AN
Elementary Treatise
ON
DIFFERENTIAL CALCULUS

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PREFACE.

The necessity of a text book on the differential calculus suitable for the Indian University underraduate has long been felt. An humble attempt as been made to make this book self-sufficient as r as possible. The books usually consulted on e subject by the Indian student are either too terse r too elaborate. This treatise has been written or the undergraduate beginner, chiefly keeping in view the difficulties that he encounters in following an altogether new subject. Absolutely rigid proofs, which would not be intelligible to a young beginner, have been avoided, but there has not been, so far as possible, any sacrifice of rigour only for the sake of simplicity.

Chapter XIV, which gives a short history of he calculus, has been added, and the author hopes that it may serve a useful purpose. The questions have been mainly selected from examination papers and some of the standard books on the subject.

My thanks are due to Dr. P. L. Srivastava, M.. A, D. Phil. (Oxon.) and Mr. R. N. Chowdhari, B. A. (Cantab.) for their kind help, and especially to a friend of mine, (who has persistently chosen to remain anonymous in spite of my repeated requests to the contrary), in consultation with whom the idea of writing this book originated, particularly for the arrangement of the chapters.

The author will only be very glad to receive any suggestions or corrections. His thanks are also due to Lala Ram Narain Lal, Publisher, in undertaking to publish this book in spite of the difficulty usually felt in getting a mathematical book published at any press in India.

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V. S. G. TIWARI.

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ERRATA.

Page 31, line 19, read "$\frac{dy}{dx}$" for "$\frac{ay}{dx}$".

,, 34, last line, read "$\frac{dy}{dx}$" for "$\frac{ay}{ax}$".

,, 38, question no. 57, read as "$\tan^{-1}\frac{1-\sqrt{1+x^2}}{1+\sqrt{1+x^2}}$.

,, ,, ,, no. 60, read as "......with respect to $\frac{\cos\sqrt{x}}{x^{\frac{5}{2}}}$.

,, 50, ,, no. 7, read as " If $y^{-2} = 1 +$ ect......".

,, 83, ,, no. 20, read as "......$+ zu_t$)."

,, 129, line 2, read "$\frac{dp}{d\alpha}$" for "$\frac{ap}{d\alpha}$"

,, 150, ,, no. 10, read as " ...... is

$$r^{\frac{m}{n-m}} = a^{\frac{m}{n-m}} \cos^{\frac{m}{m-m}} \theta.$$

,, 155, line 1, read "x" for "8".

,, 178, last line, read – " mum " for " m u m ".

Elementary Treatise
ON
Differential Calculus

CHAPTER I

Note. A.—**Zero.**—Zero is the number that divides the positive and negative numbers uniquely. It is in fact the negation of numbers and yet it is included in the category of numbers. It is qualitatively different from any other number and has a correspondingly unique character. If we measure quantity of any material substance numerically, we realise that there is vast difference between a large and a small quantity of the substance, and there is yet another kind of difference between a small quantity of it and a zero quantity. It is of no importance what the substance is, if the quantity is zero, for it is nothing in all cases. No doubt we have included zero in the category of numbers very conveniently, it does not admit of all the ordinary rules and properties of numbers. Consider the following fallacy :—

$$1 \times 0 = 0, \quad 2 \times 0 = 0$$
$$\therefore 1 \times 0 = 2 \times 0$$

and dividing both sides by 0, we get $1 = 2$.

This brings home the fact that this process is not justifiable and the fallacy is the outcome of the fact that both the sides are divided by zero. The fact of the matter is that we do not know what division by zero means. We know how to divide one rational number by any other but zero. We cannot assign any meaning for the process of dividing by zero and the formal rules of Arithmetic and

Algebra will lead to errors and meaningless results. Thus $\frac{x}{y}$ has a meaning so long as y is not zero, but $\frac{x}{0}$ has no meaning and so also $\frac{0}{0}$.

Note B.—**Infinity.**—Through the doctrine of geometrical continuity and the application of Algebra to Geometry arose the important notion of infinity as a "localised space-conception," so that mathematicians have come to speak of points at infinity, lines at infinity etc. It is said that parallel lines intersect at infinity, that all circles pass through two fixed points at infinity (the circular points) and so on. The symbol ∞ is so freely spoken of and used, as if it represented a definite quantity, that the student is inclined to take it as such, and use it as such and thus have an altogether absurd notion. He thinks it to be a definite quantity or a definite place, whereas the conception of infinity is boundless or indefiniteness.

When one has thoroughly grasped the real meaning that is connoted by the arguments which treat of infinity as if it were in the category of definite numbers, one may take advantage of such "compactness and better unity of exposition as may thus be legitimately made possible." But in the beginning, the only safe way to deal with the subject is to recognise that ∞ is not a definite quantity and does not represent any particular value. It implies greatness, becoming ever greater without restriction or possibility of finality. Consider the meaning of, that $\frac{1}{x}$ becomes infinity when x equals zero. A better way of expressing the same thing will be:—the function $\frac{1}{x}$ tends to infinity as x tends to zero, *i.e.*, the progress of the value of $\frac{1}{x}$, while x being different from zero draws ever nearer to zero, knows no bound, and no limit can be set to its numerical value. → means an operation giving a general notion of one of increasing algebraical greatness, and ∞ means without bound or limit. These verbal phrases are very cumbersome and it is convenient to have briefer phrases to use in their stead. Thus ∞ implies invariable increase, endlessness, the absence of any effectual barrier in the increase and remaining ever greater. The circumference of a circle is also endless but it is not ∞. Thus the chief characteristic is that of boundless increase and ever remaining so. All this notion is included in the simple phrase " $\frac{1}{x}$ tends to ∞."

INDEPENDENT AND DEPENDENT VARIABLES AND FUNCTION.

1. In Arithmetic we have to deal with particular known numbers, integral, fractional or irrational. Sometimes we take a typical number out of the above mentioned aggregate of numbers and denote it by x or y, etc., whose particular value does not affect the argument in which it occurs. In every problem of the infinitesimal Calculus, we have to deal with a number of magnitudes or quantities some of which may be constant, while others are regarded as variables and as admitting of continuous variation.

A variable *is a quantity or a symbol capable of assuming successively every numerical value from a given number* α *to another given number* β $(\beta > \alpha)$.

(α, β) is called the domain* or the interval of the variable.

The letter a, b, c, etc., generally denote constants and x, y, z, etc., denote variables.

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* An illustration will clear the idea of a domain. Consider the expression $\sin^{-1}x$. Here x can only lie between -1 and 1, both inclusive. Thus the variable is not capable of assuming any value beyond these limits. The function $\sin^{-1}x$ can therefore be defined for the interval $(-1, 1)$ or any other interval within this interval, for instance, it can be defined for the interval $(-\frac{1}{2}, \frac{1}{2})$ or $(0, \frac{1}{2})$, etc. Next take e^x or $\tan x$. All these can be defined for any interval.

If, on the other hand, the function $\sin^{-1}x$ be defined for $(0, \frac{1}{2})$, it will be meaningless to speak of the value of the function when $x = \frac{3}{4}$ or any value beyond the interval given. When nothing is mentioned, the maximum domain is taken.

2. Let us now consider the expression $\frac{x^4 + x^3 + 5x^2 + 3x + 8}{ax^3 - 5}$, which represents some operation performed upon x. Such operations performed on x alone may be represented for shortness sake $f(x)$ or $F(x)$ or $\phi(x)$; and is sometimes written as $y = f(x)$. Thus if certain values admissible for x are assigned to it, y will also change its value successively. Both x and y are variables. But in their relation as herein stated, there is this difference between them, that any and all values may be assigned in succession to x, lying in the domain given, while y takes in each case the value which results from the operation of the formula upon the selected value of x. The variation in x is at our choice and independent; the variation of y is limited by the relation between them. Thus x behaves as an independent variable and y as a dependent variable, but the value of the latter is also determined without doubt or ambiguity, when any value is assigned to x. The expression for which $f(x)$, etc., stand are also said to be functions of x.

3. *If α and β be any two numbers, where $\beta > \alpha$, and if to every value of x in the interval $\alpha \leqslant x \leqslant \beta$, there corresponds one or more, definite values of y, y is said to be a* **function of x** *in the interval (α, β) for x, and is written as $y = f(x)$ or $\phi(x)$ or $F(x)$, etc.*

If there corresponds to each value of x in (α, β), one and only one definite value of y, y is said to be a single valued or uniform function of x. In all what follows, we shall deal with single valued functions unless otherwise stated. Functions such as $\sin^{-1}x$, which may be any one of the functions θ, $\pi - \theta$, $2\pi + \theta$, etc., where $\sin\theta = x$, some restriction should be imposed such as $-\frac{\pi}{2} \leqslant \sin^{-1}x \leqslant \frac{\pi}{2}$, in order to make it a uniform function.

If y has a definite value, which depends upon the values of n independent variables having their n domains, y is said to be a function of n variables.

4. **Limit.**—This is an extremely common word of frequent use, but the whole of the Differential Calculus is based on it and the importance of a rigourous and clear idea of it can never be too much emphasised.

The function $f(x)$ is said to have the **Limit A,** *A being a definite number, as x tends to a, if to every arbitrarily chosen positive number ε howsoever small, there corresponds a positive number η, such that* * $| A - f(x) | < \varepsilon$, *for every value of x, for which $0 < | x - a | \leqslant \eta$, and it is written as* $\underset{x \to a}{Lt} f(x) = A$.

One advantage of arrow notation (instead of $\underset{x = a}{\text{Lt}} f(x)$, in which equality sign is rather misleading), is that it brings out clearly the fact that we say nothing about what happens when x is equal to a, and which the idea of Limit really means. In other words the Limit of $f(x)$ as $x \to a$ is not necessarily the value of the function when $x = a$.

$0 < | x - a | \leqslant \eta$ is expressly put to indicate the fact that $| A - f(x) | < \varepsilon$ for every value of x in the interval $(a - \eta, a + \eta)$ except $x = a$. Thus we notice that a

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* **Absolute value.** A real number is either positive or negative. The absolute value of any real number is its numercial value regardless of the sign. The absolute value of any real number is thus positive. $| x |$ means the absolute value of or modulus of x. We have thus $| x | = | -x |$. Algebraically $-3 < -2$, but $| -3 | > | -2 |$.

N. B.—"The absolute value of the algebraic sum of any number of terms is equal to or less than the sum of the absolute values of the separate terms". This can evidently be seen by a student himself.

limiting value of a function $f(x)$ is that quantity from which we may make $f(x)$ differ as little as we please by making x approach nearer and nearer in magnitude to some particular value without actually becoming equal to it.

Sometimes x tends to a from the right hand side only, *i.e.*, $x > a$, and then in the definition we put $0 < (x - a) \leqslant \eta$ (right hand only) instead of $0 < | x - a | \leqslant \eta$. This limit from the right hand side only is written as $\underset{x \to a+0}{\text{Lt}} f(x)$. Similarly if x tends to a from the left hand side only, *i.e.*, $x < a$, we write $0 < (a - x) \leqslant \eta$ (left hand only) instead of $0 < | x - a | \leqslant \eta$. The limit from the left hand side only is written as $\underset{x \to a-0}{\text{Lt}} f(x)$. Thus $\underset{x \to a}{\text{Lt}} f(x) = \text{A}$, means that $\underset{x \to a+0}{\text{Lt}} f(x) = \underset{x \to a-0}{\text{Lt}} f(x) = \text{A}$. $f(x)$ is also sometimes said to converge to A as $x \to a$.

Illustrations.

(1) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin x}{x} = 1.$

If ε be chosen as $\dfrac{1}{10^4}$, by a reference to the table of sines, we can take η to be $\dfrac{1}{10^2}$, in which case $\left| 1 - \dfrac{\sin x}{x} \right| < \dfrac{1}{10^4}$, for every value of x lying in the interval $\left(- \dfrac{1}{10^2}, \dfrac{1}{10^2} \right)$ excepting the value of $x = 0$.

Here we say nothing about the value of $\dfrac{\sin x}{x}$, when $x = 0$, and $\dfrac{\sin x}{x}$ takes the form $\dfrac{0}{0}$, which has no meaning.

Again if ε be chosen to be still a smaller quantity, a corresponding value of η can be found and thus $\left| 1 - \dfrac{\sin x}{x} \right|$ can be made less than any positive quantity howsoever small. Thus $\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1$.

✓(2) $\underset{x \to 3}{\text{Lt}}\ x^2 = 9$.

Here we have to shew that for any positive number ε, howsoever small, a value for η can always be found, such that $| x^2 - 9 | < \varepsilon$ when $0 < | x - 3 | \leqslant \eta$.

Since $| x - 3 | < \eta$

$\therefore$ $x = 3 + \theta\eta$ where $0 < | \theta | < |$.

Hence $x + 3 = 6 + \theta\eta$

or $| x + 3 | < | 6 | + \eta$.

Now $(x^2 - 9) = (x - 3)(x + 3)$

or
$$\begin{aligned} | x^2 - 9 | &= | x - 3 | \times | x + 3 | \\ &< \eta\ | x + 3 | \\ &< \eta (6 + \eta). \end{aligned}$$

Hence $| x^2 - 9 |$ will be less than ε when $0 < | x - 3 | < \eta$ if η be chosen such that $\eta\ (6 + \eta) \leqslant \varepsilon$.

Let ε be $\frac{1}{10^5}$, therefore η will either be equal to or less than the roots of

$$\lambda^2 + 6\lambda - \frac{1}{10^5} = 0$$

$$\text{or } \lambda = -3 + 3\left(1 + \frac{4}{36} \cdot \frac{1}{10^5}\right)^{\frac{1}{2}}$$

rejecting the negative value of λ.

Thus η can be taken less than $\frac{1}{6 \times 10^5}$.

$\therefore | x^2 - 9 | < \frac{1}{10^5}$ for every value of x when $0 < | x - 3 | < \frac{1}{6 \times 10^5}$.

Similarly we can shew that if we choose $\varepsilon = \frac{1}{10^2}$, η can be found to be $\angle \frac{1}{6 \times 10^2}$. Hence $| x^2 - 9 | \angle\ \varepsilon$ for every value of x, such that $0 < | x - 3 | < \eta$, howsoever small ε may be. Thus $\underset{x \to 3}{\text{Lt}}\, x^2 = 9$.

✓ (3) Let us now consider the limit of the function $\frac{1}{x - a}$ as $x \to a$.

$$\underset{x \to a}{\text{Lt}} \frac{1}{x - a} = \underset{h \to 0}{\text{Lt}} \frac{1}{a + h - a}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{1}{h},$$ the limiting value of which becomes numerically greater than any quantity howsoever great. It can be possible, therefore that sometimes a function diverges to $+ \infty$ or $- \infty$ as $x \to a$.

5. (*i*) If to any positive number A, howsoever large, there corresponds a positive number η, such that $f(x) > A$ for every value of x for which $0 < | x - a | \leqslant \eta$ $\underset{x \to a}{\text{Lt}} f(x) = \infty$.

(*ii*) If to any negative number $- B$ howsoever large B may be, there corresponds a positive number η, such that $f(x) < - B$, for every value of x for which $0 < | x - a | \leqslant \eta$, $\underset{x \to a}{\text{Lt}} f(x) = - \infty$.

If however $\underset{x \to a}{\text{Lt}} f(x)$ does not exist, and $f(x)$ does neither diverge to $+ \infty$ nor to $- \infty$, it is said to oscillate.

for example $\underset{x \to \infty}{\text{Lt}} \sin x$ or $\underset{x \to 0}{\text{Lt}} \cos \frac{1}{x}$.

Illustrations.

✓ (1) $\underset{x \to 0}{\text{Lt}}\, e^{\frac{1}{x}}$ is non-existent.

Evidently $\underset{x \to 0 + 0}{\text{Lt}}\, e^{\frac{1}{x}} = \infty$ and $\underset{x \to 0 - 0}{\text{Lt}}\, e^{\frac{1}{x}} = 0$.

Hence as the limits from right hand side and left hand side are not equal to each other, the limit of $e^{\frac{1}{x}}$ as $x \to 0$ does not exist. The two limits exist separately and are different.

✓(2) $\underset{x \to 1}{\text{Lt}} \sin^{-1} x$ is non-existant.

Since $\underset{x \to 1+0}{\text{Lt}} \sin^{-1} x$ does not exist, for we know no such value whose sine is ever greater than unity, but $\underset{x \to 1-0}{\text{Lt}} \sin^{-1} x$ exists and equals $\frac{\pi}{2}$.

Hence $\underset{x \to 1}{\text{Lt}} \sin^{-1} x$ and $\underset{x \to 1+0}{\text{Lt}} \sin^{-1} x$ do not exist, although

$$\underset{x \to 1-0}{\text{Lt}} \sin^{-1} x = \frac{\pi}{2}.$$

Examples.

1. If n is a positive integer, shew that

   (*a*) $\underset{x \to 0}{\text{Lt}}\, x^n = 0.$

   (*b*) $\underset{x \to \alpha}{\text{Lt}}\, (\alpha - x)^n = 0.$

2. Prove that $\underset{x \to 0}{\text{Lt}} \cos\frac{1}{x}$ is non-existent.

3. Shew that $f(x)$ is zero or 1 according as x is zero or different from zero if $f(x) = \underset{n \to \infty.}{\text{Lt}} \dfrac{x}{x + \frac{1}{n}}$

4. Shew that (*i*) $\underset{x \to a}{\text{Lt}}\, x^3 = a^3$

   (*ii*) $\underset{x \to 0}{\text{Lt}} \frac{1}{x}$ does not exist.

6. **Before proceeding to consider the continuity or discontinuity of a function, it is desirable to lay down the three fundamental theorems on Limit.**

I. **Limit of a Sum.** If $\underset{x \to a}{\text{Lt}} f(x) = \text{A}$ and $\underset{x \to a}{\text{Lt}} \phi(x) = \text{B}$, to shew that $\underset{x \to a}{\text{Lt}} [f(x) + \phi(x)] = \text{A} + \text{B}$.

Let the positive number ε be chosen, as small as we please, and since $\underset{x \to a}{\text{Lt}} f(x)$ and $\underset{x \to a}{\text{Lt}} \phi(x)$ are both existent, then to $\varepsilon/2$ there correspond say the positive numbers η_1, η_2 such that

$$|f(x) - \text{A}| \angle \varepsilon/2 \quad \text{when } 0 \angle |x - a| \leqq \eta_1$$
$$|\phi(x) - \text{B}| \angle \varepsilon/2 \quad \text{when } 0 \angle |x - a| \leqq \eta_2$$

Thus if η is not greater than η_1 or η_2

$$|f(x) + \phi(x) - \text{A} - \text{B}| \leqq |f(x) - \text{A}| + |\phi(x) - \text{B}|$$
$$\angle \varepsilon/2 + \varepsilon/2 \quad \text{when } 0 \angle |x - a| \leqq \eta$$
$$\angle \varepsilon \quad \text{when } 0 \angle |x - a| \leqq \eta.$$

Therefore $\underset{x \to a}{\text{Lt}} [f(x) + \phi(x)] = \text{A} + \text{B}$.

This result can be extended to the sum of any finite number of functions. 'The Limit of a sum is equal to the sum of the limits.'

II. **Limit of a Product.** If $\underset{x \to a}{\text{Lt}} f(x) = \text{A}$, and $\underset{x \to a}{\text{Lt}} \phi(x) = \text{B}$, to shew that $\underset{x \to a}{\text{Lt}} f(x) . \phi(x) = \text{AB}$.

Let $f(x) = \text{A} + \varepsilon_1$ where $\varepsilon_1 \to 0$ as $x \to a$ since $\underset{x \to a}{\text{Lt}} f(x) = \text{A}$.

Similarly let $\phi(x) = \text{B} + \varepsilon_2$ where $\varepsilon_2 \to 0$ as $x \to a$.

$\therefore f(x) \phi(x) = \text{AB} + \text{B}\varepsilon_1 + \text{A}\varepsilon_2 + \varepsilon_1 \varepsilon_2$.

* Applying theorem I, we get

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* Since $\underset{x \to a}{\text{Lt}} \text{A}\varepsilon_2 = 0$ $\underset{x \to a}{\text{Lt}} \text{B}\varepsilon_1 = 0$, and also $\underset{x \to a}{\text{Lt}} \varepsilon_1 \varepsilon_2 = 0$, for if we suppose

$$\underset{x \to a}{\text{Lt}}\ f(x)\ \phi\ (x) = \text{AB}.$$

This result can also be extended to any finite number of functions. The limit of a Product is equal to the Product of the Limits.

III. Limit of a quotient. If $\underset{x \to a}{\text{Lt}}\ \phi(x) = \text{B} \neq o$

to shew that $\underset{x \to a}{\text{Lt}}\ \dfrac{1}{\phi\,(x)} = \dfrac{1}{\text{B}}$

Let $\phi\,(x) = \text{B} + \varepsilon_1$ where $\varepsilon_1 \to 0$ as $x \to a$.

$$\therefore \qquad \frac{1}{\text{B}} - \frac{1}{\text{B} + \varepsilon_1} = \frac{\varepsilon_1}{\text{B}\ (\text{B} + \varepsilon_1)}$$

$$\therefore \qquad \left| \frac{1}{\text{B}} - \frac{1}{\text{B} + \varepsilon_1} \right| = \left| \frac{\varepsilon_1}{\text{B}^2 + \text{B}\varepsilon_1} \right|$$

$$\angle \left| \frac{\varepsilon_1}{\text{B}^2} \right|$$

$\therefore \qquad \text{Lt} \left(\dfrac{1}{\text{B}} - \dfrac{1}{\phi\,(x)} \right) = 0$ since $\varepsilon_1 \to 0$ as $x \to a$ and B^2 is finite.

$$\therefore \qquad \underset{x \to a}{\text{Lt}}\ \frac{1}{\phi\,(x)} = \frac{1}{\text{B}}.$$

Hence if $\underset{x \to a}{\text{Lt}}\ f(x) = \text{A}$ and $\underset{x \to a}{\text{Lt}}\ \phi(x) = \text{B} \neq 0$

$$\underset{x \to a}{\text{Lt}}\ \frac{f\,(x)}{\phi\,(x)} = \frac{\text{A}}{\text{B}}.$$

$| \varepsilon_1 - 0 | < \sqrt{\varepsilon}$ when $0 < | x - a | \leqslant \eta_1$

$| \varepsilon_2 - 0 | < \sqrt{\varepsilon}$ when $0 < | x - a | \leqslant \eta_2$

and if $\eta \ngtr \eta_1$ or η_2,

$| \varepsilon_1 \quad \varepsilon_2 | < \varepsilon$ when $0 < | x - a | \leqslant \eta$

$$\therefore \qquad \underset{x \to a}{\text{Lt}}\ \varepsilon_1\,\varepsilon_2 = 0.$$

Exercises.

Prove that

(*i*) $$\underset{x \to 0}{\text{Lt}} \frac{(1+x)^n - 1}{x} = n \text{ or } \underset{x \to 1}{\text{Lt}} \frac{x^n - 1}{x - 1} = n$$

(*ii*) $$\underset{x \to 0}{\text{Lt}} (1+x)^{\frac{1}{x}} = e$$

(*iii*) $$\underset{x \to 0}{\text{Lt}} \frac{a^x - 1}{x} = \log_e a$$

These are very important limits and will be frequently used.

7. Let us examine the curves of the figure 1.

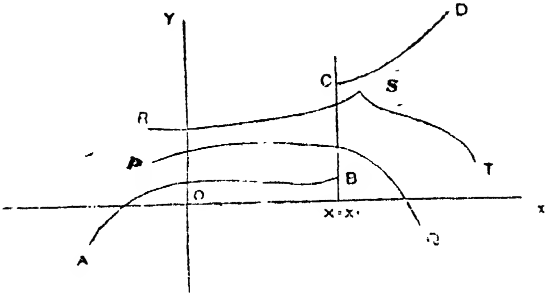

Fig. 1.

Curve PQ is one line and continuous, that is there are no breaks, and we are able to trace it from point to point without any gap. Similarly, the curve RST is also traceable continuously. But the curve ABCD comes from A upto B, and then takes a big jump from B to C, and then again goes on from point to point without any jump. For the value of the abscissa equal to say x_1, the curve ABCD does not behave in the same way as it does for other values of x, or as the curve PQ or RST behaves for even $x = x_1$. In other words, the arc AB of the curve ABCD cannot be traced beyond B, and neither can the arc CD be continued beyond the left of C.

8. *$f(x)$ is said to be* **continuous** *when $x = x_1$ if to every arbitrarily chosen positive number ε, howsoever small, there exists a positive number η, such that $| f(x) - f(x_1) | < \varepsilon$ when $| x - x_1 | \leqq \eta$.*

The definition is analogous to the definition of the limit of a function, with this difference that A in this case is the value of the function when $x = x_1$ *i. e.*, $f(x_1)$, and instead of $0 < | x - x_1 | \leqq \eta$, we have only $| x - x_1 | \leqq \eta$. Thus a function $f(x)$ is said to be continuous for $x = x_1$ if $f(x)$ has a limit when $x \to x_1$ from either side and the limit equals $f(x_1)$. In other words, for a function $f(x)$ to be continuous for $x = x_1$, it is necessary that $\operatorname*{Lt}_{x \to x_1 + 0} f(x)$ and $\operatorname*{Lt}_{x \to x_1 - 0} f(x)$ both should exist and be equal to $f(x_1)$.

9. If $f(x)$ be defined for an interval (α, β), $f(x)$ is said to be continuous in that interval if it is continuous for every value of x, such that $\alpha < x < \beta$, and provided that $\operatorname*{Lt}_{x \to \alpha + 0} f(x) = f(\alpha)$ and $\operatorname*{Lt}_{x \to \beta - 0} f(x) = f(\beta)$.

Illustrations.

✓(*i*) x^2 is continuous for every value of x.

Let us find $\operatorname*{Lt}_{x \to a} x^2$.

$$\operatorname*{Lt}_{x \to a} | x^2 - a^2 | = \operatorname*{Lt}_{h \to 0} | (a + h)^2 - a^2 |$$

$$= \operatorname*{Lt}_{h \to 0} | h (2a + h) |$$

$$= \operatorname*{Lt}_{h \to 0} | h | \; | 2a + h | ,$$ which is less than any assignable quantity as $h \to o$, and h is determined if ε is given. Since a may be any real number, x^2 is continuous for every value of x.

✓(*i*) e^x is continuous for every value of x.

We can shew that corresponding to any arbitrarily chosen ε, we can find η, such that

$$\left| e^{x} - e^{x_1} \right| < \varepsilon. \quad \text{When } | x - x_1 | \lesssim \eta \ldots (1).$$

Let $x_1 + h$ be any value of x in the interval $(x_1 - \eta, x_1 + \eta)$.

$$\left| e^{x_1 + h} - e^{x_1} \right| = \left| e^{x_1} (e^{h} - 1) \right|$$

$$\text{Now } \left| e^{h} - 1 \right| = \left| \left(1 + h + \frac{h^2}{\lfloor 2} + \ldots\right) - 1 \right|$$

$$= \left| h \left(1 + \frac{h}{\lfloor 2} + \frac{h^2}{\lfloor 3} + \ldots\right) \right|$$

$$\angle \, | h | \times \left| 1 + \frac{h}{\lfloor 1} + \frac{h^2}{\lfloor 2} + \ldots \right|$$

$$\angle \, | h | \times \left| e^{h} \right|$$

$$\therefore \left| e^{x_1 + h} - e^{x_1} \right| < \left| e^{x_1} . h . e^{h} \right| \quad \ldots \quad \ldots (2)$$

Therefore (1) will be satisfied if η be so chosen that

$$\left| e^{x_1} . \eta . e^{\eta} \right| < \varepsilon$$

$$\text{or } \eta \, e^{\eta} < \varepsilon \, e^{- x_1}$$

and since $\eta \, e^{\eta} \rightarrow 0$ as $\eta \rightarrow 0$, we can find a value of η satisfying this inequality for every value of ε, howsoever small. Thus e^{x} is continuous at x_1; and since x_1 is any finite value of x we see that e^{x} is continuous for every value of x.*

---

* Alternative Proof :— $\underset{h \rightarrow 0 + 0}{\text{Lt}} \left(e^{x_1 + h} - e^{x_1} \right) = 0$

also $\underset{h \rightarrow 0 - 0}{\text{Lt}} e^{x_1 + h} = e^{x_1}.$

i. e. $\underset{h \rightarrow 0}{\text{Lt}} e^{x_1 + h} = e^{x_1}$

Hence $f(x)$ is continuous for $x = x_1$

10. Discontinuous functions. If a function $f\ (x)$ is not continuous when $x = x_1$, it is said to be discontinuous when $x = x_1$.

Thus a function $f\ (x)$ is discontinuous when $x = x_1$, if either (*i*) $\operatorname*{Lt}_{x \to x_1 + 0} f(x)$ and $\operatorname*{Lt}_{x \to x_1 - 0} f(x)$ both exist but are different.

or (*ii*) None of the limits in (*i*) exists.

or (*iii*) One exists and the other does not exist.

or (*iv*) Both exist, are equal to each other but do not equal $f\ (x_1)$.

If the limit on one side exists and equals $f\ (x_1)$, it is said to be continuous on that side only.

Illustrations.

✓(*i*) $y = [x]$, $[x]$ meaning the greatest integral value less than x.

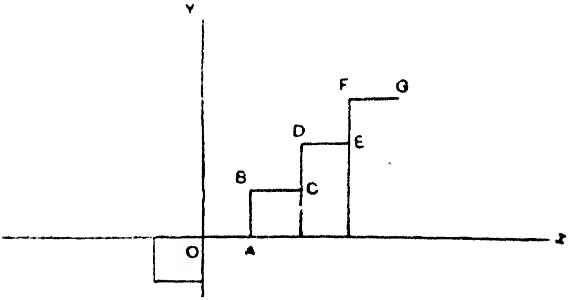

Fig. 2.

The curve is OA, BC, DE, etc.

The curve is discontinuous for all integral values of x, and continuous for other than these values.

✓ (*ii*) $e^{\frac{1}{x}}$ is discontinuous for $x = 0$.

The limits from both the sides when $x \to 0$ exist, but are different.

$$\underset{x \to 0+0}{\text{Lt}}\ e^{\frac{1}{x}} = \infty$$

$\underset{x \to 0-0}{\text{Lt}}\ e^{\frac{1}{x}} = 0$, hence $e^{\frac{1}{x}}$ is discontinuous at the origin.

✓(*iii*) $\dfrac{\text{Sin}\, x}{x}$ is discontinuous when $x = 0$.

The limit of the function when $x \to 0$ exists but it is not equal to $f(0)$, for $f(0)$ in this case becomes $\frac{0}{0}$, which is meaningless.

11. It is evident that although a function $f(x)$ may have a limit when $x \to x_1$, it may not be continuous, since while speaking of the limit of a function when $x \to x_1$ we say nothing about its value when $x = x_1$; whereas while considering the continuity, we say that the limit of the function when $x \to x_1$ exists and is equal to the value of the function when $x = x_1$.

Exercises.

1. Prove that $\sqrt{x}$ is continuous for all positive values of x. Is it continuous when $x = 0$?

2. Shew that $\frac{1}{x}$ is discontinuous for $x = 0$.

3. Shew that sin x and cos x are continuous for all values of x.

4. For what values of x are tan x, cot x, sec x and cosec x continuous or discontinuous.

5. Discuss the continuity of

$$\frac{1}{a \cos^2 x + b \sin^2 x}, \quad \sqrt{1 + \sin x}, \quad \sin \frac{1}{x}, \text{ and } x \sin \frac{1}{x}.$$

6. Shew that $x - [x]$ is discontinuous for all integral values of x.

CHAPTER II.

12. If $y = f(x)$ is defined for a certain interval (α, β), sometimes it is desirable to find the limiting value of the ratio of a change in $f(x)$, due to an indefinitely small variation in the independent variable x, to this change in x as this change tends to zero. This does not presuppose that such a limit must always exist. If the limit exists, it is called the *First Differential Co-efficient or Derivative* of the function $f(x)$ with respect to x.

Generally a small change in the variable x is denoted by δx and the corresponding change in the dependent variable by δy.

Thus $y + \delta y = f(x + \delta x)$

Therefore $\delta y = f(x + \delta x) - f(x)$

Now if the $\underset{\delta x \to 0}{\text{Lt}} \dfrac{f(x + \delta x) - f(x)}{\delta x}$ exists it is denoted as $\dfrac{d}{dx} f(x)$ or $\dfrac{d\,f(x)}{dx}$ or $\dfrac{dy}{dx}$ or $f'(x)$.

It should be clearly noted here that $\dfrac{dy}{dx}$ is only a symbol or a short method of writing $\underset{\delta x \to 0}{\text{Lt}} \dfrac{\delta y}{\delta x}$. Here dy and dx are not separate quantities as δx and δy denote, for we write $\dfrac{dy}{dx}$ only when the limit of the ratio $\dfrac{\delta y}{\delta x}$ as $\delta x \to 0$ has been attained.

If the derivative exists, it has tacitly been assumed that the function is continuous also. For

if $$\underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x) - f(x)}{\delta x} = \text{A, say}$$

$$\frac{f(x + \delta x) - f(x)}{\delta x} = \text{A} + \varepsilon_0, \text{ where } \varepsilon_0 \to 0 \text{ as } \delta x \to 0.$$

or $$f(x + \delta x) - f(x) = \delta x \, (A + \epsilon_0)$$

i. e., $$\mathop{\mathrm{Lt}}_{\delta x \to 0} f(x + \delta x) = f(x)$$

Thus we suppose that the function is continuous if the derivative exists. But *vice-versa* is not true, *i. e.*, it is not always possible to find the derivative if the function is continuous.*

13. If the limit of $\dfrac{f(x + \delta x) - f(x)}{\delta x}$ exists when $\delta x \to 0$ from the right hand side, *i. e.*, $\delta x \to 0$ through positive values alone, it is known as the right hand or Regressive differential co-efficient of $f(x)$. Similarly the limit of $\dfrac{f(x + \delta x) - f(x)}{\delta x}$ as $\delta x \to 0$ through negative values alone, *i. e.*, from left hand side, is said to be the left hand or Progressive differential co-efficient of $f(x)$, if it exists. If the two derivatives, progressive and regressive

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* After reading the article 45 , the student can understand this point by taking the case of two straight lines meeting at a point say at B in the figure 3. The tangent at B to the curve cannot be drawn.

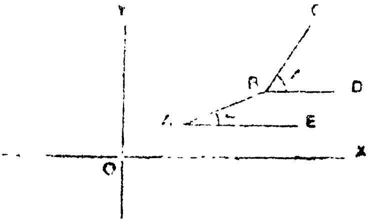

Fig. 3.

A rigorous proof of this is beyond the scope of this volume.

It may also be seen that $\delta y \to 0$ as $\delta x \to 0$ simultaneously, and $\dfrac{dy}{dx}$ is the limit of the ratio of these two infinitesimal quantities.

have the same value, it is only then and then alone that the function $f(x)$ will be said to have a differential co-efficient $\frac{dy}{dx}$, and its value will be either of the values of the two derivatives. If, however, the progressive and regressive derivatives of a function $f(x)$, for any value of x are different, or only either of them exists or none of them exists it is meaningless to talk of the differential co-efficient of the function for that value of x.

Illustration.

✓(1) Let $y = e^x$. Thus $\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{e^{x+\delta x} - e^x}{\delta x}$

or

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{e^x \; (e^{\delta x} - 1)}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{e^x}{\delta x}\left(\delta x + \frac{(\delta x)^2}{\lfloor 2} + \dots \dots\dots\dots\right)$$

$$= \underset{\delta x \to 0}{\text{Lt}} \; e^x \left(1 + \frac{\delta x}{\lfloor 2} + \text{higher powers of } \delta x\right)$$

$$= \; e^x \text{, since}$$

$$\frac{\delta x}{\lfloor 2} + \frac{(\delta x)^2}{\lfloor 3} + \dots = \delta x \left[\frac{1}{\lfloor 2} + \frac{\delta x}{\lfloor 3} + \dots\dots\dots\right]$$

$= \delta x \times$ a convergent series $\angle \; e^{\delta x}$ and this tends to 0 as $\delta x \to 0$.

Hence $\frac{d}{dx} e^x = e^x$.

✓(2) Let $y = c$ a constant

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{c - c}{\delta x} = 0$$

$$\therefore \quad \frac{d}{dx} \; (\text{ constant }) = 0.$$

14. Most of the ordinary functions which occur in the usual course of analysis can be reduced to a few standard forms. The remaining portion of this chapter will deal with the methods for finding the differential co-efficient of some of these standard forms.

I. Differential co-efficient of x^n.

Here $y = x^n$

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{(x + \delta x)^n - x^n}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{x^n \left\{ \left(1 + \frac{\delta x}{x}\right)^n - 1 \right\}}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{x^n}{\delta x} \left\{ \left(1 + \frac{n\,\delta x}{x} + \frac{n\,(n-1)}{\lfloor 2} \left(\frac{\delta x}{x}\right)^2 + \dots\right) - 1 \right\}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{n\,x^n \frac{\delta x}{x} \left\{ 1 + \frac{(n-1)}{\lfloor 2} \frac{\delta x}{x} + \dots\dots\dots\dots \right\}}{\delta x}$$

$$= n\,x^{n-1}$$

Thus

$$\frac{d}{dx}(x^n) = n\,x^{n-1}.$$

II. Differential co-efficient of sin x.

Here $y = \sin x$

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\sin\,(x + \delta x) - \sin x}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{2 \cos \left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \cos\left(x + \frac{\delta x}{2}\right) \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$= \cos x.$$

Thus

$$\frac{d}{dx}(\sin x = \cos x.$$

III. Differential co-efficient of cos x.

Here $y = \cos x$.

$$\therefore \quad \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\cos(x + \delta x) - \cos x}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{2 \sin\left(x + \frac{\delta x}{2}\right) \sin\left(-\frac{\delta x}{2}\right)}{\delta x}$$

$$= \underset{dx \to 0}{\text{Lt}} - \sin\left(x + \frac{\delta x}{2}\right) \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$= - \sin x.$$

$$\therefore \quad \frac{d}{dx}(\cos x) = - \sin x.$$

IV. Differential co-efficient of tan x.

Here $y = \tan x$.

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\tan(x + \delta x) - \tan x}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{\sin(x + \delta x) \cos x - \cos(x + \delta x) \sin x}{\delta x \cos(x + dx) \cos x.}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{\sin \delta x}{\delta x} \cdot \frac{1}{\cos x \times \cos(x + \delta x)}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$\therefore \frac{d}{dx}(\tan x) = \sec^2 x.$$

V. Differential co-efficient of cot x.

Here $y = \cot x$.

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\cot(x + \delta x) - \cot x}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{\sin x \cos(x + \delta x) - \cos x \sin(x + \delta x)}{\delta x \times \sin x \times \sin(x + \delta x)}$$

$$= \underset{\delta x \to 0}{\text{Lt}} - \frac{\sin \delta x}{\delta x} \times \frac{1}{\sin x \sin(x + \delta x)}$$

$$= - \frac{1}{\sin^2 x} = - \operatorname{cosec}^2 x.$$

$$\therefore \frac{d}{dx}(\cot x) = - \operatorname{cosec}^2 x.$$

VI. Differential co-efficient of sec x.

Here $y = \sec x$.

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\sec(x + \delta x) - \sec x}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{\cos x - \cos(x + \delta x)}{\delta x \times \cos x \times \cos(x + \delta x)}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \times \frac{\sin\left(x + \frac{\delta x}{2}\right)}{\cos x \times \cos(x + \delta x)}$$

$$= \frac{\sin x}{\cos^2 x} = \tan x \sec x$$

$$\therefore \frac{d}{dx}(\sec x) = \tan x \sec x.$$

VII. Differential co-efficient of cosec x.

Here $y = \operatorname{cosec} x$.

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\operatorname{cosec}(x + \delta x) - \operatorname{cosec} x}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{\sin x - \sin(x + \delta x)}{\delta x \sin x \times \sin(x + \delta x)}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{-\cos\left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}}{\frac{\delta x}{2} \sin x \times \sin (x + \delta x)}$$

$$= - \cos x . \frac{1}{\sin^2 x} = - \cot x \operatorname{cosec} x$$

$$\therefore \frac{d}{dx} (\operatorname{cosec} x) = - \cot x \operatorname{cosec} x.$$

VIII Differential co efficient of a^x.

Here $y = a^x$.

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{a^{x + \delta x} - a^x}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} a^x \frac{a^{\delta x} - 1}{\delta x}$$

$= a^x \log_e a$, by exercise (*iii*) pp. 13

$$\therefore \frac{d}{dx} (a^x) = a^x \log_e a$$

Cor. 1. $\frac{d}{dx}(e^x) = e^x$.

IX. Differential co-efficient of $\log_a x$.

Here $y = \log_a x$

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\log_a (x + \delta x) - \log_a x}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{1}{\delta x} \log_a \left(1 + \frac{\delta x}{x}\right)$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{1}{x} \log_a \left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

$= \frac{1}{x} \log_a e$, by exercise (*ii*) pp. 13.

$$\therefore \frac{d}{dx} (\log_a x) = \frac{1}{X} \log_a e.$$

Cor. 1. $\frac{d}{dx} (\log_e x) = \frac{1}{x}$.

Exercises.

Find the differential coefficients of the following from first principles.

1. $x \sin x$.
3. $\text{Log}_e\ (x+a)$.
2. $\sec x^2$.
4. x^{-n}.
5. $x\sqrt{x}$.

15. Differential coefficient of a sum of functions is equal to the sum of the differential coefficients of the functions, provided all of these differential coefficients exist.

Let $f\ (x)=f_1\ (x)+f_2\ (x)+f_3\ (x)+\ldots.$

$$\therefore \frac{d}{dx}f\ (x)=\underset{\delta x\to 0}{\text{Lt}}\left\{\frac{f_1(x+\delta x)-f_1(x)+f_2(x+\delta x)-f_2(x)+\ldots}{\delta x}\right\}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\left\{\frac{f_1\ (x+\delta x)-f_1\ (x)}{\delta x}+\frac{f_2(x+\delta x)-f_2\ x)}{\delta x}+\ldots.\right\}$$

$$=\frac{d}{dx}[f_1\ (x)]+\frac{d}{dx}[f_2(x\ ]+\ldots\ldots$$

16. (*a*) Differential co-efficient of a product of two functions.

Let $y=f_1\ (x).\ f_2\ (x)$.

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{f_1\ (x+\delta x).\ f_2\ (x+\delta x)-f_1(x)f_2(x)}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\frac{f_1\ (x+\delta x).\ f_2\ (x+\delta x)-f_1\ (x)f_2(x+\delta x)}{\delta x}+\frac{f_1\ (x)\ f_2\ (x+\delta x)-f_1\ (x)\ f_2\ (x)}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\left\{f_2.(x+\delta x)\frac{f_1\ (x+\delta x)-f_1\ x)}{\delta x}+f_1(x)\times\frac{f_2(x+\delta x)-f_2\ (x)}{\delta x}\right\}$$

$$=f_2(x)\frac{d}{dx}[f_1\ (x)]+f_1(x)\ \frac{d}{dx}[f_2\ x)].$$

i. e., the differential coefficient of a product of two functions = (2nd function) × (Differential co-efficient of the

1st function) + (1st function) × (Differential co-efficient of 2nd function).

(*b*) This result can be easily extended to the product of several functions.

Let $y = f_1(x)\, f_2(x)\, f_3(x)$

suppose $f_2(x)\, f_3(x) = \phi(x)$.

Then $y = f_1(x)\, \phi(x)$

and $$\frac{dy}{dx} = \phi(x)\frac{d}{dx}[f_1(x)] + f_1(x)\frac{d}{dx}[\phi(x)] \ldots\ldots\ldots\ldots(i$$

Again $$\frac{d}{dx}[\phi(x)] = f_3(x)\frac{d}{dx}[f_2(x)] + f_2(x)\frac{d}{dx}[f_3(x)].$$

Substituting this result in (*i*), we have

$$\frac{dy}{dx} = f_2(x)\, f_3(x)\frac{d}{dx}[f_1(x)] + f_1(x)\, f_3(x)\frac{d}{dx}[f_2(x)] + f_1(x)\, f_2(x)\frac{d}{dx}[f_3(x)].$$

or $$\frac{1}{y}\frac{dy}{dx} = \frac{1}{f_1(x)}\frac{d}{dx}[f_1(x)] + \frac{1}{f_2(x)}\frac{d}{dx}[f_2(x)] + \frac{1}{f_3(x)}\frac{d}{dx}[f_3(x)].$$

Thus if $f(x) = f_1(x)\, f_2(x)\, f_3(x) \ldots\ldots f_n(x)$

$$\frac{1}{f(x)}\frac{d}{dx}f(x) = \frac{1}{f_1(x)}\frac{d}{dx}[f_1(x)] + \frac{1}{f_2(x)}\frac{d}{dx}[f_2(x)] + \ldots \ldots\ldots + \frac{1}{f_n(x)}\frac{d}{dx}[f_n(x)].$$

Cor. 1. $\frac{d}{dx}[c\, f(x)] = c\frac{d}{dx}[f(x)]$.

i. e., the differential coefficient of the product of a constant and a function equals the product of the constant and the differential co-efficient of the function.

17. Differential Co-efficient of a quotient of two functions.

Let $f(x)=\frac{f_1(x)}{f_2(x)}$

$$\frac{d}{dx}[f(x)] = \underset{\delta x\to 0}{\mathrm{Lt}}\left[\frac{f_1(x+\delta x)}{f_2(x+\delta x)} - \frac{f_1(x)}{f_2(x)}\right]\frac{1}{\delta x}$$

$$= \underset{\delta x\to 0}{\mathrm{Lt}}\left[\frac{f_2(x)\,f_1(x+\delta x) - f_1(x)\,f_2(x+\delta x}{f_2(x).\ f_2(x+\delta x)}\right]\frac{1}{\delta x}$$

$$= \underset{\delta x\to 0}{\mathrm{Lt}}\left[\frac{f_2(x)\,f_1(x+\delta x)-f_1(x)\,f_2(x)}{f_2(x)\,f_2(x+\delta x)} + \frac{f_1(x)\,f_2(x)-f_1(x)\,f_2(x+\delta x)}{f_2(x)\,f_2(x+\delta x)}\right]\frac{1}{\delta x}$$

$$= \underset{\delta x\to 0}{\mathrm{Lt}}\frac{f_2(x)\frac{f_1(x+\delta x)-f_1(x)}{\delta x} - f_1(x)\frac{f_2(x+\delta x)-f_2(x)}{\delta x}}{f_2(x).\ f_2(x+\delta x)}$$

$$= \frac{f_2(x)\frac{d}{dx}[f_1(x)] - f_1(x)\frac{d}{dx}[f_2(x)]}{[f_2(x)]^2}$$

$$= \frac{f_2(x)\,f_1'(x) - f_1(x)\,f_2'(x)}{[f_2(x)]^2}$$

i. e., the differential co-efficient of the quotient of two functions is

$$\frac{\text{(Denomtr) (Diff. Coeff. of Numtr.)}-\text{(Numtr) (Diff. Coeff. of Demtr).}}{\text{Square of Denominator}}$$

Illustrations.

✓(1) $y = \sqrt{x} + \tan x.\ \sin x.$

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}(\tan x.\ \sin x).$$

Now $\frac{d}{dx}\sqrt{x} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}.\frac{1}{\sqrt{x}}$

and $\frac{d}{dx}(\tan x..\sin x) = \sin x\frac{d}{dx}(\tan x) + \tan x\frac{d}{dx}(\sin x).$

$$= \sin x.\ \sec^2 x + \tan x.\ \cos x$$
$$= \tan x \sec x + \sin x$$

Thus $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \tan x \sec x + \sin x.$

✓ (2) $y = \frac{x^5}{\tan x}$

$$\frac{dy}{dx} = \frac{\tan x . \frac{d}{dx}(x^5) - x^5 \frac{d}{dx}(\tan x)}{\tan^2 x}$$

$$= \frac{5\,x^4 \tan x - x^5 \sec^2 x}{\tan^2 x}$$

$$= x^4\,(5 \cot x - x \operatorname{cosec}^2 x).$$

✓(3) $$y = \frac{e^{x}}{1 + x}$$

$$\frac{dy}{dx} = \frac{(1 + x)\; e^{x} - e^{x}}{(1 + x)^2}$$

$$= \frac{x\, e^{x}}{(1 + x)^2}$$

Exercises.

1. Find the differential coefficients with respect to x of the following :—

(*i*) $\tan^3 x$, $x \log x$, $e^{x+a} \left|\cos(x + b)\right|$, $x^r \log \dfrac{x \tan x}{1 + \sin x}$.

(*ii*) $\dfrac{x^n}{\tan x}$, $\dfrac{x^2}{\cos^2 \frac{x}{2}}$, $\dfrac{1 + \sin x}{1 - \sin x}$, $\dfrac{x + \tan x}{x - \tan x}$.

2. Differentiate the following with respect to x,

$\dfrac{x^m - 1}{x^n - 1}$, $e^{x}\,\dfrac{\cos x - \sin x}{\cos x + \sin x}$, $\log\left[x.\ a^{x}\right]$,

$x^2 \sec x$, $\dfrac{x \tan x}{\sec x + \cos x}$.

18. Differential co-efficient of a function of a function.

Let $y = f(t)$, when $t = \phi(x)$.

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\mathrm{Lt}} \left[\frac{\delta y}{\delta t} . \frac{\delta t}{\delta x}\right]$$

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta t} \; \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta t}{\delta x}$$

Now as $t = \phi(x)$, $t + \delta t = \phi(x + \delta x)$ hence $\delta t \to 0$ as $\delta x \to 0$, since t is a continuous function.

$$\therefore \frac{dy}{dx} = \underset{\delta t \to 0}{\mathrm{Lt}} \left(\frac{\delta y}{\delta t}\right) . \underset{\delta x \to 0}{\mathrm{Lt}} \left(\frac{\delta t}{\delta x}\right)$$

$$= \frac{dy}{dt} \times \frac{dt}{dx}.$$

Cor. If
$$\begin{aligned} y &= f_1(t) \\ t &= f_2(z) \\ z &= f_3(u) \\ u &= f_4(v) \\ v &= f_5(x) \\ \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dz} \times \frac{dz}{du} \times \frac{du}{dv} \times \frac{dv}{ax}. \end{aligned}$$

Thus the theorem can be extended to any number of such functional relations.

Illustrations.

✓1. $y = e^{a \sin x}$

Putting $t = a \sin x$, we have

$$\begin{aligned} y &= e^{t} \\ \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} = e^{t} \times a \cos x. \\ &= a \cos x \times e^{a \sin x}. \end{aligned}$$

✓2. $y = \tan\left(\log \sin \frac{1}{x}\right)$. Putting $\log \sin \frac{1}{x} = t$

$\sin \frac{1}{x} = v$ and $\frac{1}{x} = u$, we have

$$y = \tan t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dv} \times \frac{dv}{du} \times \frac{du}{dx}$$

$$= \underline{\sec^2 t} \times \frac{1}{v} \times \cos u \times (-x^{-2}).$$

$$= -\frac{1}{x^2} \times \cos\frac{1}{x} \times \frac{1}{\sin\frac{1}{x}} \times \sec^2\left(\log \sin \frac{1}{x}\right)$$

$$= -\frac{\cot\frac{1}{x}}{x^2} \sec^2\left(\log \sin \frac{1}{x}\right).$$

Exercises.

1. Differentiate the following with respect to x:—

(*i*) $x\, e^{ax}$, $c^{ax} \tan(bx + c)$, $x \cos \frac{1}{x}$.

(*ii*) $\sin\left(\sin \frac{1}{x}\right)$, $\log(\cos x)$, $\log\left(\tan \frac{x}{2}\right)$, $\log\left(\log \frac{1}{x}\right)$

(*iii*) $\sin(\sqrt{x} \times e^{x})$, $\frac{(1 + x^2)\tan e^{x}}{x^2}$.

✓19. Inverse functions.

1. Differential co-efficient of $\sin^{-1} x$.

$$y = \sin^{-1} x \text{ or } \sin y = x$$

$$\therefore \frac{d}{dx}(\sin y) = 1 \text{ or } \cos y \frac{dy}{dx} = 1$$

$$\text{or} \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

2. Differential co-efficient of $\cos^{-1} x$

$$y = \cos^{-1} x \text{ or } \cos y = x$$

$$\therefore \frac{d}{dx}(\cos y) = 1 \qquad \text{or } \sin y \frac{dy}{dx} = 1$$

$$\text{or} \qquad \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

3. Differential co-efficient of $\tan^{-1} x$.

$$y = \tan^{-1} x \text{ or } \tan y = x$$

$$\therefore \sec^2 y \frac{dy}{dx} = 1 \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{1 + x^2}$$

4. Differential co-efficient of $\cot^{-1} x$.

$$y = \cot^{-1} x$$

$$\text{or} \quad y = \frac{\pi}{2} - \tan^{-1} x$$

$$\frac{dy}{dx} = -\frac{1}{1 + x^2}$$

Illustrations.

✓I. $$y = [f(x)]^{\phi(x)}$$

Taking logarithm $\log y = \phi(x) \log f(x)$.

Differentiating with respect to x,

$$\frac{1}{y}\,\frac{dy}{dx} = \phi(x)\,\frac{d}{dx}\left(\log\ f\ (x)\right) + \log f(x).\ \frac{d}{dx}\,\phi(x).$$

$$= \frac{\phi(x)}{f(x)}\,f'(x) + \phi'(x).\ \log f(x).$$

$$\therefore\ \frac{dy}{dx} = \Big[f(x)\Big]^{\phi(x)}\left\{\frac{\phi(x)}{f(x)}\,f'(x) + \phi'(x)\log f(x)\right\}.$$

II. $$y = [f(x)]\ [\phi(x)]\ [\psi(x)]\ldots\ldots$$

Taking logarithm $\log\ y = \log\ f(x) + \log\ \phi(x) + \log \psi(x) + \ldots\ldots$

Differentiating with respect to x,

$$\frac{1}{y}\,\frac{dy}{dx} = \frac{f'(x)}{f(x)} + \frac{\phi'(x)}{\phi(x)} + \frac{\psi'(x)}{\psi\ x)} + \ldots\ldots$$

$$\therefore\ \frac{dy}{dx} = [f(x)]\ [\phi(x)]\ [\psi(x)]\ldots\left\{\frac{f'(x)}{f(x)} + \frac{\phi'(x)}{\phi(x)} + \ldots\right\}$$

III. $$y = x^{1+\frac{1}{x}}$$

$$\log y = \left(1 + \frac{1}{x}\right)\log x.$$

$$\frac{1}{y}\,\frac{dy}{dx} = -\frac{1}{x^2}\log x + \left(1 + \frac{1}{x}\right)\frac{1}{x}.$$

$$\frac{dy}{dx} = \left(x^{1+\frac{1}{x}}\right)\frac{1}{x}\left\{\left(1 + \frac{1}{x}\right) - \frac{1}{x}\log x\right\}.$$

IV. $$y = \left(\sin^{-1}x\right)^{\sqrt{(1-x^2)}},\ x^{x}$$

$$\log y = \sqrt{1-x^2}\ \log\left(\sin^{-1}x\right) + x\ \log x.$$

$$\therefore\ \frac{1}{y}\,\frac{dy}{dx} = \sqrt{1-x^2}\,\frac{1}{\sin^{-1}x}.\ \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \times$$

$$\log\left(\sin^{-1}x\right) + x.\ \frac{1}{x} + \log x.$$

$$\text{or } \frac{dy}{dx} = \left(\sin^{-1} x\right)^{\sqrt{1-x^2}} \times \quad x^x \times$$

$$\left\{\frac{1}{\sin^{-1} x} - \frac{x}{\sqrt{1-x^2}} \log\left(\sin^{-1} x\right) + \log x + 1\right\}.$$

21. Differentiation of a function with respect to another function.

$$\text{Let } y = f(x)$$

$$\text{and } u = \phi\,(x)$$

To find $\frac{dy}{du}$.

$$\frac{dy}{ax} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \therefore \frac{dy}{du} = \frac{\frac{dy}{ax}}{\frac{du}{dx}} = \frac{f'(x)}{\phi'(x)}.$$

Illustration.

To differentiate $\tan x$ with respect to $\cos x$.

$$y = \tan x.$$

$$u = \cos x.$$

$$\frac{dy}{du} = \frac{\frac{dy}{dx}}{\frac{du}{dx}} = -\frac{\sec^2 x}{\sin x} = -\sec^2 x \operatorname{cosec} x.$$

22. Sometimes the function to be differentiated gets simplified by a suitable substitution for x in terms of another variable, as illustrated below.

1. $y = \tan^{-1} \frac{2x}{1 - x^2}$. Put $x = \tan\theta$.

$$y = \tan^{-1}(\tan 2\theta).$$

$$= 2\theta.$$

$$= 2 \tan^{-1} x.$$

$$\therefore \frac{ay}{ax} = \frac{2}{1 + x^2}.$$

2. $y = \sin^{-1}(x\sqrt{1 - x} - \sqrt{x}\sqrt{1 - x^2})$,

Put $\sqrt{x} = \cos\theta$.

and $x = \cos\phi$.

$$y = \sin^{-1}\left[\cos\phi\sin\theta - \sin\phi\cos\theta\right]$$

$$= \sin^{-1}\sin(\theta - \phi)$$

$$= \theta - \phi.$$

$$= \cos^{-1}\sqrt{x} - \cos^{-1}x.$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2\sqrt{x - x^2}} + \frac{1}{\sqrt{1 - x^2}}.$$

EXAMPLES ON CHAPTER II.

Differentiate the following :—

1. $\sqrt{\dfrac{1 + x}{1 - x}}$. ✓

2. $\sqrt{\dfrac{1 + x + \sin x}{1 - x + x^2}}$. ✓

3. $\dfrac{1}{(1 - x)\sqrt{1 - x^2}}$.

4. $\dfrac{\sqrt{x} + \sqrt{1 - x^2}}{\sqrt{x} + \sqrt{1 + x^2}}$.

5. $\dfrac{1}{(a^2 - x^2)^n}$.

6. $\dfrac{1}{\sqrt[3]{a^2 - x^2}}$.

7. $\dfrac{x^n}{(x^2 + 1)^n}$.

8. $\dfrac{x^2}{\sqrt{a^2 - x^2}}$.

9. $\cot^n x$.

10. $x \cot^2 x$.

11. $x^n \sqrt{\sin x}$.

12. $\sin x \times \tan (x + a)$.

13. $\dfrac{\sin^2 (ax + b)}{\cos^2 (bx + a)}$.

14. $\dfrac{\cot mx}{1 + \sec^2 nx}$.

15. $\dfrac{\sin 2x}{x^3}$.

16. $\dfrac{(a - b \cos x)}{a + b \cos x}$.

17. $\dfrac{1 + \tan \dfrac{x}{2}}{1 - \tan \dfrac{x}{2}}$.

18. $\log \dfrac{1 - \sin x}{x}$.

19. $\tan x + \frac{1}{3} \tan^3 x$.

20. $\dfrac{1}{1 + \sin^n x}$.

21. $\dfrac{1 - \sin^2 2 x}{1 + \sin^2 2 x}$.

22. $\sin 3 x \times \cos^3 3 x$.

23. $x \sin^{-1} (1 - x)$.

24. $\sin^{-1} x + \sin^{-1} \sqrt{1 - x^2}$.

25. $\cos^{-1} \dfrac{1 - x^2}{1 + x^2}$.

26. $\tan^{-1} \dfrac{p + qx}{q - px}$.

27. $\tan^{-1} \dfrac{\tan \alpha \sin x}{1 + \sec \alpha \cos x}$.

28. $\tan^{-1} (a + b \cos x)$.

29. $\dfrac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\sqrt{\dfrac{a - b}{a + b}} \tan \dfrac{x}{2} \right]$.

30. $\dfrac{\tan \alpha}{\sec \alpha + \cos x}$.

31. $\dfrac{x \sin x}{1 + \cos^2 x}$.

32. $\frac{1 + \cos x}{x + \sin x}$.

33. $\operatorname{cosec}^{-1} [x \sec^{-1} x]$.

34. $\log x + e^{\sqrt{x}}$.

35. e^{-x^2}

36. $e^{e^{\log x}}$.

37. $e^{\tan^{-1} x} \times \log (\tan x)$.

38. $\tan^{-1} \left[\frac{e^x - 1}{e^x + 1}\right]$

39. $\log \frac{x + 1}{x}$.

40. $\log (x \tan x)$

41. $\log \left[\frac{1 - \sin x}{1 + \sin x}\right]$.

Find $\frac{dy}{dx}$ in the following :—

42. $y = x^x \log x (\sin x)^{\sin x}$.

43. $y = \tan^{-1} \frac{\sqrt{1 - x^2} + \sqrt{1 + x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}}$

44. $y = (\tan x)^{\tan x} + (\cos x)^{\cos x}$.

45. $y = \sin x \times \sin 2x \times \sin 3x$.

46. $x = e^{\tan^{-1} \frac{y - x^2}{x^2}}$

47. $y = x^{y^x}$

48. $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots \mu_1, \mu_2, \mu_3$ being functions of x

and $y = \frac{\lambda_1 \times \lambda_2 \times \lambda_3 \ldots\ldots \lambda_n}{\mu_1 \times \mu_2 \times \mu_3 \ldots\ldots \mu_n}$

shew $\frac{dy}{dx} = \frac{\lambda_1 \times \lambda_2 \times \lambda_3 \ldots\ldots \lambda_n}{\mu_1 \times \mu_2 \times \mu_3 \ldots\ldots \mu_n} \left[\Sigma_{r=1}^{r=n} \frac{1}{\lambda_r} \times \frac{d\lambda_r}{dx} - \Sigma_{r=1}^{r=n} \frac{1}{\mu_r} \times \frac{d\mu_r}{dx} \right]$

49. If $f(x) = \frac{1}{1 - x}$, find the differential co-efficient of $f[f\{f(x)\}]$.

Find $\frac{dy}{dx}$ in the following :—

50. $x^m \times y^n = (x + y)^{m+n}$

51. $y = \tan^{-1}[a^{cx} \times x^{\sin x}] \dfrac{\sqrt{x}}{1+x^{\frac{3}{2}}}$

52. $y = \log^{n} x$ where $\log^{n}$ means log log log… n times.

53. $y = \left(\dfrac{1+\sqrt{x}}{1+2\sqrt{2}}\right)^{\sin e^{-x^{2}}}$

54. differentiate $\log_a x$ with respect to x^3.

55. Differentiate $e^{ax} \cos bx$ with respect to $\sin x$.

56. Differentiate $\log_e \left\{\dfrac{\alpha + \beta \tan \frac{x}{2}}{\alpha - \beta \tan \frac{x}{2}}\right\}$ with respect to $\dfrac{1}{\alpha^2 \sin^2 \frac{x}{2} - \beta^2 \cos^2 \frac{x}{2}}$

57. Differentiate $\tan^{-1} \dfrac{1-\sqrt{1+x^2}}{1+\sqrt{1+x^2}}$ with respect to $\tan^{-1} x$.

58. Differentiate $\dfrac{\sqrt{1+\alpha^2 x^2} + \sqrt{1-\beta^2 x^2}}{\sqrt{1+\alpha^2 x^2} - \sqrt{1-\beta^2 x^2}}$ with respect to $\sqrt{1-\beta^4 x^4}$.

59. Differentiate $\cos^{-1}(2x^2 - 1)$ with respect to $\sin^{-1} x$.

60. Differentiate $x^m \log(\cot^{-1} x)$ with respect to $\cos \dfrac{\sqrt{x}}{x^{\frac{5}{2}}}$.

61. Differentiate $\dfrac{(x^2+a^2)^2}{\sqrt{1+\tan x}}$ with respect to $\tan x$.

62. Differentiate $\dfrac{1}{\sqrt{(x-\alpha)(\beta-x)}}$ with respect to $\sqrt{x} \cos^2 \dfrac{\alpha}{2}$.

Find $\dfrac{dy}{dx}$ in the following :—

63. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

64. $$\left.\begin{aligned} x &= a \cos^3 \theta \\ y &= a \sin^3 \theta \end{aligned}\right\} \text{(Asteroid).}$$

65. $$\left.\begin{aligned} x &= a\,(\theta + \sin \theta) \\ y &= a\,(1 + \cos \theta) \end{aligned}\right\} \text{(Cycloid).}$$

66. $$\left.\begin{aligned} x &= at^2 \\ y &= 2at \end{aligned}\right\}.$$

67. $$\left.\begin{aligned} x &= a \cos \phi \\ y &= b \sin \phi \end{aligned}\right\} \text{where } b^2 = a^2\,(1 - e^2),\ (e < 1).$$

68. $$\left.\begin{aligned} x &= a \sin^2 t \\ y &= a\,\frac{\sin^3 t}{\cos t} \end{aligned}\right\} \text{(Cissoid of Diocles).}$$

69. Find $\dfrac{dy}{ds}$ where,

$$\left.\begin{aligned} y &= c \sec \psi \\ s &= c \tan \psi \end{aligned}\right\} \text{(Catenary).}$$

Find $\dfrac{dy}{dx}$ in the following :—

70. $$\left.\begin{aligned} x &= a \ \cos t + \frac{a}{2} \log \tan^2 \frac{t}{2} \\ y &= a \ \sin t \end{aligned}\right\} \text{(Tractrix).}$$

71. $$\left.\begin{aligned} x &= \cos \phi + \frac{\sin \alpha \sin \phi}{1 - \cos^2 \alpha \sin^2 \phi} \\ y &= \sin \phi - \frac{\sin \alpha \cos \phi}{1 - \cos^2 \alpha \sin^2 \phi} \end{aligned}\right\} \qquad \text{(Tripos 1904.)}$$

where α is a constant.

72. $$\left.\begin{aligned} x &= a\,(2 \cos \theta + \cos 2\,\theta) \\ y &= a\,(2 \sin \theta + \sin 2\,\theta) \end{aligned}\right\}.$$

73. $$\left.\begin{aligned} x &= \frac{3\,at}{1 + t^3} \\ y &= \frac{3\,at^2}{1 + t^3} \end{aligned}\right\} \text{(Folium of Descartes).}$$

74. $$\left.\begin{aligned} x &= (a + b) \cos \theta - b \cos \left(\frac{a + b}{b}\theta\right) \\ y &= (a + b) \sin \theta - b \sin \left(\frac{a + b}{b}\theta\right) \end{aligned}\right\} \text{Epicycloid.}$$

75. If $y = \sqrt{x}^{\sqrt{x}^{\sqrt{x}^{\cdot^{\cdot^{\cdot \infty}}}}}$ shew that $x\dfrac{dy}{dx} = \dfrac{y^2}{2 - y \log x}$.

✓76. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x} + \ldots\infty}}$

shew that $\dfrac{dy}{dx} = \dfrac{\sin x}{1 - 2y}$.

✓77. If $y = \cfrac{\tan x}{1 + \cfrac{\cot x}{1 + \cfrac{\tan x}{1 + \cfrac{\cot x}{1 + \ldots\infty.}}}}$

Shew that $\dfrac{dy}{dx} = \dfrac{4(\sin^2 x + y)}{\sin^2 2x\,(1 + 2y + \cot x - \tan x)}$.

✓78. If $y = \cfrac{a\sqrt{x}}{1 + \cfrac{a\sqrt{x}}{1 + \cfrac{a\sqrt{x}}{1 + \ldots\ldots\infty.}}}$

Shew that $\dfrac{dy}{dx} = \cfrac{a}{2\sqrt{x} + \cfrac{4ax}{1 + \cfrac{a\sqrt{x}}{1 + \cfrac{a\sqrt{x}}{1 + \ldots\infty.}}}}$

✓79. If $y = x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + \ldots\ldots\infty.}}}$

Shew that $\dfrac{dy}{dx} = \cfrac{2x}{2 - \cfrac{x^2}{x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + \ldots\infty.}}}}$

80. If $y = e^{x} + e^{x} + e^{x} + e.\text{....} \infty.$

Shew that $\dfrac{dy}{dx} = \dfrac{y}{1 - y}$.

81. If $y = a^{x^{a^{x \cdots\cdots \infty}}}.$

Prove that $\dfrac{dy}{dx} = \dfrac{y^2 \log y}{x\,[1 - y \log x \times \log y]}$.

82. If S_n is the sum of a G. P. to n terms of which r is the common ratio, prove that,

$$(r - 1)\frac{d\,S_n}{dr} = (n - 1)\,S_n - n\,S_{n-1} \quad \text{[coll. Ex.]}.$$

83. If $y = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$, where u_1, etc., are functions of x, shew that

$$\frac{dy}{dx} = \begin{vmatrix} \dfrac{du_1}{dx} & v_1 \\ \dfrac{du_2}{dx} & v_2 \end{vmatrix} + \begin{vmatrix} u_1 & \dfrac{dv_1}{dx} \\ u_2 & \dfrac{dv_2}{dx} \end{vmatrix}$$

84. Prove that if x be less than unity

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots \text{to } \infty.$$

$$= \frac{1}{1-x} \quad \text{[coll. Ex.]}$$

85. If x be less than unity, prove that

$$\frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \ldots\ldots \text{to } \infty$$

$$= \frac{1+2x}{1+x+x^2} \quad \text{[Edwards]}.$$

where $x \rightarrow \infty$.

86. If $C = 1 + r\cos\theta + \dfrac{r^2\cos 2\theta}{\lfloor 2} + \dfrac{r^3\cos 3\theta}{\lfloor 3} + \ldots\ldots$

and $S = r\sin\theta + \dfrac{r^2\sin 2\theta}{\lfloor 2} + \dfrac{r^3\sin 3\theta}{\lfloor 3} + \ldots\ldots$

Shew that

$$C\frac{dC}{dr} + S\frac{dS}{dr} = (C^2 + S^2)\cos\theta.$$

$$C\frac{dS}{dr} - S\frac{dC}{dr} = (C^2 + S^2)\sin\theta. \qquad \text{[Coll. Ex.]}$$

87. Determine the co-efficients A_1, A_2......so that

$$\frac{d}{dx}\left[\,\{x^m - A_1x^{m-1} + A_2x^{m-2} - \ldots\ldots + (-1)^m A_n\}\,e^x\,\right] = x^m e^x,$$

m being a positive integer. (London 1890.)

88. If $\frac{\sin\pi u}{\pi u} = \left(1 - \frac{u^2}{1^2}\right)\left(1 - \frac{u^2}{2^2}\right)\left(1 - \frac{u^2}{3^2}\right)\ldots\ldots$

Shew that

$$\frac{1}{u}\left[\frac{1}{u} - \pi\cot\pi u\right] = 2\left[\frac{1}{(1^2 - u^2)} + \frac{1}{(2^2 - u^2)} + \left(\frac{1}{(3^2 - u^2)}\right) + \ldots\ldots\right].$$

89. Shew that $-\frac{d}{du}(\cot u) = \frac{1}{u^2} + \Sigma\frac{1}{(u - m\pi)^2}$

where the summation is extended for all integral values of m excepting zero.

90. If $\frac{\sin x}{x} = \underset{n\to\infty}{\text{Lt}}\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}$

$$\ldots\ldots\ldots\cos\frac{x}{2^n}.$$

Prove that $\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \frac{1}{2^3}\tan\frac{x}{2^3}\ldots\ldots$

$$= \frac{1}{x} - \cot x$$

and $\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \frac{1}{2^6}\sec^2\frac{x}{2^3} +$

$$\frac{1}{2^8}\sec^2\frac{x}{2^4} + \ldots\ldots = \text{cosec}^2 x - \frac{1}{x^2}.$$

CHAPTER III.

SUCCESSIVE DIFFERENTIATION.

23. In the preceding chapter we dealt with the methods of finding the first differential co-efficient of a function. In general, this derived function will itself be capable of differentiation. Thus if $y = f(x)$ has a derivative $f'(x)$ and if $f'(x)$ is differentiable, *i.e.*, $\underset{\delta x \to 0}{\text{Lt}} \dfrac{f'(x + \delta x) - f'(x)}{\delta x}$ exists, it is called the first differential co-efficient of $f'(x)$ with respect to x, or the second differential co-efficient of $f(x)$ with respect to x and is generally denoted as $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ or $\dfrac{d^2y}{dx^2}$ or $f''(x)$. Again, if the function $f''(x)$ has in its turn a differential co-efficient, it is denoted as $f'''(x)$ or $\dfrac{d^3y}{dx^3}$ and is known as the third differential co-efficient of $f(x)$ with respect to x.

Similarly the n^{th} differential co-efficient of $f(x)$, if it exists, is denoted by $\dfrac{d^ny}{dx^n}$ or $f^n(x)$ or y_n.

Thus $$\frac{d^ny}{dx^n} = \underset{\delta x \to 0}{\text{Lt}} \frac{f^{n-1}(x + \delta x) - f^{n-1}(x)}{\delta x}.$$

These several differential co-efficients of a function with respect to x are known as the successive differential co-efficients of the function.

24. The following standard results will be found of great use in subsequent treatment.

I. If $y = x^n$, $y_1 = nx^{n-1}$, $y_2 = n(n - 1)x^{n-2}$, etc., and $y_n = \lfloor n$.

II. If $y = a^x$, $y_1 = a^x \times \log_e a$,

$$y_2 = a^x (\log_e a)^2$$

. . . .

$$y_n = a^x (\log_e a)^n.$$

. . . .

III. If $y = e^{ax}$, $y_1 = ae^{ax}$,

$$y_2 = a^2 e^{ax}$$

. . . .

$$y_n = a^n e^{ax}.$$

IV. If $y = \sin ax$

$$y_1 = a \cos ax = a \sin \left(ax + \frac{\pi}{2}\right)$$

$$\therefore \; y_2 = a^2 \cos \left(ax + \frac{\pi}{2}\right) = a^2 \sin \left(ax + 2\frac{\pi}{2}\right)$$

. . . .

$$y_n = a^n \sin \left(ax + n\frac{\pi}{2}\right).$$

V. If $y = \cos ax$

$$y_1 = - a \sin a x = a \cos \left(ax + \frac{\pi}{2}\right).$$

$$\therefore \; y_2 = - a^2 \sin \left(ax + \frac{\pi}{2}\right) = a^2 \cos \left(ax + 2\frac{\pi}{2}\right)$$

. . . .

$$y_n = a^n \cos \left(ax + n\frac{\pi}{2}\right).$$

VI. If $y = e^{ax} \sin b x$.

$$y_1 = a e^{ax} \sin b x + b e^{ax} \cos b x.$$

Putting $a = r \cos \phi$, and $b = r \sin \phi$, then $r = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1} \frac{b}{a}$.

$$\therefore \; y_1 = r e^{ax} \sin (b x + \phi)$$

$$= (a^2 + b^2)^{\frac{1}{2}} \; e^{ax} \sin\left(bx + \tan^{-1}\frac{b}{a}\right).$$

Similarly $y_2 = (a^2 + b^2)^{\frac{2}{2}} \; e^{ax} \sin\left(bx + 2\tan^{-1}\frac{b}{a}\right).$

……………………………………

In general, $y_n = (a^2 + b^2)^{\frac{n}{2}} \; e^{ax} \sin\left(bx + n\tan^{-1}\frac{b}{a}\right).$

VII. If $y = e^{ax} \cos bx$

$$y_1 = a\,e^{ax} \cos bx - b\,e^{ax} \sin bx.$$

Putting $a = r\cos\phi$, and $b = r\sin\phi$, we have as in VI,

$$y_1 = (a^2 + b^2)^{\frac{1}{2}} e^{ax} \cos\left(bx + \tan^{-1}\frac{b}{a}\right).$$

Similarly $y_2 = (a^2 + b^2)^{\frac{2}{2}} e^{ax} \cos\left(bx + 2\tan^{-1}\frac{b}{a}\right)$

In general $y_n = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \cos\left(bx + n\tan^{-1}\frac{b}{a}\right).$

VIII. If $y = \dfrac{1}{x + a},$

$$y_1 = -\frac{1}{(x + a)^2}$$

$$y_2 = (-1)^2 \frac{\underline{|2}}{(x + a)^3}$$

$$y_3 = (-1)^3 \frac{\underline{|3}}{(x + a)^4}$$

……………………………………

In general, $y_n = (-1)^n \dfrac{\underline{|n}}{(x + a)^{n+1}}.$

Examples.

Find y_n in the following :—

(1) $\sin x \sin 2x \sin 3x$. (2) $e^{4x} \cos 4x$.

(3) $\dfrac{1}{(2x + a)^2}$ (4) $\dfrac{1}{x^2 + x + 1}.$

(5) If $y = \sin m x + \cos m x$, shew that

$$y_r = m^r \left\{1 + (-1)^r \sin 2 m x\right\}^{\frac{1}{2}} \quad \text{[I. C. S.]}$$

25. Leibnitz's Theorem.—If $y = f(x) . \phi(x)$ be defined for a certain interval (a, b), and if all the n successive differential co-efficients of $f(x)$ and $\phi(x)$ are existent, then the n^{th} differential co-efficient of y is given by

$$y_n = f^n(x)\ \phi(x) + {}^n c_1\ f^{n-1}(x)\ \phi'(x) + {}^n c_2\ f^{n-2}(x)\ \phi''(x) + \ldots . + {}^n c_{n-1}\ f'\ (x)\ \phi^{n-1}\ (x) + f(x)\ \phi^n (x).$$

Let $y = f(x)\ \phi(x)$,

differentiating $y_1 = f'(x)\ \phi(x) + f(x)\ \phi'(x)$.

Similarly $y_2 = f''(x)\ \phi(x) + 2\ f'(x)\ \phi'(x) + f(x)\ \phi''(x)$.

and $y_3 = f'''(x)\ \phi(x) + 3\ f''(x)\ \phi'(x) + 3\ f'(x)\ \phi''(x) + f(x)\ \phi'''(x)$.

and so on..............

Here it is evident that the co-efficients of the successive terms in y_2, y_3, etc., are formed in the same way as the co-efficients in the expansion of the binomials $(x + a)^2$ and $(x + a)^3$, etc. Hence assuming generally this law of the formation of the co-efficients, let

$$y_m = f^m(x)\ \phi(x) + {}^m c_1\ f^{m-1}(x)\ \phi'(x) + {}^m c_2\ f^{m-2}(x)\ \phi''(x) + \ldots\ldots\ldots + {}^m c_{m-1}\ f'\ (x)\ \phi^{m-1}\ (x) + f(x)\phi^m\ (x).$$

Differentiating this,

$$\begin{aligned} y_{m+1} &= f^{m+1}(x)\ \phi(x) + (1 + {}^m c_1)\ f^m(x)\ \phi'(x) + \ldots . \\ &\quad + ({}^m c_{r-1} + {}^m c_r)\ f^{m+1-r}\ (x)\phi^r\ (x) + \ldots . \\ &\quad + f(x)\ \phi^{m+1}(x). \\ &= f^{m+1}(x)\ \phi(x) + {}^{m+1}c_1\ f^m(x)\ \phi'(x) \\ &\quad + {}^{m+1}c_2\ f^{m-1}(x)\ \phi''(x) + \ldots . \\ &\quad {}^{m+1}c_r\ f^{m+1-r}(x)\phi^r\ (x) + \ldots \\ &\quad + f\ (x)\ \phi^{m+1}(x). \end{aligned}$$

[Since ${}^{m}c_{r-1} + {}^{m}c_{r} = {}^{m+1}c_{r}$].

Thus assuming the theorem for the m^{th} derivative, it is observed that by actual differentiation, it is also true for the $(m+1)^{\text{th}}$ derivative. It was true in the case of second, and third derivatives, hence it holds good for the fourth, for the fifth, and hence so on universally

Thus $y_n = f^n(x, \phi\, x) + {}^{n}c_1 f^{n-1}(x)\,\phi'(x) + {}^{n}c_2 f^{n-2}(x)\,\phi''(x) + \ldots .$

$$+ {}^{n}c_r f^{n-r}(x\ \phi^r(x) + \ldots . + f(x)\,\phi^n(x)$$

Illustrations.

I. Let $y = \dfrac{\sin bx}{x}$, to find y_n.

Here supposing $f(x) \equiv \sin bx$, $\phi(x) \equiv \dfrac{1}{x}$.

$$\therefore f^n(x) = b^n \sin\left(bx + n\frac{\pi}{2}\right) \text{ and } \phi^n(x) = (-1)^n \frac{\underline{|n}}{x^{n+1}}.$$

Hence by Leibnitz Theorem, $y_n = b^n \sin\left(bx + n\dfrac{\pi}{2}\right) \times \dfrac{1}{x}$

$$- {}^{n}c_1 b^{n-1} \sin\left(bx + \overline{n-1}\frac{\pi}{2}\right)\frac{1}{x^2} + \cdots$$

$$+ {}^{n}c_r b^{n-r} \sin\left(bx + \overline{n-r}\frac{\pi}{2}\right)(-1)^r \frac{\underline{|r}}{x^{r+1}} + \cdots$$

$$\cdots + \sin bx \,.\, (-1)^n \frac{\underline{|n}}{x^{n+1}}.$$

II. Differentiate n times $y = \sin^5 x \times \cos^7 x$.

Put $\cos x + i \sin x = z$.

$$\therefore \cos x - i \sin x = \frac{1}{z}.$$

$$2\cos x = z + \frac{1}{z}.$$

$$2i \sin x = z - \frac{1}{z}.$$

$$\therefore \cos^7 x = \frac{1}{2^7}\left(z + \frac{1}{z}\right)^7$$

$$\sin^5 x = \frac{1}{2^5 i^5}\left(z - \frac{1}{z}\right)^5.$$

$$y = \frac{1}{2^{12} i^5}\left(z + \frac{1}{z}\right)^7 \left(z - \frac{1}{z}\right)^5$$

$$= \frac{1}{2^{12} \times i^5}\left[z^{12} + 2\ z^{10} - 4\, z^8 - 10\ z^6 + 5\, z^4 + 20\ z^2 - \frac{20}{z^2} - \frac{5}{z^4} + \frac{10}{z^6} + \frac{4}{z^8} - \frac{2}{z^{10}} - \frac{1}{z^{12}}.\right].$$

$$= \frac{1}{2^{12} i^5}\left[\left(z^{12} - \frac{1}{z^{12}}\right) + 2\left(z^{10} - \frac{1}{z^{10}}\right) - 4\left(z^8 - \frac{1}{z^8}\right) - 10\left(z^6 - \frac{1}{z^6}\right) + 5\left(z^4 - \frac{1}{z^4}\right) + 20\left(z^2 - \frac{1}{z^2}\right)\right].$$

Again since $2 \cos n\, x = z^n + \dfrac{1}{z^n}$

and $2\, i \sin n\, x = z^n - \dfrac{1}{z^n}$

$$\therefore\ y = \frac{1}{2^{12} i^5}\Big[\ 2\, i\ \{\sin 12\, x + 2 \sin 10\, x - 4 \sin 8\, x - 10 \sin 6\, x + 5 \sin 4\, x + 20 \sin 2\, x\,\}\Big].$$

$$= \frac{1}{2^{11}}\Big[\sin 12\, x + 2 \sin 10\, x - 4 \sin 8\, x - 10 \sin 6\, x + 5 \sin 4\, x + 20 \sin 2\, x\,\Big].$$

$$\therefore\ y_n = \frac{1}{2^{11}}\left[12^n \sin\left\{12\, x + n\frac{\pi}{2}\right\} + 2.\, 10^n \sin\left\{10\, x + n\frac{\pi}{2}\right\} - 4.\, 8^n \sin\left\{8\, x + n\frac{\pi}{2}\right\} - 10.\, 6^n \sin\left\{6\, x + n\frac{\pi}{2}\right\} + 5.\, 4^n \sin\left\{4\, x + n\frac{\pi}{2}\right\} + 20.\, 2^n \sin\left\{2\, x + n\frac{\pi}{2}\right\}\right].$$

III. Differentiate n times $y = \tan^{-1} \dfrac{x}{a}$.

$$y_1 = \frac{a}{a^2 + x^2} = \frac{1}{2\, i}\left\{\frac{1}{x - i\, a} - \frac{1}{x + i\, a}\right\}.$$

$$y_n = \frac{(-1)^{n-1}\ \lfloor\!\underline{n-1}}{2i}\left[\frac{1}{(x - i\, a)^n} - \frac{1}{(x + i\, a)^n}\right].$$

Putting $x = r\cos\theta$ ✓

$a = r\sin\theta.$ ✓

$$r^2 = x^2 + a^2,\ \tan\theta = \frac{a}{x}.$$

$$\therefore\ y_n = \frac{(-1)^{n-1}\lfloor\underline{n-1}}{2\,i.\,r^n}\left[\frac{1}{(\cos\theta - i\sin\theta)^n} - \frac{1}{(\cos\theta + i\sin\theta)^n}\right].$$

$$= \frac{(-1)^{n-1}\lfloor\underline{n-1}}{2\,i\times r^n}\left[(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin)^{-n}\right]$$

$$= \frac{(-1)^{n-1}\lfloor\underline{n-1}}{2\,i\times r^n}\left[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta\right].$$

$$= \frac{(-1)^{n-r}\lfloor\underline{n-1}}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}\lfloor\underline{n-1}\,\sin^n\theta}{a^n}\sin n\theta, \text{ where}$$

$$\theta = \tan^{-1}\frac{a}{x}.$$

IV. Differentiate n times $y = \dfrac{1}{(x-\alpha)(x-\beta)(x-\gamma)}$.

Breaking them into partial fractions,

$$y = \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\ \frac{1}{x-\alpha} + \frac{1}{(\beta-\alpha)(\beta-\gamma)}\ \frac{1}{x-\beta} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)}\ \frac{1}{x-\gamma}.$$

$$\therefore\ y_n = \frac{(-1)^n\lfloor\underline{n}.}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}\left[\frac{\gamma-\beta}{(x-\alpha)^{n+1}} + \frac{\alpha-\gamma}{(x-\beta)^{n+1}} + \frac{\beta-\alpha}{(x-\gamma)^{n+1}}\right].$$

V. $y = e^{\sin^{-1}x}$. $\qquad y_1 = e^{\sin^{-1}x}\,.\,\dfrac{1}{\sqrt{1-x^2}}$

squaring, $(1-x^2)\,y_1^2 = y^2$,

differentiating and dividing by $2\,y_1$,

$$(1-x^2)\,y_2 - x\,y_1 - y = 0.$$

Differentiating n times each term by Leibnitz theorem, we get

$$\left.\begin{array}{l}(1-x^2)\,y_{n+2} - 2\,n\,x\,y_{n+1} - n\,(n-1)\,y_n \\ \qquad\qquad\qquad - x\,y_{n+1} \qquad\qquad\qquad - n\,y_n \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - y_n\end{array}\right\} = 0.$$

$$\overline{(1-x^2)\,y_{n+2} - (2\,n+1)\,x\,y_{n+1} - (n^2+1)\,y_n = 0.}$$

EXAMPLES ON CHAPTER III.

1. Find y_4 when $y = \sin^{-1} x^2$. ✓

2. Find y_3 when $y = \tan^{-1}\sqrt{x}$.

3. Find y_7 when $y = x^8 \cos 2x$.

4. If $y = \mathrm{L}\sin a\,x + \mathrm{M}\cos a\,x$, shew that $y_2 + a^2 y = 0$.

5. If $y = \mathrm{P}\,e^{b.r} + \mathrm{Q}\,e^{-bx}$, shew that $y_2 - b^2\,y = 0$.

✓ 6. If $y = a\sin(\log x)$, prove that $x^2\,y_2 + x\,y_1 + y = 0$.

7. If $y^2 = 1 + 2\sqrt{2}\cos(2x)$, prove that $y_2 = y\,(3\,y^2 + 1)\,(7\,y^2 - 1)$. [Oxford 1889].

8. If $y - \sin x$, prove that $4\,\dfrac{d^3 \cos^7 x}{dy^3} = 105 \sin 4\,x$. [Oxford 1890].

9. If $y = \tan^{-1} x$, prove that $(1 + x^2)\,y_2 + 2\,xy_1 = 0$.

10. If $y = a\sin^{-1} b\,x$, prove that $(1 - b^2\,x^2)\,y_2 - b^2\,x\,y_1 = 0$.

11. If $y = e^{ax}\cos b\,x$, prove that $y_2 - 2\,a\,y_1 + (a^2 + b^2)\,y = 0$.

12. Find the n^{th} differential co-efficient of $e^{ax}\,\{a^2\,x^2 - 2\,n\,a\,x + n\,(n+1)\}$ [I. C. S.].

13. If $y = \sin^{-1} x$, prove that $(1 - x^2)\,y_{n+2} - (2\,n+1)\,x\,y_{n+1} - n^2\,y_n = 0$.

14. If $y = e^{-x}\sin x$, prove that $y_4 + 4\,y = 0$.

15. If $y = (x \pm \sqrt{x^2 - 1})^m$, ✓
prove that $(x^2 - 1)\,y_{n+2} + (2n+1)y_{n+1} + (n^2 - m^2)\,y_n = 0$.

16. If $P_n = \dfrac{d^n}{dx^n}\{(x^2-1)^n\}$, prove that

$$\frac{d}{dx}\left\{(x^2-1)\frac{dP_n}{dx}\right\} - n(n+1) = 0.$$

17. If $Y = sX$, $Z = tX$, and all the letters denote functions of x, then

$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$$

where the suffixes denote the derivatives with respect to x.

[Math. Tripos 1906].

18. If $y = \tan^{-1}\left\{\dfrac{e^x+1}{e^x-1}\right\}^{\frac{1}{2}}$, prove that

$$y_3 = y_1\{1 + 12y_1\}\{I + 4y_1^2\}.$$

[Math. Tripos 1907].

19. If $y = e^{a\tan^{-1}x}$, shew that

(*i*) $(1+x^2)y_2 + (2x - a)y_1 = 0.$

(*ii*) $(1+x^2)y_{n+2} + \{2(n+1)x - a\}y_{n+1} + n(n+1)y_n = 0.$

20. If $y = \sin(m\sin^{-1}x)$, shew that

(*i*) $(1-x^2)y_2 - xy_1 + m^2y = 0.$

(*ii*) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)\times y_n = 0.$

21. If u, v, w be functions of t, and suffixes denote differentiations with regard to t, prove that

$$\frac{d}{dt}\begin{vmatrix} u_1, & v_1, & w_1 \\ u_2, & v_2, & w_2 \\ u_3, & v_3, & w_3 \end{vmatrix} = \begin{vmatrix} u_1, & v_1, & w_1 \\ u_2, & v_2, & w_2 \\ u_4, & v_4, & w_4 \end{vmatrix}$$

[Coll. Exam].

Find y_n,

22. $y = \dfrac{x^3}{(1+x)^4\,(2+x)}.$

23. $$y = \frac{1}{(1 + x + x^2)}.$$

24. $$y = \frac{x^2}{a^2 - x^4}.$$

25. $$y = x^n \times \log\left(\frac{1 + x}{1 - x}\right)$$

26. $y = \dfrac{1}{x^n - a^n}$, n being a positive integer.

27. If $y = (1 + x + x^2 + x^3)^{-1}$ and $\theta = \cot^{-1}x$, shew that

$$y_n = \tfrac{1}{2}\,(-1)^n\,\underline{|n}\,\sin^{n+1}\theta\{\sin(n + 1)\theta - \cos(n + 1)\theta + (\sin\theta + \cos\theta)^{-n-1}\}.$$

[Math. Tripos].

28. Shew that if $x = \cot y$.

$$\frac{d^n}{dx^n}\,\frac{x^n}{1 + x^2} = \underline{|n}\sin y\left\{\sin y - {}^nc_1\cos y\,\sin 2y + {}^nc_2\cos^2 y\,\sin 3y - \cdots\right\},$$

[Oxford 1890.]

29. If $y = \dfrac{1}{x^3 - a^3}$, prove that

$$y_n = \frac{(-1)^n\underline{|n}}{3a^2}\,\frac{1}{(x - a)^{n+1}} + \frac{(-1)^n\underline{|n}}{3a^2} \times \frac{2\cos\left\{(n+1)\theta + \frac{2\pi}{3}\right\}}{(x^2 + ax + a^2)^{\frac{n+1}{2}}}$$

where $\tan\theta = \dfrac{a\sqrt{3}}{2x + a}$.

30. If $nA_{2k} = \dfrac{n(n-1)\cdots\cdots(n-2k+1)}{\underline{|k}}$

Shew that

$$\frac{d^n}{dx^n}\left[F(x^2)\right] = \sum_{k=0} nA_{2k}\,(2x)^{n-2k}\,[F^{n-k}(x^2)],$$

the series continuing until a zero co-efficient occurs, and $nA_0 = 1$, the indices of F denoting derivatives with respect to x^2.

[Here show that $nA_{2k} + 2(n - 2k + 2)nA_{2k-2} = (n+1)A_{2k}$ and deduce the result by induction].

Find y_n in the following.

31. $y = e^{bx^2}$.

32. $y = \sin(x^2)$.

33. If $nB_{2k} = \frac{(n+k-1)(n+k-2)\ldots(n-k)}{\lfloor k}$

Shew that

$$\frac{d_n}{dx^n}\left[F(\sqrt{x})\right] = \sum_{k=0} {}_nB_{2k}(-1)^k \times \left(\frac{1}{2\sqrt{x}}\right)^{n+k}\left[F^{n-k}(\sqrt{x})\right]$$

the summation continuing until a zero co-efficient occurs and nB_o is unity, the indices of F denoting derivatives with respect to $\sqrt{x}$.

[Here shew that $nB_{2k} + nB_{2k-2}\,2(n+k-1) = (n+1)B_{2k}$].

34. Prove that

$$\frac{d^n}{dx^n}\left(e^{a\sqrt{x}}\right) = e^{a\sqrt{x}}\left(\frac{a}{2\sqrt{x}}\right)^n \sum_{r=0}^{r=n-1}(-1)^r\left\{\frac{\lfloor n+r-1}{\lfloor r\,\lfloor n-r-1}\,\frac{1}{(2a\sqrt{x})^r}\right\}.$$

[Math. Tripos. 1886].

35. Prove that if $ac > b^2$

$$\frac{d^n}{dx^n}\,\frac{b+cx}{a+2bx+cx^2} = (-1)^n\lfloor n \times \left(\frac{c}{a+2bx+cx^2}\right)^{\frac{n+1}{2}}\cos\left[\overline{n+1}\,\tan^{-1}\frac{\sqrt{ac-b^2}}{b+cx}\right]$$

[London 1890].

36. If $\frac{1}{e^x - 1}$ be differentiated k times, the denominator of the result will be $(e^x - 1)^{k+1}$, and the sum of the co-efficients of the several powers e^x in the numerator will be $(-1)^k\, 1.2.3\ldots\ldots k$.

[Caius coll.].

37. Shew that $\tan\left(y\frac{d}{dx}\right)\sin mx = \tanh my \times \cos mx$.

[Oxford 1888].

38. Establish that if n be a positive,

$$\sin n\,x = \frac{n}{1\,.\,3\,.\,5\;\ldots\ldots\ldots(2\,n-1)} \times \left(\frac{1}{\sin\;x}\;\frac{d}{dx}\right)^{n\,-\,1}\left(\sin x\right)^{2n\,-\,1}.$$

39. Prove $\left(\frac{d}{dx}\right)^{r} e^{ax}\,x^{n} = a^{r-n}\,x^{n-r}\left(\frac{d}{dx}\right)^{n}\;e^{ax}\,x^{r}$.

[Gregory's Examples].

40. Prove that $\frac{d^n}{dx^n}\;\sin^{-1}x = \frac{1\,.\,3\,.\,5\ldots\ldots(2\,n-3)}{2^{n-1}\,(1-x)^{n-1}\,(1-x^2)^{\frac{1}{2}}}\times$

$$\left[1+\sum_{k=1}^{k=n-1}\;(-1)^{k}\;{}^{n-1}C_{k}\,T_{k}\right]$$

where $T_k = \frac{1\;.\;3\;.\;5\ldots\ldots(2\,k-1)}{(2\,n-3)\;(2\,n-5)\ldots(2\,n-2\,k-1)}\left(\frac{1-x}{1+x}\right)^{k}.$

[Frenet].

41. If $x+y=1$, prove that

$$\frac{d^n}{dx^n}\,(x^n\,y^n) = \lfloor\underline{n}\;\left(y^n - {}^{n}c_1^{\,2}\;y^{n-1}x\;+\;{}^{n}c_2^{\,2}\;y^{n-2}\;x^{2} - \ldots\ldots\ldots\right)$$

[Murphy, Electricity].

42. Prove that

$$x\log x + x\,(\log x)^2 + \frac{1}{\lfloor\underline{2}}\;\frac{d}{dx}\left\{x^2\,(\log\;x)^3\right\} + \frac{1}{\lfloor\underline{3}}\left(\frac{d}{dx}\right)^{2}\left\{x^3\left(\log x\right)^{4}\right\}+\ldots\ldots$$

to $n+1$ terms $= \frac{1}{\lfloor\underline{n+1}}\left(\frac{d}{dx}\right)^{n}\;\left\{x^{n+1}\,(\log\;x)^{n+1}\right\}.$

[Math. Tripos 1889].

CHAPTER IV.

EXPANSION.

26. Rolle's theorem.—*If a function $f(x)$ is continuous over an interval $a \leq x \leq b$ and vanishes at the ends, and has a derivative at each interior point $a < x < b$, which is continuous, there is at least one value of x, say x_1, $a < x_1 < b$, such that $f'(x_1) = 0$.*

Since $f(x)$ is continuous over the given interval and vanishes for the values of $x = a$ and $x = b$, it must do so either by increasing or decreasing from zero to a certain quantity, and then decreasing or increasing to zero respectively, at least once in the interval (a, b), unless the function is zero throughout the region The point at which the function ceases to increase and begins to decrease, or ceases to decrease and begins to increase, is known as the maximum or minimum value of the function in the neighbourhood of that point.

Suppose the function has a maximum value at $x = x_1$, then $f(x + \delta x) - f(x)$ cannot be positive if $\delta x <$ or > 0 and $\dfrac{f(x + \delta x) - f(x)}{\delta x}$ cannot be positive if $\delta x > 0$, and it cannot be negative if $\delta x < 0$. Hence in the limit when $\delta x \rightarrow 0$,

$$(i) \quad \underset{\delta x \rightarrow 0}{\mathrm{Lt}} \frac{f(x + \delta x) - f(x)}{\delta x}, \text{ cannot be positive if } \delta x > 0,$$

$$\text{and } (ii) \quad \underset{\delta x \rightarrow 0}{\mathrm{Lt}} \frac{f(x + \delta x) - (x)}{\delta x}, \text{ cannot be negative if } \delta x < 0.$$

Clearly (i) and (ii) are the regressive and progressive differential co-efficients (§ 13) of $f(x)$ at $x = x_1$ and by

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Michel Rolle was born in 1652 and died in 1719.

hypothesis these must exist and be the same; and this cannot be the case unless both of them be zero. Hence $f'(x_1) = 0$

Similar reasoning will show that $f'(x_1) = 0$ if the function has a minimum value at $x = x_1$.

27. Rolle's theorem does not apply to cases where the function or its derivative is discontinuous.

Case (*i*). When the function is discontinuous : –

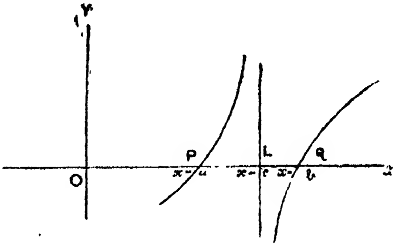

FIG. 4.

Fig. (4) Shows that the function $f(x)$ is discontinuous at $x = c$. Although $f(x)$ vanishes at $x = a$ and $x = b$ its derivative does not vanish.

Case (*ii*) when the derivative is discontinuous :—

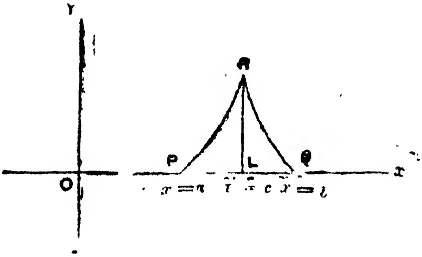

FIG. 5.

Fig. (5) shows that although $f(a)$ and $f(b)$ are zero, $f'(x)$ does not vanish for any value of x between a and b. In this case $f'(x)$ is discontinuous at $x = c$, its values being $+\infty$ and $-\infty$ according as we approach from the left hand side or the right hand side of L,

28. Mean Value theorem.—If a function $f(x)$ defined for $a \leqslant x \leqslant b$, has a derivative at each interior point and which is continuous also, there exists some point x_1, within $[a, b]$ such that,

$$\frac{f(b) - f(a)}{b - a} = f'(x_1).$$

Let us suppose that $\dfrac{f(b) - f(a)}{b - a} = \lambda$

or $f(b) - f(a) - (b - a) \lambda = 0 \ldots\ldots\ldots\ldots\ldots\ldots (i)$

Let $\phi(x) \equiv f(x) - f(a) - (x - a) \lambda \ldots\ldots\ldots (ii)$

Therefore $\phi(b) = 0$, by (i).

And also $\phi(a) = 0$, by putting $x = a$ in (ii)

Therefore by Rolle's theorem $\phi'(x_1) = 0$, where x_1 is some value of x within $[a, b]$, as $\phi(x)$ satisfies all conditions of Rolle's theorem.

Thus $\phi'(x_1) \equiv f'(x_1) - \lambda = 0$

or $\lambda = f'(x_1)$.

Hence $f(b) = f(a) + (b - a) f'(x_1) \ldots\ldots\ldots\ldots (iii)$

or $$\frac{f(b) - f(a)}{b - a} = f'(x_1) \ldots\ldots\ldots\ldots (iv)$$

Cor. 1.—Let $b = a + \delta a$, therefore $b - a = \delta a$

$\therefore\ x_1 = a + \theta \delta a$, where $0 < \theta < 1$.

Equation (iii) then becomes

$$f(a + \delta a) = f(a) + \delta a f'(a + \theta\delta a).$$

This is another form of the Mean Value theorem.

29. Mean Value theorem extension.—Let μ be defined by the following equation

$$f(b) - f(a) - (b - a) f'(a) - \tfrac{1}{2}(b - a)^2 \mu = 0 \ldots\ldots\ldots\ldots\ldots\ldots (i).$$

Now consider the function

$$\phi(x) \equiv f(x) - f(a) - (x - a) f'(a) - \tfrac{1}{2}(x - a)^2 \mu \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

$$\therefore \ \phi(b) = 0, \quad \text{by } (i)$$

and $\phi(a)$ is also identically equal to zero.

$\therefore$ under the proper[1] suppositions, by § 26

$$\phi'(x) \equiv f'(x) - f'(a) - (x - a)\mu = 0 \ldots\ldots (iii)$$

for a certain value of $x = x_1$ within $[a, b]$.

i. e., $f'(x_1) - f'(a) - (x_1 - a)\mu = 0$.

Again $\phi'(x_1) = 0$ and $\phi'(a)$ is also $= 0$.

Hence by § 26 under proper[2] suppositions, $\phi''(x)$ must vanish for some value of $x = x_2$ within $[a, x_1]$,

i. e. $\phi''(x_2) \equiv f''(x_2) - \mu = 0$

or $\mu = f''(x_2)$.

Equation (i) therefore becomes

$$f(b) = f(a) + (b - a)\, f'(a) + \tfrac{1}{2}(b - a)^2 f''(x_2) \ldots\ldots\ldots\ldots\ldots\ldots (iv).$$

This can be re-written if $b = a + \delta a$ as

$$f(a + \delta a) = f(a) + \delta a\, f'(a) + \tfrac{1}{2}\delta a^2 f''(a + \theta_2 \delta a) \quad \text{where } 0 < \theta_2 < 1.$$

Cor. 1.—Similarly defining v by the equation

$$f(b) - f(a) - (b - a)\, f'(a) - \frac{1}{\lfloor 2}(b - a)^2 f''(a) - \frac{1}{\lfloor 3}(b - a)^3 v = 0$$

We can shew that $v = f'''(x_3)$ where $a < x_3 < x_2$ and hence $f(b) = f(a) + (b - a)\, f'(a) + \frac{1}{\lfloor 2}(b - a)^2 f''(a) + \frac{1}{\lfloor 3}(b - a)^3 f'''(x_3)$. Continuing this process

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[1] *i. e.*, $\phi(x)$ has a continuous derivative at each interior point.

[2] *i. e.*, $\phi'(x)$ has a continuous derivative at each interior point or in other words $\phi(x)$ has first and second derivatives existent and continuous in the interval.

we get the general result

$$f(b) = f\ a + (b - a)\ f'(a) + \frac{1}{\lfloor 2}(b - a^2)\ f''(a) + \frac{1}{\lfloor 3}(b - a)^3\ f'''(a) + \ldots\ldots\ldots\ldots + \frac{1}{\lfloor n-1}(b - a)^{n-1}\ f^{n-1}(a) + \frac{(b - a)^n}{\lfloor n}\ f^n(x_n),$$

where $a < x_n < x_{n-1}$, etc., *i. e*, x_n lies between a and b.

30. Taylor's Theorem.—In the preceding article we proved that

$$f(b) = f(a) + (b - a)\ f'(a) + \ldots\ldots\ldots\ldots + \frac{(b - a)^n}{\lfloor n} f^n(x_n),$$ where x_n lies between a and b.

Putting $a = x$ and $b = x + h$ we have

$$f(x + h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \ldots\ldots + \frac{h^n}{\lfloor n} f^n(x + \theta h) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (i)$$

where $0 < \theta < 1$.

This is known as *Taylor's theorem*. The last term $\frac{h^n}{\lfloor n} f^n(x + \theta h)$ is known as the *remainder* in *Taylor's theorem* after n terms. If this remainder tends to zero as $n \to \infty$, Taylor's theorem can be written as

$$f(x + h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \ldots\ldots\ldots\ldots\ldots + \frac{h^n}{\lfloor n} f^n(x) + \ldots\ \text{to } \infty \ldots (ii)$$

which is known as *Taylor's series*.

31. The two series of § 30 are of great importance in the theory of differential calculus, and far reaching in their effect. The series (*i*) shows that the sum of the expression on the right hand side will be exactly equal to $f\ (x + h)$, whatever be the value of the Remainder $\frac{h^n}{\lfloor n} f^n\ (x + \theta\ h)$, whereas in the case of (*ii*), the sum of the infinite convergent series on the right hand side can be made to differ from $f\ (x + h)$ by as small a quantity as we please as $n \rightarrow \infty$.

The infinite series (*ii*) represents the function for those values of the variable and those only for which the remainder approaches zero as the number of terms increases indefinitely.

32. **Maclaurin's Theorem.**—Putting $x = 0$, and then writing x for h in the Taylor's theorem, we get

$$f\ (x) = f\,(0) + x f'\,(0) + \frac{x^2}{\lfloor 2} f''\,(0) + \ldots\ldots + \frac{x^n}{\lfloor n} f^n\ (\theta.\ x)$$

where $0 < \theta < 1$. This form of $f\,(x)$ is called Maclaurin's theorem. If the value of $\frac{x^n}{\lfloor n} f^n\ (\theta,\ x)$ tends to zero as $n \rightarrow \infty$

$$f(x) = f\ (0) + x\ f'\ (0) + \frac{x^2}{\lfloor 2} f''\,(0) + \ldots. + \frac{x^n}{\lfloor n} f^n\ (0) + \ldots\ldots \text{ to infinity,}$$

where $f^n\,(0)$ means that x is put equal to zero after differentiating $f(x)$ n times. This is known as the Maclaurin's series and gives the expansion of $f\ (x)$ in powers of x with constant co-efficients. The statements made concerning the remainder and the convergence of Taylor's series, apply with equal force to Maclaurin's series, the latter being merely a special case of the former.

33. As both Taylor's and Maclaurin's series are of fundamental importance in the development of functions, it would be advisable to recapitulate the conditions under

which both the series give an intelligible meaning. From the method of proof, it follows that all the limitations of Rolle's and Mean value theorems are also the limitations of these series together with the conditions of convergency of the remainder after the first n terms. They may be enumerated as follows:—

(1) $f(x)$ and all its derivatives must be finite within the domain given or under consideration and must be continuous as well.

(2) The reminder after the first n terms must converge to zero as $n \to \infty$.

In the case of these theorems the limitation No. 2. changes. The remainder in theorems must converge to a finite limit only.

Example.

I. To expand sin $(x + h)$ in powers of h and deduce the expansion of sin x in powers of x.

Here $f\ (x) = \sin x$

$$f'\ (x) = \cos x$$

$$f''\ (x) = \sin\left(x + \frac{2\pi}{2}\right) = -\sin x$$

$$f'''(x) = \sin\left(x + \frac{3\pi}{2}\right) = -\cos x$$

$$f^4\ (x) = \sin\left(x + \frac{4\pi}{2}\right) = \sin x$$

....

$$f^n(x) = \sin\left(x + \frac{n\pi}{2}\right)$$

$$\therefore\ {}^*\sin(x + h) = \sin x + h\cos x - \frac{h^2}{\underline{|2}}\sin x - \frac{h^3}{\underline{|3}}\cos x$$

$$+ \frac{h^4}{\underline{|3}}\sin x + \dots\dots\dots + \frac{h^n}{\underline{|n}}\sin\left(x + \frac{n\pi}{2} + \theta h\right).$$

---

* Here none of the derivatives are either infinite or discontinuous in any interval.

The Remainder after n terms is

$$\frac{h^n}{\lfloor n} \sin\left(x + \frac{n\pi}{2} + \theta h\right).$$

But $\left| \frac{h^n}{\lfloor n} \sin\left(x + \frac{n\pi}{2} + \theta h\right)\right| \leqslant \left|\frac{h^n}{\lfloor n}\right|$

and $\underset{n \to \infty}{\text{Lt}} \frac{h^n}{\lfloor n} = 0$, since h is finite.

$\therefore$ The Remainder converges to zero.

Hence $\sin(x + h) = \sin x + h \sin\left(x + \frac{\pi}{2}\right) + \frac{h^2}{\lfloor 2} \sin$

$\left(x + 2\frac{\pi}{2}\right) + \cdots\cdots + \frac{h^n}{\lfloor n} \sin\left(x + \frac{n\pi}{2}\right) + \cdots$

Now putting $x = 0$ and $h = x$, we get

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots\cdots\cdots$$

since $f(0) = f''(0) = f^4(0) = \ldots = f^{2n}(0) = 0.$

II. To Expand e^{x+h} in powers of h, and hence deduce the series for e^x.

Here $f(x) = e^x$

$f'(x) = e^x$

....

$f^n(x) = e^x$

$\therefore \quad e^{x+h} = e^x + h\, e^x + \frac{h^2}{\lfloor 2} e^x + \ldots + \frac{h^n}{\lfloor n} e^x + \cdots\cdots\cdots\cdots$

Putting 0 for x and x for h, we have $e^x = 1 + x$

$+ \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots + \frac{x^n}{\lfloor n} + \ldots$

III. Expand $(x + h)^{\frac{3}{2}}$ in powers of h.

$f(x) = x^{\frac{3}{2}}$

$f'(x) = \frac{3}{2}\, x^{\frac{1}{2}}$

$f''(x) = \frac{3}{2^2} \cdot \frac{1}{x^{\frac{1}{2}}}$

$f'''(x) = -\frac{3}{2^3} \cdot \frac{1}{x^{\frac{3}{2}}}$

$$f^4(x) = \frac{3^2}{2^4} \cdot \frac{1}{x^{\frac{5}{2}}}$$

$$f^5(x) = -\frac{3^2 \times 5}{2^5} \frac{1}{x^{\frac{7}{2}}}$$

....

We can write ∴ $(x+h)^{\frac{3}{2}} = x^{\frac{3}{2}} + \frac{3}{2} h (x + \theta_1 h)^{\frac{1}{2}} \ldots (\alpha)$

or $$(x+h)^{\frac{3}{2}} = x^{\frac{3}{2}} + \frac{3}{2} h \sqrt{x} + \frac{3}{4} \frac{h^2}{\underline{|2}} \times \frac{1}{(x + \theta_2 h)^{\frac{1}{2}}} \ldots\ldots\ldots\ldots (\beta)$$

or $$(x+h)^{\frac{3}{2}} = x^{\frac{3}{2}} + \frac{3}{2} h \sqrt{x} + \frac{3}{4} \frac{h^2}{\underline{|2}} \frac{1}{\sqrt{x}} - \frac{3}{8} \frac{h^3}{\underline{|3}} \frac{1}{x^{\frac{3}{2}}} + \ldots\ldots + (-1)^{n\ 2} \frac{3 \times 3 \times 5 \times 7 \ldots (2n-5)}{2^n} \times \frac{h^n}{\underline{|n}} \times \frac{1}{(x + \theta_n h)^{\frac{2n-3}{2}}} \ldots\ldots (i).$$

Also tho series form will be $(x+h)^{\frac{3}{2}}$

$$= x^{\frac{3}{2}} + \frac{3}{2} h \sqrt{x} + \ldots + (-1)^n \frac{3 \times 3 \times 5 \ldots (2n-5)}{2^n} \times \frac{1}{x^{\frac{2n-3}{2}}} \times \frac{h^n}{\underline{|n}} + \ldots\ldots\ldots (ii).$$

If we substitute $x = 0$ in (*ii*) the right hand expression becomes infinite, whereas the left hand assumes a definite value $h^{\frac{3}{2}}$. Hence the expansion of $(x+h)^{\frac{3}{2}}$ by Taylor's series becomes impossible when $x = 0$. But (α) gives

$$h^{\frac{3}{2}} = \frac{3}{2} h (\theta_1 h)^{\frac{1}{2}} \text{ and } (\beta) \text{ gives } h^{\frac{3}{2}} = \frac{3}{4} \frac{h^2}{\underline{|2}} \frac{1}{(\theta_2 h)^{\frac{3}{2}}}$$

for $x = 0$.

Both these give an intelligible meaning, for θ_1 and θ_2 can be properly chosen, such that the expressions on the right may equal that on the left.

Thus we find that here Taylor's theorem is true upto the second derivative only.

Examples.

1. Discuss the possibility of expanding $\sin^{-1}x$ in powers of x.
2. Shew that Maclaurin's theorem fails to give the expansion of $e^{-\frac{1}{x}}$ in powers of x.
3. Expand a^{x+h} in powers of h, by Taylor's theorem, and shew that the remainder after n terms vanishes in the limit when $n \rightarrow \infty$.
4. Expand and find the remainder after n terms of the expansion of $e^{ax} \sin bx$.

EXPANSION.

34. The importance of expanding a function in powers of the variable cannot be too greatly emphasised. The student must already be familiar with certain class of algebraical and trignometrical infinite series. Most of these can be readily deduced by means of Taylor's or Maclaurin's series, which are almost of universal application to functions which admit of such expansions. We now proceed to apply these theorems to expand a few well-known functions. As both Taylor's and Maclaurin's series give the expansion of a function in terms of the successive derivatives, our efforts will be mainly directed towards finding these derivatives either by actual differentiations or by the formation of a differential equation. Another method, which, however, is sometimes found to be more convenient in developing a function consists of integration or differentiation of a known series.

Illustrations.

Expand $y = \tan^{-1} x$.

$$y_1 = \frac{1}{1+x^2} = \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right].$$

Differentiating $(n-1)$ times we get

$$y_n = \frac{1}{2i}(-1)^{n-1} \lfloor n-1 \left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right].$$

Putting $x = r\cos\theta,\ i = r\sin\theta$

$$y_n = \frac{1}{2i}(-1)^{n-1}\frac{\lfloor\underline{n-1}}{r^n}[(\cos n\theta + i\sin n\theta) -$$

$$(\cos n\theta - i\sin n\theta) \quad [\text{by De Moivre's Theorem}]$$

$$= (-1)^{n-1}\frac{\lfloor\underline{n-1}}{r^n}\sin n\theta.$$

$$= (-1)^{n-1}\lfloor\underline{n-1}\,\sin^n\theta\,\sin n\theta, \text{ where } \cot\theta = x.$$

Now $(y_n)_{x=0} = (-1)^{n-1}\lfloor\underline{n-1}\ \sin n\frac{\pi}{2}.$

By Maclaurin's series

$$\tan^{-1}x = (y)_0 + x(y_1)_0 + \frac{x^2}{\lfloor\underline{2}}(y_2)_0 + \cdots \frac{x^n}{\lfloor\underline{n}}(y_n)_0 + \cdots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\cdots +$$

$$(-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots\cdots\cdots$$

2. Expand $y = [\log(x + \sqrt{1+x^2})]^2$

$$y_1 = 2\log(x+\sqrt{1+x^2}).\frac{1}{\sqrt{1+x^2}} = \frac{2\sqrt{y}}{\sqrt{1+x^2}}.$$

Squaring both the sides we get

$$(1+x^2)\,y_1^2 - 4y = 0.$$

Differentiating once more and dividing by $2y_1$, we get

$$(1+x^2)\,y_2 + xy_1 - 2 = 0.$$

Differentiating n times by Leibnitz theorem, we get

$$\begin{array}{rl} (1+x^2)y_{n+2} + n.\,2x.\,y_{n+1} + n(n-1)\,y_n & \\ + x\,y_{n+1} + n\,y_n & = 0 \\ \hline (1+x^2)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + n^2\,y_n & = 0 \end{array}$$

Putting $x = 0$, $(y_{n+2})_{x=0} = -\,n^2\,(y_n)_{x=0}$.

$$(y)_0 = (y_1)_0 = (y_3)_0 = (y_5)_0 = \cdots\cdots = 0.$$

$$(y_2)_0 = 2.$$

$$(y_4)_0 = -\,2^2.\,2.$$

$$(y_6)_0 = 4^2.\ 2^2.\ 2.$$
$$(y_8)_0 = -\ 6^2.\ 4^2.\ 2^2.\ 2.$$

……………………………………

$$[\log x + \sqrt{1 + x}]^2 = x^2 - 2.\ 2^2 \frac{x^4}{\lfloor 4} + 2.\ 2^2\, 4^2 \frac{x^6}{\lfloor 6} - 2.\ 2^2.\ 4^2.\ 6^2.\ \frac{x^8}{\lfloor 8} + \dots\dots$$

3. Expand $y = \sin(m \sin^{-1} x)$, and deduce the series for $\frac{\cos m\theta}{\cos\theta}$.

Here $y_1 = \cos(m \sin^{-1} x)\frac{m}{\sqrt{1 - x^2}}$

Squaring and substituting the value of $\cos(m \sin^{-1} x)$

$$(1 - x^2)\, y_1^2 = m^2 (1 - y^2).$$

Differentiating again, and dividing by $2\, y_1$, we have

$$(1 - x^2)\, y_2 - x\, y_1 + m^2\, y = 0.$$

Differentiating this n times by Leibnitz's theorem,

$$(1 - x^2)\ y_{n+2} - (2\,n + 1)\, x\, y_{n+1} + (m^2 - n^2)\, y_n = 0.$$

Putting $x = 0$,

$$(y_{n+2})_{x=0} = -\ (m^2 - n^2)\, (y_n)_{x=0}.$$

$(y)_0 = (y_2)_0 = (y_4)_0 = \dots\dots = 0$

$(y_1)_0 = m$

$(y_3)_0 = -\ m\ (m^2 - 1^2)$

$(y_5)_0 = m\ (m^2 - 1^2)\ (m^2 - 3^2)$

$(y_7)_0 = -\ m\ (m^2 - 1^2)\ (m^2 - 3^2)\ (m^2 - 5^2).$

Whence,

$$\text{Sin}\ (m \sin^{-1} x) = m\, x - \frac{m\,(m^2 - 1^2)}{\lfloor 3}\, x^3 + \frac{m\,(m^2 - 1^2)\,(m^2 - 3^2)}{\lfloor 5}\, x^5 - \dots\dots$$

Putting $x = \sin\theta$

$$\sin m\,\theta = m \sin\theta - \frac{m\,(m^2 - 1^2)}{\underline{|3}} \sin^3\theta + \frac{m\,(m^2 - 1^2)\,(m^2 - 3^2)}{\underline{|5}} \sin^5\theta - \ldots\ldots$$

Difierentiating with respect to θ

$$\frac{\cos\,m\,\theta}{\cos\theta} = 1 - \frac{(m^2 - 1^2)}{\underline{|2}} \sin^2\theta + \frac{(m^2 - 1^2)\,(m^2 - 3^2)}{\underline{|4}} \sin^4\theta - \ldots\ldots$$

EXAMPLES ON CHAPTER IV.

1. If $y = \mathrm{A} \sin m\,x + \mathrm{B} \cos m\,x$.

Prove that $y_2 + m^2\,y = 0$

$$\text{also} \quad y = \mathrm{A}\left[mx - \frac{m^3\,x^3}{\underline{|3}} + \frac{m^5\,x^5}{\underline{|5}} - \ldots\ldots \right] + \mathrm{B}\left| 1 - \frac{m^2\,x^2}{\underline{|2}} + \frac{m^4\,x^4}{\underline{|4}} - \ldots\ldots \right]$$

2. If $y = a\,x \sin x$,

prove that $(y_n)_{x=0} = -\dfrac{n}{n-2}\,(y_{n-2})_{x=0}$

Also

$$y = a\,x^2\left| 1 - \frac{x^2}{\underline{|3}} + \frac{x^4}{\underline{|5}} - \ldots \right]$$

3. Expand $\log\,(x + h)$ in powers of h and deduce the expansion of $\log\,(1 + x)$.

4. Expand $\log\,[x + \sqrt{1 + x^2}]$ in powers of x.

5. Expand $\tan h^{-1}\,x$ in powers of x.

6. Expand $\sin^{-1} x$ in powers of x.

7. Prove by Taylor's theorem that

$$\tan^{-1}(x + h) = \tan^{-1} x + (h \sin\theta) \sin\theta - \frac{(h \sin\theta)^2}{2} \sin 2\,\theta + \frac{(h \sin\theta)^3}{3} \sin 3\theta - \ldots\ldots$$

where $x = \cot\theta$.

8. Deduce from (7) that

$$\frac{\pi}{2} = \theta + \cos\theta . \sin\theta + \frac{\cos^2\theta}{2}\sin 2\theta + \frac{\cos^3\theta}{3}\sin 3\theta + \ldots\ldots$$

9. Deduce from (7) that

$$\frac{\pi}{2} = \frac{\theta}{2} + \sin\theta + \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta + \ldots\ldots$$

10. Deduce from (7)

$$\frac{\pi}{2} = \frac{\sin\theta}{\cos\theta} + \tfrac{1}{2}\frac{\sin 2\theta}{\cos^2\theta} + \tfrac{1}{3}\frac{\sin 3\theta}{\cos^3\theta} + \ldots\ldots\text{[Euler]}.$$

11. Prove $\frac{f(x+h)+f(x-h)}{2} = f(x) + \frac{h^2}{\lfloor 2} f''(x)$

$$+ \frac{h^4}{\lfloor 4} f''''(x) + \ldots + \frac{h^6}{\lfloor 6} f^6(x) + \ldots$$

12. Shew that

$$f(x) = f(0) + x\,f'(x) - \frac{x^2}{\lfloor 2} f''(x) + \frac{x^3}{\lfloor 3} f'''(x) - \ldots$$

[Bernoulli].

13. Prove that

$$f\left(\frac{x^2}{1+x}\right) = f(x) - \left(\frac{x}{1+x}\right) f'(x) + \left(\frac{x}{1+x}\right)^2 \frac{f''(x)}{\lfloor 2} - \left(\frac{x}{1+x}\right)^3 \frac{f'''(x)}{\lfloor 3} + \ldots\ldots$$

14. If $y = e^{a\sin^{-1}x}$, prove that

$$(y_{n+2})_{x=0} = (a^2+n^2)\,(y_n)_{x=0}.$$

15. Prove that

$$e^{a\sin^{-1}x} = 1 + a\,x \quad \frac{a^2x^2}{\lfloor 2} + \frac{a\,(a^2+1)\,x^3}{\lfloor 3} + \frac{a^2\,(a^2+{}_2{}^2)}{\lfloor 4}x^4 + \frac{a\,(a^2+1)\,(a^2+{}_3{}^2)}{\lfloor 5}x^5 + \ldots$$

16. From question (15) deduce the expansions

of (*i*) $\frac{\sin^{-1}x}{\lfloor 1}$

$$(ii)\ \frac{(\sin^{-1}x)^2}{\lfloor 2}$$

and $$(iii)\ \frac{(\sin^{-1}x)^3}{\lfloor 3}$$

17. Expand a^x in powers of x.

18. If $y = e^{a \log [x + \sqrt{1 + x^2}]}$, shew that

$$(y_{n+2})_{x=0} = -(n^2 - a^2)\ (y_n)_{x=0}$$

Hence find the expansion of

$$(i)\ \log [x + \sqrt{1 + x^2}]$$

and $$(ii)\ \frac{[\log (x + \sqrt{1 + x^2})]^2}{\lfloor 2}$$

19. From the relation $y = \dfrac{(1 + x)^{\frac{1}{2}}}{1 - x}$ obtain a linear differential equation with rational algebraic co-efficients and by means of it, find the expansion of y in ascending powers of x.

(I. C. S.)

20. If $\tan y = 1 + a\,x + a\,x^2$, expand y in powers of x as far as x^3.

(I. C. S.)

21. If $y = e^{\sin m x + \cos m x}$, prove that

$$(y_{n+1})_0 = m \left\{ (y_n)_0 + \sum_1^n m^r\ {}^nc_r\ (y_{n-r})_0 \times \left(\cos r \frac{\pi}{2} - \sin r \frac{\pi}{2} \right) \right\} \cdots$$

(I. C. S.)

22. If $y = e^{a \tan^{-1} x}$, shew that

$$(i)\ \frac{(\tan^{-} x)^2}{\lfloor 2} = \frac{x^2}{2} - (1 + \tfrac{1}{3}) \frac{x^4}{4} + (1 + \tfrac{1}{3} + \tfrac{1}{5}) \frac{x^6}{6} + \cdots$$

$$(ii)\ \frac{(\tan^{1-} x)^3}{\lfloor 3} = \tfrac{1}{2} \frac{x^3}{3} - \left\{ \tfrac{1}{2} + \tfrac{1}{4} (1 + \tfrac{1}{3}) \right\} \frac{x^5}{5} + \left\{ \tfrac{1}{2} + \tfrac{1}{4} (1 + \tfrac{1}{3}) + \tfrac{1}{6} (1 + \tfrac{1}{3} + \tfrac{1}{5}) \right\} \frac{x^7}{7} + \cdots$$

23. If $y = A(x + \sqrt{x^2+a^2})^n + B\,(x + \sqrt{x^2+a^2})^{-n}$

Prove that $a^2\,(y_{m+2})_{x=0} = -\,(m^2-n^2)\,(y_n)_{x=0}$.

Hence expand y in powers of x.

24. Prove that

$$\tan^{-1} x = \frac{x}{1+x^2}\left[1 + \tfrac{2}{3}\left(\frac{x^2}{1+x^2}\right) + \tfrac{2}{3} \times \tfrac{4}{5}\left(\frac{x^2}{1+x^2}\right)^2 + \dots\right]$$

(Fermat).

CHAPTER V.

PARTIAL DIFFERENTIATION.

35. So far the functions treated of were of a single independent variable. It is proposed now to give an account of functions of two or more independent variables. Thus if $u = f(x, y)$, be defined for an interval $(a \leqslant x \leqslant b, c \leqslant y \leqslant d)$ and if $\underset{\delta x \to 0}{\text{Lt}} \dfrac{f(x + \delta x, y) - f(x, y)}{\delta x}$ exists[1],

it is called the first partial differential co-efficient of $f(x, y)$ with respect to x and is written as $\dfrac{\partial f}{\partial x}$ or simply f_x. And similarly if $\underset{\delta y \to 0}{\text{Lt}} \dfrac{f(x\ y + \delta y) - f(x, y)}{\delta y}$ exists, it is called the first partial differential co-efficient of $f(x, y)$ with respect to y, and is denoted by $\dfrac{\partial f}{\partial y}$ or simply f_y.

The reason why these derivatives have been called partial is clear enough, for while finding the derivative with respect to x, the change in the function $f(x, y)$ is due to a change in the variable x by a quantity δx, while y does not change. Thus the variation in the function is partial here due to the variable x alone. Similarly if y is varying, and x does not vary, the change in the function is due to the variation in y alone.

---

[1] Continuity of functions of two or more independent variables, say $f(x, y)$ is defined as follows :—

If $f(x, y)$ is defined for a certain interval and if $\underset{\substack{x \to a \\ y \to b}}{\text{Lt}} f(x, y) = f(a, b)$, where of course a and b are values within the defined interval. Or in other words a very small change in one or some of or all the independent variables shall produce a very small change in the value of the function.

Illustrations.

1. If $u = ax^3 + by^3 + cxy.$

$$\frac{\partial u}{\partial x} = 3ax^2 + cy \quad (y \text{ treated as constant}).$$

$$\frac{\partial u}{\partial y} = 3by^2 + cx \quad (x \text{ treated as constant}).$$

2. If $u = \tan^{-1} \frac{y}{x}$

$$\frac{\partial u}{\partial x} = - \frac{y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{x}{x^2 + y^2}$$

3. If $u = x^y$

$$\frac{\partial u}{\partial x} = yx^{y-1}.$$

$$\frac{\partial u}{\partial y} = x^y \log x.$$

36. Consider a function of two variables $u \equiv f(x, y)$ both of which depend on a single independent variable say t. Let t have a small increment δt and let δx, δy and δu be the corresponding changes of x, y and u. Then

$$\begin{aligned}\delta u &= f(x+\delta x, y+\delta y) - f(x, y) \\ &= f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x, y) \\ &= \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x}\,\delta x \\ &\qquad + \frac{f(x\ y+\delta y) - f(x, y)}{\delta y}\,\delta y \quad \ldots\ldots(i)\end{aligned}$$

Now $\underset{\delta x \to 0}{\mathrm{Lt}} \dfrac{f\ x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x}$

$$= \frac{df}{dx}(x, y + \delta y)$$

$$= \frac{\partial f}{\partial x}(x, y) + \eta_1$$

*where $\eta_1 \rightarrow 0$ as $\delta y \rightarrow 0$.

$$\therefore \quad \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x}$$

$$= \frac{\partial f}{\partial x} + \eta_1 + \varepsilon_1, \text{ where}$$

$\eta_1 \rightarrow 0$ as $\delta y \rightarrow 0$ and $\varepsilon_1 \rightarrow 0$ as $\delta x \rightarrow 0$.

Similarly $\dfrac{f(x, y + \delta y) - f(x, y)}{\delta y} = \dfrac{\partial f}{\partial x} + \eta_2$

where $\eta_2 \rightarrow 0$ as $\delta y \rightarrow 0$.

Thus the equation (1) becomes

$$\delta u = \left[\frac{\partial f}{\partial x} + \eta_1 + \varepsilon_1\right]\delta x + \left[\frac{\partial f}{\partial y} + \eta_2\right]\delta y \quad \ldots\ldots (2)$$

where $\eta_1 \rightarrow 0$ and $\eta_2 \rightarrow 0$ as $\delta y \rightarrow 0$; and $\varepsilon_1 \rightarrow 0$ as $\delta x \rightarrow 0$.

Dividing both the sides of (2) by δt, we get

$$\frac{\delta u}{\delta t} = \left[\frac{\partial f}{\partial x} + \eta_1 + \varepsilon_1\right]\frac{\delta x}{\delta t} + \left[\frac{\partial f}{\partial y} + \eta_2\right]\frac{\delta y}{\delta t}.$$

---

* This is because continuity of the functions f_x and f_y are assumed.

Alternative: We can get this from the application of the Mean Value theorem. Since $f(x + \delta x) = f(x) + \delta x \frac{df}{dx} \times (x + \theta_1 \delta x)$ *i.e.*, $f(x + \delta x, y + \delta y) - f(x, y + \delta y) =$

$$= \delta x \frac{\partial f}{\partial x}(x + \theta_1 \delta x, y + \delta y)$$

($y + \delta y$ remains constant and hence the partial derivative).

Similarly $f(x, y + \delta y) - f(x, y) = \delta y \dfrac{\partial f}{\partial y}(x, y + \theta_2 \delta y)$

and from continuity of f_x and f_y, in the limit $\dfrac{\partial f}{\partial x}(x + \theta_1 \delta x, y + \delta y)$ becomes $\dfrac{\partial f}{\partial x}(x, y)$ and so also $\dfrac{\partial f}{\partial x}(x, y + \theta_2 \delta y)$ becomes $\dfrac{\partial f}{\partial x}(x, y)$.

Proceeding to the limit when $\delta t \to 0$, and therefore δx and δy both simultaneously tend to zero, we have

$$\frac{du}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

Provided both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are existent,

$\frac{du}{dt}$ is called the *Total differential co-efficient* of u with respect to t.

Now if $t \equiv x$, we get

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}$$

an expression for the total differential co-efficient of u with respect to x.

Cor.—This can be extended to functions of more than two variables *viz.* If $u = f(x, y, z)$

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

and $$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx}$$

and so on.

Note 1.—Similarly $\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\delta t}$

The proof of this is beyond the scope of this volume.

37. If $u \equiv f(x\,y) = 0$ $\frac{du}{dx} = 0$.

Hence from (B) of § 36, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Here $u = f(x, y)$ is called an implicit function of x and y and (i) gives us the means to find $\frac{dy}{dx}$ at any point (x, y).

Illustrations.

1. $u = \sin^{-1} \frac{y}{x} = 0$

$$\therefore -\frac{y}{x\sqrt{x^2 - y^2}} + \frac{1}{\sqrt{x^2 - y^2}} \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}$$

2. $u = x^3 + y^3 - 3axy = 0$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

38. Differentials. So far we have been treating of differential co-efficients. Let a new notation called differentials, which is of great use in the applications of Differential Calculus, be now introduced. By the *differential of $f(x)$*, written as $df(x)$, we mean the differential co-efficient of $f(x)$ multiplied by δx, where δx is an arbitrarily small quantity.

Hence $df(x) = f'(x) \delta x$.

In particular if $f(x) = x$, $f'(x) = 1$

$$\therefore d(x) = 1. \delta x$$

$$\therefore df(x) = f'(x) dx.$$

Hence the differential of $f(x)$ equals the differential co-efficient of $f(x)$ multiplied by the differential of x.

Now if $y = f(x)$,

$$dy = f'(x) dx.$$

or $\frac{dy}{dx} = f'(x)$.

Here dy and dx are the differentials of y and x respectively, and are such that their ratio is always equal to

$f'(x)$. dy and dx need not necessarily be small quantities tending to zero in the limit, for we have taken no account of limit whatsoever. All that can be said about them is that they are finite quantities, such that $\frac{dy}{dx} = f'(x)$, dx depending on our choice.

39. If $u = f(x, y)$, and dx and dy be the differentials of x and y, the partial differential of u with respect to x and y are defined by

$$D_x u = \frac{\partial f}{\partial x} dx$$

$$D_y u = \frac{\partial f}{\partial y} dy.$$

and the total differential of u, *i e*, du is defined as

$$\begin{aligned} du &= D_x u + D_y u \\ &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \end{aligned}$$

Cor. If $u = f(x, y, z..)$ be defined for a given interval, $du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial x} dy + \frac{\partial f}{\partial z} dz + \,..$

Successive Partial Differential Co-efficients.

40. If $u = f(x, y)$, in general $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ will be functions of x and y. If these derivatives can be further differentiated, the successive partial differential co-efficients of $\frac{\partial f}{\partial x}$ with respect to x are denoted by $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^3 f}{\partial x^3}$, .. $\frac{\partial^n f}{\partial x^n}$, . . .

Similarly successive partial differential co-efficients of $\frac{\partial f}{\partial y}$ with respect to y are denoted by $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^3 f}{\partial y^3}$, .. $\frac{\partial^n f}{\partial y^n}$, ...

If however $\frac{\partial f}{\partial x}$ is partially differentiated with respect to y, it is written as $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ or more shortly $\frac{\partial^2 f}{\partial y \partial x}$; and similarly if $\frac{\partial f}{\partial y}$ is differentiated with respect to x, it is written as $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ or more shortly $\frac{\partial^2 f}{\partial x \partial y}$. In general, differentiating partially u, first with respect to x, m times and then with respect to y, n times, it is written as $\frac{d^{m+n} f}{\partial y^n \partial x^m}$.

41. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, provided both are existent and continuous.

Let $u = f(x, y)$ be defined for a given interval, regarding y as constant and varying x alone, we get

$f(x + \delta x, y) - f(x, y) = \delta x \ f_x(x + \theta_1 \delta x, y)$,
where $0 \angle \theta_1 \angle 1$ ....(i).

Now changing y to $y + \delta y$ and maintaining x and δx as constant, the total increment of the left hand side of (i) is

$f(x + \delta x, y + \delta y) - f(x, y + \delta y) - [f(x + \delta x, y)$
$-f(x, y)]$ ....(A)

and the total increment of the right hand side is

$\delta x \ [f'_x(x + \theta_1 \delta x, y + \delta y) - f_x(x + \theta_1 \delta x, y]$
$= \delta x \ \delta y \ f_{yx}(x + \theta_1 \delta x, y + \theta_2 \delta y)$, ...$(B)$
where $0 \angle \theta_2 \angle 1$.

Thus $f(x + \delta x, y + \delta y) - f(x, y + \delta y) - [f(x + \delta x, y) - f(x, y)] = \delta x \ \delta y \ f_{yx}(x + \theta_1 \delta x, y + \theta_2 \delta y)$ ....(ii).

Similarly starting first with x as constant and y as varying, we get

$f(x + \delta x, y + \delta y) - f(x + \delta x, y) - [f(x, y + \delta y) -$

$f\ (x, y)] = \delta y\ \delta x\ f_{xy}\ (x + \theta_3\ \delta x, y + \theta_4\ \delta y)$ ....(*iii*)
where $0 \angle \theta_3 \angle 1$, and $0 \angle \theta_4 \angle 1$.

Thus from (*ii*) and (*iii*) we have $f_{xy}\ (x + \theta_3\ \delta x,\ y + \theta_4\ \delta y) = f_{yx}\ (x + \theta_1\ \delta x, y + \theta_2\ \delta y)$. Proceeding to the limit when $\delta x \rightarrow 0$ and $\delta y \rightarrow 0$ $f_{xy}\ (x, y) = f_{yx}\ (x, y)$, since the functions are assumed continuous.

$$\text{That is } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

42. $\frac{d^2y}{dx^2}$ of an implicit function $f\ (x, y) = 0$.

By § 37 $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0$....................(*i*).

Differentiating (*i*) totally with respect to x, and provided f_{xx}, f_{yy} and f_{xy} are all existent; and further f_{xy} is continuous and finite, we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} \frac{dy}{dx} + \left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \frac{dy}{dx} \right) \frac{dy}{dx} + \frac{\partial f}{\partial y} \frac{d^2 y}{dx^2} = 0 \ldots\ldots\ldots\ldots (ii).$$

Substituting for $\frac{dy}{dx}$ from (*i*) in (*ii*),

$$\frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial^2 f}{\partial x \partial y} \left(- \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \right) + \frac{\partial^2 f}{\partial y^2} \left(\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \right)^2 + \frac{\partial f}{\partial y} \frac{d^2 y}{dx^2} = 0$$

$$\text{or } \frac{\partial^2 f}{\partial x^2} \left(\frac{\partial f}{\partial y} \right)^2 - 2 \frac{\partial^2 f}{\partial x \partial y} \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial f}{\partial y} \right) + \frac{\partial^2 f}{\partial y^2} \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^3 \frac{d^2 y}{dx^2} = 0$$

$$\therefore \frac{d^2 y}{dx^2} = - \frac{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2}{f_y^3}.$$

43. Euler's Theorem.—If u is a homogeneous function of m variables x, y, z, etc., of n^{th} degree, defined for a given interval of the variables and u_x u_y u_z, etc., are all existent and finite,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+\ldots = n\,u.$$

Let $u = A_0\,x^{\alpha_0}y^{\beta_0}z^{\gamma_0}\ldots + A_1x^{\alpha_1}y^{\beta_1}z^{\gamma_1}\ldots + A_2\,x^{\alpha_2}\,y^{\beta_2}\,z^{\gamma_2}\ldots\ldots$ where $\alpha_0+\beta_0+\gamma_0+\ldots\ldots = \alpha_1+\beta_1+\gamma_1+\ldots=\ldots\ldots\ldots= n.$

$$\therefore\ \frac{\partial u}{\partial x} = A_0\alpha_0\,x^{\alpha_0-1}\,y^{\beta_0}\,z^{\gamma_0}\ldots + A_1\,\alpha_1\,x^{\alpha_1-1}\,y^{\beta_1}\,z^{\gamma_1}\ldots + A_2\,\alpha_2\,x^{\alpha_2-1}\,y^{\beta_2}\,z^{\gamma_2}\ldots\ldots$$

$$\frac{\partial u}{\partial y} = A_0\,\beta_0\,x^{\alpha_0}\,y^{\beta_0-1}\,z^{\gamma_0}\ldots\ldots + A_1\,\beta_1\,x^{\alpha_1}\,y^{\beta_1-1}\,z^{\gamma_1}\ldots + A_2\,\beta_2\,x^{\alpha_2}\,y^{\beta_2-1}\,z^{\gamma_2}\ldots\ldots$$

$$\frac{\partial u}{\partial z} = A_0\,\gamma_0\,x^{\alpha_0}y^{\beta_0}z^{\gamma_0-1}\ldots\ldots + A_1\,\gamma_1\,x^{\alpha_1}\,y^{\beta_1}z^{\gamma_1-1}\ldots + A_2\,\gamma_2\,x^{\alpha_2}y^{\beta_2}z^{\gamma_2-1}\ldots\ldots$$

and so on.

$$\text{Thus}\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+\ldots =$$

$$(\alpha_0+\beta_0+\gamma_0+\ldots)\,A_0\,x^{\alpha_0}\,y^{\beta_0}\,z^{\gamma_0}\ldots\ldots + (\alpha_1+\beta_1+\gamma_1+\ldots)\,A_1\,x^{\alpha_1}\,y^{\beta_1}\,z^{\gamma_1}\ldots + (\alpha_2+\beta_2+\gamma_2+\ldots)\,A_2\,x^{\alpha_2}\,y^{\beta_2}\,z^{\gamma_2}\ldots+\ldots\ldots$$

$$= nu.$$

44. Taylor's Theorem.—For functions of *two variables.*

$$f(x+h,\,y+k) = f(x,\,y) + \left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial x}\right)$$

$$+ \frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{\lfloor 3}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial x}\right)^3 f + \ldots$$

$$+ \frac{1}{\lfloor n}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x + \theta_1 h, y + \theta_2 k).$$

under the limitations that the function $f\,(x, y)$ together with all its partial derivatives up to a certain order n, are existent and continuous.*

Maclaurin's Theorem :—

$$f\,(x, y) = f\,(0, 0) + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)_{0,\,0} + \frac{1}{\lfloor 2}$$

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial x}\right)^2 f_{0,\,0}$$

$$+ \frac{1}{\lfloor 3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial x}\right)^3 f_{0,\,0} + \ldots$$

$$+ \frac{1}{\lfloor n}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f\,(\theta_1 x, \theta_2 y).$$

If the remainder after the first n terms tends to zero, the theorems convert into convergent series, and then they become

$$f\,(x + h, y + k) = f\,(x, y) + \left(h\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) + \frac{1}{\lfloor 2}$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f_{00} + \ldots\ldots\ldots\ldots$$

$$f\,(x, y) = f\,(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f_{00}$$

$$+ \frac{1}{\lfloor 2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f_{00} + \ldots\ldots$$

Illustrations.

1. Verify $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$ in $u = \dfrac{x^m y^m}{\sqrt{x^2 + y^2}}$

$$\frac{\partial u}{\partial x} = \frac{m\, x^{m-1}\, y^n \sqrt{x^2 + y^2} - x^{m+1}\, y^n \dfrac{1}{\sqrt{x^2 + y^2}}}{x^2 + y^2}$$

---

* The proof of this is beyond the scope of this volume.

$$= \frac{m\, x^{m-1}\, y^n\, (x^2 + y^2) - x^{m+1}\, y^n}{(x^2 + y^2)^{\frac{3}{2}}}$$

$$= \frac{(m-1)\, x^{m+1}\, y^n + m\, x^{m-1}\, y^{n+2}}{(x^2 + y^2)^{\frac{3}{2}}}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \Big\{ [n(m-1)x^{m+1}y^{n-1} + m(n+2)x^{m-1}y^{n+1}] \times$$

$$(x^2+y^2)^{3/2} - 3(x^2+y^2)^{\frac{1}{2}}\, y \times$$

$$[\,(m-1)x^{m+1}y^n + m x^{m-1}y^{n+2}\,] \Big\} \times \frac{1}{(x^2+y^2)^3}$$

$$= \frac{n(m-1)x^{m+3}y^{n-1} + (2mn - m - n + 3)x^{m+1}y^{n+1}}{(x^2+y^2)^{5/2}}$$

$$+ \frac{m(n-1)x^{m-1}y^{n+3}}{(x^2+y^2)^{5/2}}$$

Again

$$\frac{\partial u}{\partial y} = \frac{n x^m y^{n-1} \sqrt{x^2+y^2} - x^m y^{n+1} \dfrac{1}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{n x^m y^{n-1}(x^2+y^2) - x^m y^{n+1}}{(x^2+y^2)^{3/2}}$$

$$= \frac{(n-1)x^m y^{n+1} + n x^{m+2} y^{n-1}}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \Big\{ [m(n-1)x^{m-1}y^{n+1} + n(m+2)x^{m+1}y^{n-1}] \times$$

$$(x^2 + y^2)^{3/2} - 3\,(x^2 + y^2)^{\frac{1}{2}}\, x \times$$

$$[(n-1)\, x^m\, y^{n+1} + n\, x^{m+2}\, y^{n-1}] \Big\} \times \frac{1}{(x^2 + y^2)^3}$$

$$= \frac{n\,(m-1)\, x^{m+3}\, y^{n-1} + (2\, m\, n - m - n + 3)\, x^{m+1}\, y^{n+1}}{(x^2 + y^2)^{5/2}}$$

$$+ \frac{m\,(n-1)\, x^{m-1}\, y^{n+3}}{(x^2 + y^2)^{5/2}}$$

Thus $\dfrac{\partial^2 u}{\partial y\, \partial x} = \dfrac{\partial^2 u}{\partial x\, \partial y}$.

EXAMPLES ON CHAPTER V.

Find the derivative with respect to t in the following.

1. $u = x^2 + y^2$ where, $x = a \cos t$, $y = b \sin t$.

2. $u = \sin^{-1}(x-y)$, where, $x = 3\,t$, $y = 4\,t^3$.

3. $u = \cos 2\,x\,y$, where $x = \tan^{-1} t$, $y = \cot^{-1} t$.

Verify that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$ in the following.

4. $u = \sin^{-1} \frac{y}{x}$.

5. $u = \phi\left(\frac{y}{x}\right) + \psi\,(x\,y)$.

6. $u = \log\,\{x \tan^{-1} \sqrt{x^2 + y^2}\}$.

7. If $u = f(x + at) + \phi\,(x - at)$, prove that

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

8. If $u = x\,f\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right)$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2\,x\,y\,\frac{\partial^2 u}{\partial x\,\partial y} + y^2\,\frac{\partial^2 u}{\partial y^2} = 0.$$

Find $\frac{dy}{dx}$ in the following.

9. $x^3 + y^3 = 3\,a\,x\,y$

10. $a\,x^2 + 2\,h\,x\,y + b\,y^2 + 2\,f\,y + 2\,g\,x + c = 0$.

11. $(\cos x)^y = (\sin y)^x$.

12. $x^y \times y^x = x^{\cos y} + y^{\log x}$.

Find $\frac{d^2y}{dx^2}$ in the following.

13. $x^4 + y^4 + 4\,a^2\,x\,y = 0$.

14. $\text{Sin}\,(x + y) = \tan^{-1}(x - y)$.

15. $\frac{y^2}{x} + x\,(\log x - \log y) = 0$.

16. If $x = x' \cos\theta - y' \sin\theta$, $y = x' \sin\theta + y' \cos\theta$,

shew that $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial x'}\right)^2 + \left(\frac{\partial f}{\partial y'}\right)^2.$

17. If $u = e^x \sin y + e^y \sin x$, prove that,

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{2x} + e^{2y} + 2\, e^{x+y} \sin (x + y).$$

18. If $u = \log (\tan x + \tan y + \tan z)$, prove that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2\, y \frac{\partial u}{\partial y} + \sin 2\, z \frac{\partial u}{\partial z} = 2.$$

19. If $f\,(x_1, x_2, x_3, \cdots x_n)$ be any homogeneous function which becomes $F\,(X_1, X_2, X_3, \cdots X_n)$ by any linear substitution for the variables $x_1, x_2, \cdots$ in terms of X_1, X_2, $X_3, \cdots$, prove that

$x_1' f_{x\,1} + x_2' f_{x\,2} + x_3' f_{x\,3} + \dots = X'_1\, F_{x\,1} + X_2'\, F_{x\,2} + x_3'\, F_{x\,3} + \dots$ where $x'_1, x'_2, x'_3, \cdots$ and X'_1, X'_2, $X'_3, \cdots$ are simultaneous values of the two systems.

20. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, prove that

$$u_x{}^2 + u_y{}^2 + u_z{}^2 = 2\,(x\, u_x + y\, u_y + Zu).$$

[Oxford 1888].

$2u_2$

CHAPTER VI.

TANGENTS AND NORMALS.

45. Geometrical meaning of $\frac{dy}{dx}$. Let AB be the curve $y = f(x)$. Take any two points $P(x, y)$ and Q_2 $(x + \delta x_2, y + \delta y_2)$, and let the straight line PQ_2 make an angle ψ_2 with the axis of x.

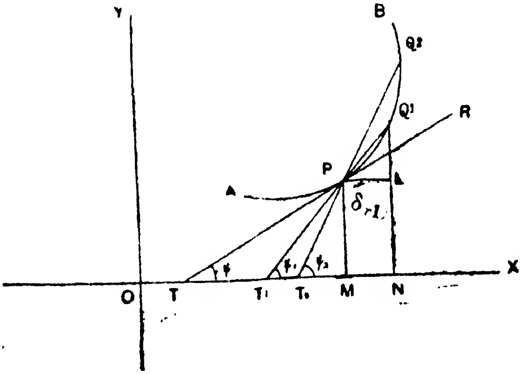

Fig. 6.

Let the point Q_2 approach P along the curve, then for different positions of Q,

$$\tan \psi_2 = \frac{\delta y_2}{\delta x_2}$$

$$\tan \psi_1 = \frac{\delta y_1}{\delta x_1},$$ and so on. Clearly as the point Q approaches P, δy and δx are each tending to zero, but in a definite ratio given by above equations. Ultimately in the limit when $Q \rightarrow P$, $\tan \psi = {}^*\underset{\delta x \rightarrow 0}{\mathrm{Lt}} \frac{\delta y}{\delta x} = \frac{dy}{dx}$, and the straight line becomes the tangent to the curve at P. Thus

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* δy also → 0, as the function $f(x)$ is taken to be continuous.

$\frac{dy}{dx}$ at any point (x, y) is equal to the tangent of the angle, which the tangent at that point to the curve makes with the axis of x, in other words $\frac{dy}{dx}$ is the slope of the curve at the point (x, y).

46. The geometrical interpretation of $\frac{dy}{dx}$ given in the last article, furnishes us an easy method of finding the equations of tangents and normals to a curve at any point on it. Thus if $f(x, y) = 0$ is any curve, the equation of any secant through (x, y), whose slope is m is given by

$$Y - y = m(X - x)$$

If the secant becomes the tangent, m becomes $\frac{dy}{dx}$.

Hence the equation of the tangent at (x, y) is given by

$$Y - y = \frac{dy}{dx}(X - x) \quad \ldots\ldots\ldots\ldots\ldots(1)$$

i e., $$Y - y = -\frac{f_x}{f_y}(X - x)$$

or $$Xf_x + Yf_y = xf_x + yf_y \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now suppose $f(x. y) = 0$ to be made homogeneous by the introduction of suitable powers of any linear parameter z, then, $$xf_x + yf_y + zf_z = nf(x, y, z) = 0 \ldots.(3)$$ combining (3) with (2) we get

$$Xf_x + Yf_y + Zf_z = 0$$

This represents the equation of the tangent at the point (x, y), in which z is to be put equal to unity after the differentiation

46. **Normal.** Since the normal at any point is perpendicular to the tangent at that point, its equation will be

$$(Y - y)\frac{dy}{dx} = - (X - x) \quad \ldots\ldots\ldots\ldots\ldots(1)$$

or $$\frac{(Y - y)}{f_y} = \frac{(X - x)}{f_x} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

47. Angle of intersection of two curves. – The angle of intersection is the same as the angle between the two tangents to both the curves at the point of their intersection.

Let the curves be

$$f(X, Y) = 0$$
$$\phi(X, Y) = 0.$$

The tangents at (x, y), the point of intersection, are

$$X f_x + Y f_y + Z f_z = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$
$$X \phi_x + Y \phi_y + Z \phi_z = 0 \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

The angle between these two straight lines is

$$\theta = \tan^{-1} \frac{-\dfrac{f_x}{f_y} + \dfrac{\phi_x}{\phi_y}}{1 + \dfrac{f_x}{f_y} \times \dfrac{\phi_x}{\phi_y}}$$

or $$\theta = \tan^{-1} \frac{\phi_x f_y - \phi_y f_x}{f_x \phi_x + f_y \phi_y}$$

Cor. (*i*) If $\dfrac{\phi_x}{\phi_y} = \dfrac{f_x}{f_y}$, the curves touch each other.

(*ii*) If $f_x \phi_x + f_y \phi_y = 0$, the curves cut orthogonally.

Some Geometrical results.

48. Let AB be the curve $f(X, Y) = 0$, and at any point P (x, y), PT, and PG be drawn tangent and normal to the curve, cutting the axis of x at T and G respectively.

Let P T cut the axis of y at S. Also let O Q and O R be perpendiculars from the origin upon P T and P G.

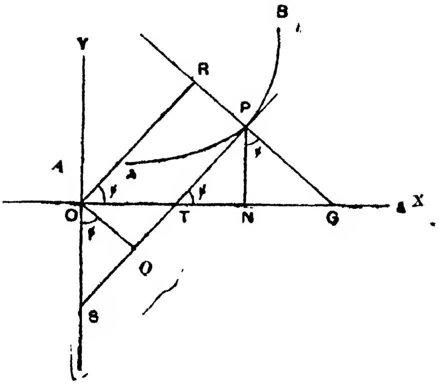

FIG. 7.

The equation of the tangent at P (x, y) is

$$(Y-y)=\frac{dy}{dx}(X-x),$$

where this cuts the x axis, $Y = 0$.

Hence the intercept on the x axis, *i e.* $OT = x - \dfrac{y}{\frac{dy}{dx}} \ldots . (i)$

Similarly when $X = 0$,

the intercept on the y axis, *i.e.*, $OS = y - x\dfrac{dy}{dx} \ldots (ii)$

The length of the tangent—*i.e.*, $PT = y \operatorname{cosec} \psi$

$$= y\sqrt{1+\cot^2\psi}$$

$$= y\frac{\sqrt{1+\tan^2\psi}}{\tan\psi}$$

$$= \frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} \ldots . (iii)$$

The length of the normal.—*i.e.* $PG = y \sec \psi$.

$$= y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \ldots (iv)$$

Sub-tangent.— *i.e.* $TN = y \cot \psi.$

$$= \frac{y}{\frac{dy}{dx}} \quad \dots\dots\dots\dots (v)$$

Sub-normal.—*i.e.* $NG = y \tan \psi.$

$$= y \frac{dy}{dx} \quad \dots\dots\dots\dots (vi)$$

$OQ = OT \sin \psi.$

$$= \left(x - \frac{y}{\frac{dy}{dx}} \right) \frac{\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

$$= \left(x \frac{dy}{dx} - y \right) \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \quad \dots\dots\dots\dots (vii)$$

$OR = PT + TQ.$

$= PT + OT \cos \psi$

$$\text{or } OR = \frac{y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} + \left(x \frac{dy}{dx} - y \right) \times \frac{1}{\frac{dy}{dx}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

$$= \frac{\left(x + y \frac{dy}{dx} \right)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \quad \dots\dots\dots\dots (viii)$$

Illustration.

1. Find equations of tangent and normal, lengths of subtangent and subnormal to the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Here $\frac{dy}{dx} = -\frac{b^2}{a^2} \times \frac{x}{y}$.

Thus the tangent at (x, y) is given by

$$(Y - y) = -\frac{b^2}{a^2}\frac{x}{y}(X - x).$$

The normal at (x, y) is given by

$$(Y - y) = \frac{a^2}{b^2}\frac{y}{x}(X - x)$$

or $$\frac{Y - y}{a^2 y} = \frac{X - x}{b^2 x}.$$

The subtangent $= \frac{y}{\frac{dy}{dx}}$

$$= -\frac{a^2}{b^2}\frac{y^2}{x}.$$

The sub-normal $= y\frac{dy}{dx}$

$$= -\frac{b^2}{a^2}x.$$

Examples.

In the following curves, find lengths of subtangent, sub-normal, tangent and normal at any point :—

1. $\left.\begin{array}{l} x = 4\,a\,\cos^3 t \\ y = 4\,a\,\sin^3 t \end{array}\right|$

2. $\left.\begin{array}{l} x = a(\cos t + t \sin t) \\ y = a(\sin t - t \cos t) \end{array}\right\}$

3. $x = \frac{3t}{1 + t^3}$
$y = \frac{3\,t^2}{1 + t^3}$

4. Find the equation of the tangent to the curve $y = b\,e^{-\frac{x}{a}}$ where it cuts the y axis.

5. Find the condition that the conics $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and

$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$ may cut orthogonally.

6. Find the angle between the parabolas

$$y^2 = 8\,x,\ x^2 = 4\,y - 12.$$

7. Find the angle between $y = x^2$, and $6\,y = 7 - x^3$ at the point (1, 1).

8. If $x \cos \alpha + y \sin \alpha = p$, touch the curve,

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1, \text{ prove that}$$

$$(a \cos \alpha)^{\frac{m}{m-1}} + (b \sin \alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}.$$

9. Shew that in the curve $\beta\, y^2 = (x + \alpha)^3$, the square of the subtangent varies as the sub-normal.

10. Show that the length of the tangent to the curve

$$x = \sqrt{c^2 - y^2} + \frac{c}{2} \log \frac{c - \sqrt{c^2 - y^2}}{c + \sqrt{c^2 - y^2}}$$

is of constant length.

49. $\frac{ds}{dx}$, $\frac{ds}{dy}$.

Let A P Q be a curve $y = f(x)$ and P (x, y) and Q $(x + \delta x, y + \delta y)$ be any two points on it.

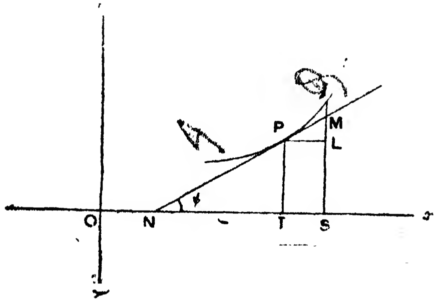

Fig. 8.

Draw P N the tangent at P making an angle ψ with the axis of x. Also let the arc of the curve be measured from some fixed point A on it, so that A P is S, and let A P Q be s + δs. Hence P Q is δs.

Thus when Q→P, chord P Q < arc P Q < P M + M Q.......................................(1)

where M is the point of intersection of the tangent P N with the ordinate of Q.

$$\text{Again P Q} = \sqrt{\delta x^2+\delta y^2}.$$

$$\text{P M+M Q} = \delta x \sec\psi + (\text{L Q}-\text{LM})$$

$$= \delta x \sec\psi + \delta y - \delta x \tan\psi$$

$$\therefore \sqrt{\delta x^2+\delta y^2} \angle \delta s < \delta x \sec\psi + \delta y - \delta x \tan\psi$$

$$\text{or} \sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2} \angle \frac{\delta s}{\delta x} \angle \sqrt{1+\left(\frac{dy}{dx}\right)^2} + \frac{\delta y}{\delta x} - \tan\psi.$$

Taking the limit when $\delta x \to 0$ and since $\frac{dy}{dx} = \tan\psi$

$$\text{we have} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \angle \frac{ds}{dx} \angle \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

which is impossible unless

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \text{ i.e., } \sec\psi \text{........}(2)$$

$$\text{Again } \frac{ds}{dy} \times \frac{dy}{dx} = \frac{ds}{dx}$$

$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \sec\psi$$

$$\therefore \frac{ds}{dy} = \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} \text{ i.e., } \operatorname{cosec}\psi \text{..........}(3)$$

Note I.—If $\frac{dx}{dy}$ and $\frac{dy}{dx}$ are both existent and none of them is either zero or infinite,

$$\frac{dy}{dx} \times \frac{dx}{dy} = \underset{\delta x \to 0}{\mathrm{Lt}} \left(\frac{\delta y}{\delta x} \times \frac{\delta x}{\delta y}\right) = 1.$$

$$\therefore \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

under similar restrictions

$$\frac{dx}{ds} = \frac{1}{\frac{ds}{dx}} = \cos \psi.$$

and $\dfrac{dy}{ds} = \dfrac{1}{\frac{ds}{dy}} = \sin \psi.$

Note II.— $\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2 = 1$; regarding dx, dy and ds as differentials

$$\frac{(dx)^2 + (dy)^2}{(ds)^2} = 1.$$

Hence the differential of the chord is equal to the differential of the arc

or $\underset{\delta s \to 0}{\mathrm{Lt}} \dfrac{PQ}{\delta s} = 1.$

50. Polar Co-ordinates.—Let P be any point (r, θ) on the curve $r = f(\theta)$ and Q another point close to P, having $(r + \delta r, \theta + \delta\theta)$ as its co-ordinates.

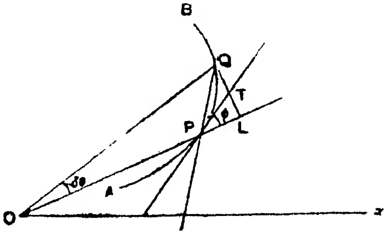

Fig. 9.

Let PT, the tangent to the curve at P make an angle ϕ with O P and Q L be the perpendicular from Q on O P.

$$\text{Now } \tan QPL = \frac{QL}{PL}$$

$$= \frac{QL}{OL - OP}$$

$$= \frac{(r+\delta r)\sin \delta\theta}{(r+\delta r)\cos\delta\theta - r}$$

$$\text{or } \tan QPL = \frac{(r+\delta r)\left[\delta\theta - \frac{(\delta\theta)^3}{\lfloor 3}\cos(\lambda_1\delta\theta)\right]}{(r+\delta r)\left[1 - \frac{(\delta\theta)^2}{\lfloor 2}\cos(\lambda_2\delta\theta)\right] - r}$$

where $0 \angle \lambda_1 \angle 1$, and also $0 < \lambda_2 < 1$.

Thus in the limit when $\delta\theta \to 0$, and so also $\delta r \to 0$, we have

$$\tan\phi = \underset{\delta\theta \to 0}{\text{Lt}} \frac{r\,\delta\theta}{\delta r},$$ neglecting $\delta r \times \delta\theta$ and terms of higher order.

$$\text{Thus } \tan\phi = r\frac{d\theta}{dr} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

51. $\frac{ds}{dr}, \frac{ds}{rd\theta}$.

Let $r = f(\theta)$ be a given curve, and let P (r, θ) and Q $(r+\delta r, \theta+\delta\theta)$ be any two points close to each other, and also let P R be the perpendicular on O Q, O being the pole.

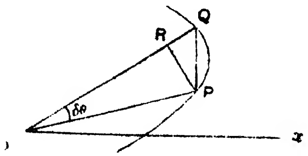

FIG. 10.

Then (chord P Q)^2 = P R^2 + R Q^2

$$= (r \sin \delta\theta)^2 + (r + \delta r - r \cos \delta\theta)^2 \ldots\ldots (1)$$

Dividing both the sides by $(\delta\theta)^2$ we get

$$\left(\frac{\text{chord PQ}}{\delta\theta}\right)^2 = r^2 \left(\frac{\sin \delta\theta}{\delta\theta}\right)^2 + \left(\frac{\delta r}{\delta\theta} + \frac{r\,(1 - \cos \delta\theta)}{\delta\theta}\right)^2$$

$$\therefore \underset{\delta\theta \to 0}{\text{Lt}} \left(\frac{\text{chord PQ}}{\delta s} \times \frac{\delta s}{\delta\theta}\right)^2 =$$

$$\underset{\delta\theta \to 0}{\text{Lt}} \left[r^2 \left(\frac{\sin \delta\theta}{\delta\theta}\right)^2 + \left\{ \frac{\delta r}{\delta\theta} + \frac{r \sin^2 \frac{\delta\theta}{2}}{\frac{\delta\theta}{2}} \right\} \right],$$

S being measured from some fixed point on the curve.

$$\therefore \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

by § 49 note II.

$$\text{or} \quad \frac{ds}{r\,d\theta} = \sqrt{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2} = \text{cosec}\,\phi \ldots. (3)$$

Similarly dividing (1) by $(\delta r)^2$ and proceeding to the limit we have

$$\left(\frac{ds}{dr}\right)^2 = 1 + r^2 \left(\frac{d\theta}{dr}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$= 1 + \tan^2\phi, \qquad \text{[by § 50,}$$

$$= \sec^2 \phi$$

$$i.e., \ \frac{ds}{dr} = \sec \phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Thus $\frac{dr}{ds} = \cos \phi$, and $r\frac{d\theta}{ds} = \sin \phi$.

Illustrations.

I. Prove for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$\frac{ds}{d\phi} = a\sqrt{1 - e^2 \cos^2 \phi}$, ϕ being the eccentric angle.

$$x = a \cos \phi$$
$$y = b \sin \phi$$
$$\frac{dx}{d\phi} = - a \sin \phi, \quad \frac{dy}{d\phi} = b \cos \phi$$
$$\frac{ds}{d\phi} = \sqrt{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2}$$
$$= \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}$$
$$= \sqrt{a^2 \sin^2 \phi + a^2 (1 - e^2) \cos^2 \phi}$$
$$= a \sqrt{1 - e^2 \cos^2 \phi}$$

11. To find ϕ and $\frac{ds}{dr}$ in $r^n = a^n \sec (n \theta + \alpha)$.

Taking the logarithmic differential co-efficient with respect to r, we get

$$\frac{1}{r} = \tan (n\theta + \alpha) \frac{d\theta}{dr}$$
$$\therefore \tan \phi = \cot (n\theta + \alpha)$$
$$\text{or } \phi = \frac{\pi}{2} - (n\theta + \alpha) \quad \dots\dots\dots\dots\dots\dots (1)$$

Also
$$\frac{ds}{dr} = \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}$$
$$= \sqrt{1 + \cot^2 (n\theta + \alpha)}$$
$$= \operatorname{cosec} (n\theta + \alpha), \quad \dots\dots\dots\dots\dots\dots (2)$$

52. Polar subtangent and subnormal.- Let AB be the given curve $r = f(\theta)$ and PT the tangent at P on it, Also let PT′ be drawn the normal at P. Through O, the pole, draw TOT′ at right angles to the radius vector OP,

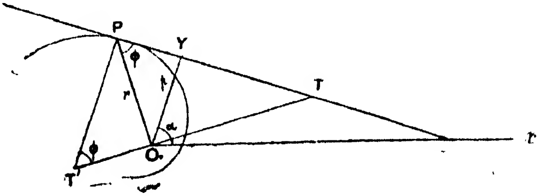

Fig. 11.

cutting the tangent and the normal at T and T′ respectively.

Then OT is called the *polar subtangent* and OT′ the *polar subnormal.*

$$\text{OT} = r \tan \phi$$
$$= r^2 \frac{d\theta}{dr} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$\text{OT}' = r \cot \phi$$
$$= \frac{dr}{d\theta}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Sometimes it is desirable to use $u = \frac{1}{r}$, that is the reciprocal of the radius vector instead of r in the equation.

$$\text{Then } \frac{du}{d\theta} = - \frac{1}{r^2} \frac{dr}{d\theta}$$
$$\text{or } \frac{1}{u^2} \frac{du}{d\theta} = - \frac{dr}{d\theta}$$

Thus from (1) and (2)

$$\textit{Polar subtangent} = r^2 \frac{d\theta}{dr} \quad \text{or} - \frac{d\theta}{du}$$
$$\textit{Polar subnormal} = \frac{dr}{d\theta} \quad \text{or} - \frac{1}{u^2} \frac{uu}{d\theta}$$

53. **Polar equation of the tangent.**—Let the polar co-ordinates of the point of contact be (r_1, θ_1). The equation of any straight line can be put down as

$$\frac{1}{r} = A \cos (\theta - \theta_1) + B \sin (\theta - \theta_1), \quad \ldots . (1)$$

where A and B are arbitrary constants. Let this represent the tangent at (r_1, θ_1).

Differentiating with respect to θ,

$$- \frac{1}{r^2} \frac{dr}{d\theta} = - A \sin (\theta - \theta_1) +$$
$$B \cos (\theta - \theta_1) \ldots\ldots\ldots\ldots (2)$$

Since the tangent touches the curve, the value of $\frac{dr}{d\theta}$ at the point of contact (r_1, θ_1) is the same for the curve as for the tangent. Hence substituting $\theta = \theta_1$ and $r = r_1$ in equations (1) and (2), we have

$$A = \frac{1}{r_1} \text{ and } B = -\frac{1}{r_1^2}\frac{dr}{d\theta}$$

whence the required equation will be

$$\frac{1}{r} = \frac{1}{r_1}\cos(\theta - \theta_1) - \frac{1}{r_1^2}\frac{dr}{d\theta}\sin(\theta - \theta_1)$$

$$\text{or} \quad \frac{r_1^2}{r} = r_1 \cos(\theta - \theta_1) - \frac{dr}{d\theta}\sin(\theta - \theta_1) \ldots\ldots (3)$$

Changing r into u, we get the equation of the tangent at (u_1, θ_1) as

$$u = u_1 \cos(\theta - \theta_1) + \frac{du}{d\theta}\sin(\theta - \theta_1) \ldots . (4)$$

54. **Polar equation of the normal.**—The equation of any straight line perpendicular to the tangent given by equation (3) of § 53 may be written as

$$\frac{c}{r} = -r_1 \sin(\theta - \theta_1) - \frac{dr}{d\theta}\cos(\theta - \theta_1)^*$$

where c is an arbitrary constant. This should pass through (r_1, θ_1) if this is to be the normal at the point of contact.

$$\text{Therefore } \frac{c}{r_1} = -\frac{dr}{d\theta}$$

$$\text{or } c = -r_1 \frac{dr}{d\theta}$$

Thus the required equation of the normal is

$$\frac{r_1 \frac{dr}{d\theta}}{r} = \frac{dr}{d\theta}\cos(\theta - \theta_1) + r_1 \sin(\theta - \theta_1) \ldots (1)$$

---

* Putting $\theta + \frac{\pi}{2}$ for θ.

Changing r into u, we get the equation of the normal at (u_1, θ_1) as

$$\frac{1}{u_1} \frac{du}{d\theta} u = \frac{du}{d\theta} \cos(\theta - \theta_1) - u_1 \sin(\theta - \theta_1), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

55. If p denotes the length of the perpendicular OY from the origin on the tangent at P, (see figure 11, § 52),

$$OY = p = r \sin \phi$$

$$\frac{1}{p^2} = \frac{1}{r^2 \sin^2 \phi}$$

$$= \frac{1}{r^2} [1 + \cot^2\phi]$$

$$= \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

or
$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Pedal Equation.

56. **p, r equation.**—The relation between the perpendicular on the tangent and the radius vector of the point of contact from the pole is known as the pedal equation.

If the equation is given in cartesian co-ordinates say $f(x, y) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

The tangent is given by $X f_x + Y f_y + Z f_z = 0, \ldots (2)$ where z is put equal to unity after differentiation.

If p be the perpendicular from the origin upon (2)

$$p = \frac{Z f_z}{\sqrt{f_x^2 + f_y^2}} \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Also $\quad r^2 = x^2 + y^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$

Eliminating x and y from equations (1), (3) and (4), we get the desired relation between p and r, which gives the pedal equation

Illustration.

I. Find the pedal equation of

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Equation of the tangent at (x, y) is

$$\frac{X}{x^{\frac{1}{3}}} + \frac{Y}{y^{\frac{1}{3}}} - a^{\frac{2}{3}} = 0.$$

Perpendicular from the origin is given by

$$p = \frac{a^{\frac{2}{3}}}{\sqrt{\frac{1}{x^{\frac{2}{3}}} + \frac{1}{y^{\frac{2}{3}}}}}$$

$$= \frac{a^{\frac{2}{3}}\, x^{\frac{1}{3}}\, y^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}$$

or $\quad p = a^{\frac{1}{3}}\, x^{\frac{1}{3}}\, y^{\frac{1}{3}} \ldots\ldots\ldots\ldots\ldots\ldots(2)$

Also $\quad r^2 = x^2 + y^2 \ldots\ldots\ldots\ldots\ldots\ldots(3)$

Now any point on the curve can be expressed as $x = a \cos^3 \phi,\ y = a \sin^3\phi$

$\therefore\ p = a \sin \phi \cos \phi \ldots\ldots\ldots\ldots\ldots\ldots(4)$

and $\quad r^2 = a^2\,[(\cos^2 \phi)^3 + (\sin^2 \phi)^3]$

$$= a^2\,[(\cos^2 \phi)^2 - \cos^2 \phi \sin^2 \phi + (\sin^2 \phi)^2]$$

$$= a^2\,[(\cos^2 \phi + \sin^2 \phi)^2 - 3 \sin^2 \phi \cos^2 \phi]$$

$$= a^2\,[1 - 3 \sin^2 \phi \cos^2 \phi]$$

or $\quad r^2 = a^2\left[1 - 3\,\frac{p^2}{a^2}\right] \quad$ by (4),

*or $\quad r^2 = a^2 - 3\,p^2.$

---

* Alternative proof :—

from (1) and (2) $x^{\frac{1}{3}} - y^{\frac{1}{3}} = \left(a^{\frac{2}{3}} - \frac{2p}{a^{\frac{1}{3}}}\right)^{\frac{1}{2}}$

$$= \frac{\sqrt{a - 2p}}{a^{\frac{1}{6}}}$$

Similarly $\quad x^{\frac{1}{3}} + y^{\frac{1}{3}} = \frac{\sqrt{a + 2p}}{a^{\frac{1}{6}}}$

$\therefore\ \ x = \left[\frac{\sqrt{a + 2p} + \sqrt{a - 2p}}{2\, a^{\frac{1}{6}}}\right]^3$ and

57. If the equation be given in polar co-ordinates say $f(r, \theta) = 0$, eliminating θ between the equation (1) of § 55 and $f(r, \theta) = 0$, we get the required relation in p and r, which is the same thing as follows :—

$$f(r, \theta) = 0 \ldots\ldots\ldots\ldots\ldots(1)$$

Also we know $p = r \sin \phi \ldots\ldots\ldots\ldots\ldots(2)$

and $\tan \phi = r \dfrac{d\theta}{dr} \ldots\ldots\ldots\ldots\ldots(3)$

Eliminating θ and ϕ between the equations (1), (2) and (3) we get the pedal equation of the curve.

Illustrations.

I. Find the pedal equation of the curve

$$\frac{L}{r} = 1 + e \cos \theta.$$

Putting $\dfrac{1}{r} = u$

$L u = 1 + e \cos \theta$, differentiating with respect to θ, we get $L \dfrac{du}{d\theta} = - e \sin \theta$.

Again from (2) of § 55

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

$$= u^2 + e^2 \frac{\sin^2 \theta}{L^2}$$

$$= u^2 + \frac{1}{L^2}\left\{ e^2 (1 - \cos^2 \theta) \right\}$$

$$= u^2 + \frac{1}{L^2}\left\{ e^2 - (Lu - 1)^2 \right\}$$

$$y = \left[\frac{\sqrt{a + 2p} - \sqrt{a - 2p}}{2\, a^{\frac{1}{6}}} \right]^3$$

$$\therefore\ 2^6 a r^2 = \{(\sqrt{a + 2p} + \sqrt{a - 2p})^2\}^3 + \{(\sqrt{a + 2p} - \sqrt{a - 2p})^2\}^3$$

$$= 64\, a\, (a^2 - 3 p^2)$$

or $r^2 = a^2 - 3 p^2$.

$$= u^2 + \frac{1}{L^2}(e^2 - L^2 u^2 - 1 + 2 L u)$$

$$\text{or} \quad \frac{1}{p^2} = \frac{e^2 - 1}{L^2} + \frac{2u}{L}$$ ✓

$$\text{or} \quad \frac{1}{p^2} = \frac{e^2 - 1}{L^2} + \frac{2}{Lr} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

In the case of a parabola $e = 1$ and $L = 2a$. Hence its pedal equation is $p^2 = ar$.

II. Find the pedal equation of $r^m = a^m \cos m\theta$.

Taking logarithmic differentiation with respect to θ· we have

$$\frac{m}{r} \frac{dr}{d\theta} = -m \frac{\sin m\theta}{\cos m\theta}$$

$$\therefore \quad \cot \phi = -\tan m\theta$$

$$= \cot\left(m\theta + \frac{\pi}{2}\right)$$

$$\therefore \quad \phi = m\theta + \frac{\pi}{2}$$

$$\text{Again } p = r \sin \phi = r \sin\left(m\theta + \frac{\pi}{2}\right)$$

$$= r \cos m\theta$$

$$= r \frac{r^m}{a^m}$$

$$\therefore \quad p = \frac{r^{m+1}}{a^m}.$$

Examples.

Find the pedal equation of the following :—

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, both with regard to the centre and with regard to a focus as pole.

2. $y^2 = 4ax$, with regard to vertex.

3. $r = d \cos \theta$.

4. $r = a e^{\theta \cot \alpha}$

5. $r^2 = a^2 \cos 2\theta$.

6. $r^2 \sin 2\theta + a^2 = 0$.

7. $r \sin \theta + a = 0$.

58. First positive Pedal curve.—If O be any fixed point, the locus of the foot of the perpendicular from O on the tangent to a curve is called the *first positive pedal* of the given curve with regard to the given point.

$$\text{Let } f(x, y) = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be any curve.

$$\text{Suppose } x \cos \alpha + y \sin \alpha = p, \ldots\ldots\ldots\ldots (2)$$

touches the given curve; also the tangent to (1) is given by

$$X f_x + y F_y + Z f_z = 0 \ldots\ldots\ldots\ldots (3)$$

If (2) and (3) represent the same line

$$\frac{f_x}{\cos\alpha} = \frac{f_y}{\sin\alpha} = \frac{f_z}{-p} = \lambda, \text{ say} \ldots\ldots\ldots\ldots (4)$$

Eliminating x, y and λ from equations (1) and (4), we get a relation between p and α only, which is the required locus, since p and α are the polar coordinates of the foot of the perpendicular. [see fig. 11 § 52].

Putting therefore, r and θ for p and α respectively, we get the equation of the first positive pedal in polar coordinates.

59. When the equation of the curve is given in polar co-ordinates, the first positive pedal can be found in the following way.

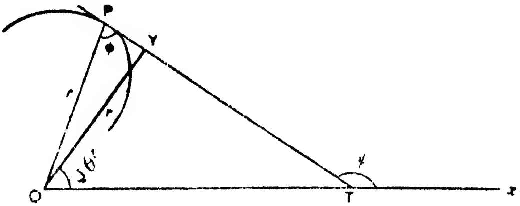

FIG. 12.

Let P be any point on the curve $f(r, \theta) = 0$, and OY the perpendicular on the tangent at P. The polar coordinates of Y are (r', θ') say.

$$\text{We have } r' = r \sin \phi \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$r \frac{d\theta}{dr} = \tan \phi \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{1}{r'^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \ldots (3)$$

$$\text{Also} \quad \theta = \psi - \phi$$

$$= \theta' + \frac{\pi}{2} - \phi \ \ldots\ldots\ldots (4)$$

Eliminating r, θ and ϕ from these equations and the equation of the curve, we get a relation between r' and θ' the polar coordinates of Y. Thus removing the dashes, we get the locus of Y *i. e*, the first positive pedal of the given curve.

Illustrations.

1. Find the first positive pedal of $y^2 = 4\,ax$.

Tangent to the curve at $(x'\ y')$ is given by

$$yy' - 2a\,(x + x') = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Also $x \cos \alpha + y \sin \alpha - p = 0, \ldots\ldots\ \ldots\ldots\ldots (2)$ touches the curve if (1) and (2) represent the same line.

$$\therefore \frac{y'}{\sin \alpha} = \frac{-2\,a}{\cos \alpha} = \frac{2\,a\,x'}{p}$$

$$\therefore \quad x' = -\frac{p}{\cos \alpha}, \quad \text{and } y' = -2\,a \tan \alpha$$

$$\text{Also} \qquad y'^2 = 4\,ax'$$

$$\therefore \quad 4\,a^2 \tan^2 \alpha = -\frac{4\,a\,p}{\cos \alpha}$$

$\therefore$ the required equation is

$$a \sin^2 \theta \sec \theta + r = 0.$$

II. Find the first positive pedal of $r^n = a^n \sin n\theta$.

Taking the logarithmic differentiation with respect to θ, we have

$$\frac{n}{r} \quad \frac{dr}{d\theta} = n \cot n\theta$$

$$\cot \phi = \cot n\theta \cdot$$

$$\therefore \quad \phi = n\theta \text{ ....}(1)$$

Also if r' and θ' be the co-ordinates of the foot of the perpendicular from the pole on the tangent,

$$\theta = \frac{\pi}{2} + \theta' - \phi$$

$$= \frac{\pi}{2} + \theta' - n\,\theta$$

$$\text{or} \quad \theta = \frac{\pi + 2\,\theta'}{2\,(n+1)}$$

Also $\qquad r' = r \sin \phi$

$$= r \sin n\,\theta$$

$$= a \sin^{\frac{n+1}{n}} n\,\theta$$ [from the equation of the curve.

$$\therefore \quad r' = a \sin^{\frac{n+1}{n}} \left[n \frac{\pi + 2\,\theta'}{2\,(n+1)} \right]$$

$$\text{or} \quad r^{\frac{n}{n+1}} = a^{\frac{n}{n+1}} \sin \left\{ \frac{n}{n+1} \left(\frac{\pi}{2} + \theta \right) \right\}$$

60. Inversion.—If O be the pole and P be any given point, and a second point P′ be taken on OP or OP produced, such that OP OP′ = k^2, where k is a constant, P′ is said to be the inverse of the point P with respect to a circle of radius k and centre O.

If P describes a curve, the curve described by P′ is known as the inverse curve of the curve described by P.

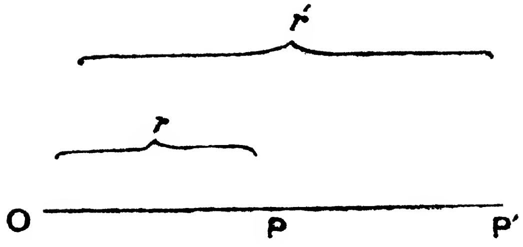

FIG. 13.

Thus if (r, θ) be the coordinates of P, the coordinates of P′ will be $\left(\frac{k^2}{r}, \theta\right)$, as $r\,r' = k^2$.

Hence if the locus of P be $f(r, \theta) = 0$.

The locus of P′ will be $f\left(\frac{k^2}{r}, \theta\right) = 0$.

Examples.

1. Find the inverse curve of $f(x, y) = 0$, with regard to a circle of radius k and centre origin.

2. Find the inverse curve of the straight line $x = a$ with regard to a circle of radius k and centre at the origin.

61. Polar Reciprocal.—If OY be the perpendicular from the pole upon the tangent to a given curve, and if any point P be taken on OY or OY produced such that OY. OP $= k^2$, the locus of P is called the *polar reciprocal* of the given curve with regard to a circle of radius k and centre at O. It is thus quite clear from this definition that the polar reciprocal of any given curve is the inverse curve of the first positive pedal.

Illustration.

I. Find the polar reciprocal of $y^2 = 4\,ax$.

The first positive pedal is $r + a \sin^2 \theta \sec \theta = 0$

The inverse of this will be given by

$$\frac{k^2}{r} + a \sin^2 \theta \sec \theta = 0$$

i. e., $k^2 + ar \sin^2 \theta \sec \theta = 0$, is the polar reciprocal with regard to a circle of radius k and centre at the origin.

Application to Mechanics.

62. Since $\frac{dy}{dx}$ is the limiting value of the rate of increase of y with respect to x, it gives us a measure of the rate of increase of the original function per unit increase of the independent variable. Consider the rectilinear motion of a point. Now the distance of a moving point from a fixed point in its path is a function of the time. If s denotes the distance from the fixed point, t will be also reckoned from some fixed time *i e.*, when the point coincided with the fixed point. If in time δt, the space described is δs, the ratio $\frac{\delta s}{\delta t}$ is called the mean velocity during the interval δt. In the limit when $\delta t \rightarrow 0$ and therefore δs also tends to zero, this mean velocity tends to a definite limit $\frac{ds}{dt}$, which is adopted as the measure of the "velocity at the instant t." Thus velocity $v = \frac{ds}{dt}$. Similarly if v be the velocity, $\frac{\delta v}{\delta t}$ is the mean acceleration during interval δt and its limit $\frac{dv}{dt}$ is the rate of change of velocity that is the acceleration at time t. Thus acceleration $= \frac{dv}{dt} = \frac{d^2s}{at^2}$ at any instant t.

Similarly $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent velocities in the directions of x and y axes respectively; and $\frac{d^2x}{dt^2}$ and

$\frac{d^2y}{dt^2}$ represent accelerations in the directions of x and y axes,

63 Suppose a particle has u as its initial velocity, v the velocity at any time t, s the space described in time t, as it is moving in a straight line with f as acceleration.

$$i.\ e.,\ \frac{d^2s}{dt^2} = f$$

$\therefore$ * $\frac{ds}{dt} = ft + k$, and from the given conditions

$$v = u + ft \ \dots\dots\dots\dots\dots\dots (1)$$

Also $\quad s = ut + \frac{1}{2} ft^2 + c,$

$\therefore$ reckoning $s = 0$ when $t = 0$, $s = ut + \frac{1}{2} ft^2 \dots .(2)$

Eliminating t between (1) and (2) we have

$$v^2 = u^2 + 2 fs \ \dots\dots\dots\dots\dots\dots (3)$$

(1), (2) and (3) are well known equations in dynamics.

EXAMPLES ON CHAPTER VI.

1. For the parabola $y^2 = 4ax$, prove that $\frac{ds}{dx}$ is equal to $\sqrt{\frac{a+x}{x}}$ and the subnormal is of constant length.

2. Find $\frac{ds}{dy}$ for the curve $y^3 = x^2$.

3. Find $\frac{ds}{dx}$ for the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, and shew that the portion of the tangent intercepted between the axes is constant.

4. Find $\frac{ds}{dx}$ for the curve $e^y \cos x = 1$.

---

* For this simple integration the student is referred to any book on integral calculus.

5. Find $\frac{ds}{dx}$ for the curve $y = a \log \sec \frac{x}{a}$, and shew that $x = a\psi$.

6. Find $\frac{ds}{dx}$ and $\frac{ds}{d\theta}$ for the cycloid

$$y = a(1 - \cos\theta),$$
$$x = a(\theta + \sin\theta).$$

7. If $r = a\,e^{\theta \cot \alpha}$,

shew that (*i*) $\phi = \alpha$.

(*ii*) $\frac{dr}{ds} = \cos\alpha$.

(*iii*) $p = r \sin\alpha$

8. If $\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\frac{a}{r}$, prove that $\cos\phi = \frac{a}{r}$.

9. If $r = a^{\theta}$, find $\frac{ds}{d\theta}$.

10. If $r^2 = a^2 \cos 2\theta$, find $\frac{ds}{d\theta}$ and $\tan\phi$.

11. For the cardioide $r = a(1 - \cos\theta)$, prove

(*i*) $\phi = \frac{\theta}{2}$

(*ii*) $p^2 = \frac{r^3}{2a}$.

(*iii*) $p = 2a \sin^3 \frac{\theta}{2}$.

(*iv*) Polar subtangent $= 2a \tan\frac{\theta}{2} \sin^2\frac{\theta}{2}$.

12. If $r = \tan\theta + \sec\theta$, find $\frac{ds}{d\theta}$.

13. In the parabola $\frac{2a}{r} = 1 - \cos\theta$, shew that

(*i*) $\phi = \pi - \frac{\theta}{2}$.

(*ii*) $p = a \operatorname{cosec} \frac{\theta}{2}$.

(*iii*) $p^2 = ar$.

(*iv*) Polar subtangent $= 2a \operatorname{cosec} \theta$.

14. If $r^n = a^n \sin n\theta$, prove that $\dfrac{ds}{d\theta} = a \operatorname{cosec}^{\frac{n-1}{n}} n\theta$

and $p = \dfrac{r^{n+1}}{a^n}$.

15. Shew that each of the curves

(*i*) $r = ae^{m\theta}$

(*ii*) $r\theta = a$

(*iii*) $r \sin n\theta = a$

(*iv*) $r \sinh n\theta = a$

has its pedal equation of the form $\dfrac{1}{p^2} = \dfrac{A}{r^2} + B$.

16. Prove that the locus of the extremity of the polar subtangent of the curve $u + f(\theta) = 0$

is $u = f'\left(\theta + \dfrac{\pi}{2}\right)$.

17. Prove that the locus of the extremity of the polar subnormal of the curve $r = f(\theta)$ is

$$r = f'\left(\theta - \frac{\pi}{2}\right).$$

Hence shew that the locus of the extremity of the polar subnormal in the equiangular spiral $r = ae^{m\theta}$ is another equiangular spiral.

18. In the curve $r = \dfrac{1 + \tan \dfrac{\theta}{2}}{m + n \tan \dfrac{\theta}{2}}$, the locus of the extremity of the polar subtangent is a cardioide.

[Prof. Wolstenholme.]

19. Find the pedal equation of $y^2 (3a - x) = (x - a)^3$

[Oxford 1889].

20. If $x = (a + b) \cos \theta - b \cos \frac{a + b}{b} \theta$

$$y = (a + b) \sin \theta - b \sin \frac{a + b}{b} \theta.$$

Shew that $p = (a + 2b) \sin \frac{a}{2b} \theta$,

$\psi = \frac{a + 2b}{2b} \theta$, and that the pedal equation

is $r^2 = a^2 + \frac{4(a + b) b}{(a + 2b)^2} p^2$

21. If r_1 and r_2 be the distances of any point P on the lemniscate $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$ from the points $\left(\pm \frac{a}{\sqrt{2}}, 0\right)$, and p_1, p_2 be the perpendiculars on the tangent at P from these points, prove that

$$\frac{p_1}{r_1^2} + \frac{p_2}{r_2^2} = \sqrt{2}\left(\frac{1}{r_1} \sim \frac{1}{r_2}\right)$$

22. Find the polar reciprocal of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with regard to a circle whose centre is the origin.

Find the first positive pedal with regard to the origin of,

23. $y^2 = 4a(x + a)$

24. $r = a(1 + \cos \theta)$

25. (*i*) $r^n = a^n \cos n\theta$ (*ii*) $r^2 = a^2 \cos 2\theta$

26. (*i*) $x^3 + y^3 = a^3$. (*ii*) $r = a e^{\theta \cot \alpha}$.
(*iii*) $A x^m + B y^m = 1$. (*iv*) $A x^2 + B y^2 = 1$.

27. Find the polar reciprocal of the curve $x^\alpha y^\beta = a^{\alpha + \beta}$ with regard to a circle whose centre is at the origin and shew that it is of the same kind.

28. Find the polar reciprocal of the curve $r^n = a^n \sin n\theta$ with regard to a circle of radius k.

29. Prove that the polar reciprocal of $(a x)^{\frac{2}{3}} + (b y)^{\frac{2}{3}} = 1$

with respect to the circle of radius $\sqrt{a^2 - b^2}$, whose centre is the origin is $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$

30. Shew that the polar reciprocal of a circle with respect to a circle with any point as centre is a conic having the centre as focus.

31. Shew that the first positive pedal of a circle with regard to any point is a Limacon $r = a + b \cos \theta$, which becomes a cardioide $r = a (1 + \cos \theta)$, when the point is on the circumference.

CHAPTER VII.

UNDETERMINED FORMS.

64. It is sometimes found in the course of mathematical analysis that a function assumes a form which is apparently meaningless, when a certain value is assigned to the independent variable For instance, $\frac{\sin x}{x}$ when x is zero assumes the form $\frac{0}{0}$, which has no meaning; again $\frac{\log(1-x)}{\frac{1}{1-x}}$ when $x = 1$ has no meaning as it becomes of the form $\frac{\infty}{\infty}$. But we know that $\underset{x \to 0}{\mathrm{Lt}} \frac{\sin x}{x} = 1$, and the $\underset{x \to 1}{\mathrm{Lt}} \frac{\log(1-x)}{\frac{1}{1-x}} = 0$. The aim of this chapter is to investigate a systematic method for finding the true values of such and other '*undetermined forms*', as they are usually called, as the independent variable tends to a certain quantity. These *undetermined forms* can be any one of the following type :—

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, \infty^0, \text{ or } 1^{\infty}.$$

65. **Form $\frac{0}{0}$.** Let $\frac{f(x)}{\phi(x)}$ be the function where $f(a) = 0$ and $\phi(a) = 0$.

It is required to find the value of $\frac{f(x)}{\phi(x)}$ when $x \to a$

$$\underset{x \to a}{\mathrm{Lt}} \frac{f(x)}{\phi(x)} = \underset{h \to 0}{\mathrm{Lt}} \frac{f(a+h)}{\phi(a+h)}$$

$$= \operatorname*{Lt}_{h \to 0} \frac{f(a) + h f'(a + \theta_1 h)^{*}}{\phi(a) + h \phi'(a + \theta_2 h)},$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$.

$$\text{Thus} \operatorname*{Lt}_{x \to a} \frac{f(x)}{\phi(x)} = \operatorname*{Lt}_{h \to 0} \frac{f'(a + \theta_1 h)}{\phi'(a + \theta_2 h)}$$

$$= \frac{f'(a)}{\phi'(a)},$$

from the continuity of these functions being assumed.

Thus if a function assumes the form $\frac{0}{0}$, we differentiate the numerator and denominator separately, and then substitute the value of the variable. If, however, $f'(a)$ and $\phi'(a)$ are also zero,

$$\operatorname*{Lt}_{x \to a} \frac{f(x)}{\phi(x)} = \operatorname*{Lt}_{h \to 0} \frac{h f'(a) + \frac{h^2}{\lfloor 2} f''(a + \theta_3 h)}{h \phi'(a) + \frac{h^2}{\lfloor 2} \phi''(a + \theta_4 h)},$$

where $0 < \theta_3 < 1$ and $0 < \theta_4 < 1$.

$$\therefore \operatorname*{Lt}_{x \to a} \frac{f(x)}{\phi(x)} = \frac{f''(a)}{\phi''(a)}.$$

Thus if $f(a) = f'(a) = \ldots . = f^r(a) = 0$

and $\phi(a) = \phi'(a) = \ldots . = \phi^r(a) = 0$

$\operatorname*{Lt}_{x \to a} \frac{f(x)}{\phi(x)} = \frac{f^{r+1}(a)}{\phi^{r+1}(a)}$, if $(r + 1)^{th}$ differential co-efficients of these functions exist and are continuous.

Illustration.

I. $\operatorname*{Lt}_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ [Form $\frac{0}{0}$

$$f(x) \equiv e^x - e^{\sin x}$$

$$\phi(x) \equiv x - \sin x.$$

$$\operatorname*{Lt}_{x \to 0} \frac{f(x)}{\phi(x)} = \operatorname*{Lt}_{x \to 0} \frac{e^x - \cos x \, e^{\sin x}}{1 - \cos x} \left[\frac{f'(0)}{\phi'(0)} = \frac{0}{0} \right.$$

$$= \operatorname*{Lt}_{x \to 0} \frac{e^x + \sin x \, e^{\sin x} - \cos^2 x \, e^{\sin x}}{\sin x} \left[\frac{0}{0} \right.$$

---

* By Taylors theorem with all its limitations.

$$= \underset{x \to 0}{\mathrm{Lt}} \frac{e^{x} + \cos x\, e^{\sin x} + 3 \sin x \cos x\, e^{\sin x} - \cos^3 x\, e^{\sin x}}{\cos x}$$

$$= 1$$

$$\therefore \underset{x \to 0}{\mathrm{Lt}} \frac{e^{x} - e^{\sin x}}{x - \sin x} = 1.$$

66. Form $\frac{\infty}{\infty}$. To find $\underset{x \to a}{\mathrm{Lt}} \frac{f(x)}{\phi(x)}$ when

$f(a) = \infty$ and $\phi(a) = \infty$.

$$\underset{x \to a}{\mathrm{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\mathrm{Lt}} \frac{\frac{1}{\phi(x)}}{\frac{1}{f(x)}} \qquad \left[\text{form } \frac{0}{0}\right.$$

$$= \underset{x \to a}{\mathrm{Lt}} \frac{\frac{\phi'(x)}{[\phi(x)]^2}}{\frac{f'(x)}{[f(x)]^2}}$$

$$= \underset{x \to a}{\mathrm{Lt}} \left[\left(\frac{\phi'(x)}{f'(x)} \right) \left\{ \frac{f(x)}{\phi(x)} \right\}^2 \right] \ldots\ldots (i)$$

Case I. If $\underset{x \to a}{\mathrm{Lt}} \frac{f(x)}{\phi(x)}$ is neither zero nor infinite, dividing both the sides of (i)

by $\underset{x \to a}{\mathrm{Lt}} \frac{f(x)}{\phi(x)}$ we have

$$1 = \underset{x \to a}{\mathrm{Lt}} \frac{\phi'(x)}{f'(x)} \cdot \underset{x \to a}{\mathrm{Lt}} \frac{f(x)}{\phi(x)}$$

$$\text{or } \underset{x \to a}{\mathrm{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\mathrm{Lt}} \frac{f'(x)}{\phi'(x)}.$$

Case II. If $\underset{x \to a}{\mathrm{Lt}} \frac{f(x)}{\phi(x)} = 0$

$$\therefore \underset{x \to a}{\mathrm{Lt}} \frac{f(x) + \phi(x)}{\phi(x)} = 1$$

Thus $\underset{x \to a}{\mathrm{Lt}} \frac{f(x) + \phi(x)}{\phi(x)}$ is neither zero nor infinite and therefore by case I

$$\underset{x \to a}{\mathrm{Lt}} \frac{f(x) + \phi(x)}{\phi(x)} = \underset{x \to a}{\mathrm{Lt}} \frac{f'(x) + \phi'(x)}{\phi'(x)}$$

or $$\underset{x \to a}{\mathrm{Lt}} \frac{f(x)}{\phi(x)} = \frac{f'(x)}{\phi'(x)}.$$

Case III. If $\underset{x \to a}{\mathrm{Lt}} \dfrac{f(x)}{\phi(x)} = \infty$

Then $\underset{x \to a}{\mathrm{Lt}} \dfrac{\phi(x)}{f(x)} = 0$

$\therefore$ by case II $\underset{x \to a}{\mathrm{Lt}} \dfrac{\phi(x)}{f(x)} = \underset{x \to a}{\mathrm{Lt}} \dfrac{\phi'(x)}{f'(x)}$

*Hence in every case

$$\underset{x \to a}{\mathrm{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\mathrm{Lt}} \frac{f'(x)}{\phi'(x)}$$

Illustration.

Find $\underset{x \to a}{\mathrm{Lt}} \dfrac{\log\left(1 - \frac{x}{a}\right)^2}{\cot^2 \frac{\pi x}{a}}$

$$\underset{x \to a}{\mathrm{Lt}} \frac{\log\left(1 - \frac{x}{a}\right)^2}{\cot^2 \frac{\pi x}{a}}$$

$$= \underset{x \to a}{\mathrm{Lt}} \frac{-\frac{2}{a - x}}{-2\frac{\pi}{a} \cot \frac{\pi x}{a} \operatorname{cosec}^2 \frac{\pi x}{a}}$$

$$= \underset{x \to a}{\mathrm{Lt}} \frac{a}{\pi} \frac{\sin^3 \pi \frac{x}{a}}{(a - x) \cos \pi \frac{x}{a}} \qquad \left[\frac{0}{0}\right.$$

$$= \underset{x \to a}{\mathrm{Lt}} \frac{a}{\pi} \frac{3\frac{\pi}{a} . \sin^2 \pi \frac{x}{a} \cos \pi \frac{x}{a}}{-\frac{\pi}{a} \sin \frac{\pi x}{a} (a - x) - \cos \pi \frac{x}{a}}$$

$$= 0.$$

---

* The method of proof assumes the existence and the continuity of every function used.

67. Form $0 \times \infty$. To find $\underset{x \to a}{\mathrm{Lt}} [f(x) . \phi(x)]$ when $f(a) = 0$ and $\phi(a) = \infty$.

Here $\underset{x \to a}{\mathrm{Lt}} [f(x) . \phi(x)] = \underset{x \to a}{\mathrm{Lt}} \dfrac{f(x)}{\dfrac{1}{\phi(x)}}$,

which becomes of the form $\frac{0}{0}$ and can thus be treated as in § 65.

68. Form $\infty - \infty$. To find $\underset{x \to a}{\mathrm{Lt}} [f(x) - \phi(x)]$,

when

$$f(a) = \infty \text{ and } \phi(a) = \infty.$$

$$\underset{x \to a}{\mathrm{Lt}} [f(x) - \phi(x)] = \underset{x \to a}{\mathrm{Lt}} f(x) \left[1 - \frac{\phi(x)}{f(x)}\right] \ldots . (i)$$

Two cases arise:—

(1) If $\underset{x \to a}{\mathrm{Lt}} \dfrac{\phi(x)}{f(x)} = 1$, (i) takes the form $\infty \times 0$, which can be treated as in § 67.

(2) If $\underset{x \to a}{\mathrm{Lt}} \dfrac{\phi(x)}{f(x)}$ is different from unity $\underset{x \to a}{\mathrm{Lt}} f(x) \left[1 - \dfrac{\phi(x)}{f(x)}\right]$ is clearly infinite.

69. Forms. 0^0, ∞^0, 1^∞:—When a function assumes any of these forms, the method of procedure is to take the logarithmic differentiation of the functions, which will reduce to the form $0 \times \infty$, already discussed in § 67.

Illustration.

I. Find the value of $\underset{x \to 0}{\mathrm{Lt}} (\cos x)^{\cot^2 x}$.

Here the form is 1^∞.

$$\text{Let } y = (\cos x)^{\cot^2 x}$$

$$\therefore \log_e y = \cot^2 x \log_e (\cos x)$$

$$\text{or} \quad \underset{x \to 0}{\text{Lt}} \log_e y = \underset{x \to 0}{\text{Lt}} \cot^2 x \log_e (\cos x) \quad [0 \times \infty$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log_e (\cos x)}{\tan^2 x} \quad \left[\frac{0}{0} \right.$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-\dfrac{\sin x}{\cos x}}{2 \tan x \sec^2 x}$$

$$= \underset{x \to 0}{\text{Lt}} - \frac{\cos^2 x}{2}$$

$$\text{or} \quad \underset{x \to 0}{\text{Lt}} \log_e y = -\tfrac{1}{2}$$

$$\therefore \underset{x \to 0}{\text{Lt}} y = e^{-\frac{1}{2}}$$

$$\textit{i. e.,} \underset{x \to 0}{\text{Lt}} (\cos x)^{\cot^2 x} = e^{-\frac{1}{2}}.$$

EXAMPLES ON CHAPTER VII.

Find the limit in the following :—

1. $\underset{x \to 1}{\text{Lt}} \dfrac{x - 1}{x^n - 1}$.

2. $\underset{x \to 0}{\text{Lt}} \dfrac{\log (1 + x^2)}{\log \cos x}$.

3. $\underset{x \to 1}{\text{Lt}} \dfrac{\log x}{x - 1}$.

4. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x}}{\sin x}$.

5. $\underset{x \to \frac{\pi}{4}}{\text{Lt}} \dfrac{1 - \tan x}{1 - \sqrt{2} \sin x}$.

6. $\underset{x \to 0}{\text{Lt}} \dfrac{(1 + t)^n - 1}{t}$.

7. $\underset{x \to 0}{\text{Lt}} (1 + x)^{\frac{1}{x}}$.

8. $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - 1}{x}$.

9. $\underset{y \to 0}{\text{Lt}} \dfrac{e^y + \sin y - 1}{\log (1 + y)}$.

10. $$\operatorname*{Lt}_{x \to 1} \frac{\log \cos (x - 1)}{1 - \sin \frac{\pi x}{2}}.$$

11. $$\operatorname*{Lt}_{x \to 0} \frac{\log_{\sin x} \cos x}{\log_{\sin \frac{x}{2}} \cos \frac{x}{2}}.$$

12. $$\operatorname*{Lt}_{x \to 0} x \log \sin x.$$

13. $\operatorname*{Lt}_{x \to 0} x^n \log x$, n being positive.

14. $$\operatorname*{Lt}_{\phi \to a} (a^2 - \phi^2) \tan \frac{\pi\phi}{2a}.$$

15. $$\operatorname*{Lt}_{\theta \to 0} \frac{\cot \theta \left[\tan^{-1} (m \tan \theta) - m \cos^2 \frac{\theta}{2}\right]}{\sin^2 \frac{\theta}{2}}.$$

16. $$\operatorname*{Lt}_{x \to 1} (1 - x^2)^{\frac{1}{\log (1 - x)}}.$$

17. $\operatorname*{Lt}_{x \to 0} (x)^{-x^m}$, m being positive.

18. $$\operatorname*{Lt}_{x \to 1} \frac{\sqrt{1 + x} - \sqrt{1 + x^2}}{\sqrt{1 - x} - \sqrt{1 - x^2}}.$$

19. $$\operatorname*{Lt}_{x \to 0} \frac{(1 + x)^{\frac{1}{x}} - e}{x}.$$

20. $$\operatorname*{Lt}_{x \to 0} \frac{(1 + x)^{\frac{1}{x}} - e + \frac{ex}{2}}{x^2}.$$

21. $$\operatorname*{Lt}_{x \to \frac{\pi}{2}} \sqrt{\frac{2 + \cos x - \sin x}{x \sin 2x + x \cos x}} - \left[\frac{\pi - 2x}{2 \sin 2x}\right]^2.$$

22. $$\operatorname*{Lt}_{x \to 0} \frac{\log (1 + x + x^2) + \log (1 - x + x^2)}{\sec x - \cos x}.$$

23. Given $x^3 + y^3 + a^3 = 3axy$, find the values of $\frac{dy}{dx}$ when $x = y = a$.

24. Find the values of $\frac{dy}{dx}$ at the origin for the curve $x^3 + y^3 = 3axy$.

25. Prove that

$$\underset{x \to 0}{\text{Lt}} \; \frac{a^x - 1}{x^n \sin x} \left(\frac{b \sin x - \sin b x}{\cos x - \cos b x} \right)^n$$

is equal to $\left(\frac{b}{3} \right)^n \log a$.

26. Prove that $\underset{h \to 0}{\text{Lt}} \; h \left[a^m + (a + h)^m + (a + 2h)^m + \ldots\ldots + (a + \overline{n - 1}\, h)^m \right]$

$$= \frac{b^{m+1} - a^{m+1}}{m + 1} \text{ where } \frac{b - a}{n} = h.$$

27. Find the value of $\underset{\theta \to 0}{\text{Lt}} \; \frac{d^2y}{dx^2}$, when $y = \frac{\theta}{\sin \theta}$ and $\theta = \cos^{-1}(1 - x)$.

28. Prove that if x is infinite and $\phi(x) = \infty$ then will

$$\underset{x \to \infty}{\text{Lt}} \; \frac{\phi(x)}{x} = \underset{x \to 0}{\text{Lt}} \; [\phi(x + 1) - \phi(x)].$$

[Todhunter.

29. If $\phi(x) = \infty$, when x is infinite

Shew that

$$\underset{x \to \infty}{\text{Lt}} \; [\phi(x)]^{\frac{1}{x}} = \underset{x \to \infty}{\text{Lt}} \; \frac{\phi(x + 1)}{\phi(x)}$$

[Todhunter.

30. Prove that $\underset{x \to \infty}{\text{Lt}} \left(\frac{x^n}{\lfloor x} \right)^{\frac{1}{x}} = e$

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CHAPTER VIII.

CURVATURE.

70. Let APQ be any curve, defined for a certain interval and any point P on it.

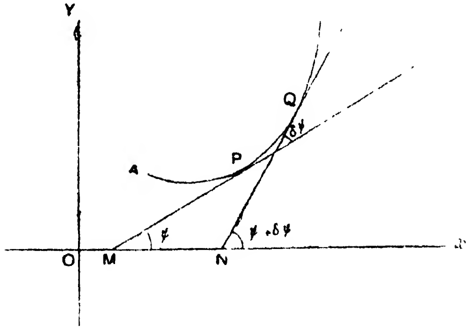

FIG. 14.

Moving along the curve in the direction of PQ, we notice that as P moves, the tangent PM to the curve at P also changes its direction Thus the direction of the tangent gives an indication of the bending of the curve, and the angle turned through by the tangent between any two positions of P seems a fit measure for the bending of the curve, if the bending has been regular.

Let AP $= s$ and AQ $= s + \delta s$ and $\delta\psi$ be angle turned through by the tangent in moving from P to Q. Then if the portion of the curve in the neighbourhood of P lies on one side of the tangent *i. e.*, P is not a point of inflexion*, the ratio $\frac{\delta\psi}{\delta s}$ is called the average curvature of the arc PQ and $\underset{\delta s \to 0}{\mathrm{Lt}} \frac{\delta\psi}{\delta s}$ *i. e.*, $\frac{d\psi}{ds}$ is called the curvature of the curve at the point P.†

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* Refer to § 94.

† Since the shape of a curve becomes defined if we know.

The *radius of curvature* at any point on the curve is defined as $\underset{\delta\psi \to 0}{\mathrm{Lt}} \frac{\delta s}{\delta\psi}$ *i. e.*, $\frac{ds}{d\psi}$ or the reciprocal of the curvature, and is written as ρ. The circle of radius ρ is given by $s = \rho\ \psi$ and the curvature of this circle is the reciprocal of its radius *i. e*, $\frac{1}{\rho}$, the same curvature* as the curve has at P. Thus the radius of curvature at any point P, is also defined as the radius of the circle which would have the same curvature as the given curve has at P.

71. Let P,Q and R be any three points close together on the curve APQRT. Let a circle LPQR pass through P,Q and R, where R → Q and P → Q.

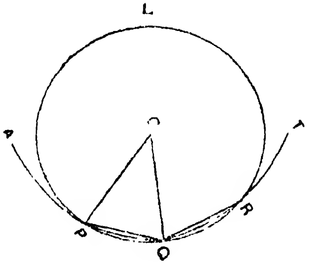

Fig. 15.

RQ and PQ both become tangents in the limit to the circle as well as the curve. Hence the angle between

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the acrual distance of any point from some fixed point on the curve and also the direction of the tangent to the curve at that point ; hence a relation between s and ψ is sufficient to define the shape of a curve, and consequently we can regard $s = f\ (\psi)$ as the equation to any curve. This is known as the intrinsic equation of the curve. For a fuller discussion of the subject the student is advised to consult any text-book on integral calculus.

* A circle has a constant curvature, *i. e.*, reciprocal of the radius throughout.

these two tangents to the circle, and the curve as well, is the same, say $\delta\psi$. Also the chord PQ of the circle, is also the chord of the curve; if therefore we measure the arcs of the circle and the curve by s' and s respectively, by note II of § 49 $ds' = ds$: and therefore $\frac{ds'}{d\psi} = \frac{ds}{d\psi}$

Now $\frac{ds}{d\psi}$ is the radius of the curvature for the curve and $\frac{ds'}{d\psi}$ is the radius of curvature for the circle *i. e.*, the radius of the circle itself. Such a limiting circle through any point on a curve, and which has the same radius as the radius of curvature for the curve, is called the *circle of curvature*, and its centre is called the centre of curvature. Any chord of the circle of curvature drawn through the point of contact in any direction is called the *chord of curvature*. Geometrically speaking the centre of curvature can be regarded as the point of intersection of two consecutive normals.

ρ in Cartesian Coordinates.

72. We know that $\frac{dy}{dx} = \tan\psi$ ....(1)

Differentiating (1) with respect to ψ, we have

$$\frac{d^2y}{dx^2} \cdot \frac{dx}{d\psi} = \sec^2\psi$$

$$\text{or} \quad \frac{dx}{d\psi} = \frac{\sec^2\psi}{\frac{d^2y}{dx^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\text{Again } \frac{dx}{d\psi} = \frac{dx}{ds} \cdot \frac{ds}{d\psi}$$

$$\therefore \text{ from (2) } \frac{ds}{d\psi} = \frac{\sec^2\psi}{\frac{d^2y}{dx^2}} \cdot \frac{ds}{dx}$$

$$= \frac{\sec^3\psi}{\frac{d^2y}{dx^2}}$$

$$\text{or} \quad \rho = \pm \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

73. x and y as functions of s We know that

$$\frac{dx}{ds} = \cos\psi \text{ and } \frac{dy}{ds} = \sin\psi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Differentiating with respect to s, we have

$$\frac{d^2x}{ds^2} = -\sin\psi \frac{d\psi}{ds} = -\frac{\sin\psi}{\rho} \quad \ldots\ldots\ldots\ldots (2)$$

$$\text{or} \quad \rho = -\frac{\frac{dy}{ds}}{\frac{d^2x}{d^2s}} \ldots\ldots\ldots\ldots (3)$$

$$\text{and} \quad \frac{d^2y}{ds^2} = \cos\psi \frac{d\psi}{ds} = \frac{\cos\psi}{\rho} \ldots\ldots\ldots\ldots\ldots (4)$$

$$\text{or } \rho = \frac{\frac{dx}{ds}}{\frac{d^2y}{ds^2}} \quad \ldots\ldots\ldots\ldots\ldots (5)$$

Squaring and adding (2) and (4), we get

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 \ldots\ldots\ldots\ldots (6)$$

74. Implicit Functions.—Let the curve be given by $f(x, y) = 0$.

$$\text{we know that } f_x + f_y \frac{dy}{dx} = 0 \ldots\ldots\ldots\ldots (1)$$

Differentiating again with respect to x, we get

$$f_{xx} + f_{xy}\frac{dy}{dx} + \left(f_{xy} + f_{yy}\frac{dy}{dx}\right)\frac{dy}{dx} + f_y \frac{d^2y}{dx^2} = 0$$

or $f_{xx} + 2 f_{xy} \cdot \frac{dy}{dx} + f_{yy} \left(\frac{dy}{dx}\right)^2 + f_y \frac{d^2y}{dx^2} = 0$

$$\text{or } \frac{d^2y}{dx^2} = -\frac{f_{xx} + 2 f_{xy} \left(\frac{dy}{dx}\right) + f_{yy} \left(\frac{dy}{ax}\right)^2}{f_y} \quad \ldots.(2)$$

Substituting these values we get

$$\rho = \pm \frac{\left(1 + \frac{f_x^{\ 2}}{f_y^{\ 2}}\right)^{\frac{3}{2}} \cdot f_y}{f_{xx} + 2 f_{xy} \left(-\frac{f_x}{f_y}\right) + f_{yy} \left(-\frac{f_x}{f_y}\right)^2}$$

$$\text{or} \quad \rho = \pm \frac{(f_x^{\ 2} + f_y^{\ 2})^{\frac{3}{2}}}{f_{xx} f_y^{\ 2} - 2 f_{xy} f_x f_y + f_{yy} f_x^{\ 2}} \quad \ldots.(3)$$

75. Sometimes the equation is expressed more conveniently when the coordinates are given as functions of a single parameter, say,

$$x = f(t)$$
$$y = \phi(t)$$

we know that $\frac{dy}{dx}$ will be equal to $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$i.\ e., \frac{dy}{dx} = \frac{\phi'(t)}{f'(t)}.$$

$$\text{Again } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$= \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \cdot \frac{dt}{dx}$$

$$= \frac{\frac{d^2y}{dt^2} \cdot \frac{dx}{dt} - \frac{d^2x}{dt^2} \cdot \frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}$$

$$= \frac{\phi''(t)\, f'(t) - f''(t)\, \phi'(t)}{[f'(t)]^3}$$

$$\therefore \quad \rho = \frac{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dt^2} \cdot \frac{dx}{dt} - \frac{d^2x}{dt^2} \cdot \frac{dy}{dt}}$$

or $\rho = \dfrac{\{x'^2 + y'^2\}^{\frac{3}{2}}}{y''\, x' - x''\, y'}$, where dashes denote the differential co-efficients with respect to the parameter

ρ can be also expressed as

$$\frac{[\{f'(t)\}^2 + \{\phi'(t)\}^2]^{\frac{3}{2}}}{\phi''(t).\, f'(t) - f''(t).\, \phi'(t)}.$$

Illustration.

I. Find ρ when $x = a \cos^3 t$, and $y = a \sin^3 t$.

Here $\dfrac{dx}{dt} = -3a \sin t \cos^2 t$

$$\frac{d^2x}{dt^2} = -3\, a\, [\cos^3 t - 2 \sin^2 t \cos t]$$

$$\frac{dy}{dt} = 3\, a \cos t \sin^2 t$$

and $\dfrac{d^2y}{dt^2} = 3\, a\, [-\sin^3 t + 2 \cos^2 t \sin t]$

$$\therefore \rho = \frac{(3\,a)^3\, [\sin^2 t \cos^4 t + \cos^2 t \sin^4 t]^{\frac{3}{2}}}{(3\,a)^2\, [\sin^2 t \cos^2 t\, (\sin^2 t - 2\cos^2 t) + \sin^2 t \cos^2 t\, (\cos^2 t - 3 \sin^2 t)]}$$

or $\rho = 3\, a.\, \sin t.\, \cos t.$

or $\rho = 3\, (a\ x\ y)^{\frac{1}{3}}$.

76. ρ for Pedal Equation. The equation of the tangent at any point $(x, y$ on a curve is given by

$$Y - y = \frac{dy}{dx}\, (X - x) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The perpendicular from the origin upon (1) is given by

$$p = \frac{x\frac{dy}{dx} - y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \ldots\ldots\ldots\ldots\ldots(2)$$

Taking the logarithmic differentiation with regard to x

$$\frac{1}{p}\frac{dp}{dx} = \frac{x\frac{d^2y}{dx^2}}{x\frac{dy}{dx} - y} - \frac{\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= \frac{\left(x + y\frac{dy}{dx}\right)\frac{d^2y}{dx^2}}{\left(x\frac{dy}{dx} - y\right)\left(1 + \left(\frac{dy}{dx}\right)^2\right)} \ldots\ldots(3)$$

Also $r^2 = x^2 + y^2$

$$\therefore \quad r\frac{dr}{dx} = x + y\frac{dy}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Dividing (4) by (3) we get

$$p\,r\,\frac{dr}{dp} = \frac{\left(x\frac{dy}{dx} - y\right)\left(1 + \left(\frac{dy}{dx}\right)^2\right)}{\frac{d^2y}{dx^2}}$$

$$\text{or} \quad r\frac{dr}{dp} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \ldots\ldots\ldots\text{by}\ldots(2)$$

$$\therefore \quad \rho = r\frac{dr}{dp} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

77. **ρ in Polar Coordinates.** We have by equation (1) of § 55,

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating with respect to p,

$$-\frac{2}{p^3} = -\frac{2}{r^3}\frac{dr}{dp} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2\frac{dr}{dp} + \frac{2}{r^4}\frac{dr}{d\theta}\cdot\frac{d^2r}{d\theta^2}\cdot\frac{d\theta}{dp}$$

or $$\frac{1}{p^3} = \left[\frac{1}{r^3} + \frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^4}\frac{d^2r}{d\theta^2}\right]\frac{dr}{dp}$$

$$\therefore\ r\frac{dr}{dp} = \frac{r^6}{p^3\left[r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right]}$$

or $$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}, \quad \ldots\ldots\ldots\ldots\ldots (2)$$

by (1).

If $r = \frac{1}{u}$,

$$\rho = \frac{\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]^{\frac{3}{2}}}{u^3\left[u + \frac{d^2u}{d\theta^2}\right]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

78. When a curve passes through the origin and either of the axes is a tangent to it at the origin, it is interesting to apply Newton's method for finding ρ. This method is equally applicable if we transform the equation referred to any other given point on the curve as origin, and choosing the axes in such a way, that one of them becomes the tangent to the curve at the new origin. Suppose the axis of x is a tangent at the origin to the curve. The y axis is then the normal and the centre of curvature lies upon it.

Let R Q P O be the circle of curvature at O, and P be a point (x, y) lying on this circle and the curve, ultimately tending to coincide with the origin.

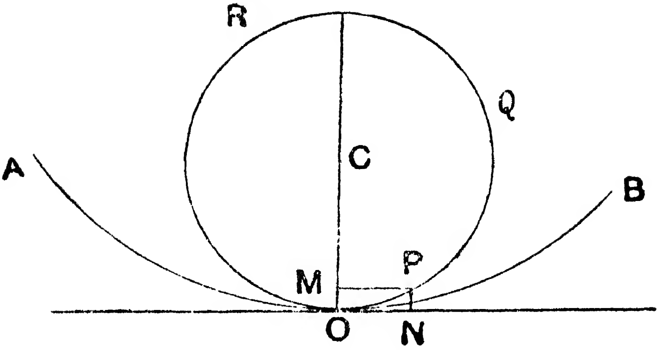

FIG 16.

Then $(2\rho - y)y = x^2$ or $2\rho - y = \frac{x^2}{y}$.

Proceeding to the limit when $y \to 0$, ρ becomes the radius of curvature at O.

$$\text{And } \rho = \frac{1}{2} \underset{y \to 0}{\mathrm{Lt}} \frac{x^2}{y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly if y axis is the tangent

$$\rho = \frac{1}{2} \underset{x \to 0}{\mathrm{Lt}} \frac{y^2}{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

79.* By the theory of envelopes we know that any curve can be regarded as the envelope of its tangents. Let the tangent to the curve at any point (x, y) be

$$x \cos \alpha + y \sin \alpha = p \ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{Also} \quad \psi = \alpha + \frac{\pi}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

For different values of α, the different members of the family of tangents will be given by (1).

Hence if a second member be given by

$$p + \delta p = x \cos (\alpha + \delta\alpha) + y \sin (\alpha + \delta\alpha); \ldots\ldots (3)$$

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* This article is to be read along with the chapter on envelopes.

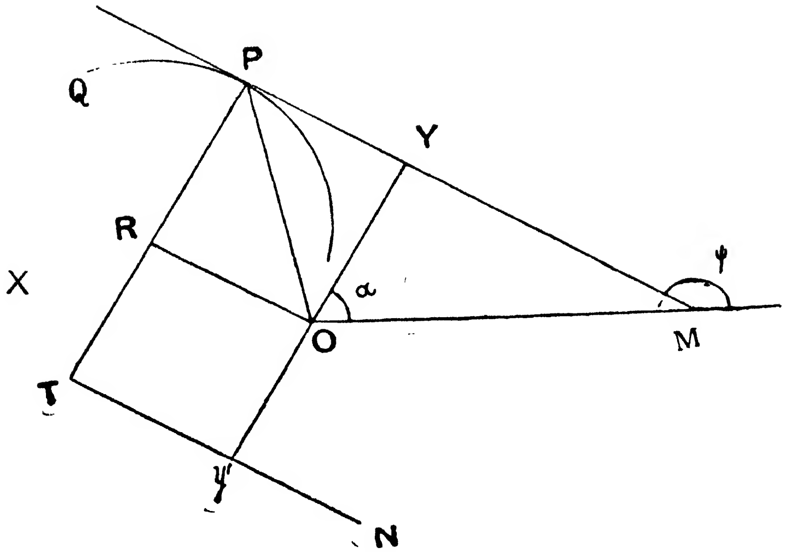

FIG. 17.

at the point of intersection of (1) and (3) in the limit when $\delta\alpha \to 0$

$$\frac{ap}{d\alpha} = - x \sin \alpha + y \cos \alpha, \ldots\ldots\ldots \quad (4)$$

must be satisfied, *i. e.*, the limiting point of intersection of (1) and (3) must be upon the curve, as well as upon (4). Comparing (1) and (4), we notice that (4) is at right angles to (1). Thus (4) represents the normal at P *i.e.*, PT

$$\therefore \frac{dp}{d\alpha} = OR$$

Similarly $\frac{d^2p}{u\alpha^2} = - x \cos \alpha - y \sin \alpha \ldots\ldots \quad (5)$

is the line perpendicular to the straight line PT and passing through say T, the point of intersection of the normals at P and say Q, a point in the neighbourhood of P ultimately tending to coincide with it. Evidently this point of intersection of two normals to a curve at any two ultimately coincident points is the centre of curvature to the curve. Hence (5) passes through the centre of curvature of the curve at P and is perpendicular to the normal at P. (5) therefore represents T N and let it intersect OY at Y′.

$$\text{Therefore } OY' = \frac{d^2p}{d\alpha^2}$$

$$\therefore \quad \rho = PT = OY + OY'$$

$$\text{or} \quad \rho = p + \frac{d^2p}{d\alpha^2}.$$

$$\text{Again } \frac{dp}{d\alpha} = \frac{dp}{d\psi}\cdot\frac{d\psi}{d\alpha} = \frac{dp}{d\psi}, \text{ from (2)}$$

$$\frac{d^2p}{d\alpha^2} = \frac{d^2p}{d\psi^2}.$$

$$\text{Thus} \quad \rho = p + \frac{d^2p}{d\psi^2}\;\ldots\ldots\ldots\ldots \quad (6)$$

Illustration.

I. Find ρ at (r, θ) of the curve $r^n = a^n \cos n\theta$, and shew that the chord of curvature through the pole is $\frac{2r}{n+1}$.

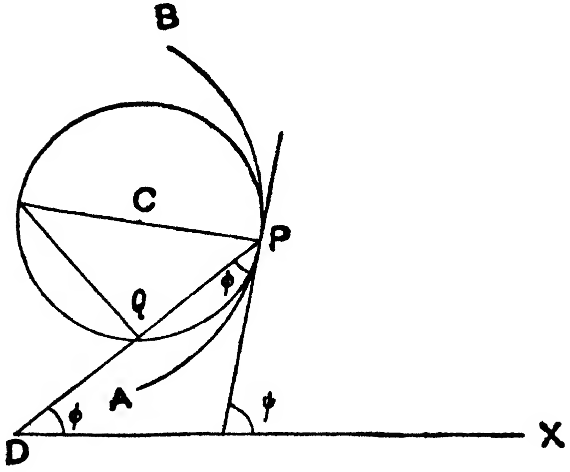

FIG. 18.

We have from the equation of the curve

$$\frac{n}{r}\ \frac{dr}{d\theta} = -\ n \tan n\theta,$$ by differentiating logarithmically with respect to θ.

$$\text{or} \quad \frac{dr}{d\theta} = -\ r \tan n\theta \ \dots\dots\dots\dots\dots\dots(1)$$

$$\therefore \quad \frac{d^2r}{d\theta^2} = -\ \tan n\theta\ \frac{dr}{d\theta} - rn \sec^2 n\theta$$

$$= r \tan^2 n\theta - rn \sec^2 n\theta$$

$$\text{Thus } \rho = \frac{[r^2 + r^2\tan^2 n\theta]^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2\tan^2 n\theta + r^2 n\sec^2 n\theta}$$

$$= \frac{r \sec^3 n\theta}{\sec^2 n\theta\ (n+1)}$$

$$\text{or} \quad \rho = \frac{a^n}{r^{n-1}(n+1)} \ \dots\dots\dots\dots\dots\dots(2)$$

chord of curvature $PQ = 2\rho \cos CPQ$

$$= 2\rho \sin \phi$$

$$\text{From (1)} \quad \tan \phi = -\ \cot n\theta$$

$$= \tan\left(\frac{\pi}{2} + n\theta\right)$$

$$\therefore \quad \phi = \frac{\pi}{2} + n\,\theta.$$

$$\text{Thus } PQ = 2\rho \sin\left(\frac{\pi}{2} + n\,\theta\right)$$

$$= 2\rho \cos n\,\theta.$$

$$= \frac{2r}{n+1}, \text{ from (2).}$$

II. Shew that for the curve $y = c \cosh \frac{x}{c}$, ρ varies as the square of the ordinate and is equal to the part of the normal intercepted between the curve and the axis of x.

$$\text{Here} \quad \frac{dy}{dx} = \sinh \frac{x}{c}$$

$$\text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{c} \cosh \frac{x}{c}$$

$$\therefore \rho = c\frac{\left[1+\sinh^2\frac{x}{c}\right]^{3/2}}{\cosh\frac{x}{c}}$$

$$\text{or } \rho = c\cosh^2\frac{x}{c}$$

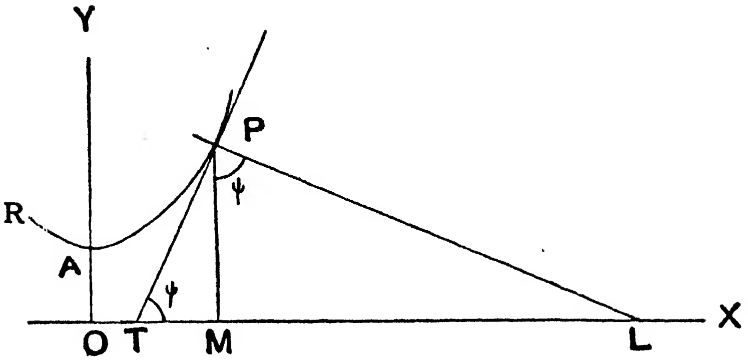

FIG. 19.

$= \frac{y^2}{c}$, thus ρ varies as the square of the ordinate.

Let PL be the intercept required.

$$PL = y\sec\psi = y\sqrt{1+\tan^2\psi}$$

$$\text{or } PL = y\sqrt{1+\sinh^2\frac{x}{c}}$$

$$= y\cosh\frac{x}{c}$$

$$= \frac{y^2}{c}$$

$$\therefore \rho = PL.$$

80. Curvature at the origin. If the curve passes through the origin, the method of finding the radius of curvature at the origin can be simplified by the use of Maclaurin's theorem instead of using Newton's methods.

If $y = f(x)$ be the curve

$$y = f'(0) . x + f''(0) \frac{x^2}{\lfloor 2} + \ldots\ldots\ldots$$

$$\text{or } y = x \left(\frac{dy}{dx}\right)_{0,0} + \frac{x^2}{\lfloor 2} \left(\frac{d^2y}{dx^2}\right)_{0,0} + \ldots\ldots\ldots\ldots$$

writing p for $\frac{dy}{dx}$ and q for $\frac{d^2y}{dx^2}$

$$\text{we have } y = p\,x + q \frac{x^2}{\lfloor 2} + \ldots\ldots\ldots$$

substituting this for y in the equation of the curve and equating the co-efficients of the like powers of x in the identity we get the values of p and q determined, and which can then he substituted in $\frac{[1 + p^2]^{\frac{3}{2}}}{q}$.

Illustration.

I. Find the radii of curvature at the origin for both the branches of the curve $y^2 (a - x) = x^2 (a + x)$.

Substituting $p\,x + q \frac{x^2}{\lfloor 2} + \ldots.$ for y we get

$$\left[p\,x + q \frac{x^2}{\lfloor 2} + \ldots\right]^2 (a - x) = x^2 (a + x)$$

$$\text{or } \left[p^2 x^2 + p\,q\,x^3 + q^2 \frac{x^4}{4} + \ldots\right](a - x) = x^2 (a + x)$$

$\therefore$ equating co-efficients of like powers we have,

$$p^2 = 1, \; apq - p^2 = 1.$$

Thus $p = \pm 1$ and $q = \pm \frac{2}{a}$.

Thus $\rho = \pm \sqrt{2a}$.

Exercises.

Find ρ in the following :--

1. $y^2 = 4\,ax$ at the point $(a, 2a)$.
2. $s = a \sec^3\psi$.

3. $x = \dfrac{t^2 + 1}{4}$

$y = \dfrac{t^3}{6}$, at the point t.

4. $x = at^2$

$y = 2\ at$, at the point t.

5. $x = a\ (\theta + \sin\theta)$

$y = a\ (1 - \cos\theta)$, at the point θ.

6. $x = 3t^2$

$y = 3t - t^3$, at the point t.

7. $r = a\ (1 + \cos\theta)$; also find the chord of curvature through the pole.

8. $r^2 = a^2 \cos 2\theta$.

9. $r = a \sin^3 \dfrac{\theta}{3}$.

10. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

11. $y^2 x = a^2 (1 - x)$, at the point $(a, 0)$.

12. $(y - x)^2 = x^5$, at $(0, 0)$.

13. $a^2 y = b\ x^2 + c\ x^2\ y$ at $(0, 0)$

14. $x^3 + y^3 = 3a\ xy$ at $(0, 0)$, for both the branches.

15. $xy = k^2$.

16. Prove that in a parabola the radius of curvature is equal to twice the part of the normal intercepted between the curve and the directrix.

17. For any curve, prove that

$$\rho = \frac{r}{\sin\theta\left(1 + \dfrac{d\phi}{d\theta}\right)},$$

where $p = r \sin\phi$.

18. Shew that in a parabola, the chord of curvature through the focus and the chord of curvature parallel to the axis are each four times the focal distance of the point.

19. Prove that the chord of curvature parallel to the axis of y in the curve $y = a \log \sec \dfrac{x}{a}$ is of constant length.

20. In the conic $ax^2 + 2h\,xy + by^2 = 1$, prove that the radius of curvature varies inversely as the cube of the central perpendicular on the tangent.

21. If ρ and ρ' be the radii of curvature at the extremities of two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that

$$(\rho^{\frac{2}{3}} + \rho'^{\frac{2}{3}})(ab)^{\frac{2}{3}} = a^2 + b^2.$$

22. Prove that for the curve $r = a \sec 2\theta$

$$\rho = -\frac{r^4}{3p^3}.$$

23. Shew that the chord of curvature through the pole of the equiangular spiral $r = ae^{\theta \cot \alpha}$ is $2r$.

24. Shew that the chord of curvature through the pole of the curve $p = f(r)$, is given by

$$\frac{2f(r)}{f'(r)}.$$

Apply it in the case of $p^2 = \frac{r^3}{2a}$.

25. Find by Newton's method the radius of curvature at the origin for the curve

$$2y^4 + 3x^4 + 4y^2x + xy - x^2 + 2y = 0.$$

26. If on the tangent at each point of a curve a constant length be measured from the point of contact, prove that the normal to the locus of the points so found passes through the corresponding centre of curvature of the given curve.

[Bertrand].

27. If on the tangent at each point of a curve, a constant length c be measured from the point of contact, shew that the radius of curvature of the locus of such points is given by

$$\rho' = \frac{[\rho^2 + c^2]^{\frac{3}{2}}}{\rho^2 + c^2 - c\frac{d\rho}{d\psi}},$$ where ρ and ψ refer to the corresponding point of the original curve.

Centre of Curvature.

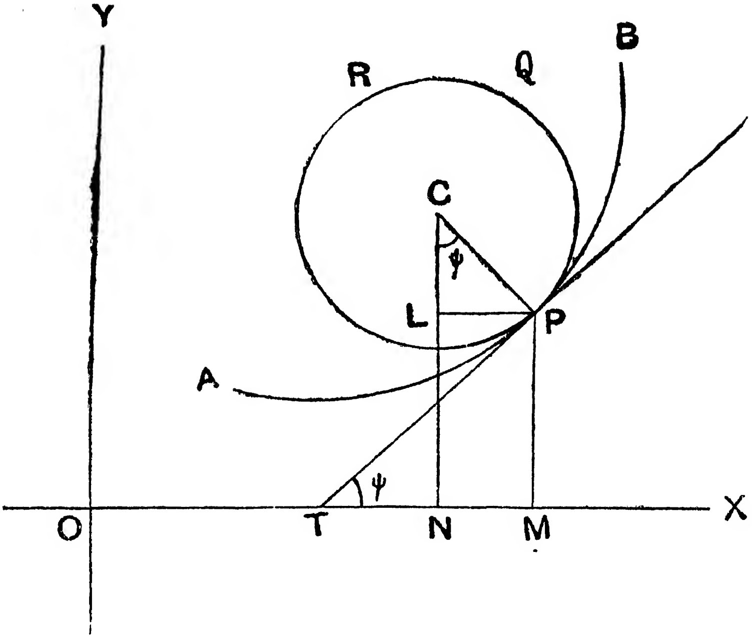

FIG. 20.

81. Let P Q R be the circle of curvature at P (x, y) on the curve APB and C (α,β) be the centre of curvature. To find (α, β)

$$\text{Here } \alpha = \text{OM} - \text{NM}$$
$$= x - \text{LP}$$
$$= x - \rho \sin \psi$$
$$= x - \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \qquad \frac{\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

$$\therefore\ \alpha = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Again

$$\beta = \mu \mathrm{P} + \mathrm{LC}$$

$$= y + \rho \cos \psi$$

$$= y + \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \cdot \frac{1}{\sqrt{1 + \left(\frac{d^2y}{dx^2}\right)}}$$

$$\therefore \beta = y + \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \dots\dots\dots\dots\dots\dots (2)$$

Evolutes and Involutes.

82. The locus of the centre of curvature of a given plane curve is called the *evolute* of the curve. If the evolute itself be regarded as the original curve, a curve of which it is the evolute, is called an involute.

If the equation of the given curve be

$(f\,x, y) = 0, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$

and (α, β) be the coordinates of the centre of curvature

$$\alpha = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \dots\dots\dots\dots (2)$$

$$\beta = y + \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \dots\dots\dots\dots\dots\dots (3)$$

Eliminating x and y between the equations (1), (2) and (3), we get the locus of (α, β) which will be the evolute of the curve $f(x, y) = 0$. A better way of finding the evolute of a curve is by regarding it as the envelope of the normal to the curve, for the discussion of which see § 85.

Illustration.

I. Find the evolute of $y = c \cosh \frac{x}{c}$. .

Here $\frac{dy}{dx} = \sinh \frac{x}{c}$, and $\frac{d^2y}{dx^2} = \frac{1}{c} \cosh \frac{x}{c}$

∵ If (α, β) be the co-ordinates of the centre of curvature

$$\alpha = x - \frac{\sinh \frac{x}{c}\left[1 + \sinh^2 \frac{x}{c}\right]}{\frac{1}{c} \cosh \frac{x}{c}}$$

$$= x - c \sinh \frac{x}{c} \cosh \frac{x}{c} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{and } \beta = y + c \cosh \frac{x}{c}$$

$$= 2\, y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\therefore \quad \alpha = x - c \cosh \frac{x}{c} \sqrt{\cosh^2 \frac{x}{c} - 1}$$

$$= x - \frac{y}{c} \sqrt{y^2 - c^2}.$$

$$\text{or} \quad \alpha = c \cosh^{-1} \frac{y}{c} - \frac{y}{c} \sqrt{y^2 - c^2}.$$

$$= c \cosh^{-1} \frac{\beta}{2c} - \frac{\beta}{2c} \sqrt{\frac{\beta^2}{4} - c^2}, \quad \text{by (2).}$$

$$\therefore \quad \alpha + \frac{\beta \sqrt{\beta^2 - 4c^2}}{4c} = c \cosh^{-1} \frac{\beta}{2c}$$

$$\text{or} \quad \beta = 2c \cosh \frac{4\, \alpha\, c + \beta \sqrt{\beta^2 - 4c^2}}{4c^2}.$$

∴ The evolute required is given by

$$y = 2c \cosh \frac{4cx + y \sqrt{y^2 - 4c^2}}{4c^2}.$$

EXAMPLES ON CHAPTER VIII.

Find the evolute of :—

1. $y^2 = 4ax.$

2. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

3. $\left.\begin{aligned} x &= a \cos t + a \log \tan \frac{t}{2} \\ y &= a \sin t. \end{aligned}\right\}$

4. Prove that the evolute of the equiangular spiral $r = ae^{m\theta}$ is also an equiangular spiral.

5. Prove that the evolute of the cardioide $r = a(1 + \cos\theta)$ is another cardioide.

6. Prove that the distance between the pole and the centre of curvature corresponding to any point on the curve $r^n = a^n \cos n\theta$ is

$$\cdot \frac{[a^{2n} + (n^2 - 1) r^{2n}]^{\frac{1}{2}}}{(n + 1) r^{n+1}}.$$

7. In the equiangular spiral $r = ae^{m\theta}$, prove that the centre of curvature is the point where the perpendicular to the radius vector through the pole intersects the normal.

CHAPTER IX.

ENVELOPES.

83. Geometrical meaning.—If there be a straight line $x \cos \alpha + y \sin \alpha = p$, its position and direction will be determined by α and p. For the same value of p, if we vary α, we shall get different straight lines. Any system of straight lines (or curves) formed in this way is called a family of curves, and α, which is a constant for any one of these straight lines, but varies as we pass from one line to another is called a variable parameter. Thus in the equation of a curve or a surface, we have some arbitrary constants besides the current co-ordinates. The shape and size of the curve in question will depend upon the value of the arbitrary constants. In the example quoted above, there is only one variable parameter. We may however, have two or more variable parameters entering into the equation. In order to give prominence to the parameter, it is generally indicated in the equation as

$$f(x, y, \alpha) = 0.$$

Again as α changes, the straight lines given by $x \cos \alpha + y \sin \alpha = p$ always remain tangent to the circle $x^2 + y^2 = p^2$, and the point of contact traces the circle. The circle is known as the envelope of the system of curves given by $x \cos \alpha + y \sin \alpha = p$ when the variable parameter is α. In general, "*if there is a curve to which the curves* of a family $f(x, y, \alpha) = 0$ are tangent and if the point of contact describes that curve as α varies, the curve is called the envelope (or part of the envelope if there are several such curves) of the family $f(x, y, \alpha) = 0$."* It is thus clear that the envelope is touched by some member of the family of curves of which it is the envelope, at

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* This definition is adapted from 'Advanced Calculus' by E. B. Wilson.

every point on it, or which is the same as saying that every member of the family of curves given by $f(x, y, \alpha) = 0$, touches its envelope at some point.

84. Envelope equation.—Let the family be given by $f(x, y, \alpha) = 0$, where α is the variable parameter, and let the equation of the envelope be

$$x = \phi(\alpha),\ y = \varepsilon(\alpha) \quad \text{together with } f(x, y, \alpha) = 0 \, .. \, (1)$$

The first set of equations indicates that the points on the envelope depend upon α, and the last equation expresses the fact that each point on the envelope lies on some curve of the family.

Differentiating $f(x, y, \alpha) = 0$ with respect to the variable parameter α, we have,

$$f_x \frac{dx}{d\alpha} + f_y \frac{dy}{d\alpha} + f_a = ..0 \, ... (2)$$

$$i.\ e,\ f_x \phi'(\alpha) + f_y \varepsilon'(\alpha) + f_a = 0 \, ... (3)$$

Now if the point of contact on the envelope is identical with the point on a certain member of the family, $\frac{dy}{dx}$ for that member is the same as that of the envelope.

$$\therefore \frac{dy}{dx} = \frac{\varepsilon'(\alpha)}{\phi'(\alpha)}, \, ... (4)$$

along the envelope.

$$\text{Also } \frac{dy}{dx} = -\frac{f_x}{f_y}, \, ... (5)$$

along the curve. From (4) and (5) we get

$$-\frac{f_x}{y} = \frac{\varepsilon'(\alpha)}{\phi'(\alpha)},$$

$$\text{or } f_x \phi'(\alpha) + f_y \varepsilon'(\alpha) = 0 \, ... (6)$$

Comparing (3) and (6) we find that

$$f'_a = 0.$$

Thus for points on the envelope we have the equations $f(x, y, \alpha)=0$ and $\frac{\partial f(x, y, \alpha)}{\partial \alpha}=0$.

satisfied. These two therefore give the parametric equations of the envelope.

If α be eliminated between these two equations, we get the cartesian equation of the envelope.

Thus to find the envelope of any family of curves given by $f(x, y, \alpha)=0$, differentiate the equation with respect to the variable parameter α, treating all other quantities involved in the equation as constants, and then find the eliminant between these two equations with respect to that variable parameter.

85. **Evolutes as envelopes of Normals.**—The equation of the normal to the curve $y=f(x)$, at the point (x, y) is

$$(X-x)+\frac{dy}{dx}(Y-y)=0 \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We have to find the envelope of (1) for different values of x and y, where they are related by the equation

$y=f(x), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

since the point (x, y) lies on the curve given.

Hence y and $\frac{dy}{dx}$ can be treated as functions of x. In other words we have to find the envelope of (1) regarding x as the variable parameter.

Differentiating (1) with respect to x.

$$-1+\frac{d^2y}{dx^2}(Y-y)-\left(\frac{dy}{dx}\right)^2=0$$

$$\text{or} \quad Y=y+\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \ldots\ldots\ldots\ldots\ldots(3)$$

Also from (1) and (3) we get

$$X = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \text{ .... } (4)$$

Eliminating x and y between equations (2), (3) and (4) we get the envelope required *i e.*, a relation between X and Y, which are clearly the co-ordinates of the centre of curvature, [§ 82]. Thus an evolute can be regarded as the envelope of the normals

86. The centre of curvature for any point (x, y) on the curve A B is given by

$$X = x - \rho \sin \psi$$
$$Y = y + \rho \cos \psi$$

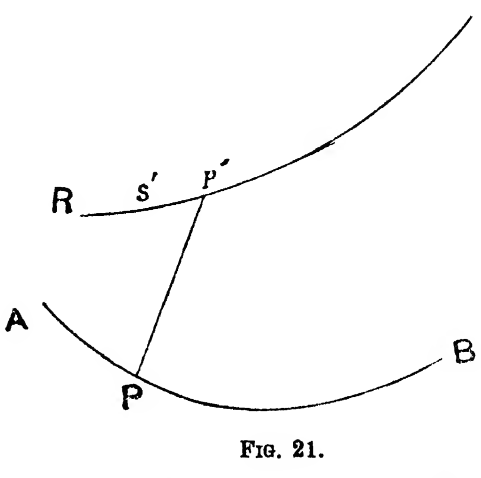

FIG. 21.

Here y, ρ and ψ being functions of x, X and Y can be regarded as a function of x.

$$\therefore \quad \frac{dX}{ax} = 1 - \frac{d\rho}{dx} \sin \psi - \rho \cos \psi \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$
$$= - \sin \psi \frac{d\rho}{dx}$$

$$\text{And } \frac{dY}{dx} = \tan\psi - \rho\sin\psi\frac{d\psi}{ds}.\frac{ds}{dx} + \frac{d\rho}{dx}\cos\psi$$

$$= \frac{d\rho}{dx}\cos\psi$$

$$\therefore \left(\frac{dX}{dx}\right)^2 + \left(\frac{dY}{dx}\right)^2 = \left(\frac{d\rho}{dx}\right)^2 \ldots\ldots\ldots\ldots(1)$$

where X and Y are the co-ordinates of the corresponding point on the evolute Also we have

$$\left(\frac{dX}{dx}\right)^2 + \left(\frac{dY}{dx}\right)^2 = \left(\frac{ds'}{dx}\right)^2, \ldots\ldots\ldots(2)$$

the arcual distance along the evolute being measured as s'.

$$\therefore \frac{ds'}{dx} = \frac{d\rho}{dx} \; i.\,e. \frac{ds'}{d\rho} = 1$$

$$\text{or} \quad ds' = d\rho.$$

87. Geometrical interpretation.—If P′, P and P″ be any 3 points on the *planer curve* A P′P P″ B, and the curvature

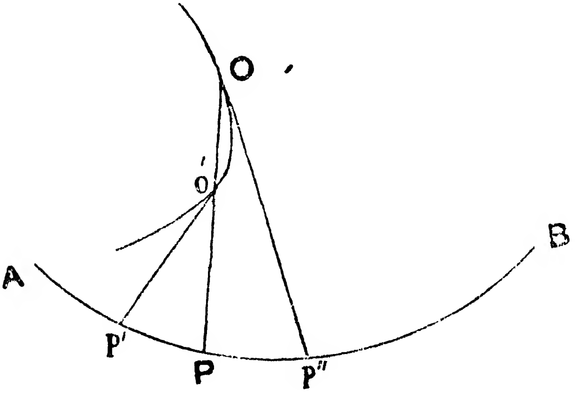

Fig. 22.

of the arc P′ P P″ be regular, the ultimate points of intersection of the normals to the curve at P′ and P when P′ → P is the centre of curvature at P′.

Thus O′ and O the centres of curvature at P′ and P, both lying upon the normal at P will try to coincide if P′ → P as well as P″ → P. Thus the normal at P will try to become the tangent at O to the locus of O. Hence the ovolute touches each and every member of the normal to the curve.

88 **Two parameters.**—In certain problems it is convenient to use two parameters which are connected by an equation. Thus let the curve by given by

$$f(x, y, \alpha, \beta) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{where } \phi(\alpha, \beta) = c \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

In reality there is only one parameter and one of these parameters could be eliminated from the equation (1) by means of the equation (2) and then the envelope can be found by the method already discussed. This procedure may at times not be convenient, and the elimination tedious.

In that case one of these parameters say β could be treated as a function of α by means of (2).

And therefore we have

$$\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial \beta}\frac{d\beta}{d\alpha} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\text{and } \frac{\partial \phi}{\partial \alpha} + \frac{\partial \phi}{\partial \beta}\frac{d\beta}{d\alpha} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Eliminating α, β and $\frac{d\beta}{d\alpha}$ between (1), (2), (3) and (4) we shall be getting the envelope equation.

89. **Indeterminate multiplier.**—Sometimes it is convenient to introduce λ an indeterminate multiplier, the process of solution remaining quite analogous to § 88. From the equations (3) and (4) of § 88, we have by eliminating $\frac{d\beta}{d\alpha}$,

$$\begin{vmatrix} \frac{\partial f}{\partial \alpha} & \frac{\partial f}{\partial \beta} \\ \frac{\partial \phi}{\partial \alpha} & \frac{\partial \phi}{\partial \beta} \end{vmatrix} = 0$$

$$\text{or} \quad \frac{\frac{\partial f}{\partial \alpha}}{\frac{\partial \phi}{\partial \alpha}} = \frac{\frac{\partial f}{\partial \beta}}{\frac{\partial \phi}{\partial \beta}} = \lambda \text{ say.}$$

$$\left.\begin{aligned} \therefore \quad \frac{\partial f}{\partial \alpha} &= \lambda \frac{\partial \phi}{\partial \alpha} \\ \text{and} \quad \frac{\partial f}{\partial \beta} &= \lambda \frac{\partial \phi}{\partial \beta} \end{aligned}\right\} \qquad (1)$$

The quantity λ is known as the "Indeterminate Multiplier." Eliminating now α, β and λ between the equations (1) and (2) of § 88 and (1) we have the required envelope.

Illustrations.

I. Find the envelope of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; where $ab = c$.

The curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ....(1)

and the relation between the parameters is

$ab = c$ ....(2)

Differentiating both with respect to a, we have

$$\frac{x^2}{a^3} + \frac{y^2}{b^3} \frac{db}{da} = 0$$

$$\text{and} \quad b + a \frac{db}{da} = 0$$

$$\therefore \quad \frac{\frac{x^{2*}}{a^3}}{b} = \frac{\frac{y^2}{b^3}}{a} \quad \text{..................................(3)}$$

$\left[\text{by eliminating } \frac{db}{da}.\right.$

---

* Applying directly the method of Indeterminate Multiplier we can get the equation (3) *i. e.*, $\frac{x^2}{a^3} = \lambda\, b$ and

$$\text{or} \quad \frac{\frac{x^2}{a^2}}{ab} = \frac{\frac{y^2}{b^2}}{ab} = \frac{1}{2c}$$

$$\therefore \quad \frac{x^2}{a^2} = \frac{1}{2}, \text{ and } \frac{y^2}{b^2} = \frac{1}{2}$$

$$i.\,e., \quad a = x\sqrt{2} \quad \text{and } b = y\sqrt{2}$$

Substituting these in (2) we get

$$2xy = c, \text{ which is the required envelope.}$$

II. Find the evolute of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

The equation to the normal at any point whose eccentric angle is ϕ is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2 \text{ ....(1)}$$

Differentiating (2) with respect to ϕ the variable parameter, we get

$$\frac{a\,x\sin\phi}{\cos^2\phi} + \frac{b\,y\cos\phi}{\sin^2\phi} = 0$$

$$\text{or} \quad \frac{\sin^3\phi}{by} + \frac{\cos^3\phi}{ax} = 0$$

$$i.\,e., \quad \frac{\sin\phi}{-\sqrt[3]{b\,y}} = \frac{\cos\phi}{\sqrt[3]{a\,x}} = \frac{1}{\sqrt{(a\,x)^{\frac{2}{3}} + (b\,y)^{\frac{2}{3}}}} \text{ ....(2)}$$

Substituting the values of $\sin\phi$ and $\cos\phi$ from (2) in (1) we have

$$\sqrt{(a\,x)^{\frac{2}{3}} + (b\,y)^{\frac{2}{3}}}\,[(a\,x)^{\frac{2}{3}} + (b\,y)^{\frac{2}{3}}] = a^2 - b^2$$

$$\text{or} \quad (a\,x)^{\frac{2}{3}} + (b\,y)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

III. Find the envelope of circles drawn on radii vectors of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, as diameter.

$$\frac{y^2}{b^2} = \lambda\,a$$

$$i.\,e., \quad \frac{\frac{x^2}{a^2}}{b} = \frac{\frac{y^2}{b^2}}{a}.$$

Any point on the curve is given by (a cos ϕ, b sin ϕ), the middle point of OP, if O is the centre origin, will be

$$\left(\frac{a \cos \phi}{2}, \frac{b \sin \phi}{2}\right).$$

The equation of the circle is therefore

$$\left(x - \frac{a \cos \phi}{2}\right)^2 + \left(y - \frac{b}{2} \sin \phi\right)^2 = \frac{a^2 \cos^2 \phi + b^2 \sin^2 \phi}{4}$$

$$\text{or} \quad x^2 + y^2 - a\,x \cos \phi - by \sin \phi = 0 \ldots\ldots\ldots\ldots(1)$$

Differentiating with respect to ϕ, we have

$$ax \sin \phi - by \cos \phi = 0$$

$$i.\,e., \ \frac{\sin \phi}{by} = \frac{\cos \phi}{ax} = \frac{1}{\sqrt{(a\,x)^2 + (b\,y)^2}} \ldots\ldots(2)$$

Substituting for ϕ from (2) in (1) we have

$$x^2 + y^2 - \frac{(a\,x)^2 + (b\,y)^2}{\sqrt{(a\,x)^2 + (b\,y)^2}} = 0$$

$$\text{or} \quad (x^2 + y^2)^2 = (a\,x)^2 + (b\,y)^2$$

Putting $x = r \cos \theta$ and $y = r \sin \theta$ we get the equation of the envelope as

$$r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta.$$

IV. Find the envelope of circles drawn on radii vectors of $r = a\,(1 + \cos \theta)$ as diameter.

If (d, α) be any point on the curve, $d = a\,(1 + \cos \alpha)$...(1)

The equation of the circle on OP as diameter is

$$r = d \cos (\theta - \alpha) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substituting for d from (1) in (2) we have,

$$r = a\,(1 + \cos \alpha) \cos (\theta - \alpha) \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Differentiating (3) with respect to α

$$- \cos (\theta - \alpha) \sin \alpha + (1 + \cos \alpha) \sin (\theta - \alpha) = 0$$

$$\text{or} \quad \frac{\sin \alpha}{1 + \cos \alpha} = \tan (\theta - \alpha)$$

$$i.\,e., \tan \frac{\alpha}{2} = \tan (\theta - \alpha)$$

$$i.e., \frac{\alpha}{2} = \theta - \alpha$$

$$\text{or} \quad \alpha = \tfrac{2}{3}\,\theta \quad \text{.................................(4)}$$

Substituting in (3) we have

The envelope as $r = a\,(1 + \cos \tfrac{2}{3}\,\theta)\cos \frac{\theta}{3}$

$$i.\ e.,\ r = 2a \cos^3 \frac{\theta}{3}.$$

90. Envelope of an equation quadratic in a parameter.

Let $A\,\alpha^2 + B\,\alpha + C = 0$ be the equation to a curve where A, B and C are functions of x and y, and α is the variable parameter. We have here,

$$2A\alpha + B = 0 \text{ or } \alpha = -\ \frac{B}{2A}.$$

Substituting in the equation to the curve we get

$$B^2 - 4\ AC = 0 \text{ as the envelope.}$$

Thus when the equation of the curve is a quadratic in the variable parameter, the discriminant of the parameter is the envelope.

EXAMPLES ON CHAPTER IX.

Find the envelopes of

1. $y = m\,x + \frac{a}{m}$, m being the parameter.

2. $\frac{x}{a} + \frac{y}{b} = 1$ when $ab = c^2$, a constant.

3. $y^2 = 2\,a\,(x - a)$, a being the parameter.

4. $y = m\,x + \sqrt{a^2\,m^2 + b^2}$, m being the parameter.

5. $\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1$, λ being the parameter.

6. $(x - \alpha)^2 + y^2 = a^2$, α being the parameter.

7. $\frac{x^2}{\alpha^2} + \frac{y^2}{b^2 - \alpha^2} = 1$, α being the parameter.

8. Find the envelope of $\frac{x}{a} + \frac{y}{b} = 1$

where (*i*) $a^m \; b^n = c^{m+n}$, a constant

(*ii*) $a + b = c$, a constant

(*iii*) $a^n + b^n = c^n$, a constant.

9. Find the envelope of $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$

where (*i*) $a^m \; b^m = k^{2m}$

(*ii*) $a^m + b^m = c^m$

10. Shew that the envelope of

$$x \cos n\,\alpha + y \sin n\,\alpha = a \cos^{\frac{n}{m}} m\,\alpha, \alpha$$

being the parameter is $r^{\frac{m}{n-m}} = a^{\frac{m}{n-m}} \cos \frac{m}{n-m}\,\theta.$

11. Find the envelope of a line of constant length a, whose extremities move along two fixed rectangular axes.

12. Shew that evolute of the parabola

$$y^2 = 4\,ax \text{ is } 27\,ay^2 = 4\,(x - 2\,a)^3$$

13. Find the envelope of the circles which pass through the origin and have their centres on the hyperbola

$$x^2 - y^2 = a^2$$

14. Find the envelope of the circles described on the radii vectors of the following curves as diameter.

(1) $y^2 = 4\,a\,(x + a)$

(2) $r^n = a^n \cos n\,\theta.$

15. Find the envelope of the circle which moves in such a way that its centre always lies on the parabola $y^2 = 4\,ax$, and its circumference passes through the vertex of the parabola.

16. Find the evolute of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

17. Shew that the envelope of the family of curves $A\lambda^3 + 3B\lambda^2 + 3C\lambda + D = 0$, where λ is the arbitrary parameter and A, B, C, D are functions of x and y is

$$(BC - AD)^2 = 4(BD - C^2)(AC - B^2)$$

18. A variable parabola is drawn having its vertex on a given parabola, the two curves having the same focus; prove that the envelope of its directrix is the curve

$$r\cos^3\frac{\theta}{3} = l, \text{ referred to the common focus as pole.}$$

Trace the curve.

[Oxford 1890.]

19. Shew that the envelope of the common chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and its circles of curvature is the curve

$$\left(\frac{x}{a} + \frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{x}{a} - \frac{y}{b}\right)^{\frac{2}{3}} = 2.$$

[Math. Tripos 1884.]

20. The envelope of the catenary $y = c\cosh\frac{x}{c}$, c being the parameter, consists of two straight lines.

21. Shew that the radius of curvature of the envelope of the line

$$x\cos\alpha + y\sin\alpha = f(\alpha)$$

is $f(\alpha) + f''(\alpha)$, and that the centre of curvature is at the point

$$x = -f'(\alpha)\sin\alpha - f''(\alpha)\cos\alpha$$
$$y = f'(\alpha)\cos\alpha - f''(\alpha)\sin\alpha.$$

22. Shew that the envelope of all cardioides described on radii vectors of the cardioide $r = a(1 + \cos\theta)$, as axes and having their cusps at the pole is $r^{\frac{1}{4}} = (2a)^{\frac{1}{4}}\cos\frac{\theta}{4}$.

CHAPTER X.

CONVEXITY, CONCAVITY AND DOUBLE POINTS.

91. It is sometimes necessary that quite a large and sufficient member of properties about the form and the nature of a curve should be known, in order to trace it. Thus it is desirable to treat the nature of some special points on a curve.

Convexity and Concavity.

92. A curve is said to be convex (or concave) at any point on it with respect to the foot of the ordinate of that point, according as the part of the curve in the immediate neighbourhood of that point lies on the other side (or the same side) of the tangent at that point than the foot of the ordinate.

Thus the curve is convex at P with respect to M in Figure 23(*a*) and concave at P with respect to M in Figure 23 (*b*).

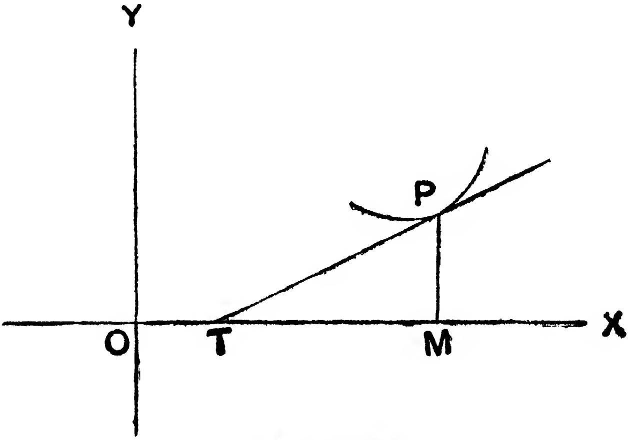

Fig 23 (*a*).

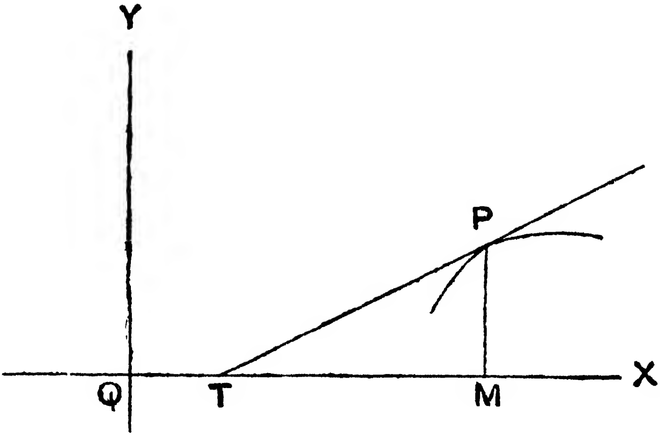

Fig. 23 (*b*).

Second Definition. A curve is said to be convex (or concave) at any point on it with respect to any given straight line according as the part of the curve in the immediate neighbourhood of that point lies within the obtuse (or the acute) angle made by the tangent to the curve at that point and the given line.

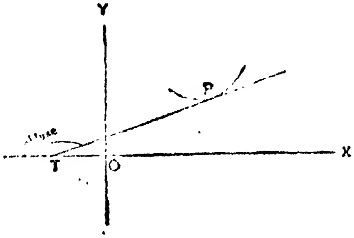

Fig. 24 (*a*).

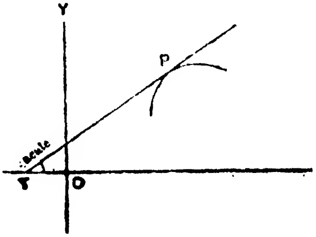

Fig. 24 (*b*).

The curve is convex in Figure 24 (*a*) at *P* with respect to the straight line OX, and in Figure 24(*b*) it is concave.

93. Test for Convexity and Concavity.—Let P (x, y) and Q $(x + \delta x, y + \delta y)$ be any two points on the curve

APQ given by $y = f(x)$ ....(1)
The equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x) \quad \ldots\ldots\ldots\ldots (2)$$

Let NQ′ the ordinate of Q intersect (2) at Q′.

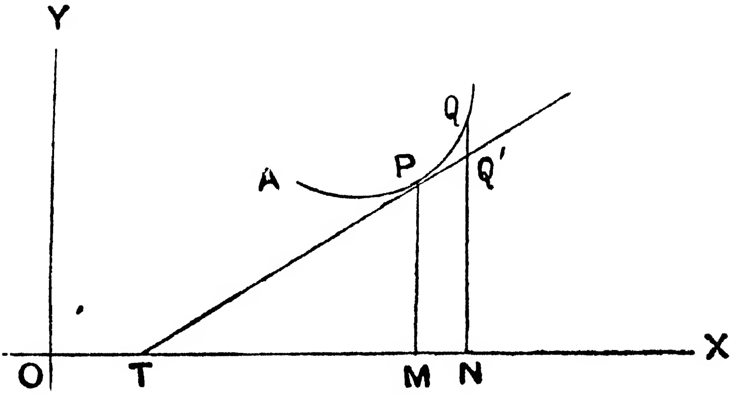

FIG. 25.

Here NQ′ $= y + \frac{dy}{dx}\delta x$ by (2) where X $= x + \delta x$.

$$= f(x) + \delta x\, f'(x) \qquad \text{from (1).}$$

Also NQ $= f(x + \delta x) = f(x) + \delta x\, f'(x)$

$$+ \frac{\delta x^2}{\lfloor 2} f''(x + \theta_1 \delta x), \text{ where } 0 < \theta_1 < 1.$$

So that NQ − NQ′ $= \frac{\delta x^2}{\lfloor 2} f''(x + \theta_1 \delta x)$........(3)

Again if $\delta x \to 0$ the sign of $f''(x + \theta_1 \delta x)$ will be the same as $f''(x)$.* Hence the sign of (NQ−NQ′) will be the same as the sign of $f''(x)$ irrespective of the sign of δx.

Case (*i*) If $f''(x)$ *is positive* NQ > NQ′ that is, the curve is *convex* at P.

Case (*ii*) If $f''(x)$ *is negative* NQ < NQ′ that is, the curve is *concave* at P.

---

* From the continuity of $f''(x)$.

If the curve lies below the axis of x both NQ and NQ′ will be negative, the curve will therefore be convex (or concave) according as $f''(x)$ is negative (or positive)

We can however combine both these cases into one *i. e.,** $f(x) f''(x)$ *is* + *ve for convexity,* and $f(x) f''(x)$ *is* — *ve for concavity.*

94. If $f''(x) = 0$ and $f'''(x) \neq 0$, the point P $(x\ y)$ is known as the ***point of inflexion*** on the curve.

Here NQ - NQ′ = $\frac{\delta x^3}{\lfloor 3} f'''(x + \theta_2 \delta x$, which will clearly change sign with δx.

∴ at a point of inflexion the curve changes from convexity to concavity or from concavity to convexity, according as (NQ - NQ′) changes sign from + *ve* to - *ve* or from - *ve* to + *ve*, as we take δx from negative to positive. The tangent to the curve crosses it at such points.

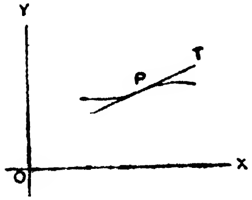

FIG. 26 (*a*).

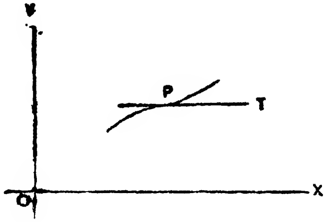

FIG. 26 (*b*).

In figure 26 (*a*) the curve changes from convexity to concavity and in figure 26 (*b*) it changes from concavity to convexity.

Hence at a point of inflexion $f''(x)$ *i. e.,* $\frac{d^2y}{dx^2} = 0$ and it must change sign as the curve passes through that point

---

* Another from of $y \frac{d^2y}{dx^2}$.

i. e., with δx. If $\frac{d^2y}{dx^2}$ does not change sign with δx, $f'''(x)$ must be equal to zero.

Thus if $f''(x) = 0$ and $f'''(x) = 0$, the point P is called a point of undulation, and the convexity or concavity is determined by the sign of $y \frac{d^4y}{dx^4}$.* Similarly if higher derivatives at any point vanish, the point will be one of a higher order of singularity.

95. **Geometrical interpretation of a point of inflexion or undulation.**—The form of the curve in the figure 26 (*a*) can be seen on a magnified scale in figure 27, where it is changing from convexity to concavity. The tangent PT here can be regarded as the limiting form of the straight line PQRT, which clearly cuts the curve in three points, all coinciding in the limit as in figure 26 (*a*). Thus the tangent to the curve at a point of inflexion accounts for three ultimately coincident points on the curve. Similarly at the point of undulation the tangent can be regarded as the limiting case of a straight line through four ultimately the coincident points on the curve. See figures 28 (*a*) and (*b*).

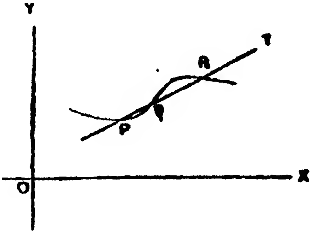

FIG. 27.

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* The discussion is beyond the scope of this volume.

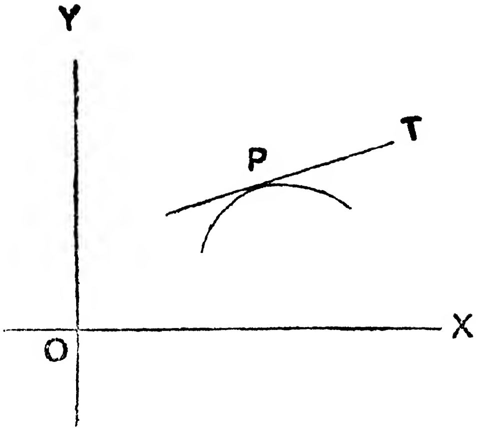


FIG. 28 (a).

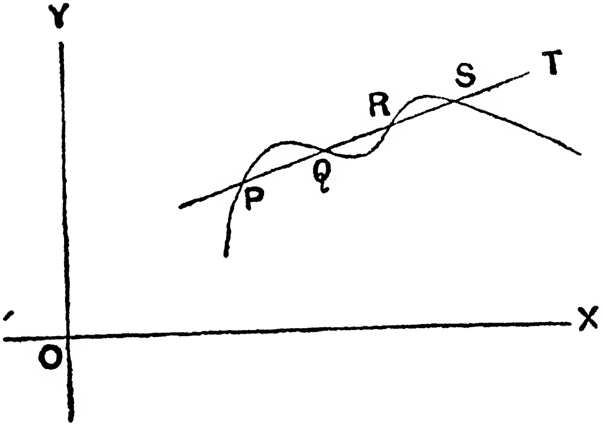


FIG. 28 (b).

(Point of Undulation).

96. Convexity and Concavity in Polar Coordinates.

To find a test of convexity or concavity at a point on a curve with respect to the pole.

Let the curve be $r = f(\theta)$ or $u = F(\theta)$.

In the figure 29, the curve is convex at P, R and Q each tending to P as $\delta\theta \rightarrow 0$ $\triangle$ OQR $>$ $\triangle$ OQP $+$ $\triangle$ OPR *i. e.* $r_1 r_2 \sin 2\delta\theta > r_1 r \sin \delta\theta + r r_2 \sin \delta\theta$ or $2 r_1 r_2 \cos \delta\theta > r(r_1 + r_2)$..........................(1)

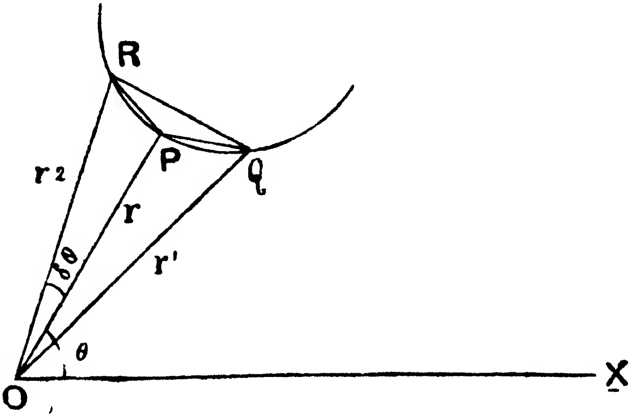


FIG. 29.

Similarly if the curve is concave as in Fig. 30.
$2 r_1 r_2 \cos \delta\theta < r(r_1 + r_2)$.. ....(2)

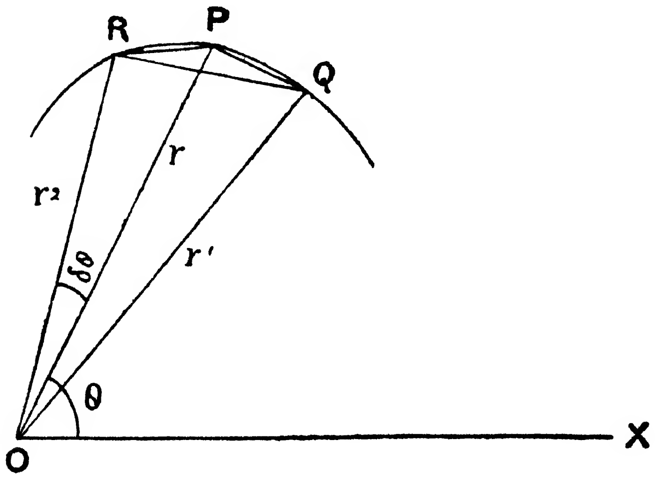


FIG. 30.

If $r = \frac{1}{u}$ etc., the curve is convex or concave at P, according as

$$2\,u \cos \delta\theta > \text{ or } < (u_1 + u_2)$$

Again $u_1 = F(\theta - \delta\theta)$

$$= F(\theta) - \delta\theta\, F'(\theta) + \frac{(\delta\theta)^2}{\lfloor 2} F''(\theta + \lambda_1 \delta\theta),$$

where $0 < \lambda_1 < 1$

Similarly $u_2 = F(\theta) + \delta\theta\, F'(\theta)$

$$+ \frac{(\delta\theta)^2}{\lfloor 2} F''(\theta + \lambda_2 \delta\theta), \text{ where } 0 < \lambda_2 < 1$$

Hence the curve is convex or concave according as

$$2\,F(\theta) \cos \delta\theta > \text{ or } < 2\,F(\theta) + \frac{(\delta\theta)^2}{\lfloor 2}$$

$$[F''(\theta + \lambda_1 \delta\theta) + F''(\theta + \lambda_2 \delta\theta)]$$

$$\text{or } 4\,F(\theta) \frac{[\cos \delta\theta - 1]}{(\delta\theta)^2} > \text{ or } <$$

$$[F''(\theta + \lambda_1 \delta\theta) + F''(\theta + \lambda_2 \delta\theta)].$$

Proceeding to the limit when

$$\delta\theta \rightarrow 0, \; -2\,F(\theta) > \text{ or } < 2\,F''(\theta)$$

Hence the curve is convex or concave with respect to the pole according as

$$F''(\theta) + F(\theta) < \text{ or } > 0$$

$$i.\,e., \frac{d^2u}{d\theta^2} + u < \text{ or } > 0.$$

Note.—If this is equal to zero, it should change sign as the curve possess through P, if the point P is a point of inflexion.

Exercises.

1. Shew that the origin is a point of inflexion on the curve $a^3y = b\,xy + cx^3 + dx^4$.

2. Shew that the curve $y = e^x$ is convex at every point with respect to the foot of the ordinate of that point.

3. Find the coordinates of the point of inflexion on the curve $x^3 - 3ax^2 + b^2y = 0$.

4. Shew that the points, in which the curve $y = c \cos \frac{x}{a}$, cuts the axis of x, are all points of inflexion.

5. Shew that the points of inflexion of the curve $y^2 = (x - a)^2 (x - b)$ lie on the line $3x + a = 4b$.

6. Find the convexity or cancavity for any value of x on the curve $y = 2\sqrt{ax}$.

7. Has the curve $x = y^3 + 3y^2$ a point of inflexion ?

Double Points.

97. Let us consider the curve in the Figure 31, drawn on a magnified scale. At such a point as P, there can be drawn two tangents one to each branch of the curve passing through it. Each tangent cuts the curve in two ultimately coincident points on one branch and incidently cuts the other branch also at a point, which also ultimately coincides in the limit with the point of contact. In the Fig. 31. PQ, the tangent at P, can be supposed as intersecting the curve at P and Q the branch to which it is the tangent and also cuts the other branch at R Each tangent therefore at such a point intersects the curve in three points ultimately coinciding in the limit, and this is due to the fact that two brances of the curve pass through P. Such points are known as *double points*. If three branches pass

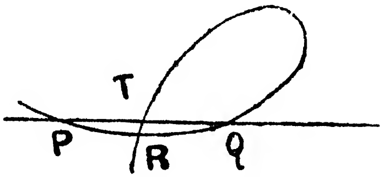


FIG. 31.

through any point, it is said to be a *triple point* or a *multiple point* of the third order, and so on.

98. Tangents at the Origin.—If the equation of a curve be rational and algebraic it can be written in the following form.

$$\begin{array}{l} a \\ + b_1 x + b_2 y \\ + c_1 x^2 + c_2 x y + c_3 y^2 \\ + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ + l_1 x^n + l_2 x^{n-1} y + \dots\dots\dots\dots\dots \\ \quad l_{n+1} y^n = 0 \dots\dots\dots\dots\dots\dots\dots\dots (1) \end{array}$$

If this be converted into polar co-ordinates, it becomes

$$\begin{array}{l} a \\ + r (b_1 \cos\theta + b_2 \sin\theta) \\ + r^2 (c_1 \cos^2\theta + c_2 \cos\theta \sin\theta + c_3 \sin^2\theta) \\ + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ + r^n (l_1 \cos^n\theta + l_2 \cos^{n-1}\theta \sin\theta + \dots\dots \\ + l_{n+1} \sin^n\theta) = 0 \dots\dots\dots\dots\dots\dots\dots (2) \end{array}$$

Let O be the pole and OA the initial line, and let a radius vector at an angle θ cut the curve given at the points P_1, P_2 etc. The roots of the equation (2), will then be OP_1, OP_2, etc., and there will be n such roots.

Case I. If $a = 0$, it is clear that the curve passes through the origin; and in this case one root of (2) is zero, and say P_1 coincides with O.

Case II. Again if in addition to this, θ is so taken that

$$b_1 \cos\theta + b_2 \sin\theta = 0, \dots\dots\dots\dots\dots\dots\dots (3)$$

a second root of the equation (2) will also be zero, and therefore we get a straight line, making an angle given by (3) *i.e.*, $\tan^{-1}\left(-\frac{b_1}{b}\right)$ with the initial line, and intersecting the given curve in two coincident points at the origin, and thus it is the tangent at the origin. Converting (3) into cartesian equation we have,

$$b_1 x + b_2 y = 0.$$

Hence if the curve passes through the origin the terms of the *first degree equated to zero gives the tangent at the origin.*

Case III. If $a = 0$, $b_1 = 0$, and $b_2 = 0$, in general it is possible to find the value of θ such that

$$c_1 \cos^2 \theta + c_2 \cos \theta \sin \theta + c_3 \sin^2 \theta = 0 \ldots\ldots\ldots\ldots (4)$$

and then three roots of (2) will be zero. Thus we get a pair of lines given by

$$c_1 x^2 + c_2 xy + c_3 y^2 = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

such that each of them cuts the curve at three coincident points at the origin. Thus there are two branches of the curve at the origin. The origin is a double point on the curve and the *terms of the lowest degree equated to zero i. e*, equation (5) gives the tangents at the origin.

Case IV. If $a = 0$, $b_1 = 0$, $b_2 = 0$, $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$, the origin is a triple point, and the tangents at the origin will similarly be given by the lowest degree terms, *i.e.*, third degree terms in equation of the curve equated to zero.

Thus in general, if the lowest degree terms existent in the equation of any curve, passing through the origin, are of the m^{th} degree, the origin is a *multiple point of the m^{th} order* on the curve, and the terms of the m^{th} degree equated to zero give the m tangents at the origin.

99. **Species of Double Points.**—Double points on a curve can be defined as points at which two tangents real, coincident or imaginary can be drawn to the curve.

Case I. If the tangents are real and different, there are two branches of the curve which pass through the point, and it is called a **node**, as in Fig. 32.

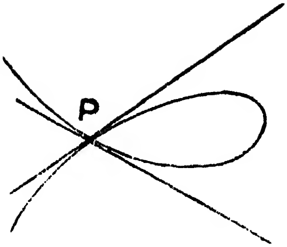


Fig. 32.

Case II. If the tangents be imaginary, there are in reality no points on the curve in the neighbourhood of that point and we are unable to trace the curve in any direction beyond that point. Such a point is only an isolated point, whose co-ordinates do satisfy the equation to the given curve, but no branch of the curve exists in the vicinity of that point. Such a point is called a conjugate point.

Case III. If the tangents are real but coincident, the two branches at the double point touch each other. The point is then called a **cusp.**

Two Species of Cusps.

100. In the figure 33 the two branches PQ and PR of the curve lie on opposite sides of the tangent at P. This point is said to be a *Cusp of the first species* In the figure 34, the two branches PQ and PR lie on the same side

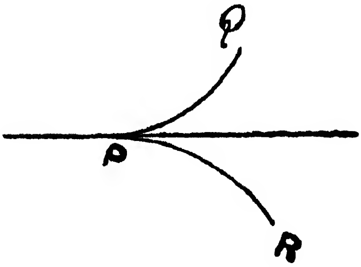


Fig. 33.

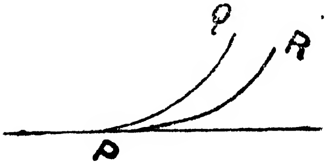


Fig. 34.

of the tangent at P. This point is said to be a *Cusp of the second species.*

101. Analytical condition of a double point. Let the equation of a straight line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=r, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

which intersects a curve of *n*th degree, whose rational algebraic equation is $f(x, y) = 0$, say.

Then $$\left.\begin{array}{l} x=\alpha+lr \\ y=\beta+mr \end{array}\right\}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and these represent the coordinates of any point on the given straight line. If the point is common to the curve as well, they will satisfy the equation of the curve *i. e.*,

$$f(\alpha+lr, \beta+mr)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

By § 44, we have

$$f(\alpha,\beta)+r\left(l\frac{df}{d\alpha}+m\frac{df}{d\beta}\right)+\frac{r^2}{\underline{\lfloor 2}}\left(l\frac{d}{d\alpha}+m\frac{d}{d\beta}\right)^2 f$$
$$+\frac{r^2}{\underline{\lfloor 3}}\left(l\frac{d}{d\alpha}+m\frac{d}{d\beta}\right)^2 f+\ldots\ldots\ldots$$
$$\frac{r^n}{\underline{\lfloor n}}\left(l\frac{d}{d\alpha}+m\frac{d}{d\beta}\right)^n f+\ldots=0 \ldots (4)$$

which gives *n* values of *r i. e.*, the co-ordinates of *n* points of intersection of the straight line with the curve.

I. If $f(\alpha, \beta) = 0$, one root of the equation (3) is zero and the point (α,β) lies on the curve.

II. If $l : m$ be now so chosen that

$$l\frac{df}{d\alpha}+m\frac{df}{d\beta}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5$$

Two roots of (3) will be zero, and (5) gives the direction in which the straight line will become the tangent at (α, β) to the curve and its equation is

$$(x-\alpha)\frac{df}{d\alpha}+(y-\beta)\frac{df}{d\beta}=0,$$

which is found by eliminating l and m from equations (1) and (5).

III. If $\frac{df}{d\alpha} = 0$ and $\frac{df}{d\beta} = 0$ along with the condition that $f(\alpha, \beta) = 0$, every straight line through (α, β) intersects the curve in two coincident points irrespective of the values of l and m. If now $l : m$ be so chosen that

$$l^2 \frac{d^2f}{d\alpha^2} + 2\,l\,m \frac{d^2f}{d\alpha d\beta} + m^2 \frac{d^2f}{d\beta^2} = 0 \ldots\ldots\ldots\ldots (6)$$

We have in general two values of $\frac{l}{m}$ *i. e.*, the two directions in which if the straight line be drawn through (α, β), it will intersect the curve in three coincident points. The point (α, β) is then a *double point* on the curve, since two branches of the curve pass through it. The equation of the two tangents is

$$(x - \alpha)^2 \frac{d^2f}{d\alpha^2} + 2\,(x - \alpha)\,(y - \beta) \frac{d^2f}{d\alpha d\beta}$$
$$+ (y - \beta)^2 \frac{d^2f}{d\beta^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

found by eliminating l and m from equations (1) and (6).

102. Species of Double points.—The equation of the tangents at a double point is given by

$$(x - \alpha)^2 \frac{d^2f}{d\alpha^2} + 2\,(x - \alpha)\,(y - \beta) \frac{d^2f}{d\alpha d\beta}$$
$$+ (y - \beta)^2 \frac{d^2f}{d\beta^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The angle between these two straight lines is the same as the angle between the two straight lines parallel to those given by (1) and passing through the origin *i, e.*, the angle between the straight lines given by*

$$x^2 \frac{d^2f}{d\alpha^2} + 2\,xy \frac{d^2f}{d\alpha d\beta} + y^2 \frac{d^2f}{d\beta^2} = 0 \ldots\ldots\ldots\ldots (2)$$

---

* Only second degree terms equated to zero.

Thus the angle θ between the tangents is given by

$$\tan\theta = \frac{2\sqrt{\left(\frac{d^2f}{d\alpha d\beta}\right)^2 - \frac{d^2f}{d\alpha^2}\cdot\frac{d^2f}{d\beta^2}}}{\frac{d^2f}{d\alpha^2}+\frac{d^2f}{d\beta^2}}$$

The tangents are real or coincident or imaginary according as $\tan\theta$ is different from zero or zero or imaginary *i e.*, according

$$\text{as}\left(\frac{d^2f}{d\alpha d\beta}\right)^2 > \text{ or } = \text{ or } < \frac{d^2f}{d\alpha^2}\cdot\frac{d^2f}{d\beta^2}.$$

Thus the point (α, β) is a node or a conjugate point according as

$$\left(\frac{d^2f}{d\alpha d\beta}\right)^2 > \text{ or } < \frac{d^2f}{d\alpha^2}\cdot\frac{d^2f}{d\beta^2}$$

and is in general a cusp if

$$\left(\frac{d^2f}{d\alpha d\beta}\right)^2 = \frac{d^2f}{d\alpha^2}\cdot\frac{d^2f}{d\beta^2}.$$

We say in general a cusp, as it will be noticed that in certain cases when this condition is satisfied, the curve becomes imaginary in the neighbourhood of that point, and which therefore is really a conjugate point. Thus further investigation should be made in this case to find out whether the point in question is really a cusp or a conjugate point.

Summing up we notice that if $f(\alpha, \beta) = 0$ and

$$\frac{df}{d\alpha} = 0 \text{ as well as } \frac{df}{d\beta} = 0,$$

the point (α, β) is a node, or a conjugate point or a cusp in general according as

$$\left(\frac{d^2f}{d\alpha d\beta}\right)^2 > \text{ or } < \text{ or } = \frac{d^2f}{d\alpha^2}\cdot\frac{d^2f}{d\beta^2}.$$

Thus the rule to search for double points on a given curve, $f(x, y) = 0$ is to find the solution of

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0,$$

and ascertain which values satisfy $f(x, y) = 0$. For these values find whether

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > \text{or} = \text{or} < \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}.$$

Note. It should be noted here that

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = \frac{\partial^2 f}{\partial x^2} \; \frac{\partial^2 f}{\partial y^2}$$

gives the curve which passes through all the cusp points on the curve $f(x,y) = 0$, and $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ passes through all the nodes of the curve $f(x,y) = 0$, at which the tangents are at right angles.

103. **Species of a cusp.** – Let the condition

$$\left(\frac{\partial^2 f}{\partial \alpha \partial \beta}\right)^2 = \frac{\partial^2 f}{\partial \alpha^2} \cdot \frac{\partial^2 f}{\partial \beta^2}$$

be satisfied with the other preliminary conditions *i e*, $f(\alpha, \beta) = 0$ and $\frac{\partial f}{\partial \alpha} = 0$ as also $\frac{\partial f}{\partial \beta} = 0$. *i. e*, the point (α, β), in general, is a cusp. Transfer the origin to the point (α, β). The transformed equation will be of the following form :—

$$(ax + by)^2 + u_3 + u_4 + u_5 + \ldots . = 0 \ldots \ldots (1)$$

Where $ax + by = 0$ is the tangent at the new origin and u_3, u_4 etc., are homogeneous rational algebraical functions of x and y of degree 3, 4 etc.

Let P be the length of the perpendicular from a point $(x'y')$ contiguous to the new origin (*i. e.*, the cusp point) upon the tangent $ax + by = 0$.

$$\therefore \quad P = \frac{ax' + by'}{\sqrt{a^2 + b^2}} \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)$$

Since $(x'y')$ lies on (1) we have

$$(a\,x' + by')^2 + u'_3 + u'_4 + \ldots\ldots = 0 \ldots\ldots(3)$$

Where u'_3, u'_4 etc. denote the expressions u_3, u_4 etc., after substituting (x', y') for (x, y). Eliminating y' between (2) and (3) and rejecting higher powers of P than the second, we shall have a quadratic in P say,

$$AP^2 + BP + C = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Where A,B and C are functions of x'. We retain only upto P^2 because we have to consider only the two small perpendiculars from points on the curve very near the origin *i. e.*, when $x' \rightarrow 0$, and hence in comparison with P^2 we can neglect higher powers of P *i. e.*, P^3, P^4 as also P^3x' etc.

If the roots of (4) be imaginary when x' is very small, the branches of the curve near the origin are non-existent and hence the origin is a *conjugate point.* If the roots be real but of opposite signs, the two perpendiculars will lie on opposite sides of the tangent and hence the origin is a cusp of the *first species.* If the roots be real and of like sign, the two perpendiculars will lie on the same side of the tangent and hence the origin is a cusp of the *second species.*

Like or unlike signs of the roots are determined by

$$P_1\,P_2 \equiv \frac{C}{A}.$$

i. e., $\frac{C}{A}$ should be + *ve* if both are of like sign and − *ve* if they are of unlike sign. If they are of like sign we have to find again the sign of $P_1 + P_2$, which will be + *ve* if both are positive or will be − *ve* if both are negative If $P_1 + P_2$ is then + *ve* the second species cusp is such that the two branches of the curve lie above the tangent and if $P_1 + P_2$ is − *ve*, both of them lie below the tangent.

Complete information is afforded by the equation (4) whether the cusp is *single or double i. e.*, whether the branches of the curve extend only in one direction of the tangent from the cusp or extend towards both the directions of the tangent. If it is single, the roots of (4) will depend upon the sign of x', on the other hand if it is double, the roots of (4) will be independent of the sign of x'. It is also possible, in the case of a double cusp, that on one side it may be of one species and on the other side of the other species, such a point is called by Cramer a point of Oscul-inflexion. All the possible five cases are drawn in the annexed figures.

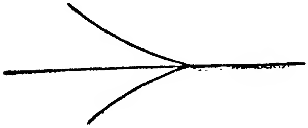


Fig. 35 single cusp, first species.

Fig. 36 single cusp, second species.

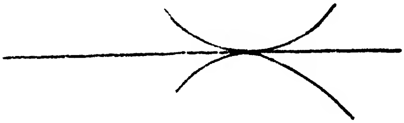


Fig. 37 double cusp, first species.

Fig. 38 double cusp, second species.

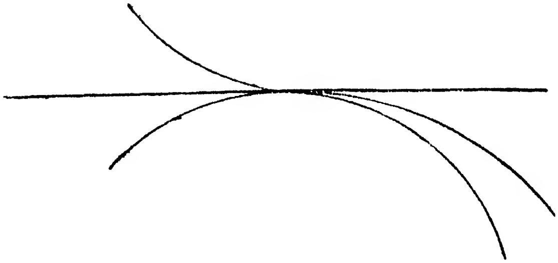


FIG. 39 double cusp—Oscul-inflexion.

Illustrations.

I. Find the double points on $x^3 + y^3 = 3axy$.

Here $\frac{\partial f}{\partial x} \equiv 3x^2 - 3ay = 0$

$$\frac{\partial f}{\partial y} \equiv 3y^2 - 3ax = 0$$

Solving these and finding which of the values satisfy the equation of the curve, we get (0, 0) as the coordinates of a double point *i. e.*, the origin is a double point.

Again $\frac{\partial^2 f}{\partial x^2} \equiv 6x.$

$$\frac{\partial^2 f}{\partial y^2} \equiv 6y$$

and $\frac{\partial^2 f}{\partial x \partial y} = -3a.$

Here $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}$ when we substitute (0, 0) for (x, y). Thus the origin is a node. The tangents at the origin are given by $xy = 0$.

II. Find the double points on the curve $(x + y)^3 - \sqrt{2}\,(x - y + 2)^2 = 0$.

Here $\frac{\partial f}{\partial x} \equiv 3\,(x + y)^2 - 2\sqrt{2}\,(x - y + 2) = 0$

$$\frac{\partial f}{\partial y} \equiv 3\,(x + y)^2 + 2\sqrt{2}\,(x - y + 2) = 0$$

i. e., $x + y = 0$

and $x - y + 2 = 0$

which gives $\left.\begin{array}{l} x = -1 \\ y = 1 \end{array}\right\}$

This point lies on the curve and is therefore a double point.

Again $\dfrac{\partial^2 f}{\partial x^2} \equiv 6(x + y) - 2\sqrt{2} = -2\sqrt{2}$ at $(-1, 1)$.

$\dfrac{\partial^2 f}{\partial y^2} \equiv 6(x + y) - 2\sqrt{2} = -2\sqrt{2}$,,

and $\dfrac{\partial^2 f}{\partial x \partial y} \equiv 6(x + y) + 2\sqrt{2} = 2\sqrt{2}$,,

Hence $\left(\dfrac{\partial^2 f}{\partial y \partial x}\right)^2 = \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2}$ at this point.

In general $(-1, 1)$ is a cusp.

Shifting the origin to $(-1, 1)$, the equation becomes

$$(x + y)^3 - \sqrt{2}(x - y)^2 = 0 \text{..................}(1)$$

Thus $x - y = 0$ is the tangent. Dropping perpendicular p from $(x' \; y')$ on $x - y = 0$

we have $\dfrac{x' - y'}{\sqrt{2}} = p$ or $x' - y' = p'$ ....(2)

Eliminating y', we get

$$(2x' - p')^3 - \sqrt{2}\,p'^2 = 0$$

or $[8x'^3 - 12x'^2 p' + 6x' p'^2 - p'^3] - \sqrt{2}\,p'^2 = 0.$

or rejecting unnecessary terms,

$$\sqrt{2}\,p'^2 + 12x'^2 p' - 8x'^3 = 0$$

or $p'^2 + 6\sqrt{2}\,x'^2 p' - 4\sqrt{2}\,x'^3 = 0.$

The roots are real if $9x'^4 + 2\sqrt{2}\,x'^3 > 0.$

Rejecting x'^4 in comparison with x'^3 as x' is very small, the sign of this will be governed by the sign of x'^3, and we have the roots real if x' be positive. The point $(-1, 1)$ is therefore a single cusp.

Again $p_1' p_2' = -4\sqrt{2}\,x'^3$ *i. e.*, $-$ *ve* when x' is $+$ *ve*, and thus the cusp is one of the first species.

EXAMPLES ON CHAPTER X.

Find the double points in the following :—

1. $(ax - by)^2 = (x - c)^5.$

2. $y^2 = x(x + k)^4.$

3. $x^2 - xy - 2y^2 + x^3 - 8y^3 = 0.$

4. $y^2(1 - y^2) = x^2.$

5. $cy^2 = x^3.$

6. $(x - 2)^2 = y(y - 1)^2.$

7. $(2y + x + 1)^2 = 4(1 - x)^5.$

8. $y^2 = x^2 \sin \frac{x}{c}.$

9. $y^2(a^2 + x^2) = x^2(a^2 - x^2).$

10. $a^3y^2 - bx^4 - x^5 = 0.$

11. $xy^2 + 2a^2y - ax^2 - 3a^2x - 3a^3 = 0.$

12. $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0.$

13. Shew that the origin is a conjugate point on the curve $x^4 - cx^2y + cxy^2 + c^2y^2 = 0.$

14. Shew that the origin is a conjugate point on the curve $y^2 = 2x^2y + x^4y - 2x^4.$

15. In the curve $a^3y^2 = 2abx^2y + x^5$, the origin is an Oscul-inflexion.

[Cramer].

16. Shew that for the cissoid $y^2 = \dfrac{x^3}{2a - x}$, origin is a cusp of the single first species.

17. Shew that the curve $y^2 = 2x^2y + x^4y + x^4$ has a double first species cusp at the origin.

18. Shew that the origin is a single second species cusp on the curve

$$x^4 - 2cx^2y - cxy^2 + c^2y^2 = 0.$$

CHAPTER XI.

MAXIMA AND MINIMA.

104. Before defining what a maximum or a minimum value of a function of one independent variable is, let us examine the curve in the figure 40.

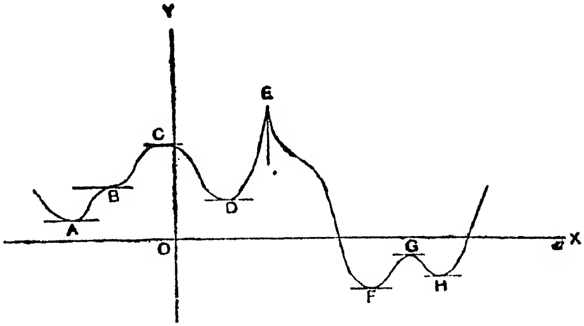


Fig. 40.

It should be noticed that A, B, C, D, E, F, G and H are points where the tangents are either parallel to x axis or y axis. A, D, F and H are such values of the function that they are least in *their respective vicinity*, and C, E and G are greatest values in *their respective vicinity*. This does not mean that these are the least and the greatest values of the function in the region for which it is defined. In other words these are only maximum and minimum values of the function in the neighbourhood of those points. At B it will be noticed that although the tangent, is parallel to x axis the function has neither a maximum nor a minimum value, the point being a point of inflexion with which the student is already familiar. Thus there can be more than one maximum or minimum value of a function, and if so they occur alternately. A maximum or a minimum value does not necessarily mean the greatest or the least value of the function. Also it should be noticed that the minimum value at D is greater than the maximum value at G.

105. $f(x)$, defined for a certain interval (α, β) is said to have a ***maximum*** (or a ***minimum***) value, when $x = a$, if it is possible to find a positive number $\varepsilon \rightarrow 0$, such that

$$f(x) - f(a) < (\text{or} >)\ 0$$

for every value of x, $0 < | x - a | \leqslant \varepsilon$

106 Suppose $f(x)$ has a maximum when $x = a$

$\therefore f(x) - f(a) < 0$ for every value of x given by $0 < | x - a | \leqslant \varepsilon$, where $\varepsilon \rightarrow 0$.

$$i.\ e.,\quad \frac{f(x) - f(a)}{x - a} < 0 \text{ or } > 0$$

according as $(x - a)$ is positive or negative.

$$\therefore \quad \underset{x \rightarrow a + 0}{\text{Lt}} \frac{f(x) - f(a)}{x - a} \ngtr 0, \text{ if the limit}$$

exists,

$$\text{and} \quad \underset{x \rightarrow a - 0}{\text{Lt}} \frac{f(x) - f(a)}{x - a} \nless 0, \text{ if the limit}$$

exists.

If now $f'(a)$ exists, these two right hand and left hand limits must be equal to each other, and since one of them $\ngtr 0$ and the other $\nless 0$, hence $f'(a) = 0$. A similar reasoning will prove that $f'(a) = 0$ if the function has a minimum when $x = a$.

Thus for a maximum or a minimum value $f'(x) = 0$.

107. If $f(x)$ and its first n derivatives are continuous and if $f'(a) = f''(a) = \ldots\ldots = f^{n-1}(a) = 0$ and $f^{n}(a) \neq 0$, $f(x)$ will have a *maximum or a minimum if n is even according as $f^{n}(a) < 0$ or > 0, and neither a maximum nor a minimum if n is odd.*

Here $f(a+h) - f(a) = \frac{h^n}{\lfloor n} f^n(a+\theta h)$,

where $0 < \theta < 1$ and $0 < |h| \leqslant \varepsilon$.

Again since $f^n(a+\theta h)$ will have the same sign as $f^n(a)$.

$\therefore$ If $\frac{h^n}{\lfloor n} f^n(a) < 0$ $f(x)$ has a maximum when $x = a$

and if $\frac{h^n}{\lfloor n} f^n(a) > 0$ $f(x)$ has a minimum when $x = a$.

Now when n is odd, $\frac{h^n}{\lfloor n} f^n(a)$ changes sign with h, and therefore there will be neither a maximum nor a minimum. But if n is even, $f(a+h) - f(a)$ has the same sign as $f^n(a)$. Hence $f(x)$ has a maximum or a minimum when $x = a$ according as $f^n(a) < 0$ or > 0.

Cor. 1. If only $f'(a) = 0$ and $f''(a) \neq 0$. $f(x)$ *has a maximum when* $x = a$ *if* $f''(a) < 0$ *and a minimum if* $f''(a) > 0$.

Cor 2. At a maximum or minimum point $f'(x)$ must change sign. Suppose $f(x)$ has a maximum or minimum when $x = a$. We know that

$$f'(a+h) = f'(a) + h f''(a+\theta h)$$

$$\text{Since } f'(a) = 0$$

$$\therefore f'(a+h) = h f''(a+\theta h).$$

The sign of the right hand depends upon the sign of $h\ f''(a)$, which changes with h. And $f''(a)$ is $-$ *ve* for maximum and $+$ *ve* for minimum.

Thus $f'(a)$ changes from $+$ *ve* to $-$ *ve* through the maximum point and $f'(a)$ changes from $-$ *ve* to $+$ *ve* through the minimum point.

108. The maximum and minimum values of a given function of a single variable can now be investigated.

Let $y = f(x)$ be the given function.

$f'(x) = 0$, will give the values of x, for which $f(x)$ is maximum or minimum. Suppose the roots of this equation to be $x_1, x_2, \ldots. x_m, \ldots\ldots$etc.

If $f''(x_m)$ is $+ ve$, $f(x)$ has a minimum when $x = x_m$ and if $f''(x_m)$ is $- ve$, $f(x)$ has a maximum when $x = x_m$.

If however $f''(x_m) = 0$ and $f'''(x_m) \neq 0, \ldots\ldots\ldots.$ $x = x_m$ will give a point of inflexion *i e.*, neither maximum nor minimum and then the maxima and minima will be given by $f^4(x) = 0$, $f''(x) = 0$ and $f'''(x) = 0$ and the sign of $f''(x)$ will indicate whether the function is a maximum or a minimum.

Illustration.

I. Find the maximum and minimum values of $2x^3 - 9x^2 + 12x - 9$.

$$\begin{aligned} \text{Here } f(x) &= 2x^3 - 9x^2 + 12x - 9 \\ \text{and } f'(x) &= 6(x^2 - 3x + 2) \\ &= 6(x-1)(x-2) \end{aligned}$$

(*i*) when $x = 1$, $f''(x) \equiv 6[2x - 3]$ is $- ve$ *i. e.*, $f(1) \equiv 14$ is a maximum value

(*ii*) when $x = 2$, $f''(x)$ is $+ ve$, *i. e.*, $f(2) \equiv -5$, is a minimum value.

109. If it be required to find the maxima and minima of a funcion of two variables, they being connected by a certain relation, the following procedure may be adopted.

$$\text{Let } u = f(x, y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{where } \psi(x, y) = 0$$

$$\text{or } \quad y = \phi(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substituting for y in (1) we get

$$u = f[x, \phi(x)], \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

a function in x alone, whose maxima and minima can be found as in the previous article.

The elimination of y from (1) and (2) may sometimes be difficult and in such cases, the following method of determination may be more conveniently used.

$$\text{Here } \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\,\frac{dy}{dx} \ldots\ldots\ldots\ldots\ldots(4)$$

$$\text{Also from (2), } \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\,\frac{dy}{dx} = 0 \ldots\ldots\ldots\ldots(5)$$

Eliminating $\frac{dy}{dx}$, we get

$$\frac{du}{ax} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\,\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}}$$

Hence for a maximum or a minimum of u

$$\frac{\partial u}{\partial x}\cdot\frac{\partial \psi}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial \psi}{\partial x} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

The values of x and y found from (2) and (6) may now be substituted in $\frac{d^2u}{dx^2}$, which will show whether u is a maximum or a minimum.

Illustration.

I. To find maxima and minima of $\frac{\cos x}{1 + \cot x}$.

Here $y = \frac{\cos x}{1 + \cot x}$.

$$\therefore \frac{dy}{dx} = \frac{-\sin x\,(1 + \cot x) + \cos x \operatorname{cosec}^2 x}{(1 + \cot x)^2} = 0$$

$$\text{or} \quad -\sin x\,(1 + \cot x) + \cos x \operatorname{cosec}^2 x = 0$$

$$i.\,e.,\ \sin^3 x = \cos x\,(1 - \sin^2 x)$$

$$\text{or} \quad \tan^3 x = 1$$

$$i.\,e.,\ x = n\pi + \frac{\pi}{4}.$$

Again $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{-\sin^3 x + \cos^3 x}{(\sin x + \cos x)^2}\right]$

$$= \frac{[-3\sin^2 x\cos x - 3\sin x\cos^2 x](1+\sin 2x)}{[1+\sin 2x]^2} - \frac{2\cos 2x[-\sin^3 x + \cos^3 x]}{[1+\sin 2x]^2}$$

$$\therefore \left(\frac{d^2y}{dx^2}\right) = \left[\frac{-\frac{3}{2}\sin 2x(\sin x + \cos x)}{(1+\sin 2x)}\right]$$

where $x = n\pi + \frac{\pi}{4}$,

as $\cos^3 x - \sin^3 x$ will be zero, and so will be $\cos 2x$.

$$\therefore \left(\frac{d^2y}{dx^2}\right) = -\frac{\frac{3}{2}\sin\left(2n\pi + \frac{\pi}{2}\right)}{1+\sin\left(2n\pi + \frac{\pi}{2}\right)}\left[\sin\left(n\pi + \frac{\pi}{4}\right) + \cos\left(n\pi + \frac{\pi}{4}\right)\right]$$

$$= -\tfrac{3}{4}\cos\left(n\pi + \frac{\pi}{4}\right)\left[1 + \tan\left(n\pi + \frac{\pi}{4}\right)\right]$$

$$= -\tfrac{3}{2}(-1)^n \cdot \frac{1}{\sqrt{2}}$$

or $\frac{d^2y}{dx^2} = \frac{3}{2}(-1)^{n+1}\frac{1}{\sqrt{2}}$, which is + ve when n is odd and negative when n is even.

$\therefore$ if $x = 2m\pi + \frac{\pi}{4}$, y is a maximum and if

$x = (2m+1)\pi + \frac{\pi}{4}$, y is a minimum.

II. Given $\frac{x}{a} + \frac{y}{b} = 1$, find the maximum and the minimum values of xy and $x^2 + y^2$ respectively.

(*i*) Let $u = xy$ where $\frac{x}{a} + \frac{y}{b} = 1$

$$\therefore \quad u = bx\left(1 - \frac{x}{a}\right)$$

and $\frac{du}{dx} = b\left(1 - \frac{x}{a}\right) - \frac{b}{a}x = 0$

$$\therefore \quad b\left(1 - \frac{2x}{a}\right) = 0$$

$$\left.\begin{aligned} \text{or} \quad x &= \frac{a}{2} \\ \text{and} \therefore \quad y &= \frac{b}{2} \end{aligned}\right\}$$

Again $\frac{d^2u}{dx^2} = -\frac{2b}{a}$, which is $-$ ve for every value of x and y. $\therefore$ $\left(\frac{a}{2}, \frac{b}{2}\right)$ gives the maximum value of xy as $\frac{ab}{4}$.

(*ii*) Let $u = x^2 + y^2$ where $\frac{x}{a} + \frac{y}{b} = 1$

$$\therefore \quad u = x^2 + b^2\left(1 - \frac{x}{a}\right)^2$$

and $\frac{du}{dx} = 2x - \frac{2b^2}{a}\left(1 - \frac{x}{a}\right) = 0.$

or $x\left(1 + \frac{b^2}{a^2}\right) = \frac{b^2}{a}$

or $x = \frac{ab^2}{a^2 + b^2}$

and $\therefore$ $y = \frac{a^2b}{a^2 + b^2}$

Again $\frac{d^2u}{dx^2} = 2\left(1 + \frac{b^2}{a^2}\right)$, which is always $+$ ve whatever be the values of x and y,

$\therefore$ $\left(\frac{ab^2}{a^2 + b^2}, \frac{a^2b}{a^2 + b^2}\right)$ gives the minimum value of $x^2 + y^2$ as $\frac{a^2b^2}{a^2 + b^2}$.

110. Lagrange's Method of undetermined multipliers.* The method is only illustrated here.†

Let $u = \phi(x_1, x_2, x_3, \ldots\ldots x_n)$ ....(1)

be a function of n variables, which are connected say by m equations

$$\left.\begin{array}{l} f_1(x_1, x_2, x_3 \ldots . x_n) = 0 \\ f_2(x_1, x_2, x_3 \ldots . x_n) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots . \ \ldots\ldots\ldots\ldots\ldots\ldots \\ f_m(x_1, x_2, x_3, \ldots . x_n) = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

, so that only $n - m$ of the variables are independent. When u is a maximum or a minimum

$$\left.\begin{array}{rl} du &= \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ldots . + \frac{\partial u}{\partial x_n} dx_n = 0 \\ \text{Also } df_1 &= \frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 + \ldots . + \frac{\partial f_1}{\partial x_n} dx_n = 0 \\ df_2 &= \frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 + \ldots . + \frac{\partial f_2}{\partial x_n} dx_n = 0 \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ df_m &= \frac{\partial f_m}{\partial x_1} dx_1 + \frac{\partial f_m}{\partial x_2} dx_2 + \ldots . + \frac{\partial f_m}{\partial x_n} dx_n = 0 \end{array}\right\} \ldots (3)$$

Multiplying these equations by 1, λ_1, λ_2 etc., respectively and adding we get

$$\left(\frac{\partial u}{\partial x_1} + \lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_1} + \ldots . + \lambda_m \frac{\partial f_m}{\partial x_1}\right) dx_1$$
$$+ \left(\frac{\partial u}{\partial x_2} + \lambda_1 \frac{\partial f_1}{\partial x_2} + \lambda_2 \frac{\partial f_2}{\partial x_2} + \ldots . + \lambda_m \frac{\partial f_m}{\partial x_2}\right) dx_2$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0,$$

which can be written as

---

* Mecanique Analytique Vol. I.

† A rigorous proof of this is beyond the scope of the present volume,

$$P_1\, dx_1 + P_2\, dx_2 + \ldots . + P_r\, dx_r + \ldots . + P_n\, dx_n = 0 \ldots .(4)$$

where $P_r = \dfrac{du}{dx_r} + \lambda_1 \dfrac{df_1}{dx_r} + \lambda_2 \dfrac{df_2}{dx_r} + \ldots . + \lambda_m \dfrac{df_m}{dx_r}$.

Let us choose λ_1, λ_2, λ_3 etc., in such a way as to satisfy the m linear equations $P_1 = P_2 = P_3 = \ldots\ldots = P_m = 0$.

The equation (4) then becomes

$$P_{m+1}\, dx_{m+1} + P_{m+2}\, dx_{m+2} + \ldots . + P_n\, dx_n = 0$$

It is indifferent which $n - m$ of the variables are independent. Let then $x_{m+1}, x_{m+2}, \ldots . x_n$, be the independent variables. Then since dx_{m+1}, $dx_{m+2} \ldots .$are all independent we have $P_{m+1} = P_{m+2} = \ldots . = P_n = 0$.

Thus the $m + n$ equations in (2) and

$$P_1 = P_2 = P_3 = P_4 = \ldots . = P_n = 0$$

are sufficient to determine the m multipliers λ_1, $\lambda_2, \ldots .\ \lambda_m$, and the values of x_1, $x_2 \ldots . x_n$, for which the maxima and minima of u are possible.

Illustration.

Find the maxima and minima of $u = x^2 + y^2 + z^2$

$$\left.\begin{aligned} \text{where} \quad ax^2 + by^2 + cz^2 - 1 &= 0 \\ \text{and} \quad lx + my + nz &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(1)$$

By Lagrange's method.

$$xdx + ydy + zdz = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$axdx + bydy + czdz = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$ldx + mdy + ndz = 0 \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Multiplying (2), (3) and (4) by 1, λ_1 and λ_2 and then adding we get

$$(x + \lambda_1 ax + \lambda_2 l)dx + (y + \lambda_1 by + \lambda_2 m)dy + (z + \lambda_1 cz + \lambda_2 m)dz = 0$$

Again equating to zero the co-efficients of dx, dy and dz, we have

$$x + \lambda_1 ax + \lambda_2 l = 0 \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$y + \lambda_1 by + \lambda_2 m = 0 \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$$z + \lambda_1 cy + \lambda_2 n = 0 \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Multiplying (5), (6) and (7) by x, y and z respectively and using the relations in (1), we have

$$u + \lambda_1 = 0 \quad \therefore \lambda_1 = -u$$

$\therefore$ from (5), (6) and (7) we get

$$x = \frac{\lambda_2 l}{au - 1}$$

$$y = \frac{\lambda_2 m}{bu - 1}$$

$$z = \frac{\lambda_2 n}{cu - 1}$$

Again since $lx + my + nz = 0$

we get $\dfrac{l^2}{(au - 1)} + \dfrac{m^2}{(bu - 1)} + \dfrac{n^2}{(cu - 1)} = 0,$

a relation giving maximum and minimum values of u.

EXAMPLES ON CHAPTER X.

Find the maxima and minima values of—

1. $x^3 - 6x^2 + 11x - 6$

2. $\dfrac{2x^3 + 3x^2 + 8}{x^2}.$

3. $\dfrac{7x^4 - 30x^3 + 11x^2 - 8}{x^2}.$

4. $\dfrac{3x^2 - a^2}{(a^2 + x^2)^3}.$

5. $4 \cos x + \cos 2x.$

6. If $x + y = k$, a constant, find the maximum value of x^3y^2, all the quantities being positive.

7. Find the maximum and minimum value of $x^2 + y^2$, where

$$ax^2 + 2hxy + by^2 = 1.$$

8. The portion of a tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, intercepted between the axes is a minimum ; find its length.

9. Shew that the shortest normal chord of the parabola $y^2 = 4ax$ is $6a\sqrt{3}$.

10. Divide a given k, into two parts such that the product of the m^{th} power of one and the n^{th} power of the other shall be a maximum.

11. Given the length of an arc of a circle find the radius of the circle when the corresponding segment has a maximum or a minimum area.

12. Find the maximum cone of given slant height.

13. Shew that the altitude of the cylinder of maximum volume that can be inscribed in a sphere of radius r is $\frac{2r}{\sqrt{3}}$.

14. Shew that the curved surface of a cylinder inscribed in a sphere of radius r is a maximum when the altitude of the cylinder is $\sqrt{2}\,r$.

15. A cylinder is to be constructed and its total surface is to be A square inches. Shew that the altitude of the cylinder of greatest volume is twice the radius of its base.

16. Two particles move uniformly along the axes of x and y with velocities u and v. They are initially at distances a and b from origin; shew that the least distance between the particles is $\frac{av - bu}{\sqrt{u^2 + v^2}}$.

17. From a fixed point A on the circumference of a circle of radius a, the perpendicular AY is let fall on the tangent at P. Prove that the greatest area that APY can have

is $\frac{3\sqrt{3}}{8}a^2$.

18. If the sum of the edges of a rectangular parallelopiped is l and the sum of the areas of the faces is $\frac{l^2}{25}$; shew that when the excess of the volume of the parallelopiped over that of a cube whose edge is its smallest edge is maximum, the, smallest edge must be $\frac{l}{20}$, and find the lengths of the other edges.

[Tripos 1903].

19. The circle of curvature at a point P of a parabola meets the curve again at Q, and its centre is C. Prove that the area of the triangle CPQ is a maximum when CP makes with the axis an angle $\tan^{-1}\dfrac{1}{\sqrt{5}}$.

[Tripos 1901].

20. Find the maximum and minimum values of

$$u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}, \text{ when}$$

$$lx + my + nz = 0$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^3} + \frac{z^2}{c^2} = 1.$$

[Oxford 1888].

21. Find the minimum value of

$$u = x^2 + y^2 + z^2, \text{ with the conditions}$$

$$ax + by + cz = 1$$

$$a'x + b'y + c'z = 1.$$

CHAPTER XII.

ASYMPTOTES.

111. Rectilinear asymptotes.—An *asymptote* to a curve is the limiting position of a tangent, whose point of contact tends to an infinite distance from the origin.

Thus it is clear that the line cannot wholly lie at infinity *i.e.*, it must pass within a finite distance from the origin.

Also a curve which has no infinite branch cannot have a real asymptote.

112. To find the asymptotes of f (x, y) = 0, where $f(x, y)$ is a polynomial of degree n in x and y.

Let the equation of any line be

$$y = mx + c \text{.....................} (1)$$

Where (1) cuts the curve, $f(x, mx + c) = 0$......(2)

Suppose (2) arranged in descending powers of x, say

$$A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} \text{........}$$
$$+ Ao = O \text{.......................} (3)$$

Which gives in general n points of intersection of (1) with the curve.

If two roots of (3) are infinite

$$A_n = 0, \text{ and } A_{n-1} = 0 \text{...............} (4)$$

A_n will be a function of m of degree $\ngtr n$ and A_{n-1} a function of m and c, the degree of c being unity. The values of m derived from $A_n = 0$, and the corresponding values of c from $A_{n-1} = 0$ when substituted in (1) will give the equations of asymptotes.

$$\text{Thus} \quad y = m_1 x + c_1$$
$$y = m_2 x + c_2$$
$$\text{.............}$$

are the asymptotes.

Cor. 1. It is evident that a curve of n^{th} degree can have only n asymptotes all real, or all imaginary, or some real and some imaginary. But since imaginary roots occur in pairs, if n be odd there is at least one root real, *i. e.*, one asymptote real or in other words, the curve must extend to infinity and thus cannot be altogether a closed one.

113. If however any value of m, say m_1 derived from $A_n = 0$, makes A_{n-1} identically zero, every straight line parallel to $y = m_1x$, will cut the curve in two points infinitely removed from the origin. In such cases the value of c is obtained by equating A_{n-2} to zero.

i. e., $A_{n-2} = 0$, corresponding to the value m_1. This in general being a quadratic in c will give two values of c and thus the lines say

$$\left.\begin{array}{r} y = m_1\,x + c_1 \\ \text{and }\ y = m_1\,x + c_2 \end{array}\right\}(1),$$

each will cut the curve at three points infinitely removed from the origin. The name asymptote then is confined to these two lines only.

If however, $A_{n-2} = 0$ fails to give c, we find it from $A_{n-3} = 0$ and so on.

Illustration.

To find the asymptotes of

$$y^3 - x^2y + 2y^2 + 4y + x = 0\ldots\ldots\ldots\ldots(1)$$

Let $y = mx + c\ldots\ldots\ldots\ldots\ldots\ldots(2)$

be an asymptote.

Substituting for y in (1) and arranging in descending powers of x, we get

$$(m^3 - m)\,x^3 + (3m^2c - c + 2m^2)\,x^2$$
$$+ (3mc^2 + 4mc + 4m + 1)\,x + c^3 + 2c^2 + 4c = 0.$$

Thus $m^3 - m = 0$ *i. e.*, $m = 0$, or 1, or -1

and $3m^2c - c + 2m^2 = 0$ *i, e.*, $c = 0$, or -1, or -1.

Thus the asymptotes are

$$\left.\begin{aligned} y &= 0 \\ y - x + 1 &= 0 \\ y + x + 1 &= 0 \end{aligned}\right\}.$$

114. Another method which is more practicable can be adopted.*

Let the equation of any curve of the n^{th} degree be so arranged that the homogeneous sets of terms be expressed together, say,

$$x^n \phi_n \left(\frac{y}{x}\right) + x^{n-1} \phi_{n-1} \left(\frac{y}{x}\right) + x^{n-2} \phi_{n-2} \left(\frac{y}{x}\right)$$
$$+ \ldots\ldots\ldots = 0 \ldots\ldots\ldots(1)$$

Let the equation of an asymptote be

$$y = \mu x + \beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substituting $\mu + \frac{\beta}{x}$ for $\frac{y}{x}$ in (1), we get

$$x^n \phi_n \left(\mu + \frac{\beta}{x}\right) + x^{n-1}\phi_{n-1} \left(\mu + \frac{\beta}{x}\right) +$$
$$x^{n-2}\phi_{n-2} \left(\mu + \frac{\beta}{x}\right) + \ldots\ldots\ldots\ldots = 0 \ldots.(3)$$

This gives the n points of intersection.

Applying Taylor's theorem, the equation (3) can be arranged as

$$x^n \phi_n(\mu) + x^{n-1} [\beta \phi'_n(\mu) + \phi_{n-1}(\mu)] +$$
$$x^{n-2} \left[\frac{\beta^2}{\underline{|2}} \phi''_n(\mu) + \beta \phi'_{n-1}(\mu) + \phi_{n-2}(\mu) \right]$$
$$+ \ldots. = 0 \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Thus $\phi_n (\mu) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$

and $\beta \phi'_n (\mu) + \phi_{n-1} (\mu) = 0 \ldots\ldots\ldots\ldots(6)$

If $\mu_1, \mu_2, \mu_3 \ldots.\mu_n$ are the n roots of (5), the corresponding values of β are given by

---

* Edward's method.

$$\beta_1 = -\frac{\phi_{n-1}(\mu_1)}{\phi'_n(\mu_1)},$$

$$\beta_2 = -\frac{\phi_{n-1}(\mu_2)}{\phi'_n(\mu_2)}, \text{ and so on.}$$

The n asymptotes are

$$\left.\begin{array}{l} y = \mu_1 x + \beta_1 \\ y = \mu_2 x + \beta_2 \\ \cdots\cdots\cdots\cdots \\ y = \mu_n x + \beta_n . \end{array}\right\}$$

115. This method is adopted, as $\phi_n(\mu)$ and $\phi_{n-1}(\mu)$ etc., can be very easily found out without actually substituting the value of y in the equation of the curve and thus avoiding a laborious process. Thus assuming the result of the § 114, we may adopt the following procedure:—

In the highest degree terms put $x = 1$, and $y = \mu$, the object of this is to form $\phi_n(\mu)$, and equate it to zero. Form $\phi_{n-1}(\mu)$ in the same way *i e*, substitute $x = 1$ and $y = \mu$ in the $(n - 1)^{th}$ degree terms. Differentiating $\phi_n(\mu)$ with respect to μ, the values of β are determined by substituting the values of μ in the formula

$$\beta = -\frac{\phi_{n-1}(\mu)}{\phi'_n(\mu)} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Illustration.

Find the asymptotes of

$$x^3 - xy^2 + 2x^2y - 2y^3 - x^2 + y^2 - 3x$$
$$-4y - 5 = 0.$$

Here $\phi_3(\mu) \equiv 1 + 2\mu - \mu^2 - 2\mu^3 = 0$

$i.\ e., (\mu + 1)(\mu - 1)(2\mu + 1) = 0$

$i.\ e., \mu = -1, 1, \text{ and } -\frac{1}{2}.$

Again $\phi_2(\mu) \equiv \mu^2 - 1$

and $\phi'_3(\mu) \equiv 2 - 2\mu - 6\mu^2$

$$\therefore \beta = \frac{\mu^2 - 1}{6\mu^2 + 2\mu - 2}.$$

Hence if $\mu = -1, \beta = 0$,
if $\mu = 1, \beta = 0$
and if $\mu = -\frac{1}{2}, \beta = \frac{1}{2}$

$$\therefore \text{ The asymptotes are } \left.\begin{array}{r} y - x = 0 \\ y + x = 0 \\ \text{and} \quad 2y + x - 1 = 0 \end{array}\right\}.$$

116. I. Suppose two roots of the equation $\phi_n(\mu) = 0$ to be equal to each other *i. e.*, say $\mu_1 = \mu_2$, then $\phi'_n(\mu_1) = 0$. If $\phi_{n-1}(\mu)$ does not contain $(\mu - \mu_1)$ as one of its factors, β becomes infinite.

The line $y = \mu_1 x + \beta_1$ therefore makes an infinite intercept on the axis of y, *i. e.*, the line lies wholly at infinity. This is not an asymptote, although it will form as one of the n theoretical asymptotes of the curve.

II. If $\phi_n(\mu) = 0$ has two equal roots *i. e.*, $\mu_1 = \mu_2$, and thus $\phi'_n(\mu_1) = 0$ also, and if $\phi_{n-1}(\mu)$ also contains $(\mu - \mu_1)$ as one of its factors, the value of β cannot be determined by (1) of § 114 We may select β so that the co-efficient of x^{n-2}

$$i.\ e.,\ \frac{\beta^2}{2}\phi''_n\ (\mu) + \beta\ \phi'_{n-1}\ (\mu\ + \phi_{n-2}\ (\mu) = 0,$$

and from which two values of β may be deduced, getting two parallel asymptotes

$$y = \mu_1 x + \beta_1$$

$$\text{and } y = \mu_1 x + \beta_2.$$

Compare (1) of § 113.

117. Assymptotes parallel to the axes.—If there is no such term as y^n in the given equation, the term m^n in $A_n = 0$ will be missing and consequently $A_n = 0$ will give one root infinite and only $(n - 1)$ finite roots of m, *i. e.*, the asymptote corresponding to the infinite value of m, will be parallel to the axis of y, and hence its equation will be of the form $x = d$. § 111 fails to give the equation of such asymptotes.

Let the equation of a given curve of degree n be arranged in the following manner:—

$$\begin{aligned} &a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots\ldots\ldots + a_n y^n \\ &\quad + b_1 x^{n-1} + b_2 x^{n-2} y + \ldots\ldots\ldots + b_n y^{n-1} \\ &\qquad + c_2 x^{n-2} + c_3 x^{n-3} y + \ldots + c_n y^{n-2} \\ &\qquad \ldots\ldots\ldots\ldots \text{etc.,} \ldots\ldots\ldots\ldots\ldots\ldots \\ &\qquad\qquad + K = 0 \ldots\ldots\ldots (1) \end{aligned}$$

Arranging it in descending powers of x, we get

$$\begin{aligned} &a_0 x^n + (a_1 y + b_1) x^{n-1} + (a_2 y^2 + b_2 y + c_2) x^{n-2} \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + K = 0 \ldots\ldots\ldots\ldots (2) \end{aligned}$$

If $a_0 = 0$, and y be so chosen that $a_1 y + b_1 = 0$, two roots of (2) will be infinite and hence $a_1 y + b_1 = 0$, is an asymptote to the curve parallel to the axis of x. If however $a_0 = 0$, $a_1 = 0$ and $b_1 = 0$, y can be so chosen that $a_2 y^2 + b_2 y + c_2 = 0$. Then three roots of (2) will be infinite and hence the two parallel asymptotes given by $a_2 y^2 + b_2 y + c_2 = 0$, will represent a pair of real or imaginary asymptotes parallel to the axis of x.

In the same way arranging (1) in descending powers of y, and if $a_n = 0$, $a_{n-1} x + b_n = 0$ is an asymptote parallel to the axis of y. Again if a_{n-1} is also equal to zero, as well as b_n,

$a_{n-2} x^2 + b_{n-1} x + c_n = 0$, gives a pair of real or imaginary asymptotes parallel to the axis of y.

If however $a_0 = a_1 = b_1 = a_2 = b_2 = c_2 = 0$

(or $a_n = a_{n-1} = b_n = b_{n-1} = a_{n-2} = c_n = 0$),

the asymptotes will be given by equating to zero the coefficient of x^{n-3} (or y^{n-3}).

Thus the asymptotes parallel to the axes are found by equating to zero the coefficients of the highest powers of x and y, in the given equation of the curve.

Illustration.

Find the asymptotes of

$$x^2y^2 = a^2(x^2 + y^2).$$

The highest power of x is x^2 and that of y is y^2.

The co-efficient of x^2 is $y^2 - a^2$ and that of y^2 is $x^2 - a^2$.

∴ The asymptotes are $\left.\begin{array}{l} x = \pm a \\ y = \mp a \end{array}\right\}.$

The equation being of 4^{th} degree, all the asymptotes to the curve have been found.

Exercises.

Find the asymptotes of :—

1. $y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0.$
2. $x^3 + 2x^2y + xy^2 - x^2 - xy + 2 = 0.$
3. $x^2y^2 = a^2(x^2 - y^2).$
4. $(x^2 - y^2)^2 - 2a^2(x^2 + y^2) + a^3x - a^4 = 0.$
5. $y^3 - 5xy^2 + 8x^2y - 4x^3 - 3y^2 + 9xy$
   $- 6x^2 + 2y - 2x - 1 = 0.$

118. If the equation of the curve $f(x, y) = 0$ can be put in the form

$$P_n + F_{n-2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where F_{n-2} is a function of x and y, and of degree $\ngtr n - 2$, and P_n is a polynomial in x and y of degree n, and if P_n can then be broken into linear factors, all different, each of the factor equated to zero will give an asymptote to the curve. Since the degree of $f(x, y)$ will be lowered by two, each of these lines will cut the curve in two points infinitely removed from the origin.

Cor. Each of the asymptotes given by $P_n = 0$ will cut the curve again in $(n - 2)$ other points which will satisfy both $P_n = 0$ and $P_n + F_{n-2} = 0$.

∴ $F_{n-2} = 0$ gives the locus of $n_{(n-2)}$ points of intersection of the asymptotes and the curve.

Illustration.

Find the asymptotes of

$$(x + y + 1)(2x + 3y + 4)(x - y)(x + 2y - 1) - 3x^2 - 4y - 3x - 5 = 0.$$

Here the asymptotes are

$$\left.\begin{array}{r} x + y + 1 = 0 \\ 2x + 3y + 4 = 0 \\ x - y = 0 \\ x + 2y - 1 = 0 \end{array}\right\}.$$

119. Let $f(x, y) = 0$ be arranged, if possible, into the form

$$(ax + by + c)\, P_{n-1} + F_{n-1} = 0$$

Any straight line parallel to $ax + by = 0$ cuts the curve at one point infinitely removed from the origin. We have to find that particular line out of all these parallel lines, which cuts the curve at a second point also infinitely removed from the origin To find this we make x and y of the curve become larger and larger in the ratio

$$\frac{x}{y} = -\frac{b}{a} \quad \text{and we get}$$

$$ax + by + c + \underset{y = -\frac{a}{b} x \to \infty}{\mathrm{Lt}} \left(\frac{F_{n-1}}{P_{n-1}}\right) = 0 \,..(1)$$

in which if the limit exists, it gives the ultimate linear form to which the curve approximates as we travel along the curve farther and farther away from the origin. The equation (1) then represents an asymptote. It will be more convenient to substitute $x = -\frac{b}{t}$ and $y = \frac{a}{t}$ and then make $t \to 0$ in the limit in the equation (1).

Illustration.

To find the asymptotes of

$$(x + y)(x^4 + y^4) - a(x^4 + a^4) = 0$$

$$x + y - \underset{y = -x \to \infty}{\mathrm{Lt}} a\,\frac{x^4 + a^4}{x^4 + y^4} = 0.$$

Putting $x = \frac{1}{t}$ and $y = -\frac{1}{t}$, we have

$$x + y - \underset{t \to 0}{\text{Lt}}\, a \frac{\frac{1}{t^4} + a^4}{\frac{1}{t^4} + \frac{1}{t^4}} = 0.$$

or $2x + 2y - a = 0.$

120. The same reasoning will show that if the curve is of the form

$$(ax + by + c)^2 P_{n-2} + F_{n-2} = 0$$

$ax + by + c = \pm \sqrt{\text{Lt}\, \frac{F_{n-2}}{P_{n-2}}}$, where x and y

tend in the limit to infinity in the ratio $\frac{y}{x} = -\frac{a}{b}$.

Cor. 1. Similarly if the form be

$$(ax + by)^2 P_{n-2} + (ax + by) F_{n-2} + \phi_{n-2} = 0.$$

Taking the limit in the direction $ax + by = 0$, we have

$$(ax + by)^2 + (ax + by).\ \text{Lt}\frac{F_{n-2}}{P_{n-2}} + \text{Lt}\frac{\phi_{n-2}}{P_{n-2}} = 0$$

we thus get a pair of parallel asymptotes

$$ax + by = r_1$$

and $ax + by = r_2$

where r_1 and r_2 are the roots of the equation

$$r^2 + r\,\text{Lt}\,\frac{F_{n-2}}{P_{n-2}} + \text{Lt}\,\frac{\phi_{n-2}}{P_{n-2}} = 0.$$

121. Curvilinear Asymptotes. If there be two curves which continually approach each other so that for a common abscissa, the limit of the difference between the ordinates is zero; or for a common ordinate, the limit of the difference between the abscissae is zero, when the common abscissa or common ordinate tends to infinity, each curve is said to be a curvilinear asymptote of the other. For instance if,

$$y = \frac{x^2}{a} - \frac{a^2}{x} \text{.....................} (1)$$

and $$y = \frac{x^2}{a} \text{..........................} (2)$$

be two curves, for any value of x the difference between the ordinates is $\frac{a^2}{x}$, which vanishes in the limit when $x \rightarrow \infty$. Hence the curves are asymptotic.

122. If however the equation of a curve can be put in the form

$$y = Ax + B + \frac{C}{x} + \frac{D}{x^2} + \text{..........}$$

$y = Ax + B$ is evidently a rectilinear asymptote. This method of finding the asymptote also indicates on which side of the asymptote the curve lies.

123. Polar Coordinates:—Let the equation of a curve be

$$r^n f_n(\theta) + r^{n-1} f_{n-1}(\theta) + \ldots + f_0(\theta) = 0 \text{........} (1)$$

Or Putting $r = \frac{1}{u}$,

$$f_n(\theta) + u f_{n-1}(\theta) + u^2 f_{n-2}(\theta) + \ldots + u^n f_0(\theta) = 0 \ldots (2)$$

The directions in which r becomes infinite or u zero, are given by

$$f_n(\theta) = 0 \text{...........................} (3)$$

and let the roots be θ_1, θ_2, θ_3 etc.

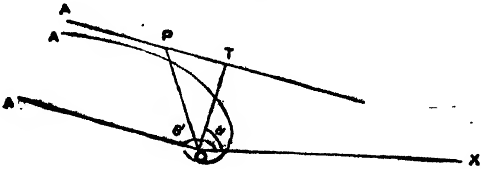


FIG. 41.

Let the angle XOA $= \theta_1$, the point where the asymptote PA meets the curve will lie in the direction OA. Let OT be drawn at right angles to the asymptote from the origin. Since OT is at right angle to PA and hence to OA also, OT is the polar subtangent

$$\therefore \quad \text{OT} = -\frac{d\theta}{du} = p \text{ say} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

If therefore the angle XOT $= \alpha$ and P (r, θ) any point on the asymptote, its equation is

$$p = r\cos(\theta - \alpha) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\text{Again } \alpha = \theta_1 - \frac{\pi}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Also we have to find the value of p *i.e.* $-\dfrac{d\theta}{du}$ when $u = 0$.

Differentiating (2) with respect to θ, and then putting $u = 0$, *i.e.*, $\theta = \theta_1$, we get

$$f'_n(\theta_1)\frac{d\theta}{du} + f_{n-1}(\theta_1) = 0$$

$$\textit{i. e.}, \ -\frac{d\theta}{du} = \frac{f_{n-1}(\theta_1)}{f'_n(\theta_1)} \ldots\ldots\ldots\ldots(7)$$

Substituting the values from (6) and (7) in (5) we get

$$r\cos\left(\theta - \theta_1 + \frac{\pi}{2}\right) = \frac{f_{n-1}(\theta_1)}{f'_n(\theta_1)}$$

$$\text{or} \quad r\sin(\theta_1 - \theta) = \frac{f_{n-1}(\theta_1)}{f'_n(\theta_1)}.$$

The remaining asymptotes are

$$r\sin(\theta_2 - \theta) = \frac{f_{n-1}\ \theta_2)}{f'_n(\theta_2)}$$

etc.

Cor. 1. If the equation is

$r\,f_1(\theta) + f_0(\theta) = 0$, the asymptotes are given by

$$r\sin(\theta_1 - \theta) = \frac{f_0(\theta)}{f'_1(\theta)}$$

etc.

where θ_1, θ_2, etc., are the values found from $f_1(\theta) = 0$.

Illustration.

Find the asymptotes of

$$r \sin 2\theta - a \cos 3\theta = 0$$

Here $f_1(\theta) \equiv \sin 2\theta = 0$

$$\therefore \quad 2\theta = 0, \pi, 2\pi \text{ etc.},$$

$$i.\,e., \quad \theta = \frac{n\pi}{2}.$$

The asymptote is

$$r \sin\left(\frac{n\pi}{2} - \theta\right) = -\left[\frac{a \cos 3\,\theta}{2 \cos 2\,\theta}\right]_{\theta} = \frac{n\pi}{2}$$

(*i*) when $n = 0$

$$r \sin \theta = \frac{a}{2}$$

(*ii*) when $n = 1,\ \theta = \frac{\pi}{2}$.

For other integral values of n we get the same two straight lines.

Hence the two asymptotes are

$$\theta = \frac{\pi}{2} \text{ and } r \sin \theta = \frac{a}{2}.$$

124. If r tends to a constant say c, when $\theta \to \infty$, the limiting equation of the curve, which takes the form $r = c$, is called an asymptotic circle of the curve.

Thus $r = a$ is an asymptotic circle of the curve

$$r = \frac{a\,\theta}{1 + \theta}.$$

EXAMPLES ON CHAPTER X.

Find the asymptotes of

1. $x^3 + y^3 = 3axy$.
2. $x^3y + 2x^2y^2 + xy^3 = a^2x^2 + b^2y^2$.
3. $y^2(x - 1)(x - 2) = x^2 + 3$.
4. $(x^2 - y^2)^2 = 2(x^2 + y^2)$.

5. $xy^3 = a^3 (a + x).$

6. $y = \dfrac{x^3 + ax^2 + a^3}{x^2 - a^2}.$

7. $x^5 + y^5 - 5a^2x^2y^2 = 0.$

8. $xy^2 = 4a^2 (2a - x)$ (Witch).

9. $y^2 (2a - x) = x^3.$ (Cissoid).

10. $r = \dfrac{a \; \theta^2}{\theta^2 - 1}.$

11. $r \, \theta \cos \theta = a \cos 2\theta$ [Oxford 1889].

12. $r^n \sin n \, \theta = a^n.$

13. $r \, \theta \cos \theta = ae^{\theta.}$

14. $r = 2a \sin \theta \tan \theta.$

15. $r = a + b \cot (n \, \theta).$

16. Shew that there is an infinite series of parallel asymptotes to the curve $r = \dfrac{a}{\theta \sin \theta} + b$; and shew that their distances from the pole are in Hormonical progression. Find also the asymptotic circle.

17. Shew that all the asymptotes of the curve

$r\tan n\theta = a$, touch the circle $r = \dfrac{a}{n}$.

18. If $u = f(\theta)$ be the equation of a curve and $f(\theta) = 0$ gives a root $\theta = \alpha$, the corresponding asymptote is

$$y = x \tan \alpha + \frac{\sec \alpha}{f'(\alpha)}$$

19. Prove that the curve

$y = x \dfrac{(x^2 + a^2)}{x^2 - a^2}$, lies above its oblique asymtote in the first quadrant.

20. Find the asymptotes of

$$r (\sin \alpha - \theta) = a \sin \alpha \cos \theta,$$

and axamine the case when α is a right angle.

(Wolstenholme.).

CHAPTER XIII.

CURVE TRACING.

125. Experience will tell how to trace any given curve by an easiest possible way and without going through any laborious processes. But the following general method may be adopted in cartesian equations.

Let us find out .—

I. Symmetry in the curve :—

(1) If there are all even powers of x, the curve is symmetrical about the axis of y.

(2) If there are all even powers of y, the curve is symmetrical about the axis of x.

(3) If all the powers of x and y both, are even, the curve is symmetrical about both the axes.

(4) If for x and y, we substitute $-x$, and $-y$, and the equation of the curve does not change, there is symmetry in opposite quadrants.

(5) If x and y are interchanged and the equation of the curve does not change, there is symmetry about the straight line $y = x$.

II. If the curve passes through the origin, find the tangents there. Also the points of intersection with the axes, or any point which obviously seems to be lying on the curve.

III. Find the rectilinear asymptotes if any.

IV. Find the double points if any and their nature, and also points of inflexion if any.

V. Find at what points $\dfrac{dy}{dx}$ vanishes or becomes infinite.

VI. Find the region, if possible, within which the curve exists.

126. Before proceeding to investigate any curve, it will be sometimes found more useful to apply Newton's method of finding the nature of the curve at the origin, which is stated here without proof.

Let the curve be given by

$$Ax^4y + Bx^3 + Cxy + Dy^3 + Exy^4 = 0 \ldots\ldots\ldots\ldots (1)$$

There is no constant term, since the curve is to be taken as passing through the origin. Also all the powers of x and y are taken positive which is always possible to take. Let us plot all the points corresponding to the indices of the terms in (1). That is let us plot the points (4, 1), (3, 0), (1, 1), (0, 3) and (1, 4), and let these points be called P, Q, R, S and T on the graph. Now let us draw a straight line through any two points out of those points, such that the origin may be on the other side of this straight line than the remaining points. That is, the origin may be on one side and the remaining, (in this case three), points may

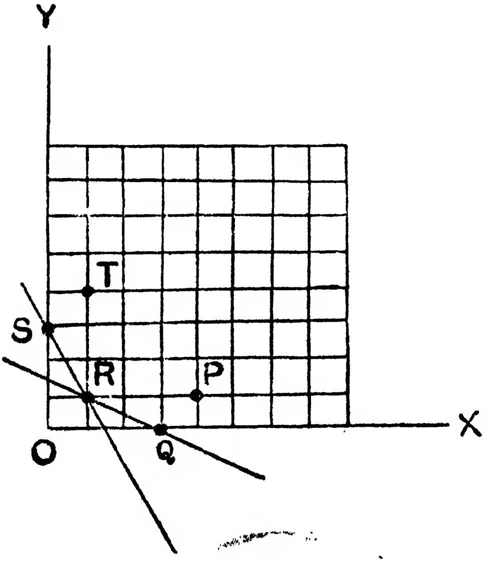


FIG 42.

be on the other side. Care must be taken that we do not take that straight line into account which passes through the origin. In the present case we get two such straight lines *i. e.*, RQ and SR. Let us take RQ first. Taking the terms corresponding to the points R and Q *i. e.*, $Bx^3 + Cxy$, we equate this to zero and this will give the nature of the curve in the neighbourhood of the origin. The student should note that this is not the form of the curve, but this is the curve to which the given curve approximates in the vicinity of the origin. Thus $Bx^3 + Cxy = 0$ gives us the nature of the curve *i. e.*, $(Bx^2 + Cy = 0$.

Similarly $(Cx + Dy^2) = 0$ also gives the nature of the curve.

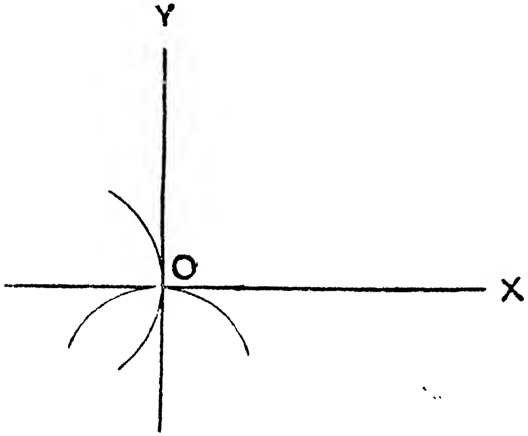
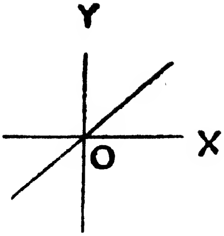


Fig. 43.

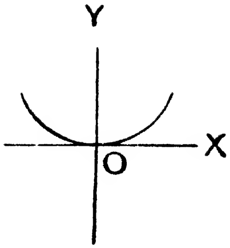
This is a very convenient way of finding the directions in which the curve deviates from the origin.

Cor. 1. If the curve does not pass through the origin, we can shift the origin to any point lying on the curve and then find out the nature of the curve at that point.

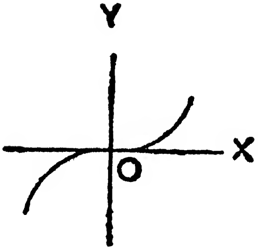
127. In order that Newton's method may be very easily applied the student is advised to be familiar with the following curves, as one of these or more may be the natures found in the case of any given curve.



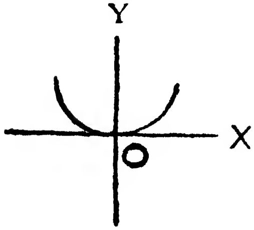
(1) $y = x$
Straight line.



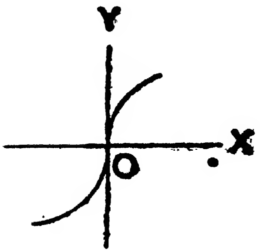
(2) $y = x^2$
Parabola.



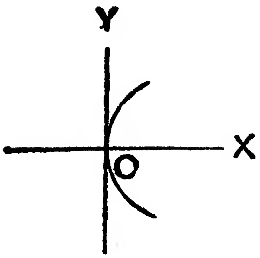
(3) $y = x^3$
Cubical Parabola
nflexion at O.



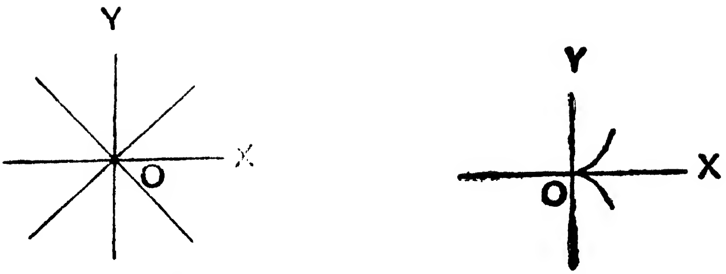
(4) $y = x^4$
Undulation at O.



5) $y = x^5$
Inflexion at .

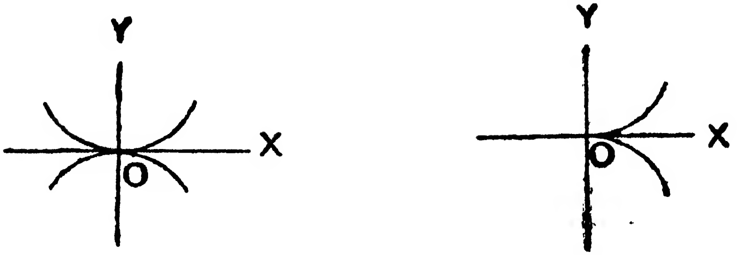


(6) $y^2 = x$
Parabola.



(7) $y^2 = x^2$
Pair of st. lines.

(8) $y^2 = x^3$
Semi-cubical Parabola cusp
at O.



(9) $y^2 = x^4$
Two Parabolas.

(10) $y^2 = x^5$
Cusp at O.

(11) $y^3 = x$
Cubical parabola
inflexion at O.

(12) $y^3 = x^2$
Semi-cubical parabola
cusp at O.

FIG. 44.

The student will be well advised to trace all the twelve curves given above himself.

Illustrations.

I. To trace the curve

$$a^3y^2 - 2ab\,x^2y - x^5 = 0.$$

1. It passes through the origin.

2. $x^2 = \frac{a^2}{2b}y$ and $x^3 = -2aby$ give the natures at the origin.

3. It does not cross the axes at any other point.

4. $y = \frac{x^2}{a^2}\left[b \pm \sqrt{b^2 + ax}\right]$, thus when x is $+ve$ infinity, y has two infinite values.

5. There is no real value of y if $x < -\frac{b^2}{a}$, *i.e.*, the curve does not exist beyond $x = -\frac{b^2}{a}$.

6. No real asymptotes.

Thus the curve is as in figure 45.

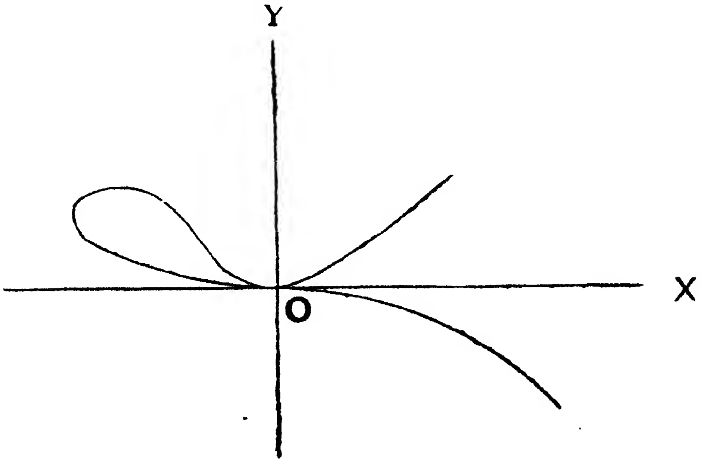


Fig. 45.

II. Trace the curve $x = (y - 1)(y - 2)(y - 3)$.

1. It does not pass through the origin, but cuts the axis of y at 1, 2, 3, and the axis of x at $- 6$.

2. If y lies between 0 and 1, x is $-$ *ve*.

   If y lies between 1 and 2, x is $+$ *ve*.

   If y lies between 2 and 3, x is $-$ *ve*.

   If y is greater than 3, x is $+$ *ve*

   and if $y \rightarrow \infty$, $x \rightarrow \infty$, as also if $y \rightarrow - \infty$, x also $\rightarrow - \infty$.

3. There are no real asymptotes.

4. There is a point of inflexion at (0, 2).

5. $\frac{dx}{dy}$ vanishes for values of y given by $3y^2 - 12y + 11 = 0$, *i. e.*, one value is between 1 and 2 and the other value is between 2 and 3.

The curve is therefore as in Figure 46.

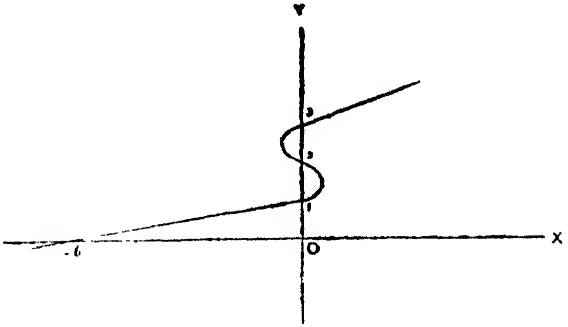


Fig. 46.

III. Trace the curve $x^3 + y^3 - 3axy = 0$.

1. The curve is symmetrical about the line $y = x$.

2. $x + y + a = 0$ is the only real asymptote.

3. The curve lies above the asymptote.

4. The curve passes through the origin and the nature is given by

$$\left.\begin{aligned} x^2 &= 3ay \\ y^2 &= 3ax \end{aligned}\right\}.$$

5. It does not cross the axes at any other point.

6. $\frac{dy}{dx}$ vanishes at points where it is intersected by the curve $x^2 = ay$, and $\frac{dy}{dx}$ is infinite where it is intersected by the curve $y^2 = ax$.

7. It intersects the line $y = x$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, where the tangent to the curve is at right angles to $y = x$.

8. Transforming to polar coordinates, we notice that the radius vector is never greater than $\frac{3a}{\sqrt{2}}$ in the first quadrant and it is zero when θ is 0 or $\frac{\pi}{2}$.

Thus the curve has a loop in the first quadrant.

The curve is as in Figure 47.

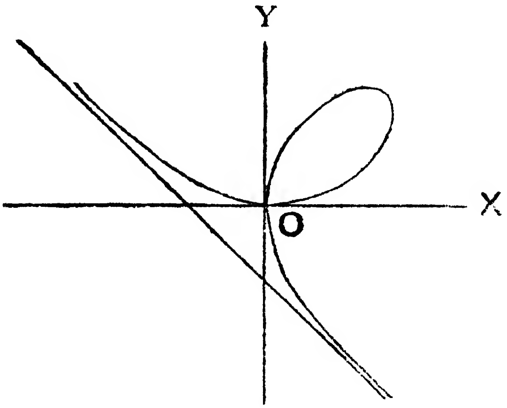


FIG. 47.

IV. To trace the curve

$$ay^2 = x^2y + x^3. \qquad \text{[A. U. B. Sc. 1930].}$$

1. There is no symmetry.

2. $x + y = a$ is the asymptote.

3. The curve passes through the origin, intersects the asymptote at $\left(\frac{a}{2}, \frac{a}{2}\right)$.

4. There is a cusp of first species at the origin.

5. $y = \frac{x^2 \pm x\sqrt{x^2 + 4ax}}{2a}$, and thus we find that the curve does not exist between the ordinates $x = 0$ and $x + 4a = 0$.

6. The curve passes through the point $(-4a, 8a)$, where if we shift the origin, we find the nature of the curve as

$$y^2 + 16a\,x = 0.$$

7. $\frac{dy}{dx}$ is zero at $\left(-\frac{9}{2}a, \frac{27}{4}a\right)$.

The curve is as below.

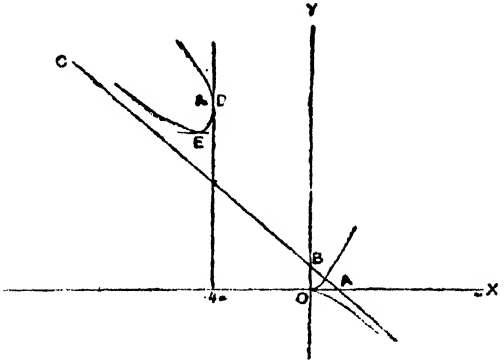


FIG. 48.

126. Polar coordinates.—The following general method may be adopted :—

I. Form a table of corresponding values of r and θ, which satisfy the curve.

II. If possible find maximum and minimum values of r. Sometimes the limits of r or θ are ascertained.

III. Notice symmetry if there is any :—

(1) If θ is changed to $-\theta$, and the equation does not change, the curve is symmetrical about the initial line.

(2) If only even powers of r exist in the equation, the curve is symmetrical about the Pole.

IV. Find tan ϕ, and thus the points where a tangent to the curve is perpendicular to the radius vector or the radius vector itself is the tangent to the curve. The conditions will be $\tan \phi = \infty$ or $\tan \phi = 0$ respectively.

V. Find rectilinear asymptotes if any.

VI. Find if there is an asymptotic circle.

VII. Find points of inflexion if any.

Illustrations.

1. Trace the curve, $r = a \cos 3\theta$.

1. The curve is symmetrical about the initial line.

2. $r \ngtr a$ and it passes through the origin.

3.

$\theta =$	0	$\frac{\pi}{6}$	$2\frac{\pi}{6}$	$3\frac{\pi}{6}$	$4\frac{\pi}{6}$	$5\frac{\pi}{6}$	π
$r =$	a	0	$-a$	0	a	0	$-a$

r for negative values of θ need not be found out as the curve is symmetrical about the initial line.

4. Radius vactor is tangent when $\theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}$, and it is at right angles to the tangent when

$$\theta = 0, \frac{2\pi}{6}, \frac{4\pi}{6}.$$

5. It has no asymptotes.

The curve is as shown in figure 49.

There are 3 loops in the curve.

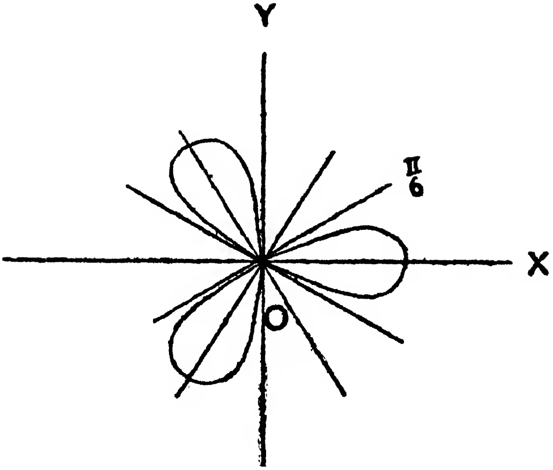


FIG. 49.

Note.—$r \cos 3\theta = a$ is the inverse curve of $ra \cos 3\theta$. The student is advised to draw it himself as well as $r = a \cos 2\theta$, in which it will be noticed that there are 4 loops.

II. Trace the curve $r = \frac{a\theta^2}{\theta^2 + 1}$.

1. The curve is symmetrical about the initial line.

2. There is an asymptotic circle $r = a$ within which the curve lies.

3. There is a cusp of the first species at the origin, and the initial line is the tangent at the origin.

The curve is as shown in the Figure 50.

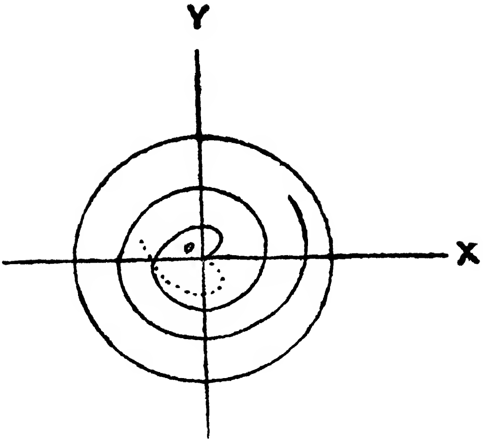


Fig. 50.

128. **Cycloid.**—When a circle rolls in a plane along a given straight line, the locus traced out by any point on the circumference of the rolling circle is called a *cycloid.*

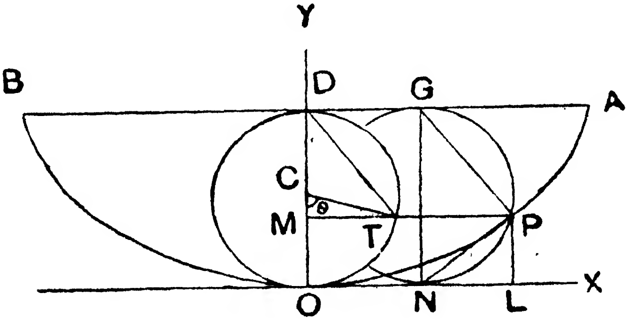


Fig. 51.

Let a circle of radius a roll on the straight AB in plane of (x, y), and A be the point where P, which traces the locus, was in contact when it started, and next time P comes in contact with AB say at B. It is clear that A and B are points of contact and that at A, B, the curve will have cusps. But we shall confine ourselves with the portion between AB only which is generally called the cycloid. Let O be the position of P when the diameter of the circle through P is at right angles to AB. This is also evident that O is the middle point of the curve between AB. Let P be any position of the moving point. Take a straight line OX through O parallel to AB as the axis of x and another straight line OY perpendicular to OX at O, as the axis of y. The circle thus moves between the rails AB and OX.

Let OD be the diameter of the fixed circle OTD, whose centre is C.

Again AB = length of the circumference of the circle.

$= 2\pi a.$

∴ AD $= \pi a$, semi circumference, and GA = arc GP

∴ DG = arc PN = arc TO.

Also DG = TP = ON.

If x and y denote the coordinates of P,

x = ON + NL

= arc OT + MT

$= a\theta + a \sin \theta$, if the angle TCM is θ, measured from CO.

Again y = LP

= OC − CM

$= a - a \cos \theta$

Thus the curve is given by

$$\left.\begin{aligned} x &= a\,(\theta + \sin\theta) \\ y &= a\,(1 - \cos\theta) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Also $\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 \left(\frac{dy}{d\theta}\right)^2$

$$= a^2 (1 + \cos\theta)^2 + a^2 \sin^2\theta$$

$$= 2a^2 (1 + \cos\theta)$$

$$= 4a^2 \cos^2 \frac{\theta}{2}$$

$$\therefore \quad ds = 2a \cos \frac{\theta}{2} d\theta$$

$$\text{or } s = 4a \sin \frac{\theta}{2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

if s be measured from o.

$$\text{Again } \frac{dy}{dx} = \frac{a \sin\theta}{a (1 + \cos\theta)}$$

$$= \tan \frac{\theta}{2}$$

$$\therefore \psi = \frac{\theta}{2}.$$

i. e., PN is the tangent to the curve at P and the intrinsic equation of the cycloid is $s = 4\, a \sin \psi$.

Examples.

Trace the following curves :—

1. $y^2 (a^2 - x^2) = x^2 (a^2 + x^2)$.
2. $x^3 + y^3 = a^3$.
3. $x^3 + y^3 = 2ax^2$.
4. $ay^2 - 2axy + x^3 = 0$.
5. $xy^2 = 4a^2 (2a - x)$. (The Witch).
6. $xy^2 = 4a^2 (x - 2a)$.
7. $y^2 (2a - x) = x^3$. (Cissoid of Diocles).
8. $y = x^2 (x - 1)$.
9. $y (x^2 - a^2) = x^3 + ax^2 + a^3$.
10. $x^5 + y^5 - 5a^2x^2y = 0$.
11. $x^5 + y^5 - 5a^2x^2y^2 = 0$.
12. $axy = x^3 - a^3$. (The Trident).
13. $a^3y^2 = x^4 (b + x)$.

14. $cy^2 = (x - a)(x - b)^2$.

15. $a^2y^2 = x^3(2a - x)$.

16. $a^2y^4 = x^4(a^2 - x^2)$.

18. $r = a(1 + \cos\theta)$.

19. $r = a + b\cos\theta$, when $a >$ or $< b$.

(Pascal's Limacon)

20. $r = a\theta$.

21. $r\theta = a$.

22. $r = ae^{m\theta}$.

23. $r = a \sin 2\theta$.

24. $r = a \sin 3\theta$.

25. $r^2 = a^2 \cos 2\theta$. (Bernoulli's Lemniscate).

26. Show that the curve

$$x^2y^2 = a^2(x^2 - y^2)$$

lies entirely between its asymptotes $y = \pm a$.

27. In the tractrix

$$x = a\left(\cos t + \log\tan\frac{t}{2}\right)$$

$$y = a \sin t.$$

prove that $s = a \log\dfrac{a}{y}$, s being measured from the cusp.

28. Trace the strophoid

$$a(x^2 - y^2) = x(x^2 + y^2)$$

and prove that it is the pedal of the parabola $y^2 = 4ax$ with respect to the point $(-a, 0)$.

29. Shew that there are two linear asymptotes and an asymptotic circle in the curve $r = \dfrac{a\theta^2}{\theta^2 - 1}$.

Also shew that there is a cusp of the first species at the origin and a point of inflexion where $\theta^2 = 3$.

30. Prove that the linear asymptote of the curve $r = \dfrac{a\theta}{1 + \theta}$, touches its asymptotic circle, and that the point of inflexion is given by the real root of $\theta^3 + \theta^2 + 2 = 0$.

31. Trace the curves

$(i)\ 2ay^2 = x(x - a)^2$ (A. U. B. Sc. 1928)

$(ii)\ a^4y^2 = x^5(2a - x)$ (A. U. B. Sc. 1928).

32. Trace the curve

$$(y^2 - 2)(x^2 + y^2 - 1) = 1 - 2x$$

(A. U. B. Sc. Hons. 1928).

CHAPTER XIV.

A SHORT HISTORY OF CALCULUS.

"There are certain epochs in History towards which the rays of the past advancement converge and from which radiate the advances of the future." Such in the history of Mathematics was the period 1650-90. During the first half of the seventeenth century, many eminent mathematicians had applied their energies in a direction, which finally led to the discovery of the infinitesimal calculus. So considerable was the advance achieved and so near were they to the threshold of the infinitesimal analysis, that it was not so much an individual discovery as the logical result of a succession of discoveries by different mathematicians. At the same time, be it said to the credit of the founders of calculus that their achievement in this subject does not lie in the use of the method of infinitesimally small quantities as employed by Archimedes,* Cavalieri,† Wallis,‡ etc ; nor in the use of the differential triangle as done by Pascal,§ Fermat‖ and others, nor in the

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* Archimedes (287 ?—212 B. C.), the greatest mathematician of antiquity was born in Syracuse. According to tradition, when the Romans took possession of the city, and a soldier approached him he called out, "Don't spoil my circles." Thereupon the soldier feeling insulted killed him.

† Cavalieri (1598—1647), professor at Bologna is celebrated for his *Geometria indivisibilibus continuorum nova quadam ratione promota, 1635.*

‡ John Wallis (1616—1703), a Cantab, was appointed Savilian professor of geometry at Oxford in 1649.

§ Blaise Pascal (1623—1662) was born at Clermont in Auvergne.

‖ Pierre de Fermat (1601—1665) studied at Toulouse. "He has left the impress of his genius upon all branches of mathematics then known.

method of determining "Tatkalika gati" *i. e.*, instantaneous motion of a planet, by Bhaskaracharya,* and which exactly corresponds to the differential of the longitude of the planet. Neither does it lie in infinite summations embodied in the principle, that area was formed either by the parallel motion of a straight line or as the sum of an infinite number of small triangles or rectangles; nor† "in Newton's method of series which demanded no more than integration of a power of the independent variable (except in the case of an index equal to—1), or its differentiation and which did not give an exact result, except when the resulting series happened to be a known one."

Nor lastly, in the invention of suitable notation by Leibniz. The discovery lies principally in the complete recognition that differentiation and integration are inverse operations, *i. e.*, the drawing of the tangent and the finding of areas are inter-related, and the presentation of the subject in a systematic and logical way as a new and definite branch of science, in other words giving the differentials and integrals of the usual functions of a dependent variable, without the necessity of expressing these as series, as well as rules for dealing with products, quotients and powers of such functions.

Calculus, as originally conceived and without modern considerations of analysis, could be divided into two portions. Firstly, an analytical one, which gave a definite

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* Bhaskaracharya was born in 1036 Salivahana era or 1114 A. D. Majority opines that he died in 1179 A. D. at the age of sixty-five. Mahamahopadhyaya Pandit Bapudeo Shastri in his article in the journal of the Asiatic society of Bengal Vol. XXVII—1858 says that his methods indicate that Bhaskaracharya was fully acquainted with the principle of differential calculus.

† Prof. J. M. Child.

and systematic treatment for differentiating known functions of a dependent variable of sums or products etc., together with the recognition of the fundamental idea that differentiation is the inverse of integration and thus to enable integration without recourse to first principles etc., etc; secondly a geometrical synthesis, embodying the same principles and methods. Before proceeding to discuss who were the real founders, let us follow hurriedlly, so far as it concerns our subject, the lives of three great mathematicians, Isaac Barrow (1630—1677), Isaac Newton (1642—1727) and Leibniz (1646—1716).

Isaac Barrow was professor of Mathematics in London and later in Cambridge. He had prepared his *Lectiones Geometricae* during the period 1663 –69 and entrusted it to the care of Newton and Collins for publication in 1669, and which was published as a supplement to the second edition of his *Lectiones Opticae* in 1670. *Lectiones Geometricae* is divided into :—

(*i*) Lectures I—V, five preliminary Lectures.

(*ii*) Lectures VI—XII, seven Lectures.

(*iii*) Several supplements which were included at the insistence of Newton, together with Newton's five examples of the use of the *a*-and *e*-method.

(*iv*) Lecture XIII—connected with Algebra.

It is the seven lectures VI—XII that form the basic of calculus. From these lectures it is quite evident that, not only did he possess a full knowledge of the subject, but was convinced of the importance of the inverse* nature

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* Lecture X Prop. 11. " Let ZGE be any curve of which the axis is AD, and let ordinates applied to this axis, AZ, PG, DE continually increase from the initial ordinate AZ. Also let AIF be a line, such that if any straight line EDF be drawn

of differentiation and integration. He was also convinced that he was in possession of knowledge, which will throw a new light on the sciences and will form a science by itself. But in 1669, be resigned his chair in favour of

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perpendicular to AD, cutting the curves in E and F, and AD in D, the rectangle contained by DF and a given line R is

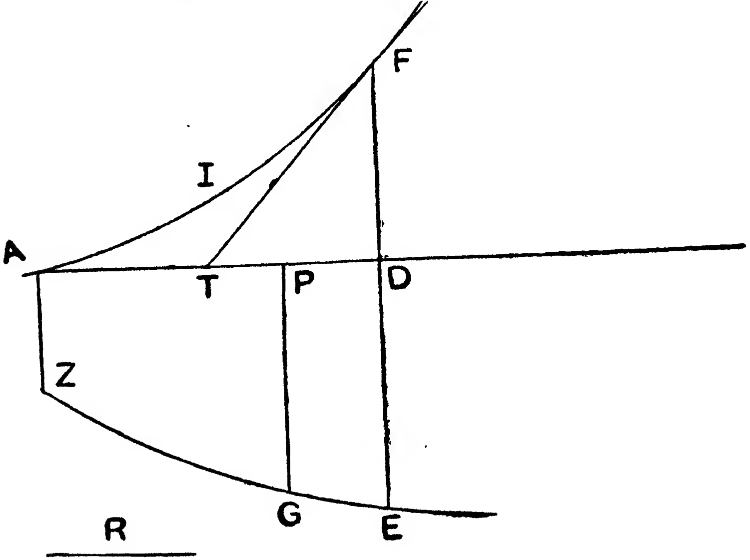

FIG. 52.

equal to the intercepted area ADEZ; also let DE : DF = R : DT, then will TF touch the curve AIF."

Translated into modern notations it will read like :—

"Let the equation of ZGE be $y = f(x)$, so that the area ADEZ $= \int f(x)\, d\, x$.

Also let the equation of AIF be $y = \phi(x)$, and let $\int f(x)\, dx = \phi(x)$ R, then if FT is the tangent to AIF, so that DF : DT $= \frac{d\,\phi(x)}{dx}$, we have

$$\frac{d\,\phi(x)}{dx} = \frac{\mathrm{DE}}{\mathrm{R}} = \frac{f(x)}{\mathrm{R}}.$$

his rising pupil and friend Isaac Newton. Canon Overton gives a quotation (source not stated) in the "Dictionary of National Biography," pointing out that Barrow's vanity and pride was offended at the neglect accorded to his work, and this made him give up for ever his mathematical investigations and turn to divinity in 1669. He had entrusted the publication of his lectures to the care of Newton and Collins, who got them printed as a supplement to the second edition of *Lectiones Opticae* in 1670.

Isaac Newton in 1664, while yet a subsizer of Trinity College, Cambridge, and who had also appeared for scholarship, came under the notice of Barrow, who becomes his tutor and practically incharge of Newton's education. In 1665 Newton retired to Woolsthorpe, his birth-place, where, professor Child suggests that Newton had retired with probably some instructions on the use of infinitesimals. In his retirement, he developed fluxions to such a pitch, that he uses them for tangents and radii of cvurature. He could differentiate an explicit function in the form of a polynomial and also integrate the same by the use of Wallis's method. During the period 1667—69, when he came back to Cambridge, he was more a friend than anything else, under the influence of Barrow, but he was chiefly engaged on optics and revised Barrow's work on the subject. In 1669, he completed his *Dc Analysi* and presented it to Barrow. The treatise, which was not published, even though Barrow insisted on it, till forty-two years later,

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If R = 1. It states that if

$$\int f(x)\,dx = \phi(x),$$

$$\text{Then } \frac{d\,\phi(x)}{dx} = f(x)$$

In lecture IX Prop. 12, lecture VIII prop 9, and lecture IX prop. 3, Barrow gives the differentiation of a product, a quotient and power + *ve*, —*ve*, integral or fractional.

shows that Newton was still using first principles for differentiation and integration. After the retirement of Barrow from the field of mathematics, Newton is considered to have carried on the work of Barrow and had calculus complete within the next two or three years.

Leibniz visited Paris in 1672, on a political mission and till then he had very little knowledge of higher mathematics. There he was initiated into higher mathematics by C. Huygens. In January 1673, Leibniz visited London and remained there till March. In a manuscript, he made a note under the heading geometry, "Tangents to all curves," probably referring to Barrow's work and had purchased a copy of it After his return to Paris, Huygens advised him to read Pascal and others.

He apparently did much work on integration, using the idea of 'Moments' during 1674-75. In October 1675, he finds integrals of x^2 and x^3 and some more theorems and from November 11 and onwards, he is using figures reminiscent of Barrow for the first time with some success. This is the year given by Leibniz for reading Barrow. He returned to Hanover in October, 1676 by way of London, and even up till now, Leibniz was in possession of the most elementary rules of formulae of the infinitesimal calculus.

But now onwards, the progress is fairly rapid. The chief point to be noticed here is that it is not till after he has really studied Barrow that he does anything substantial.

The first memorable day found from the manuscripts of Leibniz for giving the notation $\int$ is October 29th, 1675, "a notation which contributed enormously to the rapid growth and perfect development of the calculus."*

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* Cajori—A History of Mathemetics.

Leibniz has his first paper on differential calculus published in 1684 in the *Acta eruditorum*,* "nine years after the new calculus dawned upon the mind of Leibniz, and nineteen years after Newton first worked at fluxions.

Leibniz notified to H. Oldenburg, the then secretary of the Royal Society, in 1676, that he knew very general analytical methods, by which he had found theorems of great importance on quadrature of the circle by means of series. Oldenburg replied that Newton and Gregory were also in possession of methods of quadratures, which extended to circle. Leibniz desired to have these methods communicated to him, and Newton at the request of Oldenburg and Collins, wrote to the former the first of the celebrated letters on June 13th, 1676. It did not contain any exposition on the method of fluxions, but on binomial theorem and all the rest of it. Leibniz spoke in the highest terms of what Newton had done and requested further explanation. Newton in his second letter dated October 24, 1676, explains how he found the binomial theorem and mentions in the form of an anagram his method of fluxions and fluents—meaning, "Having any given equation involving never so many flowing quantities, to find the fluxions and *vice versa*."† Surely this afforded no hint. Leibniz wrote a reply to Collins, in which he explained the principle, notation and the use of differential calculus. The death of Oldenburg brought this correspondence to a close. It is alleged that Oldenburg communicated to Leibniz a copy of Newton's letter of December 10, 1672, giving fluxions method in detail. In 1850, it was shown that what Oldenburg sent to Leibniz was not Newtons' letter referred to above, but only excerpts from it, which omitted Newtons' method of drawing

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* Founded in Berlin 1682.

† Cajori.

tangents, etc., and could not possibly convey an idea of fluxions. Oldenburg's letter was found among Leibniz's manuscripts in the Royal Library at Hanover.

Thus Leibniz was never in possession of what was alleged to have been sent to him and which was not recognised by him. Newton also wrote in his *Principia* (First Edition 1687,) "In letters which went between me and that most excellent geometer, G. G. Leibniz ten years ago, when I signified that I was in the knowledge of a method of determining maxima and minima, of drawing tangents and the like, and when I concealed it in transposed letters involving this sentence (Data acquatione etc., meaning "Having any given equation etc.,") that most distinguished man wrote back that he had also fallen upon a method of the same kind and communicated his method, which hardly differed from mine, except in his forms of words and symbols."

Newton was afterwards weak enough as De Morgan suggests, "First to deny the plain and obvious meaning and secondly to omit it entirely from the third edition of the Principia" after the controversy had started between Newton and Leibniz. One thing is plain, that both were indebted to Barrow for their idea and early training and both were working independently. Leibniz, though he started late, had gained much because of his notations and he was the first to give the full benefit of the calculus to the world. Professor J. M. Child very well sums up the fact that Barrow, Newton and Leibniz each of them should be taken as the founder of calculus in the following sentence. "Although in the Mathematical Marathon, Barrow had breasted the tape before Newton had arrived at the stadium, while Leibniz was a non-starter; nevertheless some years later, when Barrow's running days were over, Newton and Leibniz had so profited by his training hints, that they made a dead-heat of it, beating Barrow's record time by minutes, not seconds."

NOTE C.

HYPERBOLIC FUNCTIONS.

I. Hyperbolic functions are defined as follows :—

$$\text{Sinh } x = \frac{e^{x} - e^{-x}}{2},$$

$$\text{Cosh } x = \frac{e^{x} + e^{-x}}{2}$$

$$\text{Tanh } x = \frac{e^{x} + e^{-x}}{e^{x} + e^{x}}, \text{ and so.}$$

Thus it is clear that

$$\cosh^{2} x - \sinh^{2} x = 1 \text{..............................(1)}$$

$$1 - \tanh^{2} x = 1 \text{..............................(2)}$$

$$\coth^{2} x - 1 = \text{cosech }^{2} x \text{..............................(3)}$$

These three formulae being analogous to the well known three formulae of the circular functions.

$$\text{Also } 2 \sinh x \cosh x = 2 \; \frac{e^{x} - e^{-x}}{2} \cdot \frac{e^{x} + e^{-x}}{2}$$

$$= \frac{e^{2x} - e^{-2x}}{2}$$

$$= \sinh 2 x$$

$$\text{and } \cosh^{2} x + \sinh^{2} x = \left(\frac{e^{x} + e^{-x}}{2}\right)^{2} + \left(\frac{e^{x} - e^{-x}}{2}\right)^{2}$$

$$= \frac{e^{2x} + e^{-2x}}{2}$$

$$= \cosh 2 x.$$

Similarly $\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y$ and $\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y$.

II. Let $y = \sin h^{-1} \frac{x}{a}$

$$\therefore \frac{x}{a} = \sin h \, y$$

$$= \frac{e^y - e^{-y}}{2}$$

$$\therefore \quad ae^{2y} - 2\,x\,e^y - a = 0$$

Taking only the positive sign

$$e^y = \frac{x + \sqrt{x^2 + a^2}}{a}$$

$$\text{or } y = \log\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right)$$

$$i.\,e.,\ \sin h^{-1}\frac{x}{a} = \log\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right)$$

Similarly

$$\cos h^{-1}\frac{x}{a} = \log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right).$$

$$\tan h^{-1}\frac{x}{a} = \frac{1}{2}\log\frac{a + x}{a - x}$$

$$\cot h^{-1}\frac{x}{a} = \frac{1}{2}\log\frac{x + a}{x - a}$$

III. Derivative of sinh x and cosh x

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\therefore \quad \frac{d \sinh x}{dx} = \cosh x$$

$$\text{Similarly } \frac{d \cosh x}{dx} = \sinh x.$$

IV. The student will be well advised to deduce the following derivatives himself.

$$y = \tanh x, \qquad \frac{dy}{dx} = \operatorname{sech}^2 x.$$

$$y = \coth x, \qquad \frac{dy}{dx} = -\operatorname{cosech}^2 x.$$

$$y = \operatorname{sech} x, \qquad \frac{dy}{dx} = -\tanh x \operatorname{sech} x.$$

$$y = \operatorname{cosech} x, \qquad \frac{dy}{dx} = -\coth x \operatorname{cosech} x.$$

$$y = \sinh^{-1} x, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}.$$

$$y = \cosh^{-1} x, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

$$y = \tanh^{-1} x, \qquad \frac{dy}{dx} = \frac{1}{1 - x^2} (x < 1).$$

$$y = \coth^{-1} x, \qquad \frac{dy}{dx} = -\frac{1}{x^2 - 1} (x > 1)$$

$$y = \operatorname{sech}^{-1} x, \qquad \frac{dy}{dx} = -\frac{1}{x\sqrt{1 - x^2}}.$$

$$y = \operatorname{cosech}^{-1} x, \qquad \frac{dy}{dx} = -\frac{1}{x\sqrt{x^2 + 1}}.$$

ANSWERS.

CHAPTER II

Page 28.

1. (*i*) $3\tan^2 x \sec^2 x$, $\log ex$, $e^{x+a}[\cos(x+b)+\sin(x+b)]$,

$$x^{r-1}\left[r\log\frac{x\tan x}{1+\sin x}+1+x\tan x+\frac{x}{\tan x(1+\sin x)}\right].$$

(*ii*)

$$x^{n-1}\left[\frac{n}{\tan x}-\frac{x}{\tan^2 x}-x\right],$$

$$\frac{x}{\cos^4\frac{x}{2}}[1+\cos x+x\sin x],$$

$$\frac{2\sec^2\frac{x}{2}\left(1+\tan\frac{x}{2}\right)}{\left(1-\tan\frac{x}{2}\right)^3},$$

$$\frac{2}{x-\tan x}\left[1+\frac{x\tan^2 x}{x-\tan x}\right].$$

2.

$$\frac{(m-n)x^{m+n-1}-m\,x^{m-1}+n\,x^{n-1}}{(x^n-1)^2},$$

$$\frac{e^x}{1+\sin 2x}(\cos 2x-2),$$

$$\frac{1}{x}(1+x\log_e a),\ x\sec x\,(2+x\tan x)$$

$$\frac{\sin x+x\cos x}{1+\cos^2 x}+\frac{2x\cos x\sin^2 x}{(1+\cos^2 x)^2}.$$

Page 35.

1. $$\frac{1}{(1-x)\sqrt{1-x^2}}.$$

2. $$\frac{1}{2(1-x+x^2)}\left\{\frac{1+\cos x}{\sqrt{1+x+\sin x}}-\frac{(2x-1)\sqrt{1+x+\sin x}}{\sqrt{1-x+x^2}}\right\}.$$

3. $\dfrac{1 + 2x}{(1 - x^2)^{\frac{3}{2}}(1 - x)}.$

4. $\left[\dfrac{1}{2\sqrt{x}} - \dfrac{x}{\sqrt{1 - x^2}}\right]\dfrac{1}{\sqrt{x} + \sqrt{1 + x^2}} -$
$\left[\dfrac{1}{2\sqrt{x}} + \dfrac{x}{\sqrt{1 + x^2}}\right]\dfrac{\sqrt{x} + \sqrt{1 - x^2}}{\sqrt{x} + \sqrt{1 + x^2}}.$

5. $\dfrac{2nx}{(a^2 - x^2)^{n+1}}.$

6. $\frac{2}{3}\dfrac{x}{(a^2 - x^2)^{\frac{4}{3}}}.$

7. $\dfrac{n x^{n-1}(1 - x^2)}{(1 + x^2)^{n+1}}.$

8. $\dfrac{(2a^2 - x^2)x}{(a^2 - x^2)^{\frac{3}{2}}}.$

9. $-n(\cot^{n-1}x + \cot^{n+1}x).$

10. $\cot^2 x - 2x\cot x - 2x\cot^2 x.$

11. $x^{a-1}\sqrt{\sin x}\,(a + \dfrac{x}{2}\cot x).$

12. $\cos x \tan(x + a) + \sin x \sec^2(x + a).$

13. $\dfrac{2\sin(ax + b)}{\cos^3(bx + a)}\{a\cos(ax + b)\cos(bx + a) +$
$b\sin(ax + b)\sin(bx + a)\}.$

14. $\dfrac{-m\,\mathrm{cosec}^2\,mx}{1 + \sec^2 nx} - \dfrac{2n\cot mx\sec^2 nx\tan nx}{(1 + \sec^2 nx)^2}.$

15. $\dfrac{1}{x^4}[2x\cos 2x - 3\sin 2x].$

16. $\dfrac{2ab\sin x}{(a + b\cos x)^2}.$

17. $\frac{1}{2}\sec^2\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right).$

18. $-\left[\dfrac{1}{x} + \dfrac{\cos x}{1 - \sin x}\right].$

19. $\sec^4 x.$

20. $- \dfrac{n \sin^{n-1} x \cos x}{(1 + \sin^n x)^2}$.

21. $- \dfrac{4 \sin 4x}{(1 + \sin^2 2x)^2}$.

22. $3 \cos^2 3x (4 \cos^2 3x - 3)$.

23. $\sin^{-1}(1 - x) - \dfrac{\sqrt{x}}{\sqrt{2 - x}}$.

24. 0.

25. $\dfrac{2}{1 + x^2}$.

26. $\dfrac{1}{1 + x^2}$.

27. $\dfrac{\tan \alpha}{\sec \alpha + \cos x}$.

28. $\dfrac{- b \sin x}{1 + a^2 + b^2 \cos^2 x + 2ab \cos x}$.

29. $\dfrac{1}{a + b \cos x}$.

30. $\dfrac{\tan \alpha \sin x}{(\sec \alpha + \cos x)^2}$.

31. $\dfrac{\sin x + x \cos x}{1 + \cos^2 x} + \dfrac{2x \sin^2 x \cos x}{(1 + \cos^2 x)^2}$.

32. $- \dfrac{x \sin x + 2 \cos x + 2}{(x + \sin x)^2}$.

33. $- \dfrac{\sqrt{x^2 - 1} \sec^{-1} x + 1}{x \sqrt{x^2 - 1} \sec^{-1} x \sqrt{(x \sec^{-1} x)^2 - 1}}$.

34. $\dfrac{1}{x} + \frac{1}{2} \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$.

35. $- 2x e^{-x^2}$.

36. $\dfrac{1}{x} e^{e^{\log x}} . e^{\log x}$.

37. $e^{\tan^{-1} x} \left\{\frac{1}{1 + x^2} \log (\tan x) + \frac{2}{\sin 2x}\right\}.$

38. $\frac{1}{e^x + e^{-x}}.$

39. $- \frac{1}{x (1 + x)}.$

40. $\frac{\tan x + x \sec^2 x}{x \tan x}.$

41. $- 2 \sec x.$

42. $x^x \log x (\sin x)^{\sin x} \left\{1 + \log x + \frac{1}{x \log x} + \cos x \right.$

$\left. + \cos x \log \sin x\right\}.$

43. $- \frac{x}{\sqrt{1 - x^4}}.$

44. $(\tan x)^{\tan x} (1 + \log \tan x) \sec^2 x - (\cos x)^{\cos x} \times$

$(1 + \log \cos x) \sin x.$

45. $\sin x \sin 2x \sin 3x (\cot x + 2 \cot 2x + 3 \cot 3x).$

46. $\frac{y^2}{x^3} + 2x.$

47. $\frac{y \log y}{1 - x \log y} \left(\frac{1 + x \log x \log y}{x \log x}\right).$

49. 1.

50. $\frac{y}{x}.$

51. $\frac{x^{\sin x} a^{cx} \left(c \log a + \frac{\sin x}{x} + \cos x \log x\right)}{1 + a^{2cx} x^{2 \sin x}} . \frac{\sqrt{x}}{1 + x^{\frac{3}{2}}}.$

$+ \tan^{-1} [a^{cx} x^{\sin x}] \left[\frac{1}{2 \sqrt{x} (1 + x^{\frac{3}{2}})} - \frac{\frac{3}{2} x}{(1 + x^{\frac{3}{2}})^2}\right].$

52. $\frac{1}{\log^{n-1} x} . \frac{1}{\log^{n-2} x} \cdots\cdots\cdots \frac{1}{\log x} . \frac{1}{x}.$

53. $$\left(\frac{1+\sqrt{x}}{1+2\sqrt{2}}\right)^{\sin e^{-x^2}} \left[\left(\log \frac{1+\sqrt{x}}{1+2\sqrt{2}}\right)\times\right.$$

$$\cos e^{-x^2}.\, e^{-x^2}(-2x)$$

$$\left.+\frac{\sin e^{-x^2}}{2\sqrt{x}\,(1+\sqrt{x})}\right].$$

54. $\frac{\log_a e}{3x^3}$.

55. $\sqrt{a^2+b^2}\; e^{ax}\cos(bx+\phi)\sec x$, where

$$\phi=\tan^{-1}\frac{b}{a}.$$

56. $$-\frac{\alpha\beta}{(\alpha^2+\beta^2)\sin\frac{x}{2}\cos^3\frac{x}{2}}\;\frac{\left(\alpha^2\sin^2\frac{x}{2}-\beta^2\cos^2\frac{x}{2}\right)^2}{\left(\alpha^2-\beta^2\tan^2\frac{x}{2}\right)}.$$

57. $-\frac{x}{(2+x^2)\sqrt{1+x^2}}$.

58.

$$-\frac{(\alpha^2+\beta^2)\sqrt{1-\beta^4x^4}}{\beta^4x^2\left\{\sqrt{1+\alpha^2x^2}-\sqrt{1-\beta^2x^2}\right\}^2(1-\beta^2x^2)^{\frac{1}{2}}\,(1+\alpha^2x^2)^{\frac{1}{2}}}$$

59. -2.

60.

$$-2x^{\frac{2m+5}{2}}\left[-\frac{m(1+x^2)\cot^{-1}x\log\cot^{-1}x-x}{\cot^{-1}x\,(1+x^2)\,(2x\sin\sqrt{x}+5\cos\sqrt{x})}\right].$$

61. $$\frac{1}{\sec^2 x}\left[\frac{4x}{a^2+x^2}-\frac{1}{2}\,\frac{\sec^2 x}{1+\tan x}\right]\frac{(a^2+x^2)^2}{\sqrt{1+\tan x}}.$$

62. $$\frac{2x-(\alpha+\beta)}{\{(x-\alpha)(\beta-x)\}^{\frac{3}{2}}}.\,\sqrt{x}.\;\sec^2\frac{\alpha}{2}.$$

63. $-\frac{b}{a}\ \frac{x}{\sqrt{a^2 - x^2}}$.

64. $-\tan\theta$.

65. $-\tan\frac{\theta}{2}$.

66. $\frac{1}{t}$.

67. $-\frac{b}{a}\cot\phi$.

68. $\frac{1}{2}\tan t\,(\sec^2 t + 2)$.

69. $\sin\psi$.

70. $\tan t$.

71. $\frac{[1-\cos^2\alpha\,\sin^2\phi]^2 - \sin\alpha\,\tan\phi\,[\cos^2\alpha\,(1+\cos^2\phi)-1]}{\sin\alpha\,[1+\cos^2\alpha\,\sin^2\phi] - \tan\phi\,[1-\cos^2\alpha\,\sin^2\phi]^2}$.

72. $-\cot\frac{3\theta}{2}$.

73. $\frac{t\,(2 - t^3)}{1 - 2\,t^3}$.

74. $\tan\left(\frac{a}{2b} + 1\right)\theta$.

CHAPTER III.

Page 45.

1. $$\frac{2^n}{4}\left[\sin\left(2x + n\frac{\pi}{2}\right) + 2^n\sin\left(4x + n\frac{\pi}{2}\right) + 3^n\sin\left(6x + n\frac{\pi}{2}\right)\right].$$

2. $$4^n\,(2)^{\frac{n}{2}}\,e^{4x}\cos\left(4x + n\frac{\pi}{4}\right).$$

3. $$(-2)^n\,\frac{\lfloor\underline{n+1}}{(2x + a)^{n+2}}.$$

4. $$\frac{2^{n+2}\lfloor\underline{n}}{3^{\frac{n+2}{2}}}\sin^{n+1}\theta\,\sin(n+1)\theta,\ \text{where } \theta = \cot^{-1}\frac{2x+1}{\sqrt{3}}.$$

Page 50.

1. $$\frac{12\,x^3\,(5 + 14\,x^4 + x^8)}{(1 - x^4)^{\frac{7}{2}}}.$$

2. $$\frac{1}{8\sqrt{x}}\left[\frac{3}{x^2} + 2x\,(1 + x)^4 + \frac{(1 - x)\,(1 + 5x)}{x(1 + x)^3}\right].$$

3. $$2\cdot3\ldots\ldots8\cdot x\cos 2x + 3\cdot4\ldots\ldots8(2\cdot7\cdot x^2)\cos\left(2x+\frac{\pi}{2}\right)$$
$$+4\cdot5\ldots\ldots8(21x^3)\cdot2^2\cdot\cos\left(2x+\frac{2\pi}{2}\right)$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+8\cdot7\cdot x^7\cdot2^7\cos\left(2x+7\,\frac{\pi}{2}\right).$$

12. $$a^{n+2}x^2\,e^{ax}\;.$$

22. $$(-1)^{n+1}\lfloor\underline{n}\left[\frac{8}{(x + 2)^{n+1}} + \frac{7(n + 1)}{(1 + x)^{n+2}}\right.$$
$$\left.+\frac{(n + 1)\,(n + 2)\,(n + 3)}{(1 + x)^{n+4}}\right]$$
$$+ (-1)^n\,4\lfloor\underline{n}\left[\frac{2}{(1 + x)^{n+1}} + \frac{(n + 1)\,(n + 2)}{(1 + x)^{n+3}}\right].$$

23. $$\frac{2^{n+2}\lfloor\underline{n}}{3^{\frac{n+2}{2}}}\sin^{n+1}\theta\,\sin\,(n+1)\theta,\text{ where }\theta = \cot^{-1}\frac{2x+1}{\sqrt{3}}.$$

24. $$\frac{(-1)^{n+1}\lfloor\underline{n}}{2}\left[\frac{\sin^{n+1}\theta\,\sin\,(n+1)\theta}{a^{\frac{n+2}{2}}} +\right.$$
$$\left.\frac{1}{2\sqrt{a}}\left\{\frac{1}{(x-\sqrt{a})^{n+1}} - \frac{1}{(x+\sqrt{a})^{n+1}}\right\}\right],$$
$$\text{where }\tan\theta = \frac{\sqrt{a}}{x}$$

25. $$\lfloor\underline{n}\left[\log\frac{1 + x}{1 - x} - {}^{n}c_1\,\frac{x}{1}\left\{\frac{1}{1+x} + \frac{1}{1 - x}\right\}\right.$$
$$+ {}^{n}c_2\,\frac{x^2}{2}\left\{\frac{(-1)^2}{(1 + x)^2} - \frac{1}{(1 - x)^2}\right\}$$
$$+ {}^{n}c_3\,\frac{x^3}{3}\left\{\frac{(-1)^3}{(1 + x)^3} - \frac{1}{(1 - x)^3}\right\}+\ldots\ldots\ldots\ldots$$

$$+ \frac{x^n}{n}\left\{\frac{(-1)^4}{(1+x)^n} - \frac{1}{(1-x)^n}\right\}\Bigg]$$

26. If n be even

$$y_n = (-1)^n \frac{\lfloor n-1}{a^{n-1}}\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right.$$

$$\left. + 2\sum_{r=1}^{r=\frac{n}{2}-1} \frac{\cos\left(\frac{2r\pi}{n} + \overline{n+1}\ \theta_r\right)}{(x^2 - 2ax\cos\frac{2r\pi}{n} + a^2)^{\frac{n+1}{2}}}\right]$$

where $\cot\theta_r + \cot\frac{2r\pi}{n} = \frac{x}{a}\operatorname{cosec}\frac{2r\pi}{n}$,

and if n be odd

$$y_n = (-1)^n \frac{\lfloor n-1}{a^{n-1}}\left[\frac{1}{(x-a)^{n+1}}\right.$$

$$\left. + 2\sum_{r=1}^{r=\frac{n-1}{2}} \frac{\cos\left(\frac{2r\pi}{n} + \overline{n+1}\ \theta_r\right)}{\left(x^2 - 2ax\cos\frac{2r\pi}{m} + a^2\right)^{\frac{n+1}{2}}}\right]$$

31. $$e^{bx^2}\left[b^n(2x)^n + \frac{n(n-1)}{\lfloor 1}b^{n-1}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{\lfloor 2}b^{n-2}(2x)^{n-4} + \text{etc.}\right]$$

32. $$\left[(2x)^n \sin\left((x^2 + n\frac{\pi}{2}\right) + \frac{n(n-1)}{\lfloor 1}(2x)^{n-2}\sin\left(x^2 + \overline{n-1}\frac{\pi}{2}\right) + \frac{n(n-1)(n-2)(n-3)}{\lfloor 2}(2x)^{n-4} \times \sin\left(x^2 + \overline{n-2}\frac{\pi}{2}\right) + \text{etc.}\right.$$

CHAPTER IV.

Page 64.

4. $$R_n = \frac{(a^2+b^2)^{\frac{n}{2}}}{\lfloor n} x^n e^{a\theta x} \sin\left(b\theta x + n\tan^{-1}\frac{b}{a}\right),$$

in powers of x.

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4. $x - \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}.\frac{3}{4}.\frac{x^5}{5}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

5. $x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

6. $x + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots\ldots\ldots\ldots\ldots$

17. $1 + x \log_e a + \frac{x^2}{\lfloor 2}(\log_e a)^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

18. (*i*) as answer No. 4.

(*ii*) $\frac{x^2}{\lfloor 3} - 2^2\frac{x^4}{\lfloor 4} + 2^2.\,4^2\frac{x^6}{\lfloor 6} - \ldots\ldots\ldots\ldots\ldots\ldots$

CHAPTER V.

Page 82.

1. $(b^2 - a^2) \sin 2t.$

2. $\frac{3}{\sqrt{1 - t^2}}.$

3. $\frac{2}{1 + t^2}[(\tan^{-1} t - \cot^{-1} t) \sin 2 \ (\tan^{-1} t.\ \ \cot^{-1} t)].$

9. $\frac{x^2 - ay}{xv - h}.$

10. $\frac{ax + hy + g}{hx + by + f}.$

11. $\frac{y \tan x + \log \sin y}{\log \cos x - x \cot y}.$

12.

$$-\frac{x^{y-1}y^{x+1} + x^y . y^x \log y - \frac{\cos y}{x} x^{\cos y} - y^{\log x} . \frac{\log y}{x}}{x^{y+1}y^{x-1} + x^y . y^x \log x + x^{\cos y} \log x.\sin y - \frac{\log x}{y} y^{\log x}}.$$

13. $\frac{2a^2 xy (x^2y^2 + 3a^4)}{(y^3 + a^2x)^3}.$

14.

$$4 \sec^2 [\sin (x + y)] \frac{[\sin (x + y) - 2 (x - y) \cos^2 (x + y)]}{[\sec^2 [\sin (x + y)] - 1]^3}.$$

15. 0.

CHAPTER VI.

Page 89.

1. (*i*) $- y \cos t$, (*ii*) $- y \tan t$,
(*iii*) $y \operatorname{cosec} t$, (*iv*) $y \sec t$,

2. (*i*) $y \cot t$, (*ii*) $y \tan t$,
(*iii*) $y \operatorname{cosec} t$, (*iv*) $y \sec t$.

3. (*i*) $\dfrac{3 t (1 - 2t^3)}{(1 + t^3) (2 - t^3)}$, (*ii*) $\dfrac{3 t^3 (2 - t^3)}{(1 + t^3) (1 - 2t^3)}$,

(*iii*) $\dfrac{3 t}{1 + t^3} \sqrt{\dfrac{(1 - 2 t^3)^2 + t^2 (2 - t^3)^2}{(2 - t^3)}}$.

(*iv*) $\dfrac{3 t^2}{1 + t^3} \sqrt{\dfrac{(1 - 2 t^3)^2 + t^2 (2 - t^3)^2}{(1 - 2 t^3)^2}}$.

4. $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

5. $a^2 - b^2 = a'^2 - b'^2$.

6. Touch at (2, 4).

7. Right angle.

Page 101.

1. $\dfrac{a^2 b^2}{p^2} + r^2 = a^2 + b^2$, (centre).

$\dfrac{b^2}{p^2} = \dfrac{2a}{r} - 1$. (focus).

2. $a^2 (r^2 - p^2)^2 = p^2 (r^2 + 4a^2) (p^2 + 4a^2)$.
3. $pd = r^2$.
4. $p = r \sin \alpha$.
5. $a^2 p = r^3$.
6. $pr = a^2$.
7. $p - a$.

Page 107.

2. $\frac{1}{2}\sqrt{4 + 9y}$.

3. $\sqrt[3]{\frac{a}{x}}$.

4. $\sec x$.

5. $\sec \frac{x}{a}$.

6. $\frac{ds}{dx} = \sec \frac{\theta}{2}$; $\frac{ds}{d\theta} = 2a \cos \frac{\theta}{2}$.

9. $a^{\theta} \sqrt{1 + (\log_e a)^2}$.

10. $\frac{ds}{d\theta} = a \sqrt{\sec 2\theta}$; $\tan \phi = - \cot 2\theta$.

12. $r \sqrt{1 + \sec^2 \theta}$.

19. $p^2 = \frac{9a^2 (r^2 - a^2)}{r^2 + 15a^2}$.

22. $a^2x^2 + b^2y^2 = k^4$.

23. tangent at the vertex.

24. $r = 2a \cos^3 \frac{\theta}{3}$.

25. (*i*) $r^{\frac{n}{n+1}} = a^{\frac{n}{n+1}} \cos \frac{n}{n+1}\theta$.

(*ii*) $r^{\frac{2}{3}} = a^{\frac{2}{3}} \cos \frac{2}{3}\theta$.

26. (*i*) $(x^2 + y^2)^{\frac{3}{2}} = a^{\frac{3}{2}} \left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$

(*ii*) $r = a \sin \alpha \, e^{\left(\frac{\pi}{2} - a\right) \cot \alpha} \, . \, e^{\theta \cot \alpha}$.

(*iii*) $r^{\frac{m}{m-1}} = \frac{\cos^{\frac{m}{m-1}} \theta}{A^{\frac{1}{m-1}}} + \frac{\sin^{\frac{m}{m-1}} \theta}{B^{\frac{1}{m-1}}}$

(*iv*) $r^2 = \frac{\cos^2 \theta}{A} + \frac{\sin^2 \theta}{B}$.

28. $k^{\frac{2n}{n+1}} = (ar)^{\frac{n}{n+1}} \sin\left[\frac{n}{n+1}\left(\theta + \frac{\pi}{2}\right)\right].$

CHAPTER VII.

Page 117.

1. $\frac{1}{n}$.

2. -2.

3. 1.

4. 2.

5. 2.

6. n.

7. e.

8. $\log_e a$.

9. 2.

10. $-\frac{4}{\pi^2}$.

11. 4.

12. 0.

13. 0.

14. $\frac{4a^2}{\pi}$.

15. $m - \frac{4}{3}m^3$.

16. e.

17. 1.

18. 0.

19. $-\frac{e}{2}$.

20. $\frac{11e}{24}$.

21. $-\frac{1}{4}$.

22. 1.

23. $\frac{1 \pm i\sqrt{3}}{2}$.

24. 0 and ∞.

CHAPTER VIII.

Page 133.

1. $4\sqrt{2}\,a$.

2. $3a\sec^3\psi\tan\psi$.

3. $\frac{t}{2}(1 + t^2)^{\frac{3}{2}}$.

4. 1.

5. $4a\cos\frac{\theta}{2}$.

6. 6.

7. $\frac{2}{3} \sqrt{2ar}$; $\frac{4r}{3}$.

8. $\frac{a^2}{3r}$.

9. $\frac{3}{4} a \sin^2 \frac{\theta}{2}$.

10. $\frac{\sqrt{a}}{2(x+y)^{\frac{3}{2}}}$.

11. $\frac{a}{2}$.

12. $\frac{1}{2}$.

13. $\frac{a^2}{2b}$.

14. $\frac{3a}{2}$.

15. $\frac{r^3}{2k^2}$.

24. $\frac{4r}{3}$.

25. 1.

Page 139.

1. $27ay^2 = 4(x - 2a)^3$.
2. $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 b^2)^{\frac{2}{3}}$.
3. $y = a \cosh \frac{x}{a}$.

CHAPTER IX.

Page 149.

1. $y^2 = 4ax$.
2. $4xy = c^2$.
3. $y = \pm \frac{x}{\sqrt{2}}$.
4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
5. $(a^2 - b^2)^2 + (x^2 + y^2)^2 = 2(x^2 - y^2)(a^2 - b^2)$.
6. $y = \pm a$.
7. $x \pm y = \pm b$.
8. (*i*) $x^m y^n = \frac{m^m n^n}{(m+n)^{m+n}} k^{m+n}$.

   (*ii*) $\sqrt{x} + \sqrt{y} = \sqrt{c}$.

   (*iii*) $x^{\frac{n}{1+n}} + y^{\frac{n}{1+n}} = k^{\frac{n}{1+n}}$.

9. (*i*) $4x^m y^m = k^{2m}$.

(*ii*) $x^{\frac{m}{2}} + y^{\frac{m}{2}} = c^{\frac{m}{2}}$.

11. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

13. $4a^2 (x^2 - y^2) = (x^2 + y^2)^2$.

14. (1) $x + a = 0$.

(2) $r^{\frac{n}{n+1}} = a^{\frac{n}{n+1}} \cos \frac{n\theta}{n+1}$.

15. $2ay^2 + x(x^2 + y^2) = 0$.

16. $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

CHAPTER X.

Page 159.

3. $\left(a, \frac{2a^3}{b^2}\right)$.

6. Concave for all + *ve* values of x to the foot of the ordinate. Negative of values of x are inadmissible.

7. yes, (2, − 1).

Page 172.

1. First species, single cusp at $\left(c, \frac{ac}{b}\right)$.

2. Conjugate point at (— k, 0).

3. Origin is a node; tangents are $x + y = 0$, and $x - 2y = 0$.

4. Origin is a node; $y = \pm x$, are the tangents.

5. Origin is a cusp of first species single.

6. Node at (2, 1), tangents are $(x - 2) = \pm (y - 1)$.

7. Single first species cusp at (1, − 1).

8. Single first species cusp at (0, 0).

9. Node at (0, 0), $y = \pm x$ are the tangents.

10. First species double cusp at (0, 0).

11. Single first species cusp at $(-a, a)$.

12. Single first species cusp at $(-1, -2)$.

CHAPTER XI.

Page 182.

1. Max. $\frac{2\sqrt{3}}{9}$ for $x = \frac{6-\sqrt{3}}{3}$

min. $\frac{-2\sqrt{3}}{9}$ for $x = \frac{6+\sqrt{3}}{3}$.

2. Min. 9 for $x = 2$.

3. Max. 20, for $x = 1$.
Min. -23, for $x = 2$.

4. Max. when $x = \pm a$.
Min. when $x = 0$.

5. Max. when $x = 2n\pi$.
Min. when $x = (2n + 1)\pi$.

6. $\frac{108}{5^5} k^5$.

7. $t^2(ab - p^2) - t(a + b) + 1 = 0$, the roots of this give the values required.

8. $a + b$.

10. $\frac{km}{m+n}, \frac{kn}{m+n}$.

11. **Max.** semi—circumference.
Min. radius → ∞.

12. radius $= \sqrt{\frac{2}{3}} \times$ slant height.

18. Each is $\frac{l}{10}$.

20. $\Sigma \frac{a^4 l^2}{1 - a^2 u^2} = 0$.

21. $$\begin{vmatrix} u & 1 & 1 \\ 1 & \Sigma a^2 & \Sigma aa' \\ 1 & \Sigma aa' & \Sigma a'^2 \end{vmatrix} = 0.$$

CHAPTER XII.

Page 191.

1. $y = x, \quad y = 2x, \quad y = 3x.$
2. $x = 0, \quad y + x = 0, \quad y + x = 1.$
3. $x = \pm a.$
4. $y = x \pm a, \quad y = -x \pm a.$
5. $y = x, y = 2x + 1, \quad y = 2x + 2.$

Page 196.

1. $x + y + a = 0.$
2. $x = 0, y = 0, y = x \pm \sqrt{a^2 + b^2}.$
3. $y = \pm 1, \quad x = 1, x = 2.$
4. $y = \pm x \pm 1.$
5. $x = 0, y = a.$
6. $x = \pm a, \quad y = x + a.$
7. $y + x = 0.$
8. $x = 0.$
9. $x = 2a.$
10. $\left.\begin{array}{l} r \sin(\theta - 1) + \dfrac{a}{2} = 0. \\ r \sin(\theta + 1) + \dfrac{a}{2} = 0 \end{array}\right\}.$
11. $r \sin\theta = a, \quad r\cos\theta = -\dfrac{2a}{(2n+1)\pi}.$
12. $\theta = \dfrac{m\pi}{n}.$
13. $r \sin\theta = a, \quad r\cos\theta = \dfrac{2a\, e^{\frac{2n+1}{2}\pi}}{(2n+1)\pi}.$
14. $r \sin\theta = 2a.$
15. $r \sin\left(\theta - k\dfrac{\pi}{n}\right) = \dfrac{b}{m}.$

THE END.

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