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APPLIED MECHANICS
DYNAMICS

by

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PREFACE

This book has been prepared as a textbook for the use of Junior Engineering students and may thus be said to be on an intermediate level. It has been planned not only to give the student certain factual information which he will need in his profession, but also to form a logical transition from the elements of Dynamics, as studied in the general physics course, to the more advanced courses in Dynamics which are now common in the engineering graduate schools.

As a textbook, the main emphasis has been on method and on the development of fundamental principles, so that the book is not to be considered as a treatise or a reference work on the subject. The fundamental principles, however, have been illustrated by a number of applications to important practical problems drawn from the various engineering fields. It is intended that the examples should not only clarify the principles but also indicate the broad applicability of the principles and methods of mechanics to all the fields of applied science.

The book begins with the simplest elements of the subject. Although the student will have had an introduction to such topics as Newton's Laws of Motion and the system of Units and Dimensions in his Freshman physics course, it is believed that the importance of a critical understanding of the elements is such that some review time can be profitably spent. The treatment of the text is intended to emphasize the fact that elementary dynamics is based upon the equation of motion and its first integrals, the equations of momentum and energy. With this objective in mind, the first part of the book consists of a concise treatment of the dynamics of a particle. This permits the principles of dynamics to be presented in their simplest forms, without being obscured by the geometry and algebra of more complicated problems. A logical extension then shows that the same methods can be applied to the treatment of systems of particles and rigid bodies.

The scope of the text is somewhat broader than has been customary for undergraduate courses in Engineering Mechanics. The elementary aspects of dimensional analysis are presented; the theory of mechanical vibrations is treated in some detail, and the mechanical-electrical analogy is discussed; the energy equation and the momentum-transfer equations for a fluid are developed; some discussion is given of electron motion, including an indication of the relativistic treatment and the equivalence of mass and energy. The final chapter is an introduction to the methods of advanced dynamics and includes a discussion of some of the simpler aspects of Lagrange's Equations of motion, the Calculus of Variations, and Hamilton's Principle. The main emphasis of the text, of course, is placed upon rigid body dynamics. The other aspects of the subject have been included to show how the methods of classical mechanics are applied to the various branches of Dynamics. It is intended that this should develop in the student a sense of the essential unity of practically all of engineering and should indicate that the branches of engineering are, for the most part, specializations of classical mechanics.

The vector notation has been utilized in the treatment of rigid bodies because of its advantages of simplicity, clarity, and generality. Vector analysis beyond the addition and multiplication of vectors is not required, however, as a prerequisite. The course in dynamics, as it is given at the California Institute of Technology, runs concurrently with a Junior course in advanced Engineering Mathematics, which includes some vector analysis and various topics in advanced calculus. There are, however, no purely mathematical topics involved in the book which cannot be explained to the student who has taken the usual Freshman and Sophomore courses in differential and integral calculus.

The material covered in the first part of the text is presented in considerable detail whereas the latter portion is given a more concise treatment, it being thought that as the student becomes familiar with the methods some of the explanatory details may be omitted. The problems are an integral part of the text, certain phases of the subject being left for the student to develop in the problem work. As a consequence, it is expected that it will be necessary for the instructor to supplement the text with considerable classroom discussion.

A number of subjects, such as fluid dynamics and the kinetics of gases, have been treated in a very brief fashion. It is realized that the treatments are superficial, and that the student will make a more complete analysis of these topics in specialized courses. It is believed, however, that the brief discussions help to combat the tendency toward a too rigorous compartmentalization of courses, which may prevent the student from attaining that broad general view of applied science which is so necessary. In all such instances care has been taken to use methods which can be extended later for the more complete treatment, and the student has been informed of the limitations of the analysis.

An attempt has been made to view the subject of dynamics from several different standpoints. Mechanics is a growing science, and there is no reason to suppose that new formulations of basic laws may not be made, which may, for certain problems, be superior to the older methods. It would be unfortunate if any student were to suppose that the subject of Dynamics is in its complete and final form—devised once and for all by some superhuman intelligence in the past and delivered to us entire, to be accepted on faith.

One of the objectives of the Mechanics course is the development in the student of skill in the formulation and numerical solution of problems. The only way in which most students can develop this skill is to solve a large number of problems. When time is available, the instructor will wish to supplement the problems in the text with additional problems suited to the interests and abilities of the students. Certain of the problems, which have been marked with an asterisk, require either more thought in the formulation, or more work in the numerical solution, than the others. Such problems should be examined by the instructor before being assigned to the students. A number of the problems involve demonstrations or proofs of general propositions. Important conclusions are thus sometimes given in the problems which are not specifically mentioned in the text. The student should always examine these problems and note the results, even though the details of the proofs are not carried through. Problems of varying degrees of difficulty are included in each group. The problems are not arranged in order of difficulty, however, as it has been found that most students need practice in deciding whether or not a problem is difficult.

The book may be adapted for shorter courses by the omission of certain chapters or sections. Chapters IV and V consist of applications, and may be partially or completely omitted. Chapters VIII and IX contain extensions of the subject and may be omitted in a shorter or more elementary course.

The authors wish to express their appreciation to Dr. F. C. Lindvall, Chairman of the Division of Engineering, California Institute of Technology, for the assistance he has given and the interest he has taken in the development of this book. Thanks are also expressed to Mr. W. L. Johnson, Mr. Seba Eldridge, Jr., and Mr. Saul Kaplun for their assistance in the preparation of problems, and to Mrs. Ruth D. Toy for her assistance with the manuscript.

G. W. H.
D. E. H.

Pasadena, Calif.
January, 1950

CONTENTS

CHAPTER I

THE GENERAL PRINCIPLES OF DYNAMICS

. . . the whole burden of philosophy seems to consist in this, from the phenomena of motions to investigate the forces of nature, and from these forces to demonstrate the other phenomena.—I. Newton, *Principia Philosophiae* (1686).

The science of mechanics has as its object the study of the motions of material bodies, and its aim is to describe the facts concerning these motions in the simplest way. From this description of observed facts, generalizations can be formulated which permit valid predictions as to the behavior of other bodies.

The motions occurring in nature are the result of interactions between the various bodies which make up the system under consideration. That portion of the subject of mechanics which describes the motion of bodies, without reference to the causes of the motion, is called *kinematics*, while that portion which studies the relationship between the mutual influences and the resulting motions is called *kinetics*. These two subjects are usually combined under the name *dynamics*, and it is this general problem that is to be treated in this book.

1. The Laws of Motion. The principles of dynamics are founded upon extensive experimental investigations. The first noteworthy experiments were performed by Galileo (1564–1642). Other investigators followed Galileo, among them being Newton (1642–1727), who, after carrying out a large number of experiments, formulated the statements which are now known as Newton's Laws of Motion:

(1) Every body persists in a state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.

(2) The time rate of change of momentum is equal to the force producing it, and the change takes place in the direction in which the force is acting

$$\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v}); \text{ or, for constant } m, \ \boldsymbol{F} = m\boldsymbol{a}$$

(3) To every action there is an equal and opposite reaction, or the mutual actions of any two bodies are always equal and oppositely directed.

These statements are interpreted as summing up the results of experimental investigations, and their validity rests upon the fact that all observations subsequent to Newton's time are in agreement with them. The study of Dynamics consists of the development of techniques for interpreting the Laws of Motion, and of an understanding of the physical significance of the properties of motion.

2. Definitions. The intuitive concepts which arise concerning such basic quantities of Dynamics as force, mass, and time must be put into a precise form before they can serve as a foundation for the development of the subject. The following definitions prescribe the sense in which these words will be used in this book.

Force and Mass. The primary objective of the science of mechanics is the study of the interactions which occur between material bodies. These interactions are of various types and might be, for example, impacts, electrical or gravitational attractions, mechanical linkages, etc. Experience shows that during these interactions the velocities of the interacting bodies are changed. We define *force*, by Newton's first law, as an action which tends to change the motion of a body. The fact that forces arise from mutual interactions and thus occur in equal and opposite pairs forms the empirical content of Newton's third law. The concept of force is made quantitatively precise by the definition that a unit force produces a unit acceleration of a specified standard body.

The *mass* of a body may now be defined by Newton's second law as the ratio of the force acting on the body to the resulting acceleration. By international agreement, the unit of mass is defined as the mass of a particular platinum-iridium cylinder, called the *international prototype kilogram*, which is in the possession of the International Committee of Weights and Measures at Sèvres, France.

The force exerted upon a body by the earth's gravitational field is called the *weight* of the body. The weight of a body is thus variable, depending upon the location of the body with respect to the earth. The magnitude of the earth's gravitational field is specified by the *acceleration of gravity* (g) which is the acceleration of an otherwise unrestrained body attracted to the earth. The gravitational acceleration has been determined experimentally and is given at a latitude ϕ and an elevation h ft by the empirical formula

$$g = 32.089(1 + 0.00524 \sin^2 \phi)(1 - 0.000000096h) \text{ ft/sec}^2$$

At sea level the maximum variation of g with latitude is of the order of 0.5%, while the variation from sea level to an altitude of 25,000 ft is of the order of 0.25%. In engineering it is customary to use a constant value of g equal to 32.2 ft/sec^2.

The mass of any body can be determined by comparing the body with the standard kilogram. In practice the mass of a body is usually determined by means of the ordinary balance. The unknown mass m_1 is balanced with a known mass m_2 so that the weights W_1 and W_2 are equal. Since $m_1 = W_1/g$ and $m_2 = W_2/g$, it follows that $m_1 = m_2$.

Experiment shows that for the bodies and motions with which the engineer is usually concerned, the mass of a body is a constant within the limits of accuracy of measurement. Experiments in atomic physics, however, show that at sufficiently high velocities the mass of a particle is not a constant, but, as predicted by the theory of relativity, is given by $m = m_0/\sqrt{1 - (v/c)^2}$, where c is the velocity of light, v the velocity of the particle, and m_0 the mass of the particle at zero velocity. Because of the large magnitude of c, it is impossible to detect the variability of mass at even the highest velocities encountered in engineering practice. It is important to note that Newton's Second Law refers to a specific body and does not refer directly to a system which is losing or gaining material.

A number of different standards of mass have been defined in terms of the prototype kilogram. In the United States the *pound-mass avoirdupois* has been defined legally by Congress as the 1/2.2046 part of the international prototype kilogram. The *pound-force* is defined as the gravitational force exerted on a standard pound-mass when g has the "standard" value of 32.174 ft/sec^2. In engineering the pound-force is taken as the unit of force, but the pound-mass is *not* taken as the unit of mass. The engineering unit of mass is that mass which is given an acceleration of 1 ft/sec^2 by a force of 1 pound-force. This unit of mass is called a *slug* and is equal to 32.174 pound-mass. Since the multiplicity of standards of force and mass sometimes leads to confusion, a summary of the definitions of a number of the commonly encountered terms is given in appendix II.

Time. The unit of time is the *second*, which is defined as the 1/86,400 part of a mean solar day. The mean solar day is the yearly average of the time intervals between successive transits of the sun past a meridian of the earth.

Length. The international standard of length is the *standard meter*, which is defined as the distance, at zero degrees centigrade, between two lines on a platinum-iridium bar in the possession of the International Committee of Weights and Measures. The *United States Yard* is defined legally by act of Congress as the 3600/3937 part of the standard meter, and the *foot* is defined as one-third of a yard.

3. Frames of Reference. In the preceding discussion of acceleration, force, and mass, it has been implied that there exists a frame of reference with respect to which measurements can be made. In engineering, unless stated to the contrary, it is always understood that measurements are to be made with respect to a coordinate system which is fixed at the earth's surface. In astronomy, distances may be measured with respect to certain stars. In any event it is always necessary to perform measurements in some coordinate system which is located with respect to some physical object.

The fact that the coordinate system may be located in various ways naturally raises the question as to the effect its position might have on the equations of motion and the solution of problems. It is possible to select a coordinate system with respect to which it is not permissible to write simply $F = ma$. An example is a coordinate system fixed with respect to an airplane which is making a turn. It then would be necessary to apply a force to a body in order to keep its observed acceleration equal to zero. The equation $F = ma$ would not give a correct description of the motion if the acceleration were measured with respect to an accelerating coordinate system, for it would be necessary to add correction terms which take into account the motion of the coordinate system. It is clearly an advantage to locate the coordinate system so that such correction terms are not required. It was formerly customary to define an absolute space and to refer all measurements to a coordinate system fixed with respect to absolute space. It is now recognized that all measurements are relative, and the concepts of absolute space and time have been discarded. The location of the coordinate system is now based upon experience. We locate the system so that the equation $F = ma$ describes the motion within the required limits of accuracy.

The difficulties associated with the ideas of absolute space, absolute time, and the location of coordinate systems might seem to be chiefly problems of philosophy. It was just these difficulties, however, which led to the formulation of the Theory of Relativity, which has been of such importance in the development of modern physics.

4. Fundamental and Derived Units. For the measurement of the physical quantities with which we are concerned, three independent units are used. In engineering it is customary to use the unit of *length* (foot), the unit of *force* (pound), and the unit of *time* (second), as the three *fundamental units*. All other quantities can be expressed in terms of the three fundamental units. The unit of acceleration, for example, is written (ft/sec^2), and the unit of mass, which, from the equation $F = ma$ is seen to be equal to force divided by acceleration, is written (lb sec^2/ft). Such units are called *derived units* to indicate the fact that they are expressed by combinations of the fundamental units. As a matter of convenience the derived units are sometimes given special names. For example, the foregoing derived unit of mass is called a slug. Many of the derived units have no special names, however, velocities being referred to as so many ft/sec, accelerations as so many ft/sec^2, etc.

The system of units described in the preceding paragraph, in which length-force-time are the fundamental units, is called the (L–F–T) or *gravitational* system of units. In physics it is customary to take length-mass-time as the fundamental units. This system is called the $(L\text{–}M\text{–}T)$ or *absolute* system of units. The words "gravitational" and "absolute" are merely the names of the systems, and it should not be inferred that there is anything absolute about a system of units.

The $(L\text{–}M\text{–}T)$ system differs from the $(L\text{–}F\text{–}T)$ system only in that the unit of mass instead of the unit of force is taken as the third fundamental unit. In the $(L\text{–}M\text{–}T)$ system the fundamental units are named the centimeter, the gram, and the second. The derived unit of acceleration is cm/sec^2, and the unit of force is that force which gives a mass of one gram an acceleration of one cm/sec^2. This derived unit of force, the gram cm/sec^2, is called the dyne.

5. Dimensions. All the physical quantities used in mechanics can be expressed in terms of the three fundamental units. The particular fundamental units required to express a quantity are called the *dimensions* of that quantity. Thus, the fact that acceleration is measured in units of length per unit of time per unit of time, is described by saying that the dimensions of acceleration are LT^{-2}. In the following table the dimensions of a number of common mechanical quantities are summarized:

Quantity	$(L\text{-}F\text{-}T)$ *System*	$(L\text{–}M\text{–}T)$ *System*
Length	L	L
Time	T	T
Force	F	MLT^{-2}
Mass	FT^2L^{-1}	M
Velocity	LT^{-1}	LT^{-1}
Acceleration	LT^{-2}	LT^{-2}
Area	L^2	L^2
Volume	L^3	L^3
Density	FT^2L^{-4}	ML^{-3}
Momentum	FT	MLT^{-1}
Work, Energy, Heat	FL	ML^2T^{-2}
Power	FLT^{-1}	ML^2T^{-3}
Pressure, Stress	FL^{-2}	$ML^{-1}T^{-2}$
Moment	FL	ML^2T^{-2}
Viscosity	$FL^{-2}T$	$ML^{-1}T^{-1}$
Angle	dimensionless	

The number of fundamental units is to a certain extent arbitrary. In the field of thermodynamics a fourth fundamental unit is commonly added. This is usually taken as temperature θ. In the field of electricity, a fourth fundamental unit is also commonly added, which is often taken as the electric charge Q. The dimensions of some common thermodynamic and electrical quantities follow:

Quantity	*(L–F–T) System*	*(L–M–T) System*
Temperature	θ	θ
Thermal Conductivity	$FT^{-1}\theta^{-1}$	$MLT^{-3}\theta^{-1}$
Entropy	$FL\theta^{-1}$	$ML^2T^{-2}\theta^{-1}$
Gas Constant	$L^2T^{-2}\theta^{-1}$	$L^2T^{-2}\theta^{-1}$
Electric Charge	Q	Q
Current	QT^{-1}	QT^{-1}
Voltage	$FQ^{-1}L$	$MQ^{-1}L^2T^{-2}$
Resistance	$FQ^{-2}LT$	$MQ^{-2}L^2T^{-1}$
Inductance	$FQ^{-2}LT^2$	$MQ^{-2}L^2$
Capacitance	$F^{-1}Q^2L^{-1}$	$M^{-1}Q^2L^{-2}T^2$

A study of the dimensions of the quantities entering into a physical problem may furnish useful information. Every equation which describes a physical process must be dimensionally correct, that is, the dimensions on one side of the equation must be the same as the dimensions on the other side. In the dimensional equations that follow, we shall indicate the fact that it is the dimensions only which are equated by enclosing the dimensional expression in square brackets. For example, the equation for the radial force F_r, acting upon a mass m which moves in a circle of radius r with velocity v is:

$$F_r = \frac{mv^2}{r}$$

Dimensionally,

$$F_r = [F]$$

$$\frac{mv^2}{r} = \frac{[FT^2L^{-1}][LT^{-1}]^2}{[L]} = [F]$$

The equation is thus dimensionally correct. If the dimensions of such an equation do not check, then we know that an error of some kind exists.

As a second example, consider the equation describing the velocity of a particle falling through a resisting medium:

$$v = \frac{W}{k}(1 - e^{-\frac{k}{m}t})$$

where:

$$
\begin{aligned}
v &= \text{velocity } [LT^{-1}] \\
W &= \text{weight of the body } [F] \\
k &= \text{resistance factor } [FTL^{-1}] \\
m &= \text{mass of the body } [FT^2L^{-1}] \\
t &= \text{time } [T]
\end{aligned}
$$

The dimensions of $\left(\frac{k}{m}t\right)$ are

$$\frac{[FTL^{-1}]}{[FT^2L^{-1}]}[T] = [F^0T^0L^0]$$

and the dimensions of $\left(\frac{W}{k}\right)$ are

$$\frac{[F]}{[FTL^{-1}]} = [LT^{-1}]$$

Thus the equation is dimensionally correct.

In the second example it will be noted that the exponent of the term $e^{-\frac{k}{m}t}$ is dimensionless. In any expression of the type $\log x$, e^x, $\sin x$, $\cosh x$, etc., the argument x can always be written as a dimensionless quantity. This follows from the fact that dimensional homogeneity of an equation involving transcendental functions can be maintained only if the arguments are dimensionless. This may be seen, for example, by examining the series expansion for a typical function of this kind:

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \cdots$$

Having the arguments in a dimensionless form has the advantage that the terms are independent of the size of the particular fundamental units used (see Problem 5).

It is always possible to write a dimensionally homogeneous equation in the form:

$$\pi_1 = \phi(\pi_2, \pi_3, \cdots)$$

where π_1, π_2, . . . represent dimensionless quantities and ϕ indicates that π_1 can be written as a function of the other dimensionless quantities. For example, taking the equation used in the second example above,

$$v = \frac{W}{k}(1 - e^{-\frac{k}{m}t})$$

we see that this can be put in the form:

$$\left(\frac{vk}{W}\right) = 1 - e^{-\frac{k}{m}t} = \phi\left(\frac{k}{m}t\right)$$

where the quantities $\left(\frac{vk}{W}\right)$ and $\left(\frac{k}{m}t\right)$ are both dimensionless.

6. Dimensional Analysis. The variables of a dimensionally homogeneous equation can always be grouped so that the equation is expressed in dimensionless terms, as is illustrated in the preceding example. By dimensional analysis it is possible to determine each of the dimensionless terms which appear in an equation without making a mathematical analysis to determine the actual functional form of the equation. Often the mathematics of a physical problem is so complex that it is not feasible to work out the exact solution. It is then advantageous to know the dimensionless terms which would appear in the solution.

Consider a problem involving n variables $v_1, v_2, \cdots, v_n$. We shall suppose there are three fundamental units L, F, and T, but it will be seen that the following method of analysis is applicable for any number of fundamental units. The exact solution of the problem would be a dimensionally homogeneous equation involving the n variables, which may be represented by

$$v_1 = \phi(v_2, v_3, \cdots, v_n)$$

It will be more convenient if we write the equation

$$\phi(v_1, v_2, \cdots, v_n) = 0$$

for now both sides of the equation are dimensionless. The equation may also be written

$$\phi(\pi_1, \pi_2, \cdots) = 0 \tag{1}$$

where each π represents a dimensionless term composed of a certain number of variables. We wish to determine the number of π terms which will appear and to determine the variables of which each π term is composed.

The dimensions of the variables can be written as:

$$\begin{aligned} v_1 &= [L^{\alpha_1}F^{\beta_1}T^{\gamma_1}] \\ v_2 &= [L^{\alpha_2}F^{\beta_2}T^{\gamma_2}] \\ &\vdots \\ v_n &= [L^{\alpha_n}F^{\beta_n}T^{\gamma_n}] \end{aligned}$$

The numerical exponents α, β, γ may be positive, negative, or zero. The variables are to be combined into terms having the dimensions $[L^0F^0T^0]$ and this can be achieved only by combining them in products of the form $\pi_1 = v_1^a v_2^b v_3^c \cdots$ or, since we do not distinguish between π_1 and π_1^a, we may write:

$$\pi_1 = v_1 v_2^b v_3^c \cdots = [L^{\alpha_1}F^{\beta_1}T^{\gamma_1}]^1[L^{\alpha_2}F^{\beta_2}T^{\gamma_2}]^b[L^{\alpha_3}F^{\beta_3}T^{\gamma_3}]^c \cdots$$

The exponents of the dimensions are additive so that the numerical values of $b, c, \ldots$ can be determined to make the resultant exponent of each dimension equal to zero. This gives the following set of three equations, one for each dimension:

$$\begin{aligned}
\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + \cdots &= 0 \\
\beta_1 + b\beta_2 + c\beta_3 + d\beta_4 + \cdots &= 0 \\
\gamma_1 + b\gamma_2 + c\gamma_3 + d\gamma_4 + \cdots &= 0
\end{aligned}$$

These are three simultaneous equations which can be used to determine three unknowns so that if we limit consideration to four variables, we obtain:

$$\begin{aligned}
\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 &= 0 \\
\beta_1 + b\beta_2 + c\beta_3 + d\beta_4 &= 0 \\
\gamma_1 + b\gamma_2 + c\gamma_3 + d\gamma_4 &= 0
\end{aligned}$$

The values of the unknowns b, c, and d are determined uniquely by these equations and they insure the existence of a dimensionless term $\pi_1 = v_1 v_2^b v_3^c v_4^d$. We conclude, therefore, that with three fundamental units four variables are required to form a dimensionless term. It may happen that in some problems certain of the α, β, γ's may be zero and hence some of the b, c, d's may be zero.* However, recognizing the possible existence of the zeros, this is consistent with the general statement that with N fundamental units, $N + 1$ variables are required to form a dimensionless term.

In order that a dimensionless term be unique it is necessary that it contain a variable that does not appear in any of the other dimensionless terms, otherwise it would be possible to express

* If one should choose as the variable v_1, for which the exponent $a = 1$, a variable which should not appear in that particular π term, one will obtain, of course, the anomalous answer $1 = 0$ from the equations.

this term by combinations of the other terms. Therefore, the total number of unique dimensionless terms which can be formed is determined by the number of variables. The first π requires four variables, thus leaving $n - 4$ unused variables which permits a total of $1 + (n - 4) = n - 3$ dimensionless terms to be formed. The procedure is to select 3 variables to appear in every π term, the remaining $n - 3$ variables then determine $n - 3$ terms.

The preceding discussion can be summarized in the form of the so-called π theorem.*

If there are n variables involving N fundamental units they can be combined to form $n - N$ dimensionless parameters, each containing $N + 1$ variables.

The following examples illustrate the use of dimensional analysis. Three points should be particularly noted: first, it is necessary to know which variables are pertinent to a problem, so that a considerable insight into the physical processes is required; second, the results of the dimensional analysis must be interpreted carefully so as to obtain as much useful information as possible; third, by its nature, dimensional analysis can never give a complete answer to the problem.

Example 1. A mass m moves in a circle of radius r with a constant velocity v. What can be concluded as to the force F which causes this motion?

Solution. The variables which enter this problem, along with their dimensions, are:

$$\begin{aligned} F &= [L^0F^1T^0] \\ m &= [L^{-1}F^1T^2] \\ r &= [L^1F^0T^0] \\ v &= [L^1F^0T^{-1}] \end{aligned}$$

There are 4 variables and 3 fundamental units so that there is $4 - 3 = 1$ dimensionless term which can be formed. This term is:

$$\begin{aligned} Fm^av^br^c &= [L^0F^1T^0]^1[L^{-1}F^1T^2]^a[L^1F^0T^{-1}]^b[L^1F^0T^0]^c \\ &= [L^{-a+b+c}F^{1+a}T^{2a-b}] \end{aligned}$$

* Buckingham, E., "On Physically Similar Systems; Illustrations of the Use of Dimensional Equations." *Physical Review* **2**, 345 (1914).

For the term to be dimensionless it is required that

$$\begin{aligned} 1 + a &= 0 \qquad & a &= -1 \\ 2a - b &= 0 & b &= -2 \\ -a + b + c &= 0 & c &= 1 \end{aligned}$$

The dimensionless term is therefore $\dfrac{Fr}{mv^2}$ and the equation describing the motion is:

$$\phi\left(\frac{Fr}{mv^2}\right) = 0$$

Since m, v, and r are constant, F must also be constant and the functional form of the equation must be

$$\frac{Fr}{mv^2} - C = 0$$

where C is a dimensionless constant (pure number). We may therefore write

$$F = C\frac{mv^2}{r}$$

If the problem were to be solved completely by using the principles of dynamics it would be found that $C = 1$, but we cannot deduce this from dimensional considerations alone.

EXAMPLE 2. Consider the problem of a vibrating mass, as illustrated in Fig. 1–1. If the mass is displaced from its equilibrium position a distance A and then released, it oscillates between $+A$ and $-A$. The spring requires a force of kx lb to deflect it x ft, that is, the spring constant is k lb/ft. Apply dimensional analysis to this problem to investigate the motion.

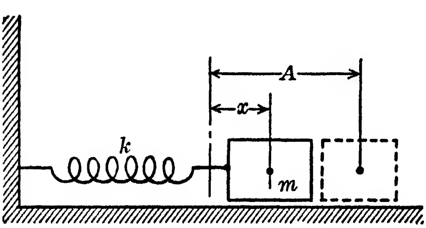


FIG. 1–1

Solution. The variables in the problem and their dimensions are:

$$\begin{aligned} x &= [L^1F^0T^0] \\ A &= [L^1F^0T^0] \\ k &= [L^{-1}F^1T^0] \\ m &= [L^{-1}F^1T^2] \\ t &= [L^0F^0T^1] \end{aligned}$$

The number of dimensionless terms which can be formed is $5 - 3 = 2$. We are chiefly interested in x and t, so that we choose A, k, and m to appear in both terms, and x and t to appear only in one term. This gives for the dimensionless terms:

$$\begin{aligned}\pi_1 = xA^a k^b m^c &= [L^1F^0T^0][L^1F^0T^0]^a[L^{-1}F^1T^0]^b[L^{-1}F^1T^2]^c \\ &= [L^{1+a-b-c}F^{b+c}T^{2c}] \\ \pi_2 = tA^a k^b m^c &= [L^{a-b-c}F^{b+c}T^{1+2c}]\end{aligned}$$

Equating the exponents to zero and solving, we obtain:

$$\pi_1 = \frac{x}{A}; \quad \pi_2 = \sqrt{\frac{k}{m}}\, t$$

The equation describing the motion is therefore:

$$\phi\left(\frac{x}{A}, \ \sqrt{\frac{k}{m}}\, t\right) = 0$$

or

$$\left(\frac{x}{A}\right) = \phi\left(\sqrt{\frac{k}{m}}\, t\right)$$

The displacement can therefore be expressed as:

$$x = A\phi\left(\sqrt{\frac{k}{m}}\, t\right)$$

Dimensional analysis cannot determine the function ϕ, but if the problem is solved by the principles of dynamics it is found that:

$$x = A \cos \sqrt{\frac{k}{m}}\, t$$

To illustrate the information that can be derived from dimensional analysis suppose that the mathematical solution of the problem was unknown and we had to construct a large oscillator weighing 1000 lb, restrained by a spring with spring constant 2000 lb/ft. Before undertaking the expense of constructing it, we would like to know precisely the motion it will have. Not knowing the mathematical solution we resort to experiment and construct a model which will have the same motion as the prototype. From the equation $x = A\phi\left(\sqrt{\frac{k}{m}}\, t\right)$ we see that if the dimensionless quantity $\left(\sqrt{\frac{k}{m}}\, t\right)$ is made the same for the model as for the prototype the model will have the same motion as the prototype. We might thus use for the model a mass weighing one pound and a spring with spring constant of 2 lb/ft.

After performing the experiment we compile the results in dimensionless form, that is, we plot the equation $\frac{x}{A} = \phi\left(\sqrt{\frac{k}{m}}\,t\right)$ as shown in Fig. 1–2. This graph describes the motion of all pos-

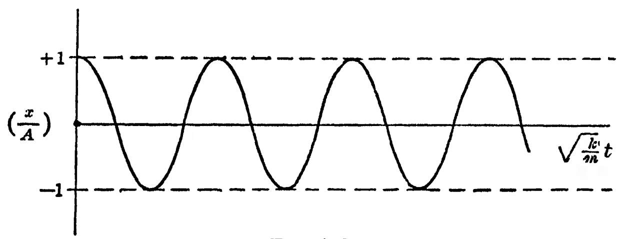


Fig. 1–2

sible oscillators with any values of A, k, and m. The experimental results can now be applied to any desired prototype. As illustrated by this problem, it is, in general, advantageous to present experimental results in dimensionless form.

Example 3. A problem of considerable practical importance is that of determining the drag force F_d acting upon a body moving through a fluid. Consider a body of specified shape moving with constant velocity v through a fluid of density ρ and viscosity μ, as in Fig. 1–3.

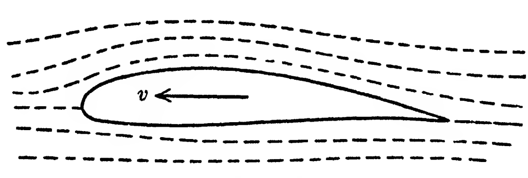


Fig. 1–3

Solution. The analysis is to apply to bodies having the specified shape so that we may take the cross-sectional area A as a measure of the size of the body. The variables entering this problem and their dimensions are:

$$\begin{aligned} F_d &= [F^1L^0T^0] \\ v &= [F^0L^1T^{-1}] \\ A &= [F^0L^2T^0] \\ \rho &= [F^1L^{-4}T^2] \\ \mu &= [F^1L^{-2}T^1] \end{aligned}$$

The number of dimensionless terms which can be formed is $5 - 3 = 2$. We are particularly concerned with the force F_d so that we shall have it appear in only one dimensionless term. Let us choose the viscosity μ as the other unique variable so that:

$$\pi_1 = F_d A^a \rho^b v^c; \quad \pi_2 = \mu A^a \rho^b v^c$$

Determining the exponents in the usual fashion we obtain:

$$\pi_1 = \frac{F_d}{A\rho v^2}; \qquad \pi_2 = \frac{vA^{\frac{1}{2}}\rho}{\mu}$$

and we may write:

$$\phi\left(\frac{F_d}{A\rho v^2}, \frac{vA^{\frac{1}{2}}\rho}{\mu}\right) = 0$$

or

$$\frac{F_d}{A\rho v^2} = \phi\left(\frac{vA^{\frac{1}{2}}\rho}{\mu}\right)$$

It is customary to write this equation as:

$$F_d = A\rho v^2 \phi\left(\frac{vA^{\frac{1}{2}}\rho}{\mu}\right) = \frac{1}{2} A\rho v^2 C_d$$

where the drag coefficient C_d is a function of Reynolds number * $\left(\frac{vA^{\frac{1}{2}}\rho}{\mu}\right)$. Usually the drag is investigated experimentally, and the results are plotted in a dimensionless form as shown in Fig. 1–4. In this form the experimental results are applicable to bodies of any size having the specified shape, and to fluids of any density and viscosity.

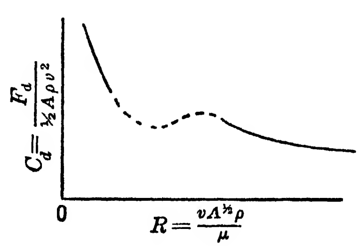
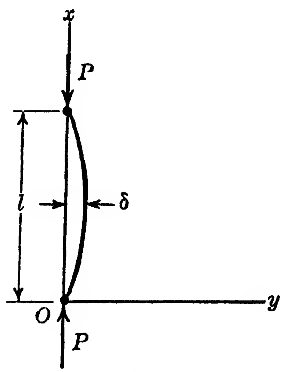


FIG. 1–4

* Named after Osburne Reynolds (1842–1912), a distinguished English investigator in the field of fluid mechanics.

PROBLEMS

1. A steel column with a moment of inertia of I in.^{4} and length of l in. is loaded with an axial force of P lb. The modulus of elasticity of the steel is E lb/in.^{2} The magnitude of P is sufficient to buckle the column into the deflected shape:



PROB. 1

$$y = \delta \sin \sqrt{\frac{P}{EI}}\, x$$

where y is the deflection in inches at point x and δ is the deflection in inches at the center. Check the dimensions of this equation.

2. A body falling through a certain resisting medium with a velocity v is subjected to a drag force that is proportional to the velocity squared, $F_d = -kv^2$. If the body has a weight W and starts from rest at time $t = 0$, its velocity at any subsequent time is

$$v = \sqrt{\frac{W}{k}} \frac{\left(e^{2g\sqrt{\frac{k}{W}}t} - 1\right)}{\left(e^{2g\sqrt{\frac{k}{W}}t} + 1\right)}$$

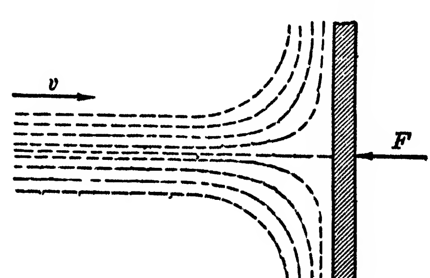
Check the dimensions of this equation.

3. A certain problem in dynamics leads to the equation

$$m\frac{d^2x}{dt^2} + \left[k - m\left(\frac{d\theta}{dt}\right)^2\right]x = my\frac{d^2\theta}{dt^2}$$

In this expression m is the mass of the body, x and y are the coordinates of the displacement, θ is an angle measured in radians, t is the time, and k is a stiffness modulus measured in pounds per foot. Check the dimensions of the equation.

4. A jet of water of cross-sectional area A and velocity v impinges upon a stationary flat plate as shown. The mass per unit volume of the water is ρ. By dimensional analysis, determine the expression for the force F exerted by the jet against the plate.

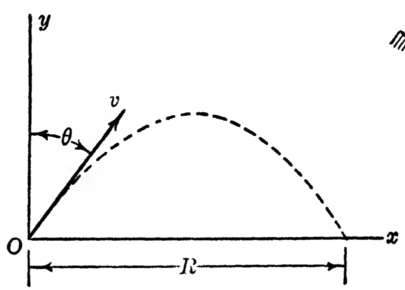


PROB. 4

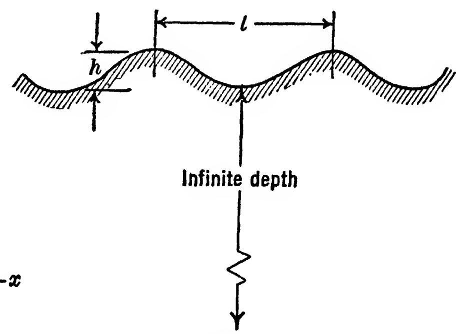
5. Show that a dimensionally homogeneous equation is independent of the size of the fundamental units, e.g. that the relation expressed by the equation is the same whether foot-pound-seconds are used or inch-ton-minutes are used. Show that this is not true of a dimensionally inhomogeneous equation.

6. (a) A gun shoots a projectile with an initial velocity v, which makes an angle of θ with the vertical. The projectile has a mass m and a range R. The gravitational acceleration is g. Find the expression for R by dimensional analysis. (Assume there is no air resistance.)

(b) Suppose the projectile of part (a) were subjected to an air resistance proportional to the velocity squared; that is, the magnitude of the retarding force is kv^2 lb. Apply dimensional analysis to obtain an expression for the range.



PROB. 6



PROB. 7

7. (a) Ocean waves have a wave length from crest to crest of l ft and a height of h ft. The density of the fluid is ρ and the acceleration of gravity is g. Solve for the velocity of the wave by dimensional analysis.

(b) If the form of the waves is preserved, that is, if $\frac{l}{h}$ is a constant, what is the solution given by dimensional analysis in terms of l, ρ, g? This problem illustrates the fact that the information that one obtains from dimensional analysis depends to a great extent upon what is known about the physical problem.

8. (a) Experiment shows that for laminar flow of a fluid through a circular pipe the significant variables are D, the diameter of the pipe, v the mean velocity of the fluid, $\frac{dp}{dx}$ the rate of change of pressure along the pipe and μ the coefficient of viscosity. Derive an expression for the velocity v by means of dimensional analysis.

(b) If the flow through the pipe of part (a) is turbulent, then experiment shows that the density of the fluid is also a significant variable. By dimensional analysis derive an expression for the rate of change of pressure for turbulent flow.

9. In a certain problem in fluid mechanics it is determined that surface tension σ is an important factor. The other significant variables in the problem are velocity v, force F, density ρ, and cross-sectional area A. By dimensional analysis derive an expression for the force F in terms of the other quantities. The dimensionless quantity $\dfrac{v^2A^{\frac{1}{2}}}{\sigma/\rho}$ which appears in the analysis is called Weber's number.

10. When a body falls through a resisting medium, such as air, its velocity increases until the drag force counterbalances the gravity force. The resultant force on the body is then zero, the velocity does not change, and the body is said to have reached its terminal velocity. Consider a spherical body of radius r falling through a very rarefied atmosphere. Let the density of the body be ρ_b, and the density of the medium ρ_m. From dimensional analysis what can be concluded as to the effect of the size of the body on the terminal velocity?

7. The Relation Between Theory and Experiment. Whenever the applications of any physical science are examined, we are impressed by the fact that perfect agreement between theory and experiment can never be obtained. By the theoretical approach, we mean the method of starting with a set of laws, axioms, or premises from which various conclusions are drawn. Thus in dynamics, conclusions are drawn from the Laws of Motion by means of mathematics. By experiment we mean the method of actually measuring the pertinent quantities and thus obtaining a solution of the problem. In the experimental work it is not possible to achieve perfect accuracy of measurement; consequently errors of a certain magnitude are always present in the measured quantities. For example, we may be interested in the motion of a mass acted upon by forces. The motion can be computed for any specified mass weighing W lb, but when measurements are made on the physical system we are unable to determine the precise weight. The best that can be done is to determine, for example, that the weight is not greater than 1.005 lb, and is not less than 0.995 lb. This can be expressed by saying that it weighs $W(1 \pm 0.005)$ lb, where the decimal figure in the parenthesis depends upon the accuracy of the measuring device and the care used in making the measurement. The same situation arises in measuring forces, velocities, and all the other quantities involved in the study of mechanics. Whereas in a theoretical analysis we may work with precise quantities such as x, v, and F, in experimental work we can only determine $x(1 \pm e_x)$, $v(1 \pm e_v)$, and $F(1 \pm e_F)$. These discrepancies will cause no serious difficulty as long as the e's can be kept sufficiently small for any particular problem.

If a number of experimental determinations can be made of a quantity, it is possible by the methods of mathematical statistics to reach some conclusions as to the most probable value of the quantity. In the following discussion we shall deal only with the upper and lower limits of error, and we shall thus have no information as to the actual error of a particular measurement, which will lie somewhere between those limits.

The preceding discussion naturally raises the question of how it is possible to tell whether there is a satisfactory agreement between theory and experiment. The procedure for investigating whether or not there is agreement is illustrated by the following example.

Suppose that a theoretical analysis of a particular problem leads to the conclusion that the relation between the force F acting on a mass m, which moves in a circular path of radius r with a velocity v is

$$F = \frac{mv^2}{r}$$

To check this experimentally, we construct some equipment, and we measure the various quantities, obtaining $F(1 \pm e_F)$, $m(1 \pm e_m)$, $v(1 \pm e_v)$, and $r(1 \pm e_r)$. If theory and experiment agree, the computed value of F should agree with the measured value, so that:

$$F(1 \pm e_F) = \frac{mv^2}{r}\left[\frac{(1 \pm e_m)(1 \pm e_v)^2}{(1 \pm e_r)}\right]$$

or

$$F = \frac{mv^2}{r}\left[\frac{(1 \pm e_m)(1 \pm e_v)^2}{(1 \pm e_F)(1 \pm e_r)}\right]$$

Expanding the expression in the square brackets, and neglecting second order and higher terms, since they are small compared to the first order terms, we obtain:

$$F = \frac{mv^2}{r}[1 \pm (e_m + 2e_v + e_F + e_r)]$$

If the measured value of F differs from $\frac{mv^2}{r}$ by more than $[1 \pm (e_m + 2e_v + e_F + e_r)]$, there is disagreement between theory and experiment. To illustrate the orders of magnitude which might be involved, suppose that each of the experimental measurements has an accuracy of 1%, that is,

$$e_m = e_v = e_F = e_r = 0.01$$

Then, substituting, we have:

$$F = \frac{mv^2}{r}(1 \pm 0.05)$$

Thus, if F does not differ from $\frac{mv^2}{r}$ by more than 5%, we would say that there is no positive disagreement.

A method of solving this problem which is sometimes more useful than the one given above is the following. Differentiating the expression $F = \frac{mv^2}{r}$, and using the usual approximation for the differentials of the variables, we have, as a first order approximation:

$$\Delta F = \frac{\partial F}{\partial m}\Delta m + \frac{\partial F}{\partial v}\Delta v + \frac{\partial F}{\partial r}\Delta r$$

In this particular problem

$$\frac{\partial F}{\partial m} = \frac{v^2}{r}; \quad \frac{\partial F}{\partial v} = \frac{2mv}{r}; \quad \frac{\partial F}{\partial r} = -\frac{mv^2}{r^2}$$

so that:

$$\Delta F = \frac{v^2}{r}\Delta m + \frac{2mv}{r}\Delta v - \frac{mv^2}{r^2}\Delta r$$

Writing

$$\frac{\Delta m}{m} = e_m, \quad \frac{\Delta v}{v} = e_v, \quad \frac{\Delta r}{r} = e_r$$

we obtain:

$$\Delta F = \frac{mv^2}{r}(e_m + 2e_v - e_r)$$

Remembering that the errors can be either plus or minus, it will be seen that this result is identical with the preceding one.

In a practical problem it may well be that an analysis of the kind given will indicate a disagreement between theory and experiment. In this event, the following possibilities should be considered:

(1) The experiment was performed improperly.
(2) A mistake was made in deriving the theoretical solution.
(3) Some significant factor was neglected which should have been included.
(4) The laws or premises upon which the theory is based do not describe the physical phenomena with the requisite accuracy.

Regarding the discrepancy between theory and experiment, it should be noted that it is very common to neglect factors when deriving a theoretical solution. For example, despite the fact that a body moving in air is always subjected to air resistance, theoretical solutions are often worked out neglecting the forces due to air resistance. By neglecting certain factors the problem can be simplified to the point where it may be relatively easy to derive a solution, whereas, if all factors were included, it might be difficult, if not impossible, to obtain the desired solution. We thus have a deliberate discrepancy between theory and experiment. It would not be proper to say that the theoretical solution is incorrect, but rather that it is not strictly applicable to the problem. In making use of such a simplified solution there will always be a discrepancy between the answer predicted by the theory and that obtained by experiment. Properly the upper limit of this discrepancy should be determined either experimentally or otherwise so that it may be said that the theoretical solution gives an answer correct to some specified degree of accuracy. *When solving a problem we should state explicitly the assumptions that have been made or the conditions that have been imposed so that it is known under what limitations the solution is valid.*

PROBLEMS

11. Show that the total error in a function of the type $y = x_1^a x_2^b x_3^c$ can be found by taking the logarithm of each side, differentiating, and writing in terms of differentials. Apply this method to the example given above, $F = \frac{mv^2}{r}$, and show that it leads to the result found there.

12.* In an impact experiment a mass m is dropped through a certain distance with an impact velocity v, and a precise value of mv is desired. The equation of motion is:

$$v = g\frac{m}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

Two procedures are possible: (1) By means of an electronic device the velocity just before impact may be measured, and the mass may also be measured; thus mv is determined within certain limits of error. (2) m, g, k, t, may be measured and the corresponding impact velocity computed. Which is the more accurate method of determining mv? The measurements of the quantities are:

Quantity	*Value*
m	$20(1 \pm 0.005)$
g	$32.18(1 \pm 0.02)$
k	$0.1(1 \pm 0.04)$
t	$2(1 \pm 0.02)$
v	$v(1 \pm 0.05)$

13. Dimensional analysis indicates that for a certain flow condition the discharge Q in cubic feet per second through a circular pipe is:

$$Q = \frac{CA^2\frac{dp}{dx}}{\mu}$$

Measurements are made over a range of values of Q, μ, A, and dp/dx and the corresponding values of C are computed. Set up the expression for the total error in C. How does one determine whether the values of C do or do not show satisfactory agreement between theory and experiment?

14. Given that the range R of a projectile fired from a gun with an angle of elevation θ is:

$$R = \frac{v_0^2}{g}\sin 2\theta$$

where v_0 is the initial velocity of the projectile and air resistance is neglected. The initial conditions are $\theta = 30° \pm 0.15°$; $v_0 = 1000$ ft/sec $\pm$ 15 ft/sec; $g = 32.18$ ft/sec^2 $\pm$ 0.08%. What is the limit of error for R in feet?

15. In a certain experiment in fluid mechanics the significant factor is Reynolds number:

$$R = \frac{vA^{\frac{1}{2}}}{\mu/\rho}$$

The limits of error for measurements of the quantities are: v, $\pm 1\%$; A, $\pm 0.5\%$; μ, $\pm 1\%$; ρ, $\pm 0.8\%$. What are the limits of error for the corresponding computed values of R?

* Problems marked with an asterisk require either more thought in the formulation, or more work in the numerical solution, than the others.

CHAPTER II

KINEMATICS: THE DESCRIPTION OF MOTION

The circumstances of mere motion, considered without reference to the bodies moved, or to the forces producing the motion, or to the forces called into action by the motion, constitute the subject of a branch of Pure Mathematics, which is called *Kinematics*.—W. Thomson and P. G. Tait, *Elements of Natural Philosophy* (1872).

For the development of Dynamics a concise and consistent notation is required for the description of the displacements, velocities, and accelerations of a body. The vector notation for these quantities will be presented first, and then various scalar components of these vectors will be developed.

8. Displacement, Velocity, and Acceleration. The displacement of a point P (Fig. 2–1) is described by the magnitude and direction of the radius vector $\boldsymbol{r}$ which extends from the origin of a

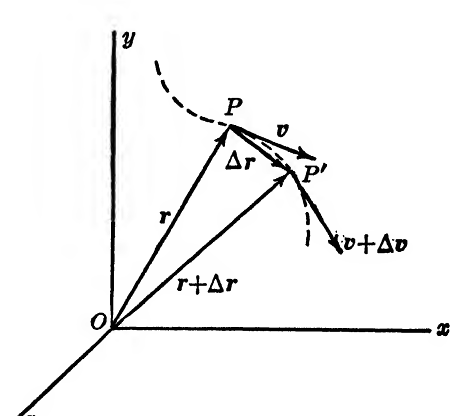


FIG. 2–1

coordinate system to the point P. At time t, let the displacement be $\boldsymbol{r}$ then at time $t + \Delta t$ the displacement is $\boldsymbol{r} + \Delta\boldsymbol{r}$ where $\Delta\boldsymbol{r}$ is the vector from P to P'. Between P and P' the average change of $\boldsymbol{r}$ per unit time is $\Delta\boldsymbol{r}/\Delta t$ and the *velocity* at P

is obtained by taking the limiting value of $\Delta \boldsymbol{r}/\Delta t$ as Δt approaches zero.

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{d\boldsymbol{r}}{dt}$$

The direction of $\boldsymbol{v}$ is tangent to the path of motion at P. At point P the velocity is $\boldsymbol{v}$ and at the point P' it is $\boldsymbol{v} + \Delta \boldsymbol{v}$. The change of velocity with time may be illustrated by a diagram in which the velocity $\boldsymbol{v}$ is drawn as a radius vector as in Fig. 2–2. The curve described by the endpoint of $\boldsymbol{v}$ in this figure is called the *hodograph* of the motion. Let the velocity be $\boldsymbol{v}$ at time t and $\boldsymbol{v} + \Delta \boldsymbol{v}$ at time $t + \Delta t$. In this interval the average change of velocity per unit time is $\Delta \boldsymbol{v}/\Delta t$, and the acceleration at time t is obtained by taking the limiting value of $\Delta \boldsymbol{v}/\Delta t$.

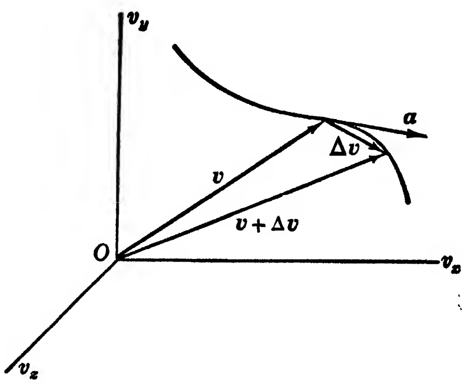


Fig. 2–2

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2}$$

The direction of the acceleration vector $\boldsymbol{a}$ coincides with the direction of the tangent to the hodograph, since the velocity of the endpoint of a vector in the hodograph plane is the time derivative of the vector. It should be noted that the acceleration is equal to zero only when both the *magnitude* and *direction* of $\boldsymbol{v}$ are constant. For example, a particle moving on a circular path can never have zero acceleration since the direction of $\boldsymbol{v}$ is always changing.

In vector notation the equation of motion is written:*

$$\boldsymbol{F} = m \frac{d^2\boldsymbol{r}}{dt^2} = m\ddot{\boldsymbol{r}} \tag{2}$$

It is often convenient to resolve the displacement, velocity and acceleration vectors into components. These components are usually taken in the principal directions of the coordinate system

* We shall use a dot placed above a letter to indicate the derivative with respect to time, and two dots to indicate the second derivative. This is the notation originally adopted by Newton.

which is most appropriate for the particular problem involved. Three commonly used sets of components will now be discussed.

(a) *Rectangular coordinates* (Fig. 2–3) are used to describe the displacement, velocity and acceleration vectors when they are resolved into components parallel to the orthogonal xyz axes. In terms of the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, which are constant:

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dt} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} \qquad (3)$$

$$\ddot{\boldsymbol{r}} = \frac{d^2\boldsymbol{r}}{dt^2} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k} \qquad (4)$$

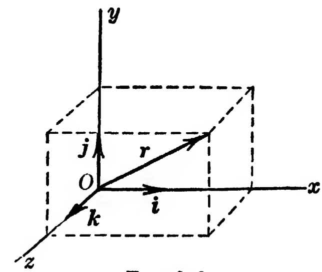


FIG. 2–3

where $\dot{x} = \dfrac{dx}{dt} = v_x$, etc., $\ddot{x} = \dfrac{d^2x}{dt^2} = a_x$, etc.

In rectangular coordinates the equation of motion is written

$$F_x = m\ddot{x}; \quad F_y = m\ddot{y}; \quad F_z = m\ddot{z} \qquad (5)$$

where F_x, F_y, F_z are the x, y, z components of the resultant force acting on the particle.

(b) *Cylindrical coordinates* z, r, ϕ are used when suited to the geometry of the problem. In this system there are three mutually perpendicular components, one parallel to the z-axis, one with a direction parallel to the radius vector in the xy plane, and the third with the direction of increasing ϕ as shown in Fig. 2–4a.

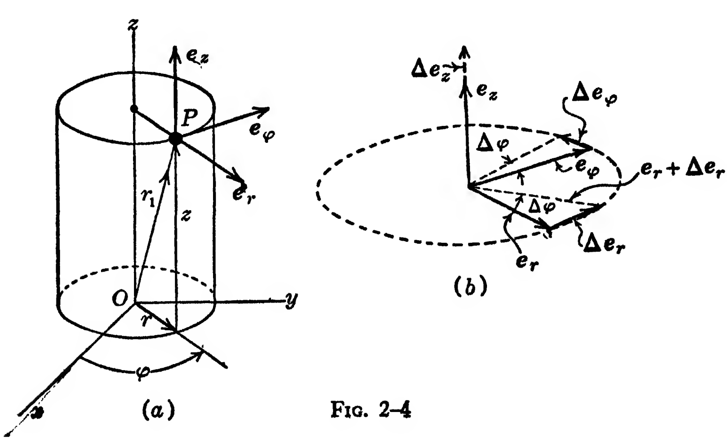


FIG. 2–4

The unit vectors specifying these directions are designated by $\boldsymbol{e}_z$, $\boldsymbol{e}_r$, and $\boldsymbol{e}_\phi$. The unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\phi$ are not constant, but change direction with time. The time derivative of a unit vector is perpendicular to the vector since the length of the vector is constant. As may be seen from Fig. 2–4b:

$$\Delta \boldsymbol{e}_r = (\Delta\phi)(1)\boldsymbol{e}_\phi \quad \text{and} \quad \Delta \boldsymbol{e}_\phi = (\Delta\phi)(1)(-\boldsymbol{e}_r)$$

so that the derivatives are:

$$\dot{\boldsymbol{e}}_r = \dot{\phi}\boldsymbol{e}_\phi; \quad \dot{\boldsymbol{e}}_\phi = -\dot{\phi}\boldsymbol{e}_r$$

The displacement of point P is:

$$\boldsymbol{r}_1 = r\boldsymbol{e}_r + z\boldsymbol{e}_z$$

The velocity is obtained by taking the derivative with respect to t:

$$\begin{aligned} \boldsymbol{v} &= \dot{r}\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r + \dot{z}\boldsymbol{e}_z + z\dot{\boldsymbol{e}}_z \\ &= \dot{r}\boldsymbol{e}_r + r\dot{\phi}\boldsymbol{e}_\phi + \dot{z}\boldsymbol{e}_z \end{aligned} \tag{6}$$

The components of the velocity in the r, ϕ, and z directions are respectively $\dot{r}$, $r\dot{\phi}$, and $\dot{z}$. The acceleration is obtained by a second differentiation.

$$\begin{aligned} \boldsymbol{a} &= \ddot{r}\boldsymbol{e}_r + \dot{r}\dot{\boldsymbol{e}}_r + \dot{r}\dot{\phi}\boldsymbol{e}_\phi + r\ddot{\phi}\boldsymbol{e}_\phi + r\dot{\phi}\dot{\boldsymbol{e}}_\phi + \ddot{z}\boldsymbol{e}_z + \dot{z}\dot{\boldsymbol{e}}_z \\ &= (\ddot{r} - r\dot{\phi}^2)\boldsymbol{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\boldsymbol{e}_\phi + \ddot{z}\boldsymbol{e}_z \end{aligned} \tag{7}$$

The equation of motion is written:

$$\begin{aligned} F_r &= m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi &= m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \\ F_z &= m\ddot{z} \end{aligned} \tag{8}$$

where F_r, F_ϕ, F_z are the components of the resultant force on the particle in the r, ϕ, z directions, and $(\ddot{r} - r\dot{\phi}^2)$, $(r\ddot{\phi} + 2\dot{r}\dot{\phi})$, and $\ddot{z}$ are the components of acceleration. Since the expressions for the acceleration components are not as simple as for rectangular components, it is not desirable to use cylindrical coordinates unless the geometry of the problem is particularly suited to their use.

(c) *Tangential and normal components* are used chiefly because they give a simple representation of acceleration in curvilinear motion. Let s be the arc length measured along the path of motion (Fig. 2–5a) and let ρ be the principal radius of curvature at the point P. The velocity is:

$$\boldsymbol{v} = \dot{s}\boldsymbol{e}_t \tag{9}$$

where the unit vector has the direction of $\boldsymbol{v}$, that is, tangent to the path of motion. The acceleration is obtained by differentiating the velocity with respect to the time:

$$\boldsymbol{a} = \ddot{s}\boldsymbol{e}_t + \dot{s}\dot{\boldsymbol{e}}_t$$

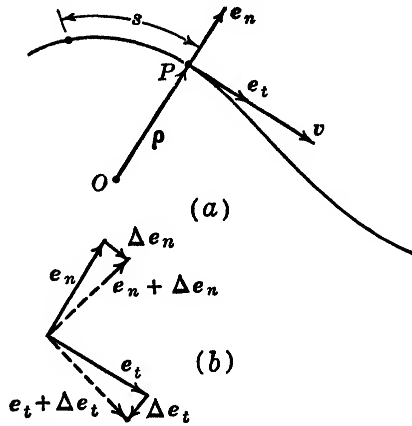


FIG. 2–5

To evaluate the time derivative of the unit vector $\boldsymbol{e}_t$ note from Fig. 2–5b that since this vector changes direction but not length, $\Delta\boldsymbol{e}_t$ is perpendicular to $\boldsymbol{e}_t$, so that:

$$\Delta\boldsymbol{e}_t = -\frac{\Delta s}{\rho}\boldsymbol{e}_n; \quad \dot{\boldsymbol{e}}_t = \lim_{\Delta t \to 0}\frac{\Delta\boldsymbol{e}_t}{\Delta t} = -\frac{1}{\rho}\boldsymbol{e}_n \lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t} = -\frac{1}{\rho}\dot{s}\boldsymbol{e}_n$$

where the minus sign indicates that $\dot{\boldsymbol{e}}_t$ is opposite in direction to $\boldsymbol{e}_n$. Substituting this value of $\dot{\boldsymbol{e}}_t$ in the foregoing expression for $\boldsymbol{a}$ gives:

$$\boldsymbol{a} = \ddot{s}\boldsymbol{e}_t - \frac{\dot{s}^2}{\rho}\boldsymbol{e}_n \tag{10}$$

The acceleration vector $\boldsymbol{a}$ may thus be resolved into two perpendicular components, a tangential acceleration of magnitude $\ddot{s}$ and a normal acceleration of magnitude $-\dot{s}^2/\rho$. The minus sign indicates that the direction of the normal acceleration is toward the center of curvature.

The equations of motion in terms of tangential and normal components of acceleration are:

$$F_t = m\ddot{s}$$
$$F_n = -\frac{m\dot{s}^2}{\rho} \tag{11}$$

In dynamics problems it is customary to make the direction of all positive vector components coincide with the positive direction of the coordinate. With this convention the sign of a component determines its direction, that is, a plus sign means the component has a positive direction.

EXAMPLE 1. A particle moves along the parabolic path $y = ax^2$ in such a way that the x-component of the velocity of the particle remains constant. Find the acceleration of the particle.

Solution. Since the conditions of the problem are stated in terms of a rectangular coordinate system, we shall probably find rectangular coordinates most convenient for this problem. Since $\dot{x} = c$, we have $\ddot{x} = 0$. Also:

$$\begin{aligned} y &= ax^2 \\ \dot{y} &= 2ax\dot{x} = 2acx \\ \ddot{y} &= 2ac\dot{x} = 2ac^2 \end{aligned}$$

so that the motion of the particle is:

$$\begin{aligned} \boldsymbol{r} &= x\boldsymbol{i} + ax^2\boldsymbol{j} \\ \dot{\boldsymbol{r}} &= c\boldsymbol{i} + 2acx\boldsymbol{j} \\ \ddot{\boldsymbol{r}} &= 2ac^2\boldsymbol{j} \end{aligned}$$

EXAMPLE 2. A particle P moves in a plane in such a way that its distance from a fixed point O is $r = a + bt^2$ and the line connecting O and P makes an angle $\phi = ct$ with a fixed line OA, as shown in Fig. 2–6. Find the acceleration of P.

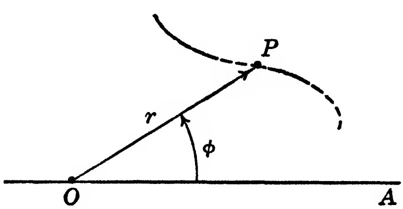


FIG. 2–6

Solution. The data for this problem are given in such a way that a plane polar coordinate system is convenient for the description of the motion. The acceleration of the point P in plane polar coordinates is found from Equation (7):

$$\boldsymbol{a} = (\ddot{r} - r\dot{\phi}^2)\boldsymbol{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\boldsymbol{e}_\phi$$

In this problem:

$$\begin{aligned} r &= a + bt^2 & \qquad \phi &= ct \\ \dot{r} &= 2bt & \dot{\phi} &= c \\ \ddot{r} &= 2b & \ddot{\phi} &= 0 \end{aligned}$$

So that:

$$\boldsymbol{a} = [2b - c^2(a + bt^2)]\boldsymbol{e}_r + 4bct\boldsymbol{e}_\phi$$

EXAMPLE 3. A particle moves along a path composed of two straight lines connected by a circular arc of radius r, as shown in Fig. 2–7. The speed along the path is given by $\dot{s} = at$. Find the maximum acceleration of the particle.

Solution. The form of the data in this problem makes the use of radial and tangential components of acceleration suitable. Using Equation (10):

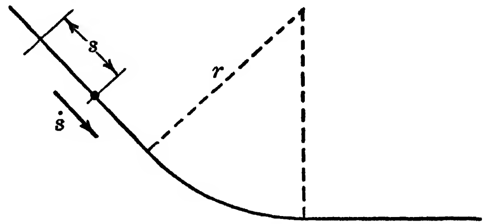


FIG. 2–7

$$\boldsymbol{a} = \ddot{s}\boldsymbol{e}_t - \frac{\dot{s}^2}{\rho}\boldsymbol{e}_n$$

we note that the normal component of acceleration is zero along the straight portion of the path, and $\frac{\dot{s}^2}{r}$ along the curved portion. The maximum acceleration will thus occur when $\dot{s}$ is a maximum on the curved path, that is, just at the end of the curve:

$$\boldsymbol{a}_{\max} = a\boldsymbol{e}_t - \frac{a^2t^2}{r}\boldsymbol{e}_n$$

PROBLEMS

16. The speed of a car increases from 2 mph to 32 mph in 10 sec. Find the time average acceleration if the car is traveling in a straight line. If $s = kt^3 + ct$, what is the acceleration at the end of 10 sec? s = the distance along the path in feet; t = time in seconds.

17. An auto starts from rest and completes one lap on a horizontal 2-mile diameter circular track in t sec. Find t if the magnitude of the average speed v' is 88 ft/sec. Find the velocity of the car at the end of the lap if the magnitude of the tangential acceleration is uniformly increasing with time. Express v as a function of v'.

18. A point moves along the x-axis with a constant acceleration a. Derive an expression for displacement as a function of acceleration and velocity if $\dot{x} = v_0$ and $x = x_0$, when $t = 0$.

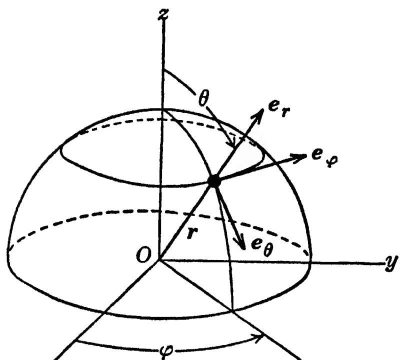
19. Derive the r, ϕ, z components of acceleration in cylindrical coordinates without using unit vectors. Do this by starting with the x, y, z components of acceleration, first showing that

$$a_r = a_x \cos\phi + a_y \sin\phi, \quad a_\phi = -a_x \sin\phi + a_y \cos\phi$$

In this way, check Equation (7).

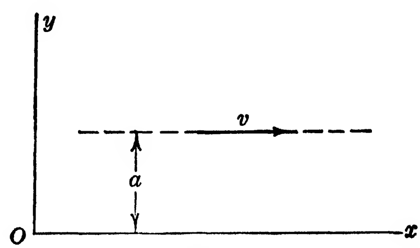
20. Derive the expressions for the tangential and normal components of acceleration without using unit vectors. Deduce them by first determining the normal and tangential components of $\Delta\mathbf{v}/\Delta t$ and then letting $\Delta t \rightarrow 0$.

21.* Referring to the figure, consider the effect of increments in ϕ and θ upon the unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\phi$, and $\boldsymbol{e}_\theta$ of a spherical coordinate system. $\boldsymbol{e}_r$ is radial, out from the pole O, $\boldsymbol{e}_\phi$ is tangential to the circle of latitude, and $\boldsymbol{e}_\theta$ is tangential to the meridian circle as shown. Derive the expressions for $\dot{\boldsymbol{e}}_r$, $\dot{\boldsymbol{e}}_\phi$, and $\dot{\boldsymbol{e}}_\theta$.



Prob. 21

22.* Using the results of Problem 21 derive the r, ϕ, and θ components of velocity and acceleration in spherical coordinates.



Prob. 23

23. A body moves in a straight line parallel to the x-axis with constant velocity v as shown in the diagram. (a) What are the components of acceleration when expressed in rectangular coordinates? (b) Referring to Equation (7), evaluate $\boldsymbol{a}$ term by term, showing that the coefficients of $\boldsymbol{e}_r$ and $\boldsymbol{e}_\phi$ vanish.

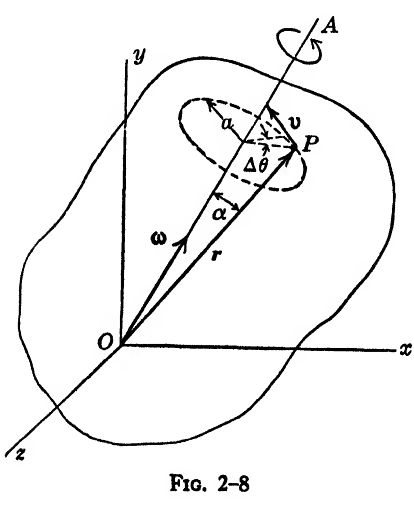


Fig. 2–8

9. Angular Velocity. Consider a rigid body rotating about an axis OA, as shown in Fig. 2–8. By the definition of rotation this means that all points of the body are, at a given instant, moving in circular paths about centers on the axis of rotation. The *angular velocity* of the body is described by the vector $\boldsymbol{\omega}$, which has the direction of the axis of rotation, as given by the right-hand screw rule, and which has a magnitude equal to the time rate of change of the angular displacement of any line in the body which is normal to the axis of rotation. Thus in Fig. 2–8, $\boldsymbol{\omega}$ would have the direction of OA, and the magnitude $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \dot{\theta}$.

There is a simple relation between the angular velocity of a rigid body rotating about a fixed axis and the linear velocity of any point in the body. Referring to Fig. 2–8, and the definition of the angular velocity, it will be seen that the velocity $\boldsymbol{v}$ of a point P located by the radius vector $\boldsymbol{r}$ is:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} \tag{12}$$

since $\boldsymbol{v}$ is perpendicular to the plane of $\boldsymbol{\omega}$ and $\boldsymbol{r}$ and

$$v = \lim_{\Delta t \to 0} \frac{a\Delta\theta}{\Delta t} = \omega r \sin \alpha$$

A summary of the algebraic properties of vector products is given in Appendix III, for the benefit of those who wish to review the subject.

PROBLEMS

24. A rigid body is rotating with an angular velocity of magnitude 500 rpm about a fixed axis which has the direction and location of the radius vector $3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$. Find the linear velocity of the point $\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ ft in the body.

25. Given an orthogonal coordinate system having unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ rotating with respect to a fixed system with an angular velocity $\boldsymbol{\omega}$, show that:

$$\dot{\boldsymbol{i}} = \boldsymbol{\omega} \times \boldsymbol{i}; \quad \dot{\boldsymbol{j}} = \boldsymbol{\omega} \times \boldsymbol{j}; \quad \dot{\boldsymbol{k}} = \boldsymbol{\omega} \times \boldsymbol{k}$$

10. Motion Referred to a Moving Coordinate System.

10. Motion Referred to a Moving Coordinate System. Suppose that the position of a point P (Fig. 2–9) is determined with respect to an xyz

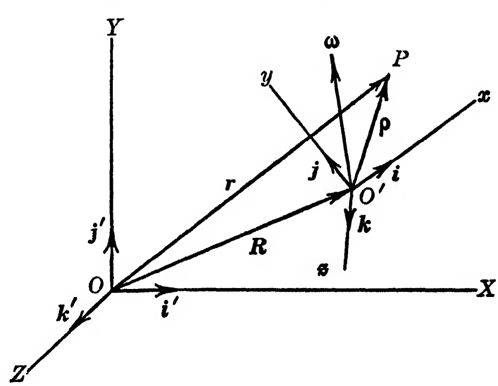


Fig. 2–9

coordinate system, while at the same time this coordinate system moves with a translational velocity $\dot{\boldsymbol{R}}$ and an angular velocity $\boldsymbol{\omega}$ with respect to a "fixed" XYZ coordinate system. This is the type of coordinate system which might become necessary, for example, in a long range ballistics problem for which the motion of the earth would have to be taken into account. In such a problem the measurements would be made with respect to the earth, and the motion of the earth relative to some coordinate system fixed with respect to certain stars would be considered.

In the analysis to follow, we shall always measure the vectors $\boldsymbol{R}$ and $\boldsymbol{r}$ in the fixed XYZ system. The unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ have always the direction of the moving coordinate axes, while the unit vectors $\boldsymbol{i}'$, $\boldsymbol{j}'$, $\boldsymbol{k}'$ have always the direction of the fixed coordinate axes.

By the *absolute displacement* $\boldsymbol{r}$ of the point P is meant the displacement measured with respect to the fixed XYZ system. By differentiating this absolute displacement we obtain the *absolute velocity* $\dot{\boldsymbol{r}}$, and the *absolute acceleration* $\ddot{\boldsymbol{r}}$.

$$\begin{aligned} \boldsymbol{r} &= X\boldsymbol{i}' + Y\boldsymbol{j}' + Z\boldsymbol{k}' \\ \dot{\boldsymbol{r}} &= \dot{X}\boldsymbol{i}' + \dot{Y}\boldsymbol{j}' + \dot{Z}\boldsymbol{k}' \\ \ddot{\boldsymbol{r}} &= \ddot{X}\boldsymbol{i}' + \ddot{Y}\boldsymbol{j}' + \ddot{Z}\boldsymbol{k}' \end{aligned} \tag{13}$$

During these differentiations, the unit vectors $\boldsymbol{i}'$, $\boldsymbol{j}'$, $\boldsymbol{k}'$ are treated as constants, since neither their magnitudes nor their directions change with time.

If we wish to express the absolute motion in terms of motion measured in the moving xyz system, we have:

$$\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{\rho} = \boldsymbol{R} + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

where the directions of the $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ unit vectors are known with respect to the fixed system. However, the unit vectors are changing direction with time, since they rotate with the xyz system. In taking the derivatives $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$, therefore, the time derivatives of these unit vectors must be included:

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}} + \dot{\boldsymbol{\rho}} = \dot{\boldsymbol{R}} + \dot{x}\boldsymbol{i} + x\dot{\boldsymbol{i}} + \dot{y}\boldsymbol{j} + y\dot{\boldsymbol{j}} + \dot{z}\boldsymbol{k} + z\dot{\boldsymbol{k}}$$

The derivatives of the unit vectors are (see Problem 25):

$$\dot{\boldsymbol{i}} = \boldsymbol{\omega} \times \boldsymbol{i}; \quad \dot{\boldsymbol{j}} = \boldsymbol{\omega} \times \boldsymbol{j}; \quad \dot{\boldsymbol{k}} = \boldsymbol{\omega} \times \boldsymbol{k}$$

so that:

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}} + (\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}) + \boldsymbol{\omega} \times (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$$

The quantity $(\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k})$ represents the velocity of the point P, measured relative to the moving coordinate system, which we shall call the *relative velocity* $\dot{\boldsymbol{\rho}}_r$. Using this notation, the expression for $\dot{\boldsymbol{r}}$ becomes:

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}} + \dot{\boldsymbol{\rho}}_r + \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{14}$$

and it is seen that $\dot{\boldsymbol{\rho}} = \dot{\boldsymbol{\rho}}_r + \boldsymbol{\omega} \times \boldsymbol{\rho}$.

The acceleration of P may be found by a second differentiation:

$$\begin{aligned}\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} + \ddot{\boldsymbol{\rho}} = \ddot{\boldsymbol{R}} &+ (\ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}) + (\dot{x}\dot{\boldsymbol{i}} + \dot{y}\dot{\boldsymbol{j}} + \dot{z}\dot{\boldsymbol{k}}) \\ &+ \dot{\boldsymbol{\omega}} \times (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) + \boldsymbol{\omega} \times (\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}) \\ &+ \boldsymbol{\omega} \times (x\dot{\boldsymbol{i}} + y\dot{\boldsymbol{j}} + z\dot{\boldsymbol{k}})\end{aligned}$$

Writing $(\ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}) = \ddot{\boldsymbol{\rho}}_r$ which we call the *relative acceleration* of the point P, the expression for $\ddot{\boldsymbol{r}}$ can be written:

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}_r + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r \tag{15}$$

The first three terms in this expression for $\ddot{\boldsymbol{r}}$ represent the absolute acceleration of a point attached to the moving coordinate system, coincident with the point P at any given time. This may be seen by noting that for a point fixed in the moving system $\dot{\boldsymbol{\rho}}_r = \ddot{\boldsymbol{\rho}}_r = 0$. The fourth term $\ddot{\boldsymbol{\rho}}_r$ represents the acceleration of P relative to the moving system. The last term $2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r$ is sometimes called the *acceleration of Coriolis*, after G. Coriolis (1792–1843), a French engineer who first called attention to the presence of this term in problems of this kind (See Problem 27).

The equation of motion in terms of the moving coordinate system may thus be written:

$$\boldsymbol{F} = m\ddot{\boldsymbol{R}} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + m\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + m\ddot{\boldsymbol{\rho}}_r + 2m\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r \tag{16}$$

Some applications of this equation will be given in the chapter on rigid body dynamics.

EXAMPLE 1. A small body of mass m slides on a rod which is the chord of a circular wheel, as in Fig. 2–10a. The wheel rotates about its center O with a clockwise angular velocity $\omega = 4$ rad/sec, and a clockwise angular acceleration $\dot{\omega} = 5$ rad/sec². The body m has a constant velocity of 6 ft/sec to the right, relative to the wheel. Find the absolute acceleration of m when $\rho = 1.5$ ft if $R = 3$ ft.

Solution. We shall fix the moving xyz coordinate system to the wheel as shown in Fig. 2–10a. The angular velocity of the co-

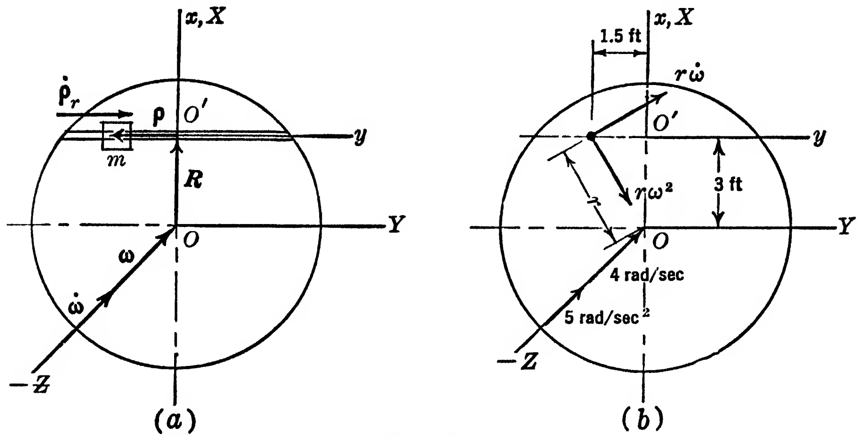


FIG. 2–10

ordinate system is $\omega = 4$ rad/sec and the angular acceleration $\dot{\omega} = 5$ rad/sec². Applying Equation (15)

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}_r + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r$$

the terms may be evaluated as follows:

$$\begin{array}{rcllcl}
\ddot{\boldsymbol{R}} & = & -(3\text{ ft})(4\text{ rad/sec})^2\boldsymbol{i} + (3\text{ ft})(5\text{ rad/sec}^2)\boldsymbol{j} & = & -48\boldsymbol{i} + 15\boldsymbol{j}\text{ ft/sec}^2 \\
\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) & = & (4\text{ rad/sec})(4\text{ rad/sec})(1.5\text{ ft})\boldsymbol{j} & = & +24\boldsymbol{j} \\
\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} & = & (5\text{ rad/sec}^2)(1.5\text{ ft})\boldsymbol{i} & = & +7.5\boldsymbol{i} \\
\ddot{\boldsymbol{\rho}}_r & = & 0 & & \\
2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r & = & -2(4\text{ rad/sec})(6\text{ ft/sec})\boldsymbol{i} & = & -48\boldsymbol{i} \\
\hline
\ddot{\boldsymbol{r}} & = & & & -88.5\boldsymbol{i} + 39\boldsymbol{j}\text{ ft/sec}^2
\end{array}$$

In some problems of this type the sum of the first three terms $\ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}$ can be computed more directly by noting that this vector sum represents the absolute acceleration of a fixed point on the moving coordinate system which is coincident with the moving point. In the present problem, for example, this coincident-point acceleration is (see Fig. 2–10b):

$$\left[(r\dot{\omega})\left(\frac{1.5}{r}\right) - (r\omega^2)\left(\frac{3}{r}\right)\right]\boldsymbol{i} + \left[r\dot{\omega}\left(\frac{3}{r}\right) + (r\omega^2)\left(\frac{1.5}{r}\right)\right]\boldsymbol{j}$$
$$= [(1.5)(5) - (3)(4)^2]\boldsymbol{i} + [(3)(5) + (1.5)(16)]\boldsymbol{j}$$
$$= -40.5\boldsymbol{i} + 39\boldsymbol{j}\text{ ft/sec}^2$$

which will be seen to be equal to the sum of the first three vectors of the solution by the other method.

EXAMPLE 2. A rigid straight bar of length l slides down a vertical wall and along a horizontal floor as shown in Fig. 2–11a.

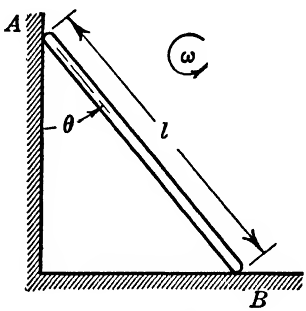


FIG. 2–11a

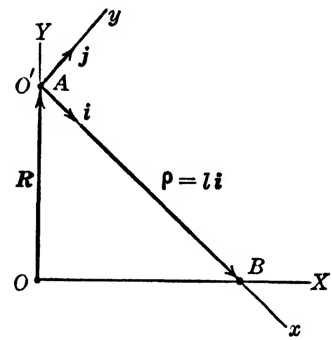


FIG. 2–11b

The end A has a constant downward vertical velocity v_A. Find the angular velocity and the angular acceleration of the bar as a function of θ.

Solution. We shall take the fixed and moving coordinate axes in this problem as shown in Fig. 2–11b. Writing the equation for the absolute velocity of the point B, we have:

$$\dot{\boldsymbol{r}}_B = \dot{\boldsymbol{R}} + \boldsymbol{\omega} \times \boldsymbol{\rho} = \boldsymbol{v}_A + \omega l \boldsymbol{j}$$

Since we know that $\boldsymbol{v}_A$ is vertical and that $\dot{\boldsymbol{r}}_B$ is horizontal, we may draw the vectors representing this equation as in Fig. 2–11c, from which we find:

$$\omega l \sin\theta = v_A$$

so that:

$$\omega = \frac{v_A}{l \sin\theta}$$

which gives the angular velocity of the bar.

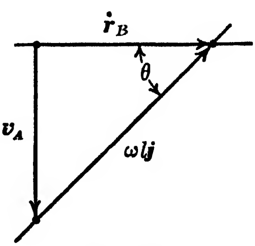


FIG. 2–11c

To find the angular acceleration we apply Equation (15):

$$\ddot{\boldsymbol{r}}_B = \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}_r + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r$$

which becomes:

$$\ddot{\boldsymbol{r}}_B = 0 - \omega^2 l \boldsymbol{i} + \dot{\omega} l \boldsymbol{j} + 0 + 0$$

Drawing the vector diagram and substituting the value of ω found above we obtain, from Fig. 2–11d,

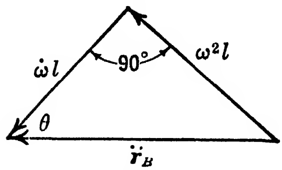


Fig. 2–11d

$$\dot{\omega} = \frac{\omega^2}{\tan\theta} = \frac{v_A{}^2}{l^2 \sin^2\theta \tan\theta}$$

$$\dot{\boldsymbol{\omega}} = -\frac{v_A{}^2}{l^2 \sin^2\theta \tan\theta}\,\boldsymbol{k}$$

which determines the angular acceleration of the bar.

Example 3. A simplified picture of the mechanism of a helicopter blade is shown in Fig. 2–12a. The blade oscillates about the horizontal axis P–P', which is carried on a rotating disk OB so that the whole assembly rotates with a constant angular

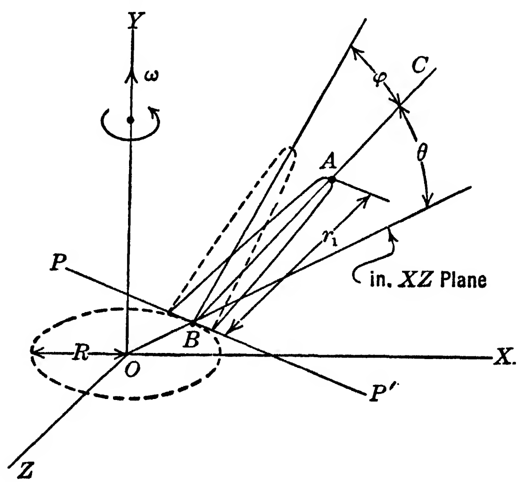


Fig. 2–12a

velocity $\boldsymbol{\omega}$. The blade has a mean position defined by the line BC, which makes an angle θ with the horizontal. At any time t the blade makes an angle ϕ with the line BC, where ϕ is given by the equation $\phi = \phi_0 \sin pt$, p being the angular frequency of the "flapping" oscillation of the blade. Find the velocity and acceleration of A, the tip of the blade, when $\phi = 0$.

Solution. We shall first find the velocity and acceleration of A relative to a coordinate system which rotates with the disk OB with an angular velocity ω, and whose origin is located at B. We orient the system so that $\boldsymbol{R}$ lies along the X-axis (Fig. 2–12b). Since

$$\phi = \phi_0 \sin pt$$

then

$$\dot{\phi} = \phi_0 p \cos pt$$
$$\ddot{\phi} = -\phi_0 p^2 \sin pt$$

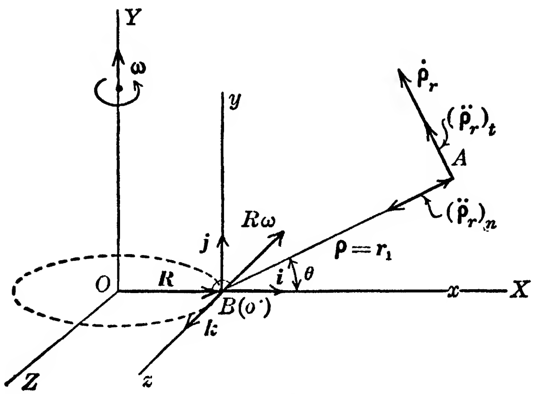


FIG. 2–12b

Thus, the magnitudes of the relative velocity and acceleration, when $\phi = 0$, are:

$$\dot{\rho}_r = r_1\dot{\phi} = r_1\phi_0 p$$
$$(\ddot{\rho}_r)_t = r_1\ddot{\phi} = 0$$
$$(\ddot{\rho}_r)_n = r_1\dot{\phi}^2 = r_1\phi_0{}^2 p^2$$

Now, using Equation (14):

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{R}} + \dot{\boldsymbol{\rho}}_r + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

where in this problem the terms become:

$$\dot{\boldsymbol{R}} = -R\omega\boldsymbol{k}$$
$$\dot{\boldsymbol{\rho}}_r = -r_1\phi_0 p \sin\theta\boldsymbol{i} + r_1\phi_0 p \cos\theta\boldsymbol{j}$$
$$\boldsymbol{\omega} \times \boldsymbol{\rho} = -r_1\omega\cos\theta\boldsymbol{k}$$

So that:

$$\dot{\boldsymbol{r}} = (-r_1\phi_0 p \sin\theta)\boldsymbol{i} + (r_1\phi_0 p \cos\theta)\boldsymbol{j} + (-R\omega - r_1\omega\cos\theta)\boldsymbol{k}$$

To find the acceleration we use Equation (15):

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}_r + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r$$

where in this problem the terms become:

$$\begin{aligned} \ddot{\boldsymbol{R}} &= -R\omega^2\boldsymbol{i} \\ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) &= -r_1\omega^2 \cos\theta\boldsymbol{i} \\ \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} &= 0 \\ \ddot{\boldsymbol{\rho}}_r &= -r_1\phi_0{}^2p^2 \cos\theta\boldsymbol{i} - r_1\phi_0{}^2p^2 \sin\theta\boldsymbol{j} \\ 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r &= 2\phi_0 r_1 p\omega \sin\theta\boldsymbol{k} \end{aligned}$$

So that:

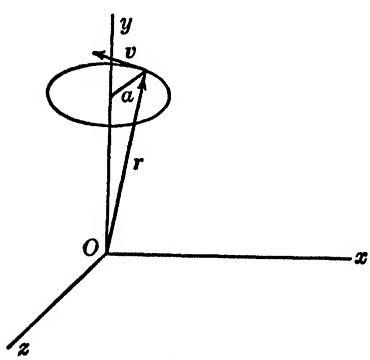
$$\ddot{\boldsymbol{r}} = (-R\omega^2 - r_1\omega^2\cos\theta - r_1\phi_0{}^2p^2\cos\theta)\boldsymbol{i} + (-r_1\phi_0{}^2p^2\sin\theta)\boldsymbol{j} + (2\phi_0 r_1 p\omega\sin\theta)\boldsymbol{k}$$

PROBLEMS

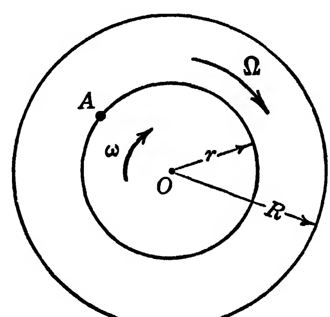
26. A particle moves in a circular path of radius a with a constant angular velocity ω as shown in the diagram. (a) Show that the acceleration term $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$ has a radial direction and a magnitude of

$$a\omega^2 = \frac{v^2}{a}.$$

(b) The magnitude of the angular velocity of the particle varies with time according to the equation $\omega = \alpha t$, where α is a constant



PROB. 26

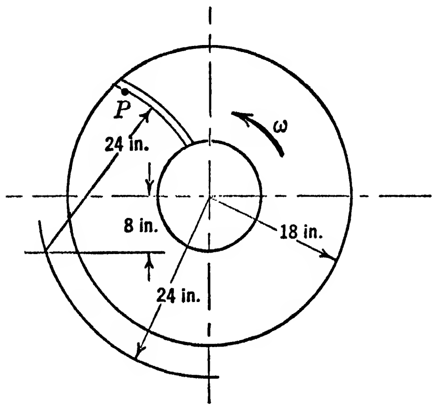


PROB. 27

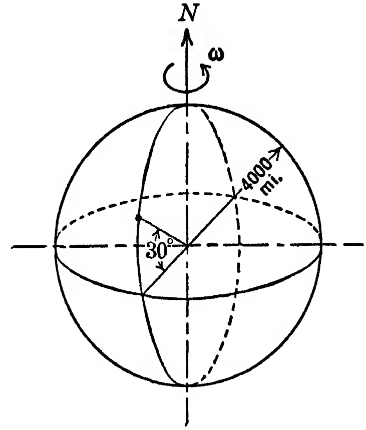
angular acceleration. The acceleration of the particle is given by $\boldsymbol{a} = \frac{d}{dt}(\boldsymbol{\omega} \times \boldsymbol{r}) = \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times \dot{\boldsymbol{r}}$. Find the magnitude and direction of each of the two terms $\dot{\boldsymbol{\omega}} \times \boldsymbol{r}$ and $\boldsymbol{\omega} \times \dot{\boldsymbol{r}}$.

27. Two concentric circular disks of radius r and R rotate about the same fixed center O. The angular velocity of the large disk, measured in a fixed system, is Ω. The angular velocity of the small disk relative to the large disk, that is, measured in a system attached to the large disk, is ω. Find the acceleration of the point A, on the circumference of the small disk. What is the Coriolis acceleration term?

28. A particle of water P moves outward along the impeller of a centrifugal pump with a constant tangential velocity of 100 ft/sec relative to the impeller. The impeller is rotating with a uniform speed of 1800 rpm in a counterclockwise direction. What is the acceleration of the particle at the point where it leaves the impeller?



PROB. 28



PROB. 31

29. Using the method of Example 2 above, find the velocity and acceleration of the midpoint of the bar. Check these answers by writing the analytical expression for the position of the center of the bar in an xy coordinate system coinciding with the floor and the wall, and by differentiating this expression.

30.* Referring to the helicopter blade of Example 3 above, find the acceleration of the blade tip A when the angle ϕ has one-half of its maximum value $\phi_0 = 6°$ and is increasing. The "coning angle" θ is 7°, the radius to the tip of the blade is $r_1 = 15$ ft, the radius of the disk is $R = 1$ ft, and the assembly rotates at 225 rpm. The blades flap once per revolution of the rotor, that is, $p = \omega$.

31. A river is flowing directly south along the surface of the earth at a uniform speed of 5 mph relative to the earth. What is the acceleration of a particle of water in the river when it crosses the 30° North latitude line?

CHAPTER III

DYNAMICS OF A PARTICLE

Newton admits nothing but what he gains from experiments and accurate observations. From this foundation, whatever is further advanced, is deduced by strict mathematical reasoning.—William Emerson, *The Principles of Mechanics* (1754).

The equation of motion as given in Chapter I is theoretically sufficient for the solution of any of the solvable problems of classical mechanics. There are several other ways, however, of presenting the basic information contained in this equation. Each of these has advantages for the solution of certain types of problems. In the present chapter we shall show first, in some simple examples, how the equation of motion can be integrated directly to give the solution of certain types of problems, and we shall then discuss several other general forms of the equation of motion.

The problems treated in this chapter will be restricted to the dynamics of a particle. If rotational effects can be neglected for a particular body, then that body can be treated as though it were a single particle with the mass of the body concentrated at one point.* If the rotational effects need to be considered, then the problem must be treated by the more general methods of rigid body dynamics. It should be noted that the same body might in one problem behave as a particle, while in another problem it might have to be treated as a rigid body. For example, a cannon ball shot through the air could be treated as a particle; the same ball rolled along the ground would have to be considered as a rigid body of a given radius.

In general, we shall consider any body as being made up of a number of particles, so that, once the basic laws describing the motion of a particle have been established, the theory may be extended to any body without the introduction of new principles. By studying first the behavior of a single particle, the various laws

* See Section 39, Chapter VI.

of dynamics can be exhibited in their simplest form, unencumbered with the purely mathematical difficulties involved in the description of complex motions.

11. Integration of the Equation of Motion for Particular Problems. In many problems the known quantities and the information desired are such that a direct integration of the equation $\mathbf{F} = m\ddot{\mathbf{r}}$, expressed in an appropriate coordinate system, will give the solution.

Example 1. Consider a particle, of mass m, which at time $t = 0$ is projected horizontally with an initial velocity $\dot{x}_0$, and is subsequently acted upon by gravity and by air resistance. Find the position and velocity of the particle at any subsequent time.

Solution. In Fig. 3–1 is shown a free-body diagram of the particle with all the forces acting. The drag force produced by air resistance has been resolved into two rectangular components, and the gravity force is shown as a downward force mg. To describe the motion we choose a rectangular xy coordinate system with the xy plane coinciding with the plane of motion. In this system the equation $\mathbf{F} = m\ddot{\mathbf{r}}$ becomes:

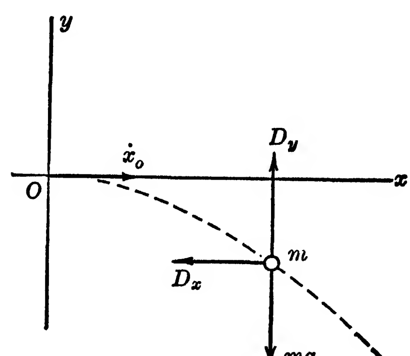


Fig. 3–1

$$F_x = m\ddot{x} = -D_x$$
$$F_y = m\ddot{y} = D_y - mg$$

In general, the drag forces D_x and D_y will be functions of the velocities $\dot{x}$ and $\dot{y}$, and these functions must be known before the equations can be integrated. In a later section we shall consider the nature of these functions and methods of integrating the resulting equations. For the present, as an illustration of the general method in its simplest form, let us suppose that the motion is taking place in a vacuum, so that $D_x = D_y = 0$. Then:

$$\ddot{x} = 0$$
$$\ddot{y} = -g$$

Integrating once:

$$\dot{x} = C_1$$
$$\dot{y} = -gt + C_2$$

The constants of integration can be determined from the initial conditions $\dot{x} = \dot{x}_0$, $\dot{y} = 0$ when $t = 0$; hence

$$C_1 = \dot{x}_0$$
$$C_2 = 0$$

Performing a second integration:

$$x = \dot{x}_0 t + C_3$$
$$y = -\frac{gt^2}{2} + C_4$$

Also, when $t = 0$; $x = 0$, $y = 0$, so $C_3 = C_4 = 0$, and we have the result:

$$x = \dot{x}_0 t$$
$$y = -\tfrac{1}{2}gt^2$$

Example 2. Suppose that the displacement of a particle is known as a function of time and that the forces which produce the motion are to be determined. Given a particle of mass m moving in a circular path of radius r with a velocity of constant magnitude v. Find the force required to maintain this motion.

Solution. We shall describe the motion in a rectangular xy coordinate system located at the center of the circular path, as in Fig. 3–2a. In this coordinate system we have:

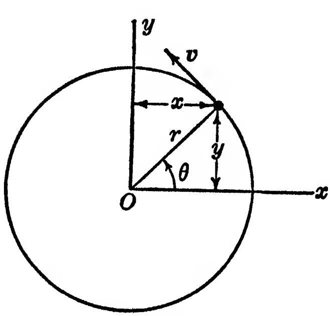


Fig. 3–2a

$$x = r\cos\theta$$
$$y = r\sin\theta$$

To find θ as a function of time, we note that $\dot{\theta} = \dfrac{v}{r} = \omega$ is a constant, so that

$$\frac{d\theta}{dt} = \omega; \quad \theta = \omega t + C$$

If $\theta = 0$ when $t = 0$, then $C = 0$ and $\theta = \omega t$ so that:

$$x = r\cos\omega t$$
$$y = r\sin\omega t$$

Differentiating these expressions twice to find the components of acceleration, we obtain:

$$\ddot{x} = -r\omega^2 \cos \omega t$$
$$\ddot{y} = -r\omega^2 \sin \omega t$$

From the equations of motion written in rectangular coordinates, we have:

$$F_x = m\ddot{x} = -mr\omega^2 \cos \omega t$$
$$F_y = m\ddot{y} = -mr\omega^2 \sin \omega t$$

The resultant force F has the magnitude and direction

$$F = \sqrt{F_x^2 + F_y^2} = mr\omega^2 = m\frac{v^2}{r}$$

$$\phi = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\tan \omega t = \theta$$

So that the force is directed radially toward the center, as in Fig. 3–2b.

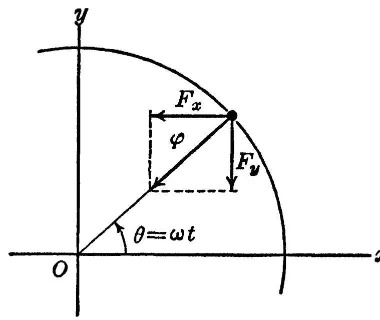


FIG. 3–2b

It should be noted that a simpler description of the motion in this problem can be obtained by the use of plane polar coordinates (see Problem 32).

PROBLEMS

32. Check the results given in Example 2 above for the uniform circular motion of a particle, by using plane polar coordinates to describe the motion.

33. (a) In rectangular coordinates the motion of a particle of mass m is given by

$$x = at - bt^3; \quad y = 0; \quad z = 0$$

What is the velocity and acceleration of the particle at any time t? What is the maximum positive displacement attained by the particle?

(b) What is the force acting upon the particle of part (a) at any time t? Find the magnitude and direction of the force at $t = 1.25$ sec, if $a = 4$ ft/sec, $b = 2$ ft/sec^3, and m has a weight of 10 lb.

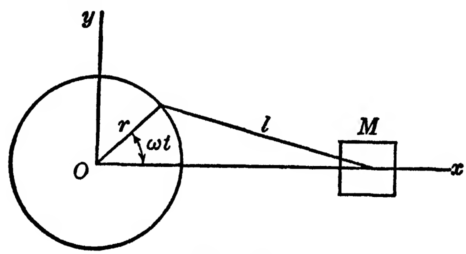
34. A particle of mass m moves along the path $y = A - Bx^2$, $z = 0$. When $x = a$, the magnitude of the velocity of the particle is v. Find the x and y components of the velocity, and the component of acceleration normal to the path of motion, at $x = a$.

35.* J. V. Poncelet (1829) concluded on the basis of tests that when a projectile of mass m is fired into earth or masonry it experiences a retarding force $F_D = -C_1 - C_2v^2$ where C_1 and C_2 depend upon the properties of the material and the shape of the projectile. If the impact velocity of the projectile is v_0, what is its total penetration?

36. An automobile of mass m travels on a straight level highway at a velocity v_0. At time $t = 0$, the brakes are locked and the automobile skids to a stop. The length of the skid is l feet, and the coefficient of sliding friction between tires and highway is μ. What is the expression for v_0 in terms of m, l, and μ?

37. If in Problem 36 the highway has an $a\%$ grade, that is, drops a ft vertically for every 100 ft horizontally, what is the expression for v_0?

38.* A wheel of radius r rotates with a uniform angular velocity ω. A massless connecting rod of length l is fastened to the wheel, and moves



PROB. 38

a piston of mass M back and forth along the x-axis as shown in the figure. Show that the x-component of the resultant force acting upon M is:

$$R_x = -Mr\omega^2\left[\cos \omega t + \frac{rl^2 \cos 2\omega t + r^3 \sin^4 \omega t}{(l^2 - r^2 \sin^2 \omega t)^{\frac{3}{2}}}\right]$$

Find an approximate solution for R_x for $\dfrac{r}{l}$ small compared to unity.

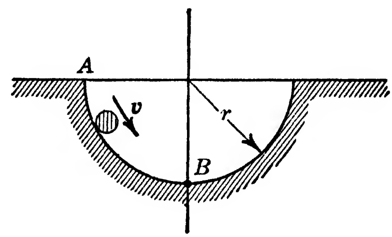
39. (a) An elevator of total weight 3000 lb starts from rest and moves upward with a constant acceleration. At the end of 3 sec it has moved a distance of 9 ft. What is the force in the supporting cable?

(b) If the maximum allowable force in the cable is 8000 lb, find the maximum allowable acceleration of the elevator.

40. (a) A mass m slides down a frictionless inclined plane which makes an angle of 30° with the horizontal. If m starts from rest and moves down the plane under the action of gravity, how far will it have traveled at the end of 10 sec?

(b) If the coefficient of friction between m and the inclined plane of part (a) is $\mu = 0.2$, what distance would be traveled down the plane in 10 sec?

41. A particle of mass m starts from rest and slides down the side of a hemispherical surface of radius r under the action of gravity. If there is no friction, and the particle starts from A, what force will be exerted on the surface by the particle at the instant when it is located at B?



PROB. 41

42. A particle of mass m is projected up a smooth inclined plane which makes an angle θ with the horizontal. If the velocity is v_0 at time $t = 0$, at what time will the velocity of m be zero? How far up the plane will the particle have traveled during this time?

43.* A rocket starts from rest and rises vertically against the action of gravity. The propulsion force is a constant P lb. The mass of the rocket is $m = m_0 - ct$, where m_0 is the initial mass and c indicates the rate at which the propellant is burned. When ct is equal to $0.6m_0$, the propulsion force ceases. What velocity will the rocket have attained at that time? What is the displacement of the rocket at any time t? (Neglect air resistance and the variation of the acceleration of gravity with height.)

12. The Equation of Impulse and Momentum.

If the force acting upon a particle is specified as a function of time, the direct integration of the equation of motion, as illustrated in the preceding section, will give the solution of the problem. If a complete specification of the forces is not available, it may still be possible to obtain some information about the motion by obtaining a general first integral of the equation of motion. Two such general integrals can be obtained and we shall first derive the time integral.

Beginning with the equation of motion in the form

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}}$$

and multiplying both sides by dt and integrating, we obtain:

$$\int_{t_1}^{t_2} \boldsymbol{F}\,dt = \int_{t_1}^{t_2} m\ddot{\boldsymbol{r}}\,dt = m\dot{\boldsymbol{r}}\Big|_1^2 = m\dot{\boldsymbol{r}}_2 - m\dot{\boldsymbol{r}}_1$$

$$\int_{t_1}^{t_2} \boldsymbol{F}\,dt = m\boldsymbol{v}_2 - m\boldsymbol{v}_1 \tag{17}$$

The term $\int_{t_1}^{t_2} \boldsymbol{F}\,dt$ is called the *impulse* of the force $\boldsymbol{F}$ and the term $m\boldsymbol{v}$ is called the *momentum* of the particle. Equation (17) thus states that the impulse is equal to the *change* in the momentum.

Both impulse and momentum are vector quantities and hence can be written in terms of components in various coordinate systems. In rectangular components, for example, the equations are:

$$\int_{t_1}^{t_2} F_x\,dt = m\dot{x}_2 - m\dot{x}_1; \quad \text{etc.}$$

It should be noted that the impulse-momentum equation is but another form of the equation of motion and that it furnishes no new information, although its use may simplify the solution of certain problems. In some problems the impulse applied to the system may be known whereas the forces are unknown. In such a problem the impulse-momentum equation gives the change of velocity directly.

Example 1. A body of mass m falls in a vacuum under the action of gravity. The velocity at $t = 0$ is 5 ft/sec downward. Find the velocity at the end of 10 sec.

Solution. In this problem the force is a constant, and the equation of impulse-momentum assumes the particularly simple form:

$$\begin{aligned} F(t_2 - t_1) &= mv_2 - mv_1 \\ mg(10\text{ sec} - 0) &= mv_2 - m(5\text{ ft/sec}) \\ v_2 &= (10\text{ sec})(32.2\text{ ft/sec}^2) + 5\text{ ft/sec} = 327\text{ ft/sec} \end{aligned}$$

Note that this solution could have been obtained by an integration of the equation $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$, but that the use of the impulse-momentum equation leads directly to the required solution.

Example 2. In some problems information concerning the forces acting can be deduced from the momentum changes which occur. Suppose, for example, that measurements have shown that the muzzle velocity of a projectile fired from a gun is v ft/sec and that Δt sec elapse from the time the shell is fired until the projectile leaves the barrel. The change in momentum of the projectile is $(mv - 0) = mv$, so that the equation of impulse-momentum is

$$\int_0^{\Delta t} F\,dt = mv$$

We cannot give a value to F since we do not know the way in which F varies with time. We can, however, compute the *time average* value of the force during the Δt sec, as:

$$(\Delta t)F_{\text{avg}} = mv$$

$$F_{\text{avg}} = \frac{mv}{\Delta t}$$

The relationship of this average force to the instantaneous force might be as shown in Fig. 3–3.

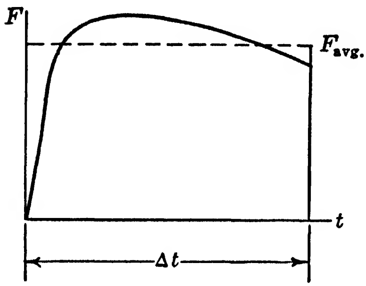


Fig. 3–3

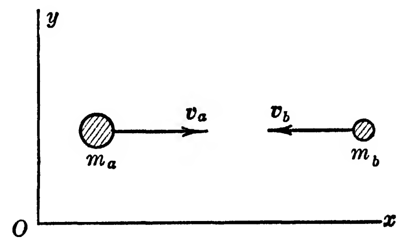


Fig. 3–4

13. Conservation of Momentum. If no force acts upon a particle, the equation of impulse-momentum is:

$$mv_2 - mv_1 = 0$$

$$mv_2 = mv_1$$

If no impulse is acting there is no change in momentum, and the momentum of the system may be said to be conserved.

Consider two particles which exert a mutual action upon each other as, for example, in a collision (Fig. 3–4). From the Third Law of Motion we know that the forces experienced by m_a and m_b during this mutual interaction are equal and opposite. The impulse acting upon m_a is therefore equal and opposite to the impulse acting upon m_b. The total impulse for the system of two particles is thus zero, and hence the total change in momentum of the system must be zero. We may thus state that the total momentum is a constant:

$$m_a\mathbf{v}_a + m_b\mathbf{v}_b = \text{constant}$$

If there are more than two particles involved and all the forces acting upon the particles are due to mutual interactions, that is, there are no external forces applied to the system, we can say:

$$\Sigma m_i \mathbf{v}_i = \text{constant} \tag{18}$$

This is a vector equation and in rectangular coordinates its components are:

$$\Sigma m_i \dot{x}_i = \text{constant, etc.}$$

This is the *Principle of the Conservation of Momentum*, which holds for any system upon which no external forces are acting.

EXAMPLE. A gun barrel is suspended from two long inextensible wires so that it is free to move through small displacements in the horizontal direction, as in Fig. 3-5. A projectile of mass m

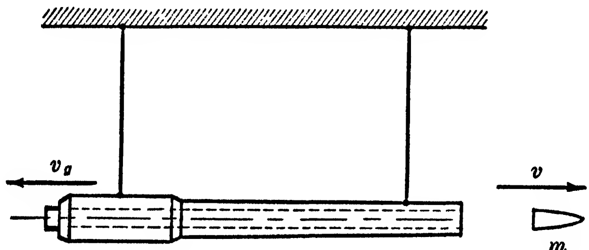


FIG. 3-5

is fired from the gun. At the instant the projectile leaves the muzzle it is observed that the gun of mass m_g has a velocity of v_g to the left. Find the muzzle velocity of the projectile.

Solution. As the explosive gases expand in the barrel the force exerted on the projectile is equal and opposite to the resultant force acting on the gun barrel. We may apply, therefore, the principle of the conservation of momentum to the system consisting of the gun, the projectile, and the gases. If we neglect the small momentum due to the mass of explosive gas, the equation becomes:

$$-m_g v_g + mv = 0$$

$$v = \frac{m_g}{m} v_g$$

PROBLEMS

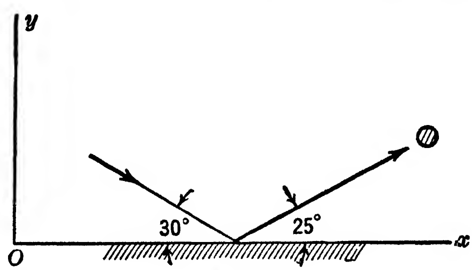
44. A particle of mass m is traveling with a velocity $\dot{x} = v_0$; $\dot{y} = 0$; $\dot{z} = 0$. At time $t = 0$, a constant force, $F_x = -P$, is applied to the particle in a direction such as to oppose the motion. What is the velocity of the mass at any subsequent time t?

45. (a) A bullet weighing 0.1 lb is fired horizontally into the side of a wood block weighing 10 lb which rests on a horizontal surface. If the block attains a velocity of 10 ft/sec, find the velocity of the bullet. (Neglect air resistance and sliding friction.)

(b) How would the solution of the problem be altered if friction were included in the analysis?

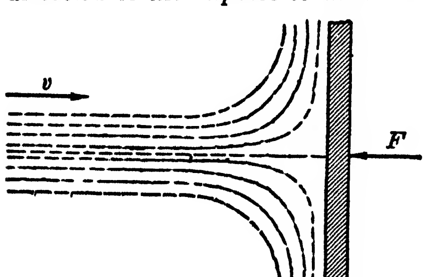
46. A particle weighing 10 lb moves in a straight line with a constant velocity $\dot{x}$ = 10 ft/sec. Starting at time t = 0, a constant force, F_x = 5 lb, acts for 3 sec. Starting at time t = 5 sec, a constant force, F_x = −4 lb, acts for 4 sec. Find the velocity of the mass at the time t = 9 sec.

47. A ball weighing 1 lb is thrown vertically upward; neglecting air resistance, find: (a) The velocity at t = 1 sec, if the velocity at t = 0 is 30 ft/sec. (b) The velocity at t = 0, given that the ball reaches its maximum height after 2.5 sec.



PROB. 48

48. A particle weighing 5 lb bounces against a surface as shown in the diagram. If the approach velocity is 20 ft/sec and the velocity of departure is 15 ft/sec, find the magnitude and direction of the impulse to which the mass is subjected.



PROB. 49

49. A jet of water impinges against a flat plate as shown in the diagram. The velocity of the water is v ft/sec, the density is ρ lb sec^2/ft^4. What is the force exerted by the jet against the plate? Taking the weight of water as 62.4 lb/ft^3, find the force for a jet having an area of 6 in.^{2} and a velocity of 30 ft/sec.

14. The Equation of Work and Energy. The first integral of the equation of motion with respect to time leads to the useful concepts of impulse and momentum. We shall now derive the first integral of the equation of motion with respect to displacement.

We start with the equation of motion in the form:

$$F = m\ddot{r}$$

Forming the dot product of each side with the displacement $\boldsymbol{dr}$, and integrating, we obtain:

$$\int_{r_1}^{r_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{r_1}^{r_2} m\ddot{\boldsymbol{r}} \cdot d\boldsymbol{r} = \int_{t_1}^{t_2} m\ddot{\boldsymbol{r}} \cdot \frac{d\boldsymbol{r}}{dt}\, dt$$

$$= \tfrac{1}{2}m \int_{t_1}^{t_2} \frac{d}{dt}(\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}})\, dt = \tfrac{1}{2}m \int_{t_1}^{t_2} \frac{d}{dt}(v^2)\, dt$$

$$\int_{r_1}^{r_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \tag{19}$$

The integral on the left side of this equation is called the *work* done by the force $\boldsymbol{F}$, and the term $\frac{1}{2}mv^2$ is called the *kinetic energy* of the particle. Thus the equation states that the work done upon the mass m by the force $\boldsymbol{F}$ is equal to the *change* in the kinetic energy of the mass.

The vector displacement $d\boldsymbol{r}$ is tangent to the path of motion of the particle, so that the scalar product $\boldsymbol{F} \cdot d\boldsymbol{r}$ represents the component of the force in the direction of the displacement multiplied by the displacement. The total work done by the force in moving along a path from A to B (Fig. 3–6) is given by the line integral $\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r}$. Expanding the dot product in terms of rectangular coordinates we have:

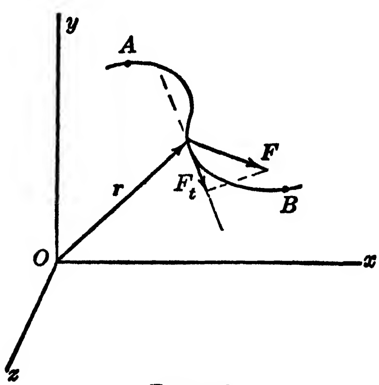


Fig. 3–6

$$Work = \int_A^B (F_x\,dx + F_y\,dy + F_z\,dz)$$

For each coordinate there will be an equation of the form,

$$\int_A^B F_x\,dx = \tfrac{1}{2}m\dot{x}_B^2 - \tfrac{1}{2}m\dot{x}_A^2$$

The rate of doing work, $\boldsymbol{F} \cdot \dfrac{d\boldsymbol{r}}{dt} = \boldsymbol{F} \cdot \boldsymbol{v}$, is called the *power*. It should be noted that work, kinetic energy, and power are scalar quantities, as defined by the dot-products, and are completely specified by their magnitudes.

Since the equation of work and energy is simply a restatement of the original law of motion, it cannot furnish any new information. In many problems, however, the work-energy equation leads directly to simple solutions.

Example. A particle of mass m falls in a vacuum under the action of gravity. Find the velocity of the particle after it has fallen 25 ft, if it starts from rest.

Solution. In this problem, the applied force mg is a constant and has the same direction as the displacement. The work-energy equation thus assumes the simple form:

$$F(x_2 - x_1) = \tfrac{1}{2}m\dot{x}_2^2 - \tfrac{1}{2}m\dot{x}_1^2$$

Setting $x_1 = 0$, $x_2 = 25$ ft, and $\dot{x}_1 = 0$, we have:

$$\begin{aligned}(mg)(25\text{ ft}) &= \tfrac{1}{2}m\dot{x}_2^2 \\ \dot{x}_2^2 &= (2)(25\text{ ft})(32.2\text{ ft/sec}^2) = 1610\text{ ft}^2/\text{sec}^2 \\ \dot{x}_2 &= 40.1\text{ ft/sec}\end{aligned}$$

This problem could also have been solved by direct integration of the equation of motion, but the work-energy relationship gives the result immediately.

PROBLEMS

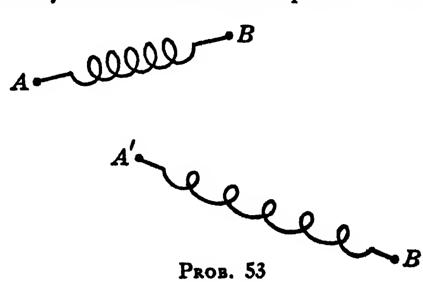
50. Integrate the equation of motion $F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k} = m\ddot{x}\boldsymbol{i} + m\ddot{y}\boldsymbol{j} + m\ddot{z}\boldsymbol{k}$ term by term to obtain the equation of work-energy expressed in rectangular coordinates.

51. A projectile is fired vertically upward. It has a velocity of 400 ft/sec when it leaves the gun barrel. What is its velocity when it has reached an elevation of 200 ft? (Neglect air resistance.)

52. A spring which is initially unstretched is elongated x feet by a force F. What is the total work done by the force F during the elongation? The spring constant of the spring is k, that is, a force of kx lb is required to stretch the spring x ft.

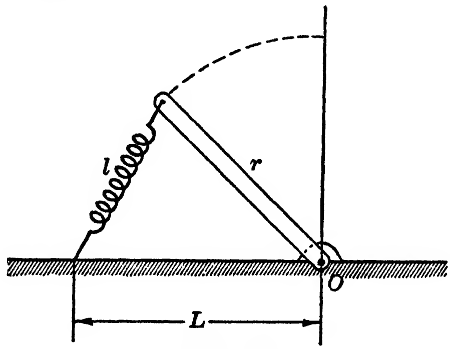
53. A spring which has been initially stretched into the position AB is elongated and displaced into the position $A'B'$ as shown in the diagram. Show that the total work done by the forces which elongate the spring depends only on the change in the length of the spring and on the average force in the spring:

$$\text{Work} = (F_{\text{avg}})(A'B' - AB)$$



Prob. 53

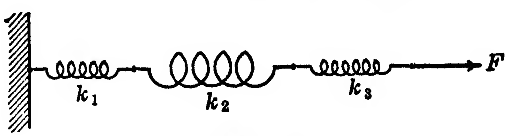
54. A spring of spring constant k whose unstretched length is l is fixed at one end, while the other end is fastened to a rigid bar of length r, as shown in the diagram. How much work will be done by the force



Prob. 54

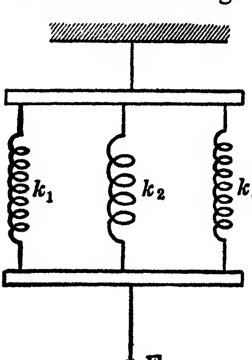
exerted by the spring on the bar as the bar is rotated about O into a vertical position?

55. A spring whose unstretched length is l requires a force of kx lb to elongate it x ft. If three such springs having spring constants k_1, k_2,



Prob. 55

and k_3 are hooked together end-to-end, how much work would be done by a force F as it elongates the system of springs through a total distance δ? In such a system the springs are said to be in series. What is the equivalent spring constant for a system of springs in series in terms of the spring constants of the individual springs?



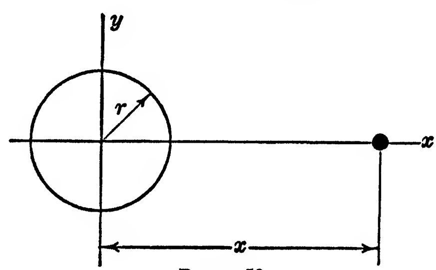
Prob. 56

56. The three springs of Problem 55 are arranged in parallel as shown in the figure. How much work is done by the force F as the assembly is stretched through a distance δ? The springs are initially in an unstretched position and the plates remain parallel. What is the equivalent spring constant for a system of springs in parallel in terms of the spring constants of the individual springs?

57. A particle of mass m is acted upon by a force whose components are $F_x = At$, $F_y = Bt$, $F_z = 0$. At time $t = 0$, the velocity of the mass is zero. What is the work done by the force in the first T sec?

58. A body is projected horizontally with a velocity of 20 ft/sec. Supposing that only gravitational forces act, find its velocity when it is 25 ft below the original position.

59. The force of gravity varies inversely as the square of the distance from the center of the earth. A projectile in space is thus acted upon by a gravitational force $F_x = -W\frac{r^2}{x^2}$, where W is the weight of the projectile at the earth's surface and r is the radius of the earth. How much work must be done against the gravitational force if the projectile is to reach a distance of $(x - r)$ from the earth's surface? Neglecting air resistance, what initial velocity must the projectile have in order to reach that distance? What initial velocity must the projectile have to escape from the earth's gravitational field? Take the radius of the earth as 4000 miles.



Prob. 59

15. Potential. The equation of work-energy for a particle of mass m acted upon by a force $\boldsymbol{F}$ is:

$$\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2$$

The right side of this equation depends only upon the velocities of the particle at the two end-points A and B. The value of the left side, however, will depend in general upon the path of integration followed between A and B. The value of the line integral $\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r}$ will depend only upon the limits of integration and not upon the path only if $\boldsymbol{F} \cdot d\boldsymbol{r}$ is an exact differential. If this is so, there exists some function Φ such that $d\Phi = \boldsymbol{F} \cdot d\boldsymbol{r}$ and

$$\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = \int_A^B d\Phi = \Phi \Big|_A^B = \Phi(B) - \Phi(A)$$

If the function Φ exists, the force $\boldsymbol{F}$ is said to be derivable from a *potential*, for

$$\begin{aligned} \boldsymbol{F} \cdot d\boldsymbol{r} &= d\Phi \\ \boldsymbol{F} \cdot d\boldsymbol{r} &= F_x dx + F_y dy + F_z dz \\ d\Phi &= \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz \end{aligned}$$

From which:

$$F_x = \frac{\partial \Phi}{\partial x}; \quad F_y = \frac{\partial \Phi}{\partial y}; \quad F_z = \frac{\partial \Phi}{\partial z} \tag{20}$$

This potential function Φ is called a *force function.*

When a force is derivable from a potential, the work done by the force is independent of the path of motion and depends only upon the end-points of the path. Since Φ is a function of the space coordinates only, the magnitude and direction of the force are completely determined when its point of application is known. This will be true if the force is a function of the displacement only. If the force is a function of velocity, it cannot be derived from a potential, and the line integral representing the work is not independent of the path of integration.

The concept of a potential function has much wider application than is suggested by the force potential. In fluid mechanics, for example, it is customary to define a velocity potential whose derivatives give the components of velocity, and in thermodynamics several potential functions are defined whose derivatives give certain thermodynamic variables.

16. Potential Energy. Suppose that a force $\boldsymbol{F}$, which is derivable from a potential, acts upon a particle that moves from point A to point B. We define the *change in the potential energy of the force*, $(V_B - V_A)$, as the negative of the work done by the force as it moves from A to B.

$$V_B - V_A = -\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} \tag{21}$$

To specify the potential energy at a point it is necessary to select a datum point at which the potential energy is arbitrarily set equal to zero. Taking some point D as the datum point we have $V_D = 0$ and

$$V_A = \int_A^D \boldsymbol{F} \cdot d\boldsymbol{r} \tag{22}$$

The datum point D is selected at any point that is convenient for the particular problem being considered.

From the definition it is seen that the potential energy is the negative of the force function since $\boldsymbol{F} \cdot d\boldsymbol{r} = -dV$. The components of the force may thus be expressed in terms of the potential energy in the same way in which they were expressed in terms of the force function, and we have:

$$F_x = -\frac{\partial V}{\partial x}; \quad F_y = -\frac{\partial V}{\partial y}; \quad F_z = -\frac{\partial V}{\partial z} \tag{23}$$

The only difference between the potential energy V and the force function Φ, other than sign, is that the potential energy usually involves an additive constant, since it is defined with respect to an arbitrarily chosen datum point; $V = -\Phi + C$. The advantage of using a potential as a description of a force is that it permits an analysis of the force without bringing into the picture the mechanism causing the force or the bodies upon which the force acts. This advantage is particularly useful for forces which act at a distance, such as gravitational and electrical forces.

If a force is not derivable from a potential function, as, for example, frictional forces or forces proportional to velocity, it is not possible to define a potential energy for the force.

17. The Conservation of Energy. If a particle is acted upon by a force which has a potential energy V, the equation of work-energy gives:

$$V_B - V_A = -\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = -\left(\tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2\right)$$

or

$$V_A + \tfrac{1}{2}mv_A^2 = V_B + \tfrac{1}{2}mv_B^2$$

This equation states that *the sum of the potential and the kinetic energy remains constant.* The energy is said to be conserved and we have:

$$V + \tfrac{1}{2}mv^2 = \text{constant} \tag{24}$$

This is the *principle of the conservation of mechanical energy.* It is valid for any system for which a potential energy can be defined. Any system to which the principle of conservation of energy applies is said to be a *conservative system,* and the forces are said to be *conservative forces.* It should be noted that the principle of the conservation of energy is a direct consequence of Newton's Laws and the definitions of the terms involved. It introduces no new physical facts into the science of mechanics.

The principle of conservation of energy is applicable only when the forces of a system have potential energies. If this is not true, for example if frictional forces are acting, the system is said to be non-conservative and the equation of work-energy must be used. The equation of work-energy is thus more general. The use of the principle of conservation of energy is, however, very convenient where conservative systems are involved.

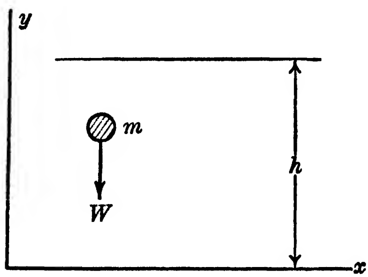


FIG. 3–7

EXAMPLE 1. Consider a body of mass m falling in a vacuum under the action of a gravity force W, as in Fig. 3–7. We shall choose the x-axis as a datum assumed to have a zero potential energy. The potential energy at any vertical distance y is then given by

$$V = -\int_0^y F_y\,dy = -\int_0^y (-W)\,dy = Wy$$

The equation of the conservation of energy is:

$$Wy + \tfrac{1}{2}mv^2 = C$$

The constant C can be evaluated if the velocity at any given value of y is known. Suppose that when $y = h$, $v = v_0$, then

$$Wh + \tfrac{1}{2}mv_0^2 = C$$

and we have

$$Wy + \tfrac{1}{2}mv^2 = Wh + \tfrac{1}{2}mv_0^2$$

Thus, if we wish to know the velocity at any point:

$$v = \sqrt{2g(h - y) + v_0^2}$$

If the body falls in air with a drag force F_d, we no longer have a conservative system, since it is not possible to define a potential energy for the frictional force. In this system energy would be dissipated, and we would have

$$Wy + \tfrac{1}{2}mv^2 = C - \int_h^y F_d\,dy$$

F_d is in general some function of velocity, and a direct evaluation of the integral in this expression is usually rather difficult.

EXAMPLE 2. Consider the motion of a mass m restrained by a massless, elastic spring, as shown in Fig. 3–8. The horizontal surface on which the mass slides is frictionless so that the only forces acting on the mass are the weight W, the vertical reaction F_N, and the spring force F_S. Take the equilibrium position of the system as the datum point for zero potential energy and as the origin of the coordinate system.

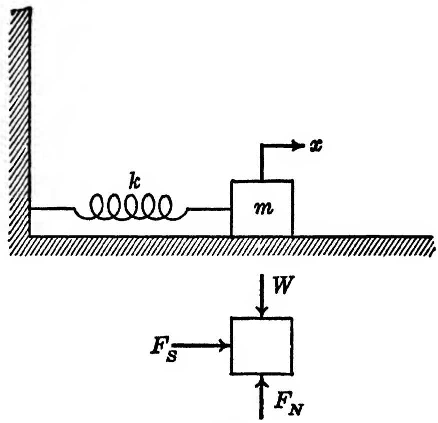


FIG. 3–8

The magnitude of the spring force is proportional to the displacement and is opposite in direction:

$$F_S = -kx$$

The potential energy of the system due to the spring force F_S is, at any position x,

$$V = -\int_0^x (-kx)\, dx = \tfrac{1}{2}kx^2$$

and the energy equation is:

$$\tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2 = C$$

If now the body is given some positive displacement $x = A$ and is then released with a zero velocity, we can evaluate the constant C:

$$\tfrac{1}{2}kA^2 + 0 = C; \quad C = \tfrac{1}{2}kA^2$$

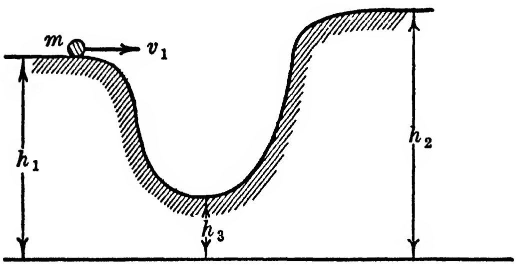
The kinetic energy at any point is thus:

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}kA^2 - \tfrac{1}{2}kx^2$$

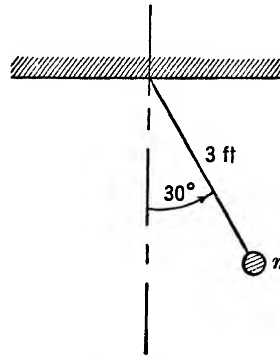
When the body is at $x = 0$, the kinetic energy is equal to $\tfrac{1}{2}kA^2$, so that the body oscillates between $x = +A$ and $x = -A$. The energy is all kinetic energy at $x = 0$ and all potential energy at $x = \pm A$. The sum of the energies is always a constant, but there is a transfer of energy between kinetic and potential.

PROBLEMS

60. A particle of mass m slides without energy loss on a surface as shown in the diagram. When it is at the elevation h_1, it has an initial velocity v_1. What should be the magnitude of v_1 if the particle is to reach the elevation h_2 with a velocity of v_2?



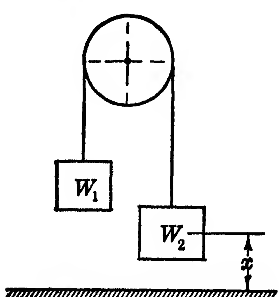
Prob. 60



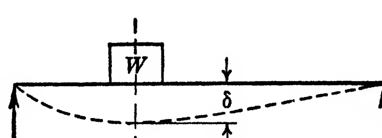
Prob. 61

61. A particle of mass m supported by a weightless string of length 3 ft is released from rest when the string makes an angle of 30° with the vertical. What is the maximum velocity attained under the action of the gravitational force? Assume no energy loss during the motion.

62. Two weights W_1 and W_2 are connected by a cable of length l which passes over a smooth shaft as shown. W_2 is larger than W_1. W_2 starts from rest and moves downward under the action of gravity. Assuming no energy loss during the motion, find the velocity of W_2 after it has moved x ft. If there is a constant friction force F_D between the cable and the shaft, what would be the velocity of W_2 after it has moved x ft?



Prob. 62



Prob. 63

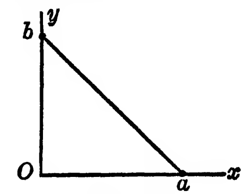
63. A beam is found to deflect δ in. under the point of application of a static load W. It is also found that the magnitude of the deflection is proportional to the load. If the weight W is raised a distance of h ft and is dropped on the beam, what is the maximum deflection of the beam under the load? (Neglect the mass of the beam.)

64.* The force acting upon a particle is given by:

$$F_x = Ax$$
$$F_y = Bx + Cy^2$$

(a) Calculate the work done by this force as the particle is moved from O around the triangular path $OabO$. Is this a conservative force system?

(b) The condition that an expression of the form $Mdx + Ndy$ is an exact differential is that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the present problem, determine whether the system is conservative or non-conservative.



PROB. 64

(c) Repeat parts (a) and (b) if the force components are:

$$F_x = Ax + By$$
$$F_y = Bx + Cy^2$$

(d) What is the potential function Φ for the system of part (c)?

18. The Solution of Problems in Dynamics. The solution of any problem in dynamics involves, in some form or another, the integration of the equation $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$. For problems in which the forces are specified and the velocities and displacements are required as a function of time, the direct integration of this equation may be the most convenient method of procedure. In other problems, some labor may be saved by using the work-energy equation or the impulse-momentum equation. The decision as to which of these forms to use for a particular problem will depend upon the given data and the desired result. As the first step in solving a dynamics problem it is well to review the given data, the required result, and the various dynamical equations from the standpoint of determining which method will be the most suitable.

Some of the characteristics of the methods which should be considered in this connection are:

(1) The impulse-momentum integral involves the velocity and force at specified times. Displacements do not appear in the expression.

(2) The impulse-momentum equation is useful for problems involving large forces of indeterminate magnitudes acting for short times. A consideration of time-average values of the forces may suffice to solve such problems.

(3) The principle of the conservation of momentum can be used only for systems not acted upon by external forces. This principle is most useful when it can be recognized that by treating several bodies together as a system certain unknown forces will occur as equal and opposite pairs and will thus cancel.

(4) The work-energy equation involves velocity and force at specified displacements. Time does not appear in the expression.

(5) The principle of the conservation of energy is applicable only to systems for which a potential energy can be defined.

(6) Potential energy is defined only for forces which can be derived from potential functions. Forces, such as friction, which cause a dissipation of energy, have no potential.

(7) The work-energy principle is more general than the principle of the conservation of energy in that it applies to nonconservative systems as well as conservative systems.

PROBLEMS

65. A 120-ton freight car on a level track hits a spring-type bumper with a velocity of 4 mph. The bumper has a spring constant of 12,000 lb per in. of compression. (a) What is the maximum compression of the spring? (b) If the brakes on the car are operated so that a constant braking force of 25 tons is set up, what is the maximum compression of the bumper spring?

66. A body weighing 10 lb is projected up an inclined plane which makes an angle of 20° with the horizontal. The coefficient of sliding friction between the body and the plane is $\mu = 0.3$. At time $t = 0$, the velocity up the plane is 20 ft/sec. What will be the velocity at the end of 3 sec?

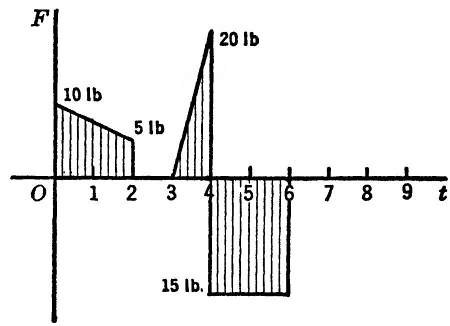
67. A projectile weighing 100 lb strikes the concrete wall of a fort with an impact velocity of 1200 ft/sec. The projectile comes to rest in 0.01 sec, having penetrated 6 ft of the 8-ft thick wall. What is the time average force exerted on the wall by the projectile?

68.* The drag force exerted by the water on a ship weighing 12,000 tons varies with speed according to the following approximate formula:

$$F_D = -kv^n$$

For the range of speeds to be considered in the present problem, n may be taken as 3. If the drag force for a particular ship has been determined to be 80 tons at a speed of 16 ft/sec, find the distance which the unpowered ship would travel as its speed decreases from 15 ft/sec to 12 ft/sec. What is the time required for this decrease?

69. The force acting on a body which weighs 150 lb and which moves in a straight line is given at any time by the accompanying graph. After $t = 6$ sec, $F = 0$. If the velocity of the particle is 25 ft/sec when $t = 0$, find the velocity of the particle and the distance that it has traveled at $t = 8$ sec.



PROB. 69

70. A projectile weighing 30 lb is fired from a gun. At the instant it emerges from the gun barrel it has a velocity of 2000 ft/sec. The distance traveled by the projectile in the gun barrel is 10 ft. What is the total work done on the projectile? What is the average force acting on the projectile over the 10-ft distance? (Assume that the projectile does not rotate in the barrel.)

71. A weight of 3 lb rests on the top of a vertical spring which is compressed 5 in. When the spring is released, the weight is projected vertically into the air. How high will the weight rise above the point at which it leaves the spring, if the spring constant is 25 lb/in.? (Neglect air resistance and assume a massless spring.)

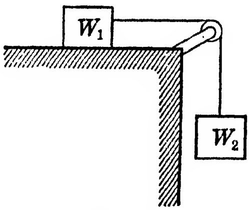
72. An automobile with a total weight of 3000 lb runs into a heavy metal power-pole. After the accident it is observed that the pole is undamaged but that the front bumper of the car is bent. Experiment shows that it requires 30,000 ft-lb of work to put such a bend in the bumper. What was the impact velocity of the automobile?

73. A particle carrying an electric charge e_1 is fixed at the origin of a coordinate system. A second particle of charge e_2 is placed at a distance r from the origin. The potential of the system is

$$\Phi = -\frac{e_1 e_2}{r}$$

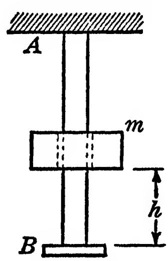
Find the radial force between the particles, and the x-component of the force exerted on the second particle.

74. Two weights, $W_1 = 10$ lb, and $W_2 = 20$ lb, are connected by an inextensible rope as shown in the diagram. W_1 moves on a smooth horizontal surface. If the system starts from rest, what will be the velocity of W_2 after it has fallen 10 ft? What change of momentum has taken place during this motion? What time is required for this displacement to take place?

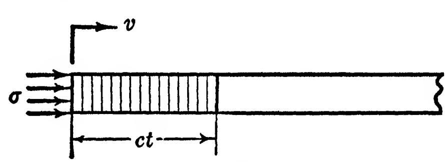


PROB. 74

75. The bar AB shown in the diagram has dimensions and elastic properties such that it requires a force of kx lb to elongate it x ft. A mass m drops through a distance of h ft and strikes the end of the bar. Find the maximum elongation of the bar and the maximum force produced in the bar. (Neglect the mass of the bar, assume that the mass remains in contact with the end of the bar, and assume that no energy is lost during the motion.) If the mass is dropped from $h = 0$, compare the elongation with that which would be produced by m acting statically.



PROB. 75



PROB. 76

76.* A long straight rod of uniform cross-sectional area is initially at rest. One end of the rod is suddenly given a velocity v, by the application of a load which sets up a uniformly distributed stress $\sigma \frac{\text{lb}}{\text{in.}^2}$ over the end of the rod. At a time t later, a length ct of the bar will be compressed, where c is the velocity of propagation of the stress wave along the rod. It will be assumed that the stress in the rod is below the elastic limit of the material so that Hooke's law can be used; hence, $\sigma = E\epsilon$, where E is the modulus of elasticity of the material, and ϵ is the strain, or unit deformation, of the rod.

By applying the principle of impulse and momentum to the strained element of the rod, find the velocity of propagation of the elastic wave in the rod. Find also the relationship between the velocity of the end of the rod and the applied stress. Examine Problem 75 from this viewpoint.

CHAPTER IV

APPLICATIONS OF PARTICLE DYNAMICS

An intelligent being who knew for a given instant all the forces by which nature is animated and possessed complete information on the state of matter of which nature consists—providing his mind were powerful enough to analyze these data—could express in the same equation the motions of the largest bodies of the universe and the motion of the smallest atoms. Nothing would be uncertain for him, and he would see the future as well as the past at one glance. —Marquis de Laplace, *Théorie Analytique des Probabilités* (1820).

The principles of particle dynamics as developed in the preceding chapter can be applied to the solution of a large number of interesting and important problems. In the present chapter the solution of such problems as the motion of a particle in a resisting medium, projectile motion, planetary motion, and some problems in electron dynamics will be given. These solutions will illustrate the application of the general principles to particular problems. It should be noted, however, that, although the principles involved in most problems in mechanics are relatively simple, the differential equations that are obtained may be of a type which cannot be integrated by elementary methods.

19. The Motion of a Body Falling Through a Resisting Medium. In the preceding chapter the equations of motion were integrated for a body falling in a vacuum. It was also indicated at that point that such a solution is approximate for a body falling through a resisting medium. Experiment shows that frictional forces exert a drag which depends upon the shape of the body, the velocity of the body, and the density and viscosity of the medium.* In general there is a certain optimum shape for which the drag is a minimum.

We shall suppose for the present example that the drag force is proportional to the velocity, $\mathbf{F}_D = -k\mathbf{v}$. Experimentally it is found that this expression is satisfactory for small velocities. The

* The drag force depends upon Reynolds number (see Fig. 1–4).

factor k must be determined experimentally. Choosing the positive z-axis in the vertical downward direction (Fig. 4–1), we have for a particle of mass m and weight W:

$$W - k\dot{z} = m\ddot{z}$$

This equation may be integrated directly by putting it into the form:

$$\ddot{z} = -\frac{k}{m}\left(\dot{z} - \frac{W}{k}\right)$$

$$\int \frac{d\left(\dot{z} - \frac{W}{k}\right)}{\left(\dot{z} - \frac{W}{k}\right)} = -\frac{k}{m}\int dt$$

$$\log\left(\dot{z} - \frac{W}{k}\right) = -\frac{k}{m}t + C$$

$$\dot{z} - \frac{W}{k} = Ce^{-\frac{k}{m}t}$$

$$\dot{z} = \frac{W}{k} + Ce^{-\frac{k}{m}t}$$

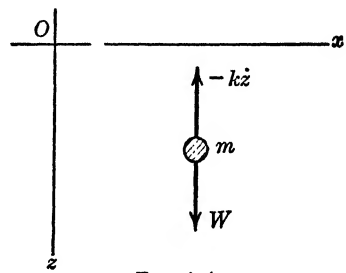


FIG. 4–1

To evaluate the constant C, we take as the initial condition $\dot{z} = 0$ when $t = 0$; this gives $C = -\frac{W}{k}$, so that the equation for the velocity at any time becomes:

$$\dot{z} = \frac{W}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

Plotting this equation in dimensionless form in Fig. 4–2, we note that as $\left(\frac{k}{m}t\right)$ increases, the quantity $\left(\frac{k}{W}\dot{z}\right)$ also increases, but

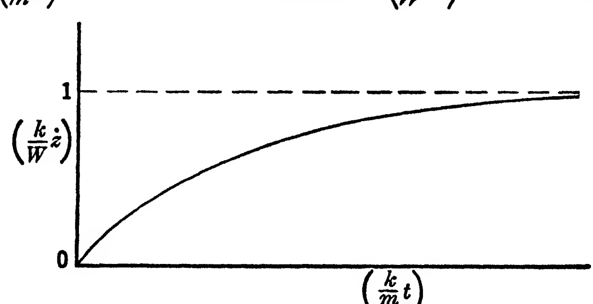


FIG. 4–2

that it approaches a limiting value of $\frac{k}{W}\dot{z} = 1$. When the limiting value $\frac{k}{W}\dot{z} = 1$ is reached, $k\dot{z} = W$; that is, the drag force has become equal to the weight of the body. The resultant force acting upon the body is then equal to zero; hence there is no acceleration and consequently no further increase in velocity. It will be seen that whenever the drag increases with velocity, a point will be reached at which the drag is equal to the weight and there will be no subsequent increase in velocity. This limiting value of velocity is called the *terminal velocity* of the body.

We have supposed, in this example, that the falling body moves through a medium of uniform density. Since the density of the atmosphere decreases with altitude, the drag force must also be a function of the altitude, and this additional factor would have to be included in the analysis if a more accurate solution were required for a large altitude range.

PROBLEMS

77. A body of weight W falls through a resisting medium in which the drag is proportional to the velocity. Find the displacement at any time, assuming that $z_0 = \dot{z}_0 = 0$ when $t = 0$.

78. For velocities above approximately 100 ft/sec the drag force is approximately described by taking it as proportional to the square of the velocity, $F_D = -kv^2$. Proceeding in the same way as was done above for drag proportional to velocity, integrate the equation of motion and find the velocity and displacement of a falling body at any time t, if $z_0 = \dot{z}_0 = 0$ when $t = 0$.

79. What is the terminal velocity for a falling body subjected to a drag proportional to velocity squared?

80. Determine the velocity with which a rain drop would strike the ground falling from a height of 1 mile if air resistance is neglected. If measurements show that the terminal velocity of the rain drop is approximately 20 ft/sec, find the drag constant k, assuming $F_D = -kv$. How far does the rain drop fall before its velocity is within 0.1% of the terminal velocity?

81.* The relation $\ddot{s} = -k(\dot{s})^n$ is assumed to describe the motion of a certain body in a viscous medium.

(a) If the body has an initial velocity, v_0, find the highest value of n for which it is brought to rest within a finite period of time. Find the time required.

(b) Find the highest value of n for which the total distance traversed by the body before it comes to rest is finite.

(c) Find the distance traveled by the body for $n = 1$.

82. An airplane weighing 20,000 lb starts from rest and accelerates along a horizontal runway. Acting on the airplane is a constant propulsion force of 2500 lb, and a drag force $F_d = -0.04v^2$ lb., where v is in ft/sec. How long a run must the plane make if it takes off at 150 mph?

20. Projectile Motion. The preparation of ballistic tables requires precise calculations of the trajectories, velocities, and times of flight of projectiles. The chief difficulty in making such calculations arises from the fact that the drag is a complicated function of the velocity. For purposes of illustration some simplified problems that are amenable to mathematical treatment will be considered. We shall suppose that the projectile remains sufficiently close to the earth so that g may be taken as a constant, and we shall neglect the rotation of the earth and any spin or other motion of the projectile as a rigid body. All these factors would have to be taken into account in a precise calculation of the trajectory of a long-range projectile or guided missile.

Consider first the two-dimensional motion of a projectile with zero drag. Let the projectile have an initial velocity v_0 making an angle θ with the x-axis of the rectangular coordinate system of Fig. 4–3. The equations of motion are:

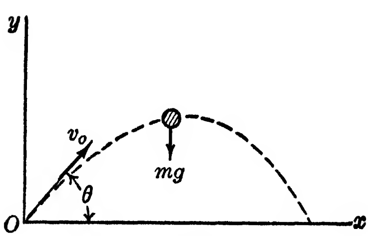


Fig. 4–3

$$m\ddot{x} = 0$$
$$m\ddot{y} = -mg$$

Integrating these equations and evaluating the constants in terms of the initial conditions $v_{0x} = \dot{x}_0$, $v_{0y} = \dot{y}_0$, $x_0 = y_0 = 0$, we have:

$$x = \dot{x}_0 t$$
$$y = -\tfrac{1}{2}gt^2 + \dot{y}_0 t$$

The equation for the trajectory is obtained by eliminating t from these equations:

$$y = -\tfrac{1}{2}g\left(\frac{x}{\dot{x}_0}\right)^2 + \dot{y}_0\left(\frac{x}{\dot{x}_0}\right)$$

which shows that the trajectory is a parabola. All the significant features of the motion can be determined from these equations.

A more practical approach to the ballistic problem must take into account the effect of air resistance. This leads to a complex mathematical problem which is usually solved by numerical integration or by special computing machines. A few special problems can be treated by simpler mathematical procedures and it is one of these which we shall treat in the following example. For an intermediate range of velocities, of approximately 100 to 1000 ft/sec, it may be assumed that the drag force is approximately proportional to the square of the velocity. If we further assume that the variation in altitude of the projectile is small, so that variations in air density may be neglected, we may write the equations of motion as:

$$
\begin{aligned}
m\ddot{x} &= -\,kv^2\cos\phi = -\,k\dot{x}^2\left[1+\left(\frac{\dot{y}}{\dot{x}}\right)^2\right]^{\frac{1}{2}}\\
m\ddot{y} &= -\,kv^2\sin\phi - mg\\
&= -\,k\dot{x}\dot{y}\left[1+\left(\frac{\dot{y}}{\dot{x}}\right)^2\right]^{\frac{1}{2}} - mg
\end{aligned}
$$

Since the problem of solving these equations is a complex one, we shall simplify them by restricting the problem to a consideration of relatively flat trajectories for which the ratio $\left(\frac{\dot{y}}{\dot{x}}\right)$ is small. This is consistent with the assumption that the variation in altitude of the projectile is small. It may be seen from Fig. 4–4 that

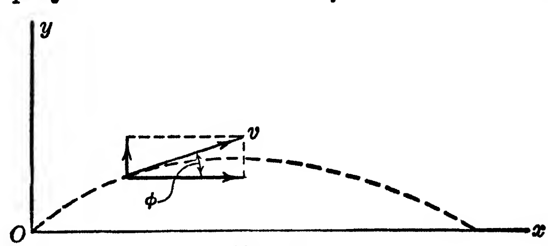


Fig. 4–4

small values of $\left(\frac{\dot{y}}{\dot{x}}\right)$ mean that the slope of the trajectory is everywhere small. With this assumption we have $\left[1+\left(\frac{\dot{y}}{\dot{x}}\right)^2\right]^{\frac{1}{2}} \approx 1$, and the differential equations become:

$$
\begin{aligned}
m\ddot{x} &= -k\dot{x}^2\\
m\ddot{y} &= -k\dot{x}\dot{y} - mg
\end{aligned}
$$

From these equations a relatively simple solution can be obtained which is satisfactory over a limited range of trajectories. The first equation can be readily solved by separating the variables:

$$\frac{d\dot{x}}{\dot{x}^2} = -\frac{k}{m}\,dt$$

$$-\frac{1}{\dot{x}} = -\frac{k}{m}t + C_1$$

When $t = 0$, $\dot{x} = \dot{x}_0$ so that $C_1 = -\frac{1}{\dot{x}_0}$, thus:

$$\dot{x} = \frac{\dot{x}_0}{\frac{k\dot{x}_0}{m}t + 1}$$

Substituting this value of $\dot{x}$ into the second differential equation gives:

$$m\ddot{y} + \frac{k\dot{x}_0}{\frac{k\dot{x}_0}{m}t + 1}\dot{y} = -mg$$

or

$$\ddot{y} + \frac{1}{t + \frac{m}{k\dot{x}_0}}\dot{y} = -g$$

This equation is a linear differential equation for which the principle of superposition is valid, that is, if two expressions are found each of which satisfies the equation, then the sum of the two expressions will satisfy the equation. The equation may thus be solved in two steps. Consider first the homogeneous equation, with the right side equal to zero instead of $-g$. The variables can be then separated and we have:

$$\frac{d\dot{y}}{\dot{y}} = -\frac{dt}{t + \frac{m}{k\dot{x}_0}}$$

$$\log \dot{y} = \log\left(t + \frac{m}{k\dot{x}_0}\right)^{-1} + \log C_2$$

$$\dot{y} = \frac{C_2}{t + \frac{m}{k\dot{x}_0}}$$

This expression is not the complete solution, however, because it gives zero instead of $-g$ when substituted into the original differential equation. We must therefore add to this solution a term which will give $-g$ when substituted into the differential equation.

If, on the basis of an inspection of the differential equation, we try an expression of the form:

$$\dot{y} = C_3\left(t + \frac{m}{k\dot{x}_0}\right)$$

we obtain upon substituting this into the differential equation:

$$C_3 + C_3 = -g; \quad C_3 = -\frac{g}{2}$$

So that the complete solution of the equation is:

$$\dot{y} = \frac{C_2}{t + \dfrac{m}{k\dot{x}_0}} - \frac{g}{2}\left(t + \frac{m}{k\dot{x}_0}\right)$$

If $\dot{y} = \dot{y}_0$ when $t = 0$, then:

$$\dot{y}_0 = \frac{C_2}{\dfrac{m}{k\dot{x}_0}} - \frac{mg}{2k\dot{x}_0}$$

and

$$C_2 = \frac{m}{k\dot{x}_0}\left(\dot{y}_0 + \frac{mg}{2k\dot{x}_0}\right)$$

So that:

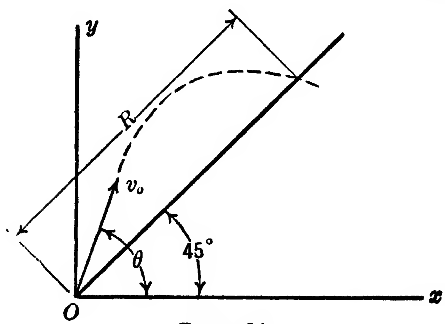
$$\dot{y} = \frac{\dot{y}_0 + \dfrac{mg}{2k\dot{x}_0}}{\left(\dfrac{k\dot{x}_0}{m}t + 1\right)} - \frac{mg}{2k\dot{x}_0}\left(\frac{k\dot{x}_0}{m}t + 1\right)$$

This expression may now be checked by substituting it into the differential equation and verifying that the equation is satisfied. The equations for $\dot{x}$ and $\dot{y}$ can be integrated to obtain expressions for x and y.

It must be kept in mind that these equations are satisfactory only for trajectories which satisfy the assumed condition that $\left(\frac{\dot{y}}{\dot{x}}\right)$ is small. The method of obtaining approximate solutions by dropping small terms from a differential equation is often a convenient procedure. The justification for it is that an analytical expression is obtained for the solution which is approximately correct over a particular range of interest in the variables. It might otherwise be necessary to perform a numerical or graphical integration which would not only be very laborious, but which would probably not exhibit the answer in a general form. Simplifications of this type will always have a physical interpretation which should be studied carefully, so that the exact nature of the limitations on the solution will be known. In the above example, the approximation is deduced by noting that there is only a small angle between the resisting force and the x-axis. From the differential equations in their simplified form we can see that this is equivalent to saying that the small vertical velocity has no effect upon the horizontal drag, but that the large horizontal velocity does have an effect upon the vertical drag. Such solutions, of course, must be used with caution.

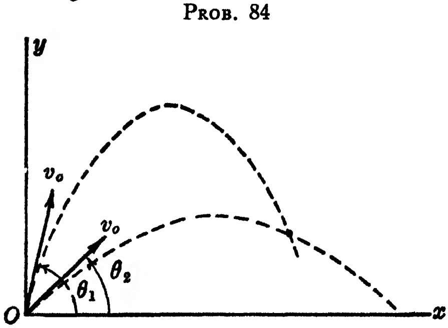
PROBLEMS

83. Find the horizontal range of a projectile having zero drag (Fig. 4–3), and find the angle θ which will make this horizontal range a maximum.



PROB. 84

84. Find the maximum range and the angle θ for the maximum, if the range is measured along a 45° slope, as shown in the diagram, and zero drag is assumed.



PROB. 85

85. Two particles are projected from the same point with the same magnitude of velocity but with different angles of elevation, as shown in the diagram. The second particle is fired a time Δt later than the first particle. What is the relation between v, θ_1, θ_2, and Δt for which the two particles will collide? (Assume zero drag.)

86. In section 20 the equations are given for the x and y components of the velocity of a projectile which is subjected to a drag force proportional to the square of the velocity. These components were worked out for a flat trajectory for which the ratio $\left(\frac{\dot{y}}{\dot{x}}\right)$ is small. From these expressions find the x and y coordinates of the projectile as a function of time, under the same assumption of a flat trajectory.

87. Assume that the total drag on an airplane is proportional to the square of the velocity. With a feathered propeller, the plane is put into a straight glide making an angle α with the horizontal. What is the expression for the velocity of the plane? What is the expression for the terminal velocity?

88. For a relatively slow-speed projectile, the air drag force can be assumed to be proportional to the velocity. Find the horizontal distance which such a projectile must travel before the tangent to the trajectory becomes horizontal.

21. Two-dimensional Harmonic Motion. In this section we shall consider the motion of a particle of mass m which can move in the xy plane, and which is restrained by equal springs in the

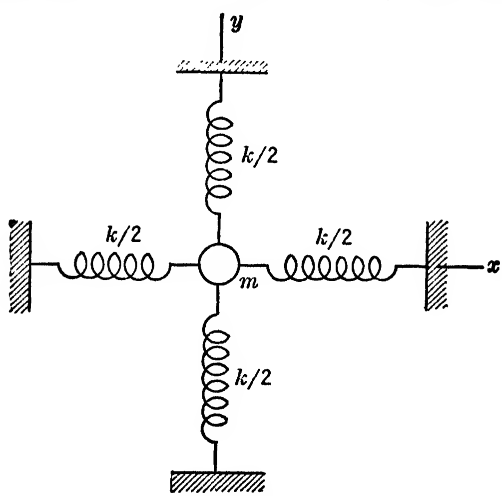


FIG. 4–5

x and y directions (Fig. 4–5). This is a problem frequently encountered in mechanical vibrations. The mass m, for example, might represent a machine, and the springs the vibration isolating mount of the machine.

The motion of the mass is to be restricted to small displacements so that the spring force in the x-direction can be considered to be independent of the y-displacement and vice versa. One method of solution is to write the differential equations of motion and integrate in the customary fashion. For purposes of illustration, however, we shall analyze the problem from the energy standpoint. The potential energy of the system is:

$$V = \tfrac{1}{2}k(x^2 + y^2) = \tfrac{1}{2}kr^2$$

where k is the spring constant of each pair of springs. The system has zero potential energy when the mass is at the origin (Fig. 4–6). The forces acting upon the mass are:

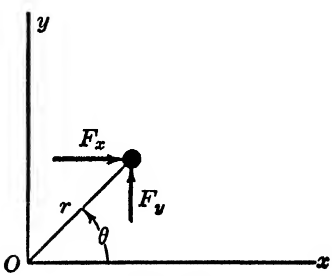


FIG. 4–6

$$F_r = -\frac{\partial V}{\partial r} = -kr$$

$$F_\theta = -\frac{\partial V}{\partial \theta} = 0$$

or

$$F_x = -\frac{\partial V}{\partial x} = -kx$$

$$F_y = -\frac{\partial V}{\partial y} = -ky$$

Since the system is conservative, the energy equation is:

$$\tfrac{1}{2}kr^2 + \tfrac{1}{2}mv^2 = C = \tfrac{1}{2}k(x^2 + y^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

If the energy equation contains no terms involving products of displacements and velocities the equations of motion in terms of the accelerations can be derived by differentiating the energy equation with respect to time as follows:

$$kx\dot{x} + ky\dot{y} + m\dot{x}\ddot{x} + m\dot{y}\ddot{y} = 0$$

Combining terms and multiplying by dt, this can be written:

$$(m\ddot{x} + kx)\,dx + (m\ddot{y} + ky)\,dy = 0$$

Since it is a condition of the problem that dx and dy are independent, that is, the mass is not constrained to follow a prescribed path, the preceding equation must be satisfied for all possible values of dx and dy. This requires each of the two expressions within parentheses to be equal to zero, that is,

$$m\ddot{x} + kx = 0$$
$$m\ddot{y} + ky = 0$$

These are the equations of motion. If dx and dy had not been independent, say the mass were constrained to move only along the path $y = f(x)$, then the values of y, $\ddot{y}$, and dy, given by this equation would be substituted into the differentiated energy equation in order to obtain the equation of motion.

The differential equations of motion may be integrated directly by means of an integrating factor. For the x-equation, for example, multiply through by $\dot{x}$ and obtain:*

$$m\dot{x}\ddot{x} + kx\dot{x} = 0 = \frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + \frac{kx^2}{2}\right)$$

Thus:

$$m\dot{x}^2 + kx^2 = C_1$$
$$\dot{x} = \sqrt{\frac{C_1}{m} - \frac{kx^2}{m}} = \frac{dx}{dt}$$
$$t = \int \frac{dx}{\sqrt{\frac{C_1}{m} - \frac{kx^2}{m}}} + C_2 = \sqrt{\frac{m}{k}} \sin^{-1} \frac{x}{\sqrt{\frac{C_1}{m}}} + C_2$$
$$x = \sqrt{\frac{C_1}{m}} \sin\left(\sqrt{\frac{k}{m}}\, t - \sqrt{\frac{k}{m}}\, C_2\right)$$

Introducing the two new constants

$$A_x = \sqrt{\frac{C_1}{m}}, \quad B_x = -\sqrt{\frac{k}{m}}\, C_2$$

which are to be determined from the initial conditions of the problem:

$$x = A_x \sin\left(\sqrt{\frac{k}{m}}\, t + B_x\right)$$

In a similar way it is found that

$$y = A_y \sin\left(\sqrt{\frac{k}{m}}\, t + B_y\right)$$

* The use of the integrating factor results in the energy equation so that we are retracing the steps by which we obtained the differential equation from the energy equation.

These expressions can be put into another form by expanding the sine and by introducing new constants, obtaining:

$$\begin{aligned} x &= C_1 \sin \sqrt{\frac{k}{m}}\, t + C_2 \cos \sqrt{\frac{k}{m}}\, t \\ y &= C_3 \sin \sqrt{\frac{k}{m}}\, t + C_4 \cos \sqrt{\frac{k}{m}}\, t \end{aligned} \tag{25}$$

A body having displacements x and y as given by these expressions is said to perform *simple harmonic motion* in two dimensions.

In Chapter V the physical significance of such solutions will be discussed in connection with vibration problems. More direct methods of integrating the equations of motion will also be taken up at that time.

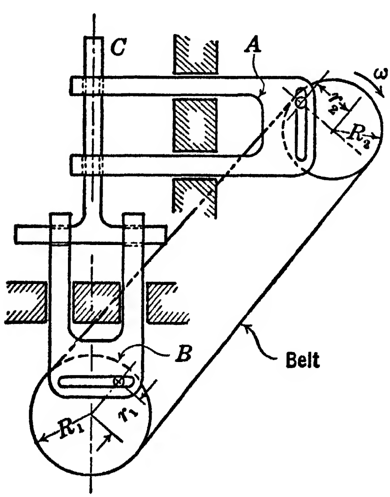
PROBLEMS

89. Show that in two-dimensional simple harmonic motion the mass m describes an elliptical path about the origin. Take as the initial conditions $y = \dot{x} = 0$, $x = x_0$, $\dot{y} = \dot{y}_0$ when $t = 0$.

90. Suppose that a particle of mass m is acted upon by a central force $F_r = -\frac{k}{r^2}$, the magnitude of the force being inversely proportional to the square of the distance from the origin. Find the potential energy of the system and write the equation of the conservation of energy. From this equation find the equations of motion in terms of accelerations in the x and y coordinates.

91. A body moves in the xy plane under the action of a force whose potential energy is $V = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2$. Derive the equations for the displacement of the system at any time. What physical problem does this potential energy describe?

92. A two-dimensional simple harmonic motion can be produced mechanically by the device shown in the diagram. The frame A can move only in the x-direction and the frame B can move only in the y-direction. Each frame is given a simple harmonic motion of amplitude r by means of a scotch-yoke mechanism as shown. The frequency ratio



PROB. 92

of the two motions is fixed by the ratio of R_1 to R_2, the radii of the two pulleys which are connected by a belt. The upper pulley is driven at a constant angular velocity ω. The end of the rod C will describe various curves, called Lissajous figures, the shape of which will depend upon the ratios r_1/r_2 and R_1/R_2. Suppose in the present problem that the pin driving the horizontal frame A is in its extreme top position when the pin driving the vertical frame B is in its extreme left position.

(a) Write the equations for the x and y motion of the point C as a function of time.

(b) What curve is described by C when $R_1 = R_2$, and $r_1 = 2r_2$?

(c) What curve is described by C when $r_1 = r_2$, $R_1 = 2R_2$?

(d) Show that if R_1 and R_2 are incommensurable, the curve traced by C never closes.

22. Planetary Motion. As a second example of two-dimensional motion under a central force, we shall consider the problem of planetary motion. This problem is particularly interesting as an example of the method of deducing general laws from experimental observations. By studying a large amount of experimental data, Kepler determined the following three facts about the motions of the planets:*

(1) *The orbit of each planet is an ellipse with the sun at one focus.*

(2) *The radius vector drawn from the sun to the planet sweeps over equal areas in equal times.*

(3) *The squares of the periods of the planets are proportional to the cubes of the semi-major axes of the elliptical orbit.*

It will be of interest to see how, from these statements of empirical facts, Newton was able to deduce the law of gravitation.†

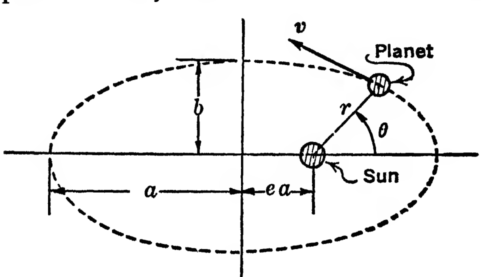


Fig. 4–7

Using the notation of Fig. 4–7, Kepler's three statements may be written analytically in the form:

(1) $$r = \frac{p}{1 + e\cos\theta}$$

where

$$p = b\sqrt{1 - e^2} = \frac{b^2}{a}$$

* J. Kepler (1571–1630). The first two statements were published in 1609, and the third in 1619.

† Certain letters of Newton indicate the methods he used.

(This is the equation of an ellipse in polar coordinates.)

(2) $$A = \int \frac{r^2}{2}\, d\theta = \int \frac{r^2}{2} \frac{d\theta}{dt}\, dt = \frac{k}{2} t$$

(3) T^2/a^3 = constant, where T is the period of a complete revolution.

In plane polar coordinates, the equations of motion of the planet are (Equations (8)):

$$F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$$
$$F_r = m(\ddot{r} - r\dot{\theta}^2)$$

Differentiating (2) above gives:

$$r^2\dot{\theta} = k$$

A second differentiation gives:

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

It thus appears that $F_\theta = 0$, and we conclude that the force on the planet must be radial.

The radial force can be determined by differentiating (1), proceeding as follows:

$$r = \frac{p}{1 + e\cos\theta}.$$
$$\dot{r} = \frac{pe\sin\theta}{(1 + e\cos\theta)^2}\dot{\theta}$$

In this expression for $\dot{r}$, substitute $r^2\dot{\theta} = k$ and $(1 + e\cos\theta) = \frac{p}{r}$ obtaining:

$$\dot{r} = \frac{ek}{p}\sin\theta$$

Differentiating this expression again, and substituting $r^2\dot{\theta} = k$, we have:

$$\ddot{r} = \frac{e}{p}\frac{k^2}{r^2}\cos\theta$$

From the equation of the orbit $\cos\theta = \frac{1}{e}\left(\frac{p}{r} - 1\right)$, therefore:

$$\ddot{r} = \frac{k^2}{pr^2}\left(\frac{p}{r} - 1\right) = \frac{k^2}{r^3} - \frac{k^2}{pr^2}$$

The expression for the radial force may thus be written:

$$F_r = m(\ddot{r} - r\dot{\theta}^2) = m\left(\frac{k^2}{r^3} - \frac{k^2}{pr^2} - \frac{k^2}{r^3}\right)$$

$$F_r = -\frac{mk^2}{pr^2}$$

and the magnitude of the radial force is *inversely proportional to the square of the distance.*

The force is now completely determined except for the factor k^2 which may depend upon the mass of the planet and the mass of the sun. From Kepler's second statement, we have for one complete period T:

$$\frac{k}{2}T = \pi ab, \quad k = \frac{2\pi ab}{T}$$

So that:

$$F_r = -\frac{m}{pr^2}\cdot\frac{4\pi^2a^2b^2}{T^2} = -4\pi^2\left(\frac{a^3}{T^2}\right)\frac{m}{r^2}$$

From Kepler's third statement, that (a^3/T^2) is the same for all of the planets, it is clear that its value depends only upon the sun. Since the force is directly proportional to the mass of the planet, we assume it is also directly proportional to the mass of the sun. Writing $\gamma m_2 = -4\pi^2(a^3/T^2)$, where m_2 is the mass of the sun, we have:

$$F_r = \gamma\frac{m_1m_2}{r^2} \tag{26}$$

where m_1 is the mass of the planet, and γ is a gravitational constant. Newton tested this result by computing, from the motion of the moon about the earth, the gravitational acceleration at the earth's surface. He then was able to check the computation against observed values.

PROBLEMS

93. Compute the value of the acceleration of gravity g from the motion of the moon about the earth, and compare this with values experimentally determined at the earth's surface. Take the radius of the earth as $R = 3950$ miles, the radius of the moon's orbit as $60R$, and the period of the moon revolving about the earth as 39,000 minutes. The gravitational attraction on a body is $W\frac{R^2}{a^2}$, where W is the weight of the body at the surface of the earth, and a is the distance of the body from the center of the earth. This attractive force is also given by the expression $4\pi^2 \frac{a^3}{T^2} \frac{m}{r^2}$. The value of g can be computed from the fact that these two expressions when equated should give $W = mg$.

94. How much energy is required to establish motion of a rocket ship of mass m in a stable circular orbit of radius r about the earth? Take the radius of the earth as R, and neglect air resistance.

95.* A particle of mass m is attracted to the origin of a coordinate system by a force which is inversely proportional to the square of the distance; $F_r = -\frac{mK}{r^2}$.

(a) Show that the equation for the conservation of energy of the system becomes:

$$\frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{mK}{r} = \text{constant} = C$$

(b) From the fact that $F_\theta = 0$, we have $r^2\dot{\theta} = k$. Substituting this into the energy expression, and making the transformation $u = \frac{1}{r}$, show that the differential equation of the orbit is:

$$\left(\frac{du}{d\theta}\right)^2 = \frac{e^2}{p^2} - \left(u - \frac{1}{p}\right)^2$$

where

$$\frac{1}{p} = \frac{K}{k^2}$$

$$e^2 = 1 + \frac{2Ck^2}{mK^2}$$

(c) Integrating directly the differential equation of the orbit, show that the equation of the orbit is:

$$r = \frac{p}{1 + e\cos\theta}$$

This is the equation of a conic having one focus at the origin. If $e < 1$, the conic is an ellipse; for $e = 1$, the conic is a parabola; and for $e > 1$, the conic is a hyperbola.

(d) Show that the form of the orbit, that is, the type of conic, depends on the total energy of the system and hence on the magnitude of the initial velocity, and not on the direction of the initial velocity.

23. Impact. The problem of impact between two bodies is characterized by the presence of forces of large magnitude and short time duration. Because of these forces sudden changes occur in the velocities of the bodies, and it is these velocity changes which are ordinarily observed and measured in impact experiments. If the forces acting on the bodies were known, the solution of an impact problem would require only the integration of the equation of motion. Experimental difficulties, however, make the precise measurement of impact forces difficult, so that a different method of solution of the problem is usually required. The motion of the bodies during impact must always satisfy the momentum equation, the energy equation, and the third law of motion, that action is equal and opposite to reaction. These are sufficient to determine all the features of the motion except those occurring during the time of impact. Since the impact time interval is usually very short, of the order of milliseconds, only small errors are introduced by assuming an instantaneous impact. If the time interval of impact is not short, the approximate nature of the solution must be kept in mind.

As an illustration of the methods used in solving impact problems, consider two smooth, spherical bodies colliding with known velocities, as in Fig. 4–8. At impact two equal and opposite forces act normal to the surface of each sphere at the point of contact. The location and direction of the forces can be determined from the geometry of the problem, and hence the location and the direction of the impulses are known. Considering the two spheres together as a system, there are no external forces acting, and the equation of the conservation of momentum can be written:

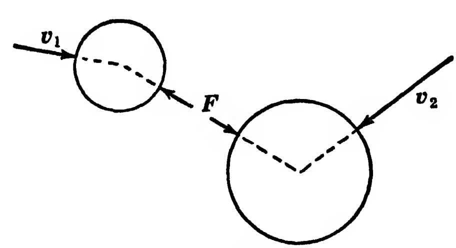


Fig. 4–8

$$m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 = m_1\boldsymbol{V}_1 + m_2\boldsymbol{V}_2 \tag{27}$$

where $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ are the velocities before and after impact, respectively. This vector equation is equivalent to three scalar equations containing six unknown velocity components. If there is no energy loss during impact the equation of the conservation of energy must be satisfied, so that

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2 \tag{28}$$

We now have four equations for determining the six unknown velocity components. The two additional equations are obtained from the known directions of the impulses which act upon the bodies. Designating the rectangular components of the impulse by I_x, I_y, I_z, we have:

$$\frac{I_x}{I_z} = k_1$$

$$\frac{I_y}{I_z} = k_2$$

where k_1 and k_2 are known from the geometry of the system (Fig. 4–9). Since the impulse is equal to the change in momentum, we may write:

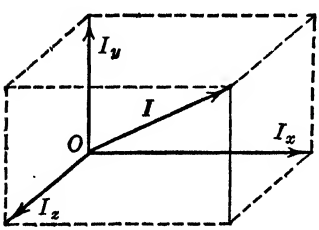


Fig. 4–9

$$\frac{V_{1x} - v_{1x}}{V_{1z} - v_{1z}} = k_1 \qquad \frac{V_{1y} - v_{1y}}{V_{1z} - v_{1z}} = k_2 \tag{29}$$

From the total of six equations now available, the six unknown velocity components can be determined. In this discussion rotation of the spheres was not involved, so that the spheres were treated as particles. If rotation were involved, as for the collision of rough spheres, six unknown angular velocity components would be introduced, and the momentum and energy equations for the angular motions would have to be considered. The problem would then involve twelve equations and twelve unknowns. In most practical impact problems, however, it will be found that only a few unknowns are involved.

Whenever there is an impact between actual bodies, there will always be some loss of energy. If the impact velocity is small, this energy loss, for many purposes, may be neglected and the equation of conservation of energy may be used as above. If, however, the impact forces are sufficiently large to produce permanent deformations of the bodies, the work done in producing these deformations represents an energy loss which may be too large to be neglected. The energy equation then becomes:

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 - E = \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2$$

where E is the energy loss during impact. It was pointed out by Newton that the information in this equation could be stated in a more useful form by the following method. Consider a direct impact where the approach and rebound velocities are parallel to the x-axis. Write first the energy and momentum equations for no energy loss:

$$\tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 = \tfrac{1}{2}m_1\dot{X}_1^2 + \tfrac{1}{2}m_2\dot{X}_2^2$$
$$\tfrac{1}{2}m_1\dot{x}_1 + \tfrac{1}{2}m_2\dot{x}_2 = \tfrac{1}{2}m_1\dot{X}_1 + \tfrac{1}{2}m_2\dot{X}_2$$

Recombining the terms gives:

$$\tfrac{1}{2}m_1(\dot{x}_1 - \dot{X}_1)(\dot{x}_1 + \dot{X}_1) = -\tfrac{1}{2}m_2(\dot{x}_2 - \dot{X}_2)(\dot{x}_2 + \dot{X}_2)$$
$$\tfrac{1}{2}m_1(\dot{x}_1 - \dot{X}_1) = -\tfrac{1}{2}m_2(\dot{x}_2 - \dot{X}_2)$$

Dividing the first equation by the second:

$$\dot{x}_1 + \dot{X}_1 = \dot{x}_2 + \dot{X}_2$$

or

$$\dot{X}_1 - \dot{X}_2 = -(\dot{x}_1 - \dot{x}_2)$$

This equation states that the relative rebound velocity after impact is equal and opposite to the relative velocity of approach. This is equivalent to stating that there is no energy loss during the impact. If there is a loss of energy, the two relative velocities will not be equal, but $(\dot{X}_1 - \dot{X}_2)$ will be smaller than $(\dot{x}_1 - \dot{x}_2)$. The energy loss, therefore, can be determined by measuring the relative velocities, and we may say:

$$(\dot{X}_1 - \dot{X}_2) = -e(\dot{x}_1 - \dot{x}_2) \tag{30}$$

where e is a number less than unity. The quantity e, called the *coefficient of restitution*, is thus a measure of the energy loss. When $e = 1$, with no energy loss, the impact is said to be perfectly elastic. If $e = 0$, the impact is said to be plastic, and the two colliding bodies remain in contact after impact with zero relative rebound velocity.

PROBLEMS

96. Show that for direct central impact, that is, direction of rebound same as direction of approach, the velocities after impact are given by the following equations if there is no energy loss:

$$V_1 = \frac{2m_2v_2 + (m_1 - m_2)v_1}{m_1 + m_2}$$

$$V_2 = \frac{2m_1v_1 + (m_2 - m_1)v_2}{m_1 + m_2}$$

97. Show that for direct central impact with coefficient of restitution e, the velocities after impact are given by the following equations:

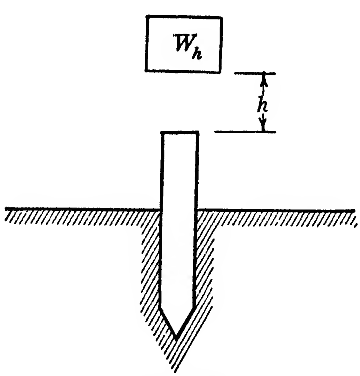
$$V_1 = \frac{m_2v_2(1 + e) + (m_1 - em_2)v_1}{m_1 + m_2}$$

$$V_2 = \frac{m_1v_1(1 + e) + (m_2 - em_1)v_2}{m_1 + m_2}$$

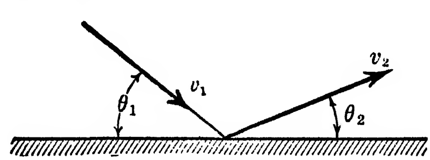
98. Compute the percentage loss in kinetic energy which takes place in a direct central impact if $m_1 = m_2$, $v_1 = -v_2$, and the coefficient of restitution is e.

99. In a pile driving operation, a hammer of weight W_h falls through a height h and makes a plastic impact with the pile of weight W_p. The penetration of the pile is resisted by a constant force R, which is chiefly due to the friction between the earth and the pile. Show that if the pile penetrates a distance x after impact then $R = \dfrac{W_h^2h}{(W_h + W_p)x}$. (This expression neglects the work done by gravity forces after impact and also assumes instantaneous impact.)

100. A golf ball dropped from rest from a height h rebounds from a steel surface to a height $0.85h$. What is the coefficient of restitution?



PROB. 99



PROB. 101

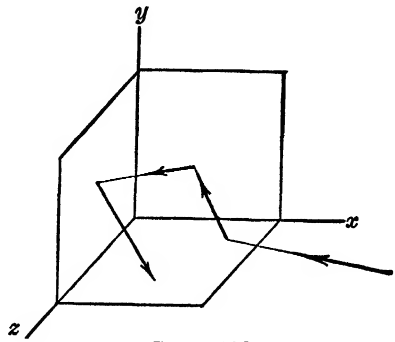
101. A particle of mass m rebounds from a flat surface. What is the relation between θ_1 and θ_2 if no energy is lost during the motion? What is the relation between θ_1 and θ_2 if the coefficient of restitution is e?

102. If in Problem 101 the coefficient of restitution is $e = 1$ but sliding contact is made with coefficient of friction μ, what is the resulting motion? (Note that e and μ specify the normal and tangential impulses.)

103. A particle of mass m rebounds from the corner of a smooth box.

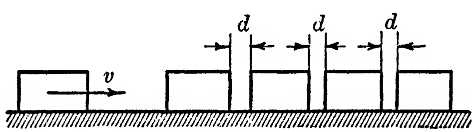
(a) What is the relation between the direction of approach and the direction of departure if no energy is lost during the motion?

(b) Find the relation between the direction of approach and the direction of departure taking account of the energy loss during impact, assuming that the coefficient of restitution is the same for all surfaces.



Prob. 103

104. Four identical bodies each of mass m are set up in a straight line on a smooth horizontal plane. A fifth body, identical with the other four, approaches with a velocity v and makes an impact with the first



Prob. 104

body. The impacts are all elastic, and all motion takes place along a straight line. (a) Describe the motion of the bodies. (b) If the distances d approach zero, what is the resulting motion? (Assume instantaneous impact, and no energy loss due to air resistance or friction.)

24. Electron Dynamics. For our present purpose, the electron may be considered to be a particle having a mass $m_0 = 9.1 \times 10^{-28}$ grams and carrying a negative electric charge $e = -4.80 \times 10^{-10}$ electrostatic units. If the electron is subjected to a known force, its motion can be computed from Newton's law written in the form $\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v})$. The equation of motion is written in this form since it has been observed that when an electron is accelerated to a high velocity the apparent mass of the electron increases according to the equation $m = m_0/\sqrt{1 - v^2/c^2}$, where c is the velocity of light in vacuum (299,800 km/sec) and m_0 is the mass of the electron when $v = 0$, the *rest mass.* Since v must exceed 40,000 km/sec for the mass to be increased by 1%, it is only for very high velocities that the effect becomes appreciable.

As a first example of a problem in electron dynamics we shall consider a simplified version of one of the basic experiments which first permitted a direct experimental verification of the variation of mass with velocity. That the apparent mass of a charged particle should increase with speed had been shown from theoretical considerations by J. J. Thomson in 1881, and a direct experimental confirmation was obtained in 1901 by Kaufmann. In 1909 experiments of Bucherer made possible the accurate determination of the relation between mass and velocity. The experi-

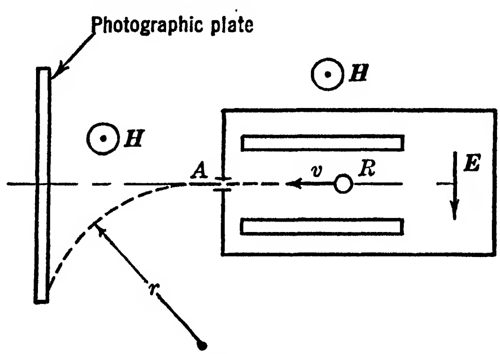


Fig. 4–10

mental method used by Bucherer is shown, in a simplified way, in Fig. 4–10. Electrons are emitted, with all directions and speeds, from a radioactive source R, located between two plates. Only those electrons having velocities along the axis of the apparatus are used in this experiment. An electric field $\boldsymbol{E}$, directed downward, is maintained between the plates and the whole apparatus is placed in a magnetic field $\boldsymbol{H}$ directed perpendicularly out of the paper. While the electron is between the two plates, it is subjected to a vertical upward force of magnitude Ee due to the electric field. The force acting on an electron moving with a velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{H}$ is $e\boldsymbol{v} \times \boldsymbol{H}$. In the present experiment $\boldsymbol{v}$ is perpendicular to $\boldsymbol{H}$, so that a vertical downward force of magnitude evH acts on the electron. If the forces due to the electric and magnetic field are just equal in magnitude, the resultant vertical force on the electron will be zero, and the electron will move horizontally with a velocity given by:

$$Ee = evH; \quad v = \frac{E}{H}$$

The apparatus is arranged with a small hole at A, so that only electrons which have this horizontal velocity v can emerge from the box. We thus have a method of producing specified electron velocities.

After leaving the electric field E, in the region between the two plates, the electron is subjected only to the force of the magnetic field. Since the force on the electron and hence the acceleration of the electron are always perpendicular to the velocity, the magnitude of the velocity does not change but the electron moves in a circular path of radius r, where:

$$evH = \frac{mv^2}{r}$$

from which

$$\frac{e}{m} = \frac{v}{rH}$$

From this equation an experimental value of $\frac{e}{m}$ for various velocities can be determined by measuring the radius of curvature r of the electron path. Now, if the expression $m = m_0\Big/\sqrt{1 - \frac{v^2}{c^2}}$ or $\frac{e}{m} = \frac{e}{m_0}\sqrt{1 - \frac{v^2}{c^2}}$ is correct, the same value of $\frac{e}{m_0}$ should be computed from the experimental measurements for different values of v. Some typical results of Bucherer's experiments are given in the following table: *

$\left(\frac{v}{c}\right)$	$\frac{e}{m_0}$ (emu/gram)
0.3173	1.752×10^7
0.3787	1.761
0.4281	1.760
0.5154	1.763
0.6870	1.767

* Bucherer, A. H., *Annalen der Physik* **28**, 513 (1909).

Since these values of $\frac{e}{m_0}$ are constant within the limits of experimental error, it appears that the experimental results are in good agreement with the expression for the variation in mass, which is derived on the basis of the special theory of relativity.

25. The Acceleration of Electrons. A number of frequently used instruments employ a stream of high-speed electrons, so that the problem of producing high-velocity electrons is of

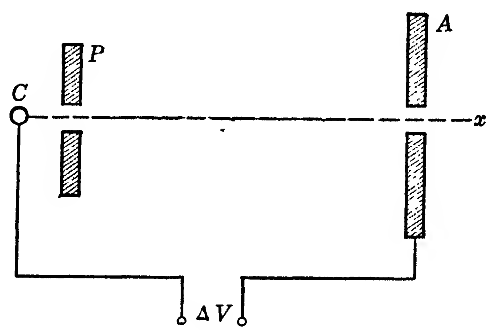


FIG. 4–11

practical importance. Consider the apparatus shown schematically in Fig. 4–11. A potential difference ΔV is maintained between a cathode C and a plate anode A. Any free electron in the field will be accelerated toward A by a force whose components are

$$eE_x = -e\frac{\partial V}{\partial x}$$
$$eE_y = -e\frac{\partial V}{\partial y}$$
$$eE_z = -e\frac{\partial V}{\partial z}$$

If the cathode C is heated, as for example, by a resistance filament, electrons will be "boiled" off and will then be accelerated by the electric field. If a plate P with a small opening is placed in front of the cathode as shown in the diagram, only those electrons traveling along the x-axis will be free to move to the anode. Such electrons will move toward the anode under the action of a force eE_x. If now a hole is arranged in the anode at the x-axis, the electrons can pass out of the electric field and continue on with the constant velocity v which they had attained at the anode. Knowing the potential difference between the cathode and the anode, the velocity may be computed by using the principle of work and energy:

$$e(V_A - V_C) = \tfrac{1}{2}mv^2$$

hence

$$v = \sqrt{\frac{2e}{m}(V_A - V_C)}$$

We can compute the velocity for a potential difference of 1 volt = $\frac{1}{300}$ esu for the electron as follows:

$$v = \left[\frac{(2)(-4.8 \times 10^{-10})}{(9.1 \times 10^{-28})}\left(-\frac{1}{300}\right)\right]^{\frac{1}{2}} = 5.94 \times 10^{7} \text{ cm/sec}$$

This velocity is of the order of 133,000 mph, which is sufficiently small compared to the velocity of light so that the variability of mass with velocity does not need to be taken into account. It is relatively easy, however, to accelerate electrons to high velocities because the mass of the electron is small compared to its charge.

26. The Cathode-ray Oscilloscope. An electronic device which has wide applications is the cathode-ray oscilloscope, represented

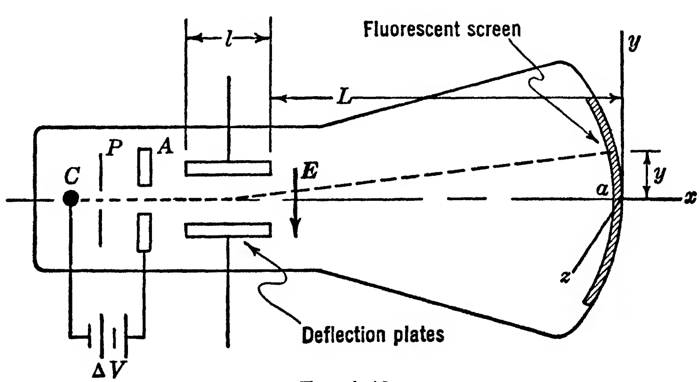


Fig. 4–12

in simplified form in Fig. 4–12. By an arrangement of cathode and anode as previously described, electrons are accelerated to a suitable velocity and are focused into a narrow beam through A along the x-axis. Two plates are arranged parallel to the x-axis so that an electric field E can be established over a length l of the beam. The whole apparatus is enclosed in an evacuated glass tube. If the electric field E is zero, the electron stream continues along the x-axis and impinges against the end of the tube at a, where a bright spot is formed on the fluorescent screen. If an electric field is set up, the electrons, as they pass between the plates, will be subjected to a force which will deflect the beam and so change the position of the bright spot on the screen.

An enlarged view of the two plates with a single electron between them is shown in Fig. 4–13. As a simplifying assumption,

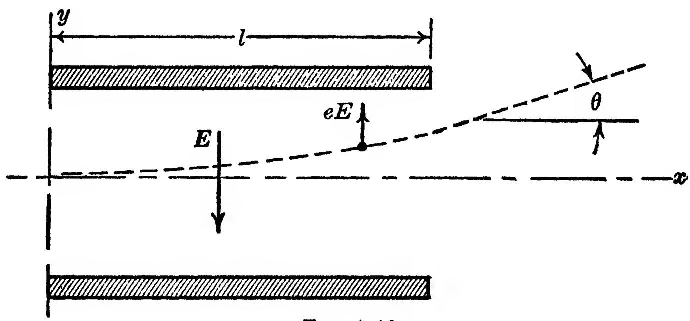


FIG. 4–13

we suppose that E is zero outside the plates and uniform between the plates. Then an upward force of eE is exerted on the electron and the equations of motion are:

$$m\ddot{x} = 0$$
$$m\ddot{y} = eE$$

subject to the conditions that $x_0 = y_0 = \dot{y}_0 = 0$ and $\dot{x}_0 = v$ when $t = 0$. After the electron has passed through the parallel plates, an integration of these equations shows that the velocity components are:

$$\dot{x} = v$$
$$\dot{y} = \frac{eEl}{mv}$$

As the electron leaves the plates, its path makes an angle θ with the x-axis, where:

$$\tan\theta = \frac{\dot{y}}{\dot{x}} = \frac{eEl}{mv^2}$$

The effect of the two plates, therefore, is to deflect an electron beam through an angle θ. After the beam leaves the plates, no force is exerted on the electrons, and the beam continues in a straight line at the angle θ. The luminous spot will thus be deflected from its zero position a distance y, where:

$$y = \frac{eEl^2}{2mv^2} + \frac{eEl}{mv^2}L$$
$$= \left[\frac{el}{mv^2}\left(L + \frac{l}{2}\right)\right]E = C_1E$$

The displacement of the luminous spot is thus directly proportional to E and can be taken as a measure of the potential difference between the plates.

If a second pair of deflecting plates, oriented at 90° to the first pair, is added to the cathode-ray tube of Fig. 4–12, the luminous spot will be deflected in the z-direction with a displacement

$$z = C_2E_z$$

where E_z is the electric field set up between the second pair of plates. The motion of the luminous spot on the face of the tube is thus given by the two equations:

$$y = C_1E_y; \quad z = C_2E_z$$

If, for example, $E_y = A \sin \omega t$ and $E_z = B \cos \omega t$, the path of the spot is the ellipse:

$$\frac{y^2}{C_1{}^2A^2} + \frac{z^2}{C_2{}^2B^2} = 1$$

which appears on the screen as a luminous line. In general, if E_y is known, E_z can be determined from the picture on the tube. The oscilloscope can thus be used to measure any quantity which can be converted into a potential difference.

27. The Equivalence of Mass and Energy. We shall now investigate some of the consequences of the fact that the apparent mass of a particle increases with velocity. We shall start with Newton's law in the form $\boldsymbol{F} = \dfrac{d}{dt}(m\boldsymbol{v})$ and shall take m as a variable, $m = m_0\Big/\sqrt{1 - \dfrac{v^2}{c^2}}$. To simplify the analysis we shall consider only one-dimensional motion starting from rest, with F and v always parallel to the x-axis.

The impulse-momentum equation will be derived first. For the particular conditions specified above, we have

$$I = \int_0^t F\,dt = mv = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Solving for v in terms of the impulse I:

$$v = c\left[\frac{1}{\sqrt{1 + \frac{m_0{}^2 c^2}{I^2}}}\right] \tag{31}$$

It is seen that as $I \to \infty$, $v \to c$, so that no matter how large the impulse, the velocity can never exceed the velocity of light.

The equation of work and energy is obtained in the usual way except that m is now a variable:

$$\int_0^x F\,dx = \int_0^x \frac{d}{dt}(mv)\,dx = \int_0^v v\,d(mv)$$

Integrating by parts the right side of this expression, and writing E_w for the work done by F, we obtain:

$$E_w = mv^2 - \int_0^v mv\,dv$$

Substituting the expression for m:

$$E_w = \frac{m_0 v^2}{\sqrt{1 - \frac{v^2}{c^2}}} - \int_0^v \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\,dv$$

Evaluating the integral, this becomes:

$$E_w = m_0 c^2\left[\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right] \tag{32}$$

which is the relativistic expression for the kinetic energy of the particle. This expression can also be written in the form:

$$\begin{aligned} E_w &= \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 \\ &= mc^2 - m_0 c^2 \end{aligned}$$

Since m_0 and c are constants, it appears from this equation that an increase in the work done on the particle requires a corresponding increase in the mass of the particle. This unexpected result gave rise to much speculation regarding our fundamental concepts of mass and energy. The Theory of Relativity which Einstein postulated in 1905 states that what we measure as mass is equivalent to energy, and that the term m_0c^2 represents the energy equivalent of the particle when it is at rest. If we write $E_w + m_0c^2 = mc^2$ and call $E_w + m_0c^2$ the total energy E, we have:

$$E = mc^2 \tag{33}$$

This is the famous expression for the equivalence of mass and energy. To obtain an idea of the magnitudes involved, we shall compute the rest energy equivalent of a mass weighing one pound.

$$E = m_0c^2 = \frac{1\ \text{lb}}{32.2\ \text{ft/sec}^2}\left(\frac{2.998 \times 10^{10}\ \text{cm/sec}}{2.54\ \text{cm/in.} \times 12\ \text{in./ft}}\right)^2 \text{ft-lb}$$

$$E = 3.0 \times 10^{16}\ \text{ft-lb}$$

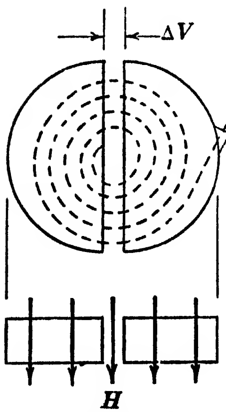
This is roughly equivalent to the energy which would be obtained from the combustion of 1,500,000 tons of coal or 300,000,000 gallons of gasoline.

The first approximate experimental verification of the equivalence of mass and energy was obtained in 1932 by J. D. Cockroft and E. T. Walton by particle bombardment of lithium. The fact that large quantities of energy can be released by nuclear fission was demonstrated by the atomic bomb in 1945. Measurements have shown that the difference in mass between the fission products and the original nucleus is just equivalent to the energy released.

PROBLEMS

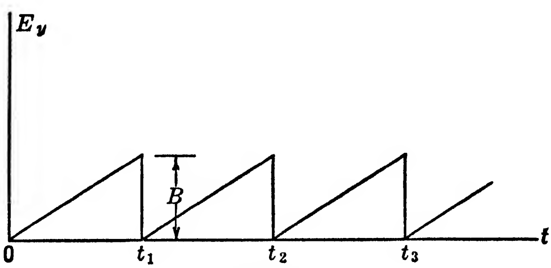
105. An electron is accelerated from rest through a potential drop of 100,000 volts. Compute its velocity assuming that its mass remains constant, and compare with the velocity obtained when the variability of mass with velocity is taken into account. This potential drop is small compared with the several million volts used in modern particle accelerators.

106. The basic elements of a cyclotron are shown in the accompanying diagram. The device consists of two halves of a cylindrical box, placed in a uniform magnetic field H as shown. If a particle having a charge e and a velocity $\boldsymbol{v}$ in the plane of the box is introduced into the box, it is subjected to the force of the magnetic field which is given by $\boldsymbol{F} = e\boldsymbol{v} \times \boldsymbol{H}$. The two halves of the box are maintained at a potential difference ΔV, so that as the particle travels from one half to the other it experiences a velocity change corresponding to ΔV. By means of an oscillator this potential is varied periodically in such a way that the particle always experiences a potential drop. Show that the particle will move in a circular path whose radius increases with v, and that the time required for one-half a revolution is independent of v. In this way the particle can be accelerated to a high velocity and can then be drawn off and used as a bombarding particle. The foregoing analysis is based on constant mass. If the velocity is so high that the variability of mass must be considered, the time of revolution is not independent of v and difficulties are encountered in synchronizing the potential drop ΔV.



PROB. 106

107. The electric field between the y deflecting plates of a cathode-ray oscilloscope tube varies as shown in the diagram:



PROB. 107

(a) If a field $E_z = E_0 \sin \omega t$ is set up between the z deflecting plates, what picture would be traced out on the screen if $t_1 = \dfrac{2\pi}{\omega}$, and the time intervals are equal?

(b) If in part (a) $t_1 = \dfrac{4\pi}{\omega}$, what picture would appear?

(c) If $E_z = f(t)$, what picture would appear on the screen during the time $t = 0$ to $t = t_1$?

108. Work out the steps in the derivation of the expression for the relativistic kinetic energy of a particle, and show that for $v \ll c$ this reduces to the familiar expression for the kinetic energy of a slow-speed particle.

CHAPTER V

DYNAMICS OF VIBRATING SYSTEMS

First of all one must observe that each pendulum has its own time of vibration, so definite and determinate that it is not possible to make it move with any other period than that which nature has given it. On the other hand one can confer motion upon even a heavy pendulum which is at rest by simply blowing against it. By repeating these blasts with a frequency which is the same as that of the pendulum one can impart considerable motion.—G. Galilei, *Discorsi a Due Nuove Scienze* (1638).

The analysis of mechanical vibrations is a problem in dynamics which is often encountered by the engineer. Such problems arise in connection with the design of almost every type of machine or structure. The vibration of high-speed machinery, aircraft flutter, the vibration of buildings during earthquakes, and the design of dynamic measuring instruments are current problems which will indicate the wide scope of the subject. It is also found that the same mathematical theory which is used for the study of mechanical vibrations is applicable to certain problems of oscillations in electrical circuits. Such a similarity between the basic equations of mechanical and electrical systems has led to several useful methods whereby the results of analysis or experimental investigations in one field have been applied to the other.

In the present chapter we shall consider only the motion of systems having one degree of freedom. Such problems are excellent examples of the methods of particle dynamics, and they will also indicate the theory behind a large number of interesting technical applications.

28. The Vibration Problem. We shall first investigate the simplest possible mechanical system which contains all the significant features of a vibration problem. Consider a mass m which has one degree of freedom, that is, its location at any time is specified by the one coordinate x. The mass is restrained by a spring k, and an external force $F(t)$ whose magnitude varies with time is applied to the mass, as shown in Fig. 5–1(a). In Fig. 5–1(b) is shown a free-body diagram of the mass, when it has a positive velocity and displacement as measured from the position of static equilibrium ($x = 0$). The fact that a friction force opposes the motion of the mass is indicated by the force F_d, which is usually some function of the velocity of the system, depending upon the nature of the contacting surfaces and the conditions of lubrication.

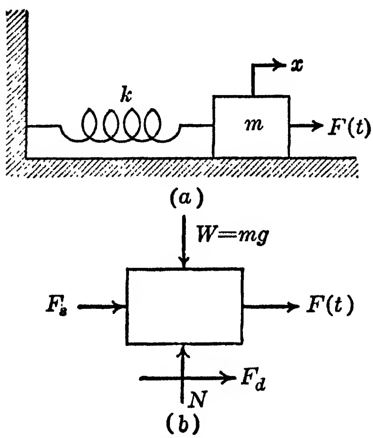


Fig. 5–1

We observe that the forces acting upon the mass belong to three general classes. First there is the *exciting force* $F(t)$, which is the externally applied force that causes the motion of the system. Second, there is the *restoring force* F_s, which is the force exerted by the spring on the mass and which tends to restore the mass to its original position. Third, there is a *damping force* F_d, which is always in such a direction that it opposes the motion of the system, and which is thus responsible for a dissipation of energy. The equation of motion can be written as:

$$m\ddot{x} = F_s + F_d + F(t) \tag{34}$$

Three such forces, along with the equation of motion, characterize the vibration problem. When an analysis of a physical problem leads to Equation (34), many of the essential features of the motion can be analyzed as in the following sections.

29. The Characteristics of the Forces. From the definition of the restoring force it is known that its direction is always toward the equilibrium position of the system. If the restoring force is produced by a spring as in Fig. 5–1, it is known that:

$$F_s = -kx$$

where the point $x = 0$ is the equilibrium position and k is the spring constant which indicates the stiffness of the spring. This spring force is called a *linear* restoring force because it is a function of x raised to the first power. In some instances the restoring force is not linear. To obtain the solution for non-linear restoring forces is difficult, so that it is customary to linearize the problem, if possible, by treating only small oscillations. For example, suppose that the restoring force is given by some function of x which can be expanded in a power series:

$$F_s = -\phi(x) = -\left(k_1 x + k_2 \frac{x^2}{2!} + k_3 \frac{x^3}{3!} + \cdots\right)$$

If x is small, the first term of this series is large compared to the sum of the following terms. Thus if the amplitude of the vibration is always sufficiently small, a satisfactory approximate solution can be obtained by taking for the restoring force only the first term of the series:

$$F_s = -k_1 x$$

We shall treat only linear restoring forces. In recent years considerable work has been done on the non-linear problem, but as yet no general solutions of a simple form have been determined.

The most important characteristic of the damping force is that its direction is always opposed to the direction of the motion. The work done by the damping force is thus always negative, and energy is dissipated from the system. In many instances the damping force is directly proportional to the velocity of the mass, so that

$$F_d = -c\dot{x}$$

Damping which can be described by this equation is called *viscous damping*, and c is called the coefficient of viscous damping. Such a damping force may arise in a number of ways. The frictional force set up between two lubricated surfaces, under the usual conditions of velocity and pressure, is approximately proportional to the velocity, and air resistance at low velocities also may be assumed to be viscous in nature. Damping forces are often intentionally introduced into a system, and this is commonly done by means of a dashpot filled with oil. Such a device can be designed to give viscous damping. In some problems in which the damping is not viscous, the concept of viscous damping may still be used, by defining an *equivalent viscous friction*, for which the coefficient of viscous friction is determined so that the total energy dissipated per cycle is the same as for the actual damping during a steady state of motion. In the analysis to follow, we shall always assume viscous damping forces, and it should therefore be realized that the solutions may be approximate for some types of mechanical systems.

Exciting forces may arise in many different ways. They may, for example, be transient forces such as would be caused by the impact of some external body, or they may be repetitive forces caused by a series of such impacts. Reciprocating or rotating machine parts often produce unbalanced alternating forces that have a sinusoidal variation. Consider the rotation of an unbalanced disk as shown in Fig. 5–2. This arrangement represents a typical vibration isolating mount for a rotating machine. The disk of mass m rotates about the center O with an angular velocity ω. The center of mass of the disk is located at a distance r from the center of rotation. The rotating system is mounted on a larger mass M which can move only in a vertical direction. M is supported on a spring having a spring constant k, and a dashpot having a coefficient of viscous damping c connects the mass to the fixed support. If we assume that the motion of M is small compared to r, then the motion of m can be taken as circular, and the acceleration of the center of mass of the disk is $r\omega^2$. There is thus a force of magnitude $mr\omega^2$ acting in a radial direction upon the large mass M. The component of the force in the y direction, that is, the component of force which causes motion of the system is $mr\omega^2 \sin\theta$. Assuming that the disk rotates with a constant speed we have for the exciting force:

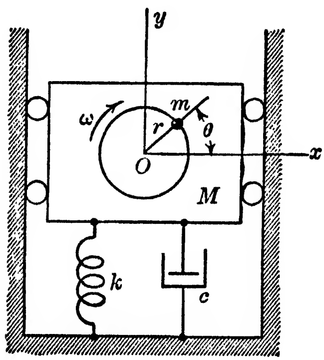


Fig. 5–2

$$F(t) = mr\omega^2 \sin \omega t$$

Since small amounts of unbalance are inevitably present in any rotating machine, sinusoidal exciting forces play an important part in vibration theory.* A more fundamental reason for the importance of the sinusoidal force is the fact that any periodic force can be represented analytically as a series of sine and cosine terms, by a Fourier series expansion. Thus, if the behavior of the system is known for a sinusoidal force, the behavior of the system can be determined for any periodic force.

30. The Differential Equation of the Vibration Problem. For the basic vibration problem we shall consider a system which consists of a linear restoring force, a viscous damping force, and a sinusoidal exciting force:

$$\begin{aligned} F_s &= -kx \\ F_d &= -c\dot{x} \\ F(t) &= F_0 \sin \omega t \end{aligned}$$

Substituting these terms into the equation of motion gives:

$$m\ddot{x} = -kx - c\dot{x} + F_0 \sin \omega t$$

We shall write this equation in the standard form

$$\ddot{x} + 2n\dot{x} + p^2 x = \frac{F_0}{m} \sin \omega t \tag{35}$$

where:

$$\frac{k}{m} = p^2 \quad \text{and} \quad \frac{c}{m} = 2n$$

The term n is called the damping factor. A system described by this equation is said to be a single degree of freedom harmonic oscillator with viscous damping. In the following sections we shall derive the solution of the equation, and we shall examine its physical significance.

31. Free Vibrations of an Undamped System. Of the three forces mentioned above, only the restoring force is necessary for the existence of a vibration problem. It may be that energy dissipation is so small that the damping force may be neglected, and the motion of the system may be started by initial displacements or velocities rather than by exciting forces. In this section we shall consider the solution of this simplest type of vibration prob-

* If the motion of M is not small, m performs two-dimensional simple harmonic motion during a steady state and the resultant exciting force is again sinusoidal (see Problem 128).

lem as an illustration of method, the nomenclature to be used, and of the physical interpretation of the results.

Setting the damping force and the exciting force equal to zero, the differential equation becomes:

$$\ddot{x} + p^2x = 0$$

This is the same equation previously solved for harmonic motion (Section 21). The solution of the equation is thus known to be:

$$x = C_1 \sin pt + C_2 \cos pt$$

where C_1 and C_2 are constants of integration which must be evaluated from the initial conditions. That this expression is in fact a solution of the differential equation may be verified by direct substitution.

When $t = 0$ let the initial displacement be x_0 and the initial velocity $\dot{x}_0$. From these two initial conditions the constants C_1 and C_2 may be found:

$$C_1 = \frac{\dot{x}_0}{p} \quad \text{and} \quad C_2 = x_0$$

The solution of the differential equation becomes:

$$x = \frac{\dot{x}_0}{p} \sin pt + x_0 \cos pt \tag{36}$$

We shall investigate the physical significance of this solution for $x_0 = A$ and $\dot{x}_0 = 0$. This means that the mass is moved a distance A from its position of equilibrium and is then released, at time $t = 0$, with zero initial velocity. The displacement is then given by:

$$x = A \cos pt$$

The motion of the mass as a function of time is shown in Fig. 5–3, where it is seen that the mass performs oscillations about the position of equilibrium. Since there is no energy loss in this ideal system, the oscillation continues indefinitely with the same amplitude A. The portion of the motion included between two points at which the mass has the same position, as B and C in Fig. 5–3, is called one *cycle* of the vibration. The time required for the completion of one cycle is called the *period*, τ, of the vibration. The number of cycles which occur in one second is called the *frequency*, f, of the vibration. To find the period, consider two displacements of the mass which are one cycle apart, as B and C in Fig. 5–3. Then:

$$A \cos pt = A \cos p(t + \tau)$$

$$\tau = \frac{2\pi}{p} = \frac{2\pi}{\sqrt{\frac{k}{m}}} \qquad (37)$$

$$f = \frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

The period can also be found by an energy method in the following way. The potential energy of the system at any position is

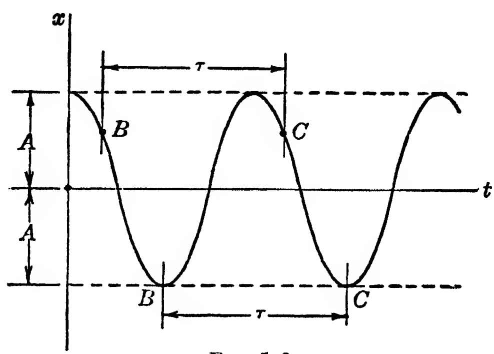


FIG. 5–3

$V = \frac{1}{2}kx^2$ and the kinetic energy is $\frac{1}{2}m\dot{x}^2$. Since the motion is known to be harmonic, the displacement and velocity can be written:

$$x = A \sin \omega t$$
$$\dot{x} = A\omega \cos \omega t$$

When $x = A$, the potential energy is equal to $\frac{1}{2}kA^2$ and the kinetic energy is zero. When $x = 0$, the kinetic energy is $\frac{1}{2}mA^2\omega^2$ and the potential energy is zero. Since energy is conserved:

$$\tfrac{1}{2}mA^2\omega^2 = \tfrac{1}{2}kA^2$$

$$\omega = \sqrt{\frac{k}{m}} = p$$

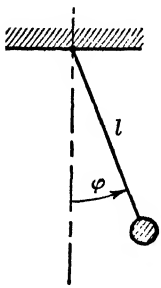
This energy method is useful for obtaining approximate frequencies in more complicated problems when it is known that the motion can be assumed to be approximately harmonic.

PROBLEMS

109. A pendulum having a mass m and a length l is supported by a string of negligible mass. Write the equation of motion for the pendulum, neglecting air damping, and show that for small oscillations this equation is:

$$\ddot{\phi} + \frac{g}{l}\phi = 0$$

Find the period of small oscillations of the pendulum. If air resistance imposes a damping force proportional to the velocity, what is the differential equation of motion for small oscillations? What force plays the part of the restoring force in this problem?



PROB. 109

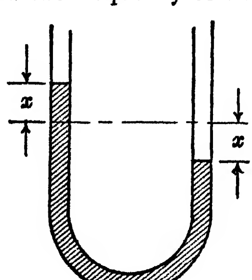
110. Write the equation of the conservation of energy for the pendulum of Problem 109. Obtain the differential equation of motion for small oscillations by differentiating the energy equation with respect to t.

111. Show that the natural frequency of free vibrations of an undamped simple harmonic oscillator is given by:

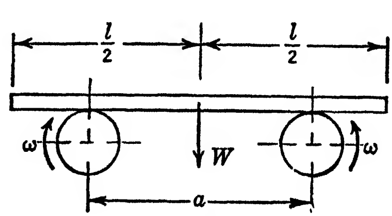
$$f = \frac{3.13}{\sqrt{\delta_{st}}} \text{ cycles per second}$$

where δ_{st} is the static deflection of the system, in inches. The static deflection of a system is defined as the deflection caused by a force of mg lb.

112. A U-tube is partially filled with mercury and is supported in a vertical position as shown in the diagram. When the system is in equilibrium, the height of the mercury is the same in each arm. The liquid in one arm is depressed a distance x thus raising the liquid a like distance in the other arm. The system is then released. Write the equation of motion for the liquid column, neglecting frictional damping forces, and find the frequency of the resulting oscillation.



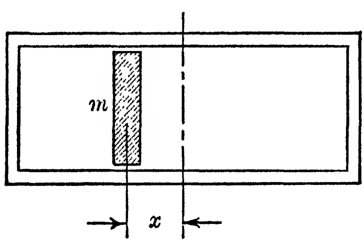
PROB. 112



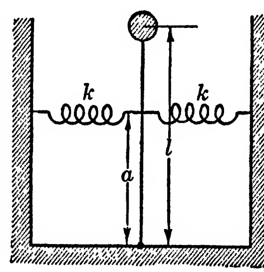
PROB. 113

113. Two parallel cylindrical rollers rotate in opposite directions as shown in the figure. The distance between the centers of the rollers is a. A straight, uniform horizontal rod of length l and weight W rests on top of the rollers. The coefficient of kinetic friction between the rod and the roller is μ. Taking x as the distance from the center of the rod to the midpoint between the rolls, write the equation of motion of the rod, supposing that it has been initially displaced from the central position. Find the frequency of the resulting vibratory motion.

114. A piston of mass m fits in a closed cylinder of cross-sectional area A. When the piston is in the central position with $x = 0$, it is in equilibrium, and the pressure on each side is p. The air in the cylinder is assumed to follow Boyle's law, that is, the pressure times the volume is equal to a constant. The piston is moved through a distance x from the position of equilibrium and is then released. Write the differential equation of motion of the system, assuming that there is viscous friction between the piston and the cylinder. Find the frequency of small oscillations of the piston, assuming that the damping force can be neglected.



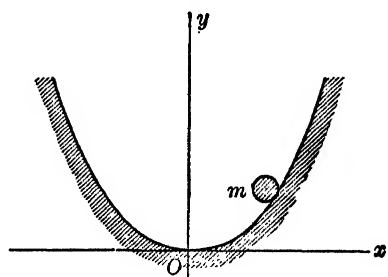
PROB. 114



PROB. 115

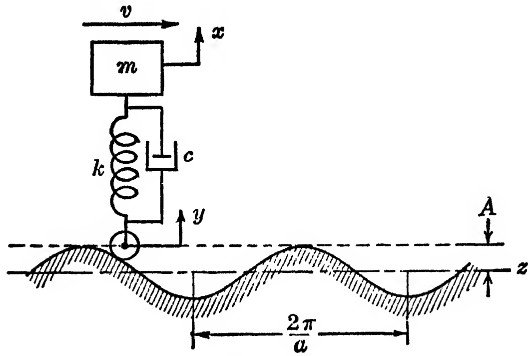
115. Find the frequency of small vibrations of an inverted pendulum restrained by two springs of spring constant k as shown in the diagram. All the mass of the pendulum is assumed to be concentrated at a distance l from the point of support, and the springs are sufficiently stiff so that the pendulum is stable.

116. A particle of mass m slides on a smooth surface whose shape is given by the equation $y = ax^2$. The particle is moved along the surface away from the position of equilibrium and is then released. Find the equation of motion of the particle, and find the frequency of small oscillations about the position of static equilibrium.



PROB. 116

117. A spring-mounted mass, supported on a wheel, as shown in the diagram, moves with a velocity v along a wavy surface which has a



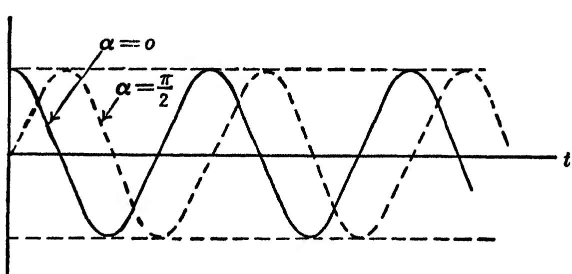
PROB. 117

sinusoidal form. The vertical displacement of the wheel y can be found from the fact that

$$y = A \sin az$$

where $z = vt$. The vertical motion of the mass is given by the coordinate x. A dashpot which introduces viscous damping into the system is connected between the wheel and the mass. Write the equation of motion for the vertical movement of the mass, noting that the forces which act upon it are the elastic force, $-k(x - y)$, and the damping force, $-c(\dot{x} - \dot{y})$. Show that this equation reduces to the general form of the vibration equation, with a sinusoidal exciting force.

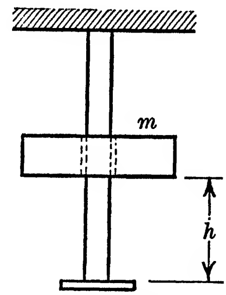
118. Show that the equation $x = C_1 \sin pt + C_2 \cos pt$ can be written in the form $x = A \cos (pt + \alpha)$ where $A = \sqrt{C_1^2 + C_2^2}$ and $\alpha =$



PROB. 118

$\tan^{-1}\left(-\frac{C_1}{C_2}\right)$. A is called the amplitude of the vibration, and α is called the phase angle. Changing the phase angle has the effect of shifting the whole curve representing the vibration to the right or left, as shown in the figure.

119. A mass m drops from rest through a height h and strikes the bottom of a rod. The rod elongates x feet when acted upon by a force of kx lb. Assuming that the mass remains in contact with the end of the rod after impact, find the motion of the mass after the impact. (Neglect the mass of the rod and assume that no energy is lost.) If m weighs 20 lb, $k = 100$ lb/ft, and $h = 2$ in., find the amplitude and the frequency of vibration.



PROB. 119

32. Damped Vibrations. In an actual vibration there will always be some damping present. Let us consider free vibrations with viscous damping, and compare the solution of this problem with that of the undamped oscillation. The differential equation of the free damped vibration is obtained from the general equation by setting the exciting force equal to zero.

$$\ddot{x} + 2n\dot{x} + p^2x = 0 \tag{38}$$

The solution of this equation must be a function which has the property that repeated differentiations do not change its form, since the function and its first and second derivatives must be added together to give zero. For such an equation, we take as a trial solution $x = Ce^{mt}$, which, upon substitution into the differential equation, gives:

$$Cm^2e^{mt} + 2Cnme^{mt} + p^2Ce^{mt} = 0$$

Cancelling the factor Ce^{mt} we have:

$$m^2 + 2nm + p^2 = 0$$

This equation has two solutions for m, each of which will make $x = Ce^{mt}$ a solution of the differential equation. The general solution of the equation may thus be written as the sum of the two: *

$$x = C_1e^{m_1t} + C_2e^{m_2t} \tag{39}$$

It should be noted that the superposition of solutions is valid only for linear differential equations, that is, equations which are linear in the dependent variable and its derivatives.

* Since the differential equation is of the *second* order in its derivatives, the solution with *two* constants of integration is the complete solution.

Solving the algebraic equation we obtain the two values:

$$m_1 = -n + \sqrt{n^2 - p^2}$$
$$m_2 = -n - \sqrt{n^2 - p^2}$$

Hence the solution is:

$$x = C_1 e^{(-n + \sqrt{n^2 - p^2})t} + C_2 e^{(-n - \sqrt{n^2 - p^2})t}$$

The physical significance of this solution depends upon the relative magnitudes of n^2 and p^2, which determine whether the exponents are real or complex quantities.

Suppose first that $n^2 > p^2$ so that the exponent is a real quantity. Physically this means a relatively large damping, since n is a measure of the damping in the system. The solution is then:

$$x = C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t} \tag{40}$$

where α_1 and α_2 are real quantities. The motion of the mass, in this event, is not oscillatory, but is an exponential subsidence. Suppose, for example, that the motion is started by giving the mass an initial displacement A and then releasing it from rest. The displacement-time curve is then as shown in Fig. 5–4. Because of the relatively large damping, the mass released from rest never passes the static equilibrium position. So much energy is dissipated by the damping force that there is not sufficient kinetic energy left to carry the mass past the equilibrium position. Such a system is said to be *overdamped.*

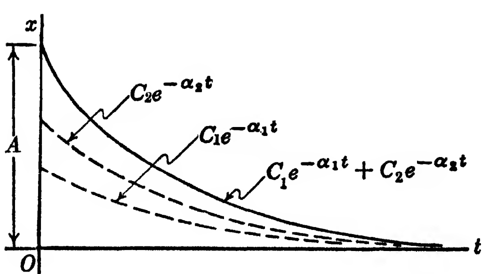


FIG. 5–4

If the damping is small, so that $n^2 < p^2$, then the term $n^2 - p^2$ is negative, and we can write:

$$x = C_1 e^{(-n + i\sqrt{p^2 - n^2})t} + C_2 e^{(-n - i\sqrt{p^2 - n^2})t}$$
$$= e^{-nt}[C_1 e^{i\sqrt{p^2 - n^2}t} + C_2 e^{-i\sqrt{p^2 - n^2}t}]$$

Using the trigonometric relation $e^{i\theta} = \cos\theta + i\sin\theta$, the displacement may be written:

$$x = e^{-nt}[(C_1 + C_2)\cos\sqrt{p^2 - n^2}\,t + i(C_1 - C_2)\sin\sqrt{p^2 - n^2}\,t]$$

Since the constants C_1 and C_2 are arbitrary and are to be determined by the initial conditions, we may simplify the expression by introducing new constants, $C_1' = i(C_1 - C_2)$ and $C_2' = C_1 + C_2$; dropping the primes:

$$x = e^{-nt}[C_1 \sin \sqrt{p^2 - n^2}\, t + C_2 \cos \sqrt{p^2 - n^2}\, t] \tag{41}$$

This equation may be checked by setting $n = 0$, thus reducing it to:

$$x = C_1 \sin pt + C_2 \cos pt$$

which was previously derived for the undamped free vibrations.

Comparing the two solutions we see that the effect of the damping is to increase the period of the vibration and to decrease the magnitudes of successive peaks of the vibration, since the amplitude of the vibration decreases exponentially with time. The motion of a typical *underdamped* oscillator is shown in Fig. 5–5.

As a convenient measure of the damping we may compute the ratio of the amplitudes of successive cycles of the vibration.

$$\frac{x_1}{x_2} = \frac{e^{-nt_1}}{e^{-n\left(t_1 + \frac{2\pi}{\sqrt{p^2 - n^2}}\right)}} = e^{\frac{2\pi n}{\sqrt{p^2 - n^2}}}$$

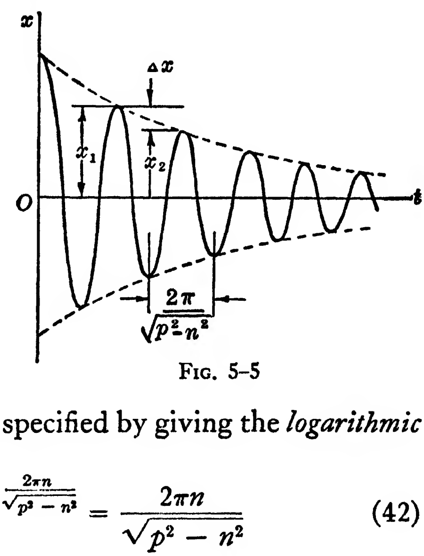


Fig. 5–5

The amount of damping is often specified by giving the *logarithmic decrement* δ, where:

$$\delta = \log \frac{x_1}{x_2} = \log e^{\frac{2\pi n}{\sqrt{p^2 - n^2}}} = \frac{2\pi n}{\sqrt{p^2 - n^2}} \tag{42}$$

For systems having small damping, a simple way of determining the logarithmic decrement from the free vibration curve is as follows:

$$\delta = \log\left(\frac{x + \Delta x}{x}\right) = \log\left(1 + \frac{\Delta x}{x}\right) = \frac{\Delta x}{x} - \frac{1}{2}\left(\frac{\Delta x}{x}\right)^2 + \frac{1}{3}\left(\frac{\Delta x}{x}\right)^3 + \cdots$$

for $\left(\frac{\Delta x}{x}\right)$ small, the higher order terms may be dropped, and:

$$\delta \approx \frac{\Delta x}{x} \tag{43}$$

Thus the logarithmic decrement is approximately equal to the fractional decrease in amplitude during one cycle of the vibration.

Another important quantity in damped vibration analysis is the energy lost per cycle due to the damping force. The total energy of the system when it is in one of its extreme positions with zero velocity is:

$$W_1 = \tfrac{1}{2}kx^2$$

The energy one cycle later is:

$$W_2 = \tfrac{1}{2}k(x - \Delta x)^2$$

Therefore the energy loss per cycle is:

$$\Delta W = W_1 - W_2 = \tfrac{1}{2}kx^2 - \tfrac{1}{2}kx^2 + kx(\Delta x) - \tfrac{1}{2}k(\Delta x)^2$$

Expressing this energy loss as a fraction of the total energy of the system gives:

$$\frac{\Delta W}{W} = 2\left(\frac{\Delta x}{x}\right) - \left(\frac{\Delta x}{x}\right)^2$$

If the damping is small the square term can be dropped, and we have:

$$\frac{\Delta W}{W} \approx 2\left(\frac{\Delta x}{x}\right) \approx 2\delta \tag{44}$$

Thus for small damping the fraction of energy lost per cycle is approximately equal to twice the logarithmic decrement.

PROBLEMS

120. At time $t = 0$, the initial displacement of a damped harmonic oscillator is x_0, and the initial velocity is $\dot{x}_0$. Show that the free vibrations of the system are described by the equation:

$$x = e^{-nt}\left[x_0 \cos\sqrt{p^2 - n^2}\,t + \frac{\dot{x}_0 + nx_0}{\sqrt{p^2 - n^2}}\sin\sqrt{p^2 - n^2}\,t\right]$$

121. Critical damping is defined as that damping for which $n = p$.

(a) If the damping is less than critical, show that the logarithmic decrement can be written:

$$\delta = \frac{2\pi\left(\dfrac{n}{n_c}\right)}{\sqrt{1 - \left(\dfrac{n}{n_c}\right)^2}}$$

where $n_c = p$ = damping factor for critical damping.

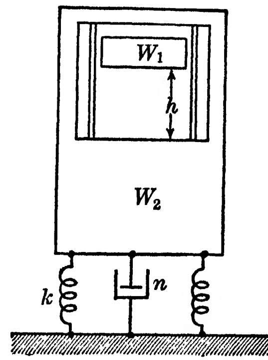
(b) Show that for small damping the logarithmic decrement can be written:

$$\delta \approx 2\pi\left(\frac{n}{n_c}\right) \tag{45}$$

122. A mass weighing 10 lb is restrained by a spring which has k = 15 lb/ft and is acted upon by a viscous damping force. It is observed that at the end of four cycles of motion the amplitude is reduced by one-half. Find the damping factor n and the period of the vibration.

123.* A drop hammer is found to transmit an objectionable shock to the surrounding ground. To eliminate this, the machine is mounted on springs, as shown in the diagram. To prevent undue vibration of the system after impact, damping is introduced as shown by the dash-pot. The constants of the system are:

W_1 = 2000 lb
W_2 = 30,000 lb
h = 8 ft
k (for all springs) = 250,000 lb/ft
n = 0.8 sec^{-1}



PROB. 123

The weight W_1 falls through a distance h and makes a plastic (no rebound) impact with W_2. The resulting motion of the system is a free vibration with damping. Find the maximum displacement of W_2, and the displacement three complete cycles after the maximum displacement occurs.

124. Solve the differential equation of motion for the critically damped oscillator, $n = p$. Evaluate the constants of integration and determine whether the displacement can change sign during a free vibration.

33. Forced Vibrations. Vibrations which are maintained by an exciting force are said to be *forced vibrations*. We shall now develop the complete solution for the motion of a damped, simple harmonic oscillator acted upon by the sinusoidal exciting force $F_0 \sin \omega t$. The differential equation of the motion (Equation 35) is:

$$\ddot{x} + 2n\dot{x} + p^2x = \frac{F_0}{m} \sin \omega t$$

The solution of this equation may be written as the sum of two terms:

$$x = e^{-nt}[C_1 \sin \sqrt{p^2 - n^2}\, t + C_2 \cos \sqrt{p^2 - n^2}\, t] + f(t)$$

for we have found from the preceding section that the first term, when substituted into the differential equation, gives zero. Therefore, a function $f(t)$ must be added of such a form that it will yield $\frac{F_0}{m} \sin \omega t$ when substituted into the equation. This second term is called the particular solution. Since two arbitrary constants already appear in the first term, no further arbitrary constants need be included.

The particular solution in the present problem may be found by taking a trial solution:

$$x = A \sin \omega t + B \cos \omega t$$

where the values of A and B are to be determined from the condition that the differential equation must be satisfied. Substituting into the equation the expressions

$$\dot{x} = A\omega \cos \omega t - B\omega \sin \omega t$$
$$\ddot{x} = -A\omega^2 \sin \omega t - B\omega^2 \cos \omega t$$

gives:

$$-A\omega^2 \sin \omega t - B\omega^2 \cos \omega t + 2nA\omega \cos \omega t - 2nB\omega \sin \omega t + p^2 A \sin \omega t + p^2 B \cos \omega t = \frac{F_0}{m} \sin \omega t$$

or:

$$(-A\omega^2 - 2n\omega B + p^2 A) \sin \omega t + (-B\omega^2 + 2nA\omega + p^2 B) \cos \omega t = \frac{F_0}{m} \sin \omega t$$

This equation must be identically satisfied, which means that the coefficient of the sin ωt on the left side of the equation must equal the coefficient of the sin ωt term on the right side of the equation, and the coefficient of the cosine term must equal zero; hence:

$$(p^2 - \omega^2)A + (-2n\omega)B = \frac{F_0}{m}$$
$$(2n\omega)A + (p^2 - \omega^2)B = 0$$

These two algebraic equations determine the proper values of A and B:

$$A = \frac{\begin{vmatrix} \left(\dfrac{F_0}{m}\right) & (-2n\omega) \\ 0 & (p^2 - \omega^2) \end{vmatrix}}{\begin{vmatrix} (p^2 - \omega^2) & (-2n\omega) \\ (2n\omega) & (p^2 - \omega^2) \end{vmatrix}} = \frac{\dfrac{F_0}{m}(p^2 - \omega^2)}{(p^2 - \omega^2)^2 + 4n^2\omega^2}$$

$$B = \frac{\begin{vmatrix} (p^2 - \omega^2) & \left(\dfrac{F_0}{m}\right) \\ (2n\omega) & 0 \end{vmatrix}}{(p^2 - \omega^2)^2 + 4n^2\omega^2} = \frac{-2n\omega \dfrac{F_0}{m}}{(p^2 - \omega^2)^2 + 4n^2\omega^2}$$

Writing the solution $x = A \sin \omega t + B \cos \omega t$ in the form $x = \sqrt{A^2 + B^2} \sin (\omega t - \phi)$, we have:

$$x = \frac{\dfrac{F_0}{m}}{\sqrt{(p^2 - \omega^2)^2 + 4n^2\omega^2}} \sin (\omega t - \phi) \tag{46}$$

where

$$\phi = \tan^{-1}\left(-\frac{B}{A}\right) = \tan^{-1}\left(\frac{2n\omega}{p^2 - \omega^2}\right) \tag{47}$$

The complete solution of the differential equation is thus:

$$x = e^{-nt}[C_1 \sin \sqrt{p^2 - n^2}\, t + C_2 \cos \sqrt{p^2 - n^2}\, t] + \frac{\dfrac{F_0}{m}}{\sqrt{(p^2 - \omega^2)^2 + 4n^2\omega^2}} \sin (\omega t - \phi) \tag{48}$$

Equation (48) represents a superposition of two motions. One has a frequency $\dfrac{1}{2\pi} \sqrt{p^2 - n^2}$ and an exponentially decreasing amplitude, and the other has a constant amplitude and the frequency $\dfrac{1}{2\pi}\,\omega$. This motion is shown in Fig. 5–6 for $p > \omega$. Because of

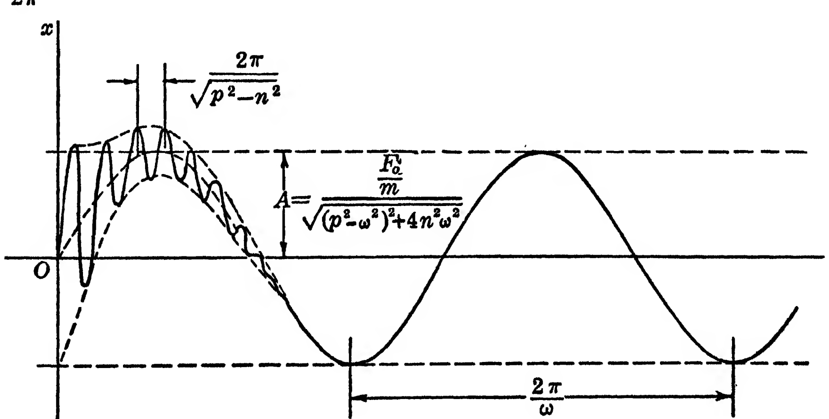


FIG. 5–6

e^{-nt} the first term of the expression decreases with time, and after a sufficient time it can be considered to be damped out, leaving the motion described by the second term. For this reason the first is called the *transient term*, and the second the *steady-state term*. The character of the transient term depends upon the initial conditions of the motion, whereas the steady state vibrations are independent of the initial conditions and depend only upon the forcing function and the parameters of the system.

The most important item in forced vibration problems usually is the amplitude of the steady forced vibration. Calling this amplitude A (Fig. 5–6), we have

$$A = \frac{\dfrac{F_0}{m}}{\sqrt{(p^2 - \omega^2)^2 + 4n^2\omega^2}}$$

Dividing numerator and denominator by p^2, and remembering that $p^2 = \dfrac{k}{m}$, we obtain:

$$A = \frac{\dfrac{F_0}{k}}{\sqrt{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2\left(\dfrac{n}{p}\right)\left(\dfrac{\omega}{p}\right)\right]^2}}$$

It is customary to express the damping as a fraction of critical damping, where critical damping n_c is defined by $n_c = p$ (see Problem 121). We write:

$$\frac{A}{\left(\dfrac{F_0}{k}\right)} = \frac{1}{\sqrt{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)\right]^2}} \tag{49}$$

It will be noted that the term $\left(\dfrac{F_0}{k}\right)$ is the deflection which the system would have under the action of a static load F_0; that is, it is the deflection of the system under a forcing function with zero frequency. The expression on the right side of the equation thus represents a dynamic amplification or magnification factor and gives the ratio between the dynamic and static deflections. The variation of this magnification factor with frequency ratio and damping ratio is shown in Fig. 5–7. The most significant feature of Fig. 5–7 is the fact that, near the frequency ratio $\left(\frac{\omega}{p}\right) = 1$, the magnification factor can become very large if the damping ratio is small. The infinite value indicated at $\left(\frac{n}{n_c}\right) = 0$ would, of course, not exist in practice, since it is impossible to

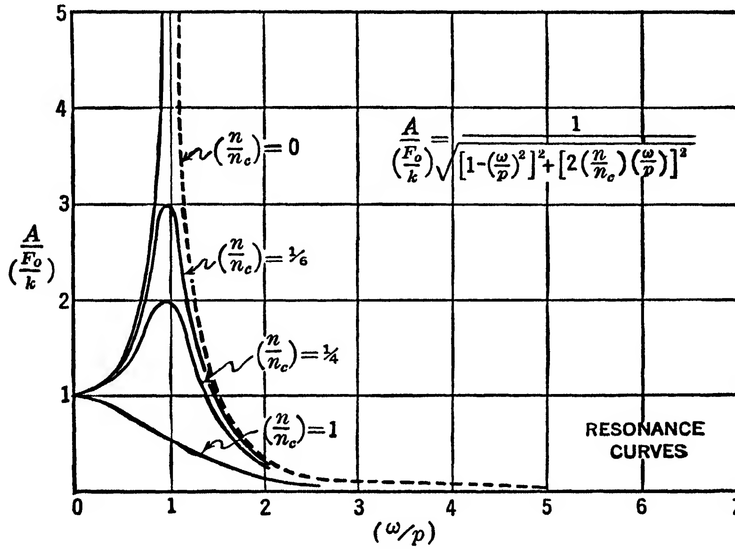


Fig. 5–7

reduce the damping to zero, and since it would require an infinite time to reach the infinite amplitude even if the damping were zero. The occurrence of large displacements near $\left(\frac{\omega}{p}\right) = 1$ is called *resonance*, and the frequency for which $\omega = p$ is called the *resonant frequency*.

If the damping is small, the maximum amplitudes occur very near $\left(\frac{\omega}{p}\right) = 1$, so that the maximum amplitude may be approximated very closely by setting $\left(\frac{\omega}{p}\right) = 1$. Thus we have:

$$A_{res} = \frac{\frac{F_0}{k}}{2\left(\frac{n}{n_c}\right)} \tag{50}$$

As an example, we might note that the damping ratio $\left(\frac{n}{n_c}\right)$ for an aircraft structure, such as a wing, has a magnitude of approximately 0.03; thus the resonant amplitude would be approximately 16 times the static deflection. This illustrates the danger of resonant conditions in structures and machines. However, if resonant vibrations of excessive amplitudes occur, it is possible to improve conditions by changing the frequency ratio or by increasing the damping in the system.

The steady forced vibration is

$$x = A \sin (\omega t - \phi)$$

The angle ϕ gives the phase relation between the motion and the exciting force. The phase angle is given by

$$\phi = \tan^{-1}\left(\frac{2n\omega}{p^2 - \omega^2}\right) = \tan^{-1} \frac{2n\frac{\omega}{p^2}}{1 - \left(\frac{\omega}{p}\right)^2} = \tan^{-1}\left[\frac{2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)}{1 - \left(\frac{\omega}{p}\right)^2}\right]$$

If $\left(\frac{\omega}{p}\right) \ll 1$, that is, if the forcing frequency is relatively low, then ϕ is small, and the motion is nearly in phase with the exciting force. If $\left(\frac{\omega}{p}\right) \gg 1$, that is, if the forcing frequency is high, ϕ is nearly 180°, showing that the motion is oppositely directed to the exciting force. At resonance $\left(\frac{\omega}{p}\right) = 1$ and $\phi = 90°$ for all values of the damping so that the exciting force is in the direction of the velocity.

PROBLEMS

125.* (a) Show that the energy input per cycle, W_i, of the exciting force is equal to

$$W_i = \int F\,dx = \int F\dot{x}\,dt = \int (F_0 \sin \omega t)[A\omega \cos (\omega t - \phi)]\,dt$$
$$= \pi F_0 A \sin \phi$$

(b) Show that the energy dissipated by the viscous damping force per cycle is

$$W_d = c\pi A^2\omega$$

(c) By equating the energy input and the energy dissipation, show that the steady state amplitude of a resonant vibration is

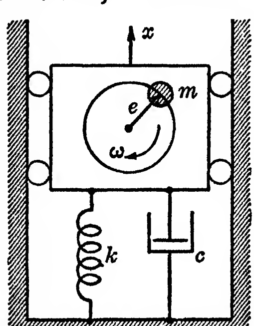
$$A_{res} = \frac{F_0}{c\omega}$$

and that this reduces to the same expression as was previously derived for A_{res}. Plot, on a graph of energy per cycle versus amplitude, the energy input and the energy dissipated at resonance and indicate the steady state amplitude.

126. A mass m restrained by a spring with a constant k is initially at rest. At time $t = 0$, it is acted upon by an exciting force $F(t) = F_0 \cos \omega t$. Assuming no damping, and given that $\frac{\omega}{2\pi} = 10$ cycles per second, m weighs 10 lb, $k = 20$ lb/ft, $F_0 = 100$ lb, find (a) the amplitude of the forced vibrations, and (b) the amplitude of the free vibrations.

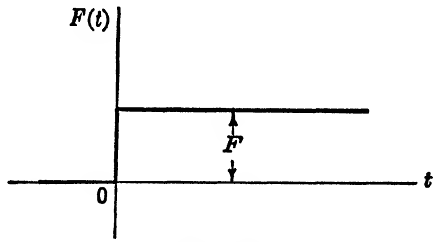
127. An undamped spring-mass system, which under gravity has a static deflection of 1 in., is acted upon by a sinusoidal exciting force which has a frequency of 4 cycles per second. What damping factor n is required to reduce the amplitude of the steady-state forced vibrations to one-half the amplitude of the undamped forced vibrations?

128. An unbalanced rotating mass is supported by a spring of constant k across which a dashpot giving a viscous damping force $c\dot{x}$ is connected. The mass rotates with an angular velocity ω, and the center of mass of the rotating body is at a distance e from the axis of rotation. Find the amplitude A of steady-state forced vibrations and plot a curve of $\left(\frac{\omega}{p}\right)$ versus $\left(\frac{A}{e}\right)$ for several values of the damping ratio $\left(\frac{n}{n_c}\right)$. The system is constrained so that the motion x is rectilinear. *Note:* The effect of the unbalanced mass is equivalent to a smaller mass m' at a larger distance r from the center of rotation. By assuming that the vibratory amplitude is small compared to r, the motion of the small mass m' can be taken as circular.



PROB. 128

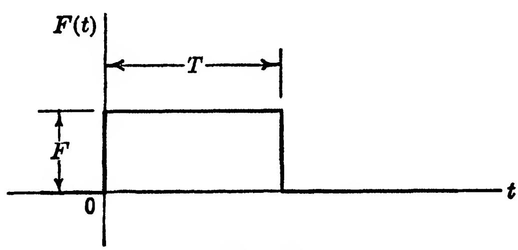
129. A step function as shown in the diagram is applied to an undamped harmonic oscillator; that is, when $t = 0$, a constant force of



PROB. 129

magnitude F is suddenly applied to the system. If the velocity and displacement of the oscillator are zero at time $t = 0$, find the subsequent motion.

130.* At time $t = 0$, a step function of the type described in Problem 129 is applied to an undamped, simple harmonic oscillator. After a time T the constant force F is suddenly removed, resulting in a forcing function of the type shown in the figure. The velocity and displacement of the mass are zero when $t = 0$.



PROB. 130

(a) Show that the displacement of the oscillator subsequent to the time T is given by:

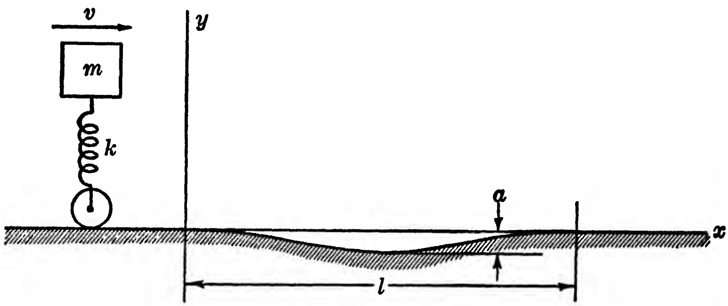
$$x = 2\frac{F}{k}\sin\left(\frac{pT}{2}\right)\sin\,(pt - \psi)$$

(b) The total impulse acting upon the system in this problem is $I = FT$. If I remains constant while T approaches zero, what is the amplitude of the resulting motion? Note that for T sufficiently small,

$$\frac{pT}{2} \approx \sin\left(\frac{pT}{2}\right)$$

Check this answer by treating the problem as a free vibration, with an initial velocity given by the impulse-momentum equation.

131. At time $t = 0$, a sinusoidal exciting force $F(t) = F_0 \sin \omega t$ is applied to an undamped, simple harmonic oscillator of mass m and spring constant k. The frequency of the exciting force is the same as the natural frequency of the system. If the mass is initially at rest, find the amplitude of vibrations as a function of time.



PROB. 132

132.* An automobile, without shock absorbers, may be represented approximately as a concentrated mass m supported by a spring having a constant k. The automobile runs with a velocity v over a hollow in the road which can be represented by the cosine curve

$$y = -\frac{a}{2}\left(1 - \cos\frac{2\pi x}{l}\right)$$

Neglecting damping, find the vertical acceleration of m when $x = l$.

34. Vibration Isolation. One of the useful applications of vibration theory is to the vibration isolation of instruments and machinery. As a first example, we shall consider the problem of mounting an instrument so as to minimize the transmission of vibration from the supporting structure to the instrument. In many applications delicate instruments must be used in structures which have appreciable amplitudes of vibration. Unless the instrument can be isolated from its support it may be impossible to make accurate measurements. Suppose that the support S in Fig. 5–8 has a motion $y_0 \sin \omega t$. The mass of the instrument is m, and it is attached to the support by a spring k. The damping in the system is represented by a dashpot

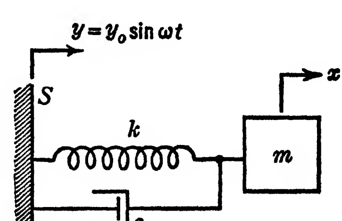


FIG. 5–8

having a viscous damping constant c. Letting x be the amplitude of motion of the instrument m, we have as the differential equation of motion:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$
$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = ky_0 \sin \omega t + cy_0\,\omega \cos \omega t$$
$$\ddot{x} + 2n\dot{x} + p^2 x = \frac{y_0}{m}(k \sin \omega t + c\omega \cos \omega t)$$

Writing the right side of the equation as:

$$\frac{y_0}{m}(k \sin \omega t + c\omega \cos \omega t) = \frac{y_0}{m}\sqrt{k^2 + (c\omega)^2}\,\sin(\omega t - \beta)$$

we see that its effect is the same as a sinusoidal exciting force, so that this is an equation of the same type as Equation (35) and the same solution can be used. Putting $F_0 = y_0\sqrt{k^2 + (c\omega)^2}$ in Equation (49), we have for the amplitude of the steady-state forced vibration:

$$A = \frac{\frac{y_0}{k}\sqrt{k^2 + (c\omega)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}}$$

$$\frac{A}{y_0} = \frac{\sqrt{1 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}} \tag{51}$$

Thus the effectiveness of the mounting in reducing the amplitude is measured by the expression on the right side of Equation (51). The appearance of this function for various values of $\left(\frac{\omega}{p}\right)$ and $\left(\frac{n}{n_c}\right)$ is shown in Fig. 5–9, where it will be noted that at any frequency ratio greater than $\sqrt{2}$ the amplitude of the mass will be less than the amplitude of the support. The main difference between this resonance curve and that given in Fig. 5–7 is that for $\left(\frac{\omega}{p}\right) > \sqrt{2}$ the damped curves are above the undamped curves. This means that the presence of damping decreases somewhat the effectiveness of the mounting. A certain amount of damping, however, is essential in order to maintain stability under transient conditions and to prevent excessive amplitudes should the vibration pass through

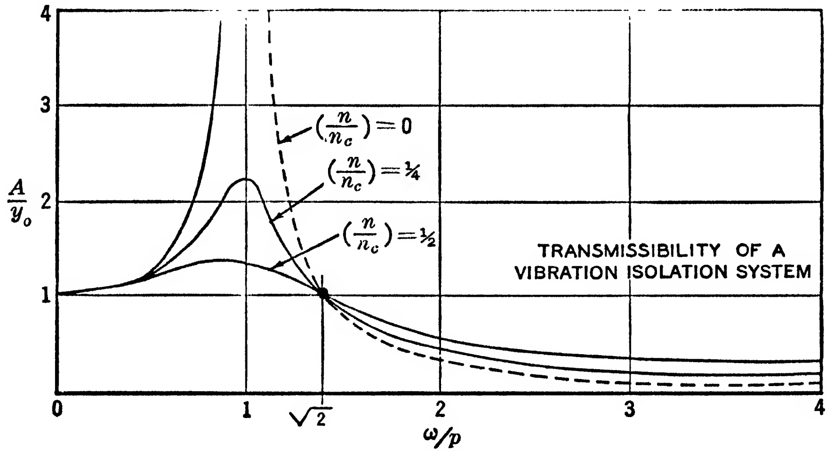


Fig. 5–9

resonance during the starting or stopping of the motion of the support.

A second type of vibration isolation problem is illustrated in Fig. 5–10. Suppose that a machine, as a result of unbalanced rotating masses, exerts an alternating force of $m'r\omega^2 \sin \omega t$ upon its foundation where the mass of the rotating unbalance is m' and the effective radius r. If the machine is rigidly fastened to the foundation, the force will be transmitted directly to the foundation and may cause objectionable vibrations. It is desirable to isolate the machine from the foundation in such a way that the transmitted force will be reduced. Letting x be the displacement of the total mass m of the machine, we have, from the analysis previously made (Problem 128):

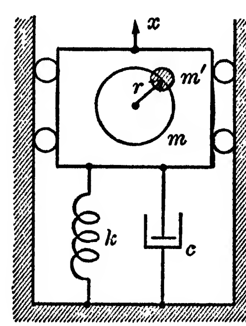


Fig. 5–10

$$x = \frac{\dfrac{m'r\omega^2}{k}}{\sqrt{\left[1 - \left(\dfrac{\omega}{p}\right)^2\right]^2 + \left[2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)\right]^2}} \sin(\omega t - \phi)$$

The only force which can be applied to the floor is the spring force kx and the damping force $c\dot{x}$; hence the total force acting on the foundation during the steady state forced vibration is:

$$F = kx + c\dot{x} = \frac{m'r\omega^2}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}} \sin(\omega t - \phi)$$

$$+ \frac{\frac{cm'r\omega^2}{k}}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}} \omega \cos(\omega t - \phi)$$

The amplitude of the resulting transmitted force is:

$$F_A = m'r\omega^2 \left[\frac{\sqrt{1 + \left(\frac{c\omega}{k}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}}\right]$$

Since $m'r\omega^2$ is the amplitude of the force which would be transmitted if the springs were infinitely rigid, we have as a measure of the effectiveness of the isolation mounting the expression:

$$\frac{F_A}{m'r\omega^2} = \frac{\sqrt{1 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}} \tag{52}$$

This is called the *transmissibility* of the system. It is exactly the same as Equation (51) obtained for the vibration isolated instrument, and Fig. 5–9 also represents the solution of the present problem. The frequency ratio $\left(\frac{\omega}{p}\right)$ and the damping have the same influence on the transmissibility as they had on the vibration isolation.

PROBLEMS

133. The amplitude of vibration in an airplane at the point at which it is desired to mount instruments is 0.015 in. and the frequency of the vibration is 1800 cycles per minute. The amplitude of the instruments is to be limited to 0.002 in. The instruments, along with the panel and mounting bracket, weigh 50 lb. Four rubber shock mounts are to be used, spaced in such a way that they are all equally loaded. Find the spring constant required for the rubber mount, assuming that damping can be neglected.

134. An instrument panel is mounted on a suspension system having a static deflection under gravity of $\frac{1}{4}$ in. It is subjected to vibrations whose frequency corresponds to one-half of the speed of an engine which runs at 2000 rpm. What percentage reduction in amplitude of vibration is to be expected from this suspension system? (Neglect the effects of damping.)

135. A machine having a total weight of 20,000 lb has an unbalance such that it is subjected to a force of amplitude 5000 lb at a frequency of 600 cycles per minute. What should be the spring constant for the supporting springs if the maximum force transmitted into the foundation due to the unbalance is to be 500 lb? (Assume that damping may be neglected.)

136. An instrument whose total weight is 20 lb is to be spring-mounted on a vibrating surface which has a sinusoidal motion of amplitude $\frac{1}{64}$ in., and frequency 60 cycles per second. If the instrument is mounted rigidly on the surface, what is the maximum force to which it is subjected? Find the spring constant for the support system which will limit the maximum acceleration of the instrument to $\frac{1}{2}$ the acceleration of gravity. (Assume that negligible damping forces have caused the transient vibrations to die out.)

137. Show that a vibration isolation system is effective only if $(\omega/p) > \sqrt{2}$.

35. The Design of Vibration Measuring Instruments. Suppose that the structure S in Fig. 5–11 is vibrating harmonically with an unknown amplitude y_0 and an unknown frequency ω. To measure y_0 and ω we may attach to the structure an instrument which consists of a mass m, a spring k, and a viscous damping c. The output of the instrument will depend upon the relative motion between the mass and the structure, since it is this relative motion which is detected and amplified by mechanical, optical, or electrical means. Taking x as the absolute displacement of the instrument mass, the output of

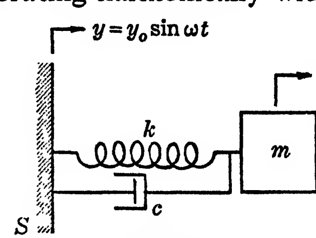


Fig. 5–11

the instrument will be proportional to $z = (x - y)$. The equation of motion of the instrument mass is:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

Subtracting $m\ddot{y}$ from each side of the equation gives:

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = my_0\omega^2 \sin \omega t$$

This equation is the same as Equation (35), so that the solution for steady forced vibrations is:

$$z = \frac{\left(\frac{\omega}{p}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{p}\right)^2\right]^2 + \left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}}\, y_0 \sin(\omega t - \phi) = Qy_0 \sin(\omega t - \phi)$$

$$\phi = \tan^{-1}\left[\frac{2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)}{1 - \left(\frac{\omega}{p}\right)^2}\right] \tag{53}$$

The instrument will read the displacement of the structure directly if $Q = 1$ and $\phi = 0$. The variation of Q with $\left(\frac{\omega}{p}\right)$ and $\left(\frac{n}{n_c}\right)$ is shown in Fig. 5–12. It is seen that if $\left(\frac{\omega}{p}\right)$ is large, Q is approximately equal to 1, and ϕ is approximately equal to zero; we conclude, therefore, that, to design a displacement pickup, $\left(\frac{\omega}{p}\right)$ should

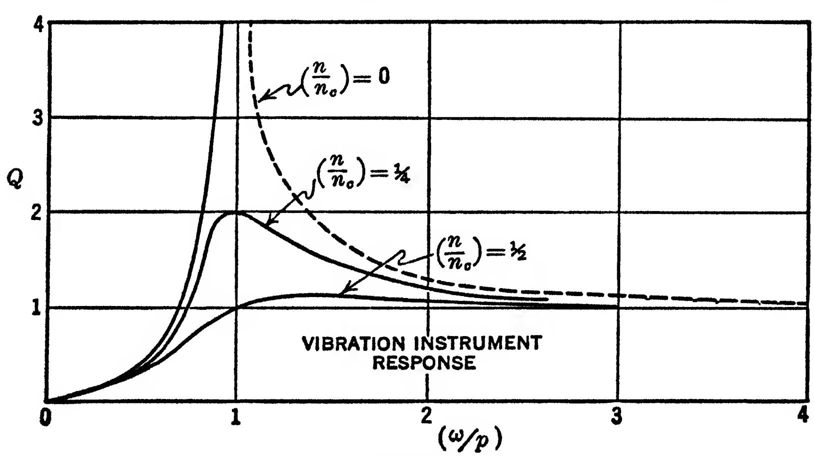


FIG. 5–12

be large, which means that the natural frequency of the instrument itself should be low compared to the frequency to be measured.

We next consider the region of the diagram where $\left(\frac{\omega}{p}\right)$ is small. ϕ is then approximately equal to zero, and the quantity

$$\frac{1}{\sqrt{\left[1-\left(\frac{\omega}{p}\right)^2\right]^2+\left[2\left(\frac{n}{n_c}\right)\left(\frac{\omega}{p}\right)\right]^2}}$$

is approximately equal to 1. The expression $z = Qy_0 \sin(\omega t - \phi)$ then becomes:

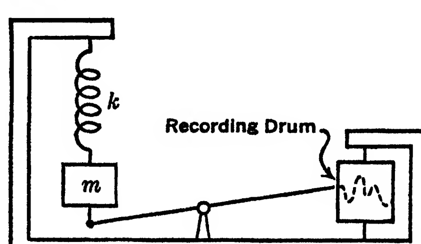
$$z = \frac{1}{p^2} y_0\omega^2 \sin \omega t$$

Since $y_0\omega^2 \sin \omega t$ is the acceleration of the structure, the instrument output is proportional to the acceleration. We thus conclude that, to design an accelerometer, $\left(\frac{\omega}{p}\right)$ should be small, which means that the natural frequency of the instrument itself should be high compared to the frequency to be measured.

Instruments designed according to the foregoing criteria will have characteristics which are independent of frequency. Such instruments can be used outside of the specified range if the exact curves of Fig. 5–12 are used.

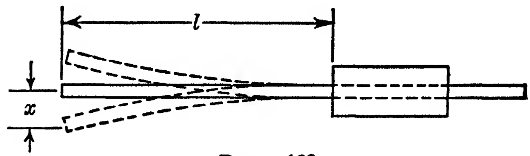
PROBLEMS

138. It is desired to design an instrument to measure the vertical oscillations of the Golden Gate Suspension Bridge. The bridge has a vertical frequency of approximately ¼ cycles per second and the amplitude may at times reach 4 to 5 ft. An instrument of the type shown in the diagram has been suggested. Would it be better to design this instrument as an accelerometer or as a displacement meter? What would be a satisfactory frequency for the spring-mass system in the instrument?



Prob. 138

139. A simple instrument for determining the frequency of vibration is constructed on the principle indicated in the diagram. A flat strip of metal is mounted as a cantilever beam of length l. The free vibrations of



PROB. 139

the strip are given by $x = A \sin pt$ where $p^2 = \frac{k}{l^4}$ and where k is a constant depending upon the proportions and material of the strip. The instrument is constructed so that the length can be varied. If the instrument is mounted upon a vibrating body whose frequency is ω, the amplitude will depend upon the ratio of the forced frequency and the natural frequency. At resonance this amplitude will be large, so that by varying l until the amplitude is a maximum the forcing frequency can be determined. Write the expression which gives the frequency of the vibrating body as a function of the length of the strip.

140. For measuring the vertical vibrations of a machine foundation, an instrument of the type shown in Fig. 5–11 is used. The spring-mass system of the instrument has been designed so that the static deflection is $\frac{3}{4}$ in. The frequency of the vibration corresponds to an engine speed of 1500 rpm. The amplitude of the relative motion between the instrument mass and the foundation is determined, from a dial gage reading, to be 0.008 in. Find the amplitude of the foundation. The damping in the instrument has a magnitude of 70% of critical damping.

36. Vibrations with Non-periodic Forces. The analysis of the preceding sections is sufficient to treat vibrations with sinusoidal exciting forces. Since any periodic forcing function can be represented in a trigonometric series, the analysis can be extended, by using the principle of superposition, to include the solution for a general periodic forcing function. For non-periodic exciting forces, however, it is desirable to develop a different method of approach. We shall limit the following analysis to undamped systems, although it is possible to extend the same method to damped systems (see Problem 145).

We shall consider first the motion of an undamped spring-mass system to which a single impulse is applied. Referring to Fig.

5–13, an impulse $F_0\Delta t$ will produce an initial velocity $\dot{x}_0$ which can be determined by the equation of impulse and momentum:

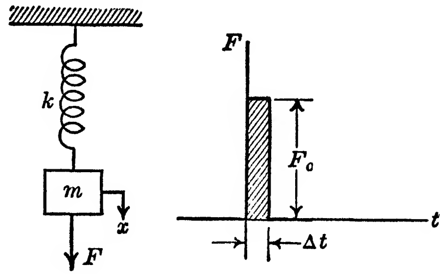


FIG. 5–13

$$F_0\Delta t = m\dot{x}_0$$

$$\dot{x}_0 = \frac{F_0\Delta t}{m}$$

The displacement x of an undamped system performing free vibrations is given by Equation (36).

$$x = \frac{\dot{x}_0}{p}\sin pt + x_0\cos pt$$

We have $\dot{x}_0 = \dfrac{F_0\Delta t}{m}$ and $x_0 = 0$ if we measure time from the point of zero deflection, so that:

$$x = \frac{F_0\Delta t}{mp}\sin pt \tag{54}$$

Having found the motion under the action of one impulse, we may now, by the principle of superposition, find the motion under the action of any arbitrary forcing function. It is only necessary

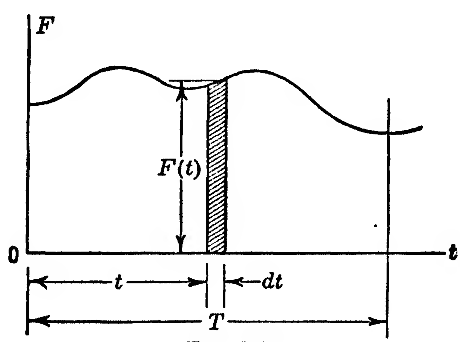


FIG. 5–14

to let the arbitrary function be represented by an infinite number of impulses. Suppose that the curve of Fig. 5–14 represents an exciting force, which is applied when $t = 0$, and that it is desired to determine the displacement at time T. Consider the force to be divided into a large number of impulses, of which one, $F(t)\,dt$, is shown in the diagram. The displacement x at the time T due to this impulse can be determined from Equation (54). In Equation (54), t represents the time which elapses between the application of the impulse and the measurement of the displacement. Thus at T, which is $(T - t)$ after the impulse is applied, we have:

$$dx = \frac{F(t)\,dt}{mp} \sin p(T - t)$$

We use the notation dx because this represents only the contribution of one impulse to the displacement x. To find the total displacement, the effects of all of the impulses from 0 to T must be added, which means that the expression for dx must be integrated, giving:

$$x = \frac{1}{mp} \int_0^T F(t) \sin p(T - t)\,dt \tag{55}$$

With this equation, the motion can be computed for any undamped system which has zero initial velocity and displacement. If $F(t)$ is given as a graph or as numerical data, instead of in analytical form, the integration can be carried out by graphical or numerical methods, and one of the advantages of the equation is its adaptability to solutions of this type.

A more formal derivation of the equation can be obtained in the following way. The differential equation of motion for an undamped system with an exciting force $F(t)$ is:

$$\ddot{x} + p^2 x = \frac{1}{m} F(t)$$

Multiplying through by the integrating factor $\sin p(T - t)$ and integrating, this becomes:

$$\int_0^T \ddot{x} \sin p(T - t)\,dt + \int_0^T p^2 x \sin p(T - t)\,dt = \frac{1}{m} \int_0^T F(t) \sin p(T - t)\,dt$$

Integrating the first term twice by parts reduces this to:

$$\dot{x} \sin p(T - t)\Big|_0^T + px \cos p(T - t)\Big|_0^T = \frac{1}{m} \int_0^T F(t) \sin p(T - t)\,dt$$

Substituting the limits of integration and solving for x gives:

$$x = \frac{1}{mp}\int_0^T F(t)\sin p(T-t)\,dt + \frac{\dot{x}_0}{p}\sin pT + x_0\cos pT \tag{56}$$

If we take as the initial conditions $x_0 = \dot{x}_0 = 0$ when $T = 0$, this expression becomes:

$$x = \frac{1}{mp}\int_0^T F(t)\sin p(T-t)\,dt$$

which is the solution derived by the superposition of impulses.

Example. To illustrate the application of the method we shall solve a problem which we have already solved by other methods. Suppose that a sinusoidal exciting force $F_0 \sin \omega t$ is applied; then x is given by:

$$x = \frac{F_0}{mp}\int_0^T \sin\omega t \sin p(T-t)\,dt$$

Making use of the trigonometric relation

$$\sin A \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$

the displacement may be written:

$$x = \frac{F_0}{2mp}\int_0^T \{\cos[(\omega+p)t - pT] - \cos[(\omega-p)t + pT]\}\,dt$$

Carrying out the integration, we obtain:

$$x = \frac{F_0}{m}\,\frac{1}{p^2\left[1-\left(\frac{\omega}{p}\right)^2\right]}\left(\sin\omega T - \frac{\omega}{p}\sin pT\right)$$

This solution represents the superposition of a free vibration of frequency $p/2\pi$ and a forced vibration of frequency $\omega/2\pi$, so that it contains both the transient and the steady-state terms. The amplitude of the steady forced vibration is:

$$A = \frac{F_0}{mp^2}\,\frac{1}{1-\left(\frac{\omega}{p}\right)^2} = \frac{F_0}{k}\left[\frac{1}{1-\left(\frac{\omega}{p}\right)^2}\right]$$

This is the same as Equation (49) with the damping set equal to zero.

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141. Carry out the integrations indicated in the preceding Example for a sinusoidal exciting force and check the result:

$$x = \frac{F_0}{mp^2} \frac{1}{\left[1 - \left(\frac{\omega}{p}\right)^2\right]} \left(\sin \omega T - \frac{\omega}{p} \sin pT\right)$$

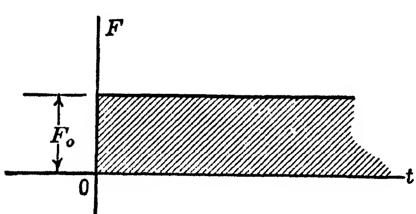
142. Show that the solution for damped forced vibrations

$$x = e^{-nt}[c_1 \sin \sqrt{p^2 - n^2}\, t + c_2 \cos \sqrt{p^2 - n^2}\, t] + \frac{\frac{F_0}{m}}{\sqrt{(p^2 - \omega^2)^2 + 4n^2\omega^2}} \sin (\omega t - \phi)$$

$$\phi = \tan^{-1}\left(\frac{2n\omega}{p^2 - \omega^2}\right)$$

reduces to the expression found in Problem (141) when damping is put equal to zero.

143. An undamped vibrating system is at rest until time $t = 0$, when a step function F_0 is applied, as shown in the diagram. Find the resulting motion by the integral method of the preceding section and show that the maximum displacement is twice the static deflection of the system.



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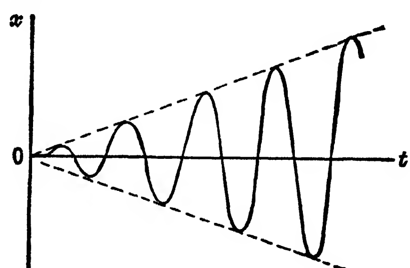

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144. Suppose that a sinusoidal exciting force $F_0 \sin pt$ having the same frequency as the natural frequency of the undamped oscillator is applied at time $t = 0$. Show that the displacement is given by:

$$x = \frac{F_0}{2mp}\left(\frac{\sin pT}{p} - T \cos pT\right)$$

Thus the resonant amplitude of the system builds up with a linearly increasing amplitude. Use the integral method for this problem.

145. Show that for a system having viscous damping, the integral solution is:

$$x = \frac{1}{m\sqrt{p^2 - n^2}} \int_0^T F(t)e^{-n(T-t)} \sin \sqrt{p^2 - n^2}(T - t)\, dt$$

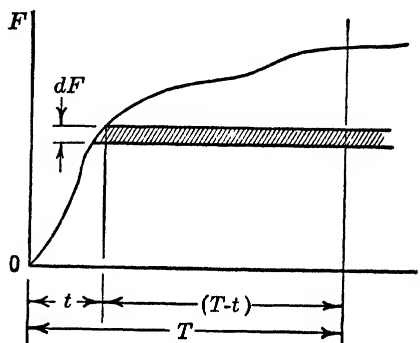
146.* The integral form of the solution for the motion of a spring-mass system can also be derived from the differential equation by use of Lagrange's method of the "variation of parameters." Carry through the solution of the equation $m\ddot{x} + kx = F(t)$ by this method, and show that Equation (55) is obtained.

147.* Show by integrating by parts that the integral solution for the undamped oscillator may be written:

$$\left(\frac{F}{k} - x\right) = \frac{1}{mp^2} \int_0^T \left(\frac{dF}{dt}\right) \cos p(T - t)\, dt$$

where $x_0 = \dot{x}_0 = 0$ and $F(t) = 0$ when $t = 0$.

Show that this method of solution is equivalent to cutting F into horizontal slices as shown in the figure, and summing the effect of the successive incremental step functions.



Prob. 147

37. Oscillations in Electric Circuits. Oscillation problems of the type treated in this chapter are also of frequent occurrence in electrical circuit analysis. Consider an electrical circuit consisting of an inductance L, a capacitance C, and a resistance R as shown in Fig. 5–15. These elements are connected in series with a source of alternating voltage with an amplitude E_0 and a frequency $\frac{\omega}{2\pi}$. The equation describing the behavior of the system is obtained by equating the applied voltage to the sum of the voltage drops across the three elements. If the current in the circuit is i, then the voltage drop across the inductance is $L\frac{di}{dt}$, that across the capacitance is

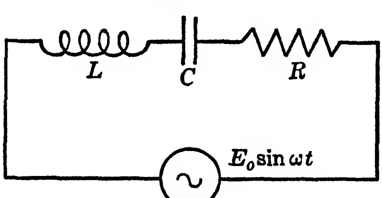


Fig. 5–15

$\frac{1}{C}\int i\,dt$, and that across the resistance is Ri. The equation thus is:

$$E_0 \sin \omega t = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt$$

If Q represents the electric charge, then

$$i = \frac{dQ}{dt}$$

and the equation may be written:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E_0 \sin \omega t \qquad (57)$$

It will be seen that this equation has exactly the same form as Equation (35), which describes the motion of a mechanical vibrating system, with the following analogous quantities:

Electrical System	*Mechanical System*
Inductance, L	Mass, m
Resistance, R	Coefficient of viscous damping, c
Reciprocal of Capacitance, $\frac{1}{C}$	Spring constant, k
Exciting voltage, E	Exciting force, F
Electrical charge, Q	Displacement, x
Current, i	Velocity, $\dot{x}$

The results of the analysis for the mechanical system can therefore be applied to the electrical system, and the solution of the differential equation is:

$$Q = e^{-\frac{R}{2L}t}\left[C_1 \sin\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\,t + C_2 \cos\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\right]t + A \sin(\omega t - \phi) \qquad (58)$$

where

$$A = \frac{\frac{E_0}{L}}{\sqrt{\left[\frac{1}{LC} - \omega^2\right]^2 + \frac{R^2}{L^2}\omega^2}}$$

and

$$\phi = \tan^{-1}\left[\frac{\frac{R\omega}{L}}{\frac{1}{LC} - \omega^2}\right]$$

Just as in mechanical systems, the solution consists of a transient term and a steady-state term. Because of the resistance in the circuit the electrical transient vibrations die out in time, leaving the forced steady-state oscillations.

To find the steady-state current i in the circuit, we write:

$$i = \frac{dQ}{dt} = \frac{E_0\omega/L}{\sqrt{\left[\frac{1}{LC} - \omega^2\right]^2 + \frac{R^2}{L^2}\omega^2}} \cos(\omega t - \phi)$$

The amplitude of the steady-state current is:

$$i_A = \frac{E_0}{\sqrt{\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2}} \tag{59}$$

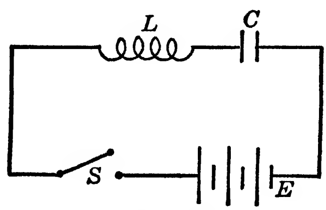
The quantity $\sqrt{\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2}$ is called the electrical impedance of the circuit, $\frac{1}{\omega C}$ is called the capacitive reactance of the circuit, and ωL is called the inductive reactance. It will be seen that resonance occurs when $\frac{1}{\omega C} = \omega L$ and that the magnitude of the resonant current is limited only by the resistance in the circuit.

Because of the analogy between electrical and mechanical problems, it is often possible to transfer solutions from one field directly to the other, thus saving duplication of work. Such analogies are also often used for experimental solutions. It is usually much easier to build an electrical circuit and to make measurements on it than it is to construct and test the analogous mechanical system. Electrical Analog Computers, which operate on this principle, have been constructed so that many different combinations of electrical elements can be set up, and in this way complex electrical, mechanical, and thermal problems have been solved.*

* See, for example, Criner, H. E., McCann, G. D., and Warren, C. E., "A New Device for the Solution of Transient Vibration Problems by the Method of Electrical-Mechanical Analog." *Journal of Applied Mechanics* **12**, 135, September 1945.

PROBLEMS

148. At time $t = 0$, the switch S in the electrical circuit shown in the diagram is closed, applying a voltage E to the series inductance and capacitance. Show, graphically, the way in which the current in the circuit varies with time, assuming that the resistance in the circuit is negligible. What would be the effect of resistance in the circuit?



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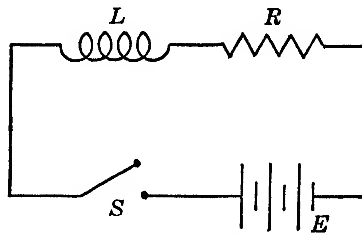
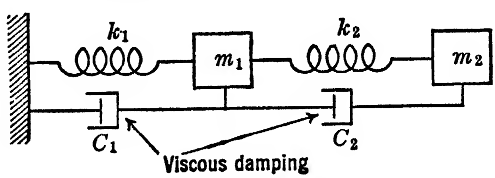


PROB. 149

149. At time $t = 0$, a switch is closed applying a voltage E to an inductance and a resistance which are in series. Find the relation between the current and time. Show that the time required for the current to reach $\left(1 - \frac{1}{e}\right)$ times its final value is equal to L/R. This is called the time constant of the circuit.

150.* Draw the circuit for the electrical analog of the mechanical system shown in the diagram.



PROB. 150

CHAPTER VI

PRINCIPLES OF DYNAMICS FOR SYSTEMS OF PARTICLES

. . . the same law takes place in a system consisting of many bodies as in a single body. For the progressive motion, whether of one single body, or of a whole system of bodies, is always to be estimated from the motion of the center of gravity.—I. Newton, *Principia Philosophiae* (1686).

In most dynamics problems it is not possible to approximate the system by a single particle, but it must be treated as a collection of particles. The system itself may be a solid body, a fluid, or a gas, but in any event it may be thought of as a collection of particles, each of which may be treated by the methods of particle dynamics. The type of interaction between the individual particles will depend upon the system being investigated, but certain general relations may be developed which apply no matter what these interactions may be. In the present chapter these general relations are developed, and in subsequent chapters the consequences of the special characteristics of the systems are treated.

38. The Equation of Motion for a System of Particles. The equation of motion for a typical particle of a system is:

$$m_i\ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i + \boldsymbol{f}_i \tag{60}$$

The subscript indicates that the equation applies to the ith particle. The resultant force acting upon the particle is written as the sum of an external force $\boldsymbol{F}$ and an internal force $\boldsymbol{f}$. The external force originates outside of the system, and represents the action of some body or agency upon the system. The internal force originates within the system in the mutual actions and reactions between the particles. The reason for distinguishing between these two types of forces is that when the system as a whole is under consideration, the sum of all the internal forces is equal to zero. This follows from the fact that internal forces always occur in equal, opposite, and collinear pairs and will thus cancel. An equation of motion for the entire system is obtained by adding the equations for the individual particles and setting $\Sigma \boldsymbol{f}_i = 0$:

$$\Sigma m_i \ddot{\boldsymbol{r}}_i = \Sigma \boldsymbol{F}_i$$

Using the notation $\Sigma \boldsymbol{F}_i = \boldsymbol{F}$, this becomes:

$$\Sigma m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F} \tag{61}$$

39. The Motion of the Center of Mass. The center of mass of a system of particles is defined as a point located by the vector $\boldsymbol{r}_c$ where

$$\boldsymbol{r}_c = \frac{\Sigma m_i \boldsymbol{r}_i}{\Sigma m_i}$$

We may introduce this quantity into the equation of motion of the system by writing equation (61) in the form:

$$\boldsymbol{F} = \frac{d^2}{dt^2}(\Sigma m_i \boldsymbol{r}_i) = \frac{d^2}{dt^2}(\boldsymbol{r}_c \Sigma m_i)$$

Setting Σm_i equal to M, the total mass of the system of particles, this becomes:

$$\boldsymbol{F} = M\ddot{\boldsymbol{r}}_c \tag{62}$$

Thus we may conclude that *the motion of the center of mass is the same as the motion of a particle, having a mass equal to the total mass of the system, acted upon by the resultant external force.* The motion of the mass center is therefore a problem in particle dynamics. This is the justification for having treated finite bodies as particles in the preceding chapters.

The equation of motion of the mass center may be integrated with respect to time and with respect to displacement, to give the impulse-momentum and the work-energy equations for the motion of the center of mass. These are:

$$\int_1^2 \boldsymbol{F}\,dt = M\dot{\boldsymbol{r}}_c \Big|_1^2 \tag{63}$$

$$\int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r}_c = \tfrac{1}{2}Mv_c^2 \Big|_1^2 \tag{64}$$

It should be noted that these equations give only information as to the motion of the center of mass of the system. The term $M\dot{\boldsymbol{r}}_c$ has the magnitude and direction of the total momentum of the system, but the location of the line of action of the total momentum vector is not determined by this expression, for it does not necessarily pass through the center of mass. The term $\frac{1}{2}Mv_c^2$ does not represent the total kinetic energy of the system since the motion of the parts of the system with respect to the center of mass will contribute additional kinetic energy.

40. The Total Kinetic Energy of a System of Particles. The total kinetic energy, T, of a system of particles is the sum of the kinetic energies of the individual particles.

$$T = \Sigma\tfrac{1}{2}m_iv_i^2$$

This expression may be put into another form, which is useful for many problems, by referring the motion of each particle to the center of mass of the system. As shown in Fig. 6–1, the vector $\boldsymbol{\rho}_i$ represents the displacement of the ith particle with respect to the center of mass.

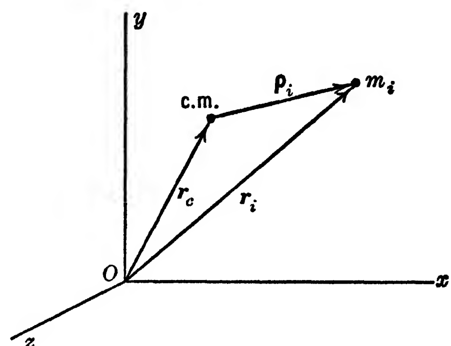


FIG. 6–1

For each particle, $\boldsymbol{r}_i = \boldsymbol{r}_c + \boldsymbol{\rho}_i$, therefore,

$$\begin{aligned} v_i^2 &= (\dot{\boldsymbol{r}}_i)\cdot(\dot{\boldsymbol{r}}_i) = (\dot{\boldsymbol{r}}_c + \dot{\boldsymbol{\rho}}_i)\cdot(\dot{\boldsymbol{r}}_c + \dot{\boldsymbol{\rho}}_i) \\ &= \dot{r}_c^2 + 2\dot{\boldsymbol{r}}_c\cdot\dot{\boldsymbol{\rho}}_i + \dot{\rho}_i^2 \end{aligned}$$

And the kinetic energy may be written:

$$T = \Sigma\tfrac{1}{2}m_i\dot{r}_c^2 + \dot{\boldsymbol{r}}_c\cdot\frac{d}{dt}(\Sigma m_i\boldsymbol{\rho}_i) + \Sigma\tfrac{1}{2}m_i\dot{\rho}_i^2$$

Since $\boldsymbol{\rho}_i$ is measured from the mass-center, we have $\Sigma m_i\boldsymbol{\rho}_i = 0$ and the second term drops out. The first term may be written:

$$\Sigma\tfrac{1}{2}m_i\dot{r}_c^2 = \dot{r}_c^2\Sigma\tfrac{1}{2}m_i = \tfrac{1}{2}M\dot{r}_c^2$$

where M is the total mass of the system. Thus the kinetic energy becomes:

$$T = \tfrac{1}{2}M\dot{r}_c^2 + \Sigma\tfrac{1}{2}m_i\dot{\rho}_i^2 \tag{65}$$

The total kinetic energy may thus be said to be the sum of the energy which would be obtained if all the mass were located at the mass-center plus the kinetic energy of the system corresponding to the motion relative to the mass-center.

The work-energy equation for a system of particles may be put into a convenient form by using the same transformation. The total work done by all the forces of the system is:

$$\Sigma \int_1^2 (\boldsymbol{F}_i + \boldsymbol{f}_i) \cdot d\boldsymbol{r}_i = \int_1^2 \Sigma(\boldsymbol{F}_i + \boldsymbol{f}_i) \cdot (d\boldsymbol{r}_c + d\boldsymbol{\rho}_i)$$
$$= \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r}_c + \int_1^2 \Sigma \boldsymbol{f}_i \cdot d\boldsymbol{r}_c + \int_1^2 \Sigma(\boldsymbol{F}_i + \boldsymbol{f}_i) \cdot d\boldsymbol{\rho}_i$$

The sum of the internal forces is zero, $\Sigma \boldsymbol{f}_i = 0$; hence the second term drops out. Equating the total work done to the change in total kinetic energy, we have:

$$\int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r}_c + \int_1^2 \Sigma(\boldsymbol{F}_i + \boldsymbol{f}_i) \cdot d\boldsymbol{\rho}_i = \tfrac{1}{2}M\dot{r}_c^2\Big|_1^2 + \Sigma \tfrac{1}{2}m_i\dot{\rho}_i^2\Big|_1^2$$

We have already shown that the first term on the left is equal to the first term on the right, so that the second term on the left must equal the second term on the right. We may thus write the two independent equations:

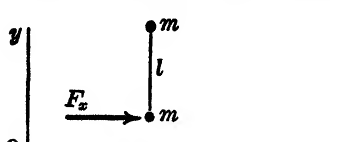
$$\int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r}_c = \tfrac{1}{2}M\dot{r}_c^2\Big|_1^2 \tag{66}$$

$$\int_1^2 \Sigma(\boldsymbol{F}_i + \boldsymbol{f}_i) \cdot d\boldsymbol{\rho}_i = \Sigma \tfrac{1}{2}m_i\dot{\rho}_i^2\Big|_1^2 \tag{67}$$

The first of these equations describes the motion of the center of mass of the system, while the second describes the motion of the system with respect to the center of mass. The fact that these two equations can be written independently of each other simplifies the solution of problems by the energy method.

PROBLEMS

151. (a) Two particles of mass m, connected by a rigid weightless rod, are acted upon by a force F_x = constant. At time $t = 0$, the system is at rest as shown. What is the subsequent motion of the mass center?



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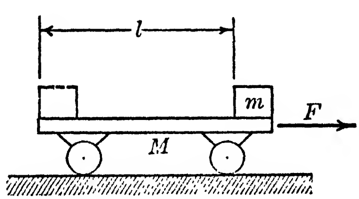
(b) Derive the motion of the mass center by setting up the equations of motion for each particle separately and integrating. This illustrates the advantage of using the principle of the motion of the mass center.

152. An explosive projectile is traveling with a constant velocity $\mathbf{v}$. At a certain instant the projectile explodes, scattering fragments in all directions. Neglecting air resistance, describe the motion of the center of mass of the system of fragments.

153. A cart of mass M, initially at rest, can move horizontally along a frictionless track. When $t = 0$, a force F is applied to the cart as shown. During the acceleration of M by the force F, a small mass m slides along the cart from the front to the rear. The coefficient of friction between m and M is μ, and it is assumed that the acceleration of M is sufficient to cause sliding.

(a) Write two equations of motion, one for m and one for M, and show that they can be combined to give the equation of motion of the mass center of the system of two bodies.

(b) Find the displacement of M at the time when m has moved a distance l along the cart.



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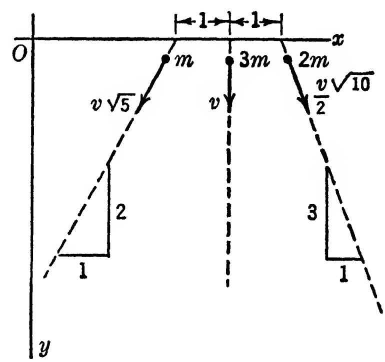


Prob. 154

154. Three particles of mass, m, $2m$, and $3m$, are moving with constant velocities in the directions shown. The motion takes place in the xy plane.

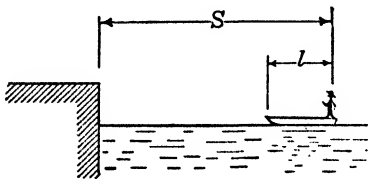
(a) Find the magnitude and the direction of the total momentum of the system of three particles.

(b) Find the total kinetic energy of the system and compare with the energy which the system would have if all of its mass were concentrated at the mass center.

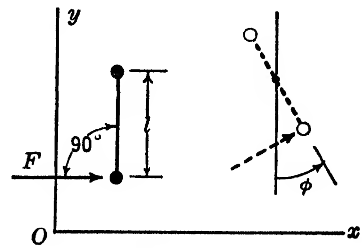
(c) If at time $t = 0$, the particles are all located on the x-axis in the positions indicated, what is the subsequent path of motion of the mass-center?

155. A handful of buckshot of total mass M is thrown against a wall. Show that the total impulse to which the wall is subjected is $2Mv_a$, where v_a is the average value of the velocity of the shot normal to the wall. (Assume no energy is lost during impact.)

156. A man of mass m stands at the rear of a boat of mass M as shown. The distance of the man from the pier is S ft. What is the distance of the man from the pier after he has walked forward in the boat a distance l? (Neglect friction between the boat and the water.)



PROB. 156

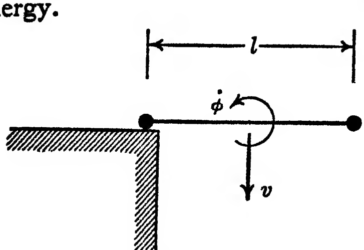


PROB. 157

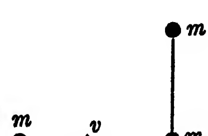
157. Two particles each having a mass m are connected by a rigid bar of length l whose mass is negligible. The system is initially at rest in the position shown. At time $t = 0$, a force F, of constant magnitude, acts normal to the bar as shown. Write the work-energy equation for the system with respect to the mass center, and show that $\dot{\phi} = \sqrt{\dfrac{2F\phi}{ml}}$ and $\phi = \dfrac{Ft^2}{2ml}$. Note that these results are obtained without considering the motion of the mass center.

158. Suppose that the system of Problem 157 has an angular velocity $\dot{\phi}$ = constant, the center of mass of the system is initially at rest, and at time $t = 0$ the system is released to fall under the action of gravity. What is the total kinetic energy of the system at a subsequent time?

159. The system of Problem 157 moves with angular velocity $\dot{\phi}$ and the linear velocity of the center of mass is v, vertically downward. When the bar is in a horizontal position it makes an elastic impact as shown. Find the subsequent motion of the system, assuming that no energy is lost during the impact and assuming no gravitational force acting. Show that there is an interchange of translational and rotational kinetic energy.



PROB. 159



PROB. 161

160. The system of Problem 157 is initially at rest when an impulse $F\Delta t$, normal to the bar, acts upon one of the masses. If Δt is an infinitesimal, find the total energy imparted to the system and describe the subsequent motion. (Assume that no gravitational force is acting.)

161. A mass m moving with a velocity v in a direction perpendicular to the bar strikes one of the masses in the system of Problem 157. Describe the subsequent motion of the mass-center of the bar, and find its angular velocity at any time, assuming that there is no energy lost during the impact and assuming no gravitational force acting.

41. Moment of Momentum. Consider a particle of mass m and momentum $m\dot{\boldsymbol{r}}$, as shown in Fig. 6–2. The *moment of momentum* * of the particle about the fixed point O is defined as the moment of the momentum vector about the point O. Calling the moment of momentum vector $\boldsymbol{H}$, we have:

$$\boldsymbol{H} = \boldsymbol{r} \times m\dot{\boldsymbol{r}}$$

The total moment of momentum of a system of particles is the sum of the moments of momentum of all the individual particles:

$$\boldsymbol{H} = \Sigma \boldsymbol{r}_i \times m\dot{\boldsymbol{r}}_i \qquad (68)$$

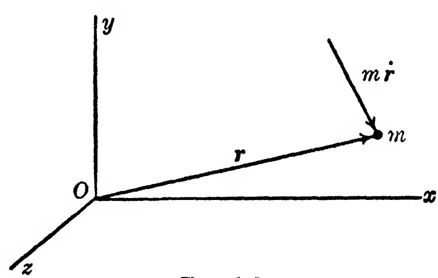


FIG. 6–2

The concept of the moment of momentum can be used to put the equation of motion into a new form, which is particularly convenient for the treatment of systems of particles. To do this, we differentiate $\boldsymbol{H}$ with respect to time, and find:

$$\dot{\boldsymbol{H}} = \Sigma \dot{\boldsymbol{r}}_i \times m\dot{\boldsymbol{r}}_i + \Sigma \boldsymbol{r}_i \times m\ddot{\boldsymbol{r}}_i$$

Since $\dot{\boldsymbol{r}}_i \times \dot{\boldsymbol{r}}_i = 0$ the first term drops out, giving:

$$\dot{\boldsymbol{H}} = \Sigma \boldsymbol{r}_i \times m\ddot{\boldsymbol{r}}_i$$

Taking the cross product of each side of the equation of motion $m\ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i + \boldsymbol{f}_i$ by $\boldsymbol{r}_i$ and summing, we obtain:

$$\Sigma \boldsymbol{r}_i \times m\ddot{\boldsymbol{r}}_i = \Sigma \boldsymbol{r}_i \times (\boldsymbol{F}_i + \boldsymbol{f}_i)$$

* The quantity moment of momentum is also called the *angular momentum.* Since no angular motion or rotation need be present in order that the moment of momentum should exist, the term *moment of momentum* is preferred.

The left side of this equation is $\dot{\boldsymbol{H}}$ and the right side represents the moment of all the forces about the fixed point O. Since the internal forces occur in equal, opposite, and collinear pairs, their moments cancel and the right side of the equation is just the sum of the moments of the external forces. Writing this moment sum as $\boldsymbol{M}$, we have:

$$\dot{\boldsymbol{H}} = \boldsymbol{M} \tag{69}$$

$\dot{\boldsymbol{H}}$, the time rate of change of the moment of momentum of the system about the fixed point O, is equal to $\boldsymbol{M}$, the resultant moment of the external forces about the same point O. This is called the equation of the moment of momentum. It is a restatement of the equation of motion in a form which, as we shall see in the next chapter, is particularly convenient for application to problems of rigid body dynamics.

When there is no external moment acting on the system, the equation of the moment of momentum takes the form:

$$\dot{\boldsymbol{H}} = 0 \quad \text{or} \quad \boldsymbol{H} = \text{constant} \tag{70}$$

This is *the principle of the conservation of moment of momentum*, which states that, if there is no external moment of force about the fixed point, the moment of momentum about that point must remain constant.

In the preceding paragraphs the moment of momentum was taken with respect to an arbitrary, *fixed* point. It is often convenient to take the moving center of mass of the system as the point about which to write the moment of momentum equation. That it is permissible to do this may be shown as follows. Referring again to Fig. 6–1, where $\boldsymbol{r}_c$ is the vector which locates the center of mass of the system, we have:

$$\boldsymbol{r}_i = \boldsymbol{r}_c + \boldsymbol{\rho}_i$$

Writing the equation of moment of momentum about the fixed point O:

$$\dot{\boldsymbol{H}} = \boldsymbol{M}$$
$$\Sigma \boldsymbol{r}_i \times m\dot{\boldsymbol{v}}_i = \Sigma \boldsymbol{r}_i \times \boldsymbol{F}_i$$
$$\Sigma(\boldsymbol{r}_c + \boldsymbol{\rho}_i) \times m\dot{\boldsymbol{v}}_i = \Sigma(\boldsymbol{r}_c + \boldsymbol{\rho}_i) \times \boldsymbol{F}_i$$
$$\boldsymbol{r}_c \times \Sigma m\dot{\boldsymbol{v}}_i + \Sigma \boldsymbol{\rho}_i \times m\dot{\boldsymbol{v}}_i = \boldsymbol{r}_c \times \Sigma \boldsymbol{F}_i + \Sigma \boldsymbol{\rho}_i \times \boldsymbol{F}_i$$

Since it has already been shown that $\Sigma m\dot{\boldsymbol{v}}_i = \Sigma \boldsymbol{F}_i$, we may write the two independent equations:

$$\boldsymbol{r}_c \times \Sigma m\dot{\boldsymbol{v}}_i = \boldsymbol{r}_c \times \boldsymbol{F}$$

and

$$\Sigma \boldsymbol{\rho}_i \times m\dot{\boldsymbol{v}}_i = \Sigma \boldsymbol{\rho}_i \times \boldsymbol{F}_i$$

The first of these is the equation of moment of momentum, about the fixed origin, of the center of mass of the system treated as a particle. In the second equation, substituting $\dot{\boldsymbol{v}}_i = \ddot{\boldsymbol{r}}_c + \ddot{\boldsymbol{\rho}}_i$, we have:

$$\Sigma[\boldsymbol{\rho}_i \times m(\ddot{\boldsymbol{r}}_c + \ddot{\boldsymbol{\rho}}_i)] = \Sigma m\boldsymbol{\rho}_i \times \ddot{\boldsymbol{r}}_c + \Sigma \boldsymbol{\rho}_i \times m\ddot{\boldsymbol{\rho}}_i = \Sigma \boldsymbol{\rho}_i \times \boldsymbol{F}_i$$

Since $\boldsymbol{\rho}_i$ is measured from the center of mass, $\Sigma m\boldsymbol{\rho}_i = 0$ and the equation becomes:

$$\Sigma \boldsymbol{\rho}_i \times m\ddot{\boldsymbol{\rho}}_i = \Sigma \boldsymbol{\rho}_i \times \boldsymbol{F}_i$$

or

$$\frac{d}{dt}\Sigma \boldsymbol{\rho}_i \times m\dot{\boldsymbol{\rho}}_i = \Sigma \boldsymbol{\rho}_i \times \boldsymbol{F}_i$$

The left side of this equation is the time derivative of the moment of momentum $\boldsymbol{H}_c$ about the center of mass, and the right side is the moment of the external forces about the center of mass, so that:

$$\dot{\boldsymbol{H}}_c = \boldsymbol{M}_c \tag{71}$$

The equation of moment of momentum in the form $\dot{\boldsymbol{H}} = \boldsymbol{M}$ *can thus be referred either to an arbitrary, fixed point or to the moving center of mass of the system.*

The equation of moment of momentum is often written in terms of rectangular coordinates as:

$$\dot{H}_x = M_x; \quad \dot{H}_y = M_y; \quad \dot{H}_z = M_z$$

For example, for a single particle acted upon by a force F:

$$\underline{\boldsymbol{H} = \boldsymbol{r} \times m\boldsymbol{v} \quad \text{and} \quad \boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}}$$

Writing $\boldsymbol{r}$, $\boldsymbol{v}$, and $\boldsymbol{F}$ in terms of their rectangular components gives:

$$\boldsymbol{H} = (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \times m(\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k})$$
$$\boldsymbol{M} = (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \times (F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k})$$

Carrying out the multiplications, remembering that $\boldsymbol{i} \times \boldsymbol{i} = 0$, $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, etc., gives:

$$\boldsymbol{H} = m(y\dot{z} - z\dot{y})\boldsymbol{i} + m(z\dot{x} - x\dot{z})\boldsymbol{j} + m(x\dot{y} - y\dot{x})\boldsymbol{k}$$
$$\boldsymbol{M} = (yF_z - zF_y)\boldsymbol{i} + (zF_x - xF_z)\boldsymbol{j} + (xF_y - yF_x)\boldsymbol{k}$$

The three component equations therefore are:

$$
\begin{aligned}
m \frac{d}{dt}(y\dot{z} - z\dot{y}) &= yF_z - zF_y \\
m \frac{d}{dt}(z\dot{x} - x\dot{z}) &= zF_x - xF_z \qquad (72) \\
m \frac{d}{dt}(x\dot{y} - y\dot{x}) &= xF_y - yF_x
\end{aligned}
$$

These equations can also be obtained by taking moments about the x-, y-, and z-axes respectively. If a system of particles is involved, these equations can be summed over all the particles.

42. Summary. It should be emphasized that the principles derived in this chapter are general in application, and that the system of particles need have no special properties. These principles are thus available for use in the analysis of rigid and deformable solid bodies, liquids, and gases. The general conclusions can be summarized in the following statements.

(1) The center of mass of any system of particles moves as though it were a particle, having a mass equal to the total mass of the system, acted upon by the resultant of the external forces applied to the system. All the methods of particle dynamics may thus be applied to the motion of the mass center.

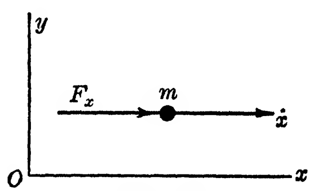
(2) The magnitude and direction of the total momentum of a system of particles are given by the product of the total mass of the system and the velocity of the mass center. The total impulse of all the external forces acting upon the system is equal to the change of the total momentum.

(3) The work-energy principle for a system of particles may be written in the form of two independent equations. One equation describes the motion of the center of mass of the system, and the other equation describes the motion of the particles of the system with respect to the center of mass.

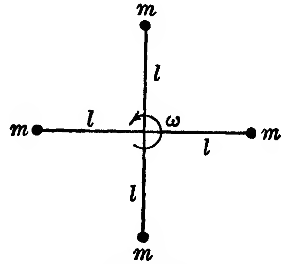
(4) The equation of moment of momentum may be written with respect to an arbitrary fixed point or with respect to the moving center of mass of the system.

PROBLEMS

162. A particle of mass m is acted upon by a force parallel to the x-axis as shown. The particle has a velocity parallel to the x-axis. Write the equation for the moment of momentum of the system, and show that this equation may be reduced to the equation of motion in the form $F_x = m\ddot{x}$.



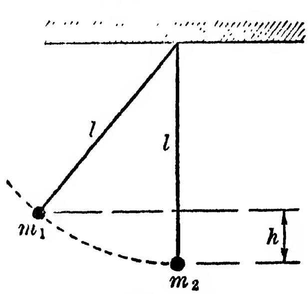
PROB. 162



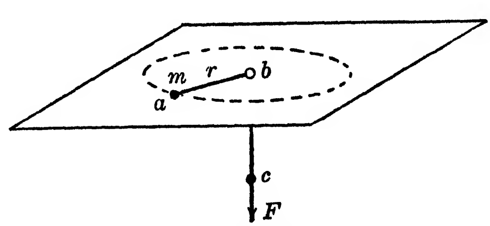
PROB. 163

163. A system of four particles of equal mass m rotates with an angular velocity ω. The particles are at equal distances l from the center of rotation, and they are spaced at equal angles as shown. Find the magnitude and direction of the vector representing the moment of momentum of the system about the point of rotation.

164. Two simple pendulums of mass m_1 and m_2 and equal lengths l are suspended from the same point. The mass m_1 is raised through a distance h as shown in the diagram and is then released. What are the velocities of m_1 and m_2 immediately after impact, assuming no energy loss during the impact?



PROB. 164



PROB. 165

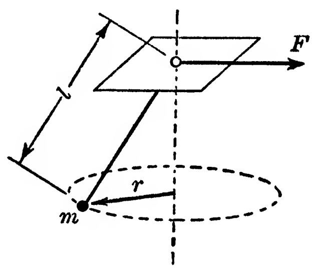
165. A particle of mass m is restrained by the string abc to move in a circle of radius r on a horizontal frictionless plane. The particle moves with a constant angular velocity ω. If the radius of the circle is reduced to r_1 by pulling on the string at c, what will be the velocity of the particle?

166. Two particles of mass m_1 and m_2 are connected by a stretched spring. The system is thrown into the air with the spring released so that the system translates, rotates, and vibrates. Describe qualitatively the motion of the mass center of the system, the rotation about the mass center, and the nature of the longitudinal vibration of the system.

167. A particle is acted upon by a force which is always directed toward a fixed point. Show that the particle moves in a plane.

168. Using the equation of moment of momentum, show that the area swept out per unit time by the radius vector drawn from the sun to a planet is a constant.

169. A particle of mass m fastened to a massless string of length l rotates in a circular path of radius r as a conical pendulum. The force F



Prob. 169

is gradually increased, thus shortening the length of the pendulum so that finally the particle moves in a circle of radius $\frac{r}{2}$. Find the velocity of the mass after the string has been shortened.

CHAPTER VII

THE DYNAMICS OF RIGID BODIES

It has been long understood that approximate solutions of problems in the ordinary branches of Natural Philosophy may be obtained by a species of abstractions, or rather limitations of the data, such as enables us easily to solve the modified form of the question, while we are well assured that the circumstances (so modified) affect the result only in a superficial manner.—W. Thomson and P. G. Tait, *Treatise on Natural Philosophy* (1872).

When applying the principles of dynamics to solid bodies it is usually assumed that the motion of the body is not influenced by the small deformations caused by the applied forces. This is equivalent to the assumption of a rigid body, and so far as the motion of the body is concerned this assumption introduces only negligible errors for the great majority of such problems encountered in engineering practice. The equations of motion for a rigid body may be developed by treating the body as a collection of particles and applying the general principles of dynamics as formulated in the preceding chapter. The condition for a rigid body, that the distances between the particles remain fixed, is then used to simplify the general equations. As the first step in deriving the required equations of motion, it will be necessary to investigate the motion of each point in a rigid body.

43. Kinematics of Rigid Body Motion. To describe the motion of a rigid body it is necessary to specify the motion of every point in the body. This is done by applying the general kinematic equations of Chapter II. (Equations 14 and 15.)

$$\begin{aligned}\dot{\boldsymbol{r}} &= \dot{\boldsymbol{R}} + \boldsymbol{\omega} \times \boldsymbol{\rho} + \dot{\boldsymbol{\rho}}_r \\ \ddot{\boldsymbol{r}} &= \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}_r + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r\end{aligned}$$

As shown in Fig. 7–1, $\boldsymbol{R}$ is the displacement of the point A in the body, that is, A is the origin of the moving xyz axes which are fixed in the body. The displacement, velocity, and acceleration of any other point B fixed in the body are:

$$\begin{aligned} \boldsymbol{r}_B &= \boldsymbol{R} + \boldsymbol{\rho} \\ \dot{\boldsymbol{r}}_B &= \dot{\boldsymbol{R}} + \boldsymbol{\omega} \times \boldsymbol{\rho} \\ \ddot{\boldsymbol{r}}_B &= \ddot{\boldsymbol{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} \end{aligned} \tag{73}$$

where $\boldsymbol{\omega}$ is the angular velocity of the body. The motion of the body is thus specified completely when $\boldsymbol{R}$ and $\boldsymbol{\omega}$ are known as

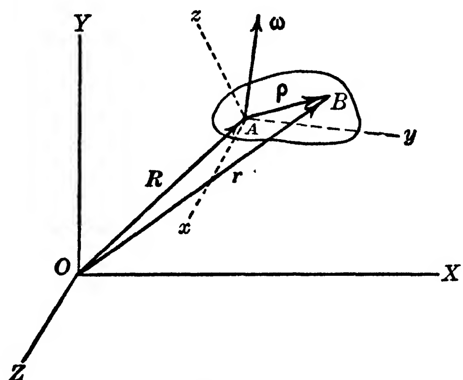


FIG. 7–1

functions of time. This is equivalent to saying that the motion of a rigid body can be described as a translation of a point in the body plus a rotation of the body about the point.*

EXAMPLE 1. A rigid body performs plane motion, that is, all points of the body move parallel to the XY plane (Fig. 7–2). When the body is in the position shown, the velocities of two points, A and B, are known. What is the angular velocity of the body at this instant?

Solution. Erect perpendiculars to $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ through points A and B and find the point of intersection C. Let $\boldsymbol{R}_C$ be the radius vector to the point C, then

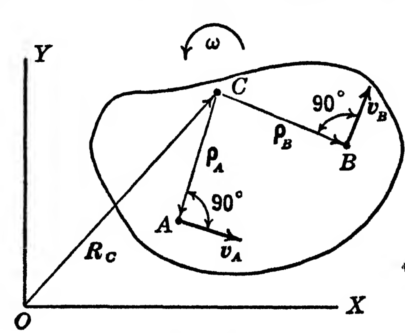


FIG. 7–2

* This is known as the *Theorem of Chasle*, after M. Chasle (1793–1880), French mathematician.

$$\boldsymbol{v}_A = \dot{\boldsymbol{R}}_C + \boldsymbol{\omega} \times \boldsymbol{\rho}_A, \quad \boldsymbol{v}_B = \dot{\boldsymbol{R}}_C + \boldsymbol{\omega} \times \boldsymbol{\rho}_B$$

But $\boldsymbol{v}_A$ and $\boldsymbol{\omega} \times \boldsymbol{\rho}_A$ have the same direction, so $\dot{\boldsymbol{R}}_C$ cannot have a direction different from $\boldsymbol{v}_A$, and neither can it have a direction different from $\boldsymbol{v}_B$. $\dot{\boldsymbol{R}}_C$ is therefore equal to zero and

$$\omega = \frac{v_A}{\rho_A} = \frac{v_B}{\rho_B}$$

The point located by $\boldsymbol{R}_C$ has an instantaneous velocity equal to zero. This point is called the *instantaneous center of rotation.* At any particular instant the velocity of every point in the body is the same as if the body were rotating about the instantaneous center.

EXAMPLE 2. A four-bar linkage is shown in Fig. 7–3. Link 1 is 5 ft long and link 4 is 3 ft long. Link 1 has an angular velocity of 2 revolutions per sec and an angular acceleration of 3 revolutions per sec^2, both clockwise. Determine the velocities and accelerations of links 2 and 3.

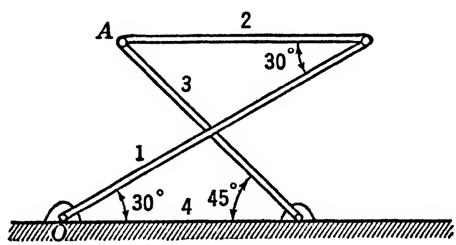


FIG. 7–3

Solution. Since each link is a rigid body, we may express the motion of point A by the following equations, taking point O as the origin.

$$\boldsymbol{r}_A = \boldsymbol{R}_1 + \boldsymbol{\rho}_2; \quad \dot{\boldsymbol{r}}_A = \dot{\boldsymbol{R}}_1 + \boldsymbol{\omega}_2 \times \boldsymbol{\rho}_2; \quad \ddot{\boldsymbol{r}}_A = \ddot{\boldsymbol{R}}_1 + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{\rho}_2 + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{\rho}_2)$$

$$\boldsymbol{r}_A = \boldsymbol{R}_4 + \boldsymbol{\rho}_3; \quad \dot{\boldsymbol{r}}_A = \dot{\boldsymbol{\rho}}_3; \quad \ddot{\boldsymbol{r}}_A = \ddot{\boldsymbol{\rho}}_3$$

These lead to three vector equations which determine the unknown velocities and accelerations: (a) $\boldsymbol{R}_1 + \boldsymbol{\rho}_2 = \boldsymbol{R}_4 + \boldsymbol{\rho}_3$, which determines the lengths of links 2 and 3; (b) $\dot{\boldsymbol{R}}_1 + \boldsymbol{\omega}_2 \times \boldsymbol{\rho}_2 = \dot{\boldsymbol{\rho}}_3$, which determines $\boldsymbol{\omega}_2$ and $\dot{\boldsymbol{\rho}}_3$; (c) $\ddot{\boldsymbol{R}}_1 + \dot{\boldsymbol{\omega}}_2 \times \boldsymbol{\rho}_2 + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \boldsymbol{\rho}_2) = \ddot{\boldsymbol{\rho}}_3$ which determines $\dot{\boldsymbol{\omega}}_2$ and $\ddot{\boldsymbol{\rho}}_3$.

Each of the preceding vector equations is equivalent to two scalar equations and determines two unknowns. The actual solution may be carried out graphically or algebraically. Fig. 7–4 shows graphical solutions where the dotted lines indicate vectors whose directions are known but whose lengths are unknown.

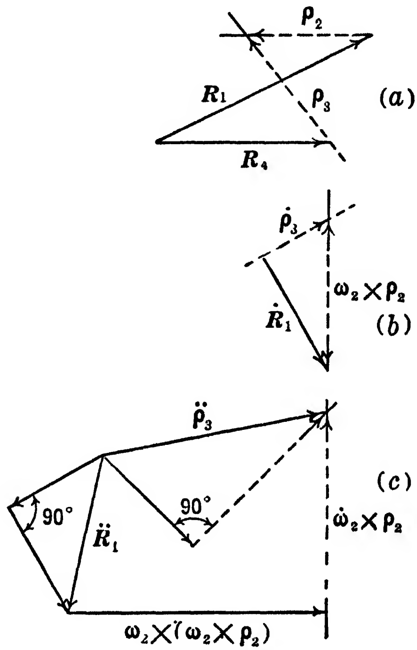
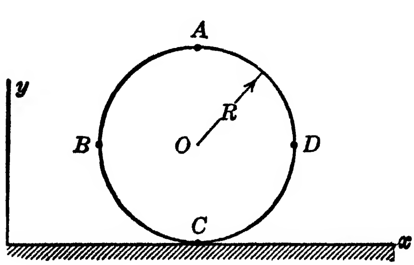


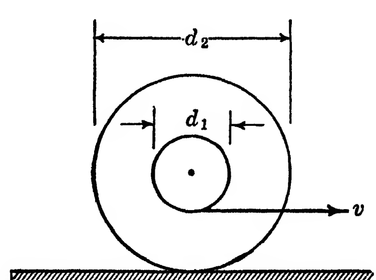
FIG. 7–4

PROBLEMS

170. A circular cylinder of radius R rolls without sliding along a horizontal plane. The horizontal velocity of the center of the cylinder is $\dot{x}_0$, and $\ddot{x}_0 = 0$. Find the velocities of the points A, B, C, and D on the periphery of the cylinder.



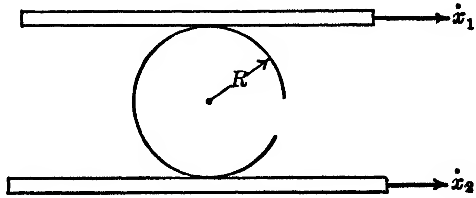
PROB. 170



PROB. 171

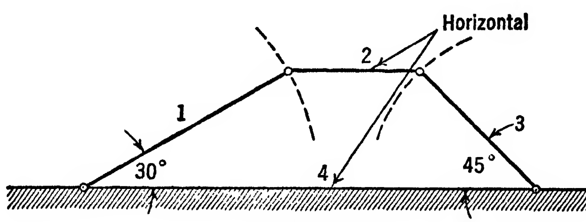
171. A pair of wheels of diameter d_2 with a rigidly attached concentric axle of diameter d_1 rolls without sliding along a horizontal plane. A rope wound around the axle is pulled with a constant horizontal velocity v as shown. Find the velocity of the center of the axle.

172. A circular cylinder of radius R is supported between two horizontal planks as shown in the diagram. The planks have horizontal velocities $\dot{x}_1$ and $\dot{x}_2$ as shown. Find the velocity of the center of the cylinder.



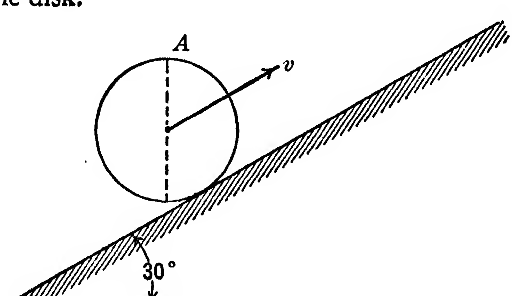
PROB. 172

173.* The mechanism of many useful machines can be reduced in its essentials to that of the four-bar linkage shown. In the particular example shown link 1 is 3 ft long, and link 4 is 6 ft long. Link 1 has a counterclockwise angular velocity of 100 rpm, and a clockwise angular acceleration of 50 rpm per minute. At the instant when link 2 is horizontal, find the angular velocities and accelerations of links 2 and 3.



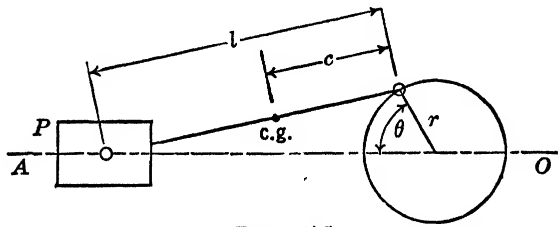
PROB. 173

174. A circular disk of 1-ft radius rolls without slipping up a 30° inclined plane. The uniform velocity of the center of the disk is 25 ft/sec parallel to the plane. Find the velocity and acceleration of the point A on the periphery of the disk, where A is the upper end of a vertical diameter of the disk.



PROB. 174

175.* A crank and connecting rod mechanism of the type commonly used in reciprocating engines is shown in the diagram, where r is the radius of the crank, and l is the length of the connecting rod. The piston P is constrained by the cylinder to move along the straight line AO. The center of gravity of the connecting rod is located a distance c from the crank pin as shown. For a particular engine $r = 4$ in., $l = 8$ in., $c = 3$ in., and the crank rotates clockwise with a constant angular velocity of 1000 rpm.

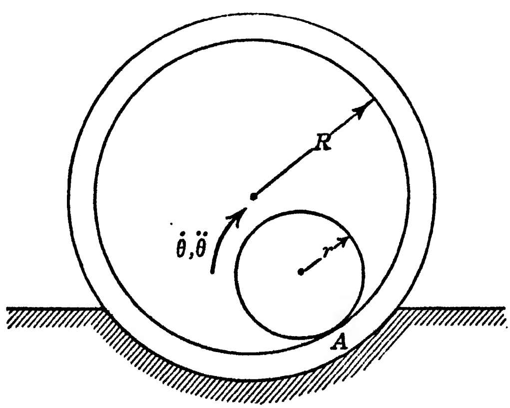


Prob. 175

(a) Find the instantaneous center of rotation of the connecting rod l at the instant when $\theta = 60°$. Using this instantaneous center, find the velocity of the center of gravity of the connecting rod. Find the required distances in this problem by laying out the diagram to scale and measuring the distances graphically.

(b) Determine the acceleration of the center of gravity of the connecting rod.

176. A small disk of radius r rolls without slipping inside a larger fixed circular ring of radius R as shown. The small disk has an angular velocity $\dot{\theta}$ and an angular acceleration $\ddot{\theta}$. Find the acceleration of the point on the small disk coinciding with the point of contact A. Find also the velocity and acceleration of the center of the small disk.



Prob. 176

44. The Moment of Momentum of a Rigid Body. When analyzing the motion of a rigid body we can write the equation of motion of the mass center:

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}}_c$$

This equation describes only the translation of the mass center and in addition we require equations describing the rotation of the body. These can be derived by applying the equation of moment of momentum:

$$\dot{\boldsymbol{H}} = \boldsymbol{M}$$

This equation may be written with respect to the origin of a coordinate system which satisfies either of the following requirements; the origin of the coordinate system is fixed in space, or the origin is at the center of mass of the body. Let us take the origin at the center of mass and first write the expression for $\boldsymbol{H}$.

Let the density of the body be ρ and let dV be a typical element of volume of the body located by the vector $\boldsymbol{r}$ (Fig. 7–5). Let the

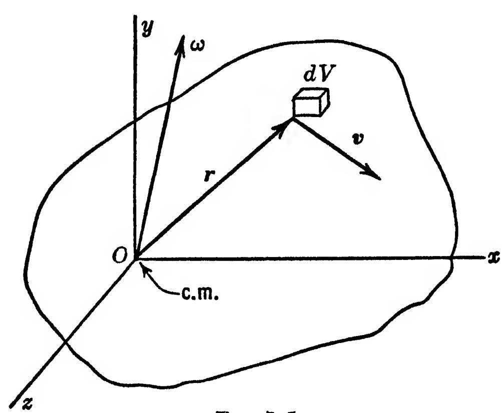


FIG. 7–5

velocity of dV relative to the origin be $\boldsymbol{v}$ and the angular velocity of the body $\boldsymbol{\omega}$. We then have:

$$d\boldsymbol{H} = \boldsymbol{r} \times \rho\, dV \boldsymbol{v}$$

$$\boldsymbol{H} = \int \rho(\boldsymbol{r} \times \boldsymbol{v})\, dV$$

The integration extends over the entire volume of the body. The velocity of an element relative to the origin is $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$, so that the equation becomes:

$$\boldsymbol{H} = \int \rho[\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})]\, dV$$

Writing $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $\boldsymbol{\omega} = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k}$ we have

$$\boldsymbol{\omega} \times \boldsymbol{r} = (z\omega_y - y\omega_z)\boldsymbol{i} + (x\omega_z - z\omega_x)\boldsymbol{j} + (y\omega_x - x\omega_y)\boldsymbol{k}$$

So that

$$\begin{aligned}\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = {} & [\omega_x(y^2 + z^2) - \omega_y xy - \omega_z xz]\boldsymbol{i} \\ & + [-\omega_x yx + \omega_y(z^2 + x^2) - \omega_z yz]\boldsymbol{j} \\ & + [-\omega_x zx - \omega_y zy + \omega_z(x^2 + y^2)]\boldsymbol{k}\end{aligned}$$

The rectangular components of the moment of momentum may thus be written:

$$\begin{aligned} H_x &= +\omega_x \int \rho(y^2 + z^2)\, dV - \omega_y \int \rho xy\, dV - \omega_z \int \rho xz\, dV \\ H_y &= -\omega_x \int \rho yx\, dV + \omega_y \int \rho(z^2 + x^2)\, dV - \omega_z \int \rho yz\, dV \\ H_z &= -\omega_x \int \rho zx\, dV - \omega_y \int \rho zy\, dV + \omega_z \int \rho(x^2 + y^2)\, dV \end{aligned} \tag{74}$$

Introducing the following notation for the integrals which appear in these expressions:

$$\int \rho(y^2 + z^2)\, dV = I_{xx}$$

$$\int \rho xy\, dV = I_{xy}, \text{ etc.}$$

the equations become:

$$\begin{aligned} H_x &= +I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{aligned} \tag{75}$$

The terms I_{xx}, I_{yy}, and I_{zz} are called the *moments of inertia*, and the terms I_{xy}, etc., are called *products of inertia*. I_{xx} is often written as I_x, etc., and it will be noted from the symmetry of the integrals that $I_{xy} = I_{yx}$, etc.

The inertia integrals are defined with respect to the xyz axes. If the coordinate axes have fixed directions in space, then as the body rotates the numerical values of the inertia integrals will change with time. On the other hand, if the coordinate axes are fixed in the body so that they rotate with it, then the inertia integrals are constants but the components H_x, H_y, and H_z are measured along rotating coordinate axes. Either type of coordinate system may be used, but in most problems the second type is more convenient. In the following sections we shall use a coordinate system that is fixed in the body and rotates with it, unless it is specifically stated to the contrary.

Before treating the general equations of motion of a rigid body we shall consider the problem of determining the inertia integrals.

45. The Calculation of Moments and Products of Inertia. Although the computation of moments and products of inertia requires only the evaluation of simple definite integrals it is found that unless the body has a very simple shape and orientation the limits of integration are such as to require an excessive amount of labor. The problem is very much simplified by the following three observations:

(1) If the moments and products of inertia are known for a particular set of axes, they can be found for any parallel set of axes by a transformation of coordinates.
(2) If the moments and products of inertia are known for a particular set of axes, they can be found for a rotated set of axes by a transformation of coordinates.
(3) The moments and products of inertia of a body of complicated shape can be found by subdividing the body into a number of simpler parts, evaluating the integrals for each of these, and then summing them.

The method of calculation is thus to subdivide the body into simple parts, choosing for each part coordinate axes which will make the integration easy. By transformation of coordinates the moments and products of inertia with respect to the desired axes can then be found. It should be noted that this and the following sections deal only with methods of calculation which shorten the labor of evaluating inertia integrals.

Before proceeding with the calculation of inertia integrals, we introduce some commonly used notation. Consider, in Fig. 7–6, any rigid body having a mass per unit volume ρ. Then by definition of the moments of inertia, we have:

$$\begin{aligned} I_x &= \int \rho(y^2 + z^2)\, dV = \int \rho a_x^2\, dV \\ I_y &= \int \rho(x^2 + z^2)\, dV = \int \rho a_y^2\, dV \\ I_z &= \int \rho(x^2 + y^2)\, dV = \int \rho a_z^2\, dV \end{aligned} \tag{76}$$

Note that a_x, a_y, a_z are the perpendicular distances from the respective axes to the volume element and are not the components of the radius vector to the element. These expressions are sometimes written as $I_x = mr_x^2$, $I_y = mr_y^2$, $I_z = mr_z^2$, where m is the total mass of the body, and the quantity r_x is called the *radius of gyration* of the body about the x-axis, etc. The radius of gyration is thus given by

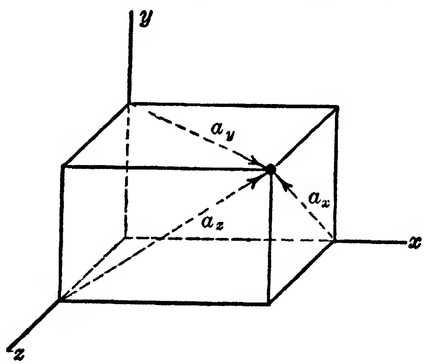


FIG. 7–6

$$r = \sqrt{\frac{I}{m}}$$

The product of inertia integrals have the form:

$$I_{xy} = \int \rho xy\, dV$$

Since x and y can be either positive or negative, the product of inertia can be either positive or negative. In particular, if the yz plane is a plane of symmetry for the body, there is a negative $\rho xy\, dV$ for each positive $\rho xy\, dV$ and the product of inertia is zero. As will be shown later, the product of inertia may be zero also when there is no plane of symmetry in the body.

46. Translation of Coordinate Axes. Suppose that the moments and products of inertia are known with respect to one set of axes, and we wish to determine the moments and products of inertia with respect to a parallel set of axes. In Fig. 7–7, the x', y', z' system has its origin located at the center of mass of the body.

The moments of inertia with respect to this centroidal system are $I_{x'}$, $I_{y'}$, and $I_{z'}$, and the products of inertia are $I_{x'y'}$, $I_{x'z'}$, $I_{y'z'}$. In the parallel xyz system the center of mass is located at the point x_c, y_c, z_c. We have then

$$
\begin{aligned}
I_z &= \int \rho(x^2 + y^2)\, dV = \int \rho[(x_c + x')^2 + (y_c + y')^2]\, dV \\
&= \int \rho(x_c^2 + 2x_cx' + x'^2 + y_c^2 + 2y_cy' + y'^2)\, dV \\
&= \int \rho(x'^2 + y'^2)\, dV + (x_c^2 + y_c^2)\int \rho\, dV + 2x_c\int \rho x'\, dV \\
&\qquad + 2y_c\int \rho y'\, dV
\end{aligned}
$$

Since the origin of the $x'y'z'$ system is at the center of mass, the integrals $\int \rho x'\, dV$ and $\int \rho y'\, dV$ are equal to zero; also:

$$
\begin{aligned}
\int \rho(x'^2 + y'^2)\, dV &= I_{z'} \\
(x_c^2 + y_c^2) &= a_{cz}^2 \\
\int \rho\, dV &= m
\end{aligned}
$$

so:

$$I_z = I_{z'} + ma_{cz}^2 \qquad (77)$$

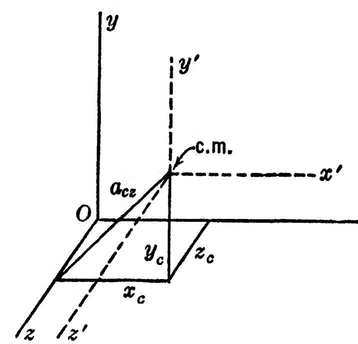


FIG. 7–7

Thus if the moments of inertia are known with respect to centroidal axes, the moments of inertia with respect to any parallel axes can be obtained.

The transformation of products of inertia for translation of coordinate axes may be derived in the same way. Referring again to Fig. 7–7, we have:

$$
\begin{aligned}
I_{xy} &= \int \rho xy\, dV = \int \rho(x_c + x')(y_c + y')\, dV \\
&= \int \rho x'y'\, dV + x_cy_c\int \rho\, dV + x_c\int \rho y'\, dV + y_c\int \rho x'\, dV
\end{aligned}
$$

And the transformation equation is:

$$I_{xy} = I_{x'y'} + mx_cy_c \qquad (78)$$

47. Rotation of Coordinate Axes. Suppose that the moments and products of inertia of a body are known with respect to an $x'y'z'$ set of axes. Let us determine the moments and products of inertia of the body with respect to an xyz set of axes which has been rotated with respect to the $x'y'z'$ axes. In Fig. 7–8, the two

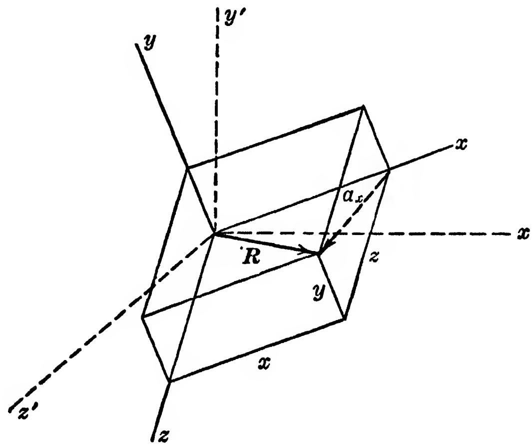


Fig. 7–8

coordinate systems are shown, and an element of volume dV of the body is located by the radius vector $\boldsymbol{R}$. Considering first the transformation of a typical moment of inertia, we have:

$$I_{xx} = \int \rho a_x^2 \, dV$$

We shall now express the integral in terms of x', y', and z'. First,

$$a_x^2 = R^2 - x^2 = x'^2 + y'^2 + z'^2 - x^2$$
$$\boldsymbol{R} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}'$$

Also:

$$x = \boldsymbol{i} \cdot \boldsymbol{R} = \boldsymbol{i} \cdot (x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}') = x'(\boldsymbol{i} \cdot \boldsymbol{i}') + y'(\boldsymbol{i} \cdot \boldsymbol{j}') + z'(\boldsymbol{i} \cdot \boldsymbol{k}')$$

We next note that the term $(\boldsymbol{i} \cdot \boldsymbol{i}')$ is equal to the cosine of the angle between the x-axis and the x'-axis. Denoting the direction cosines by l, we have:

$$l_{xx'} = \boldsymbol{i} \cdot \boldsymbol{i}'; \quad l_{yx'} = \boldsymbol{j} \cdot \boldsymbol{i}'; \quad \text{etc.}$$

so that:

$$x = x'l_{xx'} + y'l_{xy'} + z'l_{xz'}$$

With this notation the moment of inertia becomes:

$$I_{xx} = \int \rho[(x'^2 + y'^2 + z'^2) - (x'l_{xx'} + y'l_{xy'} + z'l_{xz'})^2]\, dV$$

Since $l^2_{xx'} + l^2_{xy'} + l^2_{xz'} = 1$, we may write:

$$I_{xx} = \int \rho[(x'^2 + y'^2 + z'^2)(l^2_{xx'} + l^2_{xy'} + l^2_{xz'}) - (x'l_{xx'} + y'l_{xy'} + z'l_{xz'})^2]\, dV$$

Multiplying out these expressions, and combining terms gives:

$$\begin{aligned} I_{xx} = l^2_{xx'}\int \rho(y'^2 + z'^2)\, dV + l^2_{xy'}\int \rho(x'^2 + z'^2)\, dV \\ + l^2_{xz'}\int \rho(x'^2 + y'^2)\, dV - 2l_{xx'}l_{xy'}\int \rho x'y'\, dV \\ - 2l_{xx'}l_{xz'}\int \rho x'z'\, dV - 2l_{xy'}l_{xz'}\int \rho y'z'\, dV \end{aligned}$$

or

$$\begin{aligned} I_{xx} = l^2_{xx'}I_{x'x'} + l^2_{xy'}I_{y'y'} + l^2_{xz'}I_{z'z'} \\ - 2l_{xx'}l_{xy'}I_{x'y'} - 2l_{xx'}l_{xz'}I_{x'z'} - 2l_{xy'}l_{xz'}I_{y'z'} \end{aligned} \quad (79)$$

Corresponding expressions are obtained for I_{yy} and I_{zz}.

The products of inertia can be transformed in the same manner, giving:

$$\begin{aligned} -I_{xy} = l_{xx'}l_{yx'}I_{x'x'} + l_{xy'}l_{yy'}I_{y'y'} + l_{xz'}l_{yz'}I_{z'z'} \\ -(l_{xx'}l_{yy'} + l_{xy'}l_{yx'})I_{x'y'} - (l_{xy'}l_{yz'} + l_{xz'}l_{yy'})I_{y'z'} \\ -(l_{xz'}l_{yx'} + l_{xx'}l_{yz'})I_{z'x'} \end{aligned} \quad (80)$$

The large number of terms and subscripts involved in these expressions makes desirable a systematic method of writing the transformations. Let us arrange the moments and products of inertia in rows and columns as:

$$I' = \begin{array}{c|ccc} & x' & y' & z' \\ \hline x' & +I_{x'x'} & -I_{x'y'} & -I_{x'z'} \\ y' & -I_{y'x'} & +I_{y'y'} & -I_{y'z} \\ z' & -I_{z'x'} & -I_{z'y'} & +I_{z'z'} \end{array}$$

Note the similarity between this group of terms and the terms appearing in Equation (75). Such a systematic grouping of rows and columns is called an *array*, and the purpose of the array is to present a large number of terms in an orderly and easily remembered system. The letters on the outside of the array are usually omitted, it being understood that the terms are arranged in that order. With respect to an xyz coordinate system the array is written:

$$I = \begin{bmatrix} +I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & +I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & +I_{zz} \end{bmatrix} \tag{81}$$

Note that the array is symmetrical about the main diagonal with all products of inertia terms being negative and all moments of inertia being positive.

The procedure for expressing any of the moments or products of inertia of I in terms of the moments and products of inertia I' is as follows. Let $I_{\alpha\beta}$ represent any one of the terms in the I array, where both α and β may assume the values x, y, z, depending upon the term under consideration. Similarly, let $I_{\alpha'\beta'}$ represent any term in the I' array. The direction cosines which relate the directions of the two coordinate systems will be written $l_{\alpha\beta'}$. With this notation we observe that each term in the preceding transformation has the form $l_{\alpha\beta'}l_{\beta\alpha'}I_{\alpha'\beta'}$, and the transformation between the two coordinate systems may be written:

$$I_{\alpha\beta} = \sum_{\alpha'}\sum_{\beta'} l_{\alpha\beta'}l_{\beta\alpha'}I_{\alpha'\beta'} \tag{82}$$

To illustrate the meaning of this notation, we shall evaluate the term I_{xy} which we can then check with the previously determined expression.

$$I_{xy} = \sum_{\alpha'}\sum_{\beta'} l_{x\beta'}l_{y\alpha'}I_{\alpha'\beta'}$$

Summing first with respect to β':

$$I_{xy} = \sum_{\alpha'}(l_{xx'}l_{y\alpha'}I_{\alpha'x'} + l_{xy'}l_{y\alpha'}I_{\alpha'y'} + l_{xz'}l_{y\alpha'}I_{\alpha'z'})$$

Then summing with respect to α' we have the nine terms:

$$\begin{aligned} I_{xy} = {} & l_{xx'}l_{yx'}I_{x'x'} + l_{xy'}l_{yx'}I_{x'y'} + l_{xz'}l_{yx'}I_{x'z'} \\ & + l_{xx'}l_{yy'}I_{y'x'} + l_{xy'}l_{yy'}I_{y'y'} + l_{xz'}l_{yy'}I_{y'z'} \\ & + l_{xx'}l_{yz'}I_{z'x'} + l_{xy'}l_{yz'}I_{z'y'} + l_{xz'}l_{yz'}I_{z'z'} \end{aligned}$$

Referring back to the array of inertia integrals we note that all the terms $I_{\alpha\beta}$, $\alpha \neq \beta$ have negative signs, whereas the terms for which $\alpha = \beta$ have positive signs.* Changing signs accordingly, collecting terms and remembering that $I_{\alpha'\beta'} = I_{\beta'\alpha'}$, we have:

$$\begin{aligned} -I_{xy} = {} & l_{xx'}l_{yx'}I_{x'x'} + l_{xy'}l_{yy'}I_{y'y'} + l_{xz'}l_{yz'}I_{z'z'} \\ & -(l_{xy'}l_{yx'} + l_{xx'}l_{yy'})I_{x'y'} - (l_{xz'}l_{yy'} + l_{xy'}l_{yz'})I_{y'z'} \\ & -(l_{xz'}l_{yx'} + l_{xx'}l_{yz'})I_{x'z'} \end{aligned}$$

This is the same as Equation (80).

With this transformation, the moments and products of inertia of any rigid body can be computed for any rotated coordinate axes, once the inertia integrals are known for one set of axes in the body.

The array

$$\begin{bmatrix} +I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & +I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & +I_{zz} \end{bmatrix}$$

where it is understood that the terms are defined as above, is called the *tensor of inertia*, and one may speak of transforming a tensor by means of the tensor transformation:

$$I_{\alpha\beta} = \sum_{\alpha'}\sum_{\beta'} l_{\alpha\beta'}l_{\beta\alpha'}I_{\alpha'\beta'}$$

In general, if under a transformation of coordinates an expression transforms according to this equation, it is called a tensor of the second rank.

48. Principal Axes. In the preceding transformations there were three products of inertia and three moments of inertia, so that the transformation formulas involved a correspondingly large number of terms. If, however, the initial set of axes is chosen in a special way, there is a substantial reduction in the number of terms. This is illustrated by the following considerations.

If the unit vectors in the two coordinate systems are $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ respectively, then the direction cosines are given by $l_{xx'} = \boldsymbol{i} \cdot \boldsymbol{i}'$, etc. Thus:

$$\begin{aligned} \boldsymbol{i} &= l_{xx'}\boldsymbol{i}' + l_{xy'}\boldsymbol{j}' + l_{xz'}\boldsymbol{k}' \\ \boldsymbol{j} &= l_{yx'}\boldsymbol{i}' + l_{yy'}\boldsymbol{j}' + l_{yz'}\boldsymbol{k}' \\ \boldsymbol{k} &= l_{zx'}\boldsymbol{i}' + l_{zy'}\boldsymbol{j}' + l_{zz'}\boldsymbol{k}' \end{aligned}$$

* It would have been more direct to have given the products of inertia negative signs immediately, and it is usually so done.

We have $\boldsymbol{i} \cdot \boldsymbol{i} = 1$, etc., and $\boldsymbol{i} \cdot \boldsymbol{j} = 0$, etc.; carrying out these dot products, using the above expressions for $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$, we obtain the following six relations between the direction cosines:

$$\begin{aligned}
l^2_{xx'} + l^2_{xy'} + l^2_{xz'} &= 1; \quad l_{xx'}l_{yx'} + l_{xy'}l_{yy'} + l_{xz'}l_{yz'} = 0\\
l^2_{yx'} + l^2_{yy'} + l^2_{yz'} &= 1; \quad l_{xx'}l_{zx'} + l_{xy'}l_{zy'} + l_{xz'}l_{zz'} = 0\\
l^2_{zx'} + l^2_{zy'} + l^2_{zz'} &= 1; \quad l_{yx'}l_{zx'} + l_{yy'}l_{zy'} + l_{yz'}l_{zz'} = 0
\end{aligned}$$

Since there are nine direction cosines with these six relations between them which must always be satisfied, there remain three independent relations which are required to specify the orientation of the $x'y'z'$ axes with respect to the xyz axes. We may take as these three additional relations the conditions that the three products of inertia with respect to the $x'y'z'$ axes are to be equal to zero. In this way it is possible to find a coordinate system with respect to which the three products of inertia disappear, so that the inertia tensor becomes:

$$\begin{bmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{bmatrix}$$

The coordinate axes which satisfy this condition are called *principal axes*. It is customary to use principal axes whenever possible because of the simplifications which they introduce. Since the products of inertia are all zero, the moments and products of inertia can be transformed to any other set of axes which is rotated with respect to the principal axes, by the simplified equations:

$$\begin{aligned}
I_{xx} &= l^2_{xx'}I_{x'x'} + l^2_{xy'}I_{y'y'} + l^2_{xz'}I_{z'z'}\\
I_{yy} &= l^2_{yx'}I_{x'x'} + l^2_{yy'}I_{y'y'} + l^2_{yz'}I_{z'z'}\\
I_{zz} &= l^2_{zx'}I_{x'x'} + l^2_{zy'}I_{y'y'} + l^2_{zz'}I_{z'z'}\\
I_{xy} &= -(l_{xx'}l_{yx'}I_{x'x'} + l_{xy'}l_{yy'}I_{y'y'} + l_{xz'}l_{yz'}I_{z'z'})\\
I_{yz} &= -(l_{yx'}l_{zx'}I_{x'x'} + l_{yy'}l_{zy'}I_{y'y'} + l_{yz'}l_{zz'}I_{z'z'})\\
I_{xz} &= -(l_{zx'}l_{xx'}I_{x'x'} + l_{zy'}l_{xy'}I_{y'y'} + l_{zz'}l_{xz'}I_{z'z'})
\end{aligned} \tag{83}$$

where $I_{x'x'}$, $I_{y'y'}$ and $I_{z'z'}$ are the moments of inertia about the principal axes, the *principal moments of inertia.*

Let us suppose that for a particular body the principal axes have been defined so that $I_{x'x'} > I_{y'y'} > I_{z'z'}$. Then the moment of inertia of the body about some other axis, the x-axis, is:

$$I_{xx} = l^2_{xx'}I_{x'x'} + l^2_{xy'}I_{y'y'} + l^2_{xz'}I_{z'z'}$$

Using the relation

$$l^2_{xx'} + l^2_{xy'} + l^2_{xz'} = 1$$

$$\begin{aligned} I_{xx} &= l^2_{xx'}I_{x'x'} + (1 - l^2_{xz'} - l^2_{xx'})I_{y'y'} + l^2_{xz'}I_{z'z'} \\ &= [(I_{x'x'} - I_{y'y'})l^2_{xx'}] - [(I_{y'y'} - I_{z'z'})l^2_{xz'}] + [I_{y'y'}] \end{aligned}$$

Each of the terms inside of the brackets is positive so that the maximum value of I_{xx} must occur when $l^2_{xx'}$ has its largest value of 1, and when $l^2_{xz'}$ has its smallest value of 0, so that

$$(I_{xx})_{\max} = I_{x'x'} - I_{y'y'} + I_{y'y'} = I_{x'x'}$$

Thus it is proved that the largest principal moment of inertia is also the largest moment of inertia that can be obtained by any orientation of the axes. In the same way it can be shown that the smallest principal moment of inertia is the smallest moment of inertia which can be obtained by any orientation of the axes. We thus see that the principal axes have not only the property that the products of inertia about these axes vanish, but in addition the principal moments of inertia correspond to the maximum and minimum moments of inertia for any orientation of the axes. As has been shown above, the moment of inertia of a body may be obtained by adding a term to the moment of inertia about a parallel axis through the center of mass. Therefore, the minimum principal moment of inertia with respect to a coordinate system passing through the center of mass of the body is the minimum moment of inertia for any possible axis.

If a body has two perpendicular planes of symmetry, a set of principal axes can be determined by inspection, since it is only necessary to make two of the coordinate planes coincide with the planes of symmetry in order that the products of inertia become equal to zero. If the body does not have such planes of symmetry, the orientation of the principal axes must be determined from the expressions for the products of inertia, by setting the products of inertia equal to zero.

PROBLEMS

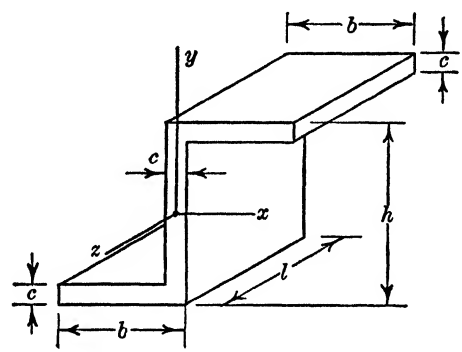
177. Calculate I_z for a homogeneous right circular cylinder of radius R and total mass m. The z axis coincides with the axis of symmetry of the figure.

178. A rectangular plate of total mass m has a length a, a width b, and a thickness c. The z-axis is normal to the plane of a and b and passes through the midpoint of the face of the plate. (a) Find I_z for the plate. (b) Find the moment of inertia of the plate about an axis through the corner of the plate parallel to the z-axis.

179. Calculate the moment of inertia of a homogeneous circular disk of radius R and thickness l about a diametral axis passing through the center of mass of the disk.

180. Calculate the moment of inertia of a slender rod about a normal axis passing through the midpoint of the rod. The rod has a uniform cross section and a uniform density.

181. Calculate the moments of inertia I_y and I_z of the z-shaped body shown. The z-axis passes through the center of mass of the body and is



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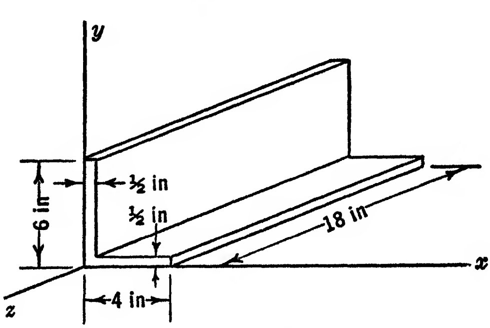
parallel to the faces of the body. The body is homogeneous and has a density ρ lb sec^2/in.^{4}; $b = 3\frac{5}{8}$ in.; $c = \frac{1}{2}$ in.; $h = 6\frac{1}{8}$ in.; $l = 10$ in.

182. A solid right circular cone has a height h, a base of radius r, and a total mass m. (a) Calculate the moment of inertia of the body about the axis of symmetry. (b) Calculate the moment of inertia about a diameter of the base.



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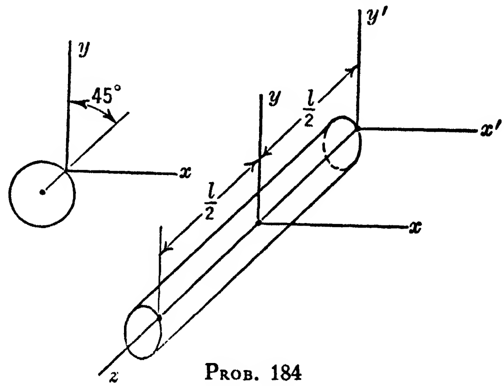
183. A steel angle has the dimensions shown. Find the moment of inertia of the body about the z-axis. Give a numerical answer in units of lb ft sec^2.

184. A homogeneous circular cylinder of radius R and length l has a total mass m. The z-axis lies along the surface of the cylinder parallel to its axis. (a) The xy plane passes through the center of the cylinder.



PROB. 184

Find I_{xy}, I_{yz} and I_{xz} for the cylinder. (b) The $x'y'$ plane coincides with the end of the cylinder. Find $I_{x'y'}$, $I_{y'z'}$, and $I_{x'z'}$.

185. Show that $l_{xx'}^2 + l_{xy'}^2 + l_{xz'}^2 = 1$.

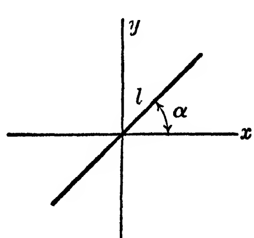
186.* Derive the transformation for I_{xy} by the first method used in the text to derive I_{xx}. (See page 155.)

187. By using the transformation formula:

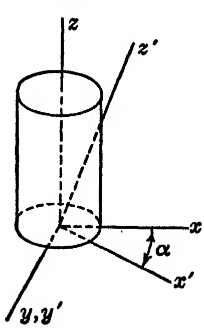
$$I_{\alpha\beta} = \sum_{\alpha'}\sum_{\beta'} l_{\alpha\beta'} l_{\beta\alpha'} I_{\alpha'\beta'}$$

derive the expression for I_{xx} which is given in the text (Equation 79).

188. Compute I_{xy} for a slender rod of length l and mass m. The rod lies in the xy plane and makes an angle α with the x-axis. For what values of α will $I_{xy} = 0$? The rod is homogeneous and of uniform cross section.



PROB. 188

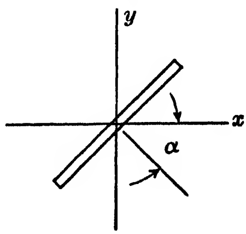


PROB. 189

189. A circular cylinder of radius R, length l, and total mass m is oriented as shown. The $x'z'$ plane coincides with the xz plane. Find $I_{z'}$.

190. Compute $I_{x'z'}$ and $I_{y'z'}$ for the cylinder of Problem 189.

191. Compute the moment of inertia of a cube about a body diagonal axis passing through two opposite corners.



PROB. 193

192. Find the products of inertia for a homogeneous cube so oriented that both the y- and z-axes are face diagonals passing through corners of the cube.

193. A thin circular disk of radius R and mass m rotates about the x-axis which passes through the center of mass of the disk. The disk is skewed on the shaft so that the normal to the disk makes an angle α with the axis of rotation. Find I_{xy} for the disk.

49. The General Equations of Motion for a Rigid Body. The general equation for the rotational motion of a rigid body is:

$$\boldsymbol{M} = \dot{\boldsymbol{H}}$$

or

$$\boldsymbol{M} = \frac{d}{dt}(H_x\boldsymbol{i} + H_y\boldsymbol{j} + H_z\boldsymbol{k})$$

where

$$\begin{aligned} H_x &= +I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}$$

The xyz axes are fixed in the body with the origin at the center of mass and are rotating with it so that I_{xx}, I_{xy}, etc., are constants. The unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ rotate with the body so that $\dot{\boldsymbol{i}}, \dot{\boldsymbol{j}}, \dot{\boldsymbol{k}}$ are not zero. The equation of motion is, therefore:

$$\boldsymbol{M} = \dot{H}_x\boldsymbol{i} + H_x\dot{\boldsymbol{i}} + \dot{H}_y\boldsymbol{j} + H_y\dot{\boldsymbol{j}} + \dot{H}_z\boldsymbol{k} + H_z\dot{\boldsymbol{k}}$$

The derivatives of the unit vectors are:

$$\dot{\boldsymbol{i}} = \boldsymbol{\omega} \times \boldsymbol{i} = (\omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k}) \times \boldsymbol{i} = \omega_z\boldsymbol{j} - \omega_y\boldsymbol{k}$$
$$\dot{\boldsymbol{j}} = \omega_x\boldsymbol{k} - \omega_z\boldsymbol{i}; \quad \dot{\boldsymbol{k}} = \omega_y\boldsymbol{i} - \omega_x\boldsymbol{j}$$

Substituting and collecting terms:

$$\boldsymbol{M} = (\dot{H}_x - \omega_z H_y + \omega_y H_z)\boldsymbol{i} + (\dot{H}_y - \omega_x H_z + \omega_z H_x)\boldsymbol{j} + (\dot{H}_z - \omega_y H_x + \omega_x H_y)\boldsymbol{k}$$

This is the general vector equation of motion. The three scalar equations of motion are:

$$\begin{aligned} M_x &= \dot{H}_x - \omega_z H_y + \omega_y H_z \\ M_y &= \dot{H}_y - \omega_x H_z + \omega_z H_x \\ M_z &= \dot{H}_z - \omega_y H_x + \omega_x H_y \end{aligned} \tag{84}$$

The appropriate expressions for H_x, H_y and H_z must be substituted in the equations. The resulting expressions are greatly simplified by locating the coordinate axes so that they coincide with the principal axes of the body. The products of inertia are then zero and $H_x = I_{xx}\omega_x$, $H_y = I_{yy}\omega_y$, $H_z = I_{zz}\omega_z$. The equations then become:

$$\begin{aligned} M_x &= I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z \\ M_y &= I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_x\omega_z \\ M_z &= I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y \end{aligned} \tag{85}$$

These are called *Euler's equations of motion of a rigid body.* It should be noted that in Euler's equations the *xyz* axes are the *principal axes* of the body, with origin at the center of mass of the body.

Since the rotation of a rigid body is described independently of the motion of the mass center by the equation $\dot{\boldsymbol{H}} = \boldsymbol{M}$, we may find the momentum and energy equations for the rotation alone by the usual first integrals. Integrating with respect to time, we obtain:

$$\int_1^2 \boldsymbol{M}\,dt = \boldsymbol{H}\Big|_1^2 \tag{86}$$

This states that the change in moment of momentum is equal to the moment of impulse.

To obtain the work energy equation we note that $\boldsymbol{\omega} \cdot \boldsymbol{M}$ is the rate of doing work and the integral with respect to time is the work done. We have therefore:

$$\int_1^2 \boldsymbol{\omega} \cdot \boldsymbol{M}\,dt = \int_1^2 \boldsymbol{\omega} \cdot \dot{\boldsymbol{H}}\,dt$$

This expression can be integrated in the following way:

$$\int \boldsymbol{\omega} \cdot \dot{\boldsymbol{H}}\,dt = \tfrac{1}{2}\int \boldsymbol{\omega} \cdot \dot{\boldsymbol{H}}\,dt + \tfrac{1}{2}\int \boldsymbol{\omega} \cdot \dot{\boldsymbol{H}}\,dt$$

Integrating one of the terms on the right by parts and collecting terms:

$$\int_1^2 \omega \cdot M\,dt = \tfrac{1}{2}\omega \cdot H\Big|_1^2 + \tfrac{1}{2}\int_1^2 (\omega \cdot \dot{H} - \dot{\omega} \cdot H)\,dt$$

The last term in this expression is equal to zero, since for a rigid body:

$$H = \int \rho r \times (\omega \times r)\,dV; \quad \dot{H} = \int \rho r \times \frac{d}{dt}(\omega \times r)\,dV$$

$$\dot{\omega} \cdot H = \int \rho\dot{\omega} \cdot r \times (\omega \times r)\,dV; \quad \omega \cdot \dot{H} = \int \rho\omega \cdot r \times (\dot{\omega} \times r)\,dV$$

Since in a double vector product the dot and the cross can be interchanged:

$$\dot{\omega} \cdot H = \int \rho(\dot{\omega} \times r) \cdot (\omega \times r)\,dV = \omega \cdot \dot{H}$$

The work-energy equation is therefore:

$$\int_1^2 \omega \cdot M\,dt = \tfrac{1}{2}\omega \cdot H\Big|_1^2 \tag{87}$$

The left side represents the work done by M, and the right side represents the change in kinetic energy.

The term $\frac{1}{2}\omega \cdot H$ represents only the kinetic energy of the motion with respect to the origin of the coordinate system. If a point in the body is fixed, and the origin is taken at that point, then $\frac{1}{2}\omega \cdot H$ is the total kinetic energy, but if the origin is at the mass center which is moving the total kinetic energy is:

$$T = \tfrac{1}{2}mv_c^2 + \tfrac{1}{2}\omega \cdot H_c$$

The first term represents the kinetic energy of translation, and the second term represents the kinetic energy of rotation about the mass center; m is the total mass of the body and v_c is the velocity of the mass center.

By expanding the dot product the kinetic energy of rotation can be written as:

$$\tfrac{1}{2}\omega \cdot H = \tfrac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_x\omega_y - 2I_{yz}\omega_y\omega_z - 2I_{zx}\omega_z\omega_x) \tag{88}$$

The preceding analysis is intended to present an overall view of the equations of motion of a rigid body. In actual practical applications we are seldom concerned with the most general motion of a rigid body. Usually we investigate only very special types of motion and the analysis and equations are then much simplified. In the following sections we shall investigate certain special types of motion.

EXAMPLE 1. From the preceding equations we can draw the following conclusions with regard to a free body. Since the moment $\boldsymbol{M}$ is zero, the moment of momentum equation states that $\boldsymbol{H}$ is a constant and the energy equation states that $\boldsymbol{\omega} \cdot \boldsymbol{H}$ is a constant. It follows, therefore, that $\boldsymbol{\omega}$ is not necessarily a constant. There may exist a component of $\boldsymbol{\omega}$ normal to $\boldsymbol{H}$ which may vary with time, although the component of $\boldsymbol{\omega}$ parallel to $\boldsymbol{H}$ is a constant. A physical example of this is the wobbling of a free spinning disk.

EXAMPLE 2. A rigid body has centroidal principal moments of inertia $I_{xx} = 10$, $I_{yy} = 5$, $I_{zz} = 2$ lb ft sec². Starting from rest, the body is acted upon by a total moment of impulse about the centroid of $30\boldsymbol{i} + 40\boldsymbol{j} + 20\boldsymbol{k}$ lb ft sec. Find the angular velocity and the kinetic energy acquired by the body.

Solution. Apply the moment of momentum equation:

$$\int \boldsymbol{M}\, dt = \boldsymbol{H}$$

$$30\boldsymbol{i} + 40\boldsymbol{j} + 20\boldsymbol{k} = 10\omega_x\boldsymbol{i} + 5\omega_y\boldsymbol{j} + 2\omega_z\boldsymbol{k}$$

$$\begin{aligned}\omega_x &= \tfrac{30}{10} = 3 \text{ rad/sec}\\ \omega_y &= \tfrac{40}{5} = 8 \text{ rad/sec}\\ \omega_z &= \tfrac{20}{2} = 10 \text{ rad/sec}\end{aligned}$$

The kinetic energy is:

$$\begin{aligned}\tfrac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{H} &= \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2\\ &= \tfrac{1}{2}[10(3)^2 + 5(8)^2 + 2(10)^2]\\ &= 315 \text{ ft-lb}\end{aligned}$$

PROBLEMS

194. Show that in the triple vector product $\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}$, the dot and the cross may be interchanged.

195. Show that $\boldsymbol{\omega} \cdot \dot{\boldsymbol{H}} = \boldsymbol{\omega} \cdot \int \rho \boldsymbol{r} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{r})\, dV$ for a rigid body.

196. Starting with the fundamental definitions:

$$T = \frac{1}{2}\int \rho v^2 \, dV$$

$$\boldsymbol{H} = \int \rho \boldsymbol{r} \times \dot{\boldsymbol{r}} \, dV$$

where $\boldsymbol{r}$ is the absolute displacement of an element of volume dV, show that the total kinetic energy of a rigid body is:

$$T = \tfrac{1}{2}mv_c^2 + \tfrac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{H}_c$$

197. Show that the rate at which a moment $\boldsymbol{M}$ does work on a rigid body with angular velocity $\boldsymbol{\omega}$ is $\boldsymbol{\omega} \cdot \boldsymbol{M}$.

198. A rigid body having a mass of 50 lb sec²/ft is whirling through space; its center of mass has a velocity of 40 ft/sec. At a given instant, the moment of momentum $\boldsymbol{H}_c$ of the body is $1500\boldsymbol{i} + 1000\boldsymbol{j} + 1200\boldsymbol{k}$ lb ft sec. The inertia integrals are given by:

$$\boldsymbol{I}_c = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 40 & -20 \\ 0 & -20 & 30 \end{bmatrix} \quad \text{lb ft sec}^2$$

(a) What is the angular velocity of the body at the given instant?
(b) What is the kinetic energy of the body at the given instant?

50. Equations of Motion for a Translating Body. The simplest type of rigid body motion is that of translation. A body having translatory motion moves in such a way that any line in the body always remains parallel to its original position, that is, the angular velocity of the body is always zero. The moment of momentum equation written about the mass center is:

$$\boldsymbol{M}_c = \dot{\boldsymbol{H}}_c$$

and since the angular velocity is zero, $\dot{\boldsymbol{H}}_c = 0$, and therefore:

$$\boldsymbol{M}_c = 0$$

The equation of motion of the mass center is:

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}}_c$$

Writing these equations in rectangular coordinates, where x_c, y_c, z_c are the coordinates of the center of mass, we obtain:

$$\begin{aligned} F_x &= m\ddot{x}_c \qquad & M_{cx} &= 0 \\ F_y &= m\ddot{y}_c \qquad & M_{cy} &= 0 \\ F_z &= m\ddot{z}_c \qquad & M_{cz} &= 0 \end{aligned} \tag{89}$$

The force equations describe the motion of the mass center and the moment equations describe the reactive forces which prevent rotation of the body.

EXAMPLE. A body of total mass m moves along a horizontal plane under the action of a force $\boldsymbol{F}$ as shown in Fig. 7–9. The

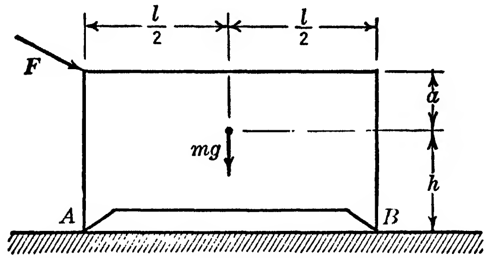


FIG. 7–9

coefficient of kinetic friction between the body and the surface is μ. Find the acceleration of the body as a function of the force $\boldsymbol{F}$, and determine the reactions exerted by the surface on the body at A and B.

Solution. We choose the x-axis in the direction of the motion of the body. The complete free-body diagram is then drawn. The

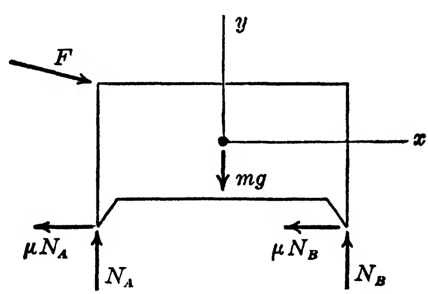


FIG. 7–10

fundamental equations $\boldsymbol{F} = m\ddot{\boldsymbol{r}}_c$ and $\boldsymbol{M}_c = \dot{\boldsymbol{H}}_c$ for this problem become:

$$F_x - \mu(N_A + N_B) = m\ddot{x}_c$$
$$N_A + N_B - mg - F_y = 0$$
$$F_y\left(\frac{l}{2}\right) - F_x a + N_B\left(\frac{l}{2}\right) - N_A\left(\frac{l}{2}\right) - \mu(N_A + N_B)h = 0$$

Eliminating $(N_A + N_B)$ between the first two equations gives the equation of motion of the body:

$$m\ddot{x}_c = F_x - \mu(mg + F_y)$$

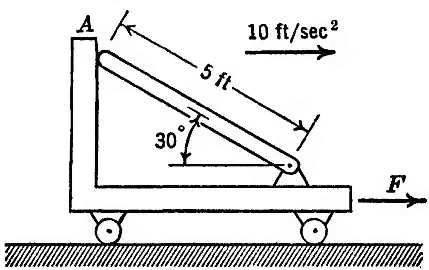
The reactions N_A and N_B are found from the second and third equations:

$$N_A = -\frac{a}{l}F_x + \left(1 - \frac{\mu h}{l}\right)F_y + mg\left(\frac{1}{2} - \frac{\mu h}{l}\right)$$

$$N_B = \frac{a}{l}F_x + \frac{\mu h}{l}F_y + mg\left(\frac{1}{2} + \frac{\mu h}{l}\right)$$

PROBLEMS

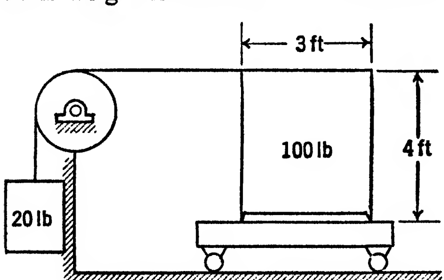
199. A uniform, straight bar weighing 50 lb is fastened with a smooth pin at one end and rests at the other end against a smooth vertical surface. The bar is 5 ft long, and the centerline makes an angle of 30°



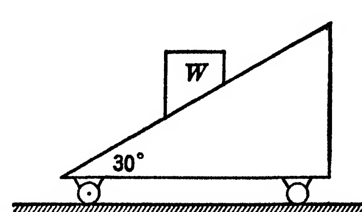
PROB. 199

with the horizontal. The whole arrangement is given an acceleration of 10 ft/sec² horizontally to the right as shown in the diagram. Determine all the forces acting on the bar.

200. A homogeneous block having the dimensions shown weighs 100 lb and rests on a car which can move along a horizontal plane. A 20-lb weight is connected to the block by means of a cable and a friction-



PROB. 200

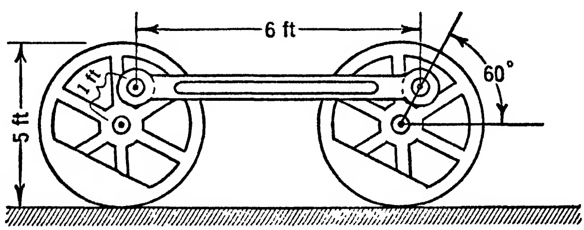


PROB. 201

less pulley as shown in the diagram. The coefficient of static friction between the block and the car is $\mu = 0.25$. If the car is given an acceleration to the right which starts from zero and gradually increases, which will occur first, slipping or tipping of the block? At what value of acceleration will this occur?

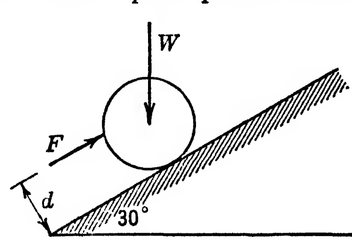
201. A body weighing W pounds rests on a 30° inclined plane as shown. The coefficient of static friction between the body and the plane is μ. What is the maximum horizontal acceleration which the whole system can have without causing the body to move on the plane?

202. The side-crank connecting rod of a locomotive drive system has the dimensions shown. Assuming that the side-crank is a straight uniform bar weighing 500 lb and that it is fastened by smooth pins at the ends, find the forces acting on the rod when the locomotive is running at 60 miles per hour, when the rod is in the position shown.



PROB. 202

203. A homogeneous circular cylinder of radius r weighing W lb rests on a 30° inclined plane as shown. The coefficient of kinetic friction between the cylinder and the plane is μ. Where should a force $\boldsymbol{F}$, parallel to the plane be applied, if the cylinder is to slide up the plane without rotation? Find the reaction of the plane on the cylinder under this condition.



PROB. 203

204. A car moves with a uniform acceleration along a horizontal surface. An instrument is required which will measure the magnitude of this acceleration. It is proposed that a small bar be mounted on a bearing attached to the car, and that the angle made by the bar with the vertical be measured as an indication of the acceleration of the car. Find the relationship between $\ddot{x}$ and ϕ, assuming that the bar is uniform and homogeneous. Would it be better to use a concentrated mass at the end of a light rod?

51. The Rotation of a Rigid Body about a Fixed Axis. Take the z-axis as the axis of rotation, and let the x and y axes be attached to the body and rotate with it (Fig. 7–11). Since the origin of this coordinate system is fixed in space, we may write the equation:

$$\boldsymbol{M} = \dot{\boldsymbol{H}}$$

In the expressions for $\boldsymbol{H}$, Equation (75), $\omega_x = \omega_y = 0$, since the rotation is about the z-axis, and the general equations (Equations 84) become:

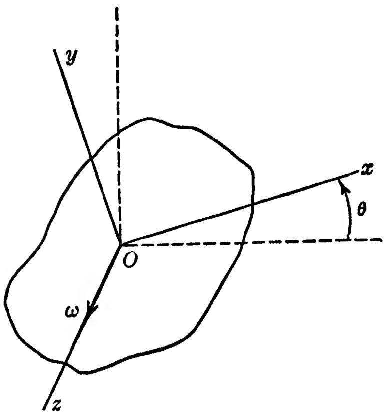


Fig. 7–11

$$M_x = \frac{d}{dt}(-I_{xz}\omega_z) - \omega_z(-I_{yz}\omega_z)$$

$$M_y = \frac{d}{dt}(-I_{yz}\omega_z) + \omega_z(-I_{xz}\omega_z)$$

$$M_z = \frac{d}{dt}(I_{zz}\omega_z)$$

Performing the differentiations, the three moment equations become:

$$\begin{aligned} M_x &= I_{yz}\omega_z^2 - I_{xz}\dot{\omega}_z \\ M_y &= -I_{xz}\omega_z^2 - I_{yz}\dot{\omega}_z \\ M_z &= I_{zz}\dot{\omega}_z \end{aligned} \tag{90}$$

In addition, the equations for the motion of the center of mass are available:

$$\begin{aligned} F_x &= m\ddot{x}_c \\ F_y &= m\ddot{y}_c \\ F_z &= 0 \end{aligned}$$

The rotation of the body about the fixed axis is described by the third moment equation, while the constraining forces which hold the axis of rotation stationary may be found from the first two moment equations and the equations of motion of the mass center.

The impulse-momentum equation and the work-energy equation for a rotating body may be derived directly from the third moment equation, $M_z = I_z\dot{\omega}$.

$$\int_1^2 M_z\,dt = I_z\omega_z\Big|_1^2 \tag{91}$$

The integral of $M_z\,dt$ is the angular impulse or moment of impulse about the axis of rotation, and the term $I_z\omega_z$ is the moment of momentum about the axis of rotation, as may be checked by referring to the general expression for H_z (Equation (75)). Similarly:

$$\int_1^2 M_z\,d\theta = \int_1^2 I_z\ddot{\theta}\frac{d\theta}{dt}\,dt = \tfrac{1}{2}I_z\omega_z^2\Big|_1^2 \tag{92}$$

The integral of $M_z\,d\theta$ represents the work done by M_z during the rotation, and the term $\frac{1}{2}I_z\omega_z^2$ represents the kinetic energy of rotation. This may be checked by the general expression for the kinetic energy, $T = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{H}$, where $\boldsymbol{H} = I_z\omega_z\boldsymbol{k}$ and therefore $T = \frac{1}{2}I_z\omega_z^2$.

EXAMPLE 1. A flywheel of radius r, having a moment of inertia I about the axis of rotation, has an angular velocity ω rad/sec (Fig. 7–12). At time $t = 0$, a brake is applied with a normal braking force P. The brake coefficient of friction is μ. How much time is required to reduce the angular velocity to ω_1 rad/sec?

Solution. If the normal braking force is P then the tangential braking force is μP and the retarding moment is $r\mu P$. Applying the equation of moment of momentum:

$$r\mu P t = I\omega - I\omega_1$$

$$t = \frac{I(\omega - \omega_1)}{r\mu P}$$

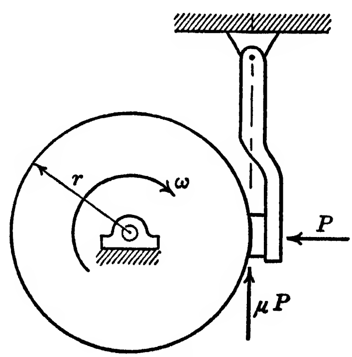


FIG. 7–12

The number of revolutions during time t can be found by applying the work-energy equation:

$$r\mu P\theta = \tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_1^2$$

$$\theta = \tfrac{1}{2}I\,\frac{(\omega^2 - \omega_1^2)}{r\mu P}$$

EXAMPLE 2. A homogeneous disk of radius R and of uniform thickness is supported by a thin rod or wire as shown in Fig. 7–13. The rod is rigidly attached to the disk and to the support. If the disk is rotated from its equilibrium position through an angle θ, the rod exerts a restoring torque on the disk which is proportional to the displacement and oppositely directed. The disk is rotated through an angle θ_0 and is then released from rest. Describe the subsequent motion of the system.

Solution. The disk will rotate about the axis of the rod under the action of a torque $-k\theta$, where k is the torsional spring constant in lb ft/rad. Writing the equation of motion about the fixed axis of rotation, we have, with I as the moment of inertia of the disk about the axis of the rod:

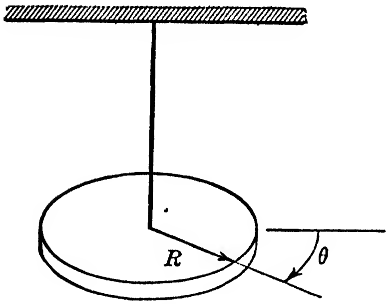


FIG. 7-13

$$I\ddot{\theta} = -k\theta$$

$$\ddot{\theta} + \frac{k}{I}\theta = 0$$

This is the differential equation of simple harmonic motion, the solution of which is:

$$\theta = C_1 \sin\sqrt{\frac{k}{I}}\,t + C_2 \cos\sqrt{\frac{k}{I}}\,t$$

When

$$t = 0, \quad \dot{\theta} = 0; \quad C_1 = 0$$
$$t = 0, \quad \theta = \theta_0; \quad C_2 = \theta_0$$

and the complete solution is:

$$\theta = \theta_0 \cos\sqrt{\frac{k}{I}}\,t$$

Thus the disk performs torsional oscillations with an amplitude θ_0, and a frequency $\frac{1}{2\pi}\sqrt{\frac{k}{I}}$ cycles per second. Such an oscillator is called a torsion pendulum. Torsional oscillation problems are in every way similar to the linear oscil ation problems treated in the chapter on vibrations and the same methods may be used. Many practical examples of such problems in engineering can be found, such as the torsional oscillations of engine crankshafts and of propeller shafts.

EXAMPLE 3. Investigate the dynamic bearing reactions caused by the rotation of an unbalanced rotor.

Solution. Consider a rotor of total weight W supported horizontally on two bearings a distance l apart as shown in Fig. 7–14. The z-axis is taken as the fixed axis of rotation, and the xy axes

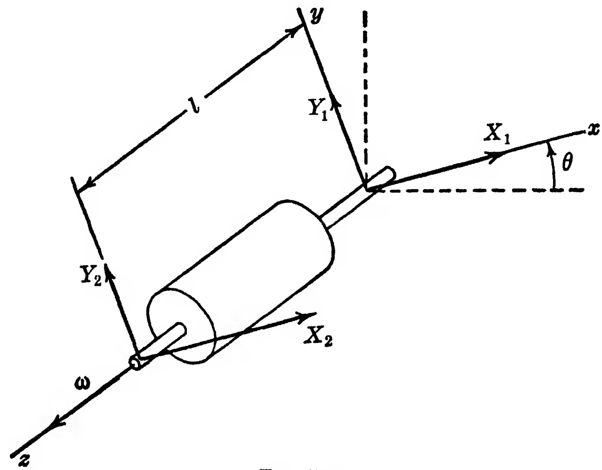


FIG. 7–14

are attached to the rotating body. During the rotation there will be dynamic reactions at the supports of the rotor. We shall determine these dynamic reactions X_1, Y_1, X_2, Y_2 in the rotating xyz system and it is to be understood that, if the total bearing reactions at any instant are required, the dynamic reactions, located in the correct direction at that instant, must be added to the static reactions caused by the weight of the rotor. The equations of motion of the mass center give us directly:

$$\Sigma F_x = m\ddot{x}_c; \quad X_1 + X_2 = -mx_c\omega^2 - my_c\dot{\omega}$$
$$\Sigma F_y = m\ddot{y}_c; \quad Y_1 + Y_2 = mx_c\dot{\omega} - my_c\omega^2$$

The moment of momentum equations give:

$$-Y_2 l = I_{yz}\omega^2 - I_{xz}\dot{\omega}$$
$$X_2 l = -I_{xz}\omega^2 - I_{yz}\dot{\omega}$$

Knowing m, ω, $\dot{\omega}$, x_c, y_c, I_{xz} and I_{yz}, we can find the four unknown dynamic reaction components from these four equations.

If the products of inertia are zero, and if the center of mass lies on the axis of rotation so that $x_c = y_c = 0$, then it will be seen that there are no dynamic reactions. The rotating body is then said to be dynamically balanced. If the center of mass lies on the axis of rotation so that the system is statically balanced, there is no gravity torque for any position of the body, but there may still be some dynamic unbalance because of the presence of products of inertia terms. Since for static balance the mass center has zero acceleration:

$$X_1 = -X_2$$
$$Y_1 = -Y_2$$

The dynamic reactions hence exert a couple on the rotor.

It is thus seen that static balancing of a rotor is not in general sufficient to remove the dynamic reactions, since a rotating dynamic couple may still be present. Complete dynamic balance is achieved by adding to the system two balance weights, so located that the dynamic reactions set up by the balance weights are equal and opposite to the dynamic reactions due to the unbalanced rotor. This is equivalent to making the products of inertia of the rotating system zero, by the addition of the extra balancing weights.

PROBLEMS

205. A rigid body is rotating about a fixed axis with an angular velocity ω. Starting with an element of volume dV of the body, show by integration over the volume of the body that the kinetic energy is $\frac{1}{2}I\omega^2$ and that the moment of momentum about the axis of rotation is $I\omega$.

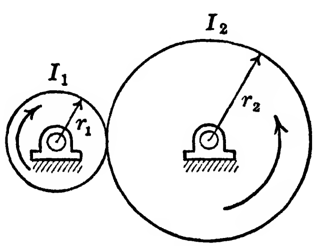
206. A rigid body rotates about a fixed axis. When $t = 0$, the angular displacement of the body measured from a fixed position is θ_0, and the angular velocity is $\dot{\theta}_0$. A torque M_z about the axis of rotation is applied to the body when $t = 0$. If $M_z = A + Bt - Ct^2$, find the angular displacement of the body at any time.

207. A flywheel with a moment of inertia I starts from rest under the action of a constant torque, M. What is the angular velocity of the disk after it has rotated through N revolutions? Do this problem in two ways, first using the equation of motion of the disk, and then using the work-energy principle.

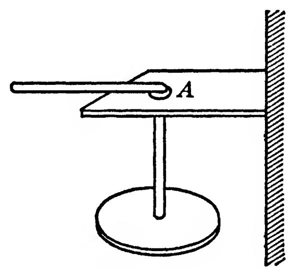
208. A homogeneous circular disk of radius R and mass m is fixed on a shaft which coincides with the geometrical axis of the disk. Acting on the shaft is a torque due to bearing friction which is proportional to the velocity of the disk and always opposes the motion, $M_F = -k\dot{\theta}$. At time $t = 0$, the disk has an angular velocity ω_0. How much time is required for the disk to come to rest, and how many revolutions of the disk are made in this time?

209. A wheel having a moment of inertia I about its axis of rotation is acted upon by a constant torque M. If the motion is resisted by a torque $M_F = -k\dot{\theta}$ due to bearing friction and air resistance, find the maximum speed which will be attained by the wheel.

210. A rotor with a moment of inertia I_1 is driven at a constant angular velocity ω_1. It is brought into contact with a second rotor I_2, which is initially at rest. There is a constant normal force of P lb between the rotors, and the coefficient of friction is μ. At first there is slipping between the rotors until the second rotor has attained the angular velocity $\omega_2 = \omega_1 \frac{r_1}{r_2}$. How much time is required for the second rotor to reach this velocity? (Assume that the coefficient of friction is independent of velocity.) If the first rotor is free to decelerate after initial contact, how much time is required to overcome slipping? What will be the final angular velocities of the two disks under these conditions?



PROB. 210



PROB. 211

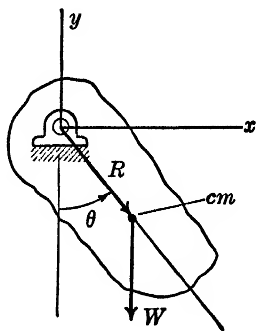
211. A torsion pendulum is mounted as shown. The point of suspension A can be rotated by means of a lever. Suppose that the pendulum is initially at rest, and that starting at time $t = 0$ the point of suspension is given a rotation

$$\theta = \theta_0 \sin \omega t$$

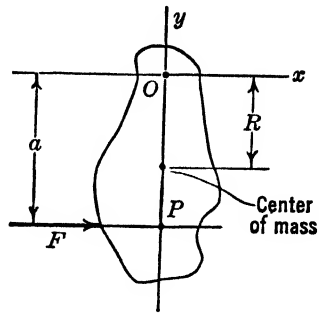
Find the resulting forced vibration of the pendulum. (Neglect damping in the system.)

212. A flywheel having a moment of inertia about its axis of rotation of 5000 lb ft sec² is driven by an electric motor which does work at the rate of $\frac{1}{4}$ horsepower. If the wheel starts from rest, what will be its kinetic energy at the end of one hour? If the flywheel is to be brought to rest in 30 sec from the speed attained at the end of one hour, what constant retarding frictional torque is required?

213. A body free to oscillate about a fixed axis under the action of gravity is called a compound pendulum. Considering the compound pendulum shown, the total weight of which is W, find the period of small vibrations of the pendulum about the equilibrium position. The distance from the point of support to the center of mass is R. What is the length of a simple pendulum with a mass concentrated at one point which would have the same period?



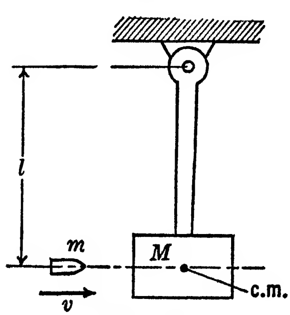
PROB. 213



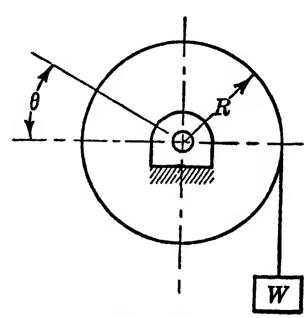
PROB. 214

214. A rigid body can rotate about a fixed point O as shown. The moment of inertia of the body about the point O is I_0, and the distance between O and the center of mass of the body is R. A force F is applied to the body perpendicular to the line joining O and the center of mass, and located at a distance a from the center of rotation. Find a so that there will be no reaction at the point of support O in the direction of the force F. The point P, located by a, is called the *center of percussion*. (Neglect gravity forces.)

215. A ballistic pendulum of mass M has a moment of inertia I about its axis of rotation. A bullet of mass m is fired into the pendulum as shown. It is observed that the pendulum then undergoes an angular displacement θ_0. What was the velocity of the bullet?



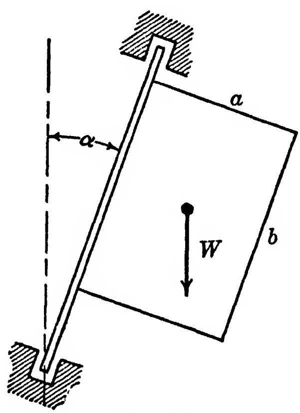
PROB. 215



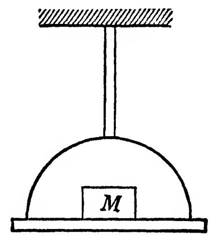
PROB. 216

216. A wheel of radius R and moment of inertia I about the axis of rotation has a rope wound around it which supports a weight W. Write the equation of conservation of energy for this system and differentiate to obtain the differential equation of motion in terms of accelerations. (Neglect the mass of the rope, and assume no energy loss during the motion.) Check the answer obtained by drawing separate free-body diagrams for the wheel and for the weight, writing the equations of motion for each body, and solving the equations simultaneously.

217. A thin, homogeneous, rectangular plate of uniform thickness is free to oscillate under the action of gravity about an inclined axis as shown. Write the equation of conservation of energy for this system and differentiate this equation to find the equation of motion in terms of accelerations. What is the period of the undamped oscillation, and how do changes in the angle α affect this period?



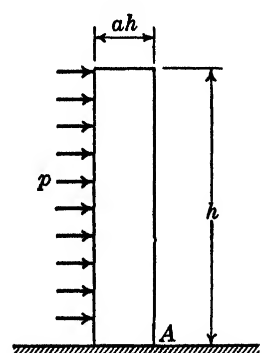
PROB. 217



PROB. 218

218. A torsional pendulum is arranged as shown, so that various weights can be placed on the pendulum disk and oscillated about the axis of the pendulum. It is observed that with a mass M of known moment of inertia I_1, a torsional frequency of oscillation f_1 is measured. If a body of unknown moment of inertia I_2 is substituted for the known mass, the frequency is observed to be f_2. The frequency of the pendulum alone, with no added weight, is f_0. Show that the unknown moment of inertia is given by:

$$I_2 = I_1 \left[\frac{\left(\frac{f_0}{f_2}\right)^2 - 1}{\left(\frac{f_0}{f_1}\right)^2 - 1} \right]$$



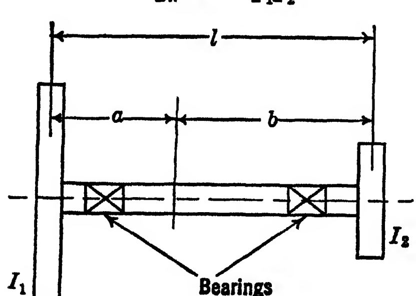
PROB. 219

219. A rigid wall of height h and width ah rests upon a horizontal surface. It is subjected to a uniform, constant blast pressure which acts for a short time Δt. With Δt given, what value of p lb/ft^2 will cause the wall to overturn about point A, assuming that there is no sliding? The wall has a weight of W lb/ft of length. The time Δt is so small that it may be assumed that there is no motion of the wall during Δt, the action of p being only to impart an initial angular velocity to the wall.

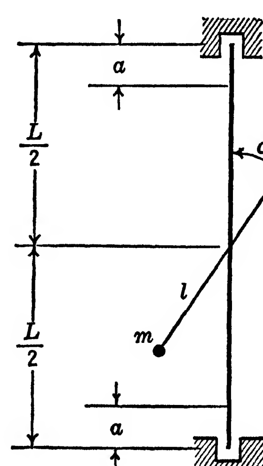
220. A rectangular door of mass m is free to swing on two hinges. It is initially at rest when it is subjected to a uniform blast pressure from a bomb. The blast pressure acts for only a small fraction of a second but reaches a high maximum value of p lb/ft^2. What is the maximum dynamic hinge reaction? A 500-lb bomb detonating at a distance of 100 ft would produce a maximum blast pressure of 8 lb/in.^{2} If the door is 2.5 ft by 7 ft, what is the maximum dynamic hinge reaction?

221. Two bodies of moment of inertia I_1 and I_2 about the axis of rotation are connected by a shaft as shown. If equal and opposite moments are applied to the bodies, the ends of the shaft will be twisted through a relative angular displacement θ, where $\theta = \dfrac{M}{k}$. If the moments are suddenly released, the two bodies will then perform torsional vibrations. Apply the principle of the conservation of moment of momentum to the system to show that the two bodies always rotate in opposite directions. From this it may be concluded that there is a certain cross section of the shaft which does not rotate during the oscillatory motion. The location of this cross section may be found by noting that, if the system is divided into two simple torsional pendulums of length a and b, the frequency of oscillation of the two must be equal. In this way show that $a = \dfrac{I_2 l}{I_1 + I_2}$ and that the frequency of vibration is:

$$f = \frac{1}{2\pi}\sqrt{\frac{k(I_1 + I_2)}{I_1 I_2}}$$



PROB. 221



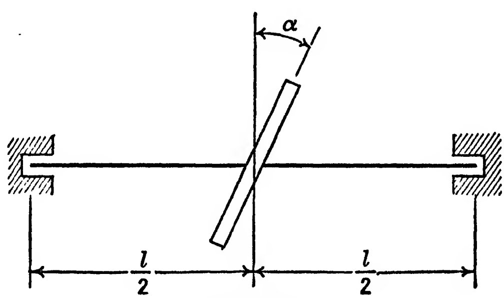
PROB. 222

222. Two equal particles of mass m are fastened to the ends of a straight rod of length $2l$ and of negligible weight. The rod is attached to the center of a vertical shaft of length L as shown. If the vertical shaft rotates at a constant angular velocity ω, find the dynamic reactions at the bearings. The system is to be dynamically balanced by the addition of two concentrated weights of mass m_1. These weights are to be located in planes at a distance a from the bearings. Show where these weights should be attached and find the radius at which they should be located.

223. Show that if a rotating body is in static balance but not dynamic balance, and if the rotational speed is constant, the dynamic bearing reactions have the magnitude:

$$R = \frac{\omega^2}{l} \sqrt{I_{yz}^2 + I_{xz}^2}$$

224. A thin circular disk of radius r is skewed a small angle α with respect to the axis of rotation, as shown. If the angular velocity of the

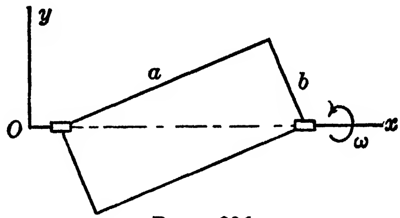


PROB. 224

system is a constant, find the dynamic reactions at the bearings. The total mass of the disk is m, and the center of mass of the disk is on the axis of rotation. If the shaft is horizontal, find the total bearing reactions in the position shown in the diagram.

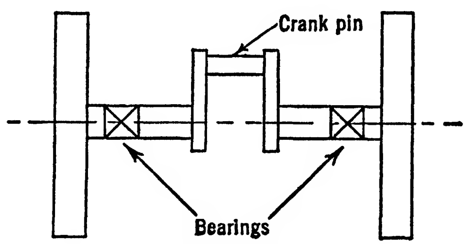
225. If in the preceding problem the rotating body is a solid cylinder of radius R and length l, find the dynamic bearing reactions. The center of mass of the cylinder is located on the axis of rotation, and the axis of the cylinder is inclined at an angle α with the axis of rotation. If the center of mass of the cylinder is located at a distance e from the axis of rotation, at the center of the shaft, in addition to the skew of the axis, what are the dynamic bearing reactions?

226. A thin, rectangular plate of mass m rotates about an axis coinciding with a diagonal of the plate. If the bearings are located at the two corners of the plate, find the dynamic bearing reactions.



Prob. 226

227. The diagram represents the two flywheels of a gas engine. The flywheels are 3 ft apart, and the center of the crank pin is located at a distance of 1 ft 4 in. from the left flywheel. The off-center crank pin and



Prob. 227

crank arms are equivalent to a concentrated weight of 80 lb at a distance of 8 in. from the bearing centerline. The system is to be balanced by two weights placed in the planes of the flywheels. If these weights are located at a radial distance of 1 ft from the center of the flywheel, what should be their magnitudes?

52. Plane Motion of a Rigid Body. If every element of a body moves parallel to a fixed plane, the body is said to have plane motion. If the xy plane is taken as the plane of motion, the angular velocity of the body is $\omega_z\boldsymbol{k}$ and the velocity of the mass center is $\dot{x}_c\boldsymbol{i} + \dot{y}_c\boldsymbol{j}$. Let the xyz coordinate axes be fixed in the body with the origin at the center of mass. The general equation of motion $\boldsymbol{M}_c = \dot{\boldsymbol{H}}_c$ then has the components:

$$
\begin{aligned}
M_x &= I_{yz}\omega_z^2 - I_{xz}\dot{\omega}_z \\
M_y &= -I_{xz}\omega_z^2 - I_{yz}\dot{\omega}_z \\
M_z &= I_z\dot{\omega}_z
\end{aligned}
\tag{93}
$$

In addition there are the independent equations for the motion of the mass center:

$$F_x = m\ddot{x}_c$$
$$F_y = m\ddot{y}_c$$
$$F_z = 0$$

It will be noted that these six equations are identical with those obtained for the rotation of a body about a fixed axis. The origin of the coordinate system, however, is located at the center of mass for plane motion, whereas for rotation it is located on the fixed axis of rotation.

Example 1. A circular cylinder of radius R and mass M is pushed along a horizontal plane by a horizontal force F. The coefficient of kinetic friction between the cylinder and the surface is μ. Describe the motion of the cylinder (Fig. 7-15).

Solution. A free-body diagram of the cylinder is first drawn. The coordinate x_c describes the linear position of the center of the cylinder measured from a fixed point,

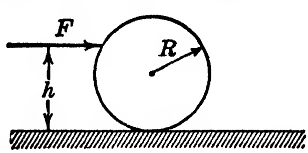


Fig. 7-15

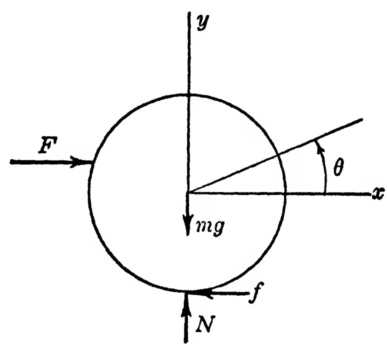


Fig. 7-16

and the coordinate θ describes the angular position. For the motion of the center of mass in the x-direction, we have:

$$\Sigma F_x = m\ddot{x}_c = F - f$$

Taking moments about the z-axis, which passes through the center of mass, we obtain:

$$M_z = I_z\ddot{\theta} = -F(h - R) - fR$$

This gives two equations relating the three unknowns x_c, θ, and f. The additional equation to be used will depend upon whether or not there is slipping between the cylinder and the plane. If there is no slipping, we may write $R\ddot{\theta} = -\ddot{x}_c$, which, together with the first two equations, gives:

$$\ddot{x}_c = \frac{FhR}{I_z + mR^2}$$

$$f = F\left(1 - \frac{hRm}{I_z + mR^2}\right)$$

If the value of the friction force f computed in this way exceeds the value μW, then slipping will occur, and $R\ddot{\theta} \neq \ddot{x}_c$. Then the third equation to be used is $f = \mu W$ and the solution is:

$$\ddot{x}_c = \frac{F - \mu W}{m}$$

$$\ddot{\theta} = \frac{F(h - R) + \mu WR}{I_z}$$

EXAMPLE 2. A cylinder of mass m, radius R, and moment of inertia I about its geometric axis rolls without slipping down a hill under the action of gravity (Fig. 7-17). If the velocity of the center of mass of the cylinder is initially v_0, find the velocity after the cylinder has dropped through a vertical distance h.

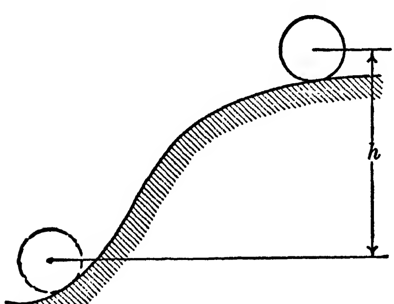


FIG. 7-17

Solution. Since there is no energy loss during the motion of the cylinder, we may write the equation of the conservation of energy, which will lead at once to the desired result. Taking the final position of the cylinder as the point of zero potential energy, we have:

$$\tfrac{1}{2}mv_0^2 + \tfrac{1}{2}I\left(\frac{v_0}{R}\right)^2 + mgh = \tfrac{1}{2}mv_h^2 + \tfrac{1}{2}I\left(\frac{v_h}{R}\right)^2$$

from which:

$$v_h = \sqrt{v_0^2 + \frac{2mgh}{\left(m + \frac{I}{R^2}\right)}}$$

EXAMPLE 3. Two uniform, homogeneous, circular disks of radius R and mass m are connected by a uniform straight bar of length l (Fig. 7–18). The mass of the straight bar is M. The assembly rolls without slipping along a horizontal plane, the center of the disk having a constant velocity v parallel to the plane. Find the forces exerted on the disks by the plane when the straight bar is parallel to the plane.

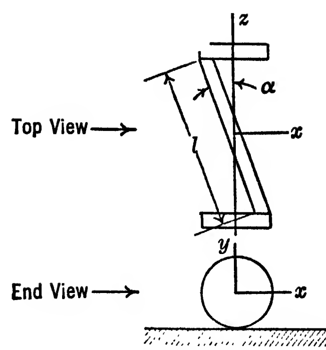


Fig. 7–18

Solution. We consider an xyz coordinate system which is attached to the body and whose origin is located at the center of mass. The free-body diagram is shown in Fig. 7–19, where the frictional forces, f_1 and f_2, and the normal forces, N_1 and N_2, are indicated. The equations of motion for the system become:

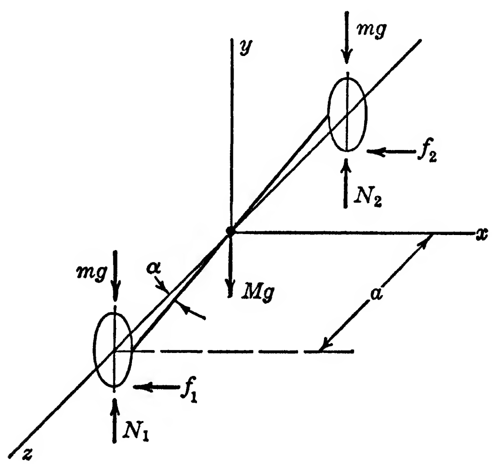


Fig. 7–19

$$
\begin{aligned}
\Sigma F_x &= m\ddot{x}_c = 0 = -f_1 - f_2 \\
\Sigma F_y &= m\ddot{y}_c = 0 = N_1 + N_2 - mg - mg - Mg \\
M_x &= I_{yz}\omega_z^2 = mga - N_1 a - mga + N_2 A = I_{yz}\left(\frac{v}{R}\right)^2 \\
M_y &= -I_{xz}\omega_z^2 = f_2 a - f_1 a = -I_{xz}\left(\frac{v}{R}\right)^2 \\
M_z &= I_{zz}\dot{\omega}_z = 0 = -\,(f_1 R + f_2 R)
\end{aligned}
$$

Evaluating the products of inertia gives:

$$I_{yz} = 0$$

$$I_{xz} = \frac{Ml^2}{12} \sin\alpha \cos\alpha$$

From the first or fifth equation, $f_1 + f_2 = 0$, and from the fourth equation

$$f_1 - f_2 = \frac{Ml^2}{12a} \frac{v^2}{R^2} \sin\alpha \cos\alpha$$

Thus

$$f_1 = -f_2 = \frac{Ml^2v^2}{24aR^2} \sin\alpha \cos\alpha$$

Since

$$a = \sqrt{\left(\frac{l}{2}\right)^2 - R^2}; \quad \sin\alpha = 2\frac{R}{l};$$

$$\cos\alpha = \frac{2}{l}\sqrt{\left(\frac{l}{2}\right)^2 - R^2}$$

we have:

$$f_1 = -f_2 = \frac{Mv^2}{6R}$$

From the second equation:

$$N_1 + N_2 = 2mg + Mg$$

From the third equation:

$$N_1 - N_2 = 0$$

hence:

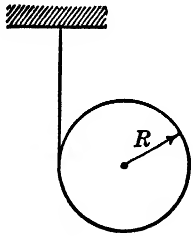
$$N_1 = N_2 = mg + \frac{Mg}{2}$$

PROBLEMS

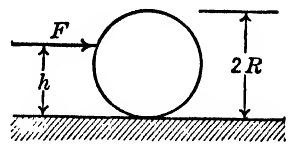
228. A uniform, circular cylinder of weight W and radius R starts from rest and rolls without slipping under the action of gravity down a plane which makes an angle α with the horizontal. Find the acceleration of the cylinder.

229. If in the preceding problem the coefficient of friction between the cylinder and the plane is μ, find the maximum angle of inclination of the plane for which the cylinder will roll without slipping.

230. A uniform, circular cylinder of weight W and radius R has a rope wrapped around it, one end of which is fixed as shown. The system is released from rest with the rope in a vertical position. Describe the subsequent motion of the system and find the force in the rope. The rope is in the plane of the mass center.



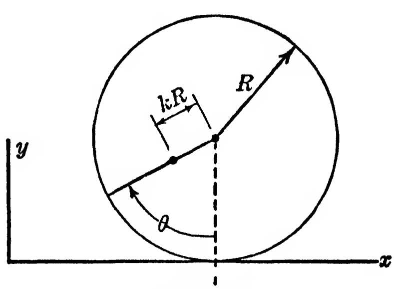
PROB. 230



PROB. 231

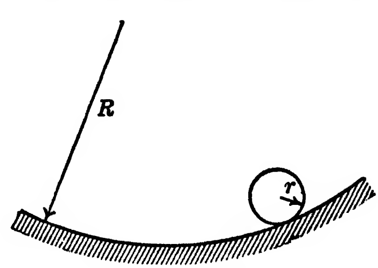
231. At what point should a billiard ball be struck with a horizontal impact in order that it will roll without sliding on a frictionless table surface?

232. A wheel of weight W is unbalanced so that its center of mass lies at a distance kR from the center of the wheel. The wheel rolls without slipping with a constant velocity v. Determine the normal force exerted by the wheel against the ground.



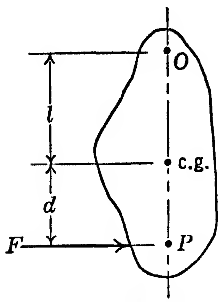
PROB. 232

233. A sphere of radius r and mass m rolls on a circular surface of radius R under the action of gravity. Find the differential equation describing small oscillations of the system about the position of equilibrium and show how this problem differs from that of the particle of mass m which slides on the surface.



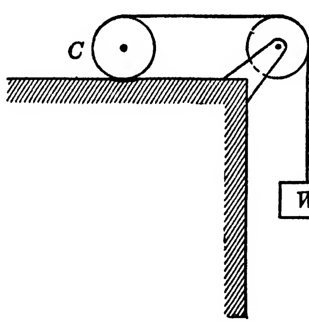
PROB. 233

234. A body of mass m and moment of inertia I_c about the center of mass is initially at rest when it is given an impulse $F\Delta t$ as shown. The body moves in a horizontal plane, and the force F is horizontal. Find the distance l from the center of mass to the point O, whose instantaneous acceleration $\ddot{x}_0$ is zero at time $t = 0$. Since the point O has a zero acceleration at $t = 0$, this point could be mounted on an axis of rotation without involving any reaction during the impact. The point P is called the center of percussion corresponding to O. Show that the point O is the center of percussion corresponding to P. Show that the period of vibration of the body as a compound pendulum acted upon by gravity is the same whether O or P is the axis of rotation. Such a compound pendulum is called *Kater's Reversible Pendulum.*

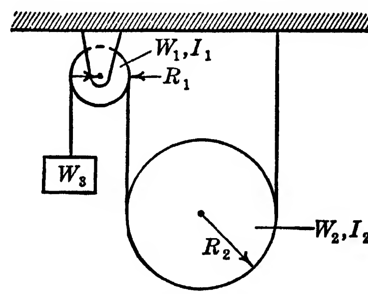


PROB. 234

235. A circular cylinder having a radius of 1 ft and a weight of 100 lb rolls without slipping along a horizontal surface. A rope wound around the cylinder passes over a frictionless pulley and supports a weight of 200 lb which moves vertically as shown in the figure. If the 200-lb weight is released from rest, find the velocity of the system at the end of 3 sec. Do this first by drawing a separate free-body diagram for each mass, in this way determining the acceleration of the system. Check the answer by applying energy principles to the whole system.



PROB. 235



PROB. 236

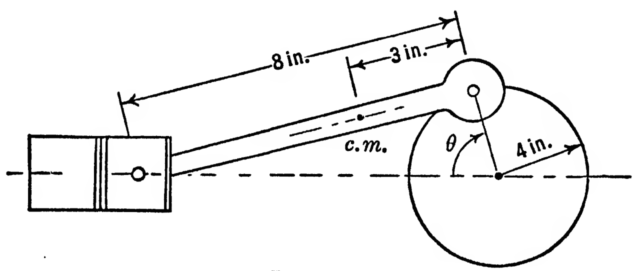
236. A fixed pulley, a moving pulley, and a weight which can move vertically are assembled as shown. The sections of rope between the pulleys are vertical, and the frictional forces in the pulleys are assumed to be negligible. Find the equation of motion of W_2 in terms of accelerations by differentiating the energy equation for the system.

237. Two identical solid spheres of mass m and radius r are free to move on a horizontal surface. If one sphere is at rest and the other sphere makes an impact with it with a velocity v, describe the resulting motion of the system. Assume that no slipping occurs between the spheres and the surface, and assume that no energy is lost during the process. The impact is a direct central impact, the direction of rebound being the same as the direction of approach.

238. In Example 3, Art. 52 suppose that the system has rotated through 90°, so that the inclined bar lies in a plane which is perpendicular to the plane on which the disks roll. Solve for the forces on the disks.

239. In Example 3, Art. 52 suppose that the centers of the disks have an acceleration $\ddot{x}$ parallel to the plane as well as a velocity $\dot{x}$. Find the forces on the disks.

240.* A crank and connecting rod mechanism has the dimensions shown in the diagram. The piston weighs 20 lb, the connecting rod 15 lb, and the center of mass of the connecting rod is located 3 in. from



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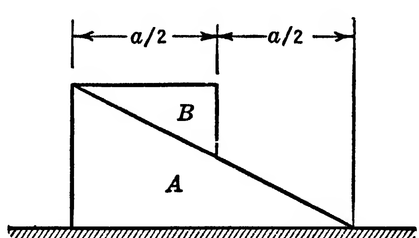
the crank-pin. The moment of inertia of the connecting rod about its center of mass is 0.2 lb ft sec². The crank is rotating at a constant speed of 1200 rpm. Find all the forces acting on the connecting rod at the instant when the crank angle $\theta = 30°$. (Neglect friction and gravity.)

241. A prism B rests on a prism A, which lies on a horizontal table as shown. The cross sections of the prisms are similar right triangles.



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A weighs four times as much as B. The prisms and the table are smooth. B slides down A until it touches the table. Find the distance through which A moves during this time.

53. Rotation about a Fixed Point. The motion of a body rotating about a fixed point is somewhat more involved than the types of motion hitherto considered. The spinning top is an example of such a motion, in which a rigid body rotates about an axis which is itself rotating. Motions of this type are best described in terms of the coordinate systems illustrated in Fig. 7–20.

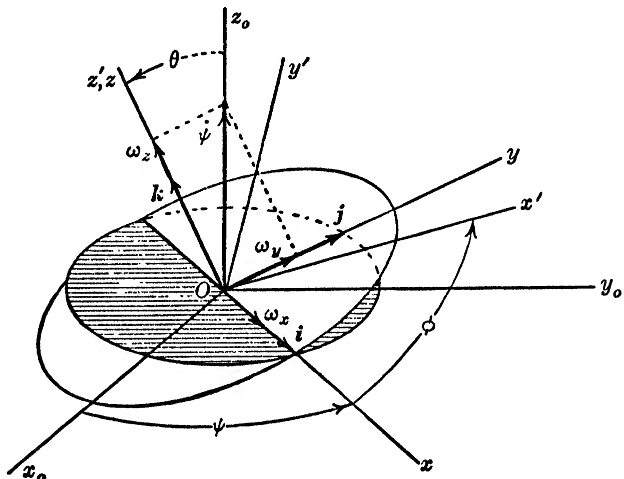


Fig. 7–20

The point O is the fixed point about which the motion occurs. The $x_0y_0z_0$ axes are fixed in direction, and the $x'y'z'$ axes are fixed in the body with the z'-axis coinciding with the axis of spin. In order to describe the motion in the simplest way we shall specify the orientation of the $x'y'z'$ axes by the so-called Eulerian angles θ, ψ, ϕ, which may be explained by the introduction of a third set of coordinate axes xyz. The z-axis coincides with the z'-axis, and the x-axis lies along the intersection of the $x'y'$ plane with the x_0y_0 plane. The angles θ and ψ describe the motion of the xyz axes, and the angle ϕ describes the motion of the $x'y'z'$ axes (the body) with respect to the xyz axes.

In the following analysis the pertinent quantities will be expressed in terms of their components in the xyz coordinate system, for which the unit vectors are $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. The angular velocity of the xyz system is:

$$\boldsymbol{\omega} = \omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + \omega_z \boldsymbol{k} = \dot{\theta} \boldsymbol{i} + \dot{\psi} \sin \theta \boldsymbol{j} + \dot{\psi} \cos \theta \boldsymbol{k}$$

Since the body has an angular velocity of spin $\dot{\phi}$ with respect to the xyz system, the angular velocity of the body is:

$$\boldsymbol{\omega}_b = \omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + (\omega_z + \dot{\phi})\boldsymbol{k}$$

The expression $(\omega_z + \dot{\phi})$ represents the total spin velocity of the body and will be designated by Ω. It should be noted that this expression for the angular velocity of the body is written in terms of components along the xyz axes, and not along the $x'y'z'$ axes which are fixed in the body.

The equations of motion are obtained in the usual manner by writing the equation of moment of momentum, $\dot{\boldsymbol{H}} = \boldsymbol{M}$, about the fixed point O. As has been shown, this equation, in the rotating xyz system, is:

$$\begin{aligned} M_x &= \dot{H}_x - H_y\omega_z + H_z\omega_y \\ M_y &= \dot{H}_y - H_z\omega_x + H_x\omega_z \\ M_z &= \dot{H}_z - H_x\omega_y + H_y\omega_x \end{aligned}$$

The ω_x, ω_y, ω_z, in these equations, are the angular velocity components of the coordinate system. H_x, H_y, H_z must be expressed in terms of the angular velocity components of the body.

$$\begin{aligned} H_x &= +I_x\omega_x - I_{xy}\omega_y - I_{xz}\Omega \\ H_y &= -I_{xy}\omega_x + I_y\omega_y - I_{yz}\Omega \\ H_z &= -I_{xz}\omega_x - I_{yz}\omega_y + I_z\Omega \end{aligned} \tag{94}$$

The general equations of motion which are obtained by substituting these values of H_x, H_y, H_z are rather complex. We shall not investigate the general equations since the applications to all the special problems which we shall now consider will involve considerable simplifications.

54. The Gyroscope. Consider a body, symmetrical about the z-axis, and mounted so as to be free to rotate about a fixed point O as shown in Fig. 7–21. Because of the symmetry $I_x = I_y$, $I_{xy} = I_{xz} = I_{yz} = 0$, and I_z is a constant even though the body is rotating with respect to the xyz axes. The components of $\boldsymbol{H}$ are, therefore:

$$H_x = I_x\omega_x; \quad H_y = I_y\omega_y; \quad H_z = I_z\Omega$$

Substituting these values into the equation of motion, and writing $I_x = I_y = I$, we have:

$$\begin{aligned} M_x &= I(\dot{\omega}_x - \omega_y\omega_z) + I_z\Omega\omega_y \\ M_y &= I(\dot{\omega}_y + \omega_z\omega_x) - I_z\Omega\omega_x \\ M_z &= I_z\dot{\Omega} \end{aligned} \tag{95}$$

Examining these moment equations we see the usual terms involving the angular velocities and the angular accelerations which would be present even if the body were not spinning about its axis. In addition, we find the terms $I_z\Omega\omega_y$ and $-I_z\Omega\omega_x$ which are consequences of the spin velocity. The moments associated with these spin components are called the *gyroscopic moments*, and if we call the resultant gyroscopic moment $\boldsymbol{M}_G$, we have:

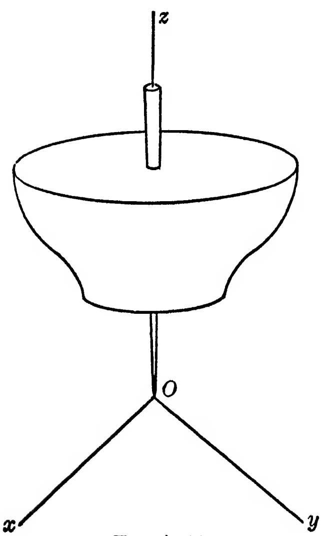


Fig. 7–21

$$\begin{aligned} \boldsymbol{M}_G &= I_z\Omega\omega_y\boldsymbol{i} - I_z\Omega\omega_x\boldsymbol{j} + I_z\dot{\Omega}\boldsymbol{k} \\ &= I_z\Omega(\omega_y\boldsymbol{i} - \omega_x\boldsymbol{j}) + I_z\dot{\Omega}\boldsymbol{k} \\ &= I_z\Omega\dot{\boldsymbol{k}} + I_z\dot{\Omega}\boldsymbol{k} \end{aligned}$$

or, $$\boldsymbol{M}_G = \frac{d}{dt}(I_z\Omega\boldsymbol{k}) = \dot{\boldsymbol{H}}_G \tag{96}$$

where $\boldsymbol{H}_G = I_z\Omega\boldsymbol{k}$, the *spin moment of momentum.*

Referring again to Fig. 7–21, let us suppose that the spin velocity Ω is a constant, and let us determine what moment should be applied to give a constant angular velocity ω_x, with $\omega_y = 0$. Substituting these conditions in equations (95), we obtain:

$$\begin{aligned} M_x &= 0 \\ M_y &= -I_z\Omega\omega_x \\ M_z &= 0 \end{aligned} \tag{97}$$

Thus it requires a moment in the y-direction to produce an angular velocity in the x-direction. A device which has such a characteristic, i.e., that the angular velocity is at right angles to the moment causing it, is said to be a *gyroscope.* The angular velocity ω_x is called the *angular velocity of precession* of the gyroscope, and it will be observed from equation (97) that as the spin velocity becomes large, the precessional velocity becomes small.

A gyroscope is often mounted in gimbals so that all rotation is about the center of mass (Fig. 7–22). Since the equation $\dot{\boldsymbol{H}} = \boldsymbol{M}$

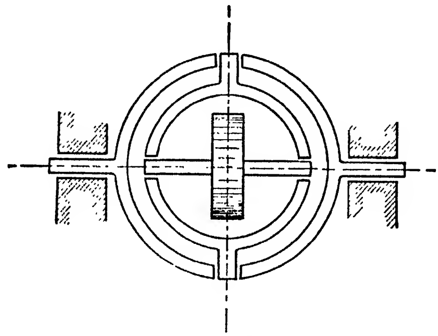


FIG. 7–22

may be written with respect to the center of mass, we see that the preceding equations are applicable to a gyroscope mounted in gimbals even though the center of mass is moving.

55. The Gyroscopic Compass. Consider a gyroscope that is mounted at the earth's surface in such a way that it is free to turn in any direction (Fig. 7–22). If no moment is applied, the axis of the gyroscope will maintain a fixed direction in space so that as the earth rotates about its axis the gyroscope axis will rotate relative to the earth. This is illustrated in Fig. 7–23a, in which we are looking due south at a gyroscope which is mounted on the equator.

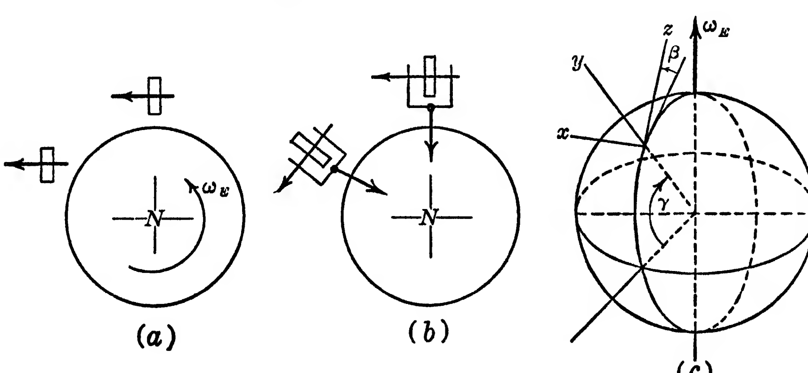


FIG. 7–23

The direction of spin of the gyro rotor is indicated by the vector. If now a small weight is attached to the spin axis below the center of mass, a moment is impressed upon the gyroscope by gravity as indicated in Fig. 7–23b. This torque, whose vector direction is parallel to the earth's axis, causes the spin axis to rotate toward the earth's axis. A device of this type will, therefore, point to true north and can be used as a compass. The preceding discussion gives a qualitative indication of the behavior of the gyrocompass, and we shall now show in more detail how the performance can be predicted from the equations of motion.

The effect of the pendulous mass is to constrain the spin axis of the gyroscope to move in a horizontal plane. We therefore take the xz plane (Fig. 7–23c) as the horizontal plane containing the spin axis, and the z-axis as the direction of spin of the rotating gyro disk. The location of the spin axis with respect to a meridian is given by the angle β, and γ is the latitude of the gyroscope, measured from the equator. The angular velocity of the earth is ω_E. The angular velocity of the xyz axes is then:

$$\begin{aligned}\omega_x &= -\omega_E \cos\gamma \sin\beta \\ \omega_y &= \omega_E \sin\gamma + \dot{\beta} \\ \omega_z &= \omega_E \cos\gamma \cos\beta\end{aligned}$$

To determine the motion of the spin axis in the horizontal plane with respect to the meridian, we use the general equation for the motion about the y-axis:

$$M_y = I_y\dot{\omega}_y + I_x\omega_x\omega_z - I_z\Omega\omega_x$$

which becomes:

$$0 = I_y\ddot{\beta} - I_x\omega_E^2 \cos^2\gamma \sin\beta \cos\beta + I_z\Omega\omega_E \cos\gamma \sin\beta$$

Since the angular velocity of the earth, ω_E, is very small compared to the spin velocity Ω, the term containing ω_E^2 may be neglected, and the equation of motion may be written:

$$\ddot{\beta} + \left(\frac{I_z}{I_y}\Omega\omega_E \cos\gamma\right)\sin\beta = 0$$

For small oscillations of the spin axis about the meridian we may put $\sin\beta \approx \beta$ and the equation becomes:

$$\ddot{\beta} + \left(\frac{I_z\Omega\omega_E \cos\gamma}{I_y}\right)\beta = 0$$

This is the equation of simple harmonic motion, from which it may be concluded that the spin axis oscillates about the meridian with a period

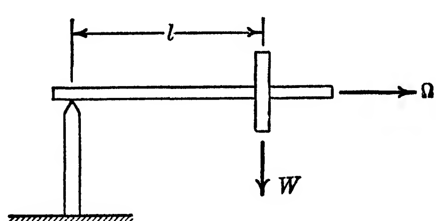
$$\tau = 2\pi \sqrt{\frac{I_y}{I_z \Omega \omega_E \cos \gamma}}$$

In practical applications of the gyro-compass, sufficient damping is introduced so that the oscillations are damped out with the spin axis finally lined up with the meridian.

The preceding discussion of gyroscopic motion illustrates a common procedure in the solution of dynamics problems. A completely general solution of such problems, which, starting from the general equations would consider all possible motions of the system, often leads to very complex analysis. Since for particular practical problems we are concerned with very special conditions, such as large spin velocities, we make use of these special conditions to simplify the equations of motion at the outset. It must always be realized, however, that such analyses are approximate and are applicable only when their conditions are satisfied.

PROBLEMS

242. A gyroscope having a rotor of weight W and moment of inertia I_z rotates about a horizontal axis which itself rotates in a horizontal plane as shown in the figure. The spin velocity of the gyro rotor is Ω, which is



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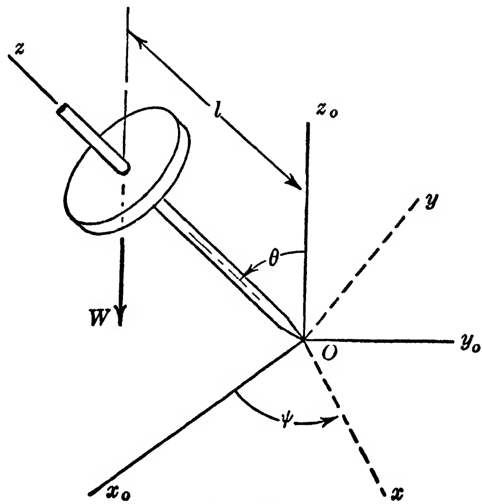
large compared with the other angular velocities of the system. Find the precessional angular velocity ω of the spin axis in the horizontal plane and show that $\omega \to 0$ as $\Omega \to \infty$.

243. A spinning top has a weight W and moment of inertia about the spin axis I_z and rotates about the fixed point O as shown in the diagram. Assuming that the spin velocity Ω of the top is large compared to the other angular velocities involved, show that the top will precess about the z_0-axis with an angular velocity $\frac{Wl}{I_z\Omega}$, if θ is constant.



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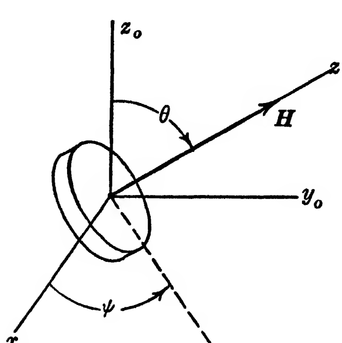
244. A gyroscope has a large constant spin velocity Ω. It may therefore be assumed that the total moment of momentum of the system is $H \approx I_z\Omega$ with a direction along the axis of spin. By computing $\frac{d\boldsymbol{H}}{dt}$ (the velocity of the end-point of the $\boldsymbol{H}$ vector) show from the general equation $\frac{d\boldsymbol{H}}{dt} = \boldsymbol{M}$ that $M_x = I_z\Omega\dot{\psi}\sin\theta$



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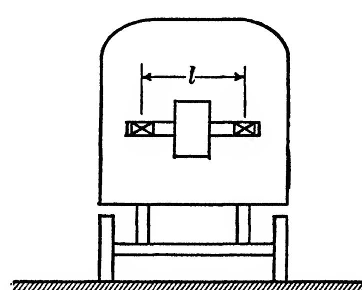

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245. The rotor of an electric motor is mounted in an electric locomotive as shown in the diagram. The locomotive travels with a velocity v around a curve of radius R. Find the gyroscopic forces exerted on the bearings of the rotor shaft.

246. Large gyroscopes have been used to stabilize ships against rolling. Show by a sketch how a gyroscope should be mounted so as to exert a stabilizing torque against rolling.

247. The propeller of an airplane has an angular velocity Ω and a moment of inertia I. The airplane is traveling with a velocity v in a horizontal circle of radius R. What are the magnitude and direction of the gyroscopic torque exerted by the propeller on the airplane?

248. When an automobile is rounding a curve at high speed, does the gyroscopic effect of the wheels tend to stabilize the car or overturn it?

249. Write the equation of conservation of energy for a gyroscope having a constant spin velocity and a potential energy V and obtain the equations:

$$\tfrac{1}{2}I_x(\omega_x^2 + \omega_y^2) + V = \text{constant}$$
$$\tfrac{1}{2}I_z\Omega^2 = \text{constant}$$

250.* Write the equation of conservation of moment of momentum about the z_0-axis of the gyroscope, and obtain the equation:

$$I \sin^2 \theta \frac{d\psi}{dt} + I_z\Omega \cos \theta = \text{constant}$$

56. D'Alembert's Principle. It was pointed out by D'Alembert that Newton's Second Law of Motion could be considered from a slightly different viewpoint by writing it in the form

$$\boldsymbol{F} + (-m\ddot{\boldsymbol{r}}) = 0$$

and treating the term $(-m\ddot{\boldsymbol{r}})$ as if it were a force. When this is done the terms $(-m\ddot{\boldsymbol{r}})$ are called *inertia forces*, and it should be particularly noted that inertia forces are not actual forces in the sense that the word "force" has been used in the preceding portion of this book. The concept of the inertia force makes it possible to apply the general methods of statics to the solution of dynamics problems, since Newton's Law may be written as $\Sigma \boldsymbol{F} = 0$ if it is understood that inertia forces are to be included in the summation.

This viewpoint can be extended to systems of particles and to rigid bodies. For any system of particles we have

$$\Sigma \boldsymbol{F}_i + \Sigma(-m_i\ddot{\boldsymbol{r}}_i) = 0$$

For a rigid body performing plane motion, the equations of motion are:

$$\Sigma \boldsymbol{F} = m\ddot{\boldsymbol{r}}_c$$
$$\Sigma \boldsymbol{M}_c = I_c\ddot{\theta}$$

These may be written:

$$\Sigma \boldsymbol{F} + (-\,m\ddot{\boldsymbol{r}}_c) = 0$$
$$\Sigma \boldsymbol{M}_c + (-\,I_c\ddot{\theta}) = 0$$

where $\ddot{\boldsymbol{r}}_c$ is the acceleration of the center of mass of the body, and I_c is the moment of inertia of the body about the center of mass. If we imply that the inertia force $(-m\ddot{\boldsymbol{r}}_c)$ and the inertia torque $(-I_c\ddot{\theta})$ are included in the summation, we have the equations in the same form as the equations of statics:

$$\Sigma \boldsymbol{F} = 0$$
$$\Sigma \boldsymbol{M} = 0$$

It should be noted that in this moment equation, the moments can be taken about any axis, as in statics, since the inertia torque has already been included with the appropriate moment of inertia about the mass center.

The introduction of the concept of an inertia force and an inertia torque does not, of course, represent any new information. For some problems, however, this method of writing the equation of motion leads to a convenient way of visualizing the dynamics of a situation, as will be illustrated in the following examples.

The concept of an inertia force can be combined with the principle of virtual displacements to give Lagrange's form of D'Alembert's principle. For example, the equation of motion for a particle is:

$$m\ddot{\boldsymbol{r}} - \boldsymbol{F} = 0$$

If, according to D'Alembert's notion, we consider this to be an equation of static equilibrium, the principle of virtual displacements states that:

$$(m\ddot{\boldsymbol{r}} - \boldsymbol{F}) \cdot \delta\boldsymbol{r} = 0$$

where $\delta \boldsymbol{r}$ is a virtual displacement. Since $\boldsymbol{F} \cdot \delta \boldsymbol{r}$ is the work done during the displacement $\delta \boldsymbol{r}$, and since

$$\frac{d}{dt}\left(\tfrac{1}{2}m\dot{r}^2\right) dt = \frac{d}{dt}\left(\tfrac{1}{2}m\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}\right) dt = m\ddot{\boldsymbol{r}} \cdot \delta \boldsymbol{r}$$

represents the increment change in kinetic energy, the equation is but another form of the work-energy equation. The general statement of D'Alembert's principle for a system of particles is:

$$\Sigma[(m\ddot{x}_i - F_{x_i})\delta x_i + (m\ddot{y}_i - F_{y_i})\delta y_i + (m\ddot{z}_i - F_{z_i})\delta z_i] = 0 \quad (98)$$

This is the equation from which Lagrange developed analytical mechanics. It introduces into dynamics the same advantage that the principle of virtual displacements introduces into statics —the conditions of equilibrium or of motion may be studied without introducing the constraining forces which may be acting.

EXAMPLE 1. Solve the problem of Example 1, Section 50, using the concept of the inertia force.

Solution. We draw a free-body diagram indicating the forces as solid vectors and the inertia forces as dotted vectors (Fig. 7–24).

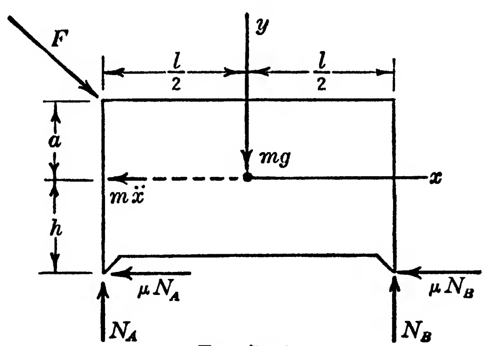


FIG. 7–24

The inertia force $-m\ddot{x}$ is shown at the center of mass of the system. Since the body is translating, there is no inertia torque acting on the system. The problem has now been reduced to a problem in statics, and the equations may be written:

$$\Sigma F_x = 0 = F_x - \mu N_A - \mu N_B - m\ddot{x}$$
$$\Sigma F_y = 0 = -F_y + N_A + N_B - mg$$
$$\Sigma M_A = 0 = -F_x(a + h) - mg\left(\frac{l}{2}\right) + lN_B + hm\ddot{x}$$

These equations lead directly to the solution given in Example 1, Section 50.

Example 2. A pulley of radius R and moment of inertia I supports two masses m_1 and m_2 fastened together by a rope as shown in Fig. 7–25. Find the equation of motion of the system.

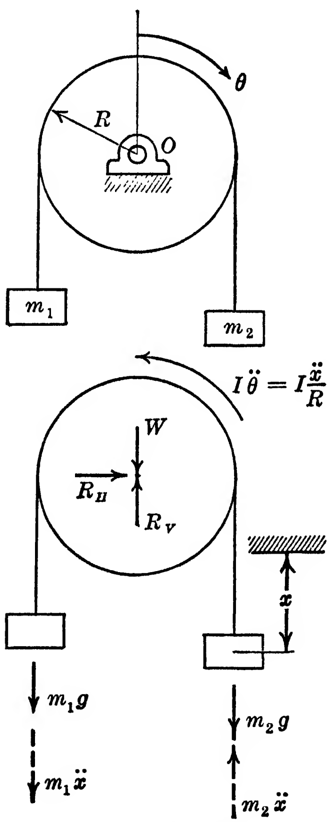


Fig. 7–25

Solution. We shall describe the motion of this single-degree of freedom system by the coordinate x, as shown in the free-body diagram. The inertia forces $-m_1\ddot{x}$ and $-m_2\ddot{x}$ are shown as dotted vectors, and the inertia torque $-I\ddot{\theta}$ is also indicated. The equation of motion can now be written:

$$\Sigma M_0 = m_2\ddot{x}R + \frac{I\ddot{x}}{R} + m_1\ddot{x}R + m_1 gR - m_2 gR = 0$$

From which:

$$\ddot{x} = \frac{(m_2 - m_1)g}{m_1 + m_2 + \dfrac{I}{R^2}}$$

Example 3. A thin hoop of radius R rotates about an axis through the center perpendicular to the plane of the hoop with a constant angular velocity ω. Find the circumferential tension force in the hoop.

Solution. We first draw a free-body diagram of one-half of the hoop (Fig. 7–26). Each element of mass of the hoop is acted upon by an inertia force directed as shown in the diagram. Consider an element of mass included by the angle $d\phi$ and let ρ be the mass per unit length of the hoop; then the inertia force is $\rho(R\omega^2)Rd\phi$ directed radially outward. We may now write:

$$\Sigma F_y = 0 = \int_0^\pi R^2\omega^2\rho \sin\phi \, d\phi - 2F$$

From which:

$$F = R^2\omega^2\rho$$

EXAMPLE 4. A type of governor mechanism used for the control of the speed of rotating shafts is shown in Fig. 7–27. A simple pendulum having a concentrated mass m and a length l is mounted on the rim of a wheel of radius R which rotates with an angular velocity ω. The pendulum is constrained by two springs which

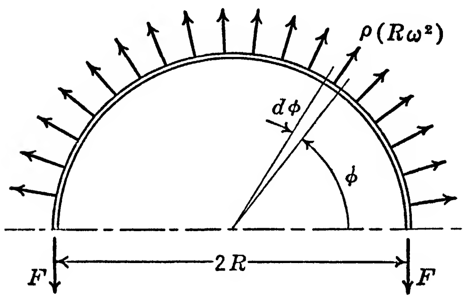


FIG. 7–26

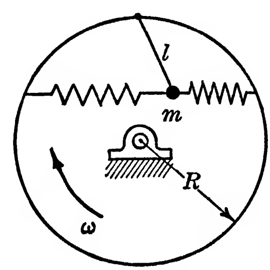


FIG. 7–27

are also attached to the rim of the wheel. For small displacements x, of the pendulum, the restoring force of the springs can be taken as $F_x = -kx$. When the wheel is rotating with a constant angular velocity, the pendulum remains in a radial position, but if the wheel accelerates or decelerates the pendulum moves to one side or the other of its neutral position. By allowing the pendulum displacement to control the power input to the system, the angular velocity can be regulated. Find the differential equation of motion of the pendulum with respect to the flywheel, in terms of angular velocity and acceleration of the flywheel.

Solution. Since we wish to express the absolute acceleration of m in terms of the relative motion, we use Equation (15):

$$\ddot{\mathbf{r}} = \ddot{\mathbf{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \ddot{\boldsymbol{\rho}}_r + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r$$

By D'Alembert's method the equation of equilibrium is:

$$\mathbf{F} - m\ddot{\mathbf{R}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) - m\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} - m\ddot{\boldsymbol{\rho}}_r - 2m\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r = 0$$

Letting the xy axes be fixed on the wheel (Fig. 7–28), each term of the equation is as follows:

$$\begin{aligned} \boldsymbol{F} &= -kl\sin\phi\,\boldsymbol{i} \\ -m\ddot{\boldsymbol{R}} &= -mR\dot{\omega}\boldsymbol{i} - mR\omega^2\boldsymbol{j} \\ -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho}) &= ml\omega^2\boldsymbol{e}_\rho \\ -m\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho} &= m\dot{\omega}l\boldsymbol{e}_\phi \\ -m\ddot{\boldsymbol{\rho}}_r &= ml\dot{\phi}^2\boldsymbol{e}_\rho - ml\ddot{\phi}\boldsymbol{e}_\phi \\ -2m\boldsymbol{\omega}\times\dot{\boldsymbol{\rho}}_r &= -2m\omega l\dot{\phi}\boldsymbol{e}_\rho \end{aligned}$$

A free-body diagram with all the inertia forces is shown in Fig. 7–29. The force in the pendulum rod now can be determined by statics and

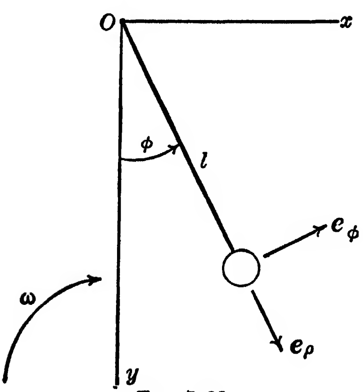


FIG. 7–28

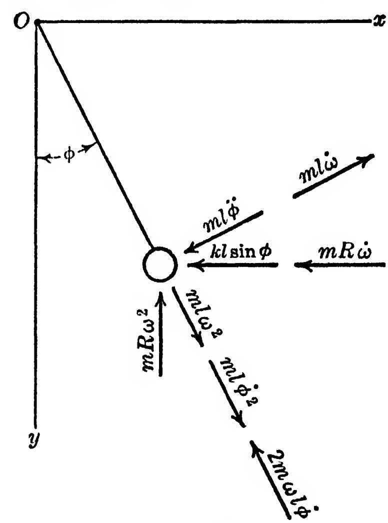


FIG. 7–29

the equation of motion can be derived by writing the moment equation $\Sigma M_0 = 0$.

$$-l(-ml\dot{\omega} + ml\ddot{\phi}) - l\cos\phi(kl\sin\phi + mR\dot{\omega}) + l\sin\phi mR\omega^2 = 0$$

For small oscillations we may set $\cos\phi \approx 1$, $\sin\phi \approx \phi$ and obtain:

$$\ddot{\phi} + \left(\frac{k}{m} - \frac{R\omega^2}{l}\right)\phi = \frac{l-R}{l}\dot{\omega}$$

This equation has the same form as the equation describing the vibration of a pendulum about a fixed point. If the spring constant is large so that $\dfrac{k}{m} > \dfrac{R\omega^2}{l}$ the pendulum will oscillate about its equilibrium position under the action of the term $\left(\dfrac{l-R}{l}\right)\dot{\omega}$, which acts like an exciting force.

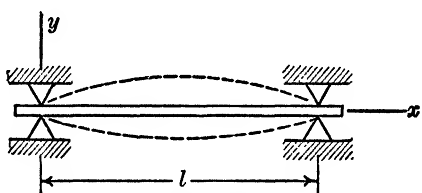
For the complete analysis of a control system of the above type, one would have to take account of the fact that the mechanisms connected to the pendulum and operated by it have dynamic characteristics which affect the behavior of the system. Considerable work has been done in recent years on control systems of all types, and much of this work is summarized in books on the theory of servo-mechanisms.

PROBLEMS

251. Derive the equation of motion and the forces at the support of a compound pendulum by using the inertia force method.

252. A slender steel rod of length l and radius r rotates about an axis through one end perpendicular to the bar with a constant angular velocity ω. Find the maximum tension force in the rod using the inertia force method. Find the numerical value of this force if $l = 3$ ft, $r = \frac{1}{4}$ in., and $\omega = 500$ rpm.

253. A steel beam of length l and weight w lb/ft is simply supported at each end. It is observed that the beam vibrates with a motion



PROB. 253

$$y = A \sin\left(\frac{\pi x}{l}\right) \sin pt$$

Find the expression for the maximum dynamic reaction which occurs at a support, using the concept of an inertia force. Find the numerical value of this force when $l = 20$ ft, $w = 20$ lb/ft, $A = \frac{1}{8}$ in., and the frequency of the vibration is 10 cycles per second.

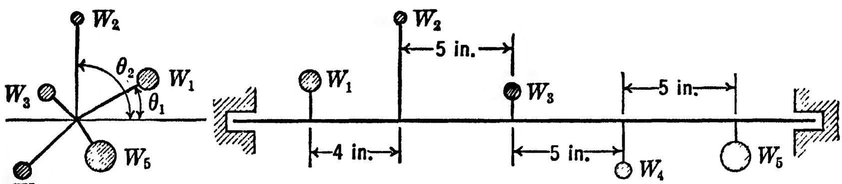
254. In tracking an airplane, a gun barrel swings with an angular velocity $\dot{\theta}$ and an angular acceleration $\ddot{\theta}$. The projectile velocity is v with respect to the gun. Find the lateral force exerted by the projectile on the gun, when the projectile is at a distance l from the axis of rotation.

255. Solve Problem 199 by using the concept of inertia force.

256. Solve Problem 200 by using the concept of inertia force.

257.* Five weights are attached to a rigid horizontal shaft as shown in the figure. The weights, radii, and angles locating the weights are given in the accompanying table. The system is to be balanced by the addition of two weights, one in the plane of W_1 and the other in the plane of W_5. Each balance weight has a radius of 2 in. Find the magnitudes and the angular positions of the balance weights. Show that the products of inertia are zero for the balanced system.

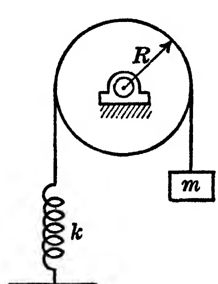
No.	W (lb)	r (in.)	θ
W_1	10	2	30°
W_2	5	3	90°
W_3	8	1	135°
W_4	8	2	225°
W_5	15	1	300°



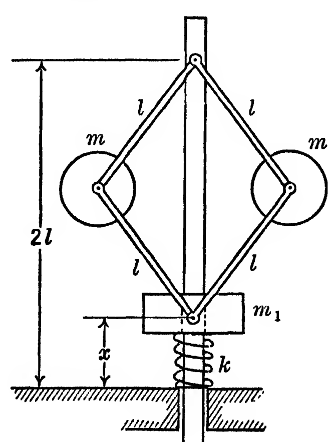
PROB. 257

258. A pulley having a moment of inertia I about its axis of rotation supports a rope which carries a mass m at one end, while the other end is connected to a spring of spring constant k as shown in the diagram. Find the period of oscillation of the system. (Assume that the rope does not slip on the pulley.)

259. A governor is constructed as is shown in the diagram. The assembly of four linked bars of equal length l rotates with an angular velocity ω about a vertical axis. The mass m_1 slides on the vertical axis and is restrained by a spring force $-kx$. Find the displacement x of the mass m_1 in terms of the constant angular velocity ω of the system.

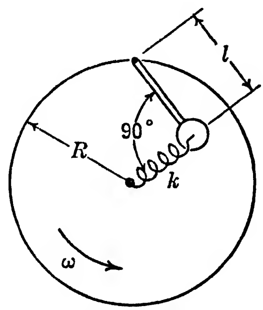


PROB. 258



PROB. 259

260. A governor is mounted on a rotating wheel as is shown in the figure. When operating in a steady-state condition (ω = constant) the angle between the spring and the pendulum is 90°. The total mass of



Prob. 260

the pendulum is m, and the moment of inertia of the pendulum about its suspension axis is I. Find the equation of motion of the pendulum for small displacements from the steady-state position.

CHAPTER VIII

NON-RIGID SYSTEMS OF PARTICLES

But I consider philosophy rather than the arts and write not concerning manual but natural powers, and consider chiefly those things which relate to gravity, levity, elastic forces, the resistance of fluids, and the like forces, whether attractive or impulsive.—I. Newton, *Principia Philosophiae* (1686).

The analysis of the dynamics of systems of particles is greatly influenced by the characteristics of the particular system being studied. For example, solid bodies, fluids, and gases are all systems of particles and as such can be treated by the general methods of dynamics which have already been discussed. The physical characteristics of these various systems differ so greatly, however, that the analysis must be handled in a distinctive fashion for the various materials. The analysis is, of course, always based upon the equations of motion, but it is developed in different ways in order to take advantage of the particular characteristics of a given system. In the following sections we shall give a brief discussion of some of the methods which can be applied to non-rigid systems of particles, such as fluids and gases.

57. The Equations of Motion of a Non-viscous Fluid. In the analysis of fluid motion it is possible to introduce certain simplifications because of the special properties of fluids. First, it is not necessary to treat the fluid as being composed of discrete particles. Instead, we consider it to be composed of elements of volume dV, an element having a mass $\rho\, dV$, where ρ is the density of the fluid. Second, we make use of the fact that the pressure at a point in a fluid is the same in all directions:

$$p_x = p_y = p_z = p$$

The equation of motion for an element of fluid is:

$$(\rho\, dV)\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} \tag{99}$$

where $\boldsymbol{v}$ is the velocity of the element and $\boldsymbol{F}$ is the resultant force acting upon the element. In rectangular coordinates the equations of motion are:

$$(\rho\, dx\, dy\, dz)\ddot{x} = F_x, \quad \text{etc.}$$

In Fig. 8–1a is shown a free-body diagram of a fluid element. If we assume a perfect or non-viscous fluid, there are no viscous shearing forces on the sides of the element, so that the only forces

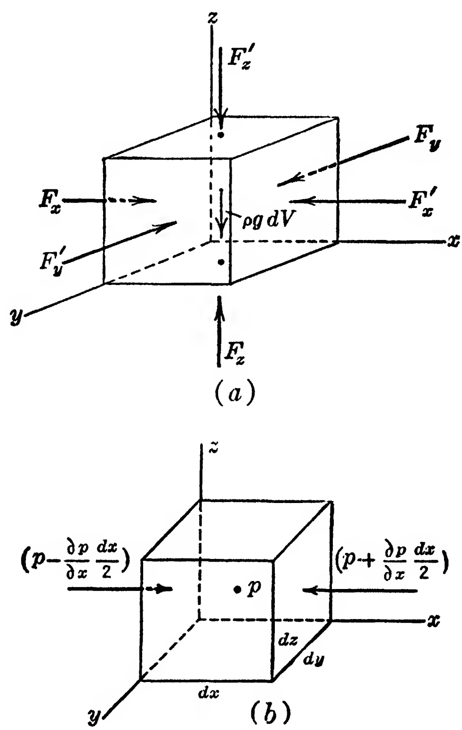


FIG. 8–1

are the normal forces acting on the faces, and the gravity force acting at the center of the element. In Fig. 8–1b are shown the pressures exerted on two faces of the element. The pressure at the center of the element is taken to be p. Then on the right face of the element, at a distance $\frac{dx}{2}$ from the center, the pressure is $\left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)$, whereas on the left face the pressure is $\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)$.

There are similar expressions for pressures on the other four sides of the element. Since the force in the x-direction is given by the pressure multiplied by the area on which it acts:

$$F_x = \left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right) dy\, dz - \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right) dy\, dz$$

$$F_x = -\frac{\partial p}{\partial x}\, dx\, dy\, dz$$

The minus sign occurs because for $\frac{\partial p}{\partial x}$ positive the resultant force acts in the negative x-direction. The x-component of the equation of motion now may be written:

$$(\rho\, dx\, dy\, dz)\ddot{x} = -\frac{\partial p}{\partial x}\, dx\, dy\, dz$$

or

$$\rho\ddot{x} = -\frac{\partial p}{\partial x}$$

In the y and z directions the equations are found in the same way to be:

$$\rho\ddot{y} = -\frac{\partial p}{\partial y}; \quad \rho\ddot{z} = -\left(\frac{\partial p}{\partial z} + \rho g\right)$$

or, in vector notation:

$$\rho\dot{\boldsymbol{v}} = -\left[\frac{\partial p}{\partial x}\boldsymbol{i} + \frac{\partial p}{\partial y}\boldsymbol{j} + \left(\frac{\partial p}{\partial z} + \rho g\right)\boldsymbol{k}\right] \tag{100}$$

With this equation it is possible to determine the motion when the pressure distribution is known, or to determine the pressure distribution when the motion is known.

There is an obvious difficulty in the practical application of this equation, since, in general, neither the motion nor the pressure in the fluid will be known. The only facts that are usually known are certain boundary values, such as the pressures at free surfaces or the directions of the velocity at a boundary, etc. The problem is to find the motion that will satisfy both the differential equation of motion and the particular boundary conditions. In general, this is a difficult problem, and many special techniques have been developed to analyze such fluid dynamics problems. These methods are treated in texts on Hydrodynamics and Aerodynamics. In the following paragraphs, we shall discuss only two of such special methods, that of the energy equation, and that of the momentum equation.

58. The Energy Equation. The equation of motion for an element of the fluid can be integrated directly to obtain the work-energy equation. Assuming an incompressible fluid, i.e., $\rho =$ constant, forming the dot product with $\boldsymbol{dr}$ and integrating, we obtain:

$$\int_1^2 \rho \dot{\boldsymbol{v}} \cdot \boldsymbol{dr} = -\int_1^2 \left[\frac{\partial p}{\partial x}\boldsymbol{i} + \frac{\partial p}{\partial y}\boldsymbol{j} + \left(\frac{\partial p}{\partial z} + \rho g\right)\boldsymbol{k}\right] \cdot \boldsymbol{dr}$$

$$\int_1^2 \rho \boldsymbol{v} \cdot \boldsymbol{dv} = -\int_1^2 \left[\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz + \rho g\, dz\right]$$

$$\tfrac{1}{2}\rho v_2^2 - \tfrac{1}{2}\rho v_1^2 = -\int_1^2 dp - \int_1^2 \rho g\, dz$$

$$= -(p_2 - p_1) - \rho g(z_2 - z_1)$$

Collecting terms, this equation may be written

$$\tfrac{1}{2}\rho v_1^2 + p_1 + \rho g z_1 = \tfrac{1}{2}\rho v_2^2 + p_2 + \rho g z_2$$

This states that the sum of the three terms is the same at all points, or:

$$\tfrac{1}{2}\rho v^2 + p + \rho g z = \text{constant} \tag{101}$$

This equation applies to an element of fluid no matter what type of fluid motion is involved. In fluid-mechanics problems, however, it is not in general possible to follow the motion of one particular element, and the usefulness of the above equation lies in the fact that, subject to certain restrictions, the equation can be applied from point to point in a fluid. The nature of these restrictions may be shown in the following way. A stream-line in a fluid is defined as a line which has at every point the direction of the velocity of the fluid at that point. If we assume steady flow of the fluid, that is, no change with time, then at any point along a stream-line, successive fluid elements will have identical characteristics. The above equation can thus be applied between any two points on a stream-line in a steady flow. It will be seen that this leads to the well-known Bernoulli's Equation, which is widely used in fluid-mechanics problems.

EXAMPLE. There is a steady flow of fluid from a reservoir through a pipe as shown in Fig. 8–2. Assuming no energy lost in

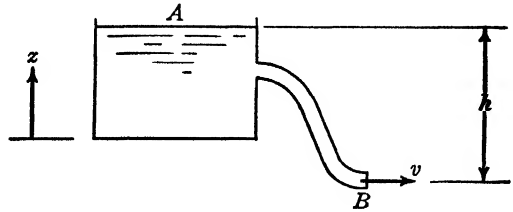


FIG. 8–2

the system, find the velocity v_B with which the fluid issues from the pipe.

Solution. Applying the energy equation to the two points A and B gives:

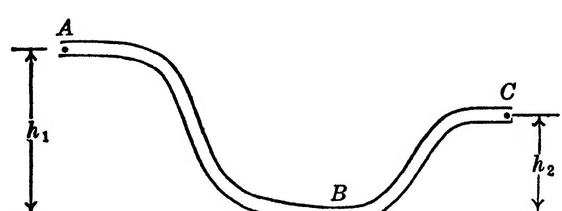
$$\tfrac{1}{2}\rho v_A^2 + p_A + \rho g z_A = \tfrac{1}{2}\rho v_B^2 + p_B + \rho g z_B$$

At point A the velocity of the fluid is very small so that to a good approximation $v_A = 0$. The pressure at A is atmospheric pressure, and we can assume with good accuracy that the pressure at B is also atmospheric. Thus:

$$\tfrac{1}{2}\rho v_B^2 = \rho g(z_A - z_B) = \rho g h$$
$$v_B = \sqrt{2gh}$$

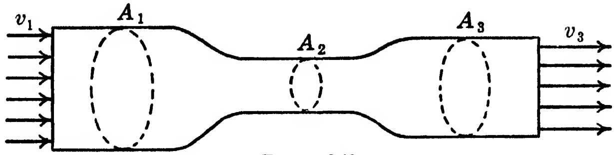
PROBLEMS

261. A non-viscous incompressible fluid flows through a pipe of uniform cross section with a velocity v. If the pressure at point A is p, find the pressure at point B and at point C.



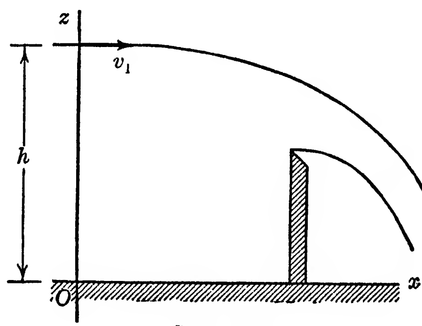
PROB. 261

262. A non-viscous incompressible fluid flows through a pipe of cross-sectional area A_1 with a velocity v_1. If the pressure at A_1 is p_1 find the pressure at a point where the area is A_2. (Neglect gravity forces in this problem.)



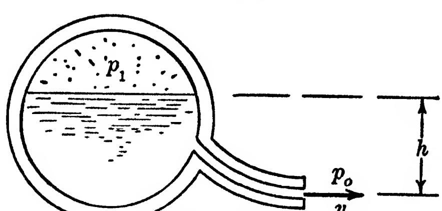
PROB. 262

263. Fluid flows over a weir as shown in the diagram. By applying the energy equation to elements in the top surface of the fluid, express the velocity at any point in the surface as a function of z.



PROB. 263

264. A pressure vessel is partly filled with liquid and partly with gas under a pressure p_1. Find the discharge velocity v.



PROB. 264

59. The Momentum Equation. In the preceding treatment of fluid motion we selected a particular element of fluid and studied the forces on and the motion of that element. This procedure is called the Lagrangian method because of the extensive use made of it by Lagrange. It is also possible to adopt a somewhat different point of view, by considering a fixed point in space and observing the motion of the fluid as it passes that point. This was the method used by Euler. For certain problems the Eulerian method has advantages, and we shall adopt it for the following development of the momentum equation. It should be noted that in this derivation no restrictions are imposed upon the compressibility or viscosity of the fluid.

The equation of motion of an element of fluid of density ρ and volume dV is:

$$(\rho\, dV)\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F}$$

where $\boldsymbol{F}$ is the resultant force acting upon the element, and $\boldsymbol{v}$ is the velocity of the element. Let us consider a number of such elements which lie along a stream-line in the fluid (Fig. 8–3). We

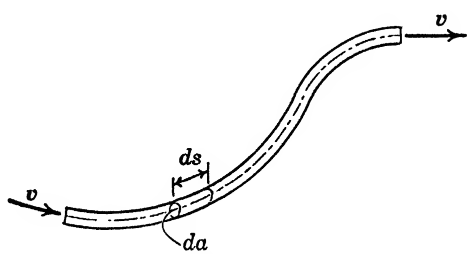


FIG. 8–3

select a volume element with cross-sectional area da, normal to the stream-line, and length ds along the stream-line. A volume having its length along a stream-line and an infinitesimally small cross-sectional area is called a stream-tube. At points along the stream-tube the velocity $\boldsymbol{v}$ is a function of both the space coordinate and the time, $\boldsymbol{v} = \boldsymbol{f}(s, t)$ so that we may write:

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{\partial \boldsymbol{v}}{\partial s}\frac{ds}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{\partial \boldsymbol{v}}{\partial s}v$$

where $\frac{ds}{dt}$ is the scalar magnitude of the velocity along the stream-line. The term $\frac{\partial \boldsymbol{v}}{\partial t}$ represents the time rate of change of $\boldsymbol{v}$ at a fixed point in space, and $\frac{\partial \boldsymbol{v}}{\partial s}v$ represents the space rate of change of velocity at a fixed time. The equation of motion may thus be written:

$$(\rho\, dV)\left(\frac{\partial \boldsymbol{v}}{\partial t} + \frac{\partial \boldsymbol{v}}{\partial s}v\right) = \boldsymbol{F}$$

Referring to Fig. 8–4, consider a definite region of space of volume V through which fluid is flowing. Integrating the preceding equation over this volume we obtain:

$$\Sigma \boldsymbol{F} = \int_V \rho \frac{\partial \boldsymbol{v}}{\partial t} \, dV + \int_V \rho v \frac{\partial \boldsymbol{v}}{\partial s} \, da \, ds$$

where $\Sigma \boldsymbol{F}$ is the resultant force acting upon the fluid within the volume V, and $da \, ds = dV$ is the volume of a length ds of the

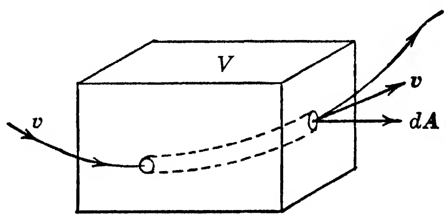


FIG. 8–4

stream-tube shown in Fig. 8–4. We shall now transform these integrals into forms which can be readily evaluated.

Using the formula for the derivative of a product, we have:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = \frac{\partial}{\partial t} (\rho \boldsymbol{v}) - \boldsymbol{v} \frac{\partial \rho}{\partial t}$$

$$\rho v \frac{\partial \boldsymbol{v}}{\partial s} \, da = \frac{\partial}{\partial s} (\rho v \boldsymbol{v} \, da) - \boldsymbol{v} \frac{\partial}{\partial s} (\rho v \, da)$$

Substituting these expressions into the integrals gives:

$$\Sigma \boldsymbol{F} = \int \left\{ \frac{\partial}{\partial t} (\rho \boldsymbol{v}) \, dV + \frac{\partial}{\partial s} (\rho v \boldsymbol{v} \, da) \, ds - \boldsymbol{v} \left[\frac{\partial \rho}{\partial t} \, dV + \frac{\partial}{\partial s} (\rho v \, da) \, ds \right] \right\}$$

It can be shown that the term in square brackets is equal to zero, by virtue of the conservation of mass. Consider an element of volume dV and of length ds as shown in Fig. 8–5. The rate at which mass is accumulating in this element, due to changing ρ, is

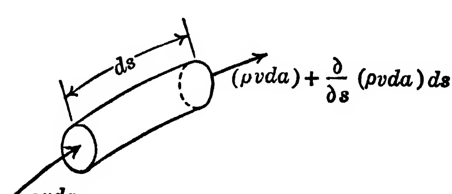


FIG. 8–5

$\frac{\partial \rho}{\partial t}\, dV$. This must be equal to the difference between the rate of inflow of mass and the rate of outflow:

$$\frac{\partial \rho}{\partial t}\, dV = \rho v\, da - \left[(\rho v\, da) + \frac{\partial}{\partial s}(\rho v\, da)\, ds\right]$$

from which:

$$\frac{\partial \rho}{\partial t}\, dV + \frac{\partial}{\partial s}(\rho v\, da)\, ds = 0$$

This is usually called the continuity equation.

The expression for the resultant force acting upon the fluid within V thus reduces to:

$$\Sigma \boldsymbol{F} = \int_V \frac{\partial}{\partial t}(\rho \boldsymbol{v})\, dV + \int_V \frac{\partial}{\partial s}(\rho v \boldsymbol{v}\, da)\, ds$$

The first integral may be written as:

$$\int \frac{\partial}{\partial t}(\rho \boldsymbol{v})\, dV = \frac{\partial}{\partial t}\int \rho \boldsymbol{v}\, dV = \frac{d\mathfrak{M}}{dt}$$

where $\mathfrak{M}$ is the resultant momentum of the fluid within V.

The second integral may be integrated along the length s of the stream-tube, to give:

$$\int \frac{\partial}{\partial s}(\rho v \boldsymbol{v}\, da)\, ds = \int (\rho v \boldsymbol{v}\, da)\Big|_{s_1}^{s_2}$$

The notation $(\rho v \boldsymbol{v}\, da)\Big|_{s_1}^{s_2}$ indicates that this term is to be evaluated at the two end points of the stream-tube; that is, at the points where the stream-tube intersects the boundary of the volume. The resulting area integral is to be evaluated, therefore, over the areas of the ends of all the stream-tubes lying on the surface of the volume. We note that the direction of a stream-tube as it intersects the boundary is in general inclined at some angle to the normal to the boundary surface. If the vector $d\boldsymbol{A}$, normal to the surface and directed out of the volume, describes the increment of surface area, then on the surface $v\, da = \boldsymbol{v} \cdot d\boldsymbol{A}$ and the area integral may be written:

$$\int_A \rho \boldsymbol{v}(\boldsymbol{v} \cdot d\boldsymbol{A})$$

Since $\rho\boldsymbol{v} \cdot d\boldsymbol{A}$ represents the mass flow out through $d\boldsymbol{A}$ per unit time, the term $\rho\boldsymbol{v}(\boldsymbol{v} \cdot d\boldsymbol{A})$ represents the rate at which momentum is flowing out through $d\boldsymbol{A}$. *The integral thus represents the net rate of outflow of momentum through the surface of V.*

We thus have obtained the following expression for the resultant force acting upon the fluid within the volume V:

$$\Sigma \boldsymbol{F} = \frac{d\mathfrak{M}}{dt} + \int_A \rho\boldsymbol{v}(\boldsymbol{v} \cdot d\boldsymbol{A}) \tag{102}$$

This equation is just Newton's law applied to a system which is losing or gaining material.

The foregoing analysis can be repeated, beginning with the equation of moment of momentum:

$$\boldsymbol{r} \times \left(\rho\, dV \frac{d\boldsymbol{v}}{dt}\right) = \boldsymbol{r} \times \boldsymbol{F}$$

and the following result will be obtained:

$$\Sigma \boldsymbol{M}_t = \frac{d\boldsymbol{H}}{dt} + \int_A \boldsymbol{r} \times \rho\boldsymbol{v}(\boldsymbol{v} \cdot d\boldsymbol{A}) \tag{103}$$

where $\Sigma \boldsymbol{M}_t$ is the resultant moment acting on the fluid within the volume V, and $\boldsymbol{H}$ is the resultant moment of momentum of the fluid within V.

The essence of the momentum flow method is that instead of observing a particular mass of fluid during its motion, attention is focused on a region of space. By noting the inflow and outflow of momentum and the change of total momentum within the volume it is possible to deduce the resultant force acting upon the fluid within the volume.

Example 1. A pipe of uniform cross-sectional area A has a horizontal offset l as shown in Fig. 8–6a. Fluid flows through the

Fig. 8–6a

pipe with a uniform velocity v. Find the moment exerted on the pipe by the fluid.

Solution. Considering the flow of momentum through the volume of the pipe indicated in the figures, and evaluating the integral $\int \rho \boldsymbol{v}(\boldsymbol{v} \cdot d\boldsymbol{A})$, we obtain the two vectors of magnitude $\rho v^2 A$ shown in Fig. 8–6a. Note that the directions are such that both these vectors are pointing out of the volume. There is thus

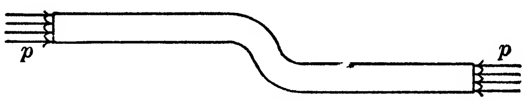


FIG. 8–6b

a counterclockwise moment of magnitude $\rho v^2 A l$ exerted upon the fluid within the volume. This moment is the resultant of the moment due to the pressure forces, and the moment M_p which is applied to the fluid by the pipe (Fig. 8–6b). Thus:

$$\rho v^2 A l = -pAl + M_p$$

The moment exerted on the pipe by the fluid is equal and opposite to M_p, and is hence a clockwise moment of magnitude $(\rho v^2 + p)Al$.

EXAMPLE 2. Fluid flows with a uniform velocity v through a pipe of uniform cross-sectional area in a horizontal plane as shown in Fig. 8–7. Find the external force which must be applied to the pipe to maintain the system in equilibrium.

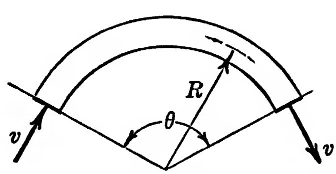


FIG. 8–7

Solution. In Fig. 8–8 are shown the momentum flow vectors $\int \rho \boldsymbol{v}(\boldsymbol{v} \cdot d\boldsymbol{A})$ and the pressure forces acting upon the volume of fluid. The resultant momentum-flow vector is equal to the force acting upon the fluid within the pipe as shown in (b). This resultant force is the sum of the two pressure forces, and the force F_p exerted by the pipe on the fluid. This force F_p can thus be found as shown in (c). The fluid exerts a force on the pipe which is equal and opposite to F_p, and hence the external force which must be applied to the pipe in order to maintain it in equilibrium is F_p, directed as shown in (c).

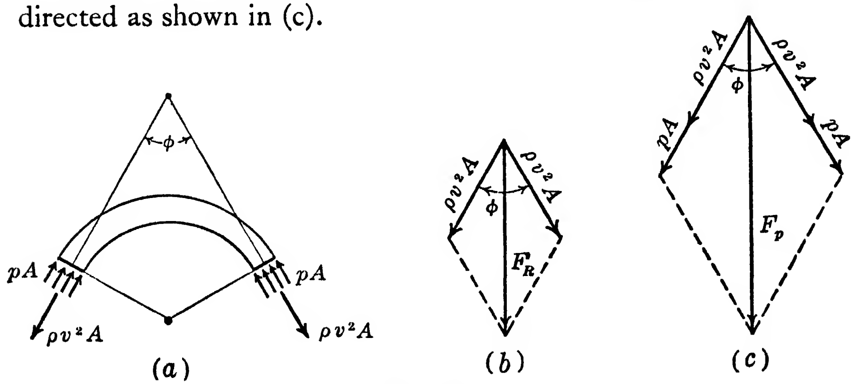


FIG. 8–8

EXAMPLE 3. Water is ejected from a container by means of gas pressure behind a piston (Fig. 8–9). What external force is required to hold the container motionless?

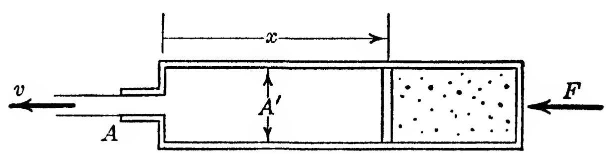


FIG. 8–9

Solution. The force F is equal to the resultant force on the fluid within the container so that we may write:

$$\boldsymbol{F} = \frac{d\mathfrak{M}}{dt} + \int \rho \boldsymbol{v}(\boldsymbol{v} \cdot d\boldsymbol{A})$$

Neglecting the momentum of the gas and piston and assuming uniform velocity in the water, the total momentum within the container is

$$\mathfrak{M} = A'\rho x\dot{x}\boldsymbol{i}$$

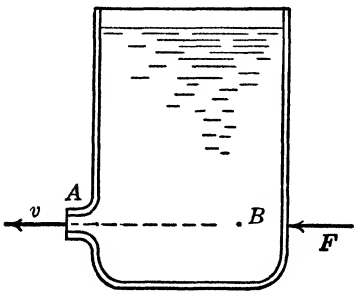
The rate of outflow of momentum is $-\rho Av^2\boldsymbol{i}$, but since $Av = A'\dot{x}$ we may write:

$$\boldsymbol{F} = \left[\frac{d}{dt}(\rho A' x\dot{x}) - \frac{\rho}{A}(A'\dot{x})^2\right]\boldsymbol{i}$$

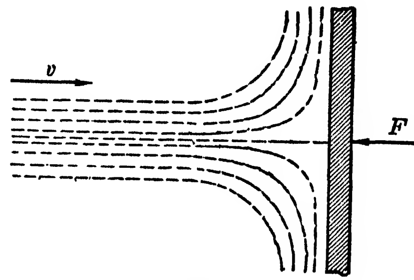
If the orifice is small, that is, if $\dfrac{A'}{A}$ is large, then the first term in the expression for $\boldsymbol{F}$ may be neglected.

PROBLEMS

265. Fluid is discharged from a tank with velocity v through an outlet of area A. Show that in order to maintain equilibrium of the system an external force $F = 2(p_B - p_A)A$ must act as shown in the figure, where p_A is the atmospheric pressure, and p_B is the pressure within the tank at the same elevation as the outlet, at a point where the fluid velocity is zero.



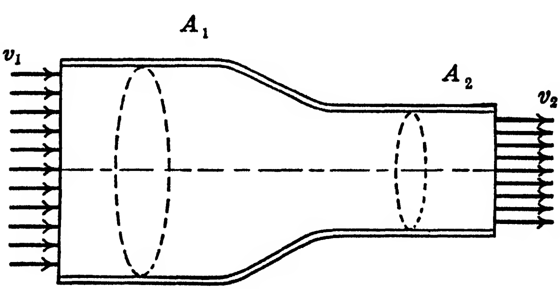
PROB. 265



PROB. 266

266. A stream of fluid impinges on a stationary surface as shown in the diagram. Find the force exerted by the fluid on the surface, by the method of the preceding section.

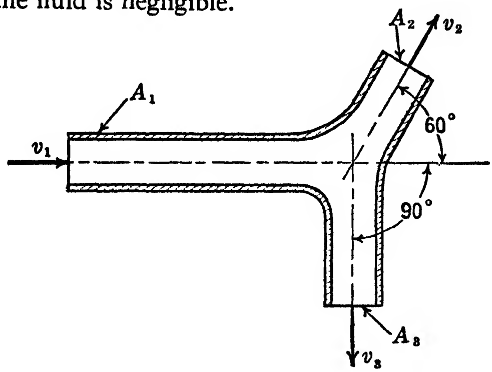
267. An incompressible fluid flows through a pipe which has a change in cross section as shown. Both ends of the pipe are at the same eleva-



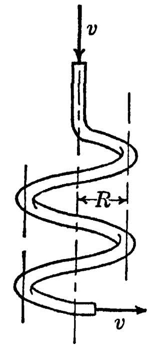
PROB. 267

tion. Assuming that the velocity is uniform across the area of both sections, find the force exerted on the pipe by the fluid.

268. Three pipes each of uniform cross section lie in a horizontal plane and converge at a point as shown in the figure. Fluid flows through A_1 with a velocity v_1, and out of A_2 and A_3 with velocities v_2 and v_3. The dimensions of the system are such that $A_2v_2 = \frac{1}{2}A_1v_1$. Find the resultant force required to hold the joint in equilibrium, assuming that the velocities are uniform across the sections of the pipe, and that the pressure in the fluid is negligible.



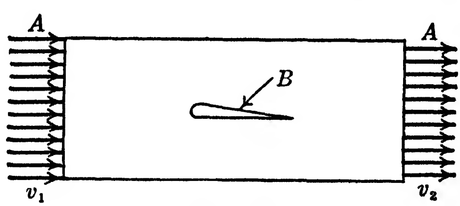
PROB. 268



PROB. 269

269. A tube of uniform cross-sectional area A is wound into a helix as shown in the figure. At the top of the helix the straight portion of the tube coincides with the axis of the helix. At the bottom the tube discharges water into the air with a horizontal velocity v normal to the radius R, at a distance R from the axis. Find the torque about the axis of the helix exerted by the fluid on the tube, if the tube is held stationary.

270. A frictionless compressible fluid flows through a straight pipe. An object B is held in a fixed position in the stream. Assuming that there



PROB. 270

is steady flow, find the relation between the force exerted by the fluid on B, and the velocities, densities, and pressures of the fluid at the two ends of the pipe.

271. A closed box of weight W is suspended from a spring balance. A bird of weight P is placed on the floor of the box. The scale then reads $W + P$. If the bird flies around in the box at a constant elevation, what will be the effect on the scale reading?

272. An airplane weighing W pounds flies with a uniform horizontal velocity v. What is the effect of this motion upon the air pressure on the ground?

273. In Example 3 of the preceding section the expression for the resultant force is

$$F = +\rho A' x\ddot{x} + \rho A'\dot{x}^2 - \frac{\rho}{A}(A'\dot{x})^2$$

Explain the physical significance of these three terms.

60. The Momentum Equation for an Accelerating Volume. In the preceding section we considered the problem of flow of a fluid through a stationary volume. We shall now consider the more general problem in which the volume itself is accelerating. This is, for example, the type of problem involved in an analysis of the dynamics of rocket flight.

Let the absolute velocity of a point in the fluid be $\boldsymbol{v}$ and consider a coordinate system fixed in the moving volume. If $\boldsymbol{v}_r$ is the velocity of a point in the fluid relative to the moving coordinate system and $\boldsymbol{u}$ is the absolute velocity of the corresponding point in the coordinate system, then $\boldsymbol{v} = \boldsymbol{v}_r + \boldsymbol{u}$. Let s be measured along a stream-line in the moving coordinate system. By a stream-line in the moving coordinate system we mean a line which has everywhere the direction of $\boldsymbol{v}_r$. We may now write:

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{\partial \boldsymbol{v}}{\partial s}\frac{ds}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + v_r \frac{\partial \boldsymbol{v}}{\partial s}$$

The resultant external force acting upon the mass contained within the volume is:

$$\Sigma \boldsymbol{F} = \int \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + v_r \frac{\partial \boldsymbol{v}}{\partial s}\right) dV$$

This integral may be transformed in the same way as the corresponding integral was transformed for the stationary volume, and the result is:

$$\Sigma \boldsymbol{F} = \frac{d\mathfrak{M}}{dt} + \int_A \rho \boldsymbol{v}(\boldsymbol{v}_r \cdot d\boldsymbol{A}) \tag{104}$$

where $\mathfrak{M}$ is the total momentum of the mass within the volume at any instant, and the integral represents the net rate of outflow of momentum from the volume.

From the moment of momentum equation, we may derive the equation:

$$\Sigma \boldsymbol{M}_t = \frac{d\boldsymbol{H}}{dt} + \int_A \boldsymbol{r} \times \rho \boldsymbol{v}(\boldsymbol{v}_r \cdot d\boldsymbol{A}) \tag{105}$$

where $\boldsymbol{H}$ represents the total moment of momentum of the mass within the volume under consideration, and $\Sigma \boldsymbol{M}_t$ is the resultant external torque acting upon the material within the volume. $\boldsymbol{H}$ and $\Sigma \boldsymbol{M}_t$ may be measured either with respect to a fixed point or with respect to the moving center of mass of the system. If $\boldsymbol{H}$ and $\Sigma \boldsymbol{M}_t$ are measured with respect to the moving center of mass, however, the velocity $\boldsymbol{v}$ appearing in Equation (105) is no longer the absolute velocity as used in the preceding analysis, but is the velocity of the element with respect to the center of mass.

Example 1. A rocket travels in straight horizontal flight. The mass of fuel burned per unit time in the rocket is k, so that the total mass of the rocket at any time is $m = m_0 - kt$. The exhaust velocity of the jet relative to the rocket is $\boldsymbol{v}_e$, and the pressure in the jet of area A is p. The mass of the shell of the rocket (total mass minus propellant mass) is M. Assuming that gravity forces and drag forces are negligible, derive the differential equation of motion of the rocket, and find the velocity of the rocket as a function of time.

Solution. We shall first find the equation of motion of the rocket by applying the momentum-flow principles to a volume which includes all the unburned propellant but excludes the shell of the rocket. We are therefore to consider only the volume enclosed in the dotted lines in Fig. 8–10. The mass within the volume is $m - M$, and we may write:

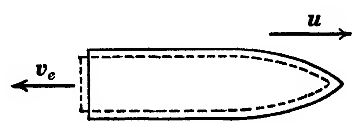


Fig. 8–10

$$\Sigma \boldsymbol{F} = \frac{d}{dt}[(m - M)\boldsymbol{v}_c] + \int_A \rho(\boldsymbol{u} - \boldsymbol{v}_e)(\boldsymbol{v}_e \cdot d\boldsymbol{A})$$

where $\boldsymbol{v}_c$ is the velocity of the center of mass of the fluid within the volume, which, to a very good approximation, may be taken as $\boldsymbol{u}$, the velocity of the rocket. The forces acting upon the rocket include the pressure force pA and the force F_s which the shell exerts upon the volume. We thus have:

$$pA + F_s = u\frac{d}{dt}(m - M) + (m - M)\dot{u} + \rho(u - v_e)v_eA$$

but $\frac{d}{dt}(m - M) = -\rho v_e A$, the rate at which mass is being ejected from the volume, so that:

$$pA + F_s = (m - M)\dot{u} - \rho v_e^2 A$$

The force $F_{s'}$ exerted by the material within the volume upon the rocket shell is equal and opposite to F_s; hence:

$$F_{s'} = pA - (m - M)\dot{u} + \rho v_e^2 A$$

This force $F_{s'}$ is just equal to the mass of the rocket shell multiplied by the acceleration of the rocket shell, so that:

$$M\dot{u} = pA - m\dot{u} + M\dot{u} + \rho v_e^2 A$$

and the equation of motion finally becomes:

$$m\dot{u} = \rho v_e^2 A + pA$$

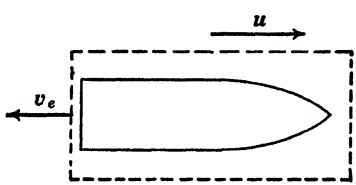


FIG. 8–11

This same equation can be obtained in a more direct manner by including the shell of the rocket within the volume under consideration, as is shown by the dotted line in Fig. 8–11. The only external force now acting on the material within the volume is the pressure force pA, and we have:

$$\Sigma \boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v}_c) + \int_A \rho(\boldsymbol{u} - \boldsymbol{v}_e)(\boldsymbol{v}_e \cdot \boldsymbol{dA})$$

$$pA = m\dot{u} - \rho v_e^2 A$$

which gives directly:

$$m\dot{u} = \rho v_e^2 A + pA \tag{106}$$

This equation may be integrated by putting it into the form:

$$du = (\rho v_e^2 A + pA)\frac{dt}{(m_0 - kt)}$$

from which, assuming that the rocket starts from rest, and putting $k = \rho v_e A$, we obtain:

$$u = \left(\frac{k^2 + \rho p A^2}{\rho A k}\right)\log\left(\frac{m_0}{m_0 - kt}\right)$$

EXAMPLE 2. A simplified water turbine made of two sections of curved pipe rotates with an angular velocity ω and discharges water at a rate of $2v_rA$ ft³/sec. The turbine does work against a torque $\boldsymbol{B}$ as shown in Fig. 8–12. If $\boldsymbol{v}_r$ and $\boldsymbol{B}$ are specified, what is the rate of doing work?

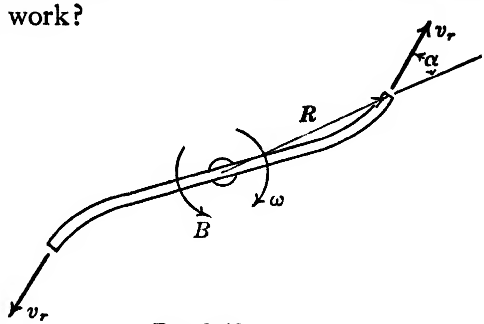


FIG. 8–12

Solution. The angular velocity $\boldsymbol{\omega}$ of the system can be determined from the equation

$$\Sigma \boldsymbol{M}_t = \frac{d\boldsymbol{H}}{dt} + \int \boldsymbol{r} \times \rho \boldsymbol{v}(\boldsymbol{v}_r \cdot d\boldsymbol{A})$$

If we consider only the steady-state motion of the system, $\boldsymbol{H} =$ constant and $\dfrac{d\boldsymbol{H}}{dt} = 0$. The equation therefore becomes:

$$\boldsymbol{B} = 2\boldsymbol{R} \times \rho \boldsymbol{v}(v_rA)$$

The absolute velocity $\boldsymbol{v}$ is given by

$$\boldsymbol{v} = (\boldsymbol{v}_r + \boldsymbol{\omega} \times \boldsymbol{R})$$

so that

$$\boldsymbol{B} = 2\boldsymbol{R} \times \rho(\boldsymbol{v}_r + \boldsymbol{\omega} \times \boldsymbol{R})v_rA$$
$$B = 2Rv_r^2A\rho \sin \alpha - 2R^2\omega v_rA\rho$$

in which the direction of $\boldsymbol{\omega}$ is taken as negative. From this equation, the angular velocity is found to be

$$\omega = \frac{1}{R}\left(v_r \sin \alpha - \frac{B}{2R\rho v_rA}\right)$$

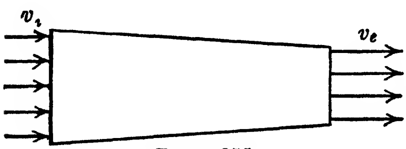
and work is being done at a rate $B\omega$. For a device such as a lawn sprinkler, $B = 0$, and the angular velocity is given by

$$\omega = \frac{v_r}{R} \sin \alpha$$

PROBLEMS

274. Show that the jet thrust force acting upon an accelerating rocket is equal to the thrust force acting on the same rocket when it is held stationary in a test stand. The exhaust velocities and exhaust pressures are assumed to be the same in each case.

275. A ramjet draws in air at the intake with a relative velocity v_i, pressure p_i, and density ρ_i. After an internal combustion process, gas is exhausted with a relative velocity v_e, pressure p_e, and density ρ_e. The intake and exhaust areas are A_i and A_e. Find the propulsive force acting on the ramjet.



Prob. 275

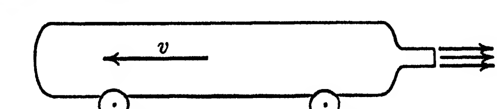
276. A rocket having a total weight of 50 lb contains 2 lb of propellant which is burned at a uniform rate in one second. The propellant has a "specific impulse" of 200 lb sec per lb of propellant; that is, a thrust force of 200 lb is produced by burning one pound of propellant in one second. Assuming that the rocket moves horizontally with negligible frictional resisting forces, find the velocity of the rocket at the end of the burning time. If the propellant is burned in 2 sec instead of one, would the velocity be different?

277. A rocket travels with a velocity v. The exhaust gas issues from the rocket with a velocity v_e relative to the rocket. If the mass of gas exhausted per second is m', then the thrust force is $m'v_e$, and the propulsion power, which is the rate at which the thrust force does work, is given by $m'v_ev$. The kinetic energy lost in the exhaust gas per unit time, which represents a power loss, is $\frac{1}{2}m'(v - v_e)^2$. Show that the propulsion efficiency is given by

$$\text{eff} = \frac{2\left(\dfrac{v}{v_e}\right)}{1 + \left(\dfrac{v}{v_e}\right)^2}.$$

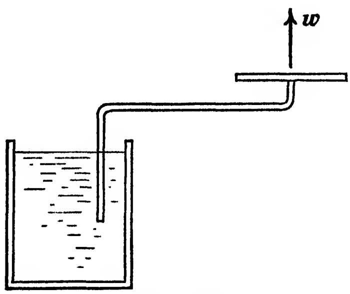
For typical propellants, v_e = 4000–6000 ft/sec. What rocket velocities must be obtained in order to have 50% efficiency?

278. A miniature jet-propelled auto is constructed using a "charged water" cartridge. The gas exhausts with a constant relative velocity v_e

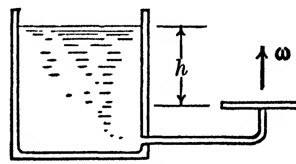


Prob. 278

into the atmosphere, and the density of the exhaust gas may be assumed to be a constant ρ. The auto travels along a horizontal track and experiences a total drag force proportional to the velocity $F_D = -kv$. Find the velocity of the auto as a function of time.



PROB. 279



PROB. 280

279. The turbine of Example 2 of the preceding section is to be used as a pump as shown in the accompanying figure. If a pumping torque B applied to the turbine gives an angular velocity ω, find the quantity of water discharged per unit time.

280. The turbine of Problem 279 discharges water under a head h. If the turbine is held stationary, what quantity of water is discharged per unit time? If the turbine is free to rotate with no energy loss, what is the discharge rate? (Assume the turbine rotates with a constant angular velocity.)

61. The Dynamics of Gases. The analysis of the dynamics of gases can be carried out in several different ways. Under certain conditions the gas can be treated as a fluid in the same manner as in the preceding sections. In this treatment, however, it would be necessary to take into account the compressibility, the viscosity, and also the effect of variations of temperature. This represents an extension of the methods of fluid dynamics. A quite different approach is that in which the gas is considered to be composed of discrete particles. The gas is supposed to consist of molecules each of which may be treated as a particle, and the dynamics of the gas is thus treated as the dynamics of a system of particles. This viewpoint leads to the kinetic theory of gases, by means of which many of the phenomena of thermodynamics may be analyzed. In the following paragraphs we shall discuss a simple problem by this method to illustrate the way in which the principles of dynamics may be applied to such systems.

Consider a rectangular box of volume V containing N molecules of a gas. Each molecule has certain velocity components v_x, v_y, v_z, as shown in Fig. 8–13. We suppose that the gas has reached a steady-state condition, that is, the center of mass of the system of molecules is at rest, and the average density is the same throughout the volume. Considering now the x-components of velocity, we shall distinguish between the molecules which travel in the positive x-direction and those which travel in the negative x-direction. On the average, $N/2$ of the molecules will have the positive direction, and $N/2$ will have the negative direction. We also note that the total momentum of the molecules moving in the $+x$-direction must equal the total momentum of those moving in the $-x$-direction, since the mass center of the system is at rest.

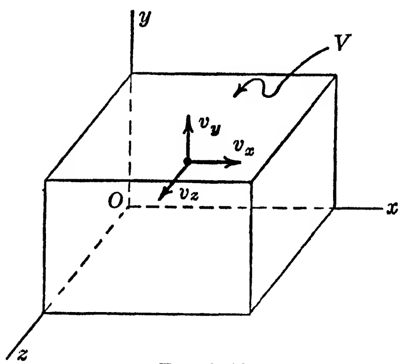


FIG. 8–13

We now examine the molecules which have velocities in the $+x$-direction. Such particles will eventually collide with the end of the box and, after the impact, will rebound with reversed

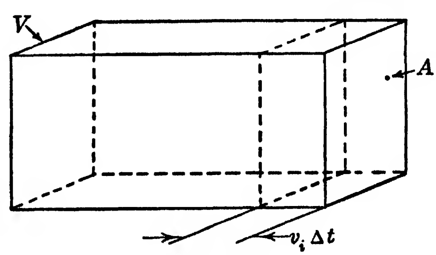


FIG. 8–14

velocities. Let us first consider all the molecules in the box which have a specific velocity v_i in the $+x$-direction. If there is a total of n_i of such molecules in the box, then on the average during a time Δt the number of molecules colliding with the end wall will be $n_i \dfrac{v_i\,\Delta t A}{V}$. This may be seen from Fig. 8–14, since any molecules which are further from the wall than $v_i \Delta t$ will not reach the wall in the time Δt. The total number reaching the wall is therefore n_i multiplied by the ratio of the volume $v_i \Delta t A$ to the total volume V. As each particle collides with the wall and rebounds, the velocity of the molecule is reversed, and the change in momentum, assuming no energy loss during impact, is $-2mv_i$ per molecule, where m is the mass of the molecule. The total change in momentum during the time Δt is:

$$-2m \frac{n_i v_i \Delta t A}{V} v_i = -\frac{2m}{V} n_i v_i^2 A \Delta t$$

From the fact that the total impulse equals the total change in momentum, we have:

$$-\frac{F}{A} \Delta t = -\frac{2m}{V} \Delta t n_i v_i^2$$

If we let p_i be the *average force per unit area* or *pressure* exerted by the molecules against the end wall, we have:

$$p_i = \frac{2m}{V} n_i v_i^2$$

A similar expression will be obtained for the molecules which have a different velocity, say v_j; summing up the p_i over all the different velocities, we obtain:

$$\Sigma p_i = \frac{2m}{V} \Sigma n_i v_i^2$$

where $\Sigma p_i = p =$ the total pressure on the end of the box, and $\Sigma n_i v_i^2$ is equal to $\frac{N}{2} (v_x^2)_{\text{avg.}}$, where $(v_x^2)_{\text{avg.}}$ is the average value of v_x^2 taken over all $\frac{N}{2}$ molecules which have velocity components $+v_x$. We may therefore write:

$$p = \frac{2m}{V} \frac{N}{2} (v_x^2)_{\text{avg.}}$$

or

$$pV = Nm(v_x^2)_{\text{avg.}}$$

We now note that the kinetic energy due to the $+x$-components of velocity is equal to:

$$\frac{1}{2} \frac{N}{2} m(v_x^2)_{\text{avg.}}$$

and that the kinetic energy of translation of all the components of velocity ($\pm v_x$, $\pm v_y$, $\pm v_z$) is equal to:

$$\tfrac{3}{2}Nm(v_x^2)_{\text{avg.}}$$

since in a steady state $(v_x^2)_{\text{avg.}} = (v_y^2)_{\text{avg.}} = (v_z^2)_{\text{avg.}}$. We may thus write the above expression for pV as:

$$pV = \tfrac{2}{3}E \tag{107}$$

where E is the total kinetic energy of translation of the molecules. It will be noted that this expression is just a statement of Boyle's law, which states that the product of the pressure and volume of a gas is constant at a constant temperature. From thermodynamics we know that the gas law for varying temperatures is:

$$pV = RT$$

where T is the absolute temperature, and R is the gas constant.* We thus obtain the following relation between temperature and the kinetic energy of a gas:

$$\tfrac{2}{3}E = RT \tag{108}$$

For example, at a temperature of 100° C or 373° abs, the kinetic energy of one mol of gas is:

$$E = (\tfrac{3}{2})(8.314)(10^7)(373) = 4.65 \times 10^{10} \text{ ergs} = 3430 \text{ ft-lb}$$

For oxygen molecules (32 grams per mol), the average value v^2 per molecule may be computed as follows:

$$\tfrac{1}{2}(32 \text{ grams})(v^2)_{\text{avg.}} \text{ cm}^2/\text{sec}^2 = (\tfrac{3}{2})(8.314)(10^7)(37.3) \text{ ergs}$$

$$(v^2)_{\text{avg.}} = 2.91 \times 10^9$$

$$\sqrt{(v^2)_{\text{avg.}}} = 5.4 \times 10^4 \text{ cm/sec} = 1770 \text{ ft/sec}$$

The root-mean-square velocity is somewhat larger than the mean velocity.

In the preceding derivation of Boyle's law it was assumed that the gas molecules traveled from wall to wall without mutual interference. This assumption seems reasonable for a gas with small density, but for a gas of high density it would be expected that there would be many collisions between molecules. The impacts

* $R = 8.314 \times 10^7$ ergs per degree per mol of gas. One mol is that volume which has a mass in grams equal to the molecular weight of the gas, and which contains 6.02×10^{23} molecules (Avogadro's Number). $R = \dfrac{8.314}{6.02} \times 10^{-16} N = kN$, where k = Boltzmann's Constant.

against the walls would then be different and if this is taken into account certain correction terms must be included in Boyle's law. Experimental evidence confirms the fact that Boyle's law is not satisfactory for high gas densities. Another factor must be considered if the gas consists of complex molecules. If, for example, each molecule is formed of several atoms, the molecule can have an appreciable kinetic energy of rotation, which must be included in the analysis. A more complete analysis of the problem, as developed in the kinetic theory of gases, shows that in the steady-state condition the total kinetic energy is divided equally among the degrees of freedom. If the molecule has the form of a rigid multi-atom body it has six degrees of freedom—three in translation and three in rotation. In a steady state $\frac{1}{6}$ of the total kinetic energy would be associated with each degree of freedom.

62. The Pressure-velocity Relations for a Jet. Using the preceding analysis, we are able to derive the relation between the exhaust velocity and the exhaust pressure of a jet. Consider gas flowing from an orifice in a container, under steady-state conditions (Fig. 8–15). This is the type of problem which we have

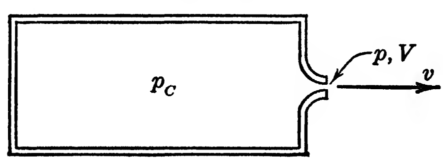


FIG. 8–15

already encountered in connection with an analysis of rocket flight. We shall suppose that the pressure inside the container remains constant at p_C, that the kinetic energy per unit mass of gas within the container has the constant value E_C, and that there is no transfer of energy into or out of the system. We now consider an element of gas having a volume V flowing from the container. For convenience we choose a volume having unit mass, so that $\rho V = 1$. We call the pressure of the gas as it leaves the container p, the velocity v, and the kinetic energy E. The work-energy equation for the element may then be written:

$$E + \int p \, dV = \text{constant}$$

where $\int p\,dV$ is the work done by the element of gas as it expands. Assuming a perfect gas for which $pV = \frac{2}{3}E$, we have:

$$\tfrac{3}{2}pV + \int p\,dV = C$$

Differentiating this equation gives:

$$\tfrac{3}{2}(p\,dV + V\,dp) + p\,dV = 0$$

Rearranging this expression, it may be written:

$$\gamma \frac{dV}{V} + \frac{dp}{p} = 0, \quad \text{where} \quad \gamma = \tfrac{5}{3}$$

Integrating directly gives the well-known equation for the adiabatic process

$$pV^{\gamma} = \text{constant} \tag{109}$$

We now recognize γ as the ratio between the specific heat of the gas at constant pressure to the specific heat at constant volume. The experimentally determined value of this ratio for monatomic gases is very close to $\frac{5}{3}$.

Consider now the equation of motion of a cylindrical element of volume V.

$$(\rho V)\frac{dv}{dt} = F$$

We shall consider an element along a stream-line in the gas as it emerges from the container (Fig. 8–16). In general, $v = f(s, t)$ so that:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial s}\frac{ds}{dt}$$

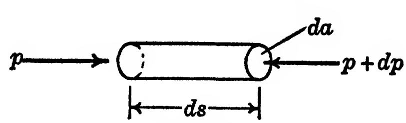


FIG. 8-16

If we restrict the analysis to steady flow, $\frac{\partial v}{\partial t} = 0$, the equation of motion becomes:

$$\rho\, ds\, da \frac{\partial v}{\partial s} v = -dp\, da$$

or

$$\rho v\, dv = -dp$$

We may use this expression for dp to modify the energy equation by observing that:

$$\int p\,dV = pV - \int V\,dp = pV + \int V\rho v\,dv = pV + \int v\,dv$$
$$= pV + \tfrac{1}{2}v^2 + C_1$$

Making this substitution the energy equation may be written:

$$E + pV + \tfrac{1}{2}v^2 = C \tag{110}$$

The term $\frac{1}{2}v^2$ represents the kinetic energy (per unit mass) of translation of the element. When the element is within the container at a pressure p_C, its velocity is zero, so that the equation may be written:

$$E + pV + \tfrac{1}{2}v^2 = E_C + p_C V_C$$

Substituting $E = \frac{3}{2}pV$ and solving for v gives:

$$v^2 = 5(p_C V_C - pV)$$

or

$$v = \left[5\frac{p_C}{\rho_C}\left(1 - \frac{pV}{p_C V_C}\right)\right]^{\frac{1}{2}}$$

Since $pV^\gamma = C$, we have

$$\frac{V}{V_C} = \left(\frac{p}{p_C}\right)^{-\frac{1}{\gamma}}$$

and hence we may write:

$$v = \left[2\frac{\gamma}{\gamma - 1}\frac{p_C}{\rho_C}\left\{1 - \left(\frac{p}{p_C}\right)^{\frac{\gamma-1}{\gamma}}\right\}\right]^{\frac{1}{2}} \tag{111}$$

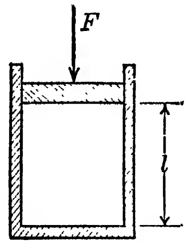
thus giving the relation between the exhaust velocity and the exhaust pressure.

PROBLEMS

281. How many foot-pounds of energy are required to give one gram of oxygen a temperature increase of one degree centigrade?

282. Show that the work that is done by a gas as it expands under a variable pressure is given by $\int_a^b p\,dV$.

283. A cylinder contains one mol of gas at a temperature T_0 degrees absolute. The volume of the cylinder is Al where A is the cross-sectional area. A piston compresses the gas uniformly (without inducing oscillations) to a volume $A(l - x)$. Find the total kinetic energy of the gas in the compressed state and find the final temperature of the gas. Assume



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that no energy is lost through the walls of the cylinder. Do this problem by considering the work-energy relation.

284. If in the preceding problem the gas is compressed very rapidly, so that an appreciable average acceleration is given to the gas molecules, then the force moving the massless piston must be larger than in the preceding problem. Suppose that the piston compresses the volume to $A(l - x)$ in such a way that twice as much work is done on the gas; find the final temperature and pressure of the gas.

285. A rocket with internal pressure p discharges a monatomic gas with exit velocity v. The gas discharges through a nozzle of area A with atmospheric pressure p_a. How does the thrust compare with that which would be obtained if there were no atmospheric pressure?

CHAPTER IX

ADVANCED METHODS IN DYNAMICS

When one obtains a simple result by means of complicated calculations, there must exist a more direct method of obtaining the result; the simplifications which occur and the terms which disappear during the course of the calculations are certain indications that a method exists for which these simplifications have already been made and in which these terms do not appear.—M. Lamé, *Théorie de Elasticité* (1866).

As the problems in dynamics become more complex it naturally becomes increasingly difficult to work out the solutions. This difficulty is associated not only with the solution of the equations of motion, but applies particularly to the derivation of the differential equations of motion. A number of methods, more powerful than those hitherto considered in this book, have been developed for deriving the differential equations of motion for complex problems. Perhaps the most generally useful of the more advanced methods is that of Lagrange, who has put the basic equations of motion in such a form that the simplifying features of a problem may be utilized most advantageously. In the present chapter we shall derive Lagrange's equations, and we shall illustrate their application by a number of examples.

63. Generalized Coordinates. One of the principal advantages of Lagrange's method is that one uses for each problem that coordinate system which most suitably describes the motion. We have already seen that the position of a particle can be described in a large number of different ways, and we have found in the problems already discussed that the choice of a proper coordinate system may introduce a considerable simplification into the solution of the problem. In general, the requirement for a system of coordinates is that the specification of the coordinates must locate completely the position of each part of the system. This means that there must be one coordinate associated with each degree of freedom of the system.* We shall restrict the following treatment to systems whose coordinates are independent, in the sense that an increment change can be given to any one of the coordinates without changing any of the other coordinates.† By the *generalized coordinates* $(q_1, q_2 \cdots q_n)$ we shall mean a set of independent coordinates, equal in number to the degrees of freedom of the system. We use the word "generalized" to emphasize the fact that such coordinates are not necessarily the simple xyz or $r\theta\phi$ coordinates, and to indicate that they are not necessarily lengths or angles but may be any quantity appropriate to the description of the position of the system.

In the application of Lagrange's method we shall find it necessary to write the kinetic energy T of a particle as a function of the generalized coordinates. It is known that the kinetic energy, expressed in terms of the xyz coordinates of the particle is:

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

We now wish to transform the coordinates from the xyz system to the $q_1q_2q_3$ system, where the q's are the generalized coordinates.

In order to make the transformation, we need the relations between the two sets of coordinates. Let us suppose that these are:‡

$$x = \phi_1(q_1, q_2, q_3); \quad y = \phi_2(q_1, q_2, q_3); \quad z = \phi_3(q_1, q_2, q_3)$$

where the ϕ's represent general functions. If, for example, we wish to transform from rectangular xyz to cylindrical $r\theta z$ coordinates, these equations have the form:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

* More exactly, there must be at least one coordinate associated with each degree of freedom. So-called "non-holonomic" systems exist, for which, due to the particular constraints involved, more coordinates are required than there are degrees of freedom. Such systems are not often encountered and will not be considered here. (See Appendix I, references 26, 27.)

† It will, in some problems, be more convenient to use coordinates which are not all independent. For methods of extending the analysis to include such problems, consult the references given in the bibliography. (See Appendix I, references 4, 8, 26, 27.)

‡ We have supposed that the relation between the coordinate systems does not involve the time. In the more general treatment in which $x = \phi(q_1, q_2, q_3, t)$ the analysis can proceed along essentially the same lines. (See Appendix I, reference 4.)

The velocities may now be written in terms of the generalized coordinates as:

$$\dot{x} = \frac{\partial\phi_1}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial\phi_1}{\partial q_2}\frac{dq_2}{dt} + \frac{\partial\phi_1}{\partial q_3}\frac{dq_3}{dt}$$

with similar expressions for $\dot{y}$ and $\dot{z}$. The kinetic energy of the particle thus becomes:

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \tfrac{1}{2}m\sum_{i=1}^{3}\left(\frac{\partial\phi_i}{\partial q_1}\dot{q}_1 + \frac{\partial\phi_i}{\partial q_2}\dot{q}_2 + \frac{\partial\phi_i}{\partial q_3}\dot{q}_3\right)^2 \qquad (112)$$

The form of this expression may be changed in the following way. We note that by taking the derivative of T with respect to $\dot{q}_1$ and then multiplying by $\dot{q}_1$ we obtain:

$$\frac{\partial T}{\partial \dot{q}_1}\dot{q}_1 = m\sum_{i=1}^{3}\left[\left(\frac{\partial\phi_i}{\partial q_1}\dot{q}_1 + \frac{\partial\phi_i}{\partial q_2}\dot{q}_2 + \frac{\partial\phi_i}{\partial q_3}\dot{q}_3\right)\left(\frac{\partial\phi_i}{\partial q_1}\dot{q}_1\right)\right]$$

Similar equations are obtained by differentiating with respect to $\dot{q}_2$ and $\dot{q}_3$. Adding these three equations gives:

$$\sum_{j=1}^{3}\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = m\sum_{i=1}^{3}\left(\frac{\partial\phi_i}{\partial q_1}\dot{q}_1 + \frac{\partial\phi_i}{\partial q_2}\dot{q}_2 + \frac{\partial\phi_i}{\partial q_3}\dot{q}_3\right)^2$$

The right side of this equation is $2T$, hence

$$T = \tfrac{1}{2}\sum_{j=1}^{3}\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j \qquad (113)$$

This expression will be useful in the succeeding analysis.*

64. Lagrange's Equation for a Particle. It is always possible to write the equations of motion in xyz coordinates and then to change the variables to any desired coordinates. This procedure, however, usually leads to rather involved algebraic manipulations. It was pointed out by Lagrange that the equations of motion in any coordinates can be written directly if the expression for the kinetic energy, in the desired coordinates, is known. To derive Lagrange's equations we note that the particle will at all times move in such a way that the change in kinetic energy is

* From the above expression for T it will be seen that T is a homogeneous quadratic function of the $\dot{q}$'s. Our result $\sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = 2T$ then follows directly from Euler's theorem for homogeneous functions.

equal to the work done; thus the work-energy relation may be written:

$$dT = dW$$

In terms of the generalized coordinates q_1, q_2, q_3, the increment of work is:

$$dW = Q_1\, dq_1 + Q_2\, dq_2 + Q_3\, dq_3 = \Sigma Q_i\, dq_i$$

where a term $Q_i dq_i$ represents the increment of work done when the coordinate q_i changes by the amount dq_i. The quantity Q_i is called the *generalized force* corresponding to the coordinate q_i. The generalized force does not necessarily have the dimensions of a force; for example, if q_i is an angle, dq_i is dimensionless and Q_i has the dimensions of foot-pounds. In this instance the generalized force is a moment.

To express dT in terms of the generalized coordinates, we note that since T is a function of the q's and the $\dot{q}$'s, we have:

$$dT = \frac{dT}{dt}\, dt = \sum_1^3 \left(\frac{\partial T}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial T}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right) dt$$

We have from the preceding article Equation (113),

$$T = \tfrac{1}{2}\Sigma \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i$$

Differentiating this we obtain:

$$2dT = \sum_1^3 \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right) dt$$

Subtracting the preceding expression for dT from this equation gives:

$$dT = \Sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}\right) dq_i$$

Equating dT and dW and collecting terms, we have:

$$\sum_i \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i\right) dq_i = 0$$

Since the three coordinates q_1, q_2, q_3 are independent so that one can be given an increment change while the other two remain unchanged, the expression within the parenthesis must be identically zero, so that:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \quad (i = 1, 2, 3) \tag{114}$$

These are Lagrange's equations for the motion of the particle. They are an expression of Newton's law in generalized coordinates.

If the force acting on the particle is conservative, the generalized force can be written $Q_i = -\frac{\partial V}{\partial q_i}$, and since the potential energy is not a function of the velocities $\dot{q}$, it follows that $\frac{\partial V}{\partial \dot{q}} = 0$, and we may write Lagrange's equations in the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (115)$$

where $L = (T - V)$, which is the difference between the kinetic and the potential energies of the system. The quantity L is often called the *Lagrangian function* or the *kinetic potential* of the system. It should be noted that Equation (115) applies only to conservative systems, whereas Equation (114) applies to non-conservative systems as well.

Because of the importance of Lagrange's equations, we shall give another derivation in essentially the form in which Lagrange originally presented it in his *Méchanique Analytique* (1788). Lagrange started with the combination of D'Alembert's principle and the principle of virtual displacements which we discussed in a preceding chapter (Equation (98)).

$$\Sigma[(m_i\ddot{x}_i - F_{x_i})\,\delta x_i + (m_i\ddot{y}_i - F_{y_i})\,\delta y_i + (m_i\ddot{z}_i - F_{z_i})\,\delta z_i] = 0$$

Writing this equation for a single particle, and rearranging the terms, we obtain:

$$F_x\delta x + F_y\,\delta y + F_z\,\delta z = m\ddot{x}\,\delta x + m\ddot{y}\,\delta y + m\ddot{z}\,\delta z \qquad (98a)$$

If we wish to transform from the xyz coordinates to the $q_1q_2q_3$ coordinates, as in the preceding derivation, we have

$$\delta x = \frac{\partial x}{\partial q_1}\,\delta q_1 + \frac{\partial x}{\partial q_2}\,\delta q_2 + \frac{\partial x}{\partial q_3}\,\delta q_3; \quad \text{etc.}$$

Let us work out the analysis, supposing that only one of the independent coordinates, q_1, has been given a virtual displacement. Then $\delta q_2 = \delta q_3 = 0$, and we have:

$$\delta x = \frac{\partial x}{\partial q_1}\,\delta q_1; \quad \delta y = \frac{\partial y}{\partial q_1}\,\delta q_1; \quad \delta z = \frac{\partial z}{\partial q_1}\,\delta q_1$$

and Equation (98a) can be written:

$$Q_1\,\delta q_1 = \left(m\ddot{x}\,\frac{\partial x}{\partial q_1} + m\ddot{y}\,\frac{\partial y}{\partial q_1} + m\ddot{z}\,\frac{\partial z}{\partial q_1}\right)\delta q_1 \qquad \text{(98b)}$$

The expression on the right side can be transformed in the following way. From the formula for the differentiation of a product:

$$\ddot{x}\,\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\dot{x}\,\frac{\partial x}{\partial q_1}\right) - \dot{x}\,\frac{d}{dt}\,\frac{\partial x}{\partial q_1}$$

Since

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial q_1}\,\dot{q}_1 + \frac{\partial x}{\partial q_2}\,\dot{q}_2 + \frac{\partial x}{\partial q_3}\,\dot{q}_3$$

we have:

$$\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1}; \quad \text{etc.}$$

Also:

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial}{\partial q_1}\left(\frac{dx}{dt}\right) = \frac{\partial \dot{x}}{\partial q_1}; \quad \text{etc.}$$

With these substitutions we may write:

$$\begin{aligned}\ddot{x}\,\frac{\partial x}{\partial q_1} &= \frac{d}{dt}\left(\dot{x}\,\frac{\partial \dot{x}}{\partial \dot{q}_1}\right) - \dot{x}\,\frac{\partial \dot{x}}{\partial q_1}\\ &= \frac{d}{dt}\,\frac{\partial}{\partial \dot{q}_1}\left(\frac{\dot{x}^2}{2}\right) - \frac{\partial}{\partial q_1}\left(\frac{\dot{x}^2}{2}\right)\end{aligned}$$

Substituting this into the Equation (98b), along with similar expressions for the terms in y and z, and remembering that $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ we obtain:

$$Q_1\delta q_1 = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1}\right]\delta q_1$$

Cancelling the δq_1, we obtain Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1$$

By permitting variations in the other coordinates instead of the q_1, two similar equations involving q_2 and q_3 will be obtained.

Several times in the preceding chapters we have derived the differential equations of motion of a system by differentiating the energy equation. It is well to note that this is just the process which is done in general by Lagrange's method.

One of the advantages of Lagrange's method is that the kinematics of a problem is simplified to the extent that accelerations are not required; it is only necessary to find the velocities in order to write the kinetic energy of the system. Since these velocities always appear squared, some possible difficulties of sign are also eliminated. These advantages, of course, apply equally to any energy method.

By the application of Lagrange's equations, we automatically obtain as many differential equations of motion as there are degrees of freedom in the system, and hence the complete solution of the problem is indicated.

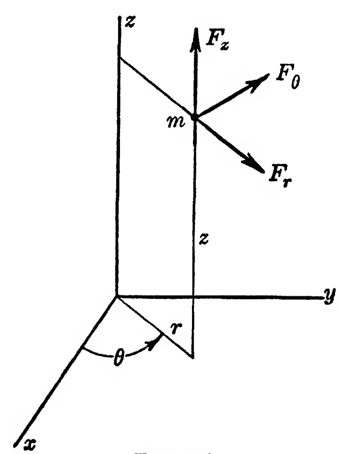


FIG. 9–1

EXAMPLE 1. Derive the equations of motion for a particle in cylindrical coordinates, using Lagrange's equations.

Solution. In this problem we have:

$$q_1 = r$$
$$q_2 = \theta$$
$$q_3 = z$$

The kinetic energy T of the particle is:

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$$

Thus, Lagrange's equation for the r-coordinate becomes:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r} = Q_r$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) = \frac{d}{dt}(m\dot{r}) = m\ddot{r}$$

$$\frac{\partial T}{\partial r} = mr\dot{\theta}^2; \quad Q_r dr = F_r dr; \quad Q_r = F_r$$

and the differential equation of motion is:

$$m(\ddot{r} - r\dot{\theta}^2) = F_r$$

For the θ-coordinate:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}(mr^2\dot{\theta}) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}$$

$$\frac{\partial T}{\partial \theta} = 0; \quad Q_\theta \, d\theta = F_\theta r \, d\theta; \quad Q_\theta = F_\theta r$$

and the differential equation of motion is:

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta$$

For the z-coordinate:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = Q_z$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = \frac{d}{dt}(m\dot{z}) = m\ddot{z}$$

$$\frac{\partial T}{\partial z} = 0; \quad Q_z \, dz = F_z \, dz; \quad Q_z = F_z$$

and the differential equation of motion is:

$$m\ddot{z} = F_z$$

It will be noted that these are the same expressions derived in Chapter II, Equation (8).

EXAMPLE 2. A simple pendulum consisting of a concentrated mass m and a weightless string of length l is mounted on a massless support which is elastically restrained by means of a spring having a spring constant k, as shown in Fig. 9–2. Write the equations of motion of the system, using Lagrange's equations.

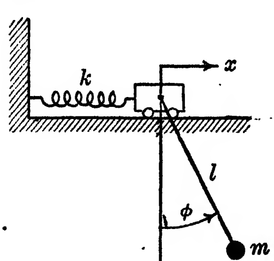


FIG. 9–2

Solution. We shall use as generalized coordinates for this problem the angle ϕ of the pendulum, and the displacement x of the point of support. The kinetic energy T of the mass is given by:

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m[\dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}l\dot{\phi}\cos\phi]$$

The potential energy V of the system is:

$$V = mgl(1 - \cos\phi) + \tfrac{1}{2}kx^2$$

Lagrange's equation for the x-coordinate is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0$$

where

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} + ml\dot{\phi}\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x} + ml\ddot{\phi}\cos\phi - ml\dot{\phi}^2\sin\phi$$

$$\frac{\partial T}{\partial x} = 0; \quad \frac{\partial V}{\partial x} = kx$$

So that the differential equation of motion becomes:

$$m\ddot{x} + ml\ddot{\phi}\cos\phi - ml\dot{\phi}^2\sin\phi + kx = 0$$

Lagrange's equation for the ϕ-coordinate is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} + \frac{\partial V}{\partial \phi} = 0$$

where

$$\frac{\partial T}{\partial \dot{\phi}} = ml^2\dot{\phi} + ml\dot{x}\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = ml^2\ddot{\phi} + ml\ddot{x}\cos\phi - ml\dot{x}\dot{\phi}\sin\phi$$

$$\frac{\partial T}{\partial \phi} = -ml\dot{x}\dot{\phi}\sin\phi; \quad \frac{\partial V}{\partial \phi} = mgl\sin\phi$$

So that the differential equation of motion is:

$$ml^2\ddot{\phi} + ml\ddot{x}\cos\phi + mgl\sin\phi = 0$$

If we limit ourselves to a consideration of small oscillations of the system, a simple solution of these equations can be obtained. Setting $\sin\phi \approx \phi$ and $\cos\phi \approx 1$, and neglecting terms of higher than second order, the two differential equations become:

$$m\ddot{x} + ml\ddot{\phi} + kx = 0$$
$$m\ddot{x} + ml\ddot{\phi} + mg\phi = 0$$

Subtracting these two equations, we have:

$$kx = mg\phi$$

Differentiating twice with respect to time, this becomes

$$k\ddot{x} = mg\ddot{\phi}$$

Eliminating the $\ddot{x}$ from the second equation by means of this expression gives:

$$\frac{mg\ddot{\phi}}{k} + l\ddot{\phi} + g\phi = 0$$

$$\ddot{\phi} + \frac{g}{\left(l + \frac{mg}{k}\right)}\phi = 0$$

Thus the pendulum executes small sinusoidal oscillations of frequency

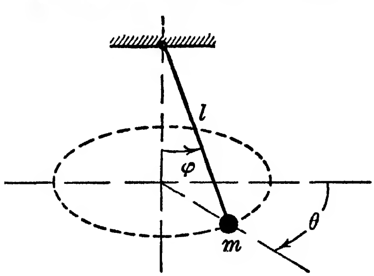
$$\omega = \sqrt{\frac{g}{l + \frac{mg}{k}}}$$

If the point of support of the pendulum is fixed, $k = \infty$ and the frequency reduces to the known value for a simple pendulum.

PROBLEMS

286. By means of Lagrange's equations derive the differential equations of motion for a particle in spherical coordinates r, θ, ϕ.

287. A spherical pendulum is formed by a particle of mass m supported by a weightless string of length l. Using the angles θ and ϕ as the coordinates, derive the two differential equations of motion by means of Lagrange's equations. Show that these equations reduce to previously known results when θ and ϕ are successively held constant.



PROB. 287

288. A particle moves in a plane and is attracted toward the origin of a coordinate system by a force which is inversely proportional to the square of the distance from the origin. Find, by means of Lagrange's equations, the differential equations of motion of the system in plane polar coordinates.

65. Lagrange's Equations for a System of Particles. The methods of the preceding section, which lead to Lagrange's equations for a single particle, can be extended directly to a system of particles. There will be $q_1, q_2 \cdots q_n$ coordinates required, where n is the number of degrees of freedom of the system. We shall, as before, consider that the coordinates are independent. By summing over the individual particles, and taking T as the total kinetic energy of the system, Lagrange's equations will be obtained in the same form as found for a single particle. The analysis proceeds in exactly the same way as before, except that the summations now extend over all the coordinates of the system which may be more than three in number. In this way the number of equations obtained will be just equal to the number of degrees of freedom of the system.

EXAMPLE 1. A rigid body rotates about a fixed axis under the action of a torque M_t. Find the equation of motion of the body by means of Lagrange's equations.

Solution. The position of the rotating rigid body is specified completely by one angle, ϕ; hence the system has one degree of freedom. As was shown in the chapter on rigid body motion the kinetic energy of a rigid rotating body is:

$$T = \tfrac{1}{2}I\dot{\phi}^2$$

where I is the moment of inertia of the body about the axis of rotation. The work done by the torque M_t as the coordinate is changed by $d\phi$ is $M_t d\phi$, therefore

$$Q_\phi = M_t$$

that is, the generalized force is the torque M_t. Lagrange's equation then is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} - Q_\phi = 0$$

$$\frac{d}{dt}(I\dot{\phi}) - M_t = 0; \quad I\ddot{\phi} = M_t$$

This checks the result of the analysis made in the chapter on rigid body dynamics.

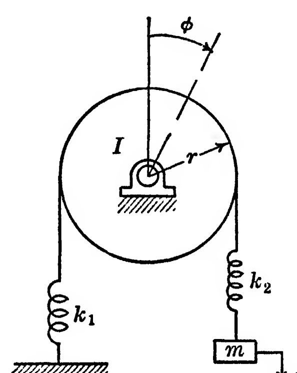


FIG. 9–3

EXAMPLE 2. A pulley of moment of inertia I about its axis of rotation is restrained by a spring of constant k_1 as shown in Fig. 9–3. From the other side of the pulley a spring of constant k_2 and a mass m are suspended, as shown. Find the equations of motion of the system.

Solution. We choose as the two coordinates the clockwise rotation of the pulley ϕ and the downward displacement x of the mass. There is no energy loss, and we may write the potential energy of the system as:

$$V = \tfrac{1}{2}k_1(r\phi)^2 + \tfrac{1}{2}k_2(x - r\phi)^2$$

The kinetic energy of the system is:

$$T = \tfrac{1}{2}I\dot{\phi}^2 + \tfrac{1}{2}m\dot{x}^2$$

Substituting these expressions directly into Lagrange's equations we obtain the two equations of motion:

$$m\ddot{x} + k_2x = k_2r\phi$$
$$I\ddot{\phi} + (k_1 + k_2)r^2\phi = k_2rx$$

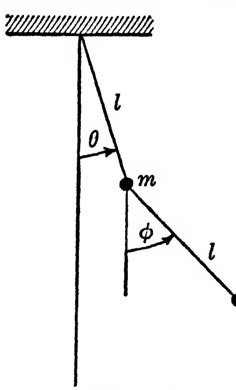
A method of finding the solution of such simultaneous differential equations will be discussed in the next section.

PROBLEMS

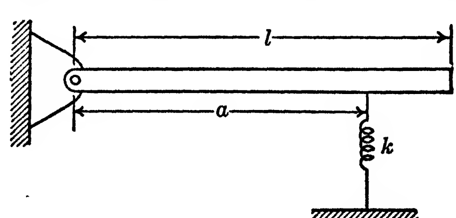
289. A rigid body oscillates about a horizontal axis as a compound pendulum. The moment of inertia of the body about the axis of rotation is I, and the distance from the axis to the center of mass of the body is a. Derive the differential equation of motion of the system by Lagrange's equations, and find the period of small oscillations.

290. A double pendulum consists of two equal masses and two strings of equal length and of negligible mass. Using Lagrange's equations, find the differential equations of motion of the system for small oscillations.

291. A rigid, straight, uniform bar of length l and mass m is pinned at one end and is supported at a distance a from the pin by a spring having a spring constant k. Find the differential equation of motion describing small oscillations of the system about the position of equilibrium, and find the period of the motion.

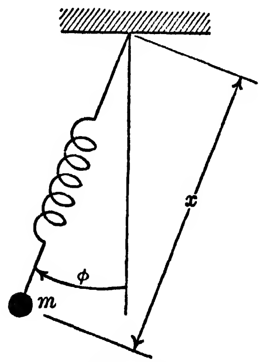


Prob. 290

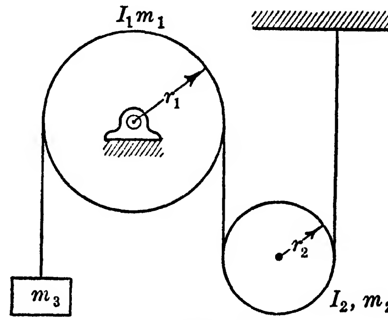


Prob. 291

292. The string of a simple pendulum is assumed to be elastic with a spring constant k. Taking as coordinates of the mass the displacement x, in the direction of the spring, and the angle ϕ made with the vertical by the spring, find the differential equations of motion describing small oscillations of the system.



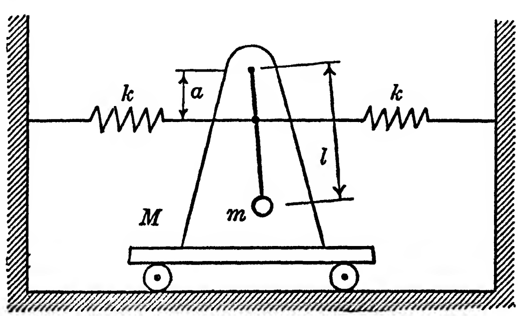
PROB. 292



PROB. 293

293. A rope of negligible mass passes over a fixed pulley with moment of inertia I_1, mass m_1, and radius r_1, and supports a movable pulley with moment of inertia I_2, mass m_2, and radius r_2 as shown in the diagram. A concentrated mass m_3 is attached to the one end of the rope, and the other end is fixed. Find the differential equation for the motion of the mass m_3, as the system moves under the action of gravity.

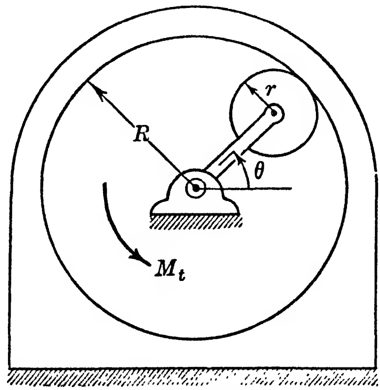
294. A car of mass M moves along a frictionless horizontal plane. The car carries a simple pendulum of length l and concentrated mass m



PROB. 294

as shown in the diagram. Two equal springs of spring constant k attach the pendulum, at a distance a from the axis of rotation, to fixed walls. Find the differential equations of motion describing small oscillations of the system about the equilibrium position.

295. A circular gear of radius r rolls around the inside of a fixed circular gear of radius R under the action of a torque M_t. What is the differential



Prob. 295

equation of motion? Consider that the driving arm of radius $(R - r)$ is weightless and rotates in a horizontal plane. The small gear has a mass m and a moment of inertia I about its axis of symmetry.

66. Oscillations of Two-degree-of-freedom Systems. Consider a conservative system consisting of two equal masses m, and three equal springs k, connected as shown in Fig. 9–4. This is the

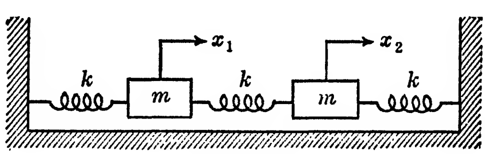


Fig. 9–4

simplest type of system which illustrates the new features which appear in a vibration problem when more than one degree of freedom is present. Taking x_1 and x_2, the displacements of the two masses from the equilibrium position, as the coordinates of the system, we have:

$$\begin{aligned}
T &= \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) \\
V &= \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}k(x_1 - x_2)^2 + \tfrac{1}{2}kx_2^2 \\
&= kx_1^2 + kx_2^2 - kx_1x_2
\end{aligned}$$

Substituting these expressions into Lagrange's equations, we obtain the two differential equations of motion:

$$\ddot{x}_1 + 2\frac{k}{m}x_1 - \frac{k}{m}x_2 = 0$$
$$\ddot{x}_2 + 2\frac{k}{m}x_2 - \frac{k}{m}x_1 = 0$$

Both equations involve x_1 and x_2. Since a displacement of x_1 changes the force applied to x_2, such coordinates are said to be statically coupled.

From the form of the equations, and from the nature of the physical problem, we expect the oscillations of the system to be harmonic, so we shall try a solution of the form $e^{i\omega t} = \cos \omega t + i \sin \omega t$. Thus we take a trial solution:

$$x_1 = A_1 e^{i\omega t}$$
$$x_2 = A_2 e^{i\omega t}$$

Substituting these into the differential equations and cancelling common factors, we obtain:

$$\left(2\frac{k}{m} - \omega^2\right)A_1 - \frac{k}{m}A_2 = 0$$
$$-\frac{k}{m}A_1 + \left(2\frac{k}{m} - \omega^2\right)A_2 = 0$$

Each of these equations gives a value for the ratio A_1/A_2; equating the ratios obtained from each equation gives:

$$\frac{k/m}{\left(2\frac{k}{m} - \omega^2\right)} = \frac{\left(2\frac{k}{m} - \omega^2\right)}{k/m}$$

This gives the frequency equation for determining ω:

$$\left(\omega^2 - 2\frac{k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0$$

From this equation, two frequencies are obtained:

$$\omega_1^2 = \frac{k}{m}$$
$$\omega_2^2 = \frac{3k}{m}$$

The frequency equation can also be obtained by noting that, since the two simultaneous algebraic equations involving A_1 and A_2 are homogeneous, they have a solution other than zero only if the determinant made up of the coefficients of the A's disappears:

$$\begin{vmatrix} \left(2\frac{k}{m} - \omega^2\right) & \left(-\frac{k}{m}\right) \\ \left(-\frac{k}{m}\right) & \left(2\frac{k}{m} - \omega^2\right) \end{vmatrix} = 0$$

This gives the same frequency equation.

To find the configuration of the system corresponding to the two natural frequencies of vibration found above, we solve for the ratio A_1/A_2 from either of the algebraic equations; for example:

$$\frac{A_1}{A_2} = \frac{k/m}{2\frac{k}{m} - \omega^2}$$

Substituting $\omega_1^2 = k/m$, we obtain:

$$\left(\frac{A_1}{A_2}\right)_1 = +1$$

or the two masses move in phase with equal amplitudes. Substituting $\omega_2^2 = 3k/m$ we obtain:

$$\left(\frac{A_1}{A_2}\right)_2 = -1$$

or the two masses move with equal amplitudes but in opposite directions, that is, with a 180° phase difference.

For this particular problem we can easily see the physical significance of these two motions. If $A_1 = A_2$ the two masses move together with equal amplitudes and the center spring connecting them is neither extended nor compressed; hence it cannot effect the motion. The frequency of this vibration should thus be the same as for one of the masses restrained by only one of the springs, that is, $\omega^2 = k/m$. For the motion in which the two masses always move in opposite directions, we note that the displacement is symmetrical and that the center point of the center spring can be considered as fixed. We can thus represent the system, for this type of motion, as shown in Fig. 9–5, which is a single-degree-of-freedom system having a spring of constant $3k$, and hence a frequency $\omega^2 = 3k/m$.

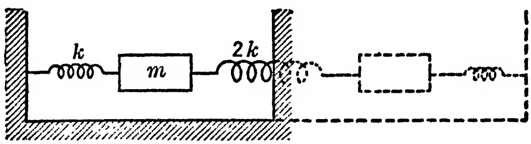


FIG. 9-5

The complete solution of the problem can be written as the sum of four terms for each coordinate:

$$\begin{aligned} x_1 &= A_1 e^{i\omega_1 t} + A_1' e^{-i\omega_1 t} + A_2 e^{i\omega_2 t} + A_2' e^{-i\omega_2 t} \\ x_2 &= A_1 e^{i\omega_1 t} + A_1' e^{-i\omega_1 t} - A_2 e^{i\omega_2 t} - A_2' e^{-i\omega_2 t} \end{aligned} \tag{116}$$

where the constants A_1 and A_2 are determined from the initial conditions of the problem. The first terms in each of the expressions represent the in-phase motion of the two masses, and the second terms represent the 180° out-of-phase motion. The most general motion of the system can thus be thought of as a superposition of the motions corresponding to the two natural frequencies.

67. Principal Modes of Vibration. The two configurations of the system in the preceding section, which correspond to motions each having one of the natural frequencies of vibration of the system, are called the two *natural modes* of vibration. In general, by the word "mode" we refer to a motion of the system which can be described by a single frequency. We have seen that the solution of the preceding problem is the superposition of two modes of vibration, each having its characteristic frequency. There is often a decided advantage in choosing for the system that set of coordinates for which the motion of each coordinate has only one frequency. In the present problem this can be done by introducing the new coordinates q_1 and q_2 where:

$$q_1 = \frac{x_1 + x_2}{2}$$

$$q_2 = \frac{x_1 - x_2}{}$$

If we write the equations:

$$x_1 = A_1e^{i\omega_1 t} + A_1'e^{-i\omega_1 t} + A_2e^{i\omega_2 t} + A_2'e^{-i\omega_2 t}$$
$$x_2 = A_1e^{i\omega_1 t} + A_1'e^{-i\omega_1 t} - A_2e^{i\omega_2 t} - A_2'e^{-i\omega_2 t}$$

in terms of these new coordinates, we have:

$$\begin{aligned} q_1 &= A_1e^{i\omega_1 t} + A_1'e^{-i\omega_1 t} \\ q_2 &= A_2e^{i\omega_2 t} + A_2'e^{-i\omega_2 t} \end{aligned} \tag{117}$$

and each coordinate involves only one frequency and one mode of vibration. Coordinates which have this property are called *normal coordinates*, and the modes of vibration corresponding to them are called the *principal modes* or *normal modes* of the system.

Once the normal coordinates of a system have been determined, the problem has been essentially reduced to a study of a set of independent single-degree-of-freedom systems. This is illustrated by the system of Fig. 9–4, with a force $F = F_0 \sin \omega t$ applied to the left mass (Fig. 9–6). Since we have already seen that the

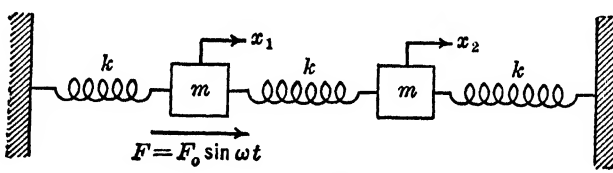


FIG. 9–6

coordinates $q_1 = \dfrac{x_1 + x_2}{2}$, $q_2 = \dfrac{x_1 - x_2}{2}$ describe the principal modes of the system, we shall use them as generalized coordinates. Substituting $x_1 = q_1 + q_2$, $x_2 = q_1 - q_2$ in the equations for kinetic and potential energy of the preceding section, we obtain:

$$T = m(\dot{q}_1^2 + \dot{q}_2^2)$$
$$V = k(q_1^2 + 3q_2^2)$$

To find the generalized force Q associated with the force F, we note that the work done by F during the displacement $dx_1 = dq_1 + dq_2$ is:

$$F\,dx_1 = F\,dq_1 + F\,dq_2 = Q_1\,dq_1 + Q_2\,dq_2$$

Hence:

$$Q_1 = F; \quad Q_2 = F$$

Lagrange's equations are now used in the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = Q$$

Note that some of the forces acting on the masses have been included in the potential energy, while some have been left as generalized forces. Substituting into this equation, we obtain the two equations of motion:

$$m\ddot{q}_1 + kq_1 = \frac{F_0}{2}\sin \omega t$$

$$m\ddot{q}_2 + 3kq_2 = \frac{F_0}{2}\sin \omega t$$

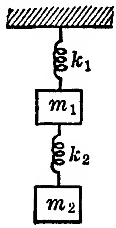
These are two independent equations, each involving only one unknown, which describe the forced oscillations of the system. They are of the same type as the equations which were treated in the chapter on vibrations, so that the conclusions reached in that chapter as to the behavior of a single-degree-of-freedom system can be applied to the motion of the present two-degree-of-freedom system. It should be noted that any system whose kinetic and potential energies are sums of squares of the coordinates and do not involve cross-product terms will have independent equations of motion. One of the ways in which the normal coordinates of a system can be found is to determine the transformation of coordinates required to make the cross-product terms in the kinetic and potential energies disappear. In most problems it is not possible to deduce the normal coordinates by inspection.

The method of this section may be extended to problems involving more than two degrees of freedom. Thus, in general, there will be as many natural frequencies of vibration as there are degrees of freedom, and each frequency will correspond to a principal mode of vibration. The mode having the lowest frequency of vibration is called the first, or fundamental, mode. The higher modes are sometimes called the higher harmonics of the system.

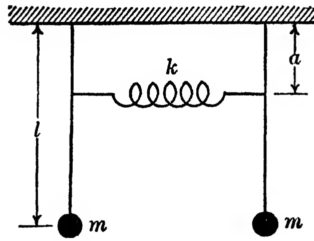
PROBLEMS

296. Derive the differential equations of motion for the system of Fig. 9-4 by a direct application of Newton's second law in the form $F = ma$.

297. Find the natural frequencies of vibration of the two-mass system shown in the figure.



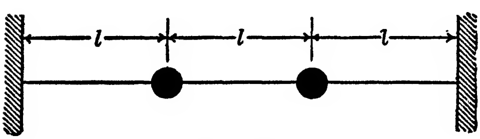
PROB. 297



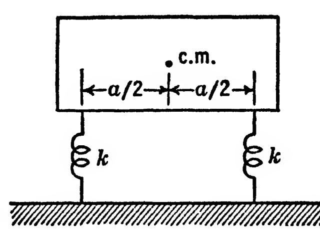
PROB. 298

298. Two simple pendulums having equal lengths l and equal concentrated masses m are connected by a spring of constant k as shown in the figure. Find the frequencies of vibration for small oscillations and the principal modes of vibration of the system. What initial conditions would be necessary to obtain free oscillations of the system in the first mode without exciting the second mode, and vice versa?

299. Two equal masses are attached to a stretched string of length $3l$ as shown in the diagram. The tensile force in the string is F and can be assumed to be constant during small transverse oscillations of the masses. Neglecting gravity forces and the mass of the string, find the principal modes of vibration of the system and their frequencies.



PROB. 299



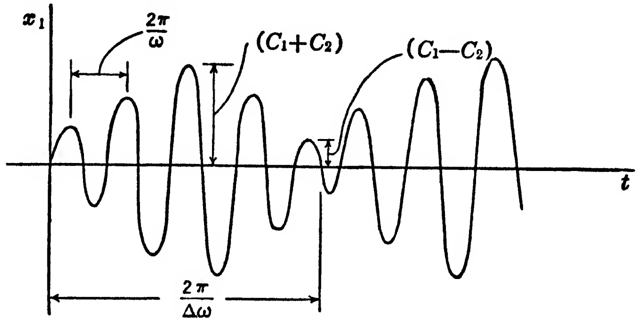
PROB. 300

300.* A mass m with a moment of inertia I about its center of mass is spring-mounted as shown in the diagram. Considering only plane oscillations of the system it will be seen that the body can vibrate vertically and can also perform rotational oscillations about an axis through the center of mass. The two identical springs have spring constants k. Find the natural frequencies of small oscillations and the principal modes.

301.* (a) Show that the displacement x_1 of Equation (116) can be written as $x_1 = C_1 \sin \omega_1 t + C_2 \sin \omega_2 t$ if $x_1 = x_2 = 0$ when $t = 0$.

(b) Show that, if ω_1 is nearly the same as ω_2, so that $\omega_1 - \omega_2 = \Delta\omega$, x_1 varies between the values $(C_1 + C_2)$ and $(C_1 - C_2)$ with a frequency

$\Delta\omega$, as shown in the diagram. $\Delta\omega$ is called the *beat frequency*. Note that if $C_1 = C_2$, x_1 passes through zero at times corresponding to the beat frequency.



PROB. 301

302.* Find the natural frequencies of vibration of the double pendulum of Problem 290.

303. Show that if the coordinates of a system are selected so that the kinetic and potential energies can be written in the form:

$$T = \tfrac{1}{2}(A\dot{q}_1^2 + B\dot{q}_2^2 + \cdots)$$
$$V = \tfrac{1}{2}(A'q_1^2 + B'q_2^2 + \cdots)$$

The equations of motion will each involve only one unknown and hence can be solved directly. Note that $q_1, q_2, \cdots$ are the normal coordinates for the system.

68. The Calculus of Variations. The techniques of the calculus of variations play an important part in the development of advanced dynamics. Although we cannot give here a detailed treatment of this subject, we can indicate the nature of the problem and the usefulness of the methods.

In ordinary maximum and minimum problems we are generally given a function of several variables in the form:

$$y = f(x_1, x_2, \cdots)$$

and we wish to find the values of the x's for which the function y has an extreme value. The necessary conditions for the solution of this problem are given by the equations:

$$\frac{\partial y}{\partial x_1} = 0; \quad \frac{\partial y}{\partial x_2} = 0; \cdots$$

In the calculus of variations we also have the problem of finding an extremum, but now the expression to be investigated no longer depends upon a finite number of independent variables, but depends instead upon the behavior of one or more dependent variables. The particular expression of this type which we shall investigate here has the form:

$$I = \int_{x_1}^{x_2} f\left(x, y, \frac{dy}{dx}\right) dx, \quad \text{where} \quad y = \phi(x) \tag{118}$$

Given the function f, we are to determine the function ϕ which will give I a stationary value.

A simple example illustrating this type of problem is that of finding the plane curve which has the minimum length between two points. Suppose, as in Fig. 9–7, that we have given the two points A and B and we wish to find the plane curve $y = \phi(x)$ joining the points, which will have the shortest length. The length L of the curve is given by:

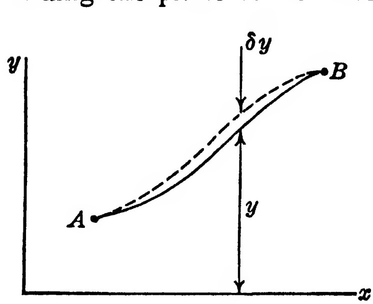


Fig. 9–7

$$L = \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

so that we are to minimize an expression of the form:

$$L = \int_A^B f\left(\frac{dy}{dx}\right) dx$$

This is a special form of Equation (118) above. We shall introduce the general method of solution and notation by solving this problem. If the length of the curve $y = \phi(x)$ is a minimum, the length of any other slightly different curve between the two points must be somewhat greater. This slightly different curve, which we have illustrated in Fig. 9–7, is called the varied path. The varied path passes through the points A and B, and has the ordinates:

$$y + \delta y = \phi(x) + \epsilon\theta(x)$$

where δy is the variation in y, and the small quantity ϵ indicates that the varied path is slightly different from the minimum path. As the path changes from the original path to the varied path, the value of the integral will change by a small amount δL, called the variation of the integral.

$$\delta L = \delta \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

We can find δL by noting that the integral sign and the variation can be interchanged, without altering the summation process, and that, since the integral is a function of $\frac{dy}{dx} = y'$ only, we have:

$$\delta L = \int_A^B \delta \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_A^B \left\{\frac{\partial}{\partial y'}(1 + y'^2)^{\frac{1}{2}}\right\} \delta y'\, dx$$

from which we obtain:

$$\delta L = \int_A^B \frac{y'}{(1 + y'^2)^{\frac{1}{2}}}\, \delta y'\, dx$$

This gives the variation in the integral due to the variation $\delta y'$. The variation $\delta y'$ in the slope of the curve is not arbitrary, but depends upon the variation δy as may be seen in the following way. The slope of the original curve is $\frac{dy}{dx} = y'$, and the slope of the varied curve is $\frac{d}{dx}(y + \delta y) = y' + \frac{d}{dx}(\delta y)$; thus the variation of the slope $\delta y'$ is equal to $\frac{d}{dx}(\delta y)$, or $\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$. The variation δ and the derivative $\frac{d}{dx}$, therefore, are commutative, and the integral can be written:

$$\delta L = \int_A^B \left[\frac{y'}{(1 + y'^2)^{\frac{1}{2}}} \frac{d}{dx}(\delta y)\right] dx$$

The integral may be transformed by integration by parts to give:

$$\delta L = \frac{y'}{(1 + y'^2)^{\frac{1}{2}}}\, \delta y \,\Big|_A^B - \int_A^B \frac{d}{dx}\left[\frac{y'}{(1 + y'^2)^{\frac{1}{2}}}\right] \delta y\, dx$$

Since by definition δy vanishes at A and B, the first term is equal to zero.

If the function $y = \phi(x)$ makes the length of the curve a minimum, then L has a stationary value for $y = \phi(x)$, and any infinitesimal variation in the curve does not change the value of the integral. The condition for the stationary value is $\delta L = 0$:

$$\int_A^B \left\{ \frac{d}{dx} \left[\frac{y'}{(1 + y'^2)^{\frac{1}{2}}} \right] \right\} \delta y \, dx = 0$$

The only way in which this integral can vanish for arbitrary variations δy is for the term in the brackets to vanish.

$$\frac{d}{dx} \left(\frac{y'}{\sqrt{1 + y'^2}} \right) = 0$$

This can be integrated once to give

$$\frac{y'}{\sqrt{1 + y'^2}} = \text{constant}$$

Solving this equation for y' gives

$$y' = \frac{dy}{dx} = C_1$$

$$y = C_1 x + C_2$$

This is the function $y = \phi(x)$ which makes the length of the curve between A and B a minimum. As expected, the required curve is a straight line.

69. Euler's Differential Equation. The procedure followed in the preceding example may be carried out for the general problem. We shall consider an integral which is a function of y, y', and x:

$$I = \int_A^B f(y, y', x) \, dx, \quad \text{where} \quad y = \phi(x)$$

The condition for an extreme value for the integral is:

$$\delta \int_A^B f(y, y', x) \, dx = 0$$

As we move to a varied curve, we have, corresponding to a particular value x, the variation δy in the ordinate and the variation $\delta y'$ in the slope; thus

$$\delta \int_A^B f(y, y', x) \, dx = \int_A^B \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right) dx = 0$$

As shown above:

$$\delta y' = \delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$$

hence we may write:

$$\int_A^B\left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y)\right)dx = 0$$

Integrating the second term by parts:

$$\int_A^B \frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y)\,dx = \frac{\partial f}{\partial y'}\delta y\Big|_A^B - \int_A^B \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y\,dx$$

Since δy is zero at both A and B, δI may be written:

$$\delta I = \int_A^B\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y\,dx = 0$$

Since δy is arbitrary, the expression within the square brackets must be zero in order to insure the vanishing of the integral, so that we have finally, as the necessary condition for the extreme value:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \qquad (119)$$

This is called Euler's differential equation, and its solution, $y = \phi(x)$, extremizes the integral:

$$I = \int_A^B f(y, y', x)\,dx$$

If the integral to be extremized is a function of more than two variables, such as:

$$I = \int_{t_1}^{t_2} f(t, x, y, z, \dot{x}, \dot{y}, \dot{z})\,dt$$

where x, y, z are dependent variables and t is the independent variable, the same reasoning can be carried through, and the following result will be obtained:

$$\delta I = \int_{t_1}^{t_2}\left\{\left[\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\right]\delta x + \left[\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]\delta y + \left[\frac{\partial f}{\partial z} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{z}}\right)\right]\delta z\right\}dt$$

If δx, δy, and δz are independent, the only way in which the integral δI can vanish is for:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) - \frac{\partial f}{\partial x} &= 0 \\ \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) - \frac{\partial f}{\partial y} &= 0 \\ \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{z}}\right) - \frac{\partial f}{\partial z} &= 0 \end{aligned} \tag{120}$$

These are Euler's equations for this problem.

If the integral to be extremized contains derivatives higher than the first, as for example:

$$I = \int_A^B f(y, y', y'', x)\, dx$$

the condition for a stationary value becomes:

$$\delta I = \int_A^B \left(\frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial y'}\,\delta y' + \frac{\partial f}{\partial y''}\,\delta y''\right) dx = 0$$

Proceeding in the same way as above, integrating the second term by parts once and the third term twice, we obtain:

$$\frac{\partial f}{\partial y'}\,\delta y\,\Big|_A^B + \frac{\partial f}{\partial y''}\,\delta y'\,\Big|_A^B - \frac{d}{dx}\frac{\partial f}{\partial y''}\,\delta y\,\Big|_A^B + \int_A^B \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\right] \delta y\, dx = 0$$

So that the necessary condition is:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0 \tag{121}$$

EXAMPLE 1. Find the equation of the plane curve which has a minimum length between two points, using Euler's differential equation.

Solution. In this problem, as shown above:

$$I = \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

So that Euler's equation becomes:

$$\frac{d}{dx}\left[\frac{\partial}{\partial y'}\sqrt{1+y'^2}\right] - \frac{\partial}{\partial y}[\sqrt{1+y'^2}] = 0$$

$$\frac{d}{dx}\left[\frac{y'}{(1+y'^2)^{\frac{1}{2}}}\right] = 0$$

$$\frac{y'}{(1+y'^2)^{\frac{1}{2}}} = C_1'$$

As was shown above, the solution of this equation is:

$$y = C_1x + C_2$$

Example 2. A particle, under the action of gravity, slides along a smooth curve which lies in a vertical plane. Find the form of the curve for which the time required for the particle to move from A to B is a minimum (Fig. 9–8). This is the famous problem

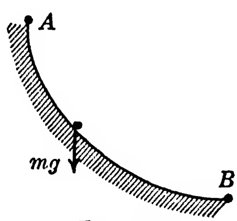


Fig. 9–8

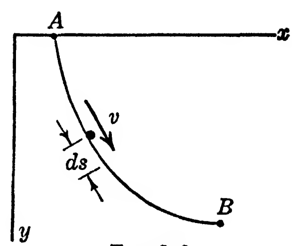


Fig. 9–9

of the *Brachistochrone*, which was formulated in 1696 by John Bernoulli and which began the modern development of the calculus of variations.

Solution. The time required for the particle to traverse an arc length ds is $\frac{ds}{v}$ (Fig. 9–9), so that the total time required for the particle to travel from A to B is given by:

$$I = \int_A^B \frac{ds}{v}$$

From the principle of the conservation of energy we have $v = \sqrt{2gy}$, so that the integral becomes:

$$I = \int_A^B \left[\frac{1+y'^2}{2gy}\right]^{\frac{1}{2}} dx$$

This integral is of the form:

$$I = \int_A^B f(y, y')\, dx$$

If Euler's differential equation is applied directly, we obtain:

$$\frac{d}{dx}\left\{\frac{\partial}{\partial y'}\left[\frac{1 + y'^2}{2gy}\right]^{\frac{1}{2}}\right\} - \frac{\partial}{\partial y}\left[\frac{1 + y'^2}{2gy}\right]^{\frac{1}{2}} = 0$$

This leads to a rather complicated differential equation, which we may avoid by noting that whenever the integral to be minimized is not a function of the independent variable x one integration of Euler's equation can be performed in general terms as follows:

$$I = \int_A^B f(y, y')\, dx$$

Substituting $f(y, y')$ in Euler's equation gives:

$$\frac{\partial^2 f}{\partial y' \partial y} y' + \frac{\partial^2 f}{\partial y'^2} y'' - \frac{\partial f}{\partial y} = 0$$

Now multiply through by y', and add and subtract the term $y'' \dfrac{\partial f}{\partial y'}$:

$$y'^2 \frac{\partial^2 f}{\partial y' \partial y} + y'y'' \frac{\partial^2 f}{\partial y'^2} + y'' \frac{\partial f}{\partial y'} - y'' \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} y' = 0$$

The first three terms of this expression are equal to $\dfrac{d}{dx}\left(y' \dfrac{\partial f}{\partial y'}\right)$ and the last two terms are equal to $-\dfrac{df}{dx}$, so that we have:

$$\frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) - \frac{df}{dx} = 0$$

Integrating this directly gives:

$$y' \frac{\partial f}{\partial y'} - f = C \tag{122}$$

For the Brachistochrone problem:

$$f = \left[\frac{1 + y'^2}{2gy}\right]^{\frac{1}{2}} \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{y'}{[2gy(1 + y'^2)]^{\frac{1}{2}}}$$

and the differential equation is:

$$\frac{(y')^2}{[2gy(1+y'^2)]^{\frac{1}{2}}} - \left[\frac{1+y'^2}{2gy}\right]^{\frac{1}{2}} = C$$

Solving for y' gives:

$$\frac{dy}{dx} = \sqrt{\frac{1-2C^2gy}{2C^2gy}}$$

This may be shown to be the equation of a cycloid which is the curve traced by a point on the circumference of a circle as the circle

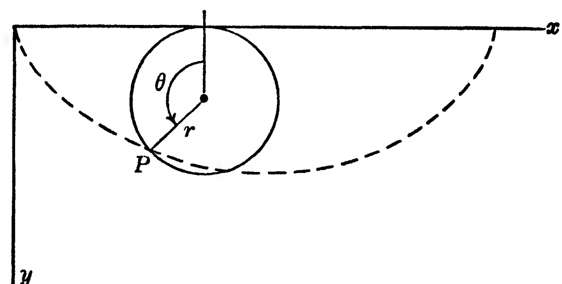


FIG. 9–10

rolls along a straight line. In the coordinate system shown in Fig. 9–10, the coordinates of the point P which traces out the cycloid are:

$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos\theta)$$

Substituting these into the equation for dy/dx, we obtain:

$$\frac{\sin\theta}{1-\cos\theta} = \left[\frac{1-2C^2gr(1-\cos\theta)}{2C^2gr(1-\cos\theta)}\right]^{\frac{1}{2}}$$

Squaring both sides and setting $\sin^2\theta = 1 - \cos^2\theta = (1-\cos\theta)$ $(1+\cos\theta)$, we have:

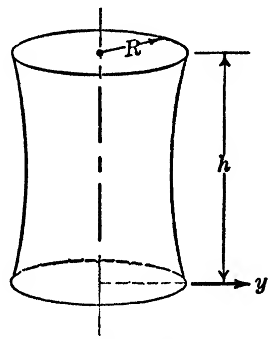
$$\frac{1+\cos\theta}{1-\cos\theta} = \frac{1-2C^2gr+2C^2gr\cos\theta}{2C^2gr(1-\cos\theta)}$$

If we put $2C^2gr = \frac{1}{2}$, this equation is identically satisfied, so that the Brachistochrone is a cycloid of radius $r = \dfrac{1}{4C^2g}$.

PROBLEMS

304. A flexible cable of span l hangs under its own weight. The shape which it assumes is such that the total potential energy is a minimum. Set up the integral for the potential energy of the system and minimize this integral to find the equation of the cable.

305. A container of height h has a circular top and bottom, each of radius R. The surface of revolution of the container is axially sym-



PROB. 305

metrical. What should be the shape of the sides to give a minimum surface area?

306.* In the Brachistochrone problem the stationary value of the integral $\int \frac{ds}{v}$ is to be found. Substitute $v = \sqrt{2gy}$ and $ds = \sqrt{1 + x'^2}\, dy$ where $x' = \frac{dx}{dy}$. Treating y as the independent variable and x as the dependent variable, show that this leads to the correct solution of the problem.

70. Hamilton's Principle. It will be observed that Euler's differential equation, Equation (120), which is the condition for extremizing a certain integral, has the same form as Lagrange's equations of motion. Writing Lagrange's equation for a conservative system in the form:

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}(T - V)\right] - \frac{\partial}{\partial q}(T - V) = 0$$

we see that an equivalent form is:

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0 \tag{123}$$

This is called *Hamilton's principle*, and it states that the actual path followed by a dynamical process is such as to make the integral of $(T - V)$ a minimum.

For non-conservative systems, Hamilton's principle may be written:

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0 \tag{124}$$

where $\delta W = \Sigma Q_i\, \delta q_i$, $Q_i\, \delta q_i$ being the increment of work corresponding to the variation δq_i of the coordinate q_i.

EXAMPLE 1. Find by Hamilton's principle the equation of motion for a single-degree-of-freedom, undamped harmonic oscillator.

Solution. For such a system, $T = \frac{1}{2}m\dot{x}^2$ and $V = \frac{1}{2}kx^2$; thus:

$$\delta \int_{t_1}^{t_2} (\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2)\, dt = 0$$

Applying Euler's differential equation to minimize the integral we have:

$$f = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2$$

hence:

$$\frac{d}{dt}(m\dot{x}) + kx = 0$$

$$m\ddot{x} + kx = 0$$

EXAMPLE 2. A straight beam having a length l, a mass per unit length ρ, a cross-sectional moment of inertia I, and a modulus of elasticity E performs free transverse vibrations as indicated in Fig. 9–11. Using Hamilton's principle, find the differential equation of motion of the system.

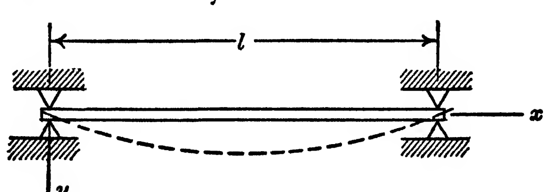


FIG. 9–11

Solution. From the theory of elasticity we know that the potential energy of bending of the beam is:

$$V = \frac{1}{2}\int_0^l \frac{M^2}{EI}\, dx$$

where M is the bending moment in the beam, and $M = EIy'' = EI\frac{\partial^2 y}{\partial x^2}$. The kinetic energy of the beam is $T = \frac{1}{2}\int_0^l \rho\dot{y}^2\,dx$ where $\dot{y} = \frac{\partial y}{\partial t}$. Hamilton's principle now states:

$$\delta\int_{t_1}^{t_2}\left[\int_0^l\left(\frac{1}{2}\rho\dot{y}^2 - \frac{EI}{2}y''^2\right)dx\right]dt = 0$$

Instead of applying Euler's differential equation to the solution of the problem, we may go through the steps which lead to Euler's equation. In some instances this procedure may be simpler than a direct substitution and solution of Euler's equation.

Performing the variation indicated by Hamilton's principle, we have:

$$\int_{t_1}^{t_2}\int_0^l(\rho\dot{y}\delta\dot{y} - EIy''\delta y'')\,dx\,dt = 0$$

Writing $\delta\dot{y} = \frac{\partial}{\partial t}\delta y$, $\delta y'' = \frac{\partial^2(\delta y)}{\partial x^2}$ and integrating by parts, we obtain:

$$\int_{t_1}^{t_2}\int_0^l(-\rho\ddot{y} - EIy^{\mathrm{IV}})\delta y\,dx\,dt = 0$$

The integrated terms disappear since they are all equal to zero. Now, since δy is arbitrary the expression within the brackets must be zero, thus giving the differential equation for the vibration of the beam:

$$EI\frac{\partial^4 y}{\partial x^4} + \rho\frac{\partial^2 y}{\partial t^2} = 0$$

As may be seen from the foregoing examples, Hamilton's principle does not lead to simplifications when applied to the problems with which we have been concerned in this book. In more advanced analysis, however, Hamilton's principle is the starting point for a number of developments in the science of mechanics. It should be noted that Hamilton's principle is a restatement of Newton's laws and cannot, of course, introduce any new information into the subject.

APPENDIX I

BIBLIOGRAPHY

The subjects discussed in this book are treated in a large number of publications. The bibliography below includes only a selection of those books which present the subject from essentially the same point of view as this book. These references may thus be of assistance to the student who desires additional explanations, illustrative examples, and problems.

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Specific chapter references are as follows:

Chapter I, Ref. 2, 9, 11, 15
Chapter II, Ref. 12, 13, 17, 22
Chapter III, Ref. 8, 12, 13, 17, 22
Chapter IV, Ref. 12, 17, 20, 21, 22, 24
Chapter V, Ref. 6, 13, 23, 24
Chapter VI, Ref. 8, 12, 13, 17, 22
Chapter VII, Ref. 8, 12, 13, 17, 22, 24
Chapter VIII, Ref. 1, 7, 10, 12, 14, 16, 17, 18, 19
Chapter IX, Ref. 4, 5, 6, 8, 12, 13, 23, 25, 26, 27

APPENDIX II

UNITS OF MASS AND FORCE

In principle a single unit of mass with its corresponding unit of force is sufficient for all purposes. In practice, however, a number of different units may be encountered. Since units are often used without a specific statement as to their definition, misunderstandings may result.

Perhaps the greatest source of difficulty is associated with the fact that the same name "pound" is given both to a unit of force and to a unit of mass. It is customary among engineers to use the name "pound," without qualification, for a force. On the other hand, the only legal definitions existing in the United States define the pound as the unit of mass. The engineer should thus understand that any "standard weight" calibrated by the U. S. Bureau of Standards will be essentially a standard of mass, and that any measurement of the weight of a body, which is made by means of a balance or spring which has been calibrated by such a "standard weight" will be actually a measure of the mass of the body. The possible error due to a confusion of the two units might seem to be negligible, since the variation of the acceleration of gravity is small. It is conceivable, however, that this error might be of concern in the calibration of high-precision testing machines or in the weighing of very expensive materials. It should be understood that the statement that one pound-mass has a weight of one pound-force is only approximate, and the fact that the error involved is small should not be permitted to obscure the fundamental difference between the concepts of force and mass.

The following definitions will indicate the precise meanings which are to be attached to the various common terms.

Units of Mass

Standard Kilogram (kg)—The international standard of mass. The mass of a particular body in the possession of the International Committee of Weights and Measures in France.

Gram (g)—One one-thousandth part of the standard kilogram.

Pound, or *Pound-Mass* (U. S. Avoirdupois) (lb, lb-m.)—Legally defined as $\dfrac{1}{2.2046}$ part of the standard kilogram. The U. S. Bureau of Standards at present uses a more accurate definition which states that the pound is equal to 453.5924277 g.

British Imperial Pound—The mass of a platinum cylinder kept in the Standard's Office, England. The legal equivalent is 453.59243 g. Hence it may be considered as practically equivalent to the U. S. Standard Avoirdupois Pound.

Slug, or *Geepound*—A unit of mass having a magnitude such that a force of 1 lb applied to a body having a mass of 1 slug would result in an acceleration of 1 ft/sec.^{2}

Units of Force

Pound, *Pound-Force*, *Pound-Weight* (lb, lb-f, lb-wt)—The force required to give a mass of 1 lb an acceleration of 32.174 ft/sec.^{2}

Poundal—The force required to give a mass of 1 lb an acceleration of 1 ft/sec.^{2}

Dyne—The force required to give a mass of 1 g an acceleration of 1 cm/sec.^{2}

Gram-Weight—The force required to give a mass of 1 g an acceleration of 980.665 cm/sec.^{2} (32.174 ft/sec.^{2})

The names of the following quantities are frequently used incorrectly. The definitions given are believed to represent the most commonly accepted engineering practice.

Specific Weight—The weight per unit volume of a material.

Specific Mass—The mass per unit volume of a material.

Density—Same as Specific Mass.

Specific Gravity—The ratio of the specific mass of a material to the specific mass of some standard material. Unless otherwise stated the standard material is taken to be water at 4° C.

APPENDIX III

VECTOR PRODUCTS

Two different vector products are defined and used in mechanics. The *scalar* or *dot* product of two vectors is defined as the scalar quantity having a magnitude equal to the product of the magnitudes of the two vectors multiplied by the cosine of the angle between the two vectors.

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab \cos \theta$$

Scalar multiplication is both commutative and distributive, that is:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$
$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$$

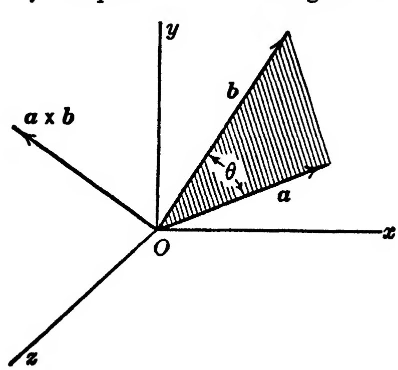
Written in terms of the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ along an orthogonal coordinate system:

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$$
$$\boldsymbol{b} = b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}$$
$$\boldsymbol{a} \cdot \boldsymbol{b} = a_xb_x + a_yb_y + a_zb_z$$

since

$$\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$$
$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{i} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{j} = 0$$

The *vector* or *cross* product of two vectors is defined as a vector whose magnitude is given by the product of the magnitudes of the two vectors



multiplied by the sine of the angle between the two vectors. The vector product is perpendicular to the plane containing the two vectors and has the direction of advance of a right-handed screw turned from the first vector to the second vector.

$$\boldsymbol{a} \times \boldsymbol{b} = ab \sin \theta \boldsymbol{e}_{\perp}$$

Vector multiplication is distributive:

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$$

but is not commutative:

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$

Written in terms of the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, we have:

$$\boldsymbol{a} \times \boldsymbol{b} = (a_y b_z - a_z b_y)\boldsymbol{i} + (a_z b_x - a_x b_z)\boldsymbol{j} + (a_x b_y - a_y b_x)\boldsymbol{k}$$

since

$$\begin{aligned} \boldsymbol{i} \times \boldsymbol{i} &= \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0 \\ \boldsymbol{i} \times \boldsymbol{j} &= -\boldsymbol{j} \times \boldsymbol{i} = \boldsymbol{k} \\ \boldsymbol{i} \times \boldsymbol{k} &= -\boldsymbol{k} \times \boldsymbol{i} = -\boldsymbol{j} \\ \boldsymbol{j} \times \boldsymbol{k} &= -\boldsymbol{k} \times \boldsymbol{j} = \boldsymbol{i} \end{aligned}$$

This may also be written as:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

APPENDIX IV

PROPERTIES OF PLANE SECTIONS

The following symbols will be used:

A = Area
x_c, y_c = Coordinates of centroid of section in xy coordinate system.
I_{x_c}, I_{y_c} = Moment of inertia about an axis through the centroid parallel to the xy axes.
r_{x_c}, r_{y_c} = Radius of gyration of the section with respect to the centroidal axes parallel to the xy axes.
$I_{x_cy_c}$ = Product of inertia with respect to the centroidal axes parallel to the xy axes.
I_x, I_y = Moment of inertia with respect to the xy axes shown.
r_x, r_y = Radius of gyration of the section with respect to the xy axes shown.
I_{xy} = Product of inertia with respect to the xy axes shown.
I_P = Polar moment of inertia about an axis passing through the centroid.
r_P = Radius of gyration of the section about the polar axis passing through the centroid.

G marks the centroid.

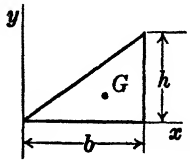
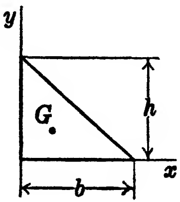
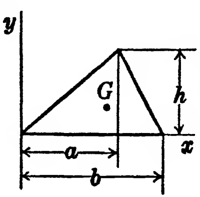
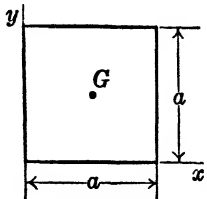
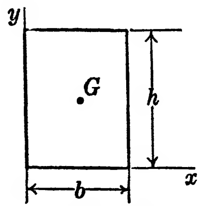
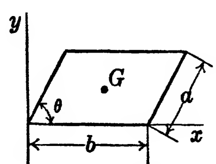
Figure	Area and Centroid	Moment of Inertia	r^2	Product of Inertia
1  Right Triangle	$A = \frac{1}{2}bh$ $x_c = \frac{2}{3}b$ $y_c = \frac{1}{3}h$	$I_{x_c} = \frac{bh^3}{36}$ $I_{y_c} = \frac{b^3h}{36}$ $I_x = \frac{bh^3}{12}$ $I_y = \frac{b^3h}{4}$	$r_{x_c}^2 = \frac{1}{18}h^2$ $r_{y_c}^2 = \frac{1}{18}b^2$ $r_x^2 = \frac{1}{6}h^2$ $r_y^2 = \frac{1}{2}b^2$	$I_{x_cy_c} = \frac{A}{36}hb = \frac{h^2b^2}{72}$ $I_{xy} = \frac{A}{4}hb = \frac{h^2b^2}{8}$
2 	$A = \frac{1}{2}bh$ $x_c = \frac{1}{3}b$ $y_c = \frac{1}{3}h$	$I_{x_c} = \frac{bh^3}{36}$ $I_{y_c} = \frac{b^3h}{36}$ $I_x = \frac{bh^3}{12}$ $I_y = \frac{b^3h}{12}$	$r_{x_c}^2 = \frac{1}{18}h^2$ $r_{y_c}^2 = \frac{1}{18}b^2$ $r_x^2 = \frac{1}{6}h^2$ $r_y^2 = \frac{1}{6}b^2$	$I_{x_cy_c} = -\frac{A}{36}hb = -\frac{h^2b^2}{72}$ $I_{xy} = \frac{A}{12}hb = \frac{h^2b^2}{24}$
3  Triangle	$A = \frac{1}{2}bh$ $x_c = \frac{1}{3}(a+b)$ $y_c = \frac{1}{3}h$	$I_{x_c} = \frac{bh^3}{36}$ $I_{y_c} = \frac{bh}{36}(b^2-ab+a^2)$ $I_x = \frac{bh^3}{12}$ $I_y = \frac{bh}{12}(b^2+ab+a^2)$	$r_{x_c}^2 = \frac{1}{18}h^2$ $r_{y_c}^2 = \frac{1}{18}(b^2-ab+a^2)$ $r_x^2 = \frac{1}{6}h^2$ $r_y^2 = \frac{1}{6}(b^2+ab+a^2)$	$I_{x_cy_c} = \frac{Ah}{36}(2a-b) = \frac{bh^2}{72}(2a-b)$ $I_{xy} = \frac{Ah}{12}(2a+b) = \frac{bh^2}{24}(2a+b)$

	Figure	Area and Centroid	Moment of Inertia	r^2	Product of Inertia
4	 Square	$A=a^2$ $x_c=\frac{1}{2}a$ $y_c=\frac{1}{2}a$	$I_{x_c}=I_{y_c}=\frac{A}{12}a^2$ $I_x=I_y=\frac{A}{3}a^2$ $I_P=\frac{A}{6}a^2$	$r_{x_c}^2=r_{y_c}^2=\frac{1}{12}a^2$ $r_x^2=r_y^2=\frac{1}{3}a^2$ $r_P^2=\frac{1}{6}a^2$	$I_{x_c y_c}=0$ $I_{xy}=\frac{A}{4}a^2=\frac{a^4}{4}$
5	 Rectangle	$A=bh$ $x_c=\frac{1}{2}b$ $y_c=\frac{1}{2}h$	$I_{x_c}=\frac{bh^3}{12}$ $I_{y_c}=\frac{b^3h}{12}$ $I_x=\frac{bh^3}{3}$ $I_y=\frac{b^3h}{3}$ $I_P=\frac{bh}{12}(b^2+h^2)$	$r_{x_c}^2=\frac{1}{12}h^2$ $r_{y_c}^2=\frac{1}{12}b^2$ $r_x^2=\frac{1}{3}h^2$ $r_y^2=\frac{1}{3}b^2$ $r_P^2=\frac{1}{12}(b^2+h^2)$	$I_{x_c y_c}=0$ $I_{xy}=\frac{A}{4}bh=\frac{b^2h^2}{4}$
6	 Parallelogram	$A=ab\sin\theta$ $x_c=\frac{1}{2}(b+a\cos\theta)$ $y_c=\frac{1}{2}(a\sin\theta)$	$I_{x_c}=\frac{a^3b}{12}\sin^3\theta$ $I_{y_c}=\frac{ab}{12}\sin\theta\,(b^2+a^2\cos^2\theta)$ $I_x=\frac{a^3b}{3}\sin^3\theta$ $I_y=\frac{ab}{3}\sin\theta\,(b+a\cos\theta)^2$ $-\frac{a^2b^2}{6}\sin\theta\cos\theta$	$r_{x_c}^2=\frac{1}{12}(a\sin\theta)^2$ $r_{y_c}^2=\frac{1}{12}(b^2+a^2\cos^2\theta)$ $r_x^2=\frac{1}{3}(a\sin\theta)^2$ $r_y^2=\frac{1}{3}(b+a\cos\theta)^2$ $-\frac{1}{6}(ab\cos\theta)$	$I_{x_c y_c}=\frac{a^3b}{12}\sin^2\theta\cos\theta$

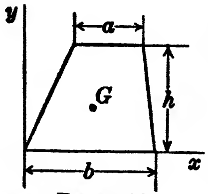
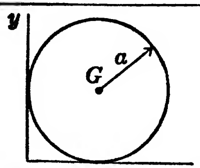
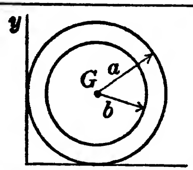
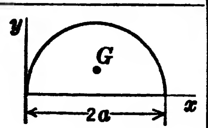
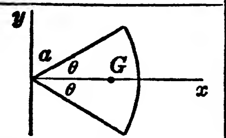
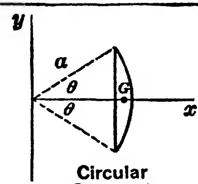
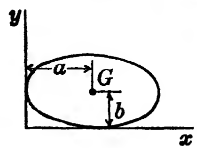
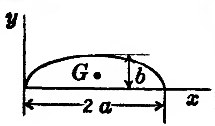
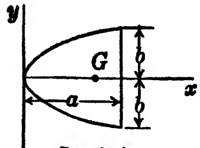
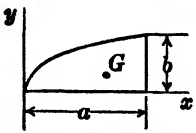
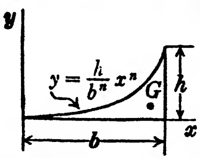
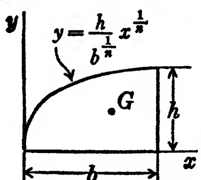
7	 Trapezoid	$A = \frac{1}{2}h(a+b)$ $y_c = \frac{1}{3}h\left(\frac{2a+b}{a+b}\right)$	$I_{x_c} = \frac{h^3(a^2+4ab+b^2)}{36(a+b)}$ $I_x = \frac{h^3(3a+b)}{12}$	${r_{x_c}}^2 = \frac{h^2(a^2+4ab+b^2)}{18(a+b)^2}$ ${r_x}^2 = \frac{h^2(3a+b)}{6(a+b)}$	
8	 Circle	$A = \pi a^2$ $x_c = a$ $y_c = a$	$I_{x_c} = \frac{1}{4}\pi a^4 = I_{y_c}$ $I_x = I_y = \frac{5}{4}\pi a^4$ $I_P = \frac{1}{2}\pi a^4$	${r_{x_c}}^2 = {r_{y_c}}^2 = \frac{1}{4}a^2$ ${r_x}^2 = {r_y}^2 = \frac{5}{4}a^2$ ${r_P}^2 = \frac{1}{2}a^2$	$I_{x_c y_c} = 0$ $I_{xy} = Aa^2$
9	 Annulus	$A = \pi(a^2 - b^2)$ $x_c = a$ $y_c = a$	$I_{x_c} = I_{y_c} = \frac{\pi}{4}(a^4 - b^4)$ $I_x = I_y = \frac{5}{4}\pi a^4 - \pi a^2 b^2 - \frac{\pi}{4}b^4$ $I_P = \frac{\pi}{2}(a^4 - b^4)$	${r_{x_c}}^2 = {r_{y_c}}^2 = \frac{1}{4}(a^2+b^2)$ ${r_x}^2 = {r_y}^2 = \frac{1}{4}(5a^2+b^2)$ ${r_P}^2 = \frac{1}{2}(a^2+b^2)$	$I_{x_c y_c} = 0$ $I_{xy} = Aa^2 = \pi a^2(a^2 - b^2)$
10	 Semi-circle	$A = \frac{1}{2}\pi a^2$ $x_c = a$ $y_c = \frac{4a}{3\pi}$	$I_{x_c} = \frac{a^4(9\pi^2 - 64)}{72\pi}$ $I_{y_c} = \frac{1}{8}\pi a^4$ $I_x = \frac{1}{8}\pi a^4$ $I_y = \frac{5}{8}\pi a^4$	${r_{x_c}}^2 = \frac{a^2(9\pi^2 - 64)}{36\pi^2}$ ${r_{y_c}}^2 = \frac{1}{4}a^2$ ${r_x}^2 = \frac{1}{4}a^2$ ${r_y}^2 = \frac{5}{4}a^2$	$I_{x_c y_c} = 0$ $I_{xy} = \frac{2}{3}a^4$

	Figure	Area and Centroid	Moment of Inertia	r^2	Product of Inertia
11	 Circular Sector	$A = a^2\theta$ $x_c = \frac{2a}{3}\frac{\sin\theta}{\theta}$ $y_c = 0$	$I_x = \frac{1}{4}a^4(\theta - \sin\theta\cos\theta)$ $I_y = \frac{1}{4}a^4(\theta + \sin\theta\cos\theta)$	$r_x^2 = \frac{1}{4}a^2\left(\frac{\theta - \sin\theta\cos\theta}{\theta}\right)$ $r_y^2 = \frac{1}{4}a^2\left(\frac{\theta + \sin\theta\cos\theta}{\theta}\right)$	$I_{x_cy_c} = 0$ $I_{xy} = 0$
12	 Circular Segment	$A = a^2(\theta - \frac{1}{2}\sin 2\theta)$ $x_c = \frac{2a}{3}\left(\frac{\sin^3\theta}{\theta - \sin\theta\cos\theta}\right)$ $y_c = 0$	$I_x = \frac{Aa^2}{4}\left[1 - \frac{2\sin^3\theta\cos\theta}{3(\theta - \sin\theta\cos\theta)}\right]$ $I_y = \frac{Aa^2}{4}\left[1 + \frac{2\sin^3\theta\cos\theta}{\theta - \sin\theta\cos\theta}\right]$	$r_x^2 = \frac{a^2}{4}\left[1 - \frac{2\sin^3\theta\cos\theta}{3(\theta - \sin\theta\cos\theta)}\right]$ $r_y^2 = \frac{a^2}{4}\left[1 + \frac{2\sin^3\theta\cos\theta}{\theta - \sin\theta\cos\theta}\right]$	$I_{x_cy_c} = 0$ $I_{xy} = 0$
13	 Ellipse	$A = \pi ab$ $x_c = a$ $y_c = b$	$I_{x_c} = \frac{\pi}{4}ab^3$ $I_{y_c} = \frac{\pi}{4}a^3b$ $I_x = \frac{5}{4}\pi ab^3$ $I_y = \frac{5}{4}\pi a^3b$ $I_P = \frac{\pi ab}{4}(a^2+b^2)$	$r_{x_c}^2 = \frac{1}{4}b^2$ $r_{y_c}^2 = \frac{1}{4}a^2$ $r_x^2 = \frac{5}{4}b^2$ $r_y^2 = \frac{5}{4}a^2$ $r_P^2 = \frac{1}{4}(a^2 + b^2)$	$I_{x_cy_c} = 0$ $I_{xy} = Aab = \pi a^2b^2$
14	 Semi-ellipse	$A = \frac{1}{2}\pi ab$ $x_c = a$ $y_c = \frac{4b}{3\pi}$	$I_{x_c} = \frac{ab^3}{72\pi}(9\pi^2 - 64)$ $I_{y_c} = \frac{\pi}{8}a^3b$ $I_x = \frac{\pi}{8}ab^3$ $I_y = \frac{5}{8}\pi a^3b$	$r_{x_c}^2 = \frac{b^2}{36\pi^2}(9\pi^2 - 64)$ $r_{y_c}^2 = \frac{1}{4}a^2$ $r_x^2 = \frac{1}{4}b^2$ $r_y^2 = \frac{5}{4}a^2$	$I_{x_cy_c} = 0$ $I_{xy} = \frac{2}{3}a^2b^2$

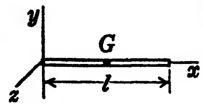
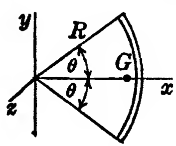
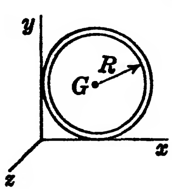
15  Parabola	$A = \frac{4}{3}ab$ $x_c = \frac{3}{5}a$ $y_c = 0$	$I_{x_c} = I_x = \frac{4}{15}ab^3$ $I_{y_c} = \frac{16}{175}a^3b$ $I_y = \frac{4}{7}a^3b$	$r_{x_c}^2 = r_x^2 = \frac{1}{5}b^2$ $r_{y_c}^2 = \frac{12}{175}a^2$ $r_y^2 = \frac{3}{7}a^2$	$I_{x_cy_c} = 0$ $I_{xy} = 0$
16  Semi-parabola	$A = \frac{2}{3}ab$ $x_c = \frac{3}{5}a$ $y_c = \frac{3}{8}b$	$I_x = \frac{2}{15}ab^3$ $I_y = \frac{2}{7}a^3b$	$r_x^2 = \frac{1}{5}b^2$ $r_y^2 = \frac{3}{7}a^2$	$I_{xy} = \frac{A}{4}ab = \frac{1}{6}a^2b^2$
17  n[th] degree parabola	$A = \frac{bh}{n+1}$ $x_c = \frac{n+1}{n+2}b$ $y_c = \frac{h}{2}\left(\frac{n+1}{2n+1}\right)$	$I_x = \frac{bh^3}{3(3n+1)}$ $I_y = \frac{hb^3}{n+3}$	$r_x^2 = \frac{h^2(n+1)}{3(3n+1)}$ $r_y^2 = \frac{n+1}{n+3}b^2$	
18  n[th] degree parabola	$A = \frac{n}{n+1}bh$ $x_c = \frac{n+1}{2n+1}b$ $y_c = \frac{n+1}{2(n+2)}h$	$I_x = \frac{n}{3(n+3)}bh^3$ $I_y = \frac{n}{3n+1}hb^3$	$r_x^2 = \frac{n+1}{3(n+3)}h^2$ $r_y^2 = \frac{n+1}{3n+1}b^2$	

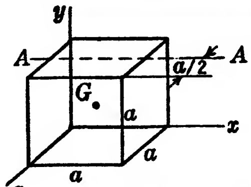
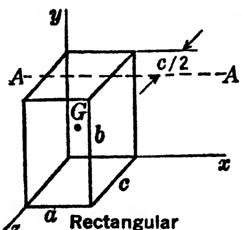
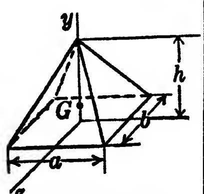
PROPERTIES OF HOMOGENEOUS BODIES

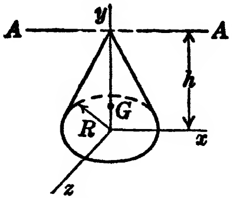
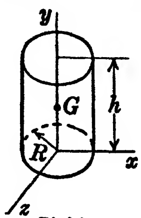
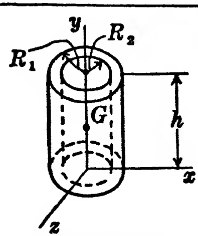
The following symbols will be used:

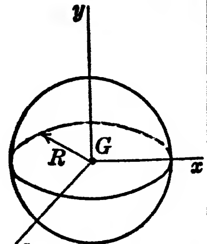
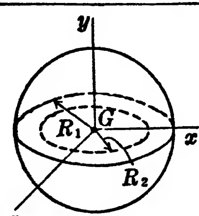
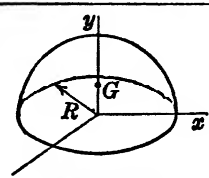
ρ = Mass density
M = Mass
x_c, y_c, z_c = Coordinates of centroid in xyz coordinate system.
$I_{x_c}, I_{y_c}, I_{z_c}$ = Moment of inertia about an axis through the centroid parallel to the xyz axes shown.
$r_{x_c}, r_{y_c}, r_{z_c}$ = Radius of gyration of the body with respect to the centroidal axes parallel to the xyz axes shown.
$I_{x_c y_c}, I_{x_c z_c}$, etc. = Product of inertia with respect to the centroidal axes parallel to the xyz axes shown.
I_x, I_y, I_z = Moment of inertia with respect to the xyz axes shown.
r_x, r_y, r_z = Radius of gyration of the body with respect to the xyz axes shown.
I_{xy}, I_{xz}, etc. = Product of inertia with respect to the xyz axes shown.
I_{AA}, r_{AA} = Moments of inertia and radii of gyration with respect to special axes shown.

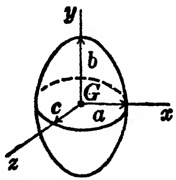
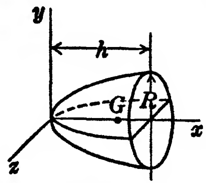
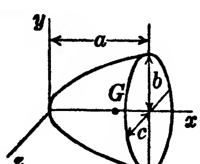
G marks the centroid

	Body	Mass and Centroid	Moment of Inertia	r^2	Product of Inertia
19	 Thin Rod	$M = \rho l$ $x_c = \frac{1}{2}l$ $y_c = 0$ $z_c = 0$	$I_x = I_{x_c} = 0$ $I_{y_c} = I_{z_c} = \frac{M}{12} l^2$ $I_y = I_z = \frac{M}{3} l^2$	$r_x^2 = r_{x_c}^2 = 0$ $r_{y_c}^2 = r_{z_c}^2 = \frac{1}{12}l^2$ $r_y^2 = r_z^2 = \frac{1}{3}l^2$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
20	 Thin Circular Rod	$M = 2\rho R\theta$ $x_c = \frac{R \sin\theta}{\theta}$ $y_c = 0$ $z_c = 0$	$I_x = I_{x_c}$ $= \frac{MR^2(\theta - \sin\theta\cos\theta)}{2\theta}$ $I_y = \frac{MR^2(\theta + \sin\theta\cos\theta)}{2\theta}$ $I_z = MR^2$	$r_x^2 = r_{x_c}^2 = \frac{R^2(\theta - \sin\theta\cos\theta)}{2\theta}$ $r_y^2 = \frac{R^2(\theta + \sin\theta\cos\theta)}{2\theta}$ $r_z^2 = R^2$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
21	 Thin Hoop	$M = 2\pi\rho R$ $x_c = R$ $y_c = R$ $z_c = 0$	$I_{x_c} = I_{y_c} = \frac{M}{2} R^2$ $I_{z_c} = MR^2$ $I_x = I_y = \frac{3}{2}MR^2$ $I_z = 3MR^2$	$r_{x_c}^2 = r_{y_c}^2 = \frac{1}{2}R^2$ $r_{z_c}^2 = R^2$ $r_x^2 = r_y^2 = \frac{3}{2}R^2$ $r_z = 3R^2$	$I_{x_c y_c}$, etc. $= 0$ $I_{xy} = MR^2$ $I_{xz} = I_{yz} = 0$

Body	Mass and Centroid	Moment of Inertia	r^2	Product of Inertia
22  Cube	$M = \rho a^3$ $x_c = \frac{1}{2}a$ $y_c = \frac{1}{2}a$ $z_c = \frac{1}{2}a$	$I_{x_c} = I_{y_c} = I_{z_c} = \frac{1}{6}Ma^2$ $I_x = I_y = I_z = \frac{2}{3}Ma^2$ $I_{AA} = \frac{5}{12}Ma^2$	$r_{x_c}^2 = r_{y_c}^2 = r_{z_c}^2 = \frac{1}{6}a^2$ $r_x^2 = r_y^2 = r_z^2 = \frac{2}{3}a^2$ $r_{AA}^2 = \frac{5}{12}a^2$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= \frac{1}{4}Ma^2$
23  Rectangular Prism	$M = \rho abc$ $x_c = \frac{1}{2}a$ $y_c = \frac{1}{2}b$ $z_c = \frac{1}{2}c$	$I_{x_c} = \frac{1}{12}M(b^2 + c^2)$ $I_x = \frac{1}{3}M(b^2 + c^2)$ $I_{AA} = \frac{1}{12}M(4b^2 + c^2)$	$r_{x_c}^2 = \frac{1}{12}(b^2 + c^2)$ $r_x^2 = \frac{1}{3}(b^2 + c^2)$ $r_{AA}^2 = \frac{1}{12}(4b^2 + c^2)$	$I_{x_c y_c}$, etc. $= 0$ $I_{xy} = \frac{1}{4}Mab$ $I_{xz} = \frac{1}{4}Mac$ $I_{yz} = \frac{1}{4}Mbc$
24  Right Rectangular Pyramid	$M = \frac{1}{3}\rho abh$ $x_c = 0$ $y_c = \frac{1}{4}h$ $z_c = 0$	$I_{x_c} = \frac{1}{80}M(4b^2 + 3h^2)$ $I_{y_c} = I_y = \frac{1}{20}M(a^2 + b^2)$ $I_x = \frac{1}{20}M(b^2 + 2h^2)$ $I_z = \frac{1}{20}M(a^2 + 2h^2)$	$r_{x_c}^2 = \frac{1}{80}(4b^2 + 3h^2)$ $r_{y_c}^2 = r_y^2 = \frac{1}{20}(a^2 + b^2)$ $r_x^2 = \frac{1}{20}(b^2 + 2h^2)$ $r_z^2 = \frac{1}{20}(a^2 + 2h^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$

25	 Right Circular Cone	$M = \frac{1}{3}\pi\rho R^2 h$ $x_c = 0$ $y_c = \frac{1}{4}h$ $z_c = 0$	$I_{x_c} = I_{z_c} = \frac{3M}{80}(4R^2 + h^2)$ $I_{y_c} = I_y = \frac{3}{10}MR^2$ $I_x = I_z = \frac{1}{20}M(3R^2 + 2h^2)$ $I_{AA} = \frac{3}{20}M(R^2 + 4h^2)$	$r_{x_c}{}^2 = r_{z_c}{}^2 = \frac{3}{80}(4R^2 + h^2)$ $r_{y_c}{}^2 = \frac{3}{10}R^2$ $r_x{}^2 = r_z{}^2 = \frac{1}{20}(3R^2 + 2h^2)$ $r_{AA}{}^2 = \frac{3}{20}(R^2 + 4h^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
26	 Right Circular Cylinder	$M = \pi\rho R^2 h$ $x_c = 0$ $y_c = \frac{1}{2}h$ $z_c = 0$	$I_{x_c} = I_{z_c} = \frac{1}{12}M(3R^2 + h^2)$ $I_{y_c} = I_y = \frac{1}{2}MR^2$ $I_x = I_z = \frac{1}{12}M(3R^2 + 4h^2)$	$r_{x_c}{}^2 = r_{z_c}{}^2 = \frac{1}{12}(3R^2 + h^2)$ $r_{y_c}{}^2 = r_y{}^2 = \frac{1}{2}R^2$ $r_x{}^2 = r_z{}^2 = \frac{1}{12}(3R^2 + 4h^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
27	 Hollow Right Circular Cylinder	$M = \pi\rho h(R_1^2 - R_2^2)$ $x_c = 0$ $y_c = \frac{1}{2}h$ $z_c = 0$	$I_{x_c} = I_{z_c}$ $= \frac{1}{12}M(3R_1^2 + 3R_2^2 + h^2)$ $I_{y_c} = I_y = \frac{1}{2}M(R_1^2 + R_2^2)$ $I_x = I_z$ $= \frac{1}{12}M(3R_1^2 + 3R_2^2 + 4h^2)$	$r_{x_c}{}^2 = r_{z_c}{}^2 = \frac{1}{12}(3R_1^2 + 3R_2^2 + h^2)$ $r_{y_c}{}^2 = r_y{}^2 = \frac{1}{2}(R_1^2 + R_2^2)$ $r_x{}^2 = r_z{}^2 = \frac{1}{12}(3R_1^2 + 3R_2^2 + 4h^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$

	Body	Mass and Centroid	Moment of Inertia	r^2	Product of Inertia
28	 Sphere	$M = \frac{4}{3}\pi\rho R^3$ $x_c = 0$ $y_c = 0$ $z_c = 0$	$I_{x_c} = I_x = \frac{2}{5}MR^2$ $I_{y_c} = I_y = \frac{2}{5}MR^2$ $I_{z_c} = I_z = \frac{2}{5}MR^2$	$r_{x_c}{}^2 = r_x{}^2 = \frac{2}{5}R^2$ $r_{y_c}{}^2 = r_y{}^2 = \frac{2}{5}R^2$ $r_{z_c}{}^2 = r_z{}^2 = \frac{2}{5}R^2$	$I_{x_cy_c}$, etc. $= 0$
29	 Hollow Sphere	$M = \frac{4}{3}\pi\rho(R_1^3 - R_2^3)$ $x_c = 0$ $y_c = 0$ $z_c = 0$	$I_x = I_y = I_z$ $= \frac{2}{5}M\frac{R_1^5 - R_2^5}{R_1^3 - R_2^3}$	$r_x{}^2 = r_y{}^2 = r_z{}^2$ $= \frac{2}{5}\frac{R_1^5 - R_2^5}{R_1^3 - R_2^3}$	I_{xy}, etc. $= 0$
30	 Hemisphere	$M = \frac{2}{3}\pi\rho R^3$ $x_c = 0$ $y_c = \frac{3}{8}R$ $z_c = 0$	$I_x = I_y = I_z = \frac{2}{5}MR^2$	$r_x{}^2 = r_y{}^2 = r_z{}^2 = \frac{2}{5}R^2$	$I_{x_cy_c}$, etc. $= 0$ I_{xy}, etc. $= 0$

31  Ellipsoid	$M = \frac{4}{3}\pi\rho abc$ $x_c = 0$ $y_c = 0$ $z_c = 0$	$I_x = \frac{1}{5}M(b^2 + c^2)$ $I_y = \frac{1}{5}M(a^2 + c^2)$ $I_z = \frac{1}{5}M(a^2 + b^2)$	$r_x^2 = \frac{1}{5}(b^2 + c^2)$ $r_y^2 = \frac{1}{5}(a^2 + c^2)$ $r_z^2 = \frac{1}{5}(a^2 + b^2)$	I_{xy}, etc. $= 0$
32  Paraboloid of Revolution	$M = \frac{1}{2}\pi\rho R^2 h$ $x_c = \frac{2}{3}h$ $y_c = 0$ $z_c = 0$	$I_{x_c} = I_x = \frac{1}{3}MR^2$ $I_{y_c} = I_{z_c} = \frac{1}{18}M(3R^2 + h^2)$ $I_y = I_z = \frac{1}{6}M(R^2 + 3h^2)$	$r_{x_c}^2 = r_x^2 = \frac{1}{3}R^2$ $r_{y_c}^2 = r_{z_c}^2 = \frac{1}{18}(3R^2 + h^2)$ $r_y^2 = r_z^2 = \frac{1}{6}(R^2 + 3h^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
33  Elliptic Paraboloid	$M = \frac{1}{2}\pi\rho abc$ $x_c = \frac{2}{3}a$ $y_c = 0$ $z_c = 0$	$I_{x_c} = I_x = \frac{1}{6}M(b^2 + c^2)$ $I_{y_c} = \frac{1}{18}M(3c^2 + a^2)$ $I_{z_c} = \frac{1}{18}M(3b^2 + a^2)$ $I_y = \frac{1}{6}M(c^2 + 3a^2)$ $I_z = \frac{1}{6}M(b^2 + 3a^2)$	$r_{x_c}^2 = r_x^2 = \frac{1}{6}(b^2 + c^2)$ $r_{y_c}^2 = \frac{1}{18}(3c^2 + a^2)$ $r_{z_c}^2 = \frac{1}{18}(3b^2 + a^2)$ $r_y^2 = \frac{1}{6}(c^2 + 3a^2)$ $r_z^2 = \frac{1}{6}(b^2 + 3a^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$

ANSWERS TO PROBLEMS

10. Terminal velocity is proportional to $\sqrt{r}$.

12. Using measured velocity, $mv(1 \pm 0.055)$
Using computed velocity, $mv(1 \pm 0.045)$

14. ± 912 ft

15. $R(1 \pm 0.0305)$

16. 4.4 ft/sec^2; 8.8 ft/sec^2

17. 377 sec; $v = 3v'$

18. $x = x_0 + \dfrac{v_0(v - v_0)}{a} + \dfrac{(v - v_0)^2}{2a}$

21. $\dot{e}_r = \dot{\theta} e_\theta + \dot{\phi} \sin \theta e_\phi$
$\dot{e}_\phi = -\dot{\phi} \cos \theta e_\theta - \dot{\phi} \sin \theta e_r$
$\dot{e}_\theta = -\dot{\theta} e_r + \dot{\phi} \cos \theta e_\phi$

22. $\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r} e_r + r\dot{\theta} e_\theta + r\dot{\phi} \sin \theta e_\phi$
$\boldsymbol{a} = \ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta) e_r$
$+ (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2 \sin \theta \cos \theta) e_\theta$
$+ (2\dot{r}\dot{\phi} \sin \theta + r\ddot{\phi} \sin \theta$
$+ 2r\dot{\phi}\dot{\theta} \cos \theta) e_\phi$

24. $\boldsymbol{v} = 56\boldsymbol{i} - 140\boldsymbol{j} - 112\boldsymbol{k}$ ft/sec

26. (b) $\alpha a e_t$; $-a\omega^2 e_n$

27. $\boldsymbol{a}_A = -(\Omega^2 r + \omega^2 r + 2\omega\Omega r) e_n$

28. $a = 79{,}900$ ft/sec^2

30. $\mathbf{a} = -8730\boldsymbol{i} - 443\boldsymbol{j} + 264\boldsymbol{k}$ ft/sec^2

31. $\mathbf{a} = 5.34 \times 10^{-4} e_\phi - 0.084 e_r - 0.049 e_\theta$ ft/sec^2

33. (a) $\dot{x} = a - 3bt^2$; $\dot{y} = \dot{z} = 0$
$\ddot{x} = -6bt$; $\ddot{y} = \ddot{z} = 0$;
$$x_{\max} = a\sqrt{\frac{a}{3b}} - b\left(\frac{a}{3b}\right)^{\frac{3}{2}}$$
(b) $F_x = -4.66$ lb; $F_y = F_z = 0$

34. $\dot{x} = \dfrac{v}{\sqrt{1 + 4a^2B^2}}$,
$\dot{y} = \dfrac{-2aBv}{\sqrt{1 + 4a^2B^2}}$;
$a_n = \dfrac{-2Bv^2}{(1 + 4a^2B^2)^{\frac{3}{2}}}$

35. $\dfrac{1}{2}\dfrac{m}{C_2} \log \left(1 + \dfrac{C_2}{C_1} v_0^2\right)$

36. $v_0 = \sqrt{2\mu g l}$

37. $v_0 = \sqrt{\dfrac{2gl}{\sqrt{10{,}000 + a^2}}(100\mu \pm a)}$

39. (a) 3186 lb; (b) 53.7 ft/sec^2

40. (a) 805 ft; (b) 526 ft

41. $3mg$

42. $t = \dfrac{v_0}{g \sin \theta}$; $l = \dfrac{1}{2}\dfrac{v_0^2}{g \sin \theta}$

43. for $t < 0.6\dfrac{m_0}{c}$,
$$y = \frac{Pt}{c} - \frac{P}{c^2}(m_0 - ct) \log \left(\frac{m_0}{m_0 - ct}\right) - \frac{gt^2}{2}$$

44. $v = v_0 - \dfrac{P}{m} t$

45. 1010 ft/sec

46. 6.78 ft/sec

47. (a) -2.2 ft/sec; (b) 80.5 ft/sec

48. 2.61 lb sec; $\theta_x = 102°50'$

49. $F = \rho A v^2$; $F = 72.6$ lb

51. 384 ft/sec

52. $\frac{1}{2}kx^2$

54. $-\dfrac{k}{2}(\sqrt{L^2 + r^2} - l)^2$

55. $k' = \dfrac{1}{\sum_i \dfrac{1}{k_i}}$

56. $k' = \sum_i k_i$

57. $\dfrac{T^4}{8m}(A^2 + B^2)$

58. 44.9 ft/sec

59. 36,900 ft/sec

60. $v_1 = \sqrt{2g(h_2 - h_1) + v_2^2}$

61. 5.09 ft/sec

62. (b) $\sqrt{2gx\left(\dfrac{W_2 - W_1 - F_D}{W_1 + W_2}\right)}$

63. $\delta\left[1 + \sqrt{1 + \dfrac{2h}{\delta}}\right]$

65. 16.1 in.; 12.4 in.

66. 3.86 ft/sec

67. 373,000 lb

68. 317 ft; 23.7 sec

69. 23.9 ft/sec; 212.8 ft

70. 1.86×10^6 ft-lb;
$F_{avg} = 186{,}000$ lb

71. 8.25 ft

72. 17.3 mph

73. $F_r = \dfrac{e_1 e_2}{r^2}$; $F_x = \dfrac{e_1 e_2 x}{r^3}$

74. 20.7 ft/sec; 14.4 lb sec; 0.965 sec

75. $y_{max} = y_{static}\left[1 + \sqrt{1 + \dfrac{2h}{y_{static}}}\right]$

76. $c = \sqrt{\dfrac{E}{\rho}}$; $v = \dfrac{\sigma}{\sqrt{E\rho}}$

77. $z = \dfrac{W}{k}\left[t - \dfrac{m}{k}\left(1 - e^{-\frac{k}{m}t}\right)\right]$

78. $z = \dfrac{m}{k}\left(\log\cosh\sqrt{\dfrac{kg}{m}}\,t\right)$

79. $\sqrt{\dfrac{W}{k}}$

80. 584 ft/sec; $k = \dfrac{W}{20}\dfrac{\text{lb sec}}{\text{ft}}$; 73.5 ft

81. (a) $n < 1$; $t = \dfrac{v_0^{(1-n)}}{k(1-n)}$

(b) $n < 2$

(c) $S_{t=\infty} = \dfrac{v_0}{k}$

82. 11,600 ft

83. $\dfrac{v_0^2}{g}\sin 2\theta$

84. $R = \dfrac{2\sqrt{2}}{g}v_0^2\cos^2\theta\,(\tan\theta - 1)$;
$\theta_{R=\max} = 67\tfrac{1}{2}°$

85. $\dfrac{\sin\left(\dfrac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\dfrac{\theta_1 + \theta_2}{2}\right)} = \dfrac{g\Delta t}{2v_0}$

86. $x = \dfrac{m}{k}\log\left(\dfrac{k\dot{x}_0}{m}t + 1\right)$

$y = \dfrac{m}{k\dot{x}_0}\left(\dot{y}_0 + \dfrac{mg}{2k\dot{x}_0}\right)\log\left(\dfrac{k\dot{x}_0}{m}t + 1\right) - \dfrac{mg}{2k\dot{x}_0}t\left(\dfrac{k\dot{x}_0}{2m}t + 1\right)$

87. $\dot{x} = \sqrt{\dfrac{mg\sin\alpha}{k}}\tanh\dfrac{\sqrt{mgk\sin\alpha}}{m}t$;

$\dot{x}_{term} = \sqrt{\dfrac{mg\sin\alpha}{k}}$

88. $x_{\dot{y}=0} = \dfrac{\dot{x}_0\dot{y}_0}{g\left(1 + \dfrac{k}{m}\dfrac{\dot{y}_0}{g}\right)}$

90. $m\ddot{x} + \dfrac{kx}{(x^2+y^2)^{\frac{3}{2}}} = 0$;

$m\ddot{y} + \dfrac{ky}{(x^2+y^2)^{\frac{3}{2}}} = 0$

91. $x = A_x\sin\left(\sqrt{\dfrac{k_1}{m}}t + B_x\right)$;

$y = A_y\sin\left(\sqrt{\dfrac{k_2}{m}}t + B_y\right)$

92. (a) $x = r_2\sin\omega t$;

$y = r_1\sin\left(\dfrac{R_2}{R_1}\omega t\right)$

(b) $y = 2x$

(c) $x = \dfrac{2y}{r_1}\sqrt{r_1^2 - y^2}$

93. $g = -32.4$ ft/sec²

94. $mgR\left(1 - \dfrac{1}{2}\dfrac{R}{r}\right)$

98. $\dfrac{\Delta T}{T} = 1 - e^2$

100. 0.922

101. $\tan\theta_2 = -e\tan\theta_1$

102. $\dot{x}_2 = \dot{x}_1 - 2\mu\dot{y}_1$
$\dot{y}_2 = -\dot{y}_1$

103. Direction of approach = direction of departure for both (a) and (b)

105. 1.88×10^{10} cm/sec classical
1.64×10^{10} cm/sec relativistic

112. $p^2 = 2g/(\text{length of liquid column})$

113. $p^2 = \dfrac{2\mu g}{a}$

114. $m\ddot{x} + c\dot{x} + 2\dfrac{pV_0A^2x}{(V_0^2 - A^2x^2)} = 0$;

$f = \dfrac{1}{2\pi}\sqrt{\dfrac{2pA^2}{mV_0}}$

115. $p^2 = \dfrac{2ka^2}{ml^2} - \dfrac{g}{l}$

116. $(1+4a^2x^2)\ddot{x}+4a^2x\dot{x}^2+2agx = 0;$
$f = \dfrac{1}{2\pi}\sqrt{2ag}$

119. 3.92 in.; 2.02 cycles per second

122. $n = 0.191\ \text{sec}^{-1}; \tau = 0.905$ sec

123. 0.0825 ft; 0.032 ft

124. $x = (C_1 + C_2t)e^{-pt}$

126. 0.083 ft

127. $n = 8.42\ \text{sec}^{-1}$

128. $\left(\dfrac{A}{e}\right) = \dfrac{\left(\dfrac{\omega}{p}\right)^2}{\sqrt{\left[1-\left(\dfrac{\omega}{p}\right)^2\right]^2+\left[2\left(\dfrac{n}{n_c}\right)\left(\dfrac{\omega}{p}\right)\right]^2}}$

129. $x = \dfrac{F}{k}(1 - \cos pt)$

131. $x = \dfrac{F_0}{2mp}\left(\dfrac{\sin pt}{p} - t\cos pt\right)$

132. $\dfrac{2ak\pi^2v^2}{(kl^2 - 4m\pi^2v^2)}\left(\cos\dfrac{l}{v}\sqrt{\dfrac{k}{m}} - 1\right)$

133. $k = 135$ lb/in.

134. 83.7%

135. $k = 18{,}500$ lb/in.

136. 115 lb; 588 lb/in.

140. 0.008 in.

153. $x = \dfrac{(F - \mu mg)l}{[F - \mu(m + M)g]}$

154. $8mv\boldsymbol{j}$; $6.5mv^2$; $5.33mv^2$; $\boldsymbol{v}_c = \frac{4}{3}v\boldsymbol{j}$;
$\boldsymbol{v}_c = \dfrac{v}{6}(\boldsymbol{i} + 11\boldsymbol{j})$

156. $\left(S - \dfrac{Ml}{m + M}\right)$

158. $m\left[\left(\dfrac{l\dot{\phi}}{2}\right)^2 + (gt)^2\right]$

159. $\omega = \dfrac{v}{\left(\dfrac{l}{2}\right)}$; $v_c = \dot{\phi}\left(\dfrac{l}{2}\right)$

160. $T = \dfrac{1}{2}\dfrac{F^2}{m}(\Delta t)^2$; $\dot{\phi} = \dfrac{F(\Delta t)}{ml}$;
$v_c = \dfrac{F(\Delta t)}{2m}$

161. $v_c = \frac{1}{2}v$; $\dot{\phi} = \dfrac{v}{l}$

163. $H = 4ml^2\omega$

164. $v_1 = \dfrac{m_1 - m_2}{m_1 + m_2}\sqrt{2gh}$;
$v_2 = \dfrac{2m_1}{m_1 + m_2}\sqrt{2gh}$

165. $v_1 = \dfrac{r_0}{r_1}v_0$

166. Deduce answer from the fact that $\boldsymbol{H}$ = constant, and momentum is $(m_1 + m_2)\boldsymbol{v}_c = (m_1 + m_2)[C_1\boldsymbol{i} + (C_2 - gt)\boldsymbol{j}]$

169. $v = 2v_0$

170. $\boldsymbol{v}_D = \dot{x}_0(\boldsymbol{i} - \boldsymbol{j})$

171. $v_C = \dfrac{v}{1 - \left(\dfrac{d_1}{d_2}\right)}$

172. $v_C = \dfrac{\dot{x}_1}{2}\left(1 + \dfrac{\dot{x}_2}{\dot{x}_1}\right)$

173. $\omega_2 = 22.6$ rad/sec;
$\omega_3 = 10.5$ rad/sec;
$\dot{\omega}_2 = 746\ \text{rad/sec}^2$;
$\dot{\omega}_3 = 945\ \text{rad/sec}^2$

174. $\boldsymbol{v}_A = 25\left(1 + \dfrac{\sqrt{3}}{2}\right)\boldsymbol{i} + 12.5\boldsymbol{j}$ ft/sec;
$\dot{\boldsymbol{v}}_A = -625\boldsymbol{j}\ \text{ft/sec}^2$

175. $v_C = 430$ in./sec;
$\dot{v}_C = 29{,}500\ \text{in./sec}^2$

176. $\boldsymbol{v}_A = 0$
$\dot{\bar{\boldsymbol{v}}}_A = -\dfrac{Rr}{(R - r)}\dot{\theta}^2\bar{\boldsymbol{e}}_r$
$\bar{\boldsymbol{v}}_c = r\dot{\theta}\bar{\boldsymbol{e}}_t$
$\dot{\bar{\boldsymbol{v}}}_c = -\dfrac{r^2\dot{\theta}^2}{(R - r)}\bar{\boldsymbol{e}}_r + r\ddot{\theta}\bar{\boldsymbol{e}}_t$

177. $\frac{1}{2}mr^2$

178. $\frac{1}{12}m(a^2 + b^2)$; $\frac{1}{3}m(a^2 + b^2)$

179. $\frac{1}{12}m(3R^2 + l^2)$

180. $\frac{1}{12}ml^2$

181. $I_y = 2190\rho$; $I_z = 476\rho\ \text{lb in. sec}^2$

182. $\frac{3}{10}mr^2$; $\frac{1}{20}m(3r^2 + 2h^2)$

183. $0.0518\ \text{lb ft sec}^2$

184. $I_{xy} = \dfrac{mR^2}{2} = I_{x'y'}$
$I_{yz} = I_{xz} = 0$
$I_{x'z'} = -\dfrac{\sqrt{2}}{4}mRl = I_{y'z'}$

188. $\frac{1}{24}ml^2\sin 2\alpha$

189. $\frac{1}{12}m(3R^2 + 4l^2)\sin^2\alpha + \frac{1}{2}mR^2\cos^2\alpha$

190. $I_{x'z'} = \frac{1}{8}M\left(R^2 - \frac{h^2}{3}\right)\sin 2\alpha;$
$I_{y'z'} = 0$

191. $\dfrac{Ma^2}{6}$

192. 0

193. $\frac{1}{8}MR^2\sin 2\alpha$

198. $\boldsymbol{\omega} = 30\boldsymbol{i} + 67.5\boldsymbol{j} + 85\boldsymbol{k}$
$T = 147{,}000$ ft lb

199. $\boldsymbol{F}_A = 51\boldsymbol{i}$ lb;
$\boldsymbol{F}_B = -35.5\boldsymbol{i} + 50\boldsymbol{j}$ lb

200. Slips at 1.34 ft/sec^2; tips at 8.05 ft/sec^2

201. $\left(\dfrac{\mu\sqrt{3} \mp 1}{\sqrt{3} \pm \mu}\right)g$

202. 9410 lb at each pin

203. $d = \left(1 - \dfrac{\mu W\sqrt{3}}{2F}\right)r$

reaction force $= \dfrac{\sqrt{3}}{2}W(1 + \mu^2)^{\frac{1}{2}}$

204. $\ddot{x} = g\tan\phi$

206. $I\theta = \dfrac{At^2}{2} + \dfrac{Bt^3}{6} - \dfrac{Ct^4}{12} + I\dot{\theta}_0 t + I\theta_0$

207. $\omega = \sqrt{\dfrac{4\pi NM}{I}}$

208. $t = \infty;\ \dfrac{I\omega_0}{2\pi k}$ revolutions

209. $\omega_{\max} = \dfrac{M}{k}$

210. $t = \dfrac{\omega_1 r_1}{\mu P}\dfrac{I_2}{r_2^2};\ t = \dfrac{\omega_1 r_1}{\mu P}\left(\dfrac{1}{\dfrac{r_2^2}{I_2} + \dfrac{r_1^2}{I_1}}\right)$

211. $A = \dfrac{\theta_0 \sin\omega t}{1 - \left(\dfrac{\omega}{p}\right)^2}$

212. 495,000 ft lb; 2350 ft lb

213. $\tau = 2\pi\sqrt{\dfrac{I}{RW}};\ l = \dfrac{I}{mR}$

214. $a = \dfrac{I_0}{mR}$

215. $v^2 = \dfrac{2(m+M)(I+ml^2)g}{m^2 l}(1-\cos\theta_0)$

216. $\left(I + \dfrac{W}{g}R^2\right)\ddot{\theta} = RW$

217. $\ddot{\theta} + \left(\dfrac{Wa}{2I}\sin\alpha\right)\theta = 0$

219. $p = \dfrac{2W}{\Delta t}\left[\dfrac{(1+a^2)(\sqrt{1+a^2}-1)}{3gh}\right]^{\frac{1}{2}}$

220. 2520 lb

222. $r = \dfrac{m}{m_1}\dfrac{l^2\sin\alpha\cos\alpha}{\left(\dfrac{L}{2} - a\right)}$

224. $Y_1 = -Y_2 = \dfrac{1}{8}\dfrac{mr^2\omega^2}{l}\sin 2\alpha$

225. $\dfrac{1}{8}\dfrac{m\omega^2}{l}\left(R^2 - \dfrac{h^2}{3}\right)\sin 2\alpha \pm \dfrac{me\omega^2}{2}$

226. $F = \dfrac{m\omega^2 ab(a^2 - b^2)}{12(a^2 + b^2)^{\frac{3}{2}}}$

227. left 29.6 lb, right 23.7 lb

228. $\ddot{\theta} = \dfrac{RW\sin\alpha}{(MR^2 + I)}$

229. $\tan\alpha = 3\mu$

230. $\ddot{y} = \frac{2}{3}g;\ F = \dfrac{W}{3}$

231. $h = \frac{7}{5}R$

232. $W\left(1 + \dfrac{kv^2}{gR}\cos\dfrac{v}{R}t\right)$

233. $\ddot{\theta} + \dfrac{5g}{7(R-r)}\theta = 0$

234. $d = \dfrac{I_c}{ml}$

235. $v_C = 40.6$ ft/sec;
$v_W = 81.2$ ft/sec

236. $\dot{v}_3 = \dfrac{\left(\dfrac{W_2}{2} - W_3\right)}{\left(\dfrac{m_2}{4} + m_3 + \dfrac{I_1}{r_1^2} + \dfrac{I_2}{4r_2^2}\right)}$

237. $\dot{x}_1 = 0;\ \dot{x}_2 = v$

238. $N = mg + \dfrac{Mg}{2} \pm \dfrac{Mv^2}{6R}$

239. $f = \dfrac{I_z\ddot{x}}{2R^2} \pm \dfrac{Mv^2}{6R}$

240. At piston, $\boldsymbol{F} = (-3200\boldsymbol{i} - 710\boldsymbol{j})$ lb; at crankpin $\boldsymbol{F} = (5600\boldsymbol{i} + 1500\boldsymbol{j})$ lb

241. $\dfrac{a}{10}$

242. $\dfrac{Wl}{I_s\Omega}$

245. $F = \dfrac{vI\Omega}{Rl}$

246. The gyro must be free to precess in the proper direction

247. Magnitude of torque $= \dfrac{vI\Omega}{R}$

248. Overturn

252. Stress in rod = 255 lb

253. 160 lb

254. $ml\ddot{\theta} + 2mv\dot{\theta}$

257. $W_1' = 12.3$ lb, $\theta_1 = 243°$; $W_5' = 7.95$ lb, $\theta_5 = 77.3°$

258. $\tau = 2\pi\left(\dfrac{I + mR^2}{kR^2}\right)^{\frac{1}{2}}$

259. $x = \dfrac{2lm\omega^2 - 2(m_1 + m)g}{2k + m\omega^2}$

260. $\ddot{\theta} + \left(\dfrac{k}{m} - \omega^2\right)\theta = 0$

261. $p_B = p + \rho g h_1$; $p_C = p + \rho g(h_1 - h_2)$

262. $p_2 = p_1 + \frac{1}{2}\rho v_1^2\left[1 - \left(\dfrac{A_1}{A_2}\right)^2\right]$

263. $v = \sqrt{v_1^2 - 2g(z - h)}$

264. $v = \sqrt{2\left(\dfrac{p_1 - p_0}{\rho}\right) + 2gh}$

266. $F = \rho v^2 A$

267. $F = p_1(A_1 - A_2) - \rho v_1^2 A_1\left[\dfrac{1}{2}\left(\dfrac{A_1}{A_2} + \dfrac{A_2}{A_1}\right) - 1\right]$

268. $\mathbf{F} = \rho v_1^2 A_1\left[\left(\dfrac{A_1}{8A_2} - 1\right)\mathbf{i} + \left(\dfrac{\sqrt{3}}{8}\dfrac{A_1}{A_2} - \dfrac{A_1}{4A_3}\right)\mathbf{j}\right]$

269. $\rho v^2 AR$

270. $F_A = [(p_1 - p_2) + (\rho_1 v_1^2 - \rho_2 v_2^2)]A$

271. No effect if bird has zero acceleration

272. Increases pressure by p_1 where $\int p_1\, dA = W$

275. $F = (p_i + \rho_i v_i^2)A_i - (p_e + \rho_e v_e^2)A_e$

276. 263 ft/sec

278. $v = \dfrac{v_e}{c}\left[1 - \left(1 - \dfrac{kt}{cm_0}\right)^c\right]$, where $c = \dfrac{k}{\rho v_c A}$

279. $Q = \dfrac{AR\omega}{\sin\alpha}\left[-1 + \sqrt{1 + \dfrac{2B\sin\alpha}{AR^3\rho\omega^2}}\right]$

280. $2A\sqrt{2gh}$; $\dfrac{2A\sqrt{2gh}}{\cos\alpha}$

281. 0.29 ft lb

283. $T = T_0\left(1 - \dfrac{x}{l}\right)^{-\frac{4}{3}}$

284. $p = \dfrac{p_0}{\left(1 - \dfrac{x}{l}\right)}\left[2\left(1 - \dfrac{x}{l}\right)^{-\frac{4}{3}} - 1\right]$

286. $F_r = m(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)$
$F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)$
$F_\phi = \dfrac{m}{r\sin\theta}\dfrac{d}{dt}(r^2\sin^2\theta\dot{\phi})$

287. $\ddot{\phi} - \sin\phi\cos\phi\dot{\theta}^2 = -\dfrac{g\sin\phi}{l}$
$\dfrac{d}{dt}(l^2\sin^2\phi\dot{\theta}) = 0$

288. $m(\ddot{r} - r\dot{\theta}^2) = -\dfrac{k}{r^2}$
$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$

289. $I\ddot{\theta} + Wa\sin\theta = 0$

290. $2\ddot{\theta} + \ddot{\phi} = -\dfrac{2g}{l}\theta$;
$\ddot{\theta} + \ddot{\phi} = -\dfrac{g}{l}\phi$

291. $\ddot{\theta} + \dfrac{3a^2k}{ml^2}\theta = 0$

292. $\ddot{\phi} + \dfrac{g}{x_0}\phi = 0$; $\ddot{x} + \dfrac{k}{m}x = \dfrac{k}{m}x_0$
where x_0 = length of spring under gravity at $\phi = 0$

293. $\dot{v}_3 = \dfrac{\left(\dfrac{W_2}{2} - W_3\right)}{\left(\dfrac{m_2}{4} + m_3 + \dfrac{I_1}{r_1^2} + \dfrac{I_2}{4r_2^2}\right)}$

294. $(M+m)\ddot{x}+2kx=-ml\ddot{\phi}-2ka\phi$
$ml\ddot{x}+2kax=-ml^2\ddot{\phi}-(mgl+2ka^2)\phi$

295. $\left(m+\frac{I}{r^2}\right)(R-r)^2\ddot{\theta}=M_t$

297. $\omega^2=\frac{1}{2}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)\pm\frac{1}{2}\sqrt{\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)^2-\frac{4k_1k_2}{m_1m_2}}$

298. $\omega^2=\frac{g}{l}+\frac{ka^2}{ml^2}\pm\frac{ka^2}{ml^2}$

299. $\omega^2=\frac{F}{ml}(2\pm 1)$

300. $\omega_1=\sqrt{\frac{2k}{m}}$; $\omega_2=\sqrt{\frac{a^2k}{2I}}$

302. $\omega^2=\frac{g}{l}(2\pm\sqrt{2})$

304. $\frac{dy}{dx}=\sqrt{\left(\frac{y}{C}\right)^2-1}$;
$y=C\cosh\left(\frac{x}{C}\right)$

305. $y=C\cosh\left(\frac{x}{C}\right)$

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