DIMENSIONAL ANALYSIS

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PREFACE

THE substance of the following pages was given as a series of five lectures to the Graduate Conference in Physics of Harvard University in the spring of 1920.

The growing use of the methods of dimensional analysis in technical physics, as well as the importance of the method in theoretical investigations, makes it desirable that every physicist should have this method of analysis at his command. There is, however, nowhere a systematic exposition of the principles of the method. Perhaps the reason for this lack is the feeling that the subject is so simple that any formal presentation is superfluous. There do, nevertheless, exist important misconceptions as to the fundamental character of the method and the details of its use. These misconceptions are so widespread, and have so profoundly influenced the character of many speculations, as I shall try to show by many illustrative examples, that I have thought an attempt to remove the misconceptions well worth the effort.

I have, therefore, attempted a systematic exposition of the principles underlying the method of dimensional analysis, and have illustrated the applications with many examples especially chosen to emphasize the points concerning which there is the most common misunderstanding, such as the nature of a dimensional formula, the proper number of fundamental units, and the nature of dimensional constants. In addition to the examples in the text, I have included at the end a number of practise problems, which I hope will be found instructive.

The introductory chapter is addressed to those who already have some acquaintance with the general method. Probably most readers will be of this class. I have tried to show in this chapter by actual examples what are the most important questions in need of discussion. The reader to whom the subject is entirely new may omit this chapter without trouble.

I am under especial obligation to the papers of Dr. Edgar Buckingham on this subject. I am also much indebted to Mr. M. D. Hersey of the Bureau of Standards, who a number of years ago presented Dr. Buckingham's results to the Conference in a series of lectures.

September, 1920.

PREFACE TO THE REVISED EDITION

IN reprinting this little book after eight years, only a few modifications, including references to recent literature, have been found necessary. The most important is a change in the proof of the Π theorem in Chapter IV. I am much indebted to Professor Warren Weaver of the University of Wisconsin for calling my attention to an error in the original proof and also for supplying the form of the proof now given. I have also profited from several suggestions of Dr. Edgar Buckingham, although I am sure that Dr. Buckingham would still take exception to many things in the book.

An appendix has been added containing the dimensional formulas of many common quantities as ordinarily defined. I hope that this will prove useful in the solution of actual problems, but at the same time I hope that the presentation of such a table will not obscure the essential fact that the dimensions there given have about them nothing of the absolute, but are merely those which experience has suggested as most likely to be of value in treating the larger part of the problems arising in practice.

Since the first printing of the book I have observed to my great surprise that in spite of what seemed to me a lucid and convincing exposition there are still differences in fundamental points of view, so that the subject cannot yet be regarded as entirely removed from the realm of controversy. Nothing that has appeared in these eight years has caused me to modify my original attitude, which is therefore given again without change, but in order that the reader may be able to form his own judgment, a few of the more important references are given here in addition to several in the body of the text.

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P. W. B.

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CHAPTER I

INTRODUCTORY

APPLICATIONS of the methods of dimensional analysis to simple problems, particularly in mechanics, are made by every student of physics. Let us analyze a few such problems in order to refresh our minds and get before us some of the questions which must be answered in a critical examination of the processes and assumptions underlying the correct application of the general method.

We consider first the illustrative problem used in nearly every introduction to this subject, that of the simple pendulum. Our endeavor is to find, without going through a detailed solution of the problem, certain relations which must be satisfied by the various measurable quantities in which we are interested. The usual procedure is as follows. We first make a list of all the quantities on which the answer may be supposed to depend; we then write down the dimensions of these quantities, and then we demand that these quantities be combined into a functional relation in such a way that the relation remains true no matter what the size of the units in terms of which the quantities are measured.

Now let us try by this method to find how the time of swing of the simple pendulum depends on the variables which determine the behavior. The time of swing may conceivably depend on the length of the pendulum, on its mass, on the acceleration of gravity, and on the amplitude of swing. Let us write down the dimensions of these various quantities, using for our fundamental system of units mass, length, and time. In the dimensional formulas the symbols of mass, length, and time will be denoted by capital letters, raised to proper powers. Our list of quantities is as follows:

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Time of swing,	t	T
Length of pendulum,	l	L
Mass of pendulum,	m	M
Acceleration of gravity,	g	LT^{-2}
Angular amplitude of swing,	θ	No dimensions.

We are to find t as a function of l, m, g, and θ, such that the functional relation still holds when the size of the fundamental units is changed in any way whatever. Suppose that we have found this relation and write

$$t = f\ (l, m, g, \theta).$$

Now the dimensional formulas show how the various fundamental units determine the numerical magnitude of the variables. The numerical magnitude of the time of swing depends only on the size of the unit of time, and is not changed when the units of mass or length are changed. Hence if the equation is to remain true when the units of mass and length are changed in any way whatever, the quantities inside the functional sign on the right-hand side of the equation must be combined in such a way that together they are also unchanged when the units of mass and length are changed. In particular, they must be unchanged when the size of the unit of mass alone is changed. Now the size of the unit of mass affects only the magnitude of the quantity m. Hence if m enters the argument of the function at all, the numerical value of the function will be changed when the size of the fundamental unit of mass is changed, and this change cannot be compensated by any corresponding change in the values of the other quantities, for these are not affected by changes in the size of the unit of mass. Hence the mass cannot enter the functional relation at all. This shows that the relation reduces to

$$t = f\ (l, g, \theta).$$

Now l and g must together enter the function in such a way that the numerical magnitude of the argument is unchanged when the size of the unit of length is changed and the unit of time is kept constant. That is, the change in the numerical value of l produced by a change in the size of the unit of length must be exactly compensated by the change produced in g by the same change. The dimensional formula shows that l must be divided by g for this to be accomplished. We now have

$$t = f\ (l/g, \theta).$$

Now a change in the size of the fundamental units produces no change in the numerical magnitude of the angular amplitude, because it is dimensionless, and hence θ may enter the unknown function in any way. But it is evident that l/g must enter the function in such a way that the combination has the dimensions of T, since these are the dimensions of t which stands alone on the left-hand side of the equation. We see by inspection that l/g must enter as the square root in order to have the dimensions of T, and the final result is to be written

$$t = \sqrt{l/g}\,\phi\,(\theta)$$

where ϕ is subject to no restriction as far as the present analysis can go. As a matter of fact, we know from elementary mechanics, that ϕ is very nearly a constant independent of θ, and is approximately equal to 2π.

A question may arise in connection with the dimensions of θ. We have said that it is dimensionless, and that its numerical magnitude does not change when the size of the fundamental units of mass, length, or time are changed. This of course is true, but it does not follow that therefore the numerical magnitude of θ is uniquely determined, as we see at once from the possibility of measuring θ in degrees or in radians. Are we therefore justified in treating θ as a constant and saying that it may enter the functional relation in any way whatever?

Now let us discuss by the same method of analysis the time of small oscillation of a small drop of liquid under its own surface tension. The drop is to be thought of as entirely outside the gravitational field, and the oscillations refer to periodic changes of figure, as from spherical to ellipsoidal and back. The time of oscillation will evidently depend on the surface tension of the liquid, on the density of the liquid, and on the radius of the undisturbed sphere. We have, as before,

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Time of oscillation,	t	T
Surface tension,	s	MT^{-2}
Density of liquid,	d	ML^{-3}
Radius of drop,	r	L

We are to find f such that

$$t = f\,(s, d, r)$$

where f is such that this relation holds true numerically whatever the size of the fundamental units in terms of which t, s, d, and r are measured. The method is exactly the same as for the pendulum problem. It is obvious that M must cancel from the right-hand side of the equation. This can occur only if s and d enter through their quotient. Hence

$$t = f\,(s/d, r).$$

Now since L does not enter t, L cannot enter f. Hence s/d and r must be combined in such a way that L cancels. Since L enters s/d to the third power, it is obvious that s/d must be divided by the cube of r in order to get rid of L. Hence

$$t = f\,(s/dr^3).$$

Now the dimensions of s/dr^3 are T^{-2}. The function must be of such a form that these dimensions are converted into T, which are the dimensions of the left-hand side. Hence the final result is

$$t = \text{Const}\,\sqrt{dr^3/s}.$$

That is, the time of oscillation is proportional to the three halves power of the radius, to the square root of the density, and inversely to the square root of the surface tension. This result is checked by experiment. The result was given by Lord Rayleigh as problem 7 in his paper in *Nature,* 95, 66, 1915.

Now let us stop to ask what we meant when in the beginning we said that the time of oscillation will "depend" only on the surface tension, density, and radius. Did we mean that the results are independent of the atomic structure of the liquid, for example? Everyone will admit that surface tension is due to the forces between the atoms in the surface layer of the liquid, and will depend in a way too complicated for us at present to exactly express on the shape and constitution of the atoms, and on the nature of the forces between them. If this is true, why should not all the elements which determine the forces between the atoms also enter our analysis, for they are certainly effective in determining the physical behavior? We might justify our procedure by some such answer as this. "Although it is true that the behavior is determined by a most complicated system of atomic forces, it will be found that these forces affect the result only in so far as they conspire to determine one property, the surface tension." This implies that if we were to measure the time of oscillation of drops of different liquids, differing as much as we pleased in atomic properties, we would find that all drops of the same radius, density, and surface tension, executed their oscillations in the same time. We add that the truth of this reply may be checked by an appeal to experiment. But our critic may not even yet be satisfied. He may ask how we were sure beforehand that among the various properties of the liquids of which the drop might be composed the surface tension was the only property affecting the time of oscillation. It may seem quite conceivable to him that the time of oscillation might depend on the viscosity or compressibility, and if we are compelled to appeal to experiment, of what value is our dimensional analysis? To which we would be forced to reply that we have indeed had a wider experimental experience than our critic, and that there are conditions under which the time of oscillation does depend on the viscosity or compressibility in addition to the surface tension, but that it will be found as a matter of experiment that if the radius of the drop is made smaller and smaller there is a point beyond which the compressibility will be found to play an imperceptibly small part, and in the same way if the viscosity of the liquid is made smaller and smaller, there will also be a point beyond which any further reduction of the viscosity will not perceptibly affect the oscillation time. And we add that it is to such conditions as these that our analysis applies. Instead of appealing to direct experiment to justify our assertions, we might, since our critic is an intelligent critic, appeal to that generalization from much experiment contained in the equations of hydrodynamics, and show by a detailed application of the equations to the present problem that compressibility and viscosity may be neglected beyond certain limiting conditions.

We shall thus ultimately be able to satisfy our critic of the correctness of our procedure, but to do it requires a considerable background of physical experience, and the exercise of a discreet judgment. The untutored savage in the bushes would probably not be able to apply the methods of dimensional analysis to this problem and obtain results which would satisfy us.

Now let us consider a third problem Given two bodies of masses m_1 and m_2 in empty space, revolving about each other in a circular orbit under their mutual gravitational attraction. We wish to find how the time of revolution depends on the other variables. We make a list of the various quantities as before.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Mass of first body,	m_1	M
Mass of second body,	m_2	M
Distance of separation,	r	L
Time of revolution,	t	T

These are evidently all the quantities physically involved, because whenever we compel two bodies of masses m_1 and m_2 to describe a circular orbit about each other under their own gravitational attraction in empty space at a distance of separation r, we find that the time of revolution is always the same, no matter what the material of which the bodies are composed, or their past history, chemical, dynamical, or otherwise. Now let us search for the functional relation, writing,

$$t = f\,(m_1, m_2, r).$$

We demand that this shall hold irrespective of the size of the fundamental units. A moment's examination confuses us, because the left-hand side involves only the element of time, and the elements of the right-hand side do not involve the time at all. Our critic at our elbow now suggests, "But you have not included all the elements on which the result depends; it is obvious that you have left out the gravitational constant." "But," say we, "how can this be? The gravitational constant can look out for itself. Nature attends to that for us. It is undeniable that two bodies of the masses m_1 and m_2 when placed at a distance r apart always revolve in the same time. We have included all the physical quantities which can be varied." But our critic insists, and to oblige him we try the effect of including the gravitational constant among the variables. We call the gravitational constant G; it obviously has the dimensions $M^{-1} L^3 T^{-2}$, since it is defined by the equation of the force between two gravitating bodies, force $= G\dfrac{m_1 m_2}{r^2}$. A "constant" which has dimensions and therefore changes in numerical magnitude when the size of the fundamental units changes is called a "dimensional" constant. We now have to find a function such that the following relation is satisfied:

$$t = f\,(m_1, m_2, G, r).$$

Now this functional equation is not quite so easy to solve by inspection as the two previous ones, and we shall have to use a little algebra on it. Let us suppose that the function is expressed in the form of a sum of products of powers of the arguments. Then we know that if the two sides of the equation are to remain equal no matter how the fundamental units are changed in magnitude, the dimensions of every one of the product terms on the right-hand side must be the same as those of the left-hand side, that is, the dimensions must be T. Assume that a typical product term is of the form

$$m_1^{\alpha}\, m_2^{\beta}\, G^{\gamma}\, r^{\delta}.$$

This must have the dimensions of T. That is,

$$M^{\alpha}\, M^{\beta}\, (M^{-1}\, L^{3}\, T^{-2})^{\gamma}\, L^{\delta} = T.$$

Writing down the conditions on the exponents gives

$$\left.\begin{aligned} \alpha + \beta - \gamma &= 0 \\ 3\gamma + \delta &= 0 \\ -2\gamma &= 1 \end{aligned}\right\}$$

Hence

$$\left.\begin{aligned} \gamma &= -\tfrac{1}{2} \\ \delta &= \tfrac{3}{2} \\ \alpha &= -\beta + \tfrac{1}{2} \end{aligned}\right\}.$$

The values of α and β are not uniquely determined, but only a relation between them is fixed. This is as we should expect, because we had only three equations of condition, and four unknown quantities to satisfy them with. The relation between α and β shows that m_1 and m_2 must enter in the form $m_2^{-\frac{1}{2}}\left(\frac{m_2}{m_1}\right)^{x}$, where there is no restriction on the value of x. Hence our unknown function is of the form

$$f = \sum A_x \frac{r^{\frac{3}{2}}}{G^{\frac{1}{2}}\, m_2^{\frac{1}{2}}}\left(\frac{m_2}{m_1}\right)^{x}$$

where x and A_x may have any arbitrary values. We may rewrite f, by factoring, in the form

$$f = \frac{r^{\frac{3}{2}}}{G^{\frac{1}{2}}\, m_2^{\frac{1}{2}}} \sum A_x \left(\frac{m_2}{m_1}\right)^{x}.$$

But now $\sum A_x \left(\frac{m_2}{m_1}\right)^{x}$, if there is no restriction on x or A_x, is merely an arbitrary function of m_2/m_1, which we write as $\phi\left(\frac{m_2}{m_1}\right)$. *Hence our final result is*

$$t = \frac{r^{\frac{3}{2}}}{G^{\frac{1}{2}}\, m_2^{\frac{1}{2}}}\, \phi\left(\frac{m_2}{m_1}\right).$$

That is, the square of the periodic time is proportional to the cube of the distance of separation, and inversely as the gravitational constant, other things being equal.

We can, by other argument, find what the nature of the unknown function ϕ is in one special case. Suppose a very heavy central body, and a satellite so light that the two together revolve approximately about the center of the heavy body. It is obvious that under these conditions the time of revolution is independent of the mass of the satellite, for if its mass is doubled, the attractive force is also doubled, and double the force acting on double the mass leaves the acceleration, and so the time of revolution, unaltered. Therefore under these special conditions the unknown function reduces to a constant, if we denote by m_1 the mass of the satellite, and the relation becomes

$$t = \text{Const}\ \frac{r^{\frac{3}{2}}}{G^{\frac{1}{2}}\, m_2^{\frac{1}{2}}}.$$

This relation we know is verified by the facts of astronomy.

Our critic seems, therefore, to have been justified by the results, and we should have included the gravitational constant in the original list. We are nevertheless left with an uncomfortable feeling because we do not see quite what was the matter with our argument, and we are disturbed by the foreboding that at some time in the future there may perhaps be a dimensional constant which we are not clever enough to think of, and which may not proclaim the impossibility of neglecting it in quite such uncompromising tones as the gravitational constant in the example. We are afraid that in such a case we will get the incorrect answer, and not know it until a Quebec bridge falls down.

Beside the matter of dimensional constants the last problem brings up another question. Why is it that we had to assume that the unknown function could be represented as a sum of products of powers of the independent variables? Certainly there are functions in mathematics which cannot be represented in this way. Is nature to be arbitrarily restricted to a small part of the functions which one of her own creatures is able to conceive?

Consider now a fourth problem, treated by Lord Rayleigh in Nature, vol. 15, 66, 1915. This is a rather famous problem in heat transfer, treated before Rayleigh by Boussinesq. A solid body, of definite geometrical shape, but variable absolute dimensions, is fixed in a stream of liquid, and maintained at a definite temperature higher than the temperature of the liquid at points remote from the body. It is required to find the rate at which heat is transferred from the body to the liquid. As before, we make a list of the various quantities involved, and their dimensions.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Rate of heat transfer,	h	HT^{-1}
Linear dimension of body,	a	L
Velocity of stream,	v	LT^{-1}
Temperature difference,	θ	θ
Heat capacity of liquid per unit volume,	c	$HL^{-3}\theta^{-1}$
Thermal conductivity of liquid,	k	$HL^{-1}T^{-1}\theta^{-1}$

This is the first heat problem which we have met, and we have introduced two new fundamental units, a quantity of heat (H), and a unit of temperature (θ). It is to be noticed that the unit of mass does not enter into the dimensional formulas of any of the quantities in this problem. If we desired, we might have introduced it, dispensing with H in so doing. Now, just as in the last example, we suppose that the rate of heat transfer, which is the quantity in which we are interested, is expressed as a sum of products of powers of the arguments, and we write one of the typical terms

$$\text{Const } a^{\alpha}\theta^{\beta}v^{\gamma}c^{\delta}k^{\epsilon}.$$

As before, we write down the conditions on the exponents imposed by the requirement that the dimensions of this product are the same as those of h. We shall thus obtain four equations, because there are four fundamental kinds of unit. The equations are

$$\begin{aligned} \delta + \epsilon &= 1 \text{ condition on exponent of H} \\ \beta - \delta - \epsilon &= 0 \text{ condition on exponent of } \theta \\ \alpha + \gamma - 3\delta - \epsilon &= 0 \text{ condition on exponent of L} \\ -\gamma - \epsilon &= -1 \text{ condition on exponent of T} \end{aligned}$$

We have five unknown quantities and only four equations, so one of the unknowns must remain arbitrary. Choose this one to be γ, and solve the equations in terms of γ. This gives

$$\begin{aligned} \alpha &= 1 + \gamma \\ \beta &= 1 \\ \delta &= \gamma \\ \epsilon &= 1 - \gamma \end{aligned}$$

Hence the product term above becomes

$$\text{Const a}\,\theta\,\text{k}\left(\frac{\text{a c v}}{\text{k}}\right)^{\gamma}.$$

The complete solution is the sum of terms of this type. As before, there is no restriction on the constant or on γ, so that all these terms together coalesce into a single arbitrary function, giving the result

$$\text{h} = \text{k a}\,\theta\,\text{F}\left(\frac{\text{a c v}}{\text{k}}\right).$$

Hence the rate of heat transfer is proportional to the temperature difference, but depends on the other quantities in a way not completely specifiable. Although the form of the function F is not known nevertheless the form of the argument of the function contains very valuable information. For instance, we are informed that the effect of changing the velocity of the fluid is exactly the same as that of changing its heat capacity. If we double the velocity, keeping the other variables fixed, we affect the rate of heat transfer precisely as we would if we doubled the heat capacity of the liquid, keeping the other variables constant.

This problem, again, is capable of raising many questions. One of these questions has been raised by D. Riabouchinsky in *Nature,* 95, 591, 1915. We quote as follows.

In *Nature* of March 18 Lord Rayleigh gives this formula, $\text{h} = \text{k a}\,\theta\,\text{F}\left(\frac{\text{a c v}}{\text{k}}\right)$, considering heat, temperature, length, and time as four "independent" quantities.

If we suppose only three of these quantities are "really independent," we obtain a different result. For example, if the temperature is defined as the mean kinetic energy of the molecules, the principle of similitude* allows us only to affirm that

$$\text{h} = \text{k a}\,\theta\,\text{F}\left(\frac{\text{v}}{\text{k a}^2}, \text{c a}^3\right).$$

* Rayleigh and other English authors use this name for dimensional analysis.

That is, instead of obtaining a result with an unknown function of only one argument, we should have obtained a function of two arguments. Now a function of two arguments is of course very much less restricted in its character than a function of only one argument. For instance, if the function is of the form suggested in two arguments, it would not follow at all that the effect of changing the velocity is the same as that of changing the heat capacity. Riabouchinsky, therefore, makes a real point.

Lord Rayleigh replies to Riabouchinsky as follows on page 644 of the same volume of *Nature*.

The question raised by Dr. Riabouchinsky belongs rather to the logic than the use of the principle of similitude, with which I was mainly concerned. It would be well worthy of further discussion. The conclusion that I gave follows on the basis of the usual Fourier equations for the conduction of heat, in which temperature and heat are regarded as *sui generis*. It would indeed be a paradox if the further knowledge of the nature of heat afforded us by molecular theory put us in a worse position than before in dealing with a particular problem. The solution would seem to be that the Fourier equations embody something as to the nature of heat and temperature which is ignored in the alternative argument of Dr. Riabouchinsky.

This reply of Lord Rayleigh is, I think, likely to leave us cold. Of course we do not question the ability of Lord Rayleigh to obtain the correct result by the use of dimensional analysis, but must we have the experience and physical intuition of Lord Rayleigh to obtain the correct result also? Might not perhaps a little examination of the logic of the method of dimensional analysis enable us to tell whether temperature and heat are "really" independent units or not, and what the proper way of choosing our fundamental units is?

Beside the prime question of the proper number of units to choose in writing our dimensional formulas, this problem of heat transfer raises many others also of a more physical nature. For instance, why are we justified in neglecting the density, or the viscosity, or the compressibility, or the thermal expansion of the liquid, or the absolute temperature? We will probably find ourselves able to justify the neglect of all these quantities, but the justification will involve real argument and a considerable physical experience with physical systems of the kind which we have been considering. The problem cannot be solved by the philosopher in his armchair, but the knowledge involved was gathered only by someone at some time soiling his hands with direct contact.

Finally, we consider a fifth problem, of somewhat different character. Let us find how the electromagnetic mass of a charge of electricity uniformly distributed throughout a sphere depends on the radius of the sphere and the amount of the charge. The charge is considered to be in empty space, so that the amount of the charge and the radius of the sphere are the only variables. We apply the method already used. The dimensions of the charge (expressed in electrostatic units) we get from the definition, which states that the numbers measuring the magnitude of two charges shall be such that the force between them is equal to their product divided by the square of the distance between them. We accordingly have the following table.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Charge,	e	$M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}$
Radius of sphere,	r	L
Electromagnetic mass,	m	M

We now write

$$m = f(e, r)$$

and try to find the form of f so that this relation is independent of the size of the fundamental units. It is obvious that T cannot enter the right-hand side of the equation, since it does not enter the left, and since T enters the right-hand side only through e, e cannot enter. But if e does not enter the right-hand side of the equation, M cannot enter either, because M enters only into e. Hence we are left with a contradiction in requirements which shows that the problem is impossible of solution. But here again our Mephistophelean critic suggests that we have left out a dimensional constant. We demur; our system is in empty space, and how can empty space require dimensional constants? But our critic insists that empty space does have properties, and when we push him, suggests that light is propagated with a definite and characteristic velocity. So we try again, including the velocity of light, c, of dimensions LT^{-1}, and now we have

$$m = f(e, r, c).$$

We now no longer encounter the previous difficulty, but immediately, with the help of our experience with more complicated examples, find the solution to be

$$m = \text{Const}\,\frac{e^2}{r\,c^2}.$$

This formula may be verified from any book on electrodynamics, and our critic is again justified. We worry over the matter of the dimensional constant, and ultimately take some comfort on recollecting that c is also the ratio of the electrostatic to the electromagnetic units, but still it is not very clear to us why this ratio should enter.

On reflecting on the solutions of the problems above, we are troubled by yet another question. Why is it that an equation which correctly describes a relation between various measurable physical quantities must in its form be independent of the size of the fundamental units? There does not seem to be any necessity for this in the nature of the measuring process itself. An equation is a description of a phenomenon, or class of phenomena. It is a statement in compact form that if we operate with a physical phenomenon in certain prescribed ways so as to obtain a set of numbers describing the results of the operations, these numbers will satisfy the equation when substituted into it. For instance, let us suppose ourselves in the position of Galileo, trying to determine the law of falling bodies. The material of our observation is all the freely falling bodies available at the surface of the earth. We use as our instruments of measurement a certain unit of length, let us say the yard, and a certain unit of time, let us say the minute. With these instruments we operate on all falling bodies according to definite rules. That is, we obtain all the pairs of numbers we can by associating for any and all of the bodies the distance which it has fallen from rest with the interval of time which has elapsed since it started to fall. And we make a great discovery in the observation that the number expressing the distance of fall of any body, no matter what its size or physical properties or the distance it has fallen, is always a fixed constant factor times the square of the number expressing the corresponding elapsed time. The numbers which we have obtained by measurement to fit into this relationship were obtained with certain definite sized units, and our description is a valid description, and our discovery is an important discovery even under the restriction that distance and time are to be measured with the same particular units as those which we originally employed.

We can write our discovery in the form of an equation

$$s = \text{Const } t^2.$$

Now an inhabitant of some other country, who uses some other system of units equally as unscientific as the yard and the minute, hears of our discovery, and tries our experiments with his measuring instruments. He verifies our result, except that he must use a different factor of proportionality in the equation. That is, the constant depends on the size of the units used in the measurements, or in other words, is a dimensional constant.

The verification of our discovery by an inhabitant of another country is reported to us, and we retire to contemplate. We at length offer the comment that this is as it should be, and that it could not well be otherwise. We offer to predict in advance just how the constant should be changed to fit with any system of measurement, and on being asked for details, make the sophisticated suggestion that we so change the constant as to exactly neutralize any change in the numbers representing the length or the time, so that we will still have essentially the same equation as before. In particular, if the unit of length is made half as large as originally, so that the number measuring a certain distance of fall is now twice as large as it was formerly, we multiply the constant by 2 so as to compensate for the factor 2 by which otherwise the left-hand side of the equation would be too large. Similarly if the unit of time is made three times as long as formerly, so that the number expressing the duration of a certain free fall becomes only $\frac{1}{3}$ of its original value, then we will multiply the constant by 9 to compensate for the factor $\frac{1}{9}$, by which otherwise the right-hand side of the equation would be too small. In other words, we give to the constant the dimensions of plus one in length, and minus two in time, and so obtain a formula valid no matter what the size of the fundamental units.

This experience emboldens us, and we try its success with other much more complicated systems. For instance, we make observations of the height of the tides at our nearest port, using a foot rule to measure the height of the water, and a clock graduated in hours as the time-measuring instrument. As the result of many observations, we find that the height of water may be represented by the formula

$$h = 5 \sin 0.5066\, t.$$

We now write this in a form to which any other observer using any other system of units may also fit his measurements by introducing two dimensional constants into the formula, which takes the form

$$h\, C_1 = 5 \sin 0.5066\, C_2\, t,$$

where C_1 has the dimensions of L^{-1}, and C_2 has the dimensions of T^{-1}.

This result immediately suggests a generalization. Any equation whatever, no matter what its form, which correctly reproduces the results of measurements made with any particular system of units on any physical system, may be thrown into such a form that it will be valid for measurements made with units of different sizes, by the simple device of introducing as a factor with each observed quantity a dimensional constant of dimensions the reciprocal of those of the factor beside which it stands, and of such a numerical value that in the original system of units it has the value unity.

Of course it may often happen that the form of the equation is such that two or more of these dimensional constants coalesce into a single factor. The first example above of the falling body is one of this kind. The general rule just given would have led to the introduction of two dimensional constants, one with s on the left-hand side of the equation, and the other with t^2 on the right-hand side of the equation, but by multiplying up, these two may be combined into a single constant.

Our query is therefore answered, and we see that every equation can be put in such a form that it holds no matter what the size of the fundamental units, but we are left in a greater quandary than ever with regard to dimensional constants. May there not be new dimensional constants appropriate to every new kind of problem, and how can we tell beforehand what the dimensional constants will be? If we cannot tell beforehand what dimensional constants enter a problem, how can we hope to apply dimensional analysis? The dimensional situation thus appears even more hopeless than at first, for we could see a sort of reason why the gravitational constant should enter the problem of two revolving bodies, and could even catch a glimmer of reasonableness in the entrance of the velocity of light into the problem of electromagnetic mass, but it is certainly difficult to discover reasonableness or predictableness if dimensional constants can be used indiscriminately as factors by the side of every measured quantity.

In our consideration of the problems above we have also made one more observation that calls for comment. We have noticed that every dimensional formula of every measurable quantity has always involved the fundamental units as products of powers. Is this necessary, or may there be other kinds of dimensional formulas for quantities measured in other ways, and if so, how will our methods apply to such quantities?

To sum up, we have met in this introductory chapter a number of important questions which we must answer before we can hope to use the methods of dimensional analysis with any certainty that our results are correct. These questions are as follows.

First and foremost, when do dimensional constants enter, and what is their form?

Is it necessary that the dimensional formula of every measured quantity be the product of powers of the fundamental kinds of unit?

What is the meaning of quantities with no dimensions?

Must the functions descriptive of phenomena be restricted to the sum of products of powers of the variables?

What kinds of quantity should we choose as the fundamentals in terms of which to measure the others? In particular, how many kinds of fundamental units are there? Is it legitimate to reduce the number of fundamental units as far as possible by the introduction of definitions in accord with experimental facts?

Finally, what is the criterion for neglecting a certain kind of quantity in any problem, as for example the viscosity in the heat flow problem, and what is the character of the result which we will get, approximate or exact? And if approximate, how good is the approximation?

CHAPTER II

DIMENSIONAL FORMULAS

In the introductory chapter we considered some special problems which raised a number of questions that must be answered before we can hope to really master the method of dimensional analysis. Let us now begin the formal development of the subject, keeping these questions in mind to be answered as we proceed.

The purpose of dimensional analysis is to give certain information about the relations which hold between the measurable quantities associated with various phenomena. The advantage of the method is that it is rapid; it enables us to dispense with making a complete analysis of the situation such as would be involved in writing down the equations of motion of a mechanical system, for example, but on the other hand it does not give as complete information as might be obtained by carrying through a detailed analysis.

Let us in the first place consider the nature of the relations between the measurable quantities in which we are interested. In dealing with any phenomenon or group of phenomena our method is somewhat as follows. We first measure certain quantities which we have some reason to expect are of importance in describing the phenomenon. These quantities which we measure are of different kinds, and for each different kind of quantity we have a different rule of operation by which we measure it, that is, associate the quantity with a number. Having obtained a sufficient array of numbers by which the different quantities are measured, we search for relations between these numbers, and if we are skillful and fortunate, we find relations which can be expressed in mathematical form. We are usually interested preëminently in one of the measured quantities and try to find it in terms of the others. Under such conditions we would search for a relation of the form

$$x_1 = f\,(x_2, x_3, x_4, \text{etc.})$$

where x_1, x_2, etc., stand for the numbers which are the measures of particular kinds of physical quantity. Thus x_1 might stand for the number which is the measure of a velocity, x_2 may stand for the number which is the measure of a viscosity, etc. By a sort of shorthand method of statement we may abbreviate this long-winded description into saying that x_1 is a velocity, but of course it really is not, but is only a number which measures velocity.

Now the first observation which we make with regard to a functional relation like the above is that the arguments fall into two groups, depending on the way in which the numbers are obtained physically. The first group of quantities we call primary quantities. These are the quantities which, according to the particular set of rules of operation by which we assign numbers characteristic of the phenomenon, are regarded as fundamental and of an irreducible simplicity. Thus in the ordinary systems of mechanics, the fundamental quantities are taken as mass, length, and time. In any functional relation such as the above, certain arguments of the function may be the numbers which are the measure of certain lengths, masses, or times. Such quantities we will agree to call primary quantities.

In the measurement of primary quantities, certain rules of operation must be set up, establishing the physical procedure by which it is possible to measure a length in terms of a particular length which we choose as the unit of length, or a time in terms of a particular interval of time selected as the standard, or in general, it is characteristic of primary quantities that there are certain rules of procedure by which it is possible to measure any primary quantity directly in terms of units of its own kind. Now it will be found that we always make a tacit requirement in selecting the rules of operation by which primary quantities are measured in terms of quantities of their own kind. This requirement for measurement of length, for example, is that if a new unit of length is chosen let us say half the length of the original unit, then the rules of operation must be such that the number which represents the measure of any particular concrete length in terms of the new unit shall be twice as large as the number which was its measure in terms of the original unit. Very little attention seems to have been given to the methodology of systems of measurement, and I do not know whether this characteristic of all our systems of measurement has been formulated or not, but it is evident on examination of any system of measurement in actual use that it has these properties. The possession of this property involves a most important consequence, which is that the ratio of the numbers expressing the measures of any two concrete lengths, for example, is independent of the size of the unit with which they are measured. This consequence is of course at once obvious, for if we change the size of the fundamental unit by any factor, by hypothesis we change the measure of every length by the reciprocal of that factor, and so leave unaltered the ratio of the measures of any two lengths. This means that the ratio of the lengths of any two particular objects has an absolute significance independent of the size of the units. This may be put into the converse form, as is evident on a minute's reflection. If we require that our system of measurement of primary kinds of quantity in terms of units of their own kind be such that the ratio of the measures of any two concrete examples shall be independent of the size of the unit, then the measures of the concrete examples must change inversely as the size of the unit.

Besides primary quantities, there is another group of quantities which we may call secondary quantities. The numerical measures of these are not obtained by some operation which compares them directly with another quantity of the same kind which is accepted as the unit, but the method of measurement is more complicated and roundabout. Quantities of the second kind are measured by making measurements of certain quantities of the first kind associated with the quantity under consideration, and then combining the measurements of the associated primary quantities according to certain rules which give a number that is defined as the measure of the secondary quantity in question. For example, a velocity as ordinarily defined is a secondary quantity. We obtain its measure by measuring a length and the time occupied in traversing this length (both of these being primary quantities), and dividing the number measuring the length by the number measuring the time (or dividing the length by the time according to our shorthand method of statement).

Now there is a certain definite restriction on the rules of operation which we are at liberty to set up in defining secondary quantities. We make the same requirement that we did for primary quantities, namely, that the ratio of the numbers measuring any two concrete examples of a secondary quantity shall be independent of the size of the fundamental units used in making the required primary measurements. That is, to say that one substance is twice as viscous as another, for example, or that one automobile is travelling three times as rapidly as another, has absolute significance, independent of the size of the fundamental primary units.

This requirement is not necessary in order to make measurement itself possible. Any rules of operation will serve as the basis of a system of measurement by which numbers may be assigned to phenomena in such a way that the particular aspect of the phenomenon on which we are concentrating attention is uniquely defined by the number in conjunction with the rules of operation. But the requirement that the ratio be constant, or we may say the requirement of the absolute significance of relative magnitude, is essential to all the systems of measurement in scientific use. In particular, it is an absolute requirement if the methods of dimensional analysis are to be applied to the results of the measurements. Dimensional analysis cannot be applied to systems which do not meet this requirement, and accordingly we consider here only such systems.

It is particularly to be noticed that the line of separation between primary and secondary quantities is not a hard and fast one imposed by natural conditions, but is to a large extent arbitrary, and depends on the particular set of rules of operation which we find convenient to adopt in defining our system of measurement. For instance, in our ordinary system of mechanics, force is a secondary quantity, and its measure is obtained by multiplying a number which measures a mass and the number which measures an acceleration (itself a secondary quantity). But physically, force is perfectly well adapted to be used as a primary quantity, since we know what we mean by saying that one force is twice another, and the physical processes are known by which force may be measured in terms of units of its own kind. It is the same way with velocities; it is possible to set up a physical procedure by which velocities may be added together directly, and which makes it possible to measure velocity in terms of units of its own kind, and so to regard velocity as a primary quantity. It is perhaps questionable whether all kinds of physical quantity are adapted to be treated, if it should suit our convenience, as primary quantities. Thus it is not at once obvious whether a physical procedure could be set up by which two viscosities could be compared directly with each other without measuring other kinds of quantity.

But this question is not essential to our progress, although of great interest in itself, and need not detain us. The facts are simply these. The assigning of numerical magnitudes to measurable quantities involves some system of rules of operation such that the quantities fall into two groups, which we call primary and secondary.

We have stated that the requirement of the absolute significance of relative magnitude imposes definite restrictions on the operations by which secondary quantities may be measured in terms of primary quantities. Let us formulate this restriction analytically. We call the primary quantities in terms of which the secondary quantity are measured α, β, γ, etc. Measurements of the primary quantities are combined in a certain way to give the measure of the secondary quantity. We represent this combination by the functional symbol f, putting the secondary quantity $= \mathrm{f}\,(\alpha, \beta, \gamma, ----)$. Now if there are two concrete examples of the secondary quantity, the associated primary quantities have different numerical magnitudes. Let us denote the set associated with the first of the concrete examples by the subscript 1, and that with the second set by the subscript 2. Then $\mathrm{f}\,(\alpha_1, \beta_1, \gamma_1, ----)$ will be the measure of the first concrete example, and $\mathrm{f}\,(\alpha_2, \beta_2, \gamma_2, ----)$ will be the measure of the second. We now change the size of the fundamental units. We make the unit in terms of which α is measured 1/xth as large. Then, as we have shown, the number measuring α will be x times as large, or $\mathrm{x}\alpha$. In the same way make the unit measuring β 1/yth as large, and the measuring number becomes $\mathrm{y}\beta$. Since our rule of operation by which the numerical measure of the secondary quantity is obtained from the associated primary quantities is independent of the size of the primary units, the number measuring the secondary quantity now becomes $\mathrm{f}\,(\mathrm{x}\alpha, \mathrm{y}\beta, ----)$. The measures of the two concrete examples of the secondary quantity will now be $\mathrm{f}\,(\mathrm{x}\alpha_1, \mathrm{y}\beta_1, ----)$ and $\mathrm{f}\,(\mathrm{x}\alpha_2, \mathrm{y}\beta_2, ----)$.

Our requirement of absolute significance of relative magnitude now becomes analytically

$$\frac{\mathrm{f}\,(\alpha_1, \beta_1, ----)}{\mathrm{f}\,(\alpha_2, \beta_2, ----)} = \frac{\mathrm{f}\,(\mathrm{x}\alpha_1, \mathrm{y}\beta_1, ----)}{\mathrm{f}\,(\mathrm{x}\alpha_2, \mathrm{y}\beta_2, ----)}.$$

This relation is to hold for all values of $\alpha_1, \beta_1, ----, \alpha_2, \beta_2, ----$ and x, y, z, ----.

We desire to solve this equation for the unknown function f. Rewrite in the form

$$\mathrm{f}\,(\mathrm{x}\alpha_1, \mathrm{y}\beta_1, ----) = \mathrm{f}\,(\mathrm{x}\alpha_2, \mathrm{y}\beta_2, ----) \times \frac{\mathrm{f}\,(\alpha_1, \beta_1, ----)}{\mathrm{f}\,(\alpha_2, \beta_2, ----)}.$$

Differentiate partially with respect to x. Use the notation f_1 to denote the partial derivative of the function with respect to the first argument, etc. Then we obtain

$$\alpha_1 f_1 (x\alpha_1, y\beta_1, ----) = \alpha_2 f_1 (x\alpha_2, y\beta_2, ----) \times \frac{f(\alpha_1, \beta_1, ----)}{f(\alpha_2, \beta_2, ----)}.$$

Now put x, y, z, etc., all equal to 1. Then we have

$$\alpha_1 \frac{f_1 (\alpha_1, \beta_1, ----)}{f (\alpha_1, \beta_1, ----)} = \alpha_2 \frac{f_1 (\alpha_2, \beta_2, ----)}{f (\alpha_2, \beta_2, ----)}.$$

This is to hold for all values of $\alpha_1, \beta_1, ----$ and $\alpha_2, \beta_2, ----$. Hence, holding $\alpha_2, \beta_2, ----$ fast, and allowing $\alpha_1, \beta_1, ----$ to vary, we have

$$\frac{\alpha}{f}\frac{\partial f}{\partial \alpha} = \text{Const},$$

or

$$\frac{1}{f}\frac{\partial f}{\partial \alpha} = \frac{\text{Const}}{\alpha},$$

which integrates to

$$f = C_1 \alpha^{\text{Const}}.$$

The factor C_1 is a function of the other variables $\beta, \gamma, ----$.

The above process may now be repeated, differentiating partially successively with respect to y, z, etc., and integrating. The final result will obviously be

$$f = C \alpha^a \beta^b \gamma^c ----$$

where a, b, c, ---- and C are constants.

Every secondary quantity, therefore, which satisfies the requirement of the absolute significance of relative magnitude must be expressible as some constant multiplied by arbitrary powers of the primary quantities. We have stated that it is only secondary quantities of this kind which are used in scientific measurement, and no other kind will be considered here.

We have now answered one of the questions of the introductory chapter as to why it was that in the dimensional formulas the fundamental units always entered as products of powers.

It is obvious that the operations by which a secondary quantity is measured in terms of primary quantities are defined mathematically by the coefficient C, and by the exponents of the powers of the various primary quantities. For the sake of simplicity, the coefficient is almost always chosen to be unity, although there is no necessity in such a choice. There are systems in use in which the factor is not always chosen as unity. Thus the so-called rational and the ordinary electrostatic units differ by a factor $\sqrt{4\pi}$. Any differences in the numerical coefficient are not important, and are always easy to deal with, but the exponents of the powers are a matter of vital importance. The exponent of the power of any particular primary quantity is by definition the "dimension" of the secondary quantity in that particular primary quantity.

The "dimensional formula" of a secondary quantity is the aggregate of the exponents of the various primary quantities which are involved in the rules of operation by which the secondary quantity is measured. In order to avoid confusion, the exponents are associated with the symbols of the primary quantities to which they belong, that symbol being itself written as raised to the power in question.

For example, a velocity is measured by definition by dividing a certain length by a certain time (do not forget that this really means dividing the number which is the measure of a certain length by the number which measures a certain time). The exponent of length is therefore plus one, and the exponent of time is minus one, and the dimensional formula of velocity is LT^{-1}. In the same way a force is defined in the ordinary Newtonian mechanical system as mass times acceleration. The dimensions of force are therefore equal to mass times the dimensions of acceleration. The dimensions of acceleration are obtained from its definition as time rate of change of velocity to be LT^{-2}, which gives for the dimensions of force MLT^{-2}.

It is to be noticed that the dimensions of any primary quantity are by a simple extension of the definition above merely the dimensional symbol of the corresponding primary quantity itself.

It is particularly to be emphasized that the dimensions of a primary quantity as defined above have no absolute significance whatever, but are defined merely with respect to that aspect of the rules of operation by which we obtain the measuring numbers associated with the physical phenomenon. The dimensional formula need not even suggest certain essential aspects of the rules of operation. For example, in the dimensional formula of force as mass times acceleration, the fact is not suggested that force and acceleration are vectors, and the components of each in the same direction must be compared. Furthermore, in our measurements of nature, the rules of operation are in our control to modify as we see fit, and we would certainly be foolish if we did not modify them to our advantage according to the particular kind of physical system or problem with which we are dealing. We shall in the following find many problems in which there is an advantage in choosing our system of measurement, that is, our rules of operation, in a particular way for the particular problem. Different systems of measurement may differ as to the kinds of quantity which we find it convenient to regard as fundamental and in terms of which we define the others, or they may even differ in the number of quantities which we choose as fundamental. All will depend on the particular problem, and it is our business to choose the system in the way best adapted to the problem in hand.

There is therefore no meaning in saying "the" dimensions of a physical quantity, until we have also specified the system of measurement with respect to which the dimensions are determined. This is not always kept clearly in mind even by those who in other conditions recognize the relative nature of a dimensional formula. As for example, Buckingham in *Phys. Rev.* 4, 357, 1914, says: ". . . Mr. Tolman's reasoning is based on the assumption that absolute temperature has the dimensions of energy, and this assumption is not permissible." Tolman,[1] in a reply, admitted the correctness of this position. My position in this matter would be that Mr. Tolman has a right to make the dimensions of temperature the dimensions of energy if it is compatible with the physical facts (as it seems to be) and if it suits his convenience.

This view of the nature of a dimensional formula is directly opposed to one which is commonly held, and frequently expressed. It is by many considered that a dimensional formula has some esoteric significance connected with the "ultimate nature" of an object, and that we are in some way getting at the ultimate nature of things in writing their dimensional formulas. Such a point of view sees something absolute in a dimensional formula and attaches a meaning to such phrases as "really" independent, as in Riabouchinsky's comments on Lord Rayleigh's analysis of a certain problem in heat transfer. For this point of view it becomes important to find the "true" dimensions, and when the "true" dimensions are found, it is expected that something new will be suggested about the physical properties of the system. To this view it is repugnant that there should be two dimensional formulas for the same physical quantity. Often a reconciliation is sought by the introduction of so-called suppressed dimensions. Such speculations have been particularly fashionable with regard to the nature of the ether, but so far as I know, no physical discovery has ever followed such speculations; we should not expect it if the view above is correct.

In the appendix of this chapter are given a number of quotations characteristic of this point of view, or others allied to it.

APPENDIX TO CHAPTER II

QUOTATIONS ILLUSTRATING VARIOUS COMMON POINTS OF VIEW WITH REGARD TO THE NATURE OF DIMENSIONAL FORMULAS

R. C. Tolman, Phys. Rev. 9 : 251, 1917.

. . . our ideas of the dimensions of a quantity as a shorthand restatement of its definition and hence as an expression of its essential physical nature.

A. W. Rücker, Phil. Mag. 27 : 104, 1889.

In the calculation of the dimensions of physical quantities we not infrequently arrive at indeterminate equations in which two or more unknowns are involved. In such cases an assumption has to be made, and in general that selected is that one of the quantities is an abstract number. In other words, the dimensions of that quantity are suppressed.

The dimensions of dependent units which are afterwards deduced from this assumption are evidently artificial, in the sense that they do not necessarily indicate their true relations to length, mass, and time. They may serve to test whether the two sides of an equation are correct, but they do not indicate the mechanical nature of the derived units to which they are assigned. On this account they are often unintelligible.

W. W. Williams, Phil. Mag. 34 : 234, 1892.

That these systems (*i.e.*, the electrostatic and the electromagnetic) are artificial appears when we consider that each apparently expresses the absolute dimensions of the different quantities, that is, their dimensions only in terms of L, M, and T; whereas we should expect that the absolute dimensions of a physical quantity could be expressed in only one way. Thus from the mechanical force between two poles we get

$$f = \frac{1}{\mu}\frac{m^2}{r^2} \therefore m = r\sqrt{\mu f}$$

and this, being a qualitative as well as a quantitative relation, involves the dimensional equality of the two sides. . . . In this way we get two different absolute dimensions for the same physical quantity, each of which involves a different physical interpretation. . . .

The dimensional formula of a physical quantity expresses the numerical dependence of the unit of that quantity upon the fundamental and secondary units from which it is derived, and the indices of the various units in the formula are termed the dimensions of the quantity with respect to those units. When used in this very restricted sense, the formulae only indicate numerical relations between the various units. It is possible, however, to regard the matter from a wider point of view, as has been emphasized by Professor Rücker in the paper referred to. The dimensional formulae may be taken as representing the physical identities of the various quantities, as indicating, in fact, how our conceptions of their physical nature (in terms, of course, of other and more fundamental conceptions) are formed, just as the formula of a chemical substance indicates its composition and chemical identity. This is evidently a more comprehensive and fundamental view of the matter, and from this point of view the primitive numerical signification of a dimensional formula as merely a change ratio between units becomes a dependent and secondary consideration.

The question then arises, what is the test of the identity of a physical quantity? Plainly it is the manner in which the unit of that quantity is built up (ultimately) from the fundamental units L, M, and T, and not merely the manner in which its magnitude changes with those units.

That the dimensional formulae are regarded from this higher standpoint, that is, regarded as being something more than mere "change ratios" between units, is shown by the fact that difficulties are felt when the dimensions of two different quantities, *e.g.*, couple and work, happen to be the same.

S. P. Thompson, Elementary Lessons in Electricity and Magnetism, p. 352.

It seems absurd that there should be two different units of electricity.

R. A. Fessenden, Phys. Rev. 10: 8, 1900.

The difference between the dimensional formula and the qualitative formula or quality of a thing is that, according to the definitions of the writers quoted above, the dimensions "are arbitrary," are "merely a matter of definition and depend entirely upon the system of units we adopt," whilst the quality is an expression of the absolute nature, and never varies, no matter what system of units we adopt. For this to be true, no qualities must be suppressed.

REFERENCES

(1) R. C. Tolman, Phys. Rev. 6, 1915, p. 226, footnote.

CHAPTER III

ON THE USE OF DIMENSIONAL FORMULAS IN CHANGING UNITS

WE saw in the last chapter how to obtain the dimensional formula of any quantitv in terms of the quantities which we chose by definition to make primary. Our method of analysis showed also the connection between the numerical magnitude of the derived quantity and the primary quantities. Thus if length enters to the first power in the dimensional formula, we saw that the number measuring that quantity is doubled when the unit of length is halved, or the numerical measures are inversely as the size of the unit, raised to the power indicated in the dimensional formula.

Let us consider a concrete example. What is a velocity of 88 feet per second when expressed in miles per hour? The dimensional formula of a velocity is LT^{-1}. Now if our unit of length is made larger in the ratio of a mile to a foot, that is, in the ratio of 5280 to 1, the velocity will be multiplied by the factor 1/5280, because length enters in the dimensional formula to the first power. And similarly, if the unit of time is made larger in the ratio of the hour to the second, that is, in the ratio of 3600 to 1, the velocity will be multiplied by the factor 3600, because time enters the dimensional formula to the inverse first power. To change from feet per second to miles per hour we therefore multiply by 3600/5280, and in this particular case the result is $88 \times 3600/5280$, or 60 miles per hour.

Now the result of these operations may be much contracted and simplified in appearance by a sort of shorthand. We write

$$
\begin{aligned}
88\,\frac{\text{ft.}}{\text{sec.}} = 88 \times \frac{1\text{ ft.}}{1\text{ sec.}} = 88 \times \frac{\frac{1}{5280}\text{ mile}}{\frac{1}{3600}\text{ hour}}
&= 88\,\frac{3600}{5280}\,\frac{\text{mile}}{\text{hour}} \\
&= 60\,\frac{\text{mile}}{\text{hour}}.
\end{aligned}
$$

A little reflection, considering the relation of the dimensional formula to the operations by which we obtain the measuring number of any physical quantity, will at once show that this procedure is general, and that we may obtain any new magnitude in terms of new units from the old magnitude by using the dimensional formula in precisely the same way. This method of use of the dimensional formula is frequently very convenient, and is the simplest and most reliable way of changing units with which I am acquainted.

In treating the dimensional formula in this way we have endowed it with a certain substantiality, substituting for the dimensional symbol of the primary unit the name of the concrete unit employed, and then replacing this concrete unit by another to which it is physically equivalent. That is, we have treated the dimensional formula as if it expressed operations actually performed on physical entities, as if we took a certain number of feet and divided them by a certain number of seconds. Of course, we actually do nothing of the sort. It is meaningless to talk of dividing a length by a time; what we actually do is to operate with numbers which are the measure of these quantities. We may, however, use this shorthand method of statement, if we like, with great advantage in treating problems of this sort, but we must not think that we are therefore actually operating with the physical things in any other than a symbolical way.[1]

This property of the dimensional formula of giving the change in the numerical magnitude of any concrete example when the size of the primary units is changed makes possible a certain point of view with regard to the nature of a dimensional formula. This view has perhaps been expressed at greatest length by James Thomson in *B. A. Rep.* 1878, 451. His point of view agrees with that taken above in recognizing that it is meaningless to say literally that a velocity, for instance, is equal to a length divided by a time. We cannot perform algebraic operations on physical lengths, just the same as we can never divide anything by a physical time. James Thomson would prefer, instead of saying velocity = length/time, to say at greater length

$$\text{Change ratio of velocity} = \frac{\text{Change ratio of length}}{\text{Change ratio of time}}.$$

Of course Thomson would not insist on this long and clumsy expression in practise, but after the matter is once understood, would allow us to write a dimensional formula in the accustomed way.

This point of view seems perfectly possible, and as far as any results go, it cannot be distinguished from that which I have adopted. However, by regarding the symbols in the dimensional formula as reminders of the rules of operation which we used physically in getting the numerical measure of the quantity, it seems to me that we are retaining a little closer contact with the actual physics of the situation than when we regard the symbols as representing the factors used in changing from one set of units to another, which after all is a more or less sophisticated thing to do, and which is not our immediate concern when first viewing a phenomenon.

Beside the sort of change of unit considered above, in which we change merely the sizes of the primary units, there is another sort of change of unit to be considered, in which we pass from one system of measurement to another in which the primary units are not only different in size, but different in character.[2] Thus, for example, in our ordinary system of units of Newtonian mechanics we regard mass, length, and time as the primary units, but it is well known that we might equally regard force, length, and time as primary. We may therefore expect to encounter problems of this sort: how shall we express a kinetic energy of 10 gm cm^2 sec^{-2} in a system in which the units are the dyne, the cm, and the sec?

There are obviously two problems involved here. One is to find the dimensional formula of kinetic energy in terms of force, length, and time, and the other is to find the new value of the numerical coefficient in that particular system in which the unit of force is the dyne, the unit of length the centimeter, and the unit of time the second.

The transformed dimensional formula is obtained easily if we observe the steps by which we pass from one system to the other. The transition is of course to be made in such a way that the two systems are consistent with each other. Thus if force is equal to mass times acceleration in one system, it is still to be equal to mass times acceleration in the other. If this were not so, we would be concerned merely with a formal change, and the thing which we might call force in the one system would not correspond to the same physical complex in the other. This relation of force and mass in the two systems is maintained by an application of simple algebra. In the first system we define force as mass times acceleration, and in the second we define mass as force divided by acceleration. Thus in each system the secondary quantity is expressed in terms of the primary quantities of the system, and the two systems are consistent.

The correct relation between the dimensional formulas in the two systems is to be maintained simply by writing down the dimensional formulas in the first system, and then inverting these formulas by solving for the quantities which are to be regarded as secondary in the second system. In the special case considered, we would have the following dimensional formulas:

In the first system	Force	$= MLT^{-2}$
In the second system	Mass	$= FL^{-1}T^{2}$.

The transformation of the numerical coefficient is to be done exactly as in the example which we have already considered by treating the dimensional symbols as the names of concrete things, and replacing the one to be eliminated by its value in terms of the one which is to replace it. Thus the complete work associated with the problem above is as follows:

$$10\,\frac{1\text{ gm }\ (1\text{ cm})^2}{(1\text{ sec})^2} = ?$$

We have to find the transformation equation of 1 gm. into terms of dynes, cm, and sec. Now

$$1\text{ dyne} = \frac{1\text{ gm }\ 1\text{ cm}}{(1\text{ sec})^2}.$$

Hence

$$1\text{ gm} = \frac{1\text{ dyne }(1\text{ sec})^2}{1\text{ cm}}$$

and substituting

$$\begin{aligned} 10\,\frac{1\text{ gm }(1\text{ cm})^2}{(1\text{ sec})^2} &= 10\,\frac{1\text{ dyne }\ (1\text{ sec})^2}{1\text{ cm}} \times \frac{(1\text{ cm})^2}{(1\text{ sec})^2} \\ &= 10\text{ dyne cm,} \end{aligned}$$

which of course is a result which we immediately recognize as true.

Let us consider the general case in which we are to change from a system of units in which the primary quantities are X_1, X_2, and X_3 to a system in which they are Y_1, Y_2, and Y_3.

We must first have the dimensional formulas of Y_1, Y_2, and Y_3 in terms of X_1, X_2, and X_3. Let us suppose the dimensions of Y_1 are a_1, a_2, and a_3, those of Y_2, b_1, b_2, and b_3, and those of Y_3, c_1, c_2, and c_3

in X_1, X_2, and X_3 respectively. Then in any concrete case we may write

$$\begin{aligned} C_1 Y_1 &= X_1^{a_1} X_2^{a_2} X_3^{a_3} \\ C_2 Y_2 &= X_1^{b_1} X_2^{b_2} X_3^{b_3} \\ C_3 Y_3 &= X_1^{c_1} X_2^{c_2} X_3^{c_3} \end{aligned}$$

where the C's are numerical coefficients. These equations are to be solved for the X's in terms of the Y's. This may be done conveniently by taking the logarithm of the equations, giving,

$$\begin{aligned} a_1 \log X_1 + a_2 \log X_2 + a_3 \log X_3 &= \log C_1 Y_1 \\ b_1 \log X_1 + b_2 \log X_2 + b_3 \log X_3 &= \log C_2 Y_2 \\ c_1 \log X_1 + c_2 \log X_2 + c_3 \log X_3 &= \log C_3 Y_3 \end{aligned}$$

These are algebraic equations in the logarithms, and may be solved immediately. The solution for X is

$$X_1 = (C_1 Y_1)^{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \div \Delta} (C_2 Y_2)^{\begin{vmatrix} c_2 & c_3 \\ a_2 & a_3 \end{vmatrix} \div \Delta} (C_3 Y_3)^{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \div \Delta}.$$

In this solution Δ stands for the determinant of the exponents.

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The values of X_2 and X_3 are to be obtained from the value for X_1 by advancing the letters.

Now let us consider an example. It is required to find what a momentum of 15 tons (mass) miles/hour becomes in that system whose fundamental units are the "2 Horsepower," the "3 ft per sec," and the "5 ergs." This ought to be sufficiently complicated. Introduce the abbreviations:

Y_1 for the "2 H.P."
Y_2 for the "3 ft per sec"
Y_3 for the "5 ergs."

$$Y_1 = 2 \text{ H.P.} = 2 \times 33000 \frac{1 \text{ lb (force) } 1 \text{ ft}}{1 \text{ min}}$$

In the first place we have to change lbs (force) to lbs (mass).

Now a pound force is that force which imparts to a mass of one pound an acceleration of 32.17 ft/sec^2.

Hence

$$1 \text{ lb force} = 32.17 \frac{1 \text{ lb (mass) } 1 \text{ ft}}{(1 \text{ sec})^2}$$

and

$$Y_1 = 66000 \times 32.17 \frac{\text{lb mass } 1 \text{ ft}}{(1 \text{ sec})^2} \cdot \frac{1 \text{ ft}}{1 \text{ min}}.$$

$$= 32.17 \times 66000 \frac{\frac{1}{2000} \text{ ton } (\frac{1}{5280} \text{ mi})^2}{(\frac{1}{3600} \text{ hour})^2 \frac{1}{60} \text{ hour}}$$

$$= 2.962 \times 10^4 \frac{\text{ton mi}^2}{\text{hour}^3}.$$

or, writing in the standard form

$$3.380 \times 10^{-5}\, Y_1 = \text{ton}^1 \text{ mi}^2 \text{ hour}^{-3}.$$

Again

$$Y_2 = 3 \frac{\text{ft}}{\text{sec}} = 3 \frac{\frac{1}{5280} \text{ mi}}{\frac{1}{3600} \text{ hour}} = \frac{45}{22} \frac{\text{mi}}{\text{hour}},$$

or

$$.4889\, Y_2 = \text{ton}^0 \text{ mi}^1 \text{ hour}^{-1}.$$

And again

$$Y_3 = 5 \text{ ergs} = 5 \frac{1 \text{ gm } (1 \text{ cm})^2}{(1 \text{ sec})^2} = 5 \frac{1.103 \times 10^{-6} \text{tons } (6.214 \times 10^{-6} \text{mi})^2}{(\frac{1}{3600} \text{ hour})^2},$$

or

$$3.622 \times 10^8\, Y_3 = \text{tons}^1 \text{ mi}^2 \text{ hour}^{-2}.$$

We rewrite these to obtain our system of equations in the standard form

$$\begin{aligned} 3.380 \times 10^{-5}\, Y_1 &= \text{ton}^1 \text{ mi}^2 \text{ hour}^{-3} \\ .4889\, Y_2 &= \text{ton}^0 \text{ mi}^1 \text{ hour}^{-1} \\ 3.622 \times 10^8\, Y_3 &= \text{ton}^1 \text{ mi}^2 \text{ hour}^{-2}. \end{aligned}$$

We now solve for the ton, mile, and hour in terms of Y_1, Y_2, and Y_3. We first find the determinant of the exponents.

$$\Delta = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 1 & 2 & -2 \end{vmatrix} = 1.$$

This is pleasingly simple.

The general scheme of solution above now gives

$$\begin{aligned}
1\ \text{ton} &= (3.380 \times 10^{-5}\ Y_1)^0\ (.4889\ Y_2)^{-2}\ (3.622 \times 10^6\ Y_3)^1 \\
1\ \text{mile} &= (\qquad \text{"} \qquad)^{-1}\ (\quad \text{"} \quad)^{+1}\ (\qquad \text{"} \qquad)^{+1} \\
1\ \text{hour} &= (\qquad \text{"} \qquad)^{-1}\ (\quad \text{"} \quad)^{0}\ (\qquad \text{"} \qquad)^{+1},
\end{aligned}$$

or simplifying,

$$\begin{aligned}
1\ \text{ton} &= 15.16 \times 10^6\ Y_2^{-2}\ Y_3 \\
1\ \text{mile} &= 5.223 \times 10^{12}\ Y_1^{-1}\ Y_2\ Y_3 \\
1\ \text{hour} &= 1.069 \times 10^{13}\ Y_1^{-1}\ Y_3.
\end{aligned}$$

And finally

$$\begin{aligned}
15\ \frac{\text{tons mi}}{\text{hour}} &= 15 \times \frac{15.16 \times 5.223 \times 10^{20}}{1.069 \times 10^{13}}\ \frac{Y_2^{-2}\ Y_3\ Y_1^{-1}\ Y_2\ Y_3}{Y_1^{-1}\ Y_3} \\
&= 1.112 \times 10^{10}\ Y_2^{-1}\ Y_3 \\
&= 1.112 \times 10^{10}\ \frac{5\ \text{ergs}}{3\ \text{ft/sec}},
\end{aligned}$$

which is the answer sought. It is to be noticed that the result involves only two of the new kind of unit instead of three, the "2 H.P." having dropped out. This of course will not in general be the case. It might at first sight appear that we might take advantage of this fact and eliminate some of the computation, but on examination this turns out not to be the case, for each of the numerical factors connecting the ton, the mile, and the hour with the new units is seen to be involved in the final result.

There are two things to be noticed in connection with the above transformations. In the first place it is not always possible to pass from a system of one kind of units to a system of another kind, but there is a certain relation which must be satisfied. This is merely the condition that the equations giving one set of units in terms of the other shall have a solution. This condition is the condition that the transformed equations, after the logarithms have been taken, shall also have a solution, and this is merely the condition that a set of algebraic equations have a solution. This condition is that the determinant of the coefficients of the algebraic equations shall not vanish. Since the coefficients of the algebraic equations in the logarithms are the exponents of the original dimensional formulas, the condition is that the determinant of the exponents of the dimensional formulas for one system of units in terms of the other system of units shall not vanish.

In attempting any such transformation as this, the first thing is to find whether it is a possible transformation, by writing down the determinant of the exponents. If this vanishes, the transformation is not possible. This means that one of the new kinds of unit in terms of which it is desired to build up the new system of measurement is not independent of the others. Thus in the example, if instead of the "5 erg" as the third unit of the new system we had chosen the "5 dyne," we would have found that the determinant of the exponents vanishes, and the transformation would not have been possible. This is at once obvious from other considerations. For a horse power is a rate of doing work, and is of the dimensions of the product of a force and a velocity, and the second unit was of the dimensions of a velocity, so that the proposed third unit, which was of the dimensions of a force, could be obtained by dividing the first unit by the second, and would therefore not be independent of them.

The second observation is that the new system of units to which we want to transform our measurement must be one in which there are the same number of kinds of fundamental unit as in the first system. If this is not true, we shall find that, except in special cases, there are either too few or too many equations to allow a solution for the new units in terms of the old. In the first case the solution is indeterminate, and in the second no solution exists.

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CHAPTER IV

THE Π THEOREM

In the second chapter we saw that the dimensional formulas of all the quantities with which we shall have to deal are expressible as products of powers of the fundamental quantities. Let us see what inferences this enables us to draw about the forms of the relations which may hold between the various measurable quantities connected with a natural phenomenon.

We also saw in the second chapter that at least sometimes the functional relation will involve certain so-called dimensional constants as well as measurable quantities. We met two examples of dimensional constants, namely, the gravitational constant, and the velocity of light in empty space, and we assigned dimensional formulas to these constants. Now it is most important to notice that these two dimensional constants had dimensional formulas of the type proved to be necessary for the measurable quantities, namely, they were expressible as products of powers of the fundamental quantities. This is no accident, but it is true of all the dimensional constants with which we shall have to deal. The proof can best be given later when we have obtained a little clearer insight into the nature of a dimensional constant. A certain apparent exception, the so-called logarithmic constant, will also be dealt with later. We may remark here, however, that one class of dimensional constant must obviously be of this form. We saw that if we start with an empirical equation which experimentally has been found to be true from measurements with a particular set of units, this equation can be made to hold for all sizes of the units by the device of introducing as a factor with each measurable quantity a dimensional constant of dimensions the reciprocal of those of the measured quantity. Since the dimensions of every measured quantity are products of powers, the dimensions of the reciprocal must also be products of powers, and the theorem is proved for this restricted class of dimensional constants. We will for the present accept as true the statement that all dimensional constants have this type of dimensional formula.

Now let us suppose that we have a functional relation between certain measured quantities and certain dimensional constants. We shall suppose that the dimensional formulas of all these quantities are known, including the dimensional constants. We shall furthermore suppose that the functional relation is of such a form that it remains true formally without any change in the form of the function when the size of the fundamental units is changed in any way whatever. An equation of such a form we shall call a "complete" equation.[1] We have seen that it is by no means necessary that an equation should be a complete equation in order to be a correct and adequate expression of the physical facts, although the contrary statement is almost always made, and is frequently made the basis of the proof of the principle of dimensional homogeneity of "physical" equations. Although every adequate equation is not necessarily complete, we have seen that every adequate equation can be made complete in a very simple way, so that the assumption of completeness is no essential restriction in our treatment, although it makes necessary a more careful examination of the question of dimensional constants.

The assumption of the completeness of the equation is absolutely essential to the treatment, and in fact dimensional analysis applies only to this type of equation. It is to be noticed that the changes of fundamental unit contemplated in the complete equation are restricted in a certain sense. We may change only the size of the fundamental units and not their character. Thus, for example, a complete equation which holds for all changes in the size of the fundamental units as long as these units are units of mass, length, and time, no longer is true, and in fact becomes meaningless in another system of units in which mass, force, length, and time are taken as fundamental.

With this preliminary, let us suppose that we have a complete equation in a certain number of measurable quantities and dimensional constants, valid for a certain system of fundamental units. Since we are concerned only with the dimensional formulas of the quantities involved, we need not distinguish in our treatment the measurable quantities from the dimensional constants. We will denote the variables by α, β, γ, ---- to n quantities, and suppose there is a functional relation

$$\phi\,(\alpha, \beta, \gamma, ----) = 0.$$

The expanded meaning of this expression is that if we substitute into the functional symbol the numbers which are the measures of the quantities α, β, etc., the functional relation will be satisfied. We use α interchangeably for the quantity itself and for its numerical measure, as already explained. Now the fact that the equation is a complete equation means that the functional relation continues to be satisfied when we substitute into it the numbers which are the numerical measure of the quantities α, β, ---- in a system of measurement whose fundamental units differ in size from those of the fundamental system in any way whatever. Now we have already employed a method, making use of the dimensional formulas, for finding how the number measuring a particular quantity changes when the size of the fundamental units changes. This was the subject of the second chapter. Let us call the fundamental units m_1, m_2, m_3, etc., to m quantities, and denote by α_1, α_2, α_3, ---- etc., the dimensions of α, by β_1, β_2, β_3, ---- etc., the corresponding dimensions of β, etc., in m_1, m_2, m_3, etc., respectively.

We now decrease the size of the fundamental units m_1, m_2, etc., by the factors x_1, x_2, etc. Then the numerical measures of α, β, etc., in terms of the new units, which we will call α', β', etc., are, as proved in Chapter III, given by

$$\left.\begin{array}{l}\alpha' = (x_1^{\alpha_1} x_2^{\alpha_2} ----)\,\alpha \\ \beta' = (x_1^{\beta_1} x_2^{\beta_2} ----)\,\beta \\ |\qquad\quad |\qquad\qquad | \\ |\qquad\quad |\qquad\qquad | \\ |\qquad\quad |\qquad\qquad |\end{array}\right\}$$

Now since the equation $\phi(\alpha, ----) = 0$ is a complete equation, it must hold when α', β', ---- etc., are substituted for α, β, ----. That is

$$\phi(\alpha', \beta', ----) = 0. \qquad \text{A}$$

or

$$\phi(x_1^{\alpha_1} x_2^{\alpha_2} ----\alpha, x_1^{\beta_1} x_2^{\beta_2} ----\beta, ----) = 0.$$

This equation must hold for all values of x_1, x_2, etc.

Now differentiate partially with respect to x_1, and after the differentiation put all the x's equal to 1. Then we obtain the following equation

$$\alpha_1 \alpha \frac{\partial \phi}{\partial \alpha} + \beta_1 \beta \frac{\partial \phi}{\partial \beta} + ---- = 0. \qquad \text{B}$$

Introduce the new independent variables

$$\alpha'' = \alpha^{\frac{1}{\alpha_1}}, \beta'' = \beta^{\frac{1}{\beta_1}}, ---- \text{etc.}$$

The effect of this substitution is obviously to make α'', β'', etc. of the first degree in the first primary quantity. If any of the quantities α_1 etc. are zero, the corresponding term in equation B does not occur, and there is no need of the transformation to α''.

With this change of variables, equation B becomes

$$\alpha'' \frac{\partial \phi}{\partial \alpha''} + \beta'' \frac{\partial \phi}{\partial \beta''} + ---- = 0. \qquad \text{C}$$

Call ζ'' the last of the n variables α'', β'', ---- and introduce $n - 1$ new variables z_1, z_2, ---- z_{n-1}, which are the ratios of α'', β'', etc. to ζ''. That is, put

$$\alpha'' = z_1 \zeta'', \beta'' = z_2 \zeta'', ---- \zeta'' = \zeta''.$$

Substituting these into the function gives

$$\phi(\alpha'', \beta'' ---- \zeta'') \equiv \phi(z_1 \zeta'', z_2 \zeta'', ---- \zeta'').$$

We can now show that the function on the right-hand side is independent of ζ''. This is done by showing that its derivative with respect to ζ'' vanishes. In fact

$$\frac{\partial \phi}{\partial \zeta''} = z_1 \phi'_1 + z_2 \phi'_2 + ---- + z_{n-1} \phi'_{n-1} + \phi'_n$$

$$= \frac{\alpha'' \phi'_1 + \beta'' \phi'_2 + ---- + \zeta'' \phi'_n}{\zeta''},$$

which is zero, for the numerator is merely the left-hand side of equation C in another form. Hence the function $\phi(z_1\zeta'', z_2\zeta'', ----)$ is actually a function only of the n — 1 z's, and we may write

$$\phi(\alpha'', \beta'', ----\zeta'') = \phi(z_1\zeta'', z_2\zeta'', ----\zeta'') = \Psi(z_1, z_2, ----z_{n-1}),$$

where Ψ is a function of n — 1 arguments, instead of n, as was ϕ. What is more, since all the quantities α'', ---- ζ'' are by construction of the first degree in the first primary quantity, the ratios z_1, z_2, etc. are dimensionless in the first primary quantity, or all the arguments of the function Ψ are dimensionless in the first primary quantity.

The argument may now be started over again from the beginning, setting $\Psi(z_1, z_2, ---- z_{n-1}) = 0$, which follows from A, since we have proved Ψ to be identically the same as ϕ. But $\Psi = 0$ is an equation of the same type as A, with the difference that one argument has disappeared from the function, and one primary quantity from the arguments. A repetition of the process above, by differentiating Ψ with respect to x_2, eliminates the second primary quantity from the arguments, and reduces the number of arguments by one more. This process may evidently be repeated until the primary quantities are entirely exhausted. Each elimination of a primary quantity is accompanied by a reduction by one of the number of arguments, so that the function that finally emerges is a function of m — n arguments.

Furthermore, an examination of the nature of the changes of variable used in effecting the reduction shows the nature of the arguments of the final function. For the changes of variable are of only two types, either raising to a power, or taking a ratio. It is obvious that combinations of such operations can give only products of powers of the original variables.

Hence we have the final result. *If the equation* $\phi(\alpha, \beta, \gamma, ----) = 0$ *is to be a complete equation, the solution has the form*

$$F(\Pi_1, \Pi_2, ----) = 0$$

where the Π's *are the n—m independent products of the arguments* α, β, ---- *etc., which are dimensionless in the fundamental units.*

The result stated in this form is known as the Π theorem, and seems to have been first explicitly stated by Buckingham,[4] although an equivalent result had been used by Jeans[3] without so explicit a statement.

The solution in the form above may be solved explicitly for any one of the products, giving the equivalent form of result

$$\alpha = \beta^{x_1} \gamma^{x_2} - - - - \Phi\ (\Pi_2, \Pi_3, - - - -)$$

where the x's are such that $\alpha\, \beta^{-x_1} \gamma^{-x_2} - - - -$ is dimensionless.

The result in this form embodies the mathematical statement of the principle of dimensional homogeneity. For the arbitrary function on the right-hand side is a function of arguments each of which is of zero dimensions, so that every term of the resulting function must itself be dimensionless. Every term of this function is to be multiplied by a term of the same dimensions as the left-hand side of the equation, with the result that every term on the right-hand side has the same dimensions as the left-hand side. The terms may now be rearranged in any way that we please, but whatever the rearrangement, the dimensions of all terms will remain the same. This is known as the principle of dimensional homogeneity.

The attempt is often made to give an off-hand proof of the principle of dimensional homogeneity from the point of view which regards a dimensional formula as an expression of the "essential physical nature" of a quantity. Thus it is said that an equation which is an adequate expression of the physical facts must remain true no matter how the fundamental units are changed in size, for a physical relationship cannot be dependent on an arbitrary choice of units, and if the equation is to remain true for all choices of units the dimensions of each term must be the same, for otherwise we would have quantities of different physical natures put equal to each other. For instance, we could not according to this view have a quantity of the dimensions of a length on the one side of an equation equal to a quantity of the dimensions of an area on the other side, for it is absurd that an area should be equal to a length.[2] The criticism of this point of view should be obvious after what has been said about an equation merely being an equation between numbers which are the numerical measures of certain physical quantities.

It is to be most carefully noticed that the work above was subject to a most important tacit restriction at the very outset. In putting $\phi\ (\alpha, \beta, - - - -) = 0$ it was tacitly assumed that this is the only relation between $\alpha, \beta, - - - -$ etc., and that the partial derivatives may be computed in the regular way on this assumption. If α, β, γ, etc., are connected by other relations than $\phi\ (\alpha, \beta, - - - -) = 0$, then the analysis above does not hold, and the results are no longer true. For it is not true in general that an equation which is a complete equation, that is, an equation which remains true when the size of the fundamental units is changed, is dimensionally homogeneous. Such an equation is dimensionally homogeneous of necessity only when there is no other numerical relation between the variables than that defined by the equation itself. Consider as an example a falling body. Let v be its velocity, s the distance of fall, t the time of fall, and g the acceleration of gravity. Now these quantities are related, and there is more than one equation of connection, because both v and s are fixed when t and g are given. The relations connecting these quantities are $v = gt$, and $s = \frac{1}{2}gt^2$. In the light of the above we would expect that a complete equation connecting v, s, g, and t need not be dimensionally homogeneous. An example can be given immediately, namely,

$$v + s = gt + \tfrac{1}{2}gt^2.$$

This is obviously a complete equation in that it is true and remains true no matter how the fundamental units of length and time are changed in size. We may, if we please, write from these elements an equation which is very much more unusual and offensive in appearance, such as

$$v\left[\sin\frac{s + gt}{v}\right]^{\sinh(s - \frac{1}{2}gt^2)} = gt\cosh(v - gt).$$

This again is a complete equation; it is not dimensionally homogeneous, and also offends our preconceived notions of what is possible in the way of transcendental functions.

The possibility of equations like those just considered is in itself a refutation of the intuitional method of proof of the principle of dimensional homogeneity sometimes given.

The equation $v + s = gt + \frac{1}{2}gt^2$ reminds one of the procedure used in vector analysis, in which three scalar equations may be replaced by a single vector equation. Obviously we may add together any number of complete equations and obtain a result which remains true. And provided that the dimensions of the original equations were all different, the resulting compound equation (complete but not dimensionally homogeneous) may be decomposed, like the vector equation, into a number of simpler equations, by picking out the parts with the same dimensions. I do not know whether this method of throwing the results into a compact form can ever be made to yield any practical advantages or not.

Let us now return to the first form in which we put the result above, namely,

$$F\,(\Pi_1, \Pi_2, ----) = 0.$$

Consider the Π's and how they are formed from the variables. Write a typical Π in the form

$$\alpha^a\ \beta^b\ \gamma^c\ ----.$$

The a, b, c, etc., are to be so chosen that this is dimensionless. Substituting now the dimensional symbols of α, β, ---- etc., gives as many equations of condition between a, b, c, etc., as there are kinds of fundamental unit. The equations are

$$\left.\begin{array}{l} \alpha_1\, a + \beta_1\, b + \gamma_1\, c + ---- = 0 \\ \alpha_2\, a + ------------- = 0 \\ | \\ | \\ \alpha_m\, a + \beta_m\, b + ------- = 0 \end{array}\right\}$$

There are m equations, each with n terms. Now the theory of the solutions of such sets of equations may be found in any standard work on algebra. In general, n will be greater than m. Under these conditions there will in general be n−m independent sets of solutions. That is, there will in general be n−m independent dimensionless products, and the arbitrary function F will be a function of n−m variables.

In certain special cases this conclusion will have to be modified. If, for instance, n = m, there will in general be no solution, but there may be in the special case that the determinant of the exponents

$$\begin{vmatrix} \alpha_1\, \beta_1 ---- \\ \alpha_2\, \beta_2 ---- \\ | \\ | \\ \alpha_m\, \beta_m ---- \end{vmatrix} \text{ vanishes.}$$

Furthermore, there may be more than n−m independent solutions if it should happen that *all* the m-rowed determinants of the exponents vanish. This, of course, will not very often occur, but we shall meet at least one example later.

In the general case, where there are n—m independent solutions, it is generally possible to select n—m of the quantities a, b, c, etc., in any convenient way, assign to them n—m sets of independent values, and solve for the remaining quantities, thus obtaining n—m sets of values which determine n—m dimensionless products Sometimes this is not possible, and the particular set of the quantities a, b, c, etc., to which arbitrary values can be assigned cannot be chosen with complete freedom. This occurs when certain determinants chosen from the array of the exponents vanish. We will not stop here to develop a general theory, but let the exceptions take care of themselves, as it is always easy to do in any special problem.

It is to be noticed that the Π theorem does not contain anything essentially new, and does not enable us to treat any problems which we could not already have handled by the methods of the introduction. The advantage of the theorem is one of convenience; it places the result in a form in which it can be used with little mental effort, and in a form of a good deal of flexibility, so that the results of the dimensional analysis may be exhibited in a variety of forms, depending on the variables in which we are particularly interested. In this way it has very important advantages.

The result of this dimensional analysis places no restrictions whatever on the form of the functions by which the results of experiments may be expressed, but the restriction is on the form of the arguments only. However complicated the function, if it is one which satisfies the fundamental requirements of the theory as developed above, it must be possible to rearrange the terms in such a way that it appears as a function of dimensionless arguments only. Now in using the theorem we are nearly always interested in expressing one of the quantities as a function of the others. This is done by solving the function for the particular dimensionless product in which the variable in question is located, and then multiplying that dimensionless product (and of course the other side of the equation as well) by the reciprocal of the other quantities which are associated with it in the dimensionless product. The result is that on the one side of the equation the variable stands alone, while on the other side is a product of certain powers of some of the other variables multiplied into an arbitrary function of the other dimensionless products. This arbitrary function may be transcendental to the worst degree; there is absolutely no restriction on it, but its arguments are dimensionless. This agrees with the result of common experience in regard to the nature of the possible functional relations. We have come to expect that any argument which appears under the sign of a transcendental function must be a dimensionless argument. This is usually expressed by saying that it makes no sense to take the hyperbolic sine, for example, of a time, but the only thing of which we can take the sinh is a number.[5] Now although the observation is correct which remarks that the arguments of the sinh functions which appear in our analysis are usually dimensionless, the reason assigned for it is not correct. There is no reason why we should not take the sinh of the number which measures a certain interval of time in hours, any more than we should not take the number which counts the apples in a peck. Both operations are equally intelligible, but the restrictions imposed by the Π theorem are such that we seldom see written the sinh of a dimensional quantity, and even if we should, it would be possible by a rearrangement of terms, as already explained, to get rid of the transcendental function of the dimensional argument by coalescing two or more such functions into a sinh of a single dimensionless argument. Thus it is perfectly correct to write the equation of a falling body in the form

$$\sinh v = \sinh gt,$$

but no one would do it, because this form is more complicated than that obtained by taking the $\sinh^{-1}$ of both sides. The equation above might be rewritten

$$\sinh v \cosh gt - \cosh v \sinh gt = 0,$$

in which form the rearrangement to get rid of the transcendental function of a dimensional argument is not so immediate, particularly if one is rusty on his trigonometric formulas. But the last form is perfectly adapted for numerical computation, in the sense that it will always give the correct result, and still holds when the size of the fundamental units is changed.

There is a corollary to these remarks about transcendental functions with respect to the exponents of powers. It is obvious that in general we cannot have an exponent which has dimensions. If such appears, it is possible to combine it with others in such a way that the dimensionality is lost. But there is absolutely no restriction whatever imposed as to numerical exponents; these may be integral, or fractional, or incommensurable. It is often felt that the dimensional formula of a quantity should not involve the fundamental quantities to fractional powers.[6] This is a part of the view that regards a dimensional formula as an expression of operations on concrete physical things, and this point of view finds it hard to assign a meaning to the two-thirds power of a time, for example. But it seems to me just as hard to assign a physical meaning to a minus second power of a time, and the possibility of such exponents is admitted by everyone.

The Π theorem as given contains all the elements of the situation. But in use there is a great deal of flexibility in the choice of the arguments of the function, as is suggested by the fact that it is possible to choose the independent solutions of a set of algebraic equations in a great number of ways. The way in which the independent solutions are chosen determines the form of the dimensionless products, and the best form for these will depend on the particular problem. We shall in chapter VI treat a number of concrete examples which will illustrate how the products are to be chosen in special cases.

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(5) E. Buckingham, Ref. 1, page 346.

> "Such expressions as log Q or sin Q do not occur in physical equations; for no purely arithmetical operator, except a simple numerical multiplier, can be applied to an operand which is not itself a dimensionless number, because we cannot assign any definite meaning to the result of such an operation."

See also in this connection page 266 of the following.

(6) S. P. Thompson, Elementary Lessons in Electricity and Magnetism, p. 352.

> "It also seems absurd that the dimensions of a unit of electricity should have fractional powers, since such quantities as $M^{\frac{1}{2}}$ and $L^{\frac{1}{2}}$ are meaningless."

W. Williams, Phil. Mag. 34, 234, 1892.

> "So long, however, as L, M, and T are fundamental units, we cannot expect fractional powers to occur. . . . Now all dynamical conceptions are built up ultimately in terms of these three

ideas, mass, length, and time, and since the process is synthetical, building up the complex from the simple, it becomes expressed in conformity with the principles of Algebra by integral powers of L, M, and T. . . . Obviously if mass, length, and time are to be ultimate physical conceptions, we cannot give interpretations to fractional powers of L, M, and T, because we cannot analyze the corresponding ideas to anything simpler. We should thus be unable, according to any physical theory, to give interpretations to formulae involving fractional powers of the fundamental units."

CHAPTER V

DIMENSIONAL CONSTANTS AND THE NUMBER OF FUNDAMENTAL UNITS

THE essential result which we have obtained in the Π theorem is in the restriction which it places on the number of arguments of the arbitrary function. The fewer the arguments, the more restricted the function, and the greater our information about the answer. Thus if the problem is such that there are four variables, and three fundamental kinds of unit, our analysis shows that there is only one dimensionless product, which we can determine, and that some function of this product is zero. This is equivalent to saying, in this special case, that the product itself is some constant, and we have complete information as to the nature of the solution, except for the numerical value of the constant. This was the nature of the solution which we found for the pendulum problem. If it had not been for the dimensional analysis, any conceivable relation between the four arguments might have been possible, and we should have had absolutely no information about the solution. Similarly, if there are two more variables than fundamental kind of quantity, there will be two dimensionless products. The solution is an arbitrary function of these two products put equal to zero, which may be solved for one of the products as a function of the other. This was the case with the heat transfer problem already treated. It certainly gives more information to know that the solution is of this form than merely to know that there is some function of the five variables which vanishes, which was all that we could say before we applied our analysis.

It is to our advantage, evidently, that the number of arguments which are to be connected by the functional relation should be as small as possible. Now the variables which enter the functional relation to which our analysis has been applied comprise all the variables which can change in numerical magnitude under the conditions of the problem. These variables are of two kinds. First are the physical variables, which are the measures of certain physical quantities, and which may change in magnitude over the domain to which our result is to apply. The numbers measuring these physical quantities may also change when the size of the fundamental units changes. In the second place, there may be other arguments of the nature of coefficients in the equation which do not change in numerical magnitude when the physical system alone changes, but which change in magnitude when the size of the fundamental measuring units changes. It is these which we have called dimensional constants. Now in any actual case we are interested only in the physical problem, and are interested in finding a relation between the physically variable quantities. The dimensional constants are to be regarded as an evil, to be tolerated only if they make possible more information about the physical variables.

We thus see that the Π theorem applies to the aggregate of physical variables and dimensional constants, whereas we are interested primarily in the physical variables alone. If the number of dimensional constants is so great that the number of arguments of the arbitrary function allowed by the Π theorem is equal to or greater than the number of physical variables alone, then we are no better off after applying our Π theorem than before. Now we have already seen that in the worst possible case the number of dimensional constants cannot exceed the number of physical variables, for any empirical equation can be made complete by the introduction of a dimensional constant with each physical variable. Furthermore, it is almost always true that the number of physical variables is equal to or greater than the number of primary units. Hence, if the number of dimensional constants is equal the number of physical variables, the number of dimensionless products is greater than or at most equal to the number of physical variables. In the general case, therefore, the Π theorem gives no new information. Hence it is of the utmost importance to keep down to the minimum the number of dimensional constants used in the equation.

When, therefore, shall we expect dimensional constants, and in any particular problem how shall we find what they are, and what are their dimensional formulas? The answer to this question is closely related to the answer to the question of how we shall choose the list of physical quantities between which we are to search for a relation. We have seen that it does not do to merely ask ourselves "Does the result depend on this or that physical quantity?" for we have seen in one problem that although the result certainly does "depend" on the action of the atomic forces, yet we do not have to consider the atomic forces in our analysis, and they do not enter the functional relation.

To answer the question of what variables to include demands a background of a great deal of physical experience. If we are to treat a certain problem by the methods of mechanics we must have enough background to be assured that the problem is a problem in mechanics, and involves essentially no elements that are not treatable by the ordinary equations of mechanics. We must know that certain aspects of the situation can be neglected, and that certain others alone are essential as far as certain features of behavior go. No one would say that in any problem of mechanics the atomic forces are not essential, but our experience shows that they combine into certain complexes, which may be sufficiently characterized by an analysis which does not go down to the ultimate component parts, and that the results of our analysis, which disregards many even essential aspects of the situation, have validity under certain conditions whose restrictions are not irksome. The experience involved in judgments of this sort reaches so far back that we know almost by instinct whether a problem is suitable for mechanical treatment or not. And if the problem is capable of mechanical treatment, we know, by the very definition of what we mean by a mechanical system, what the equations are which the motion of the component parts of the system conform to, and what the form of the equations is. In the same way, we know by instinct whether a system is a thermodynamic system, or an electrical system, or a chemical system, and in each case, because we know what we mean when we say that a phenomenon is of such or such a nature, we know what are the laws which govern the variations of the system, and the elements which must be considered in formulating the relations between the parts. But a very wide background of experience, extending over many generations, was necessary before we could say that this particular group of phenomena is mechanical or electrical, or, in general, that the phenomenon is physical.

Now my point of view is essentially that precisely the same experience which is demanded to enable us to say whether a system is mechanical or electrical is the experience which is demanded in order to enable us to make a dimensional analysis. This experience will in the first place inform us what physical variables to include in our list, and will in the second place tell us what dimensional constants are demanded in any particular problem.

Let us for the present forget what we know of dimensional analysis and imagine ourselves approaching a new problem. In the first place we decide in the light of the experience of all the ages what the nature of the problem is. Suppose that we decide that it is mechanical. Then we know that the motion of the system is governed by the laws of mechanics, and we know what these laws are. We write down certain equations of motion of the system. We are careful to include all the equations of motion, so that the system of equations by which we have described the relations between the parts of the system has a unique solution. Then we are convinced, because of our past experience, that we have essentially represented all the elements of the situation, that our equations correspond to the reality at least as far as certain aspects of the phenomenon go, and the solution of the equations will correctly represent the behavior of the system which we have thus analyzed. We are not disappointed. The fact that our predictions turn out to be verified means merely that we have become masters of a certain group of natural phenomena.

Now the astute observer (Fourier[1] was the first astute observer) notices that the equations by which the relation of the component parts of the system is analyzed are expressed in such a general form that they remain true when the size of the fundamental units is changed. For instance, the equation stating that the force acting on a particular part of our mechanical system is equal to the mass of that part times its acceleration remains true however the size of the fundamental units is changed, because in every system of units which we use for mechanical purposes, the unit of force is defined so that force has this relation to mass and acceleration. Every one of the fundamental equations of motion is in the same way a complete equation. The final solution is obtained from the equations of motion by a purely mathematical process, which has no relation to the size of the fundamental units. It follows, therefore, in general, that the final result will also be complete, in the sense that the equation expressing the final result is a complete equation.

Dimensional analysis may, therefore, be applied to the results which we obtain by solving the equations of motion. (We use equations of motion in a general sense, applying to thermodynamic and electrical as well as mechanical systems.) Now the arguments of the function which we finally obtain by solving the equations of motion can obviously be only those quantities which we put into the original equations of motion, for the mathematical operations can introduce no new arguments.

In particular, the dimensional constants which enter the final relation are those, and those only, which we had to use in writing down the equations of motion. This is the entire essence of the question of dimensional constants.

With regard to the dimensional formulas of dimensional constants, we may merely appeal to experience with the observation that all such constants are of the form of products of powers of the fundamental quantities. But it is evident on reflection, that any law of nature can be expressed in a form in which the dimensional formulas of the constants are of this type, by the device, already adopted, of introducing dimensional constants as factors with the measured quantities in such a way as to make the equation complete. We will hereafter assume that the equations of motion (which are merely expressions of the laws of nature governing phenomena) are thrown into such a form that the dimensional constants are of this type; this is seen to involve no real restriction.

It appears, therefore, that **dimensional analysis is essentially of the nature of an analysis of an analysis.** We must know enough about the situation to know what the general nature of the problem is, and what the elements are which would be introduced in writing down the equations determining the motion (in the general sense) of the system. Then, knowing the nature of the elements, we can obtain certain information about the *necessary* properties of any relations which can be deduced by mathematical manipulations with the elements. In so far as our knowledge of the underlying laws of nature is adequate we may have confidence in the result, but the result can have no validity not pertaining to the equations of motion, and is in no way different from all our other knowledge. The result is approximate, as the laws of motion are approximate, a restriction which is imposed by the very nature of knowledge itself.

The man applying dimensional analysis is not to ask himself "On what quantities does the result depend?" for this question gets nowhere, and is not pertinent. Instead we are to imagine ourselves as writing out the equations of motion at least in sufficient detail to be able to enumerate the elements which enter them. It is not necessary to actually write down the equations, still less to solve them.

Dimensional analysis then gives certain information about the necessary character of the results. It is here of course that the advantage of the method lies, for the results are applicable to systems so complicated that it would not be possible to write the equations of motion in detail.

It is to be especially noticed that the results of dimensional analysis cannot be applied to any system whose fundamental laws have not yet been formulated in a form independent of the size of the fundamental units. For instance, dimensional analysis would certainly not apply to most of the results of biological measurements, although such results may perfectly well have entire physical validity as descriptions of the phenomena. It would seem that at present biological phenomena can be described in complete equations only with the aid of as many dimensional constants as there are physical variables. In this case, we have seen, dimensional analysis has no information to give. In a certain sense, the mastery of a certain group of natural phenomena and their formulation into laws may be said to be coextensive with the discovery of a restricted group of dimensional constants adequate to coördinate all the phenomena.

Let us apply this view of the nature of dimensional constants to the problem which we have already considered of the electromagnetic mass of a spherical distribution of electricity. This is evidently a problem in electrodynamics, and must be solved by the use of the field equations. These field equations consist of certain mathematical operators operating on certain combinations of the electric and magnetic forces and the velocity of light. In this particular problem we want to solve the equations in such a form as to get the electromagnetic mass; this is the integral throughout space of a constant times the energy density, which in turn is given by the distribution of the forces, which are determined by the distribution of the charge. Hence if there is a relation of the form which we suspect, the forces will eliminate from the final result. There is, however, no reason to think that the characteristic constant "c" of the equations will also eliminate from the result, and we must therefore seek for a relation between the total charge, the mass, the radius, and the constant of the field equations, which is the velocity of light. This relation we have already found, and checked against the results of a detailed solution of the field equations applied to this particular problem.

We have seen that dimensional constants will enter the final result only in so far as they enter the equations of motion. Now a dimensional constant in an equation of motion is an expression of a physical relation which is so universal as to be characteristic of all the phenomena embraced in the particular group which we are considering. Such a universal physical relation may be treated in two ways. We may leave the dimensional constant in the equations as an explicit statement of the relation, as is done in the field equations of electrodynamics, or we may define our fundamental units with this relation in view, thus obtaining a system of units in which the dimensional constant has disappeared but in which the number of units which may be regarded as primary has been restricted in such a way that all units belonging to the system automatically bear the experimental relation to each other. The system of units so obtained is of value only in treating that group of phenomena to which the law in question applies. Thus it is a result of experience that the mass times the acceleration of a body is proportional to the force acting upon it. In this statement of the experimental facts there is no restriction whatever upon the units of mass, or length, or time, or force. The factor of proportionality will change in numerical magnitude whenever any one of the four fundamental units is changed in size. But now, instead of being bothered by a continually changing factor of proportionality, we may arbitrarily say that this factor shall be unity in all systems which we will consider, and we will bring this result about by defining the unit of force in our new system to be such that the force acting on a body is equal to the mass times the acceleration. We have in this way obtained a system of units adapted to dealing with all those physical systems in which the laws of motion involve a statement of the physical relation between force, mass, and acceleration, but if the physical system should be such that this relation is not involved in the motion of the system, then we would be unduly restricting ourselves by using the mechanical system of units.

These considerations as to the possible systems of units answer the question previously raised, as in the fourth problem of the introduction. for example, as to the number of kinds of units which we shall take as primary. The answer depends entirely upon the particular problem, and will involve the physical relations which are necessary to a complete expression of the motion of the parts. In any ordinary problem of dynamics, for example, the relation between force and mass and acceleration is essentially involved in the equations of motion. This relation may be brought into the equations either by the use of four primary kinds of unit, force, mass, length, and time, with the corresponding dimensional constant of proportionality, or by using the ordinary mechanical units of mass, length, and time, in which force is defined so that the experimental relationship is always satisfied, and the dimensional constant has disappeared. In either case the results of the dimensional analysis are the same. For the difference between the number of primary units and the number of variables, which determines the number of arguments of the unknown function, is the same in either case, because when the number of units is augmented by one by including the force, the number of variables is also augmented by one by including the dimensional constant, and the difference remains constant. If, however, the problem were such that the experimental relation between force, mass, and acceleration is not involved in the equations of motion of the system, then the ordinary mechanical units would be inappropriate, because we would obtain less information when using them. For we could in this case use four primary units without introducing a corresponding dimensional constant into the list of variables, so that the difference between the number of variables and the units would be less by one when using four than when using three primary units, and the arguments of the function would be fewer in number, which is desirable. We shall meet an example illustrating this point later.

REFERENCES

(1) Fourier, Theorie de Chaleur, 160.

As dealing with the general question of the proper number of fundamental units may be mentioned

E. Buckingham, Nat. 96, 208, and 396, 1915.

CHAPTER VI

EXAMPLES ILLUSTRATIVE OF DIMENSIONAL ANALYSIS

LET us in the first place recapitulate the results of the preceding chapter. Before undertaking a dimensional analysis we are to imagine ourselves as making an analysis to the extent of deciding the nature of the problem, and enumerating *all* the physical variables which would enter the equations of motion (in the general sense) and also all the dimensional coefficients required in writing down the equations of motion. The dimensions of all these variables are then to be written in terms of the fundamental units. These fundamental units are to be chosen for each particular problem in such a way that their number is as large as possible without involving the introduction of compensating dimensional constants into the equations of motion. The dimensionless products of the variables are then to be formed in accordance with the Π theorem, choosing the products in such a way from the great variety possible that the variables in which we are particularly interested may stand conspicuously by themselves. Having formed the products, the Π theorem gives immediately the functional relation.

In the following illustrative examples we have particularly to consider the proper number of fundamental units, and the most convenient way of choosing the dimensionless products. The matter of dimensional constants we regard as clear.

As the first example we will take the first treated by Lord Rayleigh in *Nature*.[1] Consider a wave advancing on deep water under the action of gravity. This is evidently a problem in hydrodynamics, which is merely mechanics applied to liquids. The equations of mechanics will therefore apply. Now the liquid when displaced from equilibrium is restored by the force of gravity. This will involve the density of the liquid and the intensity of gravity. Evidently these quantities will enter the equations of motion. No other properties of the liquid, such as the compressibility, will enter, because we know from a discussion of the equations of hydrodynamics that such properties are unimportant for phenomena of this scale of magnitude. Physically, of course, the compressibility affects the result to a certain extent, so that the result of our analysis will not be exact, but will be a valid approximation only to the extent that the equations of hydrodynamics are valid approximations. There are no dimensional constants entering the equations of hydrodynamics, provided that we use ordinary mechanical units, in which mass, length, and time are fundamental, for the laws of motion have entered this system of units through the definition of force. The equations, of course, are equations between the displacements and the other elements. Now it is conceivable that we might eliminate the various displacements from the equations, and come out at the end with a relation between the velocity of propagation, the density, and the intensity of gravity only, analogously to the pendulum problem.

Let us try this. Write down the variables and their dimensional formulas, as before.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Velocity of wave,	v	LT^{-1}
Density of liquid,	d	ML^{-3}
Accleration of gravity,	g	LT^{-2}

We now apply the Π theorem. We have three variables, and three fundamental kinds of unit. The difference between these numbers is zero, and therefore, according to the theorem, there are zero dimensionless products. That is, we have made some mistake, and no relation exists, unless this should be one of those exceptional cases in which a product may be formed of fewer than the normal number of factors. But an examination shows that this is no exception, and there is in fact no dimensionless product. This shows that the suggested elimination was not possible, but that some other elements or combination of elements must enter the final result. Of course the detailed analysis will give as the final result a detailed description of the motion of the water, from which we must pick out the wave motion and find its velocity. That is, along with the velocity in the final result there will be something characteristic of the particular wave. The velocity of all the waves need not be the same, but may depend on the wave length, for example. Physically, of course, we knew this in the beginning, and we were stupid only for purposes of instruction. Our experience with problems of this nature would have led us to search for a relation between the variables which we put into the analysis, the velocity, and the wave length. Let us introduce, therefore, into our list of quantities the wave length.

Wave length	λ	L

We have now four variables; the Π theorem leads us to expect one dimensionless product, and the result will be that the dimensionless product is equal to a constant.

The proof of the Π theorem also showed that one exponent in a dimensionless product can be assigned arbitrarily. Since we are particularly interested in v, let us choose its exponent as unity, and write the dimensionless product in the form $v\ d^{-\alpha}\ g^{-\beta}\ \lambda^{-\gamma}$. Putting this equal to a constant and solving for v, gives for the result

$$v = \text{Const}\ d^{\alpha}\ g^{\beta}\ \lambda^{\gamma}.$$

The dimensions of the factors on the right-hand side must together be the same as that of the velocity, which stands alone on the left-hand side. We solve for the unknown exponents of the factors of the right-hand side. Substitute the dimensional formulas for the variables

$$LT^{-1} = (ML^{-3})^{\alpha}\,(LT^{-2})^{\beta}\ L^{\gamma}.$$

Now write down in succession the condition that the exponents of M, of L, and of T be the same on the two sides of the equation. This gives

$$\begin{array}{rcll} \alpha & = & 0 & \text{condition on M} \\ -3\alpha+\beta+\gamma & = & 1 & \text{condition on L} \\ -2\beta & = & -1 & \text{condition on T.} \end{array}$$

Whence

$$\alpha = 0,\ \beta = \tfrac{1}{2},\ \gamma = \tfrac{1}{2},$$

and the final result is of the form

$$v = \text{Const}\ \sqrt{\lambda g}.$$

The velocity of a gravity wave on deep water (the reason the depth did not enter the final result was because we postulated that the water was to be deep) is therefore proportional to the square root of the wave length and the intensity of gravity, or is proportional to the velocity acquired by a body falling freely under gravity through a distance equal to the wave length.

It is to be noticed that the density of the liquid has disappeared from the final result. This might have been anticipated; if the density is doubled the gravitational force is also doubled on every element, and the accelerations and therefore all the velocities are unaltered, because the doubled force is compensated by a doubled mass of every element.

Since the density disappears from the final result we have here a dimensionless product of v, l, and g only. This is, therefore, a dimensionless product of three variables, expressed in three fundamental units. This is in general not possible, but demands some special relation between the dimensional formulas of the variables. We can see in a moment by writing out the product in terms of unknown exponents, and then writing the algebraic equations which the exponents must satisfy, that the condition that a dimensionless product exist in a number of terms just equal to the number of fundamental units is that the determinant of the exponents in the dimensional formulas of the factors vanish. This is obviously not restricted to the case of three fundamental units, but applies to any number. Conversely the condition that a particular element shall enter as a factor into a dimensionless product with a number of other factors equal in number to the fundamental units, is that the determinant of the exponents of the other factors shall not be zero; otherwise the other factors by themselves form a dimensionless product into which the factor in which we are interested does not enter.

This problem is an excellent illustration of the necessity of combining a sound physical intuition with the purely formal manipulations. That we were able to neglect the depth involved some argument convincing us that as the depth increases indefinitely the velocity approaches a limiting value independent of the depth. Further, there is still another factor which we neglected in our analysis, namely the amplitude (h) of the wave, which is obviously analogous to the amplitude of swing in the elementary pendulum problem. If we had included this in our original list of quantities, there would have been one more dimensionless product, λ/h, and if we had been perversely inspired, our result might have read $v = f(\lambda/h)\sqrt{hg}$, where f is an arbitrary function. This form is perfectly consistent with that already deduced, as may be seen on putting $f(\lambda/h) = \text{Const}\sqrt{\lambda/h}$, but it obviously gives information of very much less importance.

Consider now a second problem. An elastic pendulum is made by attaching to a weightless spring of elastic constant k a box of volume V which is filled with a liquid of density d. The mass of the liquid in the box is acted upon by gravity, and we are required to find an expression for the time of oscillation. As before we make a list of the quantities and their dimensions.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Elastic constant (force per unit displacement),	k	MT^{-2}
Time of oscillation,	t	T
Volume of box,	v	L^3
Density of liquid,	d	ML^{-3}
Acceleration of gravity,	g	LT^{-2}

The problem is obviously one in ordinary mechanics, so that we are justified in using the mechanical system of units, and there will be no dimensional constant. The variables which we have listed above are, therefore, the only ones, and are those in terms of which the problem is formulated. Here there are five quantities and three fundamental kinds of unit. There are therefore two dimensionless products. In the analysis of the last chapter we saw that in finding the dimensionless products we had to solve a system of algebraic equations. Certain of the solutions could be assigned at pleasure, and the others determined in terms of them. In this particular problem we are interested especially in t, and let us say k. Then let us choose the exponents of t and k in the dimensionless products as those which are to be assigned at pleasure and in terms of which the others are to be computed. Now the algebraic theorem showed that there were two linearly independent sets of exponents which we might assign to t and k, and that it is possible to choose these two sets in an infinite number of ways. We will try to select the two simplest. For the present purpose we will accomplish this by assigning the value 1 to the exponent of t and 0 to that of k for the one set, and 0 to the exponent of t and 1 to the exponent of k for the other set. This is certainly a simple couple of pairs, and has the effect of making both t and k appear in only one dimensionless product. We therefore have to find the two dimensionless products

$$t\, v^{\alpha_1} d^{\beta_1} g^{\gamma_1} \quad , \quad k\, v^{\alpha_2} d^{\beta_2} g^{\gamma_2}.$$

We have now two sets of algebraic equations for the two sets of unknown exponents α_1, β_1, γ_1 and α_2, β_2, γ_2. These equations are

$$\left.\begin{array}{l} 0\,\alpha_1 + \beta_1 + 0\,\gamma_1 + 0 = 0 \\ 3\,\alpha_1 - 3\,\beta_1 + \gamma_1 + 0 = 0 \\ 0\,\alpha_1 + 0\,\beta_1 - 2\,\gamma_1 + 1 = 0 \end{array}\right\} \qquad \left.\begin{array}{l} 0\,\alpha_2 + \beta_2 + 0\,\gamma_2 + 1 = 0 \\ 3\,\alpha_2 - 3\,\beta_2 + \gamma_2 + 0 = 0 \\ 0\,\alpha_2 + 0\,\beta_2 - \gamma_2 - 2 = 0 \end{array}\right\}$$

The solutions are

$$\left.\begin{aligned} \alpha_1 &= -\tfrac{1}{6} \\ \beta_1 &= 0 \\ \gamma_1 &= \tfrac{1}{2} \end{aligned}\right\} \qquad \left.\begin{aligned} \alpha_2 &= -\tfrac{2}{3} \\ \beta_2 &= -1 \\ \gamma_2 &= -1 \end{aligned}\right\}$$

Hence the dimensionless products are

$$t\, v^{-\frac{1}{6}}\, g^{\frac{1}{2}} \quad \text{and} \quad k\, v^{-\frac{2}{3}}\, d^{-1}\, g^{-1},$$

and the solution is

$$t = v^{\frac{1}{6}}\, g^{-\frac{1}{2}}\, f\left(\frac{k\, v^{-\frac{2}{3}}}{g\, d}\right),$$

where the function f is undetermined.

Now the result so obtained is undoubtedly correct as far as it goes, but an examination will show that we can do better, and obtain a form in which there is no undetermined function. This improvement can be effected by increasing the number of fundamental units. We were correct in using the ordinary mechanical units, for the equations of motion involve the dynamical relation between force, mass, and acceleration. The change is to be made in a direction at first not obvious because we are so accustomed to using the units written down. It is evident on reflection, however, that in the equations of motion governing the system no use is made of the fact that the numerical measure of the volume of the box is equal to the cube of the length of one of its linear dimensions. It is quite possible to measure volumes physically in terms of a particular volume chosen as unity by cutting up the larger volume into smaller volumes congruent with the unit, and counting the number of times that the unit is contained in the larger volume. It may then be proved that the number so obtained is proportional to the cube of the number measuring one of the linear dimensions. In fact, this is the method of proof originally adopted by Euclid in dealing with both areas and volumes. After the geometrical fact has been proved, it becomes natural to define the unit volume as that volume which is equal to a cube whose sides are unity, but this definition and restriction are of value only in those problems in which the relation between volume and length enter essentially into the result. Such is not the case here, because the volume of the box is of importance only as determining, in conjunction with the density of the liquid, the mass filling the box. We might perfectly well measure length for this problem in inches, and the volume in quarts, provided, of course, that we measure density as mass per quart.

Let us then attempt the problem again, now taking volume as an independent unit of its own kind. Then we shall have:

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Elastic constant,	k	MT^{-2}
Time of oscillation,	t	T
Volume of box,	v	V
Density of liquid,	d	MV^{-1}
Acceleration of gravity,	g	LT^{-2}

We have now five variables, but four fundamental kinds of quantity, so that there is only one dimensionless product. We are particularly interested in t, so we choose the exponent of t equal to unity, and are required to find the other exponents so that

$$t\, k^{\alpha}\, v^{\beta}\, d^{\gamma}\, g^{\delta} \text{ is dimensionless.}$$

This problem is so simple that we can solve for the unknowns by inspection, or if we prefer, write out the equations, which are:

$$\left.\begin{aligned} \alpha + \gamma &= 0 \\ \delta &= 0 \\ -2\alpha - 2\delta + 1 &= 0 \\ \beta - \gamma &= 0 \end{aligned}\right\}$$

The solution of this set of equations is

$$\alpha = 1/2, \beta = -1/2, \gamma = -1/2, \delta = 0.$$

The dimensionless product is

$$t\, k^{\frac{1}{2}}\, v^{-\frac{1}{2}}\, d^{-\frac{1}{2}},$$

and the solution is

$$t = \text{Const}\sqrt{\frac{vd}{k}}.$$

The information embodied in this solution is evidently much greater than in the more noncommittal one obtained with three units. It is seen from the new solution, for example, that the time of oscillation does not depend on the intensity of gravity. Physically, of course, this means that gravity is effective only in changing the mean position of equilibrium; as gravity increases the weight is pulled down and oscillates about a position nearer the center of attraction, but the period of oscillation is not changed thereby. It was not at all obvious or necessary from the first form of the solution that the time would be independent of gravity, but that the previous solution is not inconsistent with this one is seen by putting the f of the previous solution equal to a constant times the inverse square root of the argument, when the two solutions become identical.

Instead of increasing the number of fundamental units from three to four, we might have obtained the same result by observing that the equations of motion are concerned only with the total mass on the end of the string, and hence the volume and the density can affect the result only in so far as they enter through their product, the mass. According to this method of treatment we would have put v and d together as one quantity, so that we would have been concerned with only four quantities and three fundamental kinds of unit. The result would have been the same as by the method which we adopted. In fact, it will often be found possible by using special knowledge of the problem to obtain in this way more detailed information than would have been possible by the general analysis.

If we use the mass as one of the variables, the result assumes the form $t = \text{Const}\sqrt{\frac{m}{k}}$. Once more we have a dimensionless product of fewer than the normal number of terms.

Now let us consider a problem illustrating how it is that the result is unaffected by increasing the number of units if at the same time the number of dimensional constants is increased. We take the same problem as above, except that we now give only the mass on the end of the spring, and do not attempt to analyze the mass into volume times density. The variables will be mass (m), time of oscillation (t), and stiffness of spring (k). We can omit the intensity of gravity, because we have already seen it to be without effect. In discussing this problem we propose to use five fundamental kinds of unit, which we will choose as mass, length, time, as usual, and in addition force, and velocity. This problem is evidently one in mechanics involving, in the statement of the relations between the parts, the experimental fact that force is proportional to mass times accleration. Hence in formulating the equations of motion we will have to introduce this proportionality factor, which will appear in the analysis as a new dimensional constant. This factor is to connect force, mass, and acceleration. But now acceleration must be redefined if we are using velocity as a unit of its own kind. Acceleration will now be defined as time rate of change of velocity, and will have the dimensions VT^{-1}. The equation of motion thus written will express a relation between the force and the velocity and the time. But the force is connected with the displacement through the elastic constant, so that to solve the equations a relation is needed between displacement, velocity, and time. The experimental fact, of course, is that velocity is proportional to the quotient of distance by time. The factor of proportionality will appear in the final result as a dimensional constant. We now have our list of quantities complete. They compose three physical variables, and two dimensional constants.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Time of oscillation,	t	T
Mass at end of spring,	m	M
Stiffness of spring,	k	FL^{-1}
The force dimensional constant,	f	$FM^{-1}TV^{-1}$
The velocity dimensional constant,	v	$L^{-1}TV$

Here F is the dimensional symbol of force measured in units of force, and V the dimensional symbol of velocity. The dimensional formulas were obtained by the regular methods, noting only that the stiffness of the spring is defined as the force exerted by the spring per unit displacement of the end.

We have now to find the dimensionless products involving these five variables. We note in the first place that there are five variables, and five fundamental kinds of quantity, so that in general there would be no dimensionless product. But it may be seen on writing it out that the determinant of the exponents in the dimensional formulas vanishes, so that in this special case there is a dimensionless product with fewer than the normal number of factors. Of course we knew that this must be the case from our previous discussion. Now, as before, we select t as the quantity in which we are particularly interested, write the dimensionless product in the form

$$t\, m^{\alpha}\, k^{\beta}\, f^{\gamma}\, v^{\delta},$$

and write down the condition that the product is dimensionless. This gives

$$
\begin{aligned}
\alpha - \gamma &= 0 \quad \text{condition on M} \\
-\beta - \delta &= 0 \quad \text{condition on L} \\
\gamma + \delta + 1 &= 0 \quad \text{condition on T} \\
\beta + \gamma &= 0 \quad \text{condition on F} \\
-\gamma + \delta &= 0 \quad \text{condition on V.}
\end{aligned}
$$

The solution is

$$\alpha = -\tfrac{1}{2}, \beta = \tfrac{1}{2}, \gamma = -\tfrac{1}{2}, \delta = -\tfrac{1}{2}.$$

The dimensionless product is

$$t\, m^{-\frac{1}{2}}\, k^{\frac{1}{2}}\, f^{-\frac{1}{2}}\, v^{-\frac{1}{2}},$$

and the final solution

$$t = \text{Const} \sqrt{\frac{mfv}{k}}.$$

This is exactly the same as the solution already obtained, on putting the dimensional constants f and v equal to unity, which of course was their value in the ordinary mechanical system of units.

Although this example gives no new results, it is instructive in showing that any system of fundamental units whatever is allowable, provided only that the dimensional constants required by the special problem are also introduced.

We now consider a problem in which it is an advantage to treat force as a unit of its own kind. This is the problem of Stokes of a small sphere falling under gravity in a viscous liquid. The sphere is so small that the motion is everywhere slow, so that there is nowhere turbulence in the fluid. The elements with which we have to deal in this problem are the velocity of fall, the density of the sphere, the diameter of the sphere, the density of the liquid, the viscosity of the liquid, and the intensity of gravity. The problem is evidently one in mechanics, so that if we use the ordinary mechanical units there will be no dimensional constants to introduce. But we notice that the problem is of a very special kind for a mechanical problem. The motion is slow, and the velocity is steady, the forces acting on the sphere and the liquid being everywhere held in equilibrium by the forces called out by the viscosity of the liquid. That is, although this is a problem involving motion, it is a problem involving unaccelerated motion, and the forces are in equilibrium everywhere. The problem is essentially, therefore, one in statics, and in solving the problem we need to make no use of the fact that in those cases where there happens to be an acceleration the force is proportional to mass times acceleration. In this problem, therefore, we treat force as its own kind of quantity, and do not have to introduce a compensating dimensional constant. Our analysis of the problem is now as follows:

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Velocity of fall,	v	LT^{-1}
Diameter of sphere,	D	L
Density of sphere,	d_1	ML^{-3}
Density of liquid,	d_2	ML^{-3}
Viscosity of liquid,	μ	$FL^{-2}T$
Intensity of gravity,	g	FM^{-1}

The dimensional formula of viscosity is obtained directly from its definition as force per unit area per unit velocity gradient. The intensity of gravity is taken with the dimensions shown, because obviously the equations of motion will not mention the accelerational aspect of gravitational action, but only the intensity of the force exerted by gravity upon unit mass.

We now have six variables, and four fundamental kinds of unit. There are, therefore, two dimensionless products. One of them is evident on inspection, and is d_2/d_1. Now of the remaining quantities we are especially interested in v. We need combine this with only four other quantities to obtain a dimensionless product. We choose D, d_1, μ, and g, and seek a dimensionless product of the form

$$v\, D^{\alpha}\, d_1^{\beta}\, \mu^{\gamma}\, g^{\delta}.$$

The exponents are at once found to be

$$\alpha = -2,\ \beta = -1,\ \delta = -1,\ \gamma = 1.$$

Hence the dimensionless products are

$$v\, D^{-2}\, d_1^{-1}\, \mu\, g^{-1} \text{ and } d_2/d_1,$$

and the final solution is

$$v = \frac{D^2\, d_1\, g}{\mu}\, f\left(\frac{d_2}{d_1}\right).$$

The function f is arbitrary, so that we cannot tell how the result depends on the densities of the sphere and the liquid, but we do see that the velocity of fall varies as the square of the diameter of the sphere, and the intensity of gravity, and inversely as the viscosity of the fluid.

This problem has of course long been solved by the methods of hydrodynamics, and the solution is

$$v = \frac{8}{9}\frac{g\,D^2}{\mu}(d_1 - d_2).$$

See, for example, Millikan, *Phys. Rev.*, 2, 110, 1913. The exact solution is obtained from the more general one above by giving $\frac{8}{9}\left(1 - \frac{d_2}{d_1}\right)$ as the special value of the function.

If we had solved this problem with the ordinary mechanical units, in which force is defined as mass times acceleration, we would have had three instead of two dimensionless products, and the final result would have been of the form

$$v = \frac{\mu}{d_1 D}\,\phi\left[\left(\frac{d_1}{\mu}\right)^2 D^3 g,\ \frac{d_2}{d_1}\right].$$

In this form we evidently can say nothing about the effect on the velocity of any of the elements taken by themselves, since they all occur under the arbitrary functional symbol.

There are many problems in which some specific information about the nature of the physical system enables the information given by dimensional analysis to be supplemented so that a more restricted form of the solution can be obtained than would be possible by dimensional analysis alone. There is, of course, no law against combining dimensional analysis with any information at our command.

Let us take as a simple example the discussion of the problem of the bending of a beam. This is a problem in elasticity. Let us endeavor to find how the stiffness of the beam depends on the dimensions of the beam, and any other quantities that may be involved. Now the equations of elasticity are equations of ordinary mechanics. The mechanical system of three fundamental units is indicated. The equations of elasticity from which the solution is to be obtained will involve the elastic constants of the material. If the material is isotropic, there will be two elastic constants, which may be chosen as Young's modulus, and the shear modulus. Our analysis may now run as follows:

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Stiffness (Force/deflection),	S	MT^{-2}
Length,	l	L
Breadth,	b	L
Depth,	d	L
Young's modulus,	E	$ML^{-1}T^{-2}$
Shear modulus,	μ	$ML^{-1}T^{-2}$

There are six variables, and three fundamental kinds of unit. Hence according to the general rule there should be three dimensionless products. Three such products can obviously be written down by inspection, and are

$$b/l,\ d/l,\ \text{and}\ \mu/E.$$

Now none of these dimensionless products contains the quantity S in which we are particularly interested, and it is evident that there is something peculiar about this problem. It will in fact be found, on going back to the system of algebraic equations on which the solution depends, and writing down the matrix of the coefficients obtained from the exponents in the dimensional formulas, that each of the three rowed determinants formed out of the matrix is zero. This is evident on inspection of the matrix.

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 & -1 \\ -2 & 0 & 0 & 0 & -2 & -2 \end{vmatrix}$$

This means that in this particular case there are more dimensionless products than are given by the general rule. That such is the case should have been evident beforehand. In the first place, an inspection of the dimensional formulas shows that M and T always enter in the combination MT^{-2}, so that this combination together might have been treated as a fundamental unit itself, so that there would have been only two fundamental units instead of three, and four instead of three dimensionless products. In the second place, this is a problem in statics, in which mass and time do not enter into the results. The dimensions of all the quantities could have been given in terms of force and length as the fundamental units. This remark is the physical equivalent of the analytical observation that M and T always occur in the combination MT^{-2} (force is MT^{-2} multiplied by L).

With the knowledge that there is still another dimensionless product, we can see by inspection that it is

$$S/El,$$

so that the final solution is

$$S = E\,l\,f\left(\frac{b}{l}, \frac{d}{l}, \frac{\mu}{E}\right).$$

This solution gives no information about the variation of stiffness with the dimensions of the beam. Now it is obvious from elementary considerations of elasticity, that for slender beams, the stiffness must be approximately proportional to the breadth, other things being equal, for the boundary conditions are such that the solution for a beam of twice the breadth may be obtained approximately by simply placing beside each other two of the original beams. Hence f must be of such a form that $l\,f\left(\frac{b}{l}, \frac{d}{l}, \frac{\mu}{E}\right)$ reduces to $b\,\phi\left(\frac{d}{l}, \frac{\mu}{E}\right)$ and the value of f must obviously be $\frac{b}{l}\,\phi\left(\frac{d}{l}, \frac{\mu}{E}\right)$. The restricted solution is therefore

$$S = E\,b\,\phi\left(\frac{d}{l}, \frac{\mu}{E}\right).$$

The solution now shows that a beam of double the length can be kept of the same stiffness by doubling the depth. The detailed solution of elasticity shows that the ratio of d to l enters as the cube, as a factor of proportionality, so that the stiffness is proportional directly to the cube of the depth, inversely to the cube of the length, directly as the breadth, and to some unknown function of the elastic constants.

This method of supplementing the results of dimensional analysis by other information will often be found of value. There are numerous examples in Lord Rayleigh's treatments. Rayleigh does not always separate the analysis into a dimensional and another part, but states that a result can be proved by dimensional analysis, although it may require supplementing in some such way as above. A good example will be found in his treatment of the scattering of light by the sky.[2] The result that the scattering varies inversely as the fourth power of the wave length of the incident light is obtained by using in addition to dimensional analysis the fact that "From what we know of the dynamics of the situation i (ratio of amplitude of incident and scattered light) varies directly as T (volume of scattering particle) and inversely as r (distance of point of observation from scattering particle)."

Thus far we have considered only problems in mechanics, but of course the method is not restricted to such problems, but can be applied to any system whose laws can be formulated in a form independent of the size of the fundamental units of measurement.

Let us consider, for example, a problem from the kinetic theory of gases, and find the pressure exerted by a perfect gas. The atoms of the gas in kinetic theory are considered as perfect spheres, completely elastic, and of negligible dimensions compared with their distance apart. The only constant with dimensions required in determining the behavior of each atom is therefore its mass. The behavior of the aggregate of atoms is also evidently characterized by the density of the gas or the number of atoms per unit volume. The problem is evidently one of mechanics, and the pressure exerted by the gas is to be found by computing the change of momentum per unit time and per unit area of the atoms striking the walls of the enclosure. The mechanical system of units is therefore indicated. But in addition to the ordinary mechanical features there is the element of temperature to be considered. How does temperature enter in writing down the equations of motion of the system? Obviously through the gas constant, which gives the average kinetic energy of each atom as a function of the temperature. Our analysis of the problem therefore runs as follows:

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Pressure exerted by gas,	p	$ML^{-1}T^{-2}$
Mass of the atom,	m	M
Number of atoms per unit of volume,	N	L^{-3}
Absolute temperature,	θ	θ
Gas constant per atom,	k	$ML^{2}T^{-2}\theta^{-1}$

We have here five variables, and four kinds of units. There is, therefore, one dimensionless product. Since p is the quantity in which we are interested, we choose the exponent of this as unity. We have to find

$$p\, m^{\alpha} N^{\beta} \theta^{\gamma} k^{\delta}.$$

The work of solution is as before. The values of the exponents are

$$\alpha = 0,\ \beta = -1,\ \gamma = -1,\ \delta = -1.$$

The dimensionless product is

$$p\,N^{-1}\,\theta^{-1}\,k^{-1},$$

and the final solution is

$$p = \text{Const}\ N\,k\,\theta.$$

That is, the pressure is proportional to the gas constant, to the density of the gas, and to the absolute temperature, and does not depend on the mass of the individual atoms. The formula for pressure is, of course, one of the first obtained in any discussion of kinetic theory, and differs from the above only in that the numerical value of the constant of proportionality is determined.

In this problem, or in other problems of the same type, we could, if we preferred, eliminate temperature as an independent kind of variable and define it as equal to the energy of the atom. This amounts merely to changing the size of the degree, but does not change the ratio of any two temperatures, and is the sort of change of unit which is required according to the fundamental assumptions. If we define temperature in this way, the gas constant is of course to be put equal to unity. We would now have three fundamental units, and four variables. There is of course again only one dimensionless product, and the same result would be obtained as before. Let us go through the work; it is instructive.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Pressure exerted by gas,	p	$ML^{-1}\,T^{-2}$
Mass of atom,	m	M
Number of atoms per unit volume,	N	L^{-3}
Absolute temperature,	θ	$ML^2\,T^{-2}$

We now have to find our dimensionless product in the form

$$p\,m^{\alpha}\,N^{\beta}\,\theta^{\gamma}.$$

The exponents are at once found to be

$$\alpha = 0,\ \beta = -1,\ \gamma = -1,$$

and the solution of the problem is

$$p = \text{Const}\ N\,\theta.$$

This solution is like the one obtained previously except for the presence of the gas constant, but since the gas constant in the new system of units is unity, the two solutions are identical, as they should be.

This procedure can obviously be followed in any problem whose solution involves the gas constant. Temperature may be either chosen as an independent unit, when the gas constant will appear explicitly as a variable, or temperature may be so defined that the gas constant is always unity, and temperature has the dimensions of energy. The same procedure is not incorrect in problems not involving the gas constant in the solution. But if in this class of problem temperature is defined as equal to the kinetic energy of an atom (or more generally equal to the energy of a degree of freedom) and the gas constant is made equal to unity, the fundamental units are restricted with no compensating advantage, so that although the results are correct as far as temperature is proportional to the energy of a degree of freedom, they will not give so much information as might have been obtained by leaving the units less restricted.

It is obvious that these remarks apply immediately to the heat transfer problem of Rayleigh treated in the introductory chapter.

Many persons feel an intuitive uncertainty with regard to the dimensions to be assigned to temperature. This is perhaps because of the feeling that a dimensional formula is a statement of the physical nature of the quantity as contained in the definition. Now the absolute temperature, as we have used it above, is the thermodynamic absolute temperature, defined with relation to the second law of thermodynamics. It is difficult to see how such a complex of physical operations as is involved in the use of the second law (such as Kelvin first gave in his definition of absolute temperature) can be reproduced in a simple dimensional formula. It is, however, evident that measurements of energy, for example, are involved in an application of the second law, so that perhaps in some way the ordinary mechanical units ought to be involved in the dimensional formula. But we have seen that the dimensional formula is concerned only with an exceedingly restricted aspect of the way in which the various physical operations enter the definition, namely with the way in which the numerical measure of a quantity changes when the fundamental units change in magnitude. Now a little reflection shows that any such procedure as that of Lord Kelvin applied to the definition of absolute temperature through the use of the second law imposes on the number measuring a given concrete temperature no restriction whatever in terms of the units in which heat or energy, for example, are measured. The size of a degree of thermodynamic temperature may be fixed entirely arbitrarily so that there are any number of degrees between the freezing and the boiling points of water, for example, absolutely without reference to the size of any other unit. We are concerned in the dimensional formula with the definition in terms of the second law only in so far as this definition satisfies the principle of the absolute significance of relative magnitude, that is, the principle that the ratio of the measures of two concrete examples shall be independent of the size of the units. Now it is evident that the thermodynamic definition of absolute temperature does leave the ratio of any two concrete temperatures independent of the size of the units. The dimensional formula of temperature, therefore, need contain no other element, and temperature may be treated as having its own dimensions.

There is no necessity in using the absolute thermodynamic temperature. We might, for instance, define the number of degrees in a given temperature interval as the number of units of length which the kerosene in a certain capillary projecting from a certain bottle of kerosene moves when the bottle is brought from one temperature to another. The temperature so defined evidently satisfies the principle of the absolute significance of relative magnitude, for if the size of the unit of length measured along the capillary is cut in half, the number of degrees in every temperature interval is doubled. The advantage of the thermodynamic scale is one of simplicity; in the kerosene scale the behavior of a perfect gas could not be characterized in terms of a single constant, and the Fourier equations of heat conduction could not be written, except over a very narrow range, in terms of a single coefficient of thermal conductivity.

Besides the question of the dimensions of temperature, there is one other question connected with the application of dimensional analysis to problems in thermodynamics which is apt to be puzzling; this is the matter of the so-called logarithmic constants. In books on thermodynamics equations are very common which on first sight do not appear to be complete equations or to be dimensionally homogeneous. These equations often involve constants which cannot change in numerical magnitude by some *factor* when the size of the fundamental units is changed, but must change by the *addition* of some term. An example will be found on page 6 of Nernst's Yale lectures on the *Applications of Thermodynamics to Chemistry.* This equation is:

$$\log C = -\frac{\lambda_0}{RT} + \frac{a}{R}\log T + \frac{b}{R}T + \frac{c}{2R}T^2 + i.$$

In this equation C is a concentration of a given gaseous substance, λ_0 is a heat, a, b, and c are dimensional coefficients in the usual sense into which we need not inquire further, except that a/R is dimensionless, and i is a constant of integration. It is obvious that this formula as it stands does not allow the size of the fundamental units to be changed by making the usual sort of changes in the various quantities. But a rearrangement of terms is possible which throws the formula into the conventional form. If we group together the terms log C, a/R log T, and i into the single term

$$\log \frac{C}{T^{\frac{a}{R}}\, i'}$$

where i' is a new constant, we evidently have a complete equation in the usual sense of the word, and i' has the dimensions of $C\,T^{-\frac{a}{R}}$.

This sort of rearrangement of terms is always possible if the formula has had a theoretical derivation, as have all the formulas of these treatises, and the logarithmic constant appears only as a formal exception.

The logarithmic constant is met so often in thermodynamic formulas because in most thermodynamic expressions there is an undetermined constant of integration arising from the fact that energy, or work, or entropy, or thermodynamic potential has no absolute significance, but is only the difference between two values, and the coördinates of the initial point which fix the origin of entropy, for example, may be chosen at pleasure.

The formulas of thermodynamics also often present a strange appearance in the way that concrete quantities (that is, quantities with dimensions) appear as the arguments of transcendental functions. Thus on page 5 of the same book of Nernst's, we find the formula.

$$\lambda = R\,T^2 \frac{d \log p}{d\,T}.$$

This comes from an application of Clapeyron's equation to a substance whose vapor obeys the perfect gas law, and the volume of whose vapor is large compared with that of the liquid. In spite of the appearance of a pressure under the logarithm sign, this equation is seen on examination to be a complete equation, and holds valid for all sizes of the fundamental units. This may at once be seen on expanding $\frac{d \log p}{dT}$, which is equal to $\frac{1}{p}\frac{dp}{dT}$, and is therefore of zero dimensions in p. Expressions of this sort in which the logarithm is taken of a quantity with dimensions are particularly common in thermodynamics, and often arise from the equations of the perfect gas. The occurrence of such logarithmic terms should, it seems to me, be difficult for those to interpret who like to regard a dimensional formula as expressing a concrete physical operation on a concrete physical thing.

That the occurrence of such expressions is not contrary to the Π theorem is seen from the expanded form, $\frac{1}{p}\frac{dp}{dT}$. The slope of the curve, $\frac{dp}{dT}$, would be one of the variables in which the dimensionless products are to be expressed, and there is evidently no exception. Our theorem has merely stated that the results *are expressible* in terms of dimensionless products; we have no reason to think that the man who derived the formula was accommodating enough to write the formula so that this would appear without some rearrangement of the terms.

Let us close this chapter of special examples with several electrical examples.

As the first example consider an electric circuit possessing capacity and inductance. An oscillatory discharge is excited in it. How does the period of the discharge depend on the constants of the circuit? The solution of this problem is to be obtained from the detailed equations of the electric circuit, written in the usual form, in electromagnetic units. None of the electrostatic effects of the current, or the interactions with a magnet, have to be considered in the equations, which are of the form

$$L\frac{di}{dt} + \int \frac{i\,dt}{c} = 0.$$

Hence in establishing the units fundamental to this equation it is evidently sufficient to consider only three fundamental kinds of quantity, namely, quantity of electricity, time, and energy. Then current is to be defined as quantity per unit time, coefficient of self-induction is such that when multiplied by half the square of the current it gives an energy, and similarly the capacity is such that it gives an energy when divided into the square of a quantity. We may now make our usual analysis of the problem.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Quantity of electricity,	q	Q
Current,	i	QT^{-1}
Coefficient of self-induction,	L	$Q^{-2}T^{2}E$
Capacity,	c	$Q^{2}E^{-1}$
Periodic time,	t	T

Now the time of oscillation might conceivably involve the constants of the circuit, and the initial charge of the condenser. That is, we are to look for a relation between q, L, c, and t. Since we are especially interested in t, we try to find a dimensionless product of the form

$$t\, L^{\alpha}\, c^{\beta}\, i^{\gamma}.$$

The exponents are at once found to be

$$\alpha = -\tfrac{1}{2},\ \beta = -\tfrac{1}{2},\ \gamma = 0,$$

giving as the solution of the problem

$$t = \text{Const}\ \sqrt{L\, c}.$$

This of course is the solution which would be found by actually solving the equations of the circuit, except for the value of the constant coefficient. It is to be noticed that the initial charge does not enter. This problem is evidently the electrical analogue of the mechanical problem of the simple pendulum.

It is perhaps worth noticing again that some knowledge of the nature of the solution is necessary before dimensional analysis can be used to advantage. The Australian bushman, when attacking this problem for the first time, might be tempted to look for a relation of the dimensions of a time between the constants of the circuit, and the instantaneous current, and the instantaneous charge in the condenser. If he had included i in his list of variables, he would have found that q/i also has the dimensions of a time, and his solution would have been of the form

$$t = \sqrt{Lc}\; f\left(\frac{i^2 L c}{q^2}\right),$$

which is not incorrect, since it reduces to the form already found on putting f equal to a constant, but it gives less information than the previous form.

We now consider a problem in electrostatics. The conception of the medium introduced by Faraday tells us that it is possible to regard the medium as the seat of the essential phenomena in the electrostatic field, and that the condition of the medium at any instant is uniquely determined by the electric vector at that point. Let us seek for the connection between the space density of energy in the electrostatic field and the intensity of the field. Since this is a problem in statics, the phenomena can be adequately described in terms of two fundamental units, those of force and length. Furthermore the field equations of electrostatics contain no dimensional constants, so that the velocity of light does not enter the results, as it did the problem of the mass of the spherical distribution of charge. In terms of the two fundamental units of force and length we may make our fundamental definitions as follows. Unit electrostatic charge is that charge which at distance of unity from an equal charge in empty space exerts on it a force of unity. The electric vector is that vector which when multiplied into the charge gives the force on the charge. The dielectric constant is the ratio of the force between two charges when in empty space, and when surrounded by the medium in question. The dimensions of dielectric constant are obviously zero. The dimensions of energy with this system of units are obviously force multiplied by distance.

We now formulate the problem.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Charge,	e	$F^{1/2}L$
Field strength,	E	$F^{1/2}L^{-1}$
Energy density,	u	FL^{-2}

We are to seek for a relation between E and u. Generally there would not be a relation between these quantities, because there are two fundamental quantities and two variables. But under the special conditions of this problem a relation exists, and the result is obvious on inspection to be

$$u = \text{Const}\ E^2.$$

In treatises on electrostatics the constant is found to be ½.

If instead of the energy density in empty space we had tried to find the energy density inside a ponderable body with dielectric constant ϵ, the above result would have been modified by the appearance of an arbitrary function of the dielectric constant as a factor. Dimensional analysis can give no information as to the form of the function. As a matter of fact, the function is equal to the dielectric constant itself.

This problem is instructive in showing the variety of ways in which it is possible to choose the fundamental units. Since the problem is one which may be reduced to formulation in mechanical terms (the definitions of electrical quantities are given immediately in terms of mechanical quantities) we might have used the ordinary three units of mechanics as fundamental, and written the dimensional formulas in terms of mass, length, and time. We would have obtained the following formulation.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Charge,	e	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$
Field strength,	E	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$
Energy density,	u	$ML^{-1}T^{-2}$

Again we are to seek for a relation between the energy density and the field strength. Now here we have two variables and three kinds of fundamental units, so that again the general rule would allow no dimensionless products, and no relation, but the relation between the exponents is such that the dimensionless product does exist, and in fact is found to be exactly the same as before. The new formulation in terms of different fundamental units does not change the result, as it should not.

Many persons will object to the dimensional formulas given for these electrostatic quantities on the ground that we arbitrarily put the dielectric constant of empty space equal to unity, whereas we know nothing about its nature, and therefore have suppressed certain dimensions which are essential to a complete statement of the problem. This point of view will of course not be disturbing to the reader of this exposition, who has come to see that there is nothing absolute about dimensions, but that they may be anything consistent with a set of definitions which agree with the experimental facts. However, let us by actual example carry through this problem, including the dielectric constant of empty space explicitly as a new kind of fundamental quantity which cannot be expressed in terms of mass, length, and time. Call the dielectric constant of empty space k, and use the same letter to stand for the quantity itself, and its dimension. Then the unit of electrostatic charge is now defined by the relation, force $= e^2/k\,r^2$. Field strength is to be defined as before as $eE =$ Force. If we formulate the problem in terms of these fundamentals, the electrostatic field equations will now contain k explicitly, so that the dimensional constant k appears in the list of variables. The formulation of the problem is now as follows:

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Charge,	e	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}k^{\frac{1}{2}}$
Field strength,	E	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}k^{-\frac{1}{2}}$
Energy density,	u	$ML^{-1}T^{-2}$
Dielectric constant of empty space,	k	k

We again look for a dimensionless product in which the terms are E, u, and k, and find the result to be

$$u = \text{Const}\; k\; E^2.$$

This reduces to that previously found on putting k equal to unity, which was the value of k in the previous formulation of the problem. The form above appears somewhat more general than the form previously obtained in virtue of the factor k, but this factor does not tell us anything more about nature, but merely shows how the formal expression of the result will change when we change the formulation of the definitions at the basis of our system of equations. The inclusion of the factor k in the result and in the definitions is therefore of no advantage to us, and never can be of advantage, if our considerations are correct.

There has been much written on the "true" dimensions of k, and much speculation about the various physical pictures of the mechanical structure of the ether which follow from one or another assumption as to the "true" dimensions, but so far as I am aware, no result has been ever suggested by this method which has led to the discovery of new facts, although it cannot be denied that a number of experiments have been suggested by these considerations, as for example those of Lodge on the mechanical properties of the ether.

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> "Some attention has lately been called to the question of the dimensions of the electromagnetic units, but the following suggestion seems to have escaped notice.
>
> "The electrostatic system of units may be defined as one in which the electric inductive capacity is *assumed* to have zero dimensions, and the electromagnetic system is one in which the magnetic inductive capacity is assumed to have zero dimensions. Now if we take a system in which the dimensions of both these quantities are the same, and of the dimensions of a slowness, *i.e.*, the inverse of a velocity (T/L), the two systems become identical, as regards dimensions, and differ only by a numerical coefficient, just as centimeters and kilometers do. There seems a naturalness in this result which justifies the assumption that these inductive capacities are really of the nature of a slowness. It seems possible that they are related to the reciprocal of the square root of the mean energy of turbulence of the ether."

CHAPTER VII

APPLICATIONS OF DIMENSIONAL ANALYSIS TO MODEL EXPERIMENTS. OTHER ENGINEERING APPLICATIONS

HITHERTO we have applied dimensional analysis to problems which could be solved in other ways, and therefore have been able to check our results. There are, however, in engineering practise a large number of problems so complicated that the exact solution is not obtainable. Under these conditions dimensional analysis enables us to obtain certain information about the form of the result which could be obtained in practise only by experiments with an impossibly wide variation of the arguments of the unknown function. In order to apply dimensional analysis we merely have to know what kind of a physical system it is that we are dealing with, and what the variables are which enter the equation; we do not even have to write the equations down explicitly, much less solve them. In many cases of this sort, the partial information given by dimensional analysis may be combined with measurements on only a part of the totality of physical systems covered by the analysis, so that together all the information needed is obtained with much less trouble and expense than would otherwise be possible. This method is coming to be of more and more importance in technical studies, and has recently received a considerable impetus from the necessities of airplane design. The method has received wide use at the National Physical Laboratory in England, and at the Bureau of Standards in this country, and has been described in numerous papers. At the Bureau of Standards Dr. Edgar Buckingham has been largely instrumental in putting the results of dimensional analysis into such a form that they may be easily applied, and in making a number of important applications.

The nature of the results obtainable by this method may be illustrated by a very simple example. Suppose that it is desired to construct a very large and expensive pendulum of accurately predetermined time of swing. Dimensional analysis shows that the time of oscillation of the entire system of all pendulums is given by the formula $t = \text{Const}\ \sqrt{l/g}$. Hence to determine the time of any pendulum whatever, it is sufficient to determine by experiment only the value of the constant in the equation. The constant may evidently be found by a single experiment on a pendulum of any length whatever. The experimental pendulum may be made as simple as we please, and by measuring the time of swing of it, the time of swing of the projected large pendulum may be obtained.

The case of the pendulum is especially simple in that no arbitrary function appeared in the result. Now let us consider the more general case which may be complicated by the appearance of an arbitrary function. Suppose that the variables of the problem are denoted by Q_1, Q_2, etc., and that the dimensionless products are found, and that the result is thrown into the form

$$Q_1 = Q_2^{\alpha_1} Q_3^{\beta_1} ---- f\left(Q_2^{\alpha_2} Q_3^{\beta_2} ----, Q_2^{\alpha_3} Q_3^{\beta_3} ----\right)$$

where the arguments of the function and the factor outside embrace all the dimensionless products, so that the result as shown is general. Now in passing from one physical system to another the arbitrary function will in general change in an unknown way, so that little if any useful information could be obtained by indiscriminate model experiments. But if the models are chosen in such a restricted way that all the arguments of the unknown function have the same value for the model as for the full scale example, then the only variable in passing from model to full scale is in the factors outside the functional sign, and the manner of variation of these factors is known from the dimensional analysis.

Two systems which are so related to each other that the arguments inside the unknown functional sign are equal numerically are said to be physically similar systems.

It is evident that a model experiment can give valuable information if the model is constructed in such a way that it is physically similar to the full scale example. The condition of physical similarity involves in general not only conditions on the dimensions of the model but on all the other physical variables as well.

As an example let us consider the resistance experienced by a body of some definable shape in moving through an infinite mass of fluid. Special cases of this problem are the resistance encountered by a projectile, by an airplane, by a submarine in deep water, or by a falling raindrop. The problem is evidently one of mechanics, and involves the equations of hydrodynamics. The conditions are exceedingly complicated, and would be difficult to formulate in precise mathematical terms, but we perhaps may imagine it done by some sort of a super-being. The important thing to notice is that no dimensional constants appear in the equation of hydrodynamics if the ordinary mechanical units in terms of mass, length, and time are used, so that the result will involve only the measurable physical variables. The variables are the resistance to the motion, the velocity of motion, the shape of the body, which we may suppose specified by some absolute dimension and the ratio to it of certain other lengths (as, for instance, the shape of an ellipsoid may be specified by the length of the longest axis and the ratio to this axis of the other axes) and the pertinent constants of the fluid, which are its density, viscosity, and compressibility, the latter of which we may specify by giving the velocity of sound in the fluid. We suppose that gravity does not enter the results, that is, the body is in uniform motion at a constant level, so that no work is done by the gravitational forces. The formulation of the problem is now as follows.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Resistance,	R	MLT^{-2}
Velocity,	v	LT^{-1}
Absolute dimension,	l	L
Density of fluid,	d	ML^{-3}
Viscosity of fluid,	μ	$ML^{-1}T^{-1}$
Velocity of sound in the fluid,	v′	LT^{-1}
Shape factors, fixing shape of body,	r_1, r_2, etc.	0

We have here six variables, not counting the shape factors, which may have any number depending on the geometrical complexity of the body, so that there are three dimensionless products exclusive of the shape factors, which are already dimensionless. One of these three dimensionless products is obvious on inspection, and is v′/v. We have to find the other dimensionless products in the way best adapted to this particular problem. Since we are interested in the resistance to the motion, we choose this as the term with unit exponent in one of the products, so that we may write the result with R standing alone on the left-hand side of the equation. We find by the methods that we have used so many times that there are two dimensionless products of the forms $R\,v^{-2}\,l^{-2}\,d^{-1}$ and $\mu\,v^{-1}\,l^{-1}\,d^{-1}$, so that the final solution takes the form

$$R = v^2\,l^2\,d\,f\,(\mu/v\,l\,d,\ v'/v,\ r_1,\ r_2, ----).$$

This formula is so broad as to cover a wide range of experimental conditions. If the velocity is low, the problem reduces to one of equilibrium in which the forces on the solid body immersed in the fluid are held in equilibrium by the forces due to the viscosity in the fluid. The resistance does not depend on the density of the fluid, nor on the velocity of sound in it. Evidently if the density is to disappear from the above result, the argument μ/vld must come outside the functional sign as a factor, and for slow motion the law of resistance takes the form

$$R = v\,l\,\mu\,f\,(r_1,\ r_2, ----).$$

The resistance at low velocities is therefore proportional to the viscosity, to the velocity, and to the linear dimensions, and besides this depends only on the geometrical shape of the body. We have already met a special case in the Stoke's problem of the sphere, in the introductory chapter.

For a domain of still higher velocities the density of the fluid plays an important part, since some of the force acting on the body is due to the momentum carried away from the surface of the body by the fluid in the form of eddies (and the momentum carried away obviously depends on the density of the fluid), but the velocity of sound has not yet begun to affect the result, which means that the fluid acts sensibly like an incompressible liquid. This is the realm of velocities of interest in airplane work. Under these conditions the argument v'/v drops out of the function, therefore, and the result becomes

$$R = v^2\,l^2\,d\,f\,(\mu/v\,l\,d,\ r_1,\ r_2, ----).$$

Let us stop to inquire how the information given by this equation can be used in devising model experiments. What we desire to do is to make a measurement of the resistance encountered by the model under certain conditions, and to infer from this what would be the resistance encountered by the full size example. It is in the first place obvious that the unknown function must have the same value for the model and the original. This means, since the function is entirely unknown, that all the arguments must have the same value for the model and the original. r_1, r_2, etc., must therefore be the same for both, or in other words, the model and the original must be geometrically of the same shape. Furthermore, μ/vld must have the same numerical value for both. If the model experiment is to be performed in air, as it usually is, μ and d are the same for the model and original, so that vl must be the same for model and original. That is, if the model is one-tenth the linear dimensions of the original, then its velocity must be ten times as great as that of the original. Under these conditions the formula shows that the resistance encountered by the model is exactly the same as that encountered by the original. Now this requirement imposes such difficult conditions to meet in practise, demanding velocities in the model of the order of thousands of miles per hour, that it would seem at first sight that we had proved the impossibility of model experiments of this sort. But in practise the function of $\mu/v\,l\,d$ turns out to have such special properties that much important information can nevertheless be obtained from the model experiment. If measurements are made on the resistance of the model at various speeds, and the corresponding values of the function calculated (that is, if the measured resistances are divided by $v^2\,l^2\,d$), it will be found that at high values of the velocity the function f approaches asymptotically a constant value. This means that at high velocities the resistance approaches proportionality to the square of the velocity. It is sufficient to carry the experiment on the model only to such velocities that the asymptotical value of the function may be found, in order to obtain all the information necessary about the behavior of the full scale example, for obviously we now know that the resistance is proportional to the square of the velocity, and the model experiment has given the factor of proportionality. The only doubtful point in this proposed procedure is the question as to whether it is possible to reach with the model speeds high enough to give the asymptotic value, and this question is answered by the actual experiment in the affirmative.

The fact that at the velocities of actual airplane work the resistance has become proportional to the square of the velocity means, according to the analysis, that the viscosity no longer plays a dominant part. This means that all the work of driving the airplane is used in creating eddies in the air. Independence of viscosity and complete turbulence of motion are seen by the analysis to be the same thing. This view of the phenomena is abundantly verified by experiment.

Let us consider the possibility of making model experiments in some other medium than air. If we choose water as the medium for the model we must so choose the dimensions and the velocity of the model that $\mu/v\,d\,l$ for the model is equal to $\mu/v\,d\,l$ for the original. Now for water μ is 10^{-2} and d is 1, whereas for air μ is 170×10^{-6} and d is 1.29×10^{-3}, all at ordinary temperatures. Substituting these values shows that vl for the model must be about one-thirteenth of the value for the original. As a factor one-thirteenth is itself about the reduction in size that would be convenient for the model; this would mean that the model in water must travel at about the same rate as the original in air. Such high velocities in water are difficult, and there seems no advantage in using water over the actual procedure that is possible in air.

Consider now still higher velocities, such as those of a projectile, which may be higher than the velocity of sound in the medium, so that the medium has difficulty in getting out of the way of the body, and we have a still different order of effects. At these velocities the viscosity has entirely disappeared from the result, which now takes the form

$$R = v^2\, l^2\, d\, f\,(v'/v, r_1, r_2, ----).$$

If we are now to make model experiments, it is evident that the model projectile must be of the same shape as the original, and furthermore that v'/v must have the same value for the model and the original. If the model experiment is made in air, v' for the model will be the same as for the original, so that v must be the same also. That is, the original and the model must travel at the same speed. Under these conditions the formula shows that the resistance varies as the area of cross section of the projectile. The requirement that the model must travel at the same speed as the original means in practise that the model experiments are made with actual projectiles, the model projectile being of smaller caliber than the proposed full size projectile.

We may try to avoid the difficulty by making the experiment in another medium, such as water. But the velocity of sound in water is of the order of five times that in air, so that the conditions would require that the velocity of the model projectile in water should be five times that of the actual projectile in air, an impossible requirement.

Besides these applications to model experiments, the results of dimensional analysis may be applied in other branches of engineering. At the Bureau of Standards extensive applications have been made in discussing the performance of various kinds of technical instruments. A class of instruments for the same purpose have certain characteristics in common so that it is often possible to write down a detailed analysis applicable to all instruments of the particular type. Dimensional analysis gives certain information about what the result of such an analysis must be, so that it is possible to make inferences from the behavior of one instrument concerning the behavior of other instruments of somewhat different construction. This subject is treated at considerable length and a number of examples are given in Aeronautic Instruments Circular No. 30 of the Bureau of Standards, written by Mr. M. D. Hersey.

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CHAPTER VIII

APPLICATIONS TO THEORETICAL PHYSICS

THE methods of dimensional analysis are worthy of playing a much more important part as a tool in theoretical investigation than has hitherto been realized. No investigator should allow himself to proceed to the detailed solution of a problem until he has made a dimensional analysis of the nature of the solution which will be obtained, and convinced himself by appeal to experiment that the points of view embodied in the underlying equations are sound.

Probably one difficulty that has been particularly troublesome in theoretical applications has been the matter of dimensional constants; it is in just such theoretical investigations that dimensional constants are most likely to appear, and with no clear conception of the nature of a dimensional constant or when to expect its appearance, hesitancy is natural in applying the method. But after the discussion of the preceding pages, the matter of dimensional constants should now be readily handled in any special problem.

The indeterminateness of the numerical factors of proportionality is often also felt to be a disadvantage of the dimensional method, but in many theoretical investigations it is often possible to obtain approximate information about the numerical order of magnitude of the results. Our considerations with regard to dimensional analysis show that any numerical coefficients in the final result are the result of mathematical operations performed on the original equations of motion (in the general sense) of the system. Now it is a result of general observation that such mathematical operations usually do not introduce any very large numerical factors, or any very small ones. Any very large or small numbers in our equations almost always are the result of the substitution of the numerical value of some physical quantity, such as the number of atoms in a cubic centimeter, or the electrostatic charge on the electron, or the velocity of light. Accordingly, if the analysis is carried through with all the physical quantities kept in literal form, we may expect that the numerical coefficients will not be large or small.

This observation may be used conversely. Suppose that we suspect a connection between certain quantities, but as yet do not know enough of the nature of the physical system to be able to write down the equations of connection, or even to be sure what would be the elements which would enter an equation of connection. We assume that there is a relation between certain quantities, and then by a dimensional analysis find what the form of the relation must be. We then substitute into the relation the numerical values of the physical quantities, and thus get the numerical value of the unknown coefficient. If this coefficient is of the order of unity (which may mean not of the order of 10^{10} according to our sanguinity) the suspected relation appears as not intrinsically improbable, and we continue to think about the matter to discover what the precise relation between the elements may be. If, on the other hand, the coefficient turns out to be large or small, we discard the idea as improbable.

An exposition of this method, and an interesting example were given by Einstein[1] in the early days of the study of the specific heats of solids and their connection with quantum phenomena. The question was whether the same forces between the atoms which determine the ordinary elastic behavior of a solid might not also be the forces concerned in the infra red characteristic optical frequencies. This view evidently had important bearings on our whole conception of the nature of the forces in a solid, and the nature of optical and thermal oscillations.

For the rough analysis of the problem in these terms we may regard the solid as an array of atoms regularly spaced at the corners of cubes. In our analysis we shall evidently want to know the mass of the atoms, and their distance apart (or the number per cm^3). Furthermore, if our view of the nature of the forces is correct the nature of the forces between the atoms is sufficiently characterized by an elastic constant, which we will take as the compressibility. These elements should now be sufficient to determine the infra red characteristic frequency. We make our usual analysis of the problem.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Characteristic frequency,	ν	T^{-1}
Compressibility,	k	$M^{-1}LT^2$
Number of atoms per cm^3,	N	L^{-3}
Mass of the atom.	m	M

There should be one dimensionless product in these quantities, and it is at once found to be $k \nu^2 N^{\frac{1}{3}} m$. The final result is therefore

$$k = \text{Const}\, \nu^{-2} N^{-\frac{1}{3}} m^{-1}.$$

We now take the numerical values pertaining to some actual substance and substitute in the equation to find the numerical value of the coefficient. For copper, $k = 7 \times 10^{-13}$, $\nu = 7.5 \times 10^{12}$, $m = 1.06 \times 10^{-22}$, and $N = 7.5 \times 10^{22}$. Substituting these values gives for the constant 0.18. This is of the order of unity, and the point of view is thus far justified. It is of course now a matter of history that this point of view is the basis of Debye's analysis of the specific heat phenomena in a solid, and that it is brilliantly justified by experiment.

Another example of this sort of argument concerning the magnitude of the constants is given by Jeans.[2] The question was whether the earth has at any time in its past history passed through a stage of gravitational instability, and whether this instability has had any actual relation to the course of evolution. A preliminary examination by the method of dimensions showed Jeans what must be the form of the relation between the variables such as mean density, radius, elastic constants, etc., at the moment of gravitational instability, and then a substitution of the numerical values for the earth gave a coefficient of the order of unity. This preliminary examination showed, therefore, that it was quite conceivable that gravitational instability might be a factor at some time past or future in the earth's history, and a more detailed examination of the problem was accordingly undertaken.

Consider another application of the same argument, this time with a negative result. Let us suppose that we are trying to construct an electrodynamic theory of gravitation, and that we regard the gravitational field as in some way, as yet undiscovered, connected with the properties of the electron, to be deduced by an application of the field equations of electrodynamics. Now in the field equations there occurs a dimensional constant c, the ratio of the electromagnetic and electrostatic units, which is known to be of the dimensions of a velocity, and numerically to be the same as the velocity of light.

In searching for a relation of the sort suspected, we therefore consider as the variables the charge on the electron, the mass of the electron (for charge and mass together characterize the electron), the velocity of light, and the gravitational constant. We make the following analysis:

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Gravitational constant,	G	$M^{-1}L^{3}T^{-2}$
Charge on electron,	e	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$
Mass of electron,	m	M
Velocity of light,	c	LT^{-1}

We have four variables, and three fundamental units, so that we expect one dimensionless product. This is at once found to be Gm^2e^{-2}, the velocity of light not entering the hypothetical relation, and the final result taking the significantly simple form

$$G = \text{Const } (e/m)^2.$$

We now substitute numerical values to find the magnitude of the constant. $G = 6.658 \times 10^{-8}$, and $e/m = 5.3 \times 10^{17}$, so that Const $= 2.35 \times 10^{-43}$.

The constant is seen therefore to be impossibly small, and we give up the attempt to think how there might be a relation between these quantities, although the simplicity of the dimensional relation between G and e/m is arresting.

Identity of dimensional formulas must not be thought, therefore, to indicate an *a priori* probability of any sort of physical relation. When there are so many kinds of different physical quantities expressed in terms of a few fundamental units, there cannot help being all sorts of accidental relations between them, and without further examination we cannot say whether a dimensional relation is real or accidental. Thus the mere fact that the dimensions of the quantum are those of angular momentum does not justify us in expecting that there is a mechanism to account for the quantum consisting of something or other in rotational motion.

The converse of the theorem attempted above does hold, however. If there is a true physical connection between certain quantities, then there is also a dimensional relation. This result may be used to advantage as a tool of exploration.

Consider now a problem showing that any true physical relation must also involve a dimensional relation. Suppose that we are trying to build up a theory of thermal conduction, and are searching for a connection between the mechanism of thermal conduction and the mechanism responsible for the ordinary thermodynamic behavior of substances. The thermodynamic behavior may be considered as specified by the compressibility, thermal expansion, specific heat (all taken per unit volume), and the absolute temperature. If only those aspects of the mechanism which are responsible for the thermodynamic behavior are also effective in determining the thermal conductivity, then it must be possible to find a dimensional relation between the thermodynamic elements and the thermal conductivity. We have the following formulation of the problem.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Thermal conductivity,	μ	$MLT^{-3}\theta^{-1}$
Compressibility per unit mass,	k	$M^{-2}L^{4}T^{2}$
Thermal expansion per unit mass,	λ	$M^{-1}L^{3}\theta^{-1}$
Specific heat per unit mass,	C	$L^{2}T^{-2}\theta^{-1}$
Absolute temperature,	θ	θ

We are to seek for a dimensionless product in these variables. There are five variables, and four fundamental kinds of quantity, so that we would expect one dimensionless product. Since μ is the quantity in which we are particularly interested, we choose it as the member of the product with unity for the exponent, write the product in the form

$$\mu\, k^{\alpha}\, \lambda^{\beta}\, C^{\gamma}\, \theta^{\delta},$$

and attempt to solve for the exponents in the usual way. We soon encounter difficulties, however, for it appears that the equations are inconsistent with each other. This we verify by writing down the determinant of the exponents in the dimensional formulas for k, λ, C, and θ. The determinant is found to vanish, which means that the dimensionless product does not exist. Hence the hypothetical relation between thermal conductivity and thermodynamic data does not exist, and the mechanism of the solid must have other properties than those sufficient to account for the thermodynamic data alone.

We now give a simple discussion of the problem of radiation from a black body. A much more elaborate discussion has been given by

Jeans.[3] The paper of Jeans is also interesting because he uses a system of electrical units in which the dielectric constant of empty space is introduced explicitly. It is easy to see on a little examination that he would have obtained the same result with a system of units in which the dielectric constant of empty space is defined as unity.

Let us consider a cavity with walls which have absolutely no specific properties of their own, but are perfect reflectors of any incident radiation. Inside the cavity is a rarefied gas composed of electrons. If the gas is rarefied enough we know from such considerations as those given by Richardson in considering thermionic emission that the electrons function like a perfect gas, the effect of the space distribution of electrostatic charge being negligible in comparison with the forces due to collisions as gas particles. The electron gas in the cavity is to be maintained at a temperature θ. The electrons are acted on by two sets of forces; the collisional forces with the other electrons, which are of the nature of the forces between atoms in ordinary kinetic theory, and the radiational field in the ether. Since the electrons are continually being accelerated, they are continually radiating, and they are also continually absorbing energy from the radiational field of the ether. The system must eventually come to equilibrium with a certain energy density in the ether, the electrons possessing at the same time the kinetic energy appropriate to gas atoms at the temperature of the enclosure. The detailed solution of the problem obviously involves a most complicated piece of statistical analysis, but a dimensional analysis gives much information about the form of the result.

In solving this problem we shall have to use the field equations of electrodynamics, so that the velocity of light will be a dimensional constant in the result. The charge and the mass of the electron must be considered, the absolute temperature, and the gas constant, because this determines the kinetic energy of motion of the electrons as a function of temperature. The number of electrons per cm^3 does not enter, because we know from kinetic theory that the mean velocity of the electrons is independent of their number. The second law of thermodynamics also shows that the energy density in the enclosure is a function of the temperature, and not of the density of the electron gas.

Our formulation of the problem is now as follows:

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Energy density,	u	$ML^{-1}T^{-2}$
Velocity of light,	c	LT^{-1}
Mass of electron,	m	M
Charge of electron,	e	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$
Absolute temperature,	θ	θ
Gas constant,	k	$ML^2T^{-2}\theta^{-1}$

The ordinary electrostatic system of units is used. There are here six variables and four fundamental kinds of unit, hence two dimensionless products, unless there should be some special relation between the exponents. Since we are especially interested in u we choose this as the member of one of the products with unit exponent. We find in the usual way that two products are

$$u\, e^6\, k^{-4}\, \theta^{-4} \text{ and } k\, \theta\, m^{-1}\, c^{-2}.$$

The result therefore takes the form

$$u = k^4\, e^{-6}\, \theta^4\, f\, (k\, \theta\, m^{-1}\, c^{-2}).$$

We as yet know nothing of the nature of the arbitrary function. The argument of the function, however, is seen to have a definite physical significance. $k\theta m^{-1}$ is half the square of the velocity of the electron ($k\theta$ being its kinetic energy), so that the argument is one-half the square of the ratio of the velocity of the electron to the velocity of light. Now this quantity remains exceedingly small in the practical range of temperature, so that whatever the form of the function, we know that we have a function of a quantity which is always small. By an extension of the reasoning which we employed for the numerical value of any coefficients to be met with in dimensional analysis, we may say that the probability is that the numerical value of such a function is sensibly the same as its value for the value zero of the argument, that is, the function may be replaced by a constant for the range of values of the variable met with in practise. Hence with much plausibility we may expect the result to be of the form

$$u = \text{Const}\, k^4\, e^{-6}\, \theta^4.$$

θ is the only physical variable on the right-hand side of this equation, so that as far as physical variables go the result may be written in the form

$$u = a\, \theta^4.$$

This, of course, is the well-known Stefan's law, which checks with experiment. The result therefore justifies to a certain extent the views which led us to the result.

Our argument about the size of numerical coefficients would lead us to expect that the constant in the first form of the result could not be too large or too small. That is, if we put $a = \text{Const } k^4 e^{-6}$, the result should have a certain simplicity of form, such as might seem to be a plausible result of a mathematical operation. Now Lewis and Adams[4] have called attention to the fact that within the limits of experimental error the constant of Stefan's law may be written in the form

$$a = k^4/(4\pi e)^6.$$

Although $(4\pi)^6$ is not an especially small number in the sense of the original formulation by Einstein of the probability criterion for numerical coefficients, it is nevertheless to be regarded as small considering the size of the exponents of the quantities with which it is associated, and it is undeniable that the result is of such simplicity that it seems probable that the coefficient may be the result of a mathematical process, and is not merely due to a chance combination of elements in a dimensionally correct form.

At any rate, whatever our opinions as to the validity of the argument, the striking character of the result sticks in our minds, and we reserve judgment until the final solution is forthcoming, in the same way that the periodic classification of the elements had to be carried along with suspended judgment until the final solution was forthcoming. It may be mentioned that Lorentz and his pupils have tried a detailed analysis on these terms, with unsuccessful results.

The above analysis gives other opportunities for thought. It is significant that the quantum h does not enter the result, although it appears to be inseparably connected with the radiation processes, at least in ponderable matter. We know that h enters the formula for the spectral distribution of energy, and we also know from thermodynamics that the distribution of energy throughout the spectrum in a cavity such as the above is the same as the distribution in equilibrium with a black body composed of atoms. The spectral distribution in the cavity which we have been considering must therefore involve h. Does this mean that h can be determined in terms of the electronic constants, the gas constant, and the constants of the ether, so that no mechanism with which we are not already familiar is needed to account for h? Of course Lewis has used Planck's formula for h in terms of a, etc., in order to obtain a numerical value for h in terms of other quantities.

As an additional example of the application of dimensional analysis in theoretical investigations let us examine the possibility of explaining the mechanical behavior of substances on the basis of a particular form of the law of force between atoms. We suppose that the law of force can be written in the form

$$F = A\,r^{-2} + B\,r^{-n}.$$

A is to be intrinsically negative, and represents an attractive force, and B is positive, and represents a force of repulsion which becomes very intense on close approach of the atoms. The atoms of different substances may differ in mass and in the numerical value of the coefficients A and B, but the exponent n is to be the same for all substances. We also suppose that the temperature is so high that the quantum h plays no important part in the distribution of energy among the various degrees of freedom, but that the gas constant is sufficient in determining the distribution. The external variables which may be imposed on the system are the pressure and the temperature. When these are given the volume is also determined, and all the other properties. We have, therefore, the following list of quantities in terms of which any of the properties of the substance are to be determined.

Name of Quantity.	*Symbol.*	*Dimensional Formula.*
Pressure,	p	$ML^{-1}T^{-2}$
Temperature,	θ	θ
Mass of the atom,	m	M
Gas constant,	k	$ML^{2}T^{-2}\theta^{-1}$
A (of the law of force),	A	$ML^{3}T^{-2}$
B (of the law of force),	B	$ML^{n+1}T^{-2}$

In addition to these we will have whatever particular property of the substance is under discussion. In the above list there are six quantities in terms of four fundamental units. Therefore from this list of permanent variables there are two dimensionless products. Let us find them. We will choose one involving p and not θ, and the other θ and not p, since p and θ are the physical variables under our control. The products are at once found to be

$$p\, A^{-\frac{n+2}{n-2}}\, B^{\frac{4}{n-2}}$$

and

$$A^{-\frac{n-1}{n-2}}\, B^{\frac{1}{n-2}}\, k\, \theta.$$

The existence of these two products already gives us information about the behavior of the body in those cases in which pressure and temperature are not independently variable quantities, as they are not on the vapor pressure curve, or on the melting curve, or on the curve of equilibrium between two allotropic modifications of the solid. Under these conditions we have

$$p\, A^{-\frac{n+2}{n-2}}\, B^{\frac{4}{n-2}} = f\left(A^{-\frac{n-1}{n-2}}\, B^{\frac{1}{n-2}}\, k\, \theta\right),$$

where f is the same function for all substances. A and B vary from substance to substance. Hence this analysis shows that in terms of a new variable $p\, C_1$ for the pressure, and a new variable $\theta\, C_2$ for temperature, the equations for the equilibrium curves of all substances are the same. These new pressure and temperature variables are obtained by multiplying the ordinary pressure and temperature by constant factors, and may be called the reduced pressure and temperature. van der Waal's equation is a particular case of such an equation, which becomes the same for all substances in terms of the reduced variables.

Now consider any other physical property of the substance which is to be accounted for in terms of the variables of the analysis above. We have to form another dimensionless product in which it is involved. This dimensionless product may most conveniently be expressed in terms of the quantities m, k, A, and B, since these are physically invariable for the particular substance. The expression of any physical quantity is always dimensionally possible in terms of these quantities, unless the determinant of the exponents of m, k, A, and B vanishes, and this is seen to vanish only in the case $n = +2$, which is the trivial case of the force reducing to an attraction alone. Hence in the general case any physical property, which we may call Q, may be expressed in the form

$$Q = \text{Const}\ F\left(p\, A^{-\frac{n+2}{n-2}}\, B^{\frac{4}{n-2}},\ A^{-\frac{n-1}{n-2}}\, B^{\frac{1}{n-2}}\, k\, \theta\right),$$

where the Const may involve m, k, A, and B in any way, but does not involve p or θ. Now if we define Q/Const as the "reduced" value of Q, then we have the important result that for all substances of this type the equation connecting the reduced value of a quantity Q with the reduced pressure and temperature is the same. This applies not only to thermodynamic properties, but to all properties which are to be explained in terms of the same structure, such as thermal conductivity or viscosity.

The values of the factors by means of which the measured values of the physical variables are converted to "reduced" values will enable us to compute A, B, and m for the substance in question, if n can be otherwise determined.

It is evident on consideration of the above work that the only assumption which we have made about n is that it is dimensionless, and that we have not used the assumption stated in the beginning that n is the same for all substances. We may therefore drop this assumption, and have the theorem that for all substances whose behavior can be determined in terms of atoms which are characterized by a mass and a law of force of the form $Ar^{-2} + Br^{-n}$, with no restriction on A, B, or n, there is a law of corresponding states for all physical properties.

Evidently it would be possible to carry through an analysis like the above in which the external variables p and θ are replaced by any other two which might be convenient, such as certain of the thermodynamic potentials, and the same result would have been obtained, unless there should happen to be special relations between the dimensional exponents. Whether there are such special relations can be easily determined in any special case.

Before anyone starts on a detailed development of such a theory of the structure of matter as this, he would make a preliminary examination to see whether the properties of substances do actually obey such a law of corresponding states, and govern his future actions accordingly. The value of the advance information obtained in this way is incontestable.

The analysis above reminds one in some particulars of that of Meslin,[5] but is much more general, in that the analysis of Meslin applied only to the equation of state, and had to assume the existence of critical, or other peculiar points.

As a final application of dimensional analysis to theoretical physics we consider the determination of the so-called absolute systems of units.

The units in ordinary use are ones whose absolute size is fixed in various arbitrary ways, although the relations between the different sorts of units may have a logical ring. Thus the unit of length, the centimeter, was originally defined as bearing a certain relation to a quadrant of the earth's circumference, and the unit of mass is the mass of a quantity of water occupying the unit volume. There is something entirely arbitrary in selecting the earth and water as the particular substances which are to fix the size of the units.

We have also met in the course of our many examples dimensional constants. These constants usually are connected with some proportionality factor which enters into the expression of a law of nature empirically discovered. Such dimensional constants are the constant of gravitation, the velocity of light, the quantum, the constant of Stefan's law, etc. Now the numerical magnitude of the dimensional constants depends on the size of the fundamental units in a way fixed by the dimensional formulas. By varying the size of the fundamental units, we may vary in any way that we please the numerical magnitude of the dimensional constant. In particular, by assigning the proper magnitudes to the fundamental units we might make the numerical magnitudes of certain dimensional constants equal to unity. Now the dimensional constants are usually the expression of some *universal* law of nature. If the fundamental units are so chosen in size that the dimensional constants have the value unity, then we have determined the size of the units by reference to universal phenomena instead of by reference to such restricted phenomena as the density of water at atmospheric pressure at some fixed temperature, for instance, and the units to that extent are more significant.

There is no reason why one should be restricted to dimensional constants of universal occurrence in fixing the size of the units, but any phenomenon of universal occurrence may be used. Thus the units may be so chosen that the charge on the electron is unity.

Any system of units fixed in this way by reference to phenomena or relationships of universal occurrence and significance may be called an absolute system of units. The first system of absolute units was given by Planck[6] in his book on heat radiation. He connected the particular system which he gave with the quantum, and it might appear from Planck's treatment that before the discovery of the quantum there were not enough dimensional constants of the proper character known to make possible a universal system of units, but such is not the case. Planck was the first to think of the possibility of absolute units, and used the quantum in determining them, but there is no necessary connection with the quantum, as may be seen in the following discussion.

Let us now determine from the dimensional formulas the set of absolute units given by Planck. To fix this set of units we choose the constant of gravitation, the velocity of light, the quantum, and the gas constant. We require that the fundamental units be of such a size that each of these dimensional constants has the value unity in the new system. The discussion may be simplified for the present by omitting the gas constant, for this is the only one which involves the unit of temperature, and it is obvious that after the units of mass, length, and time have been fixed, the gas constant may be made unity by properly choosing the size of the degree. In determining the size of the new units we find it advantageous to choose the form of notation used in the third chapter in changing units. Consider, for example, the constant of gravitation. We write this as

$$\text{Constant of gravitation} = G = 6.658 \times 10^{-8}\ \text{gm}^{-1}\ \text{cm}^{3}\ \text{sec}^{-2}.$$

The value in the new system of units is to be found by substituting in the expression for G the value of the new units in terms of the old. Thus if the new unit of mass is such that it is equal to x gm, and the new unit of length is equal to y cm, and the new unit of time to z sec, we shall have as the equation to determine x, y, and z, since the numerical value of the gravitational constant is to be unity in the new system

$$6.658 \times 10^{-8}\ \text{gm}^{-1}\ \text{cm}^{3}\ \text{sec}^{-2} = 1\ (x\ \text{gm})^{-1}\ (y\ \text{cm})^{3}\ (z\ \text{sec})^{-2}.$$

The other two dimensional constants give the two additional equations needed to determine x, y, and z. These other equations are immediately written down as soon as the dimensional formulas and the numerical values of the velocity of light and the quantum are known. The equations are

$$3 \times 10^{10}\ \text{cm sec}^{-1} = 1\ (y\ \text{cm})\ (z\ \text{sec})^{-1}$$
$$6.55 \times 10^{-27}\ \text{gm cm}^{2}\ \text{sec}^{-1} = 1\ (x\ \text{gm})\ (y\ \text{cm})^{2}\ (z\ \text{sec})^{-1}.$$

This set of three equations may be readily solved, and gives x =

5.43×10^{-5}, $y = 4.02 \times 10^{-33}$, and $z = 1.34 \times 10^{-43}$. This means that

the new unit of mass is 5.43×10^{-5} gm
the new unit of length 4.02×10^{-33} cm
the new unit of time 1.34×10^{-43} sec.

So far all is plain sailing, and there can be no question with regard to what has been done. The attempt is sometimes made to go farther and see some absolute significance in the size of the units thus determined, looking on them as in some way characteristic of a mechanism which is involved in the constants entering the definition. Thus Eddington[7] says: "There are three fundamental constants of nature which stand out preëminently, the velocity of light, the constant of gravitation, and the quantum. From these we can construct a unit of length whose value is 4×10^{-33} cm. There are other natural units of length, the radii of the positive and negative charges, but these are of an altogether higher order of magnitude. With the possible exception of Osborne Reynold's theory of matter, no theory has attempted to reach such fine grainedness. But it is evident that this length must be the key to some essential structure."

Speculations such as these arouse no sympathetic vibration in the convert to my somewhat materialistic exposition. The mere fact that the dimensional formulas of the three constants used was such as to allow a determination of the new units in the way proposed seems to be the only fact of significance here, and this cannot be of much significance, because the chances are that any combination of three dimensional constants chosen at random would allow the same procedure. Until some essential connection is discovered between the mechanisms which are accountable for the gravitational constant, the velocity of light, and the quantum, it would seem that no significance whatever should be attached to the particular size of the units defined in this way, beyond the fact that the size of such units is determined by phenomena of universal occurrence.

Let us now continue with our deduction of the absolute units, and introduce the gas constant. For this we have the equation

$$\begin{aligned} \text{Gas constant} = k &= 2.06 \times 10^{-16} \text{ gm cm}^2 \text{ sec}^{-2}\, \theta^{-1} \\ &= 1\,(x \text{ gm})\,(y \text{ cm})^2\,(z \text{ sec})^{-2}\,(w\theta)^{-1}. \end{aligned}$$

x, y, and z are already determined, so that this is a single equation to determine w. The value found is 2.37×10^{32}. This means that the new degree must be equal to 2.37×10^{32} ordinary Centigrade degrees.

In the wildest speculations of the astrophysicists no such temperature has ever been suggested, yet would Professor Eddington maintain that this temperature must be the key to some fundamental cosmic phenomenon?

It must now be evident that it is possible to get up systems of absolute units in a great number of ways, depending on the universal constants or phenomena whose numerical values it is desired to simplify. With any particular selection of constants, the method in general is the same as that in the particular case above. In general there will be four fundamental kinds of unit, if we want to restrict ourselves to the electrostatic system of measuring electrical charges, and define the magnitude of the charge in such a way that the force between two charges is equal to their product divided by the square of the distance between them, or if we do not restrict ourselves to the electrostatic system, there may be five fundamental kinds of quantity. There seems to be nothing essential in the number five, which merely arises because we usually find it convenient to use the mechanical system of units in which the constant of proportionality between force and the product of mass and acceleration is always kept fixed at unity. The convenience of this system is perhaps more obvious in the case of mechanical phenomena, because of the universality of their occurrence. But if temperature effects were as universal and as familiar to us, we would also insist that we always deal only with that system of units in which the gas constant has the fixed value unity.

Having, therefore, fixed the number of fundamental units which we deem convenient, and having chosen the numerical constants whose values we wish to simplify, we proceed as above. It is evident that it will in general be necessary to assign as many constants as there are fundamental units, for otherwise there will not be enough equations to give the unknowns. Thus above, we fixed four constants, gravitational, velocity of light, quantum, and gas constant, and we had four fundamental kinds of units. Now it is important to notice that four algebraic equations in four unknowns do not always have a solution, but the coefficients must satisfy a certain condition. This condition is, when applied to the dimensional formulas into which the unknowns enter, that the determinant of the exponents must not vanish. In general, a four-rowed determinant selected at random would not be expected to vanish. In the case of the determinants obtained from the dimensional formulas of the constants of nature this is not the case, however, because the dimensional formulas are nearly all of them of considerable simplicity, and the exponents are nearly always small integers. It very often happens that the determinant of the exponents of four constants chosen at random vanishes, and the proposed scheme for determining the absolute units turns out to be impossible. The vanishing of the determinant means that all the quantities are not dimensionally independent, so that we really have not four but a smaller number of independent quantities in terms of which to determine the unknowns. For instance, we have found that the gravitational constant dimensionally has the same formula as the square of the ratio of the charge to the mass of the electron. This means that we could not set up a system of absolute units in which the gravitational constant, the charge on the electron, and the mass of the electron were all equal to unity. Now let us write down some of the important constants of nature and see what are the possibilities in the way of determining systems of absolute units.

Gravitation constant,	G	6.658×10^{-8} gm^{-1} cm^3 sec^{-2}
Velocity of light,	c	3×10^{10} cm sec^{-1}
Quantum,	h	6.547×10^{-27} gm-cm^2 sec^{-1}
Gas constant,	k	2.058×10^{-16} gm cm^2 sec^{-2} °C^{-1}
Stefan constant,	a	7.60×10^{-15} gm cm^{-1} sec^{-2} °C^{-4}
First spectral constant,	C	$0.353 \times$ gm cm^4 sec^{-3}
Second spectral constant,	a′	1.431 cm °C
Rydberg constant,	R	3.290×10^{15} sec^{-1}
Charge of the electron,	e	4.774×10^{-10} gm$^{\frac{1}{2}}$ cm$^{\frac{3}{2}}$ sec^{-1}
Mass of the electron,	m	8.8×10^{-28} gm
Avogadro number,	N	6.06×10^{23} gm^{-1}
Second Avogadro number,	N′	7.29×10^{15} gm^{-1} cm^{-2} sec^2 °C

Some of the quantities in the above list require comment. The Stefan constant "a" is defined by the relation $u = a\theta^4$, where u is the energy density in the hohlraum in equilibrium with the walls at temperature θ. The first and second spectral constants are the constants in the formula

$$E_\lambda = \frac{C}{\lambda^5}\left[e^{\frac{a}{\lambda\theta}} - 1\right]$$

for the distribution of energy in the spectrum. The Avogadro number N is defined as the number of molecules per gm molecule, and its dimensions may be obtained from the formula for it; N = (no. of molecules per gm) $\times$ (mass of molecule / mass of hydrogen molecule). Its dimensions are evidently the reciprocal of a mass, and the numerical value is merely the reciprocal of the mass of the hydrogen molecule. The second Avogadro number N′ is defined as the number of molecules per cm^3 in a perfect gas at unit temperature and at unit pressure. We know that this number is independent of the particular gas, and is therefore suited to be a universal constant. Its dimensions are evidently those of vol^{-1} $pressure^{-1}$ temp, and the numerical value may be found at once in terms of the other constants.

We have now a list of twelve dimensional constants in terms of which to define an absolute system of units. Since these constants are defined in that system in which there are four fundamental kinds of unit, in general any four of the twelve would suffice for determining the absolute system of units, but the relations are so simple that there are a large number of cases in which the determinant of the exponents vanishes, and the choice is not possible. For instance, C has dimensionally the same formula as hc^2, so that no set of four into which C, h, and c all enter is a possible set. k has the dimensions of cha'^{-1}, so that the set k, c, h, and a′ is not possible. N′ has the dimensions of k^{-1}, so that no set of four into which both k and N′ enter is possible. The examples might be continued further. The moral is that it is not safe to try for a set of absolute units in terms of any particular group of constants until one is assured that the choice is possible. For instance, one set that might seem quite fundamental turns out to be impossible. It is not possible so to choose the magnitudes of the units that the velocity of light, the quantum, the charge on the electron, and the gas constant all have the value unity.

By way of contrast, certain sets which are possible may be mentioned. It will be found that the determinant of the exponents of the following does not vanish; G, c, h, k; G, c, e, k; N, c, h, k; N, c, e, k.

If one has certain criteria of taste which make certain of the above list of quantities objectionable as universal constants, somewhat startling results may be obtained. Let us decline to consider the quantities R, m, N, and N′. There remain G, c, h, k, a, C, a′, and e.

Now it will be found that the last seven of these have the property that it is not possible to choose any four of them whose exponential determinant does not vanish. Hence any set of four quantities in terms of which the absolute system of units is to be determined, if selected from the above list of eight, must include the gravitational constant. This fact is what has made possible Tolman's Principle of Similitude.[8] It seems to me that it is not possible to ascribe any significance to the fact that there exist these relations between the various dimensional constants, but it must be regarded as an entirely fortuitous result due to the limited number of elements of which the dimensional formulas are composed, and their relative simplicity.

Another interesting speculation on the nature of the absolute units requires comment. G. N. Lewis[4] has stated it to be his conviction that any set of absolute units will be found to bear a simple numerical relation to any other possible set of absolute units. The justification of this point of view at present is not to be found in any accurate results of measurement, but is rather quasi-mystical in its character. This point of view led Lewis to notice the remarkably simple relation between the Stefan constant and the electronic charge and the gas constant, but so far as I know it has not been fruitful in other directions, and I have already indicated another possible significance of the simplicity of the relation.

Now let us examine this hypothesis of Lewis's with a numerical example. We have already found the magnitude of the fundamental units which would give the value unity to the gravitational constant, the velocity of light, the quantum, and the gas constant. Let us now find what size units would make the gravitational constant, the velocity of light, the gas constant, and the charge on the electron all equal to unity. The work is exactly the same in detail as before, and it is not necessary to write out the equations again. It will be found that the following units are required.

New unit of mass,	1.849×10^{-6} gm
New unit of length,	1.368×10^{-34} cm
New unit of time,	4.56×10^{-45} sec
New unit of temperature,	8.07×10^{30} °C

Now the ratio of all these units to the ones previously determined will be found to be 1/29.36. On the face of it, 29.36 does not appear to be a particularly simple number, but on examining the way in which it came into the formulas, it will be found that 29.36 is the approximate value of $4\pi\left(\frac{8\pi^5}{15}\right)^{\frac{1}{6}}$, and this somewhat complicated numerical expression came from Planck's relation between Stefan's constant and the spectral radiation constants. In fact, using Lewis's value for a, Planck's formula for h becomes

$$h = (4\pi)^2\left(\frac{8\pi^5}{15}\right)^{\frac{1}{3}} e^2/c.$$

It would seem that there will be considerable hesitation in calling a numerical coefficient of this form "simple." If this is simple, it is hard to see what the criterion of numerical simplicity is, and Lewis's principle, at least as a heuristic principle, becomes of exceedingly doubtful value. Lewis's[9] own feeling is that the coefficient in the above form cannot be regarded as simple, and the fact that it cannot is presumptive evidence that the formula as given by Planck can be regarded only as an approximation, and that sometime a more rigorous theory will be possible in which the number which is at present within the experimental error equal to 29.36 will be expressed in a way which will appeal to everyone as simply made up of simple integers and π's.

The justification of such speculations is thus for the future. The spirit of such speculations is evidently opposed to the spirit of this exposition, and we are for the present secure in our point of view which sees nothing mystical or esoteric in dimensional analysis.[10]

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PROBLEMS

1. THE gas constant in the equation $pv = RT$, has the value 0.08207 when the pressure p is expressed in atmospheres, the volume v is the volume in liters of 1 gm mol, and T is absolute Centigrade degrees. What is R when p is expressed in dynes/cm^2 and v is in cm^3?

2. The thermal conductivity of copper is 0.92 cal. per cm^2 per sec per 1° per cm temperature gradient. What is it in B.T.U. per hour per square foot for a temperature gradient of 1° Fahrenheit per foot? (This last is the engineering unit.)

3. If the numerical value of e^2/ch is 0.001161 in terms of the gm, cm, and sec, what is its value in terms of the ton, mile, and hour? e is the charge of the electron in E.S.U., c is the velocity of light in empty space, and h is Planck's quantum of action.

4. The thrust exerted by an air propeller varies with the number of revolutions per second and the speed of advance along the axis of revolution. Show that the critical speed of advance at which the thrust vanishes is proportional to the number of revolutions per second.

5. Show that the acceleration toward the center of a particle moving uniformly in a circle of radius r is Const v^2/r.

6. Show that the time of transverse vibration of a heavy stretched wire is Const $\times$ length $\times$ (linear density/tension)$^{1/2}$.

7. The time of longitudinal vibration of a bar is Const $\times$ length $\times$ (volume density/elastic constant)$^{1/2}$.

8. The velocity of sound in a liquid is Const $\times$ (density/modulus of compressibility)$^{1/2}$.

9. Given that the twist per unit length of a cylinder varies inversely as the elastic constant, or as the moment of the applied force, prove that it also varies inversely as the fourth power of the diameter.

10. There is a certain critical speed of rotation at which a mass of incompressible gravitating fluid becomes unstable. Prove that the angular velocity at instability is independent of the diameter and proportional to the square root of the density.

11. There is a certain size at which a solid non-rotating gravitating sphere becomes unstable under its own gravitation. Prove that the radius of instability varies directly as the square root of the elastic constant and inversely as the density.

12. Given that the velocity of advance of waves in shallow water is independent of the wave length, show that it varies directly as the square root of the depth.

13. The velocity of capillary waves varies directly as the square

root of the surface tension, and inversely as the square root of the wave length and the density.

14. A mass attached to a massless spring experiences a damping force proportional to its velocity. The mass is subjected to a periodic force. Show that the amplitude of vibration in the steady state is proportional to the force.

15. The time of contact of two equal spheres on impact is proportional to their radius. Given further that the time varies inversely as the fifth root of the relative velocity of approach, show that it varies as the 2/5th power of the density, and inversely as the 2/5th power of the elastic constant.

16. The specific heat of a perfect gas (whose atoms are characterized by their mass only) is independent of pressure and temperature.

17. Show that if a gas is considered as an assemblage of molecules of finite size exerting no mutual forces on each other except when in collision the viscosity is independent of the pressure and is proportional to the square root of the absolute temperature.

18. Show that if the thermal conductivity of the gas of problem 17 is independent of the pressure it is also proportional to the square root of the absolute temperature.

19. A periodic change of temperature is impressed on one face of a half-infinite solid. Show that the velocity of propagation of the disturbance into the solid is directly as the square root of the frequency, and the wave length is inversely as the square root of the frequency. The disturbance sinks to 1/eth of its initial value in a number of wave lengths which is independent of the frequency and the thermal constants of the material.

20. A long thin wire is immersed in a medium by which its external surface is maintained at a constant temperature. Heat is supplied to the wire by an alternating current of telephonic frequency at the rate $Q \cos \omega t$ per unit volume. Show that the amplitude of the periodic fluctuation of the average temperature of the wire is of the form $\theta = Q\, d^2/k\, f\, (\omega\, c\, d^2/k)$, where d is the diameter of the wire, k the thermal conductivity, and c the heat capacity per unit volume. If the wire is thin, show by a consideration of the numerical values of k and c for metals that θ is independent of ω and c and assumes the approximate form $\theta = \text{Const}\, Q\, d^2/k$.

21. The internal energy of a fixed quantity of a perfect gas, reckoned from 0° Abs. and 0 pressure, is independent of the pressure and proportional to the absolute temperature. Hence the internal energy reckoned from an arbitrary temperature and pressure as the initial point is independent of pressure and proportional to the excess of the absolute temperature over that of the initial point.

22. Why may not the argument of problem 21 be applied to the entropy of a fixed amount of a perfect gas?

23. T. W. Richards, Jour. Amer. Chem. Soc. 37, 1915, finds empirically the following relation for different chemical elements

$$\beta = 0.00021\ A/D^{1.33}\ (T_m - 50^\circ)$$

where $\beta = \frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_\tau$ is the compressibility, A is the atomic weight, D is the density, T_m the melting temperature on the absolute Centigrade scale. What is the minimum number of dimensional constants required to make this a complete equation, and what are their dimensions?

24. Show that the strength of the magnetic field about a magnetic doublet varies inversely as the cube of the distance, and directly as the moment of the doublet.

25. What are the dimensions of the dielectric constant of empty space in the electromagnetic system of units? What is its numerical value?

26. What are the dimensions of the magnetic permeability of empty space in the electrostatic system of units? What is its numerical value?

27. Given a half-infinite conducting medium in the plane surface of which an alternating current sheet is induced. Show that the velocity of propagation of the disturbance into the medium varies as the square root of the specific resistance divided by the periodic time, and the extinction distance varies as the square root of the product of specific resistance and the periodic time.

28. Show that the self-induction of a linear circuit is proportional to the linear dimensions.

29. A sinusoidal E.M.F. is applied to one end of an electrical line with distributed resistance, capacity, and inductance. Show that the velocity of propagation of the disturbance is inversely proportional, and the attenuation constant is directly proportional to the square root of the capacity per unit length.

30. An electron is projected with velocity v through a magnetic field at right angles to its velocity. Given that the radius of curvature of its path is directly proportional to its velocity, show that the radius of curvature is also proportional to the mass of the electron, and inversely proportional to the field and the charge.

31. In all electrodynamical problems into whose solution the velocity of light enters, the unit of time may be so defined that the velocity of light is unity, and two fundamental units, of mass and time, suffice. Write the dimensions of the various electric and magnetic quantities in terms of these units. Obtain the formula for the mass of an electron in terms of its mass and radius. . . . Problems involving the gravitational constant may also be solved with only the units of mass and length as fundamental. Discuss the formula for the mass of the electron with gravitational units.

32. The Rydberg constant (of the dimensions of a frequency) derived by Bohr's argument for a hydrogen atom is of the form $N = \text{Const}\ m e^4/h^3$, where e and m are the mass and the charge of the electron, and h is Planck's quantum of action.

APPENDIX

DIMENSIONS OF SOME COMMON QUANTITIES IN THE USUAL SYSTEMS OF MEASUREMENT

Mechanical Quantities.

Quantity	*Dimension*
Angle	0
Area	L^2
Volume	L^3
Curvature	L^{-1}
Frequency	T^{-1}
Velocity	LT^{-1}
Acceleration	LT^{-2}
Angular Velocity	T^{-1}
Angular Acceleration	T^{-2}
Density	ML^{-3}
Momentum	MLT^{-1}
Moment of Momentum	$ML^2\,T^{-1}$
Angular Momentum	$ML^2\,T^{-1}$
Force	MLT^{-2}
Moment of Couple, Torque	$ML^2\,T^{-2}$
Work, Energy	$ML^2\,T^{-2}$
Power	$ML^2\,T^{-3}$
Action	$ML^2\,T^{-1}$
Intensity of Stress, Pressure	$ML^{-1}\,T^{-2}$
Strain	0
Elastic Modulus	$ML^{-1}\,T^{-2}$
Elastic Constant	$M^{-1}\,LT^2$
Viscosity	$ML^{-1}\,T^{-1}$
Kinematic Viscosity	$L^2\,T^{-1}$
Capillary Constant	MT^{-2}

Thermal Quantities.

Quantity	*Dimension* — *Thermal Units*	*Dimension* — *Dynamical Units*
Temperature	θ	θ
Quantity of Heat	H	$ML^2 T^{-2}$
Heat Capacity per unit volume	$HM^0 L^{-3} T^0 \theta^{-1}$	$ML^{-1} T^{-2} \theta^{-1}$
Heat Capacity per unit mass	$HM^{-1} L^0 T^0 \theta^{-1}$	$M^0 L^2 T^{-2} \theta^{-1}$
Temperature Gradient	$H^0 M^0 L^{-1} T^0 \theta$	$M^0 L^{-1} T^0 \theta$
Thermal Conductivity	$HM^0 L^{-1} T^{-1} \theta^{-1}$	$MLT^{-3} \theta^{-1}$
Entropy	$HM^0 L^0 T^0 \theta^{-1}$	$ML^2 T^{-2} \theta^{-1}$
Total Heat	$HM^0 L^0 T^0 \theta^0$	$ML^2 T^{-2}$

Electrical Quantities.

Quantity	*Dimension* — *Electrostatic System*	*Dimension* — *Electromagnetic System*
Quantity of Electricity	$M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}$	$M^{\frac{1}{2}} L^{\frac{1}{2}} T^0$
Volume Density	$M^{\frac{1}{2}} L^{-\frac{3}{2}} T^{-1}$	$M^{\frac{1}{2}} L^{-\frac{5}{2}} T^0$
Surface Density	$M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}$	$M^{\frac{1}{2}} L^{-\frac{3}{2}} T^0$
Electric Field Intensity	$M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}$	$M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-2}$
Difference of Potential	$M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}$	$M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2}$
Dielectric Constant	$M^0 L^0 T^0$	$M^0 L^{-2} T^2$
Electric Displacement	$M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}$	$M^{\frac{1}{2}} L^{-\frac{3}{2}} T^0$
Capacity	$M^0 L T^0$	$M^0 L^{-1} T^2$
Current	$M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2}$	$M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}$
Current Density	$M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-2}$	$M^{\frac{1}{2}} L^{-\frac{3}{2}} T^{-1}$
Resistance	$M^0 L^{-1} T$	$M^0 L T^{-1}$
Resistivity	$M^0 L^0 T$	$M^0 L^2 T^{-1}$
Conductivity	$M^0 L^0 T^{-1}$	$M^0 L^{-2} T$
Magnetic Pole Strength	$M^{\frac{1}{2}} L^{\frac{1}{2}} T^0$	$M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}$
Magnetic Moment	$M^{\frac{1}{2}} L^{\frac{3}{2}} T^0$	$M^{\frac{1}{2}} L^{\frac{5}{2}} T^{-1}$
Magnetic Field Intensity	$M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-2}$	$M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}$
Magnetic Permeability	$M^0 L^{-2} T^2$	$M^0 L^0 T^0$
Magnetic Induction	$M^{\frac{1}{2}} L^{-\frac{3}{2}} T^0$	$M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}$
Self or Mutual Induction	$M^0 L^{-1} T^2$	$M^0 L T^0$

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