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Electric Power Transmission

M. P. Weinbach LATE PROFESSOR OF ELECTRICAL ENGINEERING
THE UNIVERSITY OF MISSOURI

1948
The Macmillan Company *New York*

PRINTED IN THE UNITED STATES OF AMERICA

Preface

The electrical transfer of energy over long distances for domestic and industrial uses is one of the major problems in the field of electrical engineering.

The dream of the radio dilettante of adapting radio techniques to wireless transmission of power will remain a dream for a long time to come. The conversion of energy from high voltage alternating current to correspondingly high voltage and direct current by means of electronic rectifiers and the consequent elimination of the reactive effects of line inductance and capacitance in the transmission of power is still in the experimental stage. In the experimental stage is also the most recent and possibly the most sanguine method of power transmission by means of wave guides.

The three-phase system of power transmission now universally used has reached its present high and efficient development in a continuous growth over a period of over half a century. The country is virtually covered with three-phase transmission networks operating at voltages ranging from 2.3 kilovolts to 300 kilovolts and transferring large amounts of electrical energy at rates ranging from a few kilowatts to hundreds of thousands of kilowatts over distances of a few miles to nearly 300 miles.

The large capital invested in present systems of transmission and the accumulated experience of half a century of operation of such systems preclude the possibility of discarding such systems for many years to come, even though wave guides or electronic high-voltage rectification prove feasible.

When the length of a transmission line, operated at the commercial frequency of 60 cps, does not exceed about 25 miles, an approximation involving the neglect of line capacitance and leakage conductance gives sufficiently accurate results whether used in the design of the line or in the predetermination of its performance.

The nominal series circuit thus obtained by the neglect of the linear admittance is wholly inadequate in the solution of problems pertaining to lines exceeding this length. It is found, however, that for power lines of lengths roughly between 25 and 100 miles operated at 60 cps, substantially accurate results are obtained by neglecting the fact that the dissipative and reactive properties of the line are uniformly distributed. Problems pertaining to lines of such length are usually solved by the use of either symmetrical T or symmetrical π circuits.

In the nominal symmetrical T circuit the entire linear admittance of the line is assumed lumped half way between the line termini. In the nominal symmetrical π circuit, half of the linear admittance of the line is assumed concentrated at each of the termini, and the whole linear impedance is thus lumped between the two. Such nominally equivalent circuits are used very extensively in the predetermination of the performance of medium long lines.

The various problems pertaining to power transmission become apparently more complicated when the line is of such a length that the effect of the linear distribution of its dissipative and reactive properties cannot be neglected. The rigorous method involving hyperbolic functions of the complex variable becomes essential either in the direct solution of transmission line problems, or through a prior conversion into truly equivalent T or π circuits.

With longer lines there came, of course, the need of higher transmission voltages and the demand for economic transfer of larger volumes of energy at correspondingly higher rates. Such transmission schemes include, in addition to the line proper, the terminal transformers needed for the conversion of voltages from those at which the energy is generated to those demanded by the length of the line, and then again to those required by the distribution system at the receiving end.

The maintenance of a definite voltage at the generating end of lines by means of automatic regulating devices, and at the receiving end or other points through the control of the reactive power by means of synchronous phase modifiers, is a problem to which justifiable attention must be given.

Dependability of service is, of course, paramount in the operation of power supply systems. Interruptions are minimized in number and in severity of effect by systematic studies of faults which might occur at important points in the system. A knowledge of the magnitude of the short-circuit currents due to such faults is quite essential to the successful operation of transmission systems.

The same may be said also with regard to the steady-state power limits and in general of stability characteristics of the system. Unstable situations may occur during sudden changes of load, during switching operations or faults of one kind or another.

All these problems are treated more or less in detail and in the sequence indicated in the Table of Contents to give the student and the practicing engineer a wholesome knowledge of the physical phenomena involved and of the mathematical processes needed in their solution.

Because of the limitation of its scope, description of generating equipment and of voltage control, of schemes of mechanical suspension of lines, of insulation structures, switching apparatus, current-limiting reactors, relay installations, lightning protection, etc., have been omitted. The emphasis is only on what is hoped to be a sound presentation and rational treatment of the electrical problem of transmission.

In the preparation of the material, the author has drawn valuable information pertinent to the various subjects discussed from technical papers, books and other publications by authorities and specialists in one or another phase of this important branch of the profession. Specific cases are given credit by being listed at the place where the material is presented. The lists of references given at the end of the various chapters should prove of great value to the reader who desires additional information on special phases of the subject.

The author wishes to call the readers' attention to the fact that all vector or vector-like quantities such as voltage, current, impedance, admittance and propagation constant, are expressed in this book by single symbols in boldface type to distinguish them from the magnitude of such quantities expressed by the same symbol, but in italics.

The work is the outgrowth of many years of teaching transmission courses at the University of Missouri. Various suggestions pertaining to the treatment of power limits, faults, and instability studies have come to the author from his associates, Professors C. M. Wallis and D. L. Waidelich, who at one time or another have attended his lectures as hearers.

The author wishes to express his gratitude to his many students, and especially to Messrs. Charles Wilhite and Bert Gastineau for checking the illustrative problems. He is particularly indebted to his former student, Stanley Stokes, now Chief Engineer of the Union Electric Company of Missouri, for the data used in the Stability problem in Chapter 10.

M. P. Weinbach

COLUMBIA, MISSOURI

Deep gratitude and appreciation are extended to Mr. Durward Brandt for his care in checking the illustrative figures and his valuable aid in reading the proofs.

Contents

Electric Power Transmission

Chapter 1 Circuit Properties of Transmission Lines

1.1. *General Classification of Lines.* A transmission line is that part of an electrical power system whose function is the transfer of electrical energy from the station where it is generated to a substation wherefrom it is distributed. A transmission line may serve also to transfer energy between two substations of the same system, or between two independent neighboring systems.

A line is said to be *direct* or *alternating* when the current transmitted is, respectively, uni-directional or alternating in character.

In its simplest form, a line consists of two solid or stranded metal conductors of a definite cross-sectional area, having a definite interaxial spacing. These conductors are properly suspended at a definite height above the ground and insulated from each other.

When such a line is supplied at the *sending end* from a single-phase alternator or from one phase of a three-phase alternator, the line is said to be a *single-phase* line. Three-phase lines consisting of either three or four conductors supplied from a three-phase source of energy are used most generally because of economy of copper, greater efficiency of transmission, and greater flexibility in the distribution and utilization of the energy for industrial purposes.

Transmission lines are classified also as *overhead* and *underground*. Underground transmission, usually required in large cities, is more expensive than overhead transmission. Such lines are necessarily short, seldom exceeding 10 to 15 miles in length. The operating voltages for underground lines are usually from 12 to 66 kv although voltages of 132 kv have been successfully used in New York City and Chicago.

1.2. *The Line as a Circuit.* Each conducting wire of a transmission line has a definite resistance which, because of the uniform cross-sectional area of the wire, is uniformly distributed and is therefore fixed in value per unit length. It is through this uniformly distributed resistance of the line conductors that a small portion of the transferred energy is converted into and dissipated as heat in the line itself.

Each conducting wire of a transmission line has also a definite self-inductance. Because of the uniform size of the conductor, of the uniform interaxial spacing between conductors, and of the magnetic homogeneity of the medium between the conductors, this inductance is uniformly distributed and fixed in value per unit length. It is through the agency of this inductance that a magnetic field is established within and about the line conductors by the currents flowing in them. By virtue of this magnetic field the space about the line becomes the seat of magnetic energy which varies in quantity at each point along its length in accordance with the variations of the conductor current at that point.

The voltage drop along a line conductor is a linear function of both the resistance, R, and of the inductive reactance, $L\omega$, of the conductor. The resistance and the inductance of each elemental length of line conductor are thought of, therefore, as though they were in series connection.

The two conducting wires of a line, in their relation to each other, may be thought of also as forming a two-element condenser. Because of the uniform size of the conducting wires, of the uniform interaxial spacing distance, and by assuming dielectric homogeneity of the separating medium, the capacitance between the wires is usually assumed also as distributed uniformly and is thus fixed in value per unit length. The capacitance between the two line conductors may be thought of as being the joint capacitance of two equal condensers in series connection, one on each side of the zero or ground potential plane to conductor. When so considered the capacitance of each conductor to ground potential is twice the capacitance between the two. It is through the agency of this uniformly distributed capacitance that an electric field is established about the line conductors by the difference of potential between them. By virtue of this electric field, the space surrounding the conductors becomes the seat of electric energy which varies in quantity at each point along the line in accordance with the variation of the difference of potential between the conductors at that point.

Although the line conductors are separated by a medium of high insulating quality, there may be, particularly at very high transmission voltages, some current leakage or even a loss of energy due to radiation. Assuming resistive homogeneity of the medium along the entire line length and because of the uniform size of the conductors and of the separating distance between them, the leakage conductance is thought of also as distributed uniformly and fixed in value per unit length. Line conductance is usually neglected in the calculation of the performance of lines operated at voltages up to about 150 kv.

Since the uniformly-distributed high insulation resistance across the line conductors may be considered as consisting of two equal resistances joined in series at the plane of zero (ground) potential, it follows that the leakage conductance per conductor is twice that between the two conductors. Furthermore, since the leakage current between the conductors is a linear function of the leakage conductance, G, and of the capacitive susceptance, $C\omega$, it follows that they must be thought of as being in parallel connection for each elemental length of line conductor.

From what has been said above it follows that, as an electric circuit, a simple two-wire line may be represented diagrammatically as shown in Fig. 1-1.

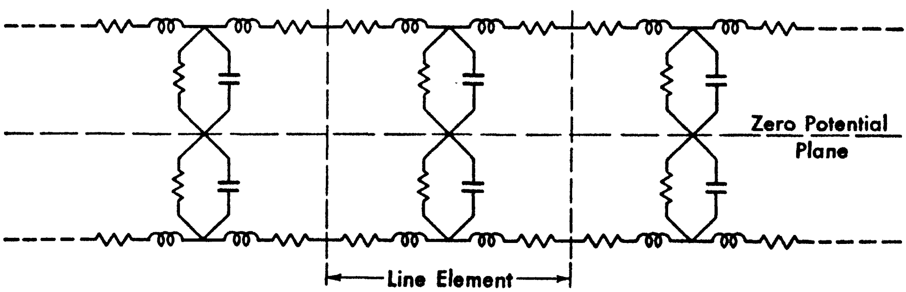


Figure 1-1

1.3. *Conductor Materials.* The most widely used metals for solid or stranded line conductors are copper and aluminum. With the exception of silver, copper has the highest conductivity and its wide use as a conductor material makes it the standard of comparison.

To conform to the specifications of the American Society of Testing Materials, electrically refined copper must have a purity of not less than 99.9 per cent copper including silver. Its electrical conductivity must be not less than 99.3 per cent of the International Annealed Copper Standard (IACS) adopted by the International Electrotechnical Commission (IEC) in 1913.

On the basis of 100 per cent conductivity, a copper wire one meter long and one square millimeter in cross-sectional area has a resistance of 17.241×10^{-8} ohms at 20° C. This is equivalent to 10.371 ohms per circular mil-foot and 54758.88 ohms per circular mil per mile.* The temperature coefficient of electrolytic copper adopted by the IEC is 0.00393 per degree centigrade at 20° C.

The density of electrolytic copper at 20° C. is 8.89 grams per centimeter cube. This is equivalent to 302.699×10^{-8} pounds per circular mil-foot and to 15983×10^{-6} pounds per circular mil per mile at 20° C. The melting point of electrolytic copper is 1083° C. Its coefficient of expansion is 0.000017 per degree centigrade. Its tensile strength is fairly high, about 34,000 pounds per square inch, with an elongation of 40 per cent before rupture.

The conductivity of aluminum is about 60 per cent that of copper. For equal conductance and equal length, an aluminum wire has a cross-sectional area 63 per cent larger than a copper wire. It weighs about half as much as the copper wire, but its diameter is 28 per cent larger. The aluminum wire has, therefore, a much larger surface area exposed to wind pressure and to snow and sleet. Since the tensile strength of aluminum wire is only about 73 per cent that of a copper wire of equal length and conductance, the sag is greater for the aluminum conductor. This necessitates higher supporting towers or poles for the same length of span or shorter spans, and thus more poles or towers.

The resistivity of aluminum is 17.084 ohms per circular mil-foot and is equivalent to 89003.35 ohms per circular-mil per mile.

Its density is 91.83×10^{-8} pounds per circular mil-foot and is equivalent to 0.48487×10^{-2} pounds per circular-mil per mile.

The disadvantages of lower tensile strength and greater cross-sectional area of aluminum conductors are offset to some extent by the use of stranded aluminum cable reinforced with steel. The central strand in a one-layer cable is of steel. If the cable consists of more than one layer, the central strand and first central layer are of steel. This type of cable conductor has found considerable favor in the construction of long lines, and is known under the trade name of A.C.S.R. (Aluminum Cable, Steel Reinforced).

1.4. *Linear Line Impedance.* The resistive property of a circuit is active as long as there is a current flowing in the circuit. The evidence of its activity is the conversion of electrical energy into heat and its dissipation as such. The inductive property of a circuit, on the other hand, comes into play only when the current in the circuit changes in value. Thus in the case of a direct-current circuit, the inductance comes into action during the short period in which the current, and its associated magnetic field, starting from zero, reach their respective maxima. The physical evidence of this inductive action is the

* General Cable Corporation, *Catalog 37.*

field itself, the stored energy therein ($\frac{1}{2} LI^2$) and the generation of a reactive emf ($d\phi/dt$), which retards the growth of both current and the field.

When the circuit carries an alternating current, its inductance is in continuous action. The physical evidence thereof is the alternating magnetic field itself, the cyclic storage and restoration of energy, and the generation of a reactive emf which limits the magnitude of the current. The value of the supplied voltage which balances this reactive emf is $L\omega I$ volts leading the current by 90°. If ϕ is the magnetic flux in webers, and N the number of current paths linking this flux, the voltage may be expressed also by $N\omega\phi$. From what has just been said it follows that

$$LI = N\phi,$$

i.e., the inductance L in henries is numerically equal to the interlinkage per ampere. In the particular case of a line conductor there is only one current path that links the flux. The above formula becomes, accordingly,

$$L = \frac{\phi}{I} \text{ henry.} \tag{1.4.1}$$

To obtain the inductance of a line conductor it is essential, therefore, to determine the current in the conductor and the flux linking it. The flux which links a line conductor may be thought of as consisting of two parts. One, ϕ_i, is due only to the current in the conductor and is confined wholly within the conductor. This part of the whole flux *does not* link with the entire current. The second part, ϕ_o, is produced jointly by the current in the conductor itself and by the currents in all neighboring conductors. It resides wholly outside the conductor and links the entire current. Accordingly the inductance L of a conductor may also be thought of as consisting of two parts. One is due to the flux-current linkage within the conductor itself and the other is due to the flux-current linkage outside the conductor. If L_i and L_o denote these two inductances, respectively, per meter length of conductor, then

$$L = L_i + L_o \text{ henries per meter.} \tag{1.4.2}$$

Designating by R_c the resistance per meter length of conductor to the flow of an alternating current of angular phase velocity ω, then

$$Z = R_c + j(L_i + L_o)\omega \text{ vector ohms/meter} \tag{1.4.3}$$

is the linear impedance per meter length of conductor.

1.5. *Determination of R_c and L_i; Skin Effect.* If the current in the conductor were unidirectional and constant, it would distribute itself uniformly over the sectional area of the conductor. Such a distribution satisfies the requirement that the energy loss, and hence the resistance, be a minimum.* Because of the reactive emf, caused by the flux linkages within the conductor when the current is sinusoidal in character, the distribution of the current is not uniform over the sectional area.

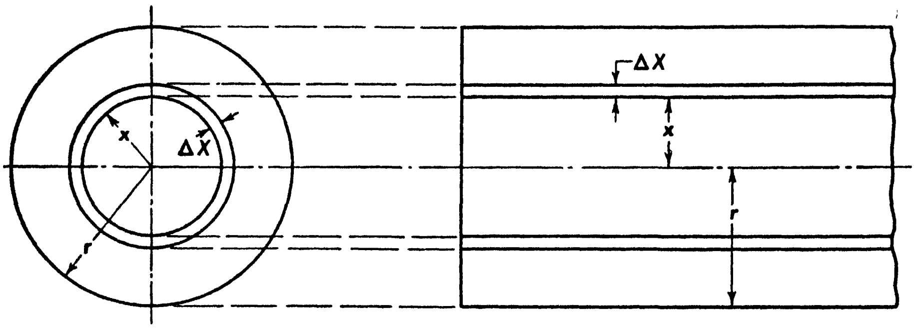


Figure 1–2

Referring to Fig. 1–2, let I'_{xd} be the density of the rms value of the current at some point x meters distant from the center of the conductor. As a function of the distance x from the axis, the current density may be represented to any required degree of accuracy by an infinite series such as

$$I'_{xd} = a + a_1x + bx^2 + a_3x^3 + \cdots$$

The current density I''_{xd} at some point x meters measured in the opposite direction is accordingly

$$I''_{xd} = a - a_1x + bx^2 - a_3x^3 + \cdots$$

But, since the current density is the same at all points equally distant from the center, it follows that the above two expressions are equal. The current density, at all points x distant from the center, is therefore

$$I_{xd} = a + bx^2 + cx^4 + \cdots \tag{1.5.1}$$

where the coefficients a, b, c, etc., are independent of x.

Let Δx be the infinitesimal thickness of a cylindrical shell at x, concentric with the axis of the conductor as shown in Fig. 1–2. Since the annular cross-sectional area $2\pi x\,\Delta x$ of this cylindrical shell is infinitesimally small, the current density I_{xd} over it may be taken as substantially uniform. The total current distributed over this small area is $2\pi I_{xd}x\,\Delta x$, and the total current I_x over the circular section of radius x is

$$I_x = \int_0^x 2\pi I_{xd}x\,dx.$$

* Maxwell, Clark, *Electricity and Magnetism*, Vol. I, p. 408.

Substituting the value of I_{xd} from (1.5.1) and integrating between the stated limits, yields

$$I_x = 2\pi\left(\frac{ax^2}{2} + \frac{bx^4}{4} + \frac{cx^6}{6} + \cdots\right). \tag{1.5.2}$$

The magnetomotive force F_x caused by this current which has only one path with reference to the flux is, in mks units,

$$F_x = I_x \text{ ampere-turns.}$$

The length of the path around which this magnetomotive force is acting is $2\pi x$ meters. The cross-sectional area, normal to the flux density for s meters length of conductor, is $s(\Delta x)$. If μ denotes the permeability of the conductor in mks units, the reluctance of the flux path at x is $2\pi x/\mu s\,\Delta x$ and its value per meter length at x is, therefore,

$$R_x = \frac{2\pi x}{\mu\,\Delta x}.$$

The flux within the cylindrical shell of thickness Δx and length one meter is

$$(\Delta\phi)_x = \frac{F_x}{R_x} = \frac{\mu I_x\,\Delta x}{2\pi x}.$$

Substituting the value of I_x from (1.5.2) gives

$$(\Delta\phi)_x = \mu\left(\frac{ax}{2} + \frac{bx^3}{4} + \frac{cx^5}{6} + \cdots\right)\Delta x.$$

The flux confined within the cylindrical shell of inside radius x and outside radius r, and which links the current I_x distributed over the section of radius x, is

$$\phi_{x-r} = \int_x^r \mu\left(\frac{ax}{2} + \frac{bx^3}{4} + \frac{cx^5}{6} + \cdots\right) dx.$$

This gives

$$\phi_{x-r} = \frac{\mu}{4}\left[ar^2 + \frac{br^4}{2^2} + \frac{cr^6}{3^2} + \cdots - \left(ax^2 + \frac{bx^4}{2^2} + \frac{cx^6}{3^2} + \cdots\right)\right] \text{ webers.} \tag{1.5.3}$$

This is the rms value of a sinusoidally varying flux, and as such it causes the generation of a reactive emf in the section of conductor of radius x. The reactive voltage drop, equal to this emf is $\omega\phi_{x-r}$ volts per meter and leads the current I_x in the section of radius x by 90°.

Let ρ designate the resistivity of the conductor material in ohms per meter cube. The resistance of the cylindrical shell of length one meter and thickness Δx is $\rho/2\pi x\,\Delta x$. Since the current density I_{xd} is substantially uniform over this small area, the total current flowing in this shell is $I_{xd}(2\pi x\,\Delta x)$.

It follows, therefore, that the voltage drop due to the resistance is ρI_{xd} and is the same per meter length at all points x distant from the center.

The voltage drop due to the resistance and to the interlinkage within the conductor shell per meter length is, therefore,

$$V_x = \rho I_{xd} + j\omega\phi_{x-r} \text{ vector volts.} \tag{1.5.4}$$

Substituting the value of ϕ_{x-r} from (1.5.3) and the value of I_{xd} from (1.5.1) gives

$$V_x = \rho(a + bx^2 + cx^4 + \cdots) + \frac{j\mu\omega}{4}\left[ar^2 + \frac{br^4}{2^2} + \frac{cr^6}{3^2} + \cdots - \left(ax^2 + \frac{bx^4}{2^2} + \frac{cx^6}{3^2} + \cdots\right)\right].$$

Separating the r and x terms, the expression becomes

$$V_x = \rho a + \frac{j\mu\omega}{4}\left(ar^2 + \frac{br^4}{2^2} + \frac{cr^6}{3^2} + \cdots\right) + \left(\rho b - \frac{j\mu\omega}{4}a\right)x^2 + \left(\rho c - \frac{j\mu\omega b}{4\cdot 2^2}\right)x^4 + \cdots \tag{1.5.5}$$

But the potential difference between two parallel sections of a conductor is the same for all points of the parallel surfaces. Hence the voltage drop V_1, due to the conductor resistance per meter length R_c, and to the interlinkage within the conductor per meter length, is independent of x and is

$$V_1 = \rho a + \frac{j\mu\omega}{4}\left(ar^2 + \frac{br^4}{2^2} + \frac{cr^6}{3^2} + \cdots\right). \tag{1.5.6}$$

The terms in x in the expression for V_x are, therefore,

$$\left(\rho b - \frac{j\mu\omega a}{4}\right)x^2 + \left(\rho c - \frac{j\mu\omega b}{4\cdot 2^2}\right)x^4 + \cdots = 0.$$

However, since x is not zero, it follows that the coefficients of x must be equal to zero, i.e.,

$$\left.\begin{aligned} b &= \frac{j\mu\omega}{4\rho}\cdot a \\ c &= \frac{j\mu\omega}{4\cdot 2^2\rho}\cdot b = \left(\frac{j\mu\omega}{4\rho}\right)^2\cdot\frac{a}{(2!)^2} \\ d &= \frac{j\mu\omega}{4\cdot 3^2\rho}\cdot c = \left(\frac{j\mu\omega}{4\rho}\right)^3\cdot\frac{a}{(3!)^2}. \end{aligned}\right\} \tag{1.5.7}$$

Setting

$$\frac{\mu\omega}{4\rho} = k,$$

the expression (1.5.6) for V_1 becomes

$$V_1 = \rho a + jk\rho\left(ar^2 + \frac{jkar^4}{(2!)^2} + \frac{(jk)^2ar^6}{(3!)^2} + \cdots\right).$$

Factoring out (ρa) gives

$$V_1 = \rho a\left(1 + jkr^2 + \frac{(jk)^2r^4}{(2!)^2} + \frac{(jk)^3r^6}{(3!)^2} + \cdots\right).$$

By setting $kr^2 = N$, the equation simplifies itself to

$$V_1 = \rho a\left(1 + jN + \frac{(jN)^2}{(2!)^2} + \frac{(jN)^3}{(3!)^2} + \cdots\right). \tag{1.5.8}$$

Equation (1.5.2) gives the current in the section of conductor of radius x. The total current in the conductor of radius r is, accordingly,

$$I = 2\pi\left(\frac{ar^2}{2} + \frac{br^4}{4} + \frac{cr^6}{6}\cdots\right). \tag{1.5.9}$$

Using the values of b, c, d, etc., given by (1.5.7) and since

$$N = \frac{\mu\omega r^2}{4\rho},$$

the above expression for I becomes

$$I = \pi ar^2\left(1 + \frac{2(jN)}{(2!)^2} + \frac{3(jN)^2}{(3!)^2} + \frac{4(jN)^3}{(4!)^2} + \cdots\right). \tag{1.5.10}$$

Solving this equation for a and substituting in equation (1.5.8) gives for the voltage drop V_1 per meter length of conductor

$$V_1 = I\frac{\rho}{\pi r^2}\left(\frac{1 + jN + \frac{(jN)^2}{(2!)^2} + \frac{(jN)^3}{(3!)^2} + \frac{(jN)^4}{(4!)^2} + \cdots}{1 + \frac{2(jN)}{(2!)^2} + \frac{3(jN)^2}{(3!)^2} + \frac{4(jN)^3}{(4!)^2} + \cdots}\right),$$

where $\rho/\pi r^2$ is the d-c resistance R_1 of the conductor per meter length. The expression may be written, therefore,

$$\frac{V_1}{I} = R_1\left(\frac{1 + jN + \frac{(jN)^2}{4} + \frac{(jN)^3}{36} + \frac{(jN)^4}{576} + \frac{(jN)^5}{14400} + \cdots}{1 + \frac{jN}{2} + \frac{(jN)^2}{12} + \frac{(jN)^3}{144} + \frac{(jN)^4}{2880} + \frac{(jN)^5}{86400} + \cdots}\right).$$

Carrying out the division gives

$$\frac{V_1}{I} = R_1\left(1 + \frac{jN}{2} - \frac{(jN)^2}{12} + \frac{(jN)^3}{48} - \frac{(jN)^4}{180} + \frac{13(jN)^5}{8640} - \cdots\right).$$

Since $j = \sqrt{-1}$, the equation becomes

$$\frac{V_1}{I} = R_1\left[\left(1 + \frac{N^2}{12} - \frac{N^4}{180} + \frac{N^6}{2440} + \cdots\right) + j\left(\frac{N}{2} - \frac{N^3}{48} + \frac{13N^5}{8640} + \cdots\right)\right]. \tag{1.5.11}$$

Furthermore, since V_1/I is of the nature of an impedance, it follows that the real part of the above expression is the resistance R_c to the flow of an alternating current per meter length of conductor. The imaginary term is the value of the reactance resulting from the internal interlinkage per meter length of conductor.

Using the value of N given above, the real and imaginary members of (1.5.11) become, respectively,

$$R_c = R_1\left[1 + \frac{1}{12}\left(\frac{\mu\omega r^2}{4\rho}\right)^2 - \frac{1}{180}\left(\frac{\mu\omega r^2}{4\rho}\right)^4 + \frac{1}{2440}\left(\frac{\mu\omega r^2}{4\rho}\right)^6 - \cdots\right] \tag{1.5.12}$$

and

$$L_i\omega = R_1\left[\frac{1}{2}\left(\frac{\mu\omega r^2}{4\rho}\right) - \frac{1}{48}\left(\frac{\mu\omega r^2}{4\rho}\right)^3 + \frac{13}{8640}\left(\frac{\mu\omega r^2}{4\rho}\right)^5 - \cdots\right], \tag{1.5.13}$$

where L_i represents the internal flux linkage per ampere.

Since $r^2 = A/\pi$ and $\mu = 4\pi/10^7$ is the permeability of copper and aluminum in rationalized mks units, the above two expressions may be written

$$R_c = R_1\left[1 + \frac{1}{12}\left(\frac{\omega A}{10^7\rho}\right)^2 - \frac{1}{180}\left(\frac{\omega A}{10^7\rho}\right)^4 + \frac{1}{2440}\left(\frac{\omega A}{10^7\rho}\right)^6 - \cdots\right] \text{ohms/meter},$$
$$L_i\omega = R_1\left[\frac{1}{2}\left(\frac{\omega A}{10^7\rho}\right) - \frac{1}{48}\left(\frac{\omega A}{10^7\rho}\right)^3 + \frac{13}{8640}\left(\frac{\omega A}{10^7\rho}\right)^5 - \cdots\right]$$
$$= \frac{R_1\omega A}{2\rho\, 10^7}\left[1 - \frac{1}{24}\left(\frac{\omega A}{10^7\rho}\right)^2 + \frac{13}{4320}\left(\frac{\omega A}{10^7\rho}\right)^4 - \cdots\right] \tag{1.5.14}$$

and since $R_1 = \rho/A$, this becomes

$$L_i = \frac{0.5}{10^7}\left[1 - \frac{1}{24}\left(\frac{\omega A}{10^7\rho}\right)^2 + \frac{13}{4320}\left(\frac{\omega A}{10^7\rho}\right)^4 - \cdots\right] \text{henry per meter}, \tag{1.5.15}$$

in which A is in square meters and ρ is in ohms per meter cube.

An analysis of equations (1.5.14) and (1.5.15) indicates that both the resistance and the internal inductance of line conductors carrying alternating currents depend upon the electric and magnetic properties of the conductor material, upon the size of the conductor, and the frequency of the supply.

For a definite conductor material such as copper or aluminum, the resistance *increases* with the frequency and with the cross-sectional area in accordance with the law defined by (1.5.14). When $\omega = O$, i.e., when the conductor carries a direct current, the resistance is a minimum. The current density is uniform and in conformity with the requirement for minimum energy loss.

From equation (1.5.15) it is seen that the inductance L_i due to internal linkages *decreases* as the frequency increases. When $\omega = O$, i.e., when the current is direct, the value of L_i is a maximum 0.5×10^{-7} henries per meter of conductor. It is the same for all conductors of nonmagnetic materials and is independent of the size of the conductor. The current density is uniform and is in conformity with the requirement for maximum energy storage.

The increase in resistance and the decrease in the internal inductance with the increase of frequency or size of conductor is equivalent to a virtual reduction in the cross-sectional area of the conductor, so that the requirement for minimum energy loss and maximum energy storage are both satisfied.

Since the flux linkage within the conductor, and hence the internal reactance, is greatest at the axis of the conductor and decreases toward the surface, the effect is as though the current density increases from the axis toward the outer surface or the *skin* of the conductor. For this reason the effect of alternating currents in altering the resistance and the inductance is called *skin effect.**

For cylindrical copper conductors having a resistivity $\rho = 17.241 \times 10^{-9}$ ohms per meter cube and whose cross-sectional areas are expressed in circular mils,† the equation (1.5.14) becomes for a frequency of 60 cycles per second

$$\left(\frac{R_c}{R_1}\right)_{cu} = 1 + 10.23(A_{cm}\,10^{-7})^2 - 83.73(A_{cm}\,10^{-7})^4 + 758.3(A_{cm}\,10^{-7})^6 - \cdots, \tag{1.5.16}$$

where A_{cm} denotes the cross-sectional area of the conductor in circular mils.

For cylindrical aluminum conductors having a resistivity $\rho = 28.28 \times 10^{-9}$ ohms per meter cube operated at 60 cps, equation (1.5.14) may be written in terms of circular mils

$$\left(\frac{R_c}{R_1}\right)_{Al} = 1 + 3.80(A_{cm}\,10^{-7})^2 - 11.56(A_{cm}\,10^{-7})^4 + 38.93(A_{cm}\,10^{-7})^6 - \cdots. \tag{1.5.17}$$

Formulas (1.5.16) and (1.5.17) give the ratio of the a-c resistance to the d-c resistance of copper and aluminum line conductors respectively, when operated at a frequency of 60 cps. The expressions are thus independent of the length of the conductor.

The value of the internal inductance L_i for copper and aluminum line conductors operated at a frequency of 60 cps may also be expressed in terms of corresponding cross-sectional areas stated in circular-mils (A_{cm}). Thus

* Maxwell, *Electricity and Magnetism*, Vol. II, Chap. 13.
Rayleigh, "On Self Induction and Resistance of Straight Conductors," *Phil. Mag.*, 1886, p. 381.
H. B. Dwight, *Transmission Line Formulas*, p. 111.
Gray, Mathews and Macrobert, *Bessel Functions*, p. 172.

† $A = \dfrac{\pi d^2(2.54)^2}{4(10)^{10}}$ where $d^2 = A_{cm}$ in circ. mils.

for copper conductors having a resistivity of 17.241 $\times$ 10^{-9} ohms per meter cube, expression (1.5.15), when multiplied by 5280/3.281, gives

$$(L_i)_{cu} = 80.47 \times 10^{-6} [1 - 5.115(A_{cm}\, 10^{-7})^2 + 45.35(A_{cm}\, 10^{-7})^4 - \cdots] \text{ henry/mile.} \quad (1.5.18)$$

To indicate the application of these formulas to the calculation of R_c/R_1 and of the internal inductance L_i, consider a #0000 AWG copper line operated at a frequency of 60 cps. The cross-sectional area of this conductor is 211,600 circ. mils. Substituting this value of A_{cm} in (1.5.16), gives

$$\frac{R_c}{R_1} = 1 + 10.23(211600 \times 10^{-7})^2 - 83.73(211600 \times 10^{-7})^4 + 758.3(211600 \times 10^{-7})^6 - \cdots$$

$$\frac{R_c}{R_1} = 1 + 0.00458 - 0.0000168.$$

The internal inductance L_i by (1.5.18) is

$$L_i = 80.47 \times 10^{-6}[1 - 5.115(2116 \times 10^{-5})^2 + 45.35(2116 \times 10^{-5})^4 - \cdots]$$

or

$$L_i = 80.47 \times 10^{-6}[1 - 0.002 + 0.000009 - \cdots] \text{ henry/mile.}$$

The result of these calculations indicates that the skin effect for line conductors up to #0000 size and operated at a frequency of 60 cps is not serious and may be neglected. The resistance of such conductors is substantially the same as given in tables. The internal inductance of such conductors is

$$L_i = 80.47 \times 10^{-6} \text{ henry/mile.} \quad (1.5.19)$$

This is substantially equal to that obtained by assuming uniform distribution of current.

Consider further a copper conductor having a cross-sectional area of 1,000,000 circular mils. In this case $(A_{cm}\, 10^{-7}) = 10^{-1}$. Hence

$$\frac{R_c}{R_1} = 1 + (10.23 \times 10^{-2}) - (83.73 \times 10^{-4}) + (758.3 \times 10^{-6}) - \cdots$$

or

$$\frac{R_c}{R_1} = 1 + 0.0947,$$

i.e., an increase of 9.47 per cent in the resistance. Similarly the value of the internal inductance L_i by (1.5.18) is

$$L_i = 80.47 \times 10^{-6} [1 - (5.115 \times 10^{-2}) + (45.35 \times 10^{-4})]$$
$$= 80.47 \times 10^{-6} [1 - 0.0466],$$

i.e., a decrease of 4.66 per cent.

It can be said in general that, for copper and aluminum conductors operating at frequencies of 60 cps or less and of sizes smaller than 300,000 circular mils, the skin effect may be neglected with impunity, but should always be corrected for larger sizes. The increase in resistance for a 300,000 circ.-mil copper conductor is somewhat less than 1 per cent and the decrease in internal inductance is less than 0.5 per cent.

1.6. *Determination of Conductor Inductance Due to the External Flux Interlinkage.* It was stated in § 1.1 that transmission lines consist of two and more generally of three or four conductors. Frequently two or three transmission lines are operated in parallel. The external magnetic flux linking any of the conductors of a system of several conductors is caused jointly by the current in each of the conductors.

Consider Fig. 1–3 in which a number of conductors are geometrically in parallel to each other and normal to the plane of the paper. Let I_1, I_2, I_3, etc., be the rms values of the currents in the conductors similarly numbered. Each of these currents establishes a magnetic field within its own conductor and in the space surrounding the conductor. The effect of the field within any of the conductors was discussed in the preceding article.

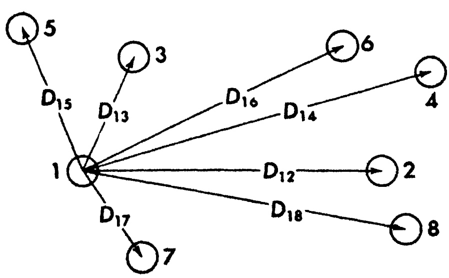


Figure 1–3

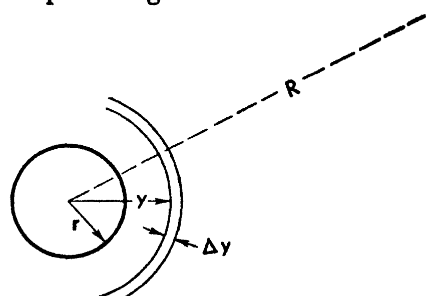


Figure 1–4

The outside flux linking any of the conductors may be obtained by considering the individual fluxes produced independently by each current. Thus, referring to Fig. 1–4, to determine the flux ϕ_{11} caused by I_1 and linking with I_1, let Δy be the width of a cylindrical shell outside the conductor at a distance y meters as indicated. The magnetomotive force acting around the shell is

$$F_y = I_1 \text{ ampere-turns.}$$

The reluctance of the flux path per meter length of conductor is

$$R_y = \frac{2\pi y}{\mu \, \Delta y},$$

where $2\pi y$ is the flux path measured in meters, Δy is the area of the flux path normal to the flux density per meter length of the shell, and μ is the permeability of the medium outside the conductors in rationalized mks units. The flux within the cylindrical shell caused by the current I_1 is

$$(\Delta\phi)_y = \frac{F_y}{R_y} = \frac{I_1\mu\,\Delta y}{2\pi y}.$$

The total flux due to I_1 within a cylindrical space coaxial with conductor (1) and of a very large radius R, per meter length of conductor is

$$\phi_{11} = \frac{\mu I_1}{2\pi}\int_{r_1}^{R}\frac{dy}{y} = \frac{\mu I_1}{2\pi}\ln\frac{R}{r_1}\text{ webers/meter.} \qquad (1.6.1)$$

In this expression r_1 is the radius of conductor (1).

Let D_{13}, D_{15}, etc., respectively, be the interaxial distances between conductors 1 and 3, 1 and 5, and so on. Then the flux produced by I_3, which links current I_1 in the same sense as ϕ_{11}, is

$$\phi_{31} = \frac{\mu I_3}{2\pi}\int_{D_{13}}^{R-D_{13}}\frac{dy}{y}.$$

This, when integrated between the stated limits, gives

$$\phi_{31} = \frac{\mu I_3}{2\pi}\ln\frac{R-D_{13}}{D_{13}}\text{ webers per meter.} \qquad (1.6.2)$$

Similarly the flux produced by the current I_5 and which links with I_1 in the same sense as ϕ_{11} is

$$\phi_{51} = \frac{\mu I_5}{2\pi}\ln\frac{R-D_{15}}{D_{15}}\text{ webers per meter.} \qquad (1.6.3)$$

The respective fluxes produced by the even-numbered currents are obtained in the same manner. Thus, the flux produced by current I_2 and which links with I_1 is

$$\phi_{21} = \frac{\mu I_2}{2\pi}\ln\frac{R-D_{12}}{D_{12}}\text{ webers per meter.} \qquad (1.6.4)$$

Taking α_1, α_2, α_3, etc., the time-phase angles of I_1, I_2, I_3, etc., respectively, to a common reference, the total external flux ϕ_{o1} linking with current I_1, is

$$\phi_{o1} = \phi_{11}\underline{/\alpha_1} + \phi_{21}\underline{/\alpha_2} + \phi_{31}\underline{/\alpha_3} + \cdots.$$

By the preceding relations, this may be written

$$\phi_{o1} = \frac{\mu}{2\pi}\left[(I_1\underline{/\alpha_1})\ln\frac{R}{r_1} + (I_2\underline{/\alpha_2})\ln\frac{R-D_{12}}{D_{12}} + (I_3\underline{/\alpha_3})\ln\frac{R-D_{13}}{D_{13}} + (I_4\underline{/\alpha_4})\ln\frac{R-D_{14}}{D_{14}} + \cdots\right].$$

However, since D_{12}, D_{13}, D_{14}, etc., are very small compared with the assumed large value of R, the preceding expression may be simplified to

$$\phi_{o1} = \frac{\mu}{2\pi}\left[(I_1/\underline{\alpha_1})\ln\frac{R}{r_1} + (I_2/\underline{\alpha_2})\ln\frac{R}{D_{12}} + (I_3/\underline{\alpha_3})\ln\frac{R}{D_{13}} + (I_4/\underline{\alpha_4})\ln\frac{R}{D_{14}} + \cdots\right]$$

webers per meter length.

Converting from meters to miles, using logarithms to the base 10, and $4\pi \times 10^{-7}$ for the permeability of the conductor material in the rationalized mks system of units, the above expression for the external flux linking current I_1 in conductor (1) is

$$\phi_{o1} = 741.13 \times 10^{-6}\left[(I_1/\underline{\alpha_1})\log\frac{R}{r_1} + (I_2/\underline{\alpha_2})\log\frac{R}{D_{12}} + (I_3/\underline{\alpha_3})\log\frac{R}{D_{13}} + \cdots\right]$$

webers per mile.

This may be written

$$\phi_{o1} = 741.13 \times 10^{-6}\,[I_1/\underline{\alpha_1}\,(\log R - \log r_1) + I_2/\underline{\alpha_2}\,(\log R - \log D_{12}) + \cdots]$$

or

$$\begin{aligned}\phi_{o1} = 741.13 \times 10^{-6}\,[&\log R(I_1/\underline{\alpha_1} + I_2/\underline{\alpha_2} + I_3/\underline{\alpha_3} + \cdots)\\ &- (I_1/\underline{\alpha_1}\log r_1 + I_2/\underline{\alpha_2}\log D_{12} + \cdots)].\end{aligned}$$

For the usual case when a complete system of n conductors is considered

$$I_1/\underline{\alpha_1} + I_2/\underline{\alpha_2} + I_3/\underline{\alpha_3} + \cdots + I_n/\underline{\alpha_n} = 0,$$

and the above expression for the flux linking with current I_1 in conductor (1) is

$$\phi_{o1} = -741.13 \times 10^{-6}[I_1/\underline{\alpha_1}\log r_1 + I_2/\underline{\alpha_2}\log D_{12} + I_3/\underline{\alpha_3}\log D_{13} + \cdots] \quad (1.6.5).$$

If L_i is the value of the internal flux-current linkage in conductor 1 per ampere under any form of current distribution, as given for instance by equation (1.5.18), the internal flux is

$$\phi_i = L_i I_1 \text{ webers.}$$

The phase of this flux is the same as the phase of I_1 with respect to the chosen reference.

The total flux linking current I_1 is accordingly

$$\phi_1 = L_i I_1/\underline{\alpha_1} - 741.13 \times 10^{-6}[I_1/\underline{\alpha_1}\log r_1 + I_2/\underline{\alpha_2}\log D_{12} + I_3/\underline{\alpha_3}\log D_{13} + \cdots]. \quad (1.6.6)$$

This expression may be simplified, however, by combining the first term with the first term in the bracket. Thus,

$$\left(L_i + 741.13 \times 10^{-6}\log\frac{1}{r_1}\right)I_1/\underline{\alpha_1} = 741.13 \times 10^{-6}\left(\frac{L_i}{741.13 \times 10^{-6}} + \log\frac{1}{r_1}\right)I_1/\underline{\alpha_1}.$$

Setting for the sake of convenience

$$\frac{L_i}{741.13 \times 10^{-6}} = a, \tag{1.6.7}$$

the preceding equation may be written

$$741.13 \times 10^{-6}\left(\log 10^a + \log \frac{1}{r}\right) = 741.13 \times 10^{-6} \log \frac{1}{r_1/10^a}. \tag{1.6.8}$$

The quantity $r_1/10^a$ is called *geomean radius* of the conductor and will be denoted in this book by the symbol r_{gm}. Its physical significance will become apparent when special cases are considered.

Using the above combination of L_i with the first term of equation (1.6.6) the formula for the flux linking current I_1 in conductor (1) of the system of n conductors becomes

$$\phi_1 = -741.13 \times 10^{-6}[I_1\underline{/\alpha_1} \log r_{1\,gm} + I_2\underline{/\alpha_2} \log D_{12} + I_3\underline{/\alpha_3} \log D_{13} + \cdots].$$

Setting for the sake of symmetry $r_{1\,gm} = D_{11}$, the expression becomes

$$\phi_1 = -741.13 \times 10^{-6}[I_1\underline{/\alpha_1} \log D_{11} + I_2\underline{/\alpha_2} \log D_{12} + I_3\underline{/\alpha_3} \log D_{13} + \cdots]. \tag{1.6.9}$$

The formula for the flux ϕ_k linking the current I_k in any conductor k of a system of n conductors, by analogy is

$$\begin{aligned}\phi_k = -741.13 \times 10^{-6}\,[&I_1\underline{/\alpha_1} \log D_{k1} + I_2\underline{/\alpha_2} \log D_{k2} + I_3\underline{/\alpha_3} \log D_{k3} + \cdots \\ &+ I_k\underline{/\alpha_k} \log D_{kk} + \cdots + I_n\underline{/\alpha_n} \log D_{kn}] \text{ webers/mile}\end{aligned} \tag{1.6.10}$$

in which

$$D_{kk} = r_{kgm} \tag{1.6.11}$$

is the geomean radius of the k^{th} conductor.

The value of the inductance L_k of any conductor k is the ratio of the flux linking it to the current flowing in it, i.e.,

$$L_k = \frac{\phi_k}{I_k} \text{ henry per mile.} \tag{1.6.12}$$

1.7. *Linear Inductance and Impedance of a Two-Wire Line.* Consider a single-phase transmission system consisting of two conductors of equal radius and with a spacing distance D. Since

$$I_1\underline{/\alpha_1} + I_2\underline{/\alpha_2} = 0,$$
$$D_{12} = D_{21} = D,$$

and

$$D_{11} = D_{22} = r_{gm}$$

the inductance of each of the conductors, by equations (1.6.10) and (1.6.12) is

$$L = 741.13 \times 10^{-6}\left(\log \frac{1}{r_{gm}} + \log D\right)$$
$$= 741.13 \times 10^{-6} \log \frac{D}{r_{gm}} \text{ henry/mile.} \qquad (1.7.1)$$

The geomean radius r_{gm} measured in the same units as the spacing distance D is, by equation (1.6.8)

$$r_{gm} = \frac{r}{10^a},$$

where, by (1.6.7),

$$a = \frac{L_i}{741.13 \times 10^{-6}}.$$

From the discussion given in § 1.5 and equation (1.5.19), (L_i) is substantially equal to 80.47 × 10^{-6} henry/mile for all solid conductors either of copper or of aluminum of sizes up to #0000 AWG and operated at a frequency of 60 cps. It follows, therefore, that

$$a = \frac{80.47}{741.13} = 0.1086, \qquad (1.7.2)$$

$$r_{gm} = \frac{r}{10^{0.1086}} = 0.7788r, \qquad (1.7.3)$$

and

$$L = 741.13 \times 10^{-6} \log \frac{D}{0.7788r} \text{ henry/mile.} \qquad (1.7.4)$$

The quantity $0.7788r = r\epsilon^{-0.25}$ is the geometric mean radius of a circular area of radius r. It may be considered, with reference to the value of L, as being the radius of a fictitious conductor within which there is no magnetic flux but which has the same inductance L as the actual conductor.

If R_1 is the resistance per mile of any copper or aluminum conductor of sizes less than 300,000 circular mils, and L, as given by equation (1.7.4), is the inductance per mile of conductor of a two-wire line, the linear line impedance per conductor at 60 cps is

$$Z = R_1 + j741.13 \times 10^{-6}\omega \log \frac{D}{0.7788r} \text{ ohms/mile,} \qquad (1.7.5)$$

where $\omega = 120\pi = 377$.

The *skin* effect on both the resistance and inductance of conductors less than 300,000 circular mils in cross-sectional area and operated at frequencies less than 60 cps is negligibly small.

For conductors of sizes larger than 300,000 circular mils, the skin effect must be taken into consideration through the modified formulas for R and L.

Thus, for a copper conductor of 1,000,000 circular mils, the value of (L_i) as calculated in § 1.5 is

$$L_i = 80.47 \times 0.9534 \times 10^{-6} = 76.728 \times 10^{-6}.$$

Hence,

$$a = \frac{76.728}{741.13} = 0.10353,$$

$$r_{gm} = \frac{r}{10^{0.1035}} = 0.7879r, \tag{1.7.6}$$

and

$$L = 741.13 \times 10^{-6} \log \frac{D}{0.7879r} \text{ henry/mile.} \tag{1.7.7}$$

The resistance per mile of this size conductor modified by the skin effect as calculated in § 1.5 for a frequency of 60 cps, is

$$R_c = 1.0947R_1 \text{ ohms,}$$

where R_1 is the ohmic resistance of the conductor per mile.

The linear line impedance per conductor of this line at a frequency of 60 cps is therefore

$$Z = 1.0947R_1 + j741.13 \times 10^{-6}\omega \log \frac{D}{0.7879r} \text{ ohms/mile.} \tag{1.7.8}$$

1.8. *Inductance and Impedance of Stranded Conductors.*

Stranded conductors or cables consisting of a central strand and one or more concentric or helical layers are frequently used for transmission line conductors because of their flexibility and greater convenience in handling. When the core of a cable consists of only one strand, the number of equal size strands in the first layer is six and the number of strands in each successive layer increases by six. Thus, the third layer has 18 strands and the total number of strands in a two-layer cable is $1 + 6 + 12 = 19$. If a represents the number of layers, and n the number of strands, then

$$n = 3a(a + 1) + 1. \tag{1.8.1}$$

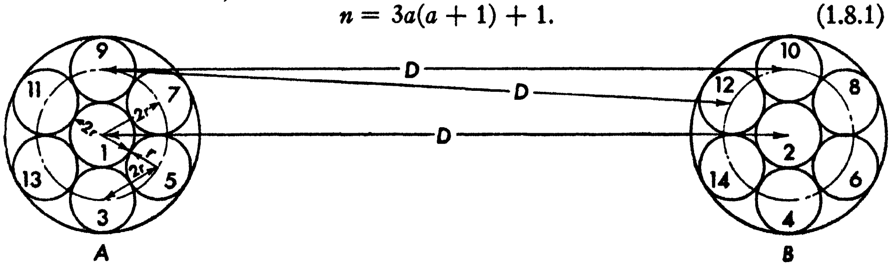


Figure 1–5

A two-cable line with a spacing distance D, as shown in Fig. 1–5, forms a system of conductors similar to the one discussed in § 1.6, provided that the strands are assumed laid concentrically. The procedure for determining the inductance of the cable conductor is as follows:

a. Obtain the inductance of each strand.

b. Adding these inductances and dividing by the number of strands gives the average inductance per strand.

c. Since the strands are electrically in parallel, the total inductance of the cable is the average inductance per strand divided by the number of strands.

The procedure just outlined is illustrated below with the determination of the inductance of a 7-strand cable. To simplify the problem let the strands in cable A in Fig. 1–5 be odd numbered and those in cable B even numbered as indicated.

Assuming uniform distribution, the currents in the strands have the same phase angle and are equal to the total current divided by the number of strands. Cable B carries the return current. From what has just been said it follows that

$$I_1 = I_3 = I_5 = I_7 = I_9 = I_{11} = I_{13}$$

and

$$I_1 = -I_2 = -I_4 = -I_6 = -I_8 = -I_{10} = -I_{12} = -I_{14}.$$

Setting for the sake of convenience

$$741.13 \times 10^{-6} = M,$$

the inductance L_1 for the strand (1) of the cable, by equation (1.6.6) is

$$L_1 = L_i + M\Big[\log\frac{1}{r_s} + \log D_{12} - \log D_{13} + \log D_{14} - \log D_{15} + \log D_{16} - \log D_{17} + \log D_{18} - \log D_{19} + \log D_{110} - \log D_{111} + \log D_{112} - \log D_{113} + \log D_{114}\Big].$$

Combining the logarithmic terms gives

$$L_1 = L_i + M\left[\log\frac{D_{12}}{r_s} + \log\frac{D_{14}}{D_{13}} + \log\frac{D_{16}}{D_{15}} + \cdots + \log\frac{D_{114}}{D_{113}}\right],$$

where r_s is the strand radius and the D's are the spacing distances from strand (1) respectively to the other strands. However, r_s is negligibly small in comparison to D_{12}, D_{14}, D_{16}, etc. Setting, accordingly, $D_{12} = D_{14} = D_{16} = \cdots = D$, and since $D_{13} = D_{15} = D_{17} = \cdots = 2r_s$, the above expression, which gives the inductance of strand (1), simplifies to

$$L_1 = L_i + M\left(\log\frac{D}{r_s} + 6\log\frac{D}{2r_s}\right) \text{ henry/mile.} \tag{1.8.2}$$

The inductance of strand 3, obtained in the same manner, is

$$L_3 = L_i + M\left(\log\frac{D_{34}}{r_s} + \log\frac{D_{36}}{D_{35}} + \log\frac{D_{38}}{D_{37}} + \log\frac{D_{310}}{D_{39}} + \log\frac{D_{312}}{D_{311}} + \log\frac{D_{314}}{D_{313}} + \log\frac{D_{32}}{D_{31}}\right).$$

For reasons given above, it may be said that $D_{36} = D_{38} = D_{310} = \cdots = D$. From the figure it will be seen also that $D_{35} = D_{313}$, $D_{311} = D_{37}$, and $D_{31} = 2r_s$. Accordingly the above expression, which gives the inductance of strand (3), becomes

$$L_3 = L_i + M\left(\log\frac{D}{r_s} + \log\frac{D}{2r_s} + 2\log\frac{D}{D_{35}} + 2\log\frac{D}{D_{37}} + \log\frac{D}{D_{39}}\right). \tag{1.8.3}$$

The equations for the inductance of each of the other five outer strands are identical to that of strand 3. The average value of the inductance L_s per strand for the seven strands of the cable is, therefore,

$$L_s = \tfrac{1}{7}(L_1 + 6L_3),$$

which, by (1.8.2) and (1.8.3), becomes

$$L_s = L_i + M\left(\log\frac{D}{r_s} + \frac{12}{7}\log\frac{D}{2r_s} + \frac{12}{7}\log\frac{D}{D_{35}} + \frac{12}{7}\log\frac{D}{D_{37}} + \frac{6}{7}\log\frac{D}{D_{39}}\right). \tag{1.8.4}$$

This may be written

$$L_s = L_i + M\log\frac{(D)(D)^{\frac{12}{7}}(D)^{\frac{30}{7}}}{(r_s)(2r_s)^{\frac{12}{7}}(D_{35})^{\frac{12}{7}}(D_{37})^{\frac{12}{7}}(D_{39})^{\frac{6}{7}}}.$$

Since the seven strands of the cable are electrically in parallel, the inductance of the seven-strand cable is

$$L_7 = \frac{L_s}{7} = \frac{L_i}{7} + M\log\frac{D}{(r_s)^{\frac{1}{7}}(2r_s)^{\frac{12}{49}}(D_{35})^{\frac{12}{49}}(D_{37})^{\frac{12}{49}}(D_{39})^{\frac{6}{49}}}. \tag{1.8.5}$$

To evaluate the denominator of the logarithmic term, note that the last three members in the denominator may be written

$$(D_{35})^{\frac{12}{49}}(D_{37})^{\frac{12}{49}}(D_{39})^{\frac{6}{49}} = \left[\sqrt[30]{(D_{35})^{12}(D_{37})^{12}(D_{39})^{6}}\right]^{\frac{30}{49}}. \tag{1.8.6}$$

The radical within the bracket will be recognized as the geometric mean of all the possible distances between the centers of the outer strands. The number of possible distances is the sum of the exponents of the D's. For the 7-strand cable it is 30.

By referring to Fig. 1–6 it will be noted also that

$$D_{35} = 2r_s$$
$$D_{37} = 4r_s \cos 30^\circ = 2r\sqrt{3},$$

and

$$D_{39} = 4r_s.$$

Substituting these values in the radical of (1.8.6) gives

$$\sqrt[30]{(D_{35})^{12}(D_{37})^{12}(D_{39})^{6}} = 2r_s(6)^{\frac{1}{5}}. \quad (1.8.7)$$

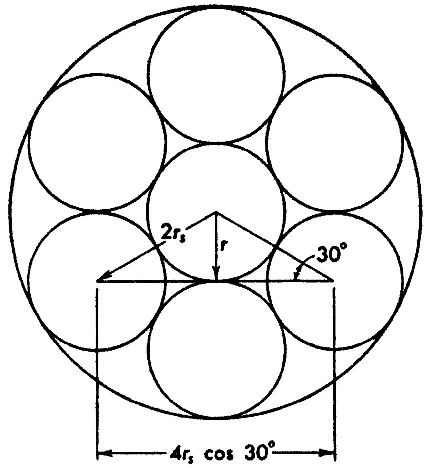


Figure 1–6

From Fig. 1–6 it will be seen also that $2r_s$ is the radius r of the circle around which the centers of the six outer strands are equally spaced. The geomean distance between six points equally spaced around a circle of radius $r = 2r_s$ therefore is given by equation (1.8.7).

More generally, the geomean distance d_{gm} between any number of points m, equally spaced around a circle of radius r is*

$$(d_{gm})_m = r(m)^{\frac{1}{m-1}}. \quad (1.8.8)$$

Thus, for the first layer of a cable of any number of layers, $m_1 = 6$, $r = 2r_s$, and the geomean distance between the centers is as given by (1.8.7). For the second layer, $m_2 = 12$, $r = 4r_s$ and the geomean distance between the centers of the strands, by (1.8.8), is

$$(d_{gm})_{12} = 4r_s(12)^{\frac{1}{11}}.$$

Similarly, for the third layer, $m_3 = 18$ and $r = 6r_s$. The geomean distance between the centers of the strands, by (1.8.8), is

$$(d_{gm})_{18} = 6r_s(18)^{\frac{1}{17}}$$

and so on.

Substituting expression (1.8.7) in (1.8.6) gives for the 7-strand cable

$$(D_{35})^{\frac{12}{49}}(D_{37})^{\frac{12}{49}}(D_{39})^{\frac{6}{49}} = \left[2r_s(6)^{\frac{1}{5}}\right]^{\frac{30}{49}}.$$

This used in equation (1.8.5) gives for the inductance L in henrys per mile of the 7-strand cable

$$L_7 = \frac{L_i}{7} + M \log \frac{D}{(r_s)^{\frac{1}{7}}(2r_s)^{\frac{12}{49}}\left[2r_s(6)^{\frac{1}{5}}\right]^{\frac{30}{49}}}.$$

* Guye, Ch. Eugène. Sur la moyenne distance géométric des éléments d'un ensemble de surfaces. *Comptes Rendue des Séances de l'Académie des Sciences*. Vol. 118, 1894. See appendix for proof of formula (1.8.8).

Factoring out the M and writing

$$\frac{L_i}{7M} = \log 10^{\frac{L_i}{7m}},$$

the equation becomes

$$L_7 = M \log \frac{D}{\dfrac{(r_s)^{\frac{1}{7}}}{(10)^{\frac{L_i}{7m}}} (2r_s)^{\frac{12}{49}} [2r_s(6)^{\frac{1}{5}}]^{\frac{30}{49}}}.$$

Using the values $L_i = 80.47 \times 10^{-6}$ from (1.5.19) and $M = 741.13 \times 10^{-6}$ gives

$$\frac{r_s}{(10)^{\frac{L_i}{M}}} = 0.7788 r_s,$$

whence

$$L_7 = M \log \frac{D}{\{(0.7788r_s)^7 (2r_s)^{2\times 6} [2r_s(6)^{\frac{1}{5}}]^{5\times 6}\}^{\frac{1}{7^2}}}. \tag{1.8.9}$$

It is important to note that the first member within the outside bracket of the denominator is *the geomean radius per strand taken as many times as the number of strands in the cable.* The second member is the geomean distance between the central strand and each strand of the six-strand layer, taken twice as many times as the number of strands in the layer. The last member is the geomean distance between the centers of the six strands in the layer, taken as many times as the number of distances, which in this case is $(m - 1)m = 5 \times 6$. The root of the denominator is the square of the number of strands in the cable, which in this case is 7.

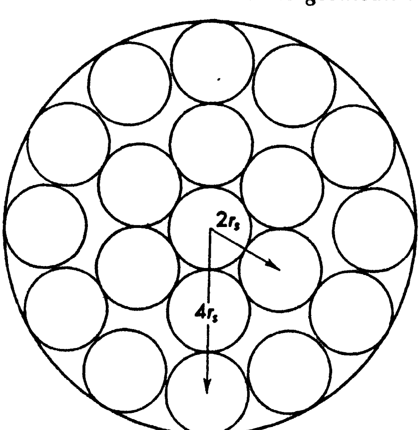


Figure 1–7

The inductance of the 19-strand cable shown in Fig. 1–7 may now be written by analogy:

$$L_{19} = M \log \frac{D}{\{(0.7788r_s)^{19} (2r_s)^{2\times 6} [2r_s(6)^{\frac{1}{5}}]^{5\times 6} (4r_s)^{12\times 7\times 2} [4r_s(12)^{\frac{1}{11}}]^{11\times 12}\}^{\frac{1}{19^2}}}, \tag{1.8.10}$$

where $M = 741.13 \times 10^{-6}$.

It will be noted that the first member within the outside bracket of the denominator is the geomean radius per strand taken as many times as the number of the strands in the cable. The second member is the geomean distance between the central strand and each strand of the first layer, taken twice times the number of strands in the first layer. The third member is the geomean distance between the centers of the strands of the first layer taken $(m_1 - 1)m_1 = 5 \times 6$ times. The fourth term is the geomean distance between the centers of each of the twelve strands of the outer layer and each of the seven strands within this layer, taken $2 \times 7 \times 12$ times.* The last term is the geomean distance between the centers of the strands of the outer layer, taken $(m_2 - 1)m_2 = 11 \times 12$ times.† The root of the denominator is the square of the number of strands.

Carrying out the calculations involved, equation (1.8.9) reduces to

$$L_7 = 741.13 \times 10^{-6} \log \frac{D}{2.176r_s} \text{ henry per mile.}$$

Designating by r_c the radius of the cable, then since $r_c = 3r_s$, the above formula becomes

$$L_7 = 741.13 \times 10^{-6} \log \frac{D}{0.7253r_c} \text{ henry per mile.}$$

The quantity $0.7253r_c$ is the geomean radius $(r_{gm})_7$ of a seven-strand cable.

The generalized formula for the inductance of concentric cables of any number of strands n is

$$L_n = 741.13 \times 10^{-6} \log \frac{D}{(r_{gm})_n} \text{ henry per mile.} \tag{1.8.11}$$

The values of the geomean radii of the more commonly used cables calculated in a manner similar to that outlined in this article are given in the following table:

TABLE I

No. of strands (n)	geomean radius $(r_{gm})_n$
3	$0.6780r_c$
7	$0.7253r_c$
19	$0.7570r_c$
37	$0.7680r_c$
61	$0.7720r_c$
91	$0.7744r_c$
127	$0.7750r_c$
solid conductor	$0.7788r_c$

* It can be shown that the geomean distance from a circle of radius r to all points within it is equal to the radius r.

† A generalized formula for the determination of the value of L for cables with any number of layers is given in H. B. Dwight's *Transmission Line Formulas*, p. 126.

1.9. ***Inductance per Conductor of Twin Single-Phase Lines.*** Consider a transmission system consisting of two single-phase lines in parallel connection as shown in Fig. 1–8. Let r_1 be the radius of conductors 1 and 2 forming one line, and

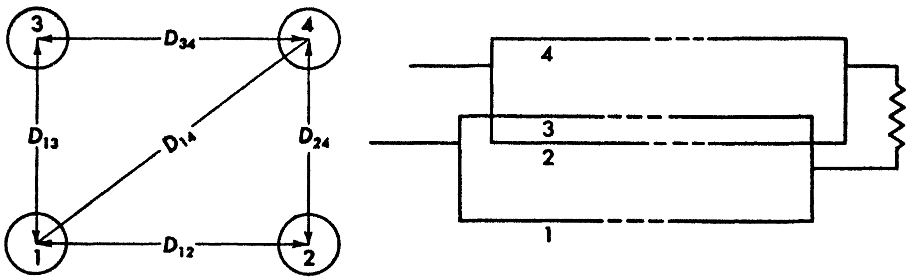


Figure 1–8

r_3 the radius of conductors 3 and 4 forming the other line. The spacing distances are as indicated in the figure. Since conductor 2 is the return of 1 and conductor 4 the return of 3, it follows that

$$I_1/\underline{\alpha_1} = -I_2/\underline{\alpha_2}$$

and

$$I_3/\underline{\alpha_3} = -I_4/\underline{\alpha_4}.$$

By direct application of the generalized formula (1.6.10) the flux ϕ_1 linking current I_1 in conductor (1) is

$$\phi_1 = 741.13 \times 10^{-6}\left(I_1/\underline{\alpha_1} \log \frac{D_{12}}{r_{1\,gm}} + I_3/\underline{\alpha_3} \log \frac{D_{14}}{D_{13}}\right). \tag{1.9.1}$$

The flux ϕ_3 linking current I_3 in conductor 3, similarly, is

$$\phi_3 = 741.13 \times 10^{-6}\left(I_3/\underline{\alpha_3} \log \frac{D_{34}}{r_{3\,gm}} + I_1/\underline{\alpha_1} \log \frac{D_{32}}{D_{31}}\right). \tag{1.9.2}$$

Referring to the figure, if $D_{12} = D_{34} = D$ and $D_{13} = D_{24}$, then setting

$$\frac{D}{D_{13}} = a,$$

the above two expressions become, respectively,

$$\phi_1 = M\left(I_1/\underline{\alpha_1} \log \frac{D}{r_{1\,gm}} + I_3/\underline{\alpha_3} \log\sqrt{a^2+1}\right) \tag{1.9.3}$$

$$\phi_3 = M\left(I_3/\underline{\alpha_3} \log \frac{D}{r_{3\,gm}} + I_1/\underline{\alpha_1} \log\sqrt{a^2+1}\right), \tag{1.9.4}$$

where $M = 741.13 \times 10^{-6}$.

Since conductors 1 and 3, in parallel with each other, are of equal length, the voltage drop per mile of each will be equal. Hence if R_1 is the resistance of conductor 1 per mile and R_3 is the resistance of conductor 3 per mile, then

$$I_1(R_1 + jL_1\omega) = I_3(R_3 + jL_3\omega). \tag{1.9.5}$$

But

$$L_1 = \frac{\phi_1}{I_1} \quad \text{and} \quad L_3 = \frac{\phi_3}{I_3},$$

where ϕ_1 and ϕ_3 are the scalar values of equations (1.9.3) and (1.9.4), respectively. It follows from the above that

$$I_1R_1 + j\phi_1\omega = I_3R_3 + j\phi_3\omega \tag{1.9.6}$$

whence

$$I_1R_1 = I_3R_3 \tag{1.9.7}$$

and

$$\phi_1 = \phi_3. \tag{1.9.8}$$

This indicates that irrespective whether the two lines are identical or not, the scalar values of the fluxes linking the currents are equal. If the conductors are of the same size, the currents will also be equal, and the inductance of the conductors will be equal. Under the conditions just stated, expressions (1.9.3) and (1.9.4) simplify to

$$\phi_1 = \phi_3 = MI\left(\log \frac{D\sqrt{a^2+1}}{r_{gm}}\right) \text{webers/mile} \tag{1.9.9}$$

and

$$L_1 = L_3 = 741.13 \times 10^{-6}\left(\log \frac{D\sqrt{a^2+1}}{r_{gm}}\right) \text{henry/mile}. \tag{1.9.10}$$

If the lines are not identical in size and in design, the flux linking the currents in the conductors will be equal, but since the currents are not equal, the inductances will be unequal. The vector values of the currents in the two parallel conductors must be known to determine the magnitude of the fluxes and the corresponding inductances.

The two lines in parallel connection may be disposed also with reference to each other in flat spacing as indicated in Fig. 1-9.

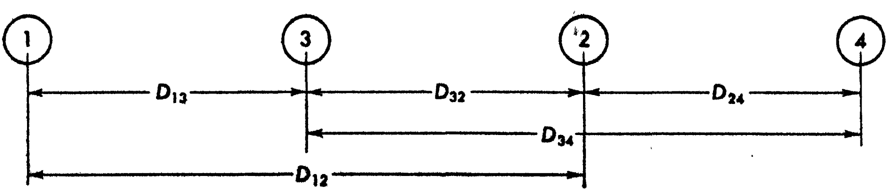


Figure 1-9

The flux ϕ_1 linking current I_1 in conductor 1 is given by equation (1.9.1) and the flux ϕ_3 linking current I_3 in conductor 3 is given by equation (1.9.2). Referring to the figure, however, let

$$D_{12} = D_{34} = D$$

and

$$D_{13} = D_{24} = aD,$$

then

$$D_{14} = D_{13} + D_{34} = D(1 + a)$$

and

$$D_{32} = D_{34} - D_{24} = D(1 - a).$$

Assuming further that the conductors are of the same size, so that their geomean radii are equal, the expressions for ϕ_1 and ϕ_3, become, respectively

$$\phi_1 = 741.13 \times 10^{-6}\left(I_1\underline{/\alpha_1}\log\frac{D}{r_{gm}} + I_3\underline{/\alpha_3}\log\frac{1+a}{a}\right) \tag{1.9.11}$$

$$\phi_3 = 741.13 \times 10^{-6}\left(I_3\underline{/\alpha_3}\log\frac{D}{r_{gm}} + I_1\underline{/\alpha_1}\log\frac{1-a}{a}\right) \tag{1.9.12}$$

It should be noted from these two expressions that, although the conductors were assumed identical, the flat-spacing arrangement causes unequal flux-current linkages per ampere in each conductor. Because of this inequality in inductance values, the linear impedance of the conductors will be unequal. The currents will be unequal, therefore, and out of phase.

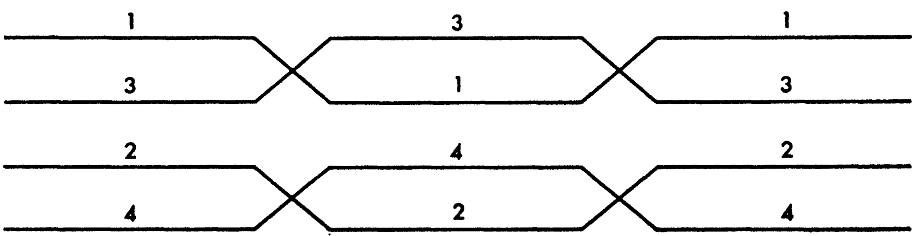


Figure 1-10

The inductance per conductor of twin single phase lines with flat spacing may be balanced more or less, and the current values equalized by proper transposition, as indicated in Fig. 1–10.

1.10. *Inductance of Nontransposed Three-Phase Lines.* Consider a nonsymmetrically spaced three-phase line having conductors of the same size as is shown in Fig. 1–11. The flux linking the currents in the three conductors may be obtained by a direct application of the generalized equation (1.6.10). Thus,

$$\phi_1 = M\left[I_1\underline{/\alpha_1}\log\frac{1}{r_{gm}} - (I_2\underline{/\alpha_2}\log D_{12} + I_3\underline{/\alpha_3}\log D_{13})\right]$$
$$\phi_2 = M\left[I_2\underline{/\alpha_2}\log\frac{1}{r_{gm}} - (I_1\underline{/\alpha_1}\log D_{21} + I_3\underline{/\alpha_3}\log D_{23})\right]$$
$$\phi_3 = M\left[I_3\underline{/\alpha_3}\log\frac{1}{r_{gm}} - (I_1\underline{/\alpha_1}\log D_{31} + I_2\underline{/\alpha_2}\log D_{32})\right]$$

where $M = 741.13 \times 10^{-6}$.

For a balanced system the currents are numerically equal and 120° apart in time phase. Taking ϕ_1 as the reference, the flux ϕ_2 lags ϕ_1 by 120°, and flux ϕ_3 lags ϕ_1 by 240°. To bring these two fluxes in phase with ϕ_1, multiply the second equation by $\underline{/120°}$ and the third by $\underline{/240°}$. The above equations may be written, accordingly:

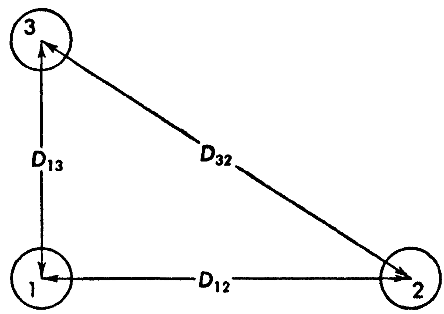


Figure 1-11

$$\phi_1 = MI_1\left[\left(\log\frac{1}{r_{gm}}\right)\underline{/0°} - (\log D_{12})\underline{/-120°} - (\log D_{13})\underline{/-240°}\right]$$
$$\phi_2 = MI_2\left[\left(\log\frac{1}{r_{gm}}\right)\underline{/0°} - (\log D_{21})\underline{/-240°} - (\log D_{23})\underline{/-120°}\right]$$
$$\phi_3 = MI_3\left[\left(\log\frac{1}{r_{gm}}\right)\underline{/0°} - (\log D_{31})\underline{/-120°} - (\log D_{32})\underline{/-240°}\right].$$

Carrying out the vector additions and collecting terms yields

$$\phi_1 = MI\left[\log\frac{(D_{12}D_{13})^{\frac{1}{2}}}{r_{gm}} + j\frac{\sqrt{3}}{2}\log\frac{D_{12}}{D_{13}}\right] \text{ webers/mile.} \quad (1.10.1)$$
$$\phi_2 = MI\left[\log\frac{(D_{21}D_{23})^{\frac{1}{2}}}{r_{gm}} + j\frac{\sqrt{3}}{2}\log\frac{D_{23}}{D_{21}}\right] \text{ webers/mile.} \quad (1.10.2)$$
$$\phi_3 = MI\left[\log\frac{(D_{31}D_{32})^{\frac{1}{2}}}{r_{gm}} + j\frac{\sqrt{3}}{2}\log\frac{D_{31}}{D_{32}}\right] \text{ webers/mile.} \quad (1.10.3)$$

The scalar values of these fluxes are of the form

$$\phi = MI\sqrt{A^2 + B^2} \quad (1.10.4)$$

where A is the real and B the quadrature member in the brackets. Thus, for

$$A_1 = \log\frac{(D_{12}D_{13})^{\frac{1}{2}}}{r_{gm}}$$

and

$$B_1 = \frac{\sqrt{3}}{2}\log\frac{D_{12}}{D_{13}}.$$

The inductance of conductor 1 is, accordingly,

$$L_1 = \frac{\phi_1}{I} = M\sqrt{A_1^2 + B_1^2}.$$

The inductances of the other two conductors are calculated in the same manner.

From the above formulas it is seen that the inductances of the conductors of a nonsymmetrically-spaced three-phase line are unequal under balanced condition of currents. Because of this inequality, the impedance of each of the three conductors will also be unequal, resulting in unbalanced voltages at the receiving end. For this reason the conductors of nonsymmetrically-spaced lines are usually transposed, as indicated in Fig. 1–12.

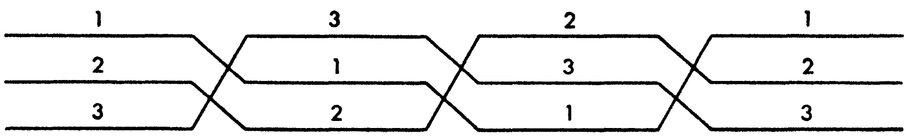


Figure 1–12

When a nonsymmetrically-spaced three-phase line is transposed, the conductors occupy each other position for one-third the length of line. The average flux linking each current is therefore

$$\phi = \frac{\phi_1 + \phi_2 + \phi_3}{3}.$$

This, by (1.10.1), (1.10.2) and (1.10.3) gives

$$\phi = MI \log \frac{(D_{12}D_{13}D_{23})^{\frac{1}{3}}}{r_{gm}} \text{ webers/mile,}$$

and the inductance of each conductor is

$$L = \frac{\phi}{I} = 741.13 \times 10^{-6} \log \frac{(D_{12}D_{13}D_{23})^{\frac{1}{3}}}{r_{gm}} \text{ henry/mile.} \qquad (1.10.5)$$

It is of interest to note that the numerator of the logarithmic term is the *geomean spacing distance* between the three conductors.

Formula (1.10.5) applies also to transposed three-phase lines when flat spacing either horizontal or vertical is used. From Fig. 1–13 it is seen that $D_{12} = D_{23} = D$ and $D_{13} = 2D$. The geomean spacing distance is

$$D_{gm} = (2D \cdot D \cdot D)^{\frac{1}{3}} = 1.26D.$$

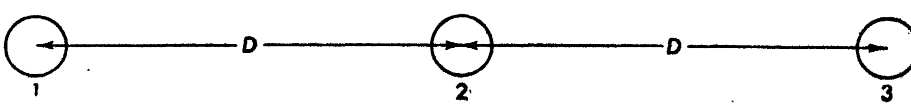


Figure 1–13

The average value of the inductance per conductor of such a flat-spaced line is therefore

$$L = 741.13 \times 10^{-6} \log \frac{1.26D}{r_{gm}} \text{ henry/mile.} \tag{1.10.6}$$

If the spacing is equilateral, i.e., if the three conductors are at the corners of an equilateral triangle, then $D_{12} = D_{13} = D_{23}$ and formula (1.10.5) becomes

$$L = 741.13 \times 10^{-6} \log \frac{D}{r_{gm}} \text{ henry/mile.} \tag{1.10.7}$$

For this reason, the geomean distance of unsymmetrically-spaced transposed three-phase lines is often referred to as *equivalent equilateral spacing.*

1.11. *Line Inductance Table.* Line inductance and more generally the inductive reactance per mile of conductor of standard sizes, and with standard equilateral or equivalent equilateral spacing distances and at commercial frequencies of 25 and 60 cps, are usually given in reference and handbooks of electrical engineering. The inductance per mile of conductor of any specific size, stranding and with any spacing may, obviously, be computed from formula (1.10.7).

A working table of line inductance values per mile of conductor for various size conductors and spacing distances, may be formulated with ease by writing formula (1.10.7) in the form

$$L = 741.13 \times 10^{-3} \log \frac{1}{r_{gm}} + 741.13 \times 10^{-3} \log D \text{ mh/mile.} \tag{1.11.1}$$

It is obvious that the first term of this expression gives the line inductance L_a per mile of conductor having a geomean radius r_{gm} feet and with equilateral or equivalent equilateral spacing distance $D = 1$ foot. The second term of the above expression may be interpreted as the line inductance L_b per mile of conductor having a geomean radius of 1 foot, and whose equilateral or equivalent spacing distance is any value D feet.

Thus Table II gives the calculated values of L_a in milli-henry for stranded copper conductors as indicated and for $D = 1$ foot.

Table III gives the values of L_b in milli-henry for conductors of $r_{gm} = 1$ foot and equilaterally spaced from 1 to 30 feet in steps of 1 foot.

Similar tables may be formulated for solid copper conductors and for solid and stranded aluminum conductors.

To illustrate the use of these tables consider a 400000 cir. mil line having an equilateral spacing distance $D = 15$ ft. From Table II, get $L_a = 1.2155$ mh; from Table III similarly get $L_b = 0.8716$ mh. The inductance per mile of conductor is therefore $L = 1.2155 + 0.8716 = 2.0871$ mh.

TABLE II

A.W.G. or Cir. Mils	Strands	r mils	r_{gm} feet	$D = 1$ foot $L_a = 0.74113 \log \frac{1}{r_{gm}}$ mh/mile
0	7	184	0.0111	1.4488
00	7	202	0.0124	1.4130
000	7	232	0.0140	1.3740
0000	7	261	0.0158	1.3350
250000	19	287.5	0.0181	1.2913
300000	19	314	0.0198	1.2624
350000	19	338	0.0213	1.2389
400000	19	362.5	0.0229	1.2155
450000	37	384	0.0246	1.1925
500000	37	407	0.0260	1.1747
600000	37	446.5	0.0285	1.1452
700000	61	482	0.0311	1.1170
750000	61	499	0.0321	1.1068
800000	61	515.5	0.0332	1.0960
900000	61	546.5	0.0352	1.0772
1000000	61	575.5	0.0370	1.0611

TABLE III

$r_{gm} = 1$ foot
$L_b = 0.74113 \log D$ mh/mile

D Feet → ↓	0	10	20
1	0	0.7718	0.9799
2	0.2231	0.7998	0.9949
3	0.3536	0.8256	1.0092
4	0.4462	0.8494	1.0229
5	0.5180	0.8716	1.0362
6	0.5767	0.8924	1.0487
7	0.6263	0.9119	1.0608
8	0.6693	0.9303	1.0725
9	0.7072	0.9477	1.0839
10	0.7411	0.9642	1.0947

1.12. *Inductance of Twin Three-Phase Lines.* Consider two nontransposed unsymmetrically-spaced three-phase lines to be operated in parallel. Assume conductors of the same size and disposed with reference to each other in clockwise sequence as indicated in Fig. 1–14. Conductors 1, 2, and 3

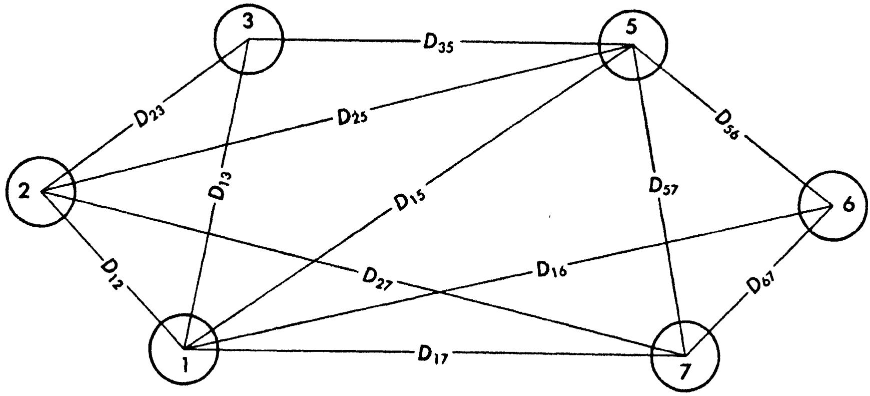


Figure 1–14

carrying respectively currents $I_1/\underline{\alpha_1}$, $I_2/\underline{\alpha_2}$, and $I_3/\underline{\alpha_3}$ form one line, and conductors 5, 6, and 7 carrying respectively currents $I_5/\underline{\alpha_5}$, $I_6/\underline{\alpha_6}$, and $I_7/\underline{\alpha_7}$, form the other line. For parallel operation, the conductors of the same phase are connected in parallel, i.e., conductors 1 and 5, 2 and 6, and 3 and 7 are, respectively, connected in parallel, as indicated schematically in Fig. 1–15.

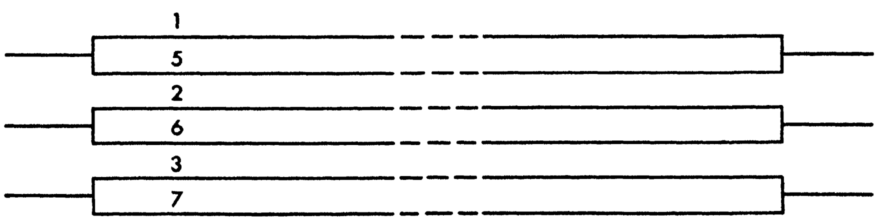


Figure 1–15

To determine the fluxes linking the conductors, apply the generalized formula (1.6.10). Assuming a balanced system, the currents are numerically equal and 120° apart for the three phases. Thus, the flux linking with the current in conductor 1, by (1.6.10) is

$$\phi_1 = MI\{(\log \frac{1}{r_{gm}}/\underline{0^\circ} - [(\log D_{12})/\underline{-120^\circ} + (\log D_{13})/\underline{-240^\circ} + (\log D_{15})/\underline{0^\circ}$$
$$+ (\log D_{16})/\underline{-120^\circ} + (\log D_{17})/\underline{-240^\circ}]\}. \quad (1.12.1)$$

Carrying out the vector additions, and combining terms, yields

$$\phi_1 = MI\left[\log \frac{1}{r_{gm}D_{15}} + \log (D_{12}D_{16})^{\frac{1}{2}} + j\frac{\sqrt{3}}{2}\log (D_{12}D_{16}) + \log (D_{13}D_{17})^{\frac{1}{2}} - j\frac{\sqrt{3}}{2}\log (D_{13}D_{17})\right].$$

This simplifies further to

$$\phi_1 = MI\left[\log \frac{(D_{12}D_{13}D_{16}D_{17})^{\frac{1}{2}}}{D_{15}r_{gm}} + j\frac{\sqrt{3}}{2}\log \frac{D_{12}D_{16}}{D_{13}D_{17}}\right]. \tag{1.12.2}$$

Note that ϕ_1 and ϕ_5 are in phase with each other. The fluxes ϕ_2 and ϕ_6 lag ϕ_1 by 120°. Similarly ϕ_3 and ϕ_7 lag ϕ_1 by 240°. To bring the five fluxes in phase with ϕ_1, multiply the equations for ϕ_2 and ϕ_6 by $\underline{/120°}$ and the equations for ϕ_3 and ϕ_7 by $\underline{/240°}$. The equations for the other five fluxes, by (1.6.10), become accordingly:

$$\phi_2 = MI\left[\log \frac{(D_{23}D_{27}D_{21}D_{25})^{\frac{1}{2}}}{D_{26}r_{gm}} + j\frac{\sqrt{3}}{2}\log \frac{D_{23}D_{27}}{D_{21}D_{25}}\right] \tag{1.12.3}$$

$$\phi_3 = MI\left[\log \frac{(D_{31}D_{35}D_{32}D_{36})^{\frac{1}{2}}}{D_{37}r_{gm}} + j\frac{\sqrt{3}}{2}\log \frac{D_{31}D_{35}}{D_{32}D_{36}}\right] \tag{1.12.4}$$

$$\phi_5 = MI\left[\log \frac{(D_{52}D_{56}D_{53}D_{57})^{\frac{1}{2}}}{D_{51}r_{gm}} + j\frac{\sqrt{3}}{2}\log \frac{D_{52}D_{56}}{D_{53}D_{57}}\right] \tag{1.12.5}$$

$$\phi_6 = MI\left[\log \frac{(D_{63}D_{67}D_{61}D_{65})^{\frac{1}{2}}}{D_{62}r_{gm}} + j\frac{\sqrt{3}}{2}\log \frac{D_{63}D_{67}}{D_{61}D_{65}}\right] \tag{1.12.6}$$

$$\phi_7 = MI\left[\log \frac{(D_{71}D_{75}D_{72}D_{76})^{\frac{1}{2}}}{D_{73}r_{gm}} + j\frac{\sqrt{3}}{2}\log \frac{D_{71}D_{75}}{D_{72}D_{76}}\right]. \tag{1.12.7}$$

The scalar values of these fluxes are of the form

$$\phi = MI\sqrt{A^2 + B^2},$$

where A is the real and B the quadrature member in the brackets. Thus, for ϕ_1

$$A_1 = \log \frac{(D_{12}D_{13}D_{16}D_{17})^{\frac{1}{2}}}{D_{15}r_{gm}}$$

and

$$B_1 = \frac{\sqrt{3}}{2}\log \frac{D_{12}D_{16}}{D_{13}D_{17}}.$$

The inductance of conductor 1 is, therefore,

$$L_1 = \frac{\phi_1}{I} = M\sqrt{A^2 + B^2},$$

where $M = 741.13 \times 10^{-6}$.

The above formulas for the fluxes linking the currents in the conductors indicate that although the currents are balanced, the inductance values of the line conductors are unequal and will, therefore, cause unbalanced line voltages at the receiving end. To equalize to some extent the inductance values, a transposition scheme such as indicated schematically in Fig. 1–16

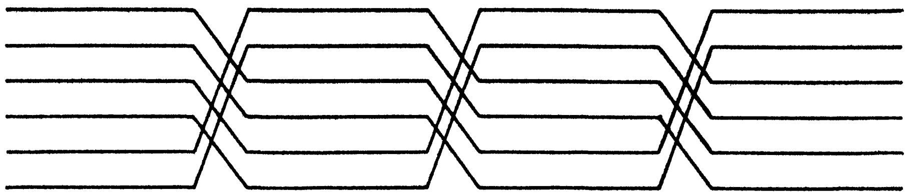


Figure 1–16

is used. The conductors occupy each other's position for one-sixth the entire length of line. The average flux linking each current is, therefore,

$$\phi = \frac{\phi_1 + \phi_2 + \phi_3 + \phi_5 + \phi_6 + \phi_7}{6}.$$

Substituting the preceding equations it is found that the sum of the j-components drops out, and collecting terms gives

$$\phi = MI \log \frac{(D_{12}D_{13}D_{16}D_{17}D_{23}D_{25}D_{27}D_{35}D_{36}D_{56}D_{57}D_{67})^{\frac{1}{6}}}{r_{gm}(D_{15}D_{26}D_{37})^{\frac{1}{3}}}. \qquad (1.12.8)$$

The value of the inductance per conductor is

$$L = \frac{\phi}{I} \text{ henry/mile.}$$

If the spacing is equilateral and equal for the two lines, then

$$D_{23} = D_{31} = D_{56} = D_{67} = D_{75} = D_{12}$$
$$D_{37} = D_{15}$$
$$D_{36} = D_{16}$$
$$D_{35} = D_{17}$$
$$D_{27} = D_{25}$$

and the formula for the inductance per conductor is

$$L = 741.13 \times 10^{-6} \log \frac{(D_{12}{}^6 D_{16}{}^2 D_{17}{}^2 D_{25}{}^2)^{\frac{1}{6}}}{r_{gm}(D_{15}{}^2 D_{26})^{\frac{1}{3}}}$$

or

$$L = 741.13 \times 10^{-6} \log \frac{D_{12}}{r_{gm}} \left(\frac{D_{16}D_{17}D_{25}}{D_{15}{}^2 D_{26}} \right)^{\frac{1}{3}} \text{ henry/mile.} \qquad (1.12.9)$$

1.13. *Capacitance. General Formula.* The capacitive properties of a circuit come into play when the difference of potential across the circuit changes in value. Thus, in the case of a circuit subject to a constant, uni-directional difference of potential, the capacitance acts only during the short interval in which the electric charge and the electric flux associated with it, starting from zero, reach their respective maxima. The physical evidence of this capacitive action is the establishment of an electric field, the stored energy ($\frac{1}{2}CV^2$) therein, and the generation of a reactive emf (ψ/C) which retards the growth of the field to its maximum value.

When the difference of potential impressed upon the circuit is alternating in character, the capacitance of the circuit is in continuous action. The physical evidence of this continuous action is the establishment of an alternating field, the cyclic storage and restoration of electric energy in and from the field, and the generation of a continuously-acting reactive emf which limits the charging current. The value of the supplied voltage which balances the reactive emf is $V = I_q/C\omega$ volts and lags the charging current I_q by 90°.

If ψ is the rms value of the sinusoidally varying electric flux measured in coulombs, the instantaneous charging current is

$$i_q = \frac{d}{dt}(\sqrt{2}\,\psi \sin \omega t).$$

This gives for the rms or effective value of the charging current

$$I_q = \omega\psi \text{ amperes}$$

which, when substituted in the voltage expression $V = I_q/C\omega$ gives

$$C = \frac{\psi}{V} \text{ farads.} \tag{1.13.1}$$

This formula defines the capacitance in farads as being numerically equal to the flux (or charge) in coulombs per volt difference of potential impressed upon the circuit. Because of the unknown value of ψ, this formula, as it stands, does not lend itself directly to the determination of the capacitance for any specific system of conductors. However, since a difference of potential is the negative line integral of the field intensity vector

$$V = -\int F\,dx, \tag{1.13.2}$$

the above formula for the capacitance may be written

$$C = \frac{\psi}{-\int F\,dx}. \tag{1.13.3}$$

In this formula F is the field intensity vector or potential gradient, measured in volts per meter at any point x in the field.

Furthermore, the ratio of the flux density ψ_d* in coulombs per sq. meter to the field intensity F in volts per meter at any point in an electric field

* To avoid confusion the symbol ψ_d is used here for electric flux density in preference to the more usual D, which designates in this book the spacing distance between conductors.

depends only upon the dielectric properties of the medium in which the field resides. The value of this ratio

$$\frac{\psi_d}{F} = \epsilon \tag{1.13.4}$$

is called the ***permittivity of the medium***. The permittivity of free space and substantially of air in the rationalized mks system of units is

$$\epsilon_o = \frac{1}{4\pi \times 9 \times 10^9}. \tag{1.13.5}$$

The permittivity of media other than air is

$$\epsilon = \epsilon_r \epsilon_o = \frac{\epsilon_r}{4\pi \times 9 \times 10^9}, \tag{1.13.6}$$

where ϵ_r, called *relative permittivity*, is numerically equal to the *dielectric constant* of the medium as given in various handbooks on electrical engineering and *Smithsonian Physical Tables*.

Substituting (1.13.4) in the relation given by (1.13.3) gives

$$C = \frac{\epsilon\psi}{-\int \psi_d \, dx}. \tag{1.13.7}$$

This expression can be evaluated with great facility because the flux density ψ_d depends upon the geometry of the electrical structure considered. This is illustrated in the next article.

1.14. *Capacitance of Single-Conductor Cables.* A single-conductor cable consists of a solid or stranded conductor properly insulated and enclosed coaxially in a cylindrical lead sheath, as shown in Fig. 1–17.

The structure forms a condenser with a definite capacitance per unit length. To determine the value of C, let r_1 be the radius of the conductor and r_2 the inside radius of the lead sheath. Let V be the potential difference between conductor and sheath and ψ the corresponding electric flux in the medium between the two per meter length of cable. If the cable is assumed to be quite long, the charge imparted to the conductor will be uniformly distributed over its length. The field

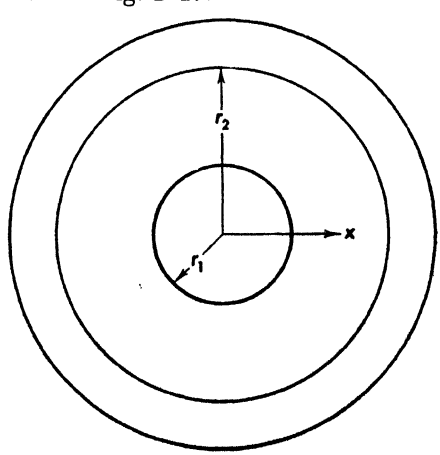


Figure 1–17

intensity vector F will be radial to the conductor axis, and the flux will be uniformly distributed over any cylindrical surface of radius x, smaller than r_2 and coaxial with the conductor. The area of such a cylindrical surface per meter length is $2\pi x$. Hence, if ψ is the electric flux uniformly distributed over the surface per meter length, its density is

$$\psi_d = \frac{\psi}{2\pi x}.$$

Substituting this in equation (1.13.7) and integrating between the limits of r_2 and r_1, gives

$$C = \frac{2\pi\epsilon}{\ln \frac{r_2}{r_1}} \text{ farads per meter.} \tag{1.14.1}$$

Changing to common logarithms and using the relative dielectric constant ϵ_r from (1.13.6), the formula becomes

$$C = \frac{\epsilon_r\, 10^{-9}}{4.6052 \times 9 \log \frac{r_2}{r_1}} \text{ farads/meter.} \tag{1.14.2}$$

Multiplying by 5280/3.281 yields

$$C = \frac{38.82 \times 10^{-9}\epsilon_r}{\log \frac{r_2}{r_1}} \text{ farads/mile.} \tag{1.14.3}$$

1.15. *Capacitance per Conductor of a System of Parallel Conductors.* Consider the general case of a system consisting of any number of charged conductors geometrically in parallel to each other as shown in Fig. 1–18. Let D_{12}, D_{23}, D_{34}, etc., designate the interaxial distances between the respective conductors of the system. Although not essential, it will be assumed for simplicity's sake that all conductors are of the same size. It will be assumed also that the interaxial distances are so large in comparison with the radii of the conductors, that the proximity effect may be neglected. The charges will be assumed, therefore, as uniformly distributed along the conductor axis. A still further assumption is that the system as a whole is shielded so that it is unaffected from outside electrical

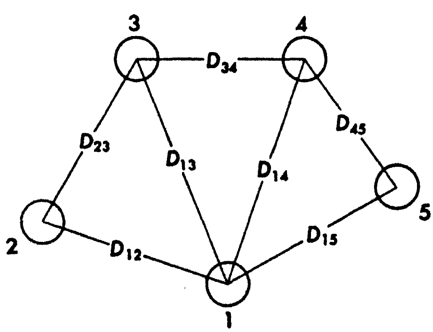


Figure 1–18

disturbances. Let charges q_1, q_2, q_3, etc., per meter length be imparted to the respective conductors so that

$$q_1 = -(q_2 + q_3 + \cdots).$$

In consequence of these charges, electric fields are established between conductor 1 and the other conductors, such that

$$\psi_1 + \psi_2 + \psi_3 + \cdots = 0. \tag{1.15.1}$$

This summation is vectorial if the corresponding voltages are alternating in character.

Due to the electric field between conductor 1 and the other conductors, differences of potential V_{12}, V_{13}, etc., are established across conductor 1 and 2, 1 and 3, and so on. If the medium in which the conductors reside is electrically homogeneous, these differences of potential will depend only upon the respective fluxes and upon the respective spacing distances. The contribution to the potential difference between any pair of conductors by each conductor in the system may be obtained by substituting equation (1.13.4) in (1.13.2).

Thus the contribution of conductor 1 to the potential difference V_{15} between conductor 1 and conductor 5, for instance, is

$$(V_{15})_1 = \frac{-\int \psi_d \, dx}{\epsilon_o}$$

where $\epsilon_o = 1/(4\pi \times 9 \times 10^9)$ is substantially the permittivity of air* in the mks system of units and ψ_d is the electric flux density due to ψ_1 at some point along the spacing distance x meters from the surface of conductor 1. The value of the flux density at this point is

$$\psi_d = \frac{\psi_1}{2\pi x}$$

where ψ_1 is the flux per meter length of conductor 1.

Substituting this value of ψ_d in the preceding expression and integrating between the limits of $D_{15} - r$ and r, yields

$$(V_{15})_1 = \frac{\psi_1}{2\pi\epsilon_o} \ln \frac{D_{15} - r}{r}. \tag{1.15.2}$$

Using the value of ϵ_o given above and converting to common logarithms, the formula becomes

$$(V_{15})_1 = 41.445 \times 10^9 \psi_1 \log \frac{D_{15}}{r}. \tag{1.15.3}$$

* The relative permittivity of air is $\epsilon_r = 1.00058$. The actual permittivity is, therefore,

$$\epsilon = \frac{1.00058}{4\pi \times 9 \times 10^9}.$$

Note that the r in the numerator of the logarithmic term is neglected in comparison with the much larger value of D_{15}.

Referring to Fig. 1–18, it is seen that the contribution of conductor 2 to the difference of potential V_{15} is

$$(V_{15})_2 = (V_{25})_2 - (V_{21})_2,$$

which by analogy to (1.15.3) may be written

$$(V_{15})_2 = 41.445 \times 10^9 \psi_2 \left(\log \frac{D_{25}}{r} - \log \frac{D_{21}}{r}\right),$$

or

$$(V_{15})_2 = 41.445 \times 10^9 \psi_2 \log \frac{D_{25}}{D_{21}}. \tag{1.15.4}$$

The contribution of conductor 3 to V_{15} is, similarly,

$$(V_{15})_3 = (V_{35})_3 - (V_{31})_3$$

or

$$(V_{15})_3 = 41.445 \times 10^9 \psi_3 \log \frac{D_{35}}{D_{31}}. \tag{1.15.5}$$

The contribution of any conductor k to V_{15} is, in general,

$$(V_{15})_k = 41.445 \times 10^9 \psi_k \log \frac{D_{k5}}{D_{k1}}. \tag{1.15.6}$$

The difference of potential V_{15} across conductors 1 and 5 due to all the conductors is, accordingly,

$$V_{15} = (V_{15})_1 + (V_{15})_2 + (V_{15})_3 + \cdots + (V_{15})_n.$$

Substituting the corresponding formulas obtained above, and assigning to the fluxes their corresponding angles, gives

$$V_{15} = 41.445 \times 10^9 \left[(\psi_1/\underline{\alpha_1}) \log \frac{D_{15}}{r} + (\psi_2/\underline{\alpha_2}) \log \frac{D_{25}}{D_{21}} + (\psi_3/\underline{\alpha_3}) \log \frac{D_{35}}{D_{31}} + \cdots + (\psi_n/\underline{\alpha_n}) \log \frac{D_{n5}}{D_{n1}} \right]. \tag{1.15.7}$$

The difference of potential between conductor 1 and any other conductor f is, by analogy,

$$V_{1f} = 41.445 \times 10^9 \left[(\psi_1/\underline{\alpha_1}) \log \frac{D_{1f}}{D_{11}} + (\psi_2/\underline{\alpha_2}) \log \frac{D_{2f}}{D_{21}} + (\psi_3/\underline{\alpha_3}) \log \frac{D_{3f}}{D_{31}} + \cdots + (\psi_n/\underline{\alpha_n}) \log \frac{D_{nf}}{D_{n1}} \right], \tag{1.15.8}$$

where for the sake of symmetry D_{11} is written for r.

Finally the difference of potential between any pair of conductors such as f and k is

$$V_{fk} = 41.445 \times 10^9 \Big[(\psi_1/\alpha_1) \log \frac{D_{1k}}{D_{1f}} + (\psi_2/\alpha_2) \log \frac{D_{2k}}{D_{2f}} + (\psi_3/\alpha_3) \log \frac{D_{3k}}{D_{3f}} + (\psi_f/\alpha_f) \log \frac{D_{fk}}{D_{ff}} + \cdots + (\psi_n/\alpha_n) \log \frac{D_{nk}}{D_{nf}} \Big], \tag{1.15.9}$$

where $D_{ff} = r_f$.

Equations (1.15.1) and (1.15.9) are sufficient for the determination of the capacitance per unit length of conductor in any system of geometrically parallel conductors.

1.16. *Capacitance per Conductor of a Two-Wire Line.*

For a two-wire line equation (1.15.9), using $f = 1$ and $k = 2$, becomes

$$V_{12} = 41.445 \times 10^9 \Big[(\psi_1/\alpha_1) \log \frac{D_{12}}{D_{11}} + (\psi_2/\alpha_2) \log \frac{D_{22}}{D_{21}} \Big].$$

However, since $\psi_1/\alpha_1 = - \psi_2/\alpha_2$,

$$D_{12} = D \quad \text{and} \quad D_{11} = D_{22} = r,$$

the above expression, when ψ_1 is the reference vector, becomes

$$V_{12} = 41.445 \times 10^9 \Big[\psi_1 \log \frac{D}{r} - \psi_1 \log \frac{r}{D} \Big]$$

or

$$V_{12} = 82.89 \times 10^9 \psi_1 \log \frac{D}{r}.$$

The capacitance of the two-wire line, therefore, is

$$C_{12} = \frac{\psi_1}{V_{12}} = \frac{10^{-9}}{82.89 \log \frac{D}{r}} \text{ farad per meter of line} \tag{1.16.1}$$

or

$$C_1 = \frac{10^{-9}}{41.445 \log \frac{D}{r}} \text{ farads per meter of conductor.} \tag{1.16.2}$$

Multiplying by 5280/3.281 yields

$$C_{12} = \frac{19.41 \times 10^{-9}}{\log \frac{D}{r}} \text{ farad/mile of line,} \tag{1.16.3}$$

or

$$C_1 = \frac{38.82 \times 10^{-9}}{\log \frac{D}{r}} \text{ farad/mile of conductor,} \tag{1.16.4}$$

where D and r must be expressed in the same units.

1.17. *Capacitance of Single Cylindrical Conductor Parallel to the Ground.* The capacitance of such a conductor is

$$C_g = \frac{38.82 \times 10^{-9}}{\log \frac{2H}{r}} \text{ farad/mile,} \tag{1.17.1}$$

where H is the height of the conductor above the earth's surface. The formula is obtained by assuming a fictitious conductor a distance H below the surface of the earth and parallel to the conductor above. The actual and fictitious conductors thus form a two-conductor line with a spacing distance equal to $2H$.

1.18. *Capacitance per Conductor of Three-Phase Lines.* Consider a three-phase three-wire line with triangular spacing as shown in Fig. 1–19. Let the interaxial spacing distances between the respective conductors be designated by D_{12}, D_{23}, and D_{31}.

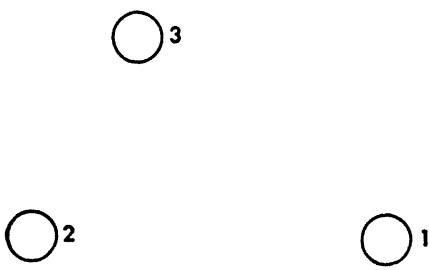


Figure 1–19

Setting for brevity's sake $41.445 \times 10^9 = N$, the differences of potential across successive pairs of conductors are, by (1.15.9),

$$V_{12} = N\left[(\psi_1/\underline{\alpha_1}) \log \frac{D_{12}}{D_{11}} + (\psi_2/\underline{\alpha_2}) \log \frac{D_{22}}{D_{21}} + (\psi_3/\underline{\alpha_3}) \log \frac{D_{32}}{D_{31}}\right] \tag{1.18.1}$$

$$V_{23} = N\left[(\psi_1/\underline{\alpha_1}) \log \frac{D_{13}}{D_{12}} + (\psi_2/\underline{\alpha_2}) \log \frac{D_{23}}{D_{22}} + (\psi_3/\underline{\alpha_3}) \log \frac{D_{33}}{D_{32}}\right] \tag{1.18.2}$$

and

$$V_{31} = N\left[(\psi_1/\underline{\alpha_1}) \log \frac{D_{11}}{D_{13}} + (\psi_2/\underline{\alpha_2}) \log \frac{D_{21}}{D_{23}} + (\psi_3/\underline{\alpha_3}) \log \frac{D_{31}}{D_{33}}\right]. \tag{1.18.3}$$

For conductors of the same size $D_{11} = D_{22} = D_{33} = r$. If the system is balanced, the voltages are equal and 120° apart in time phase, and the fluxes are equal and 120° in time phase. Taking the voltage V of conductor 1 to neutral as the reference, the above expressions may be written

$$\sqrt{3}V\underline{/30^\circ} = N\left[\log\frac{D_{12}}{r}(\psi\underline{/0^\circ} - \psi\underline{/-120^\circ}) + \left(\log\frac{D_{32}}{D_{31}}\right)(\psi\underline{/-240^\circ})\right] \quad (1.18.4)$$

$$\sqrt{3}V\underline{/-90^\circ} = N\left[\log\frac{D_{23}}{r}(\psi\underline{/-120^\circ} - \psi\underline{/-240^\circ}) + \left(\log\frac{D_{13}}{D_{12}}\right)(\psi\underline{/0^\circ})\right] \quad (1.18.5)$$

$$\sqrt{3}V\underline{/-210^\circ} = N\left[\log\frac{D_{31}}{r}(\psi\underline{/-240^\circ} - \psi\underline{/0^\circ}) + \left(\log\frac{D_{21}}{D_{23}}\right)(\psi\underline{/-120^\circ})\right]. \quad (1.18.6)$$

For equilateral spacing, $D_{12} = D_{23} = D_{31} = D$, the above expressions become, numerically,

$$\sqrt{3}V = \sqrt{3}N\psi \log\frac{D}{r}. \quad (1.18.7)$$

Using the value of $N = 41.445 \times 10^9$ gives

$$C = \frac{\psi}{V} = \frac{10^{-9}}{41.445 \log\frac{D}{r}} \text{ farad/meter of conductor} \quad (1.18.8)$$

or

$$C = \frac{38.82 \times 10^{-9}}{\log\frac{D}{r}} \text{ farad/mile of conductor.} \quad (1.18.9)$$

It follows from the above that for the particular case of an equilaterally spaced three-phase line with balanced voltages the capacitances of the conductors are equal and equal to that of the conductors of a single-phase line of the same size and spacing distance.

If the conductors are not spaced equilaterally, the capacitance to neutral of each conductor will differ in value. Because of this inequality the linear admittance per mile will be unequal, resulting in unbalanced voltages at the receiving end. The degree of unbalance is minimized by transposing the conductors in such a manner that each occupies each other's position for one-third of the entire length of the line as shown in Fig. 1–20. The improvement in balance depends largely upon the transposition distance, i.e., the number of complete transpositions in the length of line. From what has just been said it follows that the average capacitance per mile of conductor of a transposed three-phase line is one-third of the sum of the capacitances of any of the three conductors for each of its three positions.

The capacitance between conductors 1 and 2 when in position (a) is given by equation (1.18.1) in which $(V_{12})_a$ is taken as the reference vector. The capacitance between the same pair of conductors when in position (b), where conductor 1 is in the position of 2, conductor 2 in the position of 3, and conductor 3 in the position of 1 is obtained by equation (1.18.2). The voltage V_{23} lags $(V_{12})_a$ by 120°. Its vector value in phase with $(V_{12})_a$ is $(V_{12})_b = (V_{23})\underline{/120^\circ}$.

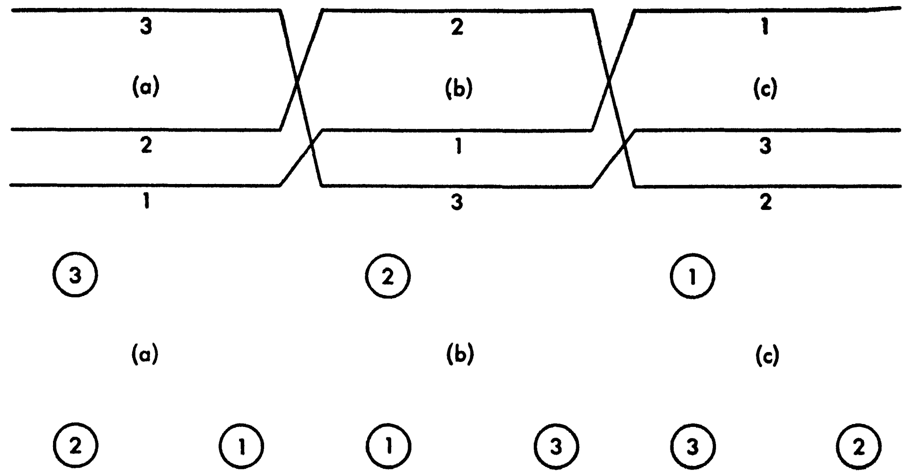


Figure 1–20

Similarly the capacitance of the same pair of conductors when in position (c) is given by equation (1.18.3). The voltage V_{31} lags $(V_{12})_a$ by 240°. Its vector value in phase with $(V_{12})_a$ is $(V_{12})_c = (V_{31})\underline{/240°}$. The three equations become accordingly, when the second is shifted by 120° and the third by 240°,

$$(V_{12})_a = N\left[(\psi\underline{/0°})\log\frac{D_{12}}{r} - (\psi\underline{/-120°})\log\frac{D_{12}}{r} + (\psi\underline{/-240})\log\frac{D_{32}}{D_{31}}\right]$$

$$(V_{12})_b = N\left[\psi\underline{/-240°}\log\frac{D_{13}}{D_{12}} + (\psi\underline{/0°})\log\frac{D_{23}}{r} - (\psi\underline{/-120°})\log\frac{D_{32}}{r}\right]$$

$$(V_{12})_c = N\left[-(\psi\underline{/-120°})\log\frac{D_{13}}{r} + (\psi\underline{/-240°})\log\frac{D_{21}}{D_{23}} + (\psi\underline{/0°})\log\frac{D_{31}}{r}\right].$$

Each of these equations gives a different value for the capacitance. The average value may be obtained from

$$V_{12} = \frac{(V_{12})_a + (V_{12})_b + (V_{12})_c}{3}.$$

In adding the right-hand side of the above three equations it is found that

$$\psi\underline{/-240°}\left(\log\frac{D_{32}}{D_{31}} + \log\frac{D_{13}}{D_{12}} + \log\frac{D_{21}}{D_{23}}\right) = 0.$$

Hence,

$$V_{12} = \frac{N}{3}\left(\log\frac{D_{12}}{r} + \log\frac{D_{23}}{r} + \log\frac{D_{31}}{r}\right)(\psi\underline{/0°} - \psi\underline{/-120°}).$$

Since in terms of the voltage V per phase $V_{12} = \sqrt{3}V$, and the numerical value of $\psi\underline{/0°} - \psi\underline{/-120°} = \sqrt{3}\psi$, the preceding expression, by assigning to N its value 41.445×10^9, becomes

$$V = 41.445 \times 10^9\psi \log\frac{(D_{12}D_{23}D_{31})^{\frac{1}{3}}}{r}. \qquad (1.18.10)$$

The capacitance per conductor is, therefore,

$$C = \frac{\psi}{V} = \frac{10^{-9}}{41.445 \log \dfrac{(D_{12}D_{23}D_{31})^{\frac{1}{3}}}{r}} \text{ farads/meter,}$$

or

$$C = \frac{38.82 \times 10^{-9}}{\log \dfrac{(D_{12}D_{23}D_{31})^{\frac{1}{3}}}{r}} \text{ farads/mile.} \tag{1.18.11}$$

The numerator of the logarithmic term in the denominator will be recognized as the geomean spacing distance between the conductors. If $D_{12} = D_{23} = D_{31} = D$, the expression becomes

$$C = \frac{38.82 \times 10^{-9}}{\log \dfrac{D}{r}} \text{ farads/mile of conductor,} \tag{1.18.12}$$

which is the same as (1.18.9). The quantity $(D_{12}D_{23}D_{31})^{\frac{1}{3}}$ is referred to as the *equivalent* spacing distance of an unsymmetrically-spaced transposed three-phase line.

For flat spacing such as shown in Fig. 1–13, where $D_{12} = D_{23} = D$ and $D_{31} = 2D$, formula (1.18.11) becomes

$$C = \frac{38.82 \times 10^{-9}}{\log \dfrac{1.26D}{r}} \text{ farad/mile of conductor,} \tag{1.18.13}$$

in which $1.26D$ is the equivalent spacing distance.

1.19. *Line Capacitance Table.*

The capacitance and, more generally, the charging current per mile of conductor at 100,000 volts to neutral ($I_c = 100000\, C\omega$) of standard sizes, and with equilateral or equivalent equilateral spacing and at frequencies of 25 and 60 cps, are usually given in reference and handbooks on electrical engineering.

The capacitance per mile of conductor of any specific size and spacing distance D in feet may, however, be computed from the general formula (1.18.12).

A working table of line capacitance per mile of conductor may be formulated by writing formula (1.18.12) in the form

$$C = \frac{38.82 \times 10^{-3}}{\log \dfrac{1}{r} + \log D} \text{ microfarads per mile.}$$

The reciprocal of this expression with C in microfarads is

$$\frac{1}{C} = \frac{10^3 \log D}{38.82} + \frac{10^3 \log \frac{1}{r}}{38.82}. \tag{1.19.1}$$

The reciprocal of the capacitance is called *elastance.* The first of this expression may be interpreted, therefore, as the elastance $1/C_a$ per mile of conductor having a radius $r = 1$ foot and whose equilateral or equivalent equilateral spacing is D feet. The second term gives the elastance $1/C_b$ per mile of conductor of radius r feet and with equilateral or equivalent equilateral spacing of $D = 1$ foot.

Table IV gives the value of $1/C_a$ with C_a in microfarads for conductors of radius $r = 1$ foot and equilaterally spaced from 1 to 30 feet in steps of 1 foot. Table V gives the value of $1/C_b$ with C_b in microfarads (μf) for stranded copper conductors as indicated and for $D = 1$ foot.

Similar tables may be formulated for solid copper conductors, and for solid and stranded aluminum conductors.

To illustrate the use of these tables, consider a 400000 cir. mil line having an equilateral spacing distance $D = 15$ feet. From Table IV, get $1/C_a = 30.296$ and from Table V, similarly get $1/C_b = 39.188$. The elastance per mile, therefore, is

$$\frac{1}{C} = 39.188 + 30.296 = 69.484$$

and

$$C = 0.01439\mu\text{f per mile of conductor.}$$

1.20. *Capacitance per Conductor of Transposed Twin Three-Phase Lines.*

The general method outlined in § 1.15 may be applied directly to the determination of the formula for the capacitance per conductor-mile of twin three-phase lines in which the conductors are arranged as shown in Fig. 1–21.

Conductors 1, 2, and 3 form one line and conductors 5, 6, and 7 form a second line, the two lines to be operated in parallel. From the symmetry of the spacing note that $D_{52} = D_{61}$; $D_{63} = D_{72}$; and $D_{53} = D_{71}$. The potential of conductors 1 and 5 to ground or neutral is assumed $V\underline{/\alpha_1}$, that of conductors 2 and 6, $V\underline{/\alpha_2}$, and that of conductors 3 and 7, $V\underline{/\alpha_3}$. The conductors of each line, respectively, are transposed in such a manner that each occupies successively the position of the other two for one-third of the entire length of line, as indicated in the figure. The improvement in voltage balance will depend largely upon the number of complete transpositions in the entire

TABLE IV

$$\frac{1}{C_a} = \frac{1000}{38.82} \log D$$

D Feet → ↓	0	10	20
1	0	26.826	34.060
2	7.7545	27.799	34.581
3	12.296	28.695	35.078
4	15.509	29.524	35.554
5	18.005	30.296	36.010
6	20.045	31.018	36.449
7	21.769	31.696	36.872
8	23.263	32.235	37.279
9	24.581	32.940	37.671
10	25.760	33.514	38.051

TABLE V

A.W.G. or Cir. Mils	Strands	r mils	r feet	$\frac{1}{C_b}$ $D = 1$ $\frac{1000}{38.82} \log \frac{1}{r}$
0	7	184	0.01533	46.741
00	7	202	0.01683	45.696
000	7	232	0.01933	44.146
0000	7	261	0.02173	42.837
250000	19	287	0.02391	41.767
300000	19	314	0.02617	40.757
350000	19	338	0.02817	39.933
400000	19	362.5	0.03011	39.188
450000	37	384	0.03200	38.507
500000	37	407	0.03392	37.855
600000	37	446.5	0.03721	36.820
700000	61	482	0.04016	35.966
750000	61	499	0.04158	35.577
800000	61	515.5	0.04296	35.212
900000	61	546.5	0.04554	34.560
1000000	61	576	0.04800	33.971

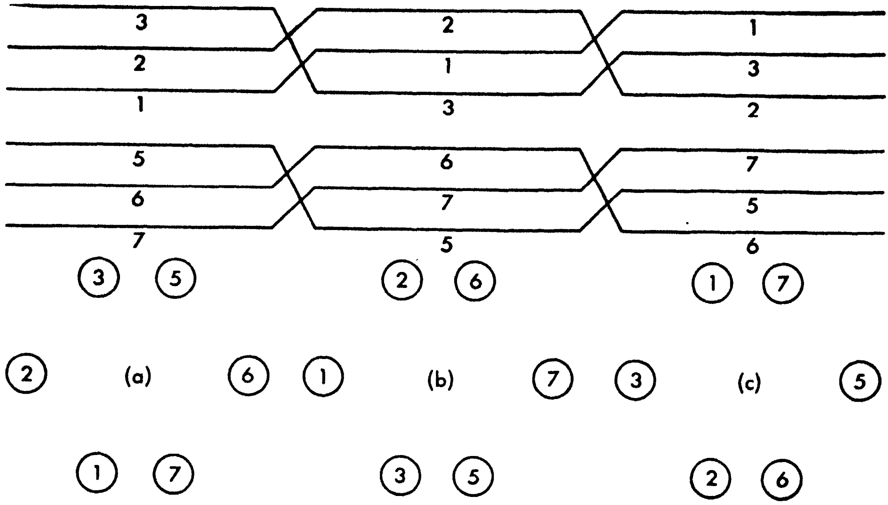


Figure 1-21

length of line. The average capacitance per mile per conductor is therefore one-third of the sum of the capacitances of any of the conductors in the system for each of its three positions. Using the general equation (1.15.9), the potential difference across any two conductors such as 1 and 2 (or 5 and 6), when in position (a), is by combining terms

$$(V_{12})_a = N\left[(\psi\underline{/0^\circ}) \log \frac{D_{12}D_{52}}{rD_{51}} + (\psi\underline{/-120^\circ}) \log \frac{rD_{62}}{D_{21}D_{61}} + (\psi\underline{/-240^\circ}) \log \frac{D_{32}D_{72}}{D_{31}D_{71}}\right].$$

The potential difference across the same conductors when in position (b), obtained in the same manner, and shifted 120° to bring it in phase with $(V_{12})_a$ is

$$(V_{12})_b = N\left[(\psi\underline{/0^\circ}) \log \frac{D_{23}D_{63}}{rD_{62}} + (\psi\underline{/-120^\circ}) \log \frac{rD_{73}}{D_{32}D_{72}} + (\psi\underline{/-240^\circ}) \log \frac{D_{13}D_{53}}{D_{12}D_{52}}\right].$$

Similarly, the potential difference across the same conductors when in position (c) and shifted in phase 240° to bring it in phase with $(V_{12})_a$ is

$$(V_{12})_c = N\left[(\psi\underline{/0^\circ}) \log \frac{D_{31}D_{71}}{rD_{73}} + (\psi\underline{/-120^\circ}) \log \frac{rD_{51}}{D_{31}D_{53}} + (\psi\underline{/-240^\circ}) \log \frac{D_{21}D_{61}}{D_{23}D_{63}}\right].$$

Each of these equations will give a different value for the capacitance per mile between conductors 1 and 2, depending upon their position in the transposition scheme. The average value of the capacitance may be obtained, however, from the average of the three equations. Referring to Fig. 1-21, (a), it is seen that, from the symmetry of the arrangement,

$$D_{23} = D_{12};\ D_{52} = D_{63} = D_{72} = D_{16};\ D_{73} = D_{15}.$$

Using these relations in the above expressions, it is found that the sum of the last three terms adds up to zero. The average of the three expressions

$$V_{12} = \frac{(V_{12})_a + (V_{12})_b + (V_{12})_c}{3}$$

is accordingly

$$V_{12} = \left[\frac{N}{3}(\psi\underline{/0°}) \log \frac{D_{12}^2 D_{16}^2 D_{13} D_{17}}{r^3 D_{15}^2 D_{26}} + (\psi\underline{/-120°}) \log \frac{r^3 D_{15}^2 D_{26}}{D_{12}^2 D_{16}^2 D_{13} D_{17}}\right].$$

Since the logarithmic terms are equal and of opposite sign, the formula may be written

$$V_{12} = \frac{N}{3} \log \frac{D_{12}^2 D_{16}^2 D_{13} D_{17}}{r^3 D_{15}^2 D_{26}} (\psi\underline{/0°} - \psi\underline{/-120°})$$
$$= \frac{N}{3} \log \frac{D_{12}^2 D_{16}^2 D_{13} D_{17}}{r^3 D_{15}^2 D_{26}} (\sqrt{3}\psi)\underline{/30°}.$$

Since in terms of the voltage V to neutral $V_{12} = \sqrt{3}V$, the formula becomes

$$V = N \log \frac{\left(\frac{D_{12}D_{16}}{D_{15}}\right)^{\frac{2}{3}}\left(\frac{D_{13}D_{17}}{D_{26}}\right)^{\frac{1}{3}}}{r} \psi.$$

The capacitance of each conductor is

$$C = \frac{\psi}{V} \text{ farads/meter.}$$

Assigning to N its value 41.445×10^9 and multiplying by 5280/3.281 yields

$$C = \frac{38.82 \times 10^{-9}}{\log \frac{\left(\frac{D_{12}D_{16}}{D_{15}}\right)^{\frac{2}{3}}\left(\frac{D_{13}D_{17}}{D_{26}}\right)^{\frac{1}{3}}}{r}} \text{ farads/mile of conductor.} \qquad (1.20.2)$$

The numerator of the logarithmic term is the geomean spacing distance between conductors and is thus the equivalent equilateral spacing between the conductors.

1.21. *Leakage Conductance.*

The insulating medium between the conductors of an overhead aerial transmission line is obviously air. The insulating medium of single-conductor cables* for underground transmission systems at voltages higher than 27 kv is generally manila or woodpulp paper impregnated with insulating compounds of mineral oil or petrolatum base. Hollow-conductor oil-filled cables† are used for voltages above 132 kv.

* Roper, D. W., and Halperin, H., *Rating of High-Tension Cables and Impregnated-Paper Insulation*, trans. A.I.E.E. 1926, p. 528.

† Shanklin, G., and Sheals, V. A., *Development of New Oil-filled Cable.* Elec. World, 1928, p. 186.

Whatever type of line is considered, the insulating medium has such a high resistivity that the conduction currents through and across this separating medium between the line conductors are usually quite small. In case of aerial lines, a certain amount of conduction currents may also *leak* over the insulators. These, as well as the conduction currents through and across the air, are called *leakage currents.* The ratio of the leakage current I_λ per conductor to the potential difference V between the conductor and neutral is called *leakage conductance.* Its value is measured in mhos and designated by the letter G.*

Accordingly, the leakage conductance per conductor is

$$G = \frac{I_\lambda}{V} \text{ mhos,} \tag{1.21.1}$$

and is numerically equal to the leakage current I_λ in amperes per volt difference of potential.

The leakage conductance may be expressed also in terms of the conductive property of the medium and the length and cross-sectional area of the leakage current path. Thus, if γ is the conductivity, i.e., the conductance per meter cube of the medium, S the length in meters, and A the cross-sectional area of the path in square meters, then

$$G = \gamma \frac{A}{S} \text{ mhos.} \tag{1.21 2}$$

Equating (1.21.1) and (1.21.2) gives

$$\gamma \frac{V}{S} = \frac{I_\lambda}{A},$$

where I_λ/A is the density I_d of the leakage current, and V/S is the potential gradient or negative electric field intensity $-F$, in volts per meter.

When the gradient is not uniform along the current path, the equation is written

$$-\gamma \frac{dV}{ds} = I_d. \tag{1.21.3}$$

This gives

$$V = \frac{-\int I_d \, ds}{\gamma} \tag{1.21.4}$$

where the integration is between the limits of the current path.

If I_λ represents the leakage current per meter length of conductor and

* The name of the international unit of conductance is the *siemens.* This name was adopted by the International Electrotechnical Commission at its meeting held in Scheveningen, Brussels, June, 1935.

V the potential difference between conductor and neutral, the leakage conductance, by (1.21.4) and (1.21.1), is

$$G = \frac{\gamma I_\lambda}{-\int I_d \, ds} \tag{1.21.5}$$

mhos per meter length of conductor.

This expression may be evaluated quite easily for such cases where the current density I_d can be determined from the geometry of the electrical structure under consideration.

It is important to note, however, that the leakage conductance and the capacitance of line conductors occur as if they were connected in shunt across the same difference of potential. Indeed, their formulas

$$G = \frac{I_\lambda}{V} \text{ mhos}$$

and

$$C = \frac{\psi}{V} \text{ farads}$$

are analogous.

Furthermore the ratio of G, as given by the general formula (1.21.5), to C, as given by the general formula (1.13.7), is

$$\frac{G}{C} = \frac{\gamma}{\epsilon} \cdot \frac{I_\lambda / \int I_d \, ds}{\psi / \int \psi_d \, ds} \tag{1.21.6}$$

where ϵ is the permittivity of the medium in the rationalized mks system of units.

It can be shown that the distribution of the leakage current through a high resistance medium, such as there is between the line conductors, is as if there are two equal and opposite charges at opposite points of the two conductors. The distribution of the leakage current in the space surrounding the conductor follows, therefore, exactly the same pattern as the electric flux distribution shown in Fig. 1–22. Under this condition it follows that

$$\frac{I_\lambda}{\int I_d \, ds} = \frac{\psi}{\int \psi_d \, ds}.$$

Equation (1.21.6) may be written, therefore,

$$\frac{G}{C} = \frac{\gamma}{\epsilon}. \tag{1.21.7}$$

This expression indicates that the leakage conductance per mile of conductor of any system of conductors, such as those discussed in the preceding

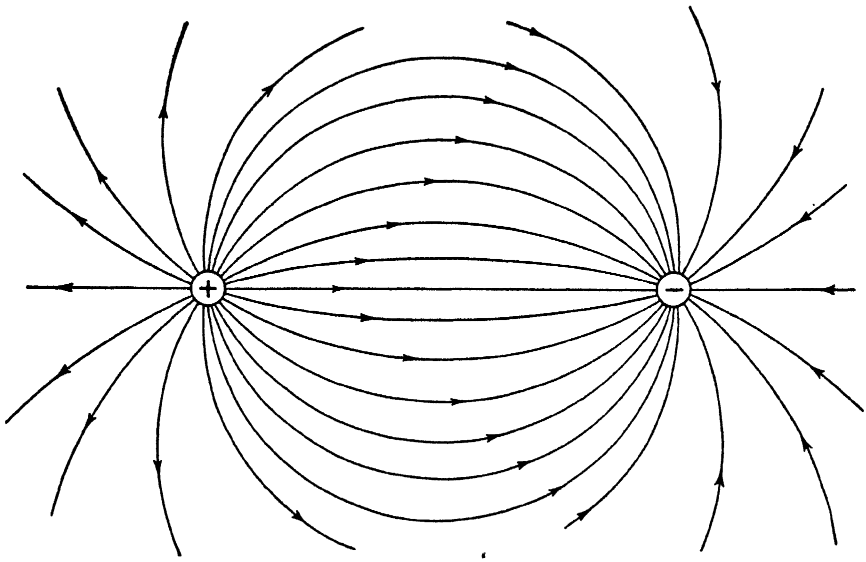


Figure 1–22

articles on capacitance, may be obtained directly by multiplying the respective capacitance by γ/ϵ. Thus the leakage conductance per conductor of a single-phase line, or per conductor of a three-phase line with equilateral spacing, by (1.21.7) and (1.18.12), is

$$G = \frac{4\pi \times 9 \times 38.82\gamma}{\log \frac{D}{r}} \text{ mhos/mile.} \qquad (1.21.8)$$

The leakage conductance from conductor to conductor per mile is one-half as much.

The conductivity of air γ which enters in the formulas for leakage conductance is not a constant, however. Its value depends upon the atmospheric pressure, upon the humidity, the temperature, and the degree of ionization of the air surrounding the line. Furthermore, since these factors, respectively, are not the same at all points along a line, there is no justification in the assumption of uniform distribution of the leakage conductance for aerial transmission lines.

The conductivity of the air, however, is so low, and the spacing distance between the conductors of high voltage lines is so large in comparison with the conductor radius that, as a rule, leakage conductance is neglected in the design or performance calculation of power lines operating at voltages not exceeding 125 kv.

1.22. *Corona Disruptive Potential Gradient and Visual Potential Gradient.* The operating voltage of long transmission lines may be sufficiently high to produce an electric field of intensity so great as to ionize the air strongly in the immediate vicinity of the conductors. Under this condition the leakage conductance becomes quite large and the conduction currents and the consequent leakage power loss are increased in proportion.

The leakage of energy from the surfaces of the conductors is associated with crackling noises, a breakdown of the air into ozone which may be detected by its characteristic odor, and by a bluish glow around the conductors, called *corona*, and which becomes visible in the dark. At very high differences of potential, the streamers of bluish light may extend to appreciable distances from the conductor surface. A more or less continuous spark or arc may actually be established at various points across the conductors. This is usually associated with a complete collapse of the insulating properties of the air.

The particular value of the difference of potential which causes the formation of the visible corona is called *visual critical voltage*, and that which causes the complete breakdown of the air is usually referred to as the *spark* or *flash over voltage.*

Research on corona formation on transmission lines and allied phenomena was pioneered some thirty years ago by the late F. W. Peek, Jr., of the General Electric Co.* His original investigations were made on experimental lines of about 500 feet in length with various size conductors, at adjustable spacing distances. From his experimentally-obtained data, Peek has shown that the actual potential gradient (field intensity) at the surface of a conductor, and at which there is just sufficient ionization taking place to cause an increase in leakage conductance, is somewhat smaller than the potential gradient at which the corona becomes visible. He called it ***disruptive potential gradient***, to distinguish it from that at which the corona becomes visible and which he called ***visual potential gradient.***

The experimental data indicated also that, at normal condition of atmospheric pressure of 76 cm. of mercury, normal temperature of 25° C. the rms value of the disruptive potential gradient at the surface of a smooth polished conductor is 53.5 kv per inch. This corresponds to 2110 kv per meter. The formulas for the potential gradient in volts per meter and for

* F. W. Peek, *Law of Corona, Proc.*, A.I.E.E., 1912, Vol. 31.
F. W. Peek, *Dielectric Phenomena in High-voltage Engineering.*
Whitehead, *Electric Strength of Air, Proc.*, A.I.E.E., 1912, Vol. 31.
Moody and Faccioli, *Corona Phenomena*, Trans., A.I.E.E., 1909, Vol. 27.

the field intensity in newtons* per coulomb are identical for all electrical systems. Their numerical values are equal and the same symbol† F will be used to designate either.

A fuller realization of the significance and magnitude of the disruptive potential gradient may be obtained by translating it in terms of its equivalent field intensity as a force per charge of one coulomb. Thus, the disruptive potential gradient of 2110 kv per meter is equivalent to 2,110,000 newtons per coulomb, or $2110000 \times 0.102 = 215220$ kilograms of force per coulomb, or $2110000 \times 0.2248 = 474{,}328$ pounds of force per coulomb or somewhat over 237 tons per coulomb!

To determine the line voltage that would give rise to the disruptive potential gradient F_d or to the visual potential gradient F_v, consider first the general relationship between voltage and potential gradient.

The field intensity or potential gradient at any point x along the interaxial distance D between two parallel conductors in air is given by equation

$$F = \frac{\psi}{2\pi\epsilon}\left(\frac{1}{x} + \frac{1}{D - x}\right) \text{ volts/meter, or newtons/coulomb,}$$

where ψ is the electric flux in coulombs per meter of conductor and $\epsilon = 1/(4\pi \times 9 \times 10^9)$ is the permittivity of air in the mks system of units. At the surface of the conductor where $x = r$, the radius of the conductor, this expression becomes

$$F = 18 \times 10^9 \psi \left(\frac{1}{r} + \frac{1}{D - r}\right).$$

However, since the interaxial distance between the conductors is very much larger than r, the equation is substantially

$$F = 18 \times 10^9 \frac{\psi}{r} \text{ volts/meter or newtons/coulomb.} \qquad (1.22.1)$$

Designating the voltage from conductor to ground or to neutral by V_o and by C the capacitance of the conductor to ground per meter of conductor, the above formula becomes

$$F = 18 \times 10^9 \frac{CV_o}{r}.$$

Using the value of C as given by equation (1.16.2) gives

$$F = \frac{V_o}{2.3r \log \frac{D}{r}} \text{ volts/meter or newtons/coulomb.} \qquad (1.22.2)$$

* The unit of force in the mks system of units is 100,000 dynes. The name *newton*, which is gaining favor among engineer and physicists, was proposed for this unit at the last meeting of the International Electrotechnical Commission held at Scheveningen, Brussels, in 1935.

† The letters g and G are frequently used to designate potential gradients. Some confusion may arise, however, because the symbol for leakage conductance is also G.

Designating by F_d the disruptive potential gradient (field intensity) in kilovolts per meter and by $(kv)_{do}$ the disruptive potential to ground or neutral in kilovolts, the equation (1.22.2) becomes

$$(kv)_{do} = 2.3F_d\, r \log \frac{D}{r}. \tag{1.22.3}$$

Substituting $F_d = 2110$ kv per meter, as determined by Peek, yields

$$(kv)_{do} = 4853r_m \log \frac{D_m}{r_m} \tag{1.22.4}$$

kilovolts to neutral. The subscript m to D and r in this expression denotes that D and r are expressed in meters.

If F_d is in kilovolts per cm. and r and D in cm., the above equation becomes

$$(kv)_{do} = 48.53r_{cm} \log \frac{D_{cm}}{r_{cm}}. \tag{1.22.5}$$

When the disruptive potential gradient is expressed in kilovolts per inch and D and r are measured in inches, the equation is

$$(kv)_{do} = 2.3 \times 21.1 \times 2.54r_i \log \frac{D_i}{r_i}$$

$$= 123r_i \log \frac{D_i}{r_i}. \tag{1.22.6}$$

Equations (1.22.4), (.5), and (.6) give the disruptive voltages to ground or neutral. The disruptive line voltage for a single-phase line is

$$(kv)_{d1} = 2(kv)_{do} \tag{1.22.7}$$

and for a three-phase line it is

$$(kv)_{d3} = \sqrt{3}(kv)_{do}. \tag{1.22.8}$$

Equations (1.22.4), (.5), and (.6) give the voltage from conductor to ground or neutral that causes a disruptive potential gradient of a rms value equal to 2110 kv per meter. The conductor must have, as previously stated, a smooth, polished surface, the barometric pressure must be normal at 76 cm. of mercury, and the temperature must be 25° C.

It was found experimentally that the voltage causing the disruptive potential gradient is somewhat smaller when the conductor surface is rough or when the line consists of stranded conductors. This is taken care of by multiplying $(kv)_{do}$ obtained above by an "irregularity" factor m_o whose value is*

1.00 for smooth, polished conductors.
0.98 to 0.93 for rough or weathered wire.
0.87 to 0.83 for seven-strand cables.
0.85 to 0.80 for concentric layer cables of 19, 37, or 61 strands.

* F. W. Peek, *Dielectric Phenomena in High-Voltage Engineering*, p. 302, McGraw-Hill Book Co.

A further correction in the value of $(kv)_{do}$ must be introduced if the atmospheric pressure, or the temperature, or both are not the standard value of 76 cm. and 25° C., respectively. This is taken care of by multiplying the radius r by the air-density factor δ, whose value is

$$\delta = \frac{3.92B}{273 + t}, \tag{1.22.9}$$

where

B = barometric pressure in cm. of mercury, and
t = temperature in degrees centigrade

or

$$\delta = \frac{17.9b}{459 + T}, \tag{1.22.10}$$

where

b = barometric pressure in inches of mercury, and
T = temperature in degrees Fahrenheit.

The complete formula for $(kv)_{do}$ when the dimensions are in centimeters, by (1.22.5), is

$$(kv)_{do} = 48.53m_o(\delta r_{cm}) \log \frac{D_{cm}}{r_{cm}}. \tag{1.22.11}$$

The complete formula for $(kv)_{do}$ when the dimensions are in inches, by (1.22.6), is

$$(kv)_{do} = 123m_o(\delta r_i) \log \frac{D_i}{r_i}. \tag{1.22.12}$$

1.23. *Visual Critical Potential Difference.* The experimental data of Peek indicate also that a visible corona is just being formed when the rms value of the disruptive potential gradient of 21.1 kv per cm. is at a distance of $(r_{cm} + 0.301\sqrt{r_{cm}})$ cm. from the axis of the conductor.

The substitution of this quantity for r_{cm} in (1.22.5) will give, therefore, the visual critical voltage $(kv)_{vo}$ to ground or neutral:

$$(kv)_{vo} = 48.53(r_{cm} + 0.301\sqrt{r_{cm}}) \log \frac{D_{cm}}{r_{cm}}.$$

This may be written

$$(kv)_{vo} = 48.53\left(1 + \frac{0.301}{\sqrt{r_{cm}}}\right) r_{cm} \log \frac{D_{cm}}{r_{cm}}. \tag{1.23.1}$$

If the dimensions are given in inches the equation becomes

$$(kv)_{vo} = 48.53\left(1 + \frac{0.301}{\sqrt{2.54r_i}}\right) 2.54\, r_i \log \frac{D_i}{r_i}$$

or

$$(kv)_{vo} = 123\left(1 + \frac{0.189}{\sqrt{r_i}}\right) r_i \log \frac{D_i}{r_i}. \tag{1.23.2}$$

Equations (1.23.1) and (.2) give the voltage from conductor to ground or neutral that causes visible corona to be formed under standard conditions, i.e., when the conductor is smooth and polished, the barometric pressure 76 cm. of mercury and the temperature 25° C.

The experimental data indicate that the equation for $(kv)_{vo}$ must be modified by the "irregularity" and the density factors given in the preceding article. If the dimensions are given in centimeters, formula (1.23.1) is modified to

$$(kv)_{vo} = 48.53\left(1 + \frac{0.301}{\sqrt{\delta r_{cm}}}\right) m_o \delta r_{cm} \log \frac{D_{cm}}{r_{cm}}. \tag{1.23.3}$$

Similarly, if the dimensions are stated in inches, formula (1.23.2) becomes

$$(kv)_{vo} = 123\left(1 + \frac{0.189}{\sqrt{\delta r_i}}\right) m_o \delta r_i \log \frac{D_i}{r_i}, \tag{1.23.4}$$

where the values of δ and m_o are given in the preceding article.

1.24. *Corona Loss and Equivalent Leakage Conductance.*

From extensive research resulting in a great mass of experimental data, Peek formulated the following empirical expression for the loss in power due to corona:

$$P = \left\{\frac{390}{\delta}(f + 25)[(kv) - (kv)_{do}]^2 \sqrt{\frac{r}{D}}\right\} 10^{-5} \tag{1.24.1}$$

kilowatts per mile of conductor.

In this formula, δ is the air density factor as calculated by (1.22.9) or (1.22.10). The term (kv) is the potential to ground in kilovolts for a single-phase line and to neutral for a balanced, equilaterally-spaced three-phase line. The quantity $(kv)_{do}$ is the disruptive potential to ground in kilovolts for a single-phase line, and to neutral for a balanced, equilaterally-spaced three-phase line and is given by either (1.22.11) or (1.22.12), depending upon what units are used. The symbol f, in the equation, stands for the frequency of the supply in cycles per second.

The above formula for the power loss P holds strictly for single-phase lines or equilaterally-spaced three-phase lines.

When a line is operated at voltage values which are conducive to the formation of corona, appreciable current leaks across the conductors. These currents are not sinusoidal, and as a consequence the corona loss as given by formula (1.24.1) will not give the correct actual leakage conductance per conductor. If the power loss due to the higher harmonics, however, is neglected, the leakage conductance is

$$G = \frac{P}{1000(kv)^2} \text{ mhos/mile of conductor.} \tag{1.24.2}$$

The value of P in this formula is given by (1.24.1) and is in kilowatts per mile, and (kv) is the voltage to neutral in kilovolts.

REFERENCES

Carroll, J. S., and Rockwell, M. M., *Emperical Methods of Calculating Corona Loss for High Voltage Transmission Lines*, Trans. Am. Inst. Elect. Engrs., 1937.

Catalogs and Publications, Aluminum Company of America.

Catalogs and Publications, General Cable Corporation, N. Y.

Copper Wire Tables, Bureau of Standards Circ. No. 31, 1914.

Cozzens, Bradley and Peterson, *Corona Experience on Transmission Lines*, Trans. Am. Inst. Elect. Engrs., 1939.

Dwight, H. B., *Skin Effect in Tubular and Flat Conductors*, Trans. Am. Inst. Elect. Engrs., 1918.

Dwight, H. B., *Transmission Line Formulas*, Van Nostrand, 1926.

Edwards, J. D., *Properties of Pure Aluminum*, Trans. Am. Electro. Chem. So., Vol. 47, 1925

Gardner, M. F., *Corona Investigation on Artificial Lines*, Trans. Am. Inst. Elect. Engrs., 1925.

Harding, C. F., *Corona Losses Between Wires at Extra High Voltages*, Trans. Am. Inst. Elect. Engrs., 1925.

Kennelly, A. E., Laws, F. A., and Pierce, P. H., *Experimental Researches on Skin Effect in Conductors*, Am. Inst. Elect. Engrs., 1915.

Loew, E. A., *Electric Power Transmission*, McGraw-Hill Book Co., 1928.

Miller, J. W., *Effective Resistance and Inductance of Iron and Bimetallic Wires*, Nat. Bur. Standards, Vol. XII, 1915.

Nesbit, William, *Electrical Characteristics of Transmission Circuits*, Westinghouse Elect. & Mfg. Co., 1926.

Overhead System Reference Book, N.E.L.A., 1927.

Peek, F. W., *Dielectric Phenomena in High-Voltage Engineering*, McGraw-Hill Book Co., 1929.

Woodruff, L. F., *Principles of Electric Power Transmission*, Wiley & Bros., 1938.

SUGGESTIVE PROBLEMS *Chapter I*

1. Using equats. (1.5.16) and (1.5.17), respectively, plot the skin effect resistance-ratio for a frequency of 60 cps as a function of the cross-sectional area in circ. mils for copper and aluminum conductors.

2. Using equat. (1.5.18) plot the internal inductance L_i in henry per mile at a frequency of 60 cps as a function of the cross-sectional area in circ. mils for copper conductors.

3. Obtain the equation for the internal inductance L_i in henry per mile at 60 cps as a function of the cross-sectional area for aluminum conductors. (See equat. (1.5.18) for copper conductors.)

4. Obtain graphs showing the variation of the inductance in henry per mile per conductor of a two-wire line (a) as a function of the spacing distance D and (b) as a function of the conductor radius for conductors of sizes smaller than 1,000,000 circ. mils.

5. Obtain a graph showing the variation of the inductance in henry per mile per conductor for stranded conductors as a function of the number of strands.

6. Calculate the average inductance in henry per mile per conductor of a twin three-phase transposed line of 1,000,000 circ. mils standard annealed copper, concentric stranded of 61 strands. Assume the equilateral spacing distance between the conductors in the same line is 10 ft. and that the spacing distance between the lines is 15 feet.

7. Derive the equation for the inductance per mile of a hollow copper conductor of internal diameter d_i and external diameter d_e. Such conductors are used in the Boulder Dam–Los Angeles transmission system.

8. Calculate the equivalent equilateral spacing (geomean spacing) distance of a three-phase flat-spaced line in which the outer conductors are 12 and 18 feet, respectively, from the middle one.

9. Using equat. (1.16.4), obtain a graph showing the variation of line capacitance in mf per mile per conductor (a) as a function of the spacing distance, and (b) as a function of the radius of the wire.

10. Calculate the average capacitance per conductor per mile of a three-phase 750,000 circ. mil line with flat spacing, assuming that the spacing between the middle and outside conductor is 10 feet.

11. Calculate the capacitance per mile per conductor of the twin three-phase line stated in Prob. 6.

12. Calculate the disruptive potential to neutral of a three-phase line whose conductors are separated by a spacing distance of 20 feet, 250,000 circ. mils in cross-section and smooth. Assume normal atmospheric pressure and temperature of 70° F.

    What is the value of the disruptive potential difference between conductors?

What would be the disruptive potential difference to neutral and between conductors if the conductor were a concentric layer cable of 61 strands, at normal atmospheric pressure and temperature of 88° F.?

Calculate the visual critical potential difference to neutral and between conductors for the two conditions stated above.

13. Using Peek's formula, calculate the corona loss per conductor mile for the conditions stated in Prob. 10, on the assumption that the line voltage is 10 per cent larger than the visual critical potential difference at 60 cps frequency.

14. Neglecting the higher harmonics, calculate the equivalent leakage conductance in mhos per mile per conductor for the line stated in Prob. 13.

Chapter 2 Transmission Line Formulas

2.1. General Considerations: The π and T Line Elements. It was shown in § 1.2 that a line conductor, in its relationship to the neutral or ground, is an electric circuit consisting of uniformly distributed resistance and inductance in series connection and uniformly distributed leakage conductance and capacitance in parallel connection, as shown schematically in Fig. 1–1.

The voltage drop in an infinitesimal conductor element of length Δx where the current is i ampere is

$$\Delta e = \left(Ri + L\frac{di}{dt}\right)\Delta x \text{ volts.}$$

The quantities R and L are the resistance and inductance, respectively, of the conductor per unit length. The space-rate of change of the voltage drop along the line conductor as $\Delta x \to 0$, is

$$\frac{de}{dx} = Ri + L\frac{di}{dt}.$$

Similarly the current through the leakage conductance and capacitance of an infinitesimal conductor element of length Δx where the potential to neutral or ground is e volts, is

$$\Delta i = \left(Ge + C\frac{de}{dt}\right)\Delta x \text{ amperes.}$$

The quantities G and C are, respectively, the leakage conductance and the capacitance of the conductor per unit length. The space-rate of change of the current along the line conductor as $\Delta x \to 0$, is

$$\frac{di}{dx} = Ge + C\frac{de}{dt}.$$

By differentiation and proper elimination, these two expressions may be written

$$\frac{d^2e}{dx^2} = LC\frac{d^2e}{dt^2} + (RC + LG)\frac{de}{dt} + RGe$$

and

$$\frac{d^2i}{dx^2} = LC\frac{d^2i}{dt^2} + (RC + LG)\frac{di}{dt} + RGi,$$

in which for a sinusoidally alternating voltage

$$e = E\epsilon^{j\omega t}$$
$$i = I\epsilon^{j(\omega t \pm \theta)}.$$

The complete solution of the above two differential equations, with the constants of integration determined to satisfy the particular limiting conditions, includes the transient values of both voltage and current incident to the energizing of the line, and the steady-state values pertaining to the steady conditions of operation.

The study of line behavior under transient conditions is beyond the scope of this book, and the line performance under steady-state conditions may be obtained by methods much simpler than that demanded by the above differentials.

Each infinitesimal part of line conductor may be thought of as consisting, at any one particular frequency, of an impedance element of the form $(R + jL\omega)$ and two admittance elements of the form $(G + jC\omega)$, as shown in Fig. 2–1. A four-terminal circuit so connected is commonly known as a π *network*. A line element may be thought of also as consisting of two impedance elements of the form $(R + jL\omega)$ and a single admittance element of the form $(G + jC\omega)$ connected as shown in Fig. 2–2. Such a four-terminal structure is referred to as a T *network*.

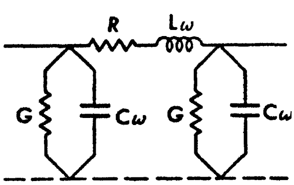


Figure 2–1

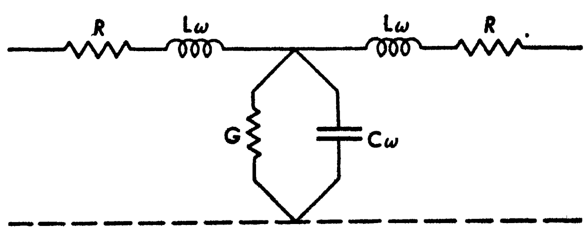


Figure 2–2

From what has just been said, it follows that since a line conductor consists electrically of recurrent symmetrical π elements or recurrent symmetrical T elements, the behavior of a conductor as a vehicle for the transfer of electrical energy may thus be obtained from an analysis of the behavior of its component elements, either of the π or of the T form. Thus one of the fundamental properties of such networks is that for *each* *value of frequency there is only one impedance which, when connected at one end of the circuit will result* in an equal impedance at the other end. The

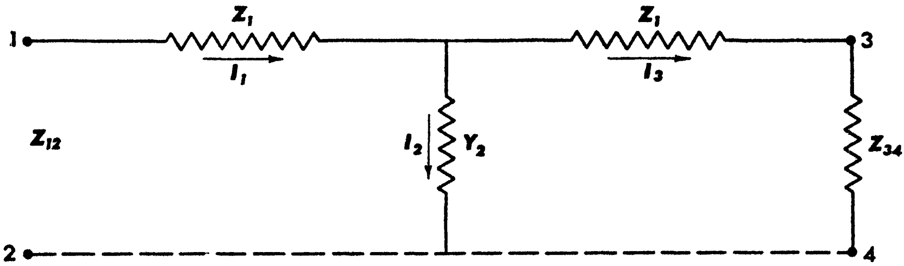


Figure 2–3

impedance of the network under this condition, therefore, is the same irrespective at which end it is measured. To determine this particular impedance value consider the symmetrical T network in Fig. 2–3. From the connection it is seen that

$$Z_{12} = Z_1 + \cfrac{1}{Y_2 + \cfrac{1}{Z_1 + Z_{34}}}.$$

Setting $Z_{12} = Z_{34} = Z_o$ and solving for Z_o, gives

$$Z_o^2 = Z_1^2 + \frac{2Z_1}{Y_2}. \tag{2.1.1}$$

This particular impedance Z_o is called *characteristic impedance* of the circuit.

Another fundamental property of the symmetrical T network is that when the network is terminated in characteristic impedance, the ratio of the currents at the two ends is fixed and equal to the respective ratio of the voltages at the two ends. Thus referring to the figure it will be noted that, since $Z_{12} = Z_{34} = Z_o$

$$V_{12} = I_1 Z_o$$

and

$$V_{34} = I_3 Z_o$$

or

$$\frac{V_{12}}{V_{34}} = \frac{I_1}{I_3}. \tag{2.1.2}$$

To obtain the value of this ratio, note that

$$I_1 = I_2 + I_3$$

and

$$\frac{I_2}{Y_2} = I_3(Z_1 + Z_o).$$

Hence

$$I_1 = I_3[1 + Y_2(Z_1 + Z_o)]$$

or, by (2.1.2),

$$\frac{V_{12}}{V_{34}} = \frac{I_1}{I_3} = 1 + Y_2(Z_1 + Z_o). \tag{2.1.3}$$

The fundamental properties of symmetrical T networks embodied in equations (2.1.1) and (2.1.3) are also fundamental to transmission lines. First, any transmission line has a *characteristic impedance*, i.e., if a network of impedance Z_o is connected at one end of the line, the impedance at the other end is the same. Second, if the receiving-end impedance is equal to Z_o, the ratio of the end currents and of end voltages are equal.

2.2. *Characteristic Impedance of Transmission Lines.*

Equations (2.1.1) and (2.1.3) obtained above hold for each elemental T of a line conductor. Thus, let

$$z = R + jL\omega \tag{2.2.1}$$
$$y = G + jC\omega \tag{2.2.2}$$

be, respectively, the linear impedance and the shunt admittance per mile of conductor.

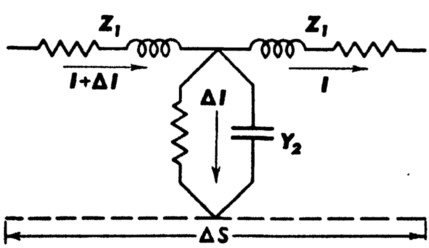


Figure 2–4

Consider a T element of length ΔS as shown in Fig. 2–4, then

$$Z_1 = \tfrac{1}{2}z\,\Delta S$$
$$Y_2 = y\,\Delta S. \tag{2.2.3}$$

Substituting these values in equation (2.1.1) for the characteristic impedance, gives

$$Z_o^2 = \tfrac{1}{4}(z\,\Delta S)^2 + \frac{z}{y}.$$

At the limit when the elemental length of the conductor is infinitesimally small, the equation becomes

$$Z_o = \sqrt{\frac{z}{y}}. \tag{2.2.4}$$

Substituting the values of z and y from (2.2.1) and (2.2.2) yields

$$Z_o = \sqrt{\frac{R + jL\omega}{G + jC\omega}} \tag{2.2.5}$$

or

$$Z_o = \left(\frac{R^2 + L^2\omega^2}{G^2 + C^2\omega^2}\right)^{\frac{1}{4}} \underline{/\zeta_o}, \tag{2.2.6}$$

where

$$\left.\begin{aligned} \zeta_o &= \frac{\alpha_1 - \alpha_2}{2} \\ \text{and} \qquad \alpha_1 &= \tan^{-1}\frac{L\omega}{R} \\ \alpha_2 &= \tan^{-1}\frac{C\omega}{G}. \end{aligned}\right\} \quad (2.2.7)$$

Equations (2.2.4), (2.2.5), and (2.2.6) represent the characteristic impedance per conductor whether the line is single phase or three phase. The angle ζ_o of the characteristic impedance Z_o is given by (2.2.7). Extreme care should be exercised in the use of proper values of line constants. These constants for copper and aluminum conductors for various types of lines and conductor arrangements are discussed in Chapter I.

The real or resistive component of the characteristic impedance is

$$R_o = Z_o \cos \zeta_o, \quad (2.2.8)$$

and the quadrature or reactive component is

$$X_o = Z_o \sin \zeta_o. \quad (2.2.9)$$

The leakage conductance G in the formula for Z_o is usually neglected when the line is operated at a voltage to ground or neutral (voltage per phase) less than the critical disruptive voltage $(kv)_{do}$ given by (1.22.12). When the operating voltage is larger than that given by this equation, the approximate value for G given by (1.24.2) should be used.

If the resistance per mile of line conductor is also negligibly small in comparison with $L\omega$, the formula (2.2.5) for the characteristic impedance becomes

$$Z_{so} = \sqrt{\frac{L}{C}}. \quad (2.2.10)$$

It is then called *surge impedance.*

For a solid conductor of a single-phase line or of an equilaterally-spaced three-phase line, the surge impedance to ground or neutral, by (1.7.4) and (1.16.2), is

$$Z_{so} = \left[\frac{741.13 \times 10^{-6} \log \dfrac{D}{0.7788r} \cdot \log \dfrac{D}{r}}{38.82 \times 10^{-9}}\right]^{\frac{1}{2}}.$$

This yields

$$Z_{so} = 138.15\left[\left(\log \frac{D}{r}\right)^2 + 0.1085 \log \frac{D}{r}\right]^{\frac{1}{2}}, \quad (2.2.11)$$

where D is the interaxial distance between the line conductors and r is the conductor radius.

If the magnetic interlinkages within the conductor are neglected, the above formula becomes

$$Z_{so} = \sqrt{\frac{L}{C}} = 138.15 \log \frac{D}{r}. \qquad (2.2.12)$$

This approximate formula is often used in the calculation of the surge impedance of transmission line. Thus, for a line whose conductors are of 1,000,000 circ. mil and spaced 25 feet, the accurate value for the surge impedance, as obtained by (2.2.11), is 392 ohms. The approximate formula yields 384 ohms, an error of 2.04 per cent.

2.3. *Propagation Constant of Transmission Lines.* Consider again the T element of the line terminated in Z_o, shown in Fig. 2–4. Let I be the current which leaves the element and enters the next one to the right, and ΔI be the current in the admittance branch of the element. The current which enters the element is, therefore, $I + \Delta I$. By (2.1.3), the ratio of the currents at the ends of the element is

$$\frac{I + \Delta I}{I} = 1 + Y_2(Z_1 + Z_o)$$

which, by (2.2.3), becomes

$$\frac{I + \Delta I}{I} = 1 + y\,\Delta S\left(\frac{z\,\Delta S}{2} + Z_o\right)$$

or

$$\frac{\Delta I}{I} = \left(\frac{yz\,\Delta S}{2} + yZ_o\right)\Delta S.$$

Using the value of Z_o, as given in (2.2.4), this equation may be written

$$\frac{\Delta I}{\Delta S} = I\left(yz\frac{\Delta S}{2} + \sqrt{yz}\right).$$

At the limit when ΔS approaches zero, i.e., when the conductor element is infinitesimal in length, the equation becomes

$$\frac{dI}{I} = \sqrt{yz}\,dS. \qquad (2.3.1)$$

Assuming that distances along the conductor are measured from the receiving end where the distance $S = 0$ and $I = I_r$, toward the station end where the distance is S and the current $I = I_s$, the equation (2.3.1) becomes

$$\int_{I_r}^{I_s} \frac{dI}{I} = \int_0^S \sqrt{yz}\,ds.$$

This, when integrated between the stated limits, gives

$$\ln \frac{I_s}{I_r} = S\sqrt{zy} \tag{2.3.2}$$

or

$$\frac{I_s}{I_r} = \epsilon^{S\sqrt{zy}}. \tag{2.3.3}$$

The quantity $\sqrt{zy}$ is called *line propagation constant* and is usually designated by the letter p. The equation may, accordingly, be written

$$\frac{I_s}{I_r} = \epsilon^{pS}. \tag{2.3.4}$$

By equation (2.1.2), this relationship holds also for the voltage ratio, i.e.,

$$\frac{V_s}{V_r} = \epsilon^{pS}. \tag{2.3.5}$$

Using the values of z and y as given by (2.2.1) and (2.2.2), the value of the propagation constant p in terms of the constants of the line conductor is

$$p = \sqrt{(R + jL\omega)(G + jC\omega)} \tag{2.3.6}$$

or

$$p = [(R^2 + L^2\omega^2)(G^2 + C^2\omega^2)]^{\frac{1}{4}}\underline{/\delta}, \tag{2.3.7}$$

where

$$\left.\begin{aligned} \delta &= \frac{\alpha_1 + \alpha_2}{2} \\ \text{and} \quad \alpha_1 &= \tan^{-1}\frac{L\omega}{R} \\ \alpha_2 &= \tan^{-1}\frac{C\omega}{G}. \end{aligned}\right\} \tag{2.3.8}$$

Like the characteristic impedance, the propagation constant of the line conductor is a vector-like quantity. Its real component, called *attenuation constant*

$$a = p\cos\delta \tag{2.3.9}$$

measures the decrease in voltage, current, and energy values per mile of line conductor as the energy is transferred (propagated) from the station end along the line. The quadrature component of p, called *phase constant*

$$\beta = p\sin\delta \tag{2.3.10}$$

measures the angular phase shift, i.e., the displacement of voltage, and of current per mile of line conductor, with reference to the respective quantities at the station end. Accordingly the vector value of the propagation constant is

$$p = a + j\beta. \tag{2.3.11}$$

As in the case of the characteristic impedance the G which enters in the formula for p may be neglected when the operating voltage to ground or neutral is less than the critical disruptive voltage $(kv)_{do}$ given by (1.22.12). The approximate formula given by (1.24.2) should be used, however, when the operating voltage to ground or neutral is larger than the critical disruptive voltage for the line under consideration.

2.4. *Velocity of Energy Transfer: Velocity of Phase Propagation.* If, in formula (2.3.6) for the propagation constant, the values of R and G are assumed negligibly small in comparison with $L\omega$ and $C\omega$ respectively, the formula becomes

$$p = \sqrt{-LC\omega^2} = j\omega\sqrt{LC}.$$

This being the quadrature component of p, it is the phase constant and should be written

$$\beta = \omega\sqrt{LC}$$

or

$$\frac{1}{\sqrt{LC}} = \frac{\omega}{\beta}. \tag{2.4.1}$$

Since the ω in this expression denotes a velocity, it follows that the quantity, $1/\sqrt{LC}$, has also the characteristics of a velocity and as such it has a particular significance and should be investigated more fully. Thus the value of L per mile of a solid conductor of a single-phase line or of an equilaterally-spaced three-phase line, by (1.7.4), is

$$L = 741.13 \times 10^{-6} \log \frac{D}{0.7788r} \text{ henry.}$$

The capacitance per mile of a solid conductor, by (1.16.4), is

$$C = \frac{38.82 \times 10^{-9}}{\log \frac{D}{r}} \text{ farads.}$$

The quantity $1/\sqrt{LC}$ has therefore a value of

$$\frac{1}{\sqrt{LC}} = 186300 \sqrt{\frac{\log \left(\frac{D}{r}\right)}{\log.\left(\frac{D}{0.7788r}\right)}}.$$

This may be simplified to

$$\frac{1}{\sqrt{LC}} = 186300 \sqrt{\frac{1}{1 + \frac{0.1086}{\log \frac{D}{r}}}}. \tag{2.4.2}$$

If the L and C are in henries and farads per mile, respectively, the quantity $1/\sqrt{LC}$ differs from the velocity of light (186,300 miles per second) by a factor equal to the radical in formula (2.4.2). Furthermore the radical would be equal to 1 if, in the calculation of L, the actual radius of the conductor were used instead of the geomean radius. In other words, if the magnetic flux within the conductor were neglected, the quantity $1/\sqrt{LC}$ would be equal to the velocity of light, and would be the same for all lines, independent of frequency, conductor size and interaxial spacing distance. For line conductors of 1,000,000 circ. mils, i.e., of 0.5 inches radius and whose interaxial spacing distance is 25 feet, the radical has a value of 0.981, indicating that the velocity of propagation for this particular line is 98.1 per cent of the velocity of electromagnetic waves through space, or 182,760 miles per second.

The quantity $1/\sqrt{LC} = v_o$, represented by equation (2.4.2), is the *velocity of energy transfer*. It is independent of the frequency or of the wave shape of the impressed voltage, but depends to a slight extent, as indicated above, upon the size of line conductors and their interaxial spacing distance.

The quantity $\omega/\beta = v_p$ has also the characteristics of a velocity. It is called *velocity of phase propagation*. It is equal to the velocity of energy transfer only when the line is hypothetically free of dissipative properties, i.e., when $R = 0$ and $G = 0$.*

The value of the propagation constant for power lines in which R is appreciable and G negligible, by (2.3.6), is

$$p = \sqrt{jC\omega(R + jL\omega)}$$

or

$$p = (C\omega\sqrt{R^2 + L^2\omega^2})^{\frac{1}{2}}\underline{/\delta}, \tag{2.4.3}$$

where

$$\delta = \frac{\tan^{-1}\dfrac{L\omega}{R} + 90^\circ}{2}. \tag{2.4.4}$$

The phase constant β, by (2.3.10), is

$$\beta = (C\omega\sqrt{R^2 + L^2\omega^2})^{\frac{1}{2}} \sin\delta. \tag{2.4.5}$$

From (2.4.4) it is found that

$$\sin\delta = \sqrt{\frac{1}{2}\left(1 + \frac{L\omega}{\sqrt{R^2 + L^2\omega^2}}\right)}.$$

This substituted in (2.4.5) and simplifying, yields

$$\beta^2 = \frac{C\omega}{2}\left(L\omega + \sqrt{R^2 + L^2\omega^2}\right).$$

* Carson, John R., *Electric Circuit Theory and the Operational Calculus*, p. 106. McGraw-Hill Book Co.

By factoring out $L\omega$, this may be put in the form

$$\beta^2 = \frac{LC\omega^2}{2}\left(1 + \sqrt{1 + \frac{R^2}{L^2\omega^2}}\right).$$

Setting

$$\frac{1}{\sqrt{LC}} = v_o,$$

the equation for the phase constant β may be written

$$\beta = \frac{\omega}{v_o}\left[\frac{1}{2}\left(1 + \sqrt{1 + \frac{R^2}{L^2\omega^2}}\right)\right]^{\frac{1}{2}}. \tag{2.4.6}$$

The velocity of phase propagation being

$$v_p = \frac{\omega}{\beta},$$

it follows that

$$\frac{v_o}{v_p} = \left[\frac{1}{2}\left(1 + \sqrt{1 + \frac{R^2}{L^2\omega^2}}\right)\right]^{\frac{1}{2}} \tag{2.4.7}$$

where v_o is the velocity of energy transfer.

From this it is easily seen that the velocity of phase propagation is smaller than the velocity of energy transfer and approaches it when the dissipative properties of the line are negligible in amount, or when the frequency is very large.

Since by definition

$$v_o = \frac{S}{t} \quad \text{and} \quad v_p = \frac{\omega}{\beta},$$

it follows that

$$\frac{v_o}{v_p} = \frac{\beta S}{\omega t}.$$

Comparing with (2.4.7), it is seen that

$$\frac{\beta S}{\omega t} = \left[\frac{1}{2}\left(1 + \sqrt{1 + \frac{R^2}{L^2\omega^2}}\right)\right]^{\frac{1}{2}}. \tag{2.4.8}$$

It will be recalled that β is the space-phase shift in radians or degrees of either the voltage or of the current at a point one mile distant from the station end with reference to the respective values at the station end. The line is assumed to be terminated with an impedance to ground or neutral equal to the characteristic impedance. Under identical conditions, the quantity βS represents, therefore, the space-phase shift (in radians or degrees) of voltage and of current at any point S miles from the station end.

The quantity ωt on the other hand is the time phase of either the voltage or of the current with respect to their respective initial steady state values. The space-phase angle βS is always larger than the time-phase angle. It will approach it in value only when the dissipative properties of the line are negligible or when ω is very large.

When βS is equal to 2π with S measured in miles and β in radians, S will be equal to one complete wave length λ. It follows, therefore, that

$$\lambda = \frac{2\pi}{\beta} \text{ miles.} \tag{2.4.9}$$

Substituting the value of β from (2.4.6) and using 186300 for v_o, there results

$$\lambda = \frac{186300}{f\left[\frac{1}{2}\left(1 + \sqrt{1 + \frac{R^2}{L^2\omega^2}}\right)\right]^{\frac{1}{2}}}, \tag{2.4.10}$$

where f is the frequency of the supply in cycles per second. Since R is rather small in comparison with $L\omega$ for long power lines, the wave length is substantially

$$\lambda = \frac{186300}{60} = 3105 \text{ miles}$$

when the frequency is 60 cps.

Since this is equivalent to 360°, it follows that commercial power lines operating at 60 cps are equivalent to a little over 8.62 miles per degree. A line 776 miles long would be equivalent to one-fourth of a wave length. The longest commercial line is about 35 degrees long.

2.5. *The Impedance Formula of Transmission Lines; Station-end Impedance to Neutral or Ground.* Consider the T element as part of a line which is terminated at the receiving end with an impedance Z_r, as shown in Fig. 2-5.

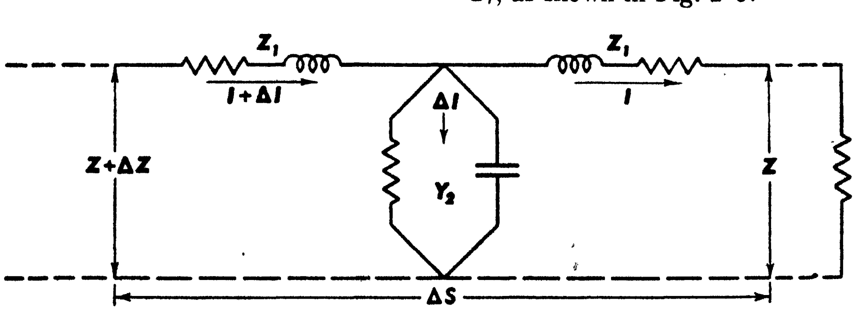


Figure 2-5

Let $\boldsymbol{Z}$ be the impedance to neutral at the output terminals of the T element and $\Delta \boldsymbol{Z}$, the impedance of the T element. The impedance at the input end of the T element is, therefore, $\boldsymbol{Z} + \Delta \boldsymbol{Z}$.

It is seen from the diagram that

$$Z + \Delta Z = Z_1 + \cfrac{1}{Y_2 + \cfrac{1}{Z_1 + Z}}.$$

This simplifies to

$$\Delta Z = \frac{Z_1^2 - Z^2 + \frac{2Z_1}{Y_2}}{Z_1 + Z + \frac{1}{Y_2}},$$

and, by (2.1.1), to

$$\Delta Z = \frac{Z_o^2 - Z^2}{Z_1 + Z + \frac{1}{Y_2}}.$$

Using the values of Z_1 and Y_2 in terms of the linear series impedance and linear shunt admittance, respectively, as given by (2.2.3), the above equation may be written

$$\Delta Z = \frac{Z_o^2 - Z^2}{\frac{1}{2}z\,\Delta S + Z + \frac{1}{y\,\Delta S}}$$

or

$$\Delta Z = \frac{(Z_o^2 - Z^2)2y\,\Delta S}{(z\,\Delta S)(y\,\Delta S) + 2Zy\,\Delta S + 2}.$$

At the limit when ΔS approaches zero, the equation becomes

$$\frac{dZ}{dS} = (Z_o^2 - Z^2)y$$

or

$$\frac{dZ}{Z_o^2 - Z^2} = ydS. \tag{2.5.1}$$

Measuring line length from the receiving end where $S = 0$ and $Z = Z_r$, to any point along the line, where $S = S$ and $Z = Z$ gives the limits to the integral

$$\int_{Z_r}^{Z} \frac{dZ}{Z_o^2 - Z^2} = \int_0^S y\,dS.$$

The integral of this expression is

$$\frac{1}{2Z_o}\left[\ln\frac{Z_o + Z}{Z_o - Z} - \ln\frac{Z_o + Z_r}{Z_o - Z_r}\right] = yS.$$

This may be written

$$\ln\frac{Z_o + Z}{Z_o - Z}\cdot\frac{Z_o - Z_r}{Z_o + Z_r} = 2Z_oyS.$$

Using the exponential form and since by (2:2.4)

$$Z_o y = \sqrt{zy} = p,$$

the above equation becomes

$$\frac{Z_o + Z}{Z_o - Z} \cdot \frac{Z_o - Z_r}{Z_o + Z_r} = \epsilon^{2pS}.$$

Solving this for Z gives

$$Z = \frac{Z_o[(Z_o + Z_r)\epsilon^{2pS} - (Z_o - Z_r)]}{(Z_o - Z_r) + (Z_o + Z_r)\epsilon^{2pS}}. \tag{2.5.2}$$

The expression may be put in the more convenient form given below by multiplying numerator and denominator by ϵ^{-pS}

$$Z = \frac{Z_o^2(\epsilon^{pS} - \epsilon^{-pS}) + Z_r Z_o(\epsilon^{pS} + \epsilon^{-pS})}{Z_o(\epsilon^{pS} + \epsilon^{-pS}) + Z_r(\epsilon^{pS} - \epsilon^{-pS})} \tag{2.5.3}$$

or

$$Z = Z_o \frac{Z_o \sinh pS + Z_r \cosh pS}{Z_o \cosh pS + Z_r \sinh pS}. \tag{2.5.4}$$

This gives the impedance to ground or neutral at any point of the line. The equation represents the impedance of the line at the station end if S designates the entire line length.

It is important to note that if the load impedance Z_r to neutral is equal to the characteristic impedance Z_o in the above two equations, they would reduce to $Z = Z_o$. This not only restates the fact that when a line is terminated in characteristic impedance, the impedance at the other end is the same, but proves also the fact that the impedance at any point of such a line is equal to Z_o.

Logically, this would be true also of an infinitely long line. For this reason, a line terminated in characteristic impedance is sometimes referred to as a portion of an infinite line or as a *smooth* line.

2.6. *Significance and Evaluation of Exponential and Hyperbolic Terms.*

The exponential term ϵ^{pS} and its reciprocal enter into many formulas pertaining to transmission lines. In equation (2.5.4) there appear also hyperbolic functions of the term pS. It will be seen, as the theory of transmission is further developed, that such hyperbolic terms enter into practically all transmission line formulas. It is important that the significance of these expressions be ascertained and procedures for evaluating them be formulated.

It was seen that the propagation constant is a vector-like quantity

$$p = a + j\beta.$$

From this it follows that expressions of the form $A\epsilon^{pS}$ may be written

$$A\epsilon^{pS} = A\epsilon^{\alpha S}\underline{/\beta S}. \tag{2.6.1}$$

Similarly

$$B\epsilon^{-pS} = B\epsilon^{-\alpha S}\underline{/-\beta S}. \tag{2.6.2}$$

If S, which designates distance along the line, is a variable, then equation (2.6.1) represents an exponentially growing vector-like quantity which rotates simultaneously counter-clockwise with an angular space-phase displacement β radians per mile as shown in Fig. 2–6.

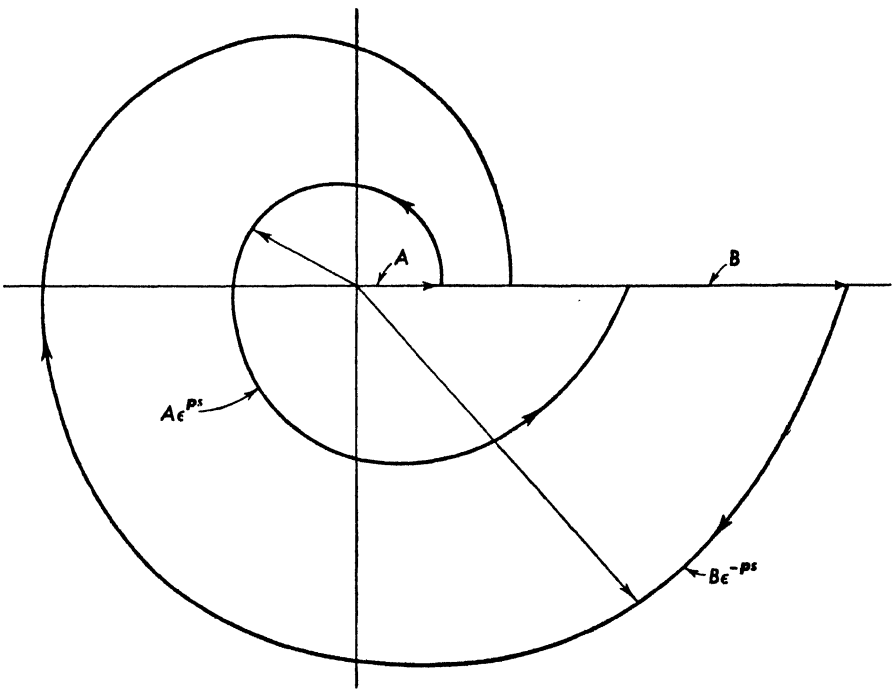


Figure 2–6

Similarly, equation (2.6.2) represents an exponentially decaying and clockwise rotating vector as indicated in the figure. The initial values of these exponentially changing and rotating vectors are A and B respectively. The numerical evaluation of such quantities is suggested by equations (2.6.1) and (2.6.2). Obviously, each may be expressed as a complex number by splitting up into a real or cosine component and a quadrature or sine component. This complex form of the exponentials is useful in evaluating the hyperbolic terms cosh pS and sinh pS.

Thus, since

$$\cosh pS = \frac{\epsilon^{pS} + \epsilon^{-pS}}{2}, \tag{2.6.3}$$

by (2.6.1) and (2.6.2) it may be put in the form

$$\cosh pS = \frac{\epsilon^{aS}}{2}\underline{/\beta S} + \frac{\epsilon^{-aS}}{2}\underline{/-\beta S}. \tag{2.6.4}$$

The equation indicates that if S is a variable, $\cosh(a + j\beta)S$ is the vector sum of two exponentially changing and rotating vectors. One of these $\frac{1}{2}\epsilon^{pS}$ is growing and rotating counter-clockwise, and the other $\frac{1}{2}\epsilon^{-pS}$ is decaying and rotating clockwise, as shown in Fig. 2–7.

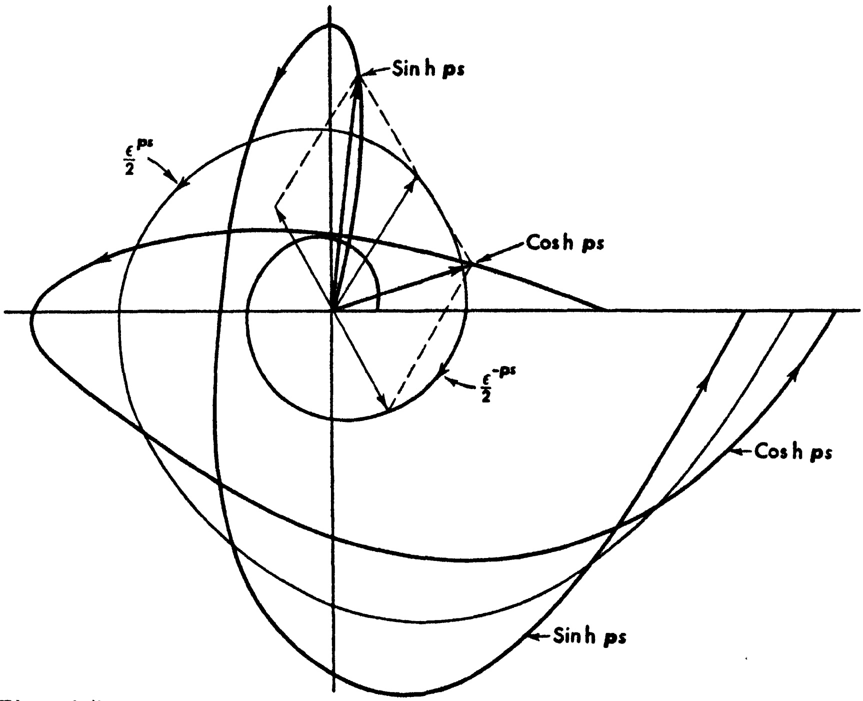


Figure 2–7

Equation (2.6.4) indicates also the procedure for evaluating $\cosh pS$ by splitting its two exponential components into complex numbers. If the equation is expanded, however, it becomes

$$\cosh pS = \cosh aS \cos \beta S + j \sinh aS \sin \beta S.$$

This simplifies readily to

$$\left.\begin{aligned} \cosh pS &= \sqrt{\sinh^2 aS + \cos^2 \beta S}\underline{/\phi} \\ \text{where} \quad \phi &= \tan^{-1}(\tanh aS \tan \beta S). \end{aligned}\right\} \tag{2.6.5}$$

It should be noted from (2.6.5) that the numerical value of $\cosh pS$, being the square root of the sums of squares, may be treated as the hypotenuse of a right triangle whose sides are sinh αS and cos βS, and thus evaluated by a trigonometric procedure.*

The hyperbolic term sinh pS is

$$\sinh(\alpha + j\beta)S = \frac{\epsilon^{pS} - \epsilon^{-pS}}{2}. \tag{2.6.6}$$

It may be written

$$\sinh(\alpha + j\beta)S = \frac{\epsilon^{\alpha S}}{2}\underline{/\beta S} - \frac{\epsilon^{-\alpha S}}{2}\underline{/-\beta S}. \tag{2.6.7}$$

This equation suggests a method of evaluating sinh pS and it indicates also that if S is a variable, the function is the vector difference of two exponentially changing and rotating vectors as shown in Fig. 2-7.

If equation (2.6.7) is expanded, however, it gives

$$\sinh pS = \sinh \alpha S \cos \beta S + j \cosh \alpha S \sin \beta S,$$

which simplifies readily to

$$\sinh pS = \sqrt{\sinh^2 \alpha S + \sin^2 \beta S}\underline{/\psi}$$

where

$$\psi = \tan^{-1}\left(\frac{\tan \beta S}{\tanh \alpha S}\right). \tag{2.6.8}$$

The numerical value of the function may be calculated easily by (2.6.8). Like the cosh pS it may be computed trigonometrically by treating the radical in (2.6.8) as the hypotenuse of a right triangle whose sides are sinh αS and sin βS.†

2.7. *The Current Formulas of Transmission Lines.* Referring to Fig. 2-5, let I be the current leaving the T element of the line conductor; Z, the impedance to neutral at that point; and ΔI, the current in the admittance branch Y_2 of the element. By observation, it is seen that

$$\Delta I = I(Z + Z_1)Y_2.$$

Hence

$$\frac{\Delta I}{I} = (Z + Z_1)Y_2.$$

Substituting for Z_1 and Y_2 their respective equivalents as given by (2.2.3), the preceding equation becomes

$$\frac{\Delta I}{I} = Zy\,\Delta S + \left(\frac{z\,\Delta S}{2}\right)(y\,\Delta S).$$

* Weinbach, M. P., *The Log Log Duplex Vector Slide Rule*, Keuffel and Esser Co.

† See Appendix 2 for the determination of $p = \alpha + j\beta$ when sinh p, cosh p, or tanh p are known.

This at the limit, when ΔS approaches zero, gives

$$\frac{dI}{I} = Zy\,dS.$$

Using the impedance formula (2.5.4) for Z puts this equation in the form

$$\frac{dI}{I} = \left(yZ_o \frac{Z_o \sinh pS + Z_r \cosh pS}{Z_o \cosh pS + Z_r \sinh pS}\right) dS.$$

It should be noted that in this expression

$$yZ_o = y\sqrt{\frac{z}{y}} = \sqrt{yz} = p$$

and that the derivative of the denominator $\frac{d}{dS}(Z_o \cosh pS + Z_r \sinh pS)$ with respect to S is equal to the numerator. The equation may be written, therefore,

$$\frac{dI}{I} = \frac{\frac{d}{dS}(Z_o \cosh pS + Z_r \sinh pS)}{Z_o \cosh pS + Z_r \sinh pS}\, dS.$$

Integrating the left member of this equation between the limits of $I = I_r$ and $I = I_s$ and the right side between the limits of $S = 0$ and $S = S$, gives

$$\ln \frac{I_s}{I_r} = \ln \frac{Z_o \cosh pS + Z_r \sinh pS}{Z_o}$$

or

$$I_s = I_r \cosh pS + \frac{I_r Z_r}{Z_o} \sinh pS. \qquad (2.7.1)$$

Since

$$I_r Z_r = V_r,$$

where V_r is the receiving-end voltage to ground or neutral, the preceding equation takes the final form

$$I_s = I_r \cosh pS + \frac{V_r}{Z_o} \sinh pS. \qquad (2.7.2)$$

This gives the current at the station end of the line conductor in terms of receiving-end quantities, and the line properties p and Z_o.

In terms of the receiving-end voltage V_r equation (2.7.2) becomes

$$I_s = V_r\left(\frac{\cosh pS}{Z_r} + \frac{\sinh pS}{Z_o}\right). \qquad (2.7.3)$$

The formula for the current at the receiving end in terms of station-end values is

$$I_r = I_s \cosh pS - \frac{V_s}{Z_o} \sinh pS, \tag{2.7.4}$$

which, in terms of V_s, becomes

$$I_r = V_s \left(\frac{\cosh pS}{Z_s} - \frac{\sinh pS}{Z_o} \right). \tag{2.7.5}$$

Taking S as the full length of the line, and using the value of Z_s as given by (2.5.4) and simplifying, yields

$$I_r = \frac{V_s}{Z_r \cosh pS + Z_o \sinh pS}. \tag{2.7.6}$$

It should be noted that for any definite line with constant impressed voltage V_s, the only variable in (2.7.6) is Z_r. When $Z_r = 0$, i.e., when the line is short-circuited at the load end, the equation becomes:

$$(I_r)_{sh} = \frac{V_s}{Z_o \sinh pS} \tag{2.7.7}$$

which is fixed in value. Also when Z_r approaches infinite value, i.e., when the line becomes open-circuited, the current I_r approaches zero. The curve marked I_r in Fig. 2–8 shows the variation of the load-end current with Z_r.

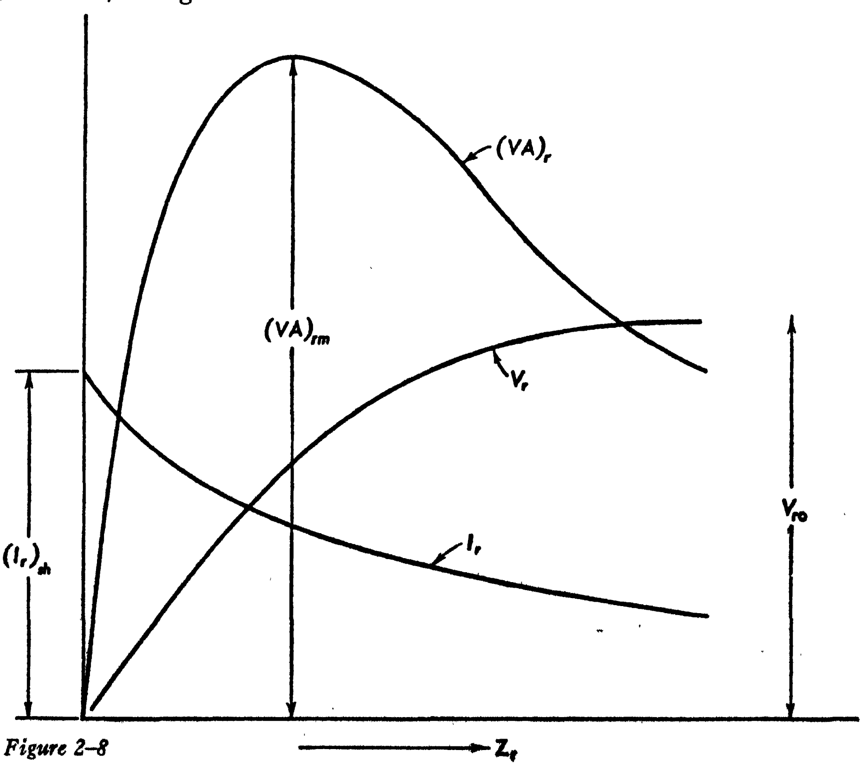


Figure 2–8

2.8. *The Voltage Formula of Transmission Lines.* If I_s is the conductor current at the station end and Z_s the impedance to ground or neutral, the voltage to ground or neutral at that end is

$$V_s = I_s Z_s.$$

Using the formula for I_s as given by (2.7.4) and the formula for Z_s as given by (2.5.4), the preceding expression would give V_s in terms of the line constants and receiving-end impedance. If formula (2.5.4) for Z_s is simultaneously multiplied and divided by I_r/Z_o it becomes

$$Z_s = \frac{I_r Z_o \sinh pS + V_r \cosh pS}{I_r \cosh pS + \dfrac{V_r}{Z_o} \sinh pS}.$$

Hence using (2.7.1) for I_s, and this formula for Z_s in the above equation for V_s, gives

$$V_s = V_r \cosh pS + I_r Z_o \sinh pS, \tag{2.8.1}$$

or the more convenient form

$$V_s = V_r \left(\cosh pS + \frac{Z_o}{Z_r} \sinh pS\right), \tag{2.8.2}$$

where S is the entire length of line.

The voltage to ground or neutral at the receiving end in terms of station-end values is by analogy

$$V_r = V_s \cosh pS - I_s Z_o \sinh pS. \tag{2.8.3}$$

Substituting V_s/Z_s for I_s, and for Z_s its formula (2.5.4) and simplifying yields the more convenient formula

$$V_r = \frac{V_s Z_r}{Z_r \cosh pS + Z_o \sinh pS}. \tag{2.8.4}$$

By putting this formula in the form

$$V_r = \frac{V_s}{\cosh pS + \dfrac{Z_o}{Z_r} \sinh pS}, \tag{2.8.5}$$

it will be observed that when the load-end impedance $Z_r = 0$, the value of $V_r = 0$. Also when Z_r approached infinity, i.e., when the receiving end becomes open-circuited, the load-end voltage approaches the constant

$$V_{ro} = \frac{V_s}{\cosh pS}. \tag{2.8.6}$$

The curve marked V_r in Fig. 2–8 shows the variation of V_r with the load impedance Z_r.

Under the condition of open-circuited receiving end, the current at that end is $I_r = 0$, and equation (2.7.2) becomes

$$I_{so} = \frac{V_{ro} \sinh pS}{Z_o}, \tag{2.8.7}$$

or, by (2.8.6),

$$I_{so} = \frac{V_s \sinh pS}{Z_o \cosh pS}. \tag{2.8.8}$$

This current is usually called the *charging current* of the line.

Formula (2.8.4) for V_r, multiplied by formula (2.7.6) for I_r, gives the formula for the receiving-end voltampere

$$(VA)_r = \frac{V_s^2 Z_r}{(Z_r \cosh pS + Z_o \sinh pS)^2}. \tag{2.8.9}$$

Since $V_r = 0$ when $Z_r = 0$ on short circuit and $I_r = 0$ when $Z_r = \infty$ on open circuit, it is reasonable to assume that $(VA)_r$ passes through a maximum value for some definite value of receiving-end load impedance Z_r. By differentiating equation (2.8.9) with respect to Z_r as the variable, equating to zero, and solving for Z_r it is found that the voltampere $(VA)_r$ at the receiving end is a maximum when the receiving-end impedance has the value

$$Z_{rm} = Z_o \frac{\sinh pS}{\cosh pS} = Z_o \tanh pS \tag{2.8.10}$$

The variation of the voltampere $(VA)_r$ with the load impedance Z_r is indicated by the curve so marked in Fig. 2-8.

2.9. *Voltage Regulation.* This is defined as the change in voltage at the receiving end from full load to no load in per cent of the full-load voltage. If V_r is the receiving-end voltage to neutral on full load and V_{ro}, as given by formula (2.8.6), is the receiving-end voltage on no load, the regulation is

$$\text{Reg.} = \frac{(V_{ro} - V_r)100}{V_r}. \tag{2.9.1}$$

2.10. *Receiving-End and Station-End Currents on Short-Circuited Receiving End.* If the load at the receiving end should suddenly become short-circuited and its equivalent impedance Z_r thus forced suddenly to approach a zero value, the current at that end would rise to a steady state value given by (2.7.7).

$$(I_r)_{sh} = \frac{V_s}{Z_o \sinh pS}. \tag{2.10.1}$$

The station-end current $(I_s)_{sh}$, assumed sustained while the line is short-circuited at the receiving end, is by (2.7.2), when $V_r = 0$ and $I_r = (I_r)_{sh}$,

$$(I_s)_{sh} = (I_r)_{sh} \cosh pS. \tag{2.10.2}$$

By (2.10.1) this becomes

$$(I_s)_{sh} = \frac{V_s \cosh pS}{Z_o \sinh pS}, \tag{2.10.3}$$

where V_s is the station-end voltage to neutral.

2.11. *Résumé of Derived Transmission Formulas.*

Linear line impedance

$$z = R + jL\omega = z/\underline{\alpha_1} \text{ vector ohms.} \tag{2.11.1}$$

Linear line admittance

$$y = G + jC\omega = y/\underline{\alpha_2} \text{ vector mhos} \tag{2.11.2}$$

for $G = 0$

$$y = C\omega/\underline{90^\circ} \text{ vector mhos.} \tag{2.11.3}$$

Characteristic impedance

$$Z_o = \sqrt{\frac{R + jL\omega}{G + jC\omega}} = \sqrt{\frac{z}{y}}/\underline{\zeta_o}, \tag{2.11.4}$$

where

$$\zeta_o = \frac{\alpha_1 - \alpha_2}{2} \text{ degrees}$$

$$\alpha_1 = \tan^{-1} \frac{L\omega}{R}$$

$$\alpha_2 = \tan^{-1} \frac{C\omega}{G}.$$

Surge impedance

$$Z_{su} = \sqrt{\frac{L}{C}} \text{ ohms.} \tag{2.11.5}$$

Propagation constant

$$p = \sqrt{(R + jL\omega)(G + jC\omega)} = \sqrt{zy}/\underline{\delta},$$

where

$$\delta = \frac{\alpha_1 + \alpha_2}{2}. \tag{2.11.6}$$

Attenuation constant

$$a = p \cos \delta. \tag{2.11.7}$$

Phase constant

$$\beta = p \sin \delta \text{ degrees/mile.} \tag{2.11.8}$$

Velocity of energy propagation

$$v_o = \sqrt{\frac{1}{LC}} \text{ miles/sec.} \tag{2.11.9}$$

Wave length

$$\lambda = \frac{2\pi}{\beta} \text{ miles.} \tag{2.11.10}$$

$$\left.\begin{aligned} \text{Sinh } pS &= (\sinh^2 aS + \sin^2 \beta S)^{\frac{1}{2}}\underline{/\psi} \\ \psi &= \tan^{-1}\left(\frac{\tan \beta S}{\tanh aS}\right). \end{aligned}\right\} \tag{2.11.11}$$

$$\left.\begin{aligned} \text{Cosh } pS &= (\sinh^2 aS + \cos^2 \beta S)^{\frac{1}{2}}\underline{/\phi} \\ \phi &= \tan^{-1}(\tan \beta S \tanh aS). \end{aligned}\right\} \tag{2.11.12}$$

Sending-end impedance of a line to ground or neutral

$$Z_s = Z_o \frac{Z_o \sinh pS + Z_r \cosh pS}{Z_o \cosh pS + Z_r \sinh pS}, \tag{2.11.13}$$

where

S = length of line

Z_r = load impedance per phase.

Station-end current

$$I_s = I_r \cosh pS + \frac{V_r}{Z_o} \sinh pS, \tag{2.11.14}$$

or in terms of V_r and Z_r,

$$I_s = V_r\left(\frac{\cosh pS}{Z_r} + \frac{\sinh pS}{Z_o}\right). \tag{2.11.15}$$

Receiving-end current in terms of V_s and Z_r

$$I_r = \frac{V_s}{Z_r \cosh pS + Z_o \sinh pS}. \tag{2.11.16}$$

Station-end voltage to neutral in terms of V_r and Z_r

$$V_s = V_r\left(\cosh pS + \frac{Z_o}{Z_r} \sinh pS\right). \tag{2.11.17}$$

Receiving-end voltage to neutral in terms of V_s and Z_r

$$V_r = \frac{V_s Z_r}{Z_r \cosh pS + Z_o \sinh pS}. \tag{2.11.18}$$

Voltage to neutral at the open-circuited end of a line, in terms of V_s

$$V_{ro} = \frac{V_s}{\cosh pS}. \tag{2.11.19}$$

Current at the station end when the load end is open-circuited, i.e., charging current

$$I_{so} = \frac{V_s \sinh pS}{Z_o \cosh pS}. \tag{2.11.20}$$

Voltamperes at the receiving end in terms of V_s and Z_r

$$(VA)_r = \frac{V_s^2 Z_r}{(Z_r \cosh pS + Z_o \sinh pS)^2}. \tag{2.11.21}$$

Receiving-end impedance for maximum voltamperes

$$Z_{rm} = Z_o \frac{\sinh pS}{\cosh pS}. \tag{2.11.22}$$

Voltage regulation

$$\text{Reg.} = \frac{(V_{ro} - V_r)100}{V_r}. \tag{2.11.23}$$

Current at the short-circuited end of a line terms of V_s

$$(I_r)_{sh} = \frac{V_s}{Z_o \sinh pS}. \tag{2.11.24}$$

Current at the station end when receiving end is short-circuited

$$(I_s)_{sh} = \frac{V_s \cosh pS}{Z_o \sinh pS}. \tag{2.11.25}$$

This by (2.11.24) becomes

$$(I_s)_{sh} = (I_r)_{sh} \cosh pS. \tag{2.11.26}$$

REFERENCES

Boyajian, Aram, *Physical Interpretation of Complex Angles*, Journal Am. Inst. Elect. Engrs., 1923.

Bradford and John, *Telephone and Power Transmission*, Wiley and Sons, 1929.

Kennelly, A. E., *Hyperbolic Functions Applied to Electrical Engineering*, McGraw-Hill Book Co., 1916.

Kennelly, A. E., *Artificial Electric Lines*, McGraw-Hill Book Co., 1917.

Loew, E. A., *Electrical Power Transmission*, McGraw-Hill Book Co., 1928.

Nesbit, William, *Electrical Characteristics of Transmission Circuits*, Westinghouse Elect. & Mfg. Co., 1926.

Waddicor, Harold, *Principles of Electric Power Transmission by Alternating Currents*, Wiley and Sons, 1928.

Weinbach, M. P., *Principles of Transmission in Telephony*, Macmillan, 1924.

Woodruff, L. F., *Principles of Electric Power Transmission*, Wiley and Sons, 1938.

Chapter 3 Applications of Transmission Formulas

The general equations developed in the preceding chapter may be applied, with proper interpretations, to the design and analysis of performance of all types of lines, d-c or a-c, single-phase or three-phase, single circuit, twin or multiple circuit. This chapter deals primarily with the application of the transmission formulas to the study of line performance.

3.1. *The Direct-Current Line.* Since $\omega = 0$ for a d-c circuit, formula (2.11.4) for the characteristic impedance becomes

$$R_o = \sqrt{\frac{R}{G}}, \tag{3.1.1}$$

where R_o may be called the *characteristic resistance* of a d-c line. Thus the characteristic resistance of a #10 A.W.G., ground-return telegraph line 100 miles long, having a resistance $R = 5.28$ ohms per mile, and a leakage conductance G of 5×10^{-6} mhos is

$$R_o = \sqrt{\frac{5.28}{5 \times 10^{-6}}} = 1026 \text{ ohms.}$$

The propagation constant, given by (2.11.6), becomes for a d-c line

$$p = a = \sqrt{RG}. \tag{3.1.2}$$

For the particular line under consideration the attenuation constant is

$$\begin{aligned} a &= \sqrt{5.28 \times 5 \times 10^{-6}} \\ &= 0.005138. \end{aligned}$$

The formula for the resistance at the station end becomes by (2.11.13)

$$R_s = R_o\left(\frac{R_o \sinh aS + R_r \cosh aS}{R_o \cosh aS + R_r \sinh aS}\right). \tag{3.1.3}$$

For the illustrative line under consideration, if terminated with a resistance of $R_r = 500$ ohms, it is

$$R_s = 1026\left(\frac{1026 \sinh 0.5138 + 500 \cosh 0.5138}{1026 \cosh 0.5138 + 500 \sinh 0.5138}\right)$$

$$R_s = 801.9 \text{ ohms.}$$

This gives for the sending-end current

$$I_s = \frac{V_s}{801.9}$$

$$= 1.247\ V_s \text{ milliamperes.}$$

The formula (2.11.18) for the voltage at the receiving end when applied to a d-c line is

$$V_r = \frac{V_s}{\cosh aS + \dfrac{R_o}{R_r}\sinh aS}. \tag{3.1.4}$$

For the illustrative problem considered it is

$$V_r = \frac{V_s}{\cosh 0.5138 + \frac{1026}{500}\sinh 0.5138}$$

or

$$V_r = 0.446\ V_s \text{ volts.}$$

The current at the receiving end is, obviously,

$$I_r = \frac{V_r}{R_r} = \frac{0.446\ V_s}{500}$$

or

$$I_r = 0.892\ V_s \text{ milliamperes.}$$

Equation (2.7.4) could, evidently, be used to determine the current at the receiving end. For the d-c line, this equation becomes

$$I_r = I_s \cosh aS - \frac{V_s}{R_o}\sinh aS. \tag{3.1.5}$$

The voltage at the open-circuited end, by (2.11.19), is

$$V_{ro} = \frac{V_s}{\cosh aS}. \tag{3.1.6}$$

For the particular line considered in the illustrative problem stated above, it is

$$V_{ro} = \frac{V_s}{1.1356}$$

$$= 0.881\ V_s \text{ volts.}$$

With a load of 500 ohms, the receiving-end voltage was found to be $V_r = 0.446$ volt per volt impressed at the station end.

If the d-c line were short-circuited at the receiving end the current at that end would be, by (2.11.24),

$$(I_r)_{sh} = \frac{V_s}{R_o \sinh aS}. \tag{3.1.7}$$

For the illustrative line this is

$$\begin{aligned}(I_r)_{sh} &= \frac{V_s}{1026 \sinh 0.5138} \\ &= 1.811\ V_s \text{ milliamperes.}\end{aligned}$$

The current at the station end when the receiving end is short-circuited, by (2.11.26), is

$$(I_s)_{sh} = (I_r)_{sh} \cosh aS. \tag{3.1.8}$$

For the illustrative line considered, it is

$$\begin{aligned}(I_s)_{sh} &= 1.811 \times 1.135\ V_s \\ &= 2.056\ V_s \text{ milliamperes.}\end{aligned}$$

The receiving-end voltamperes, which for the d-c line is numerically equal to the power in watts is

$$\begin{aligned}P_r &= V_r I_r \\ &= 0.446 \times 0.892 \times 10^{-3}\ V_s^2 \\ &= 0.398\ V_s^2 \text{ milliwatts.}\end{aligned}$$

The receiving-end resistance that would give the maximum power at the receiving end, by (2.11.22), is

$$R_{rm} = R_o \frac{\sinh aS}{\cosh aS} = R_o \tanh aS.$$

For the illustrative line used, it is

$$\begin{aligned}R_{rm} &= 1026 \times 0.473 \\ &= 485.3.\end{aligned}$$

3.2. *The Nondissipative Alternating-Current Line. Fundamental Relations.* To get a comprehensive understanding of the transmission line formulas developed in the preceding chapter it is desirable that they should be applied also to a nondissipative a-c line. The analysis of such a line in contrast with that of the nonreactive (d-c) line discussed and illustrated in the preceding article, will give not only a more complete insight into the phenomena of energy transmission, but of the performance of a-c lines under actual operating conditions. It should be kept in mind of course that a nondissipative line is nonexistent. For this reason it is sometimes referred to as an *ideal* or nonrealizable line.

The analysis is illustrated with a single-phase line consisting of two stranded hard-drawn copper cables, each of 61 strands. The cross-sectional area of the copper is 1,000,000 circ. mils. The outside diameter of the cable is 1.151 inches ($r = 0.5755$ inches). The resistance of such a cable is 0.068 ohms per mile. Since it is less than one-tenth of the inductive reactance per mile, it will be neglected and the line assumed to be substantially a nondissipative line. The interaxial spacing distance is assumed to be 24 feet, i.e., 288 inches.

The inductance and inductive reactance of standard conductors and standard spacing distances, and at commercial frequencies may be obtained from handbooks of electrical engineering, or from Tables II and III as outlined in § 1.11.

The inductance may, however, be calculated by the general formula (1.8.11)

$$L = 741.13 \times 10^{-6} \log \frac{D}{(r_{gm})_n}.$$

The geomean radius of a 61-strand cable, given in Table I is $0.7720\, r$. Accordingly the above formula becomes for the particular cable considered

$$L = 741.13 \times 10^{-6} \log \frac{288}{0.7720 \times 0.5755}$$

$$L = 0.002084 \text{ henry per mile.}$$

The inductive reactance at the commercial frequency of 60 cps is

$$L\omega = 0.002084 \times 377 = 0.786 \text{ ohms/mile.}$$

The capacitance per mile of conductor to ground or neutral is, by (1.16.4), or directly from Tables IV and V as outlined in § 1.19

$$C = \frac{38.82 \times 10^{-9}}{\log \dfrac{288}{0.5755}}$$

$$C = 14.387 \times 10^{-9} \text{ farads/mile.}$$

The capacitive susceptance is, at the frequency of 60 cps,

$$C\omega = 14.387 \times 10^{-9} \times 377$$
$$= 5.421 \times 10^{-6} \text{ mhos/mile.}$$

Under the assumption that the line is nondissipative, i.e., $R = 0$ and $G = 0$, the formula for the characteristic impedance given by (2.2.5), becomes, therefore,

$$Z_o = \sqrt{\frac{L}{C}} \underline{/0^\circ}. \tag{3.2.1}$$

The characteristic impedance of a nondissipative line is thus equal to the surge impedance, i.e., to the resonant impedance of a circuit consisting theoretically only of inductance and capacitance. All the energy that is imparted to the line is reactive in character. The phase angle of the surge impedance is zero. It indicates that the storage and restoration of the reactive energy is not between source and line but between the inductance and capacitance of the line. With reference to the source, a nondissipative line is, paradoxically, a circuit of unity power factor. The concept of storage and restoration of the energy must be interpreted to mean storage in the inductance as a magnetic field and restoration to the capacitance as an electric field, and vice versa.

The characteristic impedance of the nondissipative line under consideration is

$$\sqrt{\frac{L}{C}} = \sqrt{\frac{0.002084}{14.387 \times 10^{-9}}}$$
$$Z_o = 380 \text{ ohms.}$$

The approximate formula (2.2.12) for the surge impedance gives

$$\sqrt{\frac{L}{C}} = 138.15 \times 2.7$$
$$Z_o = 372.9 \text{ ohms.}$$

The velocity of energy propagation is

$$\sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{0.002084 \times 14.387 \times 10^{-9}}}$$
$$= 182{,}700 \text{ miles per second.}$$

On the supposition that the line is operated in fair weather, barometric pressure of 76 cm. of mercury and temperature of 25° C., the *disruptive potential to ground* would be, by (1.22.6), modified by an "irregularity factor" of 0.85 due to stranding, as indicated by (1.22.12)

$$(kv)_{do} = 123 \times 0.5755 \times 2.7 \times 0.85$$
$$= 162.3 \text{ kilovolts per phase.}$$

or $2 \times 162.3 = 324.6$ kilovolts across the two-conductor line.

On the same supposition of conductor condition and fair weather the visual critical potential to ground would be, by (1.23.2) modified by the "irregularity factor" due to stranding, as shown in (1.23.4)

$$(kv)_{vo} = 123\left(1 + \frac{0.189}{\sqrt{0.5755}}\right) \times 0.5755 \times 2.7 \times 0.85$$
$$= 202.86 \text{ kv. to ground}$$

or $2 \times 202.86 = 405.72$ kilovolts across the two-conductor line.

If the line were operated at a voltage equal to the visual critical voltage the power loss (in fair weather, $\delta = 1$) would be, by (1.24.1),

$$P = \left[390(60 + 25)(202.86 - 162.3)^2 \sqrt{\frac{0.5755}{288}}\right] 10^{-5}$$
$$= 24.36 \text{ kw per mile per conductor,}$$

or

$$2 \times 24.36 = 48.72 \text{ kw per mile of line.}$$

The propagation constant of a nondissipative line, by (2.3.6), is

$$p = \sqrt{j^2LC\omega^2} = j\omega\sqrt{LC}. \tag{3.2.2}$$

Since the propagation constant p is equal to its j-component, it follows that the attenuation constant is zero, as it should be for a nondissipative line. The phase constant is therefore

$$\beta = \omega\sqrt{LC} \text{ radians per mile.} \tag{3.2.3}$$

For the illustrative line under consideration, its value is

$$\beta = 377\sqrt{0.002084 \times 14.387 \times 10^{-9}}$$
$$= 0.002062 \text{ radians per mile}$$

or

$$\beta = 0.1184 \text{ degrees per mile.}$$

The velocity of phase propagation is

$$V_p = \frac{\omega}{\beta}.$$

For the illustrative line, it is

$$V_p = \frac{377}{0.002062} = 182700 \text{ miles/sec.}$$

It is equal to the energy velocity as it should be for a nondissipative line.

The wave length, by (2.4.9), is

$$\lambda = \frac{2\pi}{\beta} = \frac{6.28}{0.002062}$$
$$= 3045 \text{ miles.}$$

3.3. *Performance of the Nondissipative Line.*

Before applying the various transmission formulas to the calculation of the performance of the nondissipative line, it is necessary to evaluate the hyperbolic functions of the propagation constant

$$\cosh pS = \cosh (a + j\beta)S$$

and

$$\sinh pS = \sinh(a + j\beta)S,$$

for the particular condition when $a = 0$, as it is for a nondissipative line.

By expanding, the cosh function becomes

$$\cosh pS = \cosh aS \cos \beta S + j \sinh aS \sin \beta S,$$

which, when

$$a = 0, \cosh aS = 1, \sinh aS = 0$$

gives

$$\cosh pS = \cos \beta S. \tag{3.3.1}$$

Similarly, by expanding the sinh function it is found that when $a = 0$,

$$\sinh pS = j \sin \beta S.$$

Accordingly the station-end impedance, given by (2.11.13), becomes

$$\mathbf{Z}_s = \mathbf{Z}_o \frac{\mathbf{Z}_r \cos \beta S + j\mathbf{Z}_o \sin \beta S}{\mathbf{Z}_o \cos \beta S + j\mathbf{Z}_r \sin \beta S}, \tag{3.3.2}$$

where for a nondissipative line

$$\mathbf{Z}_o = \sqrt{\frac{L}{C}}.$$

To apply formula (3.3.2) to the particular illustrative line used in the preceding article, it will be assumed that the length of the line is $S = 300$ miles. The calculated value of the phase constant is $\beta = 0.1184°$. Hence

$$\beta S = 0.1184 \times 300 = 35.52°.$$

This is not quite one-tenth of a complete wave length.

The characteristic impedance was found to be

$$\sqrt{\frac{L}{C}} = 380 \text{ ohms}.$$

Assuming that the receiving-end impedance to ground is $\mathbf{Z}_r = 300\underline{/0°}$, the sending-end impedance is, by equation (3.3.2)

$$\mathbf{Z}_s = 380 \frac{300 \cos 35.52° + j\, 380 \sin 35.52°}{380 \cos 35.53° + j\, 300 \sin 35.52°}$$

or

$$= 380 \frac{329\underline{/42.15°}}{355\underline{/29.41°}}$$

$$\mathbf{Z}_s = 352\underline{/12.74°} \text{ vector ohms},$$

to the ground potential plane or a total impedance across the station-end terminals of the two-wire line equal to $704\underline{/12.74°}$ vector ohms.

It will be assumed now that the illustrative line is connected at the station end to the output terminals of a step-up transformer whose voltage is 250 kv.

Under this particular assumption, the station-end current is

$$I_s = \frac{250000}{704\underline{/12.74°}}$$
$$= 355.2\underline{/-12.74°} \text{ vector amperes.}$$

The total kva supplied to the line at the station end is

$$(kva)_s = (kv)_s I_s$$
$$= 250 \times 355.2$$
$$= 88800 \text{ kilovolt-amperes.}$$

The amount of dissipative power supplied to the line at the station end is

$$P_s = (kva)_s \cos(\theta_s)$$
$$= 88800 \cos(-12.74°)$$
$$= 86700 \text{ kilowatts.}$$

The entire amount of dissipative power supplied at the station end should reach the receiving end, because in a nondissipative line there is no loss. The power factor of the line at the station end is 97.5 per cent.

The equivalent reactive power at the station end of the line is

$$Q_s = (kva)_s \sin(\theta_s)$$
$$= 88800 \sin(-12.74°)$$
$$= -19500 \text{ kilovars.}$$

The positive angle of Z_s indicates lagging current at the station end. The negative sign of Q_s indicates lagging reactive power. There is more energy stored in the magnetic field of the line than in the electric field.

The current at the receiving end of the line is given by formula (2.11.16). For the nondissipative line it becomes

$$I_r = \frac{V_s}{Z_r \cos \beta S + jZ_o \sin \beta S}, \tag{3.3.3}$$

where

$$Z_o = \sqrt{\frac{L}{C}}.$$

For the particular illustrative line, it is

$$I_r = \frac{125000}{300 \cos 35.52° + j\, 380 \sin 35.52°}$$
$$= \frac{125000}{329\underline{/42.15°}}$$
$$= 380\underline{/-42.15°} \text{ vector amperes.}$$

The angle is with reference to the station-end voltage.

The voltage to ground-potential plane is

$$\begin{aligned} V_r &= I_r Z_r \\ &= (380\underline{/-42.15^\circ})(300\underline{/0^\circ}) \\ &= 114000\underline{/-42.15^\circ} \text{ vector volts} \end{aligned}$$

to ground-potential plane, i.e., 228 kv across the two-wire line. The angle is with reference to the station-end voltage.

The receiving-end voltage may also be calculated directly by (2.8.4), which for the nondissipative line becomes

$$V_r = \frac{V_s Z_r}{Z_r \cos \beta S + jZ_o \sin \beta S}.$$

It should be noted that because of the line capacitance, the receiving-end current is larger than the station-end current. The voltage and current at the receiving end are in time phase, because the receiving-end impedance is nonreactive.

The total power at the receiving end is

$$\begin{aligned} P_r &= (kv)_r I_r \\ &= 228 \times 380 \\ &= 86640 \; kw. \end{aligned}$$

This agrees substantially with the value obtained above.

The voltage V_{ro} at the open-circuited receiving end is given by formula (2.11.19). For the nondissipative line it becomes

$$V_{ro} = \frac{V_s}{\cos \beta S} \text{ volts}$$

to ground-potential plane.

For the illustrative line, it is

$$\begin{aligned} V_{ro} &= \frac{125000}{\cos 35.52^\circ} \\ &= 153700 \text{ volts,} \end{aligned}$$

i.e., 307.4 kv across the open-end of the line. It is important to note that this open-circuited receiving-end voltage is very much larger than the station-end voltage and is in time phase with it. The phenomenon is called *Ferranti effect*, after the engineer who first noticed it. It is due to the interaction of the inductive and capacitive properties of the line, and will occur not only in nondissipative lines but also in actual lines when the phase constant β, expressed in radians, is numerically larger than the attenuation constant α.*

* See Appendix 3 for discussion of the Ferranti effect.

The regulation of the line is by (2.11.23)

$$\text{Reg.} = \frac{(307.4 - 228)100}{228} = 34.85 \text{ per cent.}$$

This result indicates that with a station-end voltage maintained constant, the voltage at the load terminals changes greatly with load impedance variations.

If the line should become short-circuited at the receiving end, while the voltage at the station end remains constant, the current at that end would be by (2.11.24)

$$(I_r)_{sh} = \frac{V_s}{jZ_o \sin \beta S} \tag{3.3.5}$$

where

$$Z_o = \sqrt{\frac{L}{C}}.$$

For the illustrative line considered, this current is

$$\begin{aligned}(I_r)_{sh} &= \frac{125000}{j\,380 \sin 35.52^\circ}\\ &= 566\underline{/-90^\circ} \text{ vector amperes.}\end{aligned}$$

The angle is with reference to the station-end voltage.

The current at the station end when the load end is short-circuited is given either by (2.11.25) in terms of V_s and Z_o, or by (2.11.26) in terms of $(I_r)_{sh}$. For the nondissipative line, formula (2.11.26) becomes

$$(I_s)_{sh} = (I_r)_{sh} \cos \beta S. \tag{3.3.6}$$

For the illustrative line considered, this is

$$\begin{aligned}(I_s)_{sh} &= (566\underline{/-90^\circ}) \cos 35.52^\circ\\ &= 460.5\underline{/-90^\circ} \text{ vector amperes.}\end{aligned}$$

Note that the station-end current, under short-circuit condition at the load end is smaller than the load-end current. This is always the case for nondissipative lines, irrespective of line length, because under short-circuit conditions

$$\left(\frac{I_s}{I_r}\right)_{sh} = \cos \beta S$$

is always less than 1.

The receiving-end impedance that would make the volt-amperes at that end a maximum, by (2.11.22), is

$$Z_{rm} = jZ_o \tan \beta S.$$

For the illustrative line under consideration it is

$$\begin{aligned} Z_{rm} &= 380 \tan 35.52°\underline{/90°} \\ &= 271.2\underline{/90°} \text{ ohms.} \end{aligned}$$

The charging current, i.e., the sending-end current when the line is open at the receiving end, by equation (2.11.20), is

$$\begin{aligned} I_{so} &= \frac{jV_s \sin \beta S}{Z_o \cos \beta S} \\ &= 234.5\underline{/90°} \text{ vector amperes} \end{aligned}$$

with reference to the station voltage.

3.4. *The Three-Phase Line; General Relations.* The transmission formulas, developed in Chapter 2, apply directly to the calculation of the performance of three-phase lines. The line, which is used in this and in the following article to illustrate performance calculations of a three-phase line, consists of three bare concentric-layer cables each of 19 strands of standard annealed copper. The cross-sectional area of the cable is 250,000 circ. mils. The outside diameter is 0.575 inches. The line is assumed to be equilaterally spaced with a spacing distance of 10 ft. The resistance per mile of conductor is 0.263 ohms per mile at 65° C. The calculated value of the inductance per mile of conductor is 0.002032 henry and the inductive reactance at a frequency of 60 cps is

$$L\omega = 0.765 \text{ ohms per mile of conductor.}$$

The calculated capacitance is

$$C = 14.81 \times 10^{-9} \text{ farads per mile of conductor.}$$

The susceptance at 60 cps is

$$C\omega = 5.58 \times 10^{-6} \text{ mhos per mile of conductor.}$$

The velocity of energy propagation is

$$v_o = \frac{1}{\sqrt{LC}} = 182{,}300 \text{ miles per second.}$$

The linear-line impedance per mile of conductor is

$$\begin{aligned} z &= R + jL\omega \\ &= 0.263 + j\,0.765 \\ &= 0.808\underline{/71°} \text{ vector ohms.} \end{aligned}$$

Since the leakage conductance is assumed negligible, the linear-line admittance per mile of conductor is

$$\begin{aligned} y &= jC\omega \\ &= 5.58 \times 10^{-6}\underline{/90^\circ} \text{ vector mhos.} \end{aligned}$$

The characteristic impedance of line to neutral, by (2.11.4), is

$$\begin{aligned} Z_o &= \sqrt{\frac{0.808\underline{/71^\circ}}{5.58 \times 10^{-6}\underline{/90^\circ}}} \\ &= 380\underline{/-9.5^\circ} \text{ vector ohms.} \end{aligned}$$

The propagation constant, by (2.11.6), is

$$\begin{aligned} p &= \sqrt{(0.808\underline{/71^\circ})(5.58 \times 10^{-6}\underline{/90^\circ})} \\ &= 0.002123\underline{/80.5^\circ}. \end{aligned}$$

The attenuation constant, by (2.11.7), is

$$\begin{aligned} a &= 0.002123 \cos 80.5^\circ \\ &= 0.00035. \end{aligned}$$

The phase constant, by (2.11.8), is

$$\begin{aligned} \beta &= 0.002123 \sin 80.5 \\ &= 0.0020939 \text{ radians per mile} \end{aligned}$$

or

$$\beta = 0.12^\circ \text{ degrees per mile.}$$

The velocity of phase propagation, by (2.4.6), is

$$\begin{aligned} v_p &= \frac{377}{0.0020939} \\ &= 180{,}000 \text{ miles per second.} \end{aligned}$$

It is 98.9 per cent of the velocity of energy propagation of this line, and 96.8 per cent of the velocity of electromagnetic waves through space.

The wave length, by (2.11.10), is

$$\begin{aligned} \lambda &= \frac{2\pi}{0.0020939} \\ &= 3000 \text{ miles.} \end{aligned}$$

It will be assumed that the line is 300 miles long. This gives

$$pS = (0.002123\underline{/80.5^\circ})300 = 0.6369\underline{/80.5^\circ}$$

or

$$pS = 0.1050 + j\,0.628.$$

Since the quadrature component of p, i.e., the phase constant, is an angle, the quantity βS is also an angle. In the above equation it is expressed in radians. For convenience in numerical calculation, it is desirable that it be expressed in degree measure. Accordingly, the above is written for convenience in calculation

$$pS = 0.105 + j\,36°.$$

Since sinh pS and cosh pS enter into all transmission formulas, it is also convenient to have their numerical values predetermined. Thus, by (2.11.11),*

$$\sinh pS = (\sinh^2 0.105 + \sin^2 36°)^{\frac{1}{2}}\underline{/\psi},$$

where

$$\psi = \tan^{-1}\left(\frac{\tan 36°}{\tanh 0.105}\right).$$

Performing the indicated calculations gives

$$\sinh pS = 0.598\underline{/81.8°}.$$

The hyperbolic cosine, by (2.11.12), is

$$\cosh pS = (\sinh^2 0.105 + \cos^2 36°)^{\frac{1}{2}}\underline{/\phi}$$

where

$$\phi = \tan^{-1}(\tanh 0.105 \times \tan 36°).$$

Performing the indicated calculations, there results

$$\cosh pS = 0.816\underline{/4.35°}.$$

In the numerical calculations of the performance of three-phase lines it will be taken for granted that the line is connected at both ends to the high side of transformer banks, in wye connection. The low sides of the transformers may be connected in either wye or delta. The load is assumed balanced.

3.5. *Performance Calculation of Three-Phase Lines; Receiving-End Voltage and Load Known.*

A review of the general transmission formulas for voltage and for current indicates that they are in terms of the equivalent impedance $\boldsymbol{Z}_r$ of the load per phase as if it would be measured on the high side of the transformer bank at the receiving end. This equivalent impedance is easily obtained if the load in terms of kva or kw, its power factor and the voltage at which it operates are known. Conversely, if the equivalent impedance per phase $\boldsymbol{Z}_r$ is known, it is just as easy to obtain the kva or kw load, provided the voltage per phase is known.

The various aspects of the problem will be illustrated with the line whose transmission properties were obtained in the preceding article and will be based upon a receiving-end line voltage of 216.5 kv corresponding to 125 kv per phase.

* The K & E log log vector slide rule designed by the author is of great convenience in all vector calculations including hyperbolic functions of the complex variable.

There is one limiting quantity pertaining to the line behavior which can be obtained directly, since it is independent of voltage or of load. This quantity is the receiving-end impedance corresponding to the maximum voltamperes per phase, and is given by formula (2.11.22)

$$Z_{rm} = Z_o \frac{\sinh pS}{\cosh pS}.$$

For the illustrative line under consideration it is

$$\begin{aligned} Z_{rm} &= (380\underline{/-9.5^\circ})\frac{0.598\underline{/81.8^\circ}}{0.816\underline{/4.35^\circ}} \\ &= 278.5\underline{/67.95^\circ}. \end{aligned}$$

Using this as the equivalent impedance per phase, subject to the 125 kv per phase, gives

$$\begin{aligned} I_r &= \frac{125000}{278.5\underline{/67.95^\circ}} \\ &= 448.8\underline{/-67.95^\circ} \text{ vector amperes.} \end{aligned}$$

The maximum kva at the receiving end would be

$$\begin{aligned} (kva)_{rm} &= (kv)_r I_r \\ &= 125 \times 448.8 \\ &= 56100 \text{ kilovolt-ampere} \end{aligned}$$

per phase. The dissipative power for this maximum kva is

$$P_{rm} = 56100 \cos(-67.95^\circ) = 21060 \text{ kilowatts.}$$

The reactive power is

$$Q_{rm} = 56100 \sin(-67.95^\circ) = -51990 \text{ kilovars.}$$

The power factor of the load at maximum kva is

$$\cos(-67.95^\circ) = 0.375.$$

From the standpoint of low power factor, it is not desirable to operate the line at its maximum kva. There are, however, further and more fundamental reasons why it is not desirable to operate the line at its maximum kva. The load on a power line changes with the aggregate power demand at its receiving end. With these continuous changes of the load, or of the reactive character of the load, or of both, there will be corresponding changes in the value of Z_r or its angle or both. A glance at the $(VI)_r$ curve, Fig. 2–8, indicates that the maximum voltamperes corresponds to a receiving-end voltage value V_r on the steep portion of the voltage curve, and changes in Z_r below and above Z_{rm} is conducive to undesirable variations in receiving-end voltage. The curve indicates clearly that it is highly desirable to operate the line at loads whose corresponding impedances are quite larger than Z_{rm}.

As a trial calculation, it will be assumed that the equivalent impedance of the load is $Z_r = 625$, and that its phase angle is 25° positive corresponding to a lagging power factor of 90.6 per cent.

With these assumed values, the receiving-end current is

$$I_r = \frac{125000}{625\underline{/25^\circ}}$$
$$= 200\underline{/-25^\circ} \text{ vector amperes.}$$

The receiving-end kva, accordingly, is

$$(kva)_r = 125 \times 200 = 25000 \text{ kilovolt amperes.}$$

The dissipative power per phase at the receiving end is

$$P_r = (kva)_r \cos(-\theta_r)$$
$$= 25000 \cos(-25^\circ)$$
$$= 22650 \text{ kilowatts.}$$

The total dissipative power is

$$P_{rt} = 67950 \text{ kilowatts.}$$

The reactive power at the receiving end is

$$Q_r = 25000 \sin(-25^\circ)$$
$$= -10560 \text{ kilovars.}$$

The total reactive power is

$$Q_{rt} = -31680 \text{ kilovars, lagging.}$$

The current at the station end, by (2.11.15), is

$$I_s = \frac{125000(0.816\underline{/4.35^\circ})}{625\underline{/25^\circ}} + \frac{125000(0.598\underline{/81.8^\circ})}{380\underline{/-9.5^\circ}}$$
$$= 163.2\underline{/-20.65^\circ} + 196.7\underline{/91.3^\circ}$$
$$= 203\underline{/43.2^\circ} \text{ vector amperes.}$$

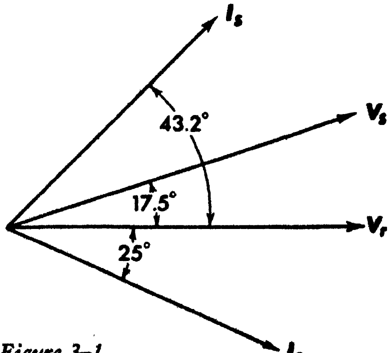


Figure 3–1

The angle is, with respect to the receiving-end voltage, as shown graphically in Fig. 3–1.

The station-end voltage is given by formula (2.11.17). For the illustrative line whose receiving-end voltage to neutral is $V_r = 125000$ volts, and whose receiving-end impedance to neutral is $Z_r = 625\underline{/25^\circ}$ the station-end voltage is

$$V_s = 125000\left[0.816\underline{/4.35^\circ} + \frac{(380\underline{/-9.5^\circ})(0.598\underline{/81.8^\circ})}{625\underline{/25^\circ}}\right]$$
$$= 102000\underline{/4.35^\circ} + 45440\underline{/47.3^\circ}$$
$$= 138900\underline{/17.5^\circ} \text{ vector volts.}$$

The station-end voltage to neutral is 138.9 kv and the line voltage is 240.3 kv. The angle is, with reference to the receiving-end voltage, as shown graphically in Fig. 3–1.

The kva supplied to the line per phase is

$$(kva)_s = (kv)_s I_s$$
$$= 138.9 \times 203 = 28190 \text{ kilovolt amperes.}$$

The phase angle between station-end voltage and station-end current is

$$\theta_s = 43.2^\circ - 17.5^\circ = 25.7^\circ.$$

The power factor of the load at the station end is

$$\cos 25.7^\circ = 0.901.$$

The positive angle of the current indicates that the load at the station end is capacitive in character, i.e., the sending-end impedance angle is negative, the current leading the voltage by 25.7°.

The dissipative power per phase at the station end is

$$P_s = (kva) \cos \theta_s$$
$$= 28190 \cos 25.7^\circ$$
$$= 25400 \text{ kilowatts.}$$

The total dissipative power is

$$P_{st} = 76200 \text{ kw.}$$

The reactive power per phase at the station end is

$$Q_s = (kva)_s \sin \theta_s$$
$$= 28190 \sin 25.7^\circ$$
$$= 12200 \text{ kilovars.}$$

The total reactive power is

$$Q_{st} = 36600 \text{ kilovars.}$$

The positive sign indicates leading or capacitive reactive power.

The efficiency of transmission in per cent is

$$\eta = \frac{100\, P_r}{P_s}$$
$$= \frac{100 \times 22650}{25400} = 89.1 \text{ per cent.}$$

The receiving-end voltage, when the receiving end is open-circuited, is given by (2.11.19). For the illustrative line under consideration it is

$$\begin{aligned} V_{ro} &= \frac{138900}{0.816} \\ &= 170.2 \text{ kilovolts.} \end{aligned}$$

The regulation is

$$\begin{aligned} \text{Reg.} &= \frac{(170.2 - 125)100}{125} \\ &= 36.1 \text{ per cent.} \end{aligned}$$

The rather poor voltage regulation indicates that the chosen value of $\boldsymbol{Z}_r = 625\underline{/25^\circ}$ is too low, i.e., the corresponding $(kva)_r$ equal to 25000 at 90.6 per cent power factor is too high.

Assume that $\boldsymbol{Z}_r = 1250\underline{/25^\circ}$ ohms. Then

$$\begin{aligned} \boldsymbol{I}_r &= 100\underline{/-25^\circ} \text{ vector amperes} \\ (kva)_r &= 12500 \text{ kilovolt ampere.} \\ P_r &= 11325 \text{ kilowatt.} \\ \boldsymbol{I}_s &= 81.6\underline{/-20.65^\circ} + 196.7\underline{/91.3^\circ} \\ &= 182.5\underline{/66.8^\circ} \text{ vector amperes} \\ \boldsymbol{V}_s &= 102000\underline{/4.35^\circ} + 22720\underline{/47.3^\circ} \\ &= 119600\underline{/11.8^\circ} \text{ vector volts.} \end{aligned}$$

The station-end voltages to neutral is 119.6 kv, and leads the phase voltage at the load end by 11.8°.

The phase angle between voltage and current at the station end is

$$\theta_s = 66.8^\circ - 11.8^\circ = 55^\circ.$$

The kva at the station end is

$$\begin{aligned} (kva)_s &= 119.6 \times 182.5 \\ &= 21820 \text{ kilovolt amperes.} \end{aligned}$$

The dissipative power per phase at the station end is

$$P_s = 21820 \cos 55^\circ = 12500 \text{ kw}$$

and the reactive power per phase at that end is

$$Q_s = 21820 \sin 55^\circ = 17850 \text{ kilovars.}$$

The efficiency, therefore, is

$$\begin{aligned} \eta &= \tfrac{11325}{12500} \\ &= 90.6 \text{ per cent.} \end{aligned}$$

The receiving-end voltage on open circuit, by (2.11.19), is

$$\begin{aligned} V_{ro} &= \frac{119600}{0.816} \\ &= 146.5 \text{ kilovolts.} \end{aligned}$$

The regulation, therefore, is

$$\text{Reg.} = \frac{(146.5 - 125)100}{125} = 17.2 \text{ per cent.}$$

It will be noted that using a receiving-end impedance of $1250\underline{/25°}$, corresponding to a load per phase of 11325 kw at 90.6 per cent power factor gave a better regulation than with a load corresponding to $\boldsymbol{Z}_r = 625\underline{/25°}$ vector ohms. It would appear that an adequate value of receiving-end impedance would be about $\boldsymbol{Z}_r = 1000\underline{/25°}$. This would give a receiving-end current of

$$\boldsymbol{I}_r = \frac{125000}{1000\underline{/25}}$$
$$= 125\underline{/-25} \text{ vector amperes.}$$

The kva at the receiving end per phase would be

$$(kva)_r = 125 \times 125 = 15625 \text{ kilovolt amperes}$$

and the receiving-end dissipative power

$$P_r = 15625 \cos 25 = 14156 \text{ kw.}$$

It is seen from the above calculations that the line whose transmission constants are stated and whose properties are formulated in § 3.4 will carry a full load per phase of about 15000 kva at 90.6 per cent power factor.

3.6. *Performance Calculation of Three-Phase Lines; Station-End Voltage and Receiving-End Impedance Known.* If the receiving-end load per phase is known in terms of the receiving-end impedance the problem is essentially similar to the one discussed in the preceding article. Thus, let the station-end voltage per phase of the illustrative line considered in § 3.5 be 125 kv and the receiving-end impedance $\boldsymbol{Z}_r = 1000\underline{/25°}$ vector ohms. The station-end impedance, by (2.11.13), is

$$\boldsymbol{Z}_s = 380\underline{/-9.5°}\left[\frac{(380\underline{/-9.5°})(0.598\underline{/81.8°}) + (1000\underline{/25°})(0.816\underline{/4.35°})}{(380\underline{/-9.5°})(0.816\underline{/4.35°}) + (1000\underline{/25°})(0.598\underline{/81.8°})}\right],$$

$$\boldsymbol{Z}_s = 380\underline{/-9.5°}\,\frac{993\underline{/38.3°}}{560.8\underline{/76.1°}},$$

$$\boldsymbol{Z}_s = 675\underline{/-47.3°} \text{ vector ohms.}$$

The current at the station-end is

$$\boldsymbol{I}_s = \frac{\boldsymbol{V}_s}{\boldsymbol{Z}_s}$$
$$= \frac{125000}{675\underline{/-47.3°}} = 185\underline{/47.3°} \text{ vector amperes.}$$

The kva supplied per phase is

$$\begin{aligned}(kva)_s &= 125 \times 185\\ &= 23120 \text{ kilovolt ampere.}\end{aligned}$$

The supplied dissipative power per phase is

$$\begin{aligned}P_s &= 23120 \cos 47.3^\circ\\ &= 15690 \text{ kw.}\end{aligned}$$

The total dissipative power supplied to the line is

$$P_{st} = 47070 \text{ kw.}$$

The current at the receiving end is given by (2.11.16). For the line under consideration it is

$$\begin{aligned}I_r &= \frac{125000}{(1000\underline{/25^\circ})(0.816\underline{/4.35^\circ}) + (380\underline{/-9.5^\circ})(0.598\underline{/81.8^\circ})}\\ &= \frac{125000}{995\underline{/38.5^\circ}}\\ &= 125.6\underline{/-38.5^\circ} \text{ vector amperes.}\end{aligned}$$

The angle is with reference to the sending-end voltage V_s.

The voltage per phase at the receiving end is given by (2.11.18). For the particular illustrative line under consideration it is given directly by

$$\begin{aligned}V_r &= I_r Z_r\\ &= (125.6\underline{/-38.5^\circ})(1000\underline{/25^\circ})\\ &= 125.6\underline{/-13.5^\circ} \text{ kv.}\end{aligned}$$

The angle is with reference to the station-end voltage.

The kva per phase at the receiving end is

$$\begin{aligned}(kva)_r &= 125.6 \times 125.6\\ &= 15775 \text{ kilovolt ampere.}\end{aligned}$$

The dissipative power per phase at the receiving end is

$$\begin{aligned}P_r &= 15775 \cos 25^\circ\\ &= 14300 \text{ kw.}\end{aligned}$$

The total dissipative power at the receiving end is

$$P_{rt} = 42900 \text{ kw.}$$

The efficiency of transmission is

$$\begin{aligned}\eta &= \frac{14300}{15690}\\ &= 91.2 \text{ per cent.}\end{aligned}$$

3.7. *Performance Calculation of Three-Phase Lines; Station-End Voltage and Receiving-End Load Known.*

The case discussed in the preceding article is hypothetical in character, for, as a rule, the value of the receiving-end impedance $\boldsymbol{Z}_r$ is not known unless the load, its power factor, and the receiving-end voltage are known. If such is the case, the problem is identical with that discussed in § 3.5.

The condition met quite frequently, however, is that in which the receiving-end load P_r and its lagging power factor $\cos\theta_r$ are known, and instead of the receiving-end voltage, the sending-end voltage V_s is known. Equation (2.11.17),

$$\boldsymbol{V}_s = \boldsymbol{V}_r\left(\cosh pS + \frac{\boldsymbol{Z}_o}{\boldsymbol{Z}_r}\sinh pS\right) \tag{3.7.1}$$

applies to this case, but with $\boldsymbol{V}_r$ and its angle δ_r with reference to $\boldsymbol{V}_s$ unknown. The vector value of $\boldsymbol{I}_r$ with reference to $\boldsymbol{V}_s$ is also unknown. In terms of receiving-end quantities, its magnitude is, however,

$$I_r = \frac{P_r}{V_r\cos\theta_r} = \frac{A}{V_r}, \tag{3.7.2}$$

where $A = P_r/\cos\theta_r$ represents the known value of the receiving-end volt-amperes. From this it follows that the numerical value of the receiving-end impedance is

$$Z_r = \frac{V_r}{I_r} = \frac{V_r^2}{A}, \tag{3.7.3}$$

and its angle is θ_r from the known value of $\cos\theta_r$.

Setting for brevity's sake

$$\left.\begin{aligned} \cosh pS &= a\underline{/\alpha} \\ Z_o\sinh pS &= b\underline{/\beta} \end{aligned}\right\} \tag{3.7.4}$$

and using vector notation, equation (3.7.1) may be written, using V_r as the reference vector

$$V_s\underline{/\delta_s} = (a\underline{/\alpha})(V_r\underline{/0^\circ}) + \frac{(A\underline{/-\theta_r})(b\underline{/\beta})}{V_r}$$

or

$$(V_s\underline{/\delta_s})V_r = (a\underline{/\alpha})V_r^2 + Ab\underline{/\beta - \theta_r},$$

where V_r is the unknown voltage at the receiving end and δ_s is the unknown phase angle of V_s with reference to V_r.

Splitting both sides of the equation into horizontal and quadrature components and equating them, respectively, yields

$$\begin{aligned} V_sV_r\cos\delta_s &= (a\cos\alpha)V_r^2 + Ab\cos(\beta - \theta_r) \\ V_sV_r\sin\delta_s &= (a\sin\alpha)V_r^2 + Ab\sin(\beta - \theta_r). \end{aligned}$$

Squaring and adding gives

$$V_s^2 V_r^2 = a^2 V_r^4 + A^2 b^2 + 2aAbV_r^2 \cos(\alpha - \beta + \theta_r).$$

For convenience in solving, the equation is put into the form

$$a^2 V_r^4 - [V_s^2 - 2abA \cos(\alpha - \beta + \theta_r)]V_r^2 = -b^2 A^2$$

or

$$V_r^4 - \left[\frac{V_s^2 - 2abA \cos(\alpha - \beta + \theta_r)}{a^2}\right] V_r^2 = -\frac{b^2 A^2}{a^2}.$$

This quadratic equation in V_r^2, when solved, gives two real values for the receiving-end voltage. The larger value, which gives more economical transmission and better voltage regulation is, of course, the correct one.

The above method for obtaining V_r, when P_r, $\cos\theta_r$, and V_s are known, although not complicated is rather tedious particularly if the receiving-end voltages are to be calculated for more than one value of receiving-end load P_r.

A more direct method and one not quite as tedious is to calculate the receiving-end currents I_r for the given values of V_s and angle θ_r, corresponding to the given power factor for three or more assumed values of Z_r. Calculated values of I_r, receiving-end voltage $V_r = I_r Z_r$, and voltamperes $V_r I_r$ may then be plotted against corresponding calculated value of receiving-end power $P_r = V_r I_r \cos\theta_r$.

Thus for the illustrative line used in preceding articles there was obtained by previous calculation

$$Z_o = 380\underline{/-9.5^\circ}$$
$$\sinh pS = 0.598\underline{/81.8^\circ}$$
$$\cosh pS = 0.816\underline{/4.35^\circ}.$$

Assume receiving-end impedances Z_r of values 500, 1000, 1500, and 2000 ohms each having a phase angle of 25° corresponding to the assumed power factor of 90.6 per cent.

The receiving-end current is given by the formula (2.11.16) and for a sending-end voltage of 125 kv, it is

$$I_r = \frac{125000}{(0.816\underline{/4.35^\circ})Z_r + (380\underline{/-9.5})(0.598\underline{/81.8})}$$
$$= \frac{125000}{(0.816\underline{/4.35^\circ})Z_r + 227.2\underline{/72.3^\circ}}.$$

For $Z_r = 500\underline{/25^\circ}$ vector ohms, the equation becomes

$$I_r = \frac{125000}{408\underline{/29.35^\circ} + 227.2\underline{/72.3^\circ}}$$
$$= 210.1\underline{/-44.43^\circ} \text{ vector amperes.}$$

The angle is with respect to the sending-end voltage. The voltage to neutral at the receiving end is

$$\begin{aligned} V_r &= (210.1\underline{/-44.43°})(500\underline{/25°}) \\ &= 105050\underline{/-19.43°} \text{ vector volts.} \end{aligned}$$

The voltampere per phase at the receiving end is

$$\frac{V_r I_r}{1000} = 105.05 \times 210.1 = 22070 \text{ kva.}$$

The dissipative power per phase at the receiving end is

$$\begin{aligned} P_r &= 22070 \cos 25° \\ &= 20000 \text{ kilowatts.} \end{aligned}$$

For $\boldsymbol{Z}_r = 1000\underline{/25°}$ vector ohms:

$$\begin{aligned} I_r &= \frac{125000}{816\underline{/29.35°} + 227.2\underline{/72.3°}} \\ &= 125.6\underline{/-38.5°} \text{ vector amperes.} \\ V_r &= (125.6\underline{/-38.5°})(1000\underline{/25°}) \\ &= 125600\underline{/-13.5°} \text{ vector volts.} \\ \frac{V_r I_r}{1000} &= 125.6 \times 125.6 \\ &= 15775 \text{ kilovolt amperes.} \\ P_r &= 15775 \cos 25° \\ &= 14300 \text{ kilowatts.} \end{aligned}$$

For $\boldsymbol{Z}_r = 1500\underline{/25°}$ vector ohms:

$$\begin{aligned} I_r &= \frac{125000}{1224\underline{/29.37°} + 227.2\underline{/72.3°}} \\ &= 89.4\underline{/-35.75°} \text{ vector amperes.} \\ V_r &= (89.4\underline{/-35.75°})(1500\underline{/25°}). \\ V_r &= 134100\underline{/-10.75°} \text{ vector volts.} \\ \frac{V_r I_r}{1000} &= 134.1 \times 89.4 \\ &= 11980 \text{ kilovolt amperes.} \\ P_r &= 11980 \cos 25° \\ &= 10860 \text{ kilowatts.} \end{aligned}$$

For $\boldsymbol{Z}_r = 2000\underline{/25°}$ vector ohms:

$$\begin{aligned} I_r &= \frac{125000}{1632\underline{/29.35°} + 227.2\underline{/72.3°}} \\ &= 69.2\underline{/-34.25°} \text{ vector amperes.} \end{aligned}$$

$$V_r = (69.2\underline{/-34.25°})(2000\underline{/25°})$$
$$= 138400\underline{/-9.25°} \text{ vector volts.}$$
$$\frac{V_r I_r}{1000} = 138.4 \times 69.2$$
$$= 9577 \text{ kilovolt amperes.}$$
$$P_r = 9577 \cos 25°$$
$$= 8680 \text{ kilowatts.}$$

The receiving-end voltage on open circuit, by (2.11.19), is

$$V_{ro} = \frac{125000}{0.816}$$
$$= 153.2 \text{ kilovolts.}$$

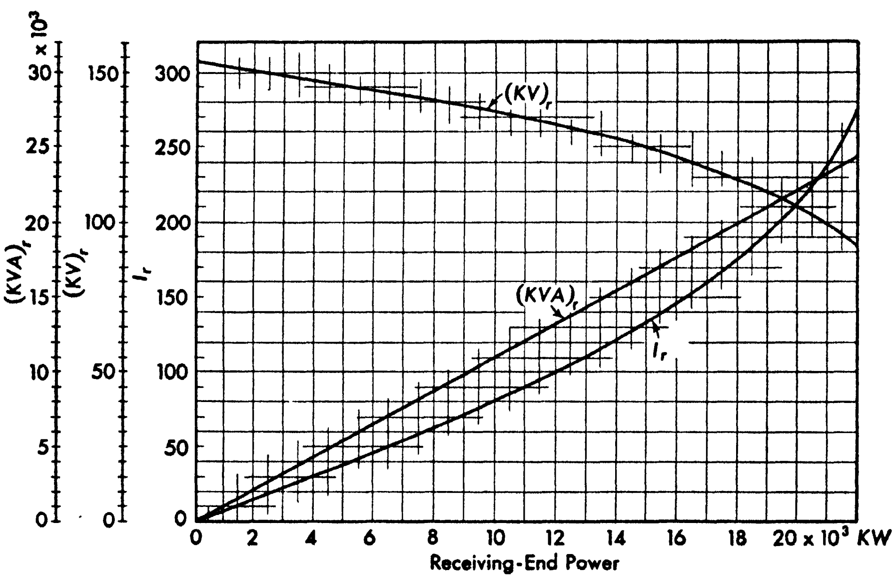


Figure 3–2

The curves I_r, $(kv)_r$, and $(kva)_r$ in Fig. 3–2, are plotted against the receiving-end power in kw. They pertain, of course, only to a station-end voltage $V_s = 125$ kv per phase and a load power factor of 90.6 per cent. Similar curves may be obtained for any other values of V_s and power factor of receiving-end load, and the values of I_r, $(kv)_r$, and $(kva)_r$ for any value of $(kw)_r$ obtained from the curves.

SUGGESTIVE PROBLEMS *Chapter 3*

1. Calculate the linear impedance and the linear admittance per mile of conductor of a three-phase line, consisting of hard drawn copper 97.3 per cent conductivity; size 750,000 circ. mils, 61 strands: equilateral spacing distance 10 ft.

2. Calculate the characteristic impedance, the propagation, attenuation and phase constant per mile to neutral of the line specified in Prob. 1.

3. Compute the surge impedance, the velocity of energy transfer, the velocity of phase propagation and the wave length of the line specified in Prob. 1.

4. A three-phase line is built of A.C.S.R. cable 397,000 circ. mils having a resistance per conductor mile of 0.235 ohms at 25° C. The cable consists of 30 strands of aluminum and 7 strands of steel and has an outside diameter of 0.806 inches. The equivalent equilateral spacing is 11 feet. The inductive reactance at 60 cps is 0.710 ohms per mile per conductor. The capacitive susceptance at the stated frequency is 5.82 micromhos.

   Assuming negligible leakage conductance, determine:

   a. The characteristic impedance to neutral.
   b. The propagation, attenuation, and phase constants to neutral.
   c. The surge impedance.
   d. Assume that the line is 300 miles long and determine the receiving-end impedance per phase for maximum receiving-end voltamperes per phase. Determine the power factor of the receiving-end load under this condition and discuss the feasibility of such a load.
   e. Assume that the load at the receiving end is 21,000 kw at 90 per cent power factor and that the line voltage at the receiving end is 220 kv. Calculate the station-end voltage and current and the efficiency of transmission.
   f. Calculate the line regulation.

5. A load of 21,000 kw at 87.5 per cent power factor lag is to be transmitted over 120 miles with a maximum loss of 12 per cent of the power delivered. The receiving-end voltage is 63.5 kv to neutral. Determine the size of conductors to be used and calculate the efficiency of transmission and the line regulation for the chosen conductor.

Chapter 4 Equivalent Circuits of Transmission Lines

4.1. *General Conditions of Equivalence.* It will be recalled that the various transmission-line equations, formulated in Chapter 2 and illustrated in the preceding chapter, were developed on the fundamental assumption that a line conductor consists essentially of infinitesimal symmetrical T or π networks. It is reasonable to presume, therefore, that since a portion of a line, however short, can be thought of as a symmetrical T or π network, a whole line, however long, could be represented also under well-defined conditions by a T or π network. In an actual line, the resistive and reactive properties are uniformly distributed along the length of the line. In a T or π network, the resistive and reactive properties are lumped or concentrated in only three branches. Complete equivalence will exist between a T or π network and a line when, under identical conditions of frequency, the circuits have identical energy transfer properties, i.e., when the characteristic impedance of the line to neutral is equal to the characteristic impedance of the T or π circuit, and when the propagation factor of the line is equal to the propagation constant of the T or π circuits, respectively.

Under these conditions, the line and its equivalent circuit have equal end values of voltage, of current, and of power for equal receiving-end impedances.

The solution of transmission line problems may be accomplished, therefore, by first converting the line into its equivalent T or π network then solving the three-branch circuit in the usual manner as an ordinary series-parallel circuit. The amount of labor involved in this method of solving transmission line problems is about the same if not greater than the direct method discussed and illustrated in the two preceding chapters. The equivalent circuit method becomes imperative, however, when an artificial circuit is to be built to simulate the behavior of the line, or when the transmission line problem includes the end transformers, as will be seen in the following chapter.

4.2. The T Equivalent of Transmission Lines. To obtain the branch impedances of a symmetrical T circuit equivalent to any line, consider the T network shown in Fig. 4–1. Let V_s and V_r be, respectively, the voltages to neutral at the station and receiving ends of the circuit as indicated. Let I_s be the station-end current and I_r the receiving-end current.

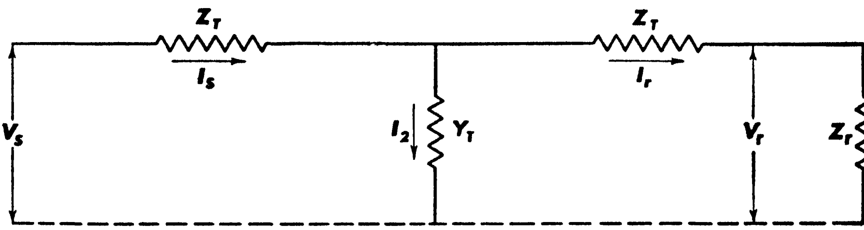


Figure 4–1

Referring to the circuit, it is seen that

$$V_s = I_s Z_T + \frac{I_2}{Y_T}$$

$$I_2 = (V_r + I_r Z_T) Y_T$$

and

$$I_s = I_r + I_2.$$

From these relations it is found that

$$V_s = V_r(1 + Z_T Y_T) + I_r(2 + Z_T Y_T) Z_T \qquad (4.2.1)$$

and

$$I_s = I_r(1 + Z_T Y_T) + V_r Y_T. \qquad (4.2.2)$$

These two equations are fundamental to symmetrical T networks, and relate end voltages and currents through the series impedance Z_T and shunt admittance Y_T of the circuit. They are called *transmission equations* of the T circuit because of their analogy to the transmission equations of a line. Under this name, the two formulas may be generalized in the form

$$\left.\begin{aligned} V_s &= AV_r + BI_r \\ I_s &= AI_r + DV_r, \end{aligned}\right\} \qquad (4.2.3)$$

in which, by (4.2.1) and (4.2.2),

$$\left.\begin{aligned} A &= 1 + Z_T Y_T \\ B &= (Z_T Y_T + 2) Z_T \\ D &= Y_T \end{aligned}\right\} \qquad (4.2.4)$$

for a symmetrical T.

If the generalized transmission equations (4.2.3) pertain to a line, then, by referring to (2.8.1) and to (2.7.2), it is seen that

$$\left.\begin{aligned} A &= \cosh pS \\ B &= Z_o \sinh pS \\ \text{and} \qquad D &= \frac{\sinh pS}{Z_o}. \end{aligned}\right\} \tag{4.2.5}$$

A symmetrical T network is the equivalent of a line when equations (4.2.4) representing the T are respectively equal to equations (4.2.5) which represent the line, i.e.,

$$Y_T = \frac{\sinh pS}{Z_o} \tag{4.2.6}$$

$$Z_T^2 Y_T + 2Z_T = Z_o \sinh pS \tag{4.2.7}$$

$$1 + Z_T Y_T = \cosh pS. \tag{4.2.8}$$

Dividing (4.2.7) by (4.2.6), yields

$$Z_o^2 = Z_T^2 + \frac{2Z_T}{Y_T}. \tag{4.2.9}$$

This gives the characteristic impedance of the line in terms of the T circuit, and shows that the characteristic impedances of the two are equal when the two are equivalent.

Substituting (4.2.6) in (4.2.8) gives

$$Z_T = \frac{Z_o(\cosh pS - 1)}{\sinh pS}. \tag{4.2.10}$$

Since by hyperbolic trigonometry

$$\sinh pS = 2 \sinh \frac{pS}{2} \cosh \frac{pS}{2}$$

and

$$\cosh pS = 2 \sinh^2 \frac{pS}{2} + 1$$

by substituting in (4.2.10) and simplifying, yields

$$Z_T = Z_o \tanh \frac{pS}{2}. \tag{4.2.11}$$

Any line may be converted into its equivalent T by the use of formulas (4.2.6) and (4.2.10) or (4.2.11), whichever is more convenient. Thus, for the illustrative 300-mile line used in the preceding chapter:

$$Z_o = 380\underline{/-9.5^\circ}$$

$$\sinh pS = 0.598\underline{/81.8^\circ}$$

$$\cosh pS = 0.816\underline{/4.35^\circ}.$$

By (4.2.6)

$$\begin{aligned} Y_T &= \frac{0.598\underline{/81.8^\circ}}{380\underline{/-9.5^\circ}} \\ &= 15.73 \times 10^{-4}\underline{/91.3^\circ} \text{ vector mhos.} \end{aligned}$$

Note that the angle of Y_T is larger than 90°. The real component, although very small, is negative, indicating that the equivalent T is non-physical in character. For all practical purposes, however, Y_T is substantially a capacitive susceptance.

$$Y_T = C\omega\underline{/90^\circ} \text{ mhos.}$$

The capacitance of the shunt branch of the equivalent T of the illustrative line at a frequency of 60 cps, therefore, is

$$\begin{aligned} C_T &= \frac{15.73 \times 10^{-4}}{377} \\ &= 4.17 \times 10^{-6} \text{ farads.} \end{aligned}$$

The value of the series impedance Z_T may be calculated by (4.2.10). Thus for the particular T equivalent of the illustrated line, it is

$$\begin{aligned} Z_T &= \frac{(380\underline{/-9.5^\circ})(0.816\underline{/4.35^\circ} - 1)}{0.598\underline{/81.8^\circ}} \\ &= 518.5\underline{/-86.95^\circ} - 635.4\underline{/-91.3^\circ} \\ &= 125.3\underline{/69.75^\circ} \text{ vector ohms.} \end{aligned}$$

The positive angle indicates that the impedance Z_T is inductively reactive. Its resistance component is

$$R_T = 43.37 \text{ ohms,}$$

and its inductive reactance is

$$L_T\omega = 117.55 \text{ ohms at 60 cps.}$$

The inductance of the series branch of the equivalent T, therefore, is

$$L_T = \frac{117.55}{377} = 0.3118 \text{ henry.}$$

In accordance with the above calculation, the equivalent T network of the 300-mile line per conductor is as shown in Fig. 4–2. It should be noted

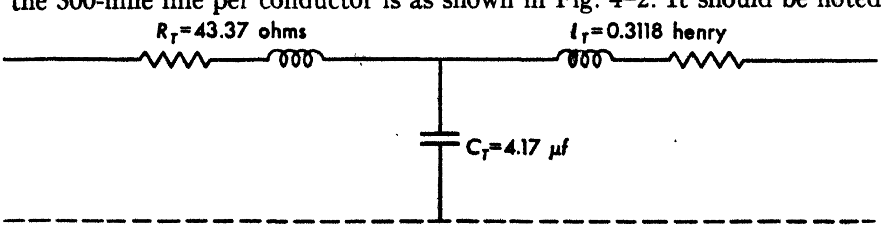


Figure 4–2

that, although this T network is a true equivalent of the line to neutral, the values of R_T, L_T, and C_T differ considerably from their corresponding values of the line itself.

4.3. *The Nominal T Equivalent of Short Lines.* When the actual values of the resistance, inductance, and capacitance of the line to neutral are used in the T circuit, i.e., when the effect of their distribution is neglected, the T network is called a *nominal* T of the line. There is a definite limit to the length of line for which a nominal T may be used without introducing serious errors. Consider equation (4.2.6) for the shunt admittance of the equivalent T. Expanding the numerator, the equation becomes

$$Y_T = \frac{\sinh aS \cos \beta S + j \cosh aS \sin \beta S}{Z_o}. \tag{4.3.1}$$

Note that when $aS < 0.1$, the $\sinh aS = aS$, and $\cosh aS = 1$, substantially. Similarly when $\beta S < 0.1$, the $\sin \beta S = \beta S$ and $\cos \beta S = 1$, substantially. It follows, therefore, that for the particular case when aS and βS are both less than 0.1, the above equation becomes

$$Y_T = \frac{(a + j\beta)S}{Z_o}. \tag{4.3.2}$$

But by (2.3.11)

$$a + j\beta = p,$$

and by (2.3.6)

$$p = \sqrt{(R + jL\omega)(G + jC\omega)}.$$

Using this equation for $a + j\beta$, and also formula (2.2.5) for Z_o, equation (4.3.2) becomes

$$Y_T = (G + jC\omega)S. \tag{4.3.3}$$

In the same manner, equation (4.2.11) becomes for the particular case when $aS < 0.1$ and $\beta S < 0.1$,

$$Z_T = \frac{Z_o (a + j\beta)S}{2} = \frac{Z_o pS}{2}.$$

Using the formulas for p and Z_o, the above equation for Z_T reduces to

$$Z_T = \frac{(R + jL\omega)S}{2}. \tag{4.3.4}$$

Equations (4.3.3) and (4.3.4) show that the distribution of R, L, G, and C may be neglected in the equivalent T only when aS and βS are smaller than 0.1. Generally, the attenuation constant a of a power transmission line is sufficiently small so that its product with the length S of the line is less than 0.1 but the value of the phase constant β is large enough to limit the use of the nominal T circuit to short lines only. Thus, for the illustrative line used in the preceding problems, $\beta = 0.0020939$ and the maximum length of line for which the nominal T could be used without serious error in the calculation of its performance is

$$S = \frac{0.1}{0.0020939} = 47.75 \text{ miles.}$$

Since this figure is used merely as a guide on whether the actual or the nominal T shall be used as the equivalent of a particular line, the scalar value of the propagation constant p may be used instead of the value of β in the determination of the limiting length of line S. Thus, for the illustrative line, $p = 0.002123$. This gives for the limiting length

$$S = \frac{0.1}{0.002123} = 47.1 \text{ miles.}$$

From what has been said above, it follows that since commercial power lines operating at 60 cps have propagation constants of about equal magnitudes, the length of line for which the nominal T could be used substantially as an equivalent is between 50 and 100 miles.

4.4. *The Equivalent π Circuit.* To obtain the actual π equivalent of a line, consider the π circuit shown in Fig. 4-3. Let V_s and I_s be respectively the voltage and current, at the station end, and V_r and I_r the voltage and current, respectively, at the receiving end. From the diagram, it is seen that

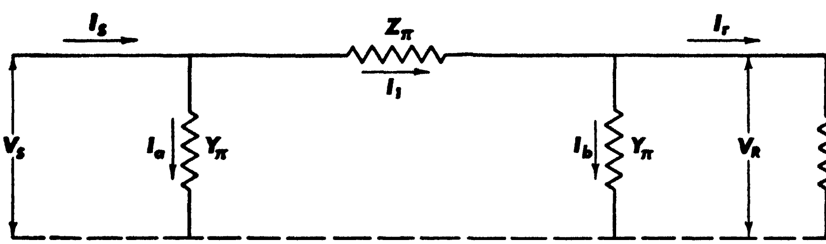


Figure 4-3

$$V_s = V_r + I_1 Z_\pi,$$
$$I_1 = I_r + I_b$$

and

$$I_b = V_r Y_\pi.$$

The expression for V_s, therefore, may be written

$$V_s = V_r(1 + Y_\pi Z_\pi) + I_r Z_\pi. \tag{4.4.1}$$

The transmission equation for the station-end voltage, by (2.11.17), is

$$V_s = V_r \cosh pS + I_r Z_o \sinh pS. \tag{4.4.2}$$

That the π network shall be an actual equivalent of the line to neutral, equations (4.4.1) and (4.4.2) must be equal, i.e.,

$$Z_\pi = Z_o \sinh pS \tag{4.4.3}$$

$$1 + Y_\pi Z_\pi = \cosh pS. \tag{4.4.4}$$

Substituting (4.4.3) for Z_π in (4.4.4) gives

$$Y_\pi = \frac{\cosh pS - 1}{Z_o \sinh pS}. \tag{4.4.5}$$

Referring this to (4.2.10) and (4.2.11), it is seen that the expression for Y_π may be written

$$Y_\pi = \frac{\tanh \dfrac{pS}{2}}{Z_o}. \tag{4.4.6}$$

Any line may be converted into an equivalent π network by the use of the formulas (4.4.3) and either (4.4.5) or (4.4.6), whichever is more convenient. Thus, for the illustrated line considered in previous problems

$$Z_o = 380\angle{-9.5^\circ}$$

$$\sinh pS = 0.598\angle{81.8^\circ}$$

$$\cosh pS = 0.816\angle{4.35^\circ}$$

By (4.4.3)

$$Z_\pi = (380\angle{-9.5^\circ})(0.598\angle{81.8^\circ})$$

$$= 227.2\angle{72.3^\circ}.$$

The positive angle indicates that Z_π is inductively reactive in character. Hence its resistance component is

$$R_\pi = 227.2 \cos 72.3^\circ = 69.07 \text{ ohms},$$

and its reactive component is

$$L_\pi \omega = 227.2 \sin 72.3^\circ = 216.4 \text{ ohms}.$$

Since the line is operated at a frequency of 60 cps, it follows that

$$L_\pi = \frac{216.4}{377} = 0.574 \text{ henry}.$$

By equation (4.4.5),

$$Y_\pi = \frac{0.816\angle{4.35^\circ} - 1}{227.2\angle{72.3^\circ}}$$

$$= 0.864 \times 10^{-3}\angle{89.33^\circ} \text{ vector mhos}.$$

The capacitive susceptance is substantially

$$C_\pi\omega = 0.864 \times 10^{-3} \text{ mhos.}$$

The capacitance, therefore, is

$$C_\pi = \frac{0.864 \times 10^{-3}}{377} = 2.295 \times 10^{-6} \text{ farads.}$$

In accordance with the above calculations, the equivalent π of the illustrative 300-mile line to neutral is as shown in Fig. 4–4. It should be

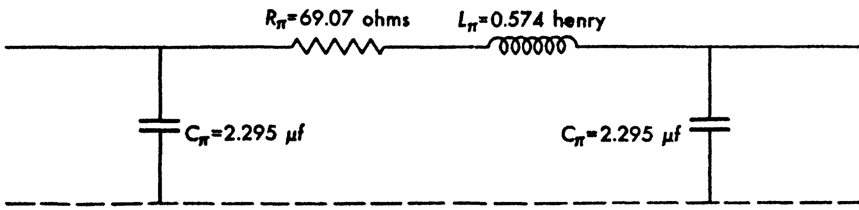


Figure 4–4

noted that although the π network is an actual equivalent of the line, the values of R, L, and C differ considerably from the corresponding values of the line itself.

4.5. *The Nominal π Equivalent of Short Lines.*

A π network in which are used the actual values of resistance, inductance, and capacitance of the line conductors, thus neglecting the effect of their respective uniform distribution, is called a ***nominal*** π of the line. By expanding $\sinh pS$ in (4.4.3), the equation becomes

$$Z_\pi = Z_o(\sinh aS \cos \beta S + j \cosh aS \sin \beta S).$$

When aS and βS are less than 0.1, then $\sinh aS = aS$, $\sin \beta S = \beta S$, $\cosh aS = 1$ and $\cos \beta S = 1$, substantially. Under this specific condition, the above expression becomes

$$Z_\pi = Z_o(a + j\beta)S$$

or

$$Z_\pi = Z_o pS. \tag{4.5.1}$$

Since

$$Z_o^2 = \frac{R + jL\omega}{G + jC\omega}$$

and

$$p^2 = (R + jL\omega)(G + jC\omega),$$

equation (4.5.1) may be written

$$Z_\pi = (R + jL\omega)S. \tag{4.5.2}$$

In a similar manner, equation (4.4.6) becomes, when aS and S are less than 0.1,

$$Y_\pi = \frac{pS}{2Z_o}. \tag{4.5.3}$$

Substituting the formulas for p and Z_o, gives

$$Y_\pi = \frac{(G + jC\omega)S}{2}. \tag{4.5.4}$$

Equations (4.5.2) and (4.5.4) show that the uniform distribution of the line constants may be neglected in the equivalent π only when βS and aS are smaller than 0.1. The condition is, as expected, identical to that pertaining to the nominal T equivalent. The nominal π may also be used, therefore, without any serious error for all lines between 50 and 100 miles long.

4.6. *Equivalence between T and π Circuits.* Since by a proper choice of circuit constants the T and π networks can be made equivalent to any line for a definite frequency and length, they must also be equivalent to each other.

Consider a T and a π circuit each equivalent to the same line as shown in Fig. 4–5.

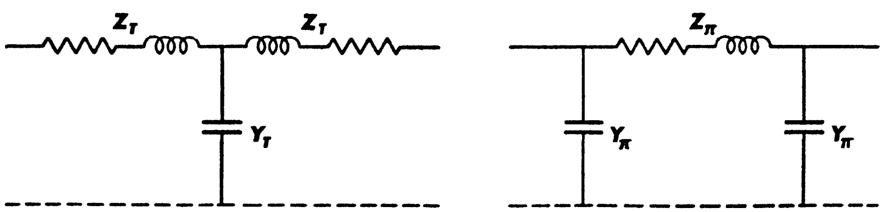


Figure 4–5

Since both are equivalent to the same line, it follows that the characteristic impedances of the two circuits are equal and equal to that of the line, i.e.,

$$Z_{oT} = Z_{o\pi} = Z_o.$$

By (4.2.6)

$$Y_T = \frac{\sinh pS}{Z_o}.$$

By (4.2.11)

$$Z_T = Z_o \tanh \frac{pS}{2}.$$

By (4.4.3)

$$Z_\pi = Z_o \sinh pS.$$

By (4.4.6)

$$Y_\pi = \frac{\tanh \frac{pS}{2}}{Z_o}.$$

By (2.1.1) the characteristic impedance of the T circuit is

$$Z_{oT}^2 = Z_T^2 + \frac{2Z_T}{Y_T}. \tag{4.6.1}$$

Dividing (4.2.11) by (4.4.6) gives

$$\frac{Z_T}{Y_\pi} = Z_o^2. \tag{4.6.2}$$

Dividing (4.4.3) by (4.2.6) gives

$$\frac{Z_\pi}{Y_T} = Z_o^2. \tag{4.6.3}$$

By the two expressions just obtained, it follows that

$$Z_T Y_T = Z_\pi Y_\pi. \tag{4.6.4}$$

Substituting for Z_T and Y_T in terms of Z_π and Y_π in (4.6.1) gives

$$Z_{o\pi}^2 = Y_\pi^2 Z_o^4 + \frac{2Y_\pi Z_o^4}{Z_\pi},$$

and since $Z_{o\pi} = Z_o$, this reduces to

$$Y_{o\pi} = \sqrt{Y_\pi^2 + \frac{2Y_\pi}{Z_\pi}}. \tag{4.6.5}$$

This formula represents the characteristic admittance of the π circuit. It is worthwhile to note the similarity between the formula (4.6.1) for the characteristic impedance Z_{oT} of the T circuit and formula (4.6.5) which gives the characteristic admittance $Y_{o\pi}$ of the π circuit.

It is sometimes convenient to have the formula for the characteristic impedance of the π circuit instead of the characteristic admittance. It can be obtained by taking the reciprocal of (4.6.5) resulting in

$$Z_{o\pi} = \sqrt{\frac{Z_\pi}{Z_\pi Y_\pi^2 + 2Y_\pi}}. \tag{4.6.6}$$

To obtain the series impedance Z_T and the shunt admittance Y_T of a T circuit equivalent to a known π circuit, combine (4.6.3) and (4.6.2) with (4.6.6). This gives

$$Z_T = \frac{Z_\pi}{Z_\pi Y_\pi + 2} \tag{4.6.7}$$

and

$$Y_T = (Z_\pi Y_\pi + 2)Y_\pi. \tag{4.6.8}$$

Similarly, to obtain the series impedance Z_π and the shunt admittance Y_π of a π circuit equivalent to a given T circuit combine (4.6.3) and (4.6.2) with (4.6.1). This gives

$$Z_\pi = (Z_T Y_T + 2)Z_T, \tag{4.6.9}$$

and

$$Y_\pi = \frac{Y_T}{Z_T Y_T + 2}. \tag{4.6.10}$$

Note the similarity between (4.6.7) and (4.6.10) and also between (4.6.8) and (4.6.9). It is also worthwhile to note that these relationships could have been obtained by treating the T circuit as a wye and the π circuit as a delta.

4.7. *The Single-Impedance Equivalent of Very Short Lines.* It was seen that the nominal T or the nominal π circuit may be used as substantially equivalent to lines when the phase shift βS does not exceed about 5.7° or 0.1 radians, i.e., when the line length does not exceed about 100 miles. Under this condition, the shunt admittance of the nominal T, given by (4.3.3), is

$$Y_T = (G + jC\omega)S,$$

and the shunt admittance of the nominal π, given by (4.5.4), is

$$Y_\pi = \frac{(G + jC\omega)S}{2}.$$

The linear line admittance (y) per phase per mile of commercial power lines is rather small (about 6×10^{-6} mhos). Hence, for short lines not exceeding about 20 miles in length, Y_T and Y_π may be omitted from their respective circuits, the effect of leakage conductance and capacitive susceptance neglected, and the line treated as a single series circuit of impedance

$$Z = (R + jL\omega)S,$$

where R is the resistance per mile of conductor and L is the inductance per mile of conductor.

4.8. *Illustrative Short-Line Calculations.* There are many transmission-line problems which cannot be solved directly by the formulas developed in the preceding articles. A few such illustrative problems are considered in this article.

Problem 1. Determine the amount of receiving-end power, of definite power factor $\cos\theta_r$, that could be transmitted over a short line with a definite station-end voltage V_s, the loss not to exceed a definite per cent of the delivered power.

Let P_r, V_r, and I_r be, respectively, the unknown values of power, voltage, and current per phase at the receiving end. Then

$$P_r = V_r I_r \cos\theta_r. \tag{4.8.1}$$

If m represents the power loss in per cent of P_r, R the resistance per mile of conductor, and S the length of the line in miles, then

$$\frac{mP_r}{100} = I_r^2 RS. \tag{4.8.2}$$

This is substantially correct if the line is fairly short, not exceeding 20 or 25 miles. For, in such case, the current at all points along the line is substantially the same as at the receiving end.

Substituting the value of P_r as given by (4.8.1) in (4.8.2) gives

$$\frac{V_r}{I_r} = \frac{100RS}{m \cos \theta_r} = Z_r. \tag{4.8.3}$$

This formula for the equivalent impedance of the load is substantially true only for very short lines.

Since the linear shunt admittance has a negligible effect in short lines, the transmission equation for such lines is

$$V_s\underline{/\delta_s} = V_r\underline{/0^\circ} + (I\underline{/\theta_r})(zS\underline{/\alpha}), \tag{4.8.4}$$

where δ_s is the unknown phase between the known station-end voltage and the unknown receiving-end voltage, I is the unknown load current lagging or leading V_r by θ_r corresponding to the known $\cos\theta_r$; z is the linear-line impedance per mile and $\alpha = \tan^{-1}(L\omega/R)$ is the phase angle of z. Using the relation given by (4.8.3), the expression (4.8.4) for V_s becomes

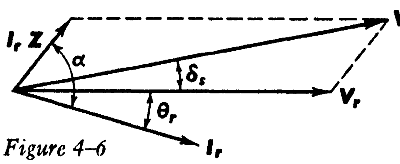


Figure 4–6

$$V_s\underline{/\delta_s} = V_r\underline{/0^\circ} + \frac{V_r}{Z_r} zS\underline{/\alpha + \theta_r}. \tag{4.8.5}$$

This equation is visualized graphically in Fig. 4–6 for the condition when θ_r is negative. Equation (4.8.5) may be written

$$V_s\underline{/\delta_s} = V_r\left(1 + \frac{zS\underline{/\alpha + \theta_r}}{Z_r}\right)$$

or

$$1 = \frac{V_r}{V_s\underline{/\delta_s}}\left(1 + \frac{zS\underline{/\alpha + \theta_r}}{Z_r}\right)$$

$$V_r\underline{/-\delta_s} = \frac{V_s}{1 + \dfrac{zS\underline{/\alpha + \theta_r}}{Z_r}}, \tag{4.8.6}$$

where Z_r is given by (4.8.3).

The formula gives the voltage to neutral at the receiving end, and its phase δ_s with reference to the station-end voltage. The required dissipative power per phase at the receiving end is

$$P_r = V_r I \cos\theta_r.$$

To illustrate numerically the above problem, let it be required to determine the amount of power that can be transmitted over a 250000 circ. mil three-phase line 20 miles long with a loss of 10 per cent of the power delivered. The line conductor is a concentric-layer copper cable of 19 strands. The equilateral spacing is 20 feet. The resistance per mile of conductor at 65° C. is 0.263 ohms. The inductive reactance at 60 cps is $L\omega = 0.765$ ohms. The impedance per conductor per mile is $z = 0.808\underline{/71°}$ vector ohms. The line being very short, the effect of the capacitive susceptance may be neglected. The power factor of the receiving-end load is assumed 90.6 per cent and lagging, corresponding to an angle of 25 degrees. The station-end voltage is 33 kv, corresponding to a phase voltage of virtually 19 kv.

The receiving-end impedance per phase, by (4.8.3), is

$$\begin{aligned} Z_r &= \frac{100 \times 0.263 \times 20}{10 \times 0.906} \underline{/25°} \\ &= 58.1\underline{/25°} \text{ vector ohms.} \end{aligned}$$

The receiving-end voltage to neutral, by (4.8.6), is

$$\begin{aligned} V_r\underline{/-\delta_s} &= \frac{19000}{1 + \dfrac{0.808 \times 20\underline{\underline{/71° - 25°}}}{58.1}} \\ &= 15660\underline{/-9.5°} \text{ vector volts.} \end{aligned}$$

The receiving-end voltage per phase is therefore 15.66 kv and lags the station-end voltage by 9.5 degrees.

The receiving-end current is

$$I_r = \frac{15660}{58.1} = 269.5 \text{ amperes,}$$

and lags the receiving-end voltage V_r by 25 degrees.

The receiving-end kilovolt amperes per phase is

$$\begin{aligned} (kva)_r &= 15.66 \times 269.5 \\ &= 4220 \text{ kilovolt amperes.} \end{aligned}$$

The dissipative power per phase at the receiving end is

$$\begin{aligned} P_r &= 4220 \cos 25° \\ &= 3825 \text{ kilowatts.} \end{aligned}$$

The total dissipative power is

$$P_{rt} = 11475 \text{ kilowatts.}$$

Since the line is short and the current is assumed the same at all points along its length, the current at the station end is

$$I_s = 269.5 \text{ amperes.}$$

The kva at the station end is

$$(kva)_s = 19.0 \times 269.5$$
$$= 5120 \text{ kilovolt amperes.}$$

The phase between the station-end current and voltage is

$$-25° - 9.5° = -34.5°.$$

The dissipative power per phase at the station end is

$$P_s = 5120 \cos 34.5°$$
$$= 4220 \text{ kw.}$$

The total dissipative power supplied to the line is

$$P_{st} = 12660 \text{ kw.}$$

The line efficiency is

$$\eta = \frac{3825}{4220}$$
$$= 90.6 \text{ per cent.}$$

The power loss per conductor is

$$\text{power loss} = 4220 - 3825$$
$$= 395 \text{ kw,}$$

or 10.32 per cent of the delivered power. This is substantially equal to the assumed per cent loss.

The voltage regulation is

$$\text{Reg.} = \frac{(19.0 - 15.66)100}{15.66}$$
$$= 21.3 \text{ per cent rise.}$$

The above method of solution gives substantially accurate results if the lines do not exceed about 25 miles and the linear-line admittance is neglected. Equation (4.8.3), which gives the equivalent load impedance, holds only for such lines.

The problem as stated above cannot be solved for lines for which the nominal or equivalent T or π are used because the station and receiving-end currents differ in value, and, as a consequence, there is no way of determining the equivalent impedance Z_r of the load. A trial calculation may be made, however, using a value of Z_r equal to or close to that obtained by (4.8.3).

Problem 2. Determine the amount of receiving-end power P_r of power factor 0.906 that could be transmitted over the line whose characteristics are stated in Prob. 1, with a loss of 15 per cent of the power delivered. The line is 100 miles long.

Formula (4.8.3) gives a rough approximation of the value of the equivalent impedance of the load. Thus

$$Z_r = \frac{100 \times 0.263 \times 100}{15 \times 0.906} \angle 25^\circ = 193.5\angle 25 \text{ vector ohms,}$$

where 0.906 is the power factor of the load.

Using V_r/Z_r for I_r in (4.2.1) the formula becomes

$$V_s = V_r\left[1 + Z_T Y_T + (Z_T Y_T + 2)\frac{Z_T}{Z_r}\right]$$

where for the first trial $Z_r = 200\angle 25^\circ$.

For the particular line under consideration the impedance per conductor per mile is $z = 0.808\angle 71^\circ$. Hence

$$Z_T = \frac{(0.808\angle 71^\circ)100}{2} = 40.4\angle 71^\circ \text{ vector}$$

From § 3.4

$$\begin{aligned}
Y_T &= CS\omega\angle 90^\circ = (5.58 \times 10^{-6}\angle 90^\circ)100 \\
&= 558 \times 10^{-6}\angle 90^\circ, \\
Z_T Y_T &= (40.4\angle 71^\circ)(558 \times 10^{-6}\angle 90^\circ) \\
&= -22540 \times 10^{-6}\angle -19^\circ, \\
(Z_T Y_T)\frac{Z_T}{Z_r} &= (-22540 \times 10^{-6}\angle -19^\circ)\frac{40.4\angle 71^\circ}{200\angle 25^\circ} \\
&= -4553 \times 10^{-6}\angle 27^\circ, \\
\frac{2Z_T}{Z_r} &= \frac{2 \times 40.4\angle 71^\circ}{200\angle 25^\circ} = 0.404\angle 46^\circ.
\end{aligned}$$

Substituting these values in the above formula for V_s, and taking V_r as the reference vector, gives

$$\begin{aligned}
V_s\angle\delta_s &= V_r\angle 0^\circ(1 - 0.02254\angle -19^\circ - 0.004553\angle 27^\circ + 0.404\angle 46^\circ) \\
&= V_r\angle 0^\circ(1.29\angle 13.3^\circ).
\end{aligned}$$

For a station-end voltage of 66 kv, corresponding to 38.05 kv per phase, the receiving-end voltage is

$$\begin{aligned}
V_r\angle -\delta_s &= \frac{38.05}{1.29\angle 13.3^\circ} \\
&= 29.46\angle -13.3^\circ \text{ kv}
\end{aligned}$$

with reference to V_s.

The current at the receiving end is

$$I_r = \frac{29460\angle 0^\circ}{200\angle 25^\circ} = 147.3\angle -25^\circ \text{ vector amperes}$$

with reference to V_r.

The kva at the receiving end is

$$(kva)_r = 29.46 \times 147.3$$
$$= 4340 \text{ kilo-voltamperes.}$$

The dissipative power per phase at the receiving end is

$$P_r = 4340 \cos 25°$$
$$= 3930 \text{ kw.}$$

The total dissipative power at the receiving end is

$$P_{rt} = 11790 \text{ kw.}$$

Equation (4.2.2) gives the current at the station end. For the particular case considered, it may be written

$$I_s = V_r\left(\frac{1 + Z_T Y_T}{Z_r} + Y_T\right),$$

where $Z_r = 200\underline{/25°}$. Substituting numerical values for the quantities involved gives

$$I_s = 134.5\underline{/-32.6°} \text{ vector amperes.}$$

The kilo-voltamperes at the station end is

$$(kva)_s = 38.05 \times 134.5$$
$$= 5120 \text{ kilo-voltamperes.}$$

The dissipative power per phase supplied to the line is

$$P_s = 5120 \cos 32.6°$$
$$= 4310 \text{ kw.}$$

The total dissipative power supplied to the line is

$$P_{st} = 12930 \text{ kw.}$$

The line efficiency is

$$\eta = \tfrac{3930}{4310}$$
$$= 91.2 \text{ per cent.}$$

The loss per conductor is 4310 − 3930 = 380, i.e., 9.68 per cent of the power delivered to the receiving end.

Problem 3. Another short transmission-line problem often met is when the load, its power factor, receiving-end voltage, and line length are known, and the size of the conductor and the sending-end voltage are to be determined for a definite permissible loss stated in per cent of the delivered power. To illustrate such a problem, determine the size of conductor and the required station-end voltage to transmit an amount of dissipative power equal to 15000 kw at 90.6 per cent power factor over a line 20 miles long. The line voltage at the receiving end is 33 kv and the permissible loss is 10 per cent of the delivered power.

From the statement of the problem, it follows that the dissipative power, P_r, per phase is 5000 kw, and that the voltage to neutral is 19 kv. The current at the receiving end, therefore, is

$$I_r = \frac{P_r}{V_r \cos \theta_r}$$

$$I_r = \frac{5000}{19 \times 0.906}$$

$$= 290.4 \text{ amperes.}$$

Since the line is very short, it will be assumed that the current is the same at all points along the conductors. Hence, if R_c is the resistance of the conductor

$$I^2R_c = \text{loss per conductor}$$

or

$$R_c = \frac{500000}{290.4^2}$$

$$R_c = 5.929 \text{ ohms.}$$

The resistance per mile is

$$R = \frac{5.929}{20} = 0.2964 \text{ ohms.}$$

The nearest size of Aluminum Cable Steel Reinforced (A.C.S.R.) is the 336,400 circ. mil cable having an outside diameter of 0.741 inches and a resistance per mile of 0.272 ohms. Using this size cable, the resistance of the conductor is

$$R_c = 0.272 \times 20 = 5.44 \text{ ohms.}$$

The spacing distance in inches, assumed equilateral, may be determined approximately by the following empyrical formula, based upon spacing distances commonly used in commercial lines

$$D = \frac{\text{span in feet}}{20} + \frac{\text{line voltage}}{1000}. \tag{4.8.7}$$

For the particular line under consideration the line voltage is 33 kv, and assuming a span of 500 feet, gives for the spacing distance

$$D = 25 + 33 = 58 \text{ inches.}$$

Using $D = 60$ inches, the inductive reactance to neutral at a frequency of 60 cps is $L\omega = 0.755$ ohms per mile, i.e., 15.1 ohms for the 20-mile conductor.

The line being short, the linear-line susceptance is neglected.

The impedance of the line to neutral is

$$\begin{aligned} \boldsymbol{Z} &= 5.44 + j\,15.1 \\ &= 16.05\underline{/70.18^\circ} \text{ vector ohms.} \end{aligned}$$

The station-end voltage to neutral is

$$\begin{aligned} \boldsymbol{V}_s\underline{/\delta_s} &= V_r\underline{/0^\circ} + (I_r\underline{/\theta_r})(Z\underline{/\alpha}) \\ &= 19000\underline{/0^\circ} + (290.4\underline{/-25^\circ})(16.05\underline{/70.18^\circ}) \\ &= 22500\underline{/8.44^\circ} \text{ vector volts to neutral.} \end{aligned}$$

The station-end line voltage is, therefore, 39 kv.

Since the line is short and the current is substantially the same at all points along the line, it follows that the station-end current is $\boldsymbol{I}_s = 290.4$ amperes lagging $\boldsymbol{V}_s$ an angle $25^\circ + 8.44^\circ = 33.44^\circ$. The kva at the station end is

$$\begin{aligned} (kva)_s &= 22.5 \times 290.4 \\ &= 6534 \text{ kilo-voltamperes.} \end{aligned}$$

The dissipative power at this end per phase is

$$P_s = 6534 \cos 33.44^\circ$$
$$P_s = 5450 \text{ kw.}$$

The total dissipative power is

$$P_{st} = 16350 \text{ kw.}$$

The line efficiency is

$$\eta = \tfrac{5000}{5450} = 91.74 \text{ per cent.}$$

The loss per conductor is 5450 − 5000 = 450 kw, i.e., 9.0 per cent of the power delivered.

The line regulation is

$$\begin{aligned} \text{Reg.} &= \frac{(22.5 - 19)100}{19} \\ &= 18.4 \text{ per cent.} \end{aligned}$$

Problem 4. A load of 30,000 kw at 90.6 per cent power factor is to be transmitted a distance of 100 miles with a maximum loss of 15 per cent of the power delivered. The receiving-end line voltage is 110 kv corresponding to 63.5 kv to neutral. Determine the size of conductor to be used and the performance of the line.

The current at the receiving end is

$$I_r = \frac{P_r}{V_r \cos \theta_r}$$

$$\begin{aligned} I_r &= \frac{10000}{63.5 \times 0.906} \\ &= 173.8 \text{ amperes.} \end{aligned}$$

The current at the station end is somewhat smaller than the current at the receiving end because of the effect of the capacitive susceptance of the line. To determine the size of conductor it is safe to assume that the line loss is due to a current equal to the receiving-end current, i.e.,

$$\text{loss per conductor} = I_r^2 R_c$$
$$1{,}500{,}000 = 173.8^2 R_c$$

or

$$R_c = 49.66 \text{ ohms.}$$

The calculated resistance per mile of conductor, therefore, is

$$R = \frac{49.66}{100} = 0.4966 \text{ ohms.}$$

A #000 A.C.S.R. has a resistance of 0.542 ohms per mile at 20° C. The next larger size is a #0000, whose resistance is 0.432 ohms per mile, or a total resistance for the 100-mile conductor equal to 43.2 ohms.

The approximate value of the equilateral spacing distance in inches for a span of 600 feet, is

$$D = \tfrac{600}{20} + \tfrac{110000}{1000}$$
$$D = 30 + 110 = 140 \text{ inches.}$$

Taking the spacing distance equal to 144 inches, the inductance is 0.00213 henry per mile* and the inductive reactance at 60 cps is $L\omega = 0.804$ ohms per mile. The total inductive reactance for the 100-mile line to neutral is 80.4 ohms. The linear impedance of the line conductor is

$$\mathbf{Z} = 43.2 + j\,80.4$$
$$\mathbf{Z} = 91.27\underline{/61.75^\circ} \text{ vector ohms.}$$

The capacitance† to neutral is 0.01415×10^{-6} farads per mile and the linear-line admittance of the 100-mile conductor to neutral at 60 cps is

$$\mathbf{Y} = 100 \times 5.34 \times 10^{-6}\underline{/90^\circ}$$
$$= 534 \times 10^{-6}\underline{/90^\circ} \text{ vector mhos.}$$

A substantially correct result will be obtained by using the nominal T. The voltage formula is

$$\mathbf{V}_s = \mathbf{V}_r(1 + \mathbf{Z}_T\mathbf{Y}_T) + \mathbf{I}_r(2 + \mathbf{Z}_T\mathbf{Y}_T)\mathbf{Z}_T.$$

* From Tables II and III.

† From Tables IV and V.

For the problem under consideration,

$$
\begin{aligned}
Z_T Y_T &= (45.63\underline{/61.75^\circ})(534 \times 10^{-6}\underline{/90^\circ}) \\
&= -0.02436\underline{/-28.25^\circ}, \\
1 + Z_T Y_T &= 1 - 0.02436\underline{/-28.25^\circ} \\
&= 0.978\underline{/0.63^\circ}. \\
2Z_T + Z_T^2 Y_T &= 91.27\underline{/61.75^\circ} - (0.02436\underline{/-28.25^\circ})(45.63\underline{/61.75^\circ}) \\
&= 90.2\underline{/62.2^\circ}.
\end{aligned}
$$

The station-end voltage, therefore, is

$$
\begin{aligned}
V_s\underline{/\delta_s} &= (63500\underline{/0^\circ})(0.978\underline{/0.63^\circ}) + (173.8\underline{/-25^\circ})(90.2\underline{/62.2^\circ}) \\
&= 75600\underline{/7.76^\circ} \text{ vector volts}
\end{aligned}
$$

per phase. The line voltage at the station end is, therefore, 130 kv.

The current at the station end is obtained by formula (4.2.2) and

$$I_s = I_r(1 + Z_T Y_T) + V_r Y_T.$$

For the particular line under consideration,

$$
\begin{aligned}
1 + Z_T Y_T &= 0.978\underline{/0.63^\circ}. \\
Y_T &= 534 \times 10^{-6}\underline{/90^\circ}
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_s &= (173.8\underline{/-25^\circ})(0.978\underline{/0.63^\circ}) + (63500\underline{/0^\circ})(534 \times 10^{-6}\underline{/90^\circ}) \\
&= 159.2\underline{/-13.22^\circ} \text{ vector amperes.}
\end{aligned}
$$

The kva per phase at the station end is

$$
\begin{aligned}
(kva)_s &= 75.6 \times 159.2 \\
&= 12200 \text{ kilo-volt amperes.}
\end{aligned}
$$

The station-end voltage leads the current at that end by

$$\theta_s = 7.76^\circ - (-13.22^\circ) = 20.98^\circ.$$

The dissipative power per phase supplied to the line is

$$P_s = 11400 \text{ kw.}$$

The total dissipative power is

$$P_{st} = 34200 \text{ kw.}$$

The efficiency of transmission is

$$
\begin{aligned}
\eta &= \frac{10000}{11400} \\
&= 87.7 \text{ per cent.}
\end{aligned}
$$

The loss per conductor is 1140 kw or 11.4 per cent of the power delivered to the receiving end.

REFERENCES

Kennelly, A. E., *Electric Lines and Nets*, McGraw-Hill Book Co., 1928.

Vaddicor, Harold, *Principles of Electric Power Transmission*, Wiley and Sons, 1928.

Weinbach, M. P., *Principles of Transmission in Telephony*, Macmillan Book Co., 1924.

SUGGESTIVE PROBLEMS *Chapter 4*

1. Obtain the equivalent T and the equivalent π of the line stated in Prob. 4, Chap. 3.

2. How much power at 87 per cent power factor lag can be transmitted over a 250,000 circ. mil copper three-phase line, 23 miles long with a loss of 12 per cent of the receiving-end power? The resistance per conductor is 0.263 per mile; the inductive reactance at 60 cps is 0.764 ohms per mile and the capacitive susceptance negligibly small. Assume the station-end line voltage equal to 33 kv.

3. Determine the size of the copper conductors and the needed station-end voltage to transmit 12,500 kw at 87 per cent power factor lag over a 3-phase 25-mile line with a loss of 12 per cent of the receiving-end power. Assume the receiving-end voltage to neutral maintained constant at 19 kv.

   What size of A.C.S.R. would be needed?

Chapter 5 Line with Transformers

5.1. *The T Equivalent of Transformers.* It was remarked in § 3.4 that three-phase transmission lines are terminated at both ends in transformer banks, normally connected in wye on the high-voltage side. The function of the transformers at the station end is to step-up the station-end voltage to that required for the line, and the function of those at the receiving end is to step down the voltage to that required by the distribution system. The transformer banks are thus part of the transmission system. Fig. 5–1 shows schematically one phase of a

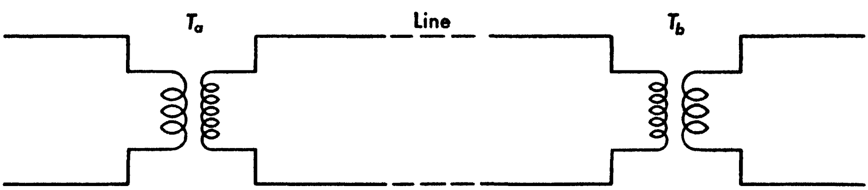


Figure 5–1

transmission system including the station- and receiving-end transformers T_a and T_b. The transmission system viewed as a whole consists, therefore, of lumped, highly inductive electric circuits coupled magnetically through mutual impedances, and interconnected through a circuit of uniformly distributed resistance, inductance, capacitance, and leakage conductance.

A comparatively simple method of approaching a solution of the complete circuit is to convert the indirect-coupled circuit of the transformers into T networks. The circuit per phase thus takes, schematically, the form shown in Fig. 5–2. Because of their nonlinear magnetic characteristics, transformers can be simulated by equivalent circuits for the particular frequency for which they are designed, for only one value of voltage and current. It is found, however, by actual calculations that the equivalent T which simulates a transformer at its rated voltage and current, may be used without serious error even when the voltage and current differ to some extent from their respective normal values.

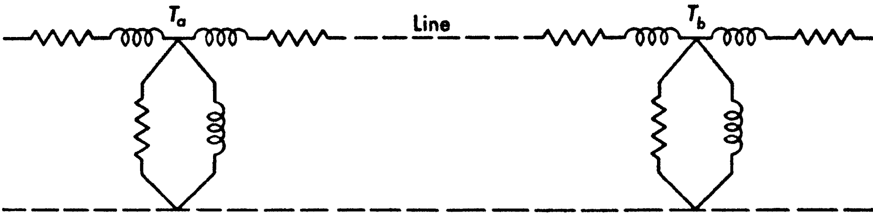


Figure 5–2

To obtain the equivalent T of a transformer let V_H be the voltage on the high side and I_{He} the input current on that side when the secondary is open-circuited. The self-impedance Z_{Hs} of the high side, therefore, is

$$Z_{Hs} = \frac{V_H}{I_{He}} \text{ vector ohms.} \tag{5.1.1}$$

In a similar manner, if V_L is the impressed voltage on the low-voltage side, and I_{Le} is the input current when the high-side is open-circuited, the self-impedance of the low-voltage side is

$$Z_{Ls} = \frac{V_L}{I_{Le}} \text{ vector ohms.} \tag{5.1.2}$$

The currents I_{He} and I_{Le} are, respectively, the exciting currents under the two stated conditions. The mutual impedance Z_m of the transformer is the vector ratio of the open-circuited voltage on one side to the exciting current on the other side, i.e.,

$$Z_m = \frac{E_L}{I_{He}} = \frac{E_H}{I_{Le}}, \text{ vector ohms,} \tag{5.1.3}$$

where E_H is the open-circuit voltage on the high-voltage side when the low-voltage side is excited, and E_L is the open-circuit voltage on the low-voltage side when the high-voltage side is excited.

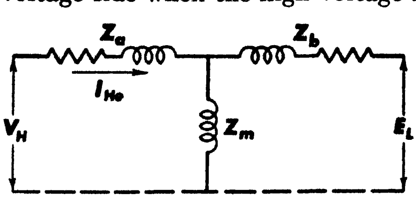


Figure 5–3

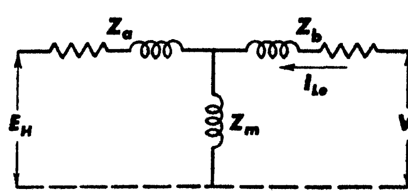


Figure 5–3a

From what has been said, it follows that a transformer may be represented by a direct inductive coupling such as the one shown diagrammatically in Fig. 5–3 and 5–3a. The two are identical except for the direction of the energy transfer. From the figures it is seen that the self-impedance on the high-voltage side is

$$Z_{Hs} = Z_a + Z_m, \tag{5.1.4}$$

and the self-impedance on the low-voltage side is

$$Z_{Ls} = Z_b + Z_m. \tag{5.1.5}$$

The quantities Z_{Hs}, Z_{Ls}, and Z_m may be obtained by direct measurement in accordance with equations (5.1.1), (5.1.2), and (5.1.3) respectively. The impedances

$$\left.\begin{aligned} Z_a &= Z_{Hs} - Z_m \\ \text{and} \qquad Z_b &= Z_{Ls} - Z_m \end{aligned}\right\} \tag{5.1.6}$$

are, respectively, the leakage impedances of the high and low-voltage windings of the transformer. The equivalent T of the transformer thus obtained, is not symmetrical. The leakage impedances Z_a and Z_b are not equal and, as a consequence, the energy transfer is not at the same potential in either direction as it would be if the T equivalent were symmetrical.

To obtain a symmetrical T equivalent of the transformer, i.e., one in which the energy transfer is at identical potentials in either direction, the transformer data must be modified to give a fictitious unity ratio transformer — one having an equal number of turns on the two windings. The self-impedance of such a transformer is obviously the same irrespective at which end it is measured, and the equivalent T is thus symmetrical.

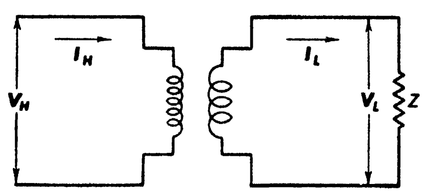


Figure 5–4

To convert a transformer circuit into such a symmetrical T with reference to the high-voltage side, consider the transformer diagram Fig. 5–4 and assume the low-voltage side loaded with an impedance Z. Let V_H be the impressed voltage across, and I_H the current in the high-voltage side taken as the primary. If I_L is the output current of the low voltage, taken as the secondary, then

$$V_H = I_H Z_{Hs} - I_L Z_m, \tag{5.1.7}$$

where Z_{Hs} is the self-impedance of the primary side and Z_m the mutual impedance.

Since the ratio of transformation is

$$a = \frac{E_H}{E_L} = \frac{I_L}{I_H}, \tag{5.1.8}$$

the above expression may be written in terms of the current in high-voltage side

$$V_H = I_H Z_{Hs} - I_H(aZ_m). \tag{5.1.9}$$

The induced emf in the low-voltage secondary is

$$E_L = I_L(Z_{Ls} + Z), \tag{5.1.10}$$

where Z is the load impedance.

Expressing both E_L and I_L, respectively, in terms of corresponding values in the high-voltage side, by (5.1.8), gives

$$\frac{E_H}{a} = aI_H(Z_{Ls} + Z)$$

or

$$E_H = I_H(Z_{Ls} + Z)a^2. \tag{5.1.11}$$

Equations (5.1.9) and (5.1.11) show that if the mutual impedance Z_m of the transformer is multiplied by the transformation ratio a and the self-impedance Z_{Ls} of the low-voltage side, and load impedance Z multiplied by the square of this ratio, the circuit will behave like a unity-ratio transformer. The self-impedances measured from either end will have the same value, indicating that the equivalent T is symmetrical as shown in Fig. 5–5. From what has just been said it follows that

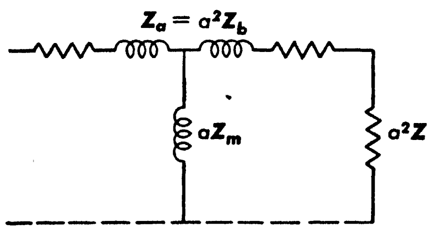


Figure 5–5

$$Z_{Hs} = a^2 Z_{Ls}. \tag{5.1.12}$$

Denoting by $Z_{\lambda H}$ and $Z_{\lambda L}$ the leakage impedances of the high and low-voltage windings, respectively, it follows from the above that for the unity-ratio transformer and its equivalent T

$$Z_{\lambda H} = Z_{\lambda L}. \tag{5.1.13}$$

The total leakage impedance of a transformer may be thought of, therefore, as being equally divided between the high and low-voltage windings in the symmetrical equivalent T of the transformer.

5.2. *Transmission Formulas of the Equivalent T of Transformers.* The relations between the sending-end and receiving-end voltage and current values of the symmetrical T circuits were obtained in § 4.2. In the generalized form these relations are

$$V_s = AV_r + BI_r \tag{5.2.1}$$
$$I_s = AI_r + DV_r, \tag{5.2.2}$$

in which, by (4.2.3) and (4.2.4),

$$A = 1 + Z_T Y_T$$
$$B = Z_T(2 + Z_T Y_T) \tag{5.2.3}$$

and

$$D = Z_T.$$

To determine these quantities when the symmetrical T simulates a transformer, consider the T equivalent of a transformer in Fig. 5–6. The quantity Z_λ is the leakage impedance of the high-voltage winding or of the low-voltage winding referred to the high-voltage side. It is, therefore, one-half of the total leakage impedance of the transformer when measured from the high-voltage side.

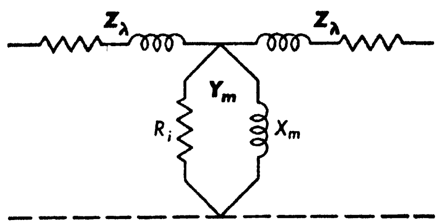


Figure 5–6

If R_i is the equivalent resistance of the iron loss and X_m the mutual reactance, both referred to the high-voltage side, and assumed in parallel connection as indicated in the figure, then

$$Y_m = \frac{1}{R_i} - \frac{j}{X_m}, \tag{5.2.4}$$

where Y_m is the mutual admittance.

In accordance with the set of equations (5.2.3), it follows that

$$Z_T = Z_\lambda$$
$$Y_T = Y_m \tag{5.2.5}$$
$$A = 1 + Z_\lambda Y_m \tag{5.2.6}$$
$$B = Z_\lambda(2 + Z_\lambda Y_m) \tag{5.2.7}$$
$$D = Y_m. \tag{5.2.8}$$

Substituting these values in equations (5.2.1) and (5.2.2) gives for the transmission equations of the transformer with the high-voltage side taken as the primary

$$V_H = (1 + Z_\lambda Y_m)V_L + Z_\lambda(2 + Z_\lambda Y_m)I_L \tag{5.2.9}$$
$$I_H = (1 + Z_\lambda Y_m)I_L + Y_m V_L, \tag{5.2.10}$$

where V_L and I_L are, respectively, the voltage and current at the secondary low-voltage side. The transmission constants of the symmetrical T of transformers, as stated by equations (5.2.6), (.7), and (.8), may be obtained by the well-known ***open- and short-circuit tests.***

Thus, with the transformer low-voltage side open, measure the supplied voltage V_H, the primary exciting current I_{He} and the power supplied P_{H1}.

Since under the conditions just stated $I_L = 0$, equations (5.2.9) and (.10) may be written

$$V_{H1} = (1 + Z_\lambda Y_m)V_L \qquad (5.2.11)$$
$$I_{H1} = Y_m V_L$$

where, if V_{H1} is taken as the reference vector, the current lags it by an angle

$$\theta_1 = \cos^{-1} \frac{P_{H1}}{(VI)_{H1}}. \qquad (5.2.12)$$

Equations (5.2.11) may be written

$$\frac{V_{H1}}{I_{H1}} = \frac{1 + Z_\lambda Y_m}{Y_m} \qquad (5.2.13)$$

$$= \frac{1}{Y_m} + Z_\lambda. \qquad (5.2.14)$$

Since Z_λ is very small compared to the mutual impedance $1/Y_m$, the above expression becomes substantially

$$Y_m = \frac{I_{H1}}{V_{H1}} \underline{/\theta_1}. \qquad (5.2.15)$$

With the low-voltage winding short-circuited through a low-impedance ammeter, and the impressed voltage on the primary reduced to a sufficiently low value to avoid excessive secondary current, measure V_{H2} the primary current I_{H2} for full load secondary current, I_{L2} and the power supplied P_{H2}.

Since under the conditions just stated $V_L = 0$, equations (5.2.9) and (.10) become

$$V_{H2} = Z_\lambda(2 + Z_\lambda Y_m)I_{L2}$$
$$I_{H2} = (1 + Z_\lambda Y_m)I_{L2}, \qquad (5.2.16)$$

where, if V_{H2} is taken as the reference vector, the current lags it by an angle

$$\theta_2 = \cos^{-1} \frac{P_{H2}}{(VI)_{H2}}. \qquad (5.2.17)$$

Equations (5.2.11) may be written, accordingly

$$\frac{V_{H2}}{I_{H2}} = \frac{Z_\lambda(2 + Z_\lambda Y_m)}{1 + Z_\lambda Y_m}. \qquad (5.2.18)$$

However, since $Z_\lambda Y_m$ is much smaller than one, the expression becomes substantially

$$2Z_\lambda = \frac{V_{H2}}{I_{H2}} \underline{/\theta_2}. \qquad (5.2.19)$$

To illustrate the above, consider the following data pertaining to one of the three similar transformers of a three-phase bank rated 110 kv/11 kv.

From the open-circuit test:

$$V_{H1} = 63.5 \text{ kv}$$
$$I_{H1} = 2.28 \text{ amperes}$$
$$P_{H1} = 33 \text{ kw.}$$

From the short-circuit test:

$$V_{H2} = 983 \text{ volts}$$
$$I_{H2} = 50 \text{ amperes}$$
$$P_{H2} = 11 \text{ kw.}$$

The mutual admittance referred to the high-voltage winding is, by equation (5.2.15),

$$Y_m = \frac{2.28}{63500} \underline{/-\cos^{-1}(33/63.5 \times 2.28)}$$
$$= 35.9 \times 10^{-6} \underline{/-76.9^\circ} \text{ mhos.}$$

The leakage reactance, referred to the high-voltage winding is, by equation (5.2.19),

$$2Z_\lambda = \tfrac{983}{50} \underline{/\cos^{-1}(11000/983 \times 50)}$$

or

$$Z_\lambda = 9.83\underline{/77.1^\circ} \text{ ohms.}$$

The transmission constants of the equivalent T of the transformer are, therefore, by equations (5.2.6), (.7), and (.8),

$$A = 1 + (9.83\underline{/77.1^\circ})(35.9 \times 10^{-6}\underline{/-76.9^\circ})$$
$$= 1 + 0.000353\underline{/0.2^\circ}$$

or, substantially

$$A = 1\underline{/0^\circ}.$$
$$B = 2Z_\lambda + Z_\lambda^2 Y_m$$
$$= 19.66\underline{/77.1^\circ} + 0.00347\underline{/77.3^\circ}$$

or, substantially

$$B = 19.66\underline{/77.1^\circ}$$

and

$$D = Y_m = 35.9 \times 10^{-6}\underline{/-76.9^\circ}.$$

The transmission equations (5.2.9) and (.10) for this particular transformer, accordingly, are

$$V_H = V_L + 19.66\underline{/77.1^\circ} I_L$$

and

$$I_H = I_L + (35.9 \times 10^{-6}\underline{/-76.9^\circ}) V_L,$$

where V_L and I_L are referred to the high-voltage side.

5.3. *High-Voltage Transformers.* The transformers used with high-voltage lines to step-up the generator voltage at the station end to that of the line voltage and to step-down the line voltage to that of the substation at the receiving end may be either single or three-phase units. As stated previously the low-voltage windings are usually connected in delta, and the high-voltage windings generally in wye. Connecting the high voltage winding in wye not only provides a higher voltage but also a point for grounding the system. This, as discussed in § 8.2, is a required provision to permit the flow of short-circuit currents in line to ground or line-line to ground short-circuit faults. A delta connection on the low side eliminates the third harmonic which otherwise may cause interference with neighboring telephone lines.

The voltage rating of transmission transformers is based not only upon the needed voltages with regard to that of the generators in the station and that required by the length of line, but also upon needed compensation for the voltage drop in the transformers and in the line. This is necessary in order to limit to some extent the range of generator excitation voltage at the station end, and the capacity of the needed phase modifier at the receiving end. As a rule the generators in the station have a voltage rating 5 per cent in excess of the transformer voltages, to compensate for the leakage impedance drop of the transformers under full load conditions. Similarly, the station-end transformers have normally a voltage rating 5 to 7 per cent higher than that of the receiving-end transformer to compensate for the voltage drop in the line.

The following data pertaining to high-voltage transformers of capacities of from 5000 to 25,000 kva are sufficiently accurate for all transmission-line calculations in the absence of manufacturers' data for specific cases.

1. The leakage impedance drop may be taken from 7 to 9 per cent of the rated voltage and at full load.
2. The copper loss is from 0.7 to 1 per cent of the rated capacity at unity power factor.
3. The iron loss is from 0.3 to 0.5 per cent of the rated output at unity power factor.
4. The exciting current is from 3 to 7 per cent of the full-load current.

5.4. *Calculation of Leakage Impedance Z_λ and of the Mutual Admittance Y_m.* Given the kva rating of the transformer and the kv to neutral at the high-voltage primary, then the primary load current is

$$I_H = \frac{(kva)}{(kv)_H}. \tag{5.4.1}$$

If Z_λ represents the leakage impedance of either the high- or the low-voltage side with reference to the high-voltage side, and m_λ is the known leakage impedance drop in per cent of the rated voltage, then

$$2I_H Z_\lambda = \frac{m_\lambda V_H}{100}$$

or

$$Z_\lambda = \frac{m_\lambda V_H}{200 I_H}. \tag{5.4.2}$$

To obtain the angle of the leakage impedance Z_λ it is necessary to calculate the resistance component of Z_λ. Its value can be obtained from the known value of the copper loss. Thus, if m_c is the copper loss in per cent of the rated capacity at unity power factor, then

$$2I_H^2 R_c = \frac{m_c(kva)1000}{100},$$

where R_c is the resistance of the high-voltage winding or of the low-voltage winding with reference to the high-voltage side. Its value is

$$R_c = \frac{5m_c(kva)}{I_H^2}. \tag{5.4.3}$$

If X_λ is the leakage reactance of the high-voltage winding or of the low-voltage winding referred to the high-voltage side, then since it is the quadrature component of Z_λ its value is

$$X_\lambda = \sqrt{Z_\lambda^2 - R_c^2}.$$

The angle of the leakage impedance is

$$\alpha = \tan^{-1}\frac{X_\lambda}{R_c}.$$

However, since

$$\alpha = \cos^{-1}\frac{R_c}{Z_\lambda}, \tag{5.4.4}$$

it follows that the quadrature component of Z_λ may be calculated conveniently by

$$X_\lambda = Z_\lambda \sin\alpha. \tag{5.4.5}$$

The impedance of the series branch of the equivalent T of the transformer, therefore, is

$$Z_T = Z_\lambda \underline{/\alpha} = R_c + jX_\lambda. \tag{5.4.6}$$

To determine the shunt admittance Y_m refer to Fig. 5–6 and note that on no load

$$\frac{V_H}{I_{He}} = Z_{Hs} \tag{5.4.7}$$

in which the self-impedance Z_{Hs} on the high-voltage side is

$$Z_{Hs} = Z_\lambda + Z_m. \tag{5.4.8}$$

If m_e is the known value of the exciting current in per cent of the full-load current, then

$$I_{He} = \frac{m_e I_H}{100}. \tag{5.4.9}$$

This, substituted in (5.4.7), gives

$$Z_{Hs} = \frac{100 V_H}{m_e I_H}. \tag{5.4.10}$$

This gives the self-impedance Z_{Hs} of the transformer in per cent of the full-load impedance V_H/I_H as measured on the high-voltage side.

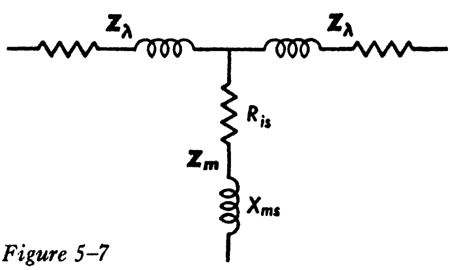


Figure 5–7

To determine the angle of Z_{Hs} it will be assumed that the components of the mutual impedance Z_m are in series instead of in parallel connection as they are usually considered. The series resistance equivalent of the iron loss, R_{is}, is thus in series with the resistance component R_c of the leakage impedance under no load condition and carries the exciting current, as shown in Fig. 5–7. It follows, therefore, that if m_i is the iron loss in per cent of the rated kva, then

$$\frac{m_i(kva)1000}{100} = I_{He}^2 R_{is}. \tag{5.4.11}$$

Hence,

$$R_{is} = \frac{10 m_i(kva)}{I_{He}^2}. \tag{5.4.12}$$

The total resistance component of the self-impedance Z_{Hs} is

$$R_{Hs} = R_c + R_{is}. \tag{5.4.13}$$

The angle of Z_{Hs} may now be calculated by

$$\theta_{Hs} = \cos^{-1} \frac{R_{Hs}}{Z_{Hs}}. \tag{5.4.14}$$

The value of the mutual impedance, by (5.4.8), is

$$Z_m \underline{/\theta_m} = Z_{Hs} \underline{/\theta_{Hs}} - Z_\lambda \underline{/\theta_\lambda}. \tag{5.4.15}$$

The mutual admittance is

$$Y_m = \frac{1}{Z_m} \underline{/-\theta_m}.$$

The shunt branch of the mutual admittance (in Fig. 5–6) representing the equivalent resistance of the iron loss with reference to the high side has a resistance

$$R_i = \frac{Z_m}{\cos\theta_m}. \tag{5.4.16}$$

The reactive branch of the mutual admittance is the mutual reactance with reference to the high side and has a value

$$X_m = \frac{Z_m}{\sin \theta_m}. \qquad (5.4.17)$$

The voltage across the shunt admittance is

$$V_e = I_{He} Z_m. \qquad (5.4.18)$$

The current through the resistive branch of Y_m is

$$I_i = \frac{V_e}{R_i}, \qquad (5.4.19)$$

and the magnetizing current I_ϕ through the mutual reactance is

$$I_\phi = \frac{V_e}{X_m}. \qquad (5.4.20)$$

5.5. *Calculation of the Equivalent T of Transformers from Given Data.* To illustrate the various relations obtained in the preceding articles, let it be required to obtain the equivalent T per transformer of a bank having a voltage rating of 132/13.8 kv, and joint capacity of 21,000 kva.

The exciting current is

$$m_e = 6.8\% \text{ of } I_H;$$

the iron loss is

$$m_i = 0.5\% \text{ of the kva};$$

the copper loss is

$$m_c = 0.85\% \text{ of the kva};$$

and the leakage impedance drop is

$$m_\lambda = 8.7\% \text{ of } V_H.$$

The voltage to neutral is

$$V_H = \frac{132}{\sqrt{3}} = 76.3 \text{ kv}.$$

The capacity per transformer is

$$\frac{21000}{3} = 7000 \text{ kva}.$$

By (5.4.1), the primary load current is

$$I_H = \frac{7000}{76.3} = 91.7 \text{ amperes}.$$

The leakage impedance of the high-voltage winding or of the low-voltage winding referred to the high side, by (5.4.2), is

$$Z_\lambda = \frac{8.7 \times 76300}{200 \times 91.8} = 36.2 \text{ ohms}.$$

The resistance of the high-voltage winding or of the low-voltage winding referred to the high side, by (5.4.3), is

$$R_c = \frac{5 \times 0.85 \times 7000}{91.7^2}$$
$$= 3.55 \text{ ohms.}$$

By (5.4.4)

$$\alpha = \cos^{-1} \frac{3.55}{36.2}$$
$$= 84.5°.$$

The leakage reactance of the high-voltage winding or of the low-voltage winding referred to the high side, by (5.4.5), is

$$X_\lambda = 36.2 \sin 84.5°$$
$$= 36.02 \text{ ohms.}$$

The impedance of the series branch of the equivalent T, therefore, is

$$Z_T = Z_\lambda$$
$$= 36.2\underline{/84.5°}$$
$$= 3.55 + j\,36.02 \text{ vector ohms.}$$

The exciting current, by (5.4.9), is

$$I_{He} = \frac{6.8 \times 91.7}{100}$$
$$= 6.24 \text{ amperes.}$$

The self-impedance at the high-voltage side, by (5.4.7), is

$$Z_{Hs} = \frac{76300}{6.24}$$
$$= 12230 \text{ ohms.}$$

The equivalent resistance of the iron loss assumed in series with R_c, by (5.4.12), is

$$R_{is} = \frac{10 \times 0.5 \times 7000}{6.24^2}$$
$$= 898.8 \text{ ohms.}$$

The total resistance of the self-impedance, therefore, is

$$R_{Hs} = 3.5 + 898.8$$
$$= 902.3 \text{ ohms.}$$

The angle of Z_{Hs} is

$$\theta_{Hs} = \cos^{-1} \frac{902.3}{12230}$$
$$= 85.8°.$$

The mutual impedance, by (5.4.8), is

$$Z_m\underline{/\theta_m} = 12230\underline{/85.8^\circ} - 36.2\underline{/84.5^\circ}$$
$$= 12190\underline{/85.7^\circ} \text{ vector ohms.}$$

Hence, the mutual admittance is

$$Y_m = 82 \times 10^{-6}\underline{/-85.7^\circ} \text{ vector mhos.}$$

The equivalent resistance R_i of the iron loss assumed in parallel with the mutual reactance and referred to the high-voltage side, by (5.4.16), is

$$R_i = \frac{12190}{\cos 85.7^\circ}$$
$$= 162530 \text{ ohms.}$$

The mutual reactance with reference to the high-voltage side, by (5.4.17), is

$$X_m = \frac{12190}{\sin 85.7^\circ}$$
$$= 12230 \text{ ohms.}$$

The voltage across the shunt admittance, by (5.4.18), is

$$V_e = 6.24 \times 12190$$
$$= 76100 \text{ volts.}$$

The equivalent T of the transformer is shown in Fig. 5–6.

The transmission constants of the transformer may be calculated by (5.2.3). Thus,

$$A = 1 + Z_T Y_T$$
$$= 1 + (36.2\underline{/84.5^\circ})(82 \times 10^{-6}\underline{/-85.7^\circ})$$
$$= 1 + 0.00296\underline{/-1.2^\circ}.$$
$$B = Z_T(2 + Z_T Y_T)$$
$$= 36.2\underline{/84.5^\circ}(2 + 0.00296\underline{/-1.2^\circ})$$
$$= 72.4\underline{/84.5^\circ} + 0.1075\underline{/83.3^\circ}$$
$$= 72.5\underline{/84.5^\circ} \text{ practically.}$$
$$D = Y_m$$
$$= 82 \times 10^{-6}\underline{/-85.7^\circ}.$$

The transmission equation of the transformer, therefore, is substantially

$$V_H = V_L + (72.5\underline{/84.5^\circ})I_L$$
$$I_H = I_L + (82 \times 10^{-6}\underline{/-85.7^\circ})V_L.$$

5.6. *Performance Formulas of Line with Transformers. Solution by Generalized Transmission Equations. Receiving-End Conditions Known.* Consider a line with station- and receiving-end transformers as indicated in Fig. 5–8.

To obtain the performance of the line including the two end transformers, the transformers are first converted into equivalent symmetrical T networks as discussed and illustrated in the preceding five articles. The system assumes schematically the

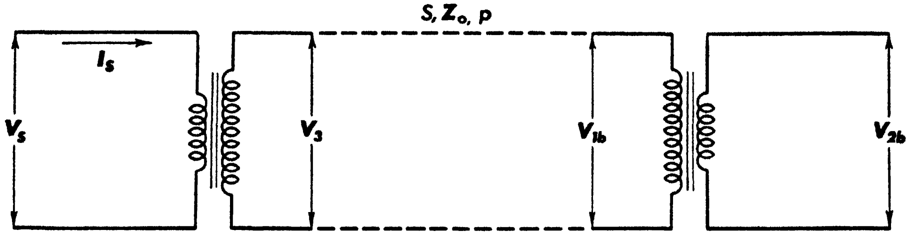


Figure 5–8

form shown in Fig. 5–8a in which Z_a, Y_a and Z_b, Y_b pertain to the sending and receiving-end transformers respectively. The line between the two transformers is defined completely by its length S miles, its characteristic impedance Z_o and its propagation constant p.

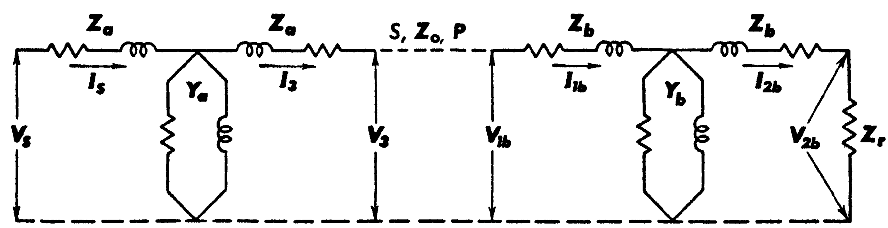


Figure 5–8a

Assuming the receiving-end voltage and current at the low-voltage side of transformer b known, and also the generalized transmission constants $\boldsymbol{A}_b$, $\boldsymbol{B}_b$, and $\boldsymbol{D}_b$ of this transformer referred to the high-voltage side, the voltage and current at the high-voltage side of this transformer are, by (5.2.9) and (5.2.10), respectively,

$$\boldsymbol{V}_{1b} = \boldsymbol{A}_b\boldsymbol{V}_{2b} + \boldsymbol{B}_b\boldsymbol{I}_{2b} \tag{5.6.1}$$

$$\boldsymbol{I}_{1b} = \boldsymbol{A}_b\boldsymbol{I}_{2b} + \boldsymbol{D}_b\boldsymbol{V}_{2b}, \tag{5.6.2}$$

where $\boldsymbol{A}_b$ is given by (5.2.6), $\boldsymbol{B}_b$ is given by (5.2.7), and $\boldsymbol{D}_b$ by (5.2.8). Referring to Fig. 5–9, it will be noted that $\boldsymbol{V}_{1b}$ and $\boldsymbol{I}_{1b}$ are the voltage and current, respectively, at the output terminals of the line. If $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{D}$ are the generalized transmission constants of the line, the voltage and current at the input terminals of the line are

$$V_3 = AV_{1b} + BI_{1b} \qquad (5.6.3)$$
$$I_3 = AI_{1b} + DV_{1b} \qquad (5.6.4)$$

where

$$A = \cosh pS$$
$$B = Z_o \sinh pS \qquad (5.6.5)$$
$$D = \frac{\sinh pS}{Z_o}.$$

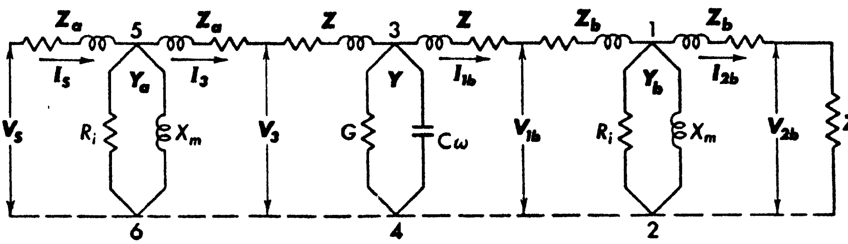


Figure 5–9

Referring again to the figure, it will be seen that V_s and I_s are the voltage and current, respectively, at the input terminals of the station-end transformer. The voltage and current at the low-voltage side of the transformer are, respectively,

$$V_s = A_aV_3 + B_aI_3 \qquad (5.6.6)$$
$$I_s = A_aI_3 + D_aV_3 \qquad (5.6.7)$$

where A_a, B_a, and D_a are the generalized transmission constants of the transformer at this end. The solution may be obtained in three steps, by solving for V_{1b} and I_{1b} using equations (5.6.1) and (5.6.2). Then solving for V_3 and I_3 using equations (5.6.3) and (5.6.4), and finally solving for V_s and I_s using equations (5.6.6) and (5.6.7). The three sets of equations may be also combined into a single set of the form

$$V_s = A_cV_{2b} + B_cI_{2b} \qquad (5.6.8)$$
$$I_s = A_cI_{2b} + D_cV_{2b} \qquad (5.6.9)$$

where A_c, B_c, and D_c are the combined transmission constants of line and transformers. Nothing is gained, however, in making this transformation. Although equations (5.6.8) and (5.6.9) appear simple, there is a good deal of labor involved in the determination of the general transmission constants of line and transformers.

5.7. ***Illustrative Problem of Line with Transformers; Receiving-End Conditions Known.*** Calculate the performance of a three-phase 300-mile line having a characteristic impedance $Z_o = 380\underline{/-9.5^\circ}$; a propagation constant $p = 2123 \times 10^{-6}\underline{/80.5^\circ}$. The transformer banks at the two ends of the line are identical, each transformer is rated at 7000 kva capacity, and 125/12.5 kv. The data of each transformer are:

Excitation current = 6.8% of the load current.
Iron losses = 0.5% of the kva.
Copper loss = 0.85% of the kva.
Leakage impedance drop = 8.7% of the high voltage.

The load current on the high side, by (5.4.1), is

$$I_H = \frac{7000}{125} = 56 \text{ amperes.}$$

The leakage impedance per winding, by (5.4.2), is

$$Z_\lambda = \frac{8.7 \times 125000}{200 \times 56} = 97.1 \text{ ohms.}$$

The resistance component of Z_λ, by (5.4.3), is

$$R_c = \frac{5 \times 0.85 \times 7000}{56^2} = 9.49 \text{ ohms.}$$

The angle of Z_λ, by (5.4.4), is

$$\alpha = \cos^{-1} \frac{9.49}{97.1} = 84.39°.$$

The leakage reactance, by (5.4.5), is

$$X_\lambda = 97.1 \sin 84.39° = 96.6 \text{ ohms.}$$

The leakage impedance may be written, therefore,

$$Z_\lambda = 97.1\underline{/84.39°} = 9.49 + j\,96.6.$$

The exciting current, by (5.4.9), is

$$I_{He} = \frac{6.8 \times 56}{100} = 3.81 \text{ amperes.}$$

The self-impedance, by (5.4.10), is

$$Z_{Hs} = \frac{125000}{3.81} = 32800 \text{ ohms.}$$

The series resistance equivalent of the iron loss, by (5.4.12), is

$$R_{is} = \frac{10 \times 0.5 \times 7000}{3.81^2} = 2410 \text{ ohms.}$$

The total resistance component of Z_{Hs}, by (5.4.13), is

$$R_{Hs} = 9.49 + 2410 = 2419.49.$$

The angle of Z_{Hs}, by (5.4.14), is

$$\begin{aligned} \theta_{Hs} &= \cos^{-1} \tfrac{2419}{32800} \\ &= 85.7^\circ. \end{aligned}$$

The mutual impedance, by (5.4.15), is

$$\begin{aligned} Z_m\underline{/\theta_m} &= 32800\underline{/85.7^\circ} - 97.1\underline{/84.39^\circ} \\ &= 32700\underline{/85.7^\circ} \text{ substantially.} \end{aligned}$$

The equivalent mutual admittance of the transformer is

$$Y_m = 30.58 \times 10^{-6}\underline{/-85.7^\circ}.$$

The transmission constants of the transformer are

$$\begin{aligned} A_{1a} = A_{1b} &= 1 + (97.1\underline{/84.39^\circ})(30.58 \times 10^{-6}\underline{/-85.7^\circ}) \\ &= 1 + 0.00297\underline{/-1.31^\circ} \\ &= 1.0\underline{/0^\circ}, \text{ substantially.} \\ B_{1a} = B_{1b} &= 2 \times 97.1\underline{/84.39^\circ} = 194.2\underline{/84.39^\circ}. \\ D_{1a} = D_{1b} &= 30.58 \times 10^{-6}\underline{/-85.7^\circ}. \end{aligned}$$

For a load of 7000 kva at 90.6 per cent power factor lag and 12.5 kv at the low-voltage side, the load current is $560\underline{/-25^\circ}$ vector amperes. With reference to the high-voltage side, the receiving-end voltage and current are

$$\begin{aligned} V_{2b} &= 125000\underline{/0^\circ} \\ I_{2b} &= 56\underline{/-25^\circ}. \end{aligned}$$

The transmission equations of the transformer, accordingly, by (5.2.1) and (5.2.2), are

$$\begin{aligned} V_{1b} &= 125000 + (194.2\underline{/84.39^\circ})(56\underline{/-25^\circ}). \\ I_{1b} &= 56\underline{/-25^\circ} + (30.58 \times 10^{-6}\underline{/-85.7^\circ})(125000\underline{/0^\circ}). \end{aligned}$$

This gives for the voltage on the high side of the receiving-end transformer

$$V_{1b} = 131000\underline{/4.1^\circ} \text{ vector volts.}$$

The angle is with reference to V_{2b}. The current at the high side of the transformer is

$$I_{1b} = 57.9\underline{/-28.3^\circ} \text{ vector amperes.}$$

This current lags the load voltage by 28.3° and the voltage on the high side by 32.4°. Referring to Fig. 5–8a, it will be seen that V_{1b} and I_{1b}, as calculated, are the voltage and current, respectively, at the input terminals of transformer b and also the voltage and current at the output terminals of the line proper.

The transmission equations for the line are given by equations (5.6.3) and (5.6.4). Assuming the values of the transmission constants of the line under consideration*

$$A = 0.816\underline{/4.35^\circ}$$
$$B = 227.2\underline{/72.3^\circ}$$
$$D = 15.73 \times 10^{-4}\underline{/90^\circ}.$$

The equation for the voltage and current at the sending end of the line, accordingly, are

$$V_3 = (0.816\underline{/4.35^\circ})(131000\underline{/4.1^\circ}) + (227.2\underline{/72.3^\circ})(57.9\underline{/-28.3^\circ})$$
$$I_3 = (0.816\underline{/4.35^\circ})(57.9\underline{/-28.3^\circ}) + (15.73 \times 10^{-4}\underline{/90^\circ})(131000\underline{/4.1^\circ}).$$

Multiplying out, these expressions become, respectively,

$$V_3 = 106890\underline{/8.45^\circ} + 13154\underline{/44.0^\circ}$$
$$I_3 = 47.2\underline{/-23.95^\circ} - 206.5\underline{/-85.9^\circ}.$$

Carrying out the calculations, yields

$$V_3 = 118000\underline{/12.17^\circ} \text{ vector volts}$$

and

$$I_3 = 188.7\underline{/81.3^\circ} \text{ vector amperes.}$$

The angles are with reference to V_{2b}. The current I_3 entering the line leads the voltage V_3 by an angle $81.3^\circ - 12.17^\circ = 69.13^\circ$.

The transmission equations for the voltage and current at the low-voltage side of the station-end transformer referred to the high-voltage side are

$$V_s = 118000\underline{/12.17^\circ} + (194.2\underline{/84.39^\circ})(188.7\underline{/81.3^\circ})$$
$$I_s = 188.7\underline{/81.3^\circ} + (30.58 \times 10^{-6}\underline{/-85.7^\circ})(118000\underline{/12.17^\circ}).$$

Carrying out the calculations involved, yields

$$V_s = 86500\underline{/23.2^\circ} \text{ vector volts}$$
$$I_s = 183\underline{/80.7^\circ} \text{ vector amperes.}$$

The angles are with reference to the receiving-end voltage V_{2b}. The station-end current leads the voltage at the station end by an angle $80.7^\circ - 23.2^\circ = 57.5^\circ$. The actual voltage and current at the low-voltage side per phase at the station end are 8650 volts and 1830 amperes, respectively.

The dissipative power per phase at the receiving end is

$$P_r = 7000 \cos 25^\circ = 6350 \text{ kilowatts.}$$

* See Art. 4.2.

The kva supplied at the station end is

$$\begin{aligned}(kva)_s &= 8.65 \times 1830 \\ &= 15820 \text{ kilovolt-amperes.}\end{aligned}$$

The dissipative power supplied per phase is

$$\begin{aligned}P_s &= 15820 \cos 57.5° \\ &= 8500 \text{ kilowatts.}\end{aligned}$$

The overall efficiency of line and transformers is

$$\begin{aligned}\eta &= \tfrac{6350}{8500} \\ &= 74.7 \text{ per cent.}\end{aligned}$$

5.8. *The Complex Circuit of Line with Transformers.* If the line in Fig. 5–8 is also converted into an equivalent T, the schematic diagram of line and transformers assumes the form shown in Fig. 5–9. If the line is long, the series impedance and the shunt admittance of the equivalent T of the line are calculated by (4.2.10) and (4.2.6), respectively,

$$\left.\begin{aligned}Z &= \frac{Z_o(\cosh pS - 1)}{\sinh pS} = Z_o \tanh \frac{pS}{2} \\ Y &= \frac{\sinh pS}{Z_o}.\end{aligned}\right\} \tag{5.8.1}$$

If the line is between 25 and 100 miles long, then

$$\left.\begin{aligned}Z &= \frac{(R + jL\omega)S}{2} \\ Y &= jC\omega.\end{aligned}\right\} \tag{5.8.2}$$

For lines shorter than 25 miles,

$$\left.\begin{aligned}Z &= (R + jL\omega)S \\ Y &= 0.\end{aligned}\right\} \tag{5.8.3}$$

The series impedance Z_a of the transformer calculated from known data, as discussed in § 5.4, is the leakage impedance per winding referred to the high-voltage side (equation (5.4.6)),

$$Z_a = Z_\lambda\underline{/\alpha} = R_c + jX_\lambda. \tag{5.8.4}$$

The shunt admittance of the transformer is the reciprocal of the mutual impedance, as calculated in § 5.4 (equation (5.4.15)). The Z_r at the receiving end is the equivalent impedance of the load per phase referred to the high-voltage side, i.e., the actual value multiplied by the square of the ratio of transformation.

Referring to Fig. 5-9 it will be noted that the admittance across junctions 1 and 2 is

$$Y_{12} = \frac{1}{Z_r + Z_b} + Y_b. \tag{5.8.5}$$

The impedance to the right of junctions 3 and 4, and not including the admittance Y, is

$$(Z_{34})_r = Z_b + Z + \frac{1}{Y_{12}}. \tag{5.8.6}$$

The admittance across junction 3–4 including the admittance Y is

$$Y_{34} = \frac{1}{(Z_{34})_r} + Y. \tag{5.8.7}$$

The impedance $(Z_{56})_r$ to the right of junction 5–6 not including the admittance Y_a is

$$(Z_{56})_r = Z + Z_a + \frac{1}{Y_{34}}. \tag{5.8.8}$$

The admittance across junction 5–6 including the admittance Y_a is

$$Y_{56} = \frac{1}{(Z_{56})_r} + Y_a. \tag{5.8.9}$$

The total impedance at the sending end of the entire circuit is

$$Z_s = Z_a + \frac{1}{Y_{56}}. \tag{5.8.10}$$

If the voltage at the station end is known, the joint performance of line and transformers may be calculated as is illustrated in the next article.

The formulas given above may be combined into a single compact formula as shown below.

Substituting the value of Y_{56}, from (5.8.9) in (5.8.10), gives

$$Z_s = Z_a + \cfrac{1}{Y_a + \cfrac{1}{(Z_{56})_r}}. \tag{5.8.11}$$

Again, substituting the value of $(Z_{56})_r$ yields

$$Z_s = Z_a + \cfrac{1}{Y_a + \cfrac{1}{Z + Z_a + \cfrac{1}{Y_{34}}}}. \tag{5.8.12}$$

Replacing Y_{34} by its value in (5.8.7) gives

$$Z_s = Z_a + \cfrac{1}{Y_a + \cfrac{1}{Z + Z_a + \cfrac{1}{Y + \cfrac{1}{(Z_{34})_r}}}}. \tag{5.8.13}$$

Similarly, replacing $(Z_{34})_r$, from (5.8.6), gives

$$Z_s = Z_a + \cfrac{1}{Y_a + \cfrac{1}{Z + Z_a + \cfrac{1}{Y + \cfrac{1}{Z_b + Z + \cfrac{1}{Y_{12}}}}}}. \tag{5.8.14}$$

Finally, replacing Y_{12} by its value given in (5.8.5) yields

$$Z_s = Z_a + \cfrac{1}{Y_a + \cfrac{1}{Z + Z_a + \cfrac{1}{Y + \cfrac{1}{Z_b + Z + \cfrac{1}{Y_b + \cfrac{1}{Z_b + Z_r}}}}}}. \tag{5.8.15}$$

This will be recognized as a "terminating continuous fraction," and lends itself very conveniently to the solution of the circuit.*

5.9. *Illustrative Problem of Line with Transformers. Receiving-End Impedance and Station-End Voltage Known.* Calculate the sending-end impedance of the line whose equivalent T per phase as calculated in § 4.2 consists of $Z_T = 125.3\underline{/69.75^\circ}$ and $Y_T = 15.73 \times 10^{-4}\underline{/90^\circ}$. The line is terminated in transformers whose equivalent T as calculated in § 5.7 consists of $Z_\lambda = 97.1\underline{/84.39^\circ}$ and $Y_m = 30.58 \times 10^{-6}\underline{/-85.7^\circ}$. Assume that the receiving-end impedance equivalent to the load per phase is $Z_r = 2230\underline{/25^\circ}$.

To obtain the sending-end impedance Z_s to neutral, refer to equation (5.8.15) and note that

$$\begin{aligned} Z_b + Z_r &= 97.1\underline{/84.39^\circ} + 2230\underline{/25^\circ} \\ &= 2280\underline{/27.1^\circ}. \\ Z_b + Z &= 97.1\underline{/84.39^\circ} + 125.3\underline{/69.75^\circ} \\ &= 220.5/76.13^\circ. \end{aligned}$$

* Bartlett, A. C., *The Theory of Electrical Artificial Lines and Filters*, Wiley and Sons, p. 41.

Substituting in (5.8.15) gives

$$Z_s = 97.1\underline{/84.39°} + \cfrac{1}{30.58 \times 10^{-6}\underline{/-85.7°} + \cfrac{1}{\frac{1}{220.5\underline{/76.13°}} + \cfrac{1}{\frac{1}{15.73 \times 10^{-4}\underline{/90°}} + \cfrac{1}{\frac{1}{220.5\underline{/76.13°}} + \cfrac{1}{30.58 \times 10^{-6}\underline{/-85.7°} + \frac{1}{2280\underline{/27.1°}}}}}}}$$

$Z_s = 465\underline{/-57.3°}$ vector ohms.

Assuming that the station-end voltage referred to the high side of the transformer is 85 kv per phase, the current at that end is

$$I_s = \frac{85000}{465\underline{/-57.3°}}$$
$$= 182.7\underline{/57.3°}.$$

The kva input per phase is

$$(kva)_s = 85 \times 182.7 = 15520 \text{ kilovolt amperes.}$$

The dissipative power per phase supplied to the station-end transformer is

$$P_s = 15520 \cos 57.3°$$
$$= 8390 \text{ kilowatts.}$$

The receiving-end voltage, current, and dissipative power may be calculated from the known values of V_s, I_s and the circuit diagram, Fig. 5–9. Thus, the voltage across junction 5–6 is

$$V_{56} = V_s - I_s Z_a.$$

The current between junctions 5 and 3 is

$$I_{53} = \frac{V_{56}}{(Z_{56})_r} = I_3$$

where $(Z_{56})_r$ is given by (5.8.8).

The voltage across junctions 3 and 4 is

$$V_{34} = V_{56} - I_{53}(Z_a + Z).$$

The current between junctions 3 and 1 is

$$I_{31} = \frac{V_{34}}{(Z_{34})_r} = I_{1b}$$

where $(Z_{34})_r$ is given by (5.8.6).

The voltage across the junctions 1 and 2 is

$$V_{12} = V_{34} - I_{31}(Z + Z_b).$$

The load current is

$$I_r = \frac{V_{12}}{Z_b + Z_r},$$

and the voltage across the load is

$$V_r = I_r Z_r = V_{12} - I_r Z_b.$$

All values are, obviously, referred to the high voltage side of the transformer.

Another method of calculating receiving-end values is by means of the transmission equations in which station-end values are known. Thus, referring to Fig. 5–9, the equations

$$\left.\begin{aligned} V_3 &= A_a V_s - B_a I_s \\ \text{and} \qquad I_3 &= A_a I_s - D_a V_s \end{aligned}\right\} \qquad (5.9.1)$$

give the voltage and current, respectively, at the output terminals of the station-end transformer, and which are also the respective values at the input terminals of the line proper.

The equations

$$\left.\begin{aligned} V_{1b} &= A V_3 - B I_3 \\ \text{and} \qquad I_{1b} &= A I_3 - D V_3 \end{aligned}\right\} \qquad (5.9.2)$$

give the voltage and current at the output terminals of the line proper, and at the input terminals of the transformer at the receiving end.

Finally, equations

$$\begin{aligned} V_{2b} &= A_b V_{1b} - B_b I_{1b} \\ \text{and} \qquad I_{2b} &= A_b V_{1b} - D_b V_{1b} \end{aligned} \qquad (5.9.3)$$

give the voltage and current at the receiving end. All quantities are, of course, referred to the high side of the transformer.

5.10. *Transformers in Parallel.* The generalized transmission formulas of transformers given in (§ 5.2) with the values of the constants given by (5.2.6), (5.2.7), and (5.2.8) afford a convenient method of obtaining the performance of transformers connected in parallel. Thus, consider two transformers a and b connected in parallel as shown in Fig. 5–10, and whose symmetrical T equivalent circuit is shown in Fig. 5–11. Let A_a, B_a, and D_a be transmission constants of transformer a,

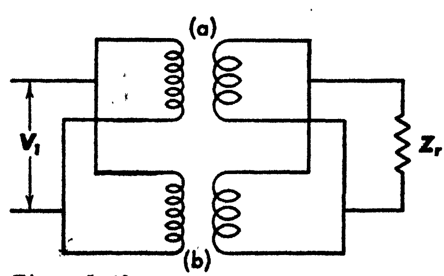


Figure 5–10

and A_b, B_b, and D_b be the transmission constants of transformer b. The transmission formulas of the two transformers, under the assumption that

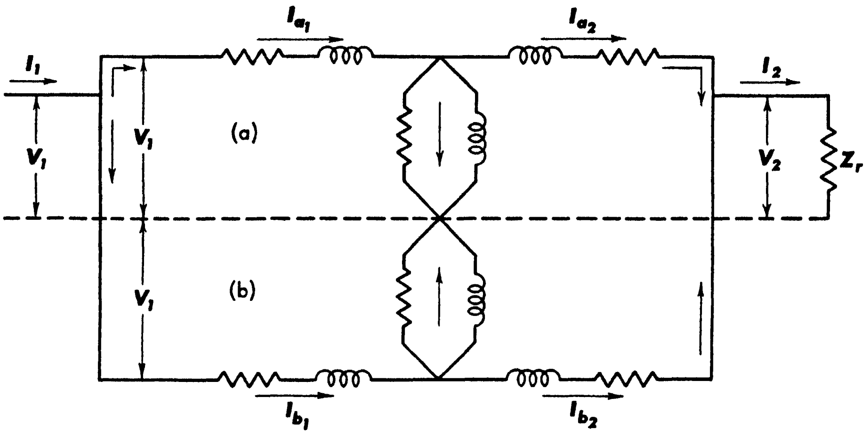


Figure 5–11

the receiving-end values with reference to the high-voltage side are known, are

$$V_1 = A_a V_2 + B_a I_{a2} \tag{5.10.1}$$

and

$$I_{a1} = A_a I_{a2} + D_a V_2 \tag{5.10.2}$$

for transformer a and

$$V_1 = A_b V_2 + B_b I_{b2} \tag{5.10.3}$$

and

$$I_{b1} = A_b I_{b2} + D_b V_2. \tag{5.10.4}$$

In these formulas

$$A_a = 1 + (Z_\lambda Y_m)_a; \qquad A_b = 1 + (Z_\lambda Y_m)_b;$$
$$B_a = Z_{\lambda a}(2 + Z_\lambda Y_m)_a; \qquad B_b = Z_{\lambda b}(2 + Z_\lambda Y_m)_b;$$
$$D_a = Y_{ma}; \qquad D_b = Y_{mb}.$$

The total secondary current is

$$I_2 = I_{a2} + I_{b2}, \tag{5.10.5}$$

and the total primary current is

$$I_1 = I_{a1} + I_{b1}. \tag{5.10.6}$$

Substituting (5.10.5) in (5.10.3), equating to (5.10.1), and solving for I_{a2} yields

$$I_{a2} = \left(\frac{A_b - A_a}{B_a + B_b}\right) V_2 + \left(\frac{B_b}{B_a + B_b}\right) I_2. \tag{5.10.7}$$

Using this value of I_{a2} in (5.10.1) and simplifying, gives

$$V_1 = \left(\frac{A_a B_b + A_b B_a}{B_a + B_b}\right) V_2 + \left(\frac{B_a B_b}{B_a + B_b}\right) I_2. \tag{5.10.8}$$

To get an expression for the current I_{a1} in the primary of the transformer a, substitute (5.10.7) in (5.10.2). This gives

$$I_{a1} = \left(\frac{A_a B_b}{B_a + B_b}\right) I_2 + \left(\frac{A_a A_b - A_a^2}{B_a + B_b} + D_a\right) V_2. \tag{5.10.9}$$

The current I_{b1} in the primary of transformer b, obtained in a similar manner, is

$$I_{b1} = \left(\frac{A_b B_a}{B_a + B_b}\right) I_2 + \left(\frac{A_a A_b - A_b^2}{B_a + B_b} + D_b\right) V_2. \tag{5.10.10}$$

The total primary current, by (5.10.6), is

$$I_1 = \left(\frac{A_a B_b + A_b B_a}{B_a + B_b}\right) I_2 - \left(\frac{(A_a - A_b)^2}{B_a + B_b} - D_a - D_b\right) V_2. \tag{5.10.11}$$

If the two transformers connected in parallel are identical, i.e., is $A_a = A_b = A$; $B_a = B_b = B$, and $D_a = D_b = D$, formulas (5.10.8) for V_1 and (5.10.11) for I_1 become, respectively,

$$V_1 = AV_2 + \left(\frac{B}{2}\right) I_2 \tag{5.10.12}$$

$$I_1 = AI_2 + 2DV_2. \tag{5.10.13}$$

In terms of primary end values, these formulas are

$$V_2 = AV_1 - \left(\frac{B}{2}\right) I_1 \tag{5.10.14}$$

$$I_2 = AI_1 - 2DV_1. \tag{5.10.15}$$

The last two sets of formulas indicate that, as expected, each transformer carries one-half of the load.

5.11. *Lines in Parallel.*

The formulas developed in the preceding article for two transformers in parallel operation, in terms of the equivalent T's of each of the transformers, hold also for two lines a and b operated in parallel. In this case, however, the constants in the transmission equations are

For line (a)	For line (b)
$A_a = \cosh (pS)_a$;	$A_b = \cosh (pS)_b$;
$B_a = Z_{oa} \sinh (pS)_a$;	$B_b = Z_{ob} \sinh (pS)_b$;
$D_a = \dfrac{\sinh (pS)_a}{Z_{oa}}$;	$D_b = \dfrac{\sinh (pS)_b}{Z_{ob}}$.

If the two lines are identical in characteristic and length, then

$$A_a = A_b = A = \cosh pS$$
$$B_a = B_b = B = Z_o \sinh pS$$
$$D_a = D_b = D = \frac{\sinh pS}{Z_o}.$$

Under this particular condition formulas (5.10.12) to (5.10.15) inclusive, indicate, as is expected, that each line carries one-half the total current.

5.12. Twin Three-Phase Lines with End Transformers. Consider Fig. 5–12, showing one phase of a twin three-phase system in which the lines including the end transformers are connected in parallel. In terms of equivalent symmetrical T's of transformers and line, the twin system is shown schematically in Fig. 5–13.

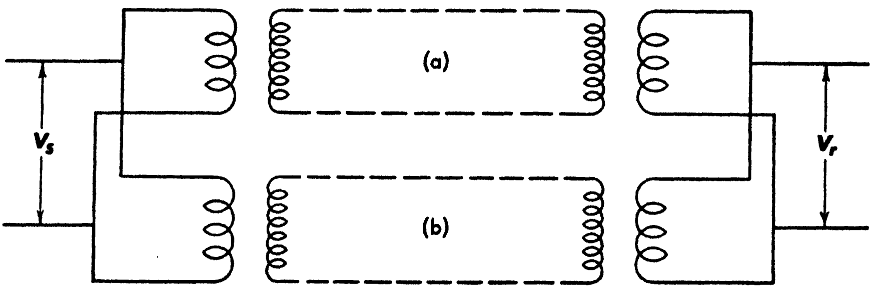


Figure 5–12

Let $A_{1a}B_{1a}D_{1a}$, $A_{2a}B_{2a}D_{2a}$, and $A_{3a}B_{3a}D_{3a}$ represent the transmission constants of the receiving-end transformer, line and station-end transformer, respectively, of line a.

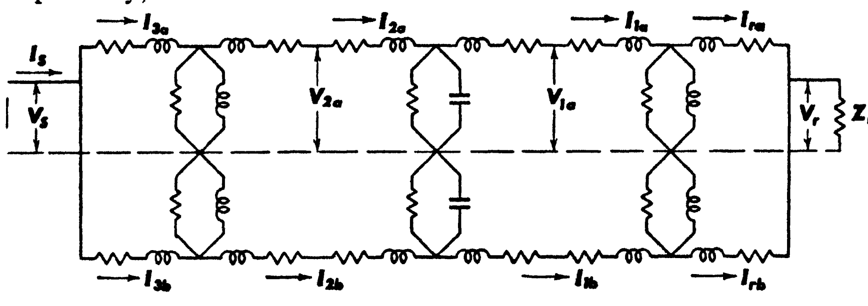


Figure 5–13

Referring to Fig. 5–13, the transmission equations of line a, are

$$V_{1a} = (A_1V_r)_a + (B_1I_r)_a \tag{5.12.1}$$

$$I_{1a} = (A_1I_r)_a + (D_1V_r)_a \tag{5.12.2}$$

$$V_{2a} = (A_2V_1)_a + (B_2I_1)_a \tag{5.12.3}$$

$$I_{2a} = (A_2I_1)_a + (D_2V_1)_a \tag{5.12.4}$$

$$V_s = (A_3V_2)_a + (B_3I_2)_a \tag{5.12.5}$$

$$I_{3a} = (A_3I_2)_a + (D_3V_2)_a. \tag{5.12.6}$$

Substituting (5.12.1) and (5.12.2) in (5.12.3) and (5.12.4), respectively, yields

$$V_{2a} = (A_4V_r)_a + (B_4I_r)_a \tag{5.12.7}$$
$$I_{2a} = (A_5I_r)_a + (D_5V_r)_a, \tag{5.12.8}$$

where

$$\left.\begin{aligned} A_{4a} &= (A_1A_2)_a + (B_2D_1)_a \\ B_{4a} &= (A_2B_1)_a + (B_2A_1)_a \\ A_{5a} &= (A_1A_2)_a + (B_1D_2)_a \\ D_{5a} &= (A_2D_1)_a + (D_2A_1)_a. \end{aligned}\right. \tag{5.12.9}$$

Substituting (5.12.7) and (5.12.8) in (5.12.5) and (5.12.6), respectively, gives

$$V_s = (A_6V_r)_a + (B_6I_r)_a \tag{5.12.10}$$

and

$$I_{3a} = (A_7I_r)_a + (D_7V_r)_a \tag{5.12.11}$$

where

$$\left.\begin{aligned} A_{6a} &= (A_3A_4)_a + (B_3D_5)_a \\ B_{6a} &= (A_3B_4)_a + (A_5B_3)_a \\ A_{7a} &= (A_3A_5)_a + (D_3B_4)_a \\ D_{7a} &= (A_3D_5)_a + (D_3A_4)_a. \end{aligned}\right\} \tag{5.12.12}$$

Similarly, if $A_{1b}B_{1b}D_{1b}$, $A_{2b}B_{2b}D_{2b}$, $A_{3b}B_{3b}D_{3b}$ represent the transmission constants of the receiving-end transformers, line and station-end transformers of line b, the station-end voltage V_s and station-end current I_{3b} for this line would be

$$V_s = (A_6V_r)_b + (B_6I_r)_b \tag{5.12.13}$$
$$I_{3b} = (A_7I_r)_b + (D_7V_r)_b. \tag{5.12.14}$$

Since the two lines with their respective transformers are in parallel, the station-end voltage V_s in equations (5.12.10) and (5.12.13) is the same, and the V_r in the two expressions are also the same. The total current supplied to the two lines is

$$I_s = I_{3a} + I_{3b}, \tag{5.12.15}$$

and the joint current delivered to the load is

$$I_r = I_{ra} + I_{rb}. \tag{5.12.16}$$

Using the method developed in § 5.10 for transformers in parallel, equations (5.12.10) and (5.12.11) and (5.12.13) and (5.12.14) may be combined to give

$$\begin{aligned} V_s &= \left(\frac{A_{6a}B_{6b} + A_{6b}B_{6a}}{B_{6a} + B_{6b}}\right)V_r + \left(\frac{B_{6a}B_{6b}}{B_{6a} + B_{6b}}\right)I_r \\ I_{3a} &= \left(\frac{A_{7a}B_{6b}}{B_{6a} + B_{6b}}\right)I_r + \left(\frac{A_{7a}(A_{6b} - A_{6a})}{B_{6a} + B_{6b}} + D_{7a}\right)V_r \\ I_{3b} &= \left(\frac{A_{7b}B_{6a}}{B_{6a} + B_{6b}}\right)I_r + \left(\frac{A_{7b}(A_{6a} - A_{6b})}{B_{6a} + B_{6b}} + D_{7b}\right)V_r. \end{aligned} \tag{5.12.17}$$

The current supplied to the two lines operated in parallel, by (5.12.15), is

$$I_s = \left(\frac{A_{7a}B_{6b} + A_{7b}B_{6a}}{B_{6a} + B_{6b}}\right) I_r + \left(\frac{(A_{7a} - A_{7b})(A_{6b} - A_{6a})}{B_{6a} + B_{6b}} + D_{7a} + D_{7b}\right) V_r. \qquad (5.12.18)$$

If the two lines including their transformers are identical in every respect, then $A_{6a} = A_{6b} = A_6$; $A_{7a} = A_{7b} = A_7$; $B_{6a} = B_{6b} = B_6$; and $D_{7a} = D_{7b} = D_7$. Under this particular condition, equations (5.12.17) and (5.12.18) become, respectively,

$$V_s = A_6 V_r + \frac{B_6}{2} I_r \qquad (5.12.19)$$

and

$$I_s = A_7 I_r + 2D_7 V_r, \qquad (5.12.20)$$

indicating that the two lines share the load equally.

REFERENCES

Bradfield and John, *Telephone and Power Transmission*, Wiley & Sons, 1929.

Dahl, O. G. C., *Electric Circuits*, Vol. I, McGraw-Hill Book Co., 1928.

Evans, R. D., and Sels, H. K., *Transmission Lines and Transformers*, Elect. Jour., Aug. 1921.

SUGGESTIVE PROBLEMS *Chapter 5*

1. Determine the equivalent T of a transformer rated 66.5/12.7 kv; 7000 kw capacity. Assume: copper loss 0.75% of rated kva; iron loss 0.32% of rated kva; exciting current 4.2% of full-load current; and leakage impedance drop 7.4% of rated voltage.

2. Calculate the performance of the line stated in Prob. 4, Chap. 3, with station and receiving-end transformers whose data are given in Prob. 1 under the assumption the line is 120 miles long and V_r to neutral is 12.7 kv. The load is assumed to be 21000 kva at 90% power factor.

3. Calculate the equivalent transmission constants of two identical parallel-connected transformers whose data are given in Prob. 1.

4. Calculate the equivalent transmission constants of a twin three-phase line whose data is specified in Prob. 4, Chap. 3.

5. Two three-phase lines whose data are specified in Prob. 4, Chap. 3, are connected in parallel and terminated with parallel-connected end transformers as shown in Fig. 5–12. The transformer data are specified in Prob. 1. Obtain the transmission equations of the transmitting system. Assume $S = 120$ miles.

Chapter 6 Voltage Control of Transmission Systems

6.1. *General Considerations.* One of the major problems in the operation of constant potential systems is that of maintaining the receiving-end voltage constant under wide variations of load. Unless controlled by some appropriate method the voltage at the receiving end of a transmission system will change with changes in load. The maintenance of a constant receiving-end voltage for short lines is accomplished by the automatic adjustment of the station-end voltage through the excitation equipment of the generating system. Line-drop compensators are frequently used with automatic voltage regulators to hold the substation bus at or as close as possible to the prescribed necessary receiving-end voltage. Transformers with multiple-ratio taps, which permit automatic changing under load conditions, are also used with short lines. It is worthwhile to note that these methods of voltage control do not eliminate the voltage change but compensate for it at the station end. Although these methods are usually satisfactory for the control of the receiving-end voltage of short lines, they are not applicable to long transmission systems because of the rather large voltage regulation of such systems.

Voltage control is also accomplished, particularly for long lines, by means of synchronous reactors installed at the receiving end. Their function is to maintain the voltage at the receiving end and at the station end constant by adjusting the reactive power of the reactor either in quantity or in character or both. The voltage at the termini of the transmission system is thus maintained constant by changing the power factor either in value or in character (lead or lag) or both. For this reason the synchronous reactors used for this specific purpose are frequently referred to as *phase modifiers*, and the method as *phase control.*

The excitation of the synchronous reactors is automatically controlled by voltage regulators actuated by the load changes.

In constant-voltage transmission systems the station-end voltage is also maintained at a fixed predetermined value by the automatic adjustment of the generator excitation by means of Tirrill regulators.

With the receiving-end voltage maintained constant by synchronous reactors, and the station-end voltage maintained constant by Tirrill regulators, the voltage at other points along the line will vary with changes in load. When the line is long, the voltages at such points may reach dangerously high values and impose damaging stresses on the insulators. To avoid this, it is advisable to have synchronous reactors installed at other points along the line.

6.2. *Leading and Lagging Reactive Power.* Before entering into a discussion of voltage control of lines by phase modifiers, it is necessary to review the mathematical definitions of leading and lagging reactive powers particularly with reference to the sign, plus or minus, that should designate each.

In terms of its components, an inductive impedance is

$$Z_L = R + jL\omega$$

or

$$Z_L = (R^2 + L^2\omega^2)^{\frac{1}{2}} \underline{/\tan^{-1}\frac{L\omega}{R}} = Z_L\underline{/\theta_L}. \qquad (6.2.1)$$

Hence, an inductive impedance is associated with a positive angle. Similarly, a capacitive impedance, in terms of its components, is

$$Z_c = R - \frac{j}{C\omega}$$

or

$$Z_c = \left(R^2 + \frac{1}{C^2\omega^2}\right)^{\frac{1}{2}} \underline{/\tan^{-1}\frac{-1}{C\omega R}} = Z\underline{/-\theta_c}. \qquad (6.2.2)$$

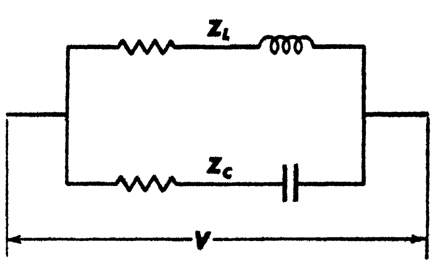


Figure 6–1

A capacitive impedance is thus associated with a negative angle.

Consider now the simple circuit shown in Fig. 6–1, consisting of an inductive branch of impedance $Z_L\underline{/\theta_L}$ vector ohms, and a capacitive branch of impedance $Z_c\underline{/-\theta_c}$ vector ohms. The circuit is simulating a constant voltage system with an inductive load in parallel with a capacitive load. The current through the inductive load is

$$\frac{V}{Z_L\underline{/\theta_L}} = I_L\underline{/-\theta_L}.$$

The voltamperes supplied to this inductive load is

$$\frac{V^2}{Z_L\underline{/\theta_L}} = VI_L\underline{/-\theta_L}.$$

This may be written in complex form

$$\frac{V^2}{Z_L\underline{/\theta_L}} = VI_L\cos\theta_L - jVI_L\sin\theta_L \tag{6.2.3}$$

$$= P_L - jQ_L. \tag{6.2.4}$$

It shows that the reactive power in an inductive load on a constant potential circuit is negative.

The current in the capacitive load, under the same condition of voltage, is

$$\frac{V}{Z_c\underline{/-\theta_c}} = I_c\underline{/\theta_c}.$$

The voltamperes supplied to the capacitive load is

$$\frac{V^2}{Z_c\underline{/-\theta_c}} = VI_c\underline{/\theta_c}$$

or

$$\frac{V^2}{Z_c\underline{/-\theta_c}} = VI_c\cos\theta_c + jVI_c\sin\theta_c \tag{6.2.5}$$

$$= P_c + jQ_c. \tag{6.2.6}$$

This last equation indicates that the reactive power in a capacitive load on a constant potential system is positive. From what has been said above, it follows that positive reactive power corresponds to a leading current and to a negative impedance angle and pertains to a capacitive circuit. Negative reactive power, on the other hand, corresponds to a lagging current, a positive impedance angle and pertains to an inductive circuit.

The positive and negative signs attached to the symbol Q used for reactive power of a synchronous reactor do *not* indicate circuit direction in the sense that when the reactor is underexcited it receives reactive energy and when overexcited it supplies reactive energy. Whether positive or negative, the reactive power of a synchronous reactor is supplied by the generating equipment of the system. The positive sign indicates that the current leads the voltage in time phase, and the negative sign indicates that the current lags the voltage in time phase. If an overexcited synchronous reactor, however, is connected in parallel with an inductive load, the lagging component of the load current, and the leading current of the synchronous reactor are in opposite time phase. This means, in effect, that during the interval that the inductive load receives its reactive power, the synchronous reactor delivers it. There is thus a total or a partial exchange of reactive energy between the synchronous reactor and the load. It should not be forgotten, however, that the exchange of the reactive energy is through the agency of the generating equipment.

6.3. *Reactive Power for Voltage Control.* Consider the simple case of a short line for which the linear line admittance $(G + jC\omega)$ is negligibly small. The transmission equations in such a case are

$$V_s = V_r + I_r Z \qquad (6.3.1)$$
$$I_s = I_r,$$

where V_s and V_r are, respectively, the station-end and receiving-end voltages per phase and $Z = zS$ is the line impedance per conductor. Taking V_r as the reference vector, the above voltage equation may be written

$$V_s\underline{/\delta_s} = V_r\underline{/0} + I_r\underline{/\theta_r}Z\underline{/\zeta}.$$

Writing the right-hand member of this expression in the complex form gives

$$V_s\underline{/\delta_s} = V_r + (I_p + jI_q)(R + jX), \qquad (6.3.2)$$

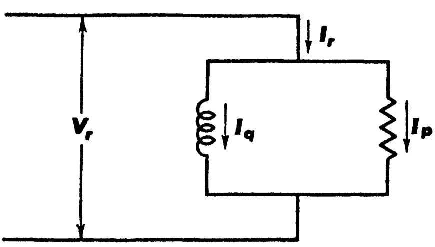


Figure 6–2

in which

$$I_p = I_r \cos\theta_r$$
$$I_q = I_r \sin\theta_r. \qquad (6.3.3)$$

The quantities R and X are the resistance and reactance, respectively, per conductor at the operating frequency of the supply. The system to which the above equations pertain is shown schematically in Fig. 6–2. Equation (6.3.2) may be written

$$V_s\underline{/\delta_s} = V_r + I_pR - I_qX + j(I_pX + I_qR). \qquad (6.3.4)$$

The value of I_p depends upon the required dissipative power at the receiving end. The quantities R and X are line constants. Hence the only two quantities that could be adjusted to maintain V_r constant are V_s and I_q. Methods of voltage control by adjustment of the station voltage V_s were mentioned in § 6.1. Such methods are imperative when the quadrature component I_q of the load current I_r is left uncontrolled.

The value and sign of I_q may be controlled, however, by means of synchronous reactors connected across the line in parallel with the load, as indicated in Fig. 6–3. The excitation of the synchronous reactor is adjusted to normal for a certain predetermined value of the receiving-end load, for which V_r is normal. When the load decreases, the receiving-end voltage V_r tends to increase. The synchronous reactor automatically adjusts its excitation to a lower value. It thus demands and takes from the line a larger and nearly 90° lagging current. This adds directly to the quadrature component I_{ql} of the load current.

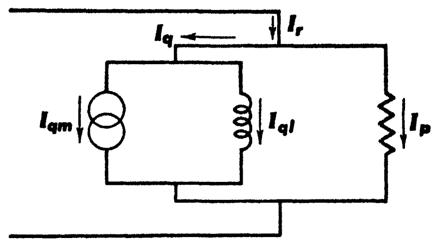


Figure 6–3

The total quadrature component I_q of the receiving-end current is thus increased, increasing the line drop I_rZ and reducing the receiving-end voltage to normal.

On the other hand if the load current increases by virtue of an increase in load demand, the tendency for V_r is to decrease because of the increase in I_rZ. This automatically brings about an increase in the excitation of the synchronous reactor. The 90° lagging current of the reactor decreases until the receiving-end voltage is increased to normal. (This automatic control of the excitation is generally so arranged that when the receiving-end load is about half normal, the excitation of the synchronous reactor is normal. Its reactive power is zero, and V_r is at its normal value.) The synchronous reactor becomes overexcited and demands leading reactive power for all receiving-end loads larger than half of the normal. Under this condition the reactive power of the reactor is positive and that of the load negative. The total reactive power at the receiving end is thus decreased and with it the receiving-end current I_r. The line drop is thus reduced and V_r is increased to normal. In contradistinction to voltage control by the adjustment of the station voltage V_s, voltage control by synchronous reactors tends to and automatically changes the line drop thus causing simultaneous adjustment in the value of the receiving-end voltage.

The fact that an overexcited synchronous reactor takes a leading, i.e., positive reactive power while the load demands lagging reactive power, has led many operators to assume erroneously that the positive and negative signs attached to reactive power connotes circuit direction of flow of such power. The positive reactive power of overexcited synchronous reactors is thus thought of, erroneously, as being produced by the reactor. The physical fact is that the reactive characteristics of reactor and load are opposite when one takes leading and the other lagging power. This means that the cyclic storage and restoration of the reactive energy are just opposite. When one stores, the other restores.

6.4. *Generalized Formula for Reactive Power to Maintain Receiving-End Voltage Constant.* The use of the general transmission equation

$$V_s = AV_r + BI_r \tag{6.4.1}$$

to determine the needed reactive power for the control of the voltage of transmission systems will result, obviously, in generalized results which could be applied to any specific case. Thus, with the proper choice of values for the transmission constants A and B, the resulting formulas will hold for lines of any length, complete systems of transmission including the terminal transformers, and for lines in parallel operation with or without the end transformers.

Using the receiving-end voltage V_r as the reference vector, and associating the angle α with A, and β with B, and assuming that θ_r is the phase of the load current I_r with respect to V_r, formula (6.4.1) when written in the polar form becomes,

$$V_s\underline{/\delta_s} = AV_r\underline{/\alpha} + BI_r\underline{/\beta + \theta_r}. \tag{6.4.2}$$

Expanding the right-side term yields

$$V_s\underline{/\delta_s} = AV_r \cos\alpha + BI_r \cos(\beta + \theta_r) + j[AV_r \sin\alpha + BI_r \sin(\beta + \theta_r)].$$

Since the numerical value of the right-side term is the square root of the sum of the squares of the horizontal and quadrature components, the above expression gives

$$V_s^2 = (AV_r)^2 + (BI_r)^2 + 2AB(VI)_r \cos(\alpha - \beta - \theta_r)$$

or

$$V_s^2 = (AV_r)^2 + (BI_r)^2 + 2AB(VI)_r \cos(\alpha - \beta)\cos\theta_r + 2AB(VI)_r \sin(\alpha - \beta)\sin\theta_r. \tag{6.4.3}$$

But

$$\left.\begin{aligned} V_rI_r\cos\theta_r &= P_r \\ V_rI_r\sin\theta_r &= Q_r. \end{aligned}\right\} \tag{6.4.4}$$

Equation (6.4.3) may be written, therefore,

$$V_s^2 = (AV_r)^2 + (BI_r)^2 + 2ABP_r \cos(\alpha - \beta) + 2ABQ_r \sin(\alpha - \beta).$$

Setting

$$2AB\cos(\alpha - \beta) = m \tag{6.4.5}$$

and

$$2AB\sin(\alpha - \beta) = n,$$

the preceding expression simplifies to

$$V_s^2 = (AV_r)^2 + (BI_r)^2 + mP_r + nQ_r. \tag{6.4.6}$$

Furthermore, since

$$I_r^2 = I_p^2 + I_q^2$$

and

$$I_p = \frac{P_r}{V_r}$$

$$I_q = \frac{Q_r}{V_r},$$

where I_p and I_q are, respectively, the dissipative and reactive components of the receiving-end current I_r, equation (6.4.6) may be written

$$V_s^2 = (AV_r)^2 + \left(\frac{BP_r}{V_r}\right)^2 + mP_r + \left(\frac{BQ_r}{V_r}\right)^2 + nQ_r$$

or

$$\left(\frac{V_sV_r}{B}\right)^2 = \left(\frac{AV_r^2}{B}\right)^2 + P_r^2 + mP_r\left(\frac{V_r}{B}\right)^2 + Q_r^2 + nQ_r\left(\frac{V_r}{B}\right)^2.$$

It should be noted that the two members in Q_r and the two members in P_r may be thought of, respectively, as members of perfect squares. Hence, by adding to the right-hand member of the preceding equation

$$\left(\frac{mV_r^2}{2B^2}\right)^2 - \left(\frac{mV_r^2}{2B^2}\right)^2 + \left(\frac{nV_r^2}{2B^2}\right)^2 - \left(\frac{nV_r^2}{2B^2}\right)^2,$$

it may be written

$$\left(\frac{V_sV_r}{B}\right)^2 = \left(\frac{AV_r^2}{B}\right)^2 + \left(P_r + \frac{mV_r^2}{2B^2}\right)^2 - \left(\frac{mV_r^2}{2B^2}\right)^2 + \left(Q_r + \frac{nV_r^2}{2B^2}\right)^2 - \left(\frac{nV_r^2}{2B^2}\right)^2$$

or

$$\left(\frac{V_sV_r}{B}\right)^2 = \left(\frac{AV_r^2}{B}\right)^2 + \left(P_r + \frac{mV_r^2}{2B^2}\right)^2 + \left(Q_r + \frac{nV_r^2}{2B^2}\right)^2 - \frac{(m^2+n^2)V_r^4}{4B^4}. \qquad (6.4.7)$$

But, by (6.4.5),

$$m^2 + n^2 = 4A^2B^2$$

whence

$$\frac{m^2+n^2}{4B^4} = \left(\frac{A}{B}\right)^2.$$

Referring this to (6.4.7), it will be seen that the first and the last members of the right-hand side cancel and the equation becomes

$$\left(\frac{V_sV_r}{B}\right)^2 = \left(P_r + \frac{mV_r^2}{2B^2}\right)^2 + \left(Q_r + \frac{nV_r^2}{2B^2}\right)^2. \qquad (6.4.8)$$

Setting

$$\frac{V_s}{V_r} = k,$$

and solving for Q_r gives

$$Q_r = \frac{-n}{2}\left(\frac{V_r}{B}\right)^2 \pm \sqrt{\left(\frac{kV_r^2}{B}\right)^2 - \left[P_r + \frac{m}{2}\left(\frac{V_r}{B}\right)^2\right]^2}. \qquad (6.4.9)$$

Using the values of m and n as given by (6.4.5), this expression becomes

$$Q_r = \frac{A}{B} V_r^2 \sin(\beta - \alpha) \pm \sqrt{\left(\frac{kV_r^2}{B}\right)^2 - \left[P_r + \frac{A}{B} V_r^2 \cos(\beta - \alpha)\right]^2}. \tag{6.4.10}$$

This expression gives the needed reactive power at the receiving end of the system to maintain a predetermined constant voltage V_r at that end in terms of the constant voltage ($V_s = kV_r$) at the sending end, the dissipative receiving-end power P_r, and the transmission properties of the system as embodied in the constants A and B.

6.5. *Generalized Formula for the Reactive Power of Phase Modifier to Maintain Constant Receiving-End Voltage.*

The expression for the receiving-end reactive power Q_r obtained in the preceding article includes the reactive power Q_l of the load and the reactive power Q_M of the phase modifier, i.e.,

$$Q_r = Q_l + Q_M \tag{6.5.1}$$

In terms of the dissipative power P_r, the reactive power of the load is

$$Q_l = P_r \tan \theta_l,$$

where θ_l is the angle corresponding to the power factor of the load.

The reactive power of the phase modifier, by equations (6.4.10) and (6.5.1), is

$$Q_M = -P_r \tan \theta_l + \frac{A}{B} V_r^2 \sin(\beta - \alpha) \pm \sqrt{\left(\frac{kV_r^2}{B}\right)^2 - \left[P_r + \frac{A}{B} V_r^2 \cos(\beta - \alpha)\right]^2} \tag{6.5.2}$$

For the particular case when $P_r = 0$, this expression becomes

$$Q_{Mo} = \frac{V_r^2}{B} [A \sin(\beta - \alpha) \pm \sqrt{k^2 - A^2 \cos^2(\beta - \alpha)}]. \tag{6.5.3}$$

There are two cases, however, for which the dissipative power P_r is zero. One is when the receiving end of the system is open-circuited. In this case, as previously shown in discussing regulation, there is a rise in voltage. The function of the modifier is to reduce the voltage by demanding lagging reactive power. This case is taken care of in equation (6.5.3) by making the radical negative. The second case when P_r might be equal to zero is when the receiving end of the system is short-circuited, in which case the receiving-end voltage decreases to zero, and the reactive power Q_{Mo} in equation (6.5.3) becomes zero. This condition, however, is quite remote, for the breakers will open the circuit. From what has been said above it follows that, under all conditions of system loading, the radical in equation (6.5.2) is negative and the equation is written

$$Q_M = -P_r \tan\theta_l + \frac{A}{B} V_r^2 \sin(\beta - \alpha) - \sqrt{\left(\frac{kV_r^2}{B}\right)^2 - \left[P_r + \frac{A}{B} V_r^2 \cos(\beta - \alpha)\right]^2}. \tag{6.5.4}$$

6.6. *Generalized Formula for the Required Voltage Ratio k. Dissipative Power Limit.* The excitation control of the phase modifier is generally adjusted to permit the phase modifier to take leading reactive power for all receiving-end loads above 50 per cent of the normal load, and lagging reactive power for receiving-end loads below 50 per cent of the normal. Under these conditions, it follows that $Q_M = 0$ when P_r is half normal load. For this specific condition, equation (6.5.4) may be written

$$-\frac{P_{rn}\tan\theta_l}{2} + \frac{A}{B} V_r^2 \sin(\beta - \alpha) = +\sqrt{k^2\left(\frac{V_r^2}{B}\right)^2 - \left[\frac{P_{rn}}{2} + \frac{A}{B} V_r^2 \cos(\beta - \alpha)\right]^2} \tag{6.6.1}$$

where P_{rn} denotes the normal dissipative load at the receiving end. With P_{rn}, V_r, $\cos\theta_l$, and the system constant $A\underline{/\alpha}$ and $B\underline{/\beta}$ known, the ratio $k = V_s/V_r$ can be determined quite easily for any system.

With V_r and k thus predetermined, the value of the reactive power Q_M of the phase modifier to maintain constant receiving-end voltage depends only upon P_r. Referring to equation (6.5.4) it is seen that when the radical in this expression is equal to zero, i.e., when the dissipative power has the particular value

$$P_{rm} = \frac{V_r^2}{B}[k - A\cos(\beta - \alpha)], \tag{6.6.2}$$

the reactive power of the phase modifier is

$$Q_{Mm} = -P_{rm}\tan\theta_l + \frac{A}{B} V_r^2 \sin(\beta - \alpha). \tag{6.6.3}$$

Note that P_{rm}, as given by (6.6.2), is the maximum dissipative power that can be transmitted over the system with the given values of V_r and k. Values of $P_r > P_{rm}$ make the radical in equation (6.5.4) imaginary.

6.7. *Voltage Control by Phase Modifiers; the Short Line.* The various formulas obtained in the preceding articles may be applied directly to short lines. The shunt admittance of such lines is neglected, hence the transmission constants are

$$A = 1\underline{/0^\circ}$$
$$B = Z\underline{/\zeta},$$

where

$$Z^2 = (R^2 + L^2\omega^2)S^2$$
$$\zeta = \tan^{-1}\frac{L\omega}{R}$$
$$S = \text{length of line in miles}$$
$$R = \text{resistance per mile of line conductor}$$
$$L\omega = \text{reactance per mile of line conductor.}$$

In accordance with the above, formula (6.4.10) for the reactive power Q_r at the receiving end of a short line, required to maintain constant terminal voltages, becomes

$$Q_r = \frac{V_r^2 \sin \zeta}{Z} - \sqrt{\left(\frac{kV_r^2}{Z}\right)^2 + \left(P_r + \frac{V_r^2 \cos \zeta}{Z}\right)^2}. \tag{6.7.1}$$

Formula (6.5.3) which gives the reactive power of the phase modifier to maintain the receiving-end voltage constant when $P_r = 0$ becomes, for short lines,

$$Q_{Mo} = \frac{V_r^2}{Z}\left[\sin \zeta - \sqrt{k^2 - \cos^2 \zeta}\right]. \tag{6.7.2}$$

Equation (6.5.4), which gives the reactive power of the phase modifier for any receiving-end dissipative load P_r of power factor $\cos \theta_l$ and constant terminal voltages, becomes for a short line

$$Q_M = -P_r \tan \theta_l + \frac{V_r^2 \sin \zeta}{Z} - \sqrt{\left(\frac{kV_r^2}{Z}\right)^2 - \left(P_r + \frac{V_r^2 \cos \zeta}{Z}\right)^2}. \tag{6.7.3}$$

The voltage ratio $k = V_s/V_r$ is calculated by (6.6.1) which, modified for a short line, becomes

$$-\frac{P_{rn} \tan \theta_l}{2} + \frac{V_r^2 \sin \zeta}{Z} = \sqrt{k^2 \left(\frac{V_r^2}{Z}\right)^2 - \left(\frac{P_{rn}}{2} + \frac{V_r^2 \cos \zeta}{Z}\right)^2}, \tag{6.7.4}$$

where P_{rn} is the normal receiving-end load. The reactive power demand by the phase modifier to maintain constant receiving-end voltage for a short line, by (6.6.3), is

$$Q_{Mm} = -P_{rm} \tan \theta_l + \frac{V_r^2 \sin \zeta}{Z}, \tag{6.7.5}$$

where

$$P_{rm} = \frac{V_r^2}{Z}(k - \cos \zeta) \tag{6.7.6}$$

is the maximum dissipative power that could be transmitted over the short line, with the terminal voltage constant.

6.8. *Illustrative Study of Voltage Control by Phase Modifiers; Short Lines.* To illustrate the material discussed in the preceding articles, consider the 250,000 circ. mil stranded copper cable three-phase line whose constants were calculated in § 3.4. Assuming that the length of the line is 25 miles, the line resistance per conductor is

$$RS = 0.263 \times 25 = 6.575 \text{ ohms.}$$

The inductive reactance per conductor at the frequency of 60 cps is

$$L\omega S = 0.765 \times 25 = 19.12.$$

The line impedance per conductor at the frequency of 60 cps is

$$\mathbf{Z} = 20.2\underline{/71^\circ} \text{ vector ohms.}$$

Hence

$$\zeta = 71^\circ.$$

Assuming that the normal receiving-end load is 15000 kw per phase at a lagging power factor of 87 per cent ($\theta_l = -29.5$), and $V_r = 19$ kv per phase, corresponding to a line voltage of 33 kv, the value of required voltage ratio may be calculated by (6.7.4). Thus,

$$\frac{P_r \tan \theta_l}{2} = \frac{15000 \times 10^3 \tan(-29.5^\circ)}{2} = -4.243 \times 10^6$$

$$\frac{V_r^2}{Z} = \frac{19^2 \times 10^6}{20.2} = 17.86 \times 10^6$$

$$\left(\frac{V_r^2}{Z}\right)^2 = 319.3 \times 10^{12}$$

$$\frac{V_r^2 \sin \zeta}{Z} = \frac{19^2 \times 10^6 \sin 71^\circ}{20.2} = 16.88 \times 10^6$$

$$\frac{V_r^2 \cos \zeta}{Z} = \frac{19^2 \times 10^6 \cos 71^\circ}{20.2} = 5.82 \times 10^6.$$

Substituting these calculated values in (6.7.4) gives

$$(4.243 \times 10^6) + (16.88 \times 10^6) = \sqrt{319.3 \times 10^{12} k^2 - (7500 \times 10^3 + 5.82 \times 10^6)^2}.$$

This gives

$$k = 1.4.$$

Using this voltage ratio in (6.7.3) gives for full load

$$\begin{aligned} Q_M &= 8.486 \times 10^6 + 16.88 \times 10^6 - \sqrt{625 \times 10^{12} - (15 + 5.82)^2 \times 10^{12}} \\ &= 11540 \text{ kilo-vars per phase, leading.} \end{aligned}$$

This is the reactive power needed to maintain $V_r = 19$ kv and $V_s = 19 \times 1.4 = 26.6$ kv to neutral when the receiving-end power is

15000 kw per phase at 87 per cent power factor lag. The reactive power of the load itself is

$$Q_l = -8480 \text{ kilo-vars per phase.}$$

The reactive power at the receiving end, therefore, is

$$\begin{aligned} Q_r &= 11540 - 8480 \\ &= 3060 \text{ kilo-vars per phase, leading.} \end{aligned}$$

The phase angle between the receiving-end current and receiving-end voltage changed from 29.5° lag to

$$\begin{aligned} \theta_r &= \tan^{-1} \tfrac{3060}{15000} \\ &= 11.5° \text{ leading.} \end{aligned}$$

The power factor at the receiving end changed from 0.87 lag to 0.981 lead.

The reactive power demand by the phase modifier to maintain the receiving-end voltage constant when the receiving-end dissipative power $P_r = 0$ on open circuit is, by (6.7.2),

$$\begin{aligned} Q_{Mo} &= 17.86 \times 10^6[\sin 71° - \sqrt{1.96 - \cos^2 71°}] \\ &= 17.86 \times 10^6(0.945 - 1.363) \\ &= -7.45 \times 10^6 \text{ vars per phase} \end{aligned}$$

or 7450 lagging kilo-vars per phase.

The maximum dissipative power that can be transmitted over the short line with terminal voltages constant is, by (6.7.6),

$$P_{rm} = 17.86 \times 10^6(1.4 - 0.325) = 19200 \text{ kw per phase.}$$

The reactive power of the phase modifier corresponding to the above maximum dissipative power of 87 per cent power factor lag is, by (6.7.5),

$$\begin{aligned} Q_{Mm} &= -19.2 \times 10^6 \tan(-29.5°) + 17.86 \times 10^6 \sin 71° \\ &= 10860 + 16880 = 27750 \text{ kilo-vars per phase.} \end{aligned}$$

6.9. *Illustrative Study of Voltage Control by Phase Modifiers; Medium Long Lines.* Substantially accurate results are obtained by the use of the nominal T for lines not exceeding about 100 miles in length. Consider the illustrative line used in § 6.8 assuming it to be 100 miles long. The resistance of the line per conductor is

$$RS = 0.263 \times 100 = 26.3 \text{ ohms.}$$

The inductive reactance at the frequency of 60 cps is

$$L\omega S = 0.765 \times 100 = 76.5 \text{ ohms.}$$

The linear line impedance, therefore, is

$$zS = 80.8\underline{/71°} \text{ vector ohms.}$$

The series impedance of the nominal T is

$$Z_T = \frac{80.8\underline{/71^\circ}}{2} = 40.4\underline{/71^\circ} \text{ vector ohms.}$$

The capacitive susceptance of the line is

$$C\omega S = 5.58 \times 10^{-6} \times 100 = 558 \times 10^{-6} \text{ mhos.}$$

The shunt admittance of the nominal T is

$$Y_T = 558 \times 10^{-6}\underline{/90^\circ} \text{ vector mhos.}$$

By equation (4.2.4), the transmission constants of the nominal T are

$$A = 1 + Z_T Y_T$$
$$B = (2 + Z_T Y_T)Z_T.$$

For the particular case under consideration

$$\begin{aligned} A &= 1 + (40.4\underline{/71^\circ})(558 \times 10^{-6}\underline{/90^\circ}) \\ &= 1 - 0.02254\underline{/-19^\circ} \\ &= 0.9787\underline{/0^\circ} \end{aligned}$$

substantially.
Hence

$$A = .9787\underline{/0^\circ}$$

and

$$\begin{aligned} \alpha &= 0^\circ \\ B &= (2 - 0.02254\underline{/-19^\circ})(40.4\underline{/71^\circ}) \\ &= 80.8\underline{/71^\circ} - 0.91\underline{/52^\circ} \\ B &= 80\underline{/71.2^\circ} \end{aligned}$$

and

$$\beta = 71.2^\circ.$$

Assume that the voltage to neutral at the receiving end is 63.5 kv corresponding to a line voltage of 110 kv; the normal receiving-end dissipative power is $P_r = 15000$ kw per phase at 87 per cent power factor. The following values may be calculated for use in (6.6.1) to obtain the voltage ratio k.

$$\frac{P_{rn} \tan \theta_l}{2} = \frac{15000 \times 10^3 \tan(-29.5^\circ)}{2} = -4.24 \times 10^6$$

$$\frac{V_r^2}{B} = \frac{63.5^2 \times 10^6}{80} = 50.4 \times 10^6$$

$$\begin{aligned} \frac{V_r^2}{B} A \sin 71.2^\circ &= 50.4 \times 10^6 \times 0.9787 \sin 71.2^\circ \\ &= 46.6 \times 10^6 \end{aligned}$$

$$\left(\frac{V_r^2}{B}\right)^2 = (50.4 \times 10^6)^2 = 2540 \times 10^{12}$$

$$\begin{aligned} \frac{V_r^2}{B} A \cos 71.2^\circ &= 50.4 \times 10^6 \times 0.9787 \cos 71.2^\circ \\ &= 15.87 \times 10^6 \end{aligned}$$

Substituting these values in (6.6.1) yields

$$(4.24 \times 10^6) + (46.6 \times 10^6) = \sqrt{2540 \times 10^{12}k^2 - [(7.5 \times 10^6) + (15.87 \times 10^6)]^2}$$
$$3121 = 2540k^2$$
$$k = 1.11.$$

Using a voltage ratio $k = 1.1$ in (6.5.4) and $P_r = 15000$ kw per phase at a power factor of 87 per cent lag gives for full load,

$$Q_M = (8.48 \times 10^6) + (46.6 \times 10^6) - \sqrt{1.21 \times 2540 \times 10^{12} - (15 + 15.87)^2 10^{12}}$$
$$= 9.08 \times 10^6 \text{ vars per phase}$$

or

$$Q_M = 9080 \text{ kilo-vars per phase,}$$

to maintain a line voltage of 110 kv at the receiving end and $1.1 \times 110 = 121$ kv at the sending end.

The reactive power of the load itself is

$$Q_l = -8480 \text{ kilo-vars per phase, lagging.}$$

The calculated reactive power of the phase modifier is

$$Q_M = 9080 \text{ kilo-vars per phase leading.}$$

The reactive power at the receiving end is

$$Q_r = 9080 - 8480$$
$$= 600 \text{ kilo-vars per phase, leading.}$$

The phase angle between the receiving-end current and voltage changed from $-29.5°$ without the phase modifier to

$$\theta_r = \tan^{-1} \tfrac{600}{15000} = 2.29°$$

with the phase modifier. The power factor at the receiving end changed from 0.87 lag to practically unity power factor.

The reactive power demand by the phase modifier to maintain the receiving-end voltage at the above values when $P_r = 0$ on open circuit is given by (6.5.3). For the particular case considered it is

$$Q_{Mo} = 50.4 \times 10^6[0.925 - \sqrt{1.21 - (0.9787 \cos 71.2°)^2}]$$
$$= 50.4 \times 10^6(0.925 - 1.053) \text{ vars per phase}$$
$$= -6450 \text{ kilo-vars per phase.}$$

The maximum dissipative power that can be transmitted over the line with the terminal voltages constant at the value stated above is, by (6.6.2),

$$P_{rm} = 50.4 \times 10^6(1.1 - 0.9787 \cos 71.2°)$$
$$= 39200 \text{ kw per phase.}$$

The reactive power demand by the phase modifier for the above dissipative power is given by (6.6.3). For the particular case considered it is

$$Q_{Mm} = 68780 \text{ kilo-vars per phase.}$$

6.10. *Illustrative Study of Voltage Control by Phase Modifiers; Long Lines.* Consider the illustrative line used in § 6.8 assuming it to be 300 miles long. The transmission constants of the line calculated in § 3.4 are:

$$A = \cosh pS = 0.816\underline{/4.35°}$$
$$B = Z_o \sinh pS = 227.2\underline{/72.3°}.$$

Assume that the voltage to neutral at the receiving end is 125 kv corresponding to a line voltage of 216.5 kv. The normal dissipative load is $P_r = 15000$ kw at 87 per cent power factor. The voltage ratio is calculated by (6.5.1) in which

$$\frac{P_{rn} \tan \theta_l}{2} = -4.24 \times 10^6$$
$$\frac{V_r^2}{B} = \frac{125^2 \times 10^6}{227.2} = 68.7 \times 10^6$$
$$A \sin(\beta - \alpha) = 0.816 \sin(72.3° - 4.35°)$$
$$= 0.7563$$
$$A \cos(\beta - \alpha) = 0.816 \cos 67.95°$$
$$= 0.3063$$
$$\frac{AV_r^2}{B} \sin(\beta - \alpha) = 68.7 \times 10^6 \times 0.7563$$
$$= 51.95 \times 10^6$$
$$\frac{AV_r^2}{B} \cos(\beta - \alpha) = 68.7 \times 10^6 \times 0.3063$$
$$= 21.04 \times 10^6$$
$$\left(\frac{V_r^2}{B}\right)^2 = 4720 \times 10^{12}.$$

uting the required values in (6.6.1) gives

$$4.24 \times 10^6 + 51.95 \times 10^6 = \sqrt{4720 \times 10^{12} k^2 - [(7.5 \times 10^6) + (21.04 \times 10^6)]^2}$$
$$k^2 = \frac{3971}{4720} = 0.8414$$
$$k = 0.919.$$

Using $k = 0.92$ in (6.5.4) for a normal dissipative power of 15000 kw at 87 per cent power factor, gives for the reactive power

$$Q_M = (8.48 \times 10^6) + (51.95 \times 10^6) - \sqrt{0.846 \times 4720 \times 10^{12} - (15 + 21.04)^2 10^{12}}$$
$$= 8530 \text{ kilo-vars},$$

per phase to maintain $V_r = 125$ kv and $V_s = 115$ kv to neutral for a normal load of 15000 kw at 87 per cent lagging power factor per phase. The reactive power of the load itself is

$$Q_l = -8480 \text{ kilo-vars per phase, lagging.}$$

The calculated reactive power of the phase modifier is

$$Q_M = 8530 \text{ kilo-vars per phase, leading.}$$

The reactive power at the receiving end is

$$\begin{aligned} Q_r &= 8530 - 8480 \\ &= 50 \text{ kilo-vars, leading.} \end{aligned}$$

The phase angle between the receiving-end current and voltage changed from $-29.5°$ without the phase modifier to

$$\theta_r = \tan^{-1} \tfrac{50}{15000} = 0.191°$$

with the phase modifier. The power factor at the receiving end changed from 87 per cent lag to substantially 100 per cent.

The reactive power demand by the phase modifier to maintain the terminal voltages constant at the above stated values when $P_r = 0$ on open circuit is given by (6.5.3). For the special case considered it is

$$\begin{aligned} Q_{Mo} &= 68.7 \times 10^6[0.7563 - \sqrt{0.8414 - 0.3063^2}] \\ &= 68.7 \times 10^6(0.7563 - 0.8680) \text{ vars per phase} \\ &= -7720 \text{ kilo-vars per phase.} \end{aligned}$$

The maximum dissipated power that could be transmitted over the line with the terminal voltages constant at the above stated values is, by (6.6.2),

$$\begin{aligned} P_{rm} &= 68.7 \times 10^6(0.92 - 0.3063) \\ &= 42160 \text{ kw per phase.} \end{aligned}$$

The reactive power demand by the phase modifier for this dissipative power to maintain the receiving-end voltages constant would be, by (6.6.3),

$$Q_{Mm} = 75790 \text{ kilo-vars per phase leading.}$$

6.11. *Dissipative and Reactive Power at the Station End of Transmission Systems with Voltage Control.* The preceding articles of this chapter deal with dissipative and reactive power conditions at the receiving end of transmission systems whose receiving-end voltage are controlled by phase modifiers installed at the receiving end. The station-end voltage of such systems is maintained at a constant predetermined value. The vector impedance $\mathbf{Z}_s$ to neutral at that end can be obtained from the known line constants and receiving-end impedance. Then if V_s is the voltage to neutral at the station end, the current at that end is

$$I_s = \frac{V_s}{Z_s/\underline{\theta_s}} \text{ vector amperes.} \tag{6.11.1}$$

The dissipative power at the station end is

$$P_s = (kv)_s I_s \cos\theta_s \text{ kw per phase,} \tag{6.11.2}$$

and the reactive power at that end is

$$Q_s = (kv)_s I_s \sin\theta_s \text{ kilo-vars per phase.} \tag{6.11.3}$$

To obtain the generalized formula for the station-end impedance in terms of the line constants consider the generalized formulas for station-end voltage and current in terms of receiving-end values

$$V_s = AV_r + BI_r$$
$$I_s = AI_r + DV_r.$$

The station-end impedance is V_s/I_s, i.e.,

$$Z_s = \frac{AV_r + BI_r}{AI_r + DV_r}. \tag{6.11.4}$$

Since $V_r = I_r Z_r$, the formula becomes

$$Z_s = \frac{AZ_r + B}{A + DZ_r}. \tag{6.11.5}$$

The quantity Z_r in this formula is the impedance at the receiving end.

For the specific case of a transmitting system with a definite receiving-end load in parallel with a phase modifier at that end, V_r is constant and known. The dissipative power P_r and the total reactive power Q_r are known for the normal load conditions. Hence, the kva at the receiving end may be calculated by

$$(kva)_r = \sqrt{P_r^2 + Q_r^2}.$$

The receiving-end current, therefore, is

$$I_r = \frac{(kva)_r}{(kv)_r}$$

and

$$Z_r = \frac{V_r}{I_r} /\underline{\tan^{-1} Q_r/P_r}.$$

This is the vector value of Z_r in formula (6.11.5).

For a short line, as shown elsewhere, $A = 1$, $D = 0$, and $B = Z/\underline{\zeta}$. Hence,

$$Z_s = Z_r/\underline{\theta_r} + Z/\underline{\zeta}. \tag{6.11.6}$$

For a line simulated by a symmetrical T network:

$$\begin{aligned} A &= 1 + Z_T Y_T \\ B &= (2 + Z_T Y_T) Z_T. \\ D &= Y_T, \end{aligned} \tag{6.11.7}$$

and for a long line

$$\begin{aligned} A &= \cosh pS \\ B &= Z_o \sinh pS \\ D &= \frac{\sinh pS}{Z_o}. \end{aligned} \tag{6.11.8}$$

The respective values of A, B, and D in equation (6.11.5) will give the required value of the impedance Z at the station end of the system.

6.12. *Illustrative Calculation of Station-End Dissipative and Reactive Powers of Transmission Systems with Phase Modifiers.*

a. Short Line. A 250,000 circ. mil stranded copper three-phase line 25 miles long is equipped with phase modifiers to maintain the receiving-end voltage constant at $V_r = 19$ kv and $V_s = 26.6$ kv to neutral ($k = 1.4$). The normal dissipative load is 15000 kw per phase at 87 per cent lagging power factor. The reactive power of the modifier for this particular load, as calculated in § 6.8, is $Q_M = 11540$ kilo-vars. Calculate the dissipative and reactive powers at the station end.

The reactive power of load and phase modifier is $Q_r = 3060$ kilo-vars and the kva at the receiving end is

$$\begin{aligned} (kva)_r &= 15000 + j\,3060 \\ &= 15300\underline{/11.54^\circ}. \end{aligned}$$

The current at the receiving end is

$$\begin{aligned} I_r &= \frac{15300\underline{/11.54^\circ}}{19} \\ &= 805\underline{/11.54^\circ} \text{ vector amperes.} \end{aligned}$$

The receiving-end impedance, therefore, is

$$\begin{aligned} Z_r &= \frac{19000}{805\underline{/11.54^\circ}} \\ &= 23.6\underline{/-11.54^\circ} \text{ vector ohms.} \end{aligned}$$

The line impedance per conductor as calculated in § 6.8 is $Z\underline{/\zeta} = 20.2\underline{/71^\circ}$. The impedance at the station end, therefore, by (6.11.6), is

$$\begin{aligned} Z_s &= 23.6\underline{/-11.54^\circ} + 20.2\underline{/71^\circ} \\ &= 33.1/25.8^\circ \text{ vector ohms.} \end{aligned}$$

The station-end current I_s is substantially 805 and lags the station-end voltage by 25.8°.

The kva at the station end is

$$(kva)_s = 26.6 \times 805\underline{/-25.8^\circ}$$
$$= 21410\underline{/-25.8^\circ}.$$

The dissipative power at the station end is

$$P_s = 21410 \cos(-25.8^\circ)$$
$$= 19250 \text{ kw per phase.}$$

The reactive power at the station end is

$$Q_s = 21410 \sin(-25.8^\circ)$$
$$= -9310 \text{ kilo-vars.}$$

The power factor at the station end is

$$\cos(-25.8^\circ) = 0.9 \text{ lagging.}$$

Note that the power factor is 97.7 per cent at the receiving end and nearly 90 per cent lag at the station end.

The efficiency of transmission with the phase modifiers is

$$\eta = \tfrac{15000}{19250}$$
$$= 77.9 \text{ per cent.}$$

b. Medium Long Line. A 250,000 circ. mil stranded copper cable three-phase line 100 miles long is equipped with phase modifiers to maintain the receiving-end voltage at 63.5 kv and the station-end voltage at 69.8 kv ($k = 1.1$). The normal dissipative load is 15000 kw per phase at 87 per cent factor. The reactive power of the phase modifier to maintain the terminal voltage constant for this particular load, as calculated in § 6.9, is $Q_M = 9080$ kilo-vars. The reactive power of load and phase modifier is $Q_r = 600$ kilo-vars. Calculate the power conditions at the station end.

The kva at the receiving end is

$$(kva)_r = 15000 + j\,600$$
$$= 15010\underline{/2.29^\circ}.$$

The current at the receiving end is

$$I_r = \frac{15010\underline{/2.29^\circ}}{63.5}$$
$$= 236.5\underline{/2.29^\circ} \text{ vector amperes.}$$

The receiving-end impedance including the phase modifier is

$$Z_r = \frac{63500}{236.5\underline{/2.29^\circ}}$$
$$= 268\underline{/-2.29^\circ} \text{ vector ohms.}$$

The transmission constants for this line, as calculated in § 6.9, are

$$A = 0.9787\underline{/0^\circ}$$
$$B = 80\underline{/71.2^\circ}$$
$$D = Y_T = 558 \times 10^{-6}\underline{/90^\circ}.$$

Using these values in (6.11.5) gives

$$Z_s = \frac{(0.9787\underline{/0^\circ})(268.0\underline{/-2.29^\circ}) + 80\underline{/71.2^\circ}}{0.9787\underline{/0^\circ} + (558 \times 10^{-6}\underline{/90^\circ})(268.0\underline{/-2.29^\circ})}$$
$$= 297.6\underline{/4.15^\circ} \text{ vector ohms.}$$

The station-end current is

$$I_s = \frac{69800}{297.6\underline{/4.15^\circ}}$$
$$= 234.8\underline{/-4.15^\circ} \text{ vector amperes.}$$

The kva at the station end is

$$(kva)_s = 69.8 \times 234.8\underline{/-4.15^\circ}$$
$$= 16370\underline{/-4.15^\circ}.$$

The dissipative power at the station end is

$$P_s = 16370 \cos(-4.15^\circ)$$
$$= 16300 \text{ kw per phase.}$$

The reactive power at the station end is

$$Q_s = 16370 \sin(-4.15^\circ)$$
$$= -1184 \text{ kilo-vars lagging.}$$

The power factor at the station end is

$$\cos(-4.15^\circ) = 0.997.$$

The efficiency of transmission is

$$\eta = \tfrac{15000}{16300}$$
$$= 92.1 \text{ per cent.}$$

c. Long Line. A 250,000 circ. mil stranded copper cable 300 miles long is equipped with phase modifiers to maintain the receiving-end voltage at 125 kv and the station-end voltage at 115.0 kv to neutral ($k = 0.92$). The normal dissipative load is 15000 kw per phase at 87 per cent power factor. The reactive power of the phase modifier, as calculated in § 6.10, is $Q_M = 8530$ kilo-vars. The reactive power of load and phase modifier is $Q_r = 50$ kilo-vars.

The kva at the receiving end is

$$(kva)_r = 15000 + j\,50$$
$$= 15000\underline{/0^\circ} \text{ kilo-voltamperes substantially.}$$

The current at the receiving end is

$$I_r = \frac{15000\underline{/0^\circ}}{125}$$
$$= 120\underline{/0^\circ} \text{ vector amperes.}$$

The impedance at the receiving end, including that of the phase modifier, is

$$Z_r = \frac{125000}{120\underline{/0^\circ}}$$
$$= 1041\underline{/0^\circ} \text{ vector ohms.}$$

The transmission constants of the line, as calculated in § 5.7, are

$$A = \cosh pS = 0.816\underline{/4.35^\circ}$$
$$B = Z_o \sinh pS = (380\underline{/-9.5^\circ})(0.598\underline{/81.8^\circ}) = 227.2\underline{/72.3^\circ}$$
$$D = \frac{\sinh pS}{Z_o} = \frac{0.598\underline{/81.8^\circ}}{380\underline{/-9.5^\circ}} = 15.73 \times 10^{-4}\underline{/90^\circ}.$$

The sending-end impedance, by (6.11.5), is

$$Z_s = \frac{(0.816\underline{/4.35^\circ})(1041\underline{/0^\circ}) + 227.2\underline{/72.3^\circ}}{0.816\underline{/4.35^\circ} + (15.73 \times 10^{-4}\underline{/90^\circ})(1041\underline{/0^\circ})}$$
$$= \frac{850\underline{/4.35^\circ} + 227.2\underline{/72.3^\circ}}{0.816\underline{/4.35^\circ} + 1.64\underline{/90^\circ}}$$
$$= 506\underline{/-47.4^\circ} \text{ vector ohms.}$$

The current at the station end is

$$I_s = \frac{115000}{506\underline{/-47.4^\circ}}$$
$$= 227\underline{/47.4^\circ} \text{ vector amperes.}$$

The kva at the station end is

$$(kva)_s = 115 \times 227\underline{/47.4^\circ}$$
$$= 26100\underline{/47.4^\circ}.$$

The dissipative power at the station end is

$$P_s = 26100 \cos 47.4^\circ$$
$$= 17660 \text{ kilowatts per phase.}$$

The reactive power at the station end is

$$Q_s = 26100 \sin 47.4^\circ$$
$$= 19200 \text{ kilo-vars per phase, leading.}$$

The power factor at the station end is

$$\cos 47.4° = 0.676.$$

The efficiency of transmission is

$$\eta = \frac{15000}{17660}$$
$$= 84.9 \text{ per cent.}$$

REFERENCES

Cisna, V. J., *Voltage Regulation Diagram for Constant Sending Voltage*, Elect. Eng., May 1937.

Dahl, O. G. C., *Electric Circuits*, Vol. I, McGraw-Hill Book Co., 1928.

Doherty, R. E., *Excitation Systems*, Am. Inst. Elect. Engrs., Quarterly Trans. No. 3, July 1928.

Dwight, H. B., *Transmission Systems with Over Compounded Voltages*, Trans. Am. Inst. Elect. Engrs., 1926.

Loew, E. A., *Electric Power Transmission*, McGraw-Hill Book Co., 1928.

Nickle, C. A., and Carothers, R. M., *Automatic Voltage Regulators*, Am. Inst. Elect. Engrs., Quarterly Trans. No. 3, July 1928.

SUGGESTIVE PROBLEMS *Chapter 6*

1. A three-phase 350,000 circ. mil copper cable line 30 miles long supplies a load of 12,000 kw per phase at a lagging power factor of 85 per cent. Assuming the receiving-end line voltage equal to 33 kv,
   Calculate:
   a. The reactive power of the load.
   b. The reactive power of a phase modifier to maintain the receiving-end line voltage at 33 kv.
   c. The combined reactive power at the receiving end of load and phase modifier.
   d. The change in power factor at the receiving end caused by the use of the phase modifier.
   e. The constant line sending-end voltage obtainable by the use of the phase modifier.
   f. The dissipative power limit that can be transmitted over the line with the sending-end voltage maintained at the value calculated in (e).
   g. The maximum reactive power of the phase modifier for the power limit calculated in (f).

2. Recalculate Prob. 1, on the assumption that the line is 112 miles long, and the receiving-end line voltage 110 kv.

3. Recalculate Prob. 1, on the assumption that the line is 278 miles long, and the receiving-end line voltage 220 kv.

Chapter 7 Steady State Power Limits

7.1. *General Considerations.* It was shown in § 2.8 that if the station-end voltage, V_s, of a transmission line is maintained constant, there is a definite value of receiving-end impedance Z_r for which the kva at that end is a maximum. This means in effect that a transmission line, through its inherent characteristics as a vehicle of energy transfer, can transmit a limited amount of power.

It was similarly shown in the preceding chapter that when the receiving-end voltage V_r of a transmission line is maintained constant, there is a definite maximum dissipative power and a definite corresponding maximum reactive power that can be transmitted over the line. It follows, therefore, that only a limited amount of power can be transmitted over a line whose receiving- and station-end voltages are both maintained constant.

The power limit in this latter case will differ in amount from that in the first case, although both will depend to a considerable degree upon the power factor of the load.

The power limit of a system of transmission will depend also upon whether the load is increased in gradual and small steps or whether the demand for power is sudden and large. This is particularly true when the load is synchronous in character, in which case the stability of operation will depend upon the maintenance of synchronism between the synchronous generating equipment at the station end and the synchronous load or control equipment at the receiving end.

The maximum amount of power that can be transferred over a transmission system when the load, either static or synchronous, is increased gradually and in small steps is called ***steady state*** power limit. The ability of a transmission system to transfer gradually increased amounts of power within the limit of the *steady state* is referred to as *steady-state stability.*

7.2. *Steady State Power Limit; Generalized Transmission Line with Constant Station-End Voltage and Static Load.* Consider a transmission system whose transmission constants A, B, and D are known. The receiving-end voltage is

$$V_r = AV_s - BI_s. \tag{7.2.1}$$

This may be written

$$V_r = \left(A - \frac{B}{Z_s}\right) V_s, \tag{7.2.2}$$

where Z_s is the station-end impedance. Substituting its value, as given by (6.11.5) in the preceding expression, gives

$$V_r = \left(\frac{A^2 - BD}{A + B/Z_r}\right) V_s. \tag{7.2.3}$$

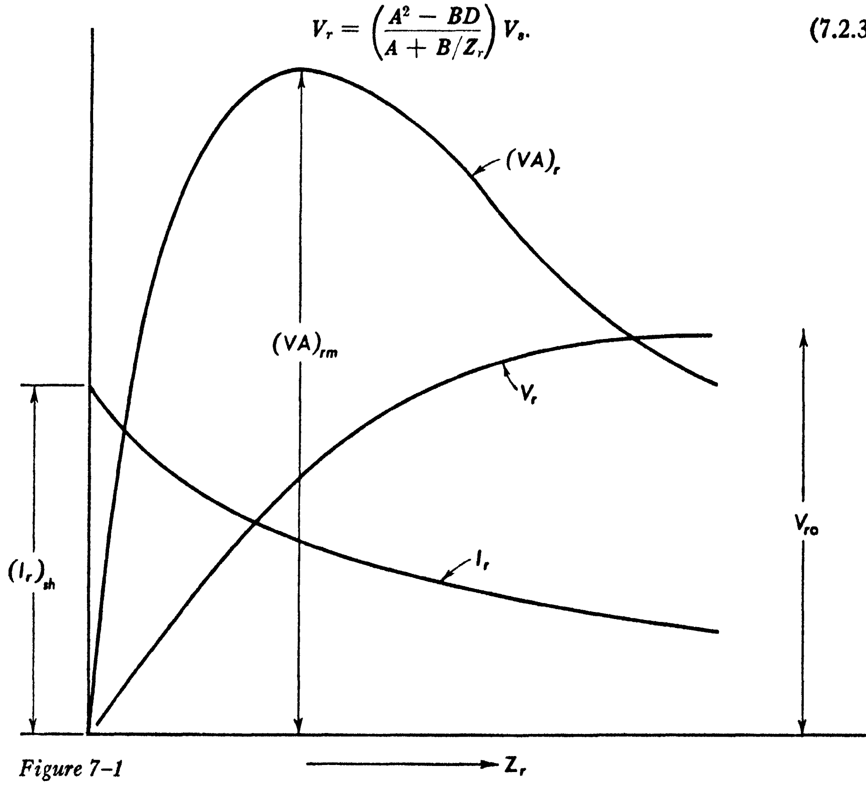


Figure 7–1

Curve marked V_r in Fig. 7–1 shows the variation of the receiving-end voltage with receiving-end impedance. The maximum value of V_r is when $Z_r \to \infty$ and is

$$V_{rm} = \frac{(A^2 - BD)V_s}{A}. \tag{7.2.4}$$

The current at the receiving end is

$$I_r = AI_s - DV_s. \tag{7.2.5}$$

This may be written

$$I_r = \left(\frac{A}{Z_s} - D\right) V_s. \tag{7.2.6}$$

Substituting (6.11.5) for Z_s gives for the current at the receiving end

$$I_r = \left(\frac{A^2 - BD}{AZ_r + B}\right) V_s. \tag{7.2.7}$$

Curve marked I_r in Fig. 7–1 shows the variation of the receiving-end current with Z_r. The current is a maximum

$$I_r = \left(\frac{A^2 - BD}{B}\right) V_s \tag{7.2.8}$$

when $Z_r = 0$.

It is evident from the shape of the V_r and I_r curves that their product $(VI)_r$ results in a curve that passes through a definite maximum as indicated. The quantity $(VI)_{rm} \cos \theta_r$ is the steady state power limit for the case considered.

The ratio of the voltage and current corresponding to the maximum $(VI)_{rm}$ gives the corresponding value of the receiving-end impedance. Its value is defined completely by the characteristics of the transmission system. Thus, multiplying (7.2.2) by (7.2.6) yields

$$V_r I_r = \frac{(AZ_s - B)(A - DZ_s)}{Z_s^2} V_s^2.$$

The maximum value of $V_r I_r$ as a function of Z_s may be obtained by solving

$$\frac{d(V_r I_r)}{dZ_s} = 0.$$

The solution shows that the maximum value of $V_r I_r$ occurs when

$$Z_s = \frac{2AB}{A^2 + BD}. \tag{7.2.9}$$

Using the value of Z_s expressed by (6.11.5) gives

$$\frac{AZ_r + B}{A + DZ_r} = \frac{2AB}{A^2 + BD},$$

which, when solved for Z_r, gives the receiving-end impedance Z_{rm} for maximum voltamperes.

$$Z_{rm} = \frac{B}{A}. \tag{7.2.10}$$

This shows that the receiving-end impedance for the power limit depends only on the constants of the transmission system with or without the transformer equipment as the case may be. Thus, for a short line, since $A = 1$ and $B = Z\underline{/\zeta}$, it follows that the power limit is that for which

$$Z_{rm} = Z\underline{/\zeta}, \tag{7.2.11}$$

i.e., the power limit of the system will be reached when the load impedance is equal to the line impedance.

If the line is simulated by a symmetrical T network, then

$$A = 1 + Z_T Y_T,$$

and

$$B = (2 + Z_T Y_T) Z_T.$$

By (7.2.10) the demand for the power limit occurs, therefore, when

$$Z_{rm} = \frac{(2 + ZY_T)Z_T}{1 + Z_T Y_T}. \tag{7.2.12}$$

Furthermore, since for a long line

$$A = \cosh pS$$
$$B = Z_o \sinh pS,$$

the demand for the power limit by (7.2.10) occurs when

$$Z_{rm} = Z_o \frac{\sinh pS}{\cosh pS}.^* \tag{7.2.13}$$

7.3. *Receiving-End Dissipative and Reactive Power in Terms of the Displacement Angle; Power-Circle Diagram.*

Consider the generalized voltage equation of a transmission system,

$$V_s = AV_r + BI_r. \tag{7.3.1}$$

In polar form this equation is

$$V_s\underline{/\delta} = AV_r\underline{/\alpha} + B\underline{/\beta} I_r\underline{/\theta_r}$$

where δ is the time-phase angle between the station-end voltage V_s and the receiving-end voltage V_r taken as the reference vector. This angle, generally referred to as a *displacement* angle, plays an important role in stability studies of transmission systems.

Solving the above equation for the current gives

$$I_r\underline{/\theta_r} = \frac{V_s}{B}\underline{/\delta - \beta} - \frac{AV_r}{B}\underline{/\alpha - \beta}.$$

The receiving-end voltamperes, therefore, is

$$(VI)_r\underline{/\theta_r} = \frac{V_s V_r}{B}\underline{/\delta - \beta} - \frac{AV_r^2}{B}\underline{/\alpha - \beta}. \tag{7.3.2}$$

* See equation (2.8.10).

Since the horizontal component of the voltamperes is the dissipative power (P_r) and the quadrature component is the reactive power (Q_r), it follows that

$$P_r = \frac{V_s V_r}{B}\left[\cos(\delta - \beta) - \frac{A V_r}{V_s}\cos(\alpha - \beta)\right] \quad (7.3.3)$$

and

$$Q_r = \frac{V_s V_r}{B}\left[\sin(\delta - \beta) - \frac{A V_r}{V_s}\sin(\alpha - \beta)\right]. \quad (7.3.4)$$

These two formulas hold for any relationship between the terminal voltages. However, if the station-end voltage V_s is maintained constant by adjustment of generator excitation, and the receiving-end voltage V_r is also maintained constant by means of phase modifiers, so that $V_s/V_r = k$, the above expressions for P_r and Q_r become, respectively,

$$P_r = \frac{V_r^2}{B}[k\cos(\delta - \beta) - A\cos(\alpha - \beta)] \quad (7.3.5)$$

and

$$Q_r = \frac{V_r^2}{B}[k\sin(\delta - \beta) - A\sin(\alpha - \beta)]. \quad (7.3.6)$$

These two equations indicate that, for any particular transmission system having constant station- and receiving-end voltages, both P_r and Q_r are functions of the displacement angle δ. In accordance with (7.3.5) the maximum value that P_r may have is

$$P_{rm} = \frac{V_r^2}{B}[k - A\cos(\alpha - \beta)] \quad (7.3.7)$$

and it occurs when $\delta = \beta$. The corresponding reactive power, in accordance with equation (7.3.6), is*

$$Q_{rm} = \frac{A V_r^2}{B}\sin(\beta - \alpha). \quad (7.3.8)$$

The variation of the receiving-end voltamperes, and of its components P_r and Q_r with the displacement angle δ, may be visualized by the polar diagram of equation (7.3.2), usually referred to as *power-circle diagram.* Note that the first term of this equation is a vector of constant magnitude and of position depending upon the displacement angle δ. Its locus is, therefore, a circle of radius ($\boldsymbol{V_s V_r/B}$) starting, when $\delta = 0$, from a position ($-\beta$) measured from the horizontal and moving counterclockwise with increases in the value of δ.

The second term of the equation is a vector of constant magnitude ($\boldsymbol{A V_r^2/B}$) and fixed position ($\alpha - \beta$) with reference to the horizontal. Since the voltamperes ($\boldsymbol{VI}$)$_r$ is the vector difference of the two terms, its locus

* See equations (6.2) and (6.3).

will also be a circle of radius $(V_s V_r/B)$ but with the center at the end of the fixed vector.

Accordingly, to construct the receiving-end power-circle diagram lay off the fixed vector

$$oa\underline{/180^\circ + (\alpha - \beta)} = -\frac{A V_r^2}{B}\underline{/\alpha - \beta}$$

to an appropriately chosen scale and with reference to the positive horizontal. Since the angle β is invariably larger than α this vector will be in the second quadrant measured from the horizontal through o and as indicated in Fig. 7–2. With point (a) as a center, draw to the same scale a circle

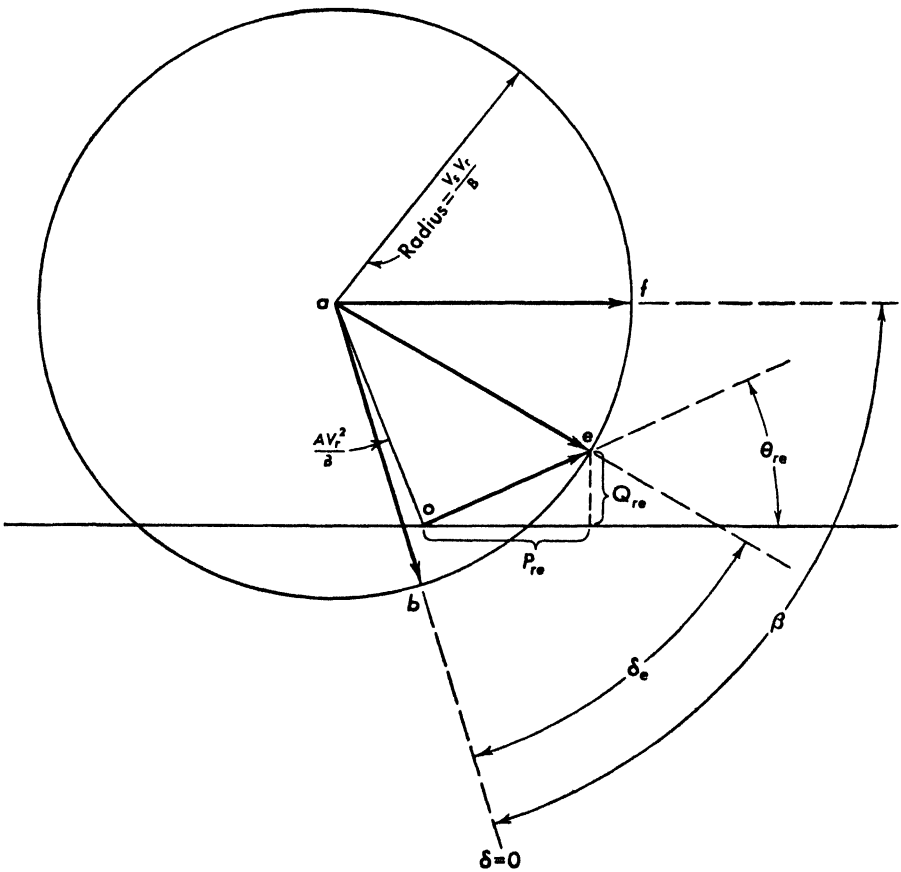


Figure 7–2. Receiving-End Power-Circle Diagram

of radius $(V_s V_r/B)$. This circle is the locus of the first term of equation (7.3.2) as the displacement angle δ is increased. The initial point of this locus occurs when $\delta = 0$, i.e., when $(V_s V_r/B)$ makes an angle $(-\beta)$ with the horizontal through the center at (a) as indicated by the line ab in the figure.

The same circle, but with o as the origin, represents the locus of the receiving-end voltamperes $(VI)_r$ as the displacement angle δ is increased. The initial point of this locus is also at (b) and the initial value of $(VI)_r$ occurring when $\delta = 0$ is ob.

Lines such as ob, oe, and of drawn from the origin at (o) to the circular locus give, therefore, the receiving-end voltamperes in terms of the chosen scale. The angles such as θ_{re} which these lines make with the horizontal through the origin at (o), are, respectively, the time-phase angles between the receiving-end current and the receiving-end voltage as the reference.

The displacement angles corresponding to the indicated values of $(VI)_r$ such as ob, oe, etc., are measured from the initial position (ab) of the (V_sV_r/B) vector as indicated in the figure by δ_e.

The horizontal projections of the $(VI)_r$ vectors represent to the chosen scale the receiving-end dissipative power P_r as given by equation (7.3.3). The power limit $P_{rm} = of_1$ is the horizontal component of $(VI)_r = of$, corresponding to the displacement angle $\delta = \beta$. Note that point (f) is the intersection of the horizontal from the center (a) to the circle locus.

The vertical projections of the $(VI)_r$ vectors represent to the chosen scale the receiving-end reactive power Q_r as given by equation (7.3.4) and illustrated by Q_{re} in Fig. 7–2. Note also that the reactive power is lagging in character when θ_r is negative and leading when θ_r is positive such as θ_{re}. The reactive power Q_{rm} (7.3.8) corresponding to the dissipative power limit is thus equal to the vertical projection from (f) and is leading in character.

The construction of the receiving-end power-circle diagram given in Fig. 7–2 is for the particular long line for which

$$A = \cosh pS = 0.816\underline{/4.35^\circ}$$
$$B = Z_o \sinh pS = (380\underline{/-9.5^\circ})(0.598\underline{/81.8^\circ}) = 227.5\underline{/72.3^\circ}$$
$$\frac{V_s}{V_r} = k = 1.$$

Substituting these values in (7.3.2) gives

$$(VI)_r\underline{/\theta_r} = \frac{V_r^2}{227.5}\underline{/\delta - 72.3^\circ} - \frac{0.816V_r^2}{227.5}\underline{/4.35^\circ - 72.3^\circ}$$

or

$$(VI)_r\underline{/\theta_r} = \frac{V_r^2}{227.5}[1.0\underline{/\delta - 72.3^\circ} + 0.816\underline{/112.05^\circ}].$$

Only the bracketed member was used in the construction of the diagram. Actual values are obtained by multiplying measured values of P_r, Q_r, and $(VI)_r$ by the factor $(V_r^2/227.5)$. Thus, if the receiving-end voltage is in kv, the measured values multiplied by $(kv_r^2/227.5)$ will give, respectively, receiving-end kva, kw, and the reactive power in kilo-vars.

The receiving-end voltamperes for a short line is substantially

$$(VI)_r\underline{/\theta_r} = \frac{V_sV_r}{Z}\underline{/\delta - \zeta} - \frac{V_r^2}{Z}\underline{/-\zeta}$$
$$= \frac{V_sV_r}{Z}\underline{/\delta - \zeta} + \frac{V_r^2}{Z}\underline{/180^\circ - \zeta} \qquad (7.3.9)$$

where $Z\underline{/\zeta}$ is the linear-line impedance.

From the similarity of this expression with (7.3.2) it may be inferred that the construction of the power-circle diagram of such a line is identical to that given above for the long line.

7.4. *Station-End Dissipative and Reactive Power in Terms of the Displacement Angle. Power-Circle Diagram.*

Consider the generalized voltage equation of a transmission system in terms of station-end values,

$$V_r = AV_s - BI_s. \qquad (7.4.1)$$

In polar notation this equation is

$$V_r\underline{/0^\circ} = AV_s\underline{/\alpha + \delta} - B\underline{/\beta}I_s\underline{/\theta_s + \delta}$$

where δ is the displacement angle between V_s and V_r, the latter being taken as the reference vector. Solving for the current gives

$$I_s\underline{/\theta_s} = \frac{AV_s}{B}\underline{/\alpha - \beta} - \frac{V_r}{B}\underline{/-(\delta + \beta)}.$$

The voltamperes at the station end, therefore, is

$$(VI)_s\underline{/\theta_s} = \frac{AV_s^2}{B}\underline{/\alpha - \beta} - \frac{V_sV_r}{B}\underline{/-(\delta + \beta)}. \qquad (7.4.2)$$

The dissipative power at the station end is the horizontal component of this expression

$$P_s = \frac{V_sV_r}{B}\left[\frac{AV_s}{V_r}\cos(\alpha - \beta) - \cos(\delta + \beta)\right]. \qquad (7.4.3)$$

The reactive component is the quadrature component of $(VI)_s$ and is

$$Q_s = \frac{V_sV_r}{B}\left[\frac{AV_s}{V_r}\sin(\alpha - \beta) + \sin(\delta + \beta)\right]. \qquad (7.4.4)$$

For constant values of V_s and V_r and $V_s/V_r = k$ the above two equations for the dissipative and reactive power at the station end, respectively, become

$$P_s = \frac{V_s^2}{B}\left[A\cos(\alpha - \beta) - \frac{1}{k}\cos(\delta + \beta)\right] \qquad (7.4.5)$$

$$Q_s = \frac{V_s^2}{B}\left[A\sin(\alpha - \beta) + \frac{1}{k}\sin(\delta + \beta)\right]. \qquad (7.4.6)$$

For any particular system having constant sending- and receiving-end voltages both P_s and Q_s are functions of the displacement angle. By

(7.4.5), it is seen that the power limit at the station end occurs when $\cos(\delta + \beta) = -1$, i.e., when

$$\delta = 180° - \beta.$$

Its value under these conditions is

$$P_{sm} = \frac{V_s^2}{B}\left[A\cos(\alpha - \beta) + \frac{1}{k}\right]. \tag{7.4.7}$$

The reactive power corresponding to the above dissipative power limit, by (7.4.6), is

$$Q_{sm} = \frac{AV_s^2}{B}\sin(\alpha - \beta). \tag{7.4.8}$$

As in the case of the receiving end, the variation of the voltamperes and of its components P_s and Q_s with the displacement angle δ may be visualized by a power-circle diagram obtained from equation (7.4.2).

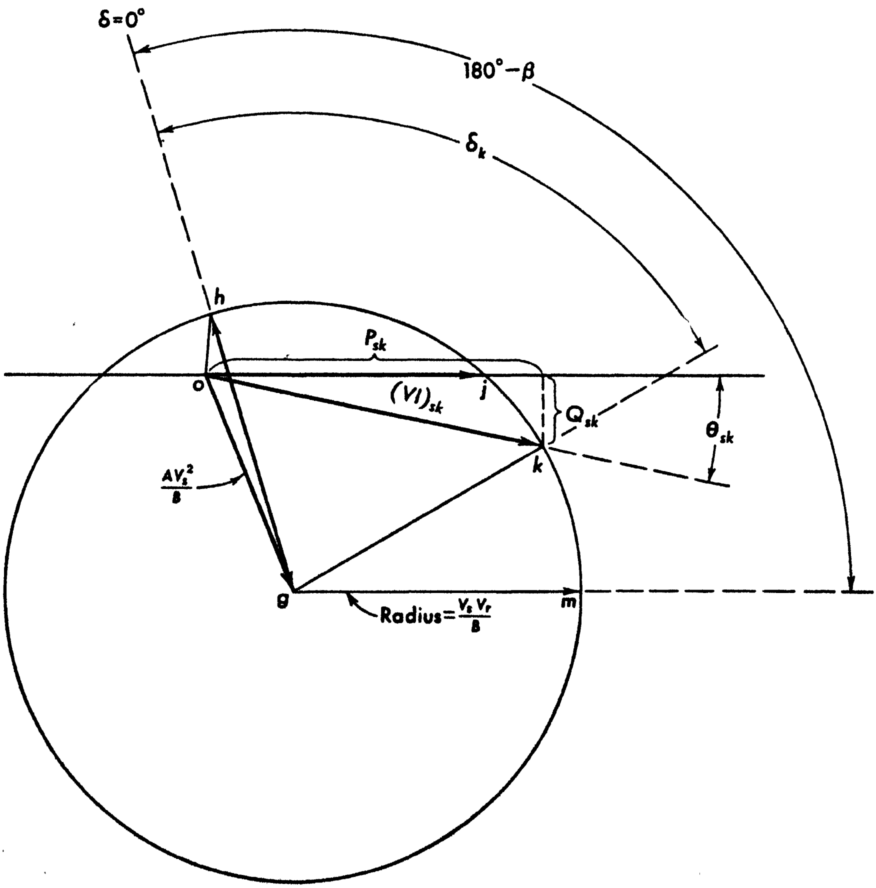


Figure 7–3. Sending-End Power-Circle Diagram

The construction is quite similar to that of the receiving end discussed in the preceding article. Thus, referring to Fig. 7–3 lay off the fixed vector

$$og\underline{/\alpha - \beta} = \frac{AV_s^2}{B}\underline{/\alpha - \beta}$$

to an appropriately chosen scale and with reference to the positive horizontal. Since the angle α is invariably smaller than β, this vector will be in the fourth quadrant as indicated in the figure. With point (g) as a center, draw to the same scale, a circle of radius (V_sV_r/B). This circle with (g) as the origin is the locus of the second term of equation (7.4.2) as the displacement angle is varied. The initial point of this locus occurs when $\delta = 0°$, i.e., when (V_sV_r/B) makes an angle ($180° - \beta$) with the horizontal through the center at (g) as indicated by the line gh.

The same circle, but with (o) as the origin, represents the locus of the voltamperes at the station end $(VI)_s$ as the displacement angle is increased. The initial point of this locus is at h and the initial value of $(VI)_s$ occurring when $\delta = 0°$ is oh.

Lines oh, oj, ok, etc., drawn from the origin at (o) to the circular locus, give values of $(VI)_s$ in terms of the chosen scale. The angles that these $(VI)_s$ vectors make with the horizontal through the origin at (o) are the respective time-phase angles between the station-end current and the station-end voltage.

The displacement angles, δ_j, δ_k, etc., corresponding to the various values of $(VI)_s$ are measured from the initial position gh of the (V_sV_r/B) vector as shown in the figure.

The horizontal projections oh_1, oj_1, ok_1, etc., represent to the chosen scale the dissipative power P_s delivered to the system at the station end. The power limit at this end is $P_{sm} = om_1$. It is the horizontal component of $(VI)_s = om$, corresponding to the displacement angle $\delta = 180° - \beta$ and is given by equation (7.4.7).

The vertical projections of the $(VI)_s$ vectors such as ho, kk_1, etc., give to the chosen scale the reactive powers at the station end. The reactive power corresponding to the power limit P_{sm} is $Q_{sm} = mm_1$ and is represented by equation (7.4.8).

The construction of the station-end power-circle diagram given in Fig. 7–3 is for the particular long line for which

$$A = 0.816\underline{/4.35°}$$

$$B = 227.5\underline{/72.3°} \quad \text{and} \quad \frac{V_s}{V_r} = k = 1.$$

Substituting these values in (7.4.2) gives

$$(VI)_s\underline{/\theta_s} = \frac{V_s^2}{227.5}(0.816\underline{/-67.95°} + 1.0\underline{/107.7° - \delta}).$$

Only the bracketed member was used in the construction of the diagram. Actual values are obtained by multiplying measured values by the multiplying factor $V_s^2/227.5$.

The voltamperes at the station end of a short line is substantially

$$(VI)_s\underline{/\theta_s} = \frac{V_s^2}{Z}\underline{/-\zeta} - \frac{V_sV_r}{Z}\underline{/-(\delta+\zeta)}. \tag{7.4.9}$$

where $Z\underline{/\zeta}$ is the linear-line impedance. From the similarity of this expression with (7.4.2), it follows that the construction of the power-circle diagram is identical.

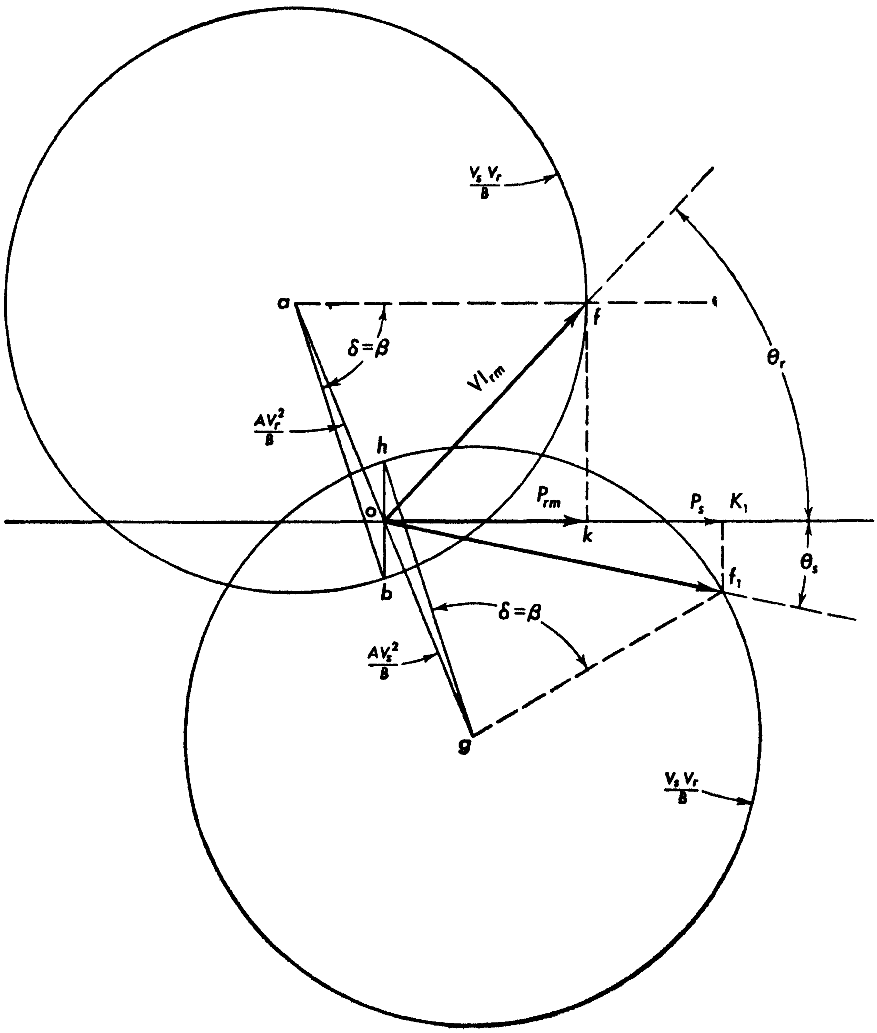


Figure 7–4. Combined Power-Circle Diagram

Using the same origin and the same scale, the power-circle diagram for the receiving end, Fig. 7–2, and that for the station end, Fig. 7–3, may be combined as shown in Fig. 7–4. By so doing, the values of voltamperes, dissipative and reactive powers at the two ends of the transmitting system for any displacement angle δ may be determined very easily. Thus, referring to the figure, it is seen that the power limit at the station end is quite larger than that at the receiving end. That at the receiving end occurs at a smaller displacement angle. The significance of this fact is discussed in § 7.8.

The $(VI)_r$ for the power limit, occurring when $\delta = \beta$, is proportional to the distance of. The corresponding $(VI)_s$ for the same value of δ is of_1. The ratio of/of_1 gives the voltampere efficiency for the receiving-end power limit.

The power limit at the receiving end is proportional to the distance ok. The corresponding dissipative power delivered at the station end is similarly proportional to the distance ok_1. The power efficiency of transmission is, therefore, ok/ok_1. The loss in the line at the power limit is proportional to the distance kk_1. Values of VI, P, losses and efficiency for other values of displacement angle may be obtained in the same manner.

7.5. *Power-Angle Curves.* Graphical methods of investigation are used frequently in system stability studies. The graphical representation of the power-angle equations (7.3.5) and (7.4.5) are of great importance in such stability studies and are called *power-angle curves.*

To illustrate power-angle curves, consider the short 25-mile line used in the illustrative problem in § 6.8. Its constants are:

$$A = 1\underline{/0^\circ};\ B = 20.2\underline{/71^\circ};\ V_r = 19 \text{ kv and } k = 1.4.$$

The power-angle equation, by (7.3.5), is

$$\begin{aligned} P_r &= 17860[1.4 \cos(\delta - 71^\circ) - \cos 71^\circ] \\ &= 25000 \cos(\delta - 71^\circ) - 5820 \text{ kw.} \end{aligned}$$

The power-angle curve for this line is shown in Fig. 7–5.

The constants of the 100-mile line used in the illustrative problem in § 6.9 are: $A = 0.9787\underline{/0^\circ}$; $B = 80\underline{/71.2^\circ}$; $V_r = 63.5$ kv and $k = 1.1$. The power-angle equation for this line in accordance with (7.3.5) is

$$\begin{aligned} P_r &= 50400[1.1 \cos(\delta - 71.2^\circ) - 0.9787 \cos 71.2^\circ] \\ &= 55440 \cos(\delta - 71.2^\circ) - 15900 \text{ kw.} \end{aligned}$$

Similarly the constants of the 300-mile line used in the illustrative problem in § 6.10 are: $A = 0.816\underline{/4.35^\circ}$; $B = 227.2\underline{/72.3^\circ}$; $V_r = 125$ kv and $k = 0.92$. The power-angle equation for this line, by (7.3.5), is

$$P_r = 68800\,[0.92 \cos(\delta - 72.3°) - 0.816 \cos 67.95°]$$
$$= 63400 \cos(\delta - 72.3°) - 21080 \text{ kw.}$$

The power-angle curves as obtained from these equations are sinusoidal, similar in shape to that shown in Fig. 7–5 for the short line.

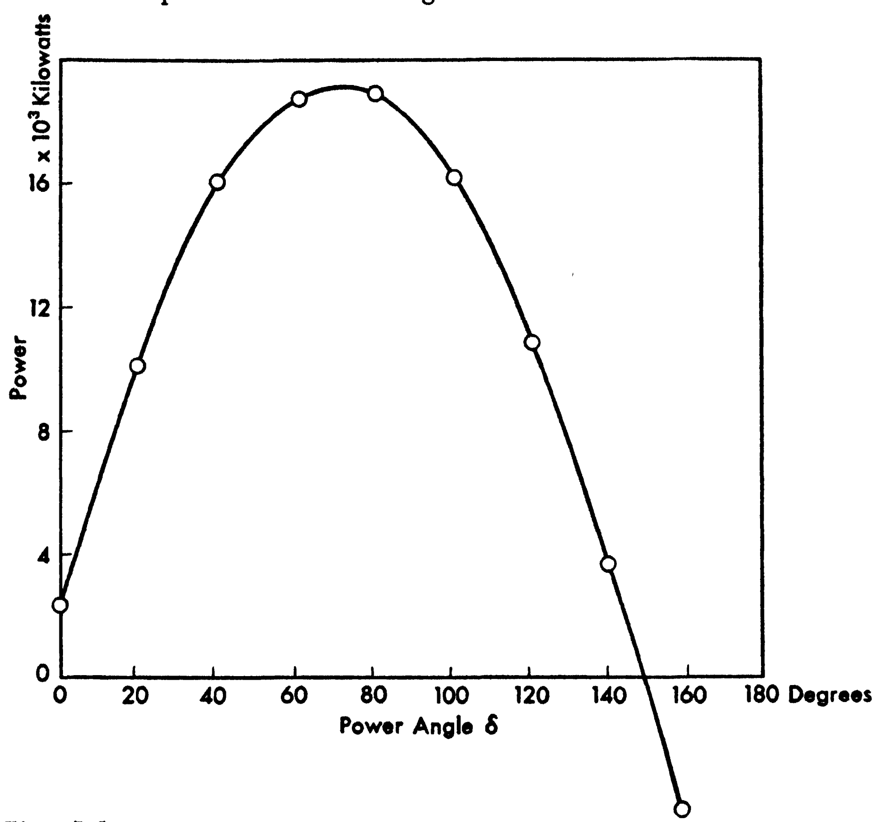


Figure 7–5

7.6. *Single Impedance Equivalent of a Transmission System.* It was shown in the previous three chapters that symmetrical T networks may be used with great convenience in the investigation of the performance of transmission systems. For short lines the shunt branch of the T equivalent of the line is assigned zero admittance. For medium long lines, the shunt branch of the T equivalent is assigned the lumped value of the linear-line admittance, and each series branch is assigned one-half of the lumped linear-line impedance. For very long lines, the shunt branch is assigned a lumped admittance $(\sinh pS)/Z_o$ and each series branch a lumped impedance equal to $Z_o \tanh(pS/2)$. Lumped impedance values for the series branches and lumped admittance values for the shunt branch may be obtained for lines with transformers or for twin lines with or without transformers as discussed in the preceding chapters.

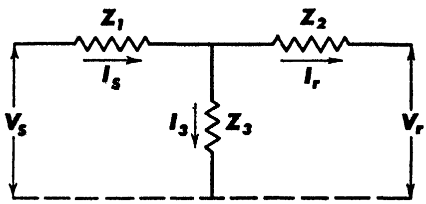


Figure 7–6

In the study of power limits and stability behavior of complete power systems or parts thereof, such as transmission lines, it is frequently desirable and often imperative to simplify further the network between the termini of the system by replacing it by a single fictitious impedance.

Thus, consider the general T network shown in Fig. 7–6. Let V_r be the potential difference to neutral across the receiving ends. Referring to the figure it is seen that

$$V_s = I_sZ_1 + I_rZ_2 + V_r. \tag{7.6.1}$$

Since

$$I_s = I_3 + I_r$$

the preceding expression may be written in terms of I_r

$$V_s = I_r(Z_1 + Z_2) + I_3Z_1 + V_r.$$

Furthermore, since

$$I_3 = \frac{V_r + I_rZ_2}{Z_3},$$

the above equation becomes

$$V_s = I_r\left(Z_1 + Z_2 + \frac{Z_1Z_2}{Z_3}\right) + \left(\frac{Z_1}{Z_3} + 1\right)V_r.$$

This simplifies to

$$V_s = I_r\left[\frac{(Z_1 + Z_3)Z_2}{Z_3} + Z_1\right] + \left(\frac{Z_1 + Z_3}{Z_3}\right)V_r,$$

or

$$\left(\frac{Z_3}{Z_1 + Z_3}\right)V_s = I_r\left(\frac{Z_1Z_3}{Z_1 + Z_3} + Z_2\right) + V_r. \tag{7.6.2}$$

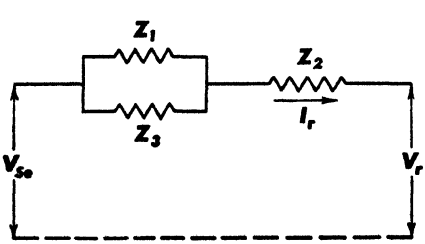


Figure 7–6a

This expression indicates that the T network, shown in Fig. 7–6, may be replaced by the one shown in Fig. 7–6a in which the series equivalent impedance replacing the T is

$$Z_{ea} = \frac{Z_1Z_3}{Z_1 + Z_3} + Z_2{}^* \tag{7.6.3}$$

* This follows also from Thevenin's theorem.

and the sending-end voltage V_s changed to

$$V_{se} = \left(\frac{Z_3}{Z_1 + Z_3}\right) V_s. \qquad (7.6.4)$$

It is important to note that in this equivalent circuit the receiving-end voltage V_r and current I_r are the same as in the actual circuit while the sending-end voltage and current differ from their respective values in the actual circuit. It follows, therefore, that the equivalent circuit in Fig. 7–6a could be used to calculate actual voltage, current, and power at the receiving end but not at the sending end.

Another series equivalent circuit may be obtained for the calculation of the actual voltage, current, and power values at the sending end. Thus, the voltage equation (7.6.1) of the T circuit may be written in terms of I_s as follows

$$V_s = I_s(Z_1 + Z_2) - I_3Z_2 + V_r.$$

Since

$$I_3 = \frac{V_s - I_sZ_1}{Z_3},$$

the expression becomes

$$V_s = I_s\left(Z_1 + Z_2 + \frac{Z_1Z_2}{Z_3}\right) - \frac{Z_2V_s}{Z_3} + V_r$$

or

$$\left(\frac{Z_2 + Z_3}{Z_3}\right) V_s = I_s\left[Z_2 + \frac{Z_1(Z_2 + Z_3)}{Z_3}\right] + V_r$$

or

$$V_s = I_s\left(Z_1 + \frac{Z_2Z_3}{Z_2 + Z_3}\right) + \left(\frac{Z_3}{Z_2 + Z_3}\right) V_r. \qquad (7.6.5)$$

This expression indicates that the T network shown in Fig. 7–6 may be replaced also by the one shown in Fig. 7–6b in which the series equivalent impedance replacing the T is

$$Z_{eb} = Z_1 + \frac{Z_2Z_3}{Z_2 + Z_3}, \qquad (7.6.6)$$

and the receiving-end voltage changed to

$$V_{re} = \left(\frac{Z_3}{Z_2 + Z_3}\right) V_r. \qquad (7.6.7)$$

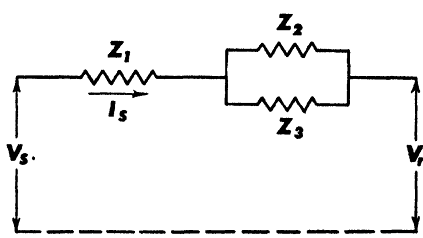


Figure 7–6b

Note again that, in the equivalent circuit shown in Fig. 7–6b, the sending-end voltage V_s and current I_s are the same as in the actual circuit while the receiving-end voltage and current differ from their respective values in the actual circuit. It follows, therefore, that the equivalent circuit in Fig. 7–6b could be used to calculate actual voltage, current, and power at the sending end but not at the receiving end.

For the specific case when the T network is symmetrical, as is the case of lines with or without transformers, $Z_1 = Z_2 = Z_T$ and $Z_3 = 1/Y_T$.

Accordingly,

$$\frac{Z_3}{Z_1 + Z_3} = \frac{1}{1 + Z_T Y_T} = a. \tag{7.6.8}$$

Equations (7.6.4) and (7.6.7) may therefore be written for the series equivalent of a symmetrical T

$$V_{se} = \frac{V_s}{1 + Z_T Y_T} = aV_s \tag{7.6.9}$$

and

$$V_{re} = \frac{V_r}{1 + Z_T Y_T} = aV_r. \tag{7.6.10}$$

Equations (7.6.3) and (7.6.6) for the series equivalent impedances of a symmetrical T are thus equal and of value

$$Z_{ea} = Z_{eb} = Z_T + \frac{Z_T}{1 + Z_T Y_T} = Z_e$$

$$Z_e = Z_T(1 + a). \tag{7.6.11}$$

To illustrate the above, consider the T equivalent of the 300-mile line used in § 4.2. The T network is shown in Fig. 7–6. By (7.6.8),

$$a = \frac{1}{1 + (125.3\underline{/69.75^\circ})(1573 \times 10^{-6}\underline{/90^\circ})}$$

$$= 1.224\underline{/-4.82^\circ}.$$

Hence, by (7.6.11),

$$Z_e = 125.3\underline{/69.75^\circ}(1 + 1.224\underline{/-4.82^\circ})$$

$$= 278\underline{/67.08^\circ}$$

and

$$V_{se} = (1.224\underline{/-4.82^\circ})V_s$$

for the equivalent circuit shown in Fig. 7–6a, and

$$V_{re} = (1.224\underline{/-4.82^\circ})V_r$$

for the equivalent circuit in Fig. 7–6b.

7.7. *Dissipative and Reactive Powers in Terms of the Single Impedance Equivalent of a Line.* From what has been said in the preceding article it follows that the line should be represented in this case by the series equivalent impedance Z_e given by (7.6.11). To obtain receiving-end values, the sending-end voltage must be changed to that given by (7.6.9). The voltage equation of the circuit is

$$V_{se}\underline{/\delta_e} = V_r\underline{/0^\circ} + (I_r\underline{/\theta_r})(Z_e\underline{/\zeta_e}).$$

Solving for the current gives

$$I_r\underline{/\theta_r} = \frac{V_{se}\underline{/\delta_e - \zeta_e}}{Z_e} - \frac{V_r\underline{/-\zeta_e}}{Z_e}.$$

The voltamperes at the receiving end are

$$(VI)_r\underline{/\theta_r} = \frac{V_{se}V_r\underline{/\delta_e - \zeta_e}}{Z_e} - \frac{V_r^2\underline{/-\zeta_e}}{Z_e}. \tag{7.7.1}$$

From the above, it follows that the dissipative and corresponding reactive powers are, respectively,

$$P_r = \frac{V_{se}V_r}{Z_e}\left[\cos(\delta_e - \zeta_e) - \frac{V_r}{V_{se}}\cos\zeta_e\right] \tag{7.7.2}$$

$$Q_r = \frac{V_{se}V_r}{Z_e}\left[\sin(\delta_e - \zeta_e) + \frac{V_r}{V_{se}}\sin\zeta_e\right] \tag{7.7.3}$$

where δ_e is the displacement angle between the receiving-end voltage as the reference and the equivalent sending-end voltage V_{se} that must be used with the single impedance equivalent of the transmission system.

It should be noted that the above expressions for P_r and Q_r are identical in form to those for a short line in which the linear-line admittance is neglected.

To obtain the dissipative and reactive powers at the sending end, the receiving-end voltage should be changed to that given by (7.6.10). Accordingly, the voltage equation of the circuit, using V_{re} as the reference

$$V_{re}\underline{/0^\circ} = V_s\underline{/\delta_e} - (I_s\underline{/\theta_s + \delta_e})(Z_e\underline{/\zeta_e}).$$

This gives for the current at the sending end

$$I_s\underline{/\theta_s} = \frac{V_s\underline{/-\zeta_e}}{Z_e} - \frac{V_{re}\underline{/-(\delta_e + \zeta_e)}}{Z_e}.$$

The voltamperes at the station end is

$$(VI)_s\underline{/\theta_s} = \frac{V_s^2\underline{/-\zeta_e}}{Z_e} - \frac{V_sV_{re}\underline{/-(\delta_e+\zeta_e)}}{Z_e}.$$

The dissipative and reactive powers at the station end are, accordingly,

$$P_s = \frac{V_sV_{re}}{Z_e}\left[\frac{V_s}{V_{re}}\cos\zeta_e - \cos(\delta_e + \zeta_e)\right] \tag{7.7.4}$$

$$Q_s = \frac{V_sV_{re}}{Z_e}\left[\sin(\delta_e + \zeta_e) - \frac{V_s}{V_{re}}\sin\zeta_e\right]. \tag{7.7.5}$$

7.8. *Steady State Power Limit of Generator and Line. Generator Excitation emf and Receiving-End Voltage Maintained Constant.* The method developed in the preceding article for the determination of the dissipative powers P_s and P_r, respectively, supplied to and delivered by a transmission system using the series impedance equivalent of the system may be extended to systems including the generator equipment at the station end.

When the bus voltage at either end of a transmission system is maintained constant and is unaffected by any change in the dissipative or in the reactive power or in both, the bus at that end is said to be an *infinite bus.* The name implies that the impedance behind such a bus is capable of receiving an infinite amount of energy provided it can get it.

The system is represented diagrammatically by Fig. 7-7 in which Z_G is the impedance per phase of the generator.

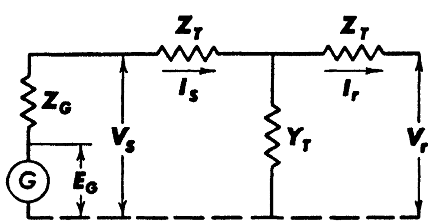


Figure 7-7

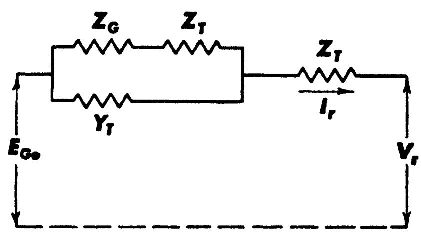


Figure 7-7a

For the determination of the power at the receiving-end where V_r is maintained constant, the system should be represented by the series impedance equivalent shown in Fig. 7-7a. This equivalent series impedance including the phase impedance Z_G of the generator is obtained by (7.6.3) in which

$$Z_1 = Z_G + Z_T$$

$$Z_3 = \frac{1}{Y_T}$$

and

$$Z_2 = Z_T.$$

This gives

$$Z_{ea} = \frac{Z_G + Z_T}{1 + (Z_G + Z_T)Y_T} + Z_T. \tag{7.8.1}$$

The generated emf at the sending end, by (7.6.9), is

$$E_{Ge} = \frac{E_G}{1 + (Z_G + Z_T)Y_T}. \tag{7.8.2}$$

By analogy to (7.7.1), the dissipative power at the receiving end is

$$P_r = \frac{E_{Ge}V_r}{Z_{ea}}\left[\cos(\delta_{ea} - \zeta_{ea}) - \frac{V_r}{E_{Ge}}\cos\zeta_{ea}\right], \quad (7.8.3)$$

where δ_{ea} is the displacement angle between V_r and E_{Ge}, and ζ_{ea} is the angle associated with the series impedance equivalent Z_{ea}.

The series circuit equivalent to the system circuit, and which should be used for the determination of the dissipative power supplied by the prime mover and converted into electrical power by the generator, is shown in Fig. 7–7b. The equivalent series impedance is, by (7.6.6),

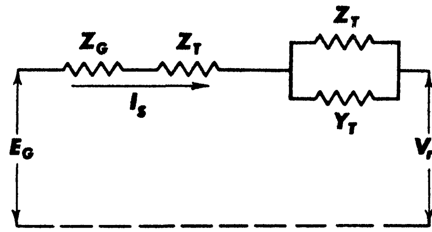


Figure 7–7b

$$Z_{eb} = (Z_G + Z_T) + \frac{Z_T}{1 + Z_T Y_T} \quad (7.8.4)$$

and the voltage at the receiving end should be changed, by (7.6.7), to

$$V_{re} = \frac{V_r}{1 + Z_T Y_T}. \quad (7.8.5)$$

By analogy to equation (7.7.2), the dissipative power converted in the generator per phase is

$$P_G = \frac{E_G V_{re}}{Z_{eb}}\left[\frac{E_G}{V_{re}}\cos\zeta_{eb} - \cos(\delta_{eb} + \zeta_{eb})\right] \quad (7.8.6)$$

where E_G is the generator excitation voltage per phase, δ_{eb} the displacement angle between E_G and V_{re}, and ζ_{eb} the angle associated with the equivalent impedance Z_{eb}.

Equation (7.8.3) indicates that the power limit at the receiving end occurs when $\cos(\delta_{ea} - \zeta_{ea}) = 1$, i.e., when $\delta_{ea} = \zeta_{ea}$, and is

$$P_{rm} = \frac{E_{Ge}V_r}{Z_{ea}}\left[1 - \frac{V_r}{E_{Ge}}\cos\zeta_{ea}\right]. \quad (7.8.7)$$

Similarly, equation (7.8.6) indicates that the maximum converted power occurs when $\cos(\delta_{eb} + \zeta_{eb}) = -1$, i.e., when

$$\delta_{eb} = 180° - \zeta_{eb}.$$

The value of the maximum converted power, accordingly, is

$$P_{Gm} = \frac{E_G V_{re}}{Z_{eb}}\left[\frac{E_G}{V_{re}}\cos\zeta_{eb} + 1\right]. \quad (7.8.8)$$

To illustrate the above, consider a transmission system for which

$$Z_T = 125.3\underline{/69.75^\circ};\ Y_T = 1573 \times 10^{-6}\underline{/90^\circ};$$

and generator phase impedance

$$Z_G = 10\underline{/90^\circ}.$$

By (7.8.1),

$$Z_{ea} = 292\underline{/67.72^\circ}.$$

By (7.8.2),

$$E_{Ge} = (1.25\underline{/-4.9^\circ})E_G.$$

By (7.8.4),

$$Z_{eb} = 288\underline{/67.86^\circ},$$

and by (7.8.5),

$$V_{re} = (1.22\underline{/-4.81^\circ})V_r.$$

For the specific case, when $E_G/V_r = 1.2$, obtain

$$\frac{V_r}{E_{Ge}} = 0.667$$

$$\frac{E_G}{V_{re}} = 0.984$$

$$E_{Ge}V_r = 1.25E_GV_r$$

$$E_GV_{re} = 1.22\, E_GV_r$$

$$\delta_{ea} = \delta - 4.9^\circ$$

$$\delta_{eb} = \delta + 4.81^\circ.$$

Substituting in the equations for P_r and P_G gives

$$P_r = \frac{1.25E_GV_r}{292}[\cos(\delta - 4.9^\circ - 67.72^\circ) - 0.667 \cos 67.72^\circ]$$

$$P_G = \frac{1.22E_GV_r}{288}[0.984 \cos 67.86^\circ - \cos(\delta + 4.81^\circ + 67.86^\circ)]$$

or

$$P_r = \frac{E_GV_r}{233.5}[\cos(\delta - 72.62^\circ) - 0.253]$$

$$P_G = \frac{E_GV_r}{236}[0.371 - \cos(\delta + 72.67^\circ)].$$

Setting

$$\frac{E_GV_r}{233.5} = N,$$

then

$$\frac{E_GV_r}{236} = 0.989N,$$

and the above expressions reduce to

$$P_r = N[\cos(\delta - 72.62^\circ) - 0.253]$$

$$P_G = N[0.367 - 0.989 \cos(\delta + 72.67^\circ)].$$

The curves in Fig. 7–8 show the variation of P_r and P_G as functions of the displacement angle δ. Note that the power limit P_{rm} at the receiving end occurs when $\delta = 72.62°$, and the power limit P_{Gm} converted by the generator for the particular excitation voltage $E_G = 1.2\ V_r$ occurs when $\delta = 180° - 72.67° = 107.33°$. The curves show that, as the power demand

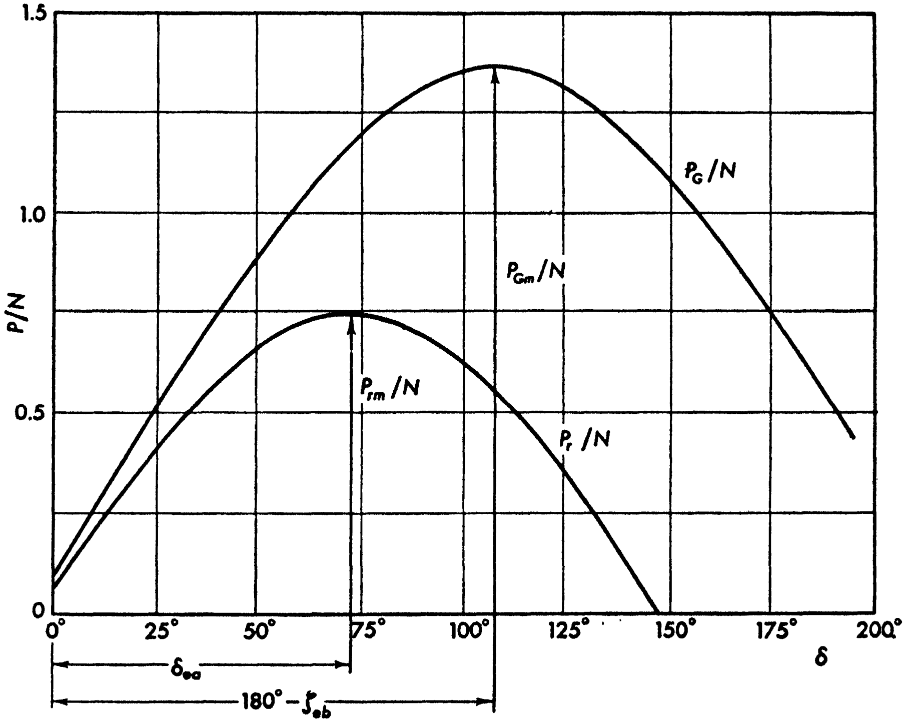


Figure 7–8

increases at the receiving end, the power supplied to and converted by the generator increases also but much more rapidly in proportion to the demand at the receiving end because of the increased losses that must be taken care of. The fact that the power limit P_{rm} at the receiving end is reached at a smaller displacement angle than that required by the power limit P_{Gm} supplied by the prime mover indicates that it is uneconomical to operate the system beyond the power limit at the receiving end. For beyond that a large amount of power is produced by the prime mover but only a small portion is delivered to the load.

The discussion of the power limits in this article pertains to a system in which the excitation voltage E_G of the alternator is maintained constant and the receiving end is an infinite bus. The fact that the generated emf E_G is maintained constant presupposes that the bus voltage at the station end is permitted to change with changes in load. Its value for any displacement angle δ between the known values of E_G and V_r may be obtained from the relations pertaining to the series equivalent of the system shown in Fig. 7–7b.

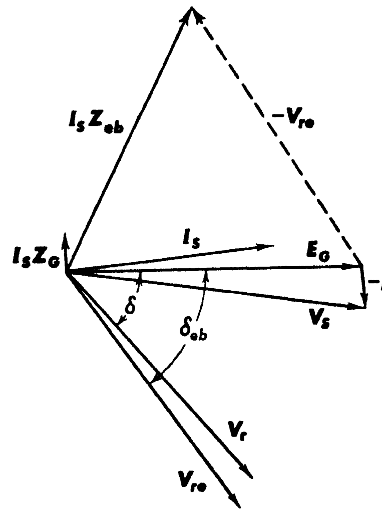


Figure 7–9

From this figure note that

$$E_G - I_sZ_{eb} = V_{re}$$

where V_{re} is obtained by (7.8.5). Hence,

$$I_s = \frac{E_G - V_{re}}{Z_{eb}}.$$

The bus voltage V_s at the station end, therefore, is

$$V_s = E_G - I_sZ_G.$$

The calculations are visualized in the vector diagram shown in Fig. 7–9 for given values of E_G, V_r, and displacement angle δ.

7.9. *Steady State Power Limit; Receiving-End Infinite Bus; Sending-End Voltage Maintained Constant by Adjustment of Excitation emf.*

In the case discussed in the preceding article, the excitation voltage E_G was maintained constant and the station-end bus voltage V_s permitted to vary as the receiving-end load increased to its limiting value. The power limit of the generator was obtained by converting the T equivalent of the system into the series equivalent Fig. 7–7b and the receiving-end power limit was similarly obtained by the use of the series equivalent shown in Fig. 7–7a. In the present case the excitation emf E_G is adjusted to maintain V_s constant as the receiving end load is continually increased. The system is represented diagrammatically in Fig. 7–7, but neither of its series equivalents, Figs. 7–7a and 7–7b, can be used in the solution of the present case. An approximate solution may be arrived at, however, by neglecting the linear-line admittance Y_T and thus assuming the same current at sending and receiving ends of the system as indicated in the diagram, Fig. 7–10.

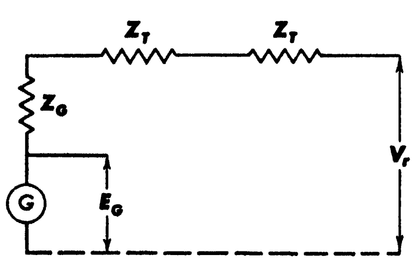


Figure 7–10

The system impedance from the receiving-end to the sending-end bus is

$$Z_{rs} = 2Z_T, \tag{7.9.1}$$

and that including the generator impedance is

$$Z_{rG} = Z_G + Z_{rs}. \tag{7.9.2}$$

The expression for the receiving-end power is

$$P_r = \frac{E_G V_r}{Z_{rG}}\left[\cos(\delta_{rG} - \zeta_{rG}) - \frac{V_r}{E_G}\cos\zeta_{rG}\right], \tag{7.9.3}$$

where δ_{rG} is the displacement angle between the receiving-end voltage V_r taken as a reference and E_G. The angle ζ_{rG} is the angle of the system impedance Z_{rG} as given by (7.9.2).

To determine the power limit by the above expression it is necessary that E_G be expressed in terms of the known quantities V_r and V_s and the displacement angle δ_{rs} between them.

Referring to Fig. 7–10 it is seen that

$$V_s\underline{/\delta_{rs}} = V_r\underline{/0°} + (I\underline{/\theta_r})(Z_{rs}\underline{/\zeta_{rs}}), \tag{7.9.4}$$

where ζ_{rs} is the angle associated with Z_{rs}; θ_r is the phase angle between I and V_r, and δ_{rs} the displacement angle between V_r and V_s.

The voltage equation of the system from the receiving end including the excitation emf E_G, similarly, is

$$E_G\underline{/\delta_{rG}} = V_r\underline{/0} + (I\underline{/\theta_r})(Z_{rG}\underline{/\zeta_{rG}}). \tag{7.9.5}$$

Solving (7.9.4) and (7.9.5), respectively, for the current I, and equating, yields

$$V_s\underline{/\delta_{rs}} = \frac{Z_{rs}E_G}{Z_{rG}}\underline{/\delta_{rG} + \zeta_{rs} - \zeta_{rG}} + \frac{V_r(Z_{rG}\underline{/\zeta_{rG}} - Z_{rs}\underline{/\zeta_{rs}})}{Z_{rG}\underline{/\zeta_{rG}}}. \tag{7.9.6}$$

Note that by (7.9.2)

$$Z_{rG}\underline{/\zeta_{rG}} - Z_{rs}\underline{/\zeta_{rs}} = Z_G\underline{/\gamma}$$

where γ is the angle of the generator phase impedance Z_G.

Setting for brevity's sake

$$\left.\begin{aligned}\delta_{rG} + \zeta_{rs} - \zeta_{rG} &= \phi_1\\ \gamma - \zeta_{rG} &= \phi_2\end{aligned}\right\} \tag{7.9.7}$$

the above voltage expression may be written conveniently,

$$V_s\underline{/\delta_{rs}} = \frac{Z_{rs}E_G}{Z_{rG}}\underline{/\phi_1} + \frac{Z_G V_r}{Z_{rG}}\underline{/\phi_2}. \tag{7.9.8}$$

The scalar relation between the quantities involved, accordingly, is

$$V_s^2 = \left[\frac{Z_{rs}E_G}{Z_{rG}}\cos\phi_1 + \frac{Z_G V_r}{Z_{rG}}\cos\phi_2\right]^2 + \left[\frac{Z_{rs}E_G}{Z_{rG}}\sin\phi_1 + \frac{Z_G V_r}{Z_{rG}}\sin\phi_2\right]^2.$$

Collecting terms, gives

$$V_s^2 = \left(\frac{Z_{rs}E_G}{Z_{rG}}\right)^2 + \left(\frac{Z_G V_r}{Z_{rG}}\right)^2 + \frac{2Z_{rs}Z_G}{Z_{rG}^2} E_G V_r \cos(\phi_1 - \phi_2). \tag{7.9.9}$$

To solve this expression for E_G, set

$$\left.\begin{aligned} \left(\frac{Z_{rG}V_s}{Z_{rs}}\right)^2 &= A \\ \left(\frac{Z_G V_s}{Z_{rs}}\right)^2 &= B \\ \text{and} \qquad \frac{2Z_G V_r}{Z_{rs}} \cos(\phi_1 - \phi_2) &= C. \end{aligned}\right\} \tag{7.9.10}$$

Equation (7.9.9) becomes, accordingly,

$$E_G^2 + CE_G = A - B$$

which, when solved, gives

$$E_G = \frac{-C}{2} \pm \sqrt{\frac{C^2}{4} + (A - B)}.$$

Substituting the relations given in (7.9.10) gives

$$E_G = \frac{-Z_G V_r \cos(\phi_1 - \phi_2)}{Z_{rs}} \pm \sqrt{\left(\frac{Z_G V_r}{Z_{rs}}\right)^2 [\cos^2(\phi_1 - \phi_2) - 1] + \left(\frac{Z_{rG}V_s}{Z_{rs}}\right)^2}.$$

Since $\cos^2(\phi_1 - \phi_2) - 1 = -\sin^2(\phi_1 - \phi_2)$ the above expression may be written

$$E_G = \frac{-Z_G V_r \cos(\phi_1 - \phi_2)}{Z_{rs}} \pm \sqrt{\left(\frac{Z_{rG}V_r}{Z_{rs}}\right)^2 - \left(\frac{Z_G V_r}{Z_{rs}}\right)^2 \sin^2(\phi_1 - \phi_2)}. \tag{7.9.11}$$

Factoring out $Z_G V_r/Z_{rs}$ and using the values of ϕ_1 and ϕ_2 from (7.9.7), this expression becomes:

$$E_G = \frac{Z_G V_r}{Z_{rs}} \left\{ \sqrt{\left(\frac{Z_{rG}V_s}{Z_G V_r}\right)^2 - \sin^2(\delta_{rG} + \zeta_{rs} - \gamma)} - \cos(\delta_{rG} + \zeta_{rs} - \gamma) \right\}. \tag{7.9.12}$$

This value of E_G should be used in expression (7.9.3) for the receiving-end power P_r.

Note, however, that the expression for P_r may be written

$$P_r = \frac{E_G V_r}{Z_{rG}} \cos(\delta_{rG} - \zeta_{rG}) - \frac{V_r^2}{Z_{rG}} \cos \zeta_{rG}.$$

Hence, using the above value of E_G, gives

$$\begin{aligned} P_r = \frac{Z_G V_r^2}{Z_{rG}Z_{rs}} \cos(\delta_{rG} - \zeta_{rG}) &\left\{ \sqrt{\left(\frac{Z_{rG}V_s}{Z_G V_r}\right)^2 - \sin^2(\delta_{rG} + \zeta_{rs} - \gamma)} \right. \\ &\left. - \cos(\delta_{rG} + \zeta_{rs} - \gamma) \right\} - \frac{V_r^2}{Z_{rG}} \cos \zeta_{rG}. \end{aligned} \tag{7.9.13}$$

Note that the only variable in this expression is the displacement angle δ_{rG} between V_r and E_G.

The power limit may now be determined by first obtaining the condition for which

$$\frac{dP_r}{d\delta_{rG}} = 0.$$

To do this set, for the sake of convenience,

$$\frac{Z_G V_r^2}{Z_{rG} Z_{rs}} \cos(\delta_{rG} - \zeta_{rG}) = X$$

$$\sqrt{\left(\frac{Z_{rG} V_s}{Z_G V_r}\right)^2 - \sin^2(\delta_{rG} + \zeta_{rs} - \gamma)} = Y$$

$$\cos(\delta_{rG} + \zeta_{rs} - \gamma) = U$$

$$\frac{V_r^2}{Z_{rG}} \cos \zeta_{rG} = K$$

$$\frac{Z_G V_r^2}{Z_{rG} Z_{rs}} = M$$

and

$$\left(\frac{Z_{rG} V_s}{Z_G V_r}\right)^2 = N.$$

Accordingly, the expression for P_r simplifies to

$$P_r = XY - XU - K.$$

Hence,

$$\frac{dP_r}{d\delta_{rG}} = \frac{X\,dY}{d\delta_{rG}} + \frac{Y\,dX}{d\delta_{rG}} - \frac{X\,dU}{d\delta_{rG}} - \frac{U\,dX}{d\delta_{rG}}$$
$$= X\left(\frac{dY}{d\delta_{rG}} - \frac{dU}{d\delta_{rG}}\right) + (Y - U)\frac{dX}{d\delta_{rG}}$$

in which

$$\frac{dX}{d\delta_{rG}} = -M \sin(\delta_{rG} - \zeta_{rG})$$

$$\frac{dY}{d\delta_{rG}} = \frac{-\cos(\delta_{rG} + \zeta_{rs} - \gamma)\sin(\delta_{rG} + \zeta_{rs} - \gamma)}{\sqrt{N - \sin^2(\delta_{rG} + \zeta_{rs} - \gamma)}}$$

and

$$\frac{dU}{d\delta_{rG}} = -\sin(\delta_{rG} + \zeta_{rs} - \gamma).$$

Substituting in the above derivative of P_r gives

$$\frac{dP_r}{d\delta_{rG}} = M\cos\alpha\left[\frac{-\cos\beta\sin\beta}{\sqrt{N - \sin^2\beta}} + \sin\beta\right] - M\sin\alpha[\sqrt{N - \sin^2\beta} - \cos\beta]$$

where, for convenience

$$\delta_{rG} + \zeta_{rs} - \gamma = \beta$$

and

$$\delta_{rG} - \zeta_{rG} = \alpha.$$

The above derivative may be further simplified to

$$\frac{dP_r}{d\delta_{rG}} = \frac{M \cos\alpha \sin\beta}{\sqrt{N - \sin^2\beta}} [\sqrt{N - \sin^2\beta} - \cos\beta] - M \sin\alpha[\sqrt{N - \sin^2\beta} - \cos\beta].$$

Factoring out the quantity in the bracket gives

$$\frac{dP_r}{d\delta_{rG}} = M[\sqrt{N - \sin^2\beta} - \cos\beta]\left[\frac{\cos\alpha \sin\beta}{\sqrt{N - \sin^2\beta}} - \sin\alpha\right]. \tag{7.9.14}$$

For $dP_r/d\delta_{rG} = 0$, either the first or the second bracket in the above expression must be equal to zero. Setting the first equal to zero gives $N = 1$. But, since

$$N = \frac{Z_{rG}V_s}{Z_G V_r} \neq 1,$$

it follows that the second bracket in the above expression is equal to zero, i.e.,

$$\cos\alpha \sin\beta = \sin\alpha \sqrt{N - \sin^2\beta}.$$

Squaring and collecting terms gives

$$(\cos^2\alpha + \sin^2\alpha)\sin^2\beta = N \sin^2\alpha$$
$$\sin\beta = \sqrt{N} \sin\alpha.$$

Using the values of α and β, the expression becomes

$$\sin(\delta_{rG} + \zeta_{rs} - \gamma) = \sqrt{N} \sin(\delta_{rG} - \zeta_{rG}).$$

Expanding and collecting terms, yields

$$\sin\delta_{rG}[\cos(\zeta_{rs} - \gamma) - \sqrt{N}\cos\zeta_{rG}] = -\cos\delta_{rG}[\sqrt{N}\sin\zeta_{rG} + \sin(\zeta_{rs} - \gamma)]$$

or

$$\tan\delta_{rG} = \frac{\sqrt{N}\sin\zeta_{rG} + \sin(\zeta_{rs} - \gamma)}{\sqrt{N}\cos\zeta_{rG} - \cos(\zeta_{rs} - \gamma)} \tag{7.9.15}$$

where, as stated above,

$$\sqrt{N} = \frac{Z_{rG}V_s}{Z_G V_r}.$$

Equation (7.9.15) gives the displacement angle between the receiving-end voltage V_r and the excitation emf E_G for the power limit at the receiving end.

The maximum value δ_{rG} can have is $180° - \zeta_{rG}$ when the generator pulls out of synchronism. It follows, therefore, that if

$$\delta_{rG} < 180° - \zeta_{rG}$$

when calculated by (7.19.15) it is an indication that the receiving-end power limit may be obtained by the use of the calculated value of δ_{rG} in the power equation (7.9.3).

If, on the other hand, the calculated value of the displacement angle is found to be

$$\delta_{rG} > 180° - \zeta_{rG}$$

it is an indication that in the adjustment of the generator excitation emf to maintain V_s constant, the generator would pull out of synchronism before the receiving-end power limit is reached. If such is the case, the receiving-end power limit is determined by the use of $180° - \zeta_{rG}$ for δ_{rG} in expression (7.9.3).

Note that the displacement angle δ_{rG} depends (a) upon the maintained ratio V_s/V_r between the sending- and receiving-end voltages; (b) the ratio of the total impedance Z_{rG} to that of the generator Z_G; and (c) the angles ζ_{rG} and γ of these impedances, and that of the system impedance ζ_{rs} between sending and receiving ends.

The angle ζ_{rs} of the series impedance of transmission systems is somewhere between 60° and 80°. The value of the generator impedance angle γ is large, almost 90°. Hence, the value of system impedance angle will be larger than ζ_{rs}, somewhere between 63° to 85°.

To illustrate the above, assume

$$\frac{Z_{rG}}{Z_G} = 2; \quad \frac{V_s}{V_r} = 1.2$$

$$\zeta_{rs} = 65°; \quad \gamma = 90° \quad \text{and} \quad \zeta_{rG} = 69°.$$

Substituting in (7.9.15) gives

$$\tan \delta_{rG} = \frac{2.4 \sin 69° + \sin (65° - 90°)}{2.4 \cos 69° - \cos (65° - 90°)} = -39.65$$

$$\delta_{rG} = 180° - 88.56° = 91.44°$$

is the displacement angle between V_r and E_G. It is smaller than $180° - 69°$ and should be used in (7.9.3) for the calculation of the receiving-end power limit.

If, on the other hand,

$$\frac{Z_{rG}}{Z_G} = 1.5; \quad \frac{V_s}{V_r} = 0.9$$

$$\zeta_{rs} = 75°; \quad \gamma = 90° \quad \text{and} \quad \zeta_{rG} = 80°$$

$$\tan \delta_{rG} = \frac{1.35 \sin 80° + \sin (75° - 90°)}{1.35 \cos 80° - \cos (75° - 90°)} = -1.477$$

$$\delta_{rG} = 180° - 55.9° = 124.1°.$$

The displacement angle at which the generator falls out of step is

$$180° - 80° = 100°.$$

Since this is smaller than the above calculated value of δ_{rG} it should be used in expression (7.9.3) for the predetermination of the receiving-end power limit.

7.10. *Vector Relations between the Constant Receiving- and Sending-End Voltages and the Adjusted emf of the Generator.* Since the sending- and receiving-end voltages for the case considered in the preceding article are constant, the equations for P_s and Q_s at the sending end are identical to (7.7.4) and (7.7.5), respectively.

$$P_s = \frac{V_s V_r}{Z_{rs}}\left[\cos(\delta_{rs} - \zeta_{rs}) - \frac{V_r}{V_s}\cos\zeta_{rs}\right] \tag{7.10.1}$$

$$Q_s = \frac{V_s V_r}{Z_{rs}}\left[\sin(\delta_{rs} - \zeta_{rs}) + \frac{V_r}{V_s}\sin\zeta_{rs}\right] \tag{7.10.2}$$

where $Z_{rs}/\underline{\zeta_{rs}}$ is the series impedance of the transmission system and δ_{rs} the displacement angle between V_s and the bus voltage V_r at the receiving end.

If the above calculated values of P_s and Q_s are expressed in terms of watts and vars, respectively, the voltampere at the sending end is

$$(VI)_s/\underline{\theta_s} = \sqrt{P_s^2 + Q_s^2}/\underline{\tan^{-1} Q_s/P_s}. \tag{7.10.3}$$

The magnitude of the current at the sending end is, therefore,

$$I = \frac{\sqrt{P_s^2 + Q_s^2}}{V_s}. \tag{7.10.4}$$

The phase angle of V_s with reference to I is

$$\theta_s = \tan^{-1}\frac{Q_s}{P_s}.$$

The generator emf per phase is, accordingly,

$$E_G/\underline{\theta_G} = V_s/\underline{\theta_s} + IZ_G/\underline{\gamma}.$$

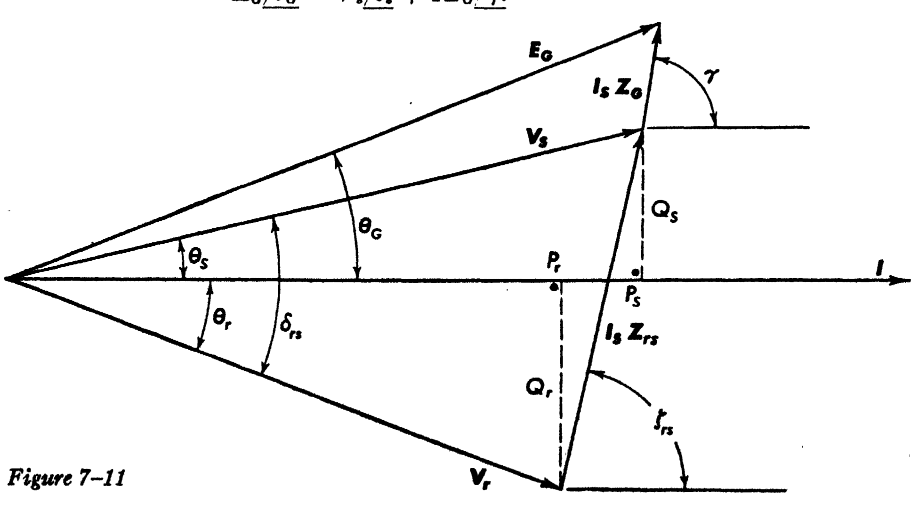


Figure 7–11

The vector relations between the quantities involved are shown in the diagram, Fig. 7–11, in which, for convenience, I taken as the reference vector is assumed leading V_r.

7.11. *Steady State Power Limit with Synchronous Motor Load. Generator and Motor emf's Specified.* This case differs from those discussed in the preceding articles by the fact that the receiving-end bus is not infinite. It is represented by the equivalent T circuit shown in Fig. 7–12 or by the series equivalent of this T circuit as indicated in Fig. 7–12a in which Z_G and Z_M are the

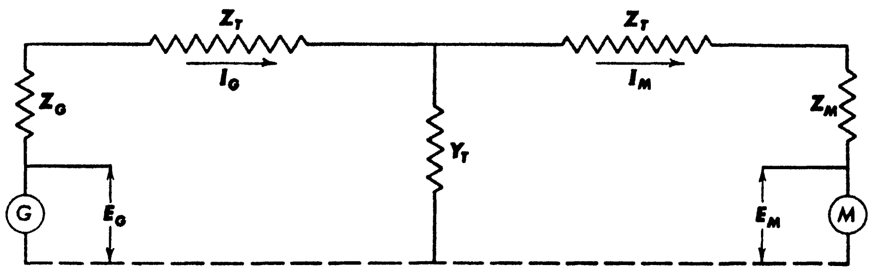


Figure 7–12

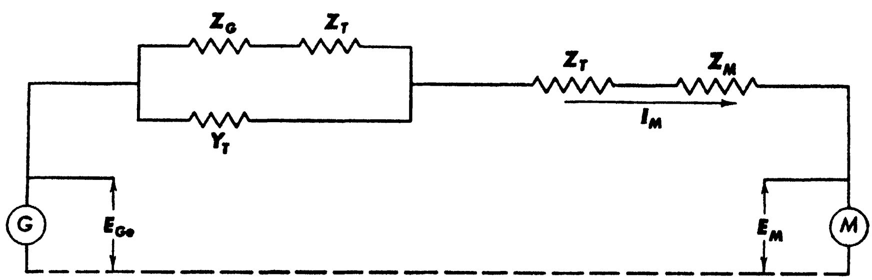


Figure 7–12a

phase impedances of generator and motor, respectively. The total series equivalent impedance of the circuit is

$$Z_e = \frac{Z_G + Z_T}{1 + (Z_G + Z_T)Y_T} + Z_T + Z_M \tag{7.11.1}$$

and the equivalent emf at the generator end is

$$E_{Ge} = \frac{E_G}{1 + (Z_G + Z_T)Y_T}. \tag{7.11.2}$$

The dissipative power per phase converted into mechanical power through the motor at the receiving end is

$$P_M = \frac{E_{Ge}E_M}{Z_e}\left[\cos(\delta_e - \zeta_e) - \frac{E_M}{E_{Ge}}\cos\zeta_e\right], \tag{7.11.3}$$

and the reactive power per phase, similarly, is

$$Q_M = \frac{E_{Ge}E_M}{Z_e}\left[\sin(\delta_e - \zeta_e) + \frac{E_M}{E_{Ge}}\sin\zeta_e\right], \tag{7.11.4}$$

where δ_e is the displacement angle between $\boldsymbol{E_{Ge}}$ and $\boldsymbol{E_M}$, and ζ_e is the angle of the series equivalent impedance of the system. Note that δ_e is the displacement angle δ between $\boldsymbol{E_G}$ and $\boldsymbol{E_M}$ modified by the angle associated with the modifying factor $1/1 + (\boldsymbol{Z_G} + \boldsymbol{Z_T})\boldsymbol{Y_T}$ as given by (7.11.2).

From expression (7.11.13) it is seen that the power limit of the motor at the receiving end is reached when

$$\delta_e = \zeta_e$$

and its value is

$$P_{Mm} = \frac{E_{Ge}E_M}{Z_e}\left[1 - \frac{E_M}{E_{Ge}}\cos\zeta_e\right]. \tag{7.11.5}$$

The voltampere per phase in the motor winding is

$$(VI)_M\underline{/\theta_M} = \sqrt{P_M^2 + Q_M^2}\underline{/\tan^{-1} Q_M/P_M}.$$

The current per phase in the motor winding is therefore

$$I_M = \frac{\sqrt{P_M^2 + Q_M^2}}{E_M}. \tag{7.11.6}$$

The angle of the voltage E_M with reference to I_M is

$$\theta_M = \tan^{-1}\frac{Q_M}{P_M}.$$

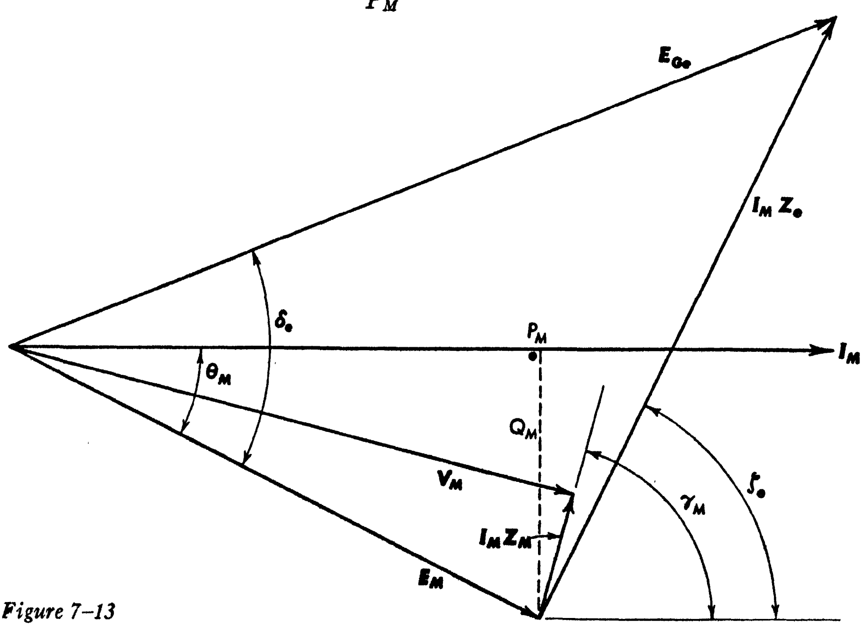


Figure 7–13

The terminal voltage per phase at the motor terminals is

$$V_M = E_M + I_M Z_M, \quad \text{or} \quad V_M = E_M\underline{/\theta_M} + I_M Z_M\underline{/\gamma_M}$$

where $Z_M\underline{/\gamma_M}$ is the impedance per phase of the motor. The vector relations between the quantities involved are shown in the diagram, Fig. 7–13, in which for convenience the current I_M, taken as the reference vector, is assumed leading E_M.

7.12. *Steady State Power Limit; System with Synchronous Motor Load and Terminal Voltages Maintained Constant.* This case presupposes that for each increase in load, the excitation voltages E_G of the generator, and E_M of the motor are automatically adjusted so as to maintain, respectively, the terminal voltages V_G and V_M at constant specified values. The power transfer from a synchronous generator to a synchronous motor whose shaft load is increased in small steps depends upon the excitation voltages

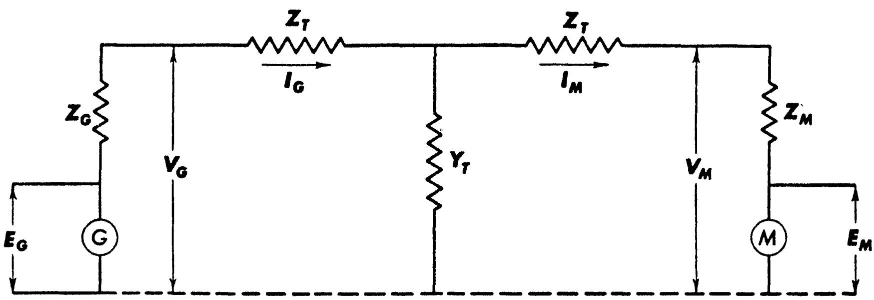


Figure 7–14

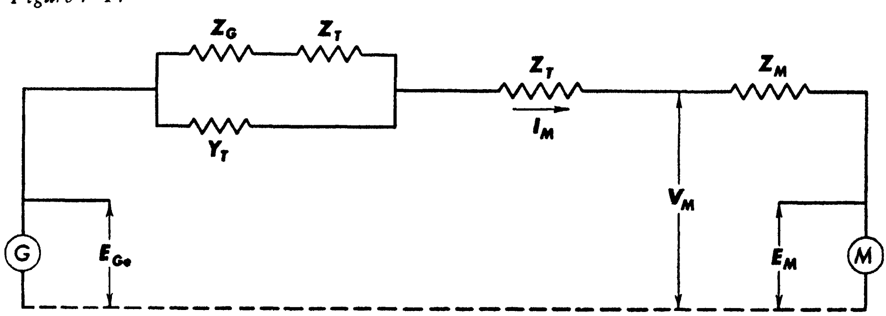


Figure 7–14a

of the machines and the displacement angle δ between these voltages. The power limit, as discussed in preceding articles, depends upon the particular values of excitation voltages E_G and E_M, corresponding, respectively, to the constant values of V_G and V_M, at the time when the displacement angle δ is equal to the system impedance angle ζ_t. The system is illustrated diagrammatically in Fig. 7–14. Since both E_G and E_M are to be determined simultaneously for a certain predetermined value of δ, none of the two possible series equivalents shown in Figs. 7–14a and 7–14b can be used. For, although

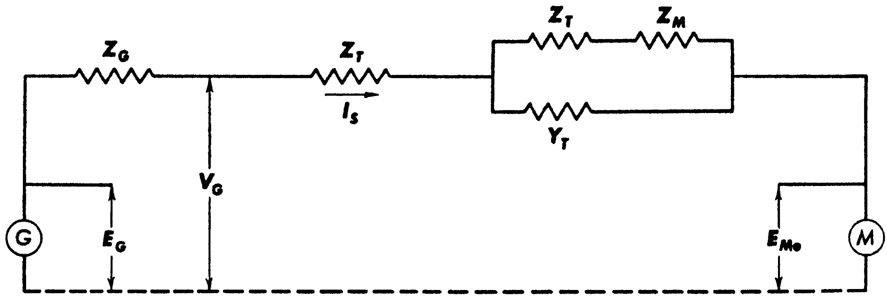


Figure 7–14b

the angle δ between E_G and E_M is known, the values of E_G and E_M are not, and if the values of V_G and V_M are known, the angle between them is not.

If the line is comparatively short, however, the linear-line admittance is neglected and the circuit may be substantially represented, as previously shown, by the series equivalent shown in Fig. 7–14c. For medium long and

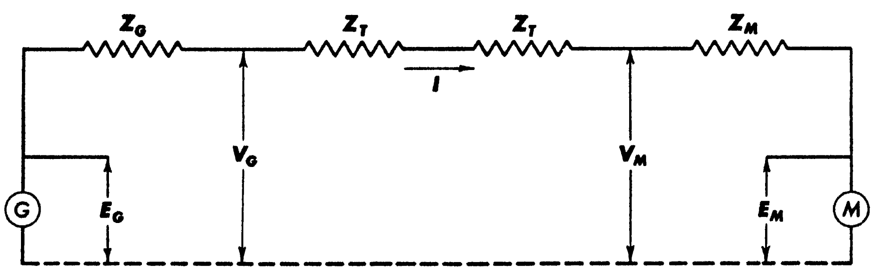


Figure 7–14c

long lines, I_s and I_r differ by the no load or charging current. The use of the circuit in Fig. 7–14c will give, therefore, only an approximate solution in the determination of E_G and E_M.

Using, therefore, the approximate diagram, Fig. 7–14c, the values of E_G and E_M for the power limit, ($\delta = \zeta_t$) may be determined graphically as indicated in Fig. 7–15.

1. Draw a horizontal line OI proportional to any assumed value of the current I through the circuit.

2. From an arbitrarily chosen point such as (a) draw a line $ab = IZ_M$ to a properly chosen scale of volts and making an angle γ_M with the horizontal. This gives the voltage drop per phase in the motor winding for the

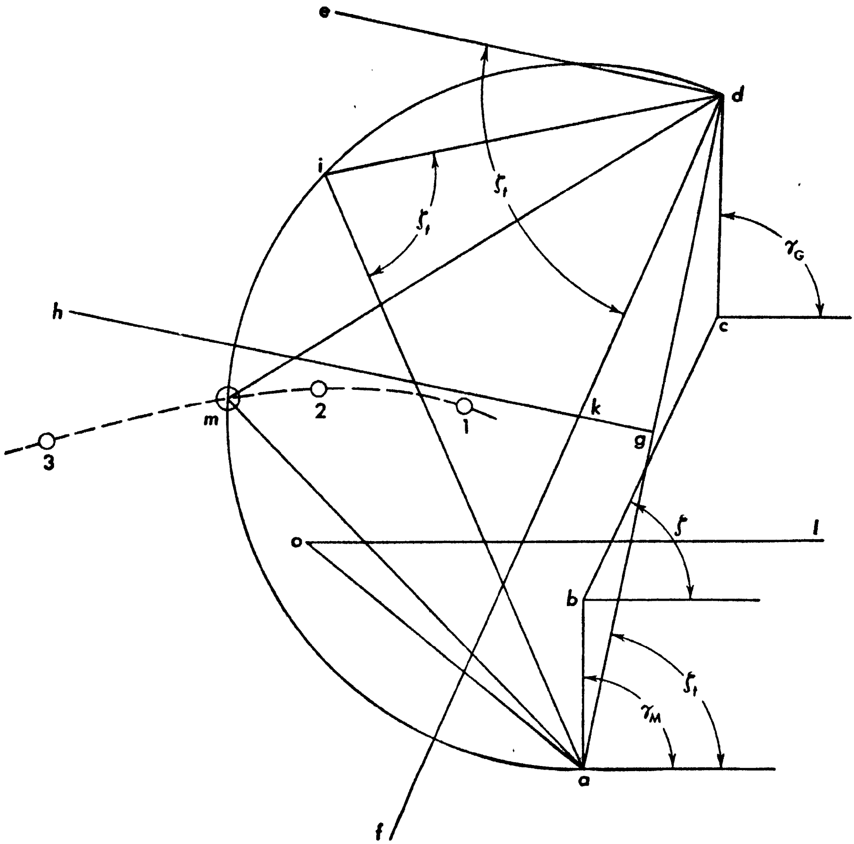


Figure 7–15

assumed value of I. It follows from the above that to the scale of IZ_M line $oa = E_M$ and $ob = V_M$ for

$$V_M = E_M + IZ_M.$$

3. At point b draw a line $bc = 2IZ_T$ the voltage drop in the line, making an angle ζ with the horizontal. Since

$$V_M + 2IZ_T = V_G,$$

it follows that a line drawn from o to c would be equal to V_G for the arbitrarily chosen current I.

4. At point c draw a line $cd = IZ_G$, the voltage drop per phase in the generator winding, making an angle γ_G with the horizontal. Since

$$V_G + IZ_G = E_G,$$

it follows that a line drawn from o to d would be equal to E_G for the arbitrarily chosen current I.

The four voltages $\boldsymbol{E_M}$, $\boldsymbol{V_M}$, $\boldsymbol{V_G}$, and $\boldsymbol{E_G}$ thus obtained are in correct vector relation to each other but do not satisfy quantitatively the actual values of V_G and V_M and the condition $\delta = \zeta_t$ for the power limit.

If the chosen current is of unit value in the above graphical construction, then $ab = Z_M$; $bc = 2Z_T$; $cd = Z_G$; and $ad = ab + bc + cd = Z_t$, the total series impedance of the circuit. The lines oa, ob, oc, and od would represent the respective voltages per unit current.

5. To determine the lengths of oa and od, i.e., of E_M and E_G for the particular condition when $\delta = \zeta_t$, erect a perpendicular de on line ad at d and draw a line df, making an angle ζ_t with de as indicated.

6. Draw a line gh perpendicular to ad at its midpoint g. Then the intersection k of line df and gh is the center of a circle which passes through points a and d. Lines drawn from any point on this circle such as i to points a and d have an angle ζ_t between them. Since the angle ζ_t is equal to the displacement angle δ between E_M and E_G for the power limit, it follows that the origin for $oa = E_M$ and $od = E_G$ must be on the circle.

7. To satisfy the condition that E_M correspond to a fixed value of V_M, and E_G correspond to a fixed value of V_G, strike an arc with a radius proportional to V_M and center at point b and an arc with a radius proportional to V_G and center at point c. Lines drawn from the intersection of these two arcs such as (1) to points b and c, respectively, will satisfy the required proportional relationship between V_M and V_G.

8. In the diagram the excitation voltages $\boldsymbol{E_G}$ and $\boldsymbol{E_M}$ and the corresponding terminal voltages $\boldsymbol{V_G}$ and $\boldsymbol{V_M}$ have a common origin, and this origin must be on the circle to satisfy the condition $\delta = \zeta_t$. To obtain, therefore, this common origin repeat the striking of arcs as directed above, using different values of V_M and V_G but keeping the ratio V_G/V_M constant. The intersections of these arcs are marked 1, 2, and 3 in the diagram. The intersection of the line drawn through these points with the circle at point m is the common origin of the voltage vectors $\boldsymbol{E_G}$, $\boldsymbol{E_M}$, $\boldsymbol{V_G}$, and $\boldsymbol{V_M}$.

9. From the construction of the diagram as outlined above, it follows that line ma is proportional to $\boldsymbol{E_M}$; line mb is proportional to $\boldsymbol{V_M}$; line mc is proportional to $\boldsymbol{V_G}$; and line md is proportional to $\boldsymbol{E_G}$.

10. The final step in this construction is the determination of the length of $ma = E_M$ and of $md = E_G$ *in volts* from the known length of either $mb = V_M$ or of $mc = V_G$ measured in volts.

The values of E_G and E_M thus obtained, substituted in

$$P_{Mm} = \frac{E_G E_M}{Z_t}\left(1 - \frac{E_M}{E_G}\cos\zeta_t\right)$$

gives substantially the power limit for the motor per phase when the line is short and approximately for medium long and long lines.*

REFERENCES

Clarke, Edith, *Steady State Stability in Transmission Systems*, Trans. Am. Inst. Elect. Engrs., 1926.

Dahl, O. G. C., *Electric Circuits*, Part I, McGraw-Hill Book Co., 1928.

Doherty, R. E., and Dewey, H. H., *Fundamental Considerations of Power Limits of Transmission Systems*, Trans. Am. Inst. Elect. Engrs., 1925.

Evans, R. D., and Sels, H. K., *Power Limitations of Transmission Systems*, Trans. Am. Inst. Elect. Engrs., 1924.

Fortescue, C. L., and Wagner, C. F., *Some Theoretical Consideration of Power Transmission Systems*, Trans. Am. Inst. Elect. Engrs., 1924.

Shand, E. B., *Limitations of Output of Power System Involving Long Transmission Lines*, Trans. Am. Inst. Elect. Engrs., 1924.

Wagner, C. F., and Evans, R. D., *Static Stability Limits and the Intermediate Condenser Station*, Trans. Am. Inst. Elect. Engrs., 1928.

SUGGESTIVE PROBLEMS *Chapter 7*

1. Construct the receiving-end power-circle diagram for the particular long line specified in Prob. 4 at end of Chap. 3.

2. Construct the station-end power-circle diagram for the line stated in Prob. 1.

3. Obtain the power-angle curve for the line specified in Prob. 2 at the end of Chap. 4.

4. Obtain the single impedance equivalent of the 300 mile A.C.S.R. stated in Prob. 4 at the end of Chap. 3.

5. Calculate the steady-state power limit of the line given in Prob. 2 under the assumption that the receiving-end line voltage is 220 kv and the voltage ratio is $k = 1.15$.

6. Calculate the power-angle curve of the 100 mile line used in the illustrative problem in § 6.8, for which $A = 1\underline{/0^\circ}$, $B = 79.9\underline{/71.2^\circ}$, assuming that the receiving-end voltage to neutral is 63.5 kv and $k = 1.25$.

* The graphical solution given above is essentially the one developed by Edith Clarke of the General Electric Co. and given in her paper on *Calculation of Steady-State Stability of Transmission Lines*, Trans. A.I.E.E., 1926.

7. Obtain the single-impedance equivalent of a T circuit, whose series impedance is $Z_T = 130\underline{/75^\circ}$ vector ohms and whose shunt admittance is $1700 \times 10^{-6}\underline{/90^\circ}$ vector mhos at 60 cps. Calculate the resistance and reactive components of the single-impedance equivalent. Is the reactive component capacitive or inductive in character?.

8. Following the graphical method outlined in § 7.12, determine the relative values of the excitation voltages of the generator at the station end, and synchronous motor at the receiving end assuming that the terminal voltages remain constant, that their ratio is 1.2, and that $Z_M = 2\underline{/80^\circ}$, $Z_G = 3\underline{/80^\circ}$ and $Z_L = 14\underline{/70^\circ}$.

Chapter 8 Faulted Transmission Systems

8.1. *General Considerations.* The term *fault* is generally applied to a partial or complete *open* or a partial or complete *short circuit* accidentally occurring at some point in the system.

The occurrence of an open, although sufficiently serious in as much as service to customers is interrupted thereby, is not as troublesome as short circuits, unless it causes undue rises in voltage at various points in the system.

The occurrence of a complete or partial short circuit at some point of a transmission system results in a change in the impedance structure of the circuit with the consequent decreases in voltage and increases in current. The magnitude of the current under a short-circuit condition depends not only upon the completeness of the short circuit but also upon its location from the source or sources of energy supply. Short-circuit faults not only cause interruption of service, but may, unless removed within reasonable time, seriously injure generators, transformers, and other parts of system equipment. A short circuit, by virtue of the large demand for current, may cause synchronous machines to fall out of synchronism resulting in more or less complete breakdown of the system.

The initial magnitude of the current due to a short circuit depends upon the particular instantaneous value of the generator emf at the instant the fault occurs. Its occurrence and calculation is not within the scope of this book. This chapter is concerned with the investigation and calculation of system behavior on sustained faults such as are usually caused, for example, by a breakdown of the insulation at some part on the system.

Such breakdowns may occur at the supporting insulators of busses or switch disconnects. The insulating equipment within an oil breaker tank or transformer tank may fail and cause injurious short circuits. The insulating compound between transformer windings may fail, or a supporting insulator of line conductors may break down. Ice formation on line conductors may cause rupture and subsequent falling of conductors against each other, against the sides of the supporting tower, or to ground. Insulation failures resulting in short circuits may be caused also by flashovers caused by sudden rises in voltage. Such rises in voltage may be due to lightning discharges, switching, ground arcing, and sudden loss of load coupled with generator overspeeding.

8.2. *Isolated and Grounded Transmission Systems; Arcing Grounds.* The insulation provided for a transmission system, including the end transformers, is determined by whether the system is to have an *isolated neutral* or a grounded neutral. The grounding of the neutral of a high voltage system, in addition to reducing the cost of the insulation requirements for transformers and line, provides also increased reliability by limiting the voltage on the insulation equipment. With grounded neutral, each insulator is subject to only $100/1.73 = 57.8$ per cent of the line voltage. These are, of course, definite reasons in favor of grounding the neutral of a system. On the other hand, with an unbalanced load on a grounded system, the ground or residual current may cause inductive interference with neighboring telephone lines. An accidental single conductor to ground connection on a grounded system is a definite short-circuit fault which may cause considerable trouble. Nongrounded isolated systems do not cause inductive interference, and an accidental direct connection between a line conductor and ground is not a short circuit. Although a line to ground short circuit of a grounded system may prove troublesome, the short-circuit current itself provides the means of operating relays for the automatic disconnection of the faulted circuit. The short-circuited current in such a case can be effectively limited in magnitude by inserting a resistance or a high reactive impedance between the neutral connection and ground. On the other hand, an accidental connection between a line conductor and ground in an ungrounded system, although in itself not a short circuit, may be conducive to serious short-circuit faults through what is generally referred to as *arcing grounds.* This is particularly the case with long lines in which the capacitive effect is appreciable as evidenced by the magnitude of the charging current.

Consider an isolated system such as the one shown schematically in Fig. 8–1, in which the line to ground capacitances are denoted by the condensers *a*, *b*, *c*. It is obvious that with the line under normal conditions of operation the line to ground capacitances form a wye-connected, approximately balanced, and nearly 90° leading load. If the conductor of phase *B* should accidentally become grounded at some point such as at *F*, the circuit direction of current flow changes from that pertaining to a normal *Y–Y* system to that shown by the arrows in the figure. There is nothing particularly serious except that the capacitive load, because of its magnetizing action, has a tendency to increase the voltages of the unaffected phases.

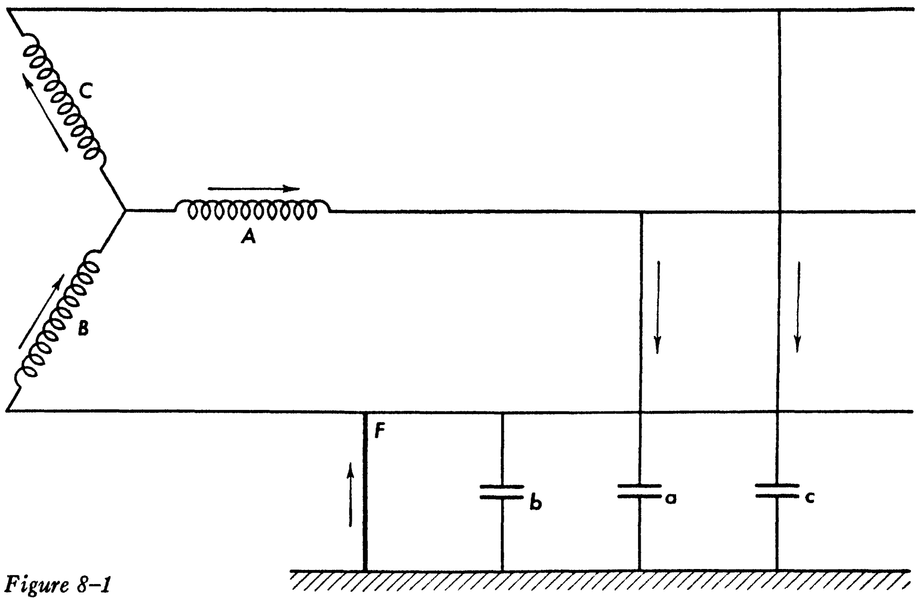


Figure 8–1

The situation is somewhat different, however, if the accidental contact between conductor *B* and the ground is, for instance, through an arc-over of a defective insulator. In this case the actual contact between conductor *B* and ground is removed when the arc goes out at the instant the current is zero. At that instant, however, the voltage across the nonaffected conductors and ground is a maximum by virtue of the leading current but is somewhat greater than the normal maximum because of the magnetizing action of the leading current. The immediate effect is that the arc is reestablished, and goes out again when the current passes again through zero. The voltage to ground of the two, as yet unaffected, conductors is further increased with the subsequent result that a more severe arc is established. The voltage thus builds up until it is three to four times the normal when it may cause a flashover at some other weak insulator in lines *A* or *C* or both. The condition is somewhat as indicated in Fig. 8–2. The second flashover conducive to a short circuit may be quite remote from the first.

The occurrence of such line-line to ground (*L–L–G*) short circuits on nongrounded high voltage lines is usually referred to as *arcing grounds*. Such arcing grounds are practically eliminated in grounded systems. In case of an arc formation between one conductor and ground, the excessive building up of voltage on the nonaffected conductors is prevented by the neutral ground.*

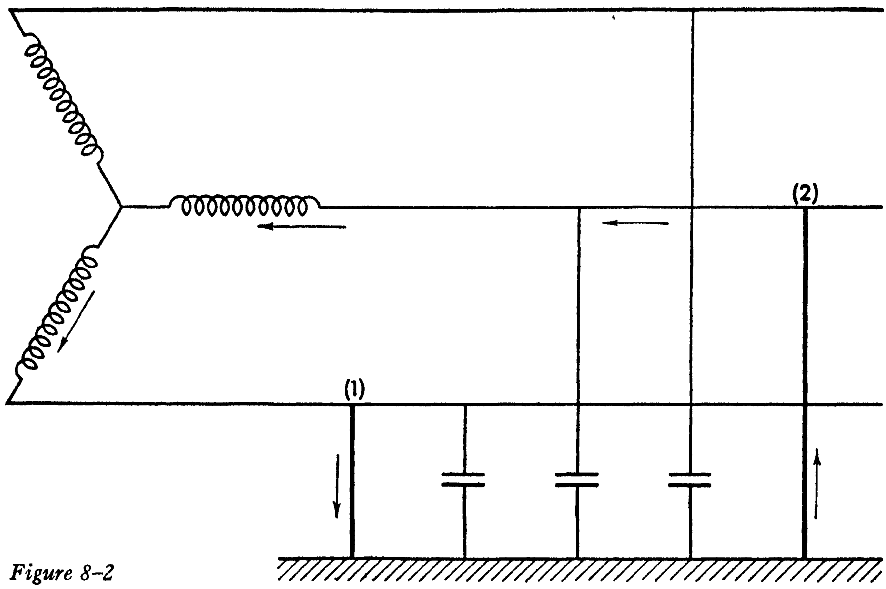


Figure 8–2

8.3. *Symmetrical Phase Components.* If adequate protection of equipment against injurious effects of short circuits is to be provided, a more or less accurate knowledge of the magnitude of short-circuit currents is important. The various factors that enter into the calculation of such currents are seldom known with any great accuracy. The circuit structure under short-circuit conditions is usually too complicated to permit a rigorous solution. Furthermore, with the single exception of a complete three-phase short circuit, all other short-circuit faults cause dissymmetry in the circuit structure. The solution of such unsymmetrical systems is greatly simplified by the use of *Symmetrical Phase Components.*† By this method, the nonsymmetrical system is split

* Clem, *Arcing Grounds and Effect of Neutral Grounding Impedance,* Trans. A.I.E.E., 1930.

† Fortescue, C. L., *Method of Symmetrical Coordinates Applied to the Solution of Polyphase Networks,* Trans. A.I.E.E., 1918.

Mackeras, H. P., *Calculation of Single-Phase Short Circuits by the Method of Symmetrical Components,* G. E. Review, Apr., 1926.

Weinbach, M. P., *A. C. Circuits,* Chap. VIII, Macmillan Co.

into three symmetrical systems each of which is solved independently, and the results superimposed to give the complete solution.

The fundamental principles underlying the method are based upon the fact that any nonsymmetrical system of vectors, such as the one shown in Fig. 8–3, may be split into three symmetrical systems of vectors: The first called *positive-phase sequence*, I_{p1}, I_{p2}, I_{p3}, shown in Fig. 8–4 in which the three vectors are equal, 120° apart and in positive phase sequence, i.e., numbered clockwise and rotating counterclockwise. The second is called *negative-phase sequence*, I_{n1}, I_{n2}, I_{n3}, and is shown in Fig. 8–4a. The three members of this phase sequence component are also equal to each other, 120° apart, but in negative phase sequence, i.e., they are numbered counterclockwise and are rotating counterclockwise. The third-phase sequence component consists of three members equal to each other and of the same phase with reference to the horizontal, i.e., of zero phase with reference to each other. For this reason it is called *zero-phase sequence* and is shown in Fig. 8–4b.

The relation between the nonsymmetrical system of vectors I_1, I_2, I_3 with the phase sequence components in which it may be split up, is such that the first member of the positive sequence component is, referring to Fig. 8–3,

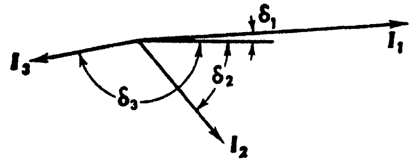


Figure 8–3

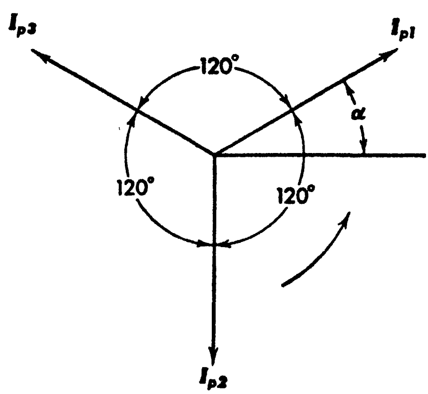


Figure 8–4

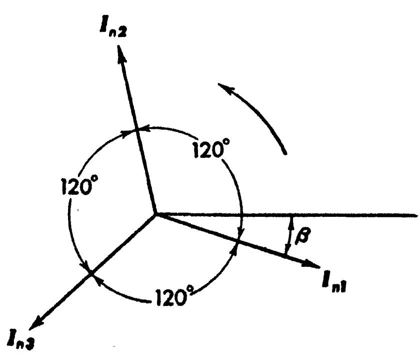


Figure 8–4a

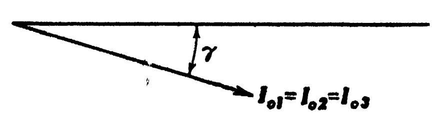


Figure 8–4b

$$I_{p1}\underline{/\alpha} = \tfrac{1}{3}(I_1\underline{/\delta_1} + I_2\underline{/\delta_2}\,\underline{/120^\circ} + I_3\underline{/\delta_3}\,\underline{/240^\circ}). \qquad (8.3.1)$$

The other two members of the positive phase component are equal to I_{p1} and lagging it by 120° and 240°, respectively, as indicated in Fig. 8–4.

The first member of the negative sequence component is

$$I_{n1}\underline{/\beta} = \tfrac{1}{3}(I_1\underline{/\delta_1} + I_2\underline{/\delta_2}\,\underline{/-120^\circ} + I_3\underline{/\delta_3}\,\underline{/-240^\circ}). \tag{8.3.2}$$

The other two members of this component are equal to I_{n1} but leading it, respectively, by 120° and 240°, as shown in Fig. 8–4a.

The zero-sequence component is

$$I_o\underline{/\gamma} = \tfrac{1}{3}(I_1\underline{/\delta_1} + I_2\underline{/\delta_2} + I_3\underline{/\delta_3}). \tag{8.3.3}$$

Conversely, if the phase sequence components are known, then

$$I_1\underline{/\delta_1} = I_p\underline{/\alpha} + I_n\underline{/\beta} + I_o\underline{/\gamma} \tag{8.3.4}$$

$$I_2\underline{/\delta_2} = I_p\underline{/\alpha}\,\underline{/-120^\circ} + I_n\underline{/\beta}\,\underline{/120^\circ} + I_o\underline{/\gamma} \tag{8.3.5}$$

and

$$I_3\underline{/\delta_3} = I_p\underline{/\alpha}\,\underline{/-240^\circ} + I_n\underline{/\beta}\,\underline{/240^\circ} + I_o\underline{/\gamma}. \tag{8.3.6}$$

In accordance with the above it follows that:

1. The voltage and current vectors of balanced three-phase systems consist only of positive sequence components.
2. The current vectors of three-phase unbalanced wye circuits consist of only positive and negative sequence components.
3. The voltage vectors of unbalanced wye circuits consist of positive, negative, and zero sequence components.
4. The line voltage vectors of unbalanced wye or delta circuit consist of positive and negative sequence components.

8.4. *Sequence Impedance.* The impedance to the flow of any sequence component current is usually designated by the particular sequence component. Thus, the impedance to the flow of a positive-sequence component current is called ***positive sequence impedance.*** To illustrate, consider the current I_1 in phase (1) of a nonsymmetrical static network. If the impedance of that particular phase is Z_1 the voltage drop across this impedance, by (8.3.4), is

$$I_1Z_1 = I_pZ_1 + I_nZ_1 + I_oZ_1$$

or

$$V_1 = V_p + V_n + V_o.$$

This indicates that the phase sequence impedances of a static network are equal and equal to the actual impedance of the network. The currents in the line conductors of nongrounded, isolated transmission systems have no zero sequence components. In the case of grounded systems, the zero sequence components of the currents in each conductor are equal and in phase. The ground return carries three times the zero sequence component. Hence, the impedance of the ground return to the zero sequence current is three times the actual impedance of the ground return. The zero-sequence impedance per phase of the line is, therefore, equal to the sum of the zero-sequence impedance of the conductor and three times the impedance of the ground return.

Because of their symmetrical structure, synchronous machines generate only positive sequence emf's. But if these emf's are impressed upon a nonsymmetrical circuit, the resulting nonsymmetrical currents will have negative sequence current components and, if the neutral is grounded, also zero sequence current components. These sequence component currents flow through the respective phase windings of the machine. Each phase offers, therefore, a negative-sequence impedance to the negative-sequence current and a zero sequence impedance to the flow of the zero-sequence current, differing in magnitude from each other and from the positive sequence.*

The impedance of two-winding transformers in each phase of a three-phase transmission system to the positive and negative sequence components of nonsymmetrical currents is the same as the actual joint impedance of the windings. This is because transformers are nonrotating apparatus. The zero sequence impedance on the other hand is either equal to the actual impedance if proper grounds are provided to complete the circuit for zero sequence currents or it is infinitely large as is the case of wye-wye connected transformers with nongrounded neutrals, or delta-delta connected transformers, there being no circuit for zero sequence currents. It is also obvious, from what was said above, that the only case of two-winding transformer for which the zero sequence impedance must be taken into consideration is a wye to wye connection with solidly grounded neutrals as shown in Fig. 8–5. For any other transformer connections the zero sequence impedance is infinite.

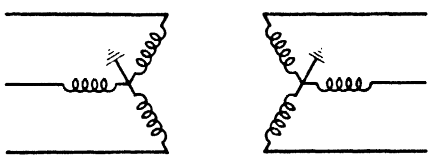


Figure 8-5

8.5. *Impedance of Single Conductor with Ground Return. Carson's Formula.*

The study and calculation of short-circuit faults in transmission systems by means of symmetrical phase components calls for a knowledge of the zero-sequence impedance of conductors and of the ground return. If the earth were a perfect conductor, the ground return from an overhead conductor would be confined, by virtue of the skin effect, to a very thin film of ground surface with the greatest density just below the overhead conductor. This of course is not the case, for the earth is far from being a

* For the experimental determination of the negative and zero sequence impedances of synchronous machinery, see Wagner and Evans, *Symmetrical Components*, McGraw-Hill.

good conductor. Substantially accurate results are obtained by assuming that the return path of the current is a conducting plane some distance below the surface of the earth.* The distance below the earth's surface depends upon the conductive character of the earth and upon the frequency. With currents of frequency 60 cps, the distance of the ground return is assumed to be vertically below the surface equal to the height of the conductor above the surface, as indicated diagrammatically in Fig. 8–6. If R_c is the resistance per mile of the conductor, X_{cg}, the reactance per mile of the grounded conductor at 60 cps, and Z_c, the impedance per mile, then

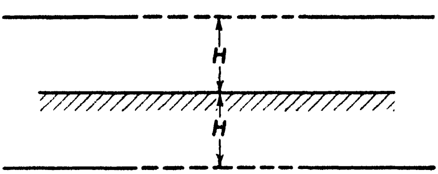


Figure 8–6

$$Z_c = R_c + jX_{cg}.$$

By (1.8.11), this is

$$Z_c = R_c + j\,741.13 \times 10^{-6}\,\omega \log \frac{2H}{r_{gm}} \text{ ohms/mile.} \tag{8.5.1}$$

The joint impedance of conductor and ground return per mile, i.e., one mile of conductor and one mile of ground return is, obviously,

$$Z_{cg} = R_c + j\,741.13 \times 10^{-6}\,\omega \log \frac{2H}{r_{gm}} + Z_g, \tag{8.5.2}$$

where Z_g is the impedance per mile of the ground return.

Theoretical studies on the determination of the impedance Z_g of ground returns were made by various investigators here and in Europe. These investigations were based on various assumed distribution patterns of alternating currents at different frequencies in the earth.†

The investigations of Carson and Pollaczek, although independent of each other, were predicated upon the identical basic assumption that the earth is perfectly homogenous and of finite resistivity, that the overhead conductor is a portion of an infinitely long one, and as a consequence the direction of current flow in the ground return is parallel to the overhead conductor.

Although the mathematical methods of handling the problem by Carson and Pollaczek are different, the results obtained for the self- and mutual impedance of ground return paths are substantially the same.

* Pollaczek, F., *Elektrische Nachrichten*, Tecknic, Sept. 3, 1926; Jan. 4, 1927.

† Breisig, F., *Telegraphen and Fernsprechtechnick*, Apr. 14, 1925.
Carson, J. R., *Wave Propagation in Overhead Wires with Ground Return*, Bell Syst. Tech. Jour., Oct., 1926.

In its simplified form, Carson's formula, for the impedance Z_g of the ground return path, is

$$Z_g = 4\omega(P + jQ) \text{ abohms/cm.}$$

Since, in this book, the symbols P and Q are used for dissipative and reactive powers, respectively, the symbols M and N are used instead, and Carson's formula is written

$$Z_g = 4\omega(M + jN) \text{ abohms/cm.} \tag{8.5.3}$$

The complex form of this expression indicates that it takes into consideration both the resistance and the reactance of the ground return path for the particular frequency stated by ω. The formula indicates also that the resistance of the ground return is a function of the frequency.

Converting the above formula in ohms per mile gives

$$Z_g = \frac{4\omega(M + jN)(5280 \times 100)}{3.281 \times 10^9}$$

or

$$Z_g = 643.9 \times 10^{-6}\omega(M + jN) \text{ ohms/mile} \tag{8.5.4}$$

of the ground return.

Carson's formulas for the quantities M and N which enter in the expression for Z_g are

$$M = \frac{\pi}{8} - \frac{p \cos\theta}{3\sqrt{2}} + \frac{p^2 \cos 2\theta}{16}\left(0.6728 + \ln\frac{2}{p}\right) + \frac{p^2\theta \sin 2\theta}{16} \tag{8.5.5}$$

$$N = -\,0.0386 + \tfrac{1}{2}\ln\frac{2}{p} + \frac{p\cos\theta}{3\sqrt{2}}. \tag{8.5.6}$$

These two formulas, in accordance with Carson's paper, hold for values of $p \leq 0.25$ in which

$$p = 2h\sqrt{\alpha} \tag{8.5.7}$$

where h is the height of the conductor above the ground expressed in centimeters, and

$$\alpha = 4\pi\lambda\omega.$$

The symbol λ in the expression stands for the conductivity of the earth expressed in abmhos per centimeter cube. The expression for p, as given above, may be written, therefore,

$$p = 2h\sqrt{4\pi\lambda\omega}$$

or

$$p = 4\sqrt{\pi}h\sqrt{\lambda\omega}. \tag{8.5.8}$$

Since the resistive properties of the earth are stated internationally in ohms per meter cube, the above formula for p should be stated in terms of the resistivity ρ of the earth in ohms per meter cube rather than in terms of the conductivity in abmhos per centimeter cube.

If ρ_1 is the resistivity of the earth in abohms per centimeter cube, then

$$\rho_1 = \frac{1}{\lambda} \text{ abohms/cm}^3.$$

or

$$= \frac{10^{-9}}{\lambda} \text{ ohms/cm}^3.$$

The resistance ρ of a body of earth 1 meter long and 1 square meter in cross-sectional area is

$$\rho = \rho_1 \frac{100}{10000} = \frac{\rho_1}{100} \text{ onms.}$$

This gives

$$\lambda = \frac{10^{-11}}{\rho} \text{ abmhos/cm}^{-3}. \tag{8.5.9}$$

Using this value of λ in the expression for p gives

$$p = 4h\sqrt{\pi} \times 3.16 \times 10^{-6} \sqrt{\frac{\omega}{\rho}}. \tag{8.5.10}$$

Expressing the height of the conductor above the earth in feet (H) instead of centimeters (h), the above expression becomes

$$p = 683.2 \times 10^{-6} H \sqrt{\frac{\omega}{\rho}}. \tag{8.5.11}$$

The formulas for M and N include also an angle θ, which, in accordance with Carson's paper, is defined as follows: Consider two conductors (a) and (b) parallel to each other, and each grounded at the far end as shown in Fig. 8–7. The angle θ in the equation for M and N is the angle between the imaginary line drawn between (a) and its image at (c) and the imaginary line drawn between (a) and the image (d) of point (b). It should be noted that the angle θ becomes larger as the spacing distance between conductors (a) and (b) is increased. When the spacing distance is infinitely large, there is left a single grounded conductor, and the angle $\theta = \pi/2$.

From what has been said above, it follows that the value of M and N to be used in equation (8.5.4) are, by (8.5.5) and (8.5.6), respectively, using $\theta = \pi/2$.

$$M = \frac{\pi}{8} - \frac{p^2}{16}\left(0.6728 + \ln \frac{2}{p}\right)$$

$$N = -0.0386 + \frac{1}{2} \ln \frac{2}{p}.$$

Using logarithms to the base 10 in lieu of the natural logarithms gives

$$M = \frac{\pi}{8} - \frac{p^2}{16}\left(0.6728 + 2.302 \log \frac{2}{p}\right) \tag{8.5.12}$$

$$N = -0.0386 + 1.151 \log \frac{2}{p}, \tag{8.5.13}$$

where for any specific case with regard to conductor height H, frequency and resistivity of the earth, p is given by (8.5.11).

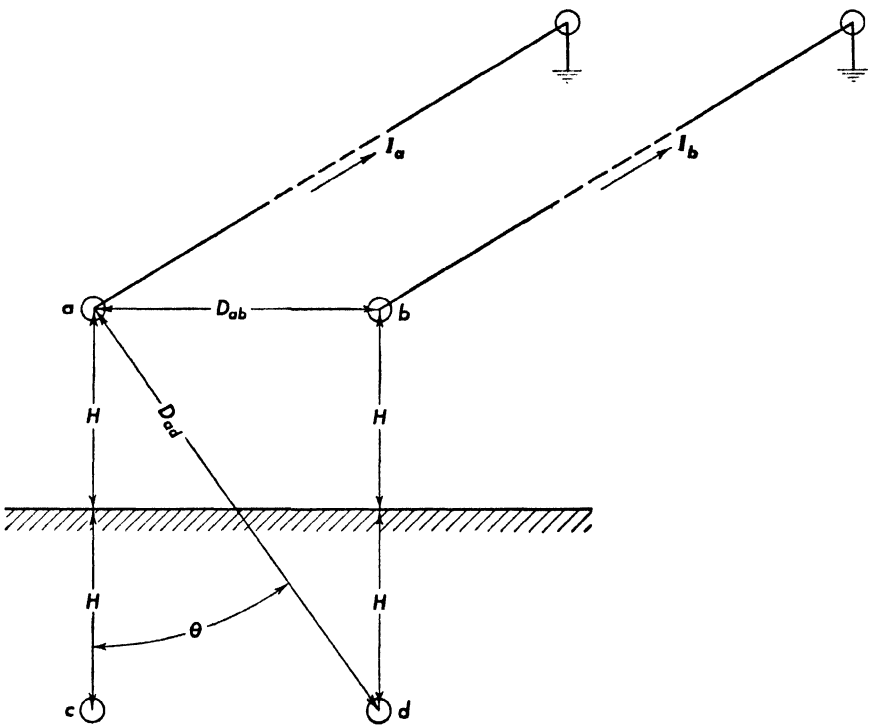


Figure 8–7

The application of Carson's formula (8.5.4) to the calculation of the impedance of the ground return of a transmission line in any one particular locality presupposes a knowledge of the resistivity of the earth in that locality.

To illustrate the application of the formula, consider a conductor at the somewhat exaggerated height of 50 feet above the ground, and operated at a frequency of 60 cps.

Using earth resistivities of ρ = 50, 100, and 2000 ohms per meter cube in (8.5.11) gives

$$p_{50} = 683.2 \times 10^{-6} \times 50\sqrt{\tfrac{377}{50}} = 0.09378$$

$$p_{100} = 683.2 \times 10^{-6} \times 50\sqrt{\tfrac{377}{100}} = 0.06631$$

$$p_{2000} = 683.2 \times 10^{-6} \times 50\sqrt{\tfrac{377}{2000}} = 0.01482.$$

Note that in each of the above cases the value of p is less than 0.25 and thus Carson's formulas for M and N hold. If these values of p for the different values of ρ are substituted in (8.5.12), it will be seen that for a single conductor with ground return operated at 60 cps, the value of M is practically independent of the resistivity of the earth, and that its value is substantially

$$M = \frac{\pi}{8} = 0.3927.$$

Consider now the value of N as given by (8.5.13). If the value of N were calculated for the above values of p, it will be found that by neglecting the negative member in the formula, the error introduced in doing so is less than 3 per cent. Considering the insufficient accuracy pertaining to the value of the resistivity of the earth, the formula for N, for a single overhead conductor with ground return may be written, therefore,

$$N = 1.151 \log \frac{2}{p}, \tag{8.5.14}$$

where p for any specific case is given by (8.5.11).

Using the value of $M = 0.3927$ and the value of N, from equation (8.5.14) in (8.5.4), gives for the impedance of the ground return path of a single overhead conductor

$$Z_g = 643.9 \times 10^{-6}\omega \left(0.3927 + j\,1.151 \log \frac{2}{p}\right)$$

or

$$Z_g = 253 \times 10^{-6}\omega + j\,741.13 \times 10^{-6}\omega \log \frac{2}{p} \text{ ohms/mile.}$$

This is the impedance per mile of the ground return path. The impedance per mile of the overhead conductor is given by (8.5.1). The impedance per mile of conductor and mile of ground return, therefore, is

$$Z_{cg} = R_c + 253 \times 10^{-6}\omega + j\,741.13 \times 10^{-6}\omega \left[\log \frac{2H}{r_{gm}} + \log \frac{2}{p}\right].$$

Substituting the value of p from equation (8.5.11) in the logarithmic member of the above expression gives

$$\log \frac{4H}{p r_{gm}} = \log \frac{4H}{683.2 \times 10^{-6} H r_{gm} \sqrt{\omega/\rho}}$$
$$= \log \frac{5858\sqrt{\rho/\omega}}{r_{gm}}.$$

The formula for the joint impedance of conductor and ground return path per mile, therefore, is

$$Z_{cg} = (R_c + 253 \times 10^{-6}\omega) + j\,741.13 \times 10^{-6}\omega \log \frac{5858\sqrt{\rho/\omega}}{r_{gm}}. \tag{8.5.15}$$

It is important to note from this formula (1) that the joint impedance of conductor and ground return path is independent of the height of the conductor above the ground; (2) that the resistance of the ground return path depends upon the frequency and paradoxically is independent of the resistivity of the earth; and (3) the joint reactance of the conductor and return path depends upon the resistivity of the earth. If the supply to this line is at a frequency of 60 cps, the formula becomes

$$Z_{cg} = R_c + 0.0954 + j\,0.2794 \log \frac{301.2\sqrt{\rho}}{r_{gm}}. \tag{8.5.16}$$

For an assumed resistivity of $\rho = 100$ ohms per meter cube, the above becomes

$$Z_{cg} = (R_c + 0.0954) + j\,0.2794 \log \frac{3012}{r_{gm}}, \tag{8.5.17}$$

where 0.0954 is the resistance of the ground return path per mile, R_c is the resistance of the overhead conductor per mile, r_{gm} is the geomean radius of the conductor, with values as given in Table I. The number 3012 may be interpreted as a fictitious spacing distance between the overhead conductor and the ground return path measured in feet. For the particular case considered, it is equivalent to a conductor height of 1506 feet above the earth's surface, and of the ground return path 1506 feet below the earth's surface!

For a conductor whose geomean radius is 0.5 inch, the joint reactance of conductor and ground return path is for $\rho = 100$ ohms per meter cube,

$$X_{cg} = 0.2794 \log \frac{3012 \times 12}{0.5}$$
$$= 1.358 \text{ ohms.}$$

For $\rho = 2000$ and $r_{gm} = 0.5$ inches,

$$X_{cg} = 0.2794 \log (7230\sqrt{2000})$$
$$= 1.54.$$

A 1900 per cent increase in ρ causes an increase in the joint reactance of a single overhead conductor of 0.5 inch geomean radius only 13.32 per cent!

It appears, therefore, that great accuracy in the value of the earth's resistivity is not required for the calculation of the joint reactance of a single overhead conductor with ground return.

8.6. *Impedance per Conductor of a System of Two Parallel Conductors with Ground Return.*

Carson's formula as applied to the determination of the impedance of a single conductor and ground return discussed in the preceding article may be extended to the formulation of the impedance equation of two conductors with ground return, shown schematically in Fig. 8–7.

Let V_a be the voltage to ground of conductor (a) at the station end and I_a the current in conductor (a). The self-impedance of conductor (a) is

$$\frac{V_a}{I_a} = (Z_{cg}S)_a \text{ ohms,} \tag{8.6.1}$$

where Z_{cg} is the joint impedance of the conductor and the ground return path, per mile, as given by formula (8.5.15), and S is the length of the conductor. Similarly, the self-impedance of conductor (b) is

$$\frac{V_b}{I_b} = (Z_{cg}S)_b \text{ ohms,} \tag{8.6.2}$$

where V_b is the voltage to ground of conductor (b) at the station end.

Carson's paper indicates also that the mutual impedance between the two parallel conductors with ground return may be expressed by

$$(Z_{ab})_m = Z_{ab} + Z_g \text{ ohms/mile,} \tag{8.6.3}$$

where Z_{ab} is the mutual impedance in ohms per mile between conductors (a) and (b) under the assumption that the common ground return path is of perfect conductivity, and Z_g is the impedance in ohms per mile of the ground return path. The formula for Z_g in ohms per mile was determined in the preceding article and is

$$Z_g = 643.9 \times 10^{-6}\omega(M + jN) \text{ ohms/mile.} \tag{8.6.4}$$

The formulas for M and N, as determined by Carson, are given by (8.5.5) and (8.5.6), respectively. Their values for the particular case under consideration are determined below.

The mutual impedance Z_{ab} between conductor (a) and conductor (b) with its ground return is*

$$Z_{ab} = j\,741.13 \times 10^{-6}\omega \log \frac{D_{ad}}{D_{ab}} \text{ ohms/mile.} \tag{8.6.5}$$

* Referring to Fig. 8–7, note that conductor (b) carrying the current I_b produces by equation (1.6.4) a magnetic flux

$$\phi_{ba} = \frac{\mu I_b}{2\pi} \ln \frac{R + D_{ab}}{D_{ab}}$$

which links with conductor (a). Similarly, the ground return d of conductor (b) carrying the current $-I_b$ produces a flux

$$\phi_{da} = \frac{-\mu I_b}{2\pi} \ln \frac{R + D_{ad}}{D_{ad}}$$

which links with conductor (a). The distances D_{ab} and D_{ad} are negligibly small in comparison with R_1 which is the large distance from either conductor (b) or its ground return d at which their magnetic effects become negligible. The mutual inductance between conductor (a) and conductor (b) with its ground return is accordingly

$$(L_{ab})_m = \frac{\phi_{ba} + \phi_{da}}{I_b} = \frac{\mu}{2\pi} \ln \frac{D_{ad}}{D_{ab}} \text{ henry/meter.}$$

Using logarithms to the base 10, $\mu = 4\pi \times 10^{-7}$ for air in the rationalized mks system of units, and the mile for the unit of conductor length, gives

$$(L_{ab})_m = 741.13 \times 10^{-6} \log \frac{D_{ad}}{D_{ab}} \text{ henry/mile.}$$

In terms of ohms per mile, formula (8.6.3) becomes, therefore,

$$(Z_{ab})_m = j\,741.13 \times 10^{-6}\omega \log \frac{D_{ad}}{D_{ab}} + 643.9 \times 10^{-6}\omega\,(M + jN). \quad (8.6.6)$$

The value of p which enters in the formulas for M and N, equations (8.5.5) and (8.5.6), for the particular case of two parallel conductors with a common ground return, as obtained by Carson, is

$$p = D'_{ad}\sqrt{\alpha}. \quad (8.6.7)$$

In this expression, D'_{ad} is the distance in centimeters between conductor (a) and the image of conductor (b) and

$$\alpha = 4\pi\lambda\omega,$$

where λ is the conductivity of the ground return in abmhos per centimeter cube. Using the resistivity in ohms per meter cube and expressing the distance D_{ad} in feet, the above formula for p becomes

$$p = 2.54 \times 12 \times D_{ad}\sqrt{\frac{4\pi\omega \times 10^{-11}}{\rho}}$$

$$= 341.6 \times 10^{-6} D_{ad}\sqrt{\frac{\omega}{\rho}}. \quad (8.6.8)$$

Referring to Fig. 8–7, it will be seen that

$$D_{ad} = \sqrt{D_{ab}^2 + 4H^2}. \quad (8.6.9)$$

To determine the value that should be assigned to the angle θ which enters in the formulas for M and N, refer again to Fig. 8–7. Note that the mutual impedance between the two conductors is larger when the spacing distance between them is smaller. It follows, therefore, that the maximum mutual impedance for the case under consideration is obtained by setting $\theta = 0$. Using this particular limiting value of θ in equations (8.5.5) for M and (8.5.6) for N gives

$$M = \frac{\pi}{8} - \frac{p}{3\sqrt{2}} + \frac{p^2}{16}\left(0.6728 + \ln\frac{2}{p}\right) \quad (8.6.10)$$

and

$$N = -\,0.0386 + \frac{1}{2}\ln\frac{2}{p} + \frac{p}{3\sqrt{2}}. \quad (8.6.11)$$

To determine what consideration should be given to these formulas in the calculation of the mutual impedance, assume a case in which $D_{ab} = 25$ ft. and $H = 50$ ft. Substituting in formula (8.6.8) gives

$$p = 341.6 \times 10^{-6}\sqrt{25^2 + 100^2}\sqrt{\frac{\omega}{\rho}}$$

$$= 35210 \times 10^{-6}\sqrt{\frac{\omega}{\rho}}.$$

For a frequency of 60 cps this becomes

$$p = \frac{0.684}{\sqrt{\rho}}.$$

For the two particular and rather extreme values of the earth's resistivities, $\rho = 100$ and $\rho = 2000$ ohms per meter cube, the values of p are

$$p_{100} = 0.0684$$

and

$$p_{2000} = 0.01528.$$

Substituting these values of p in (8.6.10) and carrying out the calculations involved, it will be found that irrespective of what values of ρ are used the last two terms in the equation for M may be neglected and the equation written

$$\begin{aligned} M &= \frac{\pi}{8} - \frac{p}{3\sqrt{2}} \\ &= 0.3927 - \frac{p}{4.242}. \end{aligned} \tag{8.6.12}$$

This expression for M was obtained by setting $\theta = 0$ in equation (8.5.5) for M. For the particular case under consideration, however, $D_{ab} = 25$ ft. and $H = 50$ ft. The angle, therefore, is

$$\begin{aligned} \theta &= \tan^{-1} \tfrac{25}{100} \\ &= 14.03°. \end{aligned}$$

Using cos 14.03° = 0.97 in equation (8.5.5), instead of 1.0, will make only a very slight difference in the result. It follows, therefore, that for all cases of two parallel conductors with ground return operated at 60 cps, irrespective of spacing distance, or height above the ground, since the earth's resistivity is not known with any great accuracy, the value of M may be taken as substantially equal to the first two terms of its formula as indicated by (8.6.12).

Using again the maximum value that $\cos\theta$ might have in formula (8.5.6) gives

$$\begin{aligned} N &= -0.0386 + \frac{1}{2} \ln \frac{2}{p} + \frac{p}{3\sqrt{2}} \\ &= -0.0386 + 1.151 \log \frac{2}{p} + \frac{p}{4.242}. \end{aligned} \tag{8.6.13}$$

For the particular case of $p_{100} = 0.0684$ ($\rho = 100$), this becomes

$$\begin{aligned} N_{100} &= -0.0386 + 1.151 \log 29.07 + \frac{0.0684}{4.242} \\ &= 1.6655. \end{aligned}$$

This result indicates that no significant error is introduced by neglecting the first and the last members in formula (8.6.13) for N, when $\rho = 100$ and the frequency is 60 cps. Similar results are obtained using the value of p for $\rho = 2000$, and by using different values of H and D_{ab} within the limits of practical applications. It may be said, therefore, that in the calculation of the mutual impedance of two parallel conductors with ground return operated at 60 cps, irrespective of height, spacing distance, the value of N as given by (8.6.13) is substantially equal to the second term only, i.e.,

$$N = 1.151 \log \frac{2}{p}. \tag{8.6.14}$$

Substituting the value of M as given by (8.6.12) and the value of N as given by (8.6.14) in the formula (8.6.6) for the mutual impedance gives

$$(Z_{ab})_m = j\,741.13 \times 10^{-6}\omega \log \frac{D_{ad}}{D_{ab}} + 643.9 \times 10^{-6}\omega \left(0.3927 - \frac{p}{4.242} + j\,1.151 \log \frac{2}{p}\right).$$

Collecting terms yields

$$\begin{aligned}(Z_{ab})_m &= 643.9 \times 10^{-6}\omega \left(0.3927 - \frac{p}{4.242}\right) + j\,741.13 \times 10^{-6}\omega \left(\log \frac{D_{ad}}{D_{ab}} + \log \frac{2}{p}\right)\\ &= 253 \times 10^{-6}\omega - 152 \times 10^{-6}\omega p + j\,741.13 \times 10^{-6}\omega \log \frac{2D_{ad}}{pD_{ab}}.\end{aligned}$$

Using the value of p as given by (8.6.8) in the second member of the above equation gives for that member

$$152 \times 10^{-6}\omega \left(341.6 \times 10^{-6} D_{ad} \sqrt{\frac{\omega}{\rho}}\right) = 51923\omega \times 10^{-12} D_{ad} \sqrt{\frac{\omega}{\rho}}.$$

For reasonable values of D_{ad}, ρ, and ω, this member in the equation for $(Z_{ab})_m$ is negligible in comparison with the first. The formula for $(Z_{ab})_m$ above becomes, accordingly,

$$(Z_{ab})_m = 253 \times 10^{-6}\omega - j\,741.13 \times 10^{-6}\omega \log \frac{2D_{ad}}{pD_{ab}}.$$

Substituting the value of p in the logarithmic member gives for that member

$$\begin{aligned}\log \frac{2D_{ad}}{pD_{ab}} &= \log \frac{2D_{ad}}{341.6 \times 10^{-6} D_{ad} \sqrt{\frac{\omega}{\rho}}\, D_{ab}}\\ &= \log \frac{2 \times 10^{6}\sqrt{\rho/\omega}}{341.6 D_{ab}}\\ &= \log \frac{5858\sqrt{\rho/\omega}}{D_{ab}}.\end{aligned}$$

The equation for the mutual impedance becomes, accordingly,

$$(Z_{ab})_m = 253 \times 10^{-6}\omega + j\,741.13 \times 10^{-6}\omega \log \frac{5858\sqrt{\rho/\omega}}{D_{ab}} \text{ ohms/mile.} \tag{8.6.15}$$

If the two conductors are identical, then the joint impedance per mile of conductor and ground return path is

$$Z_{ag} = Z_{bg} = Z_{cg} + (Z_{ab})_m$$

where Z_{cg} is given by (8.5.15). Hence,

$$Z_{ag} = Z_{bg} = R_c + 506 \times 10^{-6}\omega + j\,741.13 \times 10^{-6}\omega \log \frac{5858^2(\rho/\omega)}{D_{ab}r_{gm}} \tag{8.6.16}$$

gives the joint impedance per mile of conductor and ground return path. The impedance of the two conductors in parallel is one-half as much, i.e.,

$$Z_{abg} = \frac{R_c}{2} + 253 \times 10^{-6}\omega + j\,741.13 \times 10^{-6}\omega \log \frac{5858\sqrt{\rho/\omega}}{(D_{ab}r_{gm})^{\frac{1}{2}}}. \tag{8.6.17}$$

8.7. Zero-Sequence Impedance of a Three-Phase Line with Grounded Conductors. Consider a three-phase line completely grounded at some point S miles from the sending end as shown in Fig. 8–8. Let Z_{ag}, Z_{bg}, Z_{cg} be, respectively, the joint impedance per mile of each individual conductor and ground return. Assuming that the conductors are of the same size, these self-impedances will be equal and their values are given by equation (8.5.15).

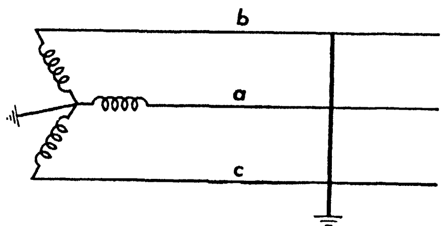


Figure 8–8

Setting for brevity's sake

$$\left.\begin{aligned} 253 \times 10^{-6}\omega &= A \\ 741.13\omega \times 10^{-6} &= B \\ 5858\sqrt{\rho/\omega} &= D_e \end{aligned}\right\} \tag{8.7.1}$$

formula (8.5.15) becomes

$$Z_{ag} = Z_{bg} = Z_{cg} = R_c + A + jB \log \frac{D_e}{r_{gm}}, \tag{8.7.2}$$

where R_c is the resistance per mile per conductor.

If $(Z_{ab})_m$ and $(Z_{ac})_m$ are the mutual impedances, respectively, between conductor a and b, and a and c per mile, their values are given by (8.6.15). By (8.7.1), they may be written in the abbreviated form

$$(Z_{ab})_m = A + jB \log \frac{D_e}{D_{ab}} \tag{8.7.3}$$

$$(Z_{ac})_m = A + jB \log \frac{D_e}{D_{ac}}. \tag{8.7.4}$$

Similarly, the mutual impedances between conductors b and a, and b and c are, respectively,

$$(Z_{ba})_m = A + jB \log \frac{D_e}{D_{ab}} \tag{8.7.5}$$

$$(Z_{bc})_m = A + jB \log \frac{D_e}{D_{bc}}. \tag{8.7.6}$$

In the same manner, the mutual impedance between conductors c and a, and c and b are

$$(Z_{ca})_m = A + jB \log \frac{D_e}{D_{ac}} \tag{8.7.7}$$

$$(Z_{cb})_m = A + jB \log \frac{D_e}{D_{bc}}. \tag{8.7.8}$$

Since the ground return current I_g of a three-phase grounded system is

$$I_g = I_a + I_b + I_c,$$

and, by (8.3.3), the zero-sequence component I_o of a three-phase unbalanced system is

$$I_o = \tfrac{1}{3}(I_a + I_b + I_c),$$

it follows that

$$I_g = 3I_o. \tag{8.7.9}$$

Each conductor of the grounded system under consideration carries a current I_o equal to the zero-sequence component. If V_{ao} is the voltage to ground of conductor a due to the zero-sequence current flowing in that conductor, it must be equal to the drop due to the impedance of conductor a and the ground return path, plus the drop due to the mutual impedance between conductors a and b, and plus the drop due to the mutual impedance between conductors a and c, i.e.,

$$V_{ao} = [I_o Z_{ag} + I_o(Z_{ab})_m + I_o(Z_{ac})_m]S,$$

where S is the length of the line to the point where the line is grounded. The above may be written

$$\frac{V_{ao}}{SI_o} = Z_{ag} + (Z_{ab})_m + (Z_{ac})_m = Z_{ao}. \tag{8.7.10}$$

This is the impedance to the zero-sequence current of conductor a. By (8.7.2), (8.7.3), and (8.7.4), this may be written

$$Z_{ao} = R_c + 3A + jB \log \frac{D_e^3}{r_{gm} D_{ab} D_{ac}}. \tag{8.7.11}$$

The formulas for the zero-sequence impedance of conductors b and c may be obtained in a similar manner and, respectively, are

$$Z_{bo} = R_c + 3A + jB \log \frac{D_e^3}{r_{gm} D_{bc} D_{ab}} \tag{8.7.12}$$

and

$$Z_{co} = R_c + 3A + jB \log \frac{D_e^3}{r_{gm} D_{ac} D_{bc}}. \tag{8.7.13}$$

The zero-sequence impedance of the three conductors in parallel is

$$Z_{o3} = \frac{Z_{ao} Z_{bo} Z_{co}}{Z_{ao} Z_{bo} + Z_{bo} Z_{co} + Z_{co} Z_{ao}}. \tag{8.7.14}$$

If the spacing is uniform, i.e., $D_{ab} = D_{bc} = D_{ac} = D$, then $Z_{ao} = Z_{bo} = Z_{co} = Z_{o1}$ and

$$Z_{o1} = R_c + 3A + jB \log \frac{D_e^3}{r_{gm} D^2} \text{ ohms/mile.} \tag{8.7.15}$$

The zero-sequence impedance Z_{o3} for the three conductors in parallel when uniformly spaced is one-third as much, i.e.,

$$Z_{o3} = \frac{Z_{o1}}{3} = \frac{R_c}{3} + A + jB \log \frac{D_e}{(r_{gm} D^2)^{\frac{1}{3}}} \text{ ohms/mile.} \tag{8.7.16}$$

Using the values of A and B stated in (8.7.1) yields

$$Z_{o3} = \frac{R_c}{3} + 253 \times 10^{-6}\omega + j\,741.13 \times 10^{-6}\omega \log \frac{5858\sqrt{\rho/\omega}}{(r_{gm} D^2)^{\frac{1}{3}}} \text{ ohms/mile.} \tag{8.7.17}$$

8.8. *Zero-Sequence Impedance of Three-Phase Transposed Line with Grounded Conductors.* In a transposed line such as the one shown in Fig. 8–9, each conductor takes successively the position of the other conductors for one-third of the line. It follows, therefore, that the zero-sequence impedance of each

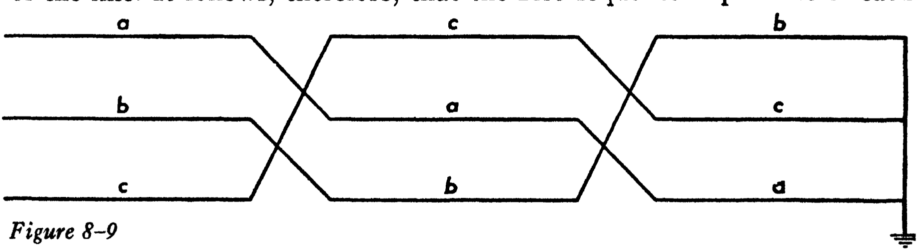


Figure 8–9

conductor is the sum of one-third of the impedance of each of the three conductors, assumed not transposed, i.e.,

$$Z_{o1} = \tfrac{1}{3}(Z_{ao} + Z_{bo} + Z_{co}) \tag{8.8.1}$$

where Z_{ao}, Z_{bo}, and Z_{co} are the zero-sequence impedances of the conductors a, b, and c, respectively, when not transposed and are given by equations (8.7.11), (8.7.12), and (8.7.13). Substituting these equations in (8.8.1) gives for the impedance per mile of transposed grounded conductor

$$Z_{o1} = \frac{1}{3}\left[3R_c + 9A + jB\left(\log\frac{D_e^3}{r_{gm}D_{ab}D_{ac}} + \log\frac{D_e^3}{r_{gm}D_{bc}D_{ab}} + \log\frac{D_e^3}{r_{gm}D_{bc}D_{ac}}\right)\right]$$

or

$$Z_{o1} = R_c + 3A + jB\log\frac{D_e^3}{(D_{ab}^2D_{bc}^2D_{ac}^2r_{gm}^3)^{\frac{1}{3}}}. \tag{8.8.2}$$

The zero-sequence impedance of the three conductors in parallel, through the common ground connection, assuming that the conductors are of the same size, is one-third that of a single conductor and is

$$Z_{o3} = \frac{R_c}{3} + A + jB\log\frac{D_e}{(D_{ab}^2D_{bc}^2D_{ac}^2r_{gm}^3)^{\frac{1}{9}}}, \tag{8.8.3}$$

where the values of A, B, and D are stated in (8.7.1).

If the conductors are spaced uniformly, i.e., if $D_{ab} = D_{bc} = D_{ac} = D$, the above equation becomes

$$Z_{o3} = \frac{R_c}{3} + A + jB\log\frac{D_e}{(D^2r_{gm})^{\frac{1}{3}}}. \tag{8.8.4}$$

Using the values of A, B, and D given by (8.7.1), the zero-sequence impedance of the three grounded conductors is

$$Z_{o3} = \frac{R_c}{3} + 253\omega \times 10^{-6} + j\,741.13 \times 10^{-6}\omega\log\frac{5858\sqrt{\rho/\omega}}{(D^2r_{gm})^{\frac{1}{3}}}\ \text{ohms/mile.} \tag{8.8.5}$$

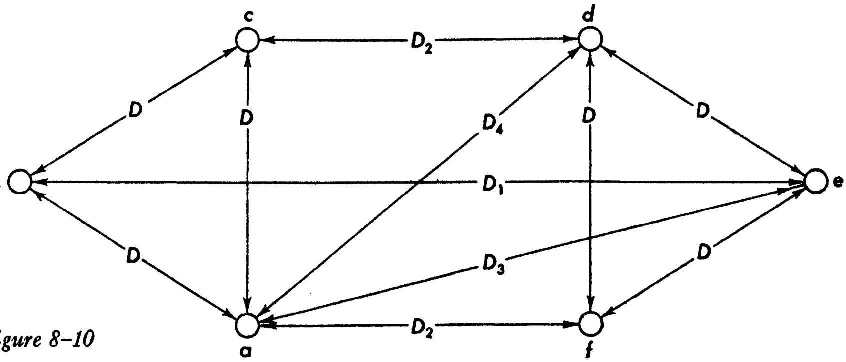


Figure 8–10

8.9. *Zero-Sequence Impedance of Non-transposed Twin Three-Phase Line with All Conductors Grounded.*

It will be assumed that each of the two lines (*abc*) and (*def*) are equally and uniformly spaced with a spacing distance D as shown in Fig. 8–10. Let the distance between the conductors in the two systems be

$$D_{be} = D_1$$
$$D_{af} = D_{cd} = D_2$$
$$D_{bd} = D_{bf} = D_{ce} = D_{ae} = D_3$$
$$D_{da} = D_{cf} = D_4.$$

Assuming that the conductors are grounded, the mutual impedances per mile between conductor a and the three conductors d, e, and f, respectively, by (8.6.15), are

$$\left.\begin{aligned}(Z_{ad})_m &= A + jB\log\frac{D_e}{D_4}\\ (Z_{ae})_m &= A + jB\log\frac{D_e}{D_3}\\ (Z_{af})_m &= A + jB\log\frac{D_e}{D_2}\end{aligned}\right\} \qquad (8.9.1)$$

where

$$A = 253 \times 10^{-6}\omega$$
$$B = 741.13 \times 10^{-6}\omega$$

and

$$D_e = 5858\sqrt{\rho/\omega}.$$

The mutual impedance of conductor a due to conductors d, e, and f jointly is

$$(Z_{am})_1 = (Z_{ad})_m + (Z_{ae})_m + (Z_{af})_m$$

or, by (8.9.1),

$$(Z_{am})_1 = 3A + jB\log\frac{D_e^3}{D_2D_3D_4}. \qquad (8.9.2)$$

The zero-sequence impedance of the grounded conductor a without the effect of conductors d, e, and f given by (8.7.15) is

$$Z_{oa} = R_c + 3A + jB\log\frac{D_e^3}{r_{gm}D^2}. \qquad (8.9.3)$$

The total zero-sequence impedance of conductor a, therefore, is

$$(Z_{oa})_t = Z_{oa} + (Z_{am})_1.$$

This, by (8.9.2) and (8.9.3), becomes

$$(Z_{oa})_t = R_c + 6A + jB\left(\log\frac{D_e^3}{r_{gm}D^2} + \log\frac{D_e^3}{D_2D_3D_4}\right)$$

or

$$(Z_{oa})_t = R_c + 6A + jB\log\frac{D_e^6}{D_2D_3D_4D^2r_{gm}}. \qquad (8.9.4)$$

The mutual impedance per mile between conductor b, and conductors d, e, and f, respectively, are found in the same manner. Thus

$$\left.\begin{aligned}(Z_{bd})_m &= A + jB\log\frac{D_e}{D_3}\\ (Z_{be})_m &= A + jB\log\frac{D_e}{D_1}\\ &\text{and}\\ (Z_{bf})_m &= A + jB\log\frac{D_e}{D_3}.\end{aligned}\right\} \qquad (8.9.5)$$

The mutual impedance per mile of conductor b due to conductors d, e, and f jointly is

$$(Z_{bm})_1 = 3A + jB \log \frac{D_e^3}{D_1 D_3^2}. \tag{8.9.6}$$

The total zero-sequence impedance per mile of conductor b, accordingly, is

$$(Z_{ob})_t = R_c + 6A + jB \log \frac{D_e^6}{D_1 D_3^2 D^2 r_{gm}}. \tag{8.9.7}$$

The mutual impedances per mile of conductor c and conductors d, e, and f, respectively, are

$$(Z_{cd})_m = A + jB \log \frac{D_e}{D_2}$$

$$(Z_{ce})_m = A + jB \log \frac{D_e}{D_3}$$

and

$$(Z_{cf})_m = A + jB \log \frac{D_e}{D_4}.$$

The mutual impedance per mile of conductor c due to conductors d, e, and f jointly is

$$(Z_{cm})_1 = 3A + jB \log \frac{D_e^3}{D_2 D_3 D_4}. \tag{8.9.8}$$

The total zero-sequence impedance per mile of conductor c, therefore, is

$$(Z_{oc})_t = R_c + 6A + jB \log \frac{D_e^6}{D_2 D_3 D_4 D^2 r_{gm}}. \tag{8.9.9}$$

The average zero-sequence impedance per mile of conductor of the line abc is the average of $(Z_{oa})_t$, $(Z_{ob})_t$, and $(Z_{oc})_t$. Thus, denoting by Z_{o1} the average value of this impedance, its value is

$$Z_{o1} = \tfrac{1}{3}[(Z_{oa})_t + (Z_{ob})_t + (Z_{oc})_t].$$

This, by (8.9.4), (8.9.7), and (8.9.9), becomes

$$Z_{o1} = R_c + 6A + jB \log \frac{D_e^6}{(r_{gm}^3 D^6 D_1 D_2^2 D_3^4 D_4^2)^{\frac{1}{3}}}. \tag{8.9.10}$$

This is average value of the zero-sequence impedance per mile per conductor.

The zero-sequence impedance per mile of the three grounded conductors of line abc is one-third of Z_{o1}, i.e.,

$$Z_{o3} = \frac{R_c}{3} + 2A + jB \log \frac{D_e^2}{(r_{gm}^3 D^6 D_1 D_2^2 D_3^4 D_4^2)^{\frac{1}{9}}}. \tag{8.9.11}$$

The zero-sequence impedance per mile of the twin lines, consisting of 6 conductors having a common ground return, is one-half of Z_{o3}, i.e.,

$$Z_{o6} = \frac{R_c}{6} + A + jB \log \frac{D_e}{(r_{gm}{}^3 D^6 D_1 D_2{}^2 D_3{}^4 D_4{}^2)^{\frac{1}{18}}} \text{ ohms/mile,} \quad (8.9.12)$$

where the values of A, B, and D are given by (8.7.1), and R_c is the resistance per mile of conductor, all six conductors being assumed of the same size.

8.10. *Zero-Sequence Impedance of Twin Transposed Three-Phase Grounded Lines.* It will be assumed that the two lines shown schematically in Fig. 8-11 have symmetrical and equal spacing distances D. By virtue of the transposition each conductor in each line takes the position of each of the other conductors in that line for one-third of the length of the line. It

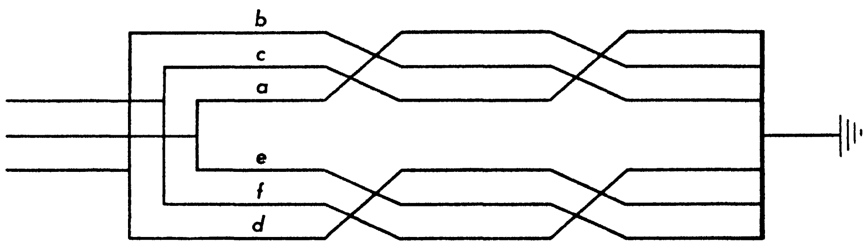


Figure 8-11

follows, therefore, that if $(Z_{am})_1$, $(Z_{bm})_1$, and $(Z_{cm})_1$ are the mutual impedances per mile of the line abc, the mutual impedance of conductor a when transposed is

$$Z_{am} = \tfrac{1}{3}[(Z_{am})_1 + (Z_{bm})_1 + (Z_{cm})_1].$$

By (8.9.2), (8.9.6), and (8.9.8), this becomes

$$Z_{am} = 3A + jB \log \frac{D_e{}^3}{(D_1 D_2{}^2 D_3{}^4 D_4{}^2)^{\frac{1}{3}}}. \quad (8.10.1)$$

The zero-sequence impedance of a conductor a without the effect of conductors d, e, and f is given by (8.7.15). The total zero-sequence impedance per mile of conductor a, therefore, is

$$(Z_{am})_t = R_c + 6A + jB\left(\log \frac{D_e{}^3}{r_{gm}D^2} + \log \frac{D_e{}^3}{(D_1 D_2{}^2 D_3{}^4 D_4{}^2)^{\frac{1}{3}}}\right)$$

or

$$(Z_{am})_t = R_c + 6A + jB \log \frac{D_e{}^6}{(r_{gm}{}^3 D^6 D_1 D_2{}^2 D_3{}^4 D_4{}^2)^{\frac{1}{3}}}. \quad (8.10.2)$$

Because of the transposition, the zero-sequence impedances of each of the three conductors are equal, i.e., $(Z_{am})_t = (Z_{bm})_t = (Z_{cm})_t$. The zero-sequence impedance of the three conductors of line *abc*, since they are in parallel, is obtained by dividing (8.10.2) by three. Thus,

$$Z_{o3} = \frac{R_c}{3} + 2A + jB \log \frac{D_e^2}{(r_{gm}{}^3 D^6 D_1 D_2{}^2 D_3{}^4 D_4{}^2)^{\frac{1}{9}}}. \tag{8.10.3}$$

The impedance of the two transposed, grounded three-phase lines, operated in parallel, is one-half as much, i.e.,

$$Z_{o6} = \frac{Z_{o3}}{2}$$

or

$$Z_{o6} = \frac{R_c}{6} + A + jB \log \frac{D_e}{(r_{gm}{}^3 D^6 D_1 D_2{}^2 D_3{}^4 D_4{}^2)^{\frac{1}{18}}} \text{ ohms/mile,} \tag{8.10.4}$$

where the values of A, B, and D are given in (8.7.1), and the spacing distances are as shown in Fig. 8–10.

8.11. *Resistivity of the Earth Return.* The study of the transmission circuits with ground return in the preceding six articles is based fundamentally upon the formulas obtained by Carson in his theoretical investigation of the subject. The impedance formulas of the various grounded transmission systems obtained in the preceding articles are substantially accurate for the commercial power frequencies of 25 and 60 cps. The approximations are due to the neglect of certain terms in Carson's formulas. The neglect of these terms accounts for the fact that the resistance component of the impedance formula of a conductor with ground return appears as independent of the resistivity of the earth.

The impedance formulas for all the systems discussed in the preceding articles contain the term

$$B \log D_e = 741.13 \times 10^{-6}\, \omega \log 5858\sqrt{\rho/\omega} \tag{8.11.1}$$

where ρ is the resistivity of the earth in ohms per meter cube. Carefully conducted tests by a joint Research and Development Subcommittee of the National Electric Light Association, now the Edison Institute, and the American Telephone and Telegraph Co.* indicate that the resistivity of the earth differs considerably over the United States of America. In accordance with the report of this research committee, the resistivity of the earth measured at a frequency of 60 cps may be anywhere from 16 to 500 ohms per meter cube, in California and Nevada; from 10 to 40 ohms in Utah, Colorado, Arizona, and New Mexico; about 10 ohms per meter cube in Texas, Oklahoma, Kansas, and Louisiana; 13 to 100 ohms in Iowa; 40 to

* Pierce, D. A., and Ferris, L. P., *Coupling Factors for Ground Return Circuits*, Eng. Report No. 14 N.E.L.A. and Bell Telephone System, Vol. II, April, 1932.

180 ohms in Missouri; 24 to 800 ohms in New Jersey; 29 ohms in Ohio; 2200 ohms in Pennsylvania; 500 ohms in Tennessee; 20 to 1200 ohms in West Virginia; 2000 ohms in Wisconsin; 5.7 ohms in New York;* swampy ground from 10 to 1000 ohms; dry earth about 1000 ohms; sea water from 0.1 to 1 ohm; sandstone 10 ohms per meter cube. The report mentioned above states specifically that the earth resistivity for the various states given above "*must not* be considered indicative of the resistivity to be expected . . . since wide variations may be found within a single state."

The term $B \log D_e$, stated in (8.11.1) and which enters in all the impedance formulas of systems of conductors with ground return, becomes for the commercial power frequency of 60 cps

$$B \log D_e = 0.279 \log 301.8\sqrt{\rho}. \tag{8.11.2}$$

The following table gives the value of this term for increasing values of ρ.

TABLE VI

	$B \log D_e$
1	0.692
10	0.832
50	0.934
100	0.972
500	1.213
1000	1.360
2000	1.430

The table shows that large changes in the value of ρ produce relatively small changes in the value of $B \log D_e$. Thus, a change in the value of ρ from 50 to 100, or 100 per cent causes an increase of only 4 per cent in the value of $B \log D_e$. A change in the value of ρ from 100 to 500, an increase of 400 per cent changes the value of $B \log D_e$ somewhat less than 25 per cent. It follows, therefore, that although accuracy in the value of ρ is not necessary, the value to be used in the impedance formula for any one particular case should be reasonably within the required degree of magnitude. Thus, the use of $\rho = 25$ for $\rho = 40$ or vice versa may not introduce serious errors, while the use of $\rho = 1000$ for $\rho = 50$ or vice versa will affect the result quite seriously.

The relationship between $B \log D_e$ and ρ for a frequency of 60 cps is shown graphically in the curves, Fig. 8–12. The average value of $B \log D_e$ for values of ρ from 1 to 2000 is in the neighborhood of 1.0 and it corre-

* See earth resistivity distribution map in Wagner and Evans, *Symmetrical Components*, p. 147, McGraw-Hill.

sponds to a value of ρ in the neighborhood of 100. From a large number of tests made by the N.E.L.A. and A.T. & T. Co.'s Research Subcommittee referred to above, it appears that the average value of the earth's resistivity

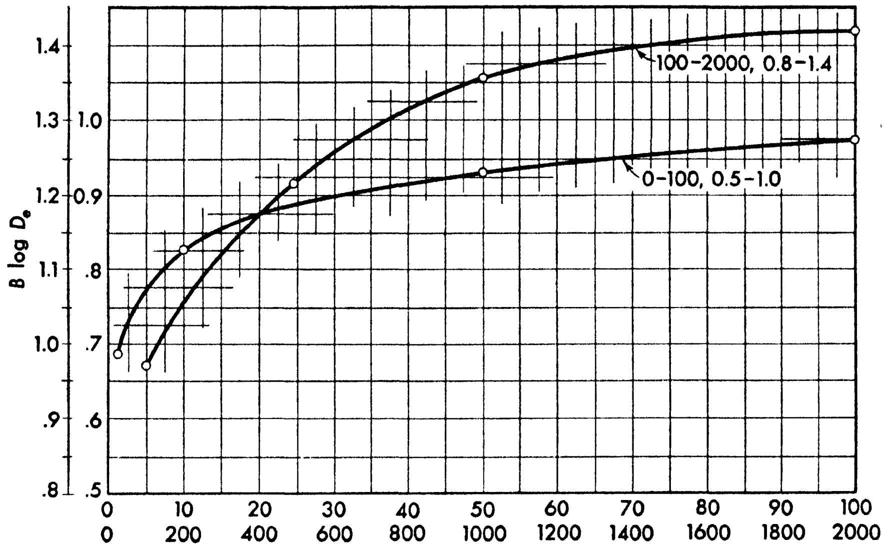


Figure 8-12

in the United States is roughly 100 ohms per meter cube. In the absence of earth resistivity data for any particular locality or the possibility of determining the resistivity by test, the value of $\rho = 100$ may be used for short-circuit calculations of transmission systems in that locality.

8.12. *Summary of Line Reactances.* The reactance of line conductors enters in all calculations of system performances under all conditions including short circuits. Tables of line reactance per conductor per mile for various size conductors and symmetrical triangular spacing have been calculated for the commercial frequencies of 25 and 60 cps and are given in handbooks of electrical engineering. It should be kept in mind that the positive and negative sequence reactances of line conductors are the same as the actual value. The symbols in the following list of formulas for the reactances of typical lines developed in this book are

$B = 741.13 \times 10^{-6}\omega$

$D =$ equilateral spacing distance in feet

$r_{gm} =$ geomean radius of conductors in feet

$D_e = 5858\sqrt{\frac{\rho}{\omega}}.$

For D_1, D_2, D_3, and D_4, which are spacing distances in feet between conductors in twin three-phase lines, refer to § 8.9.

1. The reactance per mile per conductor of three-phase line, by (1.10.7), is

$$X = B \log \frac{D}{r_{gm}}. \tag{8.12.1}$$

2. The reactance per mile per conductor of a single conductor with ground return, by (8.5.15) and (8.7.1), is

$$X_{cg} = B \log \frac{D_e}{r_{gm}}. \tag{8.12.2}$$

3. The reactance per mile per conductor of two identical conductors with ground return, by (8.6.16) and (8.7.1), is

$$\left.\begin{aligned} X_{cg} &= 2B \log \frac{D_e}{(r_{gm}D_{ab})^{\frac{1}{2}}} \\ &= B \log \frac{D_e^2}{r_{gm}D_{ab}}. \end{aligned}\right\} \tag{8.12.3}$$

4. The zero-sequence reactance per mile per conductor of symmetrically-spaced grounded three-phase line, by (8.7.15), is

$$X_{o1} = B \log \frac{D_e^3}{r_{gm}D^2} = 3B \log \frac{D_e}{(r_{gm}D^2)^{\frac{1}{3}}}. \tag{8.12.4}$$

Wagner and Evans have shown by calculated values of reactances of transmission lines with various size conductors, spacing distances, and earth resistivity that for single, three-phase lines that have no grounded neutral wires, the zero-sequence reactance per conductor is approximately 3.5 times the positive-sequence reactance of the conductor. For systems with ground wires, however, the ratio of the zero-sequence reactance to the positive sequence reactance "is much smaller, decreasing to 2.7 or 1.7, depending upon the effectiveness of the ground-wire system."

5. The average reactance per mile per conductor of transposed or non-transposed twin three-phase lines, by (1.12.9), is

$$X = B \log \frac{D\left(\dfrac{D_3^2D_2}{D_4^2D_1}\right)^{\frac{1}{3}}}{r_{gm}}. \tag{8.12.5}$$

The zero-sequence impedance per mile per conductor of transposed or nontransposed twin, three-phase lines with all conductors grounded, by (8.10.4), is

$$X_o = 6B \log \frac{D_e}{(r_{gm}{}^3D^6D_1D_2^2D_3^4D_4^2)^{\frac{1}{18}}}. \tag{8.12.6}$$

8.13. *Per Unit and Per Cent Reactance.* A knowledge of values of the current under short-circuit conditions at points of systems where such faults are likely to occur is quite essential if adequate protection of equipment and maintenance of service is to be provided for. From the discussion given in the preceding articles of this chapter, it appears that some of the factors that must be considered in the calculation of short-circuit currents are not often known with substantial accuracies. Furthermore, the circuit structures of transmission systems, quite complicated when normal, become even more so under short-circuit conditions. Rigorous solutions become impossible even by test methods such as with artificially set-up networks simulating faulted systems. Such set-up networks are called ***network calculators*** or ***calculating boards***. However, since great accuracies are not essential, the difficulties are not very serious. Because of the facts just mentioned and also because the impedance of the component parts of generating and transmitting systems have rather large angles, the calculations are simplified by neglecting the smaller resistance component of such impedances. Furthermore, in a system consisting of generators, transformers, lines, and receiving equipment each operating at different voltages, and carrying different currents, the actual ohmic reactances cannot be combined directly unless they are first converted to a common base voltage. Thus, in one phase of a three-phase system, there is the generator phase reactance X_g carrying the generator current; a step-up transformer whose low-voltage leakage reactance X_l carries the same current as the generator and whose high-voltage leakage reactance X_h carries the high-voltage current; the line reactance X_L carrying also the high-voltage current, and so on. These reactances can be combined directly only by either converting the reactances X_h and X_L on the high side to the low-voltage base by multiplying by the square of the ratio of transformation, $(V_l/V_h)^2 = a^2$, or by converting the reactances X_g and X_l on the low-voltage side to the high-voltage base by dividing by the same factor. The total reactance, when referred to the low-voltage side, is

$$X_t = X_g + X_l + a^2X_h + a^2X_L + \cdots,$$

where, if E is the generator emf per phase and I the rated current, then

$$X_t = \frac{E}{I}.$$

From the above it follows that

$$1 = \frac{X_g}{X_t} + \frac{X_l}{X_t} + \frac{a^2X_h}{X_t} + \frac{a^2X_L}{X_t} + \cdots, \tag{8.13.1}$$

where X_g/X_t is the per unit reactance of the generator phase, $(X_l+a^2X_h)/X_t$ is the per unit reactance of the transformer, and a^2X_L/X_t is the per unit reactance of the line. The above addition of the per unit reactances may be written also

$$1 = \frac{IX_g}{E} + \frac{I(X_l + a^2X_h)}{E} + \frac{Ia^2X_L}{E} + \cdots, \tag{8.13.2}$$

indicating that all reactances are converted to a common current and voltage base.

However, since

$$I = \frac{I_h}{a}$$

and

$$E = aE_h,$$

it follows that the last member of the above equation may be written

$$\frac{Ia^2X_L}{E} = \frac{I_hX_L}{E_h},$$

and the equation becomes

$$1 = \frac{IX_g}{E} + \frac{IX_T}{E} + \frac{I_hX_L}{E_h} + \cdots, \tag{8.13.3}$$

where $X_T = X_l + a^2X_h$ is the equivalent reactance of the transformer. This indicates that if a part of a circuit is in the low-voltage side, its per unit reactance is the actual voltage drop across that part divided by the total low-voltage acting across the entire circuit. Similarly, if the part is in high-voltage side, its per unit reactance is the actual voltage drop across the part divided by the equivalent high-voltage acting across the entire circuit.

Reactances of system equipment are usually expressed in per cent. Thus, when the reactance of a transformer is said to be 12 per cent, it is meant that the voltage drop across the transformer is 12 per cent of the rated voltage at rated current. From the point of view of short-circuit conditions it also means that if one winding were short-circuited while the other is subject to full rated voltage, the short-circuit current would be $\frac{100}{12}$ times the rated current.

The reactances of synchronous generators and of synchronous motors are also expressed in terms of per cent or per unit values. When the synchronous reactance of a synchronous machine is said to be 10 per cent, it is meant that there is a 10 per cent drop in voltage per phase when the machine delivers rated current. The reactance of synchronous machinery used in short-circuit calculation is called ***transient reactance***. It is defined as "the ratio of the fundamental component of reactive armature voltage due to the fundamental direct-axis alternating current component of the armature current to this component of current under suddenly applied load conditions and at rated frequency, the value of the current to be determined by the extrapolation of the envelope of the alternating current component of the current wave to the instant of the sudden application of load, neglecting the high-decrement currents during the first few cycles."* This rather involved definition implies that the transient reactance of a synchronous machine is the reactance to the initial value of the current at the instant the short circuit occurs.† The transient reactance of high-speed turbo-alternators is from 10 per cent to 25 per cent. Slow-speed water-turbine driven generators have transient reactances of 30 per cent to 40 per cent. The transient reactance of synchronous motors is about 30 per cent. In the absence of any definite data from the manufacturer for any particular case the above may be used in short-circuit calculations.

8.14. *Dependency of Per Unit Reactance on the kva Base.*

If a certain reactance X ohms is part of a circuit subject to a rated voltage V_1 and the rated current in the circuit is I_1, then, by (8.13.2), the value of X in per unit is

$$X_{u1} = \frac{I_1 X}{V_1}. \tag{8.14.1}$$

If the same value of reactance X is part of another circuit subject to a rated voltage V_2 and the rated current in the circuit is I_2, the value of X in per unit is

$$X_{u2} = \frac{I_2 X}{V_2}. \tag{8.14.2}$$

The above relations indicate that a definite reactance X ohms may have different per unit (or per cent) values, depending upon the voltage and the current in the circuit of which X is a part. The ratio of the above two equations is

$$\frac{X_{u1}}{X_{u2}} = \frac{V_2 I_1}{V_1 I_2}. \tag{8.14.3}$$

This equation gives the relation by means of which per unit values of reactance for a certain voltage rating V_1 and current rating I_1 may be changed to another voltage rating V_2 and current rating I_2. To illustrate the above, consider a transmission line conductor whose total reactance is

* Definition 10.35.120, *American Standard Definitions of Electrical Terms*, A.I.E.E. 1941, p. 69.

† Transient reactance of synchronous machinery is discussed in detail in *Effect of Armature Resistance upon Hunting* by C. F. Wagner, Trans. A.I.E.E., Vol. 29, 1930.

12 ohms. If the voltage to neutral at the sending end is 10 kv and the current is 100 amperes, the reactance of the line conductor in per unit is

$$X_{u1} = \frac{12 \times 100}{10000} = 0.12 \text{ per unit}$$

or 12 per cent with reference to the phase voltage of 10 kv and line current of 100 amperes. If the line were operated at 20 kv per phase and 50 amperes line current, the reactance in per unit, by (8.14.3), would be

$$X_{u2} = 0.12 \times \frac{10 \times 50}{20 \times 100}$$
$$= 0.03 \text{ per unit}$$

or 3 per cent.

Under the condition of equal voltages for the circuits, equation (8.14.3) becomes

$$\frac{X_{u1}}{X_{u2}} = \frac{VI_1}{VI_2} = \frac{(kva)_1}{(kva)_2}, \tag{8.14.4}$$

where X_{u1} is the per unit value of the reactance X to the $(kva)_1$ base and X_{u2} is the per unit value of the same reactance X to the $(kva)_2$ base. The above equation shows that to change the per unit (or per cent) reactance from one kva base to another, *under the same condition of voltage*, multiply the given per unit reactance by the ratio of the new base to the old base.

The reactances of the various equipment in a transmission system, such as generators, transformers, synchronous motors, phase modifiers, are usually stated in per cent to their own kva rating as a base. Before such reactances can be added they must be changed to a common base.

Thus, let

$$X_{u1} = 0.12 \text{ to } 1000 \text{ kva base}$$
$$X_{u2} = 0.26 \text{ to } 2000 \text{ kva base}$$
$$X_{u3} = 0.17 \text{ to } 5000 \text{ kva base.}$$

To add these per unit reactances, it is necessary to change all of them to a common kva base. If the chosen common base is 10000 kva, then,

$$X'_{u1} = 0.12 \times \tfrac{10000}{1000} = 1.2 \text{ to } 10000 \text{ kva base}$$
$$X'_{u2} = 0.26 \times \tfrac{10000}{2000} = 1.30 \text{ to } 10000 \text{ kva base}$$

and

$$X'_{u3} = 0.17 \times \tfrac{10000}{5000} = 0.34 \text{ to } 10000 \text{ kva base.}$$

The total reactance, therefore, is

$$X'_u = 1.2 + 1.3 + 0.34$$
$$= 2.84 \text{ in per unit to } 10000 \text{ kva base.}$$

8.15. *Per Unit Reactance of Transmission Line Conductors.* While the reactances of synchronous machinery and of transformers are usually stated in per unit or per cent of their respective kva rating, the series reactance ($L\omega$) of transmission lines is usually stated in terms of ohms per mile of conductor at the operating frequency. Line reactance is part of the system circuit structure, and as such it must be combined in short-circuit calculations with the reactances of other parts of the structure, all of which are expressed either in per unit or in per cent values to a common kva base as discussed in the preceding articles. The line reactance must be converted, therefore, from ohms to per unit value at the required kva base.

To do this, consider a line conductor whose total series reactance at the operating frequency is $L\omega S = X$ ohms. The per unit reactance, by (8.14.1), is

$$X_u = \frac{IX}{V_n}$$
$$= \frac{IX}{1000(kv)_n} \text{ per unit,}$$

where I is the actual current in the line and $(kv)_n$ is the rated voltage to neutral at which the line operates.

The expression may be written

$$X_u = \frac{(kv)_n IX}{1000(kv)_n^2}. \tag{8.15.1}$$

Using rated line voltage kv in place of the voltage to neutral, the equation becomes

$$X_u = \frac{\sqrt{3}(kv)IX}{1000(kv)^2}$$

and since $\sqrt{3}(kv)I$ is the three-phase kva rating, the per unit line reactance may be calculated by

$$X_u = \frac{X(kva)}{1000(kv)^2}. \tag{8.15.2}$$

Thus, a line of 20 ohms reactance in a three-phase system rated 33000 kva at 66 kv line voltage has a per unit reactance of

$$X_u = \frac{20 \times 33000}{1000 \times (66)^2} = 0.1515 \text{ per unit}$$

or 15.15 per cent at 33000 kva base.

8.16. *Three-Phase Short Circuits.*

This is the most severe type of fault that may occur on a transmission system. Diagrammatically it is shown in Fig. 8–13. As seen from this figure a three-phase short circuit, frequently referred to as an L–L–L fault, is equivalent to either a symmetrical delta or a symmetrical wye structure of zero impedance per branch. The system, under an L–L–L fault remains symmetrical, and the short-circuit currents in each phase are equal. They may be calculated by dividing the generated emf by the sum of the transient reactance of the generator and the conductor reactance to the point of the fault. This straightforward method cannot be used when the system includes transformers, or when the load is synchronous in character or both. In such cases the system reactances stated in per unit or per cent are subject to different voltages, and as a consequence are not additive directly. When the system is short-circuited, the synchronous machines at the receiving end act as generators and feed the fault with the energy stored in their rotors.

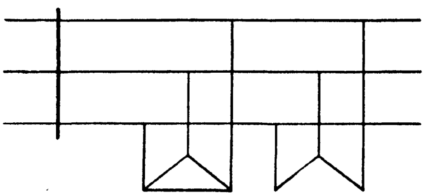


Figure 8–13

With the exception of the line, the reactances of all component parts of the system are usually stated in per cent to their own kva rating. By converting these per cent reactances, however, to a common kva base, as discussed in the preceding three articles, they become directly additive, and the combined per cent reactance to the fault easily calculated.

Assuming that the phase voltage has the same value when the system carries the short-circuit current I_{sh} and the per cent reactance to the common kva base is X per cent, as when the system carries the rated current I_r and the per cent reactance of the system is 100 per cent, then, by (8.14.3),

$$\frac{100}{X} = \frac{I_{sh}}{I_r}.$$

The short-circuit current, therefore, is

$$I_{sh} = \frac{100 I_r}{X}. \tag{8.16.1}$$

Thus, if the reactance per phase to the fault is 25 per cent, the short-circuit current is four times the normal current.

The case of three-phase to ground short-circuit (usually denoted by L–L–L–G) on a single grounded line is identical with the one discussed above.

The short circuit being symmetrical, the currents in each phase are equal and 120° apart in time phase, and the ground return current is, therefore, zero.

To illustrate the calculation of an *L–L–L* fault, consider the three-phase single line system shown in Fig. 8–14. The system is fed from two stations, *A* and *B*, and the load is not shown.

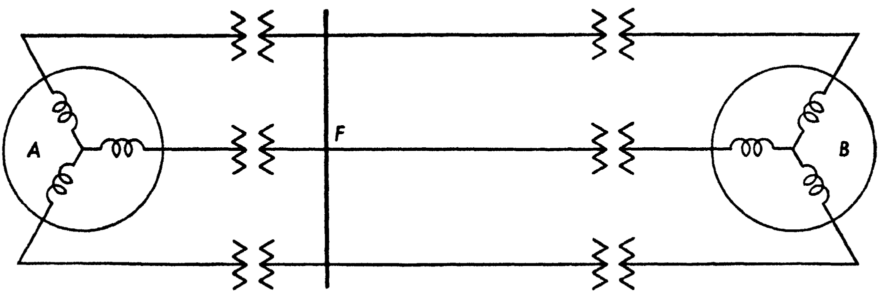


Figure 8–14

Generators *A* and *B* are identical and rated 13.8 kv, 21000 kva, and have a transient reactance of 30 per cent at own kva base.

The transformers are also identical and are rated 13.8/66 kv, 7000 kva, and have a reactance of 8.4 per cent to their own kva base.

The tie line is 50 miles long; each conductor has a reactance of 0.848 ohms per mile. The three-phase fault is assumed at *F*, 20 miles from station *A*.

Taking the rated kva of the generator as the kva base, the reactances are as follows:

$$X_A = 30 \text{ per cent}$$
$$X_B = 30 \text{ per cent.}$$

The reactance of each transformer to 21000 kva base is

$$X_T = \frac{8.4 \times 21000}{7000} = 25.2 \text{ per cent.}$$

The reactance of the line conductor to 21000 kva base at 66 kv, by (8.15.2), is

$$X_L = \frac{0.848 \times 50 \times 21000 \times 100}{1000 \times 66^2} = 20.5 \text{ per cent.}$$

The conductor reactance from station *A* to the fault is

$$X_{AF} = \tfrac{2}{5} \times 20.5 = 8.2 \text{ per cent.}$$

The conductor reactance from fault to station *B* is

$$X_{BF} = \tfrac{3}{5} \times 20.5 = 12.3 \text{ per cent.}$$

The circuit diagram per phase of the faulted system is shown in Fig. 8–14a. With reference to the fault, the reactance of the faulted system is

$$X_F = \frac{(30 + 25.2 + 8.2)(12.3 + 25.2 + 30)}{63.4 + 67.5}$$
$$= 32.7 \text{ per cent.}$$

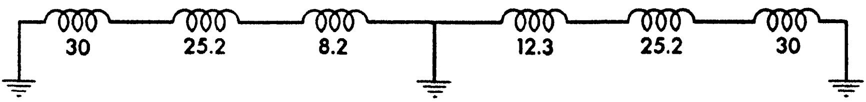


Figure 8–14a

The rated current is

$$I_r = \frac{(kva)\text{base}/3}{kv/\sqrt{3}} = \frac{(kva)\text{base}}{\sqrt{3}kv}.$$

For the particular case under consideration at 13.8 kv it is

$$I_r = \frac{21000}{13.8\sqrt{3}} = 880 \text{ amperes.}$$

By (8.16.1),

$$I_{sh} = \frac{100 I_r}{X_F}.$$

Hence, the current in the short circuit is

$$I_{sh} = \frac{88000}{32.7} = 2690 \text{ amperes,}$$

at 13.8 kv line voltage. Of this current generator A supplies

$$I_A = \frac{32.7}{63.4} \times 2690 = 1385 \text{ amperes,}$$

and generator B supplies

$$I_B = \frac{32.7}{67.5} \times 2690 = 1305 \text{ amperes.}$$

The current in the short-circuited conductor toward station A is

$$I_a = 1385 \times \frac{13.8}{66} = 290 \text{ amperes,}$$

and the current in the short-circuited conductor toward station B is

$$I_b = 1305 \times \frac{13.8}{66} = 273 \text{ amperes.}$$

To further illustrate the calculations of an L–L–L fault, consider the single-wire diagram of a twin three-phase system, shown schematically in Fig. 8–15. The figure shows only one phase to neutral.

Generator A is rated 13.8 kv, 35000 kva, and has a transient reactance of 30 per cent at own kva base. Generator B is rated 13.8 kv, 21000 kva, and has a transient reactance of 30 per cent at own kva base.

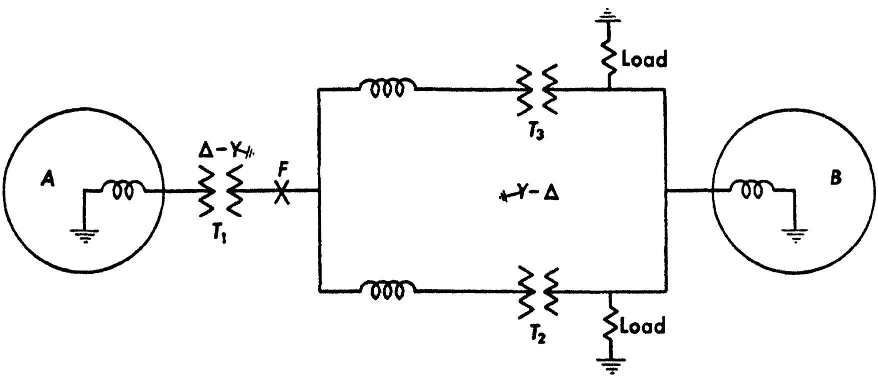


Figure 8–15

T_1 represents two identical transformers connected in parallel, each rated 13.8/66 kv, 20000 kva, having 8.4 per cent reactance at own kva base. Transformers T_2 and T_3 are identical and rated 13.8/66 kv, 20000 kva, and have 8.4 per cent reactance at own kva base.

The lines are identical, 50 miles long, #0000 copper, 16.5 feet equivalent triangular spacing, and have 0.848 ohms reactance per mile. The procedure in calculating the short-circuit currents when an L–L–L fault occurs at some point in the system such as at the high side of transformer T_1 is given below.

Select an appropriate kva base and convert all given reactances to this base. Thus taking 35000 kva which is the rating of generator A as the common base, the reactances are as follows:

$$X_A = 30 \text{ per cent}$$
$$X_B = \frac{30 \times 35000}{21000} = 50 \text{ per cent.}$$

The transformer reactances are

$$X_T = \frac{8.4 \times 35000}{20000} = 14.7 \text{ per cent.}$$

The reactances of the line per conductor in per cent to 35000 kva base at 13.8 kv, by (8.15.2), is

$$X_L = \frac{0.848 \times 50 \times 35000 \times 100}{66^2 \times 1000}$$
$$= 34.1 \text{ per cent.}$$

Neglecting the load, the circuit diagram of the faulted system is shown in Fig. 8–15a. The arrows indicate the direction of current flow toward the fault.

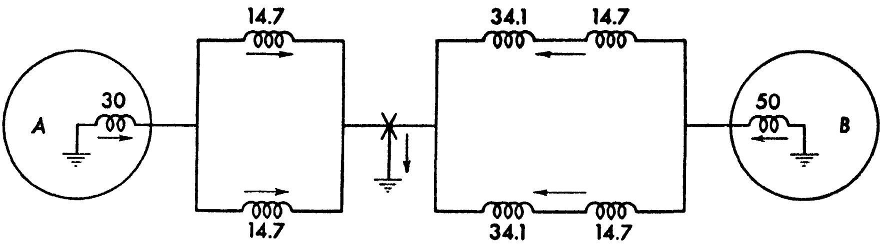


Figure 8–15a

The reactance from the neutral of generator A to the fault is

$$X_{AF} = 30 + \frac{14.7}{2} = 37.35 \text{ per cent.}$$

The reactance from the neutral of generator B to the fault is

$$X_{BF} = 50 + \frac{34.1 + 14.7}{2}$$
$$= 74.4 \text{ per cent.}$$

With reference to the fault these two reactances are in parallel and their joint value is

$$X_F = \frac{37.35 \times 74.4}{37.35 + 74.4}$$
$$= 24.9 \text{ per cent}$$

to 35000 kva base.

The rated current for 100 per cent reactance is equal to the base volt-amperes per phase divided by the phase voltage, i.e.,

$$I_r = \frac{(kva) \text{ base}}{\sqrt{3}kv}.$$

For the particular case under consideration it is

$$I_r = \frac{35000}{13.8\sqrt{3}} = 1467 \text{ amperes.}$$

The short-circuit current I_{sh}, by (8.16.1), is

$$I_{sh} = \frac{100I_r}{X_F}.$$

For the particular case under consideration it is

$$I_{sh} = \frac{146700}{24.9}$$
$$= 5890 \text{ amperes.}$$

Of this current, the generator A supplies

$$I_A = \frac{24.9}{37.35} \times 5890$$
$$= 3920 \text{ amperes,}$$

and generator B supplies

$$I_B = \frac{24.9}{74.4} \times 5830$$
$$= 1970 \text{ amperes.}$$

Each of the two lines from generator B to the fault carries a current

$$I_{LB} = \frac{1970}{2} \times \frac{13.8}{66}$$
$$= 206 \text{ amperes.}$$

The current in the fault is

$$5890 \times \frac{13.8}{66} = 1232 \text{ amperes.}$$

8.17. *Line to Line Fault.* With this nonsymmetrical short circuit, usually referred to as an L–L fault, the system structure is unbalanced as shown in the diagram, Fig. 8–16. If the system does not include transformers, and the load at the

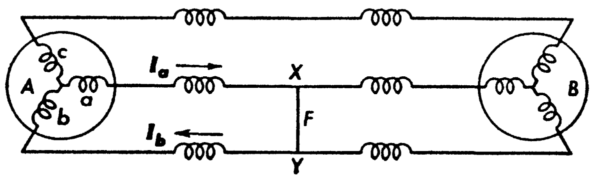


Figure 8–16

receiving end is nonsynchronous, the short-circuit current may be calculated by

$$I_{sh} = \frac{\sqrt{3}E}{2X_F}, \tag{8.17.1}$$

where E is the emf per phase and X_F is the total reactance in ohms per phase, from neutral to the fault.

The above formula cannot be used, however, if the system includes transformers. As in the case of a three-phase fault, the reactances must be converted in per unit or per cent values to a common kva base, so that they can be combined directly. Furthermore, since under the faulted condition the circuit is nonsymmetrical, it is more convenient to solve it by the method of phase-sequence components.

Referring to the figure, it is reasonable to assume that the current in the nonfaulted phase c is negligibly small in comparison with the currents in the two short-circuited phases. Furthermore, since there is no ground return with an L–L fault, the short-circuit currents have no zero-sequence components. In terms of their phase sequence components, these currents, by (8.3.4), (8.3.5), and (8.3.6), are

$$\left.\begin{aligned} I_a &= I_p + I_n \\ I_b &= I_p\underline{/-120^\circ} + I_n\underline{/120^\circ} \\ I_c &= I_p\underline{/-240^\circ} + I_n\underline{/240^\circ}. \end{aligned}\right\} \qquad (8.17.2)$$

Since, by assumption $I_c = 0$, it follows that

$$I_p\underline{/-240^\circ} = -I_n\underline{/240^\circ},$$

which, when solved for I_n, gives

$$I_n = -I_p\underline{/-120^\circ} = I_p\underline{/60^\circ}. \qquad (8.17.3)$$

This indicates that the negative and the positive sequence components of the currents under the faulted condition are equal to each other numerically and 60° apart. By (8.17.3), the set of equations given by (8.17.2) becomes

$$\begin{aligned} I_a &= I_p - I_p\underline{/-120^\circ} = \sqrt{3}I_p\underline{/30^\circ} \\ I_b &= I_p\underline{/-120^\circ} - I_p = \sqrt{3}I_p\underline{/-150^\circ} \\ I_c &= I_p\underline{/-240^\circ} - I_p\underline{/120^\circ} = 0. \end{aligned} \qquad (8.17.4)$$

This shows that the magnitude of the current in the short-circuited phases is 73 per cent larger than the positive-sequence component current.

Referring to Fig. 8–16, let V_{Fx} be the potential to neutral at point x of the fault and V_{Fy} the potential to neutral at point y of the fault, then

$$V_{Fx} = E_a - V_a \qquad (8.17.5)$$

and

$$V_{Fy} = E_b - V_b, \qquad (8.17.6)$$

where E_a and E_b are the excitation emf's in phases a and b, respectively, equal in value with E_b lagging E_a by 120°. The quantities V_a and V_b are the drops in the respective phases under the faulted condition. Keeping in mind that the excitation emf's have no negative and no zero-sequence components, the above two expressions may be written in terms of sequence components

$$\begin{aligned} V_{Fx} &= E_a - (V_p + V_n)_a \\ V_{Fy} &= E_b - (V_p + V_n)_b. \end{aligned}$$

Since the potential difference $V_{Fx} - V_{Fy} = 0$ across the fault it follows from the preceding two equations that

$$E_a - E_b = (V_p + V_n)_a - (V_p + V_n)_b$$

or

$$E_a - E_b = (I_pX_p + I_nX_n)_a - (I_pX_p + I_nX_n)_b$$

or, by (8.17.2),

$$\begin{aligned}\sqrt{3}E\underline{/30^\circ} &= I_pX_p + I_nX_n - X_pI_p\underline{/-120^\circ} - X_nI_n\underline{/120^\circ}\\ &= X_p(I_p - I_p\underline{/-120^\circ}) + X_n(I_n - I_n\underline{/120^\circ})\end{aligned}$$

or

$$\sqrt{3}E\underline{/30^\circ} = \sqrt{3}(X_pI_p\underline{/30^\circ} + X_nI_n\underline{/-30^\circ}). \tag{8.17.7}$$

Using the value of I_n as stated by (8.17.3) gives

$$\begin{aligned}E\underline{/30^\circ} &= X_pI_p\underline{/30^\circ} - X_nI_p\underline{/-120^\circ}\,\underline{/-30^\circ}\\ &= X_pI_p\underline{/30^\circ} - X_nI_p\underline{/-150^\circ}\end{aligned}$$

or

$$E = I_p(X_p + X_n). \tag{8.17.8}$$

Identical expressions can be obtained for the other two phases indicating that the nonsymmetrical L–L faulted circuit may be replaced by a symmetrical one whose reactance per phase from neutral to the fault is $X_p + X_n$ ohms and which carries the positive sequence component I_p of the short-circuit current. This means in effect that, since X_p is the actual reactance of the phase from neutral to the fault, the fault itself is replaced by a symmetrical wye circuit whose branch consists of the negative sequence of the system from neutral to the fault. This equivalent symmetrical circuit is shown in Fig. 8–16a.

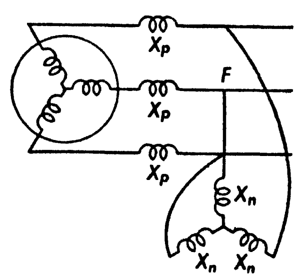


Figure 8–16a

If the reactances X_p and X_n are expressed in ohms, then equation (8.17.8) may be used directly in the calculation of the short-circuit current I_{sh}. If these reactances are stated, as they usually are, in per cent to a common kva base, then by (8.16.1) the value of I_{sh} is .

$$I_{sh} = \frac{100I_r}{(X_p + X_n)_{\text{per cent}}}. \tag{8.17.9}$$

With the value of the positive-sequence component known, the current in the short circuit may be calculated by (8.17.4).

To illustrate the calculation of the short-circuit currents caused by an L–L fault, consider the three-phase transmission system supplied by two generators, A and B, such as shown in Fig. 8–16. Assume a line to line fault across lines a and b at F, 20 miles from station A. Generators A and B are identical, are rated 21000 kva at 13.8 kv, and have a positive-sequence reactance $X_p = 30$ per cent and a negative-sequence reactance of $X_n = 15$ per cent at own kva base. The transformers are also identical and are rated 13.8/66 kv, 7000 kva, and have a reactance of 8.4 at own kva base. The line is 50 miles long; each conductor has a reactance of 0.848 ohms per mile.

Taking the kva rating of the generators as a common base, the various reactances of the system are:

$$X_{Ap} = X_{Bp} = 30 \text{ per cent}$$
$$X_{An} = X_{Bn} = 15 \text{ per cent.}$$
$$X_T = \frac{8.4 \times 21000}{7000} = 25.2 \text{ per cent.}$$

The reactance of the 50-mile tie-line per conductor is

$$X_L = \frac{0.848 \times 50 \times 21000 \times 100}{1000 \times 66^2} = 20.5 \text{ per cent.}$$

The reactance of each line conductor from station A to the fault is

$$X_{AF} = \tfrac{2}{5} \times 20.5 = 8.2 \text{ per cent.}$$

The reactance of the line conductor from fault to station B is

$$X_{BF} = \tfrac{3}{5} \times 20.5 = 12.3 \text{ per cent.}$$

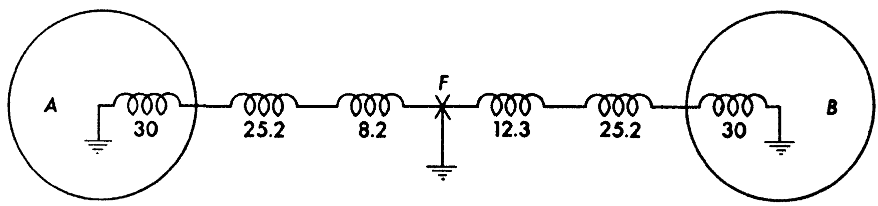


Figure 8–17

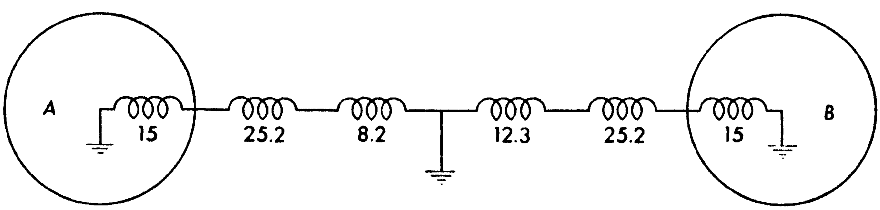


Figure 8–17a

The positive sequence reactance diagram per phase is shown in Fig. 8–17 and that of the negative sequence reactance in Fig. 8–17a. The positive sequence reactance per phase as viewed from the fault is

$$X_p = \frac{(30 + 25.2 + 8.2)(12.3 + 25.2 + 30)}{63.4 + 67.5}$$
$$\doteq 32.7 \text{ per cent.}$$

Similarly, the negative sequence reactance per phase as viewed from the fault is

$$X_n = \frac{(15 + 25.2 + 8.2)(12.3 + 25.2 + 15)}{48.4 + 52.5}$$
$$= 25.2 \text{ per cent.}$$

This is the reactance per branch of the symmetrical wye network which replaces the fault at the point of its occurrence. The equivalent circuit of the faulted system which carries the positive-sequence component current I_p of the short-circuit current is as shown in Fig. 8–17b. The reactance per

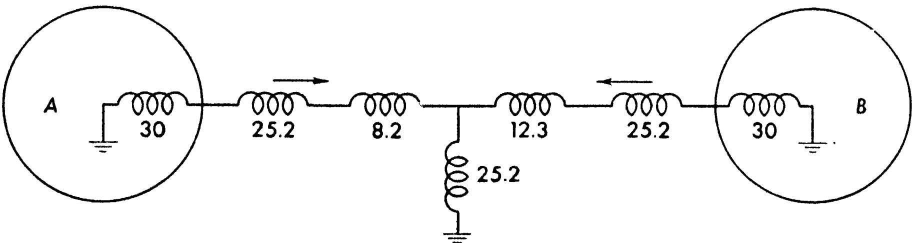


Figure 8–17b

phase of the circuit, equivalent to the faulted system, and which by (8.17.8) carries the positive-sequence component of the short-circuit current, is

$$X_F = X_p + X_n$$
$$= 32.7 + 25.2 = 57.9 \text{ per cent.}$$

The rated current is

$$I_r = \frac{(kva)_{base}}{\sqrt{3}kv}$$
$$= \frac{21000}{13.8\sqrt{3}} = 878 \text{ amperes.}$$

The positive phase sequence component of the short circuit, by (8.16.1), is

$$I_p = \frac{100 I_r}{X_F},$$

which, for the specific case under consideration, is

$$I_p = \frac{87800}{57.9} = 1520 \text{ amperes.}$$

This value of current flows in the fictitious wye circuit which replaces the fault.

The positive-sequence component of the short-circuit current supplied by generator A is

$$I_{pA} = \frac{32.7}{63.4} \times 1520 = 784 \text{ amperes.}$$

The positive-sequence component of the short-circuit current supplied by generator B is

$$I_{pB} = \frac{32.7}{67.5} \times 1520 = 736 \text{ amperes.}$$

The actual short-circuit current supplied by generators A and B, respectively, by (8.17.4), is

$$I_A = \sqrt{3} \times 784 = 1360 \text{ amperes}$$
$$I_B = \sqrt{3} \times 736 = 1272 \text{ amperes.}$$

The total current on the low-voltage side to the fault is

$$1360 + 1272 = 2632 \text{ amperes.}$$

The total current in the fault is

$$I_F = 2632 \times \frac{13.8}{66} = 551 \text{ amperes.}$$

8.18. *Line to Ground Fault. Three-Phase System.* Like the L–L fault discussed in the preceding article, this fault, usually denoted by L–G, may also be calculated directly from the constants of the circuit provided there are no transformers. It is shown in Fig. 8–18, from which it may be seen that if E_a is the excitation voltage in the phase, subject to the fault at the instant the fault occurs and X the total reactance from neutral to the fault, then

$$I_{sh} = \frac{E_a}{X} \text{ amperes.}$$

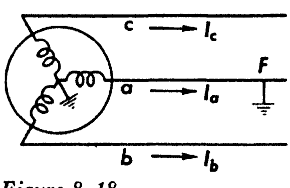


Figure 8–18

This method cannot be used if the transmission system includes transformers. Per cent reactances to a common kva base and phase-sequence components must be used in the calculation of the short-circuit currents.

In terms of phase-sequence components, the currents in the phases of the system are

$$I_a = I_p + I_n + I_o \tag{8.18.1}$$
$$I_b = I_p\underline{/-120^\circ} + I_n\underline{/120^\circ} + I_o \tag{8.18.2}$$
$$I_c = I_p\underline{/-240^\circ} + I_n\underline{/240^\circ} + I_o. \tag{8.18.3}$$

It is reasonable to assume that the load currents are negligibly small in comparison to the short-circuit current, i.e.,

$$I_b = I_c = 0$$

Accordingly, by (8.18.2) and (8.18.3),

$$I_p\underline{/-120^\circ} + I_n\underline{/120^\circ} = I_p\underline{/-240^\circ} + I_n\underline{/240^\circ}.$$

This gives

$$I_p = I_n. \tag{8.18.4}$$

Substitute this in (8.18.2) and get

$$0 = I_p\underline{/-120^\circ} + I_n\underline{/120^\circ} + I_o$$

or

$$I_o = I_p. \tag{8.18.5}$$

By (8.18.4) and (8.18.5), it follows that if the load current is neglected, the positive, negative, and zero-sequence currents are equal to each other in magnitude and phase. Equation (8.18.1) may be written, therefore,

$$I_a = 3I_p. \tag{8.18.6}$$

Now let V_F be the voltage to ground at the fault. In terms of its sequence components it is

$$V_F = V_{Fp} + V_{Fn} + V_{Fo}. \tag{8.18.7}$$

Since the negative and the zero sequence components of the excitation emf are zero, these sequence-component voltages are

$$\begin{aligned} V_{Fp} &= E - I_pX_p \\ V_{Fn} &= -I_nX_n \\ V_{Fo} &= -I_oX_o. \end{aligned} \tag{8.18.8}$$

The symbol E denotes the excitation emf per phase and is the same as the positive-sequence component. The quantities X_p, X_n, and X_o are the positive, negative, and zero sequence reactances, respectively, from the neutral of the generator to the fault. Since, by (8.18.4) and (8.18.5), $I_p = I_n = I_o$, and also since the voltage to ground at the fault $V_F = 0$, it follows, by (8.18.8), that

$$E = I_p(X_p + X_n + X_o). \tag{8.18.9}$$

Identical expressions can be obtained for the other two phases indicating that the nonsymmetrical L–G faulted circuit may be replaced by a symmetrical one whose reactance per phase from neutral to the fault is $X_p + X_n + X_o$ ohms and which, by (8.18.9), carries the positive-sequence component current. This means in effect that, since X_p is the actual reactance of the phase from neutral to the fault, the fault itself is replaced by a symmetrical wye circuit whose branches consist of the sum of the positive and negative reactance of the original system $(X_n + X_o)$ from neutral to the fault. The symmetrical circuit equivalent to the faulted circuit is shown in Fig. 8–18a and carries the positive-sequence component I_p of the short-circuit current, which, by (8.18.6), is

$$I_a = 3I_p.$$

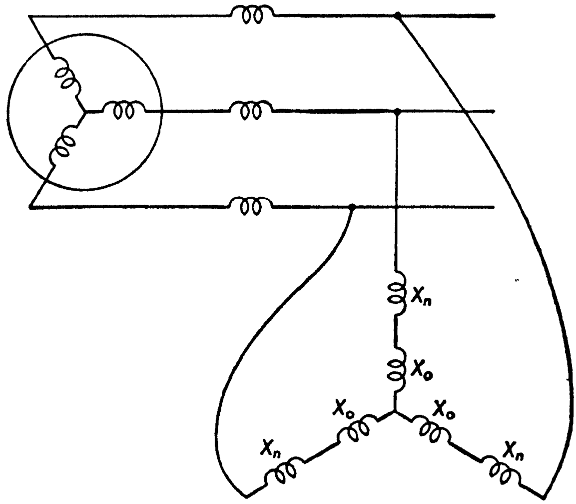


Figure 8–18a

The value of I_p may be calculated by (8.18.9) if the reactances are given in ohms. If the reactances are stated in per cent to a common kva base, then, by (8.16.1),

$$I_p = \frac{100I_r}{(X_p + X_n + X_o)},$$

where

$$I_r = \frac{(kva)_{base}}{\sqrt{3}kv}. \tag{8.18.10}$$

To illustrate the calculation of the short-circuit current caused by an L–G fault, consider the 50-mile three-phase transmission system used in the preceding article and assume an L–G fault 20 miles from station A. The various reactances to a 21000 kva base at 13.8 kv are

$$X_{Ap} = X_{Bp} = 30 \text{ per cent}$$
$$X_T = 25.2 \text{ per cent}$$
$$X_{Lp} = X_{Ln} = 20.5 \text{ per cent}$$
$$X_{An} = X_{Bn} = 15 \text{ per cent.}$$

The zero-sequence reactance of turbo-alternators ranges between 1 and 8 per cent.* For the particular case under consideration, it will be assumed

* Lewis, W. W., *Transmission Line Engineering*, p. 165, McGraw-Hill Book Co.

4 per cent at own kva base. The zero-sequence reactance of the grounded conductor must be calculated by the appropriate formula as outlined in § 8.12. For the particular case under consideration, the zero-sequence reactance per mile is given by (8.12.4) and is

$$X'_{oL} = 3 \times 741.13 \times 10^{-6}\omega \log \frac{5858\sqrt{\rho/\omega}}{(r_{gm}D^2)^{\frac{1}{3}}} \text{ ohms/mile.}$$

For a #0000 copper conductor, $r_{gm} = 0.23$ inch. Assuming a spacing distance of 16.5 feet, and $\rho = 100$, the above formula becomes for $\omega = 377$

$$X'_{oL} = 3 \times 0.279 \log \frac{3018}{\left(\frac{0.23}{12} \times (16.5)^2\right)^{\frac{1}{3}}}$$
$$= 2.71 \text{ ohms/mile.}$$

By § 8.12, the zero-sequence reactance of a grounded line conductor is approximately 3.5 times the positive-sequence reactance, i.e.,

$$X'_{oL} = 3.5 \times 0.848$$
$$= 2.96 \text{ ohms/mile}$$

as compared with the value obtained above by assuming the resistivity of the earth $\rho = 100$ ohms per meter cube.

Taking $X'_{oL} = 2.96$, the zero sequence reactance of the 50-mile line conductor in per cent to 21000 kva base, by (8.15.1), is

$$X_{oL} = \frac{2.96 \times 50 \times 21000 \times 100}{(66)^2 \times 1000}$$
$$= 71.3 \text{ per cent.}$$

The zero-sequence reactance of the 20-mile line conductor from station A to the fault is

$$(X_{oL})_{20} = \tfrac{2}{5} \times 71.3 = 28.5 \text{ per cent.}$$

The zero sequence reactance of the 30-mile conductor from the fault to station B is

$$(X_{oL})_{30} = \tfrac{3}{5} \times 71.3 = 42.8 \text{ per cent.}$$

The faulted system is shown schematically in Fig. 8–19. The positive-sequence reactance diagram per phase of the system is shown in Fig. 8–19a. The negative-sequence reactance diagram per phase of the system is as shown in Fig. 8–19b. The zero-sequence reactance diagram per phase of the system is shown in Fig. 8–19c. The zero sequence reactance as viewed from the fault is

$$X_o = \frac{(4 + 25.2 + 28.5)(42.8 + 25.2 + 4)}{57.7 + 72}$$
$$= 31.8 \text{ per cent.}$$

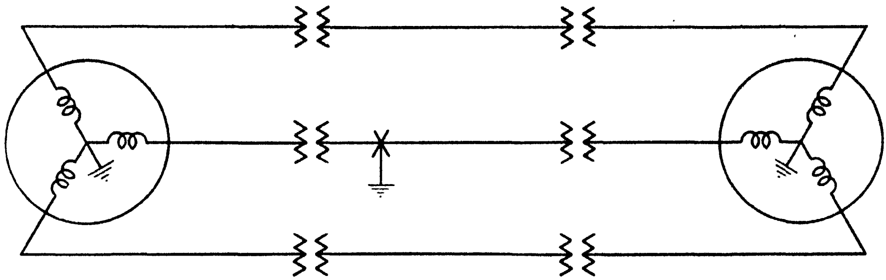


Figure 8–19

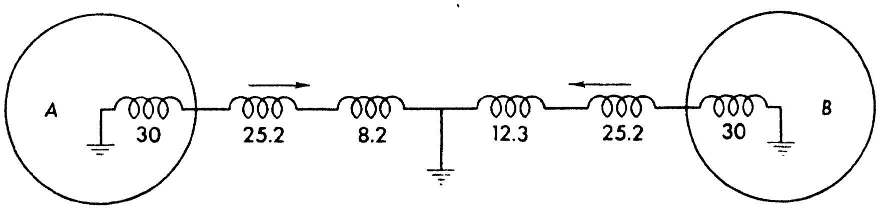


Figure 8–19a

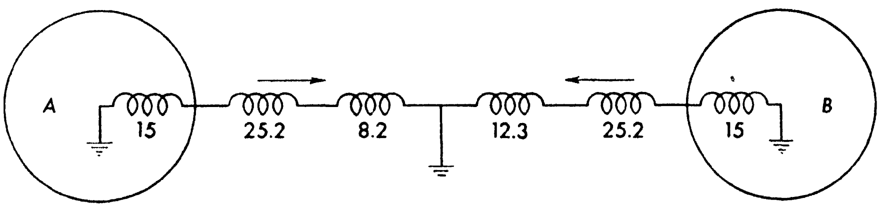


Figure 8–19b

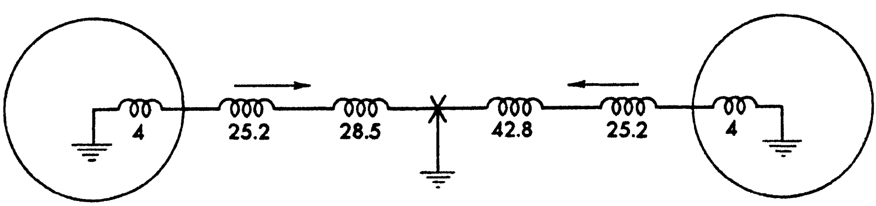


Figure 8–19c

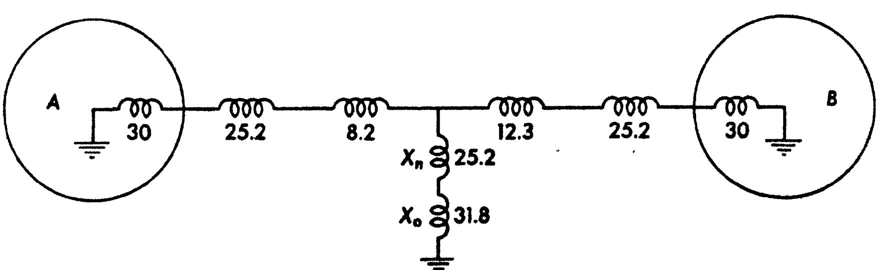


Figure 8–19d

The reactance diagram of the fictitious circuit per phase which replaces the faulted system, but carries only the positive sequence component of the short-circuit current is shown in Fig. 8–19d. The reactance of this circuit as viewed from the fault is

$$X_F = \frac{(30 + 25.2 + 8.2)(12.3 + 25.2 + 30)}{63.4 + 67.5} + 25.2 + 31.8$$
$$= 32.7 + 25.2 + 31.8 = 89.7 \text{ per cent.}$$

The normal rated current of the system, at 13.8 kv line voltage, by previous calculation is $I_r = 878$ amperes. The positive-sequence current carried by the equivalent circuit, by (8.18.10), is

$$I_p = \frac{87800}{89.7}$$
$$= 978 \text{ amperes.}$$

The positive-sequence component of the short-circuit current supplied by generator A is

$$I_{pA} = \frac{32.7}{63.4} \times 978$$
$$= 505 \text{ amperes.}$$

The positive-sequence component of the short-circuit current supplied by generator B is

$$I_{pB} = \frac{32.7}{67.5} \times 978$$
$$= 475 \text{ amperes.}$$

The actual short-circuit current supplied by generator A is

$$I_A = 3 \times 505 = 1515 \text{ amperes,}$$

and the short-circuit current supplied by generator B is

$$I_B = 3 \times 475 = 1425 \text{ amperes.}$$

The actual short-circuit on the high side of station A is

$$1515 \times \frac{13.8}{66} = 320 \text{ amperes.}$$

The actual short-circuit current on the high side of station B is

$$1425 \times \frac{13.8}{66} = 298 \text{ amperes.}$$

The actual current in the fault is 618 amperes.

8.19. *Line–Line to Ground Fault. Three-Phase System.* Consider the L–L–G fault across phases a and b of the simple system shown in Fig. 8–20.

In terms of phase-sequence components, the currents in the faulted system are

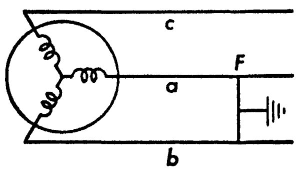


Figure 8–20

$$I_a = I_p + I_n + I_o$$
$$I_b = I_p\underline{/-120°} + I_n\underline{/120°} + I_o$$
$$I_c = I_p\underline{/-240°} + I_n\underline{/240°} + I_o. \quad (8.19.1)$$

On the supposition that the load current may be neglected in comparison with the short-circuit current, $I_c = 0$. It follows, therefore, that

$$I_p\underline{/-240°} + I_n\underline{/240°} + I_o = 0. \quad (8.19.2)$$

Let V_{Fa}, V_{Fb}, and V_{Fc} be the voltages from the respective phases to ground at the point of the fault. In terms of their respective sequence components, the voltages are

$$V_{Fa} = V_p + V_n + V_o \quad (8.19.3)$$
$$V_{Fb} = V_p\underline{/-120°} + V_n\underline{/120°} + V_o \quad (8.19.4)$$
$$V_{Fc} = V_p\underline{/-240°} + V_n\underline{/240°} + V_o \quad (8.19.5)$$

Since the fault is a short circuit to ground, it follows, however, that $V_{Fa} = 0$ and $V_{Fb} = 0$. Equations (8.19.3) and (8.19.4) may be combined, therefore, into

$$V_p + V_n = V_p\underline{/-120°} + V_n\underline{/120°}$$

which, when solved for V_n, gives

$$V_n = V_p\underline{/-120°}. \quad (8.19.6)$$

Using this value of V_n in (8.19.3) and solving for V_o gives

$$V_o = V_p\underline{/120°}. \quad (8.19.7)$$

The preceding two expressions indicate that the positive, negative, and zero-sequence components of the phase voltages at the fault form a symmetrical three-phase circuit.

In terms of the excitation emf, and keeping in mind that the excitation emf's do not have negative and zero sequence components, the above component voltages at the fault are

$$V_p = E - I_pX_p \quad (8.19.8)$$
$$V_n = -I_nX_n \quad (8.19.9)$$
$$V_o = -I_oX_o. \quad (8.19.10)$$

Using (8.19.6) and (8.19.9),

$$-I_nX_n = V_p\underline{/-120°}$$

or

$$-I_n = \frac{V_p\underline{/-120°}}{X_n}. \tag{8.19.11}$$

Similarly, using (8.19.7) and (8.19.10),

$$-I_oX_o = V_p\underline{/120°}$$

$$-I_o = \frac{V_p\underline{/120°}}{X_o}. \tag{8.19.12}$$

Substituting (8.19.11) and (8.19.12) in (8.19.2) gives

$$I_p\underline{/-240°} = \frac{V_p\underline{/-120°}\underline{/240°}}{X_n} + \frac{V_p\underline{/120°}}{X_o}$$

or

$$I_p = \frac{V_p}{X_n} + \frac{V_p}{X_o}$$

or

$$V_p = I_p\frac{X_nX_o}{X_n + X_o}.$$

Substituting this in (8.19.8) results in

$$E = I_p\left(X_p + \frac{X_n \cdot X_o}{X_n + X_o}\right). \tag{8.19.13}$$

This indicates that the nonsymmetrical L–L–G fault may be replaced by a symmetrical circuit carrying only the positive-sequence component of the short-circuit current and whose reactance per phase to neutral is

$$X_F = X_p + \frac{X_n \cdot X_o}{X_n + X_o}. \tag{8.19.14}$$

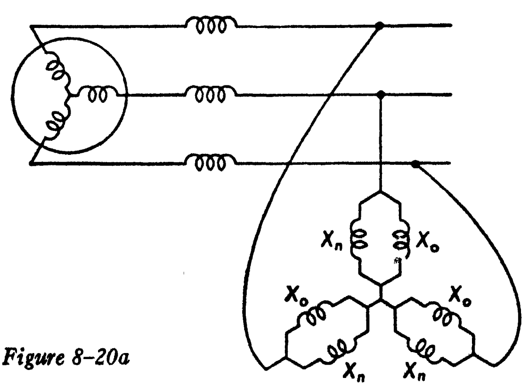


Figure 8–20a

This means in effect that, since X_p is the actual reactance per phase from neutral to the fault, the fault is replaced by a symmetrical wye network, whose branches consist of the negative and zero sequence components of the system from neutral to the fault in parallel connection as indicated in Fig. 8–20a.

The value of I_p may be determined by (8.19.13). But since the reactances are in per cent to a common kva base, the positive-sequence current, by (8.16.1), is

$$I_p = \frac{100\, I_r}{X_F} \tag{8.19.15}$$

where X_F is given by (8.19.14) in per cent values.

To determine the actual short-circuit current, note that since $I_a = -I_b$ (8.19.1) gives

$$-2I_o = I_p + I_p\underline{/-120^\circ} + I_n + I_n\underline{/120^\circ}$$

or

$$-I_o = \tfrac{1}{2}(I_p\underline{/-60^\circ} + I_n\underline{/60^\circ}). \tag{8.19.16}$$

Equation (8.19.2) similarly gives

$$-I_o = I_p\underline{/-240^\circ} + I_n\underline{/240^\circ}$$

or

$$I_o = I_p\underline{/-60^\circ} + I_n\underline{/60^\circ}. \tag{8.19.17}$$

Adding (8.19.16) and (8.19.17) gives

$$1.5I_n\underline{/60^\circ} = -1.5I_p\underline{/-60^\circ}$$

or

$$I_n = I_p\underline{/60^\circ}. \tag{8.19.18}$$

Substituting this in (8.19.17) results in

$$I_o = I_p\underline{/-60^\circ} + I_p\underline{/120^\circ} = 0. \tag{8.19.19}$$

The current in the short-circuit conductors, therefore, by (8.19.1), (8.19.18), and (8.19.19), is

$$\begin{aligned} I_a &= I_p + I_p\underline{/60^\circ} \\ &= \sqrt{3}I_p\underline{/30^\circ}. \end{aligned} \tag{8.19.20}$$

To illustrate the calculations involved in this type of fault consider the illustrative system used in the preceding article with a short-circuit fault across conductors a and b, 20 miles from station A.

The reactance diagram per phase of the circuit which replaces the faulted system and which carries only the positive sequence component of the short-circuit current is shown in Fig. 8–20b. The reactance of the circuit, by (8.19.14), is

$$\begin{aligned} X_F &= \frac{(30 + 25.2 + 8.2)(12.3 + 25.2 + 30)}{63.4 + 67.5} + \frac{25.2 \times 31.8}{25.2 + 31.8} \\ &= 32.7 + 14.02 = 46.72 \text{ per cent.} \end{aligned}$$

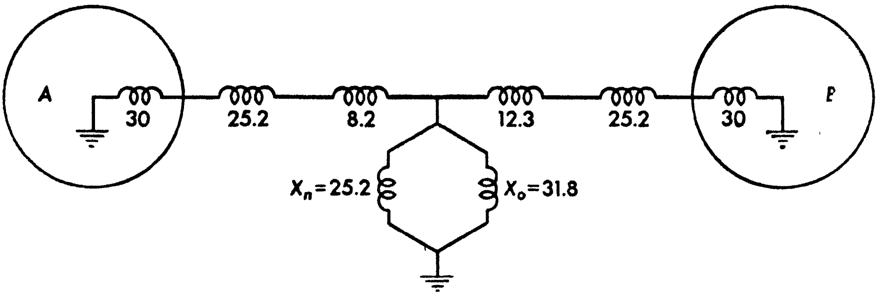


Figure 8–20b

The rated current, by previous calculation, is $I_r = 880$ amperes. The value of I_p, therefore, is

$$I_p = \frac{88000}{46.72}$$
$$= 1880 \text{ amperes.}$$

The positive-sequence component of the short-circuit current supplied by generator A is

$$I_{pA} = \frac{32.7}{63.4} \times 1880$$
$$= 970 \text{ amperes.}$$

The positive-sequence component of the short-circuit current supplied by generator B is

$$I_{pB} = \frac{32.7}{67.5} \times 1880$$
$$= 910 \text{ amperes.}$$

The actual short-circuit currents supplied by the generators A and B, respectively, by (8.19.20), are

$$I_A = \sqrt{3}\, 970 = 1680 \text{ amperes}$$
$$I_B = \sqrt{3}\, 910 = 1580 \text{ amperes.}$$

The total current supplied to the fault by the two generators is

$$1680 + 1580 = 3260 \text{ amperes.}$$

The actual current in the fault is

$$3260 \times \frac{13.8}{66} = 680 \text{ amperes.}$$

REFERENCES

Carson, John R., *Wave Propagation in Overhead Ground Wires with Ground Return*, Bell System Tech. Journal, Vol. 5, 1926.

Evans and Wagner, *Symmetrical Components*, McGraw-Hill Book Co., 1933.

Evans and Wright, *Power System Voltages and Currents Under Fault Conditions*, Elect. Eng., June 1931.

Fortescue, C. L., *Methods of Symmetrical Coordinates Applied to the Solution of Polyphase Networks*, Trans. A.I.E.E., 1918.

Frey, H. A., and Hawley, K. A., *Normal Frequency Arcover Values of Insulators as Affected by Size and Humidity*, Trans. A.I.E.E., 1932.

Ground Connections of Electrical Systems, Bureau of Standards Technical Paper No. 108, June 1918.

Harder, E. L., *Sequence Network Connections for Unbalanced Load and Fault Conditions*, Elect. Journal, December 1937.

Lewis, L. L., *Transmission Line Engineering*, McGraw-Hill Book Co., 1928.

Lyon, Waldo V., *Applications of the Method of Symmetrical Components*, McGraw-Hill Book Co., 1937.

Mackerras, U. P., *Calculation of Single Phase Short Circuits by Method of Symmetrical Components*, Gen. Elect. Rev. No. 4 and No. 7, 1926.

Peters, J. F., and Slepian, J., *Voltage Induced by Arcing Grounds*, Trans. A.I.E.E., 1923.

Shurig, O. R., *Experimental Determination of Short Circuit Currents in Electrical Power Networks*, Trans. A.I.E.E., 1923.

SUGGESTIVE PROBLEMS *Chapter 8*

1. Calculate and plot the joint impedance of a single conductor and ground return path per mile at 60 cps frequency as a function of the earth's resistivity. The conductor is a 19 strand, 250,000 circ. mils annealed copper cable having an outside diameter of 0.575 inches, and a resistance of 0.263 ohms per mile.

2. Calculate the joint impedance at 60 cps per mile of conductor and ground return path of a system of two grounded parallel conductors having the specifications as stated in Prob. 1 and a spacing distance of 12 ft. Assume the earth's resistivity $\rho = 100$ ohms per meter cube.

3. Calculate the joint impedance at 60 cps per mile of conductor and ground return path of a system of three grounded parallel conductors equilaterally spaced with a spacing distance of 12 feet. The specification of the conductors are as stated in Prob. 1, and the earth resistivity 100 ohms per meter cube.

4. Calculate the zero sequence reactance per mile per conductor at 60 cps of a symmetrically grounded three-phase line having equilateral spacing of 12 ft. and conductor specifications as stated in Prob. 1. Assume the earth's resistivity equal to 100 ohms per meter cube.

5. Calculate the per unit reactance of the line in Prob. 4, assumed 150 miles long and operated at 110 kv to neutral and at 100,000 kva base. What would be the per unit reactance at 50,000 kva base?

6. Recalculate the illustrative problem in § 8.16, on the assumption that the three-phase fault is on the high side of the transformers at the *B* terminus, Fig. 8–14.

7. Recalculate the illustrative problem in § 8.17, on the supposition that an *L–L* fault occurs at the high side of the transformers in lines (*a*) and (*b*) at the *A* terminus of the system shown in Fig. 8–14.

8. Recalculate the illustrative problem in § 8.18, on the assumption that the line-to-ground fault is at the high side of the transformer in line (*a*) at the *A* terminus of the system shown in Fig. 8–14. Assume transformers have grounded wye connection.

9. Recalculate the illustrative problem in § 8.19, on the assumption that an *L–L–G* fault occurs 20 miles from the *A* terminus of the system shown in Fig. 8–14. Assume transformers connected as in Prob. 8.

Chapter 9 Transient Stability

9.1. *General Considerations.* Transient stability is said to exist in a system if, "after an aperiodic disturbance has taken place, the system regains steady-state stability." * This definition implies that a system may become unstable during the period it is subject to a nonperiodic disturbance. It has been shown in Chap. 7 that there is a definite limit to the power that may be transferred over a transmission system, when the load is increased gradually. That limit was referred to as steady-state power limit. Aside from any economic consideration, steady-state power limit is due entirely to the physical properties of the line, the terminal voltages, the methods of their control, and the reactive character of the load as defined by the power factor.

One of the most common disturbances of transmission systems is due to the synchronous load on the system. If the shaft load is suddenly increased to a value in excess of the power limit corresponding to the initial load, there may be a falling out of step between the synchronous machinery at the termini of the system. If the excess load is not removed within a definite length of time, the machines may not be able to recover their synchronism, and service is interrupted.

Somewhat similar conditions are imposed when the system is subject to a fault. On the occurrence of a fault, the synchronous motors at the receiving end may fail to receive the power demanded by their shaft loads and thus begin to slow down, delivering the energy stored in their rotors to the fault. At the same time the power demand on the generator may become less than the mechanical input from the prime mover. If such is the case the

* *Report* of Joint Interconnection Subcommittee of the Committee on Power Generation, Protective Devices and Power Transmission and Distribution, A.I.E.E. Winter Convention, Jan., 1932.

generator will speed up. The simultaneous slowing down of the synchronous machinery at the receiving end and the speeding up of the synchronous generating machinery at the station end results, obviously, in a rapid increase in the power angle of the system. This may cause a complete loss of synchronism and subsequent interruption of service unless the condition is remedied within a definite length of time. If the disturbance is removed in time, the motors and generators will pull back in step, and the system recovers its steady-state stability. This chapter deals with the basic factors that enter into transient stability studies and their fundamental relations leading to the development of methods of solution.

9.2. *Motor-Generator System; Voltage-Current Relations.*

A convenient introduction to the basic relations which enter in transient stability studies of power systems is through the behavior of a simple system comprising a single three-phase synchronous motor of known impedance and rating supplied from a three-phase synchronous generator of known rating and impedance. One phase of such a simple system is shown in Fig. 9–1. To further simplify the

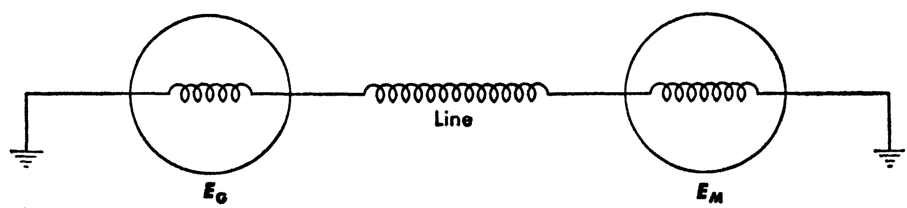


Figure 9–1

problem in its general aspects, it will be assumed (a) that the rated capacity of the generator is very large compared with that of the motor. Its speed is, therefore, independent of changes of load. (b) That the losses in both generator and motor are insignificant. This implies that the resistance of generator and motor are negligibly small. (c) That the line resistance and capacitive susceptance are also negligibly small, and (d) that the generator and motor reactances are unaffected by changes in load.

Under these particular assumptions, let E_G and E_M be the excitation emf's per phase of generator and motor, respectively, and δ the angle between them. The current flowing in the circuit is then

$$I\underline{/\theta} = \frac{E_G\underline{/\delta} - E_M\underline{/0^\circ}}{X_t\underline{/90^\circ}}, \tag{9.2.1}$$

where

$$X_t = X_G + X_L + X_M. \tag{9.2.2}$$

The angle δ is the time phase angle in electrical degrees between the generator and motor, the motor emf being taken as the reference vector. With E_G and E_M maintained constant by controlling devices, an increase in shaft load on the motor increases the space-phase angle between the rotating members of the motor and generator. The corresponding time-phase angle δ will also increase, thereby increasing the load current I to that demanded by the shaft load. If α_s represents the space-phase between the rotors of the two machines with reference to the field, then

$$\frac{\delta}{\alpha_s} = \frac{p}{2}, \tag{9.2.3}$$

where p represents the number of poles of the motor field.

9.3. *Power-Angle Curves.* Under the assumption of negligible losses in the system, the power converted in the generator through the electro-magnetic reactions taking place in its armature must be equal to the power converted by the motor and delivered as mechanical power to its shaft. Denoting by P_G the generator power and by P_M the power delivered by the motor to its shaft, it follows from what has been said above that

$$P_G = P_M,$$

i.e.,

$$E_G I \cos\theta_G = E_M I \cos\theta_M, \tag{9.3.1}$$

where θ_G and θ_M are the respective time-phase angles between the emf's and the current. For any value of shaft load P_M on the motor, there is a definite time-phase displacement δ between the vector values of E_G and E_M corresponding to a definite space-phase angle

$$\alpha_s = \frac{2\delta}{p}$$

between the rotors of the two machines. The dissipative power transferred from a generator to a motor when linear-line admittance is neglected is given by equation (7.11.3) and is

$$P = \frac{E_G E_M}{Z_t}\left[\cos(\delta - \zeta_t) - \frac{E_M}{E_G}\cos\zeta_t\right],$$

where $Z_t/\underline{\zeta_t}$ is the total circuit impedance. Under the particular assumption of negligible resistance, the above expression becomes

$$P = \frac{E_G E_M}{X_t}\left[\cos(\delta - 90°) - \frac{E_M}{E_G}\cos 90°\right],$$

i.e.,

$$P = \frac{E_G E_M}{X_t}\sin\delta. \tag{9.3.2}$$

This expression, called the *power-angle formula* represents a sine curve called the *power-angle curve.* The power limit of a system under the simplified conditions, stated in § 9.2, is, therefore, by (9.3.2),

$$P = \frac{E_G E_M}{X_t} \tag{9.3.3}$$

occurring when $\delta = 90°$, i.e., when the rotor of the motor drops behind the generator rotor $180/p$ mechanical degrees. It should be noted from (9.3.2) that $\sin \delta$ represents the power in per unit of the maximum.

9.4. *Synchronizing Power of Alternators in Parallel.* Consider two alternators connected through a tie line and feeding a common load, as indicated in Fig. 9–2. That the two machines should divide the load, their voltages at the junction must be equal, in time phase, and in opposite circuit direction with reference to each other, i.e., the two machines must be in synchronism.

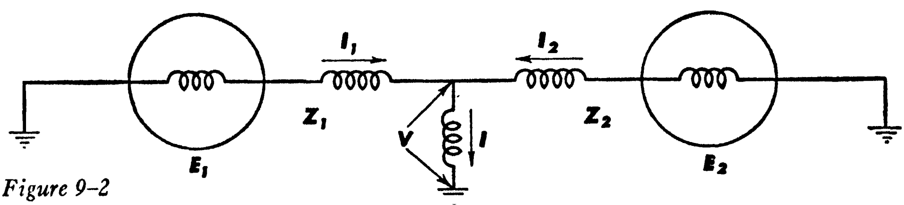


Figure 9–2

The inherent behavior of the two machines in joint operation is such that, if for any cause whatever the alternators fall apart from synchronism, there is always a tendency for its re-establishment, provided the displacement angle does not exceed a definite limit.

Let E_1 and E_2 be the excitation voltages of the machines; Z_1 and Z_2 the respective impedance of each, including the line impedance to the junction. V is the voltage to neutral at the common junction. Referring to the figure, this voltage is

$$V = E_1 - I_1 Z_1$$
$$V = E_2 - I_2 Z_2, \tag{9.4.1}$$

where I_1 and I_2 are the respective currents from the generators, and

$$I = I_1 + I_2$$

is the load current. In terms of the voltage equations given above, this load current is

$$I = \frac{E_1 - V}{Z_1} + \frac{E_2 - V}{Z_2},$$

which, when solved for V, gives

$$V = \frac{E_1 Z_2 + E_2 Z_1 - I Z_1 Z_2}{Z_1 + Z_2}.$$

Using this expression in (9.4.1) and solving for I_1 and I_2 yields

$$I_1 = \frac{E_1 - E_2}{Z_1 + Z_2} + \frac{Z_2}{Z_1 + Z_2} I \tag{9.4.2}$$

and

$$I_2 = \frac{-(E_1 - E_2)}{Z_1 + Z_2} + \frac{Z_1}{Z_1 + Z_2} I. \tag{9.4.3}$$

If the excitation voltages of the two machines are equal ($E_1 = E_2$) and in phase with each other ($\delta = 0$), the current equations above reduce to

$$I_1 = \frac{Z_2}{Z_1 + Z_2} I \tag{9.4.4}$$

and

$$I_2 = \frac{Z_1}{Z_1 + Z_2} I. \tag{9.4.5}$$

It follows, therefore, that under the stated condition ($E_1 = E_2$; $\delta = 0$) the load will be shared by the two machines in inverse proportion of the impedances from neutral to the common junction.

Now, if one of the machines, for instance generator (2), falls behind machine (1) in space-phase an angle α_s, there is immediately a corresponding time-phase displacement

$$\delta = \frac{p\alpha_s}{2}$$

between the two excitation voltages. The first members of (9.4.2) and (9.4.3) become highly significant. Generator (1) delivers a current equal to

$$I_s = \frac{E_1\underline{/0^\circ} - E_2\underline{/-\delta}}{Z_1 + Z_2} \tag{9.4.6}$$

in excess of that demanded by the load as given by (9.4.4). The first member of (9.4.3), being equal and negative, indicates that generator (2) delivers its share of the load less the excess current I_s delivered by generator (1). This means that if machine (2) falls behind in space-phase, machine (1) becomes more heavily loaded and tends to slow down. Machine (2) having been relieved of some of its load tends to speed up. The rotors of the two machines, therefore, pull back in step and synchronism is regained provided that the displacement angle does not exceed a certain limit. For this reason the current I_s given by (9.4.6) is called *synchronizing current.*

If the resistance components of the impedances are neglected, equation (9.4.6), for the synchronizing current delivered to the machine (2), becomes

$$I_s = \frac{E_1\underline{/0^\circ} - E_2\underline{/-\delta}}{jX}$$

or

$$I_s = \frac{E_2}{X}\sin\delta + \frac{j(E_2\cos\delta - E_1)}{X} \tag{9.4.7}$$

where

$$X = X_1 + X_2 \text{ ohms.}$$

The synchronizing voltamperes supplied by generator (1) is

$$E_1 I_s = \frac{E_1 E_2}{X}\sin\delta + j\,\frac{E_1 E_2\cos\delta - E_1^2}{X}. \tag{9.4.8}$$

The first member of this expression is called *synchronizing power*. It is identical to expression (9.3.2), which represents the power angle curve. The synchronizing power between two generators in parallel operation for a definite space-phase displacement is thus numerically the same as the dissipative power of the same two-machine system having an equal space-phase displacement, but with one machine functioning as a motor. Equation (9.4.8) may, accordingly, be written

$$E_G I = \frac{E_G E_M}{X}\sin\delta + j\left[\frac{E_G E_M}{X}\cos\delta - \frac{E_G^2}{X}\right]. \tag{9.4.9}$$

The real component is the ordinate of the power-angle curve and represents the dissipative power in the motor-generator system or the synchronizing power in the two-generator system. Since the losses are neglected, this member of the equation represents also the shaft load on the motor as a function of the displacement angle. The j-component is the reactive power supplied by the generator to the system reactance. Note that it is also a function of the displacement angle.

The area under the power-angle curve is

$$\int \frac{E_G E_M}{X}\sin\delta\, d\delta = \frac{-E_G E_M}{X}\cos\delta.$$

It has the characteristics of a power quantity and is 90 degrees apart from the dissipative power $E_G E_M \sin\delta/X$ as components of the voltamperes. It is, therefore, reactive in character and represents the rate of energy storage or restoration in the rotor during the transient interval when the displacement angle changes from one value to another during changes of load. The value of the above integral between proper limits of displacement angle δ is, therefore, proportional to the reactive mechanical energy stored in the rotor by virtue of its space-phase displacement.

9.5. *The Energy Stored in a Rotor.*

The rapidity with which the space-phase angle between a synchronous generator and a synchronous motor changes in value on increase of load or because of some disturbance, depends entirely upon the inertia of the rotating members of both machines. It takes a longer time for a machine of large inertia to change its relative space-phase position. If a machine with large inertia is thus less liable to fall out of step, it is also less able to swing back into synchronism. This means in effect that machines with large inertia, other factors being the same, have a larger power limit for the same initial load. The detailed effect of this inertia upon the behavior of the machine during the transitional period of load change may be investigated through the variations in the stored energy in the rotor. To determine the amount of stored energy, let

$\varepsilon =$ stored energy in watt-seconds,
$J =$ moment of inertia in meter^2-kilograms,

and

$\Omega =$ angular velocity of rotor in mechanical radians/sec,

then

$$\varepsilon = \tfrac{1}{2} J\Omega^2 \text{ watt-seconds.} \tag{9.5.1}$$

If

$m =$ mass of rotor in kilograms,

and

$\rho =$ radius of gyration in meters,

then

$$J = m\rho^2 \text{ meter}^2\text{-kilograms.} \tag{9.5.2}$$

Furthermore, if

$n =$ speed of rotor in revolutions per minute,

then

$$\Omega = \frac{2\pi n}{60} \text{ radians/sec.}$$

Substituting in (9.5.1) gives

$$\varepsilon = \tfrac{1}{2}(m\rho^2)\left(\frac{2\pi n}{60}\right)^2 \text{ watt-seconds.} \tag{9.5.3}$$

Using $\mathcal{E}$ to denote energy in kilowatt-seconds and multiplying out yields

$$\mathcal{E} = 0.547 m\rho^2 n^2\, 10^{-5} \text{ kw-seconds.}$$

If the mass is expressed in terms of W pounds and the radius of gyration in terms of r feet, then

$$\mathcal{E} = 0.547\, \frac{W}{2.202} \left(\frac{r}{3.281}\right)^2 n^2\, 10^{-5} \text{ kw-seconds.}$$

Multiplying out gives for the energy stored in a rotating rotor

$$\mathcal{E} = 0.231(Wr^2)n^2\, 10^{-6} \text{ kw-seconds.} \tag{9.5.4}$$

9.6. *Inertia Constants of Machines.* Consider a motor revolving at its rated speed n rpm and delivering its rated load P_r corresponding to a torque T_r. The amount of energy stored in the rotor is given by equation (9.5.4). If the electrical power supply to the motor is suddenly removed, while the shaft load is on, the motor will decelerate and come to rest. All the energy stored in the rotor at the time it was rotating at the rated speed must be dissipated during the period it takes the machine to come to rest.

Let P = the rate of dissipation of the stored energy in kw.
M = the time in seconds it takes the machine to come to rest, then the stored energy is

$$\mathcal{E} = \int_0^M P\,dt. \tag{9.6.1}$$

Under conditions of constant shaft torque, the speed of the rotor decreases at a uniform rate. Since the rate of energy dissipation P is at any instant proportional to the speed ($P = 2\pi Tn/60$), it follows that its value will decrease at a uniform rate, as shown by the curve, Fig. 9–3. The equation of this curve is

$$P = P_r\left(1 - \frac{t}{M}\right),$$

and the stored energy, by (9.6.1), is

$$\mathcal{E} = \int_0^M P_r\left(1 - \frac{t}{M}\right) dt.$$

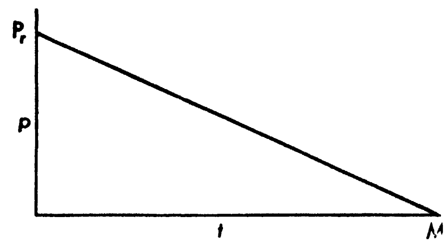


Figure 9–3

This gives, what may also be seen readily from the figure,

$$\mathcal{E} = \frac{P_r M}{2} \text{ kw-seconds.}$$

The time it takes the machine to come to rest is, therefore,

$$M = \frac{2\mathcal{E}}{P_r} \text{ seconds.} \tag{9.6.2}$$

Using the value of $\mathcal{E}$ from (9.5.4) gives

$$M = \frac{0.462(Wr^2)n^2\, 10^{-6}}{P_r} \text{ seconds.} \tag{9.6.3}$$

The quantity denoted by M is called ***inertia constant*** of the machine and represents the time in seconds that it takes a machine to come to rest from rated speed and under constant rated torque load.

The time t_v in seconds that it takes a rotor to decelerate one revolution, by (9.6.3), is

$$t_v = \frac{0.462(Wr^2)n\, 10^{-6}}{P_r} \text{ sec/rev.} \tag{9.6.4}$$

The interval in seconds t_{sd} that it takes a rotor to decelerate one space degree is $t_v/360$, i.e.,

$$t_{sd} = \frac{1.28(Wr^2)n\, 10^{-9}}{P_r} \text{ sec/degree.}$$

If p represents the number of poles of the machine, the interval in seconds t_{ed} that it takes a rotor to decelerate one electrical degree is, by (9.2.3), $2t_{sd}/p$, i.e.,

$$t_{ed} = \frac{2.56(Wr^2)n\, 10^{-9}}{pP_r} \text{ sec/elect degree.}$$

But, since the frequency

$$f = \frac{pn}{120}$$

the above may be written

$$t_{ed} = \frac{2.56(Wr^2)n^2\, 10^{-9}}{120\, fP_r} \text{ sec/elec degree.} \qquad (9.6.5)$$

The value of the energy stored in rotating machinery is usually stated in kw-seconds per kva capacity at rated speed. Thus, if a hydro-generator is said to have 2.4 kw-seconds per kva, it means that the amount of energy stored is

$$\mathcal{E} = 2.4 \times \text{kva} \quad \text{kw-seconds.}$$

The inertia constant, by (9.6.2), is

$$M = \frac{4.8\, kva}{P_r}, \qquad (9.6.6)$$

where P_r is the rated load in kw. With a unity power factor load the inertia constant would be 4.8 seconds.

The following average values of stored energy for various machinery are frequently used in stability studies:*

Hydro-generators, 1500 to 35000 kva capacity
2.4 kw-seconds per kva.

Turbo generator of 1500 to 35000 kva capacity
10.97 kw seconds per kva.

Rotary converters, 750 to 3250 kva capacity
2.0 kw-seconds per kva.

Synchronous motors, 2.0 kw-seconds per kva.

Synchronous condensers, 1000 to 40000 kva capacity
1.45 kw-seconds per kva.

* Evans and Wagner, *Trans.*, A.I.E.E., 1926.

The inertia constant of these machines may be calculated by (9.6.6). At unity power factor, the inertia constant M is numerically equal to twice the energy stored per kva capacity.

Some manufacturers of electrical machinery prefer to furnish directly values of the inertia constant M instead of the stored energy $\mathcal{E}$ in kw per kva. In the absence of accurate information, however, the following average values of time constants* may be used in stability calculations:

Turbo-generators	16 seconds
Hydro-generators	6 "
Synchronous motors	4.5 "
Synchronous condensers	3 "
Induction motors	1 "

These values differ somewhat from those calculated from stored energy values given above, obtained by different experimenters on machines of different manufacture. More recent data on inertia constants are given in the report on Power Stability by the Subcommittee of Interconnection and Stability Factors.†

9.7. *Equivalent Inertia Constant of Machines in Parallel.* Consider a number of machines of rated kw capacities P_{r1}, P_{r2}, P_{r3}, etc., all generators delivering energy jointly to the same system, or all motors receiving energy from the same source. If M_1, M_2, M_3, etc., are their respective inertia constants, and $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$, etc., the corresponding stored energies, then, by (9.6.2),

$$\mathcal{E}_1 = \frac{M_1 P_{r1}}{2}$$

$$\mathcal{E}_2 = \frac{M_2 P_{r2}}{2}$$

$$\mathcal{E}_3 = \frac{M_3 P_{r3}}{2}.$$

If P_r is the joint kw capacity of these machines, M the equivalent inertia constant, and $\mathcal{E}$ the energy stored in all, then

$$\mathcal{E} = \frac{MP_r}{2}.$$

From the above it follows that

$$M = \frac{M_1 P_{r1}}{P_r} + \frac{M_2 P_{r2}}{P_r} + \cdots \tag{9.7.1}$$

* Park and Banker, *Trans.*, A.I.E.E., 1929.

† *Electrical Engineering*, February, 1937.

This indicates that the joint or equivalent inertia constant of several machines in parallel operation is *the sum of inertia constants of the several machines reduced to a common power base*, which may be either the combined power or some other appropriate power base.

9.8. *Acceleration of Rotor.*

Let ΔT represent the accelerating torque of a rotating body in Newton-perpendicular-meters and $J = m\rho^2$, the moment of inertia in meter²-kilograms. The resulting acceleration is

$$A_s = \frac{d^2\alpha_s}{dt^2} = \frac{\Delta T}{m\rho^2} \text{ space radians/sec.}^2 \qquad (9.8.1)$$

The value of $m\rho^2$ as given by (9.5.1) and (9.5.2)

$$m\rho^2 = \frac{2\varepsilon}{\Omega^2}$$

substituted in the above equation gives

$$A_s = \frac{\Delta T \Omega^2}{2\varepsilon} \text{ space radians/sec}^2 \qquad (9.8.2)$$

where ε is the stored energy in watt-seconds and Ω is the angular velocity in space radians per second.

Expressing the energy in kw-seconds, and the power in kilowatts, gives

$$\frac{\Delta T \Omega}{1000} = \Delta P kw$$

and since

$$\frac{\varepsilon}{1000} = \mathcal{E} \text{ kw-seconds,}$$

the acceleration may accordingly be written

$$A_s = \frac{\Delta P \Omega}{2\mathcal{E}} \text{ space radians/sec}^2.$$

If n, the velocity in revolutions per minute is used instead of Ω, the equation becomes

$$A_s = \frac{2\pi n}{60} \cdot \frac{\Delta P}{2\mathcal{E}} \text{ space radians/sec}^2. \qquad (9.8.3)$$

In terms of electrical radians, this becomes

$$A = \frac{2\pi n}{60} \frac{p}{2} \frac{\Delta P}{2\mathcal{E}} \text{ elect rad/sec}^2 \qquad (9.8.4)$$

where p = number of poles on the machine. Since $f = pn/120$, the above may be written

$$A = \frac{2\pi f}{2\mathcal{E}} \Delta P. \qquad (9.8.5)$$

Expressing the energy stored $\mathcal{E}$ in terms of the inertia constant M as given by equation (9.6.2) gives

$$A = \frac{2\pi f}{M}\frac{\Delta P}{P_r} \text{ elect rad/sec}^2, \tag{9.8.6}$$

or in terms of degrees

$$A_d = \frac{360f}{M}\frac{\Delta P}{P_r} \text{ elect degrees/sec}^2 \tag{9.8.7}$$

where P_r is the rated load of the machine, or the base rated load if several machines are considered jointly.

9.9. *Equivalent Inertia Constant of Generator and Motor Combined.* Stability calculations of a two-machine system are greatly simplified by assuming that one of the machines is an infinite bus. This assumption implies that any change in the system causing a corresponding change in the displacement angle between the two machines is attributed to only one of the machines. Thus, if for any cause whatever, the motor slows down and the generator speeds up, the total change in the displacement angle may be assigned to the motor if the generator is assumed an infinite bus. This means, in effect, that the inertia constant of the motor must be altered so as to include the change that might otherwise take place in the generator.

Let J_G and J_M be, respectively, the actual moments of inertia of a generator and motor connected by a transmission line of negligible resistance. A reduction in power output of the generator due to any cause whatever, other than that demanded by a decrease in motor load, will tend to cause the generator to speed up with an acceleration given by (9.8.7)

$$A_G = \frac{360f}{M_G}\frac{\Delta P}{P_G} \text{ elect degrees/sec}^2. \tag{9.9.1}$$

Since the power supply to the motor is now reduced by virtue of the reduced power output of the generator, and its shaft load remains unchanged, the motor will immediately slow down with a deceleration

$$A_M = \frac{-360f}{M_M}\frac{\Delta P}{P_M}. \tag{9.9.2}$$

If the motor is assumed to be an infinite bus, the entire change in the acceleration is assigned to the generator. Its equivalent acceleration becomes

$$A_{Ge} = A_G - A_M$$

or, by (9.9.1) and (9.9.2),

$$A_{Ge} = 360f\,\Delta P\left(\frac{1}{M_G P_G} + \frac{1}{M_M P_M}\right). \tag{9.9.3}$$

In this expression M_G and M_M are the inertia constants of the two machines to their own rated power P_G and P_M as bases. If the inertia constants are converted to a common power base P_{rb}, and M_{Gb} and M_{Mb} are the respective inertia constants to the power base P_{rb}, then

$$M_G P_G = M_{Gb} P_{rb}$$

and

$$M_M P_M = M_{Mb} P_{rb}.$$

These, substituted in (9.9.3), gives

$$A_{Ge} = 360f \frac{\Delta P}{P_{rb}} \left(\frac{1}{M_{Gb}} + \frac{1}{M_{Mb}} \right)$$

or

$$A_{Ge} = \frac{360f}{M_e} \frac{\Delta P}{P_{rb}} \text{ elect deg/sec}^2 \tag{9.9.4}$$

where

$$M_e = \frac{M_{Gb} \cdot M_{Mb}}{M_{Gb} + M_{Mb}}. \tag{9.9.5}$$

Equation (9.9.3) gives the equivalent rate of change in speed in a two-machine system, when the change is attributed wholly to only one of the machines. Equation (9.9.5) similarly gives the equivalent inertia constant of the two machines in a two-machine system when the change in speed is associated entirely with only one of the machines.

It is important to keep in mind that to obtain the equivalent inertia constant of a group of machines, the inertia constants of each of these machines must be expressed to a common kw base.

9.10. *Determination of the Displacement Angle and of the Maximum Synchronizing Power for Any Given Load.* It was shown in § 9.3 that under condition of zero loss the power-angle formula

$$P = \frac{E_G E_M}{X} \sin \delta = P_m \sin \delta \tag{9.10.1}$$

represents the power transferred from the generator armature, where it is converted from mechanical into electrical power, to the motor armature where it is delivered to the shaft as mechanical power. In § 9.4, it was shown that the same formula represents also the synchronizing power that comes into play to pull generators operated in parallel back into synchronism, in case they have fallen out of step due to a sudden and excessive increase in load or due to some transient disturbance.

For any one particular load P on the system, there is, as previously stated, a definite displacement angle between the excitation voltages of the two machines. Furthermore, should some disturbance tend to pull the machines out of synchronism, there is immediately available a definite maximum synchronizing power to bring the machines into step again. The maximum synchronizing power for any load P is

$$P_M = \frac{E_G E_M}{X} = \frac{P}{\sin \delta}, \tag{9.10.2}$$

which indicates that its value can be obtained from the displacement angle corresponding to the load. To determine this angle, consider the two-

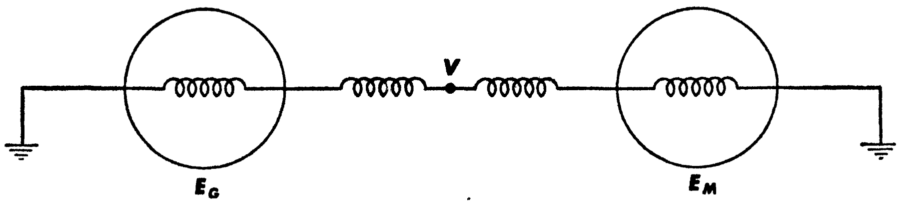


Figure 9–4

machine system shown in Fig. 9–4. A simple expression for determining the angle δ is (9.2.1), which gives

$$E_G\underline{/\delta} = E_M\underline{/0^\circ} + IX\underline{/90^\circ + \theta_M}. \tag{9.10.3}$$

In this expression θ_M is the angle between the excitation voltage E_M and the current, and X is the system reactance in ohms. A similar formula which relates δ with the other quantities is

$$E_M\underline{/-\delta} = E_G\underline{/0^\circ} - IX\underline{/90^\circ + \theta_G}. \tag{9.10.4}$$

In this expression θ_G is the angle between E_G and I, and X is the system reactance in ohms. Neither θ_M nor θ_G give what might be said to be the system power factor, particularly if both machines are generators supplying a common load. A compromise may be made, therefore, by having the phase of the current referred to the voltage V at the electrical midpoint of the system. In this case the displacement angle δ is the sum of the angles δ_1 and δ_2 between E_G and V and E_M and V, respectively. The formulas which relate these angles with the quantities involved are

$$\left.\begin{aligned} E_G\underline{/\delta_1} &= V\underline{/0^\circ} + \frac{IX}{2}\underline{/90^\circ + \theta} \\ E_M\underline{/\delta_2} &= V\underline{/0^\circ} - \frac{IX}{2}\underline{/90^\circ + \theta} \\ \text{and} \qquad \delta &= \delta_1 + \delta_2, \end{aligned}\right\} \tag{9.10.5}$$

where θ is the phase angle between the phase voltage V and the current I.

To generalize the above expressions, it is convenient to express all the quantities in per cent to a definite VI base. Thus, if X_p denotes the value x in per cent to the VI base, then

$$\frac{(IX)100}{V} = X_p.$$

The voltage drop in per cent may be written, therefore,

$$\begin{aligned} 100(IX) &= X_p(V) \\ &= \frac{X_p P}{I \cos\theta}, \end{aligned}$$

where P is the load on the system in watts corresponding to the power factor $\cos\theta$.

The per cent voltage drop to any base $(kva)_b$ is accordingly

$$100(IX) = \frac{V X_p P/\cos\theta}{(kva)_b}. \tag{9.10.6}$$

Note that if the base is in kva, the power P must be expressed in terms of kw.

Denoting per cent values by the subscript p, the voltage equations given by (9.10.5) may be written, therefore,

$$\left.\begin{aligned} (E_G)_p\underline{/\delta_1} &= 100 + \frac{(X_p P/\cos\theta)\underline{/90^\circ + \theta}}{2(kva)_b} \\ (E_M)_p\underline{/\delta_2} &= 100 - \frac{(X_p P/\cos\theta)\underline{/90^\circ + \theta}}{2(kva)_b}. \end{aligned}\right\} \tag{9.10.7}$$

The voltage $V = 100$ is thus made the base voltage of the system and $[X_pP/2\ (kva)_b \cos\theta]$ is the voltage drop in per cent corresponding to the given load P in kw from each of the termini to the electrical midpoint of the system.

Another method of calculating the value of the displacement angle δ is suggested by the graphical relation between the above two equations shown in Fig. 9–5 in which θ is assumed negative.

$$D = \frac{X_p P/\cos\theta}{2(kva)_b} \tag{9.10.8}$$

$$\alpha = 90^\circ + \theta$$

and

$$\begin{aligned} \tan\delta &= \tan(\delta_1 + \delta_2) \\ &= \frac{\tan\delta_1 + \tan\delta_2}{1 - \tan\delta_1 \tan\delta_2}. \end{aligned} \tag{9.10.9}$$

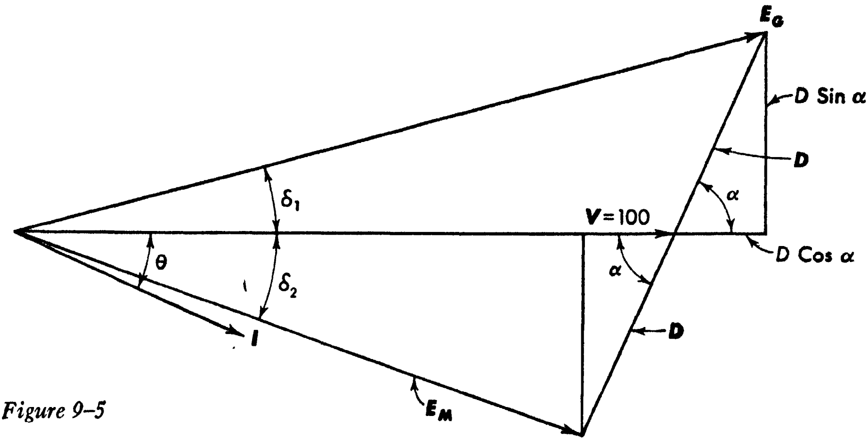


Figure 9–5

By referring to the figure it is seen that

$$\tan \delta_1 = \frac{D \sin \alpha}{100 + D \cos \alpha}$$

and

$$\tan \delta_2 = \frac{D \sin \alpha}{100 - D \cos \alpha}.$$

It follows, therefore, that

$$\tan \delta_1 + \tan \delta_2 = \frac{D \sin \alpha}{100 + D \cos \alpha} + \frac{D \sin \alpha}{100 - D \cos \alpha},$$

which brought to the same denominator and simplified yields

$$\tan \delta_1 + \tan \delta_2 = \frac{200D \sin \alpha}{10000 - (D \cos \alpha)^2}. \tag{9.10.10}$$

Similarly,

$$\begin{aligned} 1 - \tan \delta_1 \tan \delta_2 &= 1 - \frac{D \sin \alpha}{100 + D \cos \alpha} \cdot \frac{D \sin \alpha}{100 - D \cos \alpha} \\ &= 1 - \frac{(D \sin \alpha)^2}{10000 - (D \cos \alpha)^2} \\ &= \frac{10000 - D^2}{10000 - (D \cos \alpha)^2}. \end{aligned} \tag{9.10.11}$$

Substituting (9.10.10) and (9.10.11) in (9.10.9) gives

$$\begin{aligned} \tan \delta &= \frac{200D \sin \alpha}{10000 - (D \cos \alpha)^2} \times \frac{10000 - (D \cos \alpha)^2}{10000 - D^2} \\ &= \frac{200\,D \sin \alpha}{10000 - D^2} \end{aligned}$$

or, since $\alpha = 90° + \theta$

$$\tan \delta = \frac{200D \cos \theta}{10000 - D^2}, \tag{9.10.12}$$

where D is given by (9.10.8).

Knowing the value of the displacement angle δ corresponding to a load of P in kw at a given power factor from (9.10.12), the maximum available synchronizing power in case of a disturbance may be calculated by

$$P_m = \frac{P}{\sin \delta}.$$

To illustrate the above two methods of calculating δ, let the load on the above two-machine system be $P = 15000$ kw at a power factor of 86.6 per cent lag corresponding to $\theta = -30°$.

Assuming that the system reactance is 120 per cent to a 50000 kva base, formulas (9.10.7) become, respectively,

$$(E_G)_p\underline{/\delta_1} = 100 + \frac{120 \times 15000}{2 \times 50000 \times 0.866}\underline{/60°}$$

$$(E_M)_p\underline{/\delta_2} = 100 - \frac{120 \times 15000}{2 \times 50000 \times 0.866}\underline{/60°}.$$

This gives

$$(E_G)_p\underline{/\delta_1} = 100 + 20.8\underline{/60°} = 111.8\underline{/9.27°}$$

and

$$(E_M)_p\underline{/\delta_2} = 100 - 20.8\underline{/60°} = 91.4\underline{/-11.36°}.$$

The results show that $\boldsymbol{E_G}$ leads $\boldsymbol{V}$ by 9.27° and $\boldsymbol{E_M}$ lags $\boldsymbol{V}$ by 11.36°. Hence the angle δ between E_G and E_M is

$$\delta = 9.27° + 11.36° = 20.63°.$$

Using equation (9.10.12) for the calculation of δ, since $D = 20.8$, gives

$$\tan \delta = \frac{200 \times 20.8 \times 0.866}{10000 - 432.64}$$

$$\delta = 20.6°.$$

9.11. *Determination of Load Corresponding to a Definite Displacement Angle and Power Factor.*

Equation (9.10.12) for $\tan \delta$ may be written

$$D^2 + \left(\frac{200 \cos \theta}{\tan \delta}\right) D = 10000, \tag{9.11.1}$$

which when solved for D gives

$$D = 100\left[\frac{-\cos \theta \pm \sqrt{\tan^2 \delta + \cos^2 \theta}}{\tan \delta}\right]. \tag{9.11.2}$$

Using the expression for D given by (9.10.8), and solving for P, gives

$$P = \frac{200(kva)_b \cos \theta}{X_p}\left[-\frac{\cos \theta}{\tan \delta} + \sqrt{1 + \left(\frac{\cos \theta}{\tan \delta}\right)^2}\right]. \tag{9.11.3}$$

The positive sign is used with the radical, because P is always positive.

9.12. *Load Swing and Acceleration on Change of Shaft Load.* Consider a two-machine system connected through a tie line and having a total reactance X in per cent to a definite kva base. Let P_1 be the shaft load on the motor in kw and δ_1 the corresponding displacement angle, determined by either one of the two methods discussed in § 9.10. The maximum synchronizing power or power limit is calculated by

$$P_m = \frac{P_1}{\sin \delta_1}$$

and the power-angle curve plotted as shown in Fig. 9–6.

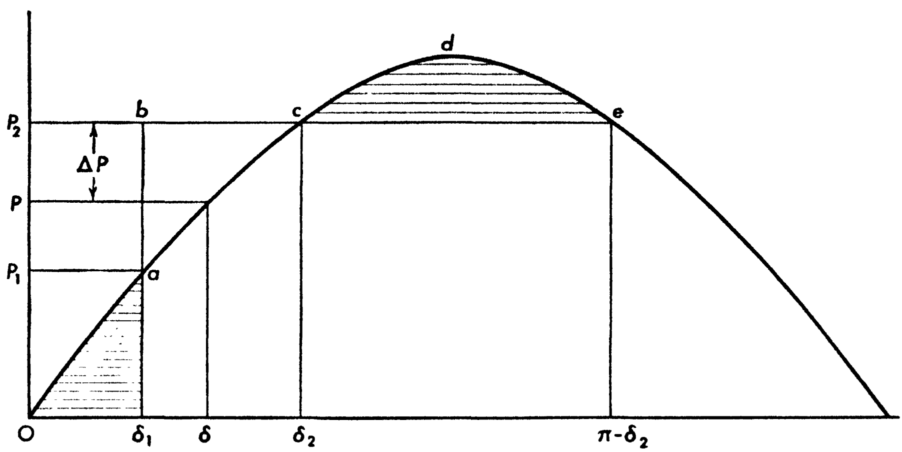


Figure 9–6

The area $oa\delta_1 o$ under the curve as stated in § 9.4 is proportional to the reactive mechanical energy stored in the rotor by virtue of its space-phase position during the transitional period when the displacement angle changed from zero to δ_1. When an additional load $P_2 - P_1$ is thrown on the system, the stored reactive mechanical energy increases to that proportional to the area $oc\delta_2 o$ which corresponds to the rotor space-phase displacement δ_2 demanded by the load P_2.

If the increase in load were gradual, the rotor would slip from one space-phase position to the other without any noticeable change in speed. Because of the inertia of the rotor, however, there is no instantaneous response to the sudden increase in power. As a consequence the motor begins to decelerate, and part of the required energy for the load and for the reactive mechanical energy demanded by the new displacement angle is temporarily supplied from the stored inertia energy ($\mathcal{E} = P_r M/2$ kw-sec) in the rotor. During the small interval that the rotor changes its space-phase position from δ_1 to δ_2, its stored reactive mechanical energy increases by an amount proportional to the area $\delta_1 ac\delta_2\delta_1$. Its stored inertia energy decreases simultaneously by an amount proportional to the area *abca*, because when the rotor in its deceleration reaches the space-phase position δ_2 its speed is somewhat lower than the normal synchronous value.

Furthermore, due to the acquired momentum, the rotor drops further back in space-phase past the displacement angle corresponding to the shaft load P_2. Referring to the power-angle curve, it is seen that the power supplied becomes greater than that demanded by the shaft load and the rotor begins therefore to gain in speed. For the interval corresponding to any slip in space-phase past the angle δ_2 the stored reactive mechanical reactive energy exceeds that demanded for the displacement angle δ_2. Although the motor gains in speed, the slip in space-phase past the angle continues until the excess energy received is larger than the amount lost from the stored inertia energy. If such is the case, the motor accelerates and the displacement angle decreases. By virtue of its momentum, however, the rotor overtravels past the required angle δ_2 and overspeeds past the normal synchronous speed. The supply becomes again smaller than the demand; the rotor slows down, then speeds up, and oscillates in speed about the synchronous value, and in space-phase about the power-angle δ_2, corresponding to the load demand P_2 with a decaying swing amplitude as shown qualitatively in Fig. 9–7.

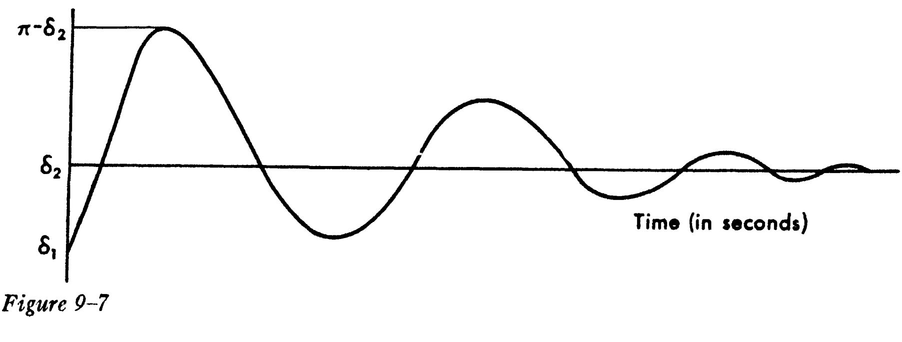


Figure 9–7

Referring to the power-angle curve, Fig. 9–6, it is seen that as the rotor slips in space-phase on sudden increase in load past the angle δ_2 the supplied power exceeds the demand. It may reach the power limit P_m when $\delta = 90°$. If the rotor slips past this position, the power supply decreases again. Note that when the space-phase position of the rotor is $\pi - \delta_2$ the power supply is again equal to that demanded by the shaft load P_2. The excess energy received by the rotor during the interval the power supply is greater than the demand is proportional to the area *cdec*.

If this area is larger than the area *abca*, the maximum excess energy received by the rotor during the interval it slips past δ_2 to $\pi - \delta_2$ is larger than that lost by the rotor from its stored inertia energy. The rotor will recover its synchronous speed and settle in its rotation at a space-phase angle corresponding to the load P_2. If, on the other hand, the area *cdec* is smaller than the area *abca*, the maximum excess energy received by the rotor is smaller than the amount it lost from the stored inertia energy. If such is the case, the rotor continues to decelerate and eventually falls out of synchronism unless the load is reduced to a safe value within a specified time.

Referring to the figure, let P represent the power at any instant during the transitional period of load change from P_1 to P_2, and δ the displacement angle corresponding to this instantaneous load.

If ΔP is the required increment in power from the instantaneous load P to the final one P_2 as indicated in the figure, then

$$\Delta P = P_2 - P.$$

But, by (9.10.1),

$$P_2 = P_m \sin \delta_2$$

and

$$P = P_m \sin \delta.$$

Hence,

$$\Delta P = P_m(\sin \delta_2 - \sin \delta). \tag{9.12.1}$$

Substituting this in equation (9.8.6) for the acceleration gives

$$A = \frac{2\pi f}{M_e} \frac{P_m}{P_{rb}} (\sin \delta_2 - \sin \delta) \text{ elect radians/sec} \tag{9.12.2}$$

where P_{rb} is the power base.

There are two general and well-defined problems in connection with transient stability studies. The first is to determine the limiting values of load which, when added to any initial load, would lead to unstable conditions and possible loss of synchronism. Such load will be referred to in what follows as ***maximum safe*** load. The second problem is the determination of the limiting time prior to which an unsafe load must be removed to prevent loss of synchronism.

9.13. *Determination of Maximum Additional Safe Load.*

It was shown in the preceding article that if the operating condition of a motor is to remain stable when its shaft load is increased, the area *cdec*, which is proportional to the maximum excess energy supplied by the generator must be either larger or at least equal to the area *abca*. This area, (*abca*), is proportional to energy lost by the motor from its stored inertia energy, during the transitional period. The criterion for the stability limit, therefore, is

$$\text{area } (abca) = \text{area } (cdec). \tag{9.13.1}$$

Let P_s represent the maximum additional safe load that may be added suddenly to an initial load P_i of a two-machine system whose power angle curve corresponding to P_i is shown in Fig. 9–8. The total safe load corresponding to P_i is

$$P_{st} = P_i + P_s. \tag{9.13.2}$$

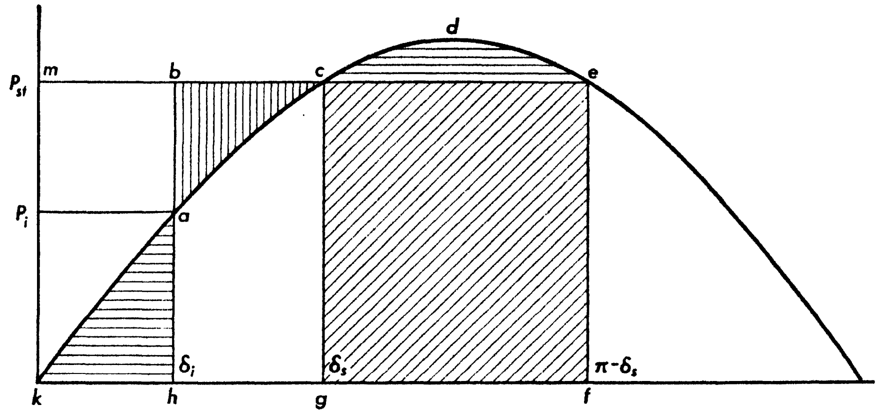


Figure 9–8

To determine the condition that would satisfy (9.13.1), refer to Fig. 9–8 and note that

$$\text{area } (abca) = \text{area } (befhb) - \text{area } (acefha). \tag{9.13.3}$$

Also

$$\begin{aligned}\text{area } (befhb) &= \text{area } (mefkm) - \text{area } (mbhkm)\\ &= (mk) \cdot (kf) - (bh) \cdot (kh).\end{aligned}$$

Since

$$bh = mk = P_{st}$$
$$kh = \delta_i$$

and

$$kf = \pi - \delta_s,$$

it follows that

$$\begin{aligned}\text{area } (befhb) &= P_{st}(\pi - \delta_s) - P_{st}\delta_i\\ &= P_{st}(\pi - \delta_s - \delta_i).\end{aligned} \tag{9.13.4}$$

Similarly area $(acefha)$ is the area under the power-angle curve between the limits of δ_i and $\pi - \delta_s$ less the area $(cdec)$. It may be written, therefore,

$$\text{area } (acefha) = \int_{\delta_i}^{\pi-\delta_s} P_m \sin\delta\, d\delta - \text{area } (cdec). \tag{9.13.5}$$

Substituting (9.13.4) and (9.13.5) in (9.13.3) gives

$$\text{area } (abca) - \text{area } (cdec) = P_{st}(\pi - \delta_s - \delta_i) - \int_{\delta_i}^{\pi-\delta_s} P_m \sin\delta\, d\delta.$$

By (9.13.1), it follows that the transient stability limit is reached when

$$P_{st}(\pi - \delta_s - \delta_i) = P_m \int_{\delta_i}^{\pi-\delta_s} \sin\delta\, d\delta. \tag{9.13.6}$$

Since

$$P_{st} = P_m \sin \delta_s, \tag{9.13.7}$$

the preceding equation becomes

$$(\pi - \delta_s - \delta_i) \sin \delta_s = \int_{\delta_i}^{\pi - \delta_s} \sin \delta \, d\delta.$$

Carrying out the integration and substituting the limits gives

$$(\pi - \delta_s - \delta_i) \sin \delta_s = \cos \delta_s + \cos \delta_i. \tag{9.13.8}$$

To determine the value of the limiting safe load, equation (9.13.7), it is necessary to obtain the angle δ_s corresponding to it from (9.13.8). This equation, it will be observed, does not lend itself to a direct algebraic solution for δ_s. The solution may, however, be obtained graphically by plotting the two sides of the equation for the known values of δ_i against assumed values of δ_s. The intersection of the two curves gives the required value of δ_s. The solution may be generalized for future use by obtaining a curve of δ_s *vs.* δ_i. Such a curve gives the maximum safe angle corresponding to a given initial angle. Furthermore, since

$$\sin \delta_s = \frac{P_{st}}{P_m}$$

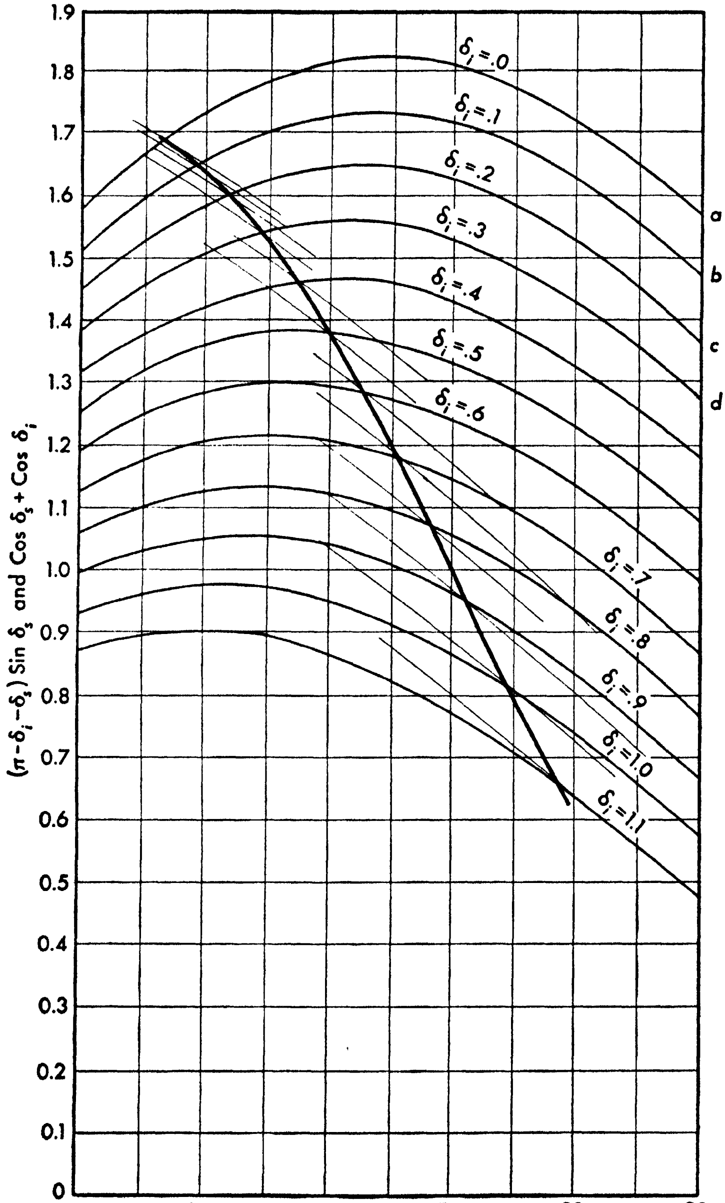
and

$$\sin \delta_i = \frac{P_i}{P_m}$$

represent, respectively, the limiting safe load, and the initial load in per unit of the maximum synchronizing power a curve 100 sin δ_s *vs.* 100 sin δ_i may be plotted which will give directly the maximum safe load as a function of the initial load both expressed in per cent of the maximum synchronizing power. Another curve, 100 (sin δ_s − sin δ_i) *vs.* 100 sin δ_i, may also be plotted. This curve gives the additional safe load that may be added to any initial load, both expressed in per cent of the maximum synchronizing power.

The curves marked *a*, *b*, *c* in Fig. 9–9 represent the left-hand side of equation (9.13.8) as a function of δ_s expressed in degrees for increasing values of δ_i in radians. Similarly, the straight lines in Fig. 9–9 represent corresponding values of the right-hand side of this equation as a function of δ_s. The curve drawn through the respective intersections gives the values of δ_s for corresponding values of δ_i and so plotted in degrees is shown in Fig. 9–10.

Curve P_{st} *vs.* P_i, in Fig. 9–11, gives the limiting safe power as a function of the initial load P_i, both expressed in per cent of the maximum synchronizing power P_m. Curve marked P_s *vs.* P_i gives the additional safe load



1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
40
50
60
70
80
90
δ_s in degrees
$(\pi-\delta_i-\delta_s)\sin\delta_s$ and $\cos\delta_s + \cos\delta_i$
$\delta_i = .0$
$\delta_i = .1$
$\delta_i = .2$
$\delta_i = .3$
$\delta_i = .4$
$\delta_i = .5$
$\delta_i = .6$
$\delta_i = .7$
$\delta_i = .8$
$\delta_i = .9$
$\delta_i = 1.0$
$\delta_i = 1.1$
a
b
c
d

Figure 9–9

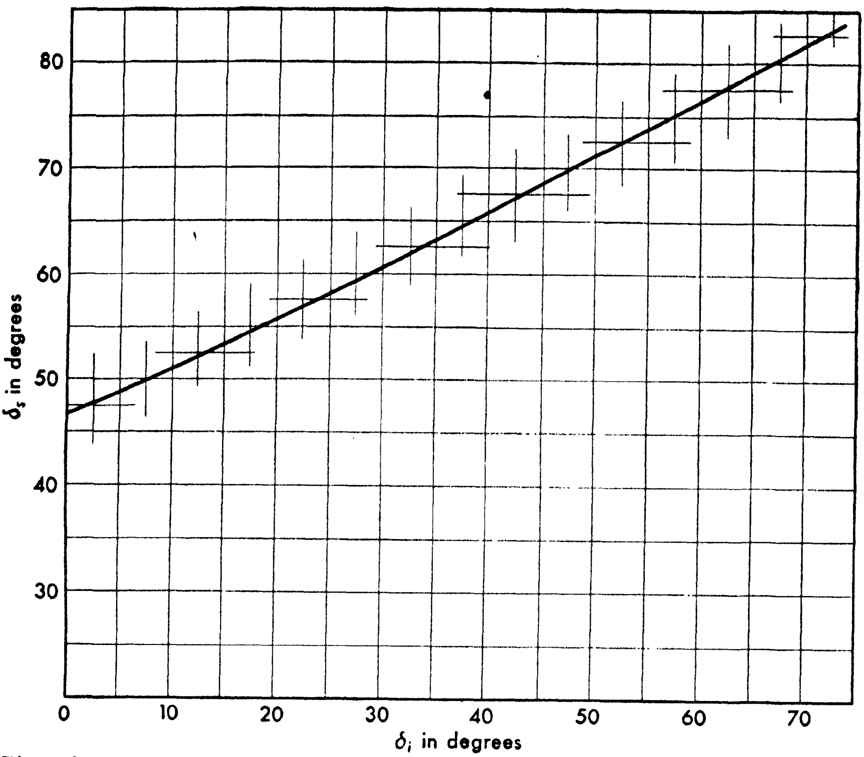


Figure 9–10

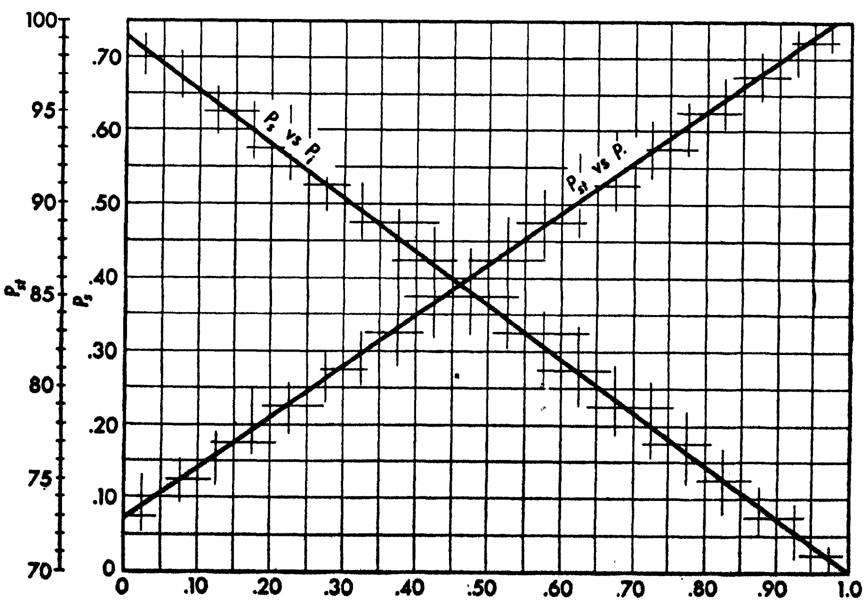


Figure 9–11

$(P_{st} - P_i)$ as a function of P_i, both expressed in per cent of the maximum synchronizing power. Thus, referring to these curves it will be noted that for an initial load of 50 per cent of the maximum P_m corresponding to a displacement angle of $\delta = \sin^{-1} 0.5 = 30°$, the additional load that may be thrown safely on the system is 36.5 per cent of the maximum. Any additional load larger than 36.5 per cent of the maximum will be conducive to loss of synchronism. The maximum safe load for the initial load is 86.5 per cent of the maximum.

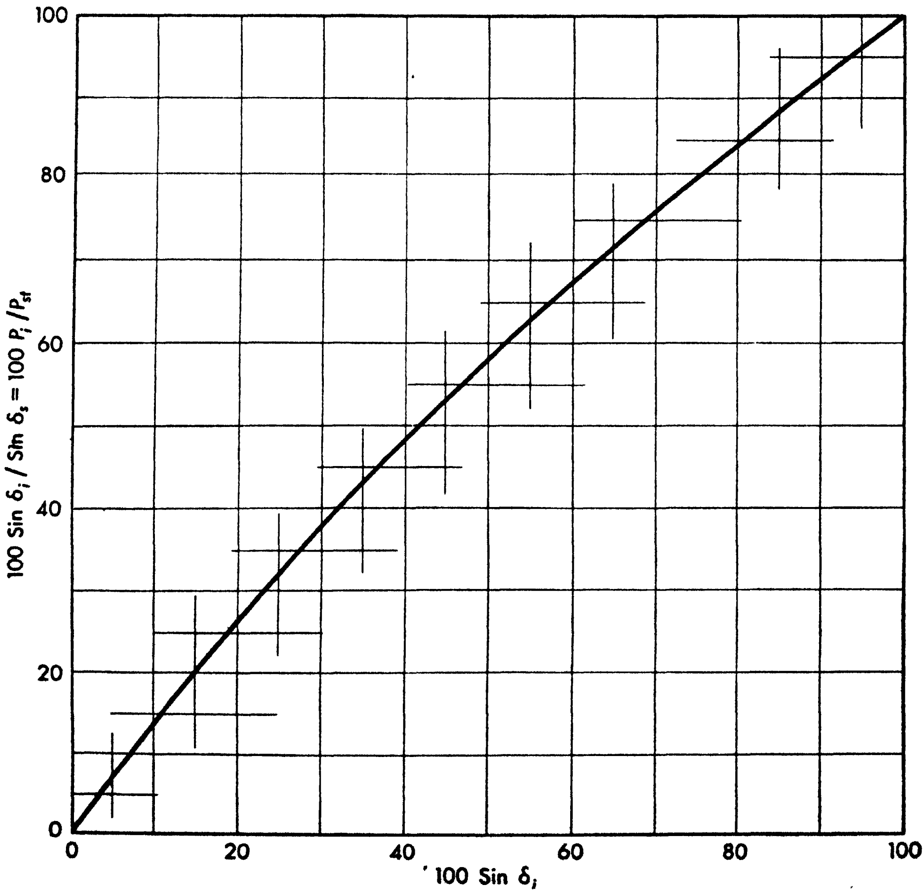


Figure 9–12

The curve marked $100\, P_i/P_{st}$ in Fig. 9–12, giving the initial load on a system in per cent of the maximum safe load $(100 \sin \delta_i/\sin \delta_s)$ as a function of the initial load in per cent of the maximum $(100 \sin \delta_i)$ will be found very convenient in the determination of limiting values of load that the system can transfer with a fault on.

9.14. *Power-Angle-Time Relations.* It was stated in the preceding articles that if the load suddenly added to any initial load of a system is larger than the safe value P_s corresponding to that initial load, the displacement angle increases beyond the safe value and synchronism will be lost unless the load is reduced to a safe value before the limiting angle is reached. This is also the case when the falling out of step is due to transient disturbances such as faults.

To determine the length of time during which the load must be reduced or the fault removed, so that stability be re-established, consider equation (9.12.2) which gives the equivalent rate of change of the speed of the two machines:

$$A = \frac{2\pi f}{M_e} \frac{P_m}{P_{rb}} (\sin \delta_f - \sin \delta). \tag{9.14.1}$$

In this expression M_e is the equivalent inertia constant of the two machines, as given by (9.9.5). The term P_m is the maximum of the power-angle curve corresponding to the initial load P_i, and its value is obtained by

$$P_m = \frac{P_i}{\sin \delta_i}.$$

The angle δ_i is obtained by formula (9.10.7) or by (9.10.12). The term P_{rb} is the rated kw base, obtained from the kva base used in the calculation of per cent reactances and the system power factor. The angle δ_f corresponds to the final load

$$P_f = P_i + P_a,$$

where P_a, the additional load, is larger than the additional safe load P_s, and is, therefore, conducive to loss of synchronism, unless it is removed in time. The angle δ in (9.14.1) corresponds to any value of load P during the transitional period when the load changes from P_i to P_f.

Setting for simplicity's sake

$$B = \frac{2\pi f}{M_e} \frac{P_m}{P_{rb}}, \tag{9.14.2}$$

and, since the acceleration is

$$A = \frac{d^2\delta}{dt^2},$$

equation (9.14.1) becomes

$$\frac{d^2\delta}{dt^2} = B(\sin \delta_f - \sin \delta). \tag{9.14.3}$$

To integrate this equation, it may be written

$$\frac{d\delta}{dt}\,dt\,\frac{d^2\delta}{dt^2} = B(\sin\delta_f - \sin\delta)\,d\delta.$$

Setting

$$\frac{d\delta}{dt} = v,$$

where v represents the rate of change of the displacement angle, the above equation becomes

$$v\,dt\,\frac{dv}{dt} = B(\sin\delta_f - \sin\delta)\,d\delta$$

or

$$v\,dv = B(\sin\delta_f - \sin\delta)\,d\delta. \tag{9.14.4}$$

The limits of this expression are determined as follows: When the load is P_i, the displacement angle is $\delta = \delta_i$, and the rate of change $d\delta_i/dt = 0$. When the load has changed to a value P, the displacement is δ, and the rate of change in δ is $d\delta/dt = v$. Using then these limits, equation (9.14.4) becomes

$$\int_0^v v\,dv = \int_{\delta_i}^{\delta} B(\sin\delta_f - \sin\delta)\,d\delta. \tag{9.14.5}$$

Integrating and substituting limits gives

$$v = \sqrt{2B}[(\delta - \delta_i)\sin\delta_f + \cos\delta - \cos\delta_i]^{\frac{1}{2}}.$$

Since $v = d\delta/dt$, the above equation becomes

$$\sqrt{2B}\,dt = \frac{d\delta}{[(\delta - \delta_i)\sin\delta_f + \cos\delta - \cos\delta_i]^{\frac{1}{2}}}. \tag{9.14.6}$$

Let δ_s be the safe limiting displacement angle, corresponding to the safe limiting interval t_s for the re-establishment of the stability of the system, measured from $t = 0$ when $\delta = \delta_i$. With these limits the above equation becomes, when the left-hand side is integrated,

$$\sqrt{2B}t_s = \int_{\delta_i}^{\delta_s} \frac{d\delta}{[(\delta - \delta_i)\sin\delta_f + \cos\delta - \cos\delta_i]^{\frac{1}{2}}}. \tag{9.14.7}$$

This expression does not lend itself to integration by usual methods.* It may be integrated, however, by a point by point method based upon the analysis given below. Let

$$\frac{1}{[(\delta - \delta_i)\sin\delta_f + \cos\delta - \cos\delta_i]^{\frac{1}{2}}} = F(\delta), \tag{9.14.8}$$

then (9.14.7) becomes

$$\sqrt{2B}t_s = \int_{\delta_i}^{\delta_s} F(\delta)\,d\delta. \tag{9.14.9}$$

* It was integrated by the use of the M.I.T. integraph and the results published. Summers, I. H., and McClure, J. B., *Progress in the Study of System Stability*, Trans., A.I.E.E., 1930.

Assume now that $F(\delta)$ is represented graphically by the curve in Fig. 9–13, so that the integral in (9.14.9) is the area under the curve. As such, the integral may be obtained by dividing the area under the curve into a large number of small areas of equal width ϕ so small in fact that any distance along the curve such as ab may be thought of as straight lines. The value of any one of these elemental areas is

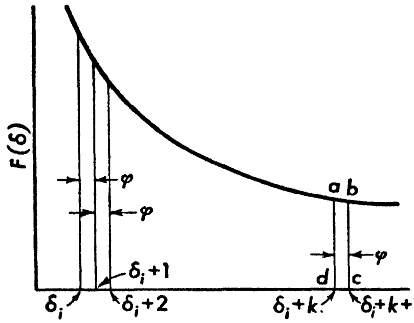


Figure 9–13

$$\text{area } (abcda) = (ad + bc)\frac{cd}{2} = a.$$

Setting

$$ad = F(\delta_i + k)$$
$$bc = F(\delta_i + k + 1)$$
$$dc = \phi,$$

the little area a may be written

$$a = [F(\delta_i + k) + F(\delta_i + k + 1)]\frac{\phi}{2}.$$

But, since

$$\delta_i + k + 1 = \delta_i + k + \phi,$$

the elemental area may be written, generally,

$$a = [F(\delta_i + k) + F(\delta_i + k + \phi)]\frac{\phi}{2}.$$

Starting with δ_i where $k = 0$, the values of the successive elemental areas are

$$a_1 = [F(\delta_i) + F(\delta_i + \phi)]\frac{\phi}{2}$$

$$a_2 = [F(\delta_i + \phi) + F(\delta_i + 2\phi)]\frac{\phi}{2}$$

$$a_3 = [F(\delta_i + 2\phi) + F(\delta_i + 3\phi)]\frac{\phi}{2}$$

$$a_k = \{F[\delta_i + (k - 1)\phi] + F(\delta_i + k\phi)\}\frac{\phi}{2}. \tag{9.14.10}$$

From what has been said above, it follows that equation (9.14.9) may be written

$$\sqrt{2B}t_s = a_1 + a_2 + a_3 + \cdots a_k + \cdots + a_n$$

or

$$\sqrt{2B}t_s = \sum_{k=1}^{k=n}\{F[\delta_i + (k - 1)\phi] + F(\delta_i + k\phi)\}\frac{\phi}{2}, \tag{9.14.11}$$

in which, by (9.14.8),

$$F[\delta_i + (k-1)\phi] = \frac{1}{\{(k-1)\phi \sin \delta_f + \cos [\delta_i + (k-1)\phi] - \cos \delta_i\}^{\frac{1}{2}}}$$

and

$$F(\delta_i - k\phi) = \frac{1}{[k\phi \sin \delta_f + \cos (\delta_i + k\phi) - \cos \delta_i]^{\frac{1}{2}}}.$$

Substituting in (9.14.11) gives

$$\sqrt{2B}t_s = \sum_{k=1}^{k=n} \left\{ \frac{\phi/2}{\{(k-1)\phi \sin \delta_f + \cos [\delta_i + (k-1)\phi] - \cos \delta_i\}^{\frac{1}{2}}} + \frac{\phi/2}{\{k\phi \sin \delta_f + \cos (\delta_i + k\phi) - \cos \delta_i\}^{\frac{1}{2}}} \right\}. \quad (9.14.12)$$

This general expression is used in the calculation of the elemental areas by giving to k successively the values 1, 2, 3, . . . n. The value of $n\phi$ should be equal to the maximum safe angle δ_s. The sum of these elemental areas thus calculated gives the value of the integral $\sqrt{2B}t_s$.

The area of the first elemental area is obtained by setting $k = 1$, resulting in

$$\sqrt{2B}t_1 = \frac{\phi/2}{(\cos \delta_i - \cos \delta_i)^{\frac{1}{2}}} + \frac{\phi/2}{[\phi \sin \delta_f + \cos (\delta_i + \phi) - \cos \delta_i]^{\frac{1}{2}}}.$$

This, it will be observed, is indeterminate as $\phi \to 0$, i.e., in the vicinity of $\delta = \delta_i$. To determine, however, the value of this first area, consider the general integral, equation (9.14.7). In the immediate vicinity of the initial angle δ_i with ϕ as the variable this equation becomes by expanding the denominator by substituting $\delta = \phi + \delta_i$

$$\sqrt{2B}t_1 = \int_0^{\phi} \frac{d\phi}{[\phi \sin \delta_f + \cos \delta_i \cos \phi - \sin \delta_i \sin \phi - \cos \delta_i]^{\frac{1}{2}}}.$$

If the limiting value of ϕ is not greater than 2.5, then $\sin \phi$ is very nearly equal to the value of ϕ in radian measure (0.0436), and $\cos \phi$ is very nearly equal to 1. The above equation becomes, accordingly, when ϕ is very small,

$$\sqrt{2B}t_1 = \int_0^{\phi} \frac{d\phi}{\sqrt{\phi\,(\sin \delta_f - \sin \delta_i)}} = \int_0^{\phi} \frac{\phi^{-\frac{1}{2}}\, d\phi}{\sqrt{\sin \delta_f - \sin \delta_i}}.$$

When integrated, this gives for the first elemental area

$$\sqrt{2B}t_1 = \frac{2\sqrt{\phi}}{\sqrt{\sin \delta_f - \sin \delta_i}} = a_1. \quad (9.14.13)$$

This relationship is substantially correct only when ϕ is not larger than 0.0436 radians corresponding to 2.5 degrees.

Inserting the value of the first elemental area just obtained in equation (9.14.12) gives

$$\sqrt{2B}t = 2\sqrt{\frac{\phi}{\sin\delta_f - \sin\delta_i}} + \sum_{k=2}^{k=\frac{\delta_s-\delta_i}{\phi}} \left\{ \frac{\phi/2}{[(k-1)\phi\sin\delta_f + \cos(\delta_i + (k-1)\phi) - \cos\delta_i]^{\frac{1}{2}}} + \frac{\phi/2}{[k\phi\sin\delta_f + \cos(\delta_i + k\phi) - \cos\delta_i]^{\frac{1}{2}}}. \right\} \quad (9.14.14)$$

A convenient value of ϕ when radian measure is used is 0.04 radians.

The actual calculations, if carried out in accordance with a definite prearranged schedule, is not as formidable as the above equation looks. To illustrate, assume a system with an initial load P_i. An additional load P_a, larger than the corresponding safe additional load, is suddenly thrown on the system. The condition is thus conducive to loss of synchronism unless the additional load is decreased or completely removed. The problem is to determine the interval of time in which this should be done. The calculation is as follows:

a. Obtain δ_i by the method outlined in § 9.10. This gives $\sin\delta_i = P_i$ in per unit of the power limit P_m. The power limit or maximum synchronizing power is calculated by

$$P_m = \frac{P_i}{\sin\delta_i}.$$

b. The final load is calculated by

$$P_f = P_i + P_a.$$

Hence,

$$\sin\delta_f = \frac{P_f}{P_m}.$$

c. The maximum safe angle δ_s is obtained from the curve δ_s *vs.* δ_i, Fig. 9–10.

d. Using $\phi = 0.04$ radians, calculate the first elemental area a_1 by (9.14.13).

e. For the other successive areas the schedule of calculations is as indicated in Table VII in which

1. Column 1 gives the successive values of $k = 1, 2, 3, \ldots (\delta_s - \delta_i)\phi$. For $\phi = 0.04$ the last value of k is $25(\delta_s - \delta_i)$, with δ_s and δ_i expressed in radian measure.
2. Column 2 gives the successive values of $k\phi$ in radians.
3. Column 3 gives the successive values of $k\phi$ in degrees.
4. Column 4 gives the successive values of $(k\phi \sin\delta_f)$.

5. Column 5 gives successive values of $(\delta_i + k\phi)$ in degrees.

6. Column 6 gives successive values of $\cos(\delta_i + k\phi)$.

7. Column 7 gives successive values of $(k\phi)\sin\delta_f + \cos(\delta_i + k\phi) - \cos\delta_i$ as obtained from Columns 4 and 6.

8. Column 8 gives the reciprocals of the values in Column 7.

9. Column 9 gives the square root of the values in Column 8.

10. Column 10 is obtained by adding successive values in Column 9; the first with second; the second with third; the third with the fourth, and so on. This column gives, therefore, twice the average height of successive elemental areas beginning with the second.

11. Column 11 is obtained by multiplying the successive values in Column 10 by $\phi/2$. The values of this column are, therefore, the elemental areas, beginning with the second.

12. Column 12 gives the summation of the area values in Column 11, i.e., the integral of equation (9.14.14) for successive values of $(\delta_i + k\phi)$ up and including the upper limit corresponding to δ_s. The last value in this Column is thus

$$\sqrt{2B}t_s = a_1 + \sum \text{Column 11}.$$

The first quantity in this column is the first elemental area a_1 obtained by (9.14.13).

The value of t_s in this equation is the interval in seconds during which the disturbance should be removed to prevent loss of synchronism. This interval is frequently stated in terms of cycles. For the operating frequency of 60 cps it is $60t_s$.

To illustrate the above, consider a two-machine system carrying an initial load P_i = 30 per cent of the maximum synchronizing power P_m and that a load P_a = 60 per cent of the maximum is suddenly added. The problem is to determine whether the system remains stable with the additional load thrown on. If the system is not stable calculate the interval t_s during which the load must be reduced so that stable operation be recovered. The calculation is as follows:

Since

$$P_i = 0.30P_m$$

it follows that

$$\sin\delta_i = 0.30$$
$$\delta_i = 17.45°.$$

The final load is

$$P_f = (0.30 + 0.60)P_m$$

corresponding to

$$\sin\delta_f = 0.9$$
$$\delta_f = 64.15°.$$

The maximum safe angle δ_s corresponding to $\delta_i = 17.45°$ is obtained from curve, Fig. 9–10, and is

$$\delta_s = 54.12 = .945 \text{ radians.}$$

Since $\delta_f > \delta_s$, the system will become unstable and will lose synchronism unless the load is reduced to safe value, less than $100 \sin \delta_s = 81.4$ per cent of the maximum synchronizing power within a specified time.

The calculation of this time interval is given below. Taking $\phi = 0.04$ radians, the first elemental area, by (9.14.13), is

$$\begin{aligned} a_1 &= 2\sqrt{\frac{0.04}{\sin \delta_f - \sin \delta_i}} \\ &= \sqrt{\frac{0.16}{0.9 - 0.3}} \\ &= 0.515 = \sqrt{2B}t_1. \end{aligned}$$

Table VII gives the tabulated calculations for the integral $\sqrt{2B}t_s$ as outlined above using $\phi = 0.04$. The calculated value of the integral is

$$\sqrt{2B}t_s = 2.105.$$

TABLE VII

1	2	3	4	5	6	7	8	9	10	11	12
	—	—	—	17.45	0.954	—	—	—	—	—	—
1	0.04	2.29	0.036	19.74	0.941	0.023	43.5	6.6	—	—	0.5150 = a_1
2	0.08	4.58	0.072	22.03	0.927	0.045	22.2	4.71	11.31	0.2262	0.7412
3	0.12	6.87	0.108	24.32	0.911	0.065	15.4	3.93	8.64	0.1728	0.9140
4	0.16	9.11	0.144	26.61	0.894	0.084	11.9	3.45	7.38	0.1476	1.0616
5	0.20	11.45	0.180	28.90	0.875	0.101	9.9	3.15	6.60	0.1220	1.1836
6	0.24	13.74	0.216	31.19	0.855	0.117	8.35	2.93	6.08	0.1216	1.3052
7	0.28	16.03	0.253	33.48	0.834	0.132	7.57	2.86	5.79	0.1158	1.4210
8	0.32	18.32	0.288	35.67	0.812	0.146	6.85	2.62	5.48	0.1096	1.5306
9	0.36	20.61	0.324	37.96	0.788	0.148	6.32	2.52	5.14	0.1028	1.6334
10	0.40	22.90	0.360	40.25	0.762	0.168	5.95	2.44	4.96	0.0992	1.7326
11	0.44	25.19	0.396	42.54	0.736	0.178	5.62	2.37	4.81	0.0962	1.8288
12	0.48	27.48	0.432	44.83	0.708	0.186	5.38	2.32	4.69	0.0938	1.9226
13	0.52	29.77	0.468	47.22	0.679	0.193	5.18	2.28	4.60	0.092	2.0146
14	0.56	32.09	0.504	49.54	0.649	0.199	5.02	2.24	4.52	0.0904	2.1050
15	0.60	34.38	0.540	51.83	0.618	0.204	4.91	2.22	4.46	0.0892	2.1942
16	0.64	36.67	0.576	54.12	0.586	0.208	4.81	2.19	4.41	0.0882	2.2824

The interval during which the load must be decreased to avoid loss of synchronism is

$$t_s = \frac{2.105}{\sqrt{2B}} \text{ seconds}$$

or, for the operating frequency of 60 cps, $60t_s$ cycles.

The value of B in the above relations is given by (9.14.2).

Curve, Fig. 9–14, shows the relation between successive values of space-phase angle δ in degrees as given by Column 5 and the corresponding values of $\sqrt{2Bt_s}$ as given by Column 12, Table VII, for the above illustrative example.

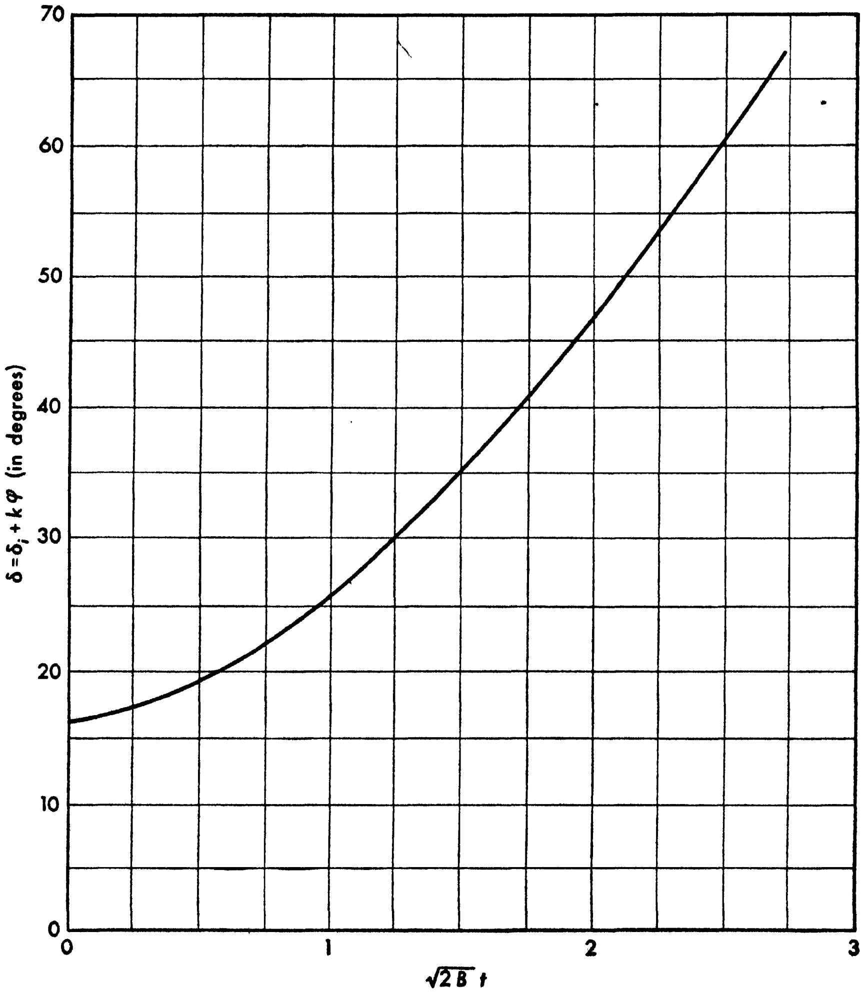


Figure 9–14

A more direct method for the calculation of the portion of the area represented by the summation in equation (9.14.14) is to plot $F(\delta)$ as given by (9.14.14) as a function of δ, between the limits $\delta = \delta_i$ and $\delta = \delta_e$. The curves have the general form shown in Fig. 9–15. The area (a) between the values δ_a and δ_e may be obtained conveniently by means of a planimeter.

The first elemental area between the limits of δ_i and δ_a is obtained by (9.14.13) in which $\phi = \delta_a - \delta_i$. This method although more direct requires as mentioned above a planimeter.

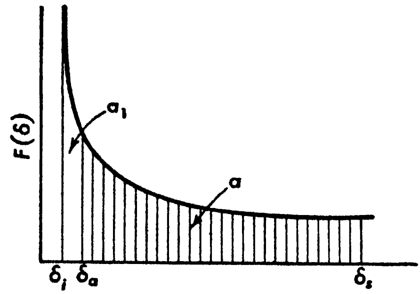


Figure 9–15

REFERENCES

A.I.E.E. Subcommittee on Interconnection and Stability Factors, *First Report on Power System Stability,* Elect. Eng., February 1937.

Bergvall, R. C., and Evans, R. D., *Experimental Analysis of Stability and Power Systems,* Trans. A.I.E.E., Vol. XLIII, 1924.

Bush, V., and Booth, R. D., *Power System Transient,* Trans. A.I.E.E., Vol. XLIV, 1925.

Byrd, H. L., and Pritchard, S. R., Jr., *Solution of Two-Machine Stability Problem,* Gen. Elec. Rev., February 1933.

Clarke, Edith, *Steady State in Transmission Systems,* Trans. A.I.E.E., Vol. XLV, 1926.

Dahl, O. G. C., *Stability of the General Two-Machine System,* Trans. A.I.E.E., 1935.

Dahl, O. G. C., *Electric Power Circuits,* Vol. II, McGraw-Hill Book Co., 1938.

Doherty, R. E., and Dewey, H. H., *Fundamental Consideration of Power Limits on Transmission Systems,* Trans. A.I.E.E., Vol. XLIV, 1925.

Evans, R. D., and Sels, H. K., *Power Limitations of Transmission Systems,* Trans. A.I.E.E., Vol. XLIII, 1924.

Fortescue, C. L., and Wagner, C. F., *Some Theoretical Considerations of Power Transmission Systems,* Trans. A.I.E.E., Vol. XLIII, 1924.

Fortescue, C. L., *Transmission Stability,* Trans. A.I.E.E., Vol. XLIV, 1925.

Hamilton, J. A., Jr., *Power System Stability,* Eng. Jour., April 1931.

Hauher, F. C., *Power Transmission,* Trans. A.I.E.E., Vol. XLIII, 1924.

Nickel, C. A., and Lawton, F. L., *An Investigation of Transmission System Power Limits,* Trans. A.I.E.E., Vol. XLV, 1926.

Park, R. H., and Bancker, E. H., *System Stability as a Design Problem,* Trans. A.I.E.E., Vol. XLVIII, 1929.

Summers, I. H., and McClure, J. B., *Progress in the Study of System Stability,* Trans. A.I.E.E., 1930.

Wilkins, Roy, *Practical Aspects of System Stability,* Trans. A.I.E.E., Vol. XLV, 1926.

SUGGESTIVE PROBLEMS *Chapter 9*

1. The inertia constant of a 2-pole, 60 cycle 25000 kva turbo-generator is 15 seconds. Calculate (a) the energy stored in the rotor; when it delivers rated load;

(b) the time in seconds that it takes the rotor to decelerate one revolution; (c) the time in seconds it takes the rotor to decelerate one electrical degree.

2. Three 4-pole 60 cycle synchronous motors rated 500, 750, and 1000 kva are operated in parallel. The measured inertia constants of these machines are 3.9, 4.2, and 4.5 seconds, respectively, at their own kw base. Calculate the equivalent inertia constant of the three motors to a common base of (a) 500, (b) 750, (c) 1000, and (d) 10000 kw.

3. Four hydro-generators each rated 13.8 kv, 21500 kw at 90% power factor have inertia constants of 3.27 seconds at own base. What is their joint inertia constant at a base of 100000 kw?

4. Calculate the equivalent inertia constant of the four generators in Prob. 3 and the three motors in Prob. 2.

5. A simple two-machine system as shown schematically in Fig. 9–4 has a load of 20000 kw at 90% power factor lag. Assume that the system reactance is 60 per cent to a 25000 kva base and calculate the displacement angle in electrical degrees between the excitation voltages at the termini of the system. What is the maximum synchronizing power for the stated load on the system? Check the results by the method based upon Fig. 9–5.

6. A two-machine system similar to the one stated in Prob. 4 carries an initial load of 25% of the maximum synchronizing power. An additional load of 50% of the maximum is suddenly thrown on the system. Determine whether the system remains stable.

Chapter 10 System Instability

10.1. *General Considerations.* It was shown in the preceding chapter that the sudden addition of load in excess of that corresponding to the safe displacement angle is conducive to loss of synchronism unless the load is reduced to a safe value within a definite time limit. The emphasis was put on the displacement angle, first because it was a convenient method of approach and second because any transient disturbance which would impair the transfer of the power may be investigated in terms of either an actual or apparent change in the displacement angle. In this manner, the problem of instability of transmission systems due to faults, switching of lines in and out of service either under normal or abnormal conditions, may be studied by adapting the principles established in the preceding chapter.

Consider, for instance, the power-angle equation

$$P = \frac{E_G E_M}{X} \sin \delta.$$

The preceding chapter dealt with the changes in the value of δ as demanded by a change in the load P, which is transferred from the source to the receiving end. The transfer of this amount of power may, however, be impaired if and when there is sudden increase in the reactance such as would occur on the switching of a line out of service or some other kind of transient change in the system. The value of P is thus decreased while the shaft load remains the same. The motor will immediately slow down, the generator will speed up, the displacement angle increases, the machines fall out of step, and synchronism is lost.

10.2. *Instability Due to Line Switching.*

Consider a two-machine system interconnected by two lines as indicated in Fig. 10–1. Let X_G and X_M be the synchronous reactances of the two machines, respectively, and X_a and X_b the reactances of the two lines including the transformers in per

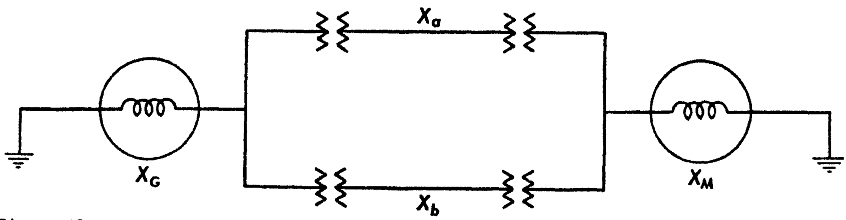


Figure 10–1

cent to a common kva base. Under normal conditions when the two lines are in service, the equivalent reactance between the two machines is

$$X_1 = X_G + \frac{X_a X_b}{X_a + X_b} + X_M. \tag{10.2.1}$$

If line b is switched out, the reactance of the system is changed from X_1 to

$$X_2 = X_G + X_a + X_M. \tag{10.2.2}$$

Since

$$X_a > \frac{X_a X_b}{X_a + X_b},$$

it follows that

$$X_2 > X_1.$$

The switching of a line out of service increases automatically the equivalent reactance of the system.

Let P_1 be the load on the system, $\cos\theta_1$ the power factor, and δ_1 the corresponding displacement angle, calculated by (9.10.12). The maximum power that can be transferred over the system with the two lines in service is

$$P_{m1} = \frac{P_1}{\sin\delta_1} = \frac{E_G E_M}{X_1} \tag{10.2.3}$$

where X_1 is the total system reactance in ohms. Under the conditions stated, the system operates on a power-angle curve whose maximum value is P_{m1} as indicated in Fig. 10–2. When line b is tripped out of service, the load remaining the same, the maximum power is immediately reduced to a value

$$P_{m2} = \frac{E_G E_M}{X_2}.$$

By (10.2.3) it follows that, in terms of per cent reactances (X_1 and X_2) to a common kva base,

$$\frac{P_{m2}}{P_{m1}} = \frac{X_1}{X_2}. \tag{10.2.4}$$

It appears from what was said above that the switching out of line (*b*) changes the operation of the system from the power-angle curve (1) to the power-angle curve (2).

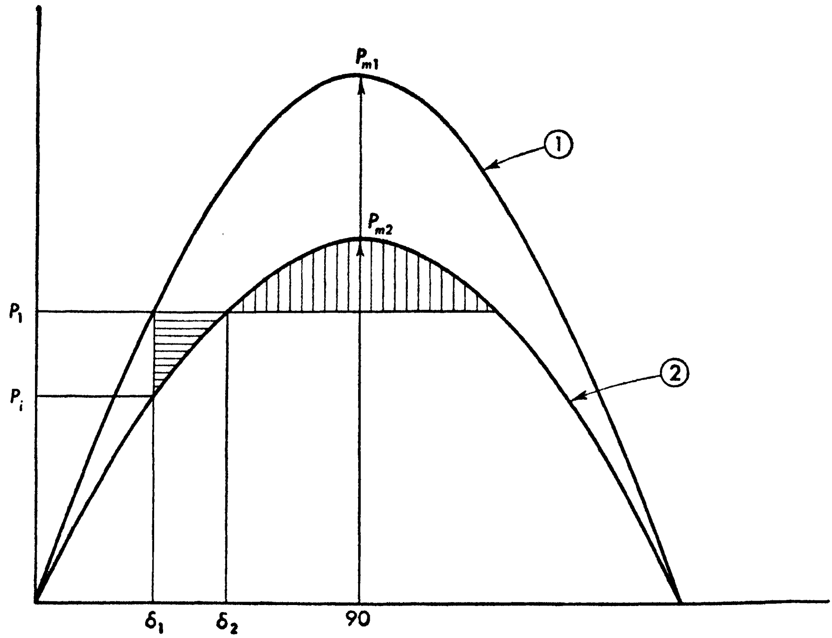


Figure 10–2

With regard to the power, the switching out of line (*b*) is identical in effect to that which would occur if the system were operating on power-angle curve (2) with an initial load

$$P_i = P_{m2} \sin \delta_1, \tag{10.2.5}$$

and the load suddenly increased, on the tripping of the line, to

$$P_1 = P_{m2} \sin \delta_2. \tag{10.2.6}$$

Two possible cases may arise in connection with the instability created by line switching. One is when the load on the system at the time of switching is less than the maximum power that can be transferred with one line, i.e., $P_1 < P_{m2}$. In this case the instability may or may not be of a transient character, i.e., the system may or may not recover its stable operation. The second case is when the load on the system at the time of switching is greater than the maximum with one line tripped, $P_1 > P_{m2}$. In this case, the instability created by switching is definitely conducive to loss of synchronism unless the tripped line is switched back into service within a definite length of time.

10.3. *Line Switching; Case I:* $P_1 < P_{m2}$. Figure 10–2 illustrates this case. The situation is, as stated above, identical to a two-machine system operating on power-angle curve (2) with an initial load P_i, and the load suddenly increased to a value P_1. The problem is to determine whether P_1 is smaller or larger than the maximum safe load corresponding to P_i as discussed in § 9.13. To do this, calculate the displacement angle δ_1, by (9.10.12). Then, since

$$P_1 = P_{m1} \sin \delta_1$$

and

$$P_1 = P_{m2} \sin \delta_2,$$

it follows that

$$\frac{P_{m2}}{P_{m1}} = \frac{\sin \delta_1}{\sin \delta_2}. \tag{10.3.1}$$

By (10.2.4), this may be written

$$\frac{\sin \delta_1}{\sin \delta_2} = \frac{X_1}{X_2}. \tag{10.3.2}$$

Calculate δ_2 from this equation and from curve δ_s *vs.* δ_i, Fig. 9–10, determine the safe displacement angle δ_s corresponding to $\delta_1 = \delta_i$. If δ_s as obtained from this curve is larger than δ_2, the instability caused by the switching is only transient. But, if $\delta_s < \delta_2$, the instability caused by the tripping out of the line is conducive to loss of synchronism unless the tripped line is replaced in service within a definitely specified time, which may be calculated as outlined in § 9.14.

From what is said above, it follows that the value of $\sin \delta_1$ corresponding to

$$\frac{X_1}{X_2} = \frac{\sin \delta_1}{\sin \delta_s} \tag{10.3.3}$$

is the load in per unit of the maximum safe value which the system can transfer with one line tripped. It may be obtained directly from curve, Fig. 9–12, which is plotted $(\sin \delta_1/\sin \delta_s)$ *vs.* $\sin \delta_1$. Curves of $\sin \delta$ *vs.* P or *vs.* P/P_r for any particular power factor may be obtained from the relation between δ and P discussed in § 9.10. These curves give the actual load or the load in per unit of rated power that can be transferred with only one line.

10.4. *Line Switching; Case II:* $P_1 > P_{m2}$. This case is illustrated graphically in Fig. 10–3 in which curve marked (1) is the power-angle curve for the normal condition of two lines in service, curve (2) is the power-angle curve for the condition when one line is tripped out. The load is P_1 and its corresponding angle is δ_1. The maximum of curve (1) is determined from

$$P_{m1} = \frac{P_1}{\sin \delta_1}$$

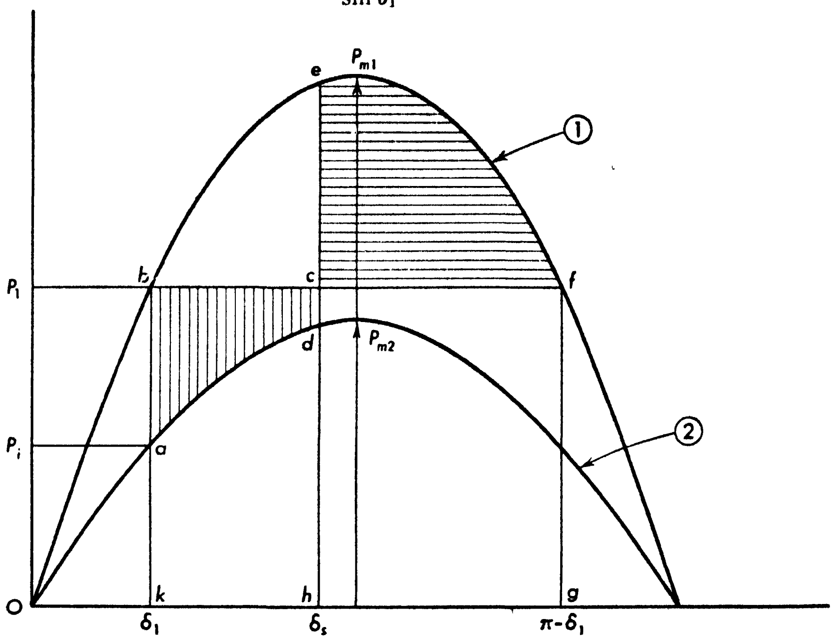


Figure 10–3

where δ_1 is first obtained by (9.10.12). The maximum of curve (2) is obtained from

$$\frac{P_{m2}}{P_{m1}} = \frac{X_1}{X_2}. \tag{10.4.1}$$

If one line is suddenly tripped out, and the line reactance changes accordingly from X_1 to X_2, the system changes its operation from the power-angle curve (1) to power-angle curve (2). As in the previous case, the effect is identical to a sudden change from a fictitious load

$$P_i = P_{m2} \sin \delta_1$$

on power-angle curve (2) to a load $P_1 > P_{m2}$. The condition is, obviously, conducive to loss of synchronism unless the tripped line is restored to service within a definite specified time.

At the instant the line is tripped out, the motor begins to slow down, the generator begins to speed up, and the two machines pull apart from synchronism. Assume now that the tripped line is restored to service at the particular instant when the displacement angle has reached the value δ_s. The operation changes to power angle curve (1). During the interval of time that the displacement angle changes from δ_1 to δ_s, the decelerating motor lost from its stored energy an amount proportional to the area *abcda*.

In its deceleration, the rotor may overtravel to the space-phase position corresponding to the displacement angle $\pi - \delta_1$. During the interval corresponding to the change in displacement angle from δ_s to $\pi - \delta_1$, the source supplies in excess to the required energy an amount proportional to the area *cefc*. If the amount of energy represented by the area *cefc* is larger than the amount of energy represented by the area *abcda*, the instability created by the switching operation is transient, and the system recovers its stability. The criterion for stable condition is, therefore, that

$$\text{area }(abcda) = \text{area }(cefc) \tag{10.4.2}$$

Referring to the figure, note that

$$\text{area }(abcda) = \text{area }(oP_1fgo) - \text{area }(oP_1bko) - \text{area }(cfghc) - \text{area }(adhka).$$

This may be written

$$\text{area }(abcda) = P_1(\pi - \delta_1) - P_1\delta_1 - P_1(\pi - \delta_1 - \delta_s) - \int_{\delta_1}^{\delta_s} P_{m2}\sin\delta\, d\delta. \tag{10.4.3}$$

Similarly,

$$\text{area }(cefc) = \text{area }(hefgh) - \text{area }(hcfgh)$$

or

$$\text{area }(cefc) = \int_{\delta}^{\pi-\delta_1} P_{m1}\sin\delta\, d\delta - P_1(\pi - \delta_1 - \delta_s). \tag{10.4.4}$$

The criterion for stability is obtained by equating (10.4.3) and (10.4.4), resulting in

$$\int_{\delta_s}^{\pi-\delta_1} P_{m1}\sin\delta\, d\delta + \int_{\delta_1}^{\delta_s} P_{m2}\sin\delta\, d\delta = P_1(\pi - 2\delta_1). \tag{10.4.5}$$

This, when integrated between the indicated limits, gives

$$P_{m1}(\cos\delta_1 + \cos\delta_s) + P_{m2}(\cos\delta_1 - \cos\delta_s) = P_1(\pi - 2\delta_1).$$

It may be written also

$$(P_{m1} + P_{m2})\cos\delta_1 + (P_{m1} - P_{m2})\cos\delta_s = P_1(\pi - 2\delta_1),$$

which, when solved for δ_s, gives

$$\cos \delta_s = \frac{P_1(\pi - 2\delta_1) - (P_{m1} + P_{m2}) \cos \delta_1}{P_{m1} - P_{m2}}, \tag{10.4.6}$$

where δ_s is the maximum safe angle. The tripped line should be restored during the interval corresponding to this angle if loss of synchronism is to be avoided. Using $P_1 = P_{m1} \sin \delta_1$, and the relation given by (10.4.1), the above equation becomes

$$\cos \delta_s = \frac{\sin \delta_1(\pi - 2\delta_1) - \left(1 + \frac{X_1}{X_2}\right) \cos \delta_1}{1 - \frac{X_1}{X_2}}.$$

Setting

$$\frac{X_1}{X_2} = m, \tag{10.4.7}$$

the final form from which the safe angle δ_s may be calculated is

$$\cos \delta_s = \frac{\sin \delta_1(\pi - 2\delta_1) - (1 + m) \cos \delta_1}{1 - m}. \tag{10.4.8}$$

The interval during which the tripped line should be restored is given by the integral (9.14.7), in which

$$\sin \delta_f = \frac{P_1}{P_{m2}}.$$

Since for the case under consideration $P_1 = P_{m1} \sin \delta_1$ and $P_{m2} = (X_1/X_2)P_{m1}$, the above may be written

$$\sin \delta_f = \frac{\sin \delta_1}{m}. \tag{10.4.9}$$

The integral given by (9.14.7) applied to this particular case becomes, therefore,

$$\sqrt{2B}t_s = \int_{\delta_1}^{\delta_s} \frac{d\delta}{\left[\frac{(\delta - \delta_1) \sin \delta_1}{m} + \cos \delta - \cos \delta_1\right]^{\frac{1}{2}}} \tag{10.4.10}$$

and its value determined by the method outlined in § 9.14.

10.5. *Instability to L–L–L–G Fault.*

It was shown in § 8.16 that a three-phase to ground fault in a system may be replaced in the diagrammatic representation of the faulted systems by a wye circuit of zero impedance per branch. Thus, consider one phase of a two-machine system interconnected through a twin line as shown in Fig. 10–4.

The reactance of the system under normal conditions before the occurrence of the fault is

$$X_1 = X_G + \frac{X_a X_b}{X_a + X_b} + X_M, \tag{10.5.1}$$

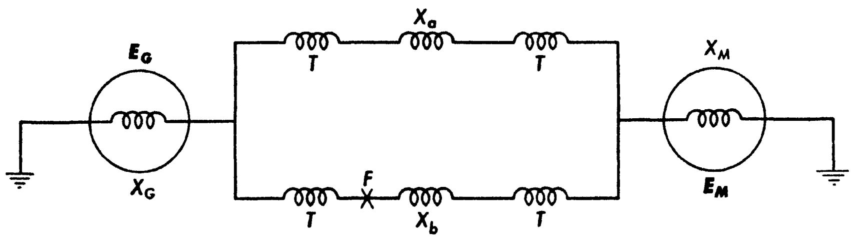


Figure 10–4

where X_G and X_M are the reactances of the generator and motor, respectively.

If P_1 is the load on the system, and $\cos\theta$ is the power factor, the corresponding displacement angle δ_1 may be calculated by (9.10.12). Then the maximum power that can be transferred normally over the system is

$$P_{m1} = \frac{P_1}{\sin \delta_1} = \frac{E_G E_M}{X_1}, \tag{10.5.2}$$

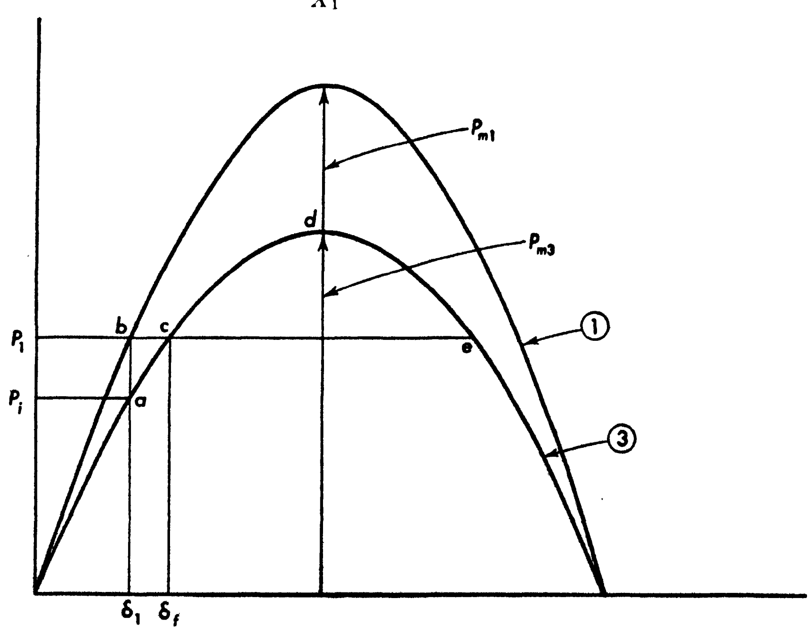


Figure 10–5

where E_G and E_M are the excitation emf's of generator and motor, respectively, and X_1 is the total reactance in ohms.

The system operates on the power-angle curve (1) whose maximum is P_{m1} as shown in Fig. 10–5. Assume now an L–L–L–G fault at the high side of transformer in line b as shown in Fig. 10–4. At the instant the fault occurs, the diagrammatic structure of the system changes to that shown in Fig. 10–6 in which X_{Gt} and X_{Mt} are now the transient reactances of the generator and motor, respectively, and X_c and X_d are the reactances of line b on either side of the fault.

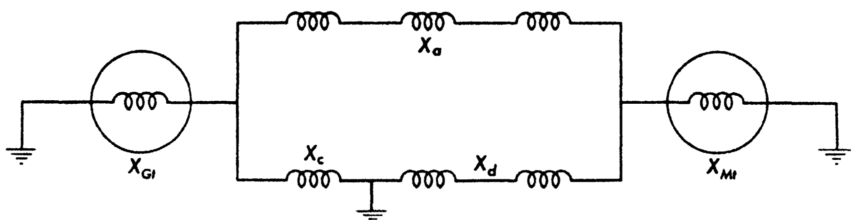


Figure 10–6

To obtain the equivalent reactance of the system under the faulted condition, convert the delta circuit, Fig. 10–6, to the wye circuit, Fig. 10–6a, in which

$$X_e = \frac{X_a X_c}{X_a + X_c + X_d} \tag{10.5.3}$$

$$X_f = \frac{X_c X_d}{X_a + X_c + X_d} \tag{10.5.4}$$

and

$$X_h = \frac{X_a X_d}{X_a + X_c + X_d}. \tag{10.5.5}$$

It was shown in § 7.6 that a circuit of the type shown in Fig. 10–6a may be further simplified by connecting the grounded branch X_f in parallel

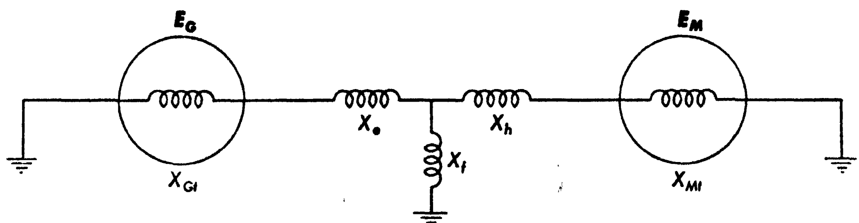


Figure 10–6a

with $(X_h + X_{Mt})$ and simultaneously changing the value of the E_M to an equivalent value

$$E_{Me} = \left(\frac{X_f}{X_f + X_h + X_{Mt}}\right) E_M. \tag{10.5.6}$$

The faulted system per phase may now be represented by the diagram, Fig. 10–6b. The reactance between the termini of the system becomes, accordingly

$$X_3 = X_{Gt} + X_e + \frac{X_f(X_h + X_{Mt})}{X_f + X_h + X_{Mt}}. \tag{10.5.7}$$

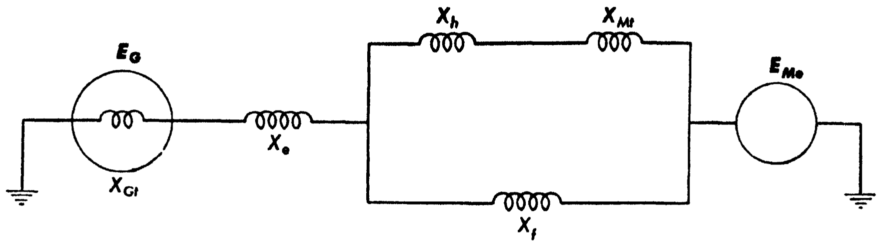


Figure 10–6b

The maximum value of the power-angle curve on which this equivalent of the faulted system operates, by (10.2.3), is

$$P_{m3} = \frac{E_G E_{Me}}{X_3}. \tag{10.5.8}$$

Setting for brevity

$$X_{Gt} + X_e = X_k$$
$$X_h + X_{Mt} = X_m,$$

the preceding equation may be written, by (10.5.7),

$$P_{m3} = \frac{X_f E_G E_M/(X_f + X_m)}{X_k + \dfrac{X_f X_m}{X_f + X_m}} = \frac{E_G E_M}{X_k + X_m + \dfrac{X_k X_m}{X_f}}. \tag{10.5.9}$$

The power-angle curve (3) in Fig. 10–5 has this maximum value.

Referring the preceding equation to (10.5.8) it is seen that the reactance of the faulted system in per cent is

$$X_3 = X_k + X_m + \frac{X_k X_m}{X_f}. \tag{10.5.10}$$

Now let the load on the system just prior to the occurrence of the fault be P_1 and the corresponding displacement angle, as calculated by (9.10.12), δ_1. The occurrence of the fault changes the operation of the system from the power-angle curve (1), whose maximum is

$$P_{m1} = \frac{P_1}{\sin \delta_1}$$

to the power-angle curve (3) whose maximum value, by (10.2.4), is

$$P_{m3} = P_{m1} \frac{X_1}{X_3}.$$

The situation is identical to that which would take place if the system were operating initially on the power-angle curve (3) with a fictitious load of

$$P_i = P_{m3} \sin \delta_1$$

and the load suddenly increased to a value

$$P_1 = P_{m3} \sin \delta_f.$$

If the load P_1 is smaller than the power limit P_{m3} of the system with the fault on, the instability created by the fault may be transient and the system carry fault and load, or the fault may be conducive to loss of synchronism. In the first case area (*cdec*) > area (*abca*) in Fig. 10–5. To determine whether the system can carry the fault and the load, obtain the safe angle δ_s corresponding to δ_i from Fig. 9–10. The value of sin δ_1 in

$$\frac{\sin \delta_1}{\sin \delta_s} = \frac{X_1}{X_3} \tag{10.5.11}$$

is the load in per unit of P_{m1} which the system can transfer with the fault on. It may be obtained directly from the curve ($\sin \delta_1/\sin \delta_s$) *vs.* $\sin \delta_1$, Fig. 9–12. The actual power in kw for any given power factor which could be transferred with the fault on may be obtained directly from the curve-plotted sin δ *vs.* P shown in Fig. 10–21. Curve sin δ *vs.* P/P_r shown in the same Fig. 10–21 gives the value of this power in per unit of the rated power P_r.

If the area (*abca*) > area (*cdec*), the fault is definitely conducive to loss of synchronism unless the fault is removed within a definite interval of time corresponding to the respective safe angle. This time interval may be calculated by integrating

$$\sqrt{2B}t_s = \int_{\delta_1}^{\delta_s} \frac{d\delta}{[(\delta - \delta_1) \sin \delta_f + \cos \delta - \cos \delta_1]^{\frac{1}{2}}}$$

in which δ_1 is the displacement angle corresponding to the load at the given power factor and is calculated by (9.10.12). The angle δ_s is the safe angle corresponding to δ_1 and is obtained directly from curve δ_s *vs.* δ_1, Fig. 9–10. The angle δ_f in the above integral is the angle which corresponds to the load P_1 on power-angle curve (3). Its value is obtained by (10.4.9) and is

$$\sin \delta_f = \frac{\sin \delta_1}{X_1/X_3}. \tag{10.5.12}$$

Setting $X_1/X_3 = m_3$, the above integral becomes

$$\sqrt{2B}t_s = \int_{\delta_1}^{\delta_s} \frac{d\delta}{\left[(\delta - \delta_1)\dfrac{\sin\delta_1}{m_3} + \cos\delta - \cos\delta_1\right]^{\frac{1}{2}}}. \qquad (10.5.13)$$

The integration is carried out as outlined in § 9.14.

If the load on the system is larger than the power limit with the fault on, i.e., when $P_1 > P_{m2}$, the disturbance is conducive to loss of synchronism, unless the fault is cleared during the interval corresponding to the respective safe angle. The analysis of this particular situation is analogous to Case II of line switching discussed in § 10.4. The return to service of the tripped line in that case corresponds to the clearing of the fault in the short-circuit case.

10.6. *Three-Phase to Ground Fault; Faulted Line Tripped.* If the three-phase fault is conducive to loss of synchronism as is the case discussed in the last part of the preceding article, stability of operation may possibly be recovered by tripping the faulted line out of service.

There are three sequent stages which must be considered. The first pertains to the system under normal operation with the two lines in service, just prior to the occurrence of the fault. The reactance of the system at this stage is X_1 as given by equation (10.5.1). The system is operating on power-angle curve (1) whose maximum is

$$P_{m1} = \frac{P_1}{\sin\delta_1}, \qquad (10.6.1)$$

as obtained in the preceding article and as shown in Fig. 10–7. The angle δ_1 corresponding to the load P_1 is calculated by (9.10.12).

The second stage pertains to the system under the faulted condition. The reactance of the system under this condition is X_3, and its value is given by (10.5.10). The power limit for this stage, by (10.4.1), is

$$P_{m3} = \frac{X_1}{X_3} P_{m1}. \qquad (10.6.2)$$

Curve (3), Fig. 10–7, is the power-angle curve for this stage.

The third stage pertains to the system with the faulted line tripped out of service. The system operates on one line only and its reactance is

$$X_2 = X_G + X_a + X_M. \qquad (10.6.3)$$

The power limit for this particular condition is

$$P_{m2} = \frac{X_1}{X_2} P_{m1}.$$

Curve (2), in Fig. 10–7, is the power-angle curve for this stage.

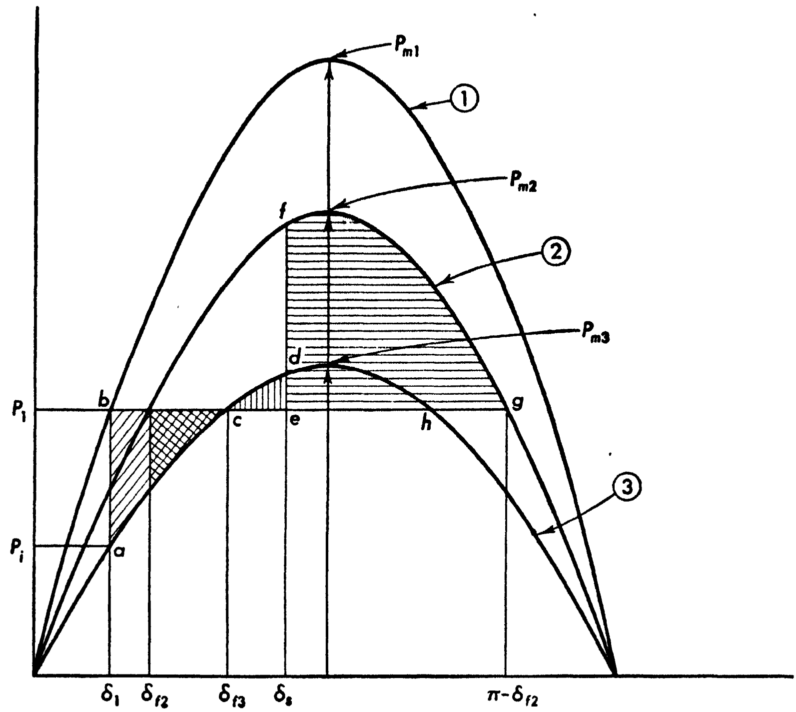


Figure 10–7

There are two general aspects to the problem. One when the actual load P_1 on the system prior to the occurrence of the fault is smaller than the power limit with the fault on, i.e., $P_1 < P_{m3}$ but the fault is conducive to loss of synchronism, and the faulted line tripped. This is the case shown in Fig. 10–7. The other case, when $P_1 > P_{m3}$. The method of analysis, however, is the same.

Considering the case illustrated in the figure, a three-phase short circuit on one of the lines is, as shown in the preceding article, identical in effect to a sudden increase in load, as if the system were operating on power-angle curve (3) with an initial load P_i suddenly changed to P_1. Assuming that area (*abca*) > area (*cdhc*), the fault is conducive to loss of synchronism. If the faulted line is switched out of service, the system immediately begins to operate on power-angle curve (2). Referring to the figure, during the interval that the motor rotor slipped from angle δ_1 to δ_{f3}, it lost an amount of energy proportional to the area (*abca*). Assuming that the faulted line is tripped when the power angle is δ_s, the energy supplied by the source is proportional to the sum of the area (*cdec*) (supplied before the line is tripped) and area (*efge*) (supplied after the faulted line is tripped). If the amount of energy supplied by the source exceeds that lost by the rotor from its stored energy, synchronism will be recovered. The criterion of recovery is, therefore, equality between the above-mentioned areas

$$\text{area }(abca) = \text{area }(cdec) + \text{area }(efge). \tag{10.6.4}$$

The angle δ_s, for which these two areas are equal, is the limiting angle by which the corresponding limiting time for tripping the line out of service may be obtained.

Referring to the figure it will be seen that

$$\begin{aligned}
\text{area }(abca) &= P_1(\pi - \delta_{f2}) - P_1\delta_1 - P_1(\pi - \delta_{f2} - \delta_{f3}) - \int_{\delta_1}^{\delta_{f3}} P_{m3}\sin\delta\, d\delta \\
\text{area }(cdec) &= \int_{\delta_{f3}}^{\delta_s} P_{m3}\sin\delta\, d\delta - P_1(\delta_s - \delta_{f3}) \\
\text{area }(efge) &= \int_{\delta_s}^{\pi - \delta_{f2}} P_{m2}\sin\delta\, d\delta - P_1(\pi - \delta_{f2} - \delta_s).
\end{aligned}$$

Using these three expressions in (10.6.4) yields

$$P_1(\pi - \delta_{f2} - \delta_1) - \int_{\delta_1}^{\delta_{f3}} P_{m3}\sin\delta\, d\delta = \int_{\delta_{f3}}^{\delta_s} P_{m3}\sin\delta\, d\delta + \int_{\delta_s}^{\pi-\delta_{f2}} P_{m2}\sin\delta\, d\delta.$$

This, when integrated between the stated limits, gives

$$\begin{aligned}
P_1(\pi - \delta_{f2} - \delta_1) + P_{m3}(\cos\delta_{f3} - \cos\delta_1) = &-P_{m3}(\cos\delta_s - \cos\delta_{f3}) \\
&+ P_{m2}(\cos\delta_s + \cos\delta_{f2}),
\end{aligned}$$

which simplifies to

$$(P_{m2} - P_{m3})\cos\delta_s = P_1(\pi - \delta_{f2} - \delta_1) - P_{m2}\cos\delta_{f2} - P_{m3}\cos\delta_1. \tag{10.6.5}$$

Since

$$\left.\begin{aligned}
P_1 &= P_{m1}\sin\delta_1 \\
P_{m2} &= \frac{X_1}{X_2}P_{m1} = M_2P_{m1} \\
P_{m3} &= \frac{X_1}{X_3}P_{m1} = M_3P_{m1},
\end{aligned}\right\} \tag{10.6.6}$$

the above equation may be written, when solved for $\cos\delta_s$,

$$\cos\delta_s = \frac{\sin\delta_1(\pi - \delta_{f2} - \delta_1) - (M_2\cos\delta_{f2} + M_3\cos\delta_1)}{M_2 - M_3} \tag{10.6.7}$$

where by (10.4.9)

$$\sin\delta_{f2} = \frac{\sin\delta_1}{X_1/X_2} = \frac{\sin\delta_1}{M_2}. \tag{10.6.8}$$

The safe limiting time t_s for tripping out the faulted line is given by the integral

$$\sqrt{2B}t_s = \int_{\delta_1}^{\delta_s} \frac{d\delta}{[(\delta - \delta_1)\sin\delta_{f3} + \cos\delta - \cos\delta_1]^{\frac{1}{2}}} \qquad (10.6.9)$$

where

$$\sin\delta_{f3} = \frac{\sin\delta_1}{X_1/X_3} = \frac{\sin\delta_1}{M_3}. \qquad (10.6.10)$$

It is important to note that for the case $P_1 > P_{m3}$, there is no actual angle δ_{f3}. Furthermore, the quantity $\sin\delta_1/M_3$ may be larger than 1. It is, therefore, more appropriate to write the above integral

$$\sqrt{2B}t_s = \int_{\delta_1}^{\delta_s} \frac{d\delta}{\left[(\delta - \delta_1)\dfrac{\sin\delta_1}{M_3} + \cos\delta - \cos\delta_1\right]^{\frac{1}{2}}}. \qquad (10.6.11)$$

10.7. *Transient Instability Due to Nonsymmetrical Faults.* Sustained nonsymmetrical faults were discussed more or less in detail in Arts. 8.17–8.19 inclusive. It was seen that nonsymmetrical faults may be replaced in the diagrammatic representation of the faulted system by a fictitious symmetrical wye-connected circuit. The reactance per branch of this fictitious wye circuit depends upon the type of fault. Thus, for an *L–L–L* fault or *L–L–L–G* fault it is zero. The schematic representation of a two-line system before the occurrence of a fault is as shown in Fig. 10–4, and the system reactance between termini is given by (10.5.1). With an *L–L–L–G* fault at the high side of the transformer at *F* in line *b*, the system is shown schematically by Figs. 10–6, 10–6a, and 10–6b. The reactance of the faulted system by diagram, Fig. 10–6b, is given by equation (10.5.10).

An *L–L* fault is replaced, for purposes of calculation, with a fictitious wye circuit whose reactance per branch is the negative sequence reactance X_n of the system as viewed from the point of fault. For the same two-line system as discussed above, but with *L–L* fault at the same place, the sche-

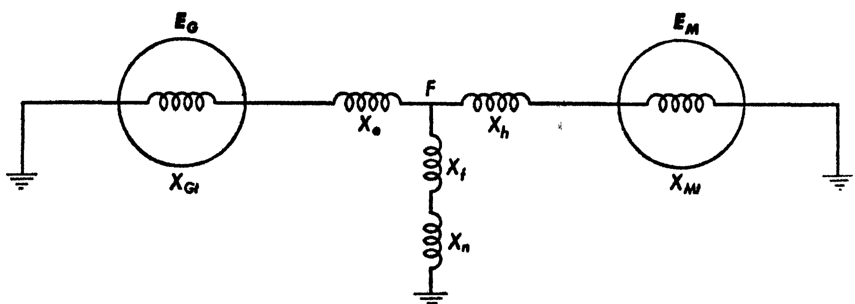


Figure 10–8

matic diagram is as shown in Fig. 10–8. The reactance between the termini of the faulted system is

$$(X_3)_{LL} = X_k + X_m + \frac{X_k X_m}{X_f + X_n} \tag{10.7.1}$$

where $X_k = X_{Gt} + X_e$ and $X_m = X_h + X_{Mt}$.

An *L–L–G* fault is replaced for purposes of calculation with a fictitious wye circuit whose reactance per branch consists of the negative and zero sequence reactances of the system as viewed from the fault, and as if connected in parallel. For the two-line system considered above and the fault at the same place, the reactance diagram per phase is shown in Fig. 10–9.

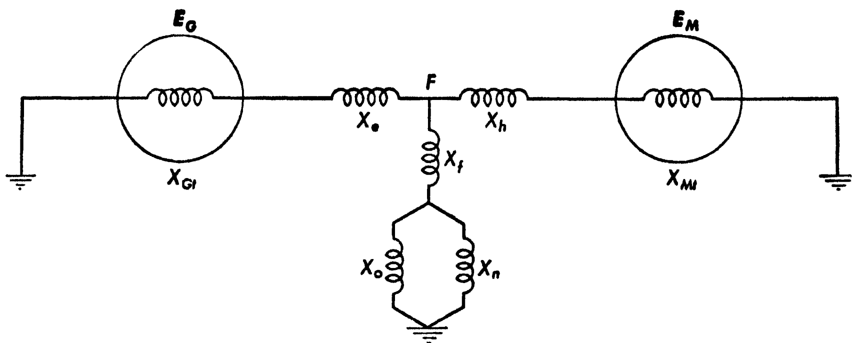


Figure 10 9

The reactance between the termini of the faulted system is

$$(X_3)_{LLG} = X_k + X_m + \frac{X_k X_m}{X_f + \dfrac{X_o X_n}{X_o + X_n}} \tag{10.7.2}$$

where X_k and X_m have the values stated above.

In the case of an *L–G* fault, the fault is replaced with a fictitious wye circuit whose reactance per branch consists of the negative and zero sequence reactances of the system as viewed from the fault and as if connected in series. For the two-line system considered and with the fault at the same place, the reactance diagram per phase is shown in Fig. 10–10. The reactance between the termini of the faulted system is

$$(X_3)_{LG} = X_k + X_m + \frac{X_k X_m}{X_f + X_o + X_n}. \tag{10.7.3}$$

From what has been said above, it follows that with the single exception that the system reactances (X_3) under the faulted conditions are different for the various faults, the methods of determining whether a fault causes a transient disturbance or not, and the calculation of the limiting time in which to remove the fault, or to switch out a faulted line are, respectively, the same as those discussed in the preceding articles of this chapter.

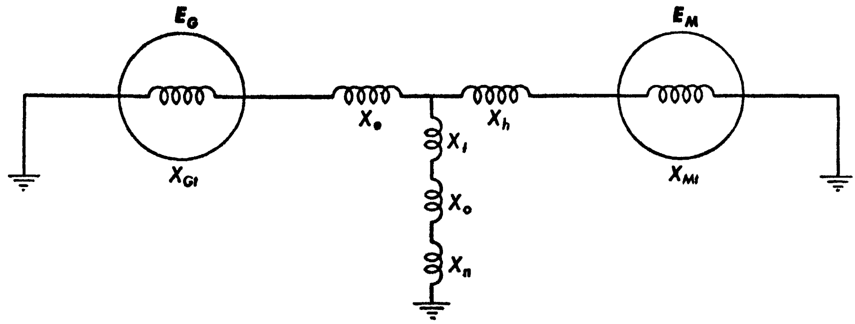


Figure 10–10

10.8. *General Assumptions in System Stability Studies.* Before applying the method developed in the preceding articles of this chapter to the study of a specific system under faulted condition, it is advisable to restate the assumptions basic to the formulation of the method, and to state a few others which are essential to the solution of the stability problem under consideration.

1. The resistances and capacitances of the component parts of the system impedances are neglected.

2. All machines in parallel connection which act jointly as generators are replaced by a single equivalent generator. Similarly, all machines in parallel connection which act jointly as motors are also replaced by a single equivalent motor. The system is thus reduced to a two-machine system.

3. The equivalent generator is assumed to supply an infinite bus through a nondissipative line. This presupposes that the combined inertia of all machines acting as generators, and of all machines functioning as motors, is concentrated in the equivalent single generator.

4. The assumption made in (3) implies also that all changes in the displacement angle (power angle) due to variations in load, line switching, or faults are attributed to the single equivalent generator. The acceleration in the equivalent generator rotor is treated as an equivalent deceleration of the motor rotor, or vice versa.

5. Local generators assisting the line to supply local loads are treated as motors.

6. The machines nearest the fault will be assumed as the generators of the system.

7. High-speed voltage regulation for all synchronous machinery is assumed, so that the excitation emf's of generators and motors are thought of as remaining constant during any disturbance created by an increase in load or the occurrence of a fault.

8. The damping of field windings and governor action in prime movers is neglected.

9. In the absence of accurate information, all local loads are assumed as consisting of 25 per cent dissipative load, 25 per cent in synchronous motors at 100 per cent power factor, and 50 per cent in induction motors at 90 per cent power factor.

10. All motors are assumed operating at 75 per cent of their respective rated capacities.

10.9 *Illustrative System Stability Study.*

(A) System Data. The system whose stability is investigated in the following pages is shown diagrammatically in Fig. 10–11, and consists of the following:

1. Six hydro generators. Each rated 13.8 kv; 21,500 kw at 90 per cent power factor, positive X_p, negative X_n and zero-sequence reactances X_o:

   $X_p =$ 30 per cent at own kva base.
   $X_n =$ 44.4 per cent at own kva base.
   X_o neglected.
   Inertia constant 3.27 seconds at own kw base.

2. 4 transformer banks. 13.8/132 kv; connected delta-wye and grounded; each transformer rated 20,000 kva.

   $X =$ 15 per cent at own kva base.

3. Twin lines. Each line 121 miles long of #0000 copper, 16.5 ft effective equilateral spacing.

   $X =$ 0.848 ohms per conductor mile.

4. Two transformer banks, 132/6.6 kv, 20,000 kva per bank, connected wye-delta and grounded.

   $X =$ 8 per cent at own kva base.

5. Two synchronous condensers, each rated 25,000 kva at 6.6 kv.

   $X_p =$ 40 per cent at own kva base.
   $X_n =$ 28 per cent at own kva base.
   Inertia constant = 1.5 seconds at own kw base.

6. Local load of 33,350 kva:

   25 per cent dissipative,
   25 per cent synchronous motor load at 100 per cent power factor,
   50 per cent induction motor load at 90 per cent power factor.

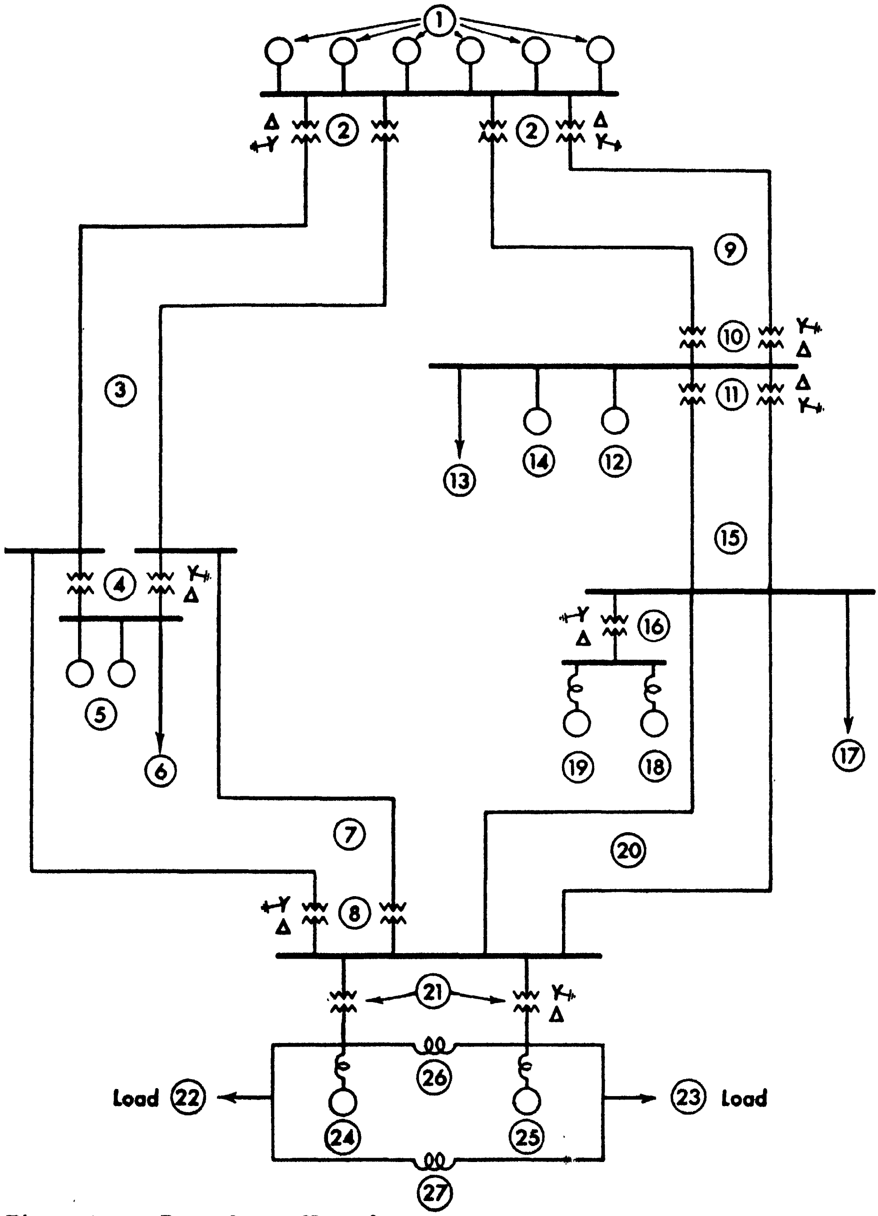


Figure 10–11. Power System Network

Synchronous motor load:

$X_p =$ 30 per cent at own kva base.

$X_n =$ 22 per cent at own kva base.

Inertia constant 4.5 seconds at own kw base.

Induction motor load:

$X_p =$ 300 per cent at own kva base.

$X_n =$ 15 per cent at own kva base.

Inertia constant = 1 second at own kw base.

7. Two lines; 58 miles long of #0000 copper, 16.5 ft effective equilateral spacing, 132 kv.

   $X =$ 0.848 ohms per conductor mile.

8. Two transformer banks. 132/6.6 kv, connected wye-delta and grounded; 12,500 kva per transformer.

   $X =$ 7.41 per cent at own kva base.

9. Twin lines; 138 miles of #0000 copper, 21 ft effective spacing.

   $X =$ 0.874 ohms per conductor mile.

10. Two transformer banks, 132/13.8 kv, 20,000 kva per transformer.

    $X =$ 15 per cent at own kva base.

11. Two transformer banks; 13.8/6.6 kv, 37,500 kva per bank;

    $X =$ 8.3 per cent per transformer at own kva base.

12 and 14. Two frequency converter sets, each rated 20,000 kw at 70 per cent power factor and 13.8 kv.

$X_p =$ 29 per cent at own kva base.

$X_n =$ 21.2 per cent at own kva base.

Inertia constant = 4.87 seconds at own kw base.

13. Local load of 16,700 kva.

15. Cable line, 8 miles long of 750,000 circ mil copper, 6.6 kv.

    $X =$ 7.1 per cent at 100,000 kva base.

16. Transformer bank, 50,000 kva per bank at 6.6/13.8 kv.

    $X =$ 8.4 per cent at own kva base.

17. Local load of 20,000 kva.

18. Generator and reactor, 35,000 kw at 90 per cent power factor, 13.8 kv.

    $X_p =$ 15 per cent at own kva base.

    $X_n =$ 12.35 per cent at own kva base.

    $X_o =$ neglected.

    Inertia constant = 10.19 seconds at own kw base.

19. Generator and reactor, 20,000 kw at 90 per cent power factor, 13.8 kv.

    $X_p =$ 15 per cent at own kva base.

    $X_n =$ 12.35 per cent at own kva base.

    $X_o =$ neglected.

    Inertia constant = 5.81 seconds at 55,000 kw base.

20. Twin line; 4.8 miles.

$X_p = 8.28$ per cent per conductor mile at 100,000 kva base.

21. Two transformer banks, each 50,000 kva, 6.6/13.8 kv.

$X = 8.4$ per cent at own kva base.

22 and 23. Local load, each 32,200 kva.

24. Generator and reactor, 35,000 kw at 90 per cent power factor, 13.8 kv.

$X_p = 15$ per cent at own kva base.
$X_n = 12.35$ per cent at own kva base.
$X_o =$ neglected.
Inertia constant = 6.58 seconds at 85,000 kw base.

25. Generator and reactor, 50,000 kw at 90 per cent power factor, 13.8 kv.

$X_p = 20.3$ per cent at own kva base.
$X_n = 17.5$ per cent at own kva base.
Inertia constant = 9.42 seconds at 85,000 kw base.

26. Bus reactor, $X = 7$ per cent at 50,000 kva base.

27. Bus reactor, $X = 5.5$ per cent at 50,000 kva base.

(B) Calculation of System Reactance to a Common kva Base of 100,000 kva, at 132 kv.

1. Generators:

$$X_p = \frac{30 \times 10^5 \times 0.9}{21500} = 125.5 \text{ per cent.}$$

$$X_n = \frac{44.4 \times 10^5 \times 0.9}{21500} = 185 \text{ per cent.}$$

$X_o =$ neglected.

2. Transformers:

$$X_p = X_n = X_o = \frac{15 \times 10^5}{20000} = 75 \text{ per cent.}$$

3. Lines:

$X_t = 0.848 \times 121 = 102.8$ ohms per conductor.

By (8.15.1),

$$X_p = X_n = \frac{102.8 \times 10^5}{132^2 \times 10^3} = 58.9 \text{ per cent.}$$

By § 8.12, following equation (8.12.4),

$X_o = 3.5X_p = 3.5 \times 58.9 = 206$ per cent.

4. Transformers:

$$X_p = X_n = X_o = \frac{8 \times 10^5}{20000/3} = 120 \text{ per cent.}$$

5. Synchronous condensers:

$$X_p = \frac{40 \times 10^5}{25000} = 160 \text{ per cent.}$$

$$X_n = \frac{28 \times 10^5}{25000} = 112 \text{ per cent.}$$

6. local load of 33,350 kva.

$$\frac{25 \times 33350}{100} = 8340 \text{ kva dissipative load.}$$

$$\frac{25 \times 33350}{100} = 8340 \text{ kva synchronous motors.}$$

$$\frac{50 \times 33350}{100} = 16680 \text{ kva induction motors.}$$

For the dissipative load

$$X_p = X_n = X_o = 0.$$

Synchronous motor capacity $\frac{8340}{0.75} = 11{,}100$ kva.

$$X_p = \frac{30 \times 10^5}{11100} = 270 \text{ per cent.}$$

$$X_n = \frac{22 \times 10^5}{11100} = 198 \text{ per cent.}$$

Induction motor capacity $\frac{16680}{0.75} = 22{,}200$ kva.

$$X_p = \frac{300 \times 10^5}{22200} = 1350 \text{ per cent.}$$

$$X_n = \frac{15 \times 10^5}{22200} = 67.5 \text{ per cent.}$$

7. Lines:

$$X_t = 58 \times 0.848 = 49.2 \text{ ohms/conductor.}$$

$$X_p = X_n = \frac{49.2 \times 10^5}{132^2 \times 10^3} = 28.3 \text{ per cent.}$$

$$X_o = 3.5 \times X_p = 99 \text{ per cent.}$$

8. Transformer:

$$X_p = X_n = X_o = \frac{7.41 \times 10^5}{12500} = 59.2 \text{ per cent.}$$

9. Line:

$$X_t = 138 \times 0.874 = 120.7 \text{ ohms/conductor.}$$

$$X_p = X_n = \frac{120.7 \times 10^5}{132^2 \times 10^3} = 69.3 \text{ per cent.}$$

$$X_o = 3.5X_p = 242.1 \text{ per cent.}$$

10. Transformers:

$$X_p = X_n = X_o = \frac{15 \times 10^5}{20000} = 75 \text{ per cent.}$$

11. Transformer:

$$X_p = X_n = X_o = \frac{8.3 \times 10^5}{37500/3} = 66.4 \text{ per cent.}$$

12 and 14. Frequency converters:

$$\text{Rated capacity } \frac{20000}{0.70} = 28{,}600 \text{ kva.}$$

$$X_p = \frac{29 \times 10^5}{28600} = 101.5 \text{ per cent.}$$

$$X_n = \frac{21.2 \times 10^5}{28600} = 74.2 \text{ per cent.}$$

13. Local load of 16,700 kva:

25 per cent or 4175 kva, resistive,
25 per cent or 4175 kva in synchronous motors,
50 per cent or 8350 kva, induction motor load.

$$\text{Synchronous motor capacity } \frac{4175}{0.75} = 5560 \text{ kva.}$$

$$X_p = \frac{30 \times 10^5}{5560} = 540 \text{ per cent.}$$

$$X_n = \frac{22 \times 10^5}{5560} = 396 \text{ per cent.}$$

$$\text{Induction motor capacity } \frac{8350}{0.75} = 11{,}120 \text{ kva:}$$

$$X_p = \frac{300 \times 10^5}{11120} = 2695 \text{ per cent.}$$

$$X_n = \frac{15 \times 10^5}{11120} = 134.7 \text{ per cent.}$$

15. Cable line:

$X_p = X_n = 7.1$ per cent.
$X_o = 3.5X_p = 24.85$ per cent.

16. Transformer:

$$X_p = X_n = X_o = \frac{8.4 \times 10^5}{50000/3} = 50.4 \text{ per cent.}$$

17. Local load of 20,000 kva:

25 per cent or 5000 kva resistive load,
25 per cent or 5000 kva in synchronous motors,
50 per cent or 10,000 kva in induction motors.

$$\text{Synchronous motor capacity, } \frac{5000}{0.75} = 6670 \text{ kva:}$$

$$X_p = \frac{30 \times 10^5}{6670} = 450 \text{ per cent.}$$

$$X_n = \frac{22 \times 10^5}{6670} = 330 \text{ per cent.}$$

Induction motor capacity $\frac{10000}{0.75} = 13{,}333$ kva:

$$X_p = \frac{300 \times 10^5}{13333} = 2250 \text{ per cent.}$$

$$X_n = \frac{15 \times 10^5}{13333} = 112.5 \text{ per cent.}$$

18. Generator and reactor:

$$X_p = \frac{15 \times 10^5}{35000/0.9} = 38.6 \text{ per cent.}$$

$$X_n = \frac{12.35 \times 10^5}{35000/0.9} = 31.7 \text{ per cent.}$$

19. Generator and reactor:

$$X_p = \frac{15 \times 10^5}{20000/0.9} = 67.5 \text{ per cent.}$$

$$X_n = \frac{12.35 \times 10^5}{20000/0.9} = 55.5 \text{ per cent.}$$

20. Line:

$$X_p = X_n = 8.28 \text{ per cent at 100,000 kva base.}$$

$$X_o = 3.5 \times 8.28 = 28.98 \text{ per cent.}$$

21. Transformer:

$$X_p = X_n = X_o = \frac{8.4 \times 10^5}{50000/3} = 50.4 \text{ per cent.}$$

22 and 23. Local loads of 32,200 kva each:

25 per cent, i.e., 8050 kva is resistive,
25 per cent, i.e., 8050 kva is in synchronous motors,
50 per cent, i.e., 16,100 kva is induction motors.

Synchronous motor capacity, $\frac{8050}{0.75} = 10{,}750$ kva:

$$X_p = \frac{30 \times 10^5}{10750} = 279 \text{ per cent.}$$

$$X_n = \frac{22 \times 10^5}{10750} = 204.5 \text{ per cent.}$$

Induction motor capacity $\frac{16100}{0.75} = 21{,}500$ kva:

$$X_p = \frac{300 \times 10^5}{21500} = 1395 \text{ per cent.}$$

$$X_n = \frac{15 \times 10^5}{21500} = 69.8 \text{ per cent.}$$

24. Generator and reactor:

$$X_p = \frac{15 \times 10^5}{35000/0.9} = 38.6 \text{ per cent.}$$

$$X_n = \frac{12.35 \times 10^5}{35000/0.9} = 31.7 \text{ per cent.}$$

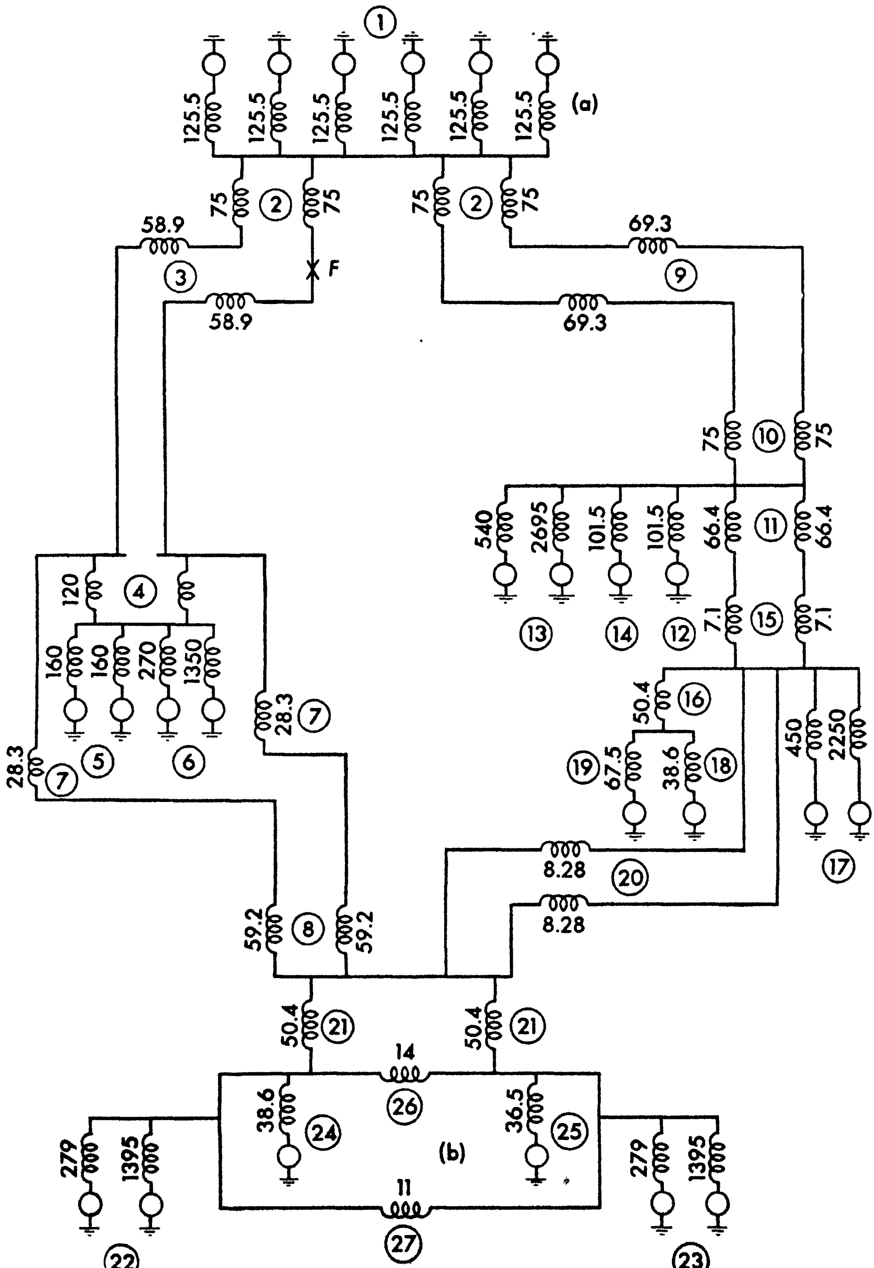


Figure 10–12. Positive Sequence System

25. Generator and reactor:

$$X_p = \frac{20.3 \times 10^5}{50000/0.9} = 36.5 \text{ per cent.}$$

$$X_n = \frac{17.5 \times 10^5}{50000/0.9} = 31.5 \text{ per cent}$$

26. Bus reactor:

$$X_p = X_n = X_o = \frac{7 \times 10^5}{50000} = 14 \text{ per cent.}$$

27. Bus reactor:

$$X_p = X_n = X_o = \frac{5.5 \times 10^5}{50000} = 11 \text{ per cent.}$$

Figure 10–12 gives the diagram of the system with the calculated values of the positive sequence reactances to 100,000 kva base indicated.

(*C*) *System Reactance under Normal Operation.* The reactance diagram of the system under normal operation is the same as the positive phase sequence reactance diagram. In accordance with the calculated values to a common kva base, the diagram per phase is given in Fig. 10–12. A faulted condition is assumed to occur on the high side of a transformer at point marked *F*. A network analyzer or calculating board would be very convenient to reduce the diagram to a single reactance between station (*a*) and station (*b*). Because of its nearness to the assumed location of the fault, station (*a*) will be taken as the generating station. All the other rotating equipment will be lumped into a single equivalent motor. By using the rather laborious process of delta-wye conversion, the system diagram was reduced to that shown in Fig. 10–13.

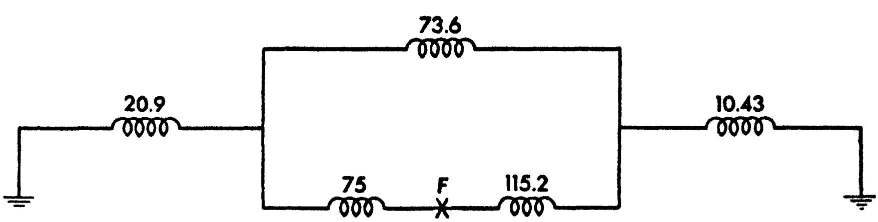


Figure 10–13

The total equivalent reactance of the system under normal conditions, therefore, is

$$X_1 = 20.9 + \frac{73.6 \times (75 + 115.2)}{73.6 + 75 + 115.2} + 10.43$$
$$= 84.5 \text{ per cent.}$$

(*D*) ***Positive Sequence Reactance Diagram of System per Phase with Fault On.*** The diagram in Fig. 10–13a is identical to that shown in Fig. 10–13, with the single exception that the point of the fault is grounded as shown.

Fig. 10–13a represents the positive sequence impedance diagram. Converting the delta to a wye gives the diagram in Fig. 10–13b for the positive sequence reactance of the faulted system.

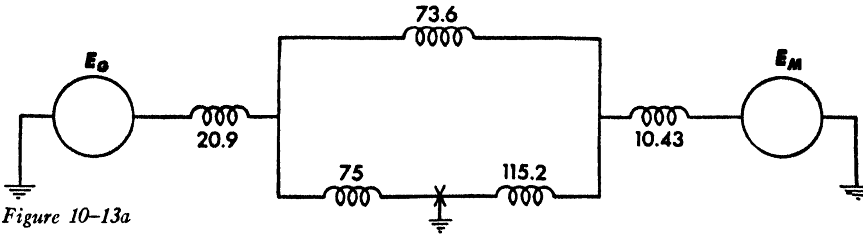


Figure 10–13a

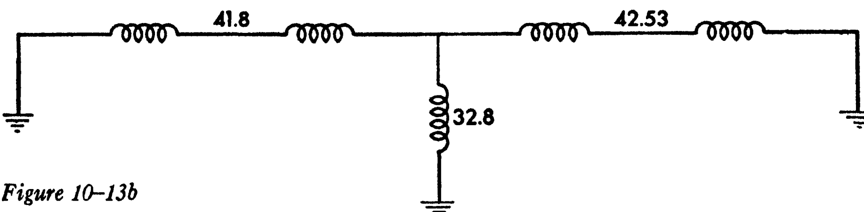


Figure 10–13b

(*E*) *Negative Sequence Reactance Diagram of the System per Phase as Viewed from the Fault.* This is structurally identical with the one shown in Fig. 10–12, but with negative sequence reactances to the common kva base as calculated in (*B*). Reduced through delta-wye conversions, it is shown in Fig. 10–14. Converting the delta to a wye circuit, the circuit is changed to that shown in Fig. 10–14a. In accordance with this reactance diagram, the negative sequence reactance of the system is

$$X_n = 31.38 + \frac{53.86 \times 40.54}{53.86 + 40.54}$$
$$= 54.4 \text{ per cent.}$$

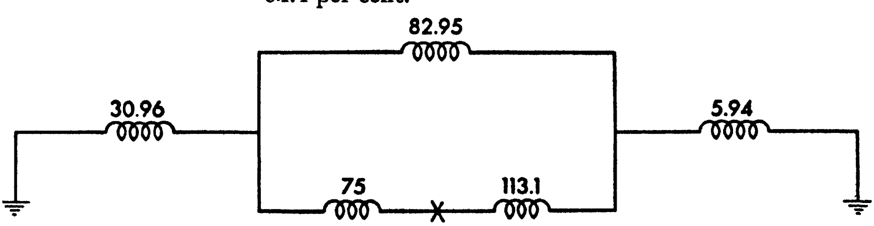


Figure 10–14

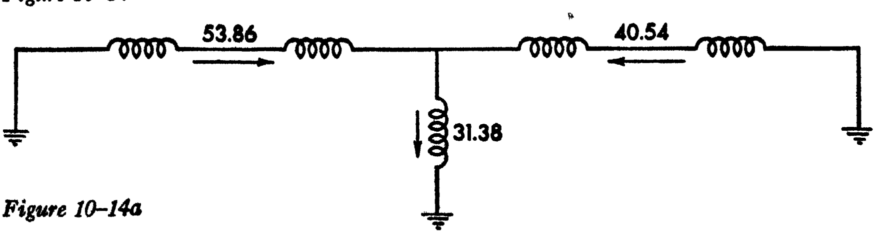


Figure 10–14a

(*F*) *Zero Sequence Reactance Diagram of the System per Phase as Viewed from Fault*, Since the neutrals of the transformers are grounded, the zero sequence reactance diagram of the system is, in accordance with the discussion in § 8.4, as indicated in Fig. 10–15. It is seen that the only portion of the system to be considered for the zero sequence reactance is the faulted line and the transformers connected to it. Using the reactance values as calculated in part (*B*), the zero sequence reactance diagram becomes as indicated in Fig. 10–15a. Referring to this diagram, it is seen that the zero sequence reactance to the right of the fault at *F* is

$$X_r = \frac{120(99 + 59.2)}{120 + 99 + 59.2} + 206 = 274.4 \text{ per cent.}$$

The zero sequence reactance as viewed from the fault, therefore, is

$$X_o = \frac{75 \times 274.4}{75 + 274.4}$$
$$= 58.9 \text{ per cent.}$$

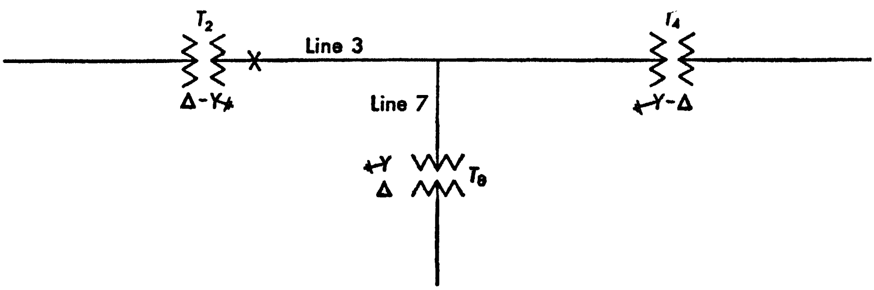


Figure 10–15

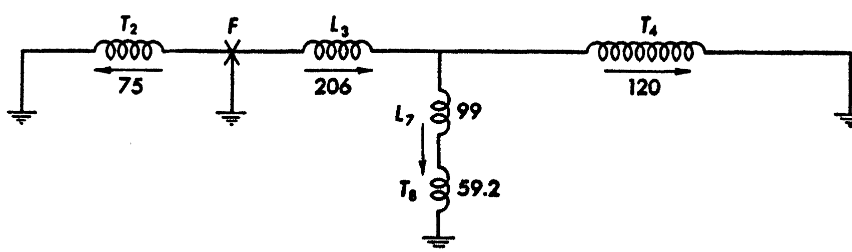


Figure 10–15a

(*G*) *Calculation of System Reactance under Faulted Condition*. By § 10.7, the reactance of a system subject to some kind of short circuit is obtained by replacing the fault at the point of its occurrence with a symmetrical wye circuit whose reactance per phase depends upon the type of fault.

(G_1) *L–L–L–G Fault*. The value of the reactance per phase which replaces at the point of the fault a three-phase short circuit is zero. Accordingly, the reactance diagram per phase is the positive sequence reactance diagram as shown in Fig. 10–13a.

By (10.5.10), the reactance of the system with the three-phase short circuit on is

$$X_{LLLG} = 41.8 + 42.53 + \frac{41.8 \times 42.53}{32.8}$$
$$= 138.6 \text{ per cent.}$$

(G_2) *L–L–G Fault.* The reactance per phase which replaces at the point of the fault a line–line to ground short circuit consists of the negative and zero sequence reactances of the system as viewed from the point of the fault, and as if they were connected in parallel. The system reactance diagram per phase with this particular fault on at the same point is, therefore,

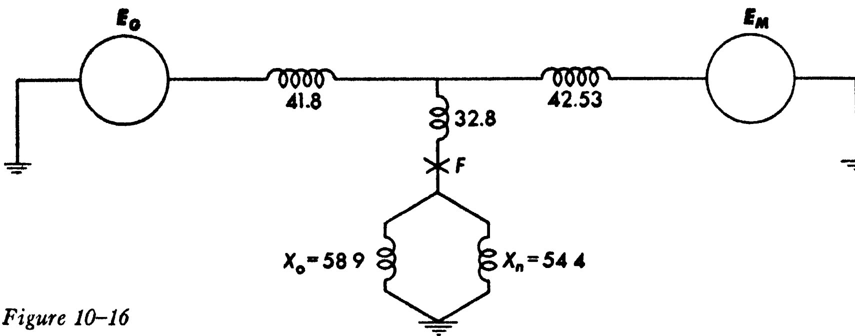


Figure 10–16

as shown in Fig. 10–16. The reactance of the system with the line–line to ground short circuit on, by (10.7.2), is

$$X_{LLG} = 41.8 + 42.53 + \frac{41.8 \times 42.53}{32.8 + \dfrac{58.9 \times 54.4}{58.9 + 54.4}}$$
$$= 113.8 \text{ per cent.}$$

(G_3) *L–L Fault.* The reactance per phase which replaces at the point of the fault a line to line short circuit consists, as previously shown, of the negative sequence reactance of the system as viewed from the point of the fault. The system reactance diagram per phase with this particular fault

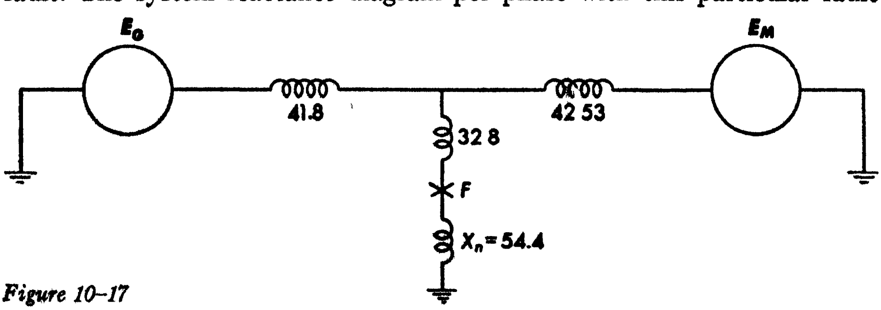


Figure 10–17

on at the same point is as shown in Fig. 10–17. The reactance of the system with the line to line short circuit is, therefore, by (10.7.1),

$$X_{LL} = 41.8 + 42.53 + \frac{41.8 \times 42.5}{32.8 + 54.4}$$
$$= 104.7 \text{ per cent.}$$

(G_4) *L–G Fault.* The reactance per phase which replaces at the point of the fault a line to ground short circuit consists of the negative and zero sequence reactances of the system as viewed from the fault, and as if they were connected in series. The reactance diagram of the system with a line to ground fault at the same point as the other faults considered is as shown in Fig. 10–18. The reactance of the system with this fault on, by (10.7.3), is

$$X_{LG} = 41.8 + 42.53 + \frac{41.8 \times 42.53}{32.8 + 58.9 + 54.4}$$
$$= 96.5 \text{ per cent.}$$

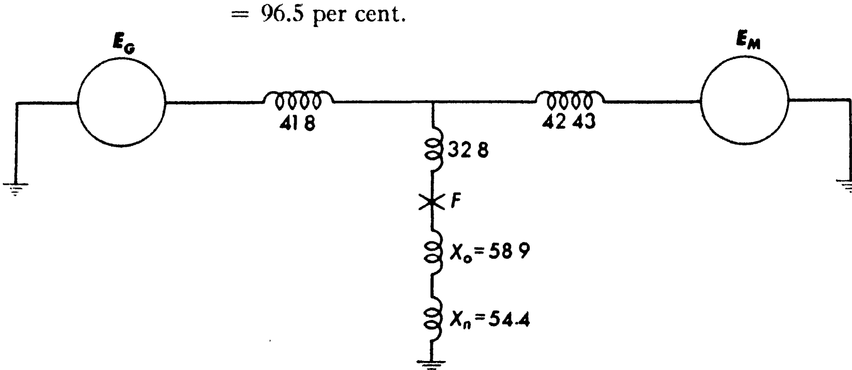


Figure 10–18

(G_5) *Reactance of System with Faulted Line Switched Out of Service.* Removing the reactance of the faulted line and all reactances in series with it in Fig. 10–12, and again using delta-wye conversions, the reactance diagram of system is as shown in Fig. 10–19. This gives

$$X_2 = 20.9 + \frac{194.6 \times 128.8}{194.6 + 128.8} + 6.55$$
$$= 105.1 \text{ per cent.}$$

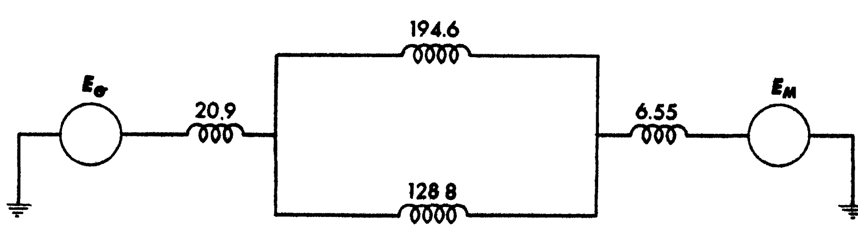


Figure 10–19

(*H*) *Calculation of the Displacement Angle Corresponding to Limiting Stable Loads.* It was shown in § 10.3, equation (10.3.3), that

$$\sin \delta_1 = \frac{X_1}{X_2} \sin \delta_s$$

is the value of the load in per unit of the corresponding maximum synchronizing power (P_1/P_{m2}) that can be transferred over the system with one line tripped out of service. It was similarly shown in § 10.5 that if X_3 is the reactance of the system under any particular fault, then

$$\sin \delta_1 = \frac{X_1}{X_3} \sin \delta_s,$$

gives the load in per unit (P_1/P_{m3}) of the corresponding maximum synchronizing power that the system can carry with the respective fault on.

To determine these limiting values of load, calculate X_1/X_3 for the various faults, and X_1/X_2 for the system with the faulted line tripped. Then since the ratio (X_1/X_2) corresponds to a definite $\sin \delta_1/\sin \delta_s$, obtain the corresponding value of $\sin \delta_1$ from curve, Fig. 9–12. Table VIII gives the values of $\sin \delta_1$ obtained in this manner for the system under consideration and under the conditions as stated.

TABLE VIII

System condition	X	X_1/X	$\sin \delta_1$ P_1/P_m	δ_1 degrees
normal	$84.5 = X_1$	1.000	1.000	90
L-L-L-G	138.6	0.609	0.530	32
L-L-G	113.8	0.742	0.685	43.2
L-L	104.7	0.807	0.755	49
L-G	96.7	0.876	0.850	58.2
faulted line tripped	$105.1 = X_2$	0.803	0.750	48.6

The angle δ_1 in this table is the displacement angle corresponding to the limiting stable load that the system could carry under the particular condition as indicated. The curve in Fig. 10–20 shows graphically the results obtained above. A three-phase short circuit at the particular point considered can be carried by the system without loss of stability for any load in per unit of the respective maximum within the range marked *A*. If the load is within the range marked *B*, and such a fault occurs on the system, stable operation will be lost unless the faulted line is tripped out. If the load on the system is within the range marked C, and a three-phase fault occurs at the point considered, it is conducive to loss of stability. If the faulted line is tripped, the load on the system is larger than the safe load for the system with one line only. Stable operation will be lost unless the load is reduced so that one line can carry it, or the fault removed within the limited specified time, if the two lines are to carry it.

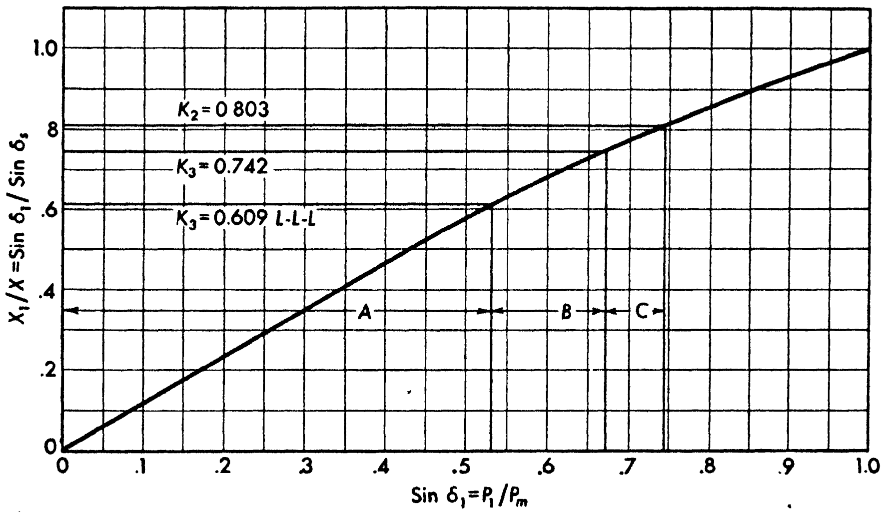


Figure 10–20

(*I*) *Calculation of Limiting Stable Loads and Maximum Synchronizing Powers under Fault Condition.* The dissipative power P in kw (or megawatts) corresponding to any displacement angle δ and for any power factor $\cos\theta$ is given by equation (9.11.3)

$$P = \frac{200\,(kva)_b \cos\theta}{X}\left[\frac{-\cos\theta}{\tan\delta} + \sqrt{1 + \left(\frac{\cos\theta}{\tan\delta}\right)^2}\right],$$

where for the particular case under consideration

$(kva)_b = 100{,}000$
$\cos\theta = 0.9$
$X =$ the reactance pertaining to condition of the system, and
$\delta =$ the displacement angle corresponding to the system condition.

Thus, for the power limit with the system under normal condition and for which $X = 84.5$ and $\delta = 90°$, the equation becomes

$$P_m = \frac{200 \times 10^5 \times 0.9}{84.5}\left[\frac{-0.9}{\tan 90°} + \sqrt{1 + \left(\frac{0.9}{\tan 90°}\right)^2}\right]$$
$$= 212900 \text{ kw}.$$

From the data on the system given in Part (*A*) the rated capacity of the system per phase at 90 per cent power factor is

$$P_r = \frac{6 \times 21000 + (2 \times 35000) + 50000 + 20000}{3}$$

$$= 88660 \text{ kw.}$$

Hence,

$$\frac{P_m}{P_r} = \frac{212900}{88660} = 2.45,$$

indicating that the power limit under normal operation is nearly 2.5 times the rated capacity of the system.

Table IX gives the calculated values of the limiting stable loads which the system can carry under the various stated conditions, the maximum synchronizing power for these stated conditions, and the loads which the system can carry in per unit of the rated capacity of the system at 90 per cent power factor. The displacement angle δ_1 used in these calculations is given in Table VIII.

TABLE IX

System condition	X	δ_1	P kw	P_m $P/\sin\delta_1$	P_m/P_r $P_m/88600$
normal	84.5	90	212900	212900	2.45
L-L-L-G	138.6	32	40800	77200	0.46
L-L-G	113.8	43.2	67200	100300	0.757
L-L	104.7	49	85500	113500	0.963
L-G	96.7	58.2	109000	128500	1.23
faulted line tripped	105.1	48.6	82600	110400	0.933

This table indicates that for a power factor of 90 per cent

1. The system can carry 2.45 times the rated capacity with no loss of synchronism.
2. The system is capable of carrying 23 per cent in excess of the rated capacity with a line to ground short at the point considered.
3. The system can carry 93.3 per cent of the rated capacity with the faulted line tripped.
4. It can carry 96.3 per cent of the rated capacity with a line to line short circuit at the point considered.
5. That with a three-phase short circuit at the point considered, the system will become unstable when the load exceeds 46 per cent of the rated capacity.

6. That with loads between 46 and 93.3 per cent of the rated capacity, the instability caused by the three-phase fault at the point considered may be removed by switching out the faulted line within a specified time.
7. That for loads larger than 93.3 per cent of rated capacity the instability caused by an *L–L–L–G* or an *L–L–G* fault at the point considered may be removed by removing the fault within a specified time.

(*J*) *Calculation of the Displacement Angle and of the Maximum Synchronizing Power for Any Load.* This may be accomplished by (9.10.12)

$$\tan \delta = \frac{200D \cos \theta}{10000 - D^2},$$

where, by (9.10.8), and for the particular case considered in which $X = 84.5$, $\cos \theta = 0.9$, and $(kva)_b = 100000$

$$D = \frac{84.5P}{2 \times 10^5 \times 0.9}.$$

This, substituted in the formula for δ, gives

$$\tan \delta = \frac{846 \times 10^{-4}P}{10000 - 22.1 \times 10^{-8}P^2}.$$

Table X gives the calculated values of the displacement angle, the maximum synchronizing power $P_m = P/\sin \delta$, and the load in per unit of the rated capacity for increasing assumed values of load P. The results are visualized in the curves, Fig. 10–21.

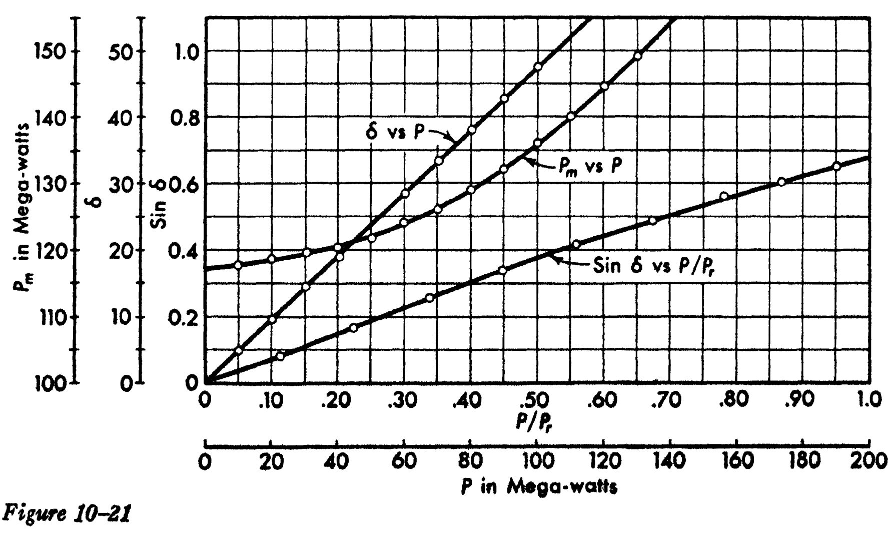


Figure 10–21

The curve $\sin \delta$ *vs.* P/P_r gives the calculated values of the load in per unit of the maximum synchronizing power as a function of the same load in per unit of rated capacity.

TABLE X

P megawatts	δ degrees	P_m megawatts	P/P_r $P/88.66$	$P/P_m = \sin \delta$
10	4.87	117.8	0.1127	0.085
20	9.69	118.8	0.2254	0.168
30	14.51	119.6	0.3380	0.254
40	19.35	120.5	0.4510	0.331
50	24.60	121.5	0.563	0.416
60	28.88	124.5	0.678	0.484
70	33.58	126.5	0.789	0.552
80	38.20	129.2	0.902	0.617
90	42.90	132.0	0.976	0.681
100	47.35	136.0	1.127	0.747
110	51.80	140.0	1.240	0.786
120	56.10	144.5	1.351	0.830
130	60.70	149.0	1.465	0.872

(*K*) *Calculation of the Safe Angle; Three-Phase to Ground Fault.* The calculation of the safe angle δ_s corresponding to the safe limiting time t_s prior to which the faulted line must be tripped out of service to avoid loss of synchronism, by (10.6.7), is

$$\cos \delta_s = \frac{\sin \delta_1(\pi - \delta_{f2} - \delta_1) - (M_2 \cos \delta_{f2} + M_3 \cos \delta_1)}{M_2 - M_3}.$$

For the particular case under consideration

$$M_2 = \frac{X_1}{X_2} = \frac{84.5}{105.1} = 0.803$$

$$M_3 = \frac{X_1}{X_{LLLG}} = \frac{84.5}{138.6} = 0.609$$

By (10.6.8),

$$\delta_{f2} = \sin^{-1}\left(\frac{\sin \delta_1}{X_1/X_2}\right) = \sin^{-1}\left(\frac{\sin \delta_1}{0.803}\right).$$

The values of $\sin \delta_1$ depend upon the load at the time the fault occurs. From Table VIII, or the corresponding curve, Fig. 10–20, it is found that stable operation is lost when the load is at some value between 0.53 and 0.75 of the respective maxima, i.e., between 77200 and 110400 kw as obtained in Table IX. To prevent loss of synchronism the faulted line must be tripped.

The curve δ_s *vs.* $\sin \delta_1$ in Fig. 10–22 is calculated from $\cos \delta_s$ above for values of $\sin \delta_1$ between 0.53 and 0.75. It gives the values of the safe angle δ_s as a function of $\sin \delta_1$, to be used in the calculation of the integral $\sqrt{2Bt_s}$.

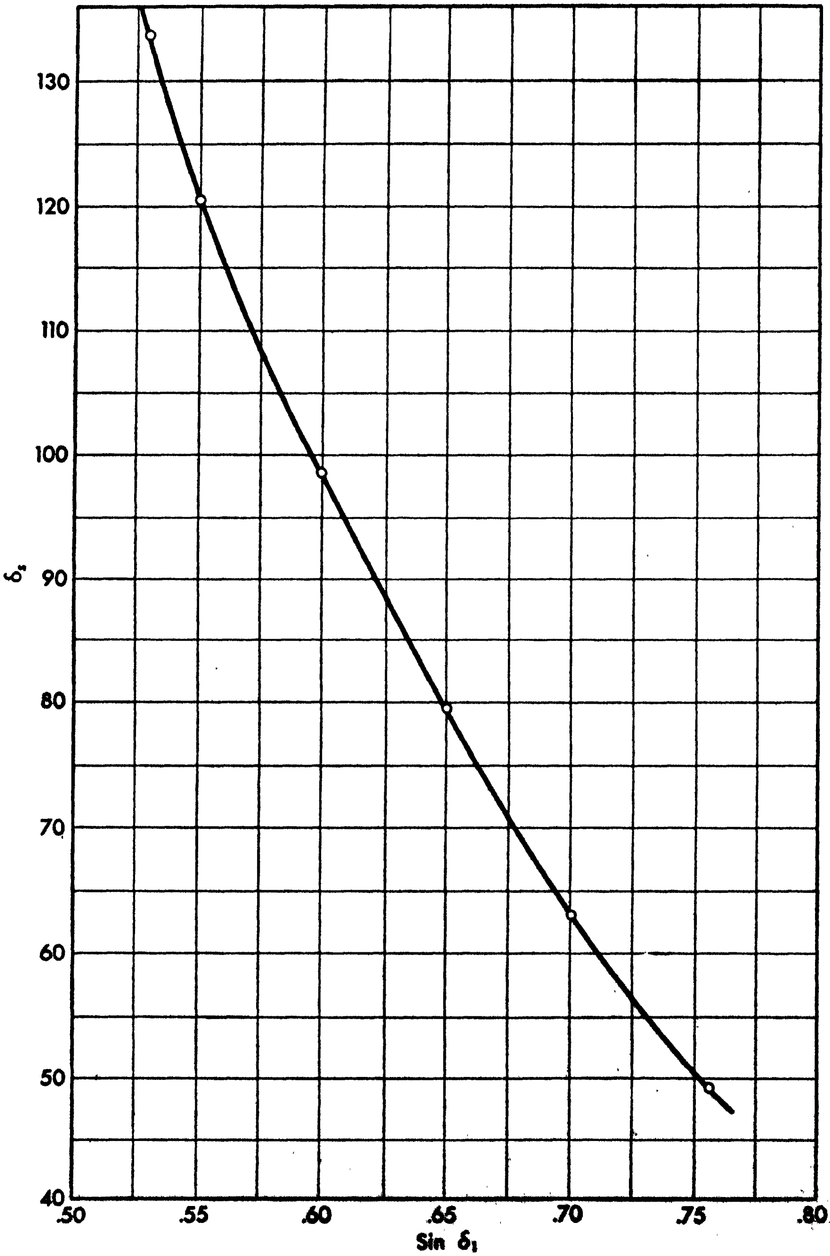


Figure 10–22

(*L*) *Calculation of the Integral* $\sqrt{2B}t_s$. The equation of this integral is given by (10.6.11)

$$\sqrt{2B}t_s = \int_{\delta_1}^{\delta_s} \frac{d\delta}{\left[(\delta - \delta_1)\frac{\sin \delta_1}{M_3} + \cos \delta - \cos \delta_1\right]^{\frac{1}{2}}}.$$

It may be integrated for given values of $\sin \delta_1$, δ_s, and fault as defined by $M_3 = X_1/X_3$, either by the point by point method or by the use of a planimeter for the measurement of the area under the curve $F(\delta)$ *vs.* δ as discussed in § 9.14. The value of $F(\delta)$ is

$$F(\delta) = \frac{1}{\left[(\delta - \delta_1)\frac{\sin \delta_1}{M_3} + \cos \delta - \cos \delta_1\right]^{\frac{1}{2}}}.$$

The value of the limiting interval t_s, to trip the faulted line, will be calculated for a three-phase short circuit at point F indicated in Fig. 10–12, for several values of load on the system, which jointly with the fault would cause loss of stability.

(L_1) $P_1 = 55$ *Per Cent of Respective Synchronizing Power* P_{m1}. *Three-Phase Fault at Point F on System Diagram, Fig. 10–12.* By curve $\sin \delta$ *vs.* P/P_r, Fig. 10–21, it is found that the assumed load P_1 is 77 per cent of the rated system capacity. Since $\sin \delta_1 = 0.55$, the corresponding angle is $\delta_1 = 33.4°$ or 0.583 radians; $\cos \delta_1 = 0.835$. The limiting safe angle corresponding to the given $\delta_1 = 33.4°$ is obtained from curve, Fig. 10–22, and is $\delta_s = 120.3° = 2.10$ radians; $M_3 = X_1/X_3 = 0.609$. The actual load corresponding to its given value in per cent of the maximum is obtained from curve δ *vs.* P, Fig. 10–21, and is 70000 kw. The maximum synchronizing power P_{m1}, corresponding to this load, is obtained from curve P_m *vs.* P, Fig. 10–21, and is 121000 kw.

Substituting the values of δ_1, $\sin \delta_1$, $\cos \delta_1$, and M_3 in the equation for $F(\delta)$ gives

$$F(\delta) = \frac{1}{\left[(\delta - 0.583)\frac{0.55}{0.609} + \cos \delta - 0.835\right]^{\frac{1}{2}}} = \frac{1}{\sqrt{N}}.$$

The calculation of this function for various values of δ is given in Table XI. The curve $F(\delta)$ *vs.* δ between the limits of $\delta = \delta_1 = 0.583$ and $\delta = \delta_s = 2.10$ radians is shown in Fig. 10–23. The area under the curve between the limits of 0.59 radians and 2.1 radians was obtained by means of a planimeter taking 10 units of $\delta.F(\delta)$ equal to one sq cm as read on the planimeter. Planimeter reading 62.9; area under the curve is, therefore, 6.29 $\delta.F(\delta)$ units.

TABLE XI

δ radians	$\delta - 0.583$ radians	$\times \frac{0.55}{0.609}$	δ degrees	$\cos \delta$	N	$F(\delta) = 1/\sqrt{N}$
0.583	0.00	0.0	33.4	0.835	0.0	
0.590	0.007	0.00632	33.8	0.831	0.00232	20.7
0.650	0.067	0.0605	37.25	0.796	0.0215	6.8
0.750	0.167	0.1510	43.0	0.731	0.0470	4.61
1.00	0.417	0.3770	57.3	0.540	0.0820	3.5
1.250	0.667	0.6030	71.6	0.316	0.0840	3.45
1.500	0.917	0.8280	86.0	0.070	0.0630	3.98
1.750	1.167	1.057	100.3	−0.179	0.043	4.80
2.100	1.517	1.373	372.25	−0.505	0.033	5.50

The area under the curve between the limits of 0.583 and 0.59, i.e., in the vicinity of δ_1, by equation (9.14.13), is

$$a_1 = 2\left(\frac{\phi}{\sin \delta_f - \sin \delta_1}\right)^{\frac{1}{2}}.$$

For the particular case under consideration

$$\phi = 0.59 - 0.583 = 0.007$$

$$\sin \delta_f = \frac{\sin \delta_1}{M_3} = \frac{0.55}{0.609}.$$

Hence,

$$a_1 = 2\left(\frac{0.007}{0.356}\right)^{\frac{1}{2}}$$
$$= 0.28.$$

The total area under the curve in $\delta.F(\delta)$ units, therefore, is

$$\sqrt{2Bt_s} = 0.28 + 6.29$$
$$= 6.57$$

(L_2) *Calculation of the Value of B in the Quantity* $\sqrt{2Bt_s}$. The value of this quantity, which enters into the calculation of the time limit for switching the faulted line out of service or for clearing a fault, is given by equation (9.14.2),

$$B = \frac{2\pi_f}{M_e}\frac{P_{m1}}{P_{rb}},$$

where

$$P_{m1} = \frac{P_1}{\sin \delta_1}$$

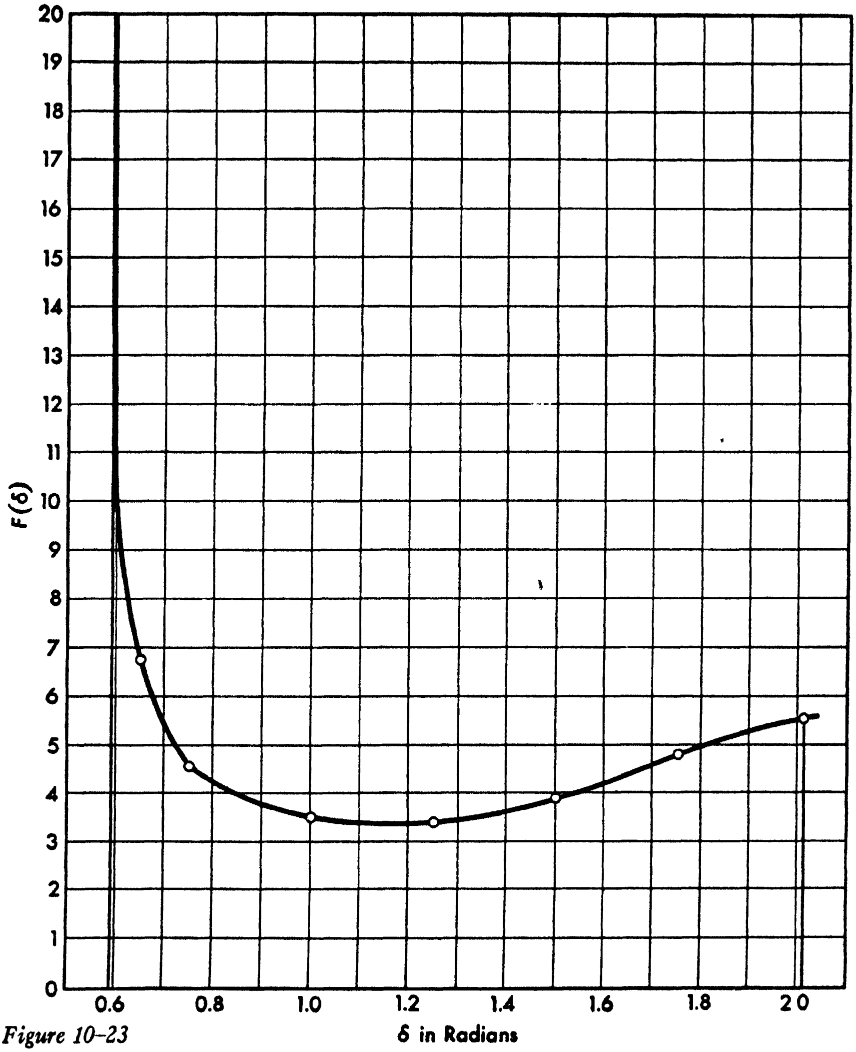


Figure 10–23

is the maximum synchronizing power in kw corresponding to the given load. For the particular problem under consideration P_1, as obtained from curve δ *vs.* P, Fig. 10–21, ($\delta_1 = 33.4°$), is 70000 kw. The corresponding maximum synchronizing power P_{m1} may be obtained from curve P_m *vs.* P, Fig. 10–21. Its value is 121000 kw.

The value of P_{rb} in the formula for B is the base rated power. For the case on hand the chosen kva base is 100000. The power factor of the system is assumed 90 per cent. The base rated power, therefore, is

$$P_{rb} = 100000 \times 0.9$$
$$= 90000 \text{ kw}$$

The formula for B, above, becomes accordingly

$$B = \frac{377 \times 121000}{90000 M_e}$$
$$= \frac{506.85}{M_e},$$

where M_e is the equivalent inertia constant of the system.

(L_3) *Calculation of the Inertia Constant of the System for the Assumed Location of the Fault.* From the given data in Part (A) the value of the inertia constant of each generator at station (a) is 3.27 seconds at own kw base of 21500 kw. For the six generators at 90000 kw base it is

1.
$$M_G = \frac{6 \times 3.27 \times 21500}{90000}$$
$$= 4.69 \text{ seconds.}$$

5. Two synchronous condensers
$$M_5 = \frac{2 \times 1.5 \times 25000}{90000}$$
$$= 0.834 \text{ seconds.}$$

6. Synchronous motor load of 8334 kva at unity power factor
$$M_{G1} = \frac{4.5 \times 8334}{90000}$$
$$= 0.4165 \text{ seconds.}$$

Induction motor load of 16668 kva at 90 per cent power factor
$$M_{G2} = \frac{1 \times 16668 \times 0.9}{90000}$$
$$= 0.167 \text{ seconds.}$$

12 and 14. Two frequency converter sets
$$M_{12} = \frac{4.87 \times 20000 \times 0.70 \times 2}{90000}$$
$$= 1.515 \text{ seconds.}$$

13. Synchronous motor load of 4175 kva at unity power factor
$$M_{131} = \frac{4.5 \times 4175}{90000}$$
$$= 0.209 \text{ seconds.}$$

Induction motor load of 8350 kva at 90 per cent power factor
$$M_{132} = \frac{1 \times 8350 \times 0.9}{90000}$$
$$= 0.0835 \text{ seconds.}$$

17. Synchronous motor load of 5000 kva at unity power factor

$$M_{171} = \frac{4.5 \times 5000}{90000}$$
$$= 0.272 \text{ seconds.}$$

Induction motor load of 10000 kva at 90 per cent power factor

$$M_{172} = \frac{1 \times 10000 \times 0.9}{90000}$$
$$= 0.1 \text{ second.}$$

18. Generator rated 35000 kw

$$M_{18} = \frac{10.19 \times 35000}{90000}$$
$$= 3.55 \text{ seconds.}$$

22 and 23. Synchronous motor load of 8050 kva at unity power factor

$$M_{22} = \frac{4.5 \times 8050}{90000}$$
$$= 0.403 \text{ seconds.}$$

Induction motor load of 16100 kva at 90 per cent power factor

$$M_{23} = \frac{1 \times 16100 \times 0.9}{90000}$$
$$= 0.161 \text{ seconds.}$$

24. Generator rated 35000 kw having an inertia constant of 6.58 seconds at 85000 kw base

$$M_{24} = \frac{6.58 \times 85000}{90000}$$
$$= 6.22 \text{ seconds.}$$

25. Generator rated 50000 kw, having an inertia constant of 9.42 seconds at 85000 kw base

$$M_{25} = \frac{9.42 \times 85000}{90000}$$
$$= 8.9 \text{ seconds.}$$

The equivalent inertia constant of all machines acting jointly as an equivalent motor is the sum of the inertia constants of all machines except the six generators at station (a):

$$M_M = 27.5425 \text{ seconds.}$$

The equivalent inertia constant of the equivalent generator and equivalent motor, with the motor taken as an infinite bus, by (9.9.5), is

$$M_e = \frac{4.69 \times 27.5425}{4.69 + 27.5425}$$
$$= 4.0 \text{ seconds.}$$

(L_4) *Calculation of the Limiting Switching Time* t_s. In Part (L_1), there was obtained

$$t_s = \frac{6.57}{\sqrt{2B}}.$$

From Part (L_2),

$$B = \frac{506.85}{M_e}.$$

From Part (L_3),

$$M_e = 4.0.$$

Substituting in the expression for B above, gives

$$B = \frac{506.85}{4.0} = 126.71.$$

Hence, the value of the limiting switching interval is

$$t_s = \frac{6.57}{\sqrt{2 \times 126.71}} = 0.412 \text{ seconds} = 24.72 \text{ cycles.}$$

(M) *Calculation of the Limiting Time* t_s *for Tripping of Faulted Line; Load on System* $P_1 = 60$ *Per Cent of Maximum Synchronizing Power*. From curve sin δ *vs.* P/P_r, Fig. 10–21, it is found that $P_1 = 87$ per cent of rated system capacity; $\sin \delta_1 = 0.60$; $\delta_1 = 36.9° = 0.644$ radians; $\cos \delta_1 = 0.8$; $M_3 = 0.609$. The limiting safe angle is obtained from Fig. 10–22 and is $\delta_s = 98.5° = 1.713$ radians. The actual load, corresponding to its given value in per cent of the maximum is from curve δ *vs.* P, Fig. 10–21, $P_1 = 77000$ kw. The maximum synchronizing power is obtained from curve P_m *vs.* P, Fig. 10–21, and is 128500 kw. Substituting the values of δ_1, $\sin \delta_1$, $\cos \delta_1$, and M_3 in the equation for $F(\delta)$ gives

$$F(\delta) = \frac{1}{[(\delta - 0.644)0.988 + \cos \delta - 0.8]^{\frac{1}{2}}}.$$

The curve $F(\delta)$ *vs.* δ is plotted in Fig. 10–24 between the limits $\delta = \delta_1 = 0.644$ and $\delta = \delta_s = 1.713$ radians. The area between the limits of $\delta = 0.66$ and 1.713 radians, measured by planimeter is

$$a = 3.78 \text{ in } \delta . F(\delta) \text{ units.}$$

The small area in the vicinity of $\delta_1 = 0.644$ radians between the limiting values of 0.644 and 0.660 radians, by (9.14.13), is

$$a_1 = 2\left(\frac{0.660 - 0.644}{0.988 - 0.60}\right)^{\frac{1}{2}} = 0.406.$$

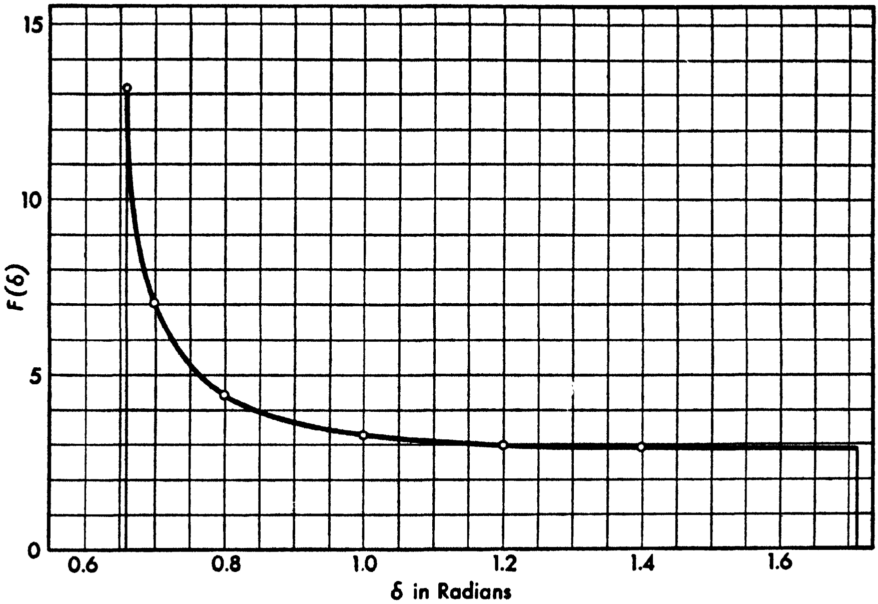


Figure 10–24

The total area equal to the integral, therefore, is

$$\sqrt{2B}t_s = 0.406 + 3.78$$
$$= 4.186 \quad \delta . F(\delta) \text{ units,}$$

where

$$B = \frac{2\pi f}{M_e} \frac{P_{m1}}{P_{rb}}$$

and

$$P_1 = 77000$$
$$P_{m1} = 128500$$
$$P_{rb} = 90000$$
$$M_e = 4.0.$$

Hence,

$$B = \frac{377}{4.0} \frac{128500}{90000}$$
$$= 134.5.$$

This gives

$$t_s = \frac{4.186}{\sqrt{2 \times 134.5}}$$
$$= 0.256 \text{ seconds}$$
$$= 15.36 \text{ cycles.}$$

(*N*) *Calculation of the Limiting Time t_s for the Tripping of Faulted Line; Load on System 65 Per Cent of Maximum Synchronizing Power.* From the curve sin δ *vs.* P_1/P_r, Fig. 10–21, obtain P_1 = 95 per cent of the rated capacity of the system, corresponding to $\sin \delta_1 = 0.65$. The angle $\delta_1 = \sin^{-1} 0.65 = 40.5° = 0.707$ radians; $\cos \delta_1 = 0.761$; and $M_3 = 0.609$. The limiting safe angle, obtained from curve, Fig. 10–22, is $\delta_s = 79.4° = 1.385$ radians. The actual load in kw obtained from curve δ *vs.* P, Fig. 10–21, is $P_1 = 83000$ kilowatts. The maximum synchronizing power, obtained from curve P_m *vs.* P, Fig. 10–21, is $P_m = 131000$ kilowatts.

Substituting the values of δ_1, $\sin \delta_1$, $\cos \delta_1$, and M_3 in the equation for $F(\delta)$ gives

$$F(\delta) = \frac{1}{[(\delta - 0.707)1.07 + \cos \delta - 0.761]^{\frac{1}{2}}}.$$

This equation, plotted between the limits of $\delta = \delta_1 = 0.707$ radians and $\delta = \delta_s = 1.385$ radians, is shown in Fig. 10–25. The area measured by planimeter between the limits of $\delta = 0.717$ and $\delta = 1.385$ radians is

$$a = 2.566 \quad \delta.F(\delta) \text{ units.}$$

The small area in the vicinity of $\delta_1 = 0.707$ radians between the limiting values of 0.707 and 0.717, by (9.14.13), is

$$a_1 = 2\left(\frac{0.717 - 0.707}{1.07 - 0.65}\right)^{\frac{1}{2}}$$
$$= 0.309 \quad \delta.F(\delta) \text{ units.}$$

The total area equal to the integral, therefore, is

$$\sqrt{2B}t_s = 0.309 + 2.566$$
$$= 2.875 \quad \delta.F(\delta) \text{ units.}$$
$$B = \frac{2\pi f}{M_e}\frac{P_{m1}}{P_{rb}}$$
$$= \frac{377}{4.0}\frac{131000}{90000}$$
$$= 137.1.$$

This gives

$$t_s = \frac{2.875}{\sqrt{2 \times 137.1}}$$
$$= 0.173 \text{ seconds}$$
$$= 7.58 \text{ cycles.}$$

From the curve in Fig. 10–20, it is found that the largest load the system could carry with a three-phase short circuit at the designated point is 53 per cent of the corresponding maximum. The displacement angle is $\delta = \sin^{-1} 0.53 = 32°$. From curve δ *vs.* P, Fig. 10–21, it is found that this load corresponds to 64000 kw.

The curve in Fig. 10–20 shows also that the maximum load the system can carry with the faulted line tripped is 75 per cent of the corresponding maximum synchronizing power. The displacement angle is $\delta = \sin^{-1} 0.75 = 48.5°$.

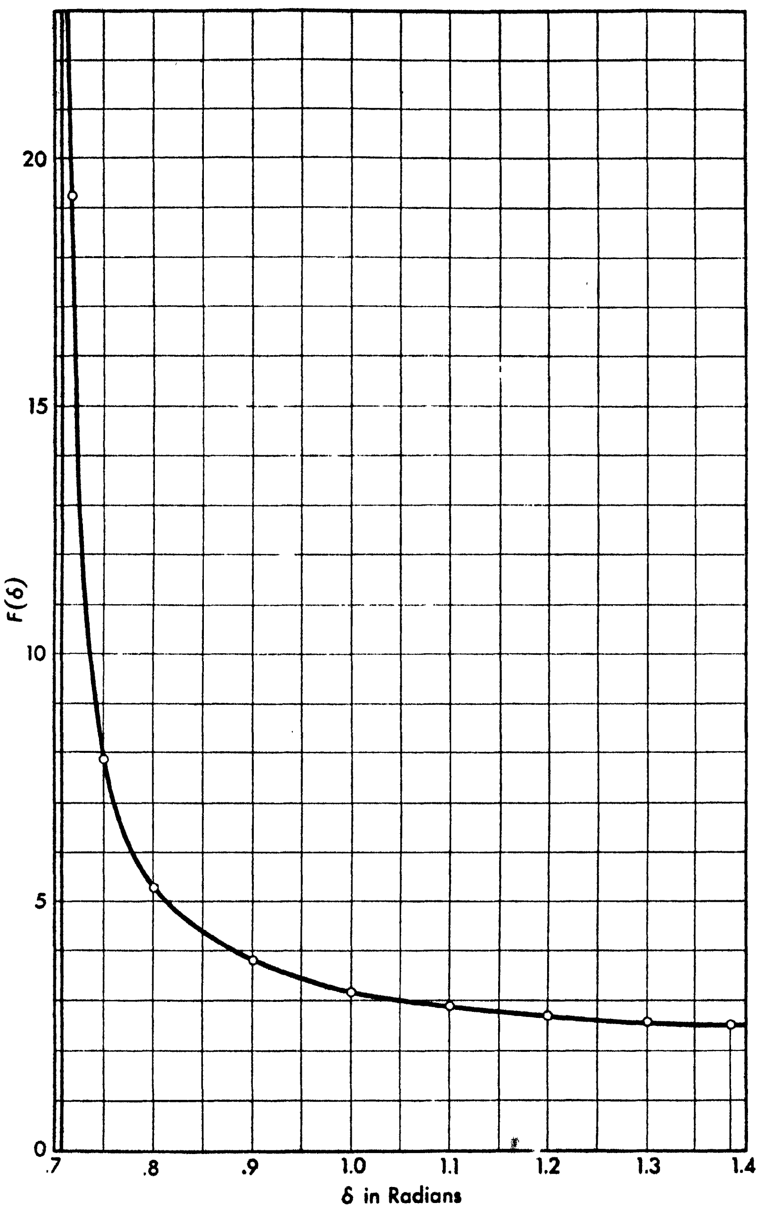


Figure 10–25

From the curve δ *vs.* P, Fig. 10–21, it is found that this corresponds to a load of 102000 kw. The curve in Fig. 10–26, plotted from the above calculated values, gives the time in cycles as a function of the load on the system, during which the line with an L–L–L fault at the designated point must be tripped out of service in order to recover the stability of operation disturbed by the fault.

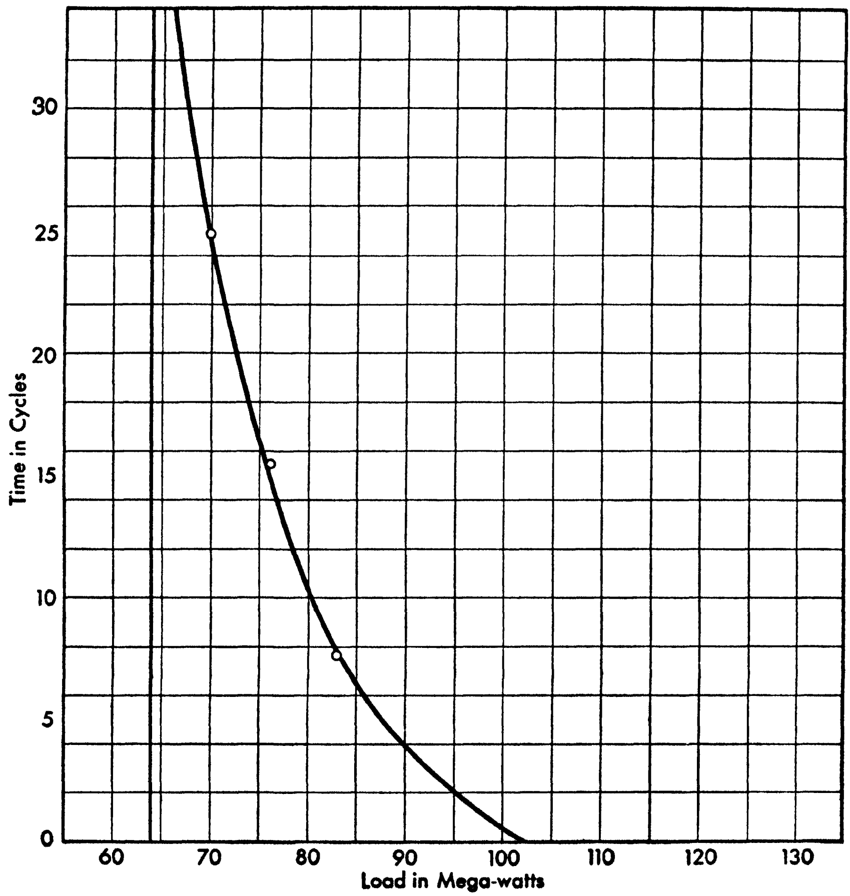


Figure 10–26

For loads larger than 102500 kw, the three-phase fault will cause loss of synchronism, unless the fault is cleared within a limited time which may be calculated in a similar manner given above in detail.

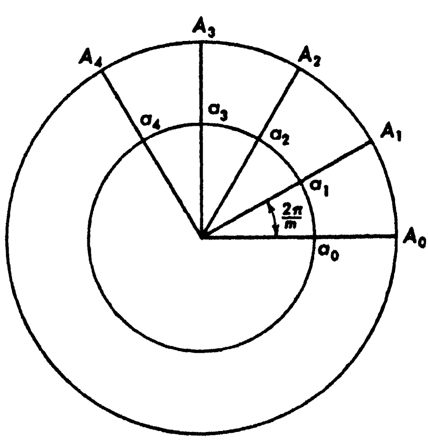
Line–line to ground, line to line and line to ground short circuits on transmission systems were discussed in preceding articles and may be investigated in the same manner as the L–L–L fault used in the illustrative problem.

SUGGESTIVE PROBLEMS *Chapter 10*

Investigate the stability of the system shown schematically in Fig. 10–11, and whose data is given in § 10.9 for a *L–L–L*, *L–L*, *L–G*, and *L–L–G* fault at any point such as:

(1) At the high side at either end of any of the two lines marked (a), or
(2) at the high-voltage side of transformer marked (16), or
(3) at the high voltage-side of either of the two transformers marked (4), or
(4) at the high-voltage side of either of the two transformers marked (8).

Appendix I The Geomean Distance between Any Number of Points Equally Spaced around a Circle of Any Radius



Consider a circle of unit radius with m points equally spaced around its circumference and marked $a_o, a_1, a_2 \ldots a_{m-1}$, as indicated in the figure.

The angular distance between the points is obviously $2\pi/m$. Let X be the radius vector to any of the m points, such as the kth.

Then

$$X_K = \epsilon^{j(2\pi/m)k} \tag{1}$$

or

$$X_K{}^m = \epsilon^{jk(2\pi)}. \tag{2}$$

Since k is a whole number, and

$$\epsilon^{jk(2\pi)} = \cos k(2\pi) + j \sin k(2\pi) = 1,$$

it follows that

$$X^m - 1 = 0. \tag{3}$$

The m roots of this expression are, referring to the above figure,

$$\left.\begin{aligned} X_o &= \epsilon^{jo} = a_o \\ X_1 &= \epsilon^{j2\pi/m} = a_1 \\ X_2 &= \epsilon^{j2(2\pi/m)} = a_1{}^2 \\ &\cdots\cdots\cdots\cdots \\ X_{m-1} &= \epsilon^{j(m-1)2\pi/m} = a_1{}^{m-1}. \end{aligned}\right\} \tag{4}$$

Expression (3) may be written, therefore,

$$X^m - 1 = (X - 1)(X - a_1)(X - a_1^2) \cdots (X - a_1^{m-1}). \tag{5}$$

This may be written

$$\frac{X^m - 1}{X - 1} = (X - a_1)(X - a_1^2)(X - a_1^3) \cdots (X - a_1^{m-1}). \tag{6}$$

By actual division the left-hand member is

$$\begin{aligned} \frac{X^m - 1}{X - 1} &= X^{m-1} + X^{m-2} + X^{m-3} + \cdots + X^{m-(m-1)} + X^{m-m} \\ &= X^{m-1} + X^{m-2} + X^{m-3} + \cdots + X + 1. \end{aligned} \tag{7}$$

It follows from (5) and (7) that

$$(X-a_1)(X-a_1^2)(X-a_1^3) \cdots (X-a_1^{m-1}) = X^{m-1} + X^{m-2} + X^{m-3} + \cdots + X + 1. \tag{8}$$

Since the magnitude of X is 1, the expression may be written

$$(1 - a_1)(1 - a_1^2)(1 - a_1^3) \cdots (1 - a_1^{m-1}) = m. \tag{9}$$

Consider now another circle of radius r, concentric with the unit circle, and assume that the radii a_o, a_1, a_2, etc., are extended to A_o, A_1, A_2, etc., respectively.

It is obvious that the magnitude of the distance A_oA_1 is the vector difference $oA_o - oA_1$, i.e.,

$$A_0A_1 = r\epsilon^{jo} - r\epsilon^{j2\pi/m},$$

which, by (4), may be written

$$A_0A_1 = r(1 - a_1). \tag{10}$$

Similarly, the magnitude of the distance A_0A_2 is the vector difference $oA_0 - oA_1$, i.e.,

$$A_0A_2 = r\epsilon^{jo} - r\epsilon^{j2(2\pi/m)}, \tag{11}$$

which, by (4), becomes

$$A_0A_2 = r(1 - a_1^2). \tag{12}$$

In the same manner, the magnitude of any distance such as A_0A_{m-1} is

$$A_0A_{m-1} = r(1 - a_1^{m-1}).$$

The product of all the distances from the point at A_0 to all the other $(m-1)$ points is

$$(A_oA_1)(A_oA_2)(A_oA_3) \cdots (A_oA_{m-1}) = r^{m-1}(1 - a_1)(1 - a_1^2)(1 - a_1^3) \cdots (1 - a_1^{m-1}).$$

By (9), this expression becomes

$$(A_oA_1)(A_oA_2)(A_oA_3) \cdots (A_oA_{m-1}) = mr^{m-1}. \tag{13}$$

This states that the products of all the distances from any one of the m points to the remaining $m - 1$ points on the circle of radius r is mr^{m-1}. There are $m - 1$ such distances. If all the distances involved are considered, there are m times as many, or a total of $m(m-1)$ distances and their product is, therefore, m times the value given by (13), i.e., mr^{m-1} raised to the mth power. The geometric mean distance, therefore, is

$$\sqrt[m(m-1)]{r^{m(m-1)}m^m} = r\sqrt[m-1]{m}. \tag{14}$$

Appendix II Evaluation of Inverse Complex Hyperbolic Functions

1. Evaluation of $a + j\beta$ when $\sinh (a + j\beta) = A/\underline{\psi}$.

From

$$\sinh (a + j\beta) = \sinh a \cos \beta + j \cosh a \sin \beta, \tag{1}$$

and

$$A/\underline{\psi} = A \cos \psi + jA \sin \psi, \tag{2}$$

it follows that

$$\sinh a \cos \beta = A \cos \psi = M_1 \tag{3}$$

and

$$\cosh a \sin \beta = A \sin \psi = M_2. \tag{4}$$

Since

$$\sinh^2 a = \cosh^2 a - 1$$

and

$$\cos^2 \beta = 1 - \sin^2 \beta,$$

equation (3) may be written

$$(\cosh^2 a - 1)(1 - \sin^2 \beta) = M_1^2$$

or

$$\cosh^2 a + \sin^2 \beta = M_1^2 + 1 + \cosh^2 a \sin^2 \beta. \tag{5}$$

Adding 2 cosh a sin β on both sides of the equality sign, gives

$$(\cosh a + \sin \beta)^2 = M_1^2 + (1 + \cosh a \sin \beta)^2,$$

which, by (4), becomes

$$\cosh a + \sin \beta = \sqrt{M_1^2 + (1 + M_2)^2}. \tag{6}$$

Subtracting $2 \cosh a \sin \beta$ from both sides of the equality sign in equation (5), similarly gives

$$(\cosh a - \sin \beta)^2 = M_1^2 + (1 - \cosh a \sin \beta)^2.$$

which, by (4), becomes

$$\cosh a - \sin \beta = \sqrt{M_1^2 + (1 - M_2)^2}. \qquad (7)$$

Subtracting equation (7) from (6) gives

$$\beta = \sin^{-1}\left(\frac{\sqrt{M_1^2+(1+M_2)^2}-\sqrt{M_1^2+(1-M_2)^2}}{2}\right). \qquad (8)$$

It is worthwhile to keep in mind that each radical in (8) may be thought of as the hypothenuse of a right triangle whose sides are M_1 and $(1 + M_2)$ for one, and M_1 and $(1 - M_2)$ for the other. As such they may be calculated conveniently by trigonometric methods.

With the value of β thus obtained by (8), calculate, by (3),

$$a = \sinh^{-1}\left(\frac{M_1}{\cos \beta}\right). \qquad (9)$$

The complex function is

$$a + j\beta = \sqrt{a^2 + \beta^2}\underline{/\tan^{-1} \beta/a}, \qquad (10)$$

where the value of β must be in radian measure.

To illustrate the above, assume

$$\sinh (a + j\beta) = 1.065\underline{/47.57^\circ}$$
$$= 0.7185 + j\,0.7861,$$

hence,

$$M_1 = 0.7185$$
$$M_2 = 0.7861.$$

By (8),

$$\beta = \sin^{-1}\left(\frac{\sqrt{0.7185^2+1.7861^2}-\sqrt{0.7185^2+0.2139^2}}{2}\right)$$
$$= 36.5^\circ = 0.638 \text{ radians.}$$

By (9),

$$a = \sinh^{-1} \frac{0.7185}{\cos 36.5}$$
$$a = 0.805,$$

whence

$$a + j\beta = 0.805 + j0.638 = 1.038\underline{/38.4^\circ}.$$

2. Evaluation of $a + j\beta$ when $\cosh (a + j\beta) = B\underline{/\phi}$.

From

$$\cosh (a + j\beta) = \cosh a \cos \beta + j \sinh a \sin \beta \qquad (11)$$

and

$$B\underline{/\phi} = B \cos \phi + jB \sin \phi \qquad (12)$$

it follows that

$$\cosh a \cos \beta = B \cos \phi = M_3 \qquad (13)$$
$$\sinh a \sin \beta = B \sin \phi = M_4. \qquad (14)$$

Since

$$\sinh^2 a = \cosh^2 a - 1$$

and

$$\sin^2 \beta = 1 - \cos^2 \beta,$$

equation (14) may be written

$$(\cosh^2 a - 1)(1 - \cos^2 \beta) = M_4^2$$

or

$$\cosh^2 a + \cos^2 \beta = M_4^2 + 1 + \cosh^2 a \cos^2 \beta. \tag{15}$$

Adding $2 \cosh a \cos \beta$ on both sides of the equality sign, gives

$$(\cosh a + \cos \beta)^2 = M_4^2 + (1 + \cosh a \cos \beta)^2.$$

This, by (13), may be written

$$\cosh a + \cos \beta = \sqrt{M_4^2 + (1 + M_3)^2}. \tag{16}$$

Subtracting $2 \cosh a \cos \beta$ on both sides of the equality sign of (15), gives, similarly,

$$\cosh a - \cos \beta = M_4^2 + (1 - M_3)^2. \tag{17}$$

Subtracting (17) from (16) gives

$$\beta = \cos^{-1}\left(\frac{\sqrt{M_4^2+(1+M_3)^2} - \sqrt{M_4^2+(1-M_3)^2}}{2}\right). \tag{18}$$

Note the similarity between this and expression (8).

With the value of β obtained by (18), calculate, by (14),

$$a = \sinh^{-1}\left(\frac{M_4}{\sin \beta}\right). \tag{19}$$

The complex function is calculated by equation (10).

To illustrate the above, let

$$\cosh(a + j\beta) = 1.2\underline{/25.75^\circ}$$
$$= 1.081 + j0.522,$$

hence,

$$M_3 = 1.081$$
$$M_4 = 0.522.$$

By (18),

$$\beta = \cos^{-1}\left[\frac{\sqrt{0.522^2 + 2.081^2} - \sqrt{0.522^2 + 0.081^2}}{2}\right]$$
$$= 35.6^\circ.$$

By (19),

$$a = \sinh^{-1} \frac{0.522}{\sin 35.6}$$
$$a = 0.809.$$

3. Evaluation of $a + j\beta$ when

$$\tanh(a + j\beta) = D\underline{/\delta} \tag{20}$$
$$= D \cos \delta + jD \sin \delta = N_1 + jN_2. \tag{21}$$

Since

$$\tanh (a + j\beta) = \frac{\sinh (a + j\beta)}{\cosh (a + j\beta)}, \tag{22}$$

it follows by (1), (3), and (4), and also by (11), (13), and (14) that

$$\tanh (a + j\beta) = \frac{M_1 + jM_2}{M_3 + jM_4},$$

which may be written

$$\begin{aligned}\tanh (a + j\beta) &= \frac{(M_1 + jM_2)(M_3 - jM_4)}{(M_3 + jM_4)(M_3 - jM_4)} \\ &= \frac{M_1M_3 + M_2M_4}{M_3^2 + M_4^2} + j\,\frac{M_2M_3 - M_1M_4}{M_3^2 + M_4^2},\end{aligned} \tag{23}$$

where, by (3) and (13), (4) and (14), (4) and (13) and (3) and (14)

$$\begin{aligned}M_1M_3 &= \sinh a \cosh a \cos^2 \beta \\ M_2M_4 &= \sinh a \cosh a \sin^2 \beta \\ M_2M_3 &= \sin \beta \cos \beta \cosh^2 a \\ M_1M_4 &= \sin \beta \cos \beta \sinh^2 a.\end{aligned}$$

Hence,

$$M_1M_3 + M_2M_4 = \sinh a \cosh a \tag{24}$$

$$M_2M_3 - M_1M_4 = \sin \beta \cos \beta. \tag{25}$$

Also

$$\begin{aligned}M_3^2 + M_4^2 &= \cosh^2 a \cos^2 \beta + \sinh^2 a \sin^2 \beta \\ &= (1 + \sinh^2 a) \cos^2 \beta + \sinh^2 a(1 - \cos^2 \beta) \\ &= \sinh^2 a + \cos^2 \beta.\end{aligned} \tag{26}$$

Substituting (24), (25), and (26) in (23) gives

$$\tanh (a + j\beta) = \frac{\sinh a \cosh a}{\sinh^2 a + \cos^2 \beta} + j\,\frac{\sin \beta \cos \beta}{\sinh^2 a + \cos^2 \beta}. \tag{27}$$

Since

$$\sinh a \cosh a = \tfrac{1}{2} \sinh 2a$$

and

$$\sin \beta \cos \beta = \tfrac{1}{2} \sin 2\beta,$$

the above equation becomes

$$\tanh (a + j\beta) = \frac{1}{2}\left[\frac{\sinh 2a}{\sinh^2 a + \cos^2 \beta} + j\,\frac{\sin 2\beta}{\sinh^2 a + \cos^2 \beta}\right]. \tag{28}$$

Referring this to (21), it is seen that

$$\frac{\sinh 2a}{\sinh^2 a + \cos^2 \beta} = 2N_1 \tag{29}$$

and

$$\frac{\sin 2\beta}{\sinh^2 a + \cos^2 \beta} = 2N_2. \tag{30}$$

Note that

$$\sqrt{\sinh^2 a + \cos^2 \beta} = \cosh (a + j\beta)$$

and

$$\sqrt{\sinh^2 a + \sin^2 \beta} = \sinh (a + j\beta).$$

It follows, therefore, by (22) and (20), that

$$\frac{\sinh^2 a + \sin^2 \beta}{\sinh^2 a + \cos^2 \beta} = D^2 \tag{31}$$

or

$$1 + \frac{\sinh^2 a + \sin^2 \beta}{\sinh^2 a + \cos^2 \beta} = 1 + D^2$$

or

$$\frac{2 \sinh^2 a + 1}{\sinh^2 a + \cos^2 \beta} = 1 + D^2$$

or

$$\frac{\cosh 2a}{\sinh^2 a + \cos^2 \beta} = 1 + D^2. \tag{32}$$

Dividing (29) by (32) gives

$$\tanh 2a = \frac{2N_1}{1 + D^2}. \tag{33}$$

Equation (31) may also be written

$$1 - \frac{\sinh^2 a + \sin^2 \beta}{\sinh^2 a + \cos^2 \beta} = 1 - D^2$$

or

$$\frac{\cos 2\beta}{\sinh^2 a + \cos^2 \beta} = 1 - D^2. \tag{34}$$

Dividing (30) by (34) gives

$$\tan 2\beta = \frac{2N_2}{1 - D^2}. \tag{35}$$

By (33) it follows that

$$a = \tfrac{1}{2} \tanh^{-1}\left(\frac{2N_1}{1 + D^2}\right) \tag{36}$$

and, by (35),

$$\beta = \tfrac{1}{2} \tan^{-1}\left(\frac{2N_2}{1 - D^2}\right). \tag{37}$$

To illustrate the above, assume

$$\begin{aligned}
\tanh (a + j\beta) &= 0.5\underline{/30^\circ} \\
&= 0.4335 + j0.25 \\
D &= 0.5 \\
N_1 &= 0.4335 \\
N_2 &= 0.25 \\
a &= \tfrac{1}{2} \tanh^{-1}\left(\frac{2 \times 0.4335}{1 + 0.25}\right) \\
a &= 0.4275
\end{aligned}$$

and

$$\begin{aligned}
\beta &= \tfrac{1}{2} \tan^{-1}\left(\frac{2 \times 0.25}{1 - 0.25}\right) \\
&= 16.85^\circ.
\end{aligned}$$

Appendix III The Ferranti Effect

The voltage at the open end of a two-wire line is given by equation (2.8.6) and is

$$V_{ro} = \frac{V_s}{\cosh pS}. \tag{1}$$

The numerical value of this voltage is

$$V_{ro} = \frac{V_s}{(\sinh^2 aS + \cos^2 \beta S)^{\frac{1}{2}}}, \tag{2}$$

where a is the attenuation constant and β the phase constant for the particular frequency at which the line is operated, and S is the length of the line.

The condition which the line must satisfy, that $V_{ro} > V_s$, generally referred to as the Ferranti effect, may be obtained by differentiating equation (2) with reference to S.

This gives

$$\begin{aligned} \frac{dV_{ro}}{dS} &= -\frac{V_s(2a \sinh aS \cosh aS - 2\beta \sin \beta S \cos \beta S)}{2(\sinh^2 aS + \cos^2 \beta S)^{\frac{3}{2}}} \\ &= \frac{V_s(\beta \sin 2\beta S - a \sinh 2aS)}{2(\sinh^2 aS + \cos^2 \beta S)^{\frac{3}{2}}}. \end{aligned} \tag{3}$$

That V_{ro} shall be greater than V_s the slope dV_{ro}/dS of the curve V_{ro} *vs.* S as given by equation (2) must be positive, and it is positive by equation (3) only when

$$\beta \sin 2\beta S > a \sinh 2aS. \tag{4}$$

However, since each member of this relation is a function of S, it is necessary to determine the slope of each as a function of S as S approaches zero. Thus,

$$\frac{d(\beta \sin 2\beta S)}{dS} = 2\beta^2 \cos 2\beta S = 2\beta^2$$

as S approaches zero, and

$$\frac{d(a \sinh 2aS)}{dS} = 2a^2 \cosh 2aS = 2a^2$$

as S approaches zero.

It follows from the above that dV_{ro}/dS is positive and hence V_{ro} increases with S when

$$\beta > a. \tag{5}$$

This shows that a transmission line is subject to the Ferranti effect at any particular frequency when the phase constant is larger than the attenuation constant for that frequency, i.e., when the angle of the propagation constant $\sqrt{zy}$ is larger than 45°.

The particular line length for which the Ferranti effect is a maximum may be obtained by setting equation (3)

$$\frac{dV_{ro}}{dS} = 0. \tag{6}$$

The solution shows that the maximum Ferranti effect occurs for the particular value of S for which

$$a \sinh 2aS = \beta \sin 2\beta S. \tag{7}$$

The intersection of the curves $(a \sinh 2aS)$ *vs.* S and $(\beta \sin 2\beta S)$ *vs.* S gives the value S for the maximum Ferranti effect.

The solutions show that only under ideal conditions when $a = 0$, does the Ferranti effect occur at the open end of a quarter-wave length line. For actual lines, the maximum Ferranti effect will occur when the line is shorter than quarter-wave length by an amount which depends upon the relative values of a and β. A quarter-wave length line at 60 cps would be a line about 750 miles long. None of the present transmission lines is that long.

Index