ $y=9, 6$. 11. $x=5, -3.$ $y=3, -5$. 12. $x=12, -11$ $y=11, -12$
13. $x=13. -9.$ $y=-9, 13$. 14. $x=-7, 13.$ $y=13, -7$. 15. $x=7, -3.$ $y=3, -7$.
16. $x=\frac{1}{2}, \frac{1}{4}.$ $y=\frac{1}{4}, \frac{1}{2}$. 17. $x=2, \frac{3}{4}.$ $y=3, 8$. 18. $x=2, -\frac{1}{5}.$ $y=1, -10$.
19. $x=6, -\frac{4}{3}.$ $y=2, -9$. 20. $x=4, 1{\cdot}6.$ $y=2, 5$. 21. $x=\pm 7, \pm 2$ $y=\pm 2, \pm 7$
22. $x=\pm 5, \pm 3.$ $y=\mp 3, \mp 5$. 23. $x=\pm 2, \pm\frac{1}{2}.$ $y=\pm 1, \pm 4$. 24. $x=\pm 3.$ $y=\pm 1$.
25. $x=\pm 2, \pm\frac{10}{3}.$ $y=\pm 5, \pm 3$. 26. $x=\pm 5, \pm 3.$ $y=\pm 2{\cdot}4, \pm 4$. 27. $x=4, 2.$ $y=2, 4$.

28. $x=\frac{1}{3},\ -\frac{1}{2}.$ $y=\frac{1}{2},\ -\frac{1}{3}.$

29. $x=5,\ 9.$ $y=9,\ 5.$

30. $x=7,\ -5.$ $y=5,\ -7.$

31 $x=1,\ -2.$ $y=-1,\ \frac{1}{2}.$

32. $x=\frac{1}{2}.$ $y=\frac{1}{3}.$

33. $x=5,\ 1\frac{1}{2}.$ $y=-2,\ -6\frac{2}{3}.$

34. $x=2,\ 1\frac{2}{5}.$ $y=1,\ 1\frac{3}{7}.$

35. $x=7,\ -2.$ $y=-2,\ 7.$

36. $x=\frac{1}{2},\ -\frac{1}{4}.$ $y=\frac{1}{4},\ -\frac{1}{2}$

37. $x=5,\ 1.$ $y=2,\ 10.$

38. $x=3,\ 0.$ $y=0,\ -9.$

39. $x=5,\ 11.$ $y=11,\ 5.$

40. $x=13,\ -12.$ $y=12,\ -13.$

41. $x=2,\ 4.$ $y=2,\ 1.$

42. $x=3,\ 1\frac{1}{3}.$ $y=-2,\ -4\frac{1}{2}.$

XXVII. b. (p. 224).

1. $x=1,\ 2.$ $y=2,\ 1.$

2. $x=4,\ -3.$ $y=3,\ -4.$

3. $x=3,\ 2.$ $y=4,\ 6.$

4. $x=5,\ 4.$ $y=-2,\ -2\frac{1}{2}.$

5. $x=1,\ \frac{2}{3}.$ $y=1,\ \frac{3}{2}.$

6. $x=4,\ -1\frac{1}{2}.$ $y=1,\ -2\frac{2}{3}.$

7. $x=\pm 1,\ \pm 2.$ $y=\mp 2,\ \pm 1.$

8. $x=\pm 5,\ \pm 4.$ $y=\pm 4,\ \pm 5.$

9. $x=\pm 4,\ \pm 3.$ $y=\pm 3,\ \pm 4$

10. $x=\pm 7,\ \pm 2.$ $y=\pm 2,\ \pm 7.$

11. $x=\frac{1}{2},\ \frac{1}{3}.$ $y=\frac{1}{3},\ \frac{1}{2}.$

12. $x=\frac{1}{4},\ -\frac{1}{5}.$ $y=\frac{1}{5},\ -\frac{1}{4}.$

13. $x=\frac{1}{5},\ \frac{2}{3}.$ $y=\frac{1}{3},\ \frac{1}{10}.$

14. $x=3,\ -15.$ $y=5,\ -1.$

15. $x=\pm\frac{1}{5},\ \pm\frac{1}{6}.$ $y=\pm\frac{1}{6},\ \pm\frac{1}{5}$

16. $x=\pm\frac{1}{2},\ \pm 1.$ $y=\pm 2,\ \pm 1.$

17. $x=\frac{1}{5},\ -\frac{1}{3}.$ $y=\frac{1}{3},\ -\frac{1}{5}.$

18. $x=4,\ \frac{1}{4}.$ $y=\frac{1}{4},\ 4.$

19. $x=2,\ -\frac{1}{2}.$ $y=\frac{1}{2},\ -2.$

20. $x=8,\ 2.$ $y=4,\ 16.$

21. $x=\frac{1}{2},\ \frac{1}{3}.$ $y=\frac{1}{3},\ \frac{1}{2}.$

22. $x=\frac{1}{5},\ -\frac{1}{4}.$ $y=\frac{1}{4},\ -\frac{1}{5}.$

23. $x=2,\ 7.$ $y=7,\ 2.$

24. $x=9,\ -3.$ $y=3,\ -9.$

25. $x=\frac{1}{2}.$ $y=1.$

26. $x=\frac{1}{2}.$ $y=\frac{1}{3}.$

XXVII. c. (p. 226).

1. $x=\pm 1.$ $y=\pm 2.$

2. $x=\pm 3,\ \pm\sqrt{2}.$ $y=\pm 2,\ \mp 4\sqrt{2}.$

3. $x=\pm 3,\ x=\mp 1.$ $y=\pm 2,\ y=\pm 2.$

4. $x=\pm 1$ (other roots imaginary). $y=\pm 1.$

5. $x=\frac{3}{5},\ 1.$ $y=\frac{14}{5},\ 2.$

6. $x=\pm 3,\ 0.$ $y=\pm 1,\ \pm 2.$

7. $x=\pm\frac{3}{\sqrt{7}}.$ $y=\pm\frac{4}{\sqrt{7}}.$

8. $x=\pm 10.$ $y=\pm 2.$

9. $x=\pm\frac{5}{\sqrt{6}}$. $y=\pm\frac{1}{\sqrt{6}}$.

10. $x=\pm 3,\ \pm\frac{5}{\sqrt{2}}$. $y=\pm 2,\ \pm\frac{1}{\sqrt{2}}$.

11. $x=\pm 2$. $y=\pm 3$.

12. $x=\pm 2$. $y=\pm 3$.

13. $\left.\begin{array}{l} x=4,\ 2. \\ y=2,\ 4. \end{array}\right\}$ other roots imaginary.

14. $x=\pm 2$. $y=\mp\frac{1}{3}$.

15. $x=\pm 2,\ \pm\frac{3}{\sqrt{2}}$. $y=\pm 1,\ \pm\frac{1}{\sqrt{2}}$.

16. $x=\pm 7$. $y=\pm 5$.

17. $x=8,\ -3$. $y=3,\ -8$.

18. $x=-7,\ 3,\ 5,\ -1$. $y=7,\ -3,\ 1,\ -5$.

19. $x=4,\ -6\frac{2}{3}$. $y=6,\ -4\frac{2}{7}$.

20. $x=\pm 5,\ \pm 2$. $y=\mp 4,\ \mp 3$.

21. $x=\pm 2$. $y=\pm 1$.

22. $x=5,\ 4$. $y=4,\ 5$.

23. $x=1,\ -3\frac{1}{2}$. $y=1,\ -\frac{2}{7}$.

24. $x=-7,\ 4$. $y=-\frac{21}{4},\ 3$.

25. $x=\frac{2}{3},\ -\frac{1}{3}$. $y=\frac{3}{2},\ 0$.

26. $x=\pm 2,\ \pm\frac{7}{\sqrt{2}}$. $y=\pm 5,\ \mp\frac{3}{\sqrt{2}}$.

27. $x=7,\ -\frac{19}{4}$. $y=3,\ -\frac{23}{8}$.

28. $x=\pm 5,\ \pm 3$. $y=\pm 3,\ \pm 5$.

29. $x=\pm 3,\ \pm\frac{8}{\sqrt{6}}$. $y=\pm 1,\ \pm\frac{1}{\sqrt{6}}$.

30. $x=2\frac{1}{2},\ -1\frac{3}{4}$. $y=-1\frac{1}{6},\ 1\frac{2}{3}$.

31. $x=2,\ 5,\ 1\pm\sqrt{6}$. $y=-5,\ -2,\ -1\pm\sqrt{6}$.

32. $x=\pm 3,\ \pm 1$. $y=\mp 1,\ \mp 3$.

33. $x=2,\ -3,\ -2\pm\sqrt{2}$. $y=3,\ 3,\ -1\pm\sqrt{2}$.

34. $\left.\begin{array}{l} x=0,\ -2. \\ y=-4,\ 2. \end{array}\right\}$ other roots imaginary.

35. $x=\pm 3,\ \pm 2,\ \pm 3,\ \pm 2$. $y=\pm 2,\ \pm 3,\ \mp 2,\ \mp 3$.

XXVII. d. (p. 229).

1. A circle, centre (0, 0), radius 6.
2. The origin.
3. ,, ,, ,, 7.
4. A circle, centre (0, 0), radius 9.
5. A circle through the origin, centre (−4, 4), radius $4\sqrt{2}$.
6. ,, ,, ,, ,, (4, 3), ,, 5.
7. A circle, centre (3, 4), radius 6.
8. A circle, centre (1, 2), radius 6.
9. ,, ,, (−2, 3), ,, 5.
10. ,, ,, (3, −3), ,, 4.
11. ,, ,, (−1, 0), ,, 4.
12. ,, ,, (2, 0), ,, 5.
13. ,, ,, (1, 0), ,, 4.
14. ,, ,, (7, 0), ,, 6.
15. A circle, centre (0, 0), radius $\sqrt{2}$.
16. ,, ,, (0, 0), ,, $\sqrt{5}$.
17. ,, ,, (0, 0), ,, $\sqrt{13}$.

18. A circle, centre (0, 0), radius $\sqrt{10}$.
19. ,, ,, (0, 0), ,, $2\sqrt{5}$.
20. ,, ,, (0, 0), ,, $\sqrt{3}$.
21. ,, ,, $(-1, -1)$, ,, $\sqrt{2}$, through the origin
22. ,, ,, (1, 0), ,, $\sqrt{2}$.
23. ,, ,, $(-2, 2)$, ,, $\sqrt{5}$.
24. ,, ,, $(-1, -1)$, ,, $\sqrt{5}$.
25. ,, ,, $(3, -2)$, ,, $\sqrt{10}$.
26. ,, ,, (0, 0), ,, $\dfrac{\sqrt{10}}{2}$.
27. ,, ,, $(1, -2)$, ,, $\sqrt{3\cdot5}$.
28. ,, ,, $(2, -1)$, ,, 1·5.
29. ,, ,, (3, 0), ,, 2·5.

XXVII. e. (p. 233).

1. $x=5{\cdot}3,\ 1{\cdot}7.$ $y=1{\cdot}7,\ 5{\cdot}3.$
2. $x=6{\cdot}56,\ 2{\cdot}44$ $y=2{\cdot}44,\ 6{\cdot}56.$
3. $x=5{\cdot}1,\ -3{\cdot}1.$ $y=3{\cdot}1,\ -5{\cdot}1.$
4. $x=5{\cdot}61,\ -1{\cdot}61.$ $y=1{\cdot}61,\ -5{\cdot}61.$
5. $x=6{\cdot}19,\ {\cdot}81.$ $y={\cdot}81,\ 6{\cdot}19.$
6. $x=4{\cdot}7,\ -1{\cdot}7.$ $y=1{\cdot}7,\ -4{\cdot}7.$
7. 4, 9.
8. 3·2, 7·8.
9. 5·73, 2·27.
10. 5·12, −3·12.
11. $x=1{\cdot}27,\ -{\cdot}47.$ $y=1{\cdot}54,\ -1{\cdot}94.$
12. $x=-2,\ 2{\cdot}8.$ $y=2,\ -{\cdot}4.$
13. $x=2{\cdot}6,\ -2.$ $y=3{\cdot}2,\ {\cdot}4.$
14. $x=1,\ -2{\cdot}2.$ $y={\cdot}5,\ 2{\cdot}1.$
15. $x={\cdot}69,\ -2{\cdot}61.$ $y=-2{\cdot}92,\ 1{\cdot}48.$
16. $x=\pm5{\cdot}29,\ \pm2{\cdot}84.$ $y=\pm2{\cdot}84,\ \pm5{\cdot}29.$
17. $x=\pm13{\cdot}8,\ \pm5{\cdot}8.$ $y=\pm5{\cdot}8,\ \pm13{\cdot}8.$
18. $x=9{\cdot}3,\ -4{\cdot}3.$ $y=8{\cdot}6,\ -18{\cdot}6.$

XXVIII. (p. 234).

1. (i) $x+y$ miles, (ii) $x-y$ miles, (iii) $\dfrac{a}{x+y}$ hours, (iv) $\dfrac{a}{x-y}$ hours
2. (i) £$\dfrac{x}{100}$, (ii) £$\dfrac{xy}{100}$, (iii) £$\dfrac{xyz}{100}$, (iv) £$\left(z+\dfrac{xyz}{100}\right)$.
3. (i) £$\dfrac{10000}{100+x}$, (ii) £$\dfrac{100a}{100+x}$, (iii) £$\dfrac{10000}{100+xy}$, (iv) £$\dfrac{100a}{100+xy}$.
4. (i) $\dfrac{1}{y}$ hours, (ii) $\dfrac{z}{y}$ hours, (iii) $\dfrac{3z}{2y}$ hours, (iv) ay miles.
5. (i) $\dfrac{x+y}{xy}$, (ii) $\dfrac{a(x+y)}{xy}$, (iii) $\dfrac{xy}{x+y}$ hours, (iv) $\dfrac{3xy}{4(x+y)}$ hours

6. (i) $\dfrac{yz+zx-xy}{xyz}$, (ii) $\dfrac{xyz}{yz+xz-xy}$ hours.

7. (i) $£\dfrac{x}{z}$, (ii) $£\dfrac{x}{yz}$, (iii) $£\dfrac{100x}{yz}$, (iv) $£\dfrac{abx}{yz}$.

8. (i) $£(z-y)$, (ii) $£\left(\dfrac{z-y}{x}\right)$, (iii) $£\dfrac{z-y}{xy}$, (iv) $£\dfrac{ab(z-y)}{xy}$,

(v) $\dfrac{100(z-y)}{xy}$ per cent.

9. (i) $\dfrac{x}{12}$ pence, (ii) $\dfrac{xy}{12}$ pence, (iii) $\dfrac{x+1}{12}$ pence,

(iv) $\dfrac{(x+1)y}{12}$ pence, (v) $\dfrac{ax}{12}$ pence, (vi) $\dfrac{a(x+1)}{12}$ pence.

10. (i) $£\dfrac{x}{100}$, (ii) $£\left(1+\dfrac{x}{100}\right)$, (iii) $£a\left(1+\dfrac{x}{100}\right)$,

(iv) $£\dfrac{ax}{100}$, (v) $£\left(1+\dfrac{x}{100}\right)^2$, (vi) $£\left(1+\dfrac{x}{100}\right)^3$,

(vii) $£\left(1+\dfrac{x}{100}\right)^n$, (viii) $£P\left(1+\dfrac{x}{100}\right)^2$, (ix) $£P\left(1+\dfrac{x}{100}\right)^3$,

(x) $£P\left(1+\dfrac{x}{100}\right)^n$, (xi) $£\left\{P\left(1+\dfrac{x}{100}\right)^n-P\right\}$.

11. (i) $£\dfrac{100x}{100+x}$, (ii) $£\dfrac{ax}{100+x}$, (iii) $£\dfrac{100xy}{100+xy}$, (iv) $£\dfrac{axy}{100+xy}$.

12. (i) $\dfrac{7}{4x}$, (ii) $\dfrac{9}{2x}$.

13. (i) $(x-y)$ miles, (ii) $a(x-y)$ miles, (iii) $\dfrac{1}{x-y}$ hours, (iv) $\dfrac{b}{x-y}$ hours.

14. $(x+2)(x+3)-x(x+1)=y$. **15.** $ax+by=\dfrac{z}{20}$. **16.** $\dfrac{ax}{12}+\dfrac{by}{10}=12z$.

17. $y^2-(y-8)^2=x$. **18.** $z^2-(z-2y)^2=a$. **19.** $\dfrac{x+b}{y-c}-\dfrac{x}{y}=a$.

20. $ax+by=(x+y)c$. **21.** $(x+a)(y+a)=2xy$. **22.** $\dfrac{3a}{x}-\dfrac{3a}{y}=n$.

23. $\dfrac{xy}{x+y}=z$. **24.** $ax=y(a-n)$. **25.** $ax+ay=n$. **26.** $ax+(a-b)y=n$.

27. $(x-1)y=1760$. **28.** $(x-1)y=1760a$ **29.** $\dfrac{x}{3}+\dfrac{x}{5}+\dfrac{x}{10}+y=x$, or $11x=30y$.

30. $y+\dfrac{xy}{100}=z$. **31.** $\dfrac{ax}{100}-\dfrac{by}{100}=c$. **32.** $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=d$. **33.** $ay-z(x-a)=20c$.

XXIX. a. (p. 239).

1. 10, 12. **2.** 16 ft., 12 ft. **3.** 16 18. **4.** 15, 16. **5.** $5\frac{1}{2}$. **6.** 7. **7.** 169. **8.** 53 yds., 106 yds. **9.** 3 ft. **10.** 12, 15. **11.** $1\frac{1}{2}$ ft. **12.** 5, 8. **13.** 15. **14.** 12. **15.** 6, 9. **16.** 72. **17.** 40 yds., 50 yds. **18.** 4. **19.** 22. **20.** 30. **21.** 10. **22.** 11. **23.** 55 and 60 miles per hr. **24.** 12 and 24 days. **25.** 2 hours, 4 hours. **26.** 25 miles per hr. **27.** $\frac{2}{5}$. **28.** 6, 7, 8, 9, 10. **29.** 95. **30.** 4 feet. **31.** 32 miles per hr. **32.** 4.

XXIX. b. (p. 239B).

1. 5, 7. **2.** 3 in. **3.** 43. **4.** 12. **5.** 93.
6. 6 yds. per sec. **7.** 14, 11. **8.** 6 miles an hour. **9.** 7.
10. 55, 60 miles an hour. **11.** 6*s*. 6*d*. **12.** 13 miles. **13.** 32.
14. 24 ft. long, 18 ft. wide, 11 ft. high.
15. 10 yds., 7 yds. square, £7, £5.
16. 30 miles an hour, 50 miles an hour.
17. 14 ft. long, 12 ft. wide, 9 ft. high. **18.** 8 miles an hour.
19. 5 miles an hour. **20.** 8 ft., $7\frac{1}{2}$ ft. **21.** 576.
22. 42*s*., 7*s*., 3*s*. 6*d*. **23.** $\frac{5}{12}$. **24.** 9 miles an hour.
25. 3*d*. for 14 lbs., 2*d*. for every extra 7 lbs. **26.** $3\frac{9}{17}$ minutes.
27. 78. **28.** 10, 7, 5 miles an hour, 70 miles. **29.** 7 ft., 18 stone.
30. 7·2 cwt., 11·25 miles. **31.** 40 yds., $60\frac{1}{2}$ yds. **32.** 7, 5.
33. 9, 4 yards. **34.** 32 yds. long, 27 yds. wide.
35. 88 in., 80 in. **36.** 10 hours, 15 hours.
37. $20\frac{1}{2}$ ft., 16 ft. **38.** 3 miles an hour. **39.** $14\frac{1}{7}$.
40. 10 minutes, 15 minutes. **41.** 3, 4, 5 miles an hour.
42. $15\frac{3}{4}$ oz., $16\frac{1}{4}$ oz. **43.** 6 miles, 8 miles an hour. **44.** £5. 14*s*.
45. $5\frac{1}{2}$, $6\frac{3}{5}$ hours. **46.** 12 miles, 3 miles an hour, 4 miles an hour.
47. 8 miles, 16 miles, $4\frac{1}{2}$ miles an hour, $7\frac{1}{2}$ miles an hour.
48. $\frac{9}{19}$. **49.** $1\frac{1}{2}$, $1\frac{1}{3}$, $1\frac{1}{4}$ minutes. **50.** 10 gallons.

XXX. a. (p. 243).

1. $\cdot 5a^3b$. **2.** $\dfrac{\cdot 01x^3}{y}$. **3.** $\cdot 5x^2y$. **4.** $\dfrac{x^5}{\cdot 08}$.
5. $2(a-b)$. **6.** $\dfrac{1}{x-3}$. **7.** $2x \pm 3y$. **8.** $1 \pm 2a^2b$.
9. $x \pm \dfrac{1}{x}$. **10.** $x \pm \dfrac{5a}{4}$. **11.** $1 \pm (a-b)$. **12.** $\dfrac{a}{b}$.
13. x. **14.** $2a$. **15.** $2x^2 \pm \dfrac{1}{2x^2}$. **16.** $2x^2 \pm \dfrac{1}{x^2}$
17. 4, 5. **18.** -3, 1. **19.** 5, 2. **20.** 4, -5.
21. 0, -5. **22.** $\pm\frac{4}{5}$. **23.** $-\frac{1}{2}$, $\frac{1}{2}$. **24.** $1\frac{1}{4}$, $2\frac{1}{3}$.
25. $1\frac{1}{2}$, $-\frac{1}{9}$. **26.** a, -3. **27.** 1. **28.** $1\frac{2}{5}$, $4\frac{2}{3}$.
29. 4, -2. **30.** -1. **31.** 1, -2. **32.** 1.
33. $1\frac{1}{2}$. **34.** $\frac{1}{2}$. **35.** 2. **36.** $\frac{1}{2}$. **37.** $\frac{1}{5}$.

XXX. b. (p. 244).

1. $\dfrac{2ax}{4x^2-9a^2}$, 0. **2.** $a+b-1$, $a^2+b^2+c^2+2ab-2ac-2bc$, $a^3+3a^2b+3ab^2+b^3$.
3. $\pm\frac{1}{2}$. **4.** 2·83, 3·61. **5.** $x=3, -2\frac{1}{3}$. **7.** $\frac{7}{18}$.
$y=1, -1\frac{2}{7}$.

XXX. c. (p. 244).

1. $\dfrac{2x}{(x-a)(x-b)(x+b)}$. 2. ± 10. 3. 3, -2. 4. $5x^2-7x+4$.
5. $x=2{\cdot}5$, $y=6{\cdot}25$. 6. $x=3$, 4, $y=4$, 3. 7. 7062.

XXX. d. (p. 245).

1. $\dfrac{a^2}{(a+2b)(a-3b)}$. 2. x^2-x-4, $a^2+4b^2+c^2-4ab+2ac-4bc$, $a^3+6a^2b+12ab^2+8b^3$.
3. 1, 2 are the roots. 4. 7·40, 7·65. 5. $x=3$, -8, $y=4$, $-1\frac{1}{7}$.
6. 1, 4. 7. Half a minute.

XXX. e. (p. 245).

1. $\dfrac{5}{(x-1)(x+2)(x+3)}$. 2. ± 12. 3. $1\frac{1}{2}$, $-1\frac{1}{3}$. 4. $x<2\frac{1}{2}>-3\frac{1}{2}$.
5. $x=\pm 1$, $y=\pm 2$. 6. $4x^2-2x+\dfrac{1}{x}$. 7. £15. 15*s*.

XXX. f. (p. 245).

1. 1. 3. $-2{\cdot}8$, $2{\cdot}3$. 4. 12·25, $-6{\cdot}25$. 5. $x=8$, $y=1$.
7. 2340.

XXX. g. (p. 246).

1. x^2. 3. 2·15, $-1{\cdot}4$. 4. 2·83. 5. $x=6$, $\frac{1}{2}$, $y=\frac{1}{2}$, $-2\frac{1}{4}$.
6. x^2-6x+1. 7. $3\frac{2}{7}$ miles an hour.

XXX. h. (p. 246).

1. $(x^2+3x+3)(x^2-3x+3)$, $(8x-1)^3(1-a)(1+a+a^2)$.
2. $\dfrac{x(a-c)}{(x+a)(x+c)}$. 3. $(x^2-y^2)^2+(x^2-y^2)z^2+z^4$. 4. 68*s*., 86*s*., 98*s*.
5. 2·6, $-1{\cdot}6$. 6. $x=\pm 1$, $y=\pm 2$. 7. 80, £32.

XXX. k. (p. 246).

1. $16a^6-36a^4b^2-108a^3b^3-162a^2b^4+486ab^5+729b^6$.
2. $\dfrac{(c+a-b)(c-a+b)(c+a-b)(c-a+b)}{4a^2b^2}$. 3. 6. 4. 2·56, $-1{\cdot}56$
5. 2.54 p.m., 1.51 p.m., 3.57 p.m. 6. $x=\pm 4$, ± 1. $y=\pm 1$, ± 4. 7. Friday.

XXX. l. (p. 247).

1. $6x^5-x^4+10x^3-14x^2-25$. 3. 5, 3. 4. $x=6{\cdot}37$, ·63, $y=$ ·63, 6·37.
5. They meet in 4 hours, 42 miles from home. They are 10 miles apart in 5 hours.
6. $x=\pm 5$, $\pm 2\sqrt{3}(=\pm 3{\cdot}46)$, $y=\pm 4$, $\mp\sqrt{3}(=\mp 1{\cdot}73)$. 7. $\dfrac{p}{q+r}$ hours

XXX. m. (p. 247).

1. $\frac{1}{(a-b)^2}$. 2. 3. 3. $\frac{x^2-3x+2a}{x-3}$.

4. £19. 18s., £41, £57. 8s. 5. $x=6,\ -2,$ $y=6,\ 2.$

6. $x=\pm 2,\ y=\pm 1.$ 7. 39 ft. long, 31 ft. wide

XXX. n. (p. 248).

1. $\frac{a-b}{a+b}$. 2. $4,\ \frac{1}{4}$. 3. $2a^2b(a+b),\ (x-2)(x-3)(x-5)$

4. $x=0,\ 4,$ $y=0,\ -8.$ 5. $12,\ -1{\cdot}5.$ 6. $x=\pm\frac{1}{2},\ \pm 9\frac{1}{2},$ $y=\mp 2,\ \pm 7.$ 7. One mile

XXX. p. (p. 248).

1. $bx+ay+1.$ 2. $-8,\ -12.$

3. $(a-b)(a+b-c)(a+b+c),\ (x^2-xy-y^2)(x^2+xy-y^2).$

4. $4{\cdot}54,\ -1{\cdot}54.$ 5. $25\frac{3}{4}$ miles an hour.

6. $x=\frac{1}{3},\ -\frac{1}{8}.\quad y=\frac{1}{4},\ -\frac{1}{7}.\quad z=\frac{1}{6},\ \frac{1}{28}.$ 7. 480 apples, 400 pears

XXXI. a. (p. 249).

1. $-a-b.$ 2. $\frac{1}{ab}$. 3. $\frac{ac}{b}$.

4. $\frac{2(a-2b)(2a-b)}{a+b}$. 5. $\frac{2}{a^2b^2+b^2c^2+c^2a^2}$. 6. $\frac{a+c}{2}$.

7. $a+b.$ 8. $\frac{a+c}{b}$. 9. $\frac{a^2+b^2}{a+b}$.

10. $abc.$ 11. $\frac{2ab}{a+b}$. 12. $\frac{pr}{q}$.

13. 0. 14. $\frac{a^2+b^2}{a+b}$. 15. $\frac{ab}{a+b}$.

16. $\frac{a^6+1}{a(a^2-a^4-2)}$. 17. $-\frac{ab}{a^2-ab+b^2}$. 18. $\frac{ad+bc-2bd}{a-b+c-d}$.

19. $\frac{2ab}{a-b}$. 20. $-\frac{a+b}{2}$. 21. $a+3b.$

22. $-\frac{a+c}{2}$. 23. $\frac{a+b}{2}$. 24. $\frac{2ab}{a+b}$. 25. $\frac{a+b}{2}$.

XXXI. b. (p. 251).

1. $x=a+1,\ y=a-1.$ 2. $x=c,\ y=-a.$

3. $x=3a-b,\ y=a+3b.$ 4. $x=\frac{s+t}{2a},\ y=\frac{s-t}{2b}$.

5. $x=\frac{a^2+ab+b^2}{a+b},\ y=\frac{ab}{a+b}$. 6. $x=a+b,\ y=a-b.$

7. $x=c,\ y=-a.$

8. $x=\frac{a-c}{a-b},\ y=\frac{a-c}{b-c}.$

9. $x=\frac{a+b}{a-b},\ y=\frac{a-b}{a+b}.$

10. $x=\frac{3b}{2},\ y=-\frac{a}{2}.$

11. $x=\frac{b+c-a}{a+b-c},\ y=\frac{a+c-b}{a+b-c}.$

12. $x=\frac{c(a^2+b^2)}{a^2-b^2},\ y=\frac{c(a^2+b^2)}{2ab},\ x=0,\ y=0.$

13. $x=\frac{c-a}{c+a},\ y=\frac{a-c}{2(c+a)}.$

14. $x=\frac{a+b+c}{a+b},\ y=\frac{a+b}{c}.$

15. $x=a,\ y=b.$

16. $x=\frac{a^2-b^2}{ap-bq},\ y=\frac{a^2-b^2}{aq-bp}.$

17. $x=a+b,\ y=a-b.$

18. $x=\frac{bc-d}{ab-1},\ y=\frac{ad-c}{ab-1}.$

19. $x=\frac{a^2-bc}{a},\ y=\frac{b^2-ac}{b}.$

20. $x=6a+b,\ y=2a-b.$

21. $x=\frac{a}{a^2+1},\ y=\frac{-1}{a^2+1}.$

22. $x=\frac{b+c-a}{2a},\ y=\frac{c+a-b}{2b},\ z=\frac{a+b-c}{2c}.$

23. $x=\frac{\pm a}{\sqrt{la^2+mb^2+nc^2}},\ y=\frac{\pm b}{\sqrt{la^2+mb^2+nc^2}},\ z=\frac{\pm c}{\sqrt{la^2+mb^2+nc^2}}.$

24. $x=\frac{2abc}{ab-bc+ac},\ y=\frac{2abc}{ab+bc-ac},\ z=\frac{2abc}{bc+ac-ab}.$

XXXI. c. (p. 252).

1. $x=5a,\ -3a.$

2. $x=2a,\ 3a.$

3. $x=\frac{1}{a},\ \frac{c}{b}.$

4. $x=a,\ b.$

5. $x=a\pm\frac{1}{a}.$

6. $x=\frac{1}{a},\ -\frac{q}{p}.$

7. $x=\pm a.$

8. $x=\frac{b}{a}.$

9. $x=\frac{1}{a},\ \frac{1}{b}.$

10. $x=-\frac{1}{a},\ \frac{1}{b}.$

11. $x=4b,\ -3b.$

12. $x=\frac{f^2}{ag}.$

13. $x=\frac{5a}{2},\ \frac{3a}{10}.$

14. $x=3a,\ \frac{3a}{2}.$

15. $x=-2a,\ 2a+2b.$

16. $x=\frac{a-b}{2},\ \frac{a+b}{2}.$

17. $x=\frac{a+b}{a-b},\ \frac{a-b}{a+b}.$

18. $x=a,\ b.$

19. $x=a+1,\ \overline{a-1}^2.$

20. $x=\frac{1}{3}[a+b+c\pm\sqrt{a^2+b^2+c^2-bc-ac-ab}].$

21. $x=\frac{1}{2}\left(a\pm\frac{1}{b}\right).$

22. $x=a+b,\ \frac{a+b}{2},$

23. $x=0,\ a+b.$

24. $x=1,\ \frac{-2ab}{a^2+2ab-b^2}.$

25. $x=b,\ 2a-b.\quad y=a,\ 2b-a.$

XXXI. d. (p. 254).

1. 11.
2. 2.
3. 7.
4. $1\frac{1}{12}.$
5. 1.
6. $\pm5.$
7. $0,\ \frac{4}{3}.$
8. 3.

9. $\frac{1}{2\sqrt{10}}$. **10.** 8. **11.** -5. **12.** -4.
13. 4. **14.** 8. **15.** $\frac{(a^2+b^2)^2}{(a+b)^2}$. **16.** ± 5.
17. $a+2b$ **18.** $-\frac{3}{28}$ **19.** $\frac{11}{7}$ **20.** $\frac{b}{a}$.
21. 16. **22.** 0. **23.** $\frac{2}{3}$. **24.** $\frac{a^2}{16}$.
25. a^2+b^2. **26.** $1, -4$. **27.** 0, 5. **28.** -1.
29. $2, -4$. **30.** $2, -\frac{4}{3}$. **31.** $\frac{1}{2}(3\pm\sqrt{5})$. **32.** $\frac{15}{2}, -1$. **33.** $2, -5$.

XXXI. e. (p. 256).

1. $x=6, 4,$ $y=4, 6,$ $z=5, 5.$

2. $x=9, 1,$ $y=3, 3,$ $z=1, 9.$

3. $x=-\frac{1}{2}, \frac{1\pm\sqrt{29}}{4}$.

4. $x=\pm\frac{3\sqrt[4]{2}}{2}, \pm 3,$ $y=\pm\sqrt[4]{2}, \pm 1.$

5. $x=\pm 4,$ $y=\pm 2.$

6. $x=a, \frac{a+b}{2},$ $y=b, \frac{a+b}{2}.$

7. $x=0, \pm\frac{2c}{\sqrt{3}},$ $y=\pm c, \mp\frac{c}{\sqrt{3}}.$

8. $x=1, 1, 2, 2, 4, 4,$ $y=2, 4, 1, 4, 1, 2,$ $z=4, 2, 4, 1, 2, 1.$

9. $x=\pm\sqrt{\frac{ab^2}{2b-a}}.$

10. $x=\frac{ab(c+d)-cd(a+b)}{ab-cd}.$

11. $x=\pm\left(\frac{1}{b}+\frac{1}{a}\right), \pm\left(\frac{1}{b}-\frac{1}{a}\right),$ $y=\pm\left(\frac{1}{b}-\frac{1}{a}\right), \pm\left(\frac{1}{b}+\frac{1}{a}\right).$

12. $x=\pm\sqrt{6},$ $y=\pm\frac{\sqrt{6}}{2},$ $z=\pm\frac{\sqrt{6}}{3}.$

13. $x=\mp\frac{23}{12},$ $y=\pm\frac{31}{12},$ $z=\pm\frac{41}{12}.$

14. $x=\pm 3, y=\pm 1.$

XXXI. f. (p. 259).

1. (0, 7)(5, 5)(10, 3)(15, 1). **2.** (0, 5)(3, 3)(6, 1).
3. (5, 1)(3, 6)(1, 11). **4.** (7, 8)(10, 1)(4, 15)(1, 22). **5.** (2, 3).
6. (11, 10)(24, 3). **7.** 7. **8.** 8. **9.** 6. **10.** 6.
12. (1, 7)(3, 4)(5, 1). **13.** (1, 13)(2, 8)(3, 3)(0, 18).
14. (0, 12)(4, 9)(8, 6)(12, 3)(16, 0). **15.** (1, 3)(8, 1).
16. (0, 10)(3, 8)(6, 6)(9, 4)(12, 2)(15, 0).
17. (2, −5)(4, −4)(6, −3)(8, −2)(10, −1). **18.** (3, −6)(6, −4)(9, −2).
19. (1, −3)(2, −2)(3, −1). **20.** (−3, −6)(−6, −4)(−9, −2).
21. (−3, −4). **22.** (−2, −10)(−4, −8)(−6, −6)(−8, −4)(−10, −2).

23. 2 at 5*s.* each, 4 at 7*s.* **24.** 6 geese, 4 turkeys.

25. 30 ways. **26.** 27, 32.

27. Give 10 four-shilling pieces, receive 2 half-crowns.

28. 4 ways. **29.** $(\frac{1}{4};\ \frac{13}{7})$, $(\frac{5}{4};\ \frac{6}{7})$. **30.** $x=13p+7,\ y=9p$.

31. 3 ways. **32.** 35, 4. **33.** 3 ways.

XXXII. (p. 266).

1. $x^2-7x+10=0$. **2.** $x^2+x-20=0$. **3.** $4x^2-1=0$.

4. $x^2+3x=0$. **5.** $x^2+ax-6a^2=0$. **6.** $x^2-2ax+a^2-1=0$.

7. $a^2x^2-2a^2x+a^2-1=0$. **8.** $x^2-2mx+n=0$. **9.** $lx^2+mx+n=0$.

10. $x^2-6x+6=0$. **11.** $25x^2-40x+13=0$. **12.** -25.

13. p^2-4q must be a perfect square. **15.** $\dfrac{p\pm\sqrt{p^2-4q}}{2}$, p, q.

16. (i) $\pm\dfrac{\sqrt{b^2-4ac}}{a}$. (ii) $\dfrac{b^2-2ac}{a^2}$. (iii) $\dfrac{b(3ac-b^2)}{a^3}$. (iv) $\dfrac{(b^2-2ac)^2}{a^4}-\dfrac{2c^2}{a^2}$.

17. $x^2-2px+4q=0$. **18.** $ax^2-bx+c=0$.

19. $ax^2+3bx+9c=0$. **20.** $acx^2-(b^2-2ac)x+ac=0$.

21. $acx^2-2(b^2-2ac)x+4ac=0$. **22.** $a^2x^2+abx+9ac-2b^2=0$.

23. $a^2cx^2-b(3ac-b^2)x+ac^2=0$. **24.** $a=-\frac{2}{3}$.

26. $x^2+4px-p^2=0$. **27.** $a^2(x^2+1)+(b^2+2a^2)x=0$.

28. $p(3q-p^2)$. **29.** $p^2x^2+px(q-r)-qr=0$. $\dfrac{(q^2-2pr)^2}{p^4}-\dfrac{2r^2}{p^2}$.

30. $a^2x^2-(b^2-2ac)x+c^2=0$. **31.** $x^2-(p^2+2q)x+q^2=0$.

32. $k=-2$. **34.** $(p'-p)(pq'-p'q)=(q-q')^2$.

39. $a^3x^2+2b(4b^2+3ac)x-c^3=0$. **40.** $\dfrac{b}{c}=\frac{32}{27}$. **43.** (i) ac. (ii) c^2.

44. $\dfrac{b^2-2ac}{a^2c^2}$. **49.** $5a$. **50.** $\frac{1}{3}$.

XXXIII. a. (p. 268).

1. $(2x-5y)(3x-4y)$, $(x^2-3xy+y^2)(x^2+3xy+y^2)$,
$(x-1)(x+1)(x^2+x+1)(x^2-x+1)$.

2. 0. **3.** $8z(2z-1)$. **4.** $x^8(x^8-y^8)$.

5. (i) $2\frac{1}{3}$, $-\frac{2}{9}$. (ii) $x=\pm2,\ \pm1$,
$y=\pm1,\ \mp2$.

6. 4 hrs. 35 min., 3 hrs. 48 min., 19·9 miles.

7. $x=-3,\ y=1\frac{1}{2},\ z=4$. **8.** $x^2+3px+2p^2+q=0$.

XXXIII. b. (p. 268).

1. $(x+7)(x+9)$, $(y-a)(y+7a)(y-6a)$,
$x(x-1)(x+1)(x-2)(x+2)(x-3)(x+3)$.

2. $3x^2-7x-2$. **3.** 4. **4.** (i) $\dfrac{1}{ab}$. (ii) 1, 13.

5. £600. **6.** 90, 81, 71, 62, 41, 21.

XXXIII. c. (p. 269).

1. $367a - 114b + 690c$, 1082. **2.** 0. **3.** $x^2 - 7x + 2$. **4.** $-\frac{1}{2}$, [illegible]

5. (i) $-\frac{ac}{b}$. (ii) $x = 2, 1\frac{1}{2}$, $y = 1, 1\frac{1}{2}$. **6.** £30. **7.** $-1, -\frac{1}{2}$.

XXXIII. d. (p. 269).

1. $x^3 - 3x^2 + 11x - 8$. **2.** $\frac{a}{b} - 1 - \frac{b}{a}$. **3.** $2x^2 + 3x - 5$.

4. $20x$ yds., $\frac{15x}{22}$ miles, $\frac{15xy}{22}$ miles, $\frac{22y}{15x}$ hours.

5. (i) $x = 0, 7, -2\frac{1}{21}$. (ii) $x = \frac{1}{3}, y = \frac{1}{2}$.

6. In $37\frac{1}{2}$ secs. **7.** $x = 1{\cdot}5$, max. value $2{\cdot}25$.

XXXIII. e. (p. 270).

2. $x^6 - y^6$. **3.** $n^2 + 3n + 1$. **4.** $-\frac{(x+y-z)^3}{2yz}$.

6. (i) $a - b$. (ii) $2{\cdot}63$, $1{\cdot}37$. **7.** 15, 12 miles per hour.

8. $\frac{2}{3}, \frac{1}{3}$.

XXXIII. f. (p. 270).

1. $x^2 + 3y^2$. **2.** $x(x-4)(4x-7)$, $(y+3)(y-3)(y^2+20)$, $(a^2+3b^2)(a^2-3ab+3b^2)(a^2+3ab+3b^2)$.

3. (i) $\frac{ab}{b-a}$. (ii) $x = \pm 4$, $y = \pm 3$. **4.** 4α, 4β.

6. 48 minutes. **7.** $-(a+b+c)$.

XXXIII. g. (p. 271).

1. $2x^3 + 3x^2 + 8x + 25$, remainder 74. **2.** $\cdot 618$.

3. 14/8, 14/-. **4.** (i) $-4(a^2+b^2)$. (ii) 0.

5. (i) $\frac{2ab}{a+b}$. (ii) $x = \frac{ac}{a+b}$, $y = \frac{bc}{a+b}$. **6.** £26, £50, £64.

7. $ax^2 - 2bx + 4c = 0$.

XXXIII. h. (p. 271).

1. $x^m(a + bx^2)$. **2.** -30.

3. (i) $a^2 - ab + b^2$. (ii) $(a^2+ab+b^2)(a^2-ab+b^2)(a^2+ab-b^2)$.

4. $x = 2$, $y = 5$ are common roots. **5.** $x^2 + 3px - 9q = 0$.

6. (i) $2, -3\frac{1}{2}$. (ii) $x = \frac{2}{3}, -\frac{1}{2}$, $y = -\frac{1}{2}, \frac{2}{3}$. **7.** 41, 28 miles per hour.

XXXIII. k. (p. 272).

2. $\cdot 31,\ -\cdot 81$ 3. $\dfrac{1}{b-c}$. 4. $\frac{20}{21}$.

5. 5/17/-, 6/6/-, 7/12/-. 6. $x=\pm(a\pm b)$, $y=\pm(a\mp b)$.

7. $ax^2+(b-2am)x+am^2-bm+c=0$.

XXXIII. l. (p. 272).

1. $(a^2-12b)(a^2+4b)$, $(a+c)(ac+b^2)$. 2. a^6-64b^6.
4. $161\frac{1}{2}$. 5. $x=1,\ y=2,\ z=3$.
6. $b^2<ac$. 7. 43, 18 miles per hour.

XXXIII. m. (p. 272).

1. $(b-c)$. 2. $(2x+7)(9x-5)$, $(a-c)(a+c-2b)$, $(x-b)(x-3b)(x-5)$
3. x^4+7x^2+2x-3. 4. 3·61.
5. (a) $x=\frac{1}{3},\ \frac{4}{3}$, $y=\frac{2}{3},\ \frac{1}{3}$. (b) $\dfrac{a}{b},\ \dfrac{b}{a}$. 7. 25, 44, 46.

XXXIII. n. (p. 273).

1. $(ac-bd)^2+(ad-bc)^2=(ac-bd)(ad-bc)$. 2. (i) 0, (ii) $\dfrac{n^3(3n^2+1)}{4}$.
3. $(3x+2)(x-2)(2x-1)(2x+1)$. 4. 3·5.
5. £800. 6. (i) $-\dfrac{bc}{a}$. (ii) $x=\pm\sqrt{\dfrac{a}{2b}}$, $y=\pm\sqrt{\dfrac{b}{2a}}$.
7. $acx^2+(ab+2ac-b^2)x+a(a-b+c)=0$.

XXXIII. p. (p. 273).

1. $x^2(x^2-1)(x^4+x^2+1)$. 2. $x^2+y^2+z^2+yz-zx+xy$.
3. $8x^6+6x^5-4x^4-37x^3-15x^2+7x+35$. 4. $-3\cdot 83$, $1\cdot 83$.
5. (i) 0, $\frac{4}{5}$. (ii) $x=-\frac{3}{4},\ 1\frac{3}{4},\ -1\frac{1}{4},\ \frac{1}{4}$, $y=-\frac{3}{4},\ 1\frac{3}{4},\ \frac{1}{4},\ -1\frac{1}{4}$. 6. 5.

XXXIII. q. (p. 274).

3. (i) $x=0,\ \dfrac{ad-bc}{a-c}$, $y=0,\ \dfrac{bc-ad}{b-d}$. (ii) $\dfrac{a-b}{2}$. 4. $(64x^6-729)(3x+2)$.
5. 9·75. 6. 35/-.

XXXIII. r. (p. 274).

1. $(x-1)(x+1)$, $(x-7)(x+1)$, $x(x-1)(x-2)$, $(3x-1)(x-2)$,
L.C.M. $x(x-1)(x+1)(x-7)(x-2)(3x-1)$.

2. (i) 3. (ii) $a+b$. **3.** $5 \cdot 53$, $-2 \cdot 53$. **4.** $2a^2-3ab+2b^2=0$.

5. A was elected by a majority of 5. **6.** $x=\pm\sqrt{2}(\pm 1 \cdot 41)$, ± 5
$y=\mp 4\sqrt{2}(5 \cdot 66)$, ± 3.

7. $\dfrac{a^3-3ab+2c}{6}$.

XXXIII. s. (p. 275).

1. $2(x^2+y^2+z^2-xy-yz-xz)$. **2.** 1, $-\dfrac{a+2b}{2a+b}$.

3. (i) n^2. (ii) $n^2+(n-1)^2$.

4. $x^2-(m+n)(p^2-2q)x+q^2(m^2+n^2)+mn(p^4-4p^2q+2q^2)=0$.

5. $x < -3\frac{1}{2}$ or $> 2\frac{1}{2}$.

6. $x=1, 1, 2, -2, 2, -2$,
$y=2, -2, 1, 1, -2, 2$,
$z=-2, 2, -2, 2, 1, 1$.

7. 14, 8 miles per hour.